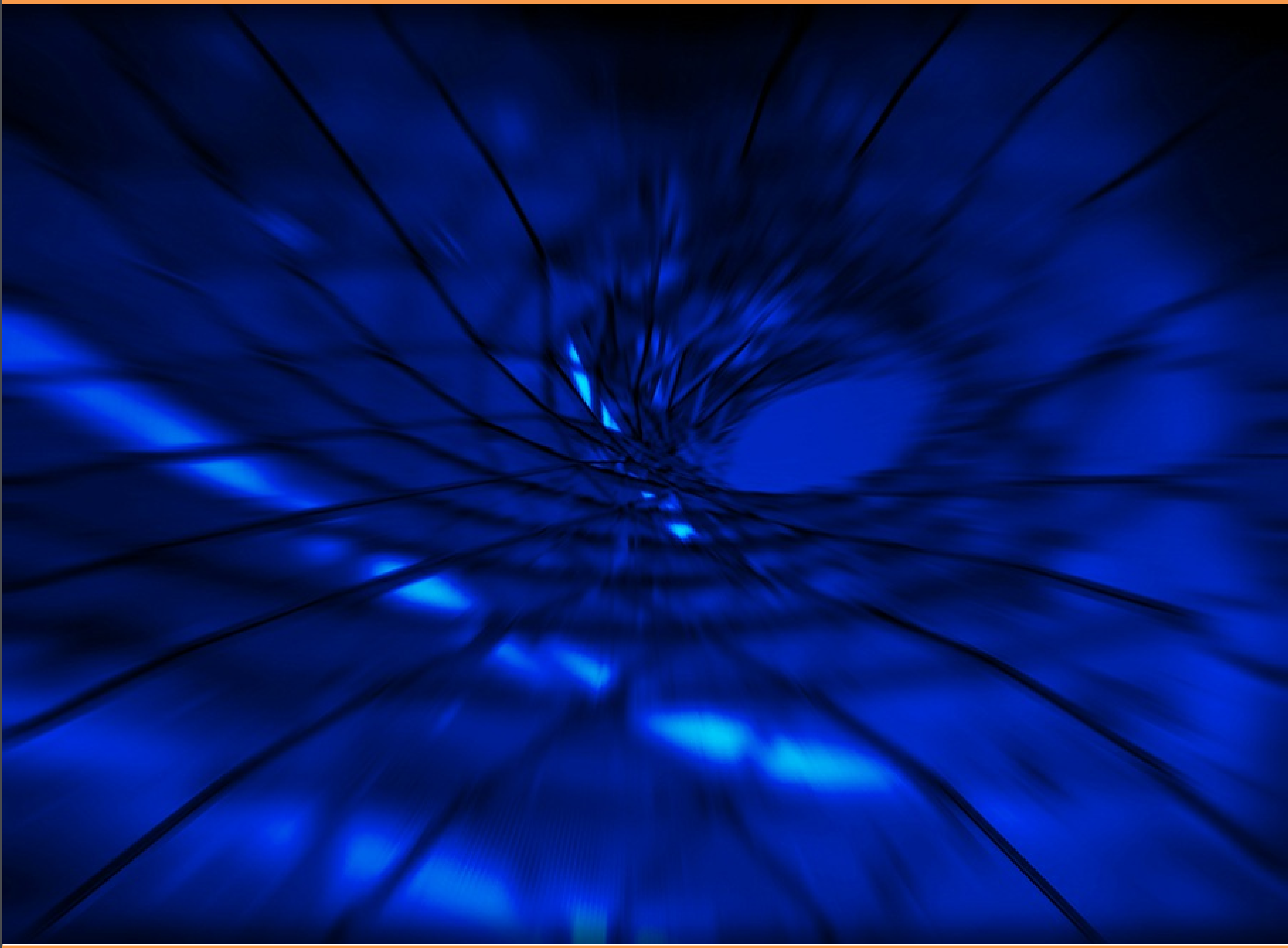
Topological and Metric Spaces, Banach Spaces...

...and Bounded Operators - Functional Analysis Examples c-2

Leif Mejlbro



Leif Mejlbro

Topological and Metric Spaces, Banach Spaces and Bounded Operators

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Contents

Introduction

This is the second volume containing examples from Functional analysis. The topics here are limited to *Topological and metric spaces, Banach spaces* and *Bounded operators*.

Unfortunately errors cannot be avoided in a first edition of a work of this type. However, the author has tried to put them on a minimum, hoping that the reader will meet with sympathy the errors which do occur in the text.

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1 Topological and metric spaces

1.1 Weierstraß's approximation theorem

Example 1.1 *Let* $\varphi \in C^1([0,1])$*. It follows from Weierstraß's approximation theorem that* $B_{n,\varphi}(\theta)$ *converges uniformly towards* $\varphi(\theta)$ *and that* $B_{n,\varphi'}(\theta)$ *converges uniformly towards* $\varphi'(\theta)$ *on* $[0,1]$*. Prove that* $B'_{n,\varphi}(\theta) \to \varphi'(\theta)$ *uniformly on* $[0,1]$*.*
HINT: *First prove that* $B'_{n,\varphi}(\theta) - B_{n-1,\varphi'}(\theta)$ *converges uniformly towards 0 on* $[0,1]$*.*
Next prove that if $\varphi \in C^\infty([0,1])$*, then we have for every* $k \in \mathbb{N}$ *that* $B^{(n)}_{n,\varphi}(\theta) \to \varphi^{(k)}(\theta)$ *uniformly on* $[0,1]$*.*

NOTATION. We use here the notation

$$B_{n,\varphi}(\theta) = \sum_{k=0}^{n} \varphi\left(\frac{k}{n}\right) \cdot \binom{n}{k} \cdot \theta^k (1-\theta)^{n-k}$$

for the so-called *Bernstein polynomials.* ◊

First write

$$B'_{n\varphi}(\theta) - B_{n-1,\varphi'}(\theta) = \sum_{k=0}^{n} \varphi\left(\frac{k}{n}\right) \cdot \binom{n}{k} \cdot \frac{d}{d\theta}\left\{\theta^k(1-\theta)^{n-k}\right\} - \sum_{k=0}^{n-1} \varphi'\left(\frac{k}{n-1}\right) \cdot \binom{n-1}{k} \cdot \theta^k(1-\theta)^{n-1-k}.$$

Here

$$\frac{d}{d\theta}\{\theta^k(1-\theta)^{n-k}\} = \begin{cases} n\theta^{n-1}, & \text{for } k=n, \\ k\cdot\theta^{k-1}(1-\theta)^{n-k}-(n-k)\theta^k(1-\theta)^{n-1-k}, & \text{for } 0<k<n, \\ -n(1-\theta)^{n-1}, & \text{for } k=0. \end{cases}$$

For $0<k<n$ we perform the calculation

$$\begin{aligned} \binom{n}{k}\frac{d}{d\theta}\left\{\theta^k(1-\theta)^{n-k}\right\} &= \frac{n!}{k!(n-k)!}\left\{k\,\theta^{k-1}(1-\theta)^{n-k}-(n-k)\theta^k(1-\theta)^{n-1-k}\right\} \\ &= \frac{n!}{(k-1)!(n-k)!}\,\theta^{k-1}(1-\theta)^{n-k}-\frac{n!}{k!(n-k-1)!}\,\theta^k(1-\theta)^{n-1-k} \\ &= n\binom{n-1}{k-1}\theta^{k-1}(1-\theta)^{n-k}-n\binom{n-1}{k}\theta^k(1-\theta)^{n-1-k}. \end{aligned}$$

Hence

$$\begin{aligned} B'_{n,\varphi}(\theta) &= \sum_{k=0}^{n}\varphi\left(\frac{k}{n}\right)\cdot\binom{n}{k}\cdot\frac{d}{d\theta}\left\{\theta^k(1-\theta)^{n-k}\right\} \\ &= \varphi(0)\cdot\left\{-n(1-\theta)^{n-1}\right\}+\varphi(1)\cdot n\theta^{n-1}+n\sum_{k=1}^{n-1}\varphi\left(\frac{k}{n}\right)\cdot\binom{n-1}{k-1}\theta^{k-1}(1-\theta)^{n-k} \\ &\qquad -n\sum_{k=1}^{n-1}\varphi\left(\frac{k}{n}\right)\cdot\binom{n-1}{k}\cdot\theta^k(1-\theta)^{n-1-k} \\ &= n\left\{\varphi(1)\cdot\theta^{n-1}-\varphi(0)\cdot(1-\theta)^{n-1}\right\}+n\sum_{k=0}^{n-2}\varphi\left(\frac{k+1}{n}\right)\cdot\binom{n-1}{k}\cdot\theta^k(1-\theta)^{n-1-k} \\ &\qquad -n\sum_{k=1}^{n-1}\varphi\left(\frac{k}{n}\right)\cdot\binom{n-1}{k}\cdot\theta^k(1-\theta)^{n-1-k} \\ &= n\sum_{k=0}^{n-1}\left\{\varphi\left(\frac{k+1}{n}\right)-\varphi\left(\frac{k}{n}\right)\right\}\cdot\binom{n-1}{k}\cdot\theta^k(1-\theta)^{n-1-k} \\ &= \sum_{k=0}^{n-1}\frac{\varphi(\frac{k+1}{n})-\varphi(\frac{k}{n})}{\frac{1}{n}}\cdot\binom{n-1}{k}\cdot\theta^k(1-\theta)^{n-1-k}. \end{aligned}$$

Whence by insertion,

$$B'_{n,\varphi}(\theta)-B_{n-.1,\varphi'}(\theta)=\sum_{k=0}^{n-1}\left\{\frac{\varphi(\frac{k+1}{n})-\varphi(\frac{k}{m})}{\frac{1}{n}}-\varphi'\left(\frac{k}{n-1}\right)\right\}\cdot\binom{n-1}{k}\cdot\theta^k(1-\theta)^{n-1-k}.$$

We have assumed from the beginning that $\varphi\in C^1([0,1])$, thus

$$\frac{\varphi(\frac{k+1}{n})-\varphi(\frac{k}{n})}{\frac{1}{n}}-\varphi'\left(\frac{k}{n-1}\right)=\frac{1}{n}\,\varepsilon\left(\frac{1}{n}\right)$$

uniformly, so the remainder term is estimated uniformly independently of k. In fact, it follows from the Mean Value Theorem that

$$\frac{\varphi(\frac{k+1}{n}) - \varphi(\frac{k}{n})}{\frac{1}{n}} = \varphi'(\xi), \qquad \text{for et passende } \xi \in \left]\frac{k}{n}\;\frac{k+1}{n}\right[,$$

and as $\dfrac{k}{n} - \dfrac{k}{n-1} = -\dfrac{k}{n(n-1)}$, we get

$$\left|\frac{k}{n} - \frac{k}{n-1}\right| \le \frac{1}{n-1},$$

and since φ' is continuous,

$$\varphi'\left(\frac{k}{n}\right) - \varphi'\left(\frac{k}{n-1}\right) \to 0 \qquad \text{ligeligt.}$$

From this follows precisely that

$$\frac{\varphi(\frac{k+1}{n}) - \varphi(\frac{k}{n})}{\frac{1}{n}} - \varphi'\left(\frac{k}{n-1}\right) = \varphi'\left(\frac{k}{n}\right) - \varphi'\left(\frac{k}{n-1}\right) 0 \frac{1}{n}\,\varepsilon\left(\frac{1}{n}\right)$$

uniformly, and the claim is proved.

Finally, we get by induction that if $\varphi \in C^k([0,1])$, then $B_{n,\varphi}^{(k)}(\theta) \to \varphi^{(k)}(\theta)$ uniformly on $[0,1]$.

Example 1.2 *Let φ be a real continuous function defined for $x \ge 0$, and assume that $\lim_{x\to\infty}\varphi(x)$ exists (and is finite). Show that for $\varepsilon > 0$ there are $n \in \mathbb{N}$ and constants a_k, $k = 0$, 1, ..., n, such that*

$$\left|\varphi(x) - \sum_{k=0}^{n} a_k\, e^{-kx}\right| \le \varepsilon$$

for all $x \ge 0$.

First note that the range of e^{-x}, $x \in [0,\infty[$, is $]0,1]$, so we have $t = e^{-x} \in]0,1]$, thus $x = \ln\dfrac{1}{t}$. The function $\psi(t)$, given by

$$\psi(t) = \begin{cases} \varphi\left(\ln\dfrac{1}{t}\right) & \text{for } t \in]0,1], \\[2mm] \lim_{x\to\infty}\varphi(x) & \text{for } t = 0, \end{cases}$$

is continuous for $t \in [0,1]$. It follows from Weierstraß's approximation theorem that there exists a polynomial $\sum_{k=0}^{n} a_k t^k$, such that

$$\left|\psi(t) - \sum_{k=0}^{n} a_k t^k\right| \le \varepsilon \qquad \text{for alle } t \in [0,1].$$

Since $\varphi(x) = \psi\left(e^{-x}\right)$ for $x \in [0,+\infty[$, we conclude that

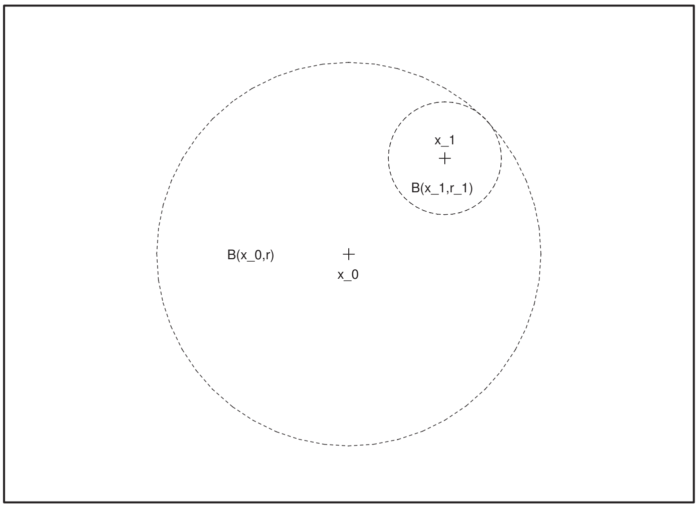
$$\left|\varphi(x) - \sum_{k=0}^{n} a_k e^{-kx}\right| \le \varepsilon \qquad \text{for every } x \in [0,+\infty[.$$

1.2 Topological and metric spaces

Example 1.3 *Let (M, d) be a metric space.*
We define the open ball *with centre x_0 and radius $r > 0$ by*

$$B(x_0, r) = \{x \in M \mid d(x, x_0) < r\}.$$

We denote a subset $A \subset M$ open*, if there for any $x_0 \in A$ is an open ball with centre x_0 contained in A.*
Show that an open ball is an open set.
Show that the open sets defined in this way is a topology on M.



Let $x_1 \in B(x_0, r)$, i.e. $d(x_0, x_1) < r$. Choose

$$r_1 = r - d(x_0, x_1) > 0.$$

We claim that

$$B(x_1, r_r) \subseteq B(x_0, r).$$

If $x \in B(x_1, r_1)$, then

$$d(x_1, x) < r_1 = r - d(x_0, x_1),$$

and it follows by the triangle inequality that

$$d(x_0, x) \le d(x_0, x_1) + d(x_1, x) < d(x_0, x_1) + r - d(x_0, x_1) = r,$$

proving that $x \in B(x_0, r)$. This holds for every $x \in B(x_1, r_1)$, so we have proved with the chosen radius r_1 that

$$B(x_1, r_1) \subseteq B(x_0, r),$$

hence every *open ball* is in fact an open set.

Then we shall prove that the system $\mathcal{T}$ generated by all open balls is a topology. Thus a set $T \in \mathcal{T}$ is characterized by the property that for every $x \in T$ there exists an $r > 0$, such that $B(x, r) \subseteq T$.

1) It is trivial that M itself is an open set.
   That $\emptyset$ is open follows from the formal definition:

   $$\forall\, x_0 \in \emptyset\, \exists\, r \in \mathbb{R}_+ : B(x_0, r) \subseteqq \emptyset.$$

   Since there is no point in $\emptyset$, the condition is trivially fulfilled.

2) Let $T = \bigcup_{j \in J} T_j$, where all $T_j \in \mathcal{T}$. If $x_0 \in T$, then there exists a $j \in J$, such that $x_0 \in T_j$. Since $T \in \mathcal{T}$, there exists an $r \in \mathbb{R}_+$, such that

   $$B(x_0, r) \subseteqq T_j \subseteqq T,$$

   thus $T \in \mathcal{T}$.

3) Let $T = \bigcap_{j=1}^{n} T_j$, where all $T_j \in \mathcal{T}$. If $T = \emptyset$, there is nothing to prove. Therefore, let $x_0 \in T$. Then x_0 must lie in every $T_j \in \mathcal{T}$, $j = 1, \dots, n$, so there are constants $r_j \in \mathbb{R}_+$, $j = 1, \dots, n$, such that $B(x_0, r_j) \subseteqq T_j$. Now put $t = \min r_j \in \mathbb{R}_+$ (notice that there is only a finite number of $r_j > 0$). Then

$$B(x_0, r) \subseteqq B(x_0, r_j) \subseteqq T_j \qquad \text{for every } j = 1, \dots, n,$$

and hence also in the intersection,

$$B(x_0, r) \subseteqq \bigcap_{j=1}^{n} T_j = T.$$

Using the definition of $\mathcal{T}$ this means that $T \in \mathcal{T}$.

We have proved that $\mathcal{T}$ is a topology.

Example 1.4 *Let (M, d) be a metric space. We say that a mapping $T : M \to M$ is* continuous *in $x_0 \in M$ if, for any $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x \in M$ we have*

$$d(x_0, x) < \delta \quad \Longrightarrow \quad d(Tx_0, Tx) < \varepsilon.$$

Show the T is continuous in x_0 if and only if

$$x_n \to x_0 \quad \Longrightarrow \quad Tx_n \to Tx_0.$$

Show that T is continuous if the open sets are defined as in Example 1.3.

Recall that $x_n \to x_0$ means that

(1) $\forall\, \delta \in \mathbb{R}_+ \, \exists\, n_0 \in \mathbb{N} \, \forall\, n \geq n_0 : d(x_n, x_0) < \delta.$

Assume that T is continuous in $x_0 \in M$ and that $x_n \to x_0$. We shall prove that $Tx_n \to Tx_0$, i.e.

$$\forall\, \varepsilon \in \mathbb{R}_+ \, \exists\, n_0 \in \mathbb{N} \, \forall\, n \geq n_0 : d(Tx_n, Tx_0) < \varepsilon.$$

Let $\varepsilon \in \mathbb{R}_+$ be arbitrary. Since T is continuous, we can find to this $\varepsilon > 0$ a constant $\delta = \delta(\varepsilon) \in \mathbb{R}_+$, such that

(2) $\forall\, x \in M : d(x_0, x) < \delta \quad \Longrightarrow \quad d(Tx_0, Tx) < \varepsilon.$

Using that $x_n \to x_0$, we get by (1) an $n_0 \in \mathbb{N}$ corresponding to $\delta = \delta(\varepsilon)$ [in fact an $n_0 \in \mathbb{N}$ corresponding to $\varepsilon \in \mathbb{R}_+$], such that

$$\forall\, n \geq n_0 : d(x_n, x_0) < \delta = \delta(\varepsilon).$$

It follows from the continuity condition (2) that $d(Tx_0, Tx_n) < \varepsilon$ for $n \geq n_0$, hence

$$\forall\, \varepsilon \in \mathbb{R}_+ \, \exists\, n_0 \in \mathbb{N} \, \forall\, n \geq n_0 : d(Tx_n, Tx_0) < \varepsilon,$$

and we have proved that if T is continuous in $x_0 \in M$, then

$$x_n \to x_0 \quad \Longrightarrow \quad Tx_n \to Tx_0.$$

Then assume that T is *not* continuous at $x_0 \in M$, thus

(3) $\exists\, \varepsilon \in \mathbb{R}_+ \,\forall\, \delta \in \mathbb{R}_+ \,\exists\, x \in M : d(x_0, x) < \delta \,\wedge\, d(Tx_0, Tx) \geq \varepsilon.$

We shall prove that there exists a sequence (x_n), such that $x_n \to x_0$, while Tx_n does not converge towards Tx_0.

Choose $\varepsilon > 0$ as in (3). Putting $\delta = \dfrac{1}{n}$ we get

$$\forall\, n \in \mathbb{N} \,\exists\, x_n \in M : d(x_0, x_n) < \frac{1}{n} \wedge d(Tx_0, Tx_n) \geq \varepsilon.$$

Then it follows that $x_n \to x_0$ and Tx_n cannot be arbitrarily close to Tx_0, thus (Tx_n) does not converge towards Tx_0.

Assume that $T^{\circ -1}(A)$ is open for every open set A. Choose $x_0 \in M$ and $A = B(Tx_0, \varepsilon)$. Then A is open, so $T^{\circ -1}(A)$ is open according to the assumption. It follows from $x_0 \in T^{\circ -1}(A)$ that there is a $\delta \in \mathbb{R}_+$, such that

$$B(x_0, \delta) \subseteqq T^{\circ -1}(A).$$

For every $x_0 \in B(x_0, \delta)$, thus $d(x, x_0) < \delta$, we get $Tx \in B(Tx_0, \varepsilon)$, hence $d(Tx, Tx_0) < \varepsilon$, and we have proved that T is continuous.

Conversely, assume that T is continuous, and let A be an open set, thus

$$\forall\, x_0 \in A \,\exists\, r \in \mathbb{R}_+ : d(x_0, x) < r \quad \Longrightarrow \quad x \in A.$$

We shall prove that $T^{\circ -1}(A)$ is open, i.e.

$$\forall y_0 \in T^{\circ -1}(A) \,\exists\, R \in \mathbb{R}_+ : B(y_0, R) \subseteqq T^{\circ -1}(A).$$

This is done INDIRECTLY. *Assume*m that

$$\exists\, y_0 \in T^{\circ -1}(A) \,\forall\, R \in \mathbb{R}_+ : B(y_0, R) \setminus T^{\circ -1}(A) \neq \emptyset,$$

thus

$$\exists\, y_0 \in T^{\circ -1}(A) \,\forall\, R \in \mathbb{R}_+ \,\exists\, y \notin T^{\circ -1}(A) : d(y_0, y) < R.$$

Since T is continuous at y_0, it follows that

$$\forall r \in \mathbb{R}_+ \exists R \in \mathbb{R}_+ \forall y \in M : d(y_0, y) < R \Longrightarrow d(Ty_0, Ty) = d(x_0, Ty) < r.$$

We conclude that $Ty \in A$ contradicting that $y \notin T^{\circ -1}(A)$, and the claim is proved.

Example 1.5 *In a set M is given a function d' from $M \times M$ to $\mathbb{R}$ that satisfies*

$$d'(x,y) = 0 \qquad \text{if and only if} \qquad x = y,$$

$$d'(x,y) \le d'(z,x) + d'(z,y) \qquad \text{for all } x,\, y,\, z \in M.$$

Show that (M, d') is a metric space.

If we choose $z = y$ in the latter assumption and then use the former one, we get

$$d'(x,y) \le d'(y,x) + d'(y,y) = d'(y,x) + 0 = d'(y,x),$$

proving that

$$d'(x,y) \le d'(y,x) \qquad \text{for all } x,\, y \in M.$$

By interchanging x and y we obtain the opposite inequality, $d'(y,x) \le d'(x,y)$, hence

$$d'(x,y) = d'(y,x) \qquad \text{for all } x,\, y \in M,$$

and d' is symmetric.
Using this result on the latter assumption we get the triangle inequality

$$d'(x,y) \le d'(x,z) + d'(z,y).$$

It only remains to prove that $d'(x,y) \ge 0$ for all x, $y \in M$ in order to conclude that d' is a metric. This follows from

$$0 = d'(x,x) \le d'(x,y) + d'(y,x) = 2d'(x,y),$$

so the two conditions of the example suffice for d' being a metric.

Example 1.6 *Let (M, d) be a metric space.*
The diameter *of a non-empty subset A of M is defined as*

$$\delta(A) = \sup_{x,\, y \in A} d(x,y) \qquad (\le \infty).$$

Show that $\delta(A) = 0$ if and only if A contains only one point.

If $A = \{x\}$ only contains one point, then

$$\delta(A) = \sup_{x,\, y \in A} d(x,y) = d(x,x) = 0.$$

If A contains at least two points, choose x, $y \in A$, where $x \ne y$, from which we conclude that

$$\delta(A) = \sup_{t,\, z \in A} d(t,z) \ge d(x,y) > 0,$$

and the claim is proved.

Example 1.7 *Let (M, d) be a metric space. Show that d_1 given by*

$$d_1(x,y) = \frac{d(x,y)}{1+d(x,y)} \qquad \text{for } x,\, y \in M$$

is a metric on M.
Show that

$$\delta_1(A) = \sup_{x,\, y \in A} d_1(x,y) \le 1$$

for all $A \subset M$.
Is it possible to find a subset A with $\delta_1(A) = 1$?
Show that $d_1(x_n, x) \to 0$ if and only if $d(x_n, x) \to 0$.

1) We shall first prove that

$$d_1(x,y) = \frac{d(x,y)}{1+d(x,y)}, \qquad x,\, y \in M,$$

is a metric.

a) It is trivial that $d_1(x,y) \geq 0$, because $d(x,y) \geq 0$.

b) Then we see that $d_1(x,y) = 0$, if and only if the numerator $d(x,y) = 0$, i.e. if and only if $x = y$.

c) The condition $d_1(x,y) = d_1(y,x)$ follows immediately from $d(x,y) = d(y,x)$.

d) It remains only to prove the triangle inequality

$$d_1(x,y) \leq d_1(x,z) + d_1(z,y).$$

Now $d(x,y) \leq d(x,z) + d(z,y)$, and the function

$$f(t) = \frac{t}{1+t} = 1 - \frac{1}{1+t}, \qquad t \geq 0,$$

is increasing. Hence

$$\begin{aligned} d_1(x,y) &= \frac{d(x,y)}{1+d(x,y)} = f(d(x,y)) \\ &\leq f(d(x,z)+d(z,y)) = \frac{d(x,z)+d(z,y)}{1+d(x,z)+d(z,y)} \\ &= \frac{d(x,z)}{1+d(x,z)+d(z,y)} + \frac{d(z,y)}{1+d(x,z)+d(z,y)} \\ &\leq \frac{d(x,z)}{1+d(x,z)} + \frac{d(z,y)}{1+d(z,y)} \\ &= d_1(x,z) + d_1(z,y). \end{aligned}$$

Summing up, we have proved that $d_1(x,y)$ is a metric on M.

2) It follows from

$$d_1(x,y) = \frac{d(x,y)}{1+d(x,y)} = 1 - \frac{1}{1+d(x,y)} \leq 1,$$

that

$$\delta_1(A) = \sup_{x,\,y \in A} d_1(x,y) \leq 1$$

for every subset A.

3) a) If the metric d is not bounded on M, then there are subsets A, such that $\delta_1(A) = 1$.
In fact, we choose to every $n \in \mathbb{N}$ points x_n, $y_n \in M$, such that

$$d(x_n, y_n) \geq n - 1 \qquad \text{for } n \in \mathbb{N}.$$

As mentioned previously, $f(t) = \dfrac{t}{1+t}$ is increasing, so

$$d_1(x_n, y_n) = f(d(x,y)) \geq f(n-1) = \frac{n-1}{n} = 1 - \frac{1}{n}.$$

Putting

$$A = \{x_n \mid n \in \mathbb{N}\} \cup \{y_n \mid n \in \mathbb{N}\},$$

it follows that $\delta_1(A) \geq 1 - \dfrac{1}{n}$ for every $n \in \mathbb{N}$, thus $\delta_1(A) \geq 1$. On the other hand, we have already proved that $\delta_1(A) \leq 1$, so we conclude that $\delta_1(A) = 1$.

b) If instead d is bounded on M, then M has itself a finite d-diameter, $\delta(M) = c < \infty$, and

$$\delta_1(M) = \frac{c}{1+c} = 1 - \frac{1}{1+c} < 1.$$

There are many examples of such metrics. The most obvious one is the well-known

$$d_0(x,y) = \begin{cases} 0 & \text{for } x = y, \\ 1 & \text{for } x \neq y, \end{cases}$$

where

$$\tilde{d}_0(x,y) = \begin{cases} 0 & \text{for } x = y, \\ \frac{1}{2} & \text{for } x \neq y. \end{cases}$$

We get another example by starting with the bounded d_1 above. Then

$$d_2(x,y) = \frac{d_1(x,y)}{1 + d_1(x,y)} = \frac{d(x,y)}{1 + 2d(x,y)},$$

with $\delta_2(A) \leq \frac{1}{2}$ for every subset $A \subseteq M$.

4) It follows from

$$d_1(x_n, x) = 1 - \frac{1}{1 + d(x_n, x)},$$

that the condition $d_1(x_n, x) \to 0$ is equivalent with $1 + d(x_n, x) \to 1$, thus with $d(x_n, x) \to 0$, and the claim is proved.

Example 1.8 *Let (M_1, d_1) and (M_2, d_2) be metric spaces.*
Show that $M_1 \times M_2$ can be made into a metric space by the following definition of a metric d:

$$d\left((x_1, x_2), (y_1, y_2)\right) = d_1(x_1, y_1) + d_2(x_2, y_2).$$

Show that also $d^\star$ given by

$$d^\star\left((x_1, x_2), (y_1, y_2)\right) = \max\left\{d_1(x_1, y_1), d_2(x_2, y_2)\right\}$$

defines a metric on $M_1 \times M_2$.

1) Clearly,

$$d\left((x_1, x_2), (y_1, y_2)\right) \geq 0 \quad \text{and} \quad d^\star\left((x_1, x_2), (y_1, y_2)\right) \geq 0.$$

2) If $(x_1, x_2) = (y_1, y_2)$, i.e. $x_1 = y_1$ and $x_2 = y_2$, then

$$d\left((x_1, x_2), (y_1, y_2)\right) = 0 \quad \text{and} \quad d^\star\left((x_1, x_2), (y_1, y_2)\right) = 0.$$

Conversely, if

$$d\left((x_1,x_2),(y_1,y_2)\right) = 0 \quad \text{or} \quad d^\star\left((x_1,x_2),(y_1,y_2)\right) = 0,$$

then both

$$d_1(x_1,y_1) = 0 \qquad \text{and} \qquad d_2(x_2,y_2) = 0,$$

and it follows that $x_1 = y_1$ and $x_2 = y_2$, and hence $(x_1,x_2) = (y_1,y_2)$.

3) The symmetry is obvious.

4) The triangle inequality holds for both d_1 and d_2. Hence, it also holds for d and $d^\star$. In fact,

$$\begin{aligned} d\left((x_1,x_2),(y_1,y_2)\right) &= d_1(x_1,y_1) + d_2(x_2,y_2) \\ &\le \{d_1(x_1,z_1)+d_1(z_1,y_1)\} + \{d_2(x_2,z_2)+d_2(z_2,y_2)\} \\ &= \{d_1(x_1,z_1)+d_2(x_2,z_2)\} + \{d_1(z_1,y_1)+d_2(z_2,y_2)\} \\ &= d\left((x_1,x_2),(z_1,z_2)\right) + d\left((z_1,z_2),(y_1,y_2)\right), \end{aligned}$$

and

$$\begin{aligned} d^\star\left((x_1,x_2),(y_1,y_2)\right) &= \max\{d_1(x_1,y_1), d_2(x_2,y_2)\} \\ &\le \max\{d_1(x_1,z_1)+d_1(z_1,y_1), d_2(x_2,z_2)+d_2(z_2,y_2)\} \\ &\le \max\{d_1(x_1,z_1), d_2(x_2,z_2)\} + \max\{d_1(z_1,y_1), d_2(z_2,y_2)\} \\ &= d^\star\left((x_1,x_2),(z_1,z_2)\right) + d^\star\left((z_1,z_2),(y_1,y_2)\right). \end{aligned}$$

Example 1.9 *Show that in any set M we can define a metric by*

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \ne y. \end{cases}$$

Then we call (M,d) for a discrete *metric space.*
Characterize the sequences in M where $d(x_n,x) \to 0$.

1) Clearly, $d(x,y) \ge 0$.

2) Clearly, $d(x,y) = 0$, if and only if $x = y$.

3) Clearly, $d(x,y) = d(y,x)$.

4) Finally, it is almost trivial that

$$d(x,y) \le d(x,z) + d(z,y),$$

because the left hand side is always ≤ 1. If the right hand side is < 1, then both $d(x,z) = 0$ and $d(z,y) = 0$, and we infer that $x = z$ and $z = y$, hence also $x = y$. This implies that the left hand side $d(x,y) = 0$, and the triangle inequality is fulfilled.

Summing up we have proved that (M, d) is a metric space.

If $d(x_n, x) \to 0$, then choose $\varepsilon = \frac{1}{2}$. There exists an $n_0 \in \mathbb{N}$, such that

$$d(x_n, x) < \varepsilon = \frac{1}{2} \qquad \text{for } n \geq n_0.$$

This is only possible, if $d(x_n, x) = 0$, i.e. if

$$x_n = x \qquad \text{for all } n \geq n_0.$$

We conclude that all the convergent sequences are constant eventually.

Example 1.10 *Let (M,d) be a metric space and consider M as a topological space with the topology stemming from the open balls (the* ball topology*).*
Recall that a set A is closed *if $M \setminus A$ is open.*
Show that $A \subset M$ is closed if and only if

$$x_n \in A, \quad x_n \to x \quad \Longrightarrow \quad x \in A.$$

Show that if (M,d) is a complete metric space and A is a closed subset of M, then (A,d) is a complete metric space.

Assume that A is closed and let $x_n \in A$ be a convergent sequence in M, i.e. $x_n \to x \in M$. We shall prove that $x \in A$.

Indirect proof. Assume that $x \notin A$, i.e. $x \in M \setminus A$, which is open.
There exists an $r > 0$, such that

$$B(x,r) \subseteqq M \setminus A, \qquad \text{i.e.} \qquad B(x,r) \cap A = \emptyset.$$

Now, $x_n \to x$, so there exists an $n_r \in \mathbb{N}$, such that

$$d(x_n, x) < r \qquad \text{for } n \geq n_r,$$

and we see that $x_n \in B(x,r) \cap A = \emptyset$, which is not possible. Hence our assumption is wrong, so we conclude that $x \in A$.

Conversely, assume for every convergent sequence $(x_n) \subseteqq A$ the limit point lies in A. We shall prove that A is closed, or equivalently that $M \setminus A$ is open.

Indirect proof. Assume that $M \setminus A$ is *not* open. There exists an $x \in M \setminus A$, such that

$$\forall\, r \in \mathbb{R}_+ \, \exists\, y \in A : d(x,y) < r.$$

If we put $r = \dfrac{1}{n}$, $n \in \mathbb{N}$, with corresponding $y = x_n$, we define a sequence in A, which converges towards x, thus $x \in A$ according to the assumption. This is contradicting the assumption that $x \in M \setminus A$. Hence this assumption must be wrong, and $x \in A$ as requested.

Finally, assume that (M,d) is a *complete* metric space and that A is a *closed* subset of M. We shall prove that (A,d) is complete.

Let (x_n) be a Cauchy sequence on A. Then (x_n) is also a Cauchy sequence on the complete metric space M, thus (x_n) converges in M towards the limit $x \in M$. However, A is a closed subset, so it follows from the previous result that $x \in A$. We have proved that every Cauchy sequence (x_n) on A has a limit $x \in A$, which means that (A,d) is complete.

Example 1.11 *Show that*

$$d(x,y) = |\arctan x - \arctan y|$$

defines a metric on $\mathbb{R}$.

The definition includes an absolute value, hence $d(x,y) \geq 0$ for all x, $y \in \mathbb{R}$.
The function $\arctan t$ is strictly increasing on $\mathbb{R}$, hence $d(x,y) = 0$, if and only if $x = y$.
Clearly, $d(x,y) = d(y,x)$.
The triangle inequality follows from

$$d(x,y) = |\arctan x - \arctan y| \leq |\arctan x - \arctan z| + |\arctan z - \arctan y| = d(x,z) + d(z,y).$$

Example 1.12 *In* $\mathbb{R}^k$ *we define*

$$d_1(x,y) = \sum_{i=1}^{k} |x_i - y_i|,$$

$$d_2(x,y) = \left(\sum_{i=1}^{k} |x_i - y_i|\right)^{\frac{1}{2}},$$

$$d_\infty(x,y) = \max_{1 \leq i \leq k} |x_i - y_i|.$$

Show that d_1, d_2 *and* d_∞ *are metrics.*
Show that

$$d_\infty(x,y) \leq d_1(x,y) \leq k\, d_\infty(x,y),$$

and find a similar inequality when d_1 *is replaced by* d_2.
Show that if a sequence (x_n) *converges to* x *in one of these metrics, then we have coordinate wise convergence:*

$$x_{ni} \to x_i \qquad \textit{for all } i = 1,\, 2,\, \dots,\, k.$$

We first prove that

$$d_1(x,y) = \sum_{i=1}^{k} |x_i - y_i|$$

is a metric:

1) Clearly, $d_1(x,y) \geq 0$.

2) Clearly, $d_1(x,y) = 0$, if and only if $x = y$.

3) Clearly, $d_1(x,y) = d_1(y,x)$.

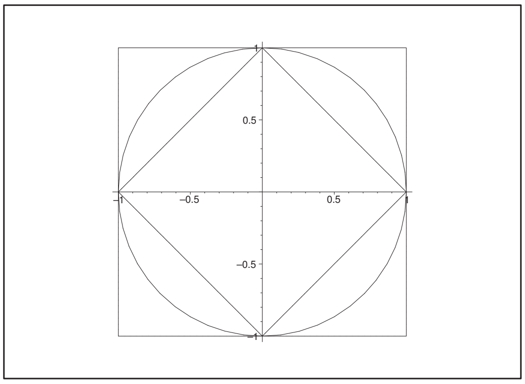


Figure 1: The three unit balls for d_1 (innermost), d_2 (the disc) and d_∞ (largest) in the case $\mathbb{R}^2$.

4) The triangle inequality follows by a small computation

$$\begin{aligned} d_1(x,y) &= \sum_{i=1}^{k}|x_i-y_i| \le \sum_{i=1}^{k}\{|x_i-z_i|+|z_i-y_i|\} \\ &= \sum_{i=1}^{k}|x_i-z_i|+\sum_{i=1}^{k}|z_i-y_i| = d_1(x,z)+d_1(z,y). \end{aligned}$$

We have proved that d_1 is a metric.

Then we prove that

$$d_2(x,y)=\left(\sum_{i=1}^{k}|x_i-y_i|^2\right)^{\frac{1}{2}}$$

is a metric. Again, the first three conditions are trivial. The triangle inequality,

$$\sqrt{\sum_{i=1}^{k}|x_i-y_i|^2} \le \sqrt{\sum_{i=1}^{k}|x_i-z_i|^2}+\sqrt{\sum_{i=1}^{k}|z_i-y_i|^2}$$

is, however, more difficult to prove. There are several proofs of the triangle inequality of d_2. Here we shall not choose the most elegant one, but instead the intuitively most obvious one.

Put $a_i=x_i-z_i$ and $b_i=z_i-y_i$, $i=1,\dots,k$. We shall prove that

$$\sqrt{\sum_{i=1}^{k}(a_i+b_i)^2} \le \sqrt{\sum_{i=1}^{k}a_i^2}+\sqrt{\sum_{i=1}^{k}b_i^2}.$$

All terms are ≥ 0, thus it is seen by squaring that we shall prove that

$$\sum_{i=1}^{k}a_i^2+\sum_{i=1}^{k}b_i^2+2\sum_{i=1}^{k}a_ib_i \le \sum_{i=1}^{k}a_i^2+\sum_{i=1}^{k}b_i^2+2\sqrt{\sum_{i=1}^{k}\sum_{j=1}^{k}a_i^2b_j^2},$$

which is reduced to the equivalent condition

$$\sum_{i=1}^{k} a_i b_i \le \sqrt{\sum_{i=1}^{k} a_i^2} \cdot \sqrt{\sum_{j=1}^{k} b_j^2}.$$

The claim follows if we can prove the CAUCHY-SCHWARZ INEQUALITY

$$\left|\sum_{i=1}^{k} a_i b_i\right| \le \sqrt{\sum_{i=1}^{k} a_i^2} \cdot \sqrt{\sum_{j=1}^{k} b_j^2}.$$

Another squaring shows that it suffices to prove that

$$\left(\sum_{i=1}^{k} a_i b_i\right) \cdot \left(\sum_{j=1}^{k} a_j b_j\right) \le \sum_{i=1}^{k}\sum_{j=1}^{k} a_i^2 b_j^2,$$

i.e.

$$\sum_{i=1}^{k} a_i^2 b_i^2 + 2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} a_i a_j b_i b_j \le \sum_{i=1}^{k} a_i^2 b_i^2 + \sum_{i=1}^{k-1}\sum_{j=i+1}^{k} \left(a_i^2 b_j^2 + a_j^2 b_i\right),$$

which again is equivalent with

$$0 \le \sum_{i=1}^{k-1}\sum_{j=i+1}^{k} \left(a_i^2 b_j^2 + a_j^2 b_i^2 - 2a_i a_j b_i b_j\right) = \sum_{i=1}^{k-1}\sum_{j=i+1}^{k} \left(a_i b_j - a_j b_i\right)^2.$$

The latter is clearly satisfied. Since we everywhere have computed "$\Longleftarrow$", the claim is proved.

Finally,

$$d_\infty(x,y) = \max_{1\le i\le k} |x_i - y_i|$$

is a metric, because the first three conditions again are trivial, and the triangle inequality follows from

$$|x_i - y_i| \le |x_i - z_i| + |z_i - y_i| \qquad \text{for every } i = 1, \dots, k,$$

thus

$$|x_i - y_i| \le d_\infty(x,z) + d_\infty(z,y) \qquad \text{for every } i = 1, \dots, k,$$

and by taking the maximum once more,

$$d_\infty(x,y) \le d_\infty(x,z) + d_\infty(z,y).$$

We have now proved that d_1, d_2 and d_∞ are all metrics.

We can find $j \in \{1,\dots,k\}$, such that

$$\begin{aligned} d_\infty(x,y) &= \max_{1\le i\le k} |x_i - y_i| = |x_j - y_j| \le \sum_{i=1}^k |x_i - y_i| = d_1(x,y) \\ &\le \sum_{i=1}^k \max |x_i - y_i| = k \cdot d_\infty(x,y). \end{aligned}$$

Analogously (with the same "maximal" j),

$$\begin{aligned} d_\infty(x,y) &= \max_{1\le i\le k} |x_i - y_i| = |x_j - y_j| = \sqrt{|x_j - y_j|^2} \\ &\le \sqrt{\sum_{i=1}^k |x_i - y_i|^2} = d_2(x,y) \le \sqrt{\sum_{i=1}^k \left\{\max_{1\le i\le k} |x_i - y_i|\right\}^2} \\ &= \sqrt{\sum_{i=1}^k \{d_\infty(x,y)\}^2} = \sqrt{k} \cdot d_\infty(x,y), \end{aligned}$$

and the wanted inequality becomes

$$d_\infty(x,y) \le d_2(x,y) \le \sqrt{k} \cdot d_\infty(x,y).$$

Remark 1.1 A simple squaring shows that $d_2(x,y) \le d_1(x,y)$, which can also be seen on the figure (the simple proof is left to the reader). This means that

$$d_\infty(x,y) \le d_2(x,y) \le d_1(x,y) \le k \cdot d_\infty(x,y). \qquad \Diamond$$

Using that $x_{ni} \to x_i$ for every $i = 1, 2, \dots, k$, if and only if $d_\infty(x_n, x) \to 0$, we conclude from the inequalities

$$d_\infty(x,y) \le d_1(x,y) \le k \cdot d_\infty(x,y),$$

$$d_\infty(x,y) \le d_2(x,y) \le \sqrt{k} \cdot d_\infty(x,y),$$

that this is fulfilled if and only if $d_1(x_n, 0) \to 0$, and if and only if $d_2(x,y) \to 0$.

Example 1.13 *Let c denote the set of convergent complex sequences $x = (x_1, x_2, \dots)$. Show that c is a complete metric space when equipped with the metric*

$$d_\infty(x,y) = \sup_i |x_i - y_i|.$$

HINT: *Show that the space of bounded complex sequences ℓ^∞ is a complete space and show then that c is a closed subset, then apply* Example 1.10.

Let $x^n = (x_1^n, x_2^n, \dots)$, where $\lim_{i\to\infty} x_i^n$ exists, be a Cauchy sequence from c, thus

$$\forall\, \varepsilon > 0 \,\exists\, N \in \mathbb{N} \,\forall\, m,\, n \geq N : d\,(x^m, x^n) < \varepsilon.$$

This means that

$$\sup_i |x_i^m - x_i^n| < \varepsilon.$$

In particular, $(x_i^n)_n$ is a Cauchy sequence on $\mathbb{R}$ for every i, hence convergent,

$$\lim_{n\to\infty} x_i^n = x_i.$$

The *Hint* is *not* used, because it is not hard to prove directly that $(x_i) \in c$. It suffices to prove that (x_i) is a Cauchy sequence, i.e.

(4) $\forall\, \varepsilon > 0 \,\exists\, I \in \mathbb{N} \,\forall\, i,\, j \geq I : |x_i - x_j| < \varepsilon.$

It follows from

$$|x_i - x_j| \leq |x_i - x_i^n| + |x_i^n - x_j^n| + |x_j^n - x_j|,$$

and $(x_i^n)_n \to x_i$, and even

$$\sup_i |x_i - x_i^n| \to 0 \qquad \text{for } n \to \infty,$$

that

$$a) \quad \forall\, \varepsilon > 0 \,\exists\, N \,\forall\, n \geq N \,\forall\, i : |x_i - x_i^n| < \frac{\varepsilon}{3},$$

$$b) \quad \forall\, \varepsilon > 0 \,\forall\, n \,\exists\, I(n) \,\forall\, i,\, j \geq I(n) : \left|x_i^n - x_j^n\right| < \frac{\varepsilon}{3}.$$

First choose N, such that a) is fulfilled.
Then choose $I = I(N)$, such that b) is fulfilled for $n = N$.
If $i,\, j \geq I = I(N)$, then

$$\begin{aligned} |x_i - x_j| &\leq \left|x_i - x_i^N\right| + \left|x_i^N - x_j^N\right| + \left|x_j^N - x_j\right| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

which is (4), and we have proved that (x_i) is a Cauchy sequence on $\mathbb{R}$, hence convergent. In particular, (x_i) is bounded, so $(x_i) \in c$, and c is complete.

Example 1.14 *In the set of bounded complex sequences ℓ^∞ equipped with the metric from* EXERCISE 12 *we consider the sets c_0 consisting of the sequences converging to 0 and c_{00} consisting of the sequences with only a finite number of elements different from 0.*
Investigate if c_0 and/or c_{00} are closed subsets of ℓ^∞.

The sequence $\left(\frac{1}{n}\right)$ belongs to ℓ^∞, though it does not belong to c_{00}. Choose

$$x^n = \left(1, \frac{1}{2}, \cdots, \frac{1}{n}, 0, 0, \cdots\right).$$

Then $x^n \in c_{00}$ and $x^n \to x = \left(\frac{1}{n}\right) \notin c_{00}$, hence c_{00} is not closed.

Let $x^n = (x_1^n, x_2^n, \cdots) \in c_0$ be convergent in ℓ^∞, i.e. $\lim_{i\to\infty} x_i^n = 0$ for every n. There exists an $x \in \ell^\infty$, such that

$$\forall\, \varepsilon > 0\, \exists\, n_0\, \forall\, n \geq n_0 : \|x - x^n\|_\infty = \sup_i |x_i - x_i^n| < \varepsilon.$$

We shall prove that $\lim_{i\to\infty} x_i = 0$. Now,

$$|x_i| \leq |x_i - x_i^n| + |x_i^n| \leq \|x - x^n\|_\infty + |x_i^n|\,.$$

First choose n, such that $\|x - x^n\|_\infty < \frac{\varepsilon}{2}$.
Then choose I, such that $|x_i^n| < \frac{\varepsilon}{2}$ for every $i \geq I$. Summing up we get for all $i \geq I$ that

$$|x_i| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

1.3 Contractions

Example 1.15 *Consider the metric space (M,d), where $M = [1,\infty[$, and d the usual distance. Let the mapping $T : M \to M$ be given by*

$$Tx = \frac{x}{2} + \frac{1}{x}.$$

Show that T is a contraction and find the minimal contraction constant α. Find also the fixed point.

First compute

$$|Tx - Ty| = \left|\frac{x}{2} + \frac{1}{x} - \frac{y}{2} - \frac{1}{y}\right| = \left|\frac{x-y}{2} + \frac{1}{x} - \frac{1}{y}\right| = \left|\frac{x-y}{2} + \frac{y-x}{xy}\right| = |x-y| \cdot \left|\frac{1}{2} - \frac{1}{xy}\right|.$$

Now, x, $y \geq 1$, so $0 < \dfrac{1}{xy} \leq 1$, and the function

$$(x,y) \mapsto \frac{1}{2} - \frac{1}{xy}$$

has the range $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right[$. We conclude that $\alpha = \dfrac{1}{2}$, so $\dfrac{1}{2}$ is the smallest α, for which

$$\left|\frac{1}{2} - \frac{1}{xy}\right| \leq \alpha.$$

The fixpoint satisfies the equation $Tx = x$, thus

$$x = \frac{x}{2} + \frac{1}{x}, \quad \text{hence } \frac{x}{2} = \frac{1}{x}, \quad \text{i.e. } x^2 = 2.$$

Since $x \geq 1$, the fixpoint must be $x = \sqrt{2}$, which also is easily seen by insertion.

Since $\alpha = \dfrac{1}{2} < 1$, it follows from the above that it is the only fixpoint.

Example 1.16 *A mapping T from a metric space (M,d) into itself is called a* weak contraction *if*

$$d(Tx,Ty) < d(x,y),$$

for all x, $y \in M$, $x \neq y$.
Show that T has at most one fixed point.
Show that T does not necessarily have a fixed point.
HINT: *One should take $Tx = x + \dfrac{1}{x}$ for $x \geq 1$.*

Let T be a weak contraction, and assume that both x and y are fixpoints, i.e. $Tx = x$ and $Ty = y$. If $x \neq y$, then

$$d(x,y) = d(Tx,Ty) < d(x,y),$$

which is not possible. Hence $y = x$, and there is at most one fixpoint.

Define $Tx = x + \frac{1}{x}$ on $[1, +\infty[$. If $x, y \in [1, +\infty[$, then

$$|Tx - Ty| = \left| x + \frac{1}{x} - y - \frac{1}{y} \right| = \left| x - y + \frac{y-x}{xy} \right| = |x-y| \cdot \left| 1 - \frac{1}{xy} \right|.$$

It follows from $0 < \frac{1}{xy} \leq 1$ for $x, y \geq 1$, that

$$|Tx - Ty| < |x - y| \qquad \text{for } x \neq y,$$

and T is a weak contraction on $[1, +\infty[$.

The weak contraction $Tx = x + \frac{1}{x}$ does not have a fixpoint, because $Tx = x$ would imply that $\frac{1}{x} = 0$, which is not possible.

Example 1.17 *It is very common in mathematical analysis to consider iterations of the form*

$$x_n = g(x_{n-1}),$$

where g is a C^1-function. Show that the sequence (x_n) is convergent for any choice of x_0 if there is an α, $0 < \alpha < 1$, such that

$$|g'(x)| \leq \alpha$$

for all $x \in \mathbb{R}$.

It follows from the Mean Value Theorem that one to any x and y can find $t = t(x, y)$ between x and y, such that

$$g(x) - g(y) = g'(t) \cdot (x - y),$$

thus

$$|g(x) - g(y)| = |g'(t)| \cdot |x - y| \leq \alpha\, |x - y|.$$

This proves that g is a contraction, and the claim follows from Banach's Fixpoint Theorem.

Example 1.18 *To approximate the solution to an equation $f(x) = 0$, we bring the equation on the form $x = g(x)$ and choose an x_0 and use the iteration $x_n = g(x_{n-1})$. Assume that g is a C^1-function on the interval $[x_0 - \delta; x_0 + \delta]$, and that $|g'(x)| \leq \alpha < 1$ for $x \in [x_0 - \delta; x_0 + \delta]$, and moreover*

$$|g(x_0) - x_0| \leq (1 - \alpha)\delta.$$

Show that there is one and only one solution $x \in [x_0 - \delta; x_0 + \delta]$ to the equation, and that $x_n \to x$.

Noticing that $|g'(x)| \leq \alpha < 1$ on the interval $[x_0 - \delta; x_0 + \delta]$, the claim follows from Banach's Fixpoint Theorem, if we only can prove that the iterative sequence (x_n) lies entirely in the interval $[x_0 - \delta, x_0 + \delta]$. We prove this by induction.

It is obvious that $x_0 \in [x_0 - \delta, x_0, \delta]$.
Assume that $x_n \in [x_0 - \delta, x_0 + \delta]$. Then we get for the following element $x_{n+1} = g(x_n)$,

$$\begin{aligned} |x_{n+1} - x_0| &= |g(x_n) - x_0| \\ &\le |g(x_n) - g(x_0)| + |g(x_0) - x_0| \\ &\le \alpha\,|x_n - x_0| + (1-\alpha)\delta \\ &\le \alpha\,\delta + (1-\alpha)\delta = \delta, \end{aligned}$$

proving that $x_{n+1} \in [x_0 - \delta, x_0 + \delta]$, and the claim follows.

Example 1.19 *Solve by iteration the equation $f(x) = 0$ for $f \in C^1([a,b])$, $f(x) < 0 < f(b)$ and f' bounded and strictly positive in $[a,b]$.*
HINT: *Take $g(x) = x - \lambda\, f(x)$ for a smart choice of λ.*

Putting

$$g(x) = x - \lambda\, f(x), \qquad \lambda \neq 0,$$

it follows that $f(x) = 0$, if and only if $g(x) = x$. Now,

$$g'(x) = 1 - \lambda\, f'(x) \qquad \text{and} \qquad 0 < k_1 \le f'(x) \le k_2,$$

so

$$1 - \lambda\, k_2 \le g'(x) \le 1 - \lambda\, k_1.$$

If we choose $\lambda = \dfrac{1}{k_2}$, then

$$0 \le g'(x) \le 1 - \frac{k_1}{k_2} = \alpha < 1,$$

and the mapping $g : [a,b] \to [a,b]$ is increasing and a contraction, so it has by Banach's Fixpoint Theorem precisely one fixpoint in $[a,b]$.

Example 1.20 *Show that it is possible to solve the equation $f(x)x^3 + x - 1 = 0$ by the iteration*

$$x_n = g(x_{n-1}) = \left(1 + x_{n-1}^2\right)^{-1}.$$

Find x_1, x_2, x_3 for $x_0 = 1$, and find an estimate for $d(x, x_n)$.

Let $g(x) = \dfrac{1}{1+x^2}$. Then $g(x) = x$ is equivalent with $x = \dfrac{1}{1+x^2}$, thus med $x(1+x^2) = 1$, which we write as

$$f(x) = x^3 + x - 1 = 0,$$

i.e. exactly the equation we want to solve.

It follows from

$$g'(x) = -\frac{2x}{(1+x^2)^2},$$

and

$$g''(x) = -\frac{2}{(1+x^2)^2} - 2x \cdot \frac{(-2)\cdot 2x}{(1+x^2)^3} = \frac{2}{(1+x^2)^3}\left\{-1-x^2+4x^2\right\} = \frac{6\left(x^2-\frac{1}{3}\right)}{(1+x^2)^3},$$

that $g''(x) = 0$ for $x = \pm\frac{1}{\sqrt{3}}$. Since $g'(x) \to 0$ for $x \to \pm\infty$, these points correspond to maximum and minimum for $g'(x)$, thus

$$|g'(x)| \le \frac{2\cdot\frac{1}{\sqrt{3}}}{\left(1+\frac{1}{3}\right)^2} = \frac{\frac{2}{\sqrt{3}}}{\frac{16}{9}} = \frac{3\sqrt{3}}{8} = \alpha \le 0.65,$$

and we have proved that g is a contraction, so the equation

$$f(x) = x^3 + x - 1 = 0$$

can be solved by the given iteration.

Let $x_0 = 1$. Then

$$\begin{aligned} x_1 &= g(x_0) = \frac{1}{1+1} = \frac{1}{2}, \\ x_2 &= g\left(\frac{1}{2}\right) = \frac{1}{1+\frac{1}{4}} = \frac{4}{5}, \\ x_3 &= g\left(\frac{4}{5}\right) = \frac{1}{1+\frac{16}{25}} = \frac{25}{41}. \end{aligned}$$

Finally,

$$|x - x_n| \leq \frac{\alpha^n}{1-\alpha} \cdot |x_1 - x_0|,$$

so

$$|x - x_n| \leq \frac{\left(\frac{3\sqrt{3}}{8}\right)^n}{1 - \frac{3\sqrt{3}}{8}} \cdot \left(1 - \frac{1}{2}\right) = \frac{4}{8 - 3\sqrt{3}} \cdot \left(\frac{3\sqrt{3}}{8}\right)^n = \frac{4}{8 - 3\sqrt{3}} \cdot \left(\frac{27}{64}\right)^{\frac{n}{2}} < \frac{3}{2} \cdot \left(\frac{27}{64}\right)^{\frac{n}{2}}.$$

When we apply the iteration above on a pocket calculator, we get

$$x = 0.682\,327\,804.$$

Remark 1.2 The iteration above can therefore be applied, though it is far from the fastest one. If the preset case we get by *Newton's iteration formula*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \frac{2}{3}\,x_n + \frac{1}{3} \cdot \frac{3 - 2x_n}{3x_n^2 + 1},$$

from which already

$$x_4 = 0.682\,327\,804. \qquad \Diamond$$

Example 1.21 *A mapping* $T : \mathbb{R} \to \mathbb{R}$ *satisfies a* Lipschitz condition *with constant* k*, if*

$$|Tx - Ty| \leq k|x - y|, \qquad \text{for all } x,\, y \in \mathbb{R}.$$

1) Is T *a contraction?*

2) If T *is a* C^1*-function with bounded derivative, show that* T *satisfies a Lipschitz condition.*

3) If T *satisfies a Lipschitz condition, is* T *then a* C^1*-function with bounded derivative?*

4) Assume that $Tx - Ty| \leq k\,|x - y|^{\alpha}$ *for some* $\alpha > 1$*. Show that* T *is a constant.*

1) If $k \geq 1$, then T is not necessarily a contraction.
   If instead $0 \leq k < 1$, then T is always a contraction.

2) It follows from the Mean Value Theorem that

   $$|T(x) - T(y)| = |T'(t)| \cdot |x - y|,$$

   where $t = t(x, y)$ lies somewhere between x and y.
   Since $|T'(t)| \leq k$, it is obvious that T fulfils a Lipschitz condition.

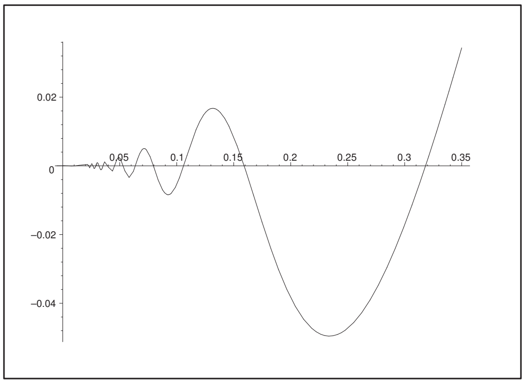


Figure 2: The graph of $f(x) = x^2 \cdot \sin \dfrac{1}{x}$ for $0 < x < 0.35$.

3) The answer is "no". Choose the function

$$f(x) = \begin{cases} x^2 \cdot \sin \dfrac{1}{x} & \text{for } x > 0, \\ \\ 0 & \text{for } x \le 0. \end{cases}$$

Then f is differentiable with the derivative

$$f'(x) = \begin{cases} 2x \cdot \sin \dfrac{1}{x} - \cos \dfrac{1}{x} & \text{for } x > 0, \\ \\ \lim_{x \to 0+} \dfrac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+} x \cdot \sin \dfrac{1}{x} = 0 & \text{for } x = 0, \\ \\ 0 & \text{for } x < 0. \end{cases}$$

Choose $x_0 > 0$, such that $f'(x_0) = 0$, and put

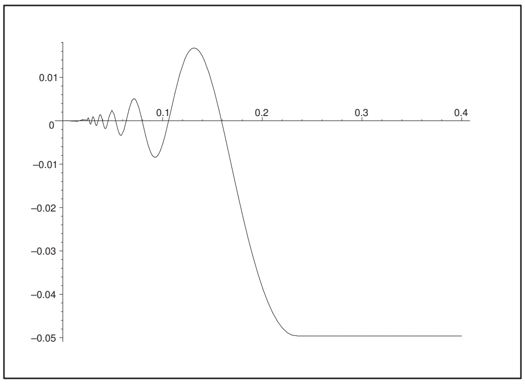


Figure 3: An example of a function $T(x)$.

$$T(x) = \begin{cases} f(x_0) & \text{for } x \geq x_0, \\[2mm] x^2 \cdot \sin \dfrac{1}{x} & \text{for } 0 < x < x_0, \\[2mm] 0 & \text{for } x \leq 0. \end{cases}$$

Then $|T'(x)| \leq 2x_0 + 1$, and $T'(x)$ is defined everywhere, though not continuous for $x = 0$, where $T'(x) = f'(x) = 2x \cdot \sin \dfrac{1}{x} - \cos \dfrac{1}{x}$ or $0 < x < x_0$ does not have a limit value for $x \to 0+$. Thus we have constructed a mapping $T \notin C^1$, which satisfies a Lipschitz condition. (It is of course possible to construct far more complicated examples).

4) Assume that there exists an $\alpha > 1$, such that

$$|Tx - Ty| \leq k\,|x - y|^{\alpha}.$$

Then

$$0 \leq \left| \lim_{y \to x} \frac{Tx - Ty}{x - y} \right| \leq \lim_{y \to x} k \cdot \frac{|x - y|^{\alpha}}{|x - y|} = k \cdot \lim_{y \to x} |y - x|^{\alpha - 1} = 0.$$

This proves that T is differentiable everywhere of the derivative 0. Then T is a constant.

Example 1.22 *Let T be a mapping from a complete metric space (M, d) into itself, and assume that there is a natural number m such that T^m is a contraction.*
Show that T has one and only one fixed point.

If T^m is a contraction, then T^m has a fixpoint x, thus $T^m x = x$. When we apply T on this equation, we get

$$T^{m+1} x = T^m(Tx) = Tx,$$

hence Tx is also a fixpoint of T^m.

Since T^m is a contraction, the fixpoint is unique, so $Tx = x$, and we have proved that x is a fixpoint for T.

Conversely, if x is a fixpoint for T, then x is also a fixpoint for T^m, because $Tx = x$ implies that

$$T^m x = T^{m-1}(Tx) = T^{m-1} = \cdots = Tx = x.$$

We have assumed that T^m is a contraction, hence the fixpoint for T^m is unique. This is true for every fixpoint x for T, hence it must be unique.

Example 1.23 *We consider the metric space* $\mathbb{R}^k$ *with the metric*

$$d_1(x, y) = \sum_{i=1}^{k} |x_i - y_i|$$

and a mapping $T : \mathbb{R}^k \to \mathbb{R}^k$ *given by* $Tx = Cx + b$*, where* $C = (c_{ij})$ *is a* $k \times k$ *matrix and* $b \in \mathbb{R}^k$*. Show that* T *is a contraction, if*

$$\sum_{i=1}^{k} |c_{ij}| < 1 \qquad \textit{for all } j = 1, 2, \dots, k.$$

If we instead use the metric

$$d_2(x, y) = \sqrt{\sum_{i=1}^{k} |x_i - y_i|^2},$$

show that T *is a contraction if*

$$\sum_{i=1}^{k} \sum_{j=1}^{k} |c_{ij}|^2 < 1.$$

First note that the i-th coordinate of Tx is

$$(Tx)_i = \sum_{j=1}^{k} c_{ij} x_j + b_i, \qquad i = 1, \dots, k.$$

Put $y = Tx$ and $w = Tz$ and

$$\alpha = \max_{1 \le j \le k} \sum_{i=1}^{k} |c_{ij}| < 1.$$

Then we get the estimates

$$\begin{aligned} d_1(Tx, Tz) &= \sum_{i=1}^{k} |y_i - w_i| = \sum_{i=1}^{k} \left| \sum_{j=1}^{k} c_{ij}(x_j - z_j) \right| \\ &\le \sum_{i=1}^{k} \sum_{j=1}^{k} |c_{ij}| \cdot |x_j - z_j| \le \alpha \sum_{j=1}^{k} |x_j - z_j| = \alpha \cdot d_1(x, z), \end{aligned}$$

and the condition $\alpha = \max_{1 \le j \le k} |c_{ij}| < 1$ assures that T is a contraction in $\left(\mathbb{R}^k, d_1\right)$.

If instead we consider the metric

$$d_2(x, y) = \sqrt{\sum_{i=1}^{k} |x_i - y_i|^2},$$

and assume that

$$\alpha^2 = \sum_{i=1}^{k} \sum_{j=1}^{k} |c_{ij}|^2 < 1,$$

then we get the following estimate

$$\begin{aligned}
\{d_2(x,y)\}^2 &= \sum_{i=1}^{k} |y_i - w_i|^2 = \sum_{i=1}^{k} \left| \sum_{j=1}^{k} c_{ij}(x_j - z_j) \right|^2 \\
&= \sum_{i=1}^{k} \left| \sum_{j=1}^{k} c_{ij}(x_j - z_j) \cdot \sum_{\ell=1}^{k} c_{i\ell}(x_\ell - z_\ell) \right| \\
&\le \sum_{i=1}^{k} \sum_{j=1}^{k} \sum_{\ell}^{k} |c_{ij}| \cdot |x_j - z_j| \cdot |c_{i\ell}| \cdot |x_\ell - z_\ell|.
\end{aligned}$$

Then apply

$$|ab| \le \frac{1}{2}\left(a^2+b^2\right),$$

which follows from the inequality $(|a|-|b|)^2 = a^2+b^2-2|ab| \ge 0$.
If we put

$$a = |c_{i\ell}|\cdot|x_j-z_j| \qquad \text{and} \qquad b = |c_{ij}\cdot|x_\ell - z_\ell|,$$

we get

$$\begin{aligned}
\{d_2(y,w)\}^2 &= \sum_{i=1}^k\sum_{j=1}^k\sum_{\ell=1}^k \frac{1}{2}\left\{|c_{i\ell}|^2|x_j-z_j|^2+|c_{ij}|^2|x_\ell-z_\ell|^2\right\}\\
&= \frac{1}{2}\sum_{i=1}^k\sum_{\ell=1}^k|c_{i\ell}|^2\cdot\sum_{j=1}^k|x_j-z_j|^2+\frac{1}{2}\sum_{i=1}^k\sum_{j=1}^k k|c_{ij}|^2\cdot\sum_{\ell=1}^k|x_\ell-z_\ell|^2\\
&\le \frac{1}{2}\,\alpha^2\{d_2(x,z)\}^2+\frac{1}{2}\,\alpha^2\{d_2(x,z)\}^2 = \alpha^2\{d_2(x,z)\}^2.
\end{aligned}$$

Since $\alpha^2<1$, and hence also $0\le\alpha<1$, and

$$d_2(y,w) = d_2(Tx,Tz) \le \alpha\cdot d_2(x,z),$$

we conclude that T is a contraction in $\left(\mathbb{R}^k,d_2\right)$.

Example 1.24 *In connection with Banach's Fixpoint Theorem, the inequality*

$$d(x,x_n) \le \frac{\alpha}{1-\alpha}\,d(x_{n-1},x_n)$$

is often mentioned. Prove this inequality.

Given that $\alpha\in\,]0,1[$, at $Tx_n = x_{n+1}$, and $x_n\to x$.

Choose to any $\varepsilon\in\mathbb{R}_+$ an N, such that we for all $p\ge N$ have $d(x,x_p)<\varepsilon$. If $p\ge N$ and $p\ge n+1$, then

$$\begin{aligned}
d(x,x_n) &\le d(x,x_p)+d(x_p,x_n) < \varepsilon + d(x_p,x_n)\\
&\le \varepsilon + d(x_p,x_{p-1})+d(x_{p-1},x_{p-2})+\cdots+d(x_{n+1},x_n)\\
&= \varepsilon + d(Tx_{p-1},Tx_{p-2})+d(Tx_{p-2},Tx_{p-3})+\cdots+d(Tx_n,Tx_{n-1})\\
&\le \varepsilon+\alpha\cdot\frac{1-\alpha^{p-n}}{1-\alpha}\cdot d(x_{n-1},x_n)\\
&\le \varepsilon+\frac{\alpha}{1-\alpha}\cdot d(x_{n-1},x_n).
\end{aligned}$$

This is true for every $\varepsilon>0$, thus

$$d(x,x_n)\le\frac{\alpha}{1-\alpha}\cdot d(x_{n-1},x_n).$$

Example 1.25 *Consider the matrix equation $Ax + b = 0$, where $A = (a_{ij})_{i,j=1}^{k}$ (and the a_{ij} real). Put $A = C - I$ and rewrite the equation as $x = Cx + b$.*
If

$$(5)\quad \sum_{j=1}^{k} |c_{ij}| < 1 \qquad \text{for } i = 1,\, 2,\, \dots,\, k,$$

then there is a unique solution x, which can be found by iteration.
Prove that the condition (5) can be formulated as the following condition of the a_{ij},

$$a_{ii} < 0, \qquad |a_{ii}| > \sum_{j=1, j\neq i}^{k} |a_{ij}|, \qquad |a_{ii}| < 2 - \sum_{j=1, j\neq i}^{k} |a_{ij}|,$$

for $i = 1,\, 2,\, \dots,\, k$.

We have $a_{ij} = c_{ij} - \delta_{ij}$, thus $c_{ij} = \delta_{ij} + a_{ij}$. In particular, $c_{ii} = 1 + a_{ii}$. Since

$$\sum_{j=1}^{k} |c_{ij}| < 1,$$

we have $|c_{ii}| < 1$, thus $a_{ii} \in]-2, 0[$.
Furthermore, $c_{ij}| = |a_{ij}|$ for $i \neq j$, so

$$\sum_{j=1}^{k} |c_{ij}| = \sum_{j=1, j\neq i}^{k} |a_{ij}| + |1 + a_{ii}| < 1.$$

It follows that

$$\sum_{j=1}^{k} |a_{ij}| < 1 - |1 + a_{ii}| = 1 - |1 - |a_{ii}|| \leq 1.$$

If

$$|a_{ii}| \leq 1 \qquad \left(< 2 - \sum_{j=1, j\neq i}^{k} |a_{ij}| \right),$$

then

$$\sum_{j=1, j\neq i}^{k} |a_{ij}| < 1 - 1 + |a_{ii}| = |a_{ii}|.$$

If

$$|a_{ii}| > 1 \qquad \left(> \sum_{j=1, j\neq i}^{k} |a_{ij}| \right),$$

then

$$\sum_{j=1,j\neq i}^{k} |a_{ij}| < 1 - |a_{ii}| + 1 = 2 - |a_{ii}|,$$

hence by a rearrangement,

$$|a_{ii}| < 2 - \sum_{j=1,j\neq i}^{k} |a_{ij}|,$$

and we derive in both cases that

$$\sum_{j=1,j\neq i}^{k} |a_{ij}| < |a_{ii}| < 2 - \sum_{j=1,j\neq i}^{k} |a_{ij}|.$$

Conversely, assume that $a_{ii} < 0$ and that

$$\sum_{j=1,j\neq i}^{k} |a_{ij}| < |a_{ii}| < 2 - \sum_{j=1,j\neq i}^{k} |a_{ij}|.$$

Then

$$\sum_{j=1,j\neq i}^{k} |a_{ij}| < 1.$$

If $|a_{ii}| \le 1$, then

$$|a_{ii}| = 1 - 1 + |a_{ii} = 1 - |1 - |a_{ii}|| = 1 - |1 + a_{ii}| = 1 - |c_{ii}|,$$

thus

$$\sum_{j=1,j\neq i}^{k} |a_{ij}| = \sum_{j=1,j\neq i}^{k} |c_{ij}| < 1 - |c_{ii}|,$$

and hence

$$\sum_{j=1}^{k} |c_{ij}| < 1.$$

If $|a_{ii}| > 1$, then

$$|a_{ii}| = 1 - 1 + |a_{ii}| = 1 + ||a_{ii}| - 1| = 1 + |a_{ii} + 1| = 1 + |c_{ii}]|,$$

hence by insertion

$$1 + |c_{ii}| < 2 - \sum_{j=1,j\neq i}^{k} |a_{ijn}| = 2 - \sum_{j=1,j\neq i}^{k} |c_{ij}|,$$

follows by a rearrangement

$$\sum_{j=1}^{k} |c_{ij}| < 1.$$

1.4 Simple integral equations

Example 1.26 *Consider the* Volterra integral equation*:*

$$x(t) - \mu \int_a^t k(t,s)x(s)\,ds = v(t), \qquad t \in [a,b],$$

where $v \in C([a,b])$, $k \in C([a,b]^2)$ and $\mu \in \mathbb{C}$.
Show that the equation has a unique solution $x \in C([a,b])$ for any $\mu \in \mathbb{C}$.
Hint*: Write the equation $x = Tx$ where*

$$Tx = v(t) + \mu \int_a^t k(t,s)x(s)\,ds.$$

Take $x_0 \in C([a,b])$ and define the iteration by $x_{n+1} = Tx_n$, then show by induction that

$$|T^m x(t) - T^m y(t)| \le |\mu|^m c^m \frac{(t-a)^m}{m!}\, d_\infty(x,y),$$

where $c = \max|k|$. Then show (by looking at $d_\infty(T^m x, T^m y)$) that T^m is a contraction for some m and argue that T then must have a unique fixed point in the metric space $(C([a,b]), d_\infty)$.

Using the given definition of T we see that the equation is equivalent with $Tx = x$. Then

$$\begin{aligned} |Tx(t) - Ty(t)| &= |\mu| \cdot \left| \int_a^t k(t,s)x(s)\,ds - \int_a^t k(t,s)y(s)\,ds \right| = |\mu| \cdot \left| \int_a^t k(t,s) \cdot \{x(s) - y(s)\}\,ds \right| \\ &\le |\mu| \cdot c \cdot d_\infty(x-y) \cdot \left| \int_a^t 1\,ds \right| = |\mu|^1 \cdot c^1 \cdot \frac{(t-a)^1}{1!}\, d_\infty(x,y), \end{aligned}$$

which shows that the inequality above holds for $m = 1$.
Assume that for some$m \in \mathbb{N}$,

$$(6)\quad |T^m x(t) - T^m y(t)| \le |\mu|^m c^m \cdot \frac{(t-a)^m}{m!}\, d_\infty(x,y).$$

Then

$$\begin{aligned} |T^{m+1}x(t) - T^{m+1}y(t)| &= |\mu| \cdot \left| \int_a^t k(t,s)\{T^m x(s) - T^m y(s)\}\,ds \right| \\ &\le |\mu| \cdot c \int_a^t |T^m x(s) - T^m y(s)|\,ds \\ &\le |\mu| \cdot c \cdot |\mu|^m \cdot c^m \cdot d_\infty(x,y) \cdot \int_a^t \frac{(s-a)^m}{m!}\,ds \\ &= |\mu|^{m+1} c^{m+1} \cdot d_\infty(x,y) \cdot \left[\frac{(s-a)^{m+1}}{(m+1)!} \right]_a^t \\ &= |\mu|^{m+1} c^{m+1} \cdot \frac{(t-a)^{m+1}}{(m+1)!} \cdot d_\infty(x,y), \end{aligned}$$

and (6) follows by induction for all $m \in \mathbb{N}$.

We infer from (6) that

$$d_\infty(T^m x, T^m y) \le |\mu|^m c^m \cdot \frac{(b-a)^m}{m!} \cdot d_\infty(x,y).$$

Now

$$\sum_{m=0}^{\infty} |\mu|^m c^m \cdot \frac{(b-a)^m}{m!} = \exp(|\mu| \cdot c \cdot (b-a))$$

is convergent, thus

$$|\mu|^m c^m \cdot \frac{(b-a)^m}{m!} \to 0 \qquad \text{for } m \to \infty.$$

There exists in particular an $M \in \mathbb{N}$, such that

$$\alpha = |\mu|^m c^m \cdot \frac{(b-a)^m}{m!} < 1 \qquad \text{for all } m \ge M.$$

Thus, if $m \ge M$, then T^m is a contraction, and T^m has a fixpoint x. An application of Example 1.22 shows that x is also a fixpoint for T, and x is the unique fixpoint of T.

Let $x_0 \in C^0([a,b])$. Define by iteration $x_{m+1} = T x_n$. Then $x_m = T^m x_0$. The sequence $(x_{m \cdot n})$ converges towards x. The same does the sequence (x_{mn+j}), where $j = 0, 1, \dots, m-1$, because

$$x_{mn+j} = T^{mn}\left(T^j x_0\right) = T^{mn+j} x_0.$$

Summing up we conclude that (x_n) itself converges towards x, and the claim is proved.

Example 1.27 *Solve by iteration the equation*

$$f(t) = u(t) = \frac{1}{2}\int_0^1 e^{t-s} f(s)\, ds, \qquad t \in [0,1],$$

(where u is a given continuous function), by choosing f_0 as u.
Find in particular the solutions in the cases

$$u(t) = 1, \qquad\qquad u(t) = t.$$

Then solve the equation directly (without using iteration), assuming that $u \in C^1([0,1])$.

If we put $f_0(t) = u(t)$, then

$$f_1(t) = u(t) + \frac{1}{2}\int_0^1 e^{t-s} u(s)\, ds = u(t) + \frac{1}{2}\left\{\int_0^1 e^{-s} u(s)\, ds\right\} \cdot e^t.$$

Putting $a = \int_0^1 e^{-s} u(s)\, ds$, we get

$$f_(t) = u(t) + \frac{a}{2}\, e^t.$$

It follows that

$$\begin{aligned} f_2(t) &= u(t) + \frac{1}{2}\int_0^1 e^{t-s} f_1(s)\, ds = u(t) + \frac{1}{2}\, e^t \left\{\int_0^1 e^{-s} u(s)\, ds + \frac{a}{2}\int_0^1 e^{-s} e^s\, ds\right\} \\ &= u(t) + e^t \left\{\frac{a}{2} + \frac{a}{4}\right\} = u(t) + \frac{3}{4}\, a \cdot e^t. \end{aligned}$$

We conclude from the structure

$$f(t) = u(t) + e^t \left\{\frac{1}{2}\int_0^1 e^{-s} f(s)\, ds\right\},$$

that a solution must have the form $f(t) = u(t) + c \cdot e^t$. We therefore guess that the n-th iteration may be written

$$f_n(t) = u(t) + a \cdot k_n e^t.$$

We get by insertion

$$\begin{aligned} f_{n+1}(t) &= u(t) + \frac{1}{2}\int_0^1 e^{y-s} f_n(s)\, ds \\ &= u(t) + \frac{1}{2}\, e^t \left\{\int_0^1 e^{-s} u(s)\, ds + a \cdot k_n \int_0^1 e^{-s} e^s\, ds\right\} \\ &= u(t) + \frac{1}{2}\, a e^t 0 \frac{1}{2}\, e^t \cdot a \cdot k_n = u(t) + a\left\{\frac{1 + k_n}{2}\right\} e^t, \end{aligned}$$

and conclude that

$$k_{n+1} = \frac{1}{2}\,(1 + k_n)\,.$$

If $k_n \in [0,1[$, then it follows that $k_n < k_{n+1} < 1$, thus (k_n) is increasing and bounded. (Notice that $k_1 = \frac{1}{2}$), thus it is convergent of the limit value k. We conclude from the equation of recursions that $k = \dfrac{1}{2}(1+k)$, thus $k=1$. Hence the solution is given by

$$f(t) = u(t) + e^t \int_0^1 e^{-s} u(s)\, ds.$$

CHECK. We get by insertion,

$$u(t) + \frac{1}{2}\int_0^1 e^{t-s} f(s)\, ds = u(t) + \frac{1}{2}\, e^t \int_0^1 e^{-s} u(s)\, ds + \frac{1}{2}\, e^t \int_0^1 e^{-s} u(s)\, ds = f(t),$$

proving that we have found a solution. ♢

If $u(t) = 1$, then

$$f(t) = 1 + e^t \int_0^q e^{-s}\, ds = 1 + e^t \left[-e^{-s}\right]_0^1 = 1 + \left(1 - \frac{1}{e}\right) e^t.$$

If $u(t) = t$, then

$$f(t) = t + e^t \int_0^1 s\, e^{-s} ds = t + e^t \left[-s\, e^{-s} - e^{-s}\right]_0^1 = t + \left(1 - \frac{2}{e}\right) e^t.$$

As mentioned above the solution must have the form $u(t) + c \cdot e^t$. Then by insertion,

$$\begin{aligned} u(t) + \frac{1}{2}\int_0^1 e^{t-s} f(s)\, ds &= u(t) + \frac{1}{2}\int_0^1 e^{t-s}\left\{u(s) + c \cdot e^s\right\} ds \\ &= u(t) + \frac{1}{2}\left\{\int_0^1 e^{-s} u(s)\, ds + c\right\} e^t = u(t) + c \cdot e^t = f(t), \end{aligned}$$

and we conclude that $c = \int_0^1 e^{-s} u(s)\, ds$.

If $u \in C^1([0,1])$, then

$$f(t) = u(t) + \left\{\frac{1}{2}\int_0^1 e^{-s} f(s)\, ds\right\} \cdot e^t \in C^1,$$

so we can ALTERNATIVELY solve the equation by differentiation with respect to t. It follows from

$$\frac{1}{2}\int_0^1 e^{t-s} f(s)\, ds = f(t) - u(t),$$

that

$$f'(t) = u'(t) + \frac{1}{2}\int_0^1 e^{t-s} f(s)\, ds = f(t) + u'(t) - u(t),$$

hence by a multiplication by e^{-t} follows by a rearrangement,

$$f'(t)\, e^{-t} - f(t)\, e^{-t} = \frac{d}{dt}\left\{e^{-t} f(t)\right\} = u'(t) e^{-t} - u(t) e^{-t} = \frac{d}{dt}\left\{e^{-t} u(t)\right\},$$

and we get by an integration

$$e^{-t}f(t) = e^{-t}u(t) + c,$$

hence

$$f(t) = u(t) + c \cdot e^{t}.$$

The constant c is determined as above. The latter variant is of course not the shortest one.

Example 1.28 *Let* $C^0([a,b])$ *be equipped with the metric*

$$d(x,y) = \max_{t\in[a,b]} |x(t) - y(t)|.$$

We define an operator (a mapping) S *by*

$$Sx(t) = \int_a^b k(t,s)x(s)\,ds,$$

where k *is a continuous function on* $[a,b]\times[a,b]$*. Let* (x_n) *be inductively given by*

(7) $x_{n+1} = u + \mu\, Sx_n,$

and put $z_n = x_n - x_{n-1}$*. Prove that (7) equivalently can be written*

(8) $z_{n+1} = \mu\, Sz_n.$

Put $x_0 = u$*, and prove that (7) implies the* Neumann series

$$x = \lim_{n\to\infty} x_n = u + \mu\, Su + \mu^2 S^2 u + \cdots .$$

We note that

$$x_{n+1}(t) = u(t) + \mu \int_a^b k(t,s)x_n(s)\,ds = u(t) + \mu\, Sx_n(t).$$

Putting $z_n = x_n - x_{n-1}$, we get

$$\begin{aligned} z_{n+1} &= x_{n+1} - x_n = u + \mu\, Sx_n - u - \mu\, Sx_{n-1} \\ &= \mu\, S(x_n - x_{n-1}) = \mu\, Sz_n. \end{aligned}$$

If $|\mu| < \dfrac{1}{(b-a)c}$, then $x_n \to x$. It follows from

$$x_n = x_n - x_{n-1} + x_{n-1} - zx_{n-2} + x_{n-2} + \cdots + x_1 - x_0 + x_0 = x_0 + z_1 + \cdots + z_n,$$

and

$$z_n = \mu\, Sz_{n-1} = \cdots = \mu^n S^n x_0,$$

that $\sum_n z_n$ is convergent, and we have

$$x = \lim_{n\to\infty} x_n = u + \mu\, Su + \mu^2 S^2 u + \cdots .$$

Example 1.29 *Solve*

$$x(t) - \mu \int_0^1 x(s)\, ds = 1$$

by means of the Neumann series, where we assume that $\|\mu| < 1$. *Try also to solve the equation directly.*

In this case, $u(t) = 1$ and $k(t,s) = 1$, $a = 0$ and $b = 1$, thus $|\mu| < 1$ is a reasonable requirement (cf. Example 1.28). It follows from Example 1.28 that

$$x = 1 + \mu\, S + \mu^2 S^2 1 + \cdots .$$

We get from $S1 = \int_0^1 1\, ds = 1$, that $S^2 1 = 1$. Then by induction, $S^n 1 = 1$, hence

$$x = 1 + \mu + \mu^2 + \cdots = \frac{1}{1-\mu}.$$

We now solve the equation directly. It follows from the rearrangement

$$x(t) = 1 + \mu \int_0^1 x(s)\, dx$$

that $x(t) = a$ must be a constant. Then by insertion,

$$a = 1 + \mu \cdot a,$$

hence

$$x(t) = a = \frac{1}{1-\mu},$$

which apparently holds for every $\mu \neq 1$, and not just for $|\mu| < 1$.

2 Banach spaces

2.1 Simple vector spaces

Example 2.1 *In the vector space $C([a,b])$ we consider the functions*

$$e_0(t),\ e_1(t),\ \dots,\ e_n(t),$$

where $e_j(t)$ is a polynomial of degree j, where $j = 0,\ 1,\ \dots,\ n$,
Show that $e_0,\ e_1,\ \dots,\ e_n$ are linearly independent.

Since $e_0(t) = e_0 \neq 0$, we infer from $a_0 e_0 = 0$ that $a_0 = 0$, and the claim is true for $k = 0$.

First let $e_k(t) = t^k$, and assume that the claim is true for $k = 0,\ 1,\ \dots,\ n$. Now let

$$a_0 + a_1 t + \cdots + a_n t^n + a_{n+1} t^{n+1} \equiv 0 \qquad \text{for } t \in [a,b].$$

We get by a differentiation,

$$a_1 + 2a_2 t + \cdots + n a_n t^{n-1} + (n+1) a_{n+1} t^n \equiv 0 \qquad \text{for } t \in [a,b],$$

thus $k a_k = 0$, $k = 1,\ 2,\ \dots,\ n+1$, according to the assumption of induction. We conclude that $a_k = 0$ for $k = 1,\ 2,\ \dots,\ n+1$, which by insertion gives the condition $a_0 = 0$. Then it follows by induction that $\{t^n \mid n \in \mathbb{N}_0\}$ are linearly independent.

Then let

$$e_k(t) = \sum_{j=0}^{k} e_{kj} t^j, \qquad e_{kk} \neq 0,$$

and assume that

$$0 \equiv \sum_{k=0}^{n} a_k e_k(t) = \sum_{k=0}^{n} \sum_{j=0}^{k} a_k e_{kj} t^j = \sum_{j=0}^{n} \left\{ \sum_{k=j}^{n} a_k e_{kj} \right\} t^j.$$

It follows from the result above that

$$\sum_{k=j}^{n} a_k e_{kj} = 0 \qquad \text{for } j = 0,\ 1,\ \dots,\ n.$$

We get for $j = n$ that $a_n e_{nn} = 0$, and since $e_{nn} \neq 0$, we must have $a_n = 0$. Since $e_{k,k+j} = 0$ for $j \geq 1$, the equation is reduced to

$$0 \equiv \sum_{j=0}^{n} \left\{ \sum_{k=j}^{n} a_k e_{kj} \right\} t^j = \sum_{j=0}^{n} \left\{ \sum_{k=j}^{n-1} a_k e_{kj} \right\} t^j = \sum_{j=0}^{n-1} \left\{ \sum_{k=j}^{n-1} a_k e_{kj} \right\} t^j,$$

where we as before conclude that $a_{n-1} = 0$. Then by recursion,

$$a_{n-2} = \cdots = a_1 = a_0 = 0.$$

Example 2.2 *Let U_1 and U_2 be subspaces of the vector space V. Show that $U_1 \cap U_2$ is a subspace. Is $U_1 \cup U_2$ always a subspace? If no, state conditions such that $U_1 \cup U_2$ is a subspace.*

If U_1 and U_2 are subspaces, then

$$\forall\, \lambda\, \forall u,\, v \in U_i : u + \lambda\, v \in U_i, \qquad i = 1,\, 2.$$

If $u,\, v \in U_1 \cap U_2$, then in particular, $u,\, v \in U_i$, $i = 1,\, 2$, thus $u + \lambda\, v \in U_i$, $i = 1,\, 2$, according to the above. It follows that $u + \lambda v \in U_1 \cap U_2$, hence $U_1 \cap U_2$ is also a subspace.

On the other hand, $U_1 \cup U_2$ is rarely a subspace. E.g. the X-axis and the Y-axis are two subspaces in $\mathbb{R}^2$, and it is obvious that the union of the two axes is not a subspace.

The condition is that $U_1 \subseteq U_2$, or $U_1 \supseteq U_2$. In fact, if one of these conditions is satisfied, then it is obvious that $U_1 \cup U_2 = U_i$, where i is one of the numbers 1, 2.
If this condition is not fulfilled, then there exist

$$u_1 \in U_1 \setminus U_2 \qquad \text{and} \qquad u_2 \in U_2 \setminus U_1.$$

Assume that $u_1 + u_2 \in U_1 \cup U_2$, e.g. $u_1 + u_2 \in U_1$. Then $u_2 = (u_1 + u_2) - u_1 \in U_1$ contradicting the assumption. Hence we conclude that $u_1 + u_2 \notin U_1 \cup U_2$, and $U_1 \cup U_2$ is not a subspace.

Example 2.3 *Let V denote the set of all real $n \times n$ matrices.*
Show that V with the usual scalar multiplication and addition is a vector space.
Is the set of all regular *$n \times n$-matrices a subspace of V?*
Is the set of all symmetric *$n \times n$ matrices a subspace of V?*

The first question is trivial: Since 0 is the zero element, and since 0 is not regular, the set of all regular matrices is not a subspace.

The set of all symmetric matrices is of course a subspace. In fact, if (a_{ij}) and (b_{ij}) are symmetric, thus $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$, then

$$\lambda(a_{ij}) + (b_{ij}) = (\lambda\, a_{ij} + b_{ij}),$$

where

$$\lambda\, a_{ij} + b_{ij} = \lambda\, a_{ji} + b_{ji},$$

hence $(\lambda\, a_{ij} + b_{ij})$ is again symmetric.

Example 2.4 *In the space $C([a, b])$ we consider the sets*

$U_1 =$ *the set of polynomials defined on* $[a, b]$.

$U_2 =$ *the set of polynomials defined on* $[a, b]$ *of degree* $\leq n$.

$U_3 =$ *the set of polynomials defined on* $[a, b]$ *of degree* $= n$.

$U_4 =$ *the set of all* $f \in C([a, b])$ *with* $f(a) = f(b) = 0$.

$U_5 = C^1([a, b])$.

Which of the U_i, $i = 1$, 2, dots, 5, are subspaces of $C([a, b])$?

$U_1 =$ the set of all polynomials is a subspace.

$U_2 =$ the set of all polynomials of degree $\leq n$ is a subspace.

$U_3 =$ the set of all polynomials of degree $= n$ is not a subspace. E.g. 0 does not belong to U_3.

$U_4 =$ the set of all $f \in C^0([a, b])$ with $f(a) = f(b) = 0$ is a subspace.

$U_5 = C^1([a, b])$ is a subspace.

Example 2.5 *In $C([-1,1])$ we consider the sets U_1 and U_2 consisting of the odd and even functions in $C([-1,1])$, respectively.*
Show that U_1 and U_2 are subspaces and that $U_1 \cap U_2 = \{0\}$.
Show that every $f \in C([-1,1])$ can be written in the form $f = f_1 + f_2$, where $f_1 \in U_1$ and $f_2 \in U_2$, and that this decomposition is unique.

If f, g are odd (even) functions, then $f + \lambda\, g$ is again an odd (even) function. Hence U_1 and U_2 are subspaces.

If $f \in U_1 \cap U_2$, then both

$$f(-t) = f(t) \qquad \text{and} \qquad f(-t) = -f(t),$$

thus $f(t) = -f(t)$ for all t, and we conclude that $2f(t) \equiv 0$. We conclude that $f \equiv 0$.

We see from

$$f(t) = \frac{f(t)+f(-t)}{2} + \frac{f(t)-f(-t)}{2},$$

where

$$\frac{f(t)+f(-t)}{2} \text{ is even,} \quad \text{and} \quad \frac{f(t)-f(-t)}{2} \text{ is odd,}$$

that such a splitting exists.

Assume that

$$f(t) = f_1(t) + f_2(t) = g_1(t) + g_2(t),$$

where f_1 and g_1 are odd, while f_2 and g_2 are even. Then

$$f_1(t) - g_1(t) = g_2(t) - f_2(t) \in U_1 \cap U_2 = \{0\},$$

hence $f_1 - g_1 = 0$ and $g_2 - f_2 = 0$. We conclude that $f_1 = g_1$ and $f_2 = g_2$, and the splitting is unique.

2.2 Normed spaces

Example 2.6 *In the space $C^1([a,b])$ we have the norm*

$$\|f\|_\infty = \sup_{t\in[a,b]} |f(t)|.$$

Show that we could take $\sup_{t\in(a,b)} |f(t)|$ instead.
Show that $C^1([a,b])$ with the sup-norm is not at Banach space.
Show that

$$\|f\|_\infty^\star = \sup_{t\in[a,b]} |f(t)| + \sup_{t\in[a,b]} |f'(t)|$$

is also a norm on $C^1([a,b])$ and that it is a Banach space with this norm.

Every $f \in C^1([a,b])$ is continuous, so

$$\sup_{t\in[a,b]} |f(t)| = \sup_{t\in(a,b)} |f(t)|,$$

and we can use any of the two sup-norms.

It follows from Weierstraß's Approximation Theorem that the set $\mathcal{P}$ of polynomials on $[a,b]$ is dense in $C^0([a,b])$ in the uniform norm. Since

$$\mathcal{P} \subset C^1([a,b]) \subset C^0([a,b])$$

and $C^1([a,b]) \neq C^0([a,b])$, we infer that $C^1([a,b])$ cannot be complete, thus $(C^1([a,b]), \|\cdot\|)$ is not a Banach space.

Then we shall prove that $\|\cdot\|_\infty^\star$ is a norm.

1) Clearly, $\|f\|_\infty^\star \geq 0$.

2) If

$$\|f\|_\infty^\star = \sup_{t\in[a,b]} |f(t)| + \sup_{t\in[a,b]} |f'(t)| = \|f\|_\infty + \|f'\|_\infty = 0,$$

then in particular $\|f\| = 0$, so $f = 0$, because f is continuous.

3)

$$\|\lambda f\|_\infty^\star = \|\lambda f\|\infty + \|\lambda f'\|_\infty = |\lambda|(\|f\|_\infty + \|f'\|_\infty) = |\lambda| \cdot \|f\|_\infty.$$

4)

$$\begin{aligned} \|f+g\|_\infty^\star &= \|f+g\|_\infty + \|f'+g'\|_\infty \leq \|f\|_\infty + \|g\|_\infty + \|f'\|_\infty + \|g'\|_\infty \\ &= (\|f\|_\infty + \|f'\|_\infty) + (\|g\|_\infty + \|g'\|_\infty) = \|f\|_\infty^\star + \|g\|_\infty^\star. \end{aligned}$$

We have proved that $\|\cdot\|_\infty^\star$ is a norm on $C^1([a,b])$.

It "only" remains to prove that $(C^1([a,b]), \|\cdot\|_\infty^\star)$ is a Banach space.
Let (f_n) be a Cauchy sequence, i.d.

$$\forall\,\varepsilon > 0\,\exists\,N \in \mathbb{N}\,\forall\,m,\,n \in \mathbb{N} : m,\,n \geq N \quad\Longrightarrow\quad \|f_m - f_n\|_\infty^\star < \varepsilon.$$

It follows from $\|f\|_\infty^\star = \|f\|_\infty + \|f'\|_\infty$, that $\|f\|_\infty \leq \|f\|_\infty^\star$ and $\|f'\|_\infty \leq \|f\|_\infty^\star$, thus (f_n) and (f_n') are Cauchy sequences in the Banach space $(C^0([a,b]), \|\cdot\|_\infty)$. Hence there are continuous functions $f,\,g \in C^0([a,b])$, such that

$$f_n \to f \qquad \text{and} \qquad f_n' \to g.$$

Notice that it is not possible from this directly to conclude that

$$a)\;\; f \in C^1([a,b]), \qquad b)\;\; f' = g.$$

A proof is required:

Define a function $h \in C^1([a,b])$ by

$$h(x) = \int_a^x g(t)\,dt + f(a), \qquad x \in [a,b].$$

We shall prove that $h(x) = f(x)$. It suffices to prove that $f_n \to h$ uniformly, because the limit function $f \in C^0([a,b])$ is unique. From $f_n \in C^1([a,b])$ follows that

$$f_n(x) = \int_a^x f_n'(t)\,dt + f_n(a), \qquad x \in [a,b],$$

hence for every $x \in [a,b]$,

$$\begin{aligned} |f_n(x) - h(x)| &= \left| \int_a^x f_n'(t)\,dt + f_n(a) - \int_a^x g(t)\,dt - f(a) \right| \\ &\le \left| \int_a^x \{f' : n(t) - g(t)\}\,dt \right| + |f_n(a) - f(a)|. \end{aligned}$$

Let $\varepsilon > 0$ be given. Since $f_n(a) \to f(a)$, and $f_n' \to g$ uniformly for $n \to +\infty$, there exists an $n_0 \in \mathbb{N}$, such that for every $n \ge n_0$,

$$|f_n(a) - f(a)| < \frac{\varepsilon}{2} \quad \text{and} \quad \sup_{t \in [a,b]} |f_n'(t) - g(t)| < \frac{\varepsilon}{2(b-a)}.$$

Therefore, if $n \ge n_0$, then for every $x \in [a,b]$,

$$|f_n(x) - h(x)| < \left| \int_a^x \frac{\varepsilon}{2(b-a)}\,dt \right| + \frac{\varepsilon}{2} \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

thus

$$\|f_n - h\|_\infty < \varepsilon \qquad \text{for all } n \ge n_0,$$

and we have proved that $f_n \to h$ uniformly, hence $f = h$. Finally, since $h' = g$, the claim is proved.

Example 2.7 *Let $f \in C([a,b])$ and consider the p-norms*

$$\|f\|_p = \left\{ \int_a^b |f(t)|^p dt \right\}^{\frac{1}{p}}, \qquad p \ge 1,$$

and

$$\|f\|_\infty = \sup_{t \in [a,b]} |f(t)|.$$

Show that $\|f\|_p \to \|f\|_\infty$ for $p \to \infty$.

The interval $[a,b]$ is bounded, so

$$\|f\|_p = \left\{ \int_a^b |f(t)|^p dt \right\}^{\frac{1}{p}} \le \left\{ \int_a^b \|f\|_\infty^p dt \right\}^{\frac{1}{p}} = \|f\|_\infty (b-a)^{\frac{1}{p}}.$$

The function f is continuous and $[a,b]$ is compact, hence there exists a $t_0 \in [a,b]$, such that

$$|f(t_0)| = \|f\|_\infty.$$

To every $\varepsilon > 0$ we can find an interval $[c_\varepsilon, d_\varepsilon] \subsetneqq [a,b]$, $c_\varepsilon < d_\varepsilon$ (independently of p), such that

$$|f(t)| \geq (1-\varepsilon)\|f\|_\infty \qquad \text{for all } t \in [c_\varepsilon, e_\varepsilon].$$

Then we get the estimate

$$\begin{aligned}\|f\|_p &= \left\{\int_a^b |f(t)|^p dt\right\}^{\frac{1}{p}} \geq \left\{\int_{c_\varepsilon}^{d_\varepsilon} |f(t)|^p dt\right\}^{\frac{1}{p}} \geq \left\{(1-\varepsilon)^p \|f\|_\infty^p \int_{c_\varepsilon}^{d_\varepsilon} dt\right\}^{\frac{1}{p}} \\ &= (1-\varepsilon)\|f\|_\infty \cdot (d_\varepsilon - e_\varepsilon)^{\frac{1}{p}}.\end{aligned}$$

Summing up we get for every $\varepsilon > 0$ that

$$(1-\varepsilon)\|f\|_\infty \cdot (d_\varepsilon - c_\varepsilon)^{\frac{1}{p}} \leq \|f\|_p \leq \|f\|_\infty \cdot (b-a)^{\frac{1}{p}}.$$

If $k > 0$ is kept fixed, we have $k^{\frac{1}{p}} \to 1$ for $p \to \infty$. To every $\varepsilon > 0$ there exists a $P_\varepsilon > 0$, such that for every $p \geq P_\varepsilon$,

$$(d_\varepsilon - c_\varepsilon)^{\frac{1}{p}} \geq 1 - \varepsilon \qquad \text{and} \qquad (b-a)^{\frac{1}{p}} \leq 1 + \varepsilon,$$

hence

$$(1-\varepsilon)^2 \|f\|_\infty \leq \|f\|_p \leq (1+\varepsilon)\|f\|_\infty \qquad \text{for every } p \geq P_\varepsilon.$$

This proves that $\lim_{p\to+\infty} \|f\|_p$ exists and that

$$\lim_{p\to+\infty} \|f\|_p = \|f\|_\infty.$$

Example 2.8 *Let V be a normed vector space and let $x_1, \dots, x_k$ be k linearly independent vectors from V. Show that there exists a positive constant m, such that for all scalars $\alpha_i \in \mathbb{C}$, $i = 1, \dots, k$, we have*

$$\|\alpha_1 x_1 + \cdots + \alpha_k x_k\| \geq m\left(|\alpha_1| + \cdots + |\alpha_k|\right).$$

Indirect proof. We assume that there exists a sequence (y_m), where

$$y_m = \sum_{i=1}^k \beta_i^{(m)} x_i, \qquad \text{where } \sum_{i=1}^k \left|\beta_i^{(m)}\right| = 1 \text{ for all } m \in \mathbb{N},$$

and where $\|y_m\| \to 0$ for $m \to +\infty$. Under these assumptions we first notice that $\left|\beta_i^{(m)}\right| \leq 1$, such that $\left(\beta_i^{(m)}\right)_{m=1}^{+\infty}$ is a bounded sequence of complex numbers. The complex numbers $\mathbb{C}$ being complete in the absolute value, there exists a convergent subsequence

$$\left(\beta_1^{(m_j^1)}\right)_{j=1}^{+\infty} \quad \text{af} \quad \left(\beta_1^{(m)}\right).$$

The trick is first to thin out $(\beta_2^{(m)})$ to the subsequence $\left(\beta_1^{(m_j^1)}\right)$, where (m_j^1) is given above.

Then thin it out once more to get a convergent subsequence

$$\left(\beta_2^{(m_j^2)}\right) \quad \text{of} \quad \left(\beta_2^{(m_j^1)}\right).$$

Because (m_j^2) is a subsequence of (m_j^1), the subsequence $\left(\beta_1^{(m_j^2)}\right)$ is also convergent.

Continue in this way. After k steps we have obtained a subsequence (m_j) from $\mathbb{N}$, such that

$$\left(\beta_i^{(m_j)}\right)_{j=1}^{+\infty} \quad \text{is convergent for all } i = 1,\, 2,\, \dots,\, k.$$

This means that (y_{m_j}) is a convergent subsequence of (y_m), hence

$$y_{m_j} \to y \qquad \text{for } j \to +\infty,$$

and

$$y = \sum_{i=1}^{k} \beta_i x_i.$$

We conclude from

$$\sum_{i=1}^{k} |\beta_i| \ge \sum_{i=1}^{k} \left|\beta_i^{(m_j)}\right| - \sum_{i=1}^{k} \left|\beta_i^{(m_j)} - \beta_i\right| = 1 - \sum_{i=1}^{k} \left|\beta_i^{(m_j)} - \beta_i\right| \to 1, \qquad \text{for } j \to +\infty,$$

and from the assumption that $x_1,\, \dots,\, x_k$ are linearly independent that $y \neq 0$. This is contradicting the assumption that $\|y_m\| \to 0$ for $m \to +\infty$.

We infer that if $\sum_{i=1}^{k} |\beta_i| = 1$, then there is a constant $c > 0$, such that

$$\left\| \sum_{i=1}^{k} \beta_i x_i \right\| \ge c.$$

We put for $(\alpha_1, \dots, \alpha_k) \neq (0, \dots, 0)$,

$$\beta_i = \frac{\alpha_i}{|\alpha_1| + \cdots + |\alpha_k|}.$$

Then the claim follows when we multiply by $|\alpha_1| + \cdots + |\alpha_k| \neq 0$.

Finally, we notice that the case $\alpha_1 = \cdots = \alpha_k = 0$ follows trivially for quite other reasons.

Example 2.9 *Let V be a vector space and let $\|\cdot\|$ and $|||\cdot|||$ be two norms on V. The norms are said to be* equivalent *if there are positive constants m and M such that*

$$m\|x\| \leq |||x||| \leq M\|x\|$$

for all $x \in V$.
Show that all norms on a finite dimensional vector space are equivalent.
Show that all equivalent norms define the same closed sets.

Let $e_1, \dots, e_k$ be a basis for V. It follows from Example 2.8 that there are constants $c_1 > 0$ and $c_2 > 0$, such that

$$\left\|\sum_{i=1}^k \alpha_i e_i\right\| \geq c_1 \sum_{i=1}^k |\alpha_i| \quad \text{and} \quad \left|\left|\left|\sum_{i=1}^k \alpha_i e_i\right|\right|\right| \geq c_2 \sum_{i=1}^k |\alpha_i|.$$

Writing $x = \sum_{i=1}^k \alpha_i e_i$, we get

$$\begin{aligned} \|x\| &= \left\|\sum_{i=1}^k \alpha_i e_i\right\| \leq \sum_{i=1}^k |\alpha_i| \cdot \|e_i\| \leq \max_{1\leq i\leq k} \|e_i\| \cdot \sum_{j=1}^k |\alpha_j| \leq \frac{1}{c_2} \max_{1\leq i\leq k} \|e_i\| \cdot \left|\left|\left|\sum_{j=1}^k \alpha_j e_j\right|\right|\right| \\ &= \frac{1}{c_2} \max_{1\leq i\leq k} \|e_i\| \cdot |||x||| \leq \frac{1}{c_2} \max 1 \leq i \leq k \|e_i\| \cdot \sum_{j=1}^k |\alpha_j| \cdot |||e_i||| \\ &\leq \frac{1}{c_2} \max_{1\leq i\leq k} \|e_i\| \cdot \max_{1\leq j\leq k} |||e_j||| \cdot \sum_{\ell=1}^k |\alpha_\ell| \leq \frac{1}{c_1} \cdot \frac{1}{c_2} \max \|e_i\| \cdot \max |||e_j||| \cdot \|x\|. \end{aligned}$$

Thus we have proved that

$$\|x\| \leq a \cdot |||x||| \leq b \cdot \|x\|,$$

where

$$a = \frac{1}{c_2} \max_{i \leq i \leq k} \|e_i\| > 0 \quad \text{and} \quad b = a \cdot \frac{1}{c_1} \max_{1 \leq j \leq k} |||e_j||| > 0.$$

When we divide by $a > 0$, we get

$$m\|x\| = \frac{1}{a}\,\|x\| \leq |||x||| \leq \frac{b}{a}\,\|x\| = M\|x\|,$$

and we have proved that any two norms on a finite dimensional subspace are equivalent.

Since

$$m\|x\| \leq |||x||| \leq M\|x\|, \qquad 0 < m \leq M,$$

and

$$\frac{1}{M}\,|||x||| \leq \|x\| \leq \frac{1}{m}\,|||x|||,$$

are equivalent, it suffices to prove that if U is closed with respect to $\|\cdot\|$, then U is also closed with respect to $|||\cdot|||$.

It is well-known (cf. Example 1.10) that U is closed, if and only if

$$x_n \in U \text{ and } x_n \to x \quad \Longrightarrow \quad x \in U.$$

Assume that U is closed with respect to $\|\cdot\|$, and let $(x_n) \subseteqq U$ be a sequence for which

$$|||x_n||| \to 0 \qquad \text{for } n \to +\infty,$$

thus (x_n) is convergent with respect to the norm $|||\cdot|||$. We shall prove that $x \in U$. However,

$$\|x_n - x\| \leq \frac{1}{m}\,|||x_n - x||| \to 0 \qquad \text{for } n \to +\infty,$$

so also $x_n \to x$ with respect to the norm $\|\cdot\|$. It follows from the condition of Example 1.10 (applied with respect to $\|\cdot\|$) that $x \in U$, and the claim is proved.

Example 2.10 *Show that a compact set in a normed vector space V is closed and bounded. If V is finite dimensional, show that a closed and bounded set is compact.*

Assume that U is compact in V, i.e. every sequence $(x_n) \subseteqq U$ has a subsequence (y_n), which converges towards an element y in U. We shall prove that U is closed and bounded.

Assume that $(x_n) \subseteqq U$ is convergent in V, thus $x_n \to x \in V$. It follows from Example 1.10 that U is closed, if we can prove that also $x \in U$.

According to the assumption there is a subsequence (y_n) of (x_n), such that $y_n \to y \in U$. However, since $x_n \to x$, also $y_n \to x$, and since the limit value is unique in normed spaces, we conclude that $x = y \in U$, and it follows that U is closed.

Then we shall prove that if U is compact, then U is bounded. *Indirect proof.* Assume that U is unbounded. Let $x_1 \in U$ be arbitrarily chosen. There exists an $x_2 \in U$, such that

$$\|x_2\| \geq 1 + \|x_1\|.$$

Choose inductively a sequence $(x_n) \subseteqq U$, such that

$$\|x_{n+1}\| \geq 1 + \|x_n\|.$$

Then note that if x_n and x_{n+p}, $p \in \mathbb{N}$ are any two elements, then

$$\|x_{n+p}\| \geq 1 + \|x_{n+p-1}\| \geq 2 + \|x_{n+p-2}\| \geq \cdots \geq p + \|x_n\|,$$

hence

$$\|x_{n+p} - x_n\| \geq \|x_{n+p}\| - \|x_n\| \geq 0 \geq 1 \qquad \text{for alle } p \in \mathbb{N},$$

proving that no subsequence of (x_n) is convergent, and U is not compact.

We get by contraposition that if U is compact, then U is bounded.

Assume now that V is finite dimensional and that U is bounded and closed. Let $e_1 \dots, e_k$ denote a basis for V, and let the constant $c > 0$ be chosen as in Example 2.8, such that

$$\left\|\sum_{i=1}^{k} \alpha_: i e_i\right\| \geq c\left(|\alpha_1| + \cdots + |\alpha_k|\right) = c\sum_{i=1}^{k}|\alpha_i|.$$

Let $x_n \in U$, $x_n = \sum_{i=1}^{k} \alpha_i^n e_i$, be any sequence. It follows from U being bounded that $\|x\| \leq B$ for every $x \in U$, i.e.

$$|\alpha_i| \leq \sum_{i=1}^{k}|\alpha_i| \leq \frac{1}{c}\left\|\sum_{i=1}^{k}\alpha_i e_i\right\| \leq \frac{B}{c}$$

for all $i = 1, \dots, k$. Hence the sequence $(\alpha_1^n)_n$ is bounded, and it has therefore a convergent subsequence $\left(\alpha_1^{n_j^1}\right)$.

Since $\left(\alpha_2^{n_j^1}\right)$ is a bounded sequence, it has a convergent subsequence $\left(\alpha_2^{n_j^2}\right)$, etc..
After k steps we have found a sequence (n_j), for which $\left(\alpha_i^{n_j}\right)_j$ is convergent for $j \to +\infty$ for every $i = 1, \dots, k$, of limit value α_i.

Putting

$$y_j = \sum_{i=1}^{k}\alpha_i^{n_j} e_i,$$

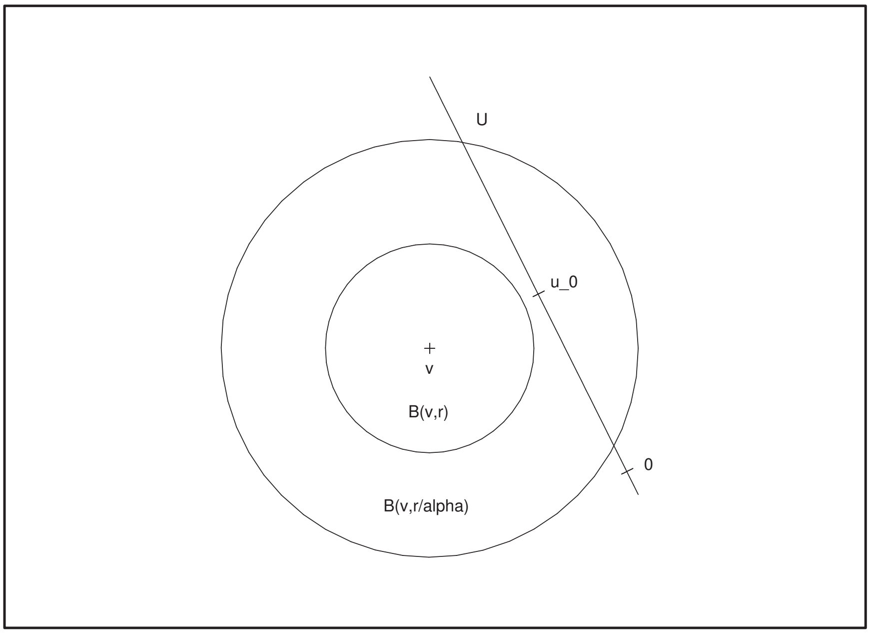
we get that (y_j) is convergent of limit

$$y_j \to y = \sum_{i=1}^{k}\alpha_i e_i.$$

Since $y_j \in U$, and U is closed, we get $y \in U$ according to Example 1.10, and the claim is proved.

Example 2.11 *Riesz's lemma. Let V be a normed vector space and let U be a closed subspace of V, $U \neq V$. Let α, $0 < \alpha < 1$, be given. Show that there is a $v \in V$, such that*

$$\|v\| = 1 \quad \textit{and} \quad \|v - u\| \geq \alpha \qquad \textit{for all } u \in U.$$



It follows from $U \neq V$, that there exists a $v \in V \setminus U$.
The set U is closed, so $V \setminus U$ is open. Hence there exists an $r > 0$, such that $B(v,r) \cap U = \emptyset$, where $B(v,r)$ denotes the open ball of centre v and radius r. This means that

(9) $\|v - u\| \geq r \qquad$ for all $u \in U$.

Choose r sufficiently large such that (cf. the figure)

$$B(v,r) \cap U = \emptyset \qquad \text{and} \qquad B\left(v, \frac{1}{\alpha}\, r\right) \cap U \neq \emptyset.$$

Then for every $u_0 \in B\left(v, \frac{1}{\alpha}\, r\right) \cap U$,

(10) $r \leq \|v - u_0\| \leq \dfrac{1}{\alpha}\, r$.

If we put

$$w = \frac{v - u_0}{\|v - u_0\|},$$

then $\|w\| = 1$.

We have for any $u \in U$ that

$$\|w-u\| = \left\| \frac{v-u_0}{\|v-u_0\|} - u \right\| = \frac{1}{\|v-u_0\|} \, \|v-u_0-\|v-u_0\|\,u\| \,.$$

Now u, $u_0 \in U$, and U is a subspace, hence $u_0 + \|v-u_0\|\,u \in U$. By applying (9) with $u_0 + \|v-u_0\|\,u$ instead of u, it follows from (10) that

$$\|w-u\| = \frac{1}{\|v-u_0\|} \, \|v-(u_0+\|v-u_0\|\,u)\| \geq \frac{r}{\|v-u_0\|} \geq \frac{r}{\frac{1}{\alpha}\,r} = \alpha.$$

We have proved that $w \in V$ satisfies

$$\|w\| = 1 \quad \text{and} \quad \|w-u\| \geq \alpha \qquad \text{for eevery } u \in U.$$

Example 2.12 *In ℓ^∞, the vector space of bounded sequences, we consider the sets U_1 and U_2, where U_1 denotes the set of sequences with only finitely many elements different from 0 and U_2 the set of sequences with all but the N first elements equal to 0.*
Are U_1 and/or U_2 closed subspaces in ℓ^∞?
Are U_1 and/or U_2 finite dimensional?

It follows from

$$x_n = \left(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, 0, \dots\right) \in U_1,$$

and

$$x_n \to \left(\frac{1}{k}\right)_{k\in\mathbb{N}} \notin U_1,$$

that U_1 is not closed.

Of course U_1 is a subspace, and since every *finite dimensional* subspace is closed (which U_1 is not), we conclude that U_1 is not finite dimensional.

On the other hand, U_2 and $\mathbb{R}^N$ are isomorphic, so U_2 er is a closed and finite dimensional vector space, $\dim U_2 = N$.

Example 2.13 *Let $(V, \|\cdot\|)$ be a normed vector space, and let U be the* unit ball,

$$U = \{x \in V \mid \|x\| \le 1\}.$$

Prove that U is compact, if and only if V is finite dimensional.

Obviously, U is closed and bounded. If V is finite dimensional, then it follows from EXAMPLE 2.10 that U is compact. It remains to be proved that if U is compact, then V is finite dimensional.

INDIRECT PROOF. Assume that V is not finite dimensional. Choose any $x_1 \in U$, such that $\|x_1\| = 1$. Then x_1 generates a subspace V_1. Then by Riesz's lemma (EXAMPLE 2.11) there exists an $x_2 \in U$, such that

$$\|x_2\| = 1 \quad \text{and} \quad \|x_2 - \lambda x_1\| \ge \frac{1}{2} \qquad \text{for all } \lambda.$$

By induction, using Riesz's lemma in each step, we obtain a sequence $x_n \in U$ of unit vectors, $\|x_n\| = 1$, such that

$$\left\| x_n - \sum_{j=1}^{n-1} \lambda_j x_j \right\| \ge \frac{1}{2} \qquad \text{for any } \lambda_j.$$

We have in particular,

$$\|x_n - x_m\| \ge \frac{1}{2} \qquad \text{for } n \ne m,$$

proving that (x_n) does not contain any convergent subsequence. Hence U is not compact.

We get by contraposition that if the unit ball U is compact, then the vector space V is finite dimensional.

Example 2.14 *Consider in* ℓ^p *(where* $1 \le p \le +\infty$*) the subspace* U *consisting of all sequences which are 0 eventually.*

1) If $1 \le p < +\infty$*, is the subspace* U *then dense in* ℓ^p*?*

2) If $p = +\infty$*, is the subspace* U *then dense in* ℓ^∞*?*

1) The answer is 'yes'. In fact, if $(x_j)_{j\in\mathbb{N}} \in \ell^p$, then

$$\sum_{j=1}^{+\infty} |x_j|^p < +\infty.$$

To every $\varepsilon > 0$ there is an N, such that

$$\sum_{j=N+1}^{+\infty} |x_j|^p < \varepsilon^p.$$

Putting $x^N = (x_1, \dots, x_N, 0, 0, \dots) \in U$, we get

$$\|x - x^N\|_p = \left\{ \sum_{j=N+1}^{+\infty} |x_j|^p \right\}^{\frac{1}{p}} < \{\varepsilon^p\}^{\frac{1}{p}} = \varepsilon.$$

2) In this case the answer is 'no'. In fact, if $x = (1, 1, 1, \dots) \in \ell^\infty$, then

$$\|x - y\|_\infty \ge 1 \qquad \text{for every } y \in U.$$

Example 2.15 *On $C([a,b])$ we introduce the norm*

$$\|f\|_p = \left\{\int_a^b |f(t)|^p\,dt\right\}^{\frac{1}{p}}, \qquad p \in]1,+\infty[.$$

Let $g \in C([a,b])$, and let q be given by $\dfrac{1}{p}+\dfrac{1}{q}=1$. Prove that we by

$$T_g f = \int_a^b f(t)\overline{g(t)}\,dt$$

define a linear functional on $C([a,b])$, and that

$$\|T_g\| = \|g\|_q \qquad \left(=\left\{\int_a^b |g(t)|^q\,dt\right\}^{\frac{1}{q}}\right).$$

Most of the claims have already been proved, included the estimate $\|T_g\| \le \|g\|_1$. We shall only proof that we even get equality. The trick is to choose a suitable $f \in C([a,b])$. We have

$$T_g f = \int_a^b f(t)\overline{g(t)}\,dt.$$

Since $g(t)$ is continuous, we get

$$g(t) = e^{i\,\varphi(t)}\,|g(t)|,$$

where $\varphi(t)$ can be chosen continuous in every interval, in which $g(t) \neq 0$.

Choosing

$$f(t) = e^{i\,\varphi(t)}\,|g(t)|^{\frac{q}{p}},$$

f is again continuous and

$$\|f\|_p^p = \int_a^b |g(t)|^q\,dt = \|g\|_q^q, \qquad \text{thus} \qquad \|f\|_p = \|g\|_q^{\frac{q}{p}} = \|g\|_q^{q-1},$$

and

$$\begin{aligned} T_g f &= \int_a^b f(t)\overline{g(t)}\,dt = \int_a^b e^{i\,\varphi(t)}\,|g(t)|^{\frac{q}{p}}\,e^{-i\,\varphi(t)}\,|g(t)|\,dt \\ &= \int_a^b |g(t)|^{\frac{q}{p}+1}\,dt = \int_a^b |g(t)|^{q\left(\frac{1}{p}+\frac{1}{q}\right)}\,dt = \int_a^b |g(t)|^q\,dt \\ &= \|g\|_q^q = \|g\|_q \cdot \|g\|_q^{q-1} = \|g\|_q \cdot \|f\|_p. \end{aligned}$$

It follows from

$$|T_g f| = T_g f = \|g\|_q \|f\|_p \le \|T_g\| \cdot \|f\|_p,$$

that $\|g\|_q \le \|T_g\|$. Since already $\|T_g\| \le \|g\|_q$, we must have $\|T_g\| = \|g\|_q$.

2.3 Banach spaces

Example 2.16 *Show that a closed subspace of a Banach space is itself a Banach space.*

Let U be a closed subspace of a Banach space V. Since V is complete, it follows from EXAMPLE 1.10 that U is also complete, hence U is a Banach space.

Example 2.17 *Let V_i, $i = 1, 2, \dots, n$, be normed vector spaces, with norms $\|\cdot\|_i$, $i = 1, 2, \dots, n$. The product space $V_1 \times V_2 \times \cdots \times V_n = \bigotimes_{i=1}^n V_i$ is defined by*

$$\bigotimes_{i=1}^n V_i = \{(x_1, x_2, \dots, x_n) \mid x_i \in V_i,\ i = 1, 2, \dots, n\}.$$

In $\bigotimes_{i=1}^n V_i$ we use coordinate wise addition:

$$(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n),$$

and scalar multiplication:

$$\lambda(x_1, x_2, \dots, x_n) = (\lambda x_1, \lambda x_2, \dots, \lambda x_n),$$

and we define the norm by

$$\|(x_1, x_2, \dots, x_n)\| = \sum_{i=1}^n \|x_i\|_i.$$

Show that $\bigotimes_{i=1}^n V_i$ with this norm is a normed vector space, and show that if all the spaces V_i with their respective norms are Banach spaces, then $\bigotimes_{i=1}^n V_i$ is a Banach space.

We shall prove the claim by induction over n. For $n = 1$ there is nothing to prove.

If $n = 2$, then clearly $V_1 \times V_2$ is a vector space with the operations addition and scalar multiplication defined above. Then we shall prove that

$$|(x_1, x_2)\| = \|x_1\|_1 + \|x_2\|_2$$

is a norm.
Clearly, $\|(x_1, x_2)\| \geq 0$, and if $\|(x_1, x_2)\| = \|x_1\|_1 + \|x_2\|_2 = 0$, then both $\|x_1\|_1 = 0$ and $\|x_2\|_2 = 0$, thus $x_1 = 0$ og $x_2 = 0$.
Furthermore,

$$\|\lambda(x_1, x_2)\| = \|(\lambda x_1, \lambda x_2)\| = \|\lambda x_1\|_1 + \|\lambda x_2\|_2 = |\lambda| \left(\|x_1\|_1 + \|x_2\|_2\right) = |\lambda| \cdot \|(x_1, x_2)\|.$$

Finally,

$$\begin{aligned}
\|(x_1, x_2) + (y_1, y_2)\| &= \|(x_1 + y_1, x_2 + y_2)\| = \|x_1 + y_1\|_1 + \|x_2 + y_2\|_2 \\
&\leq \|x_1\|_1 + \|y_1\|_1 + \|x_2\|_2 + \|y_2\|_2 \\
&= (\|x_1\|_1 + \|x_2\|_2) + (\|y_1\|_1 + \|y_2\|_2) \\
&= \|(x_1, x_2)\| + \|(y_1, y_2)\|,
\end{aligned}$$

and we have proved that $\|\cdot\|$ is a norm on $V_1 \times V_2$.

Then assume that both V_1 and V_2 are complete, and let $((x_1^n, x_2^n))_n$ be a Cauchy sequence on $V_1 \times V_2$. It follows from

$$\|x_i^n - x_i^m\|_i \le \|(x_1^n - x_1^m, x_2^n - x_2^m)\| = \|(x_1^n, x_2^n) - (x_1^m, x_2^m)\|, \qquad t = 1,\, 2,$$

that $(x_i^n)_n$ are Cauchy sequences on V_i, $i = 1,\, 2$, hence convergent with limit values x_i, $i = 1,\, 2$. By this construction we then get

$$\|(x_1, x_2) - (x_1^n, x_2^n)\| = \|x_1 - x_1^n\|_1 + \|x_2 - x_2^n\|_2 \to 0 \qquad \text{for } n \to +\infty,$$

proving that $(x_1^n, x_2^n) \to (x_1, x_2) \in V_1 \times V_2$. We have proved that $V_1 \times V_2$ is complete, thus $(V_1 \times V_2, \|\cdot\|)$ is a Banach space.

Assume that the claims are true for some $n \in \mathbb{N}$ (this is true by the above for $n = 1$ and for $n = 2$), and consider $\bigotimes_{i=1}^{n+1} U_i$, where each U_i is a normed vector space (a Banach space). We define

$$V_1 = \bigotimes_{i=1}^{n} U_i \qquad \text{and} \qquad V_2 = U_{n+1}.$$

It follows from the assumption of the induction that $(V_1, \|\cdot\|_n^\star)$ is a normed vector space (or a Banach space) under the given assumptions, and the same is true for the space $(V_2, \|\cdot\|_{n+1})$. It only remains to notice that

$$\|(x_1, x_2, \dots, x_n)\|_n^\star = \|x_1\|_1 + \|x_2\|_2 + \cdots \|x_n\|_n,$$

hence

$$\|(x_1, \dots, x_n, x_{n+1})\| = \|(x_1, \dots, x_n)\|_n^\star + \|x_{n+1}\|_{n+1}.$$

It follows that $\bigoplus_{i=1}^{n+1} U_i$ is a normed vector space (or a Banach space) under the given assumptions.

Example 2.18 *Assume that V and U are normed spaces and $f : V \to U$ is a continuous mapping, and assume that $X \subset V$ is a compact subset. Show that the image $f(X) \subset U$ is compact.*
Show that a real function attains both maximum and minimum on a compact set.

There are several definitions of compactness. We shall here use *sequential compactness*, which is defined by X being sequential compact, if every sequence on X has a convergent subsequence.

We shall prove that if $f : V \to U$ is continuous, and $X \subset V$ is compact, then the image $f(X) \subset U$ is also compact.

Let $(y_n) \subset f(X)$ be any sequence on the image $f(X)$. There exists a sequence $(x_n) \subset X$, such that $y_n = f(x_n)$ for every $n \in \mathbb{N}$. Since X is compact, (x_n) has a convergent subsequence $(x_n') \subseteqq (x_n)$, where $x_n' \to x_0 \in X$ for $n \to +\infty$.

Now, f is continuous at $x_0 \in X$, so to every $\varepsilon > 0$ there exists a $\delta > 0$, such that

$$\|f(x_n') - f(x_0)\|_U < \varepsilon \qquad \text{for } \|x_n' - x_0\|_V < \delta.$$

Then $(x_n') \to x_0$ implies that there exists an $n_0 \in \mathbb{N}$, such that

$$\|x_n' - x_0\|_V < \delta \qquad \text{for all } n \geq n_0.$$

We have for the same n_0 that

$$\|f(x_n') - f(x_0)\|_U < \varepsilon \qquad \text{for all } n \geq n_0,$$

which means that $(f(x_n'))$ converges towards $f(x_0)$, thus every sequence $(y_n) = (f(x_n)) \subseteq f(X)$ has s convergent subsequence $(y_n') = (f(x_n'))$. Note for the limit point that $f(x_0) \in f(X)$.

Assume that $f : X \to \mathbb{R}$ is continuous, where X is a compact subset of a normed space. It follows from the above that $f(X) \subseteq \mathbb{R}$ is compact, thus closed and bounded in $\mathbb{R}$. In particular, f has both a maximum value and a minimum value.

Example 2.19 *Show that any finite dimensional subspace of a normed vector space is a Banach space.*

Let $(V, \|\cdot\|)$ be the normed space, and let U be a finite dimensional subspace of V. Let $e_1, \dots, e_k$, denote a for U. It follows from EXAMPLE 2.8 that there exists a constant $c > 0$ (corresponding to the basis $e_1, \dots, e_k$), such that

$$\left\| \sum_{i=1}^{k} \alpha_i e_i \right\| \geq c\left(|\alpha_1| + \cdots + |\alpha_k|\right).$$

Let $x^n = \sum_{i=1}^{k} \alpha_i^n e_i$ denote a Cauchy sequence on U, thus

$$\forall \varepsilon > 0\, \exists N\, \forall m,\, n \geq N : \|x^m - x^n\| = \left\| \sum_{i=1}^{k} \left(\alpha_i^m - \alpha_i^n\right) e_i \right\| < \varepsilon.$$

Then in particular,

$$|\alpha_i^m - \alpha_i^n| \leq \sum_{i=1}^{k} |\alpha_i^m - \alpha_i^n| \leq \frac{1}{c} \left\| \sum_{i=1}^{k} \left(\alpha_i^m - \alpha_i^n\right) e_i \right\| < \frac{\varepsilon}{c} \qquad \text{for } m,\, n \geq N.$$

It follows that $\left(\alpha_i^n\right)_n$ is a Cauchy sequence on $\mathbb{C}$ for every $i = 1, \dots, k$, hence convergent, $\alpha_i^n \to \alpha_i$ for $n \to +\infty$.
In this way we construct an element

$$x = \sum_{i=1}^{k} \alpha_i e_i \in U.$$

It remains to be proved that $x^n \to x$ for $n \to +\infty$. However,

$$\|x - x^n\| = \left\| \sum_{i=1}^{k} \left(\alpha_i - \alpha_i^n\right) e_i \right\| \leq \sum_{i=1}^{k} |\alpha_i - \alpha_i^n| \cdot \|e_i\| \to 0 \quad \text{for } n \to +\infty,$$

because every term in the finite sum tends towards 0 for $n \to +\infty$. This proves that every finite dimensional subspace of a normed vector space is a Banach space.

Example 2.20 *Let V be a Banach space. A series $\sum_{k=0}^{\infty} x_k$, $x_k \in V$, is* convergent *if the sequence (s_n), where*

$$s_n = \sum_{k=0}^{n} x_k,$$

is convergent in V.
Show that $\sum_{k=0}^{\infty} \|x_k\| < \infty$ implies that $\sum_{k=0}^{\infty} x_k$ is convergent.
Does the convergence of $\sum_{k=0}^{\infty} x_k$ imply that $\sum_{k=0}^{\infty} \|x_k\| < \infty$?
What if the space V is only assumed to be a normed space?

1) Given a Banach space V. It suffices to prove that (s_n) is a Cauchy sequence.
Let $\varepsilon > 0$ be given. Since

$$\sum_{k=0}^{\infty} \|x_k\| < +\infty,$$

is finite, there exists an N, such that

$$\sum_{k=N}^{\infty} \|x_k\| < \varepsilon.$$

It holds for $n > m \geq N$ that

$$\|s_n - s_m\| = \left\| \sum_{k=0}^{n} x_k - \sum_{k=1}^{m} x_k \right\| = \left\| \sum_{k=m+1}^{n} x_k \right\| \leq \sum_{k=m+1}^{n} \|x_k\| \leq \sum_{k=N}^{\infty} \|x_k\| < \varepsilon,$$

thus (s_n) is a Cauchy sequence in a Banach space, hence also convergent.

2) It is well-known that the claim does not hold in the simplest possible Banach space $(\mathbb{R}, |\cdot|)$, because there exist conditional convergent series like e.g.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2,$$

which are not absolutely convergent,

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

3) This is not true, either. Denote by c the vector space consisting of real sequences (x_n), where $x_n = 0$ eventually, e.g. for $n \geq N(x)$. Choose as norm,

$$\|x\| = \sqrt{\sum_{n=1}^{\infty} x_n^2}.$$

Then c is dense in ℓ^2, and $c \neq \ell^2$.

Choose $x_n = \dfrac{1}{n}\, e_n$. Then

$$\left\| \sum_{n=1}^{\infty} x_n \right\|^2 = \sum_{n=1}^{\infty} \|x_n\|^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

så $\sum_{n=1}^{\infty} x_n \in \ell^2$.

Clearly, $\sum_{n=1}^{\infty} x_n$ is not zero, eventually, while all $s_n = \sum_{k=1}^{n} x_k$ have this property. Hence

$$c \ni s_n \to \sum_{n=1}^{\infty} x_n \in \ell^2 \setminus c.$$

Example 2.21 *Let* $(V, \|\cdot\|)$ *denote a normed space. Let* V' *denote the set of all bounded linear functionals on* $(V, \|\cdot\|)$*. The set* V' *is organized as a vector space by the operations*

$$(f+g)(x) = f(x) + g(x), \qquad \textit{for all } x \in V,$$

$$(\alpha f)(x) = \alpha\, f(x), \qquad \textit{for all } x \in V,$$

and we introduce a norm on V' *by*

$$\|f\|' = \sum_{\|x\| \le 1} |f(x)|.$$

Prove that $(V', \|\cdot\|')$ *is a Banach space. It is called the* dual space V.

We shall first show that $\|\cdot\|'$ is a norm on V'. It is obvious that $\|f\|' \ge 0$. If $\|f\|' = 0$, then

$$\sup_{\|x\| \le 1} |f(x)| = 0.$$

Then we have $\left\| \dfrac{x}{\|x\|} \right\| = 1$ for arbitrary $x \neq 0$, hence

$$|f(x)| = \left| f\left(\|x\| \cdot \frac{x}{\|x\|} \right) \right| = \|x\| \cdot \left| f\left(\frac{x}{\|x\|} \right) \right| = 0.$$

It follows from $f(0) = 0$ that $f(x) = 0$ for every $x \in V$, thus $f \equiv 0$. Furthermore,

$$\|\alpha f\|' = \sup_{\|x\| \le 1} |\alpha f(x)| = |\alpha| \cdot \sup_{\|x\| \le 1} |f(x)| = |\alpha| \cdot \|f\|',$$

and finally,

$$\begin{aligned} \|f+g\|' &= \sup_{\|x\| \le 1} |f(x) + g(x)| \le \sup_{\|x\| \le 1} (|f(x)| + |g(x)|) \\ &\le \sup_{\|x\| \le 1} |f(x)| + \sup_{\|x\| \le 1} |g(x)| = \|f\|' + \|g\|', \end{aligned}$$

and we have proved that $\|\cdot\|'$ is a norm.

Assume that (f_n) is a Cauchy sequence on V', i.e.

$$\forall \varepsilon > 0 \, \exists N \in \mathbb{N} \, \forall m, n \geq N : \|f_n - f_m\| < \varepsilon.$$

This means that

$$\|f_n - f_m\|' = \sup_{\|x\| \leq 1} |f_n(x) - f_m(x)| < \varepsilon \qquad \text{for all } m, n \geq N,$$

i.e. we have for every x, for which $\|x\| \leq 1$ that $(f_n(x))$ is a Cauchy sequence in $\mathbb{C}$, hence convergent.

For any $x \neq 0$ it follows that $\dfrac{x}{\|x\|}$ is a unit vector, thus

$$\forall \varepsilon > 0 \, \exists N_x \in \mathbb{N} \, \forall m, n \geq N_x : \|f_n - f_m\|' < \frac{\varepsilon}{\|x\|},$$

which only means that

$$|f_n(x) - f_m(x)| = \|x\| \cdot \left| f_n\left(\frac{x}{\|x\|}\right) - f_m\left(\frac{x}{\|x\|}\right) \right| < \|x\| \cdot \frac{\varepsilon}{\|x\|} = \varepsilon,$$

so $(f_n(x))$ is convergent for every $x \in V \setminus \{0\}$. If $x = 0$, we just get $f_n(0) = 0 \to 0$ for $n \to +\infty$. If we put

$$f(x) = \lim_{n \to +\infty} f_n(x),$$

then we have defined a functional on V for which in particular $f(0) = 0$. It remains only to prove that 1) f is linear, and at 2) f is bounded. However,

$$f(x+\lambda y) = \lim_{n\to+\infty} f_n(x+\lambda y) = \lim_{n\to+\infty} \{f_n(x) + \lambda f_n(y)\} = f(x) + \lambda f(y),$$

proving the linearity. Then

$$\begin{aligned} (11)\|f\|' &= \sup_{\|x\|\le 1} |f(x)| = \sup_{\|x\|\le 1} |f(x) - f_n(x) + f_n(x)| \\ &\le \sup_{\|x\|\le 1} |f(x) - f_n(x)| + \sup_{\|x\|\le 1} |f_n(x)| \\ &= \sup_{\|x\|\le 1} |f(x) - f_n(x)| + \|f_n\|'. \end{aligned}$$

Choose n, such that for all $m \ge n$,

$$\|f_n - f_m\|' = \sup_{\|x\|\le 1} |f_n(x) - f_m(x)| < 1.$$

Then $f_m(x) \in B(f_n(x), 1)$ for every x, for which $\|x\| \le 1$. Since $f_m(x) \to f(x)$ for $m \to +\infty$, we have $f(x) \in \overline{B(f_n(x), 1)}$, so $|f_n(x) - f(x)| \le 1$ for all x, for which $\|x\| \le 1$. From this we infer that

$$\sup_{\|x\|\le 1} |f(x) - f_n(x)| \le 1.$$

Therefore, if n is chosen as above, then it follows from (11) that $\|f\|' \le 1 + \|f_n\|'$, hence f is bounded, and we have proved that every Cauchy sequence on V' is convergent, i.e. V' is a Banach space.

2.4 The Lebesgue integral

'n

Example 2.22 *Let $f \in L^1(\mathbb{R})$.*

1) Can we conclude that $f(x) \to 0$ for $|x| \to \infty$?

2) Can we find $a, b \in \mathbb{R}$ such that $|f(x)| \le b$ for $|x| \ge a$?

In both cases the answer is 'no'. For example, $g(x) = x \cdot 1_{\mathbb{Z}}(x)$ fulfils none of the conditions, and

$$\int |g(x)|\, dx = 0.$$

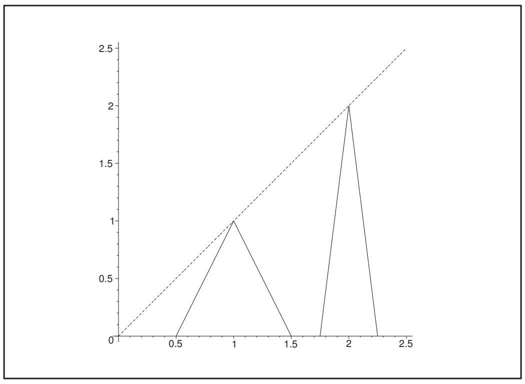


Figure 4: The graph of a *continuous function* $f(x)$, which does not fulfil the two requirements.

We shall now construct a function f, which is *continuous* and Lebesgue integrable, and which does not fulfil any of the two requirements above. Let

$$f(x) = \begin{cases} n & \text{for } x = n, \quad n \in \mathbb{N}, \\ 0 & \text{for } x = n \pm 2^{-n}, \quad n \in \mathbb{N}, \\ \text{piecewise linear}, & \text{otherwise}. \end{cases}$$

Clearly, f is continuous and satisfies neither (1) nor (2). We shall only prove that f is integrable. Now, $f \ge 0$, so

$$\int_{-\infty}^{+\infty} f(x)\, dx = \sum_{n=1}^{+\infty} \frac{1}{2}\, n \cdot 2 \cdot 2^{-n} = \sum_{n=1}^{+\infty} n\, 2^{-n} < +\infty,$$

and the claim is proved.

Remark 2.1 For completeness we here add the full proof. We have

$$\begin{aligned} \sum_{n=1}^{+\infty} n \cdot 2^{-n} &= 2\sum_{n=1}^{+\infty} n \cdot 2^{-(n+1)} = 2\sum_{n=2}^{+\infty}(n-1)2^{-n} = 2\sum_{n=2}^{+\infty} n \cdot 2^{-n} - 2\sum_{n=2}^{+\infty} 2^{-n} \\ &= 2\sum_{n=1}^{+\infty} n \cdot 2^{-n} - 2 \cdot 1 \cdot 2^{-1} - \sum_{n=1}^{+\infty} 2^{-n} = 2\sum_{n=1}^{+\infty} n \cdot 2^{-n} - 1 - 1, \end{aligned}$$

hence by a rearrangement,

$$\sum_{n=1}^{+\infty} n \cdot 2^{-n} = 2.$$

ALTERNATIVELY one may exploit that

$$\frac{d}{dz}\left(\frac{1}{1-z}\right) = \frac{1}{(1-z)^2} = \frac{d}{dz}\left(\sum_{n=0}^{+\infty} z^n\right) = \sum_{n=1} +\infty n\, z^{n-1},$$

for $|z| < 1$. When we insert $z = \frac{1}{2}$, we easily get the result.$\Diamond$

Example 2.23 *Prove that if $f : \mathbb{R} \to \mathbb{R}$ is monotonous, then f has at most countably many points of discontinuity.*

We may assume that f is increasing, thus $f(x) \geq f(y)$ for $x > y$. We may even restrict ourselves to the interval $[0, 1]$, because the number of intervals of the form $[n, n + 1]$, $n \in \mathbb{Z}$, is countable. This means that we may assume that $f(x) = 0$ for $x \leq 0$, and $f(x) = 1$ for $x \geq 1$.

Let $\{x_j \mid j \in J\}$ be the set of all points of discontinuity in $[0, 1]$. Then to any x_j we can find an interval I_j with interior points on the Y-axis, such that $f(x) \notin I_j$ for all $x \in [0, 1]$, i.e. one jumps over the values in I_j over.

Every I_j can be "numbered" by a rational number $q_j \in I_j$, because $\mathbb{Q}$ is dense in $\mathbb{R}$. This means that $\{x_j \mid j \in J\}$ contains just as many elements, as there are different elements in

$$\{q_j \mid j \in J\} \subseteqq \mathbb{Q}.$$

Now, $\mathbb{Q}$ is countable, so $\{q_j \mid j \in J\}$ is countable, and thus $\{x_j \mid j \in J\}$ is at most countable.

Define

$$f(x) = 2^{-n+1} \qquad \text{for } x \in \left]\frac{1}{n+1}, \frac{1}{n}\right], \qquad n \in \mathbb{N}.$$

Then f is monotonous of the countably many points of discontinuity $\left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{N} \setminus \{1\} \right\}$, showing that there exist monotonous functions with a countable number of points of discontinuity.

An ALTERNATIVE proof is the following: We may as before assume that f is increasing on the interval $[0, 1]$ with $f(x) = 0$ for $x \leq 0$ and $f(x) = 1$ for $x \geq 1$.

If x_0 is a point of discontinuity, then $f(x) \leq f(x_0)$ for every $x \leq x_0$. Hence, if $x_n \nearrow x_0$, then $(f(x_n))$ is an increasing bounded sequence of numbers, so $(f(x_n))$ is convergent with the limit value c.

Let $y_n \nearrow x_0$ be another such sequence of numbers. Then $(f(y_n)) \to c'$. We shall prove that $c = c'$. This is done INDIRECTLY.

Assume (e.g.) that $c < c'$, and let $0 < \varepsilon < c' - c$. Corresponding to this ε there exists an N, such that

$$|c' - f(y_n)| = c' - f(y_n) < \varepsilon \qquad \text{for all } n \geq N.$$

To any y_n we can find an x_m, such that $y_n < x_m < x_0$, hence

$$f(y_n) \le f(x_m) \qquad [\le c].$$

Then it follows that

$$\varepsilon < |c' - c| = c' - c = c' - f(y_n) + f(y_n) - c < \varepsilon + f(y_n) - c,$$

so $f(y_n) - c > 0$, and we have come to the contradiction

$$c < f(y_n) \le f(x_m) \le c \qquad \text{for } n \ge N.$$

We therefore conclude that $c' = c$.

Since the limit value is the same, no matter how $x_n \nearrow x_0$ is chosen, we conclude that

$$c = \lim_{x \to x_0 -} f(x).$$

We prove in a similar way that $\lim_{x \to x_0 +} f(x)$ exists, and that these two values are different at any point of discontinuity.

Define the jump at a point of discontinuity x_0 as

$$\sigma_0 = \lim_{x \to x_0 +} f(x) - \lim_{x \to x_0 -} f(x) > 0.$$

If $x_0 < x_1$ are both points of discontinuity, then it follows from that the function is monotonous that

$$\lim_{x \to x_0 +} f(x) \le \lim_{x \to x_1 -} f(x).$$

Let $\{x_j \mid j \in J\}$ denote the set of point of discontinuity in $[0,1]$. The image is contained in $[0,1]$, hence

$$\sum_{x_j} \sigma_j \le 1,$$

and the sum is finite. Every $\sigma_j > 0$, so the sum is at most countable, thus $J \subseteq \mathbb{N}$, and the claim is proved.

Example 2.24 *Prove that* $f(x) = \dfrac{|\sin x|}{x}$ *is not Lebesgue integrable on* $[\pi, +\infty[$*, thus* $f \notin L^1([\pi, +\infty[)$.
Hint*: Consider*

$$f_n(x) = \begin{cases} \dfrac{|\sin x|}{x}, & \pi \le x \le n\pi, \\ 0, & \textit{otherwise}, \end{cases}$$

and exploit that $f_n(x) \nearrow f(x)$ *and* $\int_\pi^\infty f_n(x)\,dx \ge \dfrac{1}{3}\sum_{k=2}^n \dfrac{1}{k}$.

Let f_n be given as above. Then clearly,

$$0 \le f_n(x) \nearrow f(x).$$

Furthermore,

$$\begin{aligned}\int_\pi^\infty f_n(x)\,dx &= \int_\pi^{n\pi} \frac{|\sin x|}{x}\,dx = \sum_{k=2}^n \int_{(k-1)\pi}^{k\pi} \frac{|\sin x|}{x}\,dx \\ &\geq \sum_{k=2}^n \int_{(k-1)\pi}^{k\pi} \frac{1}{k\pi}\cdot|\sin x|\,dx = \sum_{k=2}^n \frac{1}{k\pi}\left|\int_{(k-1)\pi}^{k\pi} \sin x\,dx\right| \\ &= \sum_{k=2}^n \frac{1}{k\pi}\left|\int_0^\pi \sin x\,dx\right| = \sum_{k=2}^n \frac{2}{k\pi} = \frac{2}{\pi}\sum_{k=2}^n \frac{1}{k} \to +\infty \qquad \text{for } n\to+\infty,\end{aligned}$$

and we infer that f is not Lebesgue integrable, i.e. f does not belong to $L^1([\pi,\infty[)$.

Example 2.25 *Give a simple proof of Hölder's inequality in the case of $p = q = 2$ for the spaces of sequences.*

We shall more precisely prove (Bohnenblust-Bunjakovski)-Cauchy-Schwarz-(Sobčyk)'s inequality

$$\sum_{i=1}^{+\infty} |x_i\overline{y}_i| = \sum_{i=1}^{\infty} |x_i|\cdot|y_i| \leq \|x\|_2\cdot\|y\|_2,$$

if $x,\, y\in\ell^2$.

Using that $x + \lambda y \in \ell^2$ for every $\lambda \in \mathbb{R}$, we get

$$\begin{aligned} 0 &\leq \|x+\lambda y\|_2^2 = \sum_{i=1}^{+\infty}(x_i+\lambda y_i)\cdot(\overline{x}_i+\lambda\,\overline{y}_i) \\ &= \sum_{i=1}^{+\infty}\left\{|x_i|^2+\lambda^2|y_i|^2+\lambda\,\overline{x}_i y_i+\lambda\,x_i\overline{y}_i\right\} \\ &= \lambda^2\sum_{i=1}^{+\infty}|y_i|^2+\lambda\left\{\sum_{i=1}^{+\infty}\overline{x}_i y_i+\sum_{i=1}^{+\infty}x_i\overline{y}_i\right\}+\sum_{i=1}^{+\infty}|x_i|^2, \end{aligned}$$

which we write in the form

$$\lambda^2\cdot|y\|_2^2+\lambda\left\{\sum_{i=1}^{+\infty}\overline{x}_i y_i+\sum_{i=1}^{+\infty}x_i\overline{y}_i\right\}+\|x\|_2^2\geq 0.$$

This must hold for every real $\lambda \in \mathbb{R}$, so we must have

$$\begin{aligned} 0 &\geq B^2-4AC=\left\{\sum_{i=1}^{\infty}\overline{x}_i y_i+\sum_{i=1}^{+\infty}x_i\overline{u}_i\right\}^2-4\|x\|_2^2\|y\|_2^2 \\ &= 4\left(\operatorname{Re}\left\{\sum_{i=1}^{+\infty}\overline{x}_i y_i\right\}-\{\|x\|_2\|y\|_2\}^2\right), \end{aligned}$$

hence

$$\left|\operatorname{Re}\left\{\sum_{i=1}^{\infty}\overline{x}_i y_i\right\}\right|\leq\|x\|_2\|y\|_2.$$

When x_i and y_i are all real, the inequality follows immediately.

In general,

$$\begin{aligned} \sum_{i=1}^{+\infty}|x_i\overline{y}_i| &= \sum_{i=1}^{+\infty}|x_i|\cdot|\overline{y}_i|\leq\|\,|x|\,\|_2\cdot\|\,|y|\,\|_2 \\ &= \left\{\sum_{i=1}^{+\infty}|x_i|^2\right\}^{\frac{1}{2}}\left\{\sum_{i=1}^{\infty}|y_i|^2\right\}^{\frac{1}{2}}=\|x\|_2\|y\|_2, \end{aligned}$$

and the claim is proved.

Example 2.26 *Let $w(t) \geq 0$ be a non-negative function on $\mathbb{R}$. We define a linear functional I_w by*

$$I_w(f) = \int_{\mathbb{R}} f(t)\, w(t)\, dt,$$

for $f\,w \in L^1(\mathbb{R})$.
Assume that $|f|^p w$ and $|g|^q w$ are in $L^1(\mathbb{R})$, where f and g are (measurable) functions and $1 < p$, $q < \infty$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

1. *Show the generalized Hölder's inequality*

$$|I_w(fg)| \leq \{I_w(|f|^p)\}^{\frac{1}{p}} \{I_w(|g|^q)\}^{\frac{1}{q}},$$

where the inequality for $w = 1$ can be taken to be valid.

Now recall the Gamma function,

$$\Gamma(x) = \int_0^{\infty} t^{x-1}\, e^{-t}\, dt, \qquad x > 0,$$

with the property $\Gamma(x+1) = x\,\Gamma(x)$ for $x > 0$.

2. *Use the generalized Hölder's inequality with*

$$w(t) = t^{n-1}\, e^{-t}, \quad 0 < t < \infty, \qquad \text{and} \qquad p = q = 1,$$

to show that

$$\Gamma\left(n + \frac{1}{2}\right) \leq \frac{n!}{\sqrt{n}}, \qquad n \in \mathbb{N}.$$

3. *Give a similar estimation of $\Gamma(n+1)$ by taking*

$$w(t) = t^{n-\frac{1}{2}}\, e^{-t}, \quad 0 < t < \infty, \qquad \text{and} \qquad p = q = 2,$$

and deduce that

$$\frac{n!}{\sqrt{n + \frac{1}{2}}} \leq \Gamma\left(n + \frac{1}{2}\right) \leq \frac{n!}{\sqrt{n}}, \qquad n \in \mathbb{N}.$$

1) We get from $w(t) \geq 0$ that both $w^{1/p}$ and $w^{1/q}$ are defined and that $w^{1/p} \cdot w^{1/q} = w$, and $f \cdot w^{1/p} \in L^p(\mathbb{R})$ and $g \cdot w^{1/q} \in L^q(\mathbb{R})$. Applying the usual Hölder's inequality we get

$$\begin{aligned} |I_w(f \cdot g)| &= \left| \int_{-\infty}^{+\infty} f(t)\, g(t)\, w(t)\, dt \right| \leq \int_{-\infty}^{+\infty} \left| f(t)\, w^{\frac{1}{p}}(t) \right| \cdot \left| g(t)\, w^{\frac{1}{q}}(t) \right| dt \\ &\leq \left\{ \int_{-\infty}^{+\infty} |f(t)|^p w(t)\, dt \right\}^{\frac{1}{p}} \left\{ \int_{-\infty}^{+\infty} |g(t)|^q w(t)\, dt \right\}^{\frac{1}{q}} = \{I_w(|f|^p)\}^{\frac{1}{p}} \{I_w(|g|^q)\}^{\frac{1}{q}}, \end{aligned}$$

and we have proved the generalized Hölder's inequality.

2) Then apply this generalized inequality on $f(t) = \sqrt{t} \cdot 1_{\mathbb{R}_+}(t)$ and $g(t) = 1$, and $w(t) = t^{n-1}\, e^{-1} \cdot 1_{\mathbb{R}_+}(t)$, we get

$$\begin{aligned}
\Gamma\left(n+\frac{1}{2}\right) &= \int_0^{+\infty} \sqrt{t} \cdot 1 \cdot t^{n-1}\, e^{-t}\, dt \leq \{I_w(t)\}^{\frac{1}{2}}\, \{I_w(1)\}^{\frac{1}{2}} \\
&= \left\{\int_0^{+\infty} t \cdot t^{n-1}\, e^{-t}\, dt\right\}^{\frac{1}{2}} \left\{\int_0^{+\infty} 1 \cdot t^{n-1}\, e^{-t}\, dt\right\}^{\frac{1}{2}} \\
&= \left\{\int_0^{+\infty} t^n\, e^{-t}\, dt\right\}^{\frac{1}{2}} \left\{\int_0^{+\infty} t^{n-1}\, e^{-t}\, dt\right\}^{\frac{1}{2}} \\
&= \{\Gamma(n+1)\}^{\frac{1}{2}}\{\Gamma(n)\}^{\frac{1}{2}} = \{n!(n-1)!\}^{\frac{1}{2}} = \left\{\frac{(n!)^2}{n}\right\}^{\frac{1}{2}} = \frac{n!}{\sqrt{n}}.
\end{aligned}$$

3) Finitely, let $f(t) = \sqrt{t} \cdot 1_{\mathbb{R}_+}(t)$ and $g(t) = 1$, and $w(t) = t^{n-\frac{1}{2}}\, e^{-t} \cdot 1_{\mathbb{R}_+}(t)$. Then we get with $p = q = 2$,

$$\begin{aligned}
n! &= \Gamma(n+1) = \int_0^{+\infty} t^n\, e^{-t}\, dt = \int_0^{+\infty} \sqrt{t} \cdot 1 \cdot t^{n-\frac{1}{2}}\, e^{-t}\, dt \\
&\leq \left\{\int_0^{+\infty} t^{t+\frac{1}{2}}\, e^{-t}\, dt\right\}^{\frac{1}{2}} \left\{\int_0^{+\infty} t^{n-\frac{1}{2}}\, e^{-t}\, dt\right\}^{\frac{1}{2}} \\
&= \left\{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\frac{1}{2}\right)\right\}^{\frac{1}{2}} = \left\{\left(n+\frac{1}{2}\right)\left[\Gamma\left(n+\frac{1}{2}\right)\right]^2\right\}^{\frac{1}{2}} = \sqrt{n+\frac{1}{2}} \cdot \Gamma\left(n+\frac{1}{2}\right),
\end{aligned}$$

and we have

$$\frac{n!}{\sqrt{n+\frac{1}{2}}} \leq \Gamma\left(n+\frac{1}{2}\right) \leq \frac{n!}{\sqrt{n}}.$$

Remark 2.2 Furthermore, if we use that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, it follows from the functional equation that

$$\begin{aligned}
\Gamma\left(n+\frac{1}{2}\right) &= \left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right) = \cdots = \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) \\
&= \frac{(2n-1)(2n-3)\cdots 3\cdot 1}{2\cdot 2\cdots 2\cdot 2}\sqrt{\pi} \\
&= \frac{\sqrt{\pi}}{2^n}\cdot\frac{2n}{2n}\cdot\frac{2n-1}{1}\cdot\frac{2n-2}{2(n-1)}\cdots\frac{4}{2\cdot 2}\cdot\frac{3}{1}\cdot\frac{2}{2\cdot 1}\cdot\frac{1}{1} \\
&= \frac{\sqrt{\pi}}{2^n}\cdot\frac{(2n)!}{2^n\cdot n!} = \frac{\sqrt{\pi}}{4^n}\binom{2n}{n} n!,
\end{aligned}$$

hence by insertion

$$\frac{n!}{\sqrt{n+\frac{1}{2}}} \leq \frac{\sqrt{\pi}}{4^n}\binom{2n}{n} n! \leq \frac{n!}{\sqrt{n}},$$

thus

$$\frac{4^n}{\sqrt{\pi\left(n+\frac{1}{2}\right)}} \le \binom{2n}{n} \le \frac{4^n}{\sqrt{\pi\, n}},$$

which is in agreement with Stirling's formula

$$n! \sim \sqrt{2\pi}\cdot n^{n+\frac{1}{2}}\, e^{-n},$$

because

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2} \sim \frac{\sqrt{2\pi}\cdot(2n)^{2n+\frac{1}{2}}\, e^{-2n}}{\left\{\sqrt{2\pi}\cdot n^{n+\frac{1}{2}}\cdot e^{-n}\right\}^2} = \frac{1}{\sqrt{2\pi}}\cdot\frac{(2n)^{2n+\frac{1}{2}}}{n^{2n+1}} = \frac{2^{2n}\sqrt{2}}{\sqrt{2\pi\, n}} = \frac{4^n}{\sqrt{\pi\, n}}. \qquad \Diamond$$

Example 2.27 *Let*

$$F = \{f \in C^2([0,1]) \mid f(0) = f(1) = 0\} \subsetneqq L^2([0,1]).$$

1) Show that $\|f'\|^2 \leq \|f\| \cdot \|f''\|$ *for* $f \in F$.

2) Let $f \in F$. *Show that* $|f(x)| \leq \|f'\| \sqrt{x}$ *for* $0 \leq x \leq 1$, *and deduce that*

$$\|f\| \leq \frac{1}{\sqrt{2}} \|f'\| .$$

3) Show that for $f \in C^2([0,1])$ *with* $f(0) = f(1)$ *we have*

$$\|f'\| \leq \frac{1}{\sqrt{2}} \|f''\| .$$

4) Show by a counterexample that the result from question (3) is not valid for general $f \in C^2([0,1])$.

1) We deduce from $f \in C^2([0,1])$ and $f(0) = f(1) = 0$ and a partial integration, followed by an application of the Cauchy-Schwarz inequality that

$$\begin{aligned} \|f'\|_2^2 &= \int_0^1 f'(t)\,\overline{f'(t)}\,dt = \left[f(t)\,\overline{f'(t)}\right]_0^1 - \int_0^1 f(t)\,\overline{f''(t)}\,dt \\ &\leq 0 + \int_0^1 |f(t)| \cdot |f''(t)|\;dt \leq \|f\|_2 \cdot \|f''\|_2 . \end{aligned}$$

2) From

$$f(x) = f(0) + \int_0^x f'(t)\,dt = \int_0^1 1_{[0,x]}(t)\,f'(t)\,dt$$

follows by Cauchy-Schwarz's inequality that

$$|f(x)| = \left|\int_0^1 1_{[0,x]}(t)\,f'(t)\,dt\right| \leq \left\|1_{[0,x]}\right\|_2 \cdot \|f'\|_2 = \sqrt{x} \cdot \|f'\|_2 ,$$

where we have used that

$$\left\|1_{[0,x]}\right\|_2 = \sqrt{\int_0^1 1_{[0,x]}(t)\,dt} = \sqrt{\int_0^x 1\,dt} = \sqrt{x}.$$

3) Let $f \in F$. It follows from (1) and (2) that

$$\begin{aligned} \|f'\|_2^2 &\leq \|f\|_2 \cdot \|f''\|_2 = \left\{\int_0^1 |f(x)|^2 dx\right\}^{\frac{1}{2}} \cdot \|f''\|_2 \leq \left\{\int_0^1 x\;\|fd'\|_2^2\;dx\right\}^{\frac{1}{2}} \|f''\|_2 \\ &= \left\{\int_0^1 x\,dx\right\}^{\frac{1}{2}} \|f'\|_2 \cdot \|f''\|_2 = \frac{1}{\sqrt{2}}\;\|f'\|_2\,\|f''\|_2 . \end{aligned}$$

If $\|f'\|_2 = 0$, the inequality is obvious.
If $\|f'\|_2 > 0$, we obtain the inequality when we divide by $|f'\|_2$.

We derived the above by assuming that $f \in F$, thus $f(0) = f(1) = 1$.

Now, let $f(0) = f(1) = c$. Then $f(x) - c \in F$, hence

$$\|f'\|_2 = \|(f-x)'\|_2 \leq \frac{1}{\sqrt{2}} \|(f-c)''\}_2 = \frac{1}{\sqrt{2}} \|f''\|_2 .$$

4) Finally, let $f(x) = a\,x$. Then $f'(x) = a$ and $f''(x) = 0$, hence

$$\|f'\|_2 = |a| \qquad \text{og} \qquad \|f''\|_2 = 0,$$

and the inequality is not fulfilled for any $a \neq 0$.

Example 2.28 *1) Let $1 \leq p \leq q \leq \infty$. Show that $\ell^p \subset \ell^q$.*

2) Let $1 \leq r < p < 2r$ and assume that the sequence (x_n) satisfies

$$\sum_{n=1}^{\infty} n\;|x_n|^p < \infty.$$

Show that $(x_n) \in \ell^r$.

1) If $(x_n) \in \ell^p$, then $\sum_{n=1}^{+\infty} |x_n|^p < +\infty$. In particular, $x_n \to 0$ for $n \to +\infty$, hence there exists an $N \in \mathbb{N}$, such that $|x_n| < 1$ for all $n \geq N+1$.

For $p = q$ there is nothing to prove. If $1 \leq p < q < +\infty$, then

$$\sum_{n=1}^{+\infty} |x_n\|^q = \sum_{n=1}^{N} |x_n|^q + \sum_{n=N+1}^{+\infty} |x_n|^p \cdot |x_n|^{q-p} < \sum_{n=1}^{N} |x_n|^q + \sum_{n=N+1}^{+\infty} |x_n|^p < +\infty,$$

showing that $(x_n) \in \ell^q$.

If $1 \leq p < q = +\infty$, then clearly

$$\sup_{n \in \mathbb{N}} |x_n| \leq \max\{1, \sup\{|x_n| \mid n = 1, \dots, N\}\} < +\infty,$$

and we conclude that $(x_n) \in \ell^\infty$.

2) Then let $1 \leq r < p < 2r$ and assume that

$$\sum_{n=1}^{+\infty} n\;|x_n|^p < +\infty.$$

Let $0 < s < 1$. We shall somehow way apply Hölder's inequality with $\tilde{p} = \dfrac{1}{s} > 1$ and $\tilde{q} = \dfrac{1}{1-s} > 1$. The assumption shall also be applied later os, so we get by a reasonable rewriting and an application of Hölder's inequality,

$$\sum_{n=1}^{+\infty} |x_n|^r = \sum_{n=1}^{+\infty} \{n\,|x_n|^p\}^s \left\{\frac{1}{n^s}\,|x_n|^{r-sp}\right\} \le \left\{\sum_{n=1}^{+\infty} n\,|x_n|^p\right\}^s \cdot \left\{\sum_{n=1}^{+\infty} n^{-\frac{s}{1-s}}\,|x_n|^{\frac{r-sp}{1-s}}\right\}^{1-s}.$$

By the assumption, the former factor is finite for every $s \in]0,1[$. The task is to choose s in this interval, such that the latter factor also becomes finite.

Using that $2r > p$, we get $\dfrac{r-sp}{1-s} = 0$ for $s = \dfrac{r}{p} > \dfrac{1}{2}$. We get with this s that $\alpha = \dfrac{s}{1-s} > 1$ and

$$\sum_{n=1}^{+\infty} n^{-\frac{s}{1-s}}\,|x_n|^{\frac{r-sp}{1-s}} = \sum_{n=1}^{+\infty} \frac{1}{n^\alpha} \cdot |x_n|^0 = \sum_{n=1}^{+\infty} \frac{1}{n^\alpha} < +\infty,$$

and the latter factor in the estimate above is finite for this particular $s = \dfrac{r}{p} \in \left]\dfrac{1}{2}, 1\right[$. Now, s does not occur in the sum, we are estimating, so we conclude that

$$\sum_{n=1}^{+\infty} |x_n|^r < +\infty,$$

and we have proved that $(x_n) \in \ell^r$.

Example 2.29 *Define in* $\mathbb{R}^2$ *the function*

$$\|x\| = \|(x_1, x_2)\| = \left(\sqrt{|x_1|} + \sqrt{|x_2|}\right)^2.$$

Is it a norm?
Sketch the set $\{(x_1, x_2) \mid \|(x_1, x_2)\| \le 1\}$.

First note that $\|x\| = \|x\|_p$, where $p = \dfrac{1}{2} < 1$.

The first two conditions of a norm are trivially fulfilled, so we shall only consider the triangle inequality. We shall prove that it is *not* satisfied. It suffices to find two vectors x and y, for which the triangle inequality does not hold.

Choose $x = (1, 0)$ and $y = (0, 1)$. Then $\|x\| = \|y\| = 1$, and

$$\|x + y\| = \|(1, 1)^{\|} = (\sqrt{1} + \sqrt{1})^2 = 4,$$

hence

$$\|x + y\| = 4 > 2 = \|x\| + \|y\|,$$

and the triangle inequality is not fulfilled, and $\|\cdot\|$ is not a norm.

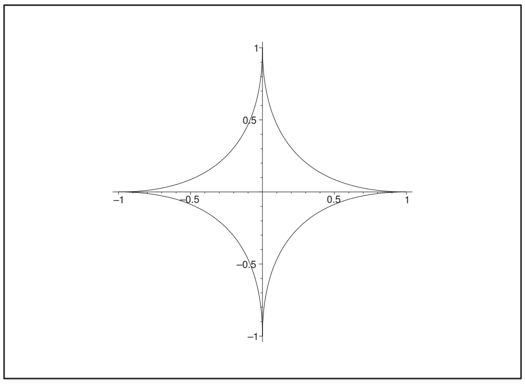


Figure 5: The unit “ball” corresponding to $\|\cdot\|$.

Remark 2.3 It is not hard to prove that *if* $\|\cdot\|$ is a norm, then the corresponding unit ball is convex. (However, not every convex set will induce a norm).

Since the set, which should be the unit ball clearly is *not* convex (cf. the figure), $\|\cdot\|$ is *not* a norm. ◊

Remark 2.4 Even if $\|\cdot\|_{\frac{1}{2}}$ is not a norm in the usual sense, there exist some applications of it, e.g. in the theory of H^p spaces in Complex Function Theory, and the “norm” of such functions can nevertheless be given a reasonable interpretation.◊

3 Bounded operators

Example 3.1 *Let T be a linear operator from a normed space V into a normed space W.*
Show that the image $T(V)$ is a subspace of W.
Show that the kernel (or null-space) $\ker(T)$ is a subspace of V.
If T is bounded, is it true that $T(V)$ and/or $\ker(T)$ are closed?

1) Let $w_1,\, w_2 \in T(V) \subseteq W$, and let λ be a scalar. We shall prove that

$$w_1 + \lambda w_2 \in T(V).$$

Remark 3.1 It is here of paramount importance that the field of the scalars is the same both places. If e.g. $T : V \to W$ is given by

$$Tx = x + i \cdot 0,$$

where $V = (\mathbb{R}, +, \cdot, \|\cdot\|, \mathbb{R})$ and $W = (\mathbb{C}, +, \cdot, \|\cdot\|, \mathbb{C})$, then T is linear, and $T(V)$ is a subspace of the 2-dimensional space $(\mathbb{C}, +, \cdot, \|\cdot\|, \mathbb{R})$ over $\mathbb{R}$. It is, however, not a subspace of the 1-dimensional space $W = (\mathbb{C}, +, \cdot, \|\cdot\|, \mathbb{C})$ over $\mathbb{C}$, so the claim is not true in this case. $\Diamond$

It follows from the assumption $w_1,\, w_2 \in T(V)$ that there exist v_1 and $v_2 \in V$, such that $w_1 = Tv_1$ and $w_2 = Tv_2$.
If we put $v = v_1 + \lambda v_2 \in V$, then

$$T(V) \ni Tv = T(v_1 + \lambda v_2) = Tv_1 + \lambda\, Tv_2 = w_1 + \lambda\, w_2.$$

2) Now $\ker(T) = \{v \in V \mid Tv = 0\}$, and T is linear. Hence, if $v_1,\, v_2 \in \ker(T)$, and λ is a scalar, then

$$T(v_1 + \lambda v_2) = Tv_1 + \lambda\, Tv_2 = 0 + \lambda \cdot 0 = 0,$$

thus $v_1 + \lambda\, v_2 \in \ker(T)$, and $\ker(T)$ is a subspace.

3) If T is bounded, then T is continuous. Now $\{0\} \subset W$ is closed, so $\ker(T) = T^{\circ -1}(\{0\})$ is closed.

On the other hand, $T(V)$ need not be closed, which is demonstrated by the example below.

Choose $V = W = C^0([0,1])$ with the norm $\|\cdot\|_\infty$, and let $T : V \to W$ be given by

$$Tf(t) = \int_0^t f(s)\, dx, \qquad t \in [0,1].$$

Then T is bounded,

$$|Tf(t)| = \left|\int_0^t f(s)\, ds\right| \le \int_0^t |f(s)|\, ds \le \int_0^1 |f(s)|\, ds \le 1 \cdot \|f\|_{infty}, \qquad t \in [0,1],$$

hence

$$\|Tf\|_\infty \le 1 \cdot \|f\|_\infty, \qquad \|T\| \le 1.$$

Furthermore,

$$T(V) = \{w \in C^1([0,1]) \mid w(0) = 0\}$$

is dense in

$$\{w \in C^0([0,1]) \mid w(0) = 0\} \subset W,$$

without being equal to it.

That $T(V)$ is dense, is seen in the following way: Every polynomial of constant term 0 lies in $T(V)$. The claim then follows by a suitable variant of Weierstraß's Approximation Theorem.

There exist of course C^0-functions which are not of class C^1, hence $T(V)$ is not equal to the smallest closed subspace

$$\{w \in C^0([0,1]) \mid w(0) = 0\}$$

which contains $T(V)$ (because $T(V)$ is dense in this space).

Example 3.2 *In the Banach space ℓ^p, $1 \le p \le \infty$, we have a sequence (x_n) converging to an element x, where*

$$x_n = (x_{n1}, x_{n2}, \dots) \qquad \text{and} \qquad x = (x_1, x_2, \dots).$$

Show that if $x_n \to x$ in ℓ^p, then $x_{nk} \to x_k$ for all $k \in \mathbb{N}$.
If $x_{nk} \to x_k$ for all $k \in \mathbb{N}$, is it true that $x_n \to x$ in ℓ^p?

Let $x_n \to x$ in ℓ^p, $1 \le p < \infty$, thus $\|x - x_n\|_p \to 0$ for $n \to \infty$, i.e.

$$\sum_{k=1}^{\infty} |x_k - x_{nk}|^p = \|x - x_n\|_p^p \to 0 \qquad \text{for } n \to \infty.$$

If $p = \infty$, then $x_n \to x$ in ℓ^∞ means that

$$\|x - x_n\|_\infty = \sup_k |x_k - x_{nk}| \to 0 \qquad \text{for } n \to \infty.$$

In both cases we get for every fixed k that

$$|x_k - x_{nk}| \le \|x - x_n\|_p \to 0 \qquad \text{for } n \to \infty,$$

thus $x_{nk} \to x_k$ for $n \to \infty$, and the first claim is proved.

On the other hand, if $x_{nk} \to x_k$ for every fixed k, then we cannot conclude that $x_n \to x$ in ℓ^p. Just choose

$$x_n = (\delta_{nk}) = (0, \dots, 0, 1, 0, \dots)$$

with 1 on place number n, and 0 otherwise.

We have for this sequence that $x_{nk} \to 0$ for every fixed k, thus $x = 0$.

On the other hand,

$$\|x_n\|_p = \|x_n - 0\|_p = \left\{\sum_{k=1}^{\infty} |\delta_{nk}|^p\right\}^{\frac{1}{p}} = 1 \qquad \text{for } 1 \le p < +\infty,$$

and

$$\|x_n\|_\infty = \|x_n - 0\|_\infty = 1,$$

so none of these sequences converges towards, i.e. the sequence does not converge in any ℓ^p, $1 \le p \le +\infty$.

Example 3.3 *Let T be a linear mapping from $\mathbb{R}^m$ to $\mathbb{R}^n$, both equipped with the 2-norm. Let (a_{ij}) denote a real $n \times m$ matrix corresponding to T. Show that T is a bounded linear operator with $\|T\|^2 \le \sum_i \sum_j a_{ij}^2$.*

We get (cf. Example 1.23)

$$\begin{aligned}\|Tx\|_2^2 &= \left\|\left(\sum_{j=1}^{m} a_{ij}x_j\right)_{i\in\mathbb{N}}\right\|_2^2 = \sum_{i=1}^{n}\left\{\sum_{j=1}^{m} a_{ij}x_k\right\}^2 = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{m} a_{ij}x_j a_{ik}x_k \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{m} (a_{ij}x_k)\cdot(a_{ik}x_j)\,.\end{aligned}$$

Then note that

$$|a_{ij}x_k| \cdot |a_{ik}x_j| \le \frac{1}{2}\,a_{ij}^2x_k^2 + \frac{1}{2}\,a_{ik}^2x_j^2.$$

By insertion of this inequality,

$$\begin{aligned}\|Tx\|_2^2 &= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{m}(a_{ij}x_k)\cdot(a_{ik}x_j) \le \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{m} a_{ij}^2x_k^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{m} a_{ik}^2x_j^2 \\ &= 2\cdot\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}^2 \cdot \sum_{k=1}^{m} x_k^2 = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}^2 \cdot \|z\|_1^2.\end{aligned}$$

Since $\|T\|^2$ is the smallest constant, for which we have such an estimate, we have

$$\|T\|^2 \le \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2.$$

Example 3.4 *Let T be a linear operator from a normed space V into a normed space W, and assume that V is finite dimensional. Show that T must be bounded.*

The space V is finite dimensional, thus we can choose a basis $e_1, \dots, e_n$ for V, where $\|e_k\|_V = 1$. Then for every $v \in V$,

$$\begin{aligned}\|Tv\|_W &= \left\| T\left(\sum_{j=1}^n \lambda_j e_j\right)\right\|_W = \left\|\sum_{j=1}^n \lambda_j T e_j\right\|_W \le \sum_{j=1}^n |\lambda_j| \cdot \|Te_j\|_W \\ &\le \max\left\{\|Te_j\|_W \;\middle|\; j = 1, \cdots, n\right\} \cdot \sum_{j=1}^n |\lambda_j|.\end{aligned}$$

If we can prove that there exists a constant $c > 0$, such that

$$(12) \quad \sum_{j=1}^n |\lambda_j| \le c \left\|\sum_{j=1}^n \lambda_j e_j\right\|_V \qquad \text{for every } \lambda_1, \dots, \lambda_n,$$

then

$$\|Tv\|_W \le c \cdot \max_j \|Te_j\|_W \cdot \|v\|_V,$$

which shows that T is bounded

$$\|T\| \le c \cdot \max_{j=1,\dots,n} \|Te_j\|_W.$$

We shall therefore only prove (12).

Indirect proof. Assume that (12) does *not* hold, i.e. assume that

$$(13)\ \forall N \in \mathbb{N}\, \exists\, \lambda_{N,1}, \dots, \lambda_{N,n} : \sum_{j=1}^{n} |\lambda_{N,j}| > N \left\| \sum_{j=1}^{n} \lambda_{N,j} e_j \right\|_V .$$

Due to the homogeneity we may assume that $\lambda_{N,j}$ is chosen, such that

$$\sum_{j=1}^{n} |\lambda_{N,j}| = 1.$$

Then it follows from (13) that $\|v_N\|_V \le \frac{1}{N}$, hence

$$v_N = \sum_{j=1}^{n} \lambda_{N,j} e_j \to 0 \qquad \text{for } N \to \infty.$$

Now, $e_1, \dots, e_n$ is a basis for V, hence $\lambda_{N,j} \to 0$ for $N \to \infty$ for every $j = 1, \dots, n$. In particular, there is an $N_0 \in \mathbb{N}$, such that for every $N \ge N_0$ we have $|\lambda_{N,j}| < \frac{1}{2n}$. This gives us the following *contradiction*

$$1 = \sum_{j=1}^{n} |\lambda_{N,j}| < \sum_{j=1}^{n} \frac{1}{2n} = \frac{1}{2}.$$

We have now proved that (13) does *not* hold, hence (12) holds instead, and as proved previously (12) implies that T is bounded, and the claim is proved.

Example 3.5 *Let T be a linear operator from a finite dimensional vector space into itself. Show that T is injective if and only if T is surjective.*

Let $T : V \to V$ be linear, where $\dim V = n$. Let $e_1, \dots, e_n$ form a basis. Now, T is linear, so T is injective, if and only if $Tu = Tv$, i.e. $T(u-v) = 0$ implies that $u = v$, or put in another way, $u - v = 0$. Thus T is injective, if and only if

$$(14)\ Tv = 0 \quad \Longrightarrow \quad v = 0.$$

Now assume that T is injective. We shall prove that $Te_1, \dots, Te_n \in V$ are linearly independent.

Assume that $\lambda_1 Te_1 + \cdots + \lambda_n Te_n = 0$. Then by the linearity,

$$0 = \lambda_1 Te_1 + \cdots + \lambda_n Te_n = T\left(\lambda_1 e_1 + \cdots + \lambda_n e_n\right),$$

and we conclude using (14) that

$$\lambda_1 e_1 + \cdots + \lambda_n e_n = 0.$$

Since $e_1, \dots, e_n$ is a basis for V, we must have $\lambda_1 = \cdots = \lambda_n = 0$, and it follows that $Te_1, \dots, Te_n$ are n linearly independent vectors in the image $T(V)$. Then

$$n \ge \dim T(V) \ge n, \qquad \text{thus} \qquad \dim T(V) = n,$$

hence $T(V) = V$, and we have proved that T is surjective.

Assume conversely that T is surjective. To the basis formed by $e_1, \dots, e_n \in V$ corresponds the vectors $f_1, \dots, f_n \in V$, where

$$Tf_1 = e_1, \quad \dots, \quad Tf_n = e_n.$$

If $\lambda_1 f_1 + \cdots + \lambda_n f_n = 0$, then we conclude that

$$0 = T\left(\lambda_1 f_1 + \cdots + \lambda_n f_n\right) = \lambda_1 Tf_1 + \cdots + \lambda_n Tf_n = \lambda_1 e_1 + \cdots + \lambda_n e_n.$$

Using again that $e_1, \dots, e_n$ form a basis for V, we infer that $\lambda_1 = \cdots = \lambda_n = 0$, which again implies that $f_1, \dots, f_n$ form a basis for V.

If $v = \lambda_1 f_1 + \cdots + \lambda_n f_n$ satisfies $Tv = 0$, then

$$0 = Tv = T(\lambda_1 f_1 + \cdots + \lambda_n f_n) = \lambda_1 Tf_1 + \cdots + \lambda_n Tf_n = \lambda_1 e_1 + \cdots + \lambda_n e_n,$$

and we infer again that $\lambda_1 = \cdots = \lambda_n = 0$, hence $v = 0$, and (14) is fulfilled, so T is injective.

Example 3.6 *Let T be the linear mapping from $C^\infty(\mathbb{R})$ into itself given by $Tf = f'$.*
Show that T is surjective?
Is T injective?

Let $f \in C^\infty(\mathbb{R})$. Define $g \in C^\infty(\mathbb{R})$ by

$$g(t) = \int_0^t f(s)\,ds, \qquad t \in \mathbb{R}.$$

Clearly, $Tg = g$, so $T(V) = C^\infty(\mathbb{R})$, and T is surjective.

Define instead

$$g_1(t) = 1 + \int_0^t f(s)\,ds = 1 + g(t) \in C^\infty(\mathbb{R}).$$

Then

$$Tg_1 = f = Tg,$$

and since $g_1 \neq g$, it follows that T is not injective.

Example 3.7 *Let $I = [a, b]$ be a bounded interval and consider the linear mapping T from $C([a, b])$ into itself, given by*

$$Tf(t) = \int_a^t f(s)\, ds.$$

We assume that $C([a, b])$ is equipped with the sup-norm.
Show that T is bounded and find $\|T\|$.
Show that T is injective and find $T^{-1} : T(C([a, b])) \to C([a, b])$.
Is T^{-1} bounded?

When

$$Tf(t) = \int_a^t f(s)\, ds \qquad \text{for } t \in [a, b],$$

then

$$|Tf(t)| = \left| \int_a^t f(s)\, ds \right| \le \int_a^t |f(s)|\, ds \le \|f\|_\infty \int_a^t ds = (t-a)\|f\|_\infty \le (b-a) \cdot \|f\|_\infty,$$

thus

$$\|Tf\|_\infty \le (b-a) \cdot \|f\|_\infty,$$

proving that T is bounded and $\|T\| \le b - a$.

Choose $f(t) = 1$ for every $t \in [a, b]$. Then $\|f\|_\infty = 1$, and

$$Tf(t) = \int_a^t ds = t - a \qquad \text{for } t \in [a, b],$$

hence

$$\|Tf\|_\infty = \sup_{t \in [a,b]} (t - a) = b - a,$$

and we conclude that $\|T\| \ge b - a$, whence by the previously proved result, $\|T\| = b - a$.

Assume that

$$Tf(t) = \int_a^t f(s)\, ds \equiv 0.$$

Since $f \in C([a, b])$, we have $Tf \in C^1([a, b])$ with

$$\frac{d}{dt}\, Tf(t) = f(t) \equiv 0,$$

which shows that $f \equiv 0$, so T is injective.

It follows from the above that $T(C([a, b])) \subseteqq C^1([a, b])$. We get from $Tf(a) = 0$ that even

$$T(C([a, b])) \subseteqq \{g \in C^1([a, b]) \mid g(a) = 0\}.$$

Conversely, if $g \in C^1([a,b])$ and $g(a) = 0$, then $f = g' \in C([a,b])$, and $Tf = g$, and the image becomes

$$T(C([a,b])) = \{g \in C^1([a,b]) \mid g(a) = 0\}.$$

Finally, it is immediately seen that

$$T^{-1} : T(C([a,b])) \to C([a,b])$$

is given by $T^{-1}g = g'$.

The operator T^{-1} is not bounded. We have e.g. that $(t-a)^n \in T(C([a,b]))$, and

$$\|(t-a)^n\|_\infty = \sup_{t \in [a,b]} |(t-a)^n| = (b-a)^n.$$

It follows from $T^{-1}(t-a)^n = n(t-a)^{n-1}$ that

$$\|T^{-1}(t-a)^n\|_\infty = n(b-a)^{n-1} = \frac{n}{b-a}\,\|(t-a)^n\|_\infty,$$

proving that there is no constant $c > 0$, such that

$$\|T^{-1}f\|_\infty \leq c\,\|f\|_\infty, \qquad \text{for all } f \in T(C([a,b])),$$

and T is not bounded.

Example 3.8 *Let T be a bounded linear operator from a normed vector space V into a normed vector space W, and assume that T is surjective. Assume that there is a $c > 0$, such that*

$$\|Tx\| \geq c\,\|x\| \qquad \text{for all } x \in V.$$

show that T^{-1} exists and that $T^{-1} \in B(W,V)$.

We require that T^{-1} exists, so we shall first prove that T is injective, i.e. if $Tx = Ty$, then $x = y$.

The mapping T is linear, so this is equivalent with that $T(x-y) = 0$ implies that $x - y = 0$, or by a slight change of notation:

Assume that $Tx = 0$. Prove that $x = 0$.

When $Tx = 0$, then it follows from the assumption that

$$0 \leq \|x\| \leq \frac{1}{c}\,\|Tx\| = 0, \qquad \text{thus } \|x| = 0, \text{hence } x = 0,$$

and the claim is proved.

We have proved that T is injective, thus T^{-1} exists. Now $T(V) = W$, so $T^{-1} : W \to V$, and T^{-1} is defined on all of W. It remains only to be proved that T^{-1} is bounded.

Let $y \in W$. Then $x = T^{-1}y$ is defined. It follows from the assumption that

$$\|T^{-1}y\| = \|x\| \leq \frac{1}{c}\,\|Tx\| = \frac{1}{c}\,\|T(T^{-1}y)\| = \frac{1}{c}\,\|y\|,$$

which shows that T^{-1} is bounded, $\|T^{-1}\| \leq \dfrac{1}{c}$, and it follows that $T^{-1} \in B(W,V)$.

Example 3.9 *Let V and W be two normed spaces. Prove that $B(V,W)$ is a normed vector space and that $B(V,W)$ is a Banach space, if W is a Banach space.*

It is well-known that $B(V,W)$ is a vector space.
Define $\|T\|$ by

$$\|T\| = \sup\{\|Tx\|_W \mid \|x\|_V \leq 1\}.$$

Then clearly, $\|T\| \geq 0$. If $T \neq 0$, then there exists an $x \in V$, such that $Tx \neq 0$, and we conclude that $\|T\| = 0$, if and only if $T = 0$.

Furthermore,

$$\|\alpha T\| = \sup\{\|\alpha\,Tx\|_W \mid \|x\|_V \leq 1\} = |\alpha| \cdot \sup\{\|Tx\|_W \mid \|x\|_V \leq 1\} = |\alpha| \cdot \|T\|.$$

Finally,

$$\begin{aligned}\|T_1 + T_2\| &= \sup\{\|(T_1+T_2)x\|_W \mid \|x\|_V \leq 1\} \leq \sup\{\|Tx\|_W + \|T_2 x\|_W \mid \|x\|_V \leq 1\} \\ &\leq \sup\{\|T_1 x\|_W \mid \|x\|_V \leq 1\} + \sup\{\|T_2 x\|_W \mid \|x\|_V \leq 1\} = \|T_1\| + \|T_2\|,\end{aligned}$$

and we have proved that $\|\cdot\|$ is a norm on $B(V,W)$, and $B(V,W)$ is a normed vector space.

We now assume that W is a Banach space, thus every Cauchy sequence on W is convergent. We shall prove that $B(V,W)$ becomes a Banach space with the norm introduced above. Let (T_n) be a Cauchy sequence on $B(V,W)$, i.e.

$$\forall\,\varepsilon>0\,\exists\,N\in\mathbb{N}\,\forall\,m,\,n\geq N:\|T_m-T_n\|<\varepsilon.$$

Then it follows from the definition that

$$\|T_m-T_n\|=\sup\{\|(T_m-T_n)x\|_W\mid\|x\|_V\leq 1\}=\sup\{\|T_m-T_nx\|_W\mid\|x\|_V\leq 1\}<\varepsilon.$$

In particular, we have for every $x\in V$, for which $\|x\|_V\leq 1$ that

$$\forall\,\varepsilon>0\,\exists\,N\in\mathbb{N}\,\forall\,m,\,n\geq N:\|T_mx-T_nx\|_W<\varepsilon,$$

which is the condition for (T_nx) being a Cauchy sequence on W. We assumed that W was a Banach space, so it is complete. This implies that (T_nx) is convergent, and it follows that $(T_n(\lambda x))=(\lambda\,T_nx)$ is also convergent in W for every λ, and the condition $\|x\|_V\leq 1$ has become superfluous.

Define an operator $T:V\to W$ by

$$Tx=\lim_{n\to+\infty}T_nx,\qquad x\in V.$$

Then

$$T(x+\lambda y)=\lim_{n\to+\infty}T_n(x+\lambda y)=\lim_{n\to+\infty}\{T_nx+\lambda T_ny\}=\lim_{n\to+\infty}T_nx+\lambda\lim_{n\to+\infty}T_ny=Tx+\lambda\,Ty,$$

which shows that T is linear.

It remains to be proved that $T\in B(V,W)$, i.e. that T is bounded. If $x\in V$ with $\|x\|_V\leq 1$, then

$$\|Tx\|=\left\|\lim_{n\to+\infty}T_nx\right\|\leq\sup_{n\in\mathbb{N}}\|T_nx\|\leq\sup_{n\in\mathbb{N}}\|T_n\|.$$

Since (T_n) is a Cauchy sequence, we have $\sup_{n\in\mathbb{N}}\|T_n\|<+\infty$, and we conclude that $T\in B(V,W)$. Thus we have proved that the Cauchy sequence $(T_n)\subseteqq B(V,W)$ converges towards $T\in B(V,W)$, and we have proved that $B(V,W)$ is a Banach space.

Example 3.10 *Let S, $T\in B(V,V)$. Prove that the composite mapping ST (defined by $(ST)x=S(Tx)$ for $x\in V$) belongs to $B(V,V)$, and that*

$$\|ST\|\leq\|S\|\cdot\|T\|.$$

When S, $T\in B(V,V)$, the composition ST is defined (and linear) on all of V. We shall only prove that ST is bounded. Now, for every $x\in V$,

$$\|(ST)x\|_V=\|S(Tx)\|_V\leq\|S\|\cdot\|Tx\|_V\leq\|S\|\cdot\|T\|\cdot\|x\|_V,$$

so

$$\|ST\|=\sup\{\|(ST)x\|_V\mid\|x\|_V\leq 1\}\leq\sup\{\|S\|\cdot\|T\|\cdot\|x\|_V\mid\|x\|_V\leq 1\}=\|S\|\cdot\|T\|.$$

Example 3.11 *Let V be a Banach space and let $T \in B(V)$ be such that T^{-1} exists and belongs to $B(V)$.*
Show that if $\|T\|$ and $\left\|T^{-1}\right\| \leq 1$, then

$$\|T\| = \left\|T^{-1}\right\| = 1,$$

and $\|Tf\| = \|f\|$ for all $f \in V$.

It follows from the assumptions that T is bijective,

(15) $Tf = g, \qquad T^{-1}g = f.$

We first prove that

$$\|Tf\| = \|f\| \qquad \text{for every } f \in V.$$

This follows from

$$\|Tf\| \leq \|T\| \cdot \|f\| = \|f\| = \left\|T^{-1}f\right\| \leq \left\|T^{-1}\right\| \cdot \|g\| = \|g\| = \|Tf\|.$$

Hence we must have equality everywhere, and in particular,

$$\|Tf\| = \|f\| \qquad \text{for all } f \in V,$$

and

$$\left\|T^{-1}g\right\| = \|g\| \qquad \text{for all } g \in V.$$

Finally, we get

$$\|T\| = \sum\{\|Tf\| \mid \|f\| = 1\} = \sup\{\|f\| \mid \|f\| = 1\} = 1,$$

and

$$\left\|T^{-1}\right\| = \sup\{\|T^{-1}g\| \mid \|g\| = 1\} = \sup\{\|g\| \mid \|g\| = 1\} = 1.$$

Example 3.12 *Let H denote a Hilbert space and let $T \in B(H)$ and assume that there is a positive c such that*

$$|(Tx, x)| \geq c\,\|x\|^2 \qquad \textit{for all } x \in H.$$

Show that T^{-1} exists and belongs to $B(H)$.

Assume that $Tx = 0$. Then

$$0 = |(Tx, x)| \geq c\,\|x\|^2 \geq 0,$$

from which we conclude that $x = 0$, and we have proved that T is injective, so T^{-1} exists.

If $x = T^{-1}y$ for some $y \in H$, then it follows from the estimate

$$c\,\|x\|^2 = c\,\left\|T^{-1}y\right\|^2 \le \left|(y, T^{-1}y)\right| \le \|y\| \cdot \left\|T^{-1}y\right\|,$$

that $\left\|T^{-1}\right\| \le \dfrac{1}{c}$, so T^{-1} is bounded on the image $T(H)$.

It remains to prove that the image $T(H)$ is all of H. Let $x \perp T(H)$. Then we get again that

$$0 = |(Tx, x)| \ge c\,\|x\|^2,$$

which proves that $x = 0$ is the only vector, which is perpendicular to the image, so $\overline{T(H)} = H$. Since T^{-1} is bounded, it has a continuous extension to $\overline{T(H)} = H$, and it follows that $T^{-1} \in B(H)$.

Example 3.13 *Let $p > 1$ and let $f(x,t) \geq 0$ be a (measurable) function on $\mathbb{R}^2$ such that*

$$g(t) = \left\{ \int_{\mathbb{R}} f(x,t)\,dx \right\}^{p-1}$$

exists.

1) Put $q = \dfrac{p}{p-1}$ and show that

$$\left\| \int_{\mathbb{R}} f(x,\cdot)\,dx \right\|_p^p \leq \|g\|_q \int_{\mathbb{R}} \|f(x,\cdot)\|_p\,dx.$$

2) Let $f(x,t)$ be a (measurable) function on $\mathbb{R}^2$ such that the function

$$x \mapsto \|f(x,\cdot)\|_p$$

belongs to $L^1(\mathbb{R})$. Use question 1 to show the inequality

$$\left\| \int_{\mathbb{R}} f(x,\cdot)\,dx \right\|_p \leq \int_{\mathbb{R}} \|f(x,\cdot)\|_p\,dx,$$

first for $p > 1$, and then for $p = 1$.

3) Let $g \in L^p(\mathbb{R})$ and $h \in L^1(\mathbb{R})$. We define the convolution *$g \star h$ by*

$$g \star h(t) = \int_{\mathbb{R}} g(t-x)\,h(x)\,dx.$$

Show that convolution with an $L^1(\mathbb{R})$-function is a linear and bounded mapping from $L^p(\mathbb{R})$ into $L^p(\mathbb{R})$ for any $p > 1$.

1) We get

$$\begin{aligned}
\left\| \int_{\mathbb{R}} f(x,\cdot)\,dx \right\|_p^p &= \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} f(x,t)\,dx \right\}^p dt = \int_{\mathbb{R}} g(t) \left\{ \int_{\mathbb{R}} f(x,t)\,dx \right\} dt \\
&= \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} g(t) \cdot f(x,t)\,dt \right\} dx \leq \int_{\mathbb{R}} \|g\|_q \, \|f(x,\cdot)\|_p \, dx \\
&= \|g\|_q \int_{\mathbb{R}} \|f(x,\cdot)\|_p\,dx.
\end{aligned}$$

2) We may of course assume that $f(x,t) \geq 0$, because we can in general replace f by $|f|$, which gives a more "narrow" estimate. Then we can use the result from 1.

Let $p > 1$. Then

$$\begin{aligned}
\|g\|_q &= \left\{ \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,t)\,dx \right)^{(p-1)\cdot\frac{p}{p-1}} dt \right\}^{\frac{p-1}{p}} = \left(\left\{ \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,t)\,dx \right)^p dt \right\}^{\frac{1}{p}} \right)^{p-1} \\
&= \left\| \int_{\mathbb{R}} f(x,\cdot)\,dx \right\|_p^{p-1},
\end{aligned}$$

which inserted into the result of 1) gives

$$\left\|\int_{\mathbb{R}} f(x,\cdot)\,dx\right\|_p^p \leq \left\|\int_{\mathbb{R}} f(x,\cdot)\,dx\right\|_o^{p-1} \cdot \int_{\mathbb{R}} \|f(x,\cdot)\|_p\,dx.$$

Since $p>1$, this is reduced to

$$\left\|\int_{\mathbb{R}} f(x,\cdot)\,dx\right\|_p \leq \int_{\mathbb{R}} \|f(x,\cdot)\|_p\,dx.$$

When $p=1$, then we get instead by interchanging the order of integration

$$\left\|\int_{\mathbb{R}} f(x,\cdot)\,dx\right\|_1 = \int_{\mathbb{R}}\left\{\int_{\mathbb{R}} f(x,t)\,dx\right\}dt = \int_{\mathbb{R}}\left\{\int_{\mathbb{R}} f(x,t)\,dt\right\}dx = \int_{\mathbb{R}} \|f(x,\cdot)\|_1\,dt.$$

For a general f we get

$$\left\|\int_{\mathbb{R}} f(x,\cdot)\,dx\right\|_1 \leq \left\|\int_{\mathbb{R}} |f(x,\cdot)|\,dx\right\|_p \leq \int_{\mathbb{R}} \|f(x,\cdot)\|_p\,dx,$$

because $\|\,|f(x,\cdot)\|\,\|_p = \|f(x,\cdot)\|_p$.

3) Given $h \in L^1(\mathbb{R})$- Define an operator T by

$$Tg(x) = g\star h(x),$$

for the $g \in L^p(\mathbb{R})$, $p>1$, for which this expression makes sense. Then clearly, T is linear.

Let $g \in L^p(\mathbb{R})$. Using 2) above we get the following estimate, where we allow ourselves to write $\|g\star h\|$ before we have proved that it makes sense,

$$\begin{aligned}\|Tg\|_p &= \|g\star h\|_p = \left\|\int_{\mathbb{R}} g(\star - x)\,h(x)\,dx\right\|_p \\ &\leq \int_{\mathbb{R}} \|g(\star-x)\|_p \cdot h(x)\,dx = \|g\|_p\cdot\|h\|_1 < \infty.\end{aligned}$$

This estimate shows that $g\star h \in L^p(\mathbb{R})$ is defined and that the mapping T is bounded of norm $\|T\| \leq \|h\|_1$.

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