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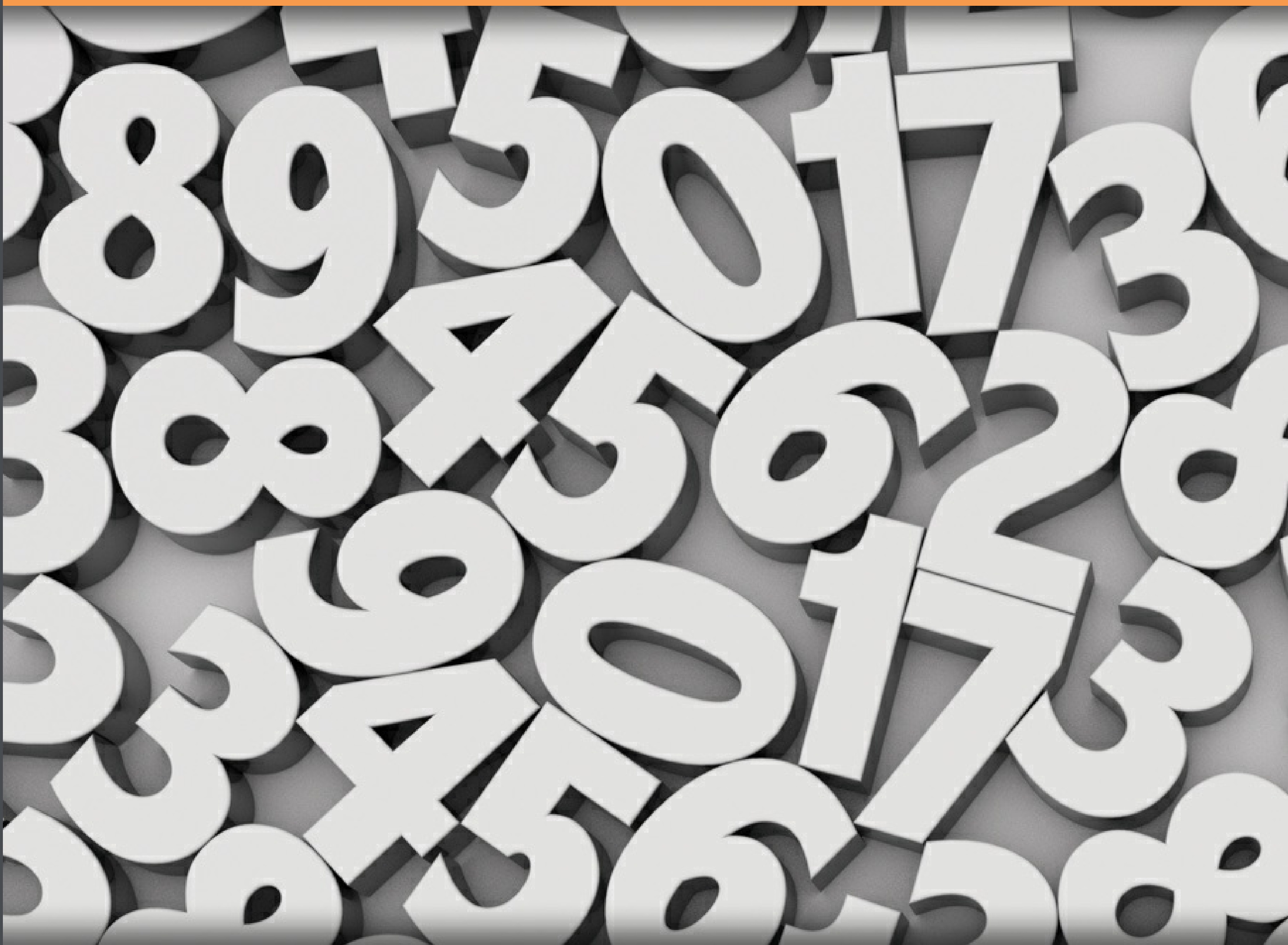
Stability Theory of Large-Scale Dynamical Systems

A. A. Martynyuk; V. G. Miladzhanov



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Contents

Everything should be made as simple as possible,
but not simpler.

ALBERT EINSTEIN

PREFACE

The present monograph deals with some topical problems of stability theory of nonlinear large-scale systems. The purpose of this book is to describe some new applications of Liapunov matrix-valued functions method to the theory of stability of evolution problems governed by nonlinear equations with structural perturbations.

The concept of structural perturbations has extended the possibilities of engineering simulation of the classes of real world phenomena. We have written this book for the broadest audience of potentially interested learners: applied mathematicians, applied physicists, control and electrical engineers, commmunication network specialists, performance analysts, operations researchers, etc., who deal with qualitative analysis of ordinary differential equations, difference equations, impulsive equations, and singular perturbed equations.

To accomplish our aims, we have thought it necessary to make the analysis:

(i) general enough to apply to the many variety of applications which arise in science and engineering, and

(ii) simple enough so that it can be understood by persons whose mathematical training does not extend beyond the classical methods of stability theories which were popular at the end of the twentieth century.

Of course, we understood that it is not possible to achive generality and simplicity in a perfect union but, in fact, the new generalization of direct Liapunov's method give us new possibilities in the direction.

In this monograph the concept of structural perturbations is developed in the framework of four classes of systems of nonlinear equations mentioned above. The direct Liapunov method being one of the main methods of qualitative analysis of solutions to nonlinear systems is used in this monograph in the direction of its generalization in terms of matrix-valued auxiliary functions.

Thus, the concept of structural perturbations combined with the method of Liapunov matrix-valued functions is a methodological base for the new direction of investigations in nonlinear systems dynamics.

The monograph is arranged as follows.

Chapter 1 provides an overview of recent results for four classes of systems of equations (continuous, discrete-time, impulsive, and singular perturbed systems), which are a necessary introduction to the qualitative theory of the same classes of systems of equations but under structural perturbations.

Chapters 2 – 5 expose the mathematical stability theory of equations under structural perturbations. The sufficient existence conditions for various dynamical properties of solutions to the classes of systems of equations under consideration are obtained in terms of the matrix-valued Liapunov functions and are easily available for practical applications. All main results are illustrated by many examples from mechanics, power engineering and automatical control theory.

Final Sections of Chapters 2 – 5 deal with the discussion of some directions of further generalization of obtained results and their applications. To this end new problems of nonlinear dynamics and system theory are involved.

Some of the important features of the monograph are as follows. This is the first book that

(i) treats the stability theory of large scale dynamical systems via matrix-valued Lyapunov functions;
(ii) demonstrates that developing of the direct Lyapunov method for time-continuous, discrete-time, impulsive and singularly perturbed large scale systems via matrix auxiliary functions is a powerful technique for the qualitative study of large scale systems;
(iii) presents sufficient stability conditions in terms of sign definiteness of special matrices;
(iv) shows that utilizing of the matrix-valued Lyapunov functions in investigating the stability theory of large scale dynamical systems is significantly more useful.

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V. G. Miladzhanov

NOTATION

R — the set of all real numbers
$R_+ = [0, +\infty) \subset R$ — the set of all nonnegative numbers
R^k — k-th dimensional real vector space
$R \times R^n$ — the Cartesian product of R and R^n
$G_1 \times G_1$ — topological product
(a, b) — open interval $a < t < b$
$[a, b]$ — closed interval $a \le t \le b$
$A \cup B$ — union of sets A and B
$A \cap B$ — intersection of sets A and B
$\overline{D}$ — closure of set D
∂D — boundary of set D
$N_\tau^+ \triangleq \{\tau_0, \dots, \tau_0 + k, \dots\}, \quad \tau_0 \ge 0, \quad k = 1, 2, \dots$
$\{x\colon \Phi(x)\}$ — set of x's for which the proposition Φ is true
$\mathcal{T} = [-\infty, +\infty] = \{t\colon -\infty \le t \le +\infty\}$ — the largest time interval
$\mathcal{T}_\tau = [\tau, +\infty) = \{t\colon \tau \le t < +\infty\}$ — the right semi-open unbounded interval associated with τ
$\mathcal{T}_i \subseteq R$ — a time interval of all initial moments t_0 under consideration (or, all admissible t_0)
$\mathcal{T}_0 = [t_0, +\infty) = \{t\colon t_0 \le t < +\infty\}$ — the right semi-open unbounded interval associated with t_0
$\|x\|$ — the Euclidean norm of vector x in R^n
$\chi(t; t_0, x_0)$ — a motion of a system at $t \in R$ iff $x(t_0) = x_0$, $\chi(t_0; t_0, x_0) \equiv x_0$
$B_\varepsilon = \{x \in R^n\colon \|x\| < \varepsilon\}$ — open ball with center at the origin and radius $\varepsilon > 0$
$\delta_M(t_0, \varepsilon) = \max\{\delta\colon \delta = \delta(t_0, \varepsilon) \ni x_0 \in B_\delta(t_0, \varepsilon) \Rightarrow \chi(t; t_0, x_0) \in B_\varepsilon, \forall t \in \mathcal{T}_0\}$ — the maximal δ obeying the definition of stability
$\Delta_M(t_0) = \max\{\Delta\colon \Delta = \Delta(t_0), \forall \rho > 0, \forall x_0 \in B_\Delta, \exists \tau(t_0, x_0, \rho) \in (0, +\infty) \ni \chi(t; t_0, x_0) \in B_\rho, \forall t \in \mathcal{T}_\tau\}$ — the maximal Δ obeying the definition of attractivity
$\tau_m(t_0, x_0, \rho) = \min\{\tau\colon \tau = \tau(t_0, x_0, \rho) \ni \chi(t; t_0, x_0) \in B_\rho, \forall t \in \mathcal{T}_\tau\}$ — the minimal τ satisfying the definition of attractivity
$\mathcal{N}$ — a time-invariant neighborhood of original of R^n
$f\colon R \times \mathcal{N} \to R^n$ — a vector function mapping $R \times \mathcal{N}$ into R^n
$C(\mathcal{T}_\tau \times \mathcal{N})$ — the family of all functions continuous on $\mathcal{T}_\tau \times \mathcal{N}$
$C^{(i,j)}(\mathcal{T}_\tau \times \mathcal{N})$ — the family of all functions i-times differentiable on $\mathcal{T}_\tau$ and j-times differentiable on $\mathcal{N}$
$\mathcal{C} = C([-\tau, 0], R^n)$ — the space of continuous functions which map $[-\tau, 0]$ into R^n
$U(t, x)$, $U\colon \mathcal{T}_\tau \times R^n \to R^{s \times s}$ — matrix-valued Liapunov function, $s = 2, 3, \dots, m$
$V(t, x)$, $V\colon \mathcal{T}_\tau \times R^n \to R^s$ — vector Liapunov function
$v(t, x)$, $v\colon \mathcal{T}_\tau \times R^n \to R_+$ — scalar Liapunov function
$D^+ v(t, x)\,(D^- v(t, x))$ — the upper right (left) Dini derivative of v along $\chi(t; t_0, x_0)$ at (t, x)
$D_+ v(t, x)\,(D_- v(t, x))$ — the lower right (left) Dini derivative of v along $\chi(t; t_0, x_0)$ at (t, x)

$D^*v(t,x)$ — denotes that both $D^+v(t,x)$ and $D_+v(t,x)$ can be used
$Dv(t,x)$ — the Eulerian derivative of v along $\chi(t;t_0,x_0)$ at (t,x)
$\lambda_i(\cdot)$ — the i-th eigenvalue of a matrix $(\cdot)$
$\lambda_M(\cdot)$ — the maximal eigenvalue of a matrix $(\cdot)$
$\lambda_m(\cdot)$ — the minimal eigenvalue of a matrix $(\cdot)$

1

GENERALITIES

1.1 Introduction

This Chapter contains description of some classes of large scale dynamical systems and a concept of nonclassical structural perturbations. These types of systems are investigated in Chapters 2 − 5 for the same classes of large scale systems of equations which, however, contain nonclassical structural perturbations.

The Chapter is arranged as follows.

Section 1.2 deals with description of stability problems for continuous, discrete-time, impulsive and singularly perturbed large scale dynamical systems. The definitions for various types of motion stability of nonautonomous and nonlinear systems are presented.

Section 1.3 presents some approaches to qualitative analysis of nonlinear systems under structural perturbations.

Section 1.4 exposes general concept of stability under nonclassical structural perturbations.

Section 1.5 sets out a version of generalization of the Liapunov direct method via matrix-valued Liapunov functions as a main approach to stability analysis under nonclassical structural perturbations in the book.

Finaly, in Section 1.6 there are some comments to Chapter 1.

1.2 Some Types of Large-Scale Dynamical Systems

In this Section the notions of motion stability corresponding to the motion properties of nonautonomous systems are presented being necessary in subsequent presentation. Basic notions of the method of matrix-valued Liapunov functions are discussed and general theorems and some corollaries are set out.

Throughout this Section, real systems of ordinary differential equations will be considered. Notations will be used.

1.2.1 Ordinary differential large-scale systems We start with a general description of a dynamic system of ordinary differential equations

$$\frac{dy_i}{dt} = Y_i(t,\, y_1, \dots, y_n), \quad i = 1, 2, \dots, n, \tag{1.2.1}$$

or in the equivalent vector form

$$\frac{dy}{dt} = Y(t, y), \tag{1.2.2}$$

where $x \in R^n$, $Y(t,y) = (Y_1(t,y), \dots, Y_n(t,y))^{\mathrm{T}}$, $Y\colon \mathcal{T} \times R^n \to R^n$. We will assume that the right-hand part of (1.2.2) satisfies the solution existence and uniqueness conditions of the Cauchy problem

$$(1.2.3)\qquad \frac{dy}{dt} = Y(t,y), \quad y(t_0) = y_0,$$

for any $(t_0, y_0) \in \mathcal{T} \times \Omega$, $0 \in \Omega$ and Ω is an open connected subset of R^n.

Let $y(t) = \psi(t; t_0, y_0)$ be the solution of system (1.2.2), definite on the interval $[t_0, \tau)$ and noncontinuable behind the point τ, i.e. $y(t)$ is not definite for $t = \tau,$. Then

$$(1.2.4)\qquad \overline{\lim}\,\|y(t)\| = +\infty \quad \text{as} \quad t \to \tau - 0.$$

Using solution $y(t)$ and the right-hand part of system (1.2.2) we construct the vector-function

$$(1.2.5)\qquad f(t,x) = Y(t,\, x + \psi(t)) - Y(t, \psi(t))$$

and consider the system

$$(1.2.6)\qquad \frac{dx}{dt} = f(t,x).$$

It is easy to verify that the solutions of systems (1.2.2) and (1.2.6) are correlated as

$$x(t) = y(t) - \psi(t)$$

on the general interval of existence of solutions $y(t)$ and $\psi(t)$. It is clear that system (1.2.6) has a trivial solution $x(t) \equiv 0$. This solution corresponds to the solution $y(t) = \psi(t)$ of system (1.2.2). Obviously, the reduction of system (1.2.2) to system (1.2.6) is possible only when the solution $y(t) = \psi(t)$ is known.

Qualitative investigation of solutions of system (1.2.2) relatively solution $\psi(t)$ is reduced to the investigation of behavior of solution $x(t)$ to system (1.2.6) which differs “little” from the trivial one for $t = t_0$.

In motion stability theory system (1.2.6) is called the system of perturbed motion equations.

Since equations (1.2.6) can generally not be solved analytically in closed from, the qualitative properties of the equilibrium state are of great practical interest. We start with a series of definitions.

Definition 1.2.1 The equilibrium state $x = 0$ of the system (1.2.6) is:

(i) *stable* iff for every $t_0 \in \mathcal{T}_i$ and every $\varepsilon > 0$ there exists $\delta(t_0, \varepsilon) > 0$, such that $\|x_0\| < \delta(t_0, \varepsilon)$ implies

$$\|x(t; t_0, x_0)\| < \varepsilon \quad \text{for all} \quad t \in \mathcal{T}_0;$$

(ii) *uniformly stable* iff both (i) holds and for every $\varepsilon > 0$ the corresponding maximal δ_M obeying (i) satisfies

$$\inf\,[\delta_M(t,\varepsilon)\colon\ t \in \mathcal{T}_i] > 0;$$

(iii) *stable in the whole* iff both (i) holds and

$$\delta_M(t,\varepsilon) \to +\infty \quad \text{as} \quad \varepsilon \to +\infty \quad \text{for all} \quad t \in R;$$

(iv) *uniformly stable in the whole* iff both (ii) and (iii) hold;

(v) *unstable* iff there are $t_0 \in \mathcal{T}_i$, $\varepsilon \in (0, +\infty)$ and $\tau \in \mathcal{T}_0$, $\tau > t_0$, such that for every $\delta \in (0, +\infty)$ there is x_0, $\|x_0\| < \delta$, for which

$$\|x(\tau; t_0, x_0)\| \geq \varepsilon.$$

Definition 1.2.2 The equilibrium state $x = 0$ of the system (1.2.6) is:

(i) *attractive* iff for every $t_0 \in \mathcal{T}_i$ there exists $\Delta(t_0) > 0$ and for every $\zeta > 0$ there exists $\tau(t_0; x_0, \zeta) \in [0, +\infty)$ such that $\|x_0\| < \Delta(t_0)$ implies $\|x(t; t_0, x_0)\| < \zeta$ for all $t \in (t_0 + \tau(t_0; x_0, \zeta), +\infty)$;

(ii) x_0-*uniformly attractive* iff both (i) is true and for every $t_0 \in R$ there exists $\Delta(t_0) > 0$ and for every $\zeta \in (0, +\infty)$ there exists $\tau_u[t_0, \Delta(t_0), \zeta] \in [0, +\infty)$ such that

$$\sup\,[\tau_m(t_0; x_0, \zeta)\colon\ x_0 \in B_\Delta(t_0)] = \tau_u(t_0, x_0, \zeta);$$

(iii) t_0-*uniformly attractive* iff both (i) is true, there is $\Delta > 0$ and for every $(x_0, \zeta) \in B_\Delta \times (0, +\infty)$ there exists $\tau_u(R, x_0, \zeta) \in [0, +\infty)$ such that

$$\sup\,[\tau_m(t_0); x_0, \zeta)\colon\ t_0 \in \mathcal{T}_i] = \tau_u(\mathcal{T}_i, x_0, \zeta);$$

(iv) *uniformly attractive* iff both (ii) and (iii) hold, that is, that (i) is true, there exists $\Delta > 0$ and for every $\zeta \in (0, +\infty)$ there is $\tau_u(R, \Delta, \zeta) \in [0, +\infty)$ such that

$$\sup\,[\tau_m(t_0; x_0, \zeta)\colon\ (t_0, x_0) \in \mathcal{T}_i \times B_\Delta] = \tau(\mathcal{T}_i, \Delta, \zeta);$$

The properties (i) – (iv) hold *"in the whole"* iff (i) is true for every $\Delta(t_0) \in (0, +\infty)$ and every $t_0 \in \mathcal{T}_i$.

Definition 1.2.3 The equilibrium state $x = 0$ of the system (1.2.6) is:

(i) *asymptotically stable* iff it is both stable and attractive;

(ii) *equi-asymptotically stable* iff it is both stable and x_0-uniformly attractive;

(iii) *quasi-uniformly asymptotically stable* iff it is both uniformly stable and t_0-uniformly attractive;

(iv) *uniformly asymptotically stable* iff it is both uniformly stable and uniformly attractive;

(v) The properties (i) – (iv) hold *"in the whole"* iff both the corresponding stability of $x = 0$ and the corresponding attraction of $x = 0$ hold in the whole;

(vi) *exponentially stable* iff there are $\Delta > 0$ and real numbers $\alpha \geq 1$ and $\beta > 0$ such that $\|x_0\| < \Delta$ implies

$$\|x(t; t_0, x_0)\| \leq \alpha \|x_0\| \exp[-\beta(t - t_0)], \quad \text{for all} \quad t \in \mathcal{T}_0, \quad \text{for all} \quad t_0 \in \mathcal{T}_i.$$

This holds *in the whole* iff it is true for $\Delta = +\infty$.

In the investigation of both system (1.2.2) and (1.2.11) the solution $x(t)$ is assumed to be definite for all $t \in \mathcal{T}$ (for all $t \in \mathcal{T}_0$).

Further, with reference to system (1.2.6) we introduce the notations

$$\begin{gathered} x^{\mathrm{T}} = (x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, \dots, x_m^{\mathrm{T}})^{\mathrm{T}} \in R^n, \quad x_s \in R^{n_s}, \\ f^{\mathrm{T}}(t, x) = (f_1^{\mathrm{T}}(t, x_1), \dots, f_m^{\mathrm{T}}(t, x_m))^{\mathrm{T}}, \\ g^{\mathrm{T}}(t, x) = (g_1^{\mathrm{T}}(t, x), \dots, g_m^{\mathrm{T}}(t, x))^{\mathrm{T}}, \quad \sum_{s=1}^{m} n_s = n. \end{gathered} \tag{1.2.7}$$

System (1.2.6) has the meaning of a large scale system, if for its dimensions being large enough the decomposition to the form

$$\frac{dx_s}{dt} = f_s(t, x_s) + g_s(t, x_1, \dots, x_n), \quad s = 1, 2, \dots, m, \tag{1.2.8}$$

with the independent subsystems

$$\frac{dx_s}{dt} = f_s(t, x_s), \quad s = 1, 2, \dots, m, \tag{1.2.9}$$

and interconnection functions

$$g_s\colon \; g_s(t, x_1, \dots, x_n), \quad s = 1, 2, \dots, m, \tag{1.2.10}$$

simplifies the procedure of qualitative analysis of its solutions.

The decomposition is correct if systems (1.2.6) and (1.2.8) are equivalent by their dynamical properties.

Since the decomposition of system (1.2.6) to (1.2.8) can be accomplished in several ways, the dynamical properties of its independent subsystems (1.2.9) may differ. Besides, the interconnection functions (1.2.10) can influnce essentially the dynamical properties of subsystems (1.2.8).

Note that if subsystems (1.2.9) possess strong stability, for example, the zero solution of subsystems (1.2.9) is uniformly asymptotically stable or exponentially stable, then for bounded at each instant of time interconnection functions (1.2.10) the solution of system (1.2.7) possesses the same type of stability even in the case of $g_s(t, x_1, \dots, x_n)$ not equal to zero for $x_1 = x_2 = \dots = x_m = 0$, though being small at each instant of time on semiaxis.

1.2.2 Ordinary difference large-scale systems Consider a system with finite number of degrees of freedom described by the system of difference equations in the form

$$x(\tau + 1) = f(\tau,\, x(\tau)), \tag{1.2.11}$$

where $\tau \in N_\tau^+ \triangleq \{\tau_0,\, \dots,\, \tau_0 + k,\, \dots\}$, $\tau_0 \geq 0$, $k = 1, 2, \dots,$ $x \in R^n$, $f\colon N_\tau^+ \times R^n \to R^n$, $f(\tau, x)$ is continuous in x. Let solution $x(\tau; \tau_0, x_0)$ of system (1.2.11) be definite for all $\tau \in N_\tau^+$ and $x(\tau_0; \tau_0, x_0) = x_0$. Assume that $f(\tau,\, x) = x$ for all $\tau \in N_\tau^+$ iff $x = 0$. Besides, system (1.2.11) admits zero solution $x = 0$ and it corresponds to the unique equilibrium state of system (1.2.11).

The definitions of the dynamical properties of solutions of system (1.2.11) are obtained by replacing the independent variable $t \in R$ by $\tau \in N_\tau^+$ in Definitions 1.2.1 − 1.2.3 and so are omitted.

Stability (instability) of the equilibrium state $x = 0$ of system (1.2.11) is sometimes studied by means of reducing this system to the form

$$x(\tau + 1) = Ax(\tau) + g(\tau, x(\tau)), \tag{1.2.12}$$

where A is a constant $n \times n$–matrix, $g\colon N_\tau^+ \times R^n \to R^n$ is a vector-function continuous in x and satisfies certain conditions of smallness.

In this case, under some additional restrictions on the properties of matrix A, stability (instability) of the state $x = 0$ of system (1.2.12) can be studied in terms of the first order approximation equations.

Of essential interest is the case when the order of system (1.2.11) is rather high, or when this system is a composition of more simple subsystems. In this case the finite-dimensional system of equations of the type of

$$\begin{gathered} x_i(\tau + 1) = f_i(\tau, x_i(\tau)) + g_i(\tau, x_1(\tau), \dots, x_m(\tau)), \\ i = 1, 2, \dots, m, \end{gathered} \tag{1.2.13}$$

is considered, where $x_i \in R^{n_i}$, $f_i\colon N_\tau^+ \times R^{n_i}$, $g_i\colon N_\tau^+ \times R^{n_1} \times \dots \times R^{n_m} \to R^{n_i}$.

Via designation (1.2.7) system (1.2.13) can be presented in the vector form

$$x(\tau + 1) = f(\tau, x(\tau)) + g(\tau, x(\tau)) \triangleq H(\tau, x(\tau)). \tag{1.2.14}$$

Formally system (1.2.14) coinsides in form with system (1.2.11). However, if $g(\tau, x(\tau)) \equiv 0$, this system falls into the independent subsystems

$$x_i(\tau + 1) = f_i(\tau, x_i(\tau)), \quad i = 1, 2, \dots, m, \tag{1.2.15}$$

each of the latter can possess the same degree of complexity of the solution behavior as the system (1.2.11).

1.2.3 Ordinary impulsive large-scale systems The *impulsive system* of differential equations of general type

$$\frac{dx}{dt} = f(t,x), \quad t \neq \tau_k(x), \qquad (1.2.16)$$
$$\Delta x = I_k(x), \quad t = \tau_k(x), \quad k = 1, 2, \dots,$$

has the meaning of a large scale impulsive system, if it can be decomposed into m *interconnected impulsive subsystems*

$$\frac{dx_j}{dt} = f_j(t, x_j) + f_j^*(t,x), \quad t \neq \tau_k(x), \quad j = 1, 2, \dots, m, \qquad (1.2.17)$$
$$\Delta x_j = I_{kj}(x_j) + I_{kj}^*(x), \quad t = \tau_k(x), \quad k = 1, 2, \dots.$$

We assume on system (1.2.16) that

(1) $x \in R^n$, $f(t,x) = 0$ iff $x = 0$;
(2) $0 < \tau_k(x) < \tau_{k+1}(x)$, $\tau_k(x) \to +\infty$ as $k \to \infty$;
(3) $I_k\colon R^n \to R^n$ and $I_k = 0$ iff $x = 0$;
(4) functions $f(t,x)$ and $I_k(x)$ are definite and continuous in the domain

$$\mathcal{T}_0 \times \mathcal{S}(\rho) = [t_0, \infty) \times \{x\colon \|x\| \leq \rho \leq \rho_0\}, \quad t_0 \geq 0;$$

(5) functions $\tau_k(x)$, $k = 1, 2, \dots$, and number ρ satisfy conditions excluding beating of solutions of system (1.2.16) against the hypersurfaces $S_i\colon t = \tau_k(x)$, $k = 1, 2, \dots$, $t \geq 0$.

We assume on system (1.2.17) that

(1) $x_j = (0, \dots, 0, x_j^{\mathrm{T}}, 0, \dots, 0)^{\mathrm{T}} \in R^n,\ x_j \in R^{n_j}$,
$f = (f_1^{\mathrm{T}}, \dots, f_m^{\mathrm{T}})^{\mathrm{T}},\ f_j^*(t,x) = f_j(t,x) - f_j(t,x_j)$;

(2) $I_{kj} = (I_{k1}^{\mathrm{T}}, I_{k2}^{\mathrm{T}}, \dots, I_{km}^{\mathrm{T}})^{\mathrm{T}},\ I_{kj}^*(x) = I_{kj}(x) - I_{kj}(x_j),\ n = n_1 + \dots + n_m$.

The state of the j-th *noninteracting impulsive subsystem* is described by the equations

$$\frac{dx_j}{dt} = f_j(t,x_j), \quad t \neq \tau_k(x_j); \qquad \Delta x_j = I_{kj}(x_j), \quad t = \tau_k(x_j). \tag{1.2.18}$$

The problem on stability for large scale impulsive system (1.2.16) is formulated as follows:

To establish conditions under which stability of equilibrium state $x = 0$ of system (1.2.17) is derived from the properties of stability of impulse subsystems (1.2.18) and properties of connection functions $f_j^*(t,x)$ and $I_{kj}^*(x)$.

Let $x_0(t) = x(t;t_0,y_0)$ $(y_0 \neq x_0)$ be a given solution of the system (1.2.16). Since the times of impulsive effects on solution $x_0(t)$ may not coincide with those on any neighboring solution $x(t)$ of system (1.2.16), the smallness requirement for the difference $\|x(t) - x_0(t)\|$ for all $t \geq t_0$ seems not natural.

Therefore the stability definitions presented in Section 1.2.1 for the system of ordinary differential equations should be adapted to system (1.2.16).

We designate by Ξ a set of functions continuous from the left with discontinuities of the first kind, defined on R_+ with the values in R^n. Let the set of the discontinuity point of each of these functions be no more than countable and do not contain finite limit points in R^1. Let $\zeta \geq 0$ be a fixed number.

Definition 1.2.4 A *function* $y(t) \in \Xi$ is in ζ*-neighborhood of function* $x(t) \in \Xi$, if

(1) discontinuity points of function $y(t)$ are in ζ-neighborhoods of discontinuity points of function $x(t)$;

(2) for all $t \in R_+$, that do not belong to ζ-neighborhoods of discontinuity points of function $x(t)$, the inequality $\|x(t) - y(t)\| < \zeta$ is satisfied.

The totality of ζ-neighborhoods, $\zeta \in (0,\infty)$, of all elements of the set Ξ forms the basis of topology, which is referred to as *B-topology*.

Let $x(t)$ be a solution of system (1.2.16), and $t = \tau_k$, $k \in \mathcal{Z}$, be an ordered sequence of discontinuity points of this solution.

Definition 1.2.5 *Solution* $x(t)$ of system (1.2.16) satisfies

(1) *α-condition*, if there exists a number $\vartheta \in R_+,\ \vartheta > 0$, such that for all $k \in \mathcal{Z}$: $\tau_{k+1} - \tau_k \geq \vartheta$;
(2) *β-condition*, if there exists a $k \geq 0$ such that every unit segment of the real axis R_+ contains no more than k points of sequence τ_k.

Let the solution $x(t)$ satisfy one of the conditions (α or β) and be definite on $[a,\infty)$, $a \in R$. Besides, the solution $x(t)$ is referred to as *unboundedly continuable to the right*.

Let the solution $x_0(t) = x(t; t_0, y_0)$ of system (4.2.1) exist for all $t \geq t_0$ and be unperturbed. We assume that $x_0(t)$ reaches the surface $S_k\colon\ t = \tau_k(x)$ at times $t_k,\ t_{k+1} > t_k$ and $t_k \to \infty$ as $k \to \infty$.

Definition 1.2.6 *Solution* $x_0(t)$ of system (1.2.16) is

(i) *stable*, if for any tolerance $\varepsilon > 0,\ \Delta > 0,\ t_0 \in R_+$ a $\delta = \delta(t_0, \varepsilon, \Delta) > 0$ exists such that condition $\|x_0 - y_0\| < \delta$ implies $\|x(t) - x_0(t)\| < \varepsilon$ for all $t \geq t_0$ and $|t - t_k| > \Delta$, where $x(t)$ is an arbitrary solution of system (1.2.16) existing on interval $[t_0, \infty)$;
(ii) *uniformly stable*, if δ in condition (1) of Definition 1.2.6 does not depend on t_0;
(iii) *attractive*, if for any tolerance $\varepsilon > 0,\ \Delta > 0,\ t_0 \in R_+$ there exist $\delta_0 = \delta_0(t_0) > 0$ and $T = T(t_0, \varepsilon, \Delta) > 0$ such that whenever $\|x_0 - y_0\| < \delta_0$, then $\|x(t) - x_0(t)\| < \varepsilon$ for $t \geq t_0 + T$ and $|t - t_k| > \Delta$;
(iv) *uniformly attractive*, if δ_0 and T in condition (3) of Definition 1.2.6 do not depend on t_0;
(v) *asymptotically stable*, if conditions (1) and (3) of Definition 1.2.6 hold;
(vi) *uniformly asymptotically stable*, if conditions (2) and (3) of Definition 1.2.6 hold.

Remark 1.2.1 If $f(t, 0) = 0$ and $I_k(0) = 0,\ k \in \mathcal{Z}$, then system (1.2.16) admits zero solution. Moreover, if $\tau_k(x) \equiv t_k,\ k \in \mathcal{Z}$, are such that $\tau_k(x)$ do not depend on x, then any solution of system (1.2.16) undergoes the impulsive effect at one and the same time. This situation shows that the notion of stability for system (1.2.16) is an ordinary one.

Remark 1.2.2 Actually the condition (1) of Definition 1.2.6 means that for the solution $x_0(t)$ of system (1.2.16) to be stable in the sense of Liapunov, it is necessary that for $\|x(t_0) - x_0(t_0)\| < \delta$ any solution $x(t)$ of the system remain in the neighborhood of solution $x_0(t)$ for all $t \in [t_0, \infty)$, and point t_0 is not to be the discontinuity point of solutions $x(t)$ and $x_0(t)$.

1.2.4 Ordinary singularly perturbed large-scale systems The perturbed equations of motion of a singularly perturbed large-scale system are

$$\frac{dx_i}{dt} = f_i(t, x, y), \quad i = 1, 2, \dots, q, \tag{1.2.19}$$

$$\mu_j \frac{dy_j}{dt} = g_j(t, x, y, M), \quad j = 1, 2, \dots, r, \tag{1.2.20}$$

where $x_i \in R^{n_i}$, $n_1 + n_2 + \dots + n_q = n$, $y_j \in R^{m_j}$, $m_1 + m_2 + \dots + m_r = m$ and $q + r = s$; f_i and g_j are continuous vector functions of the corresponding dimensions, μ_j are positional parameters, taking arbitrary small values, $\mu_j \in \,]0, 1]$, and $M = \operatorname{diag}\{\mu_1, \dots, \mu_r\}$. The set of all admissible values of M is denoted by

$$\mathcal{M} = \{M\colon\ 0 < M \le I\} \quad I = \operatorname{diag}\{1, 1, \dots, 1\}$$

and then

$$\mathcal{M}_m = \{M\colon\ 0 < \mu_j < \mu_{jm} \quad \forall\, j \in [1, r]\},$$

where μ_{jm} is the upper admissible value of μ_j. If the small parameters μ_j are not interconnected then the system (1.2.19), (1.2.20) has r essentially independent timescales t_j:

$$t_j = \frac{t - t_0}{\mu_j}, \quad j = 1, 2, \dots, r. \tag{1.2.21}$$

In this case the timescale is graduated nonuniformly. The timescales t_j can be interconnected through values τ_j:

(1.2.22) $$\frac{t_j}{t_1} = \tau_j, \quad j = 1, 2, \dots, r,$$

varying within the limits

(1.2.23) $$\tau_j \in [\underline{\tau}_j, \overline{\tau}_j], \quad j = 1, 2, \dots, r,$$

where $0 < \underline{\tau}_j \leq \overline{\tau}_j < +\infty$ for all $j \in [1, r]$.

In the case (1.2.23), (1.2.24) we have uniform graduation of the timescale. This implies that

(1.2.24) $$\tau_j = \frac{\mu_1}{\mu_j}, \quad j = 1, 2, \dots, r.$$

It is clear then that $\underline{\tau}_1 = \tau_1 = \overline{\tau}_1 = 1$.

The interconnected i-th singularly perturbed subsystem S_i of the system (1.2.19, (1.2.20) is described by the equations

(1.2.25) $$\frac{dx_i}{dt} = f_i(t, x, y),$$

(1.2.26) $$\mu_i \frac{dy_i}{dt} = g_i(t, x, y, M),$$

and the independent i-th singularly perturbed subsystem $\widehat{S}_i$ is described by the equations

(1.2.27) $$\frac{dx_i}{dt} = f_i(t, x^i, y^i),$$

(1.2.28) $$\mu_i \frac{dy_i}{dt} = g_i(t, x^i, y^i, M),$$

where

$$x^i = (0, 0, \dots, 0, x_i, 0, \dots, 0)^{\mathrm{T}} \in R^n, \quad x_i \in R^{n_i},$$
$$y^i = (0, 0, \dots, 0, y_i, 0, \dots, 0)^{\mathrm{T}} \in R^n, \quad y_i \in R^{m_i}.$$

When $q = r$, we can consider the equations

(1.2.29) $$\frac{dx_i}{dt} = f_i(t, x^i, y^i),$$

(1.2.30) $$0 = g_i(t, x^i, y^i, 0),$$

which are referred to as the *equations of the i-th degenerate independent subsystem* $\widehat{S}_{i0}$ of the system (1.2.19), (1.2.20), and the equations

(1.2.31) $$\mu_j \frac{dy_i}{dt} = g_i(\alpha, b^i, y^i, 0)$$

of the *j-th independent subsystem of the boundary layer (fast subsystem)* S_j of the system (1.2.19), (1.2.20). In the system (1.2.31) $\alpha \in R$, $b^i = (0, 0, \dots, 0, b_i, 0, \dots,)^{\mathrm{T}} \in R^n$, $b_i \in R^{n_i}$.

If in (1.2.19), (1.2.20) all μ_j (formally) comprise a zero set then the equations

$$\frac{dx_i}{dt} = f_i(t, x, y), \quad i = 1, 2, \dots, q, \tag{1.2.32}$$

$$0 = g_j(t, x, y, 0), \quad j = 1, 2, \dots, r, \tag{1.2.33}$$

are called an *interconnected degenerate subsystem* S_0 of the system (1.2.19), (1.2.20), and the equations

$$\mu_j \frac{dy_j}{dt_1} = g_j(\alpha, b, y, 0), \quad j = 1, 2, \dots, r, \tag{1.2.34}$$

are said to be an *interconnected fast subsystem* S_t (a *boundary layer*) of the system (1.2.19), (1.2.20). Here $\alpha \in R$ and $b \in R^n$.

We suppose that the equations $0 = g_j(t, x, y, 0)$ for all $(t, x, y) \in R \times N_x \times N_y$ are satisfied iff $y = 0$ and $0 = g_j(t, x^i, y^j, 0)$ for all $(t, x^i, y^j) \in R \times N_{ix} \times N_{jy}$ iff $y^j = 0$. Therefore the systems (1.2.29), (1.2.30) and (1.2.32), (1.2.33) are equivalent to the systems

$$\frac{dx_i}{dt} = f_i(t, x^i, 0), \quad i = 1, 2, \dots, q, \tag{1.2.35}$$

$$\frac{dx_i}{dt} = f_i(t, x, 0), \quad i = 1, 2, \dots, q, \tag{1.2.36}$$

respectively.

The separation of the time scales in the investigation of the stability of the system (1.2.19), (1.2.20) is essential since the analysis of the degenerate system (1.2.29), (1.2.30) and the fast system (1.2.31) is simpler problem than that of general problem of stability of the system (1.2.19), (1.2.20).

Stability analysis of systems of (1.2.19) and (1.2.20) type under nonclassical structural perturbations is the subject of Chapter 5. In this chapter the development of the direct Liapunov method in terms of matrix-valued functions is proposed.

1.3 Structural Perturbations of Dynamical Systems

The processes and phenomena of the real world are modeled correctly by the systems of equations or inequalities only when the model admits small changes. In other words the phenomenon model is correct provided that it allows some uncertainties in definitions of both the parameters and the external effects on the real system or process and at the same time displays the main properties of the modeled process.

1.3.1 Classical structural perturbations Let, for example, the differential equation

$$\frac{dx}{dt} = g(x), \quad x \in M, \tag{1.3.1}$$

determine the vector field on the compact manifold M. The naive speculations above lead to the following notion of structural stability.

Definition 1.3.1 (see Arnol'd [1]) System (1.3.1) is *structurally stable*, if for arbitrary small changes of the vector field the obtained system is equivalent to the initial one in the sense of fixed dynamical property.

Andronov and Pontriagin [1] considered dynamical system on the disk D_2 and said that a system X is "rough" if, by perturbing it slightly in the C_1, one gets a system YX and the corresponding homeomorphism can be made arbitrarily small by taking Y close enough to X. They gave a set of conditions as being necessary and sufficient for X to be rough (see de Baggis [1]).

It seems that Lefschetz [1] was the first who translated "rough" by the much better sounding "structurally stable". He exhibited the true meaning of the new concept, namely a fusion of the two concepts of stability and qualitative behavior in the sense of topological equivalence.

Let ρ be a metric in $\mathfrak{X}$, and we assume that there is also a metric in M^n.

Definition 1.3.2 (see, e.g., Peixoto [1]) On a *compact differentiable manifold* M^n *a vector field* $X \in \mathfrak{X}$ is said to be *structurally stable*, if given $\varepsilon > 0$, one may find $\delta > 0$ such that wherenever $\rho(X, Y) < \delta$, then $Y \sim X$ and the corresponding homeomorphism is within ε from the identity.

The problems of structural stability in one-dimensional case (M-circle), systems on two-dimensional sphere, equations on torus and U-systems are discussed in the book by Arnol'd [1]. The Anosov's theorem on structural stability of torus automorphism and the Grobman-Hartman theorem on structural stability of saddle are presented in this book as well. The readers who are interested in the results in this direction can find some references in the survey by Sell [1].

In our monograph we apply the model of dynamical system under non-classical structural perturbation which appeared in motion stability theory of large scale systems. This model originates from one idea of Chetaev [1], presented below and the notion of structural perturbations introduced earlier in the works by Siljak [1 – 3].

1.3.2 An idea of parametric perturbations Chetaev [1] proposed a constructive realization of the Andronov-Pontryagin idea of motion stability investigation of rough systems within the framework of the Liapunov direct method. The general Chetaev's approach is as follows.

Let the motion of system with finite degrees of freedom in "linear" approximation be described by the equations

$$\frac{dx}{dt} = Px, \quad x(t_0) = x_0, \tag{1.3.2}$$

where $x \in R^n$ and $P = C + \varepsilon F(t, x)$, C is a constant matrix, $F(t, x)$ is unknown in general matrix function with bounded real elements in the domain $R_+ \times \Omega$, $\Omega \subset R^n$, ε is a real parameter. The fact that the system (1.3.2) is of the form

$$\frac{dx}{dt} = Cx + \varepsilon F(t, x)x \tag{1.3.3}$$

yields that for $\varepsilon = 0$ the system (1.3.3) does not have structural perturbations and the properties of its equilibrium state $x = 0$ are completely determined by signs of real parts of roots of the characteristic equation

$$\det (C - \lambda E) = 0. \tag{1.3.4}$$

In the case when all $\operatorname{Re} \lambda_i(C) < 0$, $i = 1, 2, \dots, n$, under certain conditions the equilibrium state $x = 0$ of (1.3.2) possesses the same type of asymptotic stability as the system (1.3.3) for $\varepsilon = 0$.

A key idea in this approach is that the mathematical models of a real system with structural perturbations is "decomposed" into a "stationary" part and the terms bearing the information on structural and/or parametric perturbations. Anyway the parametric perturbations must be small and such that the solutions of system (1.3.3) must exist on the interval not smaller than that on which the dynamics of system (1.3.3) is studied.

This Chetaev's idea is used in the implicit form in modern nonlinear dynamics of systems with uncertain parameter values.

1.3.3 Šiljak's idea of connective stability D.D. Siljak [1−3] proposed a description of structural perturbations which appear in stability investigation of large scale systems. In his model some dynamical system

$$\frac{dx}{dt} = f(t,x,E), \tag{1.3.5}$$

where $x \in R^n$, $f\colon\ R_+ \times R^n \to R^n$, is decomposed into m interconnected subsystems

$$\frac{dx_i}{dt} = f_i(t,x_i) + g_i(t,x), \quad i = 1,2,\dots,m, \tag{1.3.6}$$

where $x_i \in R^{n_i}$, $g_i\colon\ R_+ \times R^{n_i} \to R^{n_i}$, $g_i\colon\ R_+ \times R^n \to R^{n_i}$.

It is assumed that system (1.3.5) and the free subsystems

$$\frac{dx_i}{dt} = f_i(t,x_i), \quad i = 1,2,\dots,m, \tag{1.3.7}$$

satisfy the existence conditions for solutions $x(t,t_0,x_0)$ for all $(t_0,x_0) \in R_+ \times R^n$ and $f(t,0) = f_i(t,0) = 0$ for all $t \in R_+$, i.e. the motions of system (1.3.5) and subsystems (1.3.7) can be realized on any given time interval.

In order to take into account the mutual interaction between subsystems (1.3.7) in system (1.3.5) and the dynamical properties of the initial system (1.3.5) the binary elements e_{ij} of the interaction matrix E are introduced in the form

$$e_{ij} = \begin{cases} 1, & i\text{-subsystem acts on } j\text{-subsystem},\\ 0, & i\text{-subsystem does not act on } j\text{-subsystem}. \end{cases}$$

In this case the interconnection functions $g_i(t,x)$ are represented as

$$g_i(t,x) = g_i(t, e_{i1}x_1,\, e_{i2}x_2,\, \dots,\, e_{im}x_m), \quad i = 1,2,\dots,m. \tag{1.3.8}$$

As a result structural perturbations to be considered in this connection are such that any number of existing interconnections among the subsystems (1.3.6) can be ON or OFF as arbitrary functions of the state $x(t) \in R^n$ and/or time $t \in \mathcal{T}_0$. At each instant of time $t \in \mathcal{T}_0$ there is an interconnection matrix E which describes the structure of system (1.3.5).

In terms of the above model of structural perturbations various stability problems are investigated for the system (1.3.5) and its generalizations in the sense of the following definition (see Siljak [4, 5]).

Definition 1.3.3 The equilibrium state $x = 0$ of a free dynamical system (1.3.5) is *connectively stable* if and only if it is stable in the sense of Liapunov for all interconnection matrices E.

It should be noted that in the above model the action of structural perturbations "is revealed" as a result of the analysis of the initial systems (1.3.5) decomposed into a series of the independent subsystems (1.3.7). Besides, the right-hand side of the system (1.3.5) does not undergo any changes.

Before we finish these comments we note that connective stability is a Liapunov-type stability, and the differences between stability under structural perturbations and structural stability (catastrophe theory) are between stability in the sense of Liapunov and structural stability in the sense of Andronov and Pontriagin. A system can be structurally stable, yet unstable in the sense of Liapunov! For the details see Thom [1].

1.4 Stability under Nonclassical Structural Perturbations

The concept of stability under nonclassical structural perturbations is set out using the example of large scale system of ordinary differential equations.

Let the behaviour of a mechanical or other nature system be described by differential equations of the form

$$\frac{dx}{dt} = Q(t, x, P, S), \tag{1.4.1}$$

where $x(t) \in R^n$ for all $t \in (-\infty, +\infty)$, $P \in \mathcal{P}$, $S \in \mathcal{S}$, $Q\colon R \times R^n \times \mathcal{P} \times \mathcal{S} \to R^n$. Here $\mathcal{P}$ is a compact set in R^m describing parametric perturbations and $\mathcal{S} = (S_1, \dots, S_n)$ is a finite set characteristic of admissible structures S_k of system (1.4.1).

Further in Chapters 2–5 these sets will be concretized which is necessary for constructing algorithms of motion stability analysis of the appropriate systems of equations.

Associated with (1.4.1) we consider the initial value problem given by

$$\frac{dx}{dt} = Q(t, x, P, S), \quad x(t_0) = x_0, \tag{1.4.2}$$

where $t_0 \in R_+$ and $x_0 \in \Omega$.

The function $\psi = \psi(t; t_0, x_0)$ is a solution to initial value problem (1.4.2) for any $(P, S) \in \mathcal{P} \times \mathcal{S}$ if and only if ψ is a solution of the integral equation

$$\psi(t) = x_0 + \int_{t_0}^{t} Q(\tau, \psi(\tau), P, S)\, d\tau \tag{1.4.3}$$

for all $t \in [t_0, b)$ and any $(P, S) \in \mathcal{P} \times \mathcal{S}$.

On the product $C([t_0, b), R^n) \times \mathcal{P} \times \mathcal{S}$ we determine the operator

$$(T\psi)(t) = x_0 + \int_{t_0}^{t} Q(\tau, \psi(\tau), P, S)\, d\tau. \tag{1.4.4}$$

Function ψ is a solution of the system (1.4.2) if and only if ψ is a fixed point of the operator T, i.e. the condition (see Miller [1])

$$\psi = T\psi$$

is to be satisfied for any $(P, S) \in \mathcal{P} \times \mathcal{S}$.

Let P^* be fixed parameter values and S^* be a given structure of system (1.4.1). Consider *nominal system*

$$\frac{dx}{dt} = Q(t, x, P^*, S^*) \tag{1.4.5}$$

and the transformed system (1.4.1)

$$\frac{dy}{dt} = Q(t, x, P^*, S^*) + \Delta Q(t, x, P, S), \tag{1.4.6}$$

where $\Delta Q(t, x, P, S) = Q(t, x, P, S) - Q(t, x, P^*, S^*)$.

We introduce some assumptions on systems (1.4.1) and (1.4.5).

H_1. Vector-function Q is given for all $t \in (-\infty, +\infty)$, $x \in \Omega \subset R^n$, $P^* \in \mathcal{P}$, $S^* \in \mathcal{S}$, and is real and continuous.

H_2. For every $t_0 \in (-\infty, +\infty)$, $x_0 \in \Omega$, $P^* \in \mathcal{P}$ and $S^* \in \mathcal{S}$ positive numbers a, b, c and K can be found such that the sphere $\|x - x_0\| \le b$ is contained in the domain Ω and the sphere $\|P^*\| \le c$ is embedded into the set $\mathcal{P}$ and the Lipschitz condition

$$\|Q(t, x', P^*, S^*) - Q(t, x'', P^*, S^*)\| \le K\|x' - x''\|$$

is satisfied for $|t - t_0| \le a$, $\|x - x_0\| \le b$, $\|P - P^*\| \le c$, for given $S^* \in \mathcal{S}$.

H_3. For any $(P, S) \in \mathcal{P} \times \mathcal{S}$ $\delta_2 = \max(\|\Delta Q(t, x, P, S)\|$ for $|t - t_0| \le h) < k < +\infty$.

Proposition 1.4.1 Under conditions $H_1 - H_3$ there exists a unique solution $x(t) = x(t, t_0, x_0, P, S)$ of system (1.4.1) determined for $|t - t_0| \le h,\ h = \min(a, b/M)$ where

$$M = \max(\|Q(t, x, P^*, S^*)\| \quad \text{for} \quad |t-t_0| \le a,\ \|x-x_0\| \le b,\ \|P-P^*\| \le c)$$

for $S^* \in \mathcal{S}$, which satisfies condition $x(t) = x_0$ for $t = t_0$. This solution is a continuous function of parameters $P \in \mathcal{P}$ in closed domain $\|P-P^*\| \le c$ for given structure $S^* \in \mathcal{S}$.

Proof of this assertion is based on the fundamental inequality

$$\|x(t) - y(t)\| \le \delta_1 e^{L(t-t_0)} + \left(\frac{\delta_2}{K}\right)\left(e^{L(t-t_0)} - 1\right), \tag{1.4.7}$$

were the value δ_1 characterizes the initial deviations of solutions $x(t)$ and $y(t)$ of systems (1.4.5) and (1.4.6) for $t = t_0$, i.e. $\|x_0 - y_0\| \le \delta_1$.

Estimate (1.4.7) allows one to show that the solutions of systems (1.4.5) and (1.4.6) depend continuously on the system structure and/or parameter only on the finite time interval. Hence it follows closeness of the appropriate solutions on finite interval.

The problem on closeness of solutions to systems (1.4.5) and (1.4.6) on infinite interval is a subject of special investigation of theory of stability under nonclassical structural perturbations which is basic in this monograph.

We add one more assumption to $H_1 - H_3$.

H_4. System (1.4.1) possesses a trivial solution $x = 0$, which is preserved for any $(P, S) \in \mathcal{P} \times \mathcal{S}$.

Since further solutions of system (1.4.1) are considered on the infinite time interval, we recall some conditions ensuring the existence of such solutions.

Proposition 1.4.2 Let vector-function $Q(t, x, P, S)$ be definite and continuous in the domain of values $(t, x) \in R_+ \times R^n$ for any $(P, S) \in \mathcal{P} \times \mathcal{S}$ and in this domain the inequality

$$\|Q(t, x, P, S)\| \le L(\|x\|) \quad \text{for} \quad (P, S) \in \mathcal{P} \times \mathcal{S}$$

holds true, where $L(r)$ is a continuous function of r satisfying the condition

$$\int_{r_0}^{r} \frac{dr}{L(r)} \to +\infty \quad \text{for} \quad r \to +\infty.$$

Moreover, the vector-function $Q(t, x, P, S)$ satisfies the Lipschits condition in x in any domain $\{x \in R^n \colon\ \|x\| \le N\}$ with constant K.

Then any solution $x(t)$ of system (1.4.1) can be extended for all values $t_0 \le t < +\infty$.

Note that the constant K can depend on the value N and also on $(P, S) \in \mathcal{P} \times \mathcal{S}$, i.e. $K = K(N, P, S)$.

This assertion is proved by a slight modification of the proof of Theorem 2.1.2 by Lakshmikantham and Leela [1].

System (1.4.1) is called *the system with nonclassical structural perturbations* if for it assumptions H_1–H_4 are satisfied and any of its solutions has an extension on the interval $[t_0, +\infty)$.

In the investigation of the dynamical behavior of solutions to system (1.4.1) under nonclassical structural perturbations we shall use definitions obtained in terms of Definitions 1.2.1 – 1.2.3.

Definition 1.4.1 The equilibrium state $x = 0$ of system (1.4.1) is

(i) *stable* (in the whole) *under nonclassical structural perturbations* if and only if it is stable (in the whole) in the sense of Liapunov (in the sense of Barbashin-Krasovskii) for any $(P, S) \in (\mathcal{P}, \mathcal{S})$ respectively;

(ii) *unstable under nonclassical structural perturbations* if and only if it is unstable in the sense of Liapunov for at least one pair $(P, S) \in (\mathcal{P}, \mathcal{S})$.

Definitions of other types of stability are formulated in the same way as Definition 1.4.1(i) and are presented in the book when necessary.

Remark 1.4.1 The concept of stability under nonclassical structural perturbations is not identical with the concept of connected stability introduced by Siljak [1–3] and is further development of the notion of stability in mathematical system theory.

Remark 1.4.2 Further on for the sake of briefness alongside the expression “under nonclassical structural perturbations” a more short expression “on $\mathcal{P} \times \mathcal{S}$” is used.

1.5 Method of Stability Analysis of Motion

The main method of stability analysis of systems of (1.4.1) type is the method of Liapunov functions. In the monograph by Grujić *et al.* [1] the results of stability analysis of systems under nonclassical structural perturbations are presented obtained in terms of vector Liapunov functions. Besides new aggregation forms are presented for large-scale systems and conditions for different types of motion stability are established. Models of large-scale Lurie-Postnikov systems and power systems are considered as examples.

In this monograph we propose to apply the matrix-valued Liapunov functions for stability analysis of large-scale systems mentioned in Section 1.2. This method is developed recently in qualitative theory of equations and is set out in Martynyuk [1, 2]. We shall recall some notions of this technique.

Presently the Liapunov direct method (see Liapunov [1]) in terms of three classes of auxiliary functions: scalar, vector and matrix ones is intensively applied in qualitative theory. In this point we shall present the description of the matrix-valued auxiliary functions.

For the system (1.2.6) we shall consider a continuous matrix-valued function

$$U(t,x) = [v_{ij}(t,x)], \quad i,j = 1,2,\dots,m, \tag{1.5.1}$$

where $v_{ij} \in C(\mathcal{T}_\tau \times R^n, R)$ for all $i,j = 1,2,\dots,m$. We assume that the following conditions are fulfilled

(i) $v_{ij}(t,x)$, $i,j = 1,2,\dots,m$, are locally Lipschitzian in x;
(ii) $v_{ij}(t,0) = 0$ for all $t \in R_+$ $(t \in \mathcal{T}_\tau)$, $i,j = 1,2,\dots,m$;
(iii) $v_{ij}(t,x) = v_{ji}(t,x)$ in any open connected neighborhood $\mathcal{N}$ of point $x = 0$ for all $t \in R_+$ $(t \in \mathcal{T}_\tau)$.

Definition 1.5.1 All functions of the type

$$v(t,x,\alpha) = \alpha^{\mathrm{T}} U(t,x)\alpha, \quad \alpha \in R^m, \tag{1.5.2}$$

where $U \in C(\mathcal{T}_\tau \times \mathcal{N},\, R^{m\times m})$, are attributed to the *class SL*.

Here the vector α can be specified as follows:

(i) $\alpha = y \in R^m$, $y \neq 0$;
(ii) $\alpha = \xi \in C(R^n,\, R^m_+)$, $\xi(0) = 0$;
(iii) $\alpha = \psi \in C(\mathcal{T}_\tau \times R^n,\, R^m_+)$, $\psi(t,0) = 0$;
(iv) $\alpha = \eta \in R^m_+$, $\eta > 0$.

Note that the choice of vector α can influence the property of having a fixed sign of function (1.5.1) and its total derivative along solutions of system (1.2.6).

For the functions of the class *SL* we shall cite some definitions which are applied in the investigation of dynamics of system in the book.

Definition 1.5.2 The *matrix-valued function* $U\colon \mathcal{T}_\tau \times R^n \to R^{m\times m}$ is:

(i) *positive semi-definite on* $\mathcal{T}_\tau = [\tau, +\infty),\ \tau \in R,$ iff there are time-invariant connected neighborhood $\mathcal{N}$ of $x = 0,\ \mathcal{N} \subseteq R^n,$ and vector $y \in R^m,\ y \neq 0,$ such that

  (a) $v(t, x, y)$ is continuous in $(t, x) \in \mathcal{T}_\tau \times \mathcal{N} \times R^m$;

  (b) $v(t, x, y)$ is non-negative on $\mathcal{N},\ v(t, x, y) \geq 0$ for all $(t, x, y \neq 0) \in \mathcal{T}_\tau \times \mathcal{N} \times R^m,$ and

  (c) vanishes at the origin: $v(t, 0, y) = 0$ for all $t \in \mathcal{T}_\tau \times R^m$;

  (d) iff the conditions (a)−(c) hold and for every $t \in \mathcal{T}_\tau$, there is $w \in \mathcal{N}$ such that $v(t, w, y) > 0,$ then v is strictly positive semi-definite on $\mathcal{T}_\tau$.

The expression "*on* $\mathcal{T}_\tau$" is omitted iff all corresponding requirements hold for every $\tau \in R$.

Definition 1.5.3 The *matrix-valued function* $U\colon \mathcal{T}_\tau \times R^n \to R^{m\times m}$ is:

(i) *positive definite on* $\mathcal{T}_\tau,\ \tau \in R$, iff there are a time-invariant connected neighborhood $\mathcal{N}$ of $x = 0,\ \mathcal{N} \subseteq R^n$ and a vector $y \in R^m$, $y \neq 0,$ such that both it is positive semi-definite on $\mathcal{T}_\tau \times \mathcal{N}$ and there exists a positive definite function w on $\mathcal{N},\ w\colon R^n \to R_+$, obeying $w(x) \leq v(t, x, y)$ for all $(t, x, y) \in \mathcal{T}_\tau \times \mathcal{N} \times R^m$;

(ii) *negative definite (in the whole) on* $\mathcal{T}_\tau$ (*on* $\mathcal{T}_\tau \times \mathcal{N} \times R^m$) iff $(-v)$ is positive definite (in the whole) on $\mathcal{T}_\tau$ (on $\mathcal{T}_\tau \times \mathcal{N} \times R^m$) respectively.

The expression "*on* $\mathcal{T}_\tau$" is omitted iff all corresponding requirements hold for every $\tau \in R$.

The set $v_\zeta(t)$ is *the largest connected neighborhood of* $x = 0$ *at* $t \in R$ which can be associated with a function $U\colon R \times R^n \to R^{m\times m}$ so that $x \in v_\zeta(t)$ implies $v(t, x, y) < \zeta,\ y \in R^m$.

Definition 1.5.4 The matrix-valued function $U\colon R \times R^n \to R^{s\times s}$ is:

(i) *decreasing on* $\mathcal{T}_\tau,\ \tau \in R,$ iff there is a time-invariant neighborhood $\mathcal{N}$ of $x = 0$ and a positive definite function w on $\mathcal{N},\ w\colon R^n \to R_+$, such that $y^{\mathrm{T}} U(t, x) y \leq w(x)$ for all $(t, x) \in \mathcal{T}_\tau \times \mathcal{N}$;

(ii) *decreasing in the whole on* $\mathcal{T}_\tau$ iff (i) holds for $\mathcal{N} = R^n$.

The expression *"on* $\mathcal{T}_\tau$*"* is omitted iff all corresponding conditions still hold for every $\tau \in R$.

Definition 1.5.5 The matrix-valued function $U\colon R \times R^n \to R^{m\times m}$ is:

(i) *radially unbounded on* $\mathcal{T}_\tau, \tau \in R$, iff $\|x\| \to \infty$ implies $y^{\mathrm{T}} U(t, x) y \to +\infty$ for all $t \in \mathcal{T}_\tau,\ y \in R^m,\ y \neq 0$;

(ii) *radially unbounded*, iff $\|x\| \to \infty$ implies $y^{\mathrm{T}} U(t, x) y \to +\infty$ for all $t \in \mathcal{T}_\tau$ for all $\tau \in R,\ y \in R^m,\ y \neq 0.$

According to Liapunov [1] function (1.5.2) is applied in motion investigation of system (1.2.6) together with its total derivative along solutions $x(t) = x(t; t_0, x_0)$ of system (1.2.6). Assume that each element $v_{ij}(t, x)$ of the matrix-valued function (1.5.2) is definite on the open set $\mathcal{T}_\tau \times \mathcal{N}$, $\mathcal{N} \subset R^n$, i.e. $v_{ij}(t, x) \in C(\mathcal{T}_\tau \times \mathcal{N},\, R)$.

If $\gamma(t; t_0, x_0)$ is a solution of system (1.2.6) with the initial conditions $x(t_0) = x_0$, i.e. $\gamma(t_0; t_0, x_0) = x_0,$, the right-hand upper derivative of function (1.5.2) for $\alpha = y,\ y \in R^m,$ with respect to t along the solution of (1.2.6) is determined by the formula

$$D^+v(t,x,y) = y^{\mathrm{T}} D^+U(t,x)y, \tag{1.5.3}$$

where $D^+U(t,x) = [D^+v_{ij}(t,x)],\ i,j = 1,2,\dots,m,$ and

$$\begin{aligned} D^+v_{ij}(t,x) = \lim\sup\Big\{ &\sup_{\gamma(t,t,x)=x} [v_{ij}(t+\sigma,\, \gamma(t+\sigma,\, t,x)) \\ &- v_{ij}(t,x)]\,\sigma^{-1}\colon\ \sigma\to 0^+\Big\}, \quad i,j = 1,2,\dots,m. \end{aligned} \tag{1.5.4}$$

If the matrix-valued function $U(t,x) \in C^{1,1}(\mathcal{T}_\tau \times \mathcal{N},\ R^{m\times m})$, i.e. all its elements $v_{ij}(t,x)$ are functions continuously differentiable in t and x, then the expression (1.5.4) is equivalent to

$$Dv_{ij}(t,x) = \frac{\partial v_{ij}}{\partial t}(t,x) + \sum_{s=1}^{n} \frac{\partial v_{ij}}{\partial x_s}(t,x)\, f_s(t,x), \tag{1.5.5}$$

where $f_s(t,x)$ are components of the vector-function $f(t,x) = (f_1(t,x),\dots,$ $f_n(t,x))^{\mathrm{T}}$.

In Chapter 2, Sections 2.1 – 2.5, we will establish the sufficient conditions for asymptotic stability (in the whole), uniform asymptotic stability (in the whole), exponential stability (in the whole), and instability of solutions of nonlinear large scale systems under nonclassical structural perturbations by applying Liapunov's matrix functions (1.5.1) and its derivative (1.5.3) or (1.5.5)

1.6 Notes and References

Section 1.2 The problem of motion stability arises whenever the engineering or physical problem is formulated as a mathematical problem of qualitative analysis of equations. Poincare and Liapunov laid a background for the method of auxiliary functions for continuous systems which allow not to integrate the motion equations for their qualitative analysis. The ideas of Poincare and Liapunov were further developed and applied in many branches of modern natural sciences.

The results of Liapunov [1], Chetaev [1], Persidskii [1], Malkin [1], Ascoli [1], Barbasin and Krasovskii [1], Massera [1], and Zubov [1], were a base for Definitions 1.2.1 – 1.2.3 (ad hoc see Grujić *et al.* [1], pp. 8 – 12 and cf. Rao Mohana Rao [1], Yoshizawa [1], Rouche *et al.* [1], Antosiewicz [1], Lakshmikantham an Leela [1], Hahn [2], etc.). For Definitions 1.2.4 – 1.2.7, and 1.2.13 see Hahn [2], and Martynyuk [9]. Definitions 1.2.8 – 1.2.12 are based on some results by Liapunov [1], Hahn [2], Barbashin and Krasovskii [1] (see and cf. Djordjevic [1], Grujić [3], and Martynyuk [2, 3, 5, 10, 13, 17]).

Discrete systems appear to be efficient mathematical models in the investigation of many real world processes and phenomena (see Samarskii and Gulia [1]). Note that yet in the works by Euler and Lagrange the so-called recurrent series and some problems of probability theory were studied being described by discrete (finite difference) equations. The active investigation of discrete systems (for the last three decades) is stipulated by new problems of the technical progress. Discrete equations prove to be the most efficient model in description of the mechanical system with impulse perturbations as well as the systems comprising digital computing devices. Recently the discrete systems have been applied in the modelling of processes in population dynamics, macro-economy, chaotic dynamics of economic systems, modelling of recurrent neuron networks, chemical reactions, dynamics of discrete Markov processes, finite and probably automatic machines and computing processes.

The dynamics of discrete-time systems is in the focus of attention of many experts (see, for example, Aulbach [1], Diamond [1], Elaydi and Peterson [1], Luca and Talpalaru [1], Maslovskaya [1], etc.).

Many evolution processes are characterized by the fact that at certain moments of time they experience a change of state abruptly. This is due to short term perturbations whose duration is negligible in comparison with the duration of the process. It is natural, therefore, to assume that such perturbations act instantaneously, that is, in the form of impulses. Thus impulsive differential equations, namely, differential equations involving impulse effects, appear as natural description of observed evolution phenomenon of several real-world problems. Of course, the theory of impulsive differential equations is much richer than the corresponding theory of differential equations without impulse effects (see Blaquiere [1], Krylov and Bogoliubov [1], Mil'man and Myshkis [1], Myshkis and Samoilenko [1], etc.).

For Definitions 1.4.1 – 1.4.3 see Lakshmikantham, Bainov, *et al.* [1], Samoilenko and Perestyuk [1], Simeonov and Bainov [1], etc.

Original results and the surveys of some directions of investigations are presented in the monographs by Lakshmikantham, Leela, and Martynyuk [1, 2], Pandit and Deo [1], and in many papers.

The physical system can consist of subsystems that react differently to the external impacts (see Pontryagin [1], Tikhonov [1], Volosov [1], Hopensteadt [1], Grujić, *et al.* [1], etc.). Moreover, each of the subsystems has its own scale of natural time. In the case when the subsystems are not interconnected, the dynamical properties of each subsystem are examined in terms of the corresponding time scale. It turned out that it is reasonable to use such information when the additional conditions on the subsystems are formulated in the investigation of large scale systems. The existence of various time scales related to the separated subsystems is mathematically expressed by arbitrarily small positive parameters μ_i present at the part of the higher derivatives in differential equation. If the parameters μ_i vanish, the number of differential equations of the large scale system is diminished and, hence the appearance of algebraic equations. This is just the singular case allowing the consideration of various peculiarities of the system with different time scales.

Modern analytical and qualitative methods of analysis of singularly perturbed systems are based on some ideas and results of the classical works by Tikhonov and Pontryagin. The development of general ideas in the direction is presented in the papers and monographs by Vasil'eva and Butuzov [1], Mishchenko and Rozov [1], Eckhaus [1], Carrier [1], O'Malley [1], Kokotovic and Khalil [1], Miranker [1], Chang and Howes [1], etc.

Section 1.3 Various problems of the stability theory under classical structural perturbations were studied in many papers (see, e.g. Aeppli and Markus [1], Arnol'd [1], Bowen and Ruelle [1], Conley and Zehnder [1], Coppel [1], Cronin [1], Hale [1], Hirsch [1], Kneser [1], Kaplan [1], Markus [1], Moser [1], Pilugin [1], Shub [1], Zeeman [1], etc.).

This Section encorporates some results by Arnol'd [1], Sell [1], Lefshetz [1], Peixoto [1], Šiljak [1], and Chetaev [1], etc.

Section 1.4 We focused main attention on the concept of stability under non-classical structural perturbations in the sense of Liapunov. We used in the point the results from monograph by Grujić, Martynyuk and Ribbens-Pavella [1].

Section 1.5 For the details of the method of matrix-valued Liapunov functions see Martynyuk [1–3] and Djordjević [1]. This method has been developed at the Stability of Processes Department of the Institute of Mechanics of NAS of Ukraine since 1979 (see Ph.D. thesises by Shegai [1], Miladzhanov [1], Azimov [1], Begmuratov [1], Martynyuk-Chernienko [1], Slyn'ko [1], Lykyanova [1]).

For the recent papers concerning the topics of Sections 1.2 – 1.5 see Kramer and Hofman [1].

We note that the two-index system of functions (1.5.1) being suitable for construction of the Liapunov functions allows to involve more wide classes of functions as compared with those usually applied in motion stability theory. For example, the bilinear forms prove to be natural non-diagonal elements of matrix-valued functions. Another peculiar feature of the approach being of importance is the fact that the application of the matrix-valued function in the investigation of multidimensional systems enables to allow for the interconnections between the subsystems in their natural form, i.e. not necessarily as the destabilizing factor. Finally, for the determination of the property of having a fixed sign of the total derivative of auxiliary function along solutions of the system under consideration it is not necessary to encorporate the estimation functions with the quasi-monotonicity property. Naturally, the awkwardness of calculations in this case is the price.

2

CONTINUOUS LARGE-SCALE SYSTEMS

2.1 Introduction

Qualitative analysis of nonlinear systems by Liapunov's direct (second) method (see Liapunov [1]) can be effectively done only when there is an algorithm of construction of an appropriate function for the system under consideration. A series of investigations simplify the initial problem so that stability properties are defined not immediatelly, but via investigation of an intermediate system. Here we study large scale nonlinear continuous systems under nonclassical structural perturbations in context with method of Liapunov matrix-valued functions.

The purpose of this Chapter is to obtain sufficient conditions for asymptotic stability (in the whole), uniform asymptotic stability (in the whole), exponential stability (in the whole), and instability of solutions of nonlinear large scale systems under nonclassical structural perturbations by applying matrix Liapunov's functions method.

The present chapter is arranged as follows.

In Section 2.2 the composition of continuous large scale system under given models of connectedness is described.

Section 2.3 provides necessary information about the matrix-valued functions which are applied in the investigation of large scale continuous systems under nonclassical structural perturbations.

Section 2.4 is focussed on the new sufficient conditions for various types of stability of nonlinear systems under nonclassical structural perturbations. These conditions were established while solving Problem C_A and Problem C_B.

In Section 2.5 the method of choosing the elements of the matrix-valued function is concretized and the results of stability investigation of linear system under nonclassical structural perturbations are presented. General results are illustrated by the numerical examples.

The final Section 2.6 indicates some possible trends of the further development of the method of matrix-valued functions and their applications. Namely, in point 2.6.1 Liapunov's matrix-valued function is applied in stability investigation with respect to two measures under nonclassical structural perturbations. In point 2.6.2 the problem of stability of large scale power system under nonclassical structural perturbations is discussed.

2.2 Nonclassical Structural Perturbations in Time-Continuous Systems

We consider nonlinear continuous systems whose description is based on the assumptions below. Furtheron the systems, subsystems, of this class are designated by C, C_i, respectively.

H_1. The imaginary mechanical or other system C consists of m interacting subsystems C_i, whose behaviour is described by continuous systems of ordinary differential equations the order of which is not changed on the interval of the system functionning.

H_2. The internal (e.g., parametric) or external perturbations of the C are characterized by the matrix $P = (p_1^{\mathrm{T}}, p_2^{\mathrm{T}}, \dots, p_m^{\mathrm{T}})^{\mathrm{T}} \in R^{m\times q}$. The set of all admissible matrices P is designated by

$$\mathcal{P} = \{P\colon\ P_1 \le P(t) \le P_2, \quad \text{for all} \quad t \in R\}, \tag{2.2.1}$$

where P_1 and P_2 are the prescribed constant matrices.

H_3. The family $\mathcal{F}$, is determined consisting of the vector functions f_1, $f_2, \dots, f_m$ for which $f_i^k \in C(\mathcal{T} \times R^n \times R^{1\times q},\ R^{n_i})$, for all $k = 1, 2, \dots, N$, where N is a real number, $n = n_1 + n_2 + \cdots + n_m$, and $i = 1, 2, \dots, m$.

H_4. The dynamics of the interconnected subsystem C_i in system C is described by the equations

$$\frac{dx_i}{dt} = f_i(t, x, p_i), \quad i = 1, 2, \dots, m, \tag{2.2.2}$$

where $x_i \in R^{n_i}$, $f_i \in \mathcal{F}_i$, $\mathcal{F}_i = \{f_i^1, f_i^2, \dots, f_i^N\}$, $x = (x_1^{\mathrm{T}}, \dots, x_i^{\mathrm{T}}, \dots, x_m^{\mathrm{T}})^{\mathrm{T}}$. The functions f_i in system (2.2.2) satisfy the condition $f_i(t, 0, 0) = 0$ for all $t \in \mathcal{T}$.

H_5. The dynamics of the i-th isolated subsystem $\widehat{C}_i$ is descibed by the equations

$$\frac{dx_i}{dt} = g_i(t, x_i), \quad x_i(t_0) = x_i^0. \tag{2.2.3}$$

Here $x_i \in R^{n_i}$, the state vector of the subsystem $\widehat{C}_i$, and the functions $g_i\colon \mathcal{T} \times R^{n_i} \to R^{n_i}$ are determined by the correlations

$$g_i(t, x_i) = f_i(t, x^i, 0), \quad i = 1, 2, \dots, m,$$

where $x^i = (0, \dots, 0, x_i^{\mathrm{T}}, 0, \dots, 0)^{\mathrm{T}}$.

The subsystems (2.2.3) do not contain structural and/or parametric perturbations and bear the main information on the dynamical properties of subsystems $\widehat{C}_i$, while the functions

$$h_i(t, x, p_i) = f_i(t, x, p_i) - g_i(t, x_i), \quad i = 1, 2, \dots, m$$

in the system

$$\text{(2.2.4)} \qquad \frac{dx_i}{dt} = g_i(t, x_i) + h_i(t, x, p_i), \quad i = 1, 2, \dots, m,$$

describe the effect of the subsystems $C_1, \dots, C_{i-1}, C_{i+1}, \dots, C_m$ of system C on the subsystem C_i.

Designate by $\mathcal{H}_i$ the set of all possible h_i, from

$$h_i^j(t, x, p_i) = f_i^j(t, x, p_i) - g_i(t, x_i), \quad j = 1, 2, \dots, N, \quad i = 1, 2, \dots, m.$$

The fact that $f_i^j(t, x, p_i) \in \mathcal{F}_i$ implies that $h_i^j(t, x, p_i) \in \mathcal{H}_i$ for all $i = 1, 2, \dots, m$.

The binary function $s_{ij}\colon \mathcal{T} \to \{0, 1\}$ is applied as a structural parameter of system ($s_{ij}\colon \mathcal{T} \to [0, 1]$). This function represents the (i, j)-th element of the structural matrix $S_i\colon R \to R^{n_i \times N_{n_i}}$ of the i-th interconnecting subsystem S_i.

If we designate

$$\mathcal{S} = \left\{ S\colon \ S = \begin{pmatrix} S_1 & 0_{12} & \dots & 0_{1m} \\ 0_{21} & S_2 & \dots & 0_{2m} \\ \dots & \dots & \dots & \dots \\ 0_{m1} & 0_{m2} & \dots & S_m \end{pmatrix} \right\}, \qquad 0_{ij} \in R^{n_i \times n_j},$$

where $S_i = (s_{i1} I_i, s_{i2} I_i, \dots, s_{iN} I_i)$, $s_{ij} \in \{0, 1\}$, $I_i = \operatorname{diag}\{1, 1, \dots, 1\} \in R^{n_i} \times R^{n_i}$, the dynamics of the i-th interconnecting subsystem C_i can be deccribed by the equations

$$\text{(2.2.5)} \qquad \frac{dx_i}{dt} = g_i(t, x_i) + S_i(t) h_i(t, x, p_i), \quad i = 1, 2, \dots, m.$$

where $h_i \in C(\mathcal{T} \times R^n \times R^{1 \times q}, , \operatorname{Re}^{N_{n_i}})$.

In general the dynamics of the system C can be represented by the vector differential equation

$$\text{(2.2.6)} \qquad \frac{dx}{dt} = g(t, x) + S(t) h(t, x, P), \quad P \in \mathcal{P}, \quad S(t) \in \mathcal{S},$$

where

$$x \in R^n, \quad g(t, x) = \left(g_1^{\mathrm{T}}(t, x_1), \dots, g_m^{\mathrm{T}}(t, x_m)\right)^{\mathrm{T}},$$
$$h = \left(h_1^{\mathrm{T}}(t, x, p_1), \dots, h_m^{\mathrm{T}}(t, x, p_m)\right)^{\mathrm{T}}.$$

Remark 2.2.1 On the set $\mathcal{N} = \{1, \dots, N\}$ the variation of the exponent $k(t) \in \mathcal{N}$ for all $t \in R$ describes structural changes of system C. System C is structurally invariant if and only if $k(t) = \text{const}$, or if the set $\mathcal{N}$ is unitary. Thus, N indicates the number of all possible structures of the system C.

Remark 2.2.2 The set $\mathcal{P}$ can be either singleton, i.e. $\mathcal{P} \triangleq p$, $p \in \Delta \subset R^1$, Δ is a compact in R^1, or empty $(P_1 \equiv P_2 \equiv 0)$. In the case when $\mathcal{P} = \varnothing$ the system C does not have parametric perturbations, but it can have structural changes, since $f \in \mathcal{F}$.

Remark 2.2.3 It is easy to notice that the proposed formalization of motion equations for continuous multidimansional system C and their representations in the form of (2.2.5) or the vector form (2.2.6) is one of possible realizations of the general Chetayev's idea [1] described above.

2.3 Estimates of Matrix-Valued Functions

Together with (2.2.6) we consider a matrix-valued function

$$U(t, x) = [v_{ij}(t, x)] \quad \text{for all} \quad (i, j) = 1, 2, \dots, m, \tag{2.3.1}$$

where $v_{ii} \in C(R_+ \times R^n, R_+)$ for all $i = 1, 2, \dots, m$ and $v_{ij} \in C(R_+ \times R^n, R)$ for all $i \neq j$, $i, j = 1, 2, \dots, m$. By means of (2.3.1) a scalar function

$$v(t, x, \psi) = \psi^{\mathrm{T}} U(t, x) \psi \tag{2.3.2}$$

is introduced with $\psi = (\psi_1, \psi_2, \dots, \psi_m)^{\mathrm T}$, $\psi_i \neq 0$, $i = 1, 2, \dots, m$. Note, that if $\psi = (1, 1, \dots, 1)^{\mathrm T} \in R^m_+$ then (2.3.2) becomes

$$v(t,x) = \sum_{i,j=1}^{m} v_{ij}(t,x). \tag{2.3.3}$$

Let $v_{ii} = v_{ii}(t, x_i)$ correspond to subsystems (2.2.3) and $v_{ij} = v_{ji} = v_{ij}(t, x_i, x_j)$ take into consideration connections $S_i(t)h_i(t, x, p_i)$ between the equations (2.2.3) for all $v \neq j$, $i, j = 1, 2, \dots, m$.

Assumption 2.3.1 There exist

(1) open connected neighbourhoods $\mathcal{N}_{ix} \subseteq R^{n_i}$ of the states $(x_i = 0) \in R^{n_i}$ for all $i = 1, 2, \dots, m$;
(2) functions $\varphi_{ik} \colon \mathcal{N}_{ix} \to R_+$, for all $i = 1, 2, \dots, m$; $k = 1, 2$, $\varphi_{ik} \in K$ ($\varphi_{ik} \in KR$);
(3) constants $\underline{\alpha}_{ij}, \overline{\alpha}_{ij}$, $i, j = 1, 2, \dots, m$, and a function $\Delta(t) \in C(R, R_+)$, $\Delta(t) \geq c > 0$, and
(4) a matrix-valued function $U(t,x)$ with elements $v_{ii}(t, x_i)$, $v_{ii}(t, 0) = 0$ for all $t \in R_+$, and $v_{ij}(t, x_i, x_j)$, $v_{ij}(t, 0, 0) = 0$ for all $i \neq j$ and for all $t \in R_+$ satisfying the estimates:

(a) $\underline{\alpha}_{ii}\varphi^2_{i1}(\|x_i\|)\Delta(t) \leq v_{ii}(t, x_i) \leq \overline{\alpha}_{ii}\varphi^2_{i2}(\|x_i\|)$
for all $(t, x_i) \in R_+ \times \mathcal{N}_{ix}$ (for all $(t, x_i) \in R_+ \times R^{n_i}$),
$i = 1, 2, \dots, m$;

(b) $\underline{\alpha}_{ij}\varphi_{i1}(\|x_i\|)\varphi_{j1}(\|x_j\|)\Delta(t) \leq v_{ij}(t, x_i, x_j)$
$\leq \overline{\alpha}_{ij}\varphi_{i1}(\|x_i\|)\varphi_{j2}(\|x_j\|)$ for all $(t, x_i, x_j) \in R_+ \times \mathcal{N}_{ix} \times \mathcal{N}_{jx}$
(for all $(t, x_i, x_j) \in R_+ \times R^{n_i} \times R^{n_j}$) for all $i \neq j$,
$i, j = 1, 2, \dots, m$.

If we can find a matrtix-valued function $U(t,x)$ which satisfies the conditions in Assumption 2.3.1, we can prove the following assertion.

Proposition 2.3.1 *If all conditions of Assumption 2.3.1 hold for function (2.3.1), then*

$$\begin{gathered}\Delta(t)\Phi_1^{\mathrm T} H^{\mathrm T} A H \Phi_1 \leq v(t, x, \psi) \leq \Phi_2^{\mathrm T} H^{\mathrm T} B H \Phi_2 \\ \textit{for all} \quad (t, x_i, x_j) \in R_+ \times \mathcal{N}_{ix} \times \mathcal{N}_{jx} \\ (\textit{for all} \quad (t, x_i, x_j) \in R_+ \times R^{n_i} \times R^{n_j}),\end{gathered} \tag{2.3.7}$$

where

$$\begin{gathered}\Phi_1^{\mathrm T} = (\varphi_{11}(\|x_1\|),\, \varphi_{21}(\|x_2\|),\, \dots,\, \varphi_{m1}(\|x_m\|)), \\ \Phi_2^{\mathrm T} = (\varphi_{12}(\|x_1\|),\, \varphi_{22}(\|x_2\|),\, \dots,\, \varphi_{m2}(\|x_m\|)), \\ H = \operatorname{diag}[\psi_1, \psi_2, \dots, \psi_m], \\ A = [\underline{\alpha}_{ij}], \quad \underline{\alpha}_{ij} = \underline{\alpha}_{ji}, \\ B = [\overline{\alpha}_{ij}], \quad \overline{\alpha}_{ij} = \overline{\alpha}_{ji}, \quad i, j = 1, 2, \dots, m.\end{gathered}$$

Proof Let all conditions of Assumption 2.3.1 be satisfied. Simple algebraic transformations of the expression (2.3.2) lead to the estimates

$$v(t,x,\psi) = \sum_{i=1}^{m} \psi_i^2 v_{ii}(t,x_i) + 2\sum_{i=1}^{m}\sum_{\substack{j=2\\ j>i}}^{s} \psi_i\psi_j v_{ij}(t,x_i,x_j)$$

$$\geq \sum_{i=1}^{m} \psi_i^2 \underline{\alpha}_{ii}\varphi_{i1}^2(\|x_i\|)\Delta(t) + 2\sum_{i=1}^{m}\sum_{\substack{j=2\\ j>i}}^{s} \psi_i\psi_j \underline{\alpha}_{ij}\varphi_{i1}(\|x_i\|)\varphi_{j1}(\|x_j\|)\Delta(t)$$

$$= \Delta(t)(\varphi_{11}(\|x_1\|),\, \varphi_{21}(\|x_2\|),\, \dots,\, \varphi_{m1}(\|x_m\|))^{\mathrm{T}}$$

$$\times \operatorname{diag}[\psi_1,\psi_2,\dots,\psi_m] \begin{pmatrix} \underline{\alpha}_{11} & \underline{\alpha}_{12} & \cdots & \underline{\alpha}_{1s} \\ \cdots & \cdots & \cdots & \cdots \\ \underline{\alpha}_{1m} & \underline{\alpha}_{2m} & \cdots & \underline{\alpha}_{mm} \end{pmatrix} \operatorname{diag}[\psi_1,\psi_2,\dots,\psi_m]$$

$$\times(\varphi_{11}(\|x_1\|),\, \varphi_{21}(\|x_2\|),\, \dots,\, \varphi_{m1}(\|x_m\|)) = \Delta(t)\Phi_1^{\mathrm{T}} H^{\mathrm{T}} A H \Phi_1$$

for all $(t,x_i,x_j) \in R_+ \times \mathcal{N}_{ix} \times \mathcal{N}_{jx}$ (for all $(t,x_i,x_j) \in R_+ \times R^{n_i} \times R^{n_j}$).

The estimate from above in inequality (2.3.7) is proved similarly.

2.4 Tests for Stability Analysis

2.4.1 The Problem $\mathrm{C_A}$ This section gives a solution of the following problem.

Problem $\mathrm{C_A}$. Let the continuous dynamical system C be obtained as a result of composition of the interacting subsystems (2.2) according to the adopted model of generalized connectedness. It is necessary to establish sufficient conditions of various types of stability for the equilibrium state $x = 0$ of the system (2.6) in terms of the dynamical characteristics of the isolated subsystems (2.3) and qualitative estimates of the interconnection functions between these subsystems.

But first, for the reader's convenience, we recall some definitions.

Definition 2.4.1 The equilibrium $x = 0$ of system (2.2.6) possesses certain dynamical property under parametric and/or nonclassical structural perturbations if and only if this equilibrium state possesses the corresponding dynamical property for any $(P,S) \in \mathcal{P} \times \mathcal{S}$.

According to some results from Section 1.5 we will use the following definitions.

Definition 2.4.2 The function

$$D^+v(t,x,\psi)$$
$$= \lim_{\delta\to 0^+} \sup\{[v(t+\delta,\, x+\delta(g(t,x)+S(t)h(t,x,P)),\, \psi) - v(t,x,\psi)]\,\delta^{-1}\}$$

for all $(t,x) \in R_+ \times R^n$ and $(P,S) \in \mathcal{P} \times \mathcal{S}$ is called *total derivative of the matrix-valued function* (2.3.2) along solutions of the system (2.2.6).

We designate $\mathcal{N}_{ix0} = \{x_i\colon\ x_i \in \mathcal{N}_{ix} \subseteq R^{n_i},\ x_i \neq 0\}$, and formulate one more assumption.

Assumption 2.4.1 There exist

(1) open connected neighbourhoods $\mathcal{N}_{ix} \subseteq R^{n_i}$ of the states $0 \in R^{n_i}$ for all $i = 1, 2, \dots, m$ and a connected neighbourhood $\mathcal{N}_x \subseteq \mathcal{N}_{1x} \times \mathcal{N}_{2x} \times \dots \times \mathcal{N}_{mx}$ of the state $x = 0$;

(2) functions $\varphi_i \colon \mathcal{N}_{ix} \to R_+$, $i = 1, 2, \dots, m$, $\varphi_i \in K\,(KR)$ and the functions v_{ij}, $i, j = 1, 2, \dots, m$, mentioned in Assumption 2.3.1 and moreover

(a) $v_{ii} \in C(R_+ \times \mathcal{N}_{ix0}, R_+)$ $(v_{ii} \in C(R_+ \times R^{n_i}, R_+))$ for all $i = 1, 2, \dots, m$, and

(b) $v_{ij} \in C(R_+ \times \mathcal{N}_{ix0} \times \mathcal{N}_{jx0}, R)$ $(v_{ij} \in C(R_+ \times R^{n_i} \times R^{n_j}, R))$ for all $i \neq j$, $i, j = 1, 2, \dots, m$;

(3) constants ρ^i_{1i}, ρ^j_{2i}, ρ^i_{3j}, $\rho^i_{4i}(P, S)$, $\rho^j_{mi}(P, S)$, $\rho^i_{qj}(P, S)$, ρ_{kij}, $\rho_{lij}(P, S)$, $\rho_{rij}(P, S)$, $k = 1, 2$; $l = 3, 4$; $r = 6, 7$; $m = 5, 7$; $q = 6, 8$; $i, j = 1, 2, \dots, m$, and conditions

(a) $D^+_t v_{ii} + (D^+_{x_i} v_{ii})^{\mathrm{T}} g_i(t, x_i) \leq \rho^i_{1i} \varphi^2_i(\|x_i\|)$ for all $(t, x_i) \in R_+ \times \mathcal{N}_{ix0}$ (for all $(t, x_i) \in R_+ \times R^{n_i}$), $i = 1, 2, \dots, m$;

(b) $D_t^+ v_{ij} + (D_{x_i}^+ v_{ij})^{\mathrm{T}} g_i(t, x_i) \le \rho_{2i}^j \varphi_i^2(\|x_i\|)$
$+ \rho_{1ij}\varphi_i(\|x_i\|)\varphi_j(\|x_j\|)$ for all $(t, x_i, x_j) \in R_+ \times \mathcal{N}_{ix0} \times \mathcal{N}_{jx0}$
(for all $(t, x_i, x_j) \in R_+ \times R^{n_i} \times R^{n_j}$) for all $i \neq j$,
$i, j = 1, 2, \dots, m$;

(c) $(D_{x_j}^+ v_{ij})^{\mathrm{T}} g_j(t, x_j) \le \rho_{3j}^i \varphi_j^2(\|x_j\|) + \rho_{2ij}\varphi_i(\|x_i\|)\varphi_j(\|x_j\|)$
for all $(t, x_i, x_j) \in R_+ \times \mathcal{N}_{ix0} \times \mathcal{N}_{jx0}$ (for all $(t, x_i, x_j) \in$
$R_+ \times R^{n_i} \times R^{n_j}$) for all $i \neq j$, $i, j = 1, 2, \dots, m$;

(d) $(D_{x_i}^+ v_{ii})^{\mathrm{T}} S_i(t) h_i(t, x, p_i) \le \rho_{4i}^i(P, S)\varphi_i^2(\|x_i\|)$

$$+ \sum_{\substack{j=1\\ j\neq i}}^{m} \rho_{3ij}(P, S)\varphi_i(\|x_i\|)\varphi_j(\|x_j\|), \quad i = 1, 2, \dots, m,$$

for all $(t, x_i, x_j) \in R_+ \times \mathcal{N}_{ix0} \times \mathcal{N}_{jx0}$
(for all $(t, x_i, x_j) \in R_+ \times R^{n_i} \times R^{n_j}$),
for all $(P, S) \in \mathcal{P} \times \mathcal{S}$;

(e) $(D_{x_i}^+ v_{ij})^{\mathrm{T}} S_i(t) h_i(t, x, p_i) \le \rho_{5i}^j(P, S)\varphi_i^2(\|x_i\|)$

$$+ \sum_{\substack{i=1\\ i\neq j}}^{m} \rho_{4ij}(P, S)\varphi_i(\|x_i\|)\varphi_j(\|x_j\|)$$

$$+ \sum_{\substack{j=1\\ j\neq i}}^{m} \rho_{5ij}(P, S)\varphi_i(\|x_i\|)\varphi_j(\|x_j\|) + \rho_{7i}^j(P, S)\varphi_j^2(\|x_j\|),$$

for all $(t, x_i, x_j) \in R_+ \times \mathcal{N}_{ixo} \times \mathcal{N}_{jx0}$
(for all $(t, x_i, x_j) \in R_+ \times R^{n_i} \times R^{n_j}$), for all $(P, S) \in \mathcal{P} \times \mathcal{S}$,
for all $i \neq j$, $i, j = 1, 2, \dots, m$;

(f) $(D_{x_j}^+ v_{ij})^{\mathrm{T}} S_j(t) h_j(t, x, p_j) \le \rho_{6j}^i(P, S)\varphi_i^2(\|x_i\|)$

$$+ \sum_{i=1}^{s} \rho_{6ij}(P, S)\varphi_i(\|x_i\|)\varphi_j(\|x_j\|)$$

$$+ \sum_{\substack{j=1\\ j\neq i}}^{m} \rho_{7ij}(P, S)\varphi_i(\|x_i\|)\varphi_j(\|x_j\|) + \rho_{8j}^i(P, S)\varphi_j^2(\|x_j\|),$$

for all $(t, x_i, x_j) \in R_+ \times \mathcal{N}_{ixo} \times \mathcal{N}_{jx0}$
(for all $(t, x_i, x_j) \in R_+ \times R^{n_i} \times R^{n_j}$), for all $(P, S) \in \mathcal{P} \times \mathcal{S}$,
for all $i \neq j$, $i, j = 1, 2, \dots, m$,

hold true.

Proposition 2.4.1 *If all conditions of Assumption 2.4.1 are satisfied then*

(2.4.1) $$D^+ v(t, x, \psi) \le w^{\mathrm{T}} Q(P, S) w$$

for all $(t, x_i, x_j) \in R_+ \times \mathcal{N}_{ix0} \times \mathcal{N}_{jx0}$ *(for all* $(t, x_i, x_j) \in R_+ \times R^{n_i} \times R^{n_j}$*)* *and for all* $(P, S) \in \mathcal{P} \times \mathcal{S}$*, where*

$$w^{\mathrm{T}} = (\varphi_1(\|x_1\|),\, \varphi_2(\|x_2\|),\, \dots,\, \varphi_m(\|x_m\|)),$$
$$Q(P,S) = [q_{ij}(P,S)], \quad q_{ij} = q_{ji}, \quad i,j = 1,2,\dots,m$$

and

$$q_{ii}(P,S) = \psi_i^2(\rho_{1i}^i + \rho_{4i}^i(P,S)) + 2\sum_{\substack{j=1\\ j\neq i}}^{m} \psi_i\psi_j\{\rho_{2i}^j + \rho_{3j}^i$$
$$+ \rho_{5i}^j(P,S) + \rho_{7i}^j(P,S) + \rho_{6j}^i(P,S) + \rho_{8j}^i(P,S)\}, \quad i = 1,2,\dots,m;$$
$$q_{ij}(P,S) = \frac{1}{2}\,\psi_i^2(\rho_{3ij}(P,S) + \rho_{3ji}(P,S))$$
$$+ \psi_i\psi_j\Bigg\{\rho_{1ij} + \rho_{2ij} + \sum_{\substack{l=1\\ l\neq j}}^{m}\sum_{\substack{q=1\\ q\neq l}}^{m}(\rho_{4lq}(P,S) + \rho_{6lq}(P,S))$$
$$+ \sum_{\substack{q=1\\ q\neq i}}^{m}\sum_{\substack{l=1\\ l\neq q}}^{m}(\rho_{5lq}(P,S) + \rho_{7lq}(P,S))\Bigg\}, \quad i \neq j, \quad i,j = 1,2,\dots,m.$$

Proof Let all conditions of Assumption 2.4.1 be satisfied. Then for the expression $D^+v(t,x,\psi)$ we have

$$D^+v(t,x,\psi) = \psi^{\mathrm{T}} D^+U(t,x)\psi = \sum_{i=1}^{m}\psi_i^2 D^+v_{ii}(t,x_i)$$
$$+ 2\sum_{i=1}^{m}\sum_{\substack{j=2\\ j>i}}^{m}\psi_i\psi_j D^+v_{ij}(t,x_i,x_j). \tag{2.4.2}$$

Furthermore, by conditions (3)(a)−(f), we have

$$D^+v(t,x,\psi) \le \sum_{i=1}^{m}\Bigg\{\psi_i^2(\rho_{1i}^i + \rho_{4i}^i(P,S)) + 2\sum_{\substack{j=1\\ j\neq i}}^{m}\psi_i\psi_j(\rho_{2i}^j + \rho_{3j}^i$$
$$+ \rho_{5i}^j(P,S) + \rho_{7i}^j(P,S) + \rho_{6j}^i(P,S) + \rho_{8j}^i(P,S))\Bigg\}\varphi_i^2(\|x_i\|)$$
$$+ 2\sum_{i=1}^{m}\sum_{\substack{j=1\\ j>i}}^{m}\Bigg\{\frac{1}{2}\,\psi_i^2(\rho_{3ij}(P,S) + \rho_{3ji}(P,S))$$
$$+ \psi_i\psi_j\Bigg(\rho_{1ij} + \rho_{2ij} + \sum_{\substack{l=1\\ l\neq j}}^{m}(\rho_{4lj}(P,S) + \rho_{6lj}(P,S))$$
$$+ \sum_{\substack{q=1\\ q\neq i}}^{m}(\rho_{5iq}(P,S) + \rho_{7iq}(P,S))\Bigg)\Bigg\}\varphi_i(\|x_i\|)\varphi_j(\|x_j\|)$$
$$= \sum_{i=1}^{m}\xi_{ii}(P,S)\varphi_i^2(\|x_i\|) + 2\sum_{i=1}^{m}\sum_{\substack{j=2\\ j>i}}^{m}\xi_{ij}(P,S)\varphi_i(\|x_i\|)\varphi_j(\|x_j\|)$$
$$= w^{\mathrm{T}}Q(P,S)w$$

for all $(t, x_i, x_j) \in R_+ \times \mathcal{N}_{ix0} \times \mathcal{N}_{jx0}$ (for all $(t, x_i, x_j) \in R_+ \times R^{n_i} \times R^{n_j}$) and for all $(P, S) \in \mathcal{P} \times \mathcal{S}$.

This completes the proof of Proposition 2.4.1.

By means of $\widetilde{U}(t,x) = \operatorname{diag} U(t,x) = [v_{11}(t,x_1),\dots,v_{mm}(t,x_m)]$, a matrix K and a vector $\psi \in R^m$ we construct the vector function

$$L(t,x,\psi) = K\widetilde{U}(t,x)\psi \tag{2.4.3}$$

with components $L_1(t,x,\psi)$, $L_2(t,x,\psi)$, ..., $L_m(t,x,\psi)$.

Following Grujić *et al.* [1] we will use the next definitions.

Definition 2.4.3 We say that the set $L_\zeta(t)$ is a *maximal connected neighborhood* of the origin for each $\zeta \in (0,+\infty)$ and $t \in R$ if $x \in L_\zeta(t)$ implies $L_\zeta(t,x,\psi) < \zeta$.

Let $\mathcal{O}$ be a set $\{x : x = 0\}$.

Definition 2.4.4 We say that time-varying set $\Pi(t)$ is *asymptotically contractive* (or *asymptotically contracts to* $\mathcal{O}$) iff:

(1) there exists a $\tau \in R$ such that $\Pi(t)$ is a neighbourhood of the origin for any $t \leq \tau$ and

(2) $\lim\,[\Pi(t) : t \to +\infty] = \mathcal{O}$.

We note that the notions of asymptotically contractive sets with respect to functions were discovered by Grujić [1].

The following theorem provides our main characterization of stability under nonclassical structural perturbations.

Theorem 2.4.1 *Assume that the perturbed motion equations (2.2.10) are such that all conditions of Assumptions 2.3.1 and 2.4.1 are satisfied except for upper estimates of the functions* $v_{ii}(t,x_i)$ *and* $v_{ij}(t,x_i,x_j)$ *for all* $i,j = 1,2,\dots,m$ *and moreover*

(1) *there exist positive numbers* ξ_i *(or* $\xi_i = +\infty$*) such that the sets* $L_{i\zeta_i}(t)$ *are asymptotically contractive for any* $\zeta_i \in (0,\xi_i)$ *and any* $i = 1,2,\dots,m;$
(2) *the matrix A is positive definite;*
(3) *there exists a negative definite matrix* $\overline{Q} \in R^{m\times m}$ *such that*

$$\frac{1}{2}\,(Q(P,S) + Q^{\mathrm{T}}(P,S)) \le \overline{Q} \quad \textit{for all} \quad (P,S) \in \mathcal{P}\times\mathcal{S}.$$

Then the equilibrium state $x = 0$ *of the system (2.2.10) is asymptotically stable on* $\mathcal{P}\times\mathcal{S}$

If all hypotheses of the theorem hold for $\mathcal{N}_{ix} = R^{n_i}$*, for radially unbounded functions* $v_{ii}(t,x_i)$ *and* $v_{ij}(t,x_i,x_j)$ *and for* $\xi_i = +\infty$ *for any* $i = 1,2,\dots,m,$ *then the equilibrium state* $x = 0$ *of the system (2.2.10) is asymptotically stable in the whole on* $\mathcal{P}\times\mathcal{S}$.

Proof If Assumption 2.3.1, Proposition 2.3.1 and hypothesis (b) of the theorem are satisfied, the function (2.3.2) is positive definite on $\mathcal{N}_{1x}\times\mathcal{N}_{2x}\times\ldots\times\mathcal{N}_{mx}$. Hypothesis (a) of the theorem ensures asymptotic contraction of the set $L_{1\zeta_1}(t)\times L_{2\zeta_2}(t)\times\ldots\times L_{m\zeta_m}(t)$ for every $\zeta_i \in (0,\xi_i)$ for all $i = 1,2,\dots,m$. Let $L_\zeta(t)$ be the largest connected neighbourhood of $x = 0$, so that $L_\zeta(t,x,\psi) < \zeta$ for all $x \in L_\zeta(t)$ and for all $t \in R_+$. Therefore, the set $L_\zeta(t)$ is asymptotically contractive for every $\zeta \in (0,\xi)$, where $\xi = \min\{\psi_i^2\xi_i\colon\ i = 1,2,\dots,m\}$.

Further by Assumption 2.4.1 there exists a connected neighbourhood $\mathcal{N}_x$ of the state $x = 0$: $\mathcal{N}_x \subseteq \mathcal{N}_{1x}\times\mathcal{N}_{2x}\times\ldots\times\mathcal{N}_{mx}$ such that the conditions of Proposition 2.4.1 hold. By hypothesis (c) of the theorem the expression $D^+v(t,x,\psi)$ is negative definite for any $(P,S)\in\mathcal{P}\times\mathcal{S}$. As is known (see Theorem 1.2.3) these conditions are sufficient for asymptotic stability of equilibrium state of the system (2.2.1) on $\mathcal{P}\times\mathcal{S}$.

In case $\mathcal{N}_{ix} = R^{n_i}$ the function (2.3.2) is positive definite and radially unbounded. This fact together with other hypotheses of Theorem 2.4.1 proves its second assertion. This completes the proof of Theorem 2.4.1.

Theorem 2.4.2 *Assume that the perturbed motion equations (2.2.10) are such that all conditions of Assumptions 2.3.1 and 2.4.1 are satisfied with* $\Delta(t) = 1$ *for all* $t \in R_+$ *and*

(1) *the matrices A and B are positive definite;*
(2) *there exists a negative definite matrix* $\overline{Q} \in R^{m\times m}$ *such that*

$$\frac{1}{2}\,(Q(P,S) + Q^{\mathrm{T}}(P,S)) \le \overline{Q} \quad \textit{for all} \quad (P,S) \in \mathcal{P}\times\mathcal{S}.$$

Then the equilibrium state $x = 0$ *of the system (2.2.10) is uniformly asymptotically stable on* $\mathcal{P}\times\mathcal{S}$.

If moreover for $\mathcal{N}_{ix} = R^{n_i}$ *the functions* $v_{ii}(t,x_i)$ *and* $v_{ij}(t,x_i,x_j)$ *are radially unbounded and the functions* $\varphi_i \in KR,$ *then the equilibrium state* $x = 0$ *of the system (2.2.10) is uniformly asymptotically stable in the whole on* $\mathcal{P}\times\mathcal{S}$.

The Proof of Theorem 2.4.2 is similar to the proof of Theorem 2.4.1.

Theorem 2.4.3 *Assume that the perturbed motion equations (2.2.10) are such that all conditions of Assumptions 2.3.1 and 2.4.1 are satisfied with* $\Delta(t) = 1$ *for all* $t \in R_+$ *and*

(1) *for given functions* $\varphi_i(\|x_i\|)$ *there exist positive numbers* β_i *and* γ_i *such that*

$$\beta_i \|x_i\| \le \varphi_i(\|x_i\|) \le \gamma_i \|x_i\|$$

for all $x_i \in \mathcal{N}_{ix}$ *(for all* $x_i \in R^{n_i}$*) and all* $i = 1, 2, \dots, m$*;*

(2) *the matrices*

$$A^* = \left[\underline{\alpha}^*_{ij}\right], \quad \underline{\alpha}^*_{ij} = \underline{\alpha}^*_{ji};$$
$$B^* = \left[\overline{\alpha}^*_{ij}\right], \quad \overline{\alpha}^*_{ij} = \overline{\alpha}^*_{ji}, \quad i, j = 1, 2, \dots, m,$$

with elements

$$\underline{\alpha}^*_{ii} = \underline{\alpha}_{ii}\beta_i^2, \quad \overline{\alpha}^*_{ii} = \overline{\alpha}_{ii}\gamma_i^2,$$
$$\underline{\alpha}^*_{ij} = \begin{cases} \underline{\alpha}_{ij}\beta_i\beta_j & \text{if } \underline{\alpha}_{ij} \ge 0 \text{ for all } i \ne j; \\ \underline{\alpha}_{ij}\gamma_i\gamma_j & \text{if } \underline{\alpha}_{ij} < 0 \text{ for all } i \ne j; \end{cases}$$
$$\overline{\alpha}^*_{ij} = \begin{cases} \overline{\alpha}_{ij}\gamma_i\gamma_j & \text{if } \overline{\alpha}_{ij} \ge 0 \text{ for all } i \ne j; \\ \overline{\alpha}_{ij}\beta_i\beta_j & \text{if } \overline{\alpha}_{ij} < 0 \text{ for all } i \ne j, \end{cases}$$

are positive definite;

(3) *there exists a negative definite matrix* $\overline{Q}^* \in R^{m\times m}$ *such that*

$$\frac{1}{2}\left(Q^*(P,S) + Q^{*\mathrm{T}}(P,S)\right) \le \overline{Q}^* \quad \text{for all} \quad (P,S) \in \mathcal{P}\times\mathcal{S}.$$

Here $Q^*(P,S) = [\sigma^*_{ij}(P,S)]$*,* $\sigma^*_{ij} = \sigma^*_{ji}$ *for all* $i \ne j$*,* $i, j = 1, 2, \dots, m$*,*

$$\begin{aligned}\sigma^*_{ii}(P,S) &= \psi_i^2\left(\rho^i_{1i}k_{i1} + \rho^i_{4i}k_{i2}\right) \\ &\quad + 2\sum_{\substack{j=1\\ j\ne i}}^{s}\psi_i\psi_j\big\{\rho^j_{2i}k_{i3} + \rho^i_{3j}k_{j4} + \rho^j_{5i}(P,S)k_{i5} \\ &\quad + \rho^j_{7i}(P,S)k_{j6} + \rho^i_{6j}(P,S)k_{i7} + \rho^i_{8j}(P,S)k_{j8}\big\};\\ \sigma^*_{ij}(P,S) &= \psi_i\psi_j\left(\rho_{1ij}k^1_{ij} + \rho_{2ij}k^2_{ij}\right) + \frac{1}{2}\psi_i^2\rho_{3ij}(P,S)k^3_{ij} \\ &\quad + \sum_{\substack{l=1\\ l\ne j}}^{s}\psi_l\psi_j\left\{\rho_{4lj}(P,S)k^4_{lj} + \rho_{6lj}(P,S)k^5_{lj}\right\} \\ &\quad + \sum_{\substack{m=1\\ m\ne i}}^{s}\psi_i\psi_m\left\{\rho_{5im}(P,S)k^6_{im} + \rho_{7im}(P,S)k^7_{im}\right\},\end{aligned}$$

for all $i, j = 1, 2, \dots, m$ *and* $r = 1, 2, \dots, 8$

$$k_{ir} = \begin{cases} \gamma_i^2 & (\textit{or } \gamma_j^2) \textit{ if the corresponding multiplier} \\ & \varphi_i^2(\|x_i\|) \quad (\textit{or } \varphi_j^2(\|x_j\|)) \textit{ is positive,} \\ \beta_i^2 & (\textit{or } \beta_j^2) \textit{ if the corresponding multiplier} \\ & \varphi_i^2(\|x_i\|) \quad (\textit{or } \varphi_j^2(\|x_j\|)) \textit{ is negative;} \end{cases}$$

for all $i, j = 1, 2, \dots, m$ *and* $q = 1, 2, \dots, 7$

$$k_{ij}^q = \begin{cases} \gamma_i\gamma_j & \textit{if the corresponding multiplier} \\ & \varphi_i^2(\|x_i\|)\varphi_j^2(\|x_j\|) \textit{ is positive,} \\ \beta_i\beta_j & \textit{if the corresponding multiplier} \\ & \varphi_i^2(\|x_i\|)\varphi_j^2(\|x_j\|) \textit{ is negative.} \end{cases}$$

Then the equilibrium state $x = 0$ *of (2.2.10) is exponentially stable on* $\mathcal{P} \times \mathcal{S}$.

If all hypotheses of the theorem are satisfied for $\mathcal{N}_{ix} = R^{n_i}$ *then the equilibrium state* $x = 0$ *of (2.2.10) is exponentially stable in the whole on* $\mathcal{P} \times \mathcal{S}$.

Proof Provided that Assumption 2.3.1 and Proposition 2.3.3, and hypotheses (a), (b) of Theorem 2.4.3 are satisfied, we have for function (2.3.2)

$$\lambda_m(H^{\mathrm{T}}A^*H)\|x\|^2 \le v(t, x, \psi) \le \lambda_M(H^{\mathrm{T}}B^*H)\|x\|^2$$

for all $x \in \mathcal{N}_x \subseteq R^n$, where $\lambda_m(\cdot)$ is a minimal eigenvalue of the matrix $H^{\mathrm{T}}A^*H$, and $\lambda_M(\cdot)$ is a maximal eigenvalue of the matrix $H^{\mathrm{T}}B^*H$, $x = (x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, \dots, x_m^{\mathrm{T}})^{\mathrm{T}}$, and $H = \mathrm{diag}\,[\psi_1, \psi_2, \dots, \psi_m]$.

If all conditions of Assumption 2.4.1 and hypothesis (c) of Theorem 2.4.3 are satisfied, then for $D^+v(t,x,\psi)$ the estimate

$$D^+v(t,x,\psi) \le \lambda_M(\overline{Q}_1^*)\|x\|^2 \quad \text{for all} \quad (t,x,P,S) \in R \times \mathcal{N}_x \times \mathcal{P} \times \mathcal{S}$$

is valid, where $\lambda_M(\overline{Q}_1^*) < 0$, and $\overline{Q}_1^* = \frac{1}{2}\left(\overline{Q}^* + \overline{Q}^{*\mathrm{T}}\right)$. Therefore the equilibrium state $x = 0$ of (2.2.10) is exponentially stable on $\mathcal{P} \times \mathcal{S}$.

If $\mathcal{N}_{ix} = R^{n_i}$ for all $i = 1, 2, \dots, m$, then $\mathcal{N}_x = R^n$ and all hypotheses of Theorem 2.5.6 by Martynyuk [13] are satisfied for all $(P,S) \in \mathcal{P} \times \mathcal{S}$. Hence, the equilibrium state $x = 0$ of (2.2.10) is exponentially stable in the whole on $\mathcal{P} \times \mathcal{S}$. This completes the proof of Theorem 2.4.3.

Remark 2.4.1. If $\varphi_i(\|x_i\|) = \alpha\|x_i\|$, $\alpha \in R_+$, then Theorem 2.4.3 remains valid for $A^* = \alpha A$, $B^* = \alpha B$ and $\overline{Q}_1^* = \frac{1}{2}\,\alpha\left(\overline{Q}^* + \overline{Q}^{*\mathrm{T}}\right)$.

Assumption 2.4.2 Let in the inequalities (3) (a) – (f) of Assumption 2.4.1 sign "$\le$" be reversed, i.e. "$\ge$".

Proposition 2.4.2 *If all hypotheses of Assumption 2.4.2 hold, then*

$$D^+v(t,x,\psi) \ge w^{\mathrm{T}}Q(P,S)w$$

for all $(t,x) \in R_+ \times \mathcal{N}_{x0}$ *and for all* $(P,S) \in \mathcal{P} \times \mathcal{S}$.

Here the vector w and matrix $Q(P,S)$ are defined in the same way as in Proposition 2.4.1.

The Proof is similar to that of Proposition 2.1.2.

Theorem 2.4.4 *Assume that the perturbed motion equations (2.2.10) are such that all conditions of Assumptions 2.3.1 and 2.4.2 are satisfied with function* $\Delta(t) = 1$ *for all* $t \in R_+$ *and*

(1) *matrices* A *and* B *are positive definite,*
(2) *there exists a positive definite matrix* $D \in R^{m\times m}$ *such that for matrix* $Q(P,S)$ *the estimate*

$$\frac{1}{2}\left(Q(P,S) + Q^{\mathrm{T}}(P,S)\right) \ge D$$

holds at least for one pair $(P,S) \in \mathcal{P} \times \mathcal{S}$.

Then the equilibrium state $x = 0$ *of (2.2.10) is unstable on* $\mathcal{P} \times \mathcal{S}$.

Proof We construct a scalar function (2.3.2) based on a matrix-valued function $U(t,x)$. Due to Assumption 2.3.1, Proposition 2.3.3 and conditions (a) of Theorem 2.4.4 the function $v(t,x,\psi)$ is positive definite and admits infinitely small upper bound on $\mathcal{N}_x$. By Assumption 2.4.2, Proposition 2.4.2 and conditions (b) of Theorem 2.4.4 the function $D^+v(t,x,\psi)$ is positive definite at least for one pair $(P,S) \in \mathcal{P} \times \mathcal{S}$.

These conditions are sufficient (see Theorem 2.5.7 by Martynyuk [13]) for instability of the equilibrium state $x = 0$ of (2.2.10).

Example 2.4.1 Let a fourth order system be given, which consists of two subsystems of second order

$$\begin{aligned} \frac{dx_1}{dt} &= \frac{1}{1+t^2}\Big\{-(1+t)x_1 + s_{11}(t)\begin{pmatrix}-\operatorname{sat} 0.1x_{11}\\ 0\end{pmatrix} \\ &\quad + s_{12}(t)\begin{pmatrix}0.5x_{22}\\ 0\end{pmatrix} + s_{13}(t)\begin{pmatrix}0\\ 0.1x_{21}\end{pmatrix}\Big\}; \\ \frac{dx_2}{dt} &= \frac{1}{1+t^2}\Big\{-(2+t)x_2 + s_{21}(t)\begin{pmatrix}0.4x_{11}\\ 0\end{pmatrix} \\ &\quad + s_{22}(t)\begin{pmatrix}0\\ 0.4x_{12}\end{pmatrix} + s_{23}(t)\begin{pmatrix}0\\ \operatorname{sat} 0.2x_{22}\end{pmatrix}\Big\}, \end{aligned} \tag{2.4.4}$$

where $x_1 = (x_{11}, x_{12})^{\mathrm{T}}$, $x_2 = (x_{21}, x_{22})^{\mathrm{T}}$, $\operatorname{sat}\xi = \xi$ for $|\xi| \le 1$ and $\operatorname{sat}\xi = \operatorname{sign}\xi$ for $|\xi| > 1$.

In this example $\mathcal{P} = \{0\}$ and the structural matrices $S_i(t)$ have the form

$$S_i(t) = \begin{pmatrix} s_{i1}(t) & 0 & s_{i2}(t) & 0 & s_{i3}(t) & 0 \\ 0 & s_{i1}(t) & 0 & s_{i2}(t) & 0 & s_{i3}(t)\end{pmatrix},$$

$$S(t) = \begin{pmatrix} S_1(t) & 0 \\ 0 & S_2(t)\end{pmatrix}, \qquad i = 1, 2.$$

The structural set of the system (2.4.4) is defined as

$$\mathcal{S} = \Big\{S(t)\colon\; S(t) = \begin{pmatrix} S_1(t) & 0 \\ 0 & S_2(t)\end{pmatrix}, \quad S_i(t) = (s_{i1}(t)I_2,\, s_{i2}(t)I_2,\, s_{i3}(t)I_2),$$

$$s_{ij}(t) \in \{0, 1\} \quad \text{for all} \quad t \in R, \quad \text{for all} \quad i = 1, 2, \;\; j = 1, 2, 3\Big\}.$$

Note that structural changes of the given system within structural set $\mathcal{S}$ are inadmissible in the frames of connected stability (see Šiljak [1]) and equality $s_{ij} = s_{mn}(t)$ is admissible for all $i, m = 1, 2$ and $j, n = 1, 2, 3$.

All possible interactions are described by means of the matrices

$$S_1(t)h_1(t, x) \quad \text{and} \quad S_2(t)h_2(t, x),$$

where

$$\begin{aligned} h_1(t, x) &= (-\operatorname{sat} 0.1x_{11},\, 0,\, 0.5x_{22},\, 0,\, 0,\, 0.1x_{21})^{\mathrm{T}}, \\ h_2(t, x) &= (0.4x_{11},\, 0,\, 0,\, 0.4x_{12},\, 0,\, \operatorname{sat} 0.2x_{22})^{\mathrm{T}} \end{aligned}$$

and "sat" is the saturation nonlinearity.

We connect with the independent subsystems

$$\frac{dx_1}{dt} = -\frac{1+t}{1+t^2}\, x_1, \qquad \frac{dx_2}{dt} = -\frac{2+t}{1+t^2}\, x_2 \tag{2.4.5}$$

the functions

$$v_{ii}(t, x_i) = (1+t^2)x_i^2, \quad i = 1, 2. \tag{2.4.6}$$

The sets $L_{i\xi}(t)$ are defined as follows

$$L_{i\xi}(t) = \left\{ x_i \colon \; x_{i1}^2 + x_{i2}^2 < \frac{\xi}{1+t^2} \right\}, \quad \text{for} \quad i = 1, 2.$$

It is clear that they are asymptotically contractive for any $\zeta \in (0, +\infty)$. Further we define the functions $v_{ij}(t, x_i, x_j)$ as follows

$$v_{21}(t, x_1, x_2) = v_{12}(t, x_1, x_2) = 0.1(1+t^2)\, x_1 x_2. \tag{2.4.7}$$

For the functions (2.4.6) and (2.4.7) the estimates

$$v_{ii}(t, x_i) \geq (1+t^2)\|x_i\|^2, \quad i = 1, 2;$$
$$v_{ij}(t, x_1, x_2) \geq -0.1(1+t^2)\|x_1\|\,\|x_2\|$$

are satisfied and matrix $\widetilde{A}$ corresponding to matrix A of Proposition 2.3.1

is positive definite and the function $\Delta(t) = 1 + t^2 \geq 1 > 0$ for all $t \in R_+$.

Let $\psi = (1,1)^{\mathrm{T}}$, then for given choice of the elements v_{ij}, $i,j = 1,2$, of the matrix-valued function $U(t,x)$, the elements of the matrix $\widetilde{Q}(S)$ corresponding to the matrix $Q(P,S)$ in Proposition 2.4.1 take the values

$$\widetilde{\sigma}_{11}(S) = -2 + 0.1s_{11} + 0.025s_{12} + 0.005s_{13},$$

$$\widetilde{\sigma}_{22}(S) = -4 + 0.25s_{23} + 0.04s_{22},$$

$$\widetilde{\sigma}_{12}(S) = \widetilde{\sigma}_{21}(S) = 0.25s_{12} + 0.2\sqrt{s_{21}^2 + s_{22}^2} + 0.3 + 0.01s_{11} + 0.04s_{21},$$

and the matrix $\widetilde{Q}$ corresponding to matrix $\overline{Q}$ in Theorem 2.4.1

$$\widetilde{Q} = \begin{pmatrix} -1.87 & 0.6 + 0.2\sqrt{2} \\ 0.6 + 0.2\sqrt{2} & -3.76 \end{pmatrix}$$

is negative definite.

Since all hypotheses of Theorem 2.4.1 are satisfied, the state $x = 0$ $(x \in R^4)$ of (2.4.4) is asymptotically stable in the whole on $\mathcal{S}$.

Remark 2.4.2 Example 2.4.1 was studied by Grujić, *et al.* [1] by means of Liapunov vector function with the components

$$v_i(t,x_i) = (1+t^2)(|x_{i1}| + |x_{i2}|), \quad i = 1,2.$$

By means of this function the aggregation matrix for Example 2.4.1 was obtained as follows

$$A(S) = \begin{pmatrix} -1 + 0.1s_{11} & 0.5s_{12} + 0.1s_{13} \\ 0.4(s_{21} + s_{22}) & -2 + 0.2s_{23} \end{pmatrix}$$

for which

$$A(S) \le \begin{pmatrix} -0.9 & 0.6 \\ 0.8 & -1.8 \end{pmatrix} = A, \quad \text{for all} \quad S \in \mathcal{S}.$$

Having compared the matrix $\widetilde{Q}$ obtained in terms of the matrix-valued function with elements (2.4.6), (2.4.7) with the matrix A obtained in terms of the vector-function one can easily see that the estimates of total derivative of the auxiliary function along solutions of system (2.4.4) based on the matrix Q extend the possibilities of the Liapunov direct method in the investigation of this system.

2.4.2 The Problem $\mathrm{C_B}$ In this section we propose a solution of Problem $\mathrm{C_B}$ which is formulated as follows.

Problem $\mathrm{C_B}$**.** Let the continuous dynamical system (C) be obtained as a result of composition of the interacting subsystems (2.2) according to the adopted generalized model of connectedness. It is necessary to establish sufficient conditions of various types of stability for the equilibrium state $x = 0$ of system (2.6) in terms of the dynamical characteristics of the interacting subsystems (2.2) when there is no information on the dynamical properties of the isolated subsystems (2.3).

Let with interconnected subsystems (2.2.5) the elements of matrix-valued function $U(t,x)$ be connected for which Assumption 2.2.1 holds. Now we shall formulate some more assumptions.

Assumption 2.4.3 There exist

(1) open connected neighbourhoods $\mathcal{N}_{ix} \subseteq R^{n_i}$ of the states $(x_i = 0) \in R^{n_i}$, for all $i = 1, 2, \dots, m$;

(2) the functions $\varphi_i \colon \mathcal{N}_{ix} \to R_+$, $i = 1, 2, \dots, m$ $(\varphi_i \in K(KR))$ and the functions $v_{ij}(t, \cdot)$ mentioned in Assumption 2.3.1 and besides

(a) the functions $v_{ii} \in C(R_+ \times \mathcal{N}_{ix0}, R_+)$ or $v_{ii} \in C(R_+ \times R^{n_i}, R_+)$ for all $i = 1, 2, \dots, m$;

(b) the functions $v_{ij} \in C(R_+ \times \mathcal{N}_{ix0} \times \mathcal{N}_{jx0}, R)$ or $v_{ij} \in C(R_+ \times R^{n_i} \times R^{n_j}, R)$ for all $i \neq j$, $i, j = 1, 2, \dots, m$;

(3) positive definite function $\beta(t, x)$, $\beta \colon R_+ \times R^n \to R_+$ and constants ρ_{2ij}, $\rho_{ki}(P, S)$, $\rho_{3j}(P, S)$, $\rho_{rij}(P, S)$, $k = 1, 2$, $r = 1, 3, 4$, $i, j = 1, 2, \dots, m$, $i \neq j$, for which the following conditions hold

(a)
$$D_t^+ v_{ii} + (D_{x_i}^+ v_{ii})^{\mathrm{T}} g_i(t, x_i) + (D_{x_i}^+ v_{ii})^{\mathrm{T}} S_i h_i(t, x, p_i) \leq \Big\{ \rho_{1i}(P, S) \varphi_i^2(\|x_i\|) + \sum_{\substack{j=1 \\ j \neq i}}^{s} \rho_{1ij}(P, S) \varphi_i(\|x_i\|) \varphi_j(\|x_j\|) \Big\} \beta(t, x)$$

for all $(t, x_i, x_j) \in R_+ \times \mathcal{N}_{ix0} \times \mathcal{N}_{jx0}$ (for all $(t, x_i, x_j) \in R_+ \times R^{n_i} \times R^{n_j}$) and for all $(P, S) \in \mathcal{P} \times \mathcal{S}$, $i = 1, 2, \dots, m$;

(b)
$$\begin{aligned} &D_t^+ v_{ij} + (D_{x_i}^+ v_{ij})^{\mathrm{T}} g_i(t, x_i) + (D_{x_i}^+ v_{ij})^{\mathrm{T}} S_i h_i(t, x, p_i) \\ &+ (D_{x_j}^+ v_{ij})^{\mathrm{T}} g_j(t, x_j) + (D_{x_j}^+ v_{ij})^{\mathrm{T}} S_i h_i(t, x, p_i) \\ &\leq \Big\{ \rho_{2i}(P, S) \varphi_i^2(\|x_i\|) + \rho_{3j}(P, S) \varphi_j^2(\|x_j\|) \\ &+ \rho_{2ij}(P, S) \varphi_{i2}(\|x_i\|) \varphi_{j2}(\|x_j\|) + \sum_{\substack{l=1 \\ l \neq j}}^{s} \rho_{3lj}(P, S) \varphi_l(\|x_l\|) \varphi_j(\|x_j\|) \\ &+ \sum_{\substack{m=1 \\ m \neq i}}^{s} \rho_{1im}(P, S) \varphi_i(\|x_i\|) \varphi_m(\|x_m\|) \Big\} \beta(t, x) \end{aligned}$$

for all $(t, x_i, x_j) \in R_+ \times \mathcal{N}_{ix0} \times \mathcal{N}_{jx0}$ (for all $(t, x_i, x_j) \in R_+ \times R^{n_i} \times R^{n_j}$) and for all $(P, S) \in \mathcal{P} \times \mathcal{S}$ for $i \neq j$, $i, j = 1, 2, \dots, m$.

Proposition 2.4.3 *If all conditions of Assumption 2.4.2 hold, then*

$$D^+ v(t, x, \psi) \leq w^{\mathrm{T}} \Theta(P, S) w |\beta(t, x)|$$

for all $(t, x) \in R_+ \times \mathcal{N}_{x0}$ *(for all* $(t, x) \in R_+ \times R^n$*) and for all* $(P, S) \in \mathcal{P} \times \mathcal{S}$*, where*

$$\begin{aligned} w^{\mathrm{T}} &= (\varphi_1(\|x_1\|), \varphi_2(\|x_2\|), \dots, \varphi_m(\|x_m\|)); \\ \Theta(P, S) &= [\theta_{ij}(P, S)], \quad \theta_{ij} = \theta_{ji}, \quad i, j = 1, 2, \dots, m; \\ \theta_{ii}(P, S) &= \psi_i^2 \rho_{1i}(P, S) + 2 \sum_{\substack{j=1 \\ j \neq i}}^{s} \psi_i \psi_j [\rho_{2i}(P, S) + \\ &\quad + \rho_{3j}(P, S)], \quad \text{for all} \quad i = 1, 2, \dots, m; \end{aligned}$$

$$\theta_{ij}(P,S) = \frac{1}{2}\psi_i^2[\rho_{1ij}(P,S) + \rho_{1ji}(P,S)] + \psi_i\psi_j\left[\rho_{2ij} + \sum_{\substack{l=1\\ l\neq j}}^{s}\rho_{3lj}(P,S) + \sum_{\substack{m=1\\ m\neq i}}^{s}\rho_{4im}(P,S)\right]$$
$$\text{for all} \quad i \neq j, \quad i,j = 1,2,\dots,m.$$

The Proof of Proposition 2.4.3 is similar to that of Proposition 2.4.1.

We now adapt the matrix-valued Liapunov functions method to the system under nonclassical structural perturbation (2.2.1).

Theorem 2.4.5 *Assume that the perturbed motion equations (2.2.10) are such that all conditions of Assumptions 2.3.1 and 2.4.3 are satisfied except for upper estimates of the functions $v_{ii}(t,x_i)$ and $v_{ij}(t,x_i,x_j)$ for all $i,j = 1,2,\dots,m$ and moreover*

(1) *there exist positive numbers ξ_i (or $\xi_i = +\infty$) such that sets $L_{i\xi_i}(t)$ are asymptotically contractive for any $\zeta_i \in (0,\xi_i)$ and every $i = 1,2,\dots,m$;*
(2) *the matrix A in the inequalities (2.3.7) is positive definite;*
(3) *there exists a negative definite matrix $H \in R^{m\times m}$ such that*

$$\frac{1}{2}(\theta(P,S) + \theta^{\mathrm{T}}(P,S)) \leq H \quad \textit{for all} \quad (P,S) \in \mathcal{P}\times\mathcal{S}$$

is satisfied component-wise.

Then the equilibrium state $x = 0$ *of the system (2.2.10) is asymptotically stable on* $\mathcal{P} \times \mathcal{S}$.

If all hypotheses of Theorem 2.4.5 hold for $\mathcal{N}_{ix} = R^{n_i}$ *for radially unbounded functions* $v_{ij}(t,\cdot)$, $i,j = 1,2,\dots,m$, *then the equilibrium state* $x = 0$ *of the system (2.2.10) is asymptotically stable in the whole on* $\mathcal{P} \times \mathcal{S}$.

The Proof of Theorem 2.4.5 is similar to that of Theorem 2.4.1.

Theorem 2.4.6 *Assume that the perturbed motion equations (2.2.10) are such that all conditions of Assumptions 2.3.1 and 2.3.2 are satisfied for* $\beta(t,x) \equiv 1$ *and*

(1) *the conditions (a) and (c) of Theorem 2.3.1 are satisfied;*
(2) *the matrices A and B in the inequalities (2.3.7) are positive definite.*

Then the equilibrium state $x = 0$ *of the system (2.2.10) is uniformly asymptotically stable on* $\mathcal{P} \times \mathcal{S}$.

If moreover $\mathcal{N}_{ix} = R^{n_i}$ *for all* $i = 1,2,\dots,m$, *the functions* $v_{ij}(t,\cdot)$ *are radially unbounded and functions* $\varphi_i \in KR$, *then the equilibrium state* $x = 0$ *of the system (2.2.10) is uniformly asymptotically stable in the whole on* $\mathcal{P} \times \mathcal{S}$.

The Proof of Theorem 2.4.6 is similar to that of Theorem 2.4.2.

Theorem 2.4.7 *Assume that the perturbed motion equations (2.2.10) are such that all conditions of Assumptions 2.4.1 and 2.4.3 and hypotheses (a) and (b) of Theorem 2.4.5 are satisfied and moreover, there exist*

(1) *positive numbers* β_i *and* γ_i *such that*
$$\beta_i\|x_i\| \le \varphi_i(\|x_i\|) \le \gamma_i\|x_i\|$$
for all $x_i \in \mathcal{N}_{ix}$ *(for all* $x_i \in R^{n_i}$*) and for all* $i = 1,2,\dots,m$;
(2) *the matrix A in the inequalities (2.3.7) is equal to* $A^* = [\underline{\alpha}^*_{ij}]$, $\underline{\alpha}^*_{ij} = \underline{\alpha}^*_{ji}$ *and is positive definite;*
(3) *a symmetric negative definite matrix* $H^* \in R^{m\times m}$ *such that*
$$\Theta^*(P,S) \le H^* \quad \textit{for all} \quad (P,S) \in \mathcal{P} \times \mathcal{S},$$
where
$$\Theta^*(P,S) = \left[\theta^*_{ij}(P,S)\right], \quad \theta^*_{ij} = \theta^*_{ji} \quad \textit{for all} \quad i,j = 1,2,\dots,m;$$
$$\theta^*_{ii}(P,S) = \psi_i^2\rho_{1i}(P,S)k_{i1} + 2\sum_{\substack{j=1\\ j\ne i}}^{s}\psi_i\psi_j(\rho_{2i}(P,S)k_{i2} + \rho_{3j}(P,S)k_{j3});$$
$$\theta^*_{ij}(P,S) = \frac{1}{2}\,\psi_i^2[\rho_{1ij}(P,S) + \rho_{1ji}(P,S)]k^1_{ij} + \psi_i\psi_j\Big[\rho_{2ij}k^2_{ij} + \sum_{\substack{l=1\\ l\ne j}}^{s}\rho_{3lj}(P,S)k^3_{lj} + \sum_{\substack{m=1\\ m\ne i}}^{s}\rho_{4im}(P,S)k^4_{im}\Big]$$
$$\textit{for all} \quad i \ne j, \quad i,j = 1,2,\dots,m,$$
$$k_{ijr} = \begin{cases} \gamma_i^2 \quad (\textit{or } \gamma_j^2) \quad \textit{if the corresponding multiplier} \\ \qquad \varphi_i^2(\|x_i\|) \ (\textit{or } \varphi_j^2(\|x_j\|)) \quad \textit{is positive}, \\ \beta_i^2 \quad (\textit{or } \beta_j^2) \quad \textit{if the corresponding multiplier} \\ \qquad \varphi_i^2(\|x_i\|) \ (\textit{or } \varphi_j^2(\|x_j\|)) \quad \textit{is negative}, \end{cases}$$
for all $i,j = 1,2,\dots,m$ *and* $r = 1,2,3$;

$$k_{ij}^q = \begin{cases} \gamma_i\gamma_j & \textit{if the corresponding multiplier} \\ & \varphi_i(\|x_i\|)\varphi_j(\|x_j\|) \quad \textit{is positive}, \\ \beta_i\beta_j & \textit{if the corresponding multiplier} \\ & \varphi_i(\|x_i\|)\varphi_j(\|x_j\|) \quad \textit{is negative}, \end{cases}$$
$$\textit{for all} \quad i,j = 1,2,\dots,m \quad \textit{and} \quad q = 1,2,3,4;$$

(4) *constant* $\alpha > 0$ *such that*

$$|\beta(t,x)| \ge \alpha \quad \textit{for all} \quad (t,x) \in R_+ \times \mathcal{N}_x, \quad \mathcal{N}_x \subseteq R^n.$$

Then the equilibrium state $x = 0$ *of the system (2.2.10) is exponentially stable on* $\mathcal{P} \times \mathcal{S}$.

If all hypotheses of Theorem 2.4.7 are satisfied for $\mathcal{N}_{ix} = R^{n_i}$, *then the equilibrium state* $x = 0$ *of the system (2.2.10) is exponentially stable in the whole on* $\mathcal{P} \times \mathcal{S}$.

The Proof of Theorem 2.4.7 is similar to that of Theorem 2.4.3.

Example 2.4.2 We consider a fourth order system consisting of two subsystems of the second order

(2.4.8)
$$\begin{aligned} \frac{dx_1}{dt} &= \begin{pmatrix} 0 & -1 \\ -1 & -2 \end{pmatrix} x_1 \\ &+ \begin{pmatrix} -\dfrac{5+t+5t^2}{1+t^2}\, x_{11} + \dfrac{1}{3}\, P_{11}S_{11}x_{21} + \dfrac{1}{6}\, P_{11}S_{11}x_{22} \\ -\dfrac{1}{1+t^2}\, x_{12} + \dfrac{1}{6}\, P_{11}S_{11}x_{21} + \dfrac{1}{3}\, P_{11}S_{11}x_{22} \end{pmatrix}; \\ \frac{dx_2}{dt} &= \begin{pmatrix} 0 & -0.5 \\ -0.5 & -1 \end{pmatrix} x_2 \\ &+ \begin{pmatrix} \dfrac{1}{2}\, P_{21}S_{21}x_{11} + \dfrac{1}{4}\, P_{21}S_{21}x_{12} - \dfrac{4+t+4t^2}{1+t^2}\, x_{21} \\ \dfrac{1}{4}\, P_{21}S_{21}x_{11} + \dfrac{1}{2}\, P_{21}S_{21}x_{12} - \dfrac{t}{1+t^2}\, x_{22} \end{pmatrix}, \end{aligned}$$

where $x_1 = (x_{11}, x_{12})^{\mathrm{T}}$ and $x_2 = (x_{21}, x_{22})^{\mathrm{T}}$.

Structural matrices and structural set of the system are defined as

$$S = \begin{pmatrix} S_{11} & 0 & 0 & 0 \\ 0 & S_{11} & 0 & 0 \\ 0 & 0 & S_{11} & 0 \\ 0 & 0 & 0 & S_{11} \end{pmatrix} = \begin{pmatrix} S_{11}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & S_{21}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$$
$$= \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}, \qquad \mathcal{S} = \{S\colon\ 0 \le S \le I_4\}.$$

Parametric perturbation matrix has the form

$$P = (P_{11}, P_{21})^{\mathrm{T}}$$

and the set $\mathcal{P}$ of addmissible perturbations is described as

$$\mathcal{P} = \left\{P\colon \quad \begin{pmatrix} -0.75 \\ -0.25 \end{pmatrix} \le P \le \begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix}\right\}.$$

The elements of matrix-valued function $U(t,x)$ are defined by

$$v_{ii}(t,x_i) = (1+t^2)x_i^2, \quad i=1,2;$$
$$v_{12}(t,x_1,x_2) = v_{21}(t,x_1,x_2) = 0.1(1+t^2)x_1x_2.$$

and for them estimates

$$v_{ii}(t,x_i) \geq (1+t^2)\|x_i^2\|, \quad i=1,2;$$
$$v_{12}(t,x_1,x_2) \geq -0.1(1+t^2)\|x_1\|\,\|x_2\|$$

hold.

The matrix

$$A = \begin{pmatrix} 1 & -0.1 \\ -0.1 & 1 \end{pmatrix}$$

is positive definite and the function $\varphi(t) = 1+t^2 \geq 1 > 0$.

Let $\eta = (1,1)^{\mathrm{T}}$, then given choice of elements $v_{ii}(t,x_i)$, $i=1,2$, $v_{12}(t,x_1,x_2)$ of matrix-valued function $U(t,x)$, the elements of matrix $\overline{G}(P,S)$ are defined as

$$\overline{\sigma}_{11}(P,S) = -3.39 + \frac{1}{2}\,|P_{21}S_{21}|;$$
$$\overline{\sigma}_{22}(P,S) = -1.8 + \frac{1}{3}\,|P_{11}S_{11}|;$$
$$\overline{\sigma}_{12}(P,S) = 0.88 + |P_{11}S_{11}| + \frac{3}{2}\,|P_{21}S_{21}|.$$

For such a definition of elements of matrix $\overline{G}(P,S)$ the matrix

$$\overline{G}(P,S) \le \overline{G} = \begin{pmatrix} -3.265 & 2.005 \\ 2.005 & -1.55 \end{pmatrix}$$

is negative definite and the function

$$\psi(t,x) = \varphi(t) = 1 + t^2 \ge 1 > 0 \quad \text{for all} \quad t \in R_+.$$

The sets $v_{i\zeta}(t)$ defined by

$$v_{i\zeta}(t) = \left\{x_i\colon\ x_{i1}^2 + x_{i2}^2 < \frac{\zeta}{1+t^2}\right\}, \qquad i = 1,2,$$

are asymptotically contractive for all $\zeta \in (0,+\infty)$. Therefore $\xi_i = +\infty$ for $i = 1,2$. Thus, all hypotheses of Theorem 2.4.5 are satisfied and equilibrium state $(x = 0) \in R^4$ of the system in question is asymptotically stable in the whole on $\mathcal{P} \times \mathcal{S}$.

Assumption 2.4.4 Assume that

(1) the conditions (1) and (2) of Assumption 2.4.3 are satisfied;
(2) there exist constants $\rho_{ii}(P,S)$ and $\rho_{ij}(P,S)$, $i = 1,2,\dots,m$, $j = 2,3,\dots,m$, $i < j$, such that

$$\begin{aligned}
&\sum_{i=1}^{m} \eta_i^2 \{D_t^+ v_{ii} + (D_{x_i}^+ v_{ii})^{\mathrm{T}} f_i(t,x^i,0) + (D_{x_i}^+ v_{ii})^{\mathrm{T}} + S_i h_i(t,x,p_i)\} \\
&+ 2\sum_{i=1}^{m-1} \sum_{\substack{j=2 \\ i<j}}^{m} \eta_i\eta_j \{D_t^+ v_{ij} + (D_{x_i}^+ v_{ij})^{\mathrm{T}} f_i(t,x^i,0) + (D_{x_j}^+ v_{ij})^{\mathrm{T}} f_j(t,x^j,0) \\
&\qquad + (D_{x_i}^+ v_{ij})^{\mathrm{T}} S_i h_i(t,x,p_i) + (D_{x_j}^+ v_{ij})^{\mathrm{T}} S_j h_j(t,x,p_j)\} \\
&\qquad \le \sum_{i=1}^{m} \rho_{ii}(P,S)\varphi_i^2(x_i) + \sum_{i=1}^{m}\sum_{j=2}^{m} \rho_{ij}(P,S)\varphi_i(x_i)\varphi_j(x_j)
\end{aligned}$$

for all $(t,x_i,x_j) \in R_+ \times \mathcal{N}_{ix0} \times \mathcal{N}_{jx0}$ and for all $(P,S) \in \mathcal{P} \times \mathcal{S}_S$.

Proposition 2.4.4 *If Assumption 2.4.4 holds, then estimate*

$$D^+ v(t,x,\eta) \le u^{\mathrm{T}} \widehat{G}(P,S) u \quad \textit{for all} \quad (t,x,P,S) \in R_+ \times \mathcal{N}_{x0} \times \mathcal{P} \times \mathcal{S}$$

is valid, where

$$\begin{aligned}
u^{\mathrm{T}} &= (\varphi_1(x_1),\, \varphi_2(x_2),\, \dots,\, \varphi_m(x_m)), \\
\widehat{G}(P,S) &= [\hat\sigma_{ij}(P,S)], \quad i,j = 1,2,\dots,m, \\
\hat\sigma_{ii}(P,S) &= \rho_{ii}(P,S), \quad i = 1,2,\dots,m, \\
\hat\sigma_{ij}(P,S) &= \frac{1}{2}\,\rho_{ij}(P,S), \quad i = 1,2,\dots,m, \quad j = 2,3,\dots,m, \\
&\qquad i < j.
\end{aligned}$$

Theorem 2.4.8 *Let perturbed motion equations (2.2.10) be such that all conditions of Assumptions 2.3.1 and 2.4.4 are satisfied, except for the estimate from above of functions* $v(t,x,\eta)$.

If hypotheses (a) and (b) of Theorem 2.4.5 are satisfied and there exists a negative definite matrix $\widehat{G} \in R^{m\times m}$ *such that for matrix* $\widehat{G}(P,S)$ *the estimate*

$$\frac{1}{2}(\widehat{G}(P,S)+\widehat{G}^{\mathrm{T}}(P,S)) \le \widehat{G} \quad \textit{for all} \quad (P,S)\in\mathcal{P}\times\mathcal{S}$$

holds elementwise, then the equilibrium state $x=0$ *of system (2.2.10) is asymptotically stable on* $\mathcal{P}\times\mathcal{S}$.

If all hypotheses of Theorem 2.4.8 are satisfied for $\mathcal{N}_{ix}=R^{n_i}$ *and for radially unbounded functions* v_{ij} *and for* $\xi_i=+\infty$ *when each* $i=1,2,\dots,m$, *then the equilibrium state* $x=0$ *of (2.2.10) is asymptotically stable in the whole on* $\mathcal{P}\times\mathcal{S}$.

The Proof of Theorem 2.4.8 is similar to that of Theorem 2.4.1.

Theorem 2.4.9 *Let perturbed motion equations (2.2.10) be such that all conditions of Assumptions 2.3.1 and 2.4.4 are satisfied for* $\varphi(t)\equiv 1$ *and*

(1) *the matrices* A *and* B *are positive definite;*
(2) *there exists a negative definite matrix* $\widehat{G}\in R^{m\times m}$ *such that for matrix* $\widehat{G}(P,S)$ *the estimate*

$$\frac{1}{2}\,(\widehat{G}(P,S)+\widehat{G}^{\mathrm{T}}(P,S)) \le \widehat{G} \quad \textit{for all} \quad (P,S)\in\mathcal{P}\times\mathcal{S}$$

holds.

Then the equilibrium state $x=0$ *of (2.2.10) is uniformly asymptotically stable on* $\mathcal{P}\times\mathcal{S}$.

If, moreover $\mathcal{N}_{ix}=R^{n_i}$, *functions* v_{ij} *are radially unbounded and functions* φ_i *are of Hahn class* KR, *then the equilibrium state* $x=0$ *of (2.2.10) is uniformly asymptotically stable in the whole on* $\mathcal{P}\times\mathcal{S}$.

The Proof of Theorem 2.4.9 is similar to that of Theorem 2.4.2.

Theorem 2.4.10 *Let perturbed motion equations (2.2.10) be such that all conditions of Assumptions 2.3.1 and 2.4.4 are satisfied for* $\varphi(t)\equiv 1$ *and hypotheses (a) and (b) of Theorem 2.4.3 hold and, moreover, there exists a symmetric negative definite matrix* $\widehat{G}^*\in R^{m\times m}$ *such that for matrix* $\widehat{G}^*(P,S)$ *the estimate*

$$\frac{1}{2}\,(\widehat{G}^*(P,S)+\widehat{G}^{*\mathrm{T}}(P,S)) \le \widehat{G}^* \quad \textit{for all} \quad (P,S)\in\mathcal{P}\times\mathcal{S}$$

is valid, where

$$\widehat{G}^*(P,S)=k_{ii}\hat{\sigma}_{ii}(P,S), \quad \hat{\sigma}^*_{ij}(P,S)=k_{ij}\hat{\sigma}_{ij}(P,S), \quad i\ne j=1,2,\dots,m,$$

$$k_{ii}=\begin{cases}\gamma_i^2, & \textit{if} \quad \hat{\sigma}_{ii}(P,S)>0,\\ \beta_i^2, & \textit{if} \quad \hat{\sigma}_{ii}(P,S)<0,\end{cases} \qquad i=1,2,\dots,m,$$

$$k_{ij}=\begin{cases}\gamma_i\gamma_j, & \textit{if} \quad \hat{\sigma}_{ij}(P,S)>0,\\ \beta_i\beta_j, & \textit{if} \quad \hat{\sigma}_{ij}(P,S)<0,\end{cases} \qquad i,j=1,2,\dots,m, \quad j\ne j.$$

Then the equilibrium state $x = 0$ *of system (2.2.10) is exponentially stable in the whole on* $\mathcal{P} \times \mathcal{S}$.

If all conditions of Theorem 2.4.10 hold for $\mathcal{N}_{ix} = R^{n_i}$, *then the equilibrium state* $x = 0$ *of (2.2.10) is exponentially stable in the whole on* $\mathcal{P} \times \mathcal{S}$.

The Proof of Theorem 2.4.10 is similar to that of Theorem 2.4.3.

Example 2.4.3 Consider the forth order system (S) consisting of two subsystems (S_i) of the second order

$$\frac{dx_i}{dt} = -\alpha x_i + s_{i1}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \varphi_i(\sigma_i), \qquad \sigma_i = (-4, -2)(2x_i - x_j), \qquad i, j = 1, 2, \quad i \neq j \tag{2.4.9}$$

when the conditions

$$\varphi_i(\sigma_i)\sigma_i^{-1} \in [0, +\infty), \quad \text{for all} \quad \sigma_i \in R, \quad \alpha \in (0, +\infty), \quad \alpha = \text{const}$$

hold.

We suppose that

$$\mathcal{S} = \left\{ S\colon \; S = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix} = \begin{pmatrix} s_{11} & 0 & 0 & 0 \\ 0 & s_{22} & 0 & 0 \\ 0 & 0 & s_{22} & 0 \\ 0 & 0 & 0 & s_{22} \end{pmatrix} \right\}, \qquad s_{i1} \in \{0, 1\}, \quad i = 1, 2.$$

Let $v_{ii} = 2x_i^2$, $v_{12} = v_{21} = -x_1x_2$.

Matrix $\hat{A}$, corresponding to matrix A in the estimate from Proposition 2.3.1, has the form

$$\hat{A} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

and is positive definite.

We set $\eta = (1, 1)^{\mathrm{T}}$, then

$$Dv(t, x) = \sum_{i=1}^{2} Dv_{ii} + 2Dv_{12} = -4\alpha x_1^2 - 4\alpha x_2^2 - s_{11}\varphi_1(\sigma_1)\sigma_1$$
$$- s_{21}\varphi_2(\sigma_2)\sigma_2 + 4\alpha x_1 x_2 \le -4\alpha\|x_1\|^2 + 4\alpha\|x_1\|\,\|x_2\| - 4\alpha\|x_2\|^2$$

and the matrix

$$\widehat{G} = \begin{pmatrix} -4\alpha & 2\alpha \\ 2\alpha & -4\alpha \end{pmatrix} = 2\alpha \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

is negative definite.

Thus, all hypotheses of Theorem 2.4.9 are satisfied and the state $x = 0$ of (2.4.9) is uniformly asymptotically stable in the whole on $\mathcal{S}$.

2.4.3 Instability conditions Some modifications of Assumptions 2.4.1 and 2.4.3 allow to apply the obtained inequalities for establishing instability of the equilibrium state $x = 0$ of system (2.2.1).

Assumption 2.4.5 Let in the inequalities (iii) (a) – (b) of Assumption 2.4.3 the sign "$\le$" be reversed, i.e. "$\ge$".

Proposition 2.4.5 *If all conditions of Assumption 2.4.5 are satisfied, then*

$$D^+v(t, x, \psi) \ge w^{\mathrm{T}}\Xi(P, S)w|\beta(t, x)|$$

for all $(t, x) \in R_+ \times \mathcal{N}_{x0}$ *(for all* $(t, x) \in R_+ \times R^n$*) and for all* $(P, S) \in \mathcal{P} \times \mathcal{S}$.

Here the function $\beta(t, x)$, vector w and matrix $\Theta(P, S)$ are defined as in Proposition 2.4.3.

Theorem 2.4.11 *Assume that the perturbed motion equations (2.2.10) are such that all conditions of Assumptions 2.3.1 and 2.4.5 are satisfied with function* $\beta(t, x) = 1$ *for all* $t \in R_+$ *and*

(1) *the matrices* A *and* B *are positive definite;*
(2) *there exists a positive definite matrix* $D^* \in R^{m\times m}$ *such that*

$$\frac{1}{2}\left(\Xi(P, S) + \Xi^{\mathrm{T}}(P, S)\right) \ge D^*$$

at least for one pair $(P, S) \in \mathcal{P} \times \mathcal{S}$.

Then the equilibrium state $x = 0$ *of the system (2.2.10) is unstable.*

Proof We construct a scalar function (2.3.2) based on a matrix-valued function (2.3.2). Due to Assumption 2.3.1, Proposition 2.3.1 and condition (a) of Theorem 2.4.11 the function (2.3.2) is positive definite and admits infinitely small upper bound on $\mathcal{N}_x \subseteq R^n$. By Assumption 2.4.5, Proposition 2.4.4 and condition (b) of Theorem 2.4.11 the function $D^+v(t,x,\psi)$ is positive definite at least for one pair $(P,S) \in \mathcal{P} \times \mathcal{S}$. These conditions (see Theorem 2.5.7 by Martynyuk [13]) are sufficient for the instability of the equilibrium state $x = 0$ of (2.2.10).

Assumption 2.4.6 Let in inequalities of Assumption 2.4.4 the sign "$\le$" be reversed, i.e. "$\ge$".

Proposition 2.4.6 *If all conditions of Assumption 2.4.4 are satisfied, then for $D^+v(t,x,\psi)$ the estimate*

$$D^+v(t,x,\psi) \ge u^{\mathrm{T}} \widehat{G}(P,S) u$$

takes place for all $(t,x) \in R \times \mathcal{N}_{x0}$, and for all $(P,S) \in \mathcal{P} \times \mathcal{S}$.

Here the vector u and the matrix $\widehat{G}(P,S)$ are defined in the same way as in Proposition 2.4.4.

The Proof of Proposition 2.4.6 is similar to that of Proposition 2.4.5.

Theorem 2.4.12 *Let perturbed motion equations (2.2.10) be such that all conditions of Assumptions 2.3.1 and 2.4.6 are satisfied and*

(1) *the matrices A and B are positive definite;*
(2) *there exists a positive definite matrix $\widehat{G} \in R^{m \times m}$, such that the estimate*

$$\frac{1}{2}\,(\widehat{G}(P,S) + \widehat{G}^{\mathrm{T}}(P,S)) \ge \widehat{G} \quad \text{for all} \quad (P,S) \in \mathcal{P} \times \mathcal{S}$$

is true.

Then the equilibrium state $x = 0$ of (2.2.10) is unstable on $\mathcal{P} \times \mathcal{S}$.

The Proof of Theorem 2.4.7 is similar to that of Theorem 2.4.11.

2.5 Linear Systems Analysis

We consider the linear system

$$\frac{dx_i}{dt} = A_i x_i + \sum_{\substack{j=1 \\ j \ne i}}^{m} s_{ij} A_{ij} x_j, \quad i = 1, 2, \dots, m, \tag{2.5.1}$$

where A_{ij} are constant matrices of the corresponding order,

$$\begin{gathered} \mathcal{S} = \{S\colon\ S = \operatorname{diag}[S_1, S_2, \dots, S_s]\}, \\ S_i = [s_{i1}, \dots, s_{i,i-1}, I, s_{i,i+1}, \dots, s_{is}], \\ 0 \le s_{ij} \le I, \end{gathered}$$

where I is a unique matrix of the corresponding dimension, $n_1 + n_2 + \dots + n_m = n$, $x = (x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, \dots, x_m^{\mathrm{T}})^{\mathrm{T}} \in R^n$.

For the system (2.5.1) we construct a matrix function

$$U(x) = \big[v_{ij}(x_i, x_j)\big], \quad v_{ij} = v_{ji} \quad \text{for all} \quad i, j = 1, 2, \dots, m \tag{2.5.2}$$

with the elements

$$\begin{aligned} v_{ii}(x_i) &= x_i^{\mathrm{T}} B_{ii} x_i \quad \text{for all} \quad i = 1, 2, \dots, m, \\ v_{ij}(x_i, x_j) &= x_i^{\mathrm{T}} B_{ij} x_j \quad \text{for all} \quad i \neq j, \quad i, j = 1, 2, \dots, m, \end{aligned} \tag{2.5.3}$$

where B_{ii} are symmetric positive definite matrices and B_{ij} are constant matrices for all $i \neq j$. It can be easily verified that for the functions (2.5.3) the estimates (cf. Krasovskii [1], and Djordjevic [1])

$$\lambda_m(B_{ii})\|x_i\|^2 \le v_{ii}(x_i) \le \lambda_M(B_{ii})\|x_i\|^2$$
$$\text{for all} \quad x_i \in \mathcal{N}_{ix0} \quad \text{and} \quad i = 1, 2, \dots, m;$$
$$-\lambda_M^{1/2}(B_{ij}B_{ij}^{\mathrm{T}})\|x_i\|\|x_j\| \le v_{ij}(x_i, x_j) \le \lambda_M^{1/2}(B_{ij}B_{ij}^{\mathrm{T}})\|x_i\|\,\|x_j\|$$
$$\text{for all} \quad (x_i, x_j) \in \mathcal{N}_{ix0} \times \mathcal{N}_{jx0} \quad \text{and} \quad i \neq j, \quad i, j = 1, 2, \dots, m$$

hold true, where $\lambda_m(B_{ii})$ and $\lambda_M(B_{ii})$ are minimal and maximal eigenvalues of the matrices B_{ii} for all $i = 1, 2, \dots, m$ and $\lambda_M^{1/2}(B_{ij}B_{ij}^{\mathrm{T}})$ are norms of matrices B_{ij} for $i \neq j$, $i, j = 1, 2, \dots, m$.

We introduce the function

$$v(x, \psi) = \psi^{\mathrm{T}} U(x)\psi, \quad \psi \in R_+^m, \quad \psi > 0. \tag{2.5.4}$$

For the function $v(x,\psi)$ in view of Proposition 2.3.1 we have

$$\eta^{\mathrm{T}} H^{\mathrm{T}} A H \eta \le v(x,\psi) \le \eta^{\mathrm{T}} H^{\mathrm{T}} B H \eta, \tag{2.5.5}$$

where

$$\begin{gathered}
\eta^{\mathrm{T}} = (\|x_1\|, \|x_2\|, \dots, \|x_s\|),\\
H = \operatorname{diag}[\psi_1, \psi_2, \dots, \psi_s],\\
A = [a_{ij}], \quad B = [b_{ij}], \quad i,j = 1,2,\dots,m,\\
a_{ii} = \lambda_m(B_{ii}), \quad b_{ii} = \lambda_M(B_{ii}), \quad i = 1,2,\dots,m,\\
a_{ij} = -b_{ij} = -\lambda_M^{1/2}(B_{ij}B_{ij}^{\mathrm{T}}), \quad i = 1,2,\dots, m-1, \quad j = 1,2,\dots,m,\\
a_{ij} = a_{ji}, \quad b_{ij} = b_{ji} \quad \text{for all} \quad i \ne j.
\end{gathered}$$

Together with the function $v(x,\psi)$ in (2.5.4) its total derivative

$$D^+ v(x,\psi) = \psi^{\mathrm{T}} D^+ U(x) \psi, \quad \psi \in R_+^m, \quad \psi > 0, \tag{2.5.6}$$

along solutions of the system (2.5.1) is considered.

Proposition 2.5.1 *If for the system (2.5.1) there exists the matrix-valued function (2.5.2) with elements (2.5.3), then the total derivative of (2.5.3) by virtue of system (2.5.1) satisfies the estimates*

(a) $$\psi_i^2 (D_{x_i}^+ v_{ii}(x_i))^{\mathrm{T}} \frac{dx_i}{dt} \le \rho_{1i}\|x_i\|^2 + 2\sum_{\substack{j=1\\ j\ne i}}^{m} \rho_{1ij}(S)\|x_i\|\,\|x_j\|$$

for all $(x_i, x_j) \in R^{n_i} \times R^{n_j}$, $i = 1,2,\dots,m$, $S \in \mathcal{S}; d$

(b) $$2\psi_i\psi_j\left[(D_{x_i}^+ v_{ij}(x_i,x_j))^{\mathrm{T}} \frac{dx_i}{dt} + (D_{x_j}^+ v_{ij}(x_i,x_j))^{\mathrm{T}} \frac{dx_j}{dt}\right] \le \rho_{2i}(S)\|x_i\|^2 + (\rho_{2ij} + \rho_{3ij}(S))\|x_i\|\,\|x_j\|$$

for all $(x_i, x_j) \in R^{n_i} \times R^{n_j}$, *for* $i \ne j$, $i,j = 1,2,\dots,m$,

where

$$\rho_{1i} = \lambda_M[\psi_i^2(B_{ii}A_i + A_i^{\mathrm{T}} B_{ii})], \quad i = 1,2,\dots,m,$$

$$\rho_{2i}(S) = \lambda_M\left[\sum_{j=1}^{i-1} \psi_i\psi_j(B_{ji}^{\mathrm{T}} S_{ji} A_{ji} + (S_{ji}A_{ji})^{\mathrm{T}} B_{ji}) + \sum_{j=i+1}^{m} \psi_i\psi_j(B_{ij}^{\mathrm{T}} S_{ji} A_{ji} + (S_{ji}A_{ji})^{\mathrm{T}} B_{ij})\right]$$

for all $i \ne j$, $i,j = 1,2,\dots,m$,

λ_M *are minimal eigenvalues of matrices* $(\,\cdot\,)$ *respectively, and*

$$\rho_{1ij}(S) = \left\| \frac{1}{2}\,\psi_i^2\left(B_{ii}S_{ij}A_{ij} + (S_{ij}A_{ij})^{\mathrm{T}}\right) \right\|,$$

$$\rho_{2ij} = \|\psi_i\psi_j(a_i^{\mathrm{T}} B_{ij} + B_{ij}A_j)\|,$$

$$\rho_{3ij}(S) = \left\| \sum_{k=1}^{i-1} \psi_k(\psi_i B_{ki}^{\mathrm{T}} S_{kj} A_{kj} + (S_{kj}A_{kj})^{\mathrm{T}} \psi_j B_{kj}) \right.$$

$$+ \sum_{k=i+1}^{m} \psi_k(\psi_i B_{ik}^{\mathrm{T}} S_{kj} A_{kj} + (S_{ki} A_{ki})^{\mathrm{T}} \psi_j B_{kj})$$
$$+ \sum_{k=j+1}^{m} \psi_k(\psi_i B_{ik}^{\mathrm{T}} S_{kj} A_{kj} + (S_{ki} A_{ki})^{\mathrm{T}} \psi_j B_{jk})\Big\|$$
for all $i \neq j, \quad i,j = 1,2,\dots,m.$

Proof Let for the system (2.5.1) the matrix-valued function (2.5.2) be constructed with elements (2.5.3). Then we have in case (a)

$$\psi_i^2 \left(D_{x_i}^+ v_{ii}(x_i)\right)^{\mathrm{T}} \frac{dx_i}{dt} = \psi_i^2 \Big[A_i x_i + \sum_{\substack{j=1\\ j\neq i}}^{m} S_{ij} A_{ij} x_j\Big]^{\mathrm{T}} B_{ii} x_i$$
$$+ \psi_i^2 x_i^{\mathrm{T}} B_{ii} \Big[A_i x_i + \sum_{\substack{j=1\\ j\neq i}}^{m} S_{ij} A_{ij} x_j\Big] = x_i^{\mathrm{T}} \psi_i^2 [B_{ii} A_i + A_i^{\mathrm{T}} B_{ii}] x_i$$
$$+ 2 \sum_{\substack{j=1\\ j\neq i}}^{m} x_i^{\mathrm{T}} \frac{1}{2} \psi_i^2 [B_{ii} S_{ij} A_{ij} + (S_{ij} A_{ij})^{\mathrm{T}} B_{ii}] x_j$$
$$\leq \rho_{1i} \|x_i\|^2 + 2 \sum_{\substack{j=1\\ j\neq i}}^{m} \rho_{1ij}(S) \|x_i\|\, \|x_j\|$$
for all $(x_i, x_j) \in R^{n_i} \times R^{n_j}$, for all $S \in \mathcal{S}$.

The estimate (b) is proved similarly.

Proposition 2.5.2 *If estimates (a) and (b) of Proposition 2.5.1 hold, then the total derivative (2.5.6) of the function (2.5.4) by virtue of the system (2.5.1) is estimated by the inequality*

(2.5.7) $$D^+ v(x,\psi) \leq \eta^{\mathrm{T}} \Omega(S) \eta \quad \text{for all} \quad x \in R^n \quad \text{and} \quad S \in \mathcal{S},$$

where

$$\Omega(S) = [\sigma_{ij}(S)], \quad \sigma_{ij} = \sigma_{ji} \quad i,j = 1,2,\dots,m,$$
$$\sigma_{ii}(S) = \rho_{1i} + \rho_{2i}(S), \quad i = 1,2,\dots,m,$$
$$\sigma_{ij}(S) = \frac{1}{2}\left(\rho_{1ij}(S) + \rho_{1ji}(S)\right) + \rho_{2ij} + \rho_{3ij}(S)$$
for all $i \neq j, \quad i,j = 1,2,\dots,m$ *and* $S \in \mathcal{S}$.

The Proof of Proposition 2.5.2 is similar to that of Proposition 2.3.1.

Theorem 2.5.1 *Assume that the system (2.5.1) is such that*

(1) *there exists a matrix-valued function (2.5.2) with elements (2.5.3);*
(2) *the matrix A in (2.5.5) is positive definite;*

(3) *there exists a negative definite matrix* $\overline{\Omega} \in R^{m\times m}$ *such that*

$$\frac{1}{2}\left(\Omega^{\mathrm T}(S) + \Omega(S)\right) \le \overline{\Omega} \quad \textit{for all} \quad S \in \mathcal{S}.$$

Then the equilibrium state $x = 0$ *of the system (2.5.1) is asymptotically stable in the whole on* $\mathcal{S}$.

Remark 2.5.1 If in the system (2.5.1) there are no structural perturbations, i.e. $S_{ij} = I$, then for the total derivative of the function (2.5.4), by virtue of the system (2.5.1), the following estimate should be applied

$$D^+ v(x,\psi) \le \sum_{i=1}^{m} \lambda_M(C_{ii})\|x_i\|^2 + 2\sum_{i=1}^{m}\sum_{j=2}^{m} \lambda_M^{1/2}(C_{ij}C_{ij}^{\mathrm T})\|x_i\|\,\|x_j\|$$
$$\text{for all} \quad (x_i, x_j) \in R^{n_i} \times R^{n_j},$$

where

$$C_{ii} = \psi_i^2(B_{ii}A_i + A_i^{\mathrm T}B_{ii}) + \sum_{j=1}^{i-1} \psi_i\psi_j\left(A_{ji}^{\mathrm T}B_{ji} + B_{ji}^{\mathrm T}A_{ji}\right)$$
$$+ \sum_{j=i+1}^{m} \psi_i\psi_j\left(B_{ij}A_{ij} + A_{ij}^{\mathrm T}A_{ij}^{\mathrm T}\right), \quad i = 1, 2, \dots, m,$$
$$C_{ij} = \frac{1}{2}\,\psi_i^2\left(B_{ii}A_{ij} + A_{ij}^{\mathrm T}B_{ii}\right) + \psi_i\psi_j\left(A_i^{\mathrm T}B_{ij} + B_{ij}A_j\right)$$

$$
\begin{aligned}
&+\sum_{k=1}^{i-1}\psi_k\left(\psi_i B_{ik}^{\mathrm T}A_{kj}+A_{ki}^{\mathrm T}\eta_j B_{kj}\right)\\
&+\sum_{k=i}^{j-1}\psi_k\left(\psi_i B_{ik}^{\mathrm T}A_{kj}+A_{ki}^{\mathrm T}\eta_j B_{kj}\right)\\
&+\sum_{k=j}^{m}\psi_k\left(\psi_i B_{ik}^{\mathrm T}A_{kj}+A_{ki}^{\mathrm T}\eta_j B_{kj}\right),\\
&i=1,2,\dots,m-1,\quad j=2,3,\dots,m,\quad i<j,\\
&\qquad C_{ij}=C_{ji}\quad \text{for all}\quad i\neq j.
\end{aligned}
$$

In this case matrix Ω has the form

$$\Omega=[\overline{\sigma}_{ij}],\quad \overline{\sigma}_{ij}=\overline{\sigma}_{ji},\quad i,j=1,2,\dots,m,$$

where $\overline{\sigma}_{ii}=\lambda_M(C_{ii})$ and $\overline{\sigma}_{ij}=\lambda_M^{1/2}(C_{ij}C_{ij}^{\mathrm T})$ are maximal eigenvalues of matrices C_{ii} and $\lambda_M^{1/2}(\cdot)$ are norms of matrices $C_{ij},\ i,j=1,2,\dots,m.$

Example 2.5.1 We consider a linear fourth order system consisting of two second order subsystems

$$
\begin{aligned}
\frac{dx_1}{dt}&=\begin{pmatrix}-1 & 0.5\\ -0.5 & -2\end{pmatrix}x_1+\begin{pmatrix}s_{21} & 0\\ 0 & s_{21}\end{pmatrix}\begin{pmatrix}0.5 & 1\\ -1 & 0.5\end{pmatrix}x_2,\\
\frac{dx_2}{dt}&=\begin{pmatrix}-2 & -1\\ 0.5 & -3\end{pmatrix}x_2+\begin{pmatrix}s_{11} & 0\\ 0 & s_{11}\end{pmatrix}\begin{pmatrix}0.1 & -1\\ 1 & 0.1\end{pmatrix}x_1,
\end{aligned}
\tag{2.5.8}
$$

where $x_i\in R^2,\ i=1,2.$ Structural matrices S, S_i, and $S_{ij},\ i,j=1,2,$ are of the form

$$
\begin{aligned}
&S=\mathrm{diag}\,\{S_1,S_2\},\quad S_1=[Is_{12}],\quad S_2=[Is_{21}],\\
&s_{12}=s_{21}I_2,\quad s_{21}=s_{11}I_2,\quad I_2=\mathrm{diag}\,\{1,1\}.
\end{aligned}
$$

The structural set $\mathcal{S}$ is defined by the formula

$$\mathcal{S}=\{S\colon\ 0\le S\le I_4\},\quad I_4=\mathrm{diag}\,\{1,1,1,1\}.$$

We construct the matrix-valued function (2.5.2) with the elements

$$
\begin{aligned}
v_{ii}&=x_i^{\mathrm T}\,\mathrm{diag}\,\{2,2\}x_i,\quad i=1,2,\\
v_{12}(x_1,x_2)=v_{21}(x_1,x_2)&=x_1^{\mathrm T}\,\mathrm{diag}\,\{0.1,0.1\}x_2.
\end{aligned}
$$

For this function the following estimates

$$
\begin{aligned}
v_{ii}(x_i)&\ge 2\|x_i\|^2\quad \text{for all}\quad x_i\in R^2,\quad i=1,2;\\
v_{12}(x_1,x_2)&\ge -0.1\|x_1\|\,\|x_2\|\quad \text{for all}\quad (x_1,x_2)\in R^2\times R^2
\end{aligned}
$$

hold.

If $\psi^{\mathrm T}=(1,1),$ then the matrix A has the form

$$A=\begin{pmatrix}2 & -0.1\\ -0.1 & 2\end{pmatrix}$$

and is positive definite.

Elements of the matrix Ω are

$$\sigma_{11}(S) = -2 + 0.02s_{11}, \quad \sigma_{22}(S) = -3.59 + 0.1s_{21},$$
$$\sigma_{12}(S) = \sigma_{21}(S) = 0.274 + 0.1s_{11} + 0.5s_{21}.$$

For such a definition of the elements of $\Omega(S)$ we have

$$\Omega(S) \le \overline{\Omega} = \begin{pmatrix} -1.98 & 0.874 \\ 0.874 & -3.49 \end{pmatrix}.$$

The matrix $\overline{\Omega}$ is negative definite. Thus the equilibrium state $x = 0$ of the system (2.5.7) is asymptotically stable in the whole on $\mathcal{S}$.

2.6 Certain Trends of Generalizations and Applications

2.6.1 Stability analysis with respect to two measures For the reader's convinience we shall recall some notions of stability theory where the motion properties are studied with the application of two measures. Further system (2.2.7) is considered under all but one assumptions made in Section 2.2. In this subsection the right-hand side of system (2.2.7) is not assumed vanishing for $x = 0$.

The state of system (2.2.7) is characterised by means of two measures $\rho_0(t, x)$ and $\rho(t, x)$, taking their values from the sets

$$\mathcal{M} = \bigl\{\rho \in C(\mathcal{T} \times R^{2k}, R_+) \colon \inf_{(t,x)} \rho(t, x) = 0\bigr\},$$
$$\mathcal{M}_0 = \bigl\{\rho \in M \colon \inf_{x} \rho(t, x) = 0 \quad \text{for all} \quad t \in \mathcal{T}\bigr\}.$$

Let $U(t, x)$ be a matrix-valued function, i.e. $U \colon \mathcal{T} \times R^n \to R^{m\times m}$ with the elements

$$u_{ij} \colon \mathcal{T} \times R^n \to R, \quad (i, j) = 1, 2, \dots, m. \tag{2.6.1}$$

The property of having a fixed sign of the matrix-valued function $U(t, x)$ with respect to measure $\rho(t, x)$ is determined as follows.

Let $w \in R^m$ and $v \colon \mathcal{T} \times R^n \times R^m \to R$ be determined by w and U by the formula

$$v(t, x, w) = w^{\mathrm{T}} U(t, x) w. \tag{2.6.2}$$

Definition 2.6.1 The matrix-valued function $U\colon R \times R^n \to R^{m\times m}$ is

(1) *ρ-positive definite on $\mathcal{T}_\tau$, $\tau \in \mathcal{R}$,* if there exist an open connected subset $\mathcal{N} \subseteq R^n$, $0 \in \mathcal{N}$, a comparison function $a \in K$ and a constant $\Delta_1 > 0$ such that

$$a(\rho(t,x)) \le v(t,x,w) \quad \text{for all} \quad (t,x,w \neq 0) \in \mathcal{T}_\tau \times \mathcal{N} \times R^m \tag{2.6.3}$$

whenever $\rho(t,x) < \Delta_1$;

(2) *ρ-positive definite on $\mathcal{T}_\tau \times \mathcal{S}$,* if all conditions of definition (1) are satisfied for $\mathcal{N} = \mathcal{S}$;

(3) *ρ-positive definite in the whole on $\mathcal{T}_\tau$*, if conditions of definition (1) are satisfied for $\mathcal{N} = R^n$, $a \in KR$ and $\Delta_1 = +\infty$;

(4) *ρ-negative definite (in the whole) on $\mathcal{T}_\tau$ (on $\mathcal{T}_\tau \times \mathcal{N}$)*, if $(-v)$ is ρ-positive definite (in the whole) on $\mathcal{T}_\tau$ (on $\mathcal{T}_\tau \times \mathcal{N}$).

The expression "*on $\mathcal{T}_\tau$*" is omitted in Definitions (1)–(4) if and only if the conditions of these definitions are satisfied for $\tau \in \mathcal{R}$.

Proposition 2.6.1 *For the matrix-valied function $U\colon R\times R^n \to R^{m\times m}$ to be ρ-positive definite on $\mathcal{T}_\tau$, it is necessary and sufficient that it can be represented as*

$$\begin{gathered} v(t,x,w) = w^{\mathrm{T}}U_+(t,x)w + a(\rho(t,x)) \\ \text{for} \quad \rho(t,x) < \Delta_1 \quad \text{and} \quad (t,x,w\neq 0) \in \mathcal{T}_\tau \times \mathcal{N} \times R^m, \end{gathered} \tag{2.6.4}$$

where $U_+(t,x)$ is positive semi-definite on $\mathcal{T}_\tau$ matrix-valued function and $a \in K$.

For the the proof see Martynyuk [8].

Definition 2.6.2 The matrix-valued function $U\colon R \times R^n \to R^{m\times m}$ is

(1) *ρ-decreasing on $\mathcal{T}_\tau$, $\tau \in R$,* if there exist an open connected subset $\mathcal{N} \subseteq R^n$, $0 \in \mathcal{N}$, a comparison function $b \in K$ and a constant $\Delta_2 > 0$ such that

$$v(t,x,w) \le b(\rho(t,x)) \quad \text{for all} \quad (t,x,w \ne 0) \in \mathcal{T}_\tau \times \mathcal{N} \times R^m$$

whenever $\rho(t,x) < \Delta_2$;

(2) *ρ-decreasing on $\mathcal{T}_\tau \times \mathcal{S}$,* if all conditions of definition (1) are satisfied for $\mathcal{N} = \mathcal{S}$;

(3) *ρ-decreasing in the whole on $\mathcal{T}_\tau$,* if all conditions of definition (1) are satisfied for $\mathcal{N} = R^n$, $b \in KR$ and $\Delta_2 = +\infty$;

(4) *weakly ρ-decreasing on $\mathcal{T}_\tau$,* if all conditions of definition (1) are satisfied with the comparison function b of class CK, i.e.

$$v(t,x,w) \le b(t,\rho(t,x)) \quad \text{for all} \quad (t,x,w \ne 0) \in \mathcal{T}_\tau \times \mathcal{N} \times R^m$$

whenever $\rho(t,x) < \Delta_3$, $\Delta_3 > 0$;

(5) *asymptotically ρ-decreasing on $\mathcal{T}_\tau$,* if all conditions of definition (1) are satisfied with the comparison function b of class KL, i.e

$$v(t,x,w) \le b(\rho(t,x),t) \quad \text{for all} \quad (t,x,w \ne 0) \in \mathcal{T}_\tau \times \mathcal{N} \times R^m$$

whenever $\rho(t,x) < \Delta_4$, $\Delta_4 > 0$.

The expression "*on $\mathcal{T}_\tau$*" in definitions (1)–(3) is omitted, if all conditions are satisfied for $\tau \in \mathcal{R}$.

Proposition 2.6.2 *For the matrix-valued function $U\colon R\times R^n \to R^{m\times m}$ to be ρ-decreasing on $\mathcal{T}_\tau$, it is necessary and sufficient that it can be represented as*

$$v(t,x,w) = w^{\mathrm{T}} U_-(t,x) w + b(\rho(t,x)) \quad \text{for} \quad \rho(t,x) < \Delta_2 \quad \text{and} \quad (t,x,w \ne 0) \in \mathcal{T}_\tau \times \mathcal{N} \times R^m, \tag{2.6.5}$$

where $U_-(t,x)$ is negative semi-definite on $\mathcal{T}_\tau$ matrix-valued function and function b is of class K.

The proof is similar to the proof of Proposition 2.6.1 with the function $w^{\mathrm{T}} U_-(t,x) w = v(t,x,w) - b(\rho(t,x))$.

Definition 2.6.3 The matrix-valued function $U\colon R \times R^n \to R^{m\times m}$ is

(1) *radially ρ-unbounded on $\mathcal{T}_\tau$,* if for $\rho(t,x) \to +\infty$

$$v(t,x,w) \to +\infty \quad \text{for all} \quad t \in \mathcal{T}_\tau;$$

(2) *radially ρ-unbounded,* if for $\rho(t,x) \to +\infty$

$$v(t,x,w) \to +\infty \quad \text{for all} \quad t \in \mathcal{T}_\tau \quad \text{for all} \quad \tau \in \mathcal{R}.$$

For the measures $\rho(t,x)$ and $\rho_0(t,x)$ taking values from the sets $\mathcal{M}$ and $\mathcal{M}_0$ respectively the notions below are the generalizations of property of Movchan's metrices (cf. Movchan [1], and Lakshmikantham, Leela, and Martynyuk [1]).

Definition 2.6.4 Let $\rho_0, \rho \in \mathcal{M}$. We claim that

(1) *the measure* $\rho(t,x)$ *is continuous with respect to the measure* $\rho_0(t,x)$, if there exist a constant $\delta_1 > 0$ and a comparison function $\varphi \in CK$ such that

$$\rho(t,x) < \varphi(t, \rho_0(t,x))$$

whenever $\rho_0(t,x) < \delta_1$;

(2) *the measure* $\rho(t,x)$ *is uniformly continuous with respect to the measure* $\rho_0(t,x)$, if in definition (1) the comparison function $\varphi \in K$, i.e. φ does not depend on t;

(3) *the measure* $\rho(t,x)$ *is asymptotically continuous with respect to the measure* $\rho_0(t,x)$, if there exist a constant $\delta_2 > 0$ and a comparison function $\psi \in KL$ such that

$$\rho(t,x) < \psi(t, \rho_0(t,x))$$

whenever $\rho_0(t,x) < \delta_2$.

We return now to the system (2.2.7) and assume that the operator (2.2.9) is contractive for all $(P,S) \in \mathcal{P} \times \mathcal{S}$. The solution $x(t;t_0,x_0)$ of system (2.2.7) is designated by $x(t)$ and its dependence on $(P,S) \in \mathcal{P} \times \mathcal{S}$ is taken into account.

Definition 2.6.5 System (2.2.7) is

(1) (ρ_0,ρ)-*stable* on $\mathcal{P} \times \mathcal{S}$, if for every $t_0 \in \mathcal{T}_i$ and $\varepsilon > 0$ there exists a positive function $\delta(t_0,\varepsilon)$, continuous in t_0 for each ε so that for $\rho_0(t_0,x_0) < \delta$ the inequality $\rho(t,x(t)) < \varepsilon$ holds for all $t \in \mathcal{T}_0$ and all $(P,S) \in \mathcal{P} \times \mathcal{S}$;

(2) (ρ_0,ρ)-*attractive* on $\mathcal{P} \times \mathcal{S}$, if for any $t_0 \in \mathcal{T}_i$ and any $\zeta > 0$ there exist $\Delta(t_0) > 0$ and $\tau = \tau(t_0,x_0,\zeta) \in [0,+\infty)$ such that for $\rho_0(t_0,x_0) < \Delta(t_0)$ the inequality $\rho(t,x(t)) < \zeta$ holds for all $t \in (t_0+\tau,\infty)$ when all $(P,S) \in \mathcal{P} \times \mathcal{S}$;

(3) *asymptotically* (ρ_0,ρ)-*stable* on $\mathcal{P} \times \mathcal{S}$, if it is (ρ_0,ρ)-stable on $\mathcal{P} \times \mathcal{S}$ and (ρ_0,ρ)-attractive on $\mathcal{P} \times \mathcal{S}$.

The definitions of other types of dynamical properties of system (2.2.5) with respect to two measures under nonclassical structural perturbations can be formulated in terms of Definition 2.6.5 and the corresponding Definitions 1.2.1 – 1.2.3.

2.6.1.2 Test for stability analysis The application of matrix-valued Liapunov function and two measures allows one to extend the set of the dynamical properties of the system under consideration which can be investigated by the Liapunov direct method. Moreover, it is possible to use less strict assumptions on the components of auxiliary matrix-valued function.

Theorem 2.6.1 *In system (2.2.7) let the vector-function* Q *be continuous on* $R \times \Omega \times \mathcal{P} \times \mathcal{S}$. *If*

(1) *the measures* ρ_0 *and* ρ *are of class* $\mathcal{M}$;

(2) *there exist a matrix-valued function* $U(t,x)$ *and a vector* $w \in R^m$ *such that* $v(t,x,w) \in C(R \times \mathcal{S} \times R^m,\ R_+)$ *and is locally Lipschitzian in* x;

(3) *function* $v(t, x, w)$ *satisfies the estimates*

 (a) $a(\rho(t, x)) \le v(t, x, w) \le b(t, \rho_0(t, x))$ *for all* $(t, x, w) \in S(\rho, H) \times R^m$ *or*

 (b) $a(\rho(t, x)) \le v(t, x, w) \le c(\rho_0(t, x))$ *for all* $(t, x, w) \in S(\rho, H) \times R^m$,

 where a *and* c *are of class* K *and* b *is of class* CK*;*

(4) *there exists a matrix* $\Phi(P, S)$ *such that*

$$D^+ v(t, x, w)|_{(\cdot)} \le e^{\mathrm{T}} \widehat{\Phi}(P, S) e \quad \text{for all} \quad (P, S) \in \mathcal{P} \times \mathcal{S},$$

where $e = (1, 1, \ldots, 1)^{\mathrm{T}} \in R^m$ *and* $\widehat{\Phi}(P, S) = \dfrac{1}{2}(\Phi(P, S) + \Phi^{\mathrm{T}}(P, S))$*;*

(5) *there exists a constant* $m \times m$ *matrix* $\overline{\Phi}$ *such that* $\widehat{\Phi}(P, S) \le \overline{\Phi}$ *for all* $(P, S) \in \mathcal{P} \times \mathcal{S}$.

Then

(1) *system (2.2.7) is* (ρ_0, ρ)*-stable on* $\mathcal{P} \times \mathcal{S}$*, if the matrix* $\overline{\Phi}$ *is negative semi-definite, the measure* ρ *is continuous with respect to measure* ρ_0 *and condition (3a) is satisfied;*

(2) *system (2.2.7) is uniformly* (ρ_0, ρ)*-stable on* $\mathcal{P} \times \mathcal{S}$*, if the matrix* $\overline{\Phi}$ *is negative semi-definite, the measure* ρ *is uniformly continuous with respect to measure* ρ_0 *and condition (3b) is satisfied.*

Proof Note that the function $v(t, x, w)$ determined by the formula (2.6.2) is scalar pseudoquadratic with respect to $w \in R^m_+$. Therefore the property of having a fixed sign of function (2.6.2) with respect to measure ρ does not requre the ρ-signdefiniteness of elements $u_{ij}(t, x)$ of matrix (2.6.1).

First we shall prove the assertion (1) of Theorem 2.6.1. Conditions (1), (2) and (4a) imply that the function $v(t,x,w)$ is weakly ρ_0-decreasing. Therefore there exists a constant $\Delta_0 = \Delta_0(t_0) > 0$ for $t_0 \in R$ such that for $\rho_0(t_0,x_0) < \Delta_0$

$$v(t_0,x_0,w) \le b(t_0,\rho_0(t_0,x_0)). \tag{2.6.6}$$

Also, by condition (4a) there exists $\Delta_1 \in (0,H)$ such that

$$a(\rho(t,x)) \le v(t,x,w) \quad \text{for} \quad \rho(t,x) \le \Delta_1. \tag{2.6.7}$$

The facf that the measure ρ is continuous with respect to the measure ρ_0 yields the existence of a function $\varphi \in CK$ and a constant $\Delta_2 = \Delta_2(t_0) > 0$ such that

$$\rho(t_0,x_0) \le \varphi(t_0,\rho_0(t_0,x_0)) \quad \text{for} \quad \rho_0(t_0,x_0) < \Delta_2, \tag{2.6.8}$$

where Δ_2 is taken so that

$$\varphi(t_0,\Delta_2) < \Delta_1. \tag{2.6.9}$$

Let $\varepsilon \in (0,\Delta_0)$ and $t_0 \in R$. Since the functions $a \in K$ and $b \in CK$, given ε and t_0, one can take $\Delta_3 = \Delta_3(t_0,\varepsilon) > 0$ so that

$$b(t_0,\Delta_3) < a(\varepsilon). \tag{2.6.10}$$

We take $\delta(t_0) = \min(\Delta_1,\Delta_2,\Delta_3)$. Conditions (2.6.6)–(2.6.10) imply that for $\rho_0(t_0,x_0) < \delta$

$$a(\rho(t_0,x_0)) \le v(t_0,x_0,w) \le b(t_0,\rho_0(t_0,x_0)) < a(\varepsilon),$$

from which it follows that

$$\rho(t_0,x_0) < \varepsilon.$$

Let $x(t;t_0,x_0) = x(t)$ be a solution of system (2.2.7) with the initial conditions for which $\rho_0(t_0,x_0) < \delta$. Let us verify that under conditions of Theorem 2.6.1 the estimate

$$\rho(t,x(t)) < \varepsilon \quad \text{for all} \quad t \ge t_0 \quad \text{and for all} \quad (P,S) \in \mathcal{P} \times \mathcal{S} \tag{2.6.11}$$

holds true.

Assume that there exists $t_1 \ge t_0$ such that

$$\rho(t_1,x(t)) = \varepsilon \quad \text{and} \quad \rho(t,x(t)) < \varepsilon, \quad t \in [t_0,t_1),$$

for the solution $x(t)$ with the initial conditions $\rho_0(t_0,x_0) < \delta$. Condition (3) and the fact that the matrix $\widehat{\Phi}(P,S)$ is negative semi-definite in the domain $S(\rho,H)$ imply that the roots $\lambda_i = \lambda_i(\widehat{\Phi}(P,S))$ of the equation

$$\det[\widehat{\Phi} - \lambda E] = 0 \tag{2.6.12}$$

satisfy the condition $\lambda_i(\widehat{\Phi}(P,S)) \le 0$, $i = 1,2,\dots,m$, in the domain $S(\rho,H)$. Therefore

$$D^+v(t,x,w)|_{(2.2.7)} \le e^{\mathrm{T}}\widehat{\Phi}(P,S)e \le 0 \quad \text{for all} \quad (P,S) \in \mathcal{P} \times \mathcal{S}$$

and for all $t \in [t_0, t_1]$. Hence it follows that

$$a(\varepsilon) = a(\rho(t_1, x(t_1))) \le v(t, x, w) \le v(t_0, x_0, w) \\ \le b(t_0, \rho_0(t_0, x_0)) < a(\varepsilon).$$

The contradiction obtained shows that the assertion $t_1 \in [t_0, +\infty)$ is false. Consequently, system (2.2.7) is (ρ_0, ρ)–stable on $\mathcal{P} \times \mathcal{S}$.

The assertion (2) of Theorem 2.6.1 is proved in a similar manner. Besides, it is taken into account that condition (4b) is satisfied and the measure ρ is uniformly continuous with respect to the measure ρ_0, and the value δ can be taken independent of $t_0 \in R$ $(t_0 \in \mathcal{T}_\tau)$. Hence, system (2.2.7) is uniformly (ρ_0, ρ)–stable on $\mathcal{P} \times \mathcal{S}$.

The assertion below is an analogue of the Liapunov's theorem on asymptotic stability in the framework of stability investigation of system (2.2.7) with respect to two measures.

Theorem 2.6.2 *In system (2.2.7) let the vector-function Q be continuous on $R_+ \times \Omega \times \mathcal{P} \times \mathcal{S}$. If*

(1) *the measures ρ_0 and ρ are of class $\mathcal{M}$;*

(2) *there exist a matrix-valued function $U \in C(R \times \mathcal{S}, R^{m\times m})$ and a vector $w \in R^m_+$ such that the function $v(t, x, w)$ is locally Lipschitzian in x and satisfies estimates*

$$a(\rho(t, x)) \le v(t, x, w) \le c(\rho_0(t, x))$$

for all $(t, x, w) \in S(\rho, H) \times R^m_+$, where $a, c \in K$;

(3) *there exists a constant $m \times m$–matrix $B(P, S)$, such that*

$$D^+ v(t, x, w) \le u^{\mathrm{T}} B(P, S) u \quad \textit{for all} \quad (P, S) \in \mathcal{P} \times \mathcal{S}$$

where $u^{\mathrm{T}} = (\rho^{1/2}_{01}(t, x), \ldots, \rho^{1/2}_{0m}(t, x))$ and $\rho^{1/2}_{0s}(t, x) = s\rho^{1/2}_0(t, x)$, $s = 1, 2, \ldots, m$;

(4) *there exists a constant $m \times m$–matrix $\overline{B}$ such that*

$$\widehat{B}(P, S) = \frac{1}{2}(B(P, S) + B^{\mathrm{T}}(P, S)) \le \overline{B} \quad \textit{for all} \quad (P, S) \in \mathcal{P} \times \mathcal{S}.$$

Then system (2.2.7) is uniformly asymptotically (ρ_0, ρ)–stable on $\mathcal{P}\times\mathcal{S}$, if the matrix $\overline{B}$ is negative definite and the measure ρ is uniformly continuous with respect to the measure ρ_0.

Proof Condition (2) of Theorem 2.6.2 implies that for the function

$$v(t, x, w) = w^{\mathrm{T}} U(t, x) w, \quad w \in R^m_+$$

the constants $0 < H_0 \le H$ and $\Delta_0 > 0$ exist so that

$$a(\rho(t, x)) \le v(t, x, w) \quad \text{for all} \quad (t, x, w) \in S(\rho, H_0) \times R^m_+$$

and

$$v(t, x, w) \le b(\rho_0(t, x)) \quad \text{for} \quad \rho_0(t, x) < \Delta_0, \quad w \in R^m_+.$$

Having compared the conditions of Theorem 2.6.1 and those of Theorem 2.6.2 we conclude that system (2.2.7) is uniformly (ρ_0, ρ)–stable on $\mathcal{P} \times \mathcal{S}$. Hence it follows that for $\varepsilon = H_0$ one can take $\Delta_1 = \Delta_1(H_0)$ so that the inequality $\rho(t, x(t)) < H_0$ holds true whenever $\rho_0(t_0, x_0) < \Delta_1$ for any solution $x(t) = x(t; t_0, x_0)$ of system (2.2.7).

By condition (3) of Theorem (2.2.6) we have

(2.6.13)
$$\begin{aligned} D^+ v(t, x, w) &\le u^{\mathrm{T}} B(P, S) u \le \lambda_M(\overline{B}) u^{\mathrm{T}} u \\ &= \lambda_M(\overline{B}) \sum_{s=1}^{m} \rho_{0s}(t, x) \quad \text{for all} \quad (P, S) \in \mathcal{P} \times \mathcal{S}. \end{aligned}$$

Since the matrix $\overline{B}$ is symmetric and negative definite, $\lambda_M(\overline{B}) < 0$. The measure ρ_0 is of class $\mathcal{M}$, so there exists a function $\psi \in K$ such that

(2.6.14)
$$\psi(\rho_0(t, x)) \ge \sum_{s=1}^{m} \rho_{0s}(t, x).$$

Therefore

(2.6.15)
$$D^+ v(t, x, w) \le -\lambda_M(\overline{B}) \psi(\rho_0(t, x))$$

for all $(t, x, w) \in S(\rho, H) \times R^m_+$ and for all $(P, S) \in \mathcal{P} \times \mathcal{S}$.

Further for arbitrary $0 < \varepsilon < H_0$ we take $\delta = \delta(\varepsilon)$ being the same as in the definition of uniform (ρ_0, ρ)–stability. Assume that $\rho_0(t_0, x_0) < \delta^* = \min\{\Delta_0, \Delta_1\}$ and take

$$T(\varepsilon) = \frac{b(\delta^*)}{\lambda_M(\overline{B})\psi(\delta)} + 1,$$

where $\lambda_M(\overline{B})$ is the maximal eigenvalue of the symmetric matrix $\overline{B}(w)$ and the function ψ is of class K. We shall prove uniform asymptotic (ρ_0, ρ)–stability on $\mathcal{P} \times \mathcal{S}$ of system (2.2.7), if we make sure that a $t^* \in [t_0,\ t_0 + T]$ exists such that

(2.6.16) $$\rho_0(t^*,\ x(t^*)) < \delta.$$

If this is not true, then there exists a solution $x(t) = x(t; t_0, x_0)$ of system (2.2.7) with local values $\rho_0(t_0, x_0) < \delta^*$ for which

(2.6.17) $$\rho_0(t, x(t)) \geq \delta \quad \text{for all} \quad t^* \in [t_0, t_0 + T].$$

We have from (2.6.16)
(2.6.18)
$$\lambda_M(\overline{B}) \int\limits_{t_0}^{t_0+T} \psi(\rho_0(s, x(s)))\, ds \leq v(t_0, x_0, w) \leq b(\rho_0(t_0, x_0) \leq b(\delta^*).$$

In view of (2.6.16) we have from (2.6.13)

(2.6.19) $$\lambda_M(\overline{B}) \int\limits_{t_0}^{t_0+T} \psi(\rho_0(s, x(s)))\, ds \geq \lambda_m(B)\psi(\delta)T > b(\delta^*)$$

for the above choice of T. Inequality (2.6.19) contradicts inequality (2.6.18). This proves Theorem 2.6.2.

Remark 2.6.1 The further development of stability theory of nonlinear systems under nonclassical structural perturbations with respect to two measures (ρ_0, ρ) is associated with the construction of the matrix-valued functions which satisfy the conditions of sign-definiteness with respect to a given measure. In the investigation of the dynamical properties of system (2.2.7) it is reasonable to consider the following measures;

(1) in the investigation of stability of the state $x = 0$ in the sense of Liapunov the two measures ρ_0 and ρ are taken as follows: $\rho(t, x) = \|x\|$ and $\rho_0(t_0, x_0) = \|x_0\|$;
(2) in the investigation of stability of the prescribed motion $x_0(t)$ of system (2.2.7) the two measures are taken as $\rho(t, x) = \rho_0(t, x) = \|x - x_0(t)\|$;
(3) in the stability investigation of the zero solution of system (2.2.7) with respect to a part of variables the two measures are taken as $\rho(t, x) = \|x\|_s,\ 1 \leq s < n,$ and $\rho_0(t, x) = \|x\|$;
(4) if $\mathcal{T}_i = \mathcal{R},\ \rho(t, x) = \rho_0(t, x) = \|x\| + \sigma(t)$, where σ is of class L, then Definition 2.6.5 (a) characterizes stability of the asymptotically invariant set $\{0\}$;

(5) let $A \subset R^n$, $\mathcal{T}_i = \mathcal{R}$ and $\rho(t,x) = \rho_0(t,x) = d(x,A)$, where $d(x,A)$ is the distance from the set A to the point x. Then the two given measures characterize stability of the invariant set A;

(6) let $A \subset B \subset R^n$, $\mathcal{T}_i = \mathcal{R}$ and $\rho(t,x) = d(x,B)$, $\rho_0(t,x) = d(x,A)$. Then the two given measures characterize stability of the invariant set B with respect to the set A;

(7) let the k-dimensional integral manifold M of system (2.2.7) contain the point $x = 0$ and the vector-function $Q(t,x,P,S) = 0$ for $x = 0$, and also $\mathcal{T}_i = \mathcal{R}$, $\rho(t,x) = \rho_0(t,x) = \|x\|_{n+k} + d(x,M)$. Then the two given measures are characteristics of conditional stability of the state $x = 0$ of system (2.2.7) under nonclassical structural perturbations;

(8) let system (2.2.7) have a periodic solution and C be a closed orbit in the phase space. If $\rho(t,x) = \rho_0(t,x) = d(x,C)$ and $\mathcal{T}_i = \mathcal{R}$, then the two given measures are characteristics of orbital stability of the periodic motion under nonclassical structural perturbations.

2.6.2 Large-scale power systems In this Section, besides the general notation used throughout the book, the following symbols will be used (see Grujić, *et al.* [1])

N is a number of system's generators (or machines);

n, $n = N - 1$, contrary to the previous Sections n does not denote here the dimension of system's state;

$A_{ij} = E_i E_j Y_{ij}$;

D_i is a mechanical damping coefficient of the i-th generator;

D_{ij} is an electromagnetic damping coefficient between the i-th and j-th generators;

E_i is a modulus of the i-th generator's internal electromotive force (voltage);

$k_i = M_i^{-1}$;

M_i is an inertia coefficient of the i-th generator;

P_{mi} is a mechanical power delivered to the i-th generator from its turbine;

P_{ei} is an electrical power delivered by the i-th generator to the network;

p_{mi} is a variation of the mechanical power of the i-th generator;

P_{mi}^0 is a steady state value of P_{mi};

p_{mi}^0 is a steady state value of p_{mi};

$p_i = p_{mi} - p_{mi}^0$;

$p_{iN} = p_i - p_N$;

Y is an admittance matrix of the network reduced at the internal generator nodes;

Y_{ij} is a modulus of the ij-th element of Y, $Y_{ij} = Y_{ji}$;

$\alpha_i \mu_i^{-1}$ is a gain of the first order proportional regulator of the i-th generator;

$\beta_{ij} = A_{ij} M_i^{-1}$;

$\Gamma_i = \lambda_i + \lambda_{Ni} + \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} \lambda_{ij}$;

δ_i is a rotor angle of the i-th generator relative to a reference;

$\delta_{ij} = \delta_i - \delta_j$;

δ_i^0 is the equilibrium under consideration of the i-th generator;
δ_{iN}^0 is a value of δ_{iN} at the equilibrium state;
θ_{ij} is an argument of the ij-th element of Y, $\theta_{ij} = \theta_{ji}$;
$\lambda_i = D_i M_i^{-1}$, if $\lambda_i = \lambda$, constant for $i = 1, 2, \dots, N$;
λ is a "uniform" (mechanical) damping;
$\lambda_{ij} = D_{ij} M_i^{-1}$;
$\Lambda_{ij} = \lambda_{ij} - \lambda_{Nj}$ for all $i = 1, 2, \dots, n$;
$\Lambda_{Ni} = \lambda_N - \lambda_i + \lambda_{iN}$;
μ_i^{-1} is a time constant of the first order proportional speed regulator of the i-th generator;
$\sigma_{iN} = \delta_{iN} - \delta_{iN}^0$;
$\sigma_{ij} = \sigma_{iN} - \sigma_{jN}$;
Ω_i is a rotor speed of the i-th generator above the synchronous speed: $\Omega_i = \dot{\delta}_i$;
Ω_i^0 is the value of Ω_i at the steady state operation called "equilibrium state";
$\Omega_{ij} = \Omega_i - \Omega_j$;
$\omega_i = \Omega_i - \Omega_i^0$;
$\omega_{ij} = \omega_i - \omega_j$.

According to Shaaban [1], Shaaban and Grujic [1], and Grujic, *et al.* [1], pp. 341 − 345, an N-machine power system is decomposed into subsystems, each consisting of two machines in addition to the comparison machine. The system is decomposed into $(N-1)/2$ interconnected subsystems for odd number N. When considering transfer conductances, mechanical damping, electromagnetic damping and speed governor action, the mathematical model of the system is derived, and it is decomposed into $(N-1)/2$ sixth-order and one second-order interconnected subsystems. If N is even, then the system is decomposed into $(N-2)/2$ sixth-order, one third-order and one second-order sybsystems.

In this section N is odd, without loss of generality. The system

$$\begin{aligned}\frac{d\omega_i}{dt} &= -\lambda_i\omega_i - \sum_{j=1}^{N}\lambda_{ij}(\omega_i-\omega_j) + k_i\Big(p_i - \sum_{j=1}^{N}A_{ij}\phi_{ij}\Big),\\ \frac{dp_i}{dt} &= -\mu_i p_i - \alpha_i\omega_i, \quad i=1,2,\dots,n, \quad N=n+1,\end{aligned} \tag{2.6.20}$$

is decomposed into $(N-1)/2$ interconnected subsystems, each consisting of two machines and the comparison machine, using the triplewise decomposition. It is to be noted that none of the system machines (except for the comparison machine) can be included in more than one subsystem.

Now, by introducing the set $J_I = \{i_I,\, i_I+1\}$ and defining the state vectors x_I and x_N as follows:

$$\begin{aligned}x_I &= [\sigma_{i_I N}, \sigma_{i_I+1,N}, \omega_{i_I N}, \omega_{i_I+1,N}, P_{i_I N}, P_{i_I+1,N}]^{\mathrm{T}}\\ &= [x_{I1},\, x_{I2},\, x_{I3},\, x_{I4},\, x_{I5},\, x_{I6}]^{\mathrm{T}},\\ x_N &= [\omega_N,\, P_N]^{\mathrm{T}} = [x_{N1},\, x_{N2}]^{\mathrm{T}},\end{aligned} \tag{2.6.21}$$

we can decompose the system mathematical model

$$x = [\sigma_{1N},\, \omega_1,\, p_1,\, \sigma_{2N},\, \omega_2,\, p_2,\, \dots,\, \sigma_{nN},\, \omega_n,\, p_n,\, \omega_N,\, p_N]^{\mathrm{T}} \tag{2.6.22}$$

into $s = (N-1)/2$ sixth-order interconnected subsystems and the second-order interconnected subsystem, which has the general form

$$\frac{dx_N}{dt} = P_N x_N + h_N(x), \tag{2.6.23}$$

where

$$P_N = \begin{pmatrix} -\lambda_N & M_N^{-1} \\ -\alpha_N & \mu_N \end{pmatrix},$$

and

$$h_N = \begin{pmatrix} \displaystyle\sum_{j=1}^{N-1}\{\lambda_{Nj}\omega_{jN} - M_N^{-1}A_{Nj}\phi_{Nj}(\sigma_{Nj})\} \\ 0 \end{pmatrix}.$$

Each of the sixth-order subsystems may be written in the general form

$$\frac{dx_I}{dt} = P_I x_I + B_I f_I(\sigma_I) + h_I(x) \quad \text{for} \quad I = 1,2,\dots,m, \tag{2.6.24}$$

and it can be decomposed into the free (disconnected) subsystems given by

$$\frac{dx_I}{dt} = P_I x_I + B_I f_I(\sigma_I), \tag{2.6.25}$$

where

$$\sigma_I = C_I^{\mathrm{T}} x_I \quad \text{for} \quad I = 1, 2, \ldots, m \tag{2.6.26}$$

and the interconnections $h_I(x)$.

In (2.6.25), the matrices P_I, B_I and C_I^{T} are constant matrices, and $f_I(\sigma_I)$ is a nonlinear vector function. Referring to (2.6.22), we can define the matrix P_I as

$$P_i = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\Gamma_I & \Lambda_I & M_{i_I}^{-1} & 0 \\ 0 & 0 & \bar{\Lambda}_I & -\bar{\Gamma}_I & 0 & M_{i_I+1}^{-1} \\ 0 & 0 & -\alpha_{i_I} & 0 & -\mu_{i_I} & 0 \\ 0 & 0 & 0 & -\alpha_{i_I+1} & 0 & -\mu_{i_I+1} \end{pmatrix}. \tag{2.6.27}$$

Assuming that the free subsystem (2.6.25) contains the six nonlinearities given by

(2.6.28)

$$\begin{aligned}
\phi_{I1}(\sigma_{I1}) &= \cos(\sigma_{i_I N} + \delta^\circ_{i_I N} - \theta_{i_I N}) - \cos(\delta^\circ_{i_I N} - \theta_{i_I N}), \\
\phi_{I2}(\sigma_{I2}) &= \cos(\sigma_{i_I+1,N} + \delta^\circ_{i_I+1,N} - \theta_{i_I+1,N}) - \cos(\delta^\circ_{i_I+1,N} - \theta_{i_I+1,N}), \\
\phi_{I3}(\sigma_{I3}) &= \cos(\sigma_{i_I,i_I+1} + \delta^\circ_{i_I,i_I+1} - \theta_{i_I,i_I+1}) - \cos(\delta^\circ_{i_I,i_I+1} - \theta_{i_I,i_I+1}), \\
\phi_{I4}(\sigma_{I4}) &= \cos(\sigma_{i_I+1,i_I} + \delta^\circ_{i_I+1,i_I} - \theta_{i_I+1,i_I}) - \cos(\delta^\circ_{i_I+1,i_I} - \theta_{i_I+1,i_I}), \\
\phi_{I5}(\sigma_{I5}) &= \cos(\sigma_{N i_I} + \delta^\circ_{N i_I} - \theta_{i_I N}) - \cos(\delta^\circ_{N i_I} - \theta_{i_I N}), \\
\phi_{I6}(\sigma_{I6}) &= \cos(\sigma_{N,i_I+1} + \delta^\circ_{N,i_I+1} - \theta_{i_I+1,N}) - \cos(\delta^\circ_{N,i_I+1} - \theta_{i_I+1,N}),
\end{aligned}$$

we can define the following matrices

$$\begin{aligned} f_I(\sigma_I) = [&\phi_{I1}(\sigma_{I1}),\, \phi_{I2}(\sigma_{I2}),\, \phi_{I3}(\sigma_{I3}), \\ &\phi_{I4}(\sigma_{I4}),\, \phi_{I5}(\sigma_{I5}),\, \phi_{I6}(\sigma_{I6})]^{\mathrm{T}}, \end{aligned} \tag{2.6.29}$$

(2.6.30)

$$B_I = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -M_{i_I}^{-1} A_I & 0 & -M_{i_I}^{-1} \widetilde{A}_I & 0 & M_N^{-1} A_I & M_N^{-1} \bar{A}_I \\ 0 & -M_{i_I+1}^{-1} A_I & 0 & -M_{i_I+1}^{-1} \widetilde{A}_I & M_N^{-1} A_I & M_N^{-1} \bar{A}_I \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

(2.6.31)
$$C_I^{\mathrm{T}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

(2.6.32)
$$h_I(x) = \begin{pmatrix} 0 \\ 0 \\ \tau_I x_{N1} + \eta_I x_{N2} + \sum\limits_{j \in J_I}^{N-1} \Big\{ \Lambda_{Ij}\omega_{jN} \\ \quad - \dfrac{A_{i_I j}\phi_{i_I j}(\sigma_{i_I j})}{M_{i_I}} + \dfrac{A_{Nj}\phi_{Nj}(\sigma_{Nj})}{M_N} \Big\} \\ \bar{\tau}_I x_{N1} + \bar{\eta}_I x_{N2} + \sum\limits_{j \notin J_I}^{N-1} \Big\{ \bar{\Lambda}_{Ij}\omega_{jN} \\ \quad - \dfrac{A_{i_I+1,j}\phi_{i_I+1,j}(\sigma_{i_I+1,j})}{M_{i_I+1}} + \dfrac{A_{Nj}\phi_{Nj}(\sigma_{Nj})}{M_N} \Big\} \\ -\alpha_I x_{N1} - \mu_I x_{N2} \\ -\bar{\alpha}_I x_{N1} - \bar{\mu}_I x_{N2} \end{pmatrix}.$$

In (2.6.27) – (2.6.32) the following additional notation is used:

$$
\begin{gathered}
A_{ij} = E_i E_j Y_{ij},\\
A_I = E_{i_I} E_N Y_{i_I N}, \quad \bar{A}_I = E_{i_I+1} E_N Y_{i_I+1,N}, \quad \widetilde{A}_I = E_{i_I} E_{i_I+1} Y_{i_I,i_I+1},\\
\sigma_{i_I N} = \delta_{i_I N} - \delta^{\circ}_{i_I N}, \quad \sigma_{i_I+1,N} = \delta_{i_I+1,N} - \delta^{\circ}_{i_I+1,N},\\
\sigma_{i_I,i_I+1} = \sigma_{i_I N} - \sigma_{i_I+1,N} = \delta_{i_I,i_I+1} - \delta^{\circ}_{i_I,i_I+1},\\
\omega_{i_I N} = \omega_{i_I} - \omega_N, \quad \omega_{i_I+1,N} = \delta_{i_I+1} - \omega_N,\\
\tau_I = \lambda_N - \lambda_{i_I}, \quad \bar{\tau}_I = \lambda_N - \lambda_{i_I+1},\\
\Lambda_I = \lambda_{i_I,i_I+1} - \lambda_{N,i_I+1}, \quad \bar{\Lambda}_I = \lambda_{i_I+1,i_I} - \lambda_{N i_I},\\
\Lambda_{Ij} = \lambda_{i_I j} - \lambda_{Nj}, \quad \bar{\Lambda}_{Ij} = \lambda_{i_I+1,j} - \lambda_{Nj},\\
\gamma_I = \lambda_{i_I} + \lambda_{N i_I} + \sum_{j \neq i_I}^{N} \lambda_{i_I j}, \quad \bar{\gamma}_I = \lambda_{i_I+1} + \lambda_{N,i_I+!} + \sum_{j \neq i_I+1}^{N} \lambda_{i_I+1,j},\\
\mu_I = \mu_{i_I} - \mu_N, \quad \bar{\mu}_I = \mu_{i_I+1} - \mu_N,\\
\alpha_I = \alpha_{i_I} - \alpha_N, \quad \bar{\alpha}_I = \alpha_{i_I+1} - \alpha_N,\\
\eta_I = M^{-1}_{i_I} - M^{-1}_N, \quad \bar{\eta}_I = M^{-1}_{i_I+1} - M^{-1}_N.
\end{gathered}
$$

It is obvious that the state vector of the whole system is given now by

$$x = [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, \dots, x_m^{\mathrm{T}}, x_N^{\mathrm{T}}]^{\mathrm{T}}. \tag{2.6.33}$$

For system (2.6.24) the problem of stability is formulated as follows. Find the conditions under which stability of the equilibrium state $x = 0$ of system (2.6.30) is deduced from the stability properties of the free subsystems

$$\frac{dx_I}{dt} = P_I x_I + B_I F_I(\sigma_I) \quad \text{for all} \quad I = 1, 2, \dots, m \tag{2.6.34}$$

and the properties of the interconnection functions.

According to the general method of investigation of the large scale systems we shall consider together with system (2.6.25) and subsystems (2.6.34) the matrix function $U(x)$ whose elements are defined as follows

$$
\begin{aligned}
v_{II}(x_I) &= x_I^{\mathrm{T}} Q_{II} x_I && \text{for all} \quad I = 1, 2, \dots, m,\\
v_{IJ}(x_I, x_J) &= x_I^{\mathrm{T}} Q_{IJ} x_J && \text{for all} \quad I, J = 1, 2, \dots, m, \quad I \neq J,\\
v_{IJ}(x_I, x_J) &= v_{JI}(x_I, x_J) && \text{for all} \quad I = 1, 2, \dots, m, \quad I \neq J,
\end{aligned} \tag{2.6.35}
$$

where $Q_{II} = [q^I_{\alpha\beta}]$, $q^I_{\alpha\beta} = q^I_{\beta\alpha}$, $\alpha, \beta = 1, \dots, 5$, are symmetric positive definite matrices of the order of 5×5; $Q_{IJ} = [q^{IJ}_{\alpha\beta}]$, $\alpha, \beta = 1, \dots, 5$ are constant matrices of the order of 5×5.

It is easy to verify that for the functions (2.6.41) the bilaterial inequalities

(2.6.36)

$$
\begin{gathered}
\lambda_m(Q_{II})\|x_I\|^2 \leq v_{II}(x_I) \leq \lambda_M(Q_{II})\|x_I\|^2\\
\text{for all} \quad x_I \in N_{Ix}, \quad I = 1, 2, \dots, m;\\
-\lambda_M^{1/2}(Q_{IJ}^{\mathrm{T}} Q_{IJ})\|x_I\|\,\|x_J\| \leq v_{IJ}(x_I, x_J) \leq \lambda_M^{1/2}(Q_{IJ}^{\mathrm{T}} Q_{IJ})\|x_I\|\,\|x_J\|\\
\text{for all} \quad (x_I, x_J) \in N_{Ix} \times N_{Jx}, \quad I, J = 1, 2, \dots, m, \quad I \neq J
\end{gathered}
$$

are satisfied, where $\lambda_m(Q_{II})$ are minimal and $\lambda_M(Q_{II})$ are maximal eigenvalues of the matrices Q_{II}; $\lambda_M^{1/2}(Q_{IJ}^{\mathrm{T}} Q_{IJ})$ are the norms of the matrices Q_{IJ} respectively, N_{Ix} is the neighborhood of the state $x_I = 0$.

For the function

(2.6.37) $$v(x,\eta) = \eta^{\mathrm T} U(x)\eta, \qquad \eta \in R^m_+$$

the estimates

(2.6.38) $$u^{\mathrm T} H^{\mathrm T} \underline{Q} H u \le v(x,\eta) \le u^{\mathrm T} H^{\mathrm T} \overline{Q} H u$$

are valid for all $x \in N_x \subseteq N_{1x} \times N_{2x} \times \ldots \times N_{3x}$, where

$$\underline{Q} = [\underline{q}_{IJ}], \quad \overline{Q} = [\overline{q}_{IJ}], \quad I,J = 1,2,\ldots,m,$$
$$\underline{q}_{II} = \lambda_m(Q_{II}), \quad \overline{q}_{II} = \lambda_M(Q_{II}), \quad I,J = 1,2,\ldots,m,$$
$$\underline{q}_{IJ} = -\overline{q}_{IJ} = -\lambda_M^{1/2}(Q^{\mathrm T}_{IJ}, Q_{IJ}), \quad I,J = 1,2,\ldots,m, \quad I \ne J.$$

For the total derivatives of functions (2.6.35) along solutions of system (2.6.24) we get

(2.6.39)

(a) $\eta_I^2 Dv_{II}(x_I)\big|_{(2.6.34)} \le \rho_I^{(1)} \|x_I\|^2,$

for all $x_I \in N_{Ix0}, \quad I = 1,2,\ldots,m,$

(b) $$\sum_{I=1}^m \eta_I^2 Dv_{II}(x_I)^{\mathrm T} h_I(x,S) + 2\sum_{I=1}^m \sum_{J=2}^m \eta_I\eta_J Dv_{IJ}(x_I,x_J)\big|_{(2.6.24)}$$
$$\le \sum_{I=1}^m \rho_I^{(2)} + 2\sum_{I=1}^m \sum_{\substack{J=2\\J>1}}^m \rho_{IJ}\|x_I\|\,\|x_J\|,$$

for all $(x_I,x_J) \in N_{Ix0} \times N_{Jx0}, \quad S_{ij} \in G_{ij}, \quad i,j = 1,2,\ldots,N,$

where $\rho_I^{(1)}$ and $\rho_I^{(2)}$ are maximal eigenvalues of the matrices

$$Q_{II}P_I + P_I^{\mathrm T} Q_{II} + A_{1I}(S^*) A^{\mathrm T}_{1I}(S^*),$$
$$A_{2I}(S^*) + A^{\mathrm T}_{2I}(S^*) + \sum_{\substack{J=1\\J\ne I}}^m A^J_{2I}(S^*) + A^{J\mathrm T}_{2I}(S^*)$$

respectively; ρ_{IJ} are the norms of the matrices

$$\frac12 \big(A_{3I}(S^*) + A^{\mathrm T}_{3I}(S^*) + A_{3J}(S^*) + A^{\mathrm T}_{3J}(S^*) + Q_{IJ}P_J$$
$$+ P_J^{\mathrm T} Q_{IJ} + A^J_{1I}(S^*) + A^{I\mathrm T}_{1J}(S^*) + A^J_{3I}(S^*) + A^{I\mathrm T}_{3J}(S^*)\big).$$

Here $S^* \in G_{ij}$ is the constant matrix such that

$$A_{kI}(S) \le A_{kI}(S^*), \qquad A^r_{kl}(S) \le A^r_{kl}(S^*),$$
$$r,\, l = I, J, \quad k = 1,2,3;$$
$$A_{kI}(S) = \begin{pmatrix} a^{kI}_{11} & a^{kI}_{12} & 0 & 0 & 0\\ a^{kI}_{21} & a^{kI}_{22} & 0 & 0 & 0\\ a^{kI}_{31} & a^{kI}_{32} & 0 & 0 & 0\\ a^{kI}_{41} & a^{kI}_{42} & 0 & 0 & 0\\ a^{kI}_{51} & a^{kI}_{52} & 0 & 0 & 0 \end{pmatrix},$$

$$A^r_{kl}(S) = \begin{pmatrix} a^{rkl}_{11} & a^{rkl}_{12} & 0 & 0 & 0 \\ a^{rkl}_{21} & a^{rkl}_{22} & 0 & 0 & 0 \\ a^{rkl}_{31} & a^{rkl}_{32} & 0 & 0 & 0 \\ a^{rkl}_{41} & a^{rkl}_{42} & 0 & 0 & 0 \\ a^{rkl}_{51} & a^{rkl}_{52} & 0 & 0 & 0 \end{pmatrix},$$

$$r,\, l = I, J, \quad k = 1, 2, 3.$$

The elements a^{kI}_{ij}, a^{rkl}_{ij}, $i = 1, \dots, 5$, $j = 1, 2$, $k = 1, 2, 3$, $r, l = I, J$ are determined by means of the constants found in estimates (2.6.45).

It is easy to show that for the function (2.6.37) the estimate

(2.6.40)
$$Dv(x, \eta)\big|_{(2.6.24)} \le u^{\mathrm{T}} \overline{C} u, \quad \text{for all} \quad x \in N_{x0}, \quad S_{ij} \in G_{ij}, \quad i, j = 1, 2, \dots, N,$$

holds true, where

$$u^{\mathrm{T}} = (\|x_1\|, \|x_2\|, \dots, \|x_m\|), \quad \overline{C} = [\overline{c}_{IJ}], \quad \overline{c}_{IJ} = \overline{c}_{JI},$$
$$I, J = 1, 2, \dots, m, \quad \overline{c}_{II} = \rho^{(1)}_I + \rho^{(2)}_I, \quad I = 1, 2, \dots, m,$$
$$\overline{c}_{IJ} = \rho_{IJ}, \quad J > I, \quad I = 1, 2, \dots, m, \quad J = 2, 3, \dots, m,$$
$$N_{x0} \subseteq N_{1x0} \times N_{2x0} \times \dots \times N_{mx0}.$$

Sufficient conditions of asymptotic stability of system (2.6.24) under non-classical structural perturbations are as follows.

Theorem 2.6.3 *For the large scale power system (2.6.24) let matrix-valued function $U(x)$ with elements (2.6.35) be constructed. If the matrix $H^{\mathrm{T}}QH$ is positive definite and the matrix $\overline{C}$ is negative definite, then the equilibrium state $x = 0$ of system (2.6.24) is asymptotically stable under nonclassical structural perturbations.*

The Proof of this theorem follows immediately from the proof of Theorem 2.4.1.

Remark 2.6.2 System (2.6.24) was studied earlier be means of the vector Liapunov function. The comparison of results of the above monograph with the approach based on the matrix-valued function shows that the aggregation matrices in both cases are of the same order. The difference between the aggregation matrices is that in the determination of the diagonal elements of the aggregation matrix $\overline{C}$ the matrices

$$A^J_{2I}(S^*) + A^{J\mathrm{T}}_{2I}(S^*) \tag{2.6.41}$$

are added, while in the determination of non-diagonal elements the matrices

$$Q_{IJ}P_JP_I^{\mathrm{T}}Q_{IJ} + A^J_{1I}(S^*) + A^{J\mathrm{T}}_{1I}(S^*) + A^J_{3I}(S^*) + A^{J\mathrm{T}}_{3I}(S^*) \tag{2.6.42}$$

are required. The matrices (2.6.41) and (2.6.42) appear because of the presence of the functions

$$v_{IJ}(x_I, x_J) = x_I^{\mathrm{T}}Q_{IJ}x_J,$$

$I, J = 1, 2, \dots, m,\ I \neq J$. If we allow for the fact that Q_{IJ} are only symmetric positive definite matrices and $Q_{IJ}\ (I \neq J)$ are arbitrary constant ones such that $\lambda_m(H^{\mathrm{T}}QH) > 0$, then one can "influence" the choice of the matrix-valued function $U(x)$ by choosing the matrices Q_{IJ}. This makes possible to extend the classes of functions suitable for establishing the criteria for construction of sufficient stability conditions for system (2.6.30). Besides, the matrix $\overline{C}$ can be constructed so that the absolute values of the diagonal elements will be maximally large while the non-diagonal elements will be minimal. This fact allows to get more refined results as compared with those obtained by means of the vector Liapunov function. However we note that for a large number of subsystems it becomes difficult on the analytical level to choose matrices $Q_{IJ}\ (I \neq J)$ and to verify the property of having a fixed sign of the aggregation matrix $\overline{C}$. In such a case it is expedient to employ computer programmes in the direction.

2.6.3 Large-scale Lur'e-Postnikov systems In this section for the analysis of absolute stability under structural perturbations a special Liapunov function is constructed in terms of matrix-valued function. This allows to obtain new sufficient conditions for structural absolute stability of the equilibrium state of the system under consideration.

Consider a large-scale Lur'e system decomposed into s subsystems

$$\begin{aligned} \frac{dx_i}{dt} &= \sum_{l=1}^{s} S_{il}^{(1)}A_{il}x_l + \sum_{l=1}^{s} S_{il}^{(2)}q_{il}f_{il}\sigma_{il}, \\ \sigma_{il} &= C_{il}^{\mathrm{T}}x, \quad i = 1, 2, \dots, s, \end{aligned} \tag{2.6.43}$$

where $\sigma_{il}^{-1} f_{il}(\sigma_{il}) \in [0, k_{il}] \subseteq R_+$, A_{il} are constant matrices, $x_i \in R^{n_i}$, $n_1 + n_2 + \cdots + n_s = n$, k_{il} are constants. Here all matrices and vectors are of the corresponding dimensions, and $S_{il}^{(1)}$ and $S_{il}^{(2)}$ are diagonal matrices. By means of the structural matrices

$$S_i = \begin{pmatrix} S_{i1}^{(1)} & S_{i2}^{(1)} & \dots & S_{i,i-1}^{(1)} & I & S_{i,i+1}^{(1)} & S_{is}^{(1)} \\ S_{i1}^{(2)} & S_{i2}^{(2)} & \dots & S_{i,i-1}^{(2)} & I & S_{i,i+1}^{(2)} & S_{is}^{(2)} \end{pmatrix},$$
$$S = \operatorname{diag}\{S_1, S_2, \dots, S_s\},$$

the structural set $\mathcal{S}$ is determined by the formula

$$\mathcal{S} = \{S\colon\ 0 \le S_{il}^{(k)} \le I,\ S_{ii}^{(1)} = I,\ i, l = 1, 2, \dots, s,\ k = 1, 2\},$$

where I is an identity matrix of corresponding dimensions.

Independent subsystems corresponding to system (2.6.43) are obtained as a result of substitution by the vector x^i for x:

$$\frac{dx_i}{dt} = A_{ii} x_i + S_{ii}^{(2)} q_{ii} f_{ii}(\tilde{s}_{ii}), \tag{2.6.44}$$

where $\tilde{\sigma}_{ii} = C_{ii}^{\mathrm{T}} x^i$, $x^i = (0^{\mathrm{T}}, 0^{\mathrm{T}}, \dots, 0^{\mathrm{T}}, x_i^{\mathrm{T}}, 0^{\mathrm{T}}, \dots 0^{\mathrm{T}})^{\mathrm{T}} \in R^n$, $i \in [1, s]$.

We introduce the designations

$$f_i(x^i) = A_{ii} x_i + S_{ii}^{(2)} q_{ii} f_{ii}(\tilde{\sigma}_{ii});$$
$$f_i^*(x, S) = \sum_{\substack{l=1\\ l \ne i}}^{s} S_{il}^{(1)} A_{il} x_l + \sum_{\substack{l=1\\ l \ne i}}^{s} S_{il}^{(2)} q_{il} f_{il}(\sigma_l) + S_{ii}^{(2)} q_{ii} [f_{ii}(\sigma_{ii}) + f_{ii}(\tilde{\sigma})];$$
$$\sigma_{il} = C_{il}^{\mathrm{T}} x, \quad i \in [1, s].$$

Then system (2.6.43) becomes

$$\frac{dx_i}{dt} = f_i(x^i) + f_i^*(x, S), \quad i = 1, 2, \dots, s. \tag{2.6.45}$$

Alongside system (2.6.43) and subsystems (2.6.44) we shall consider the matrix-valued function (2.5.2) with elements (2.5.3) for which estimates (2.5.5) hold.

Together with function (2.5.4) its total derivative

$$D^+ v(x, \psi) = \psi^{\mathrm{T}} D^+ U(x) \psi \tag{2.6.46}$$

along solutions of system (2.6.43) is considered.

Proposition 2.6.2 *If for system (2.6.43) the matrix-valued function (2.5.2) with elements (2.5.3) is constructed, then for the Dini derivatives of function (2.5.4) along solutions of system (2.6.43) the following estimates hold*

(a) $\psi_i^2 (D_{x_i}^+ v_{ii})^{\mathrm{T}} f_i(x^i) \le \rho_i^{(1)}(S)\|x_i\|^2$ *for all* $x_i \in \mathcal{N}_{ix0}$, $i \in [1, s]$;

(b) $$\sum_{i=1}^{s} \psi_i^2 (D_{x_i}^+ v_{ii})^{\mathrm{T}} f_i^*(x, S) + 2 \sum_{i=1}^{s} \sum_{\substack{j=2 \\ j>i}}^{s} \psi_i \psi_j \{(D_{x_i}^+)^{\mathrm{T}} (f_i(x^i) + f_i^*(x, S))$$

$$+ (D_{x_j}^+ v_{ij})^{\mathrm{T}} (f_j(x^j) + f_j^*(x, S)) \le \sum_{i=1}^{s} \rho_i^{(2)}(S)\|x_i\|^2$$

$$+ 2 \sum_{i=1}^{s} \sum_{\substack{j=2 \\ j>i}}^{s} \rho_{ij}(S)\|x_i\|\,\|x_j\| \quad \textit{for all} \quad (x_i, x_j, S) \in \mathcal{N}_{ix_0} \times \mathcal{N}_{jx_0} \times \mathcal{S},$$

where $\rho_i^{(k)}(S)$, $k = 1, 2$, $i \in [1, s]$ *are maximal eigenvalues of the matrices*

$$\psi_i^2 [P_{ii} A_{ii} + A_{ii}^{\mathrm{T}} P_{ii} + P_{ii} S_{ii}^{(2)} q_{ii} k_{ii}^* (c_{ii}^i)^{\mathrm{T}} + (S_{ii}^{(2)} q_{ii} k_{ii}^* (c_{ii}^i)^{\mathrm{T}})^{\mathrm{T}} P_{ii}];$$

$$\sum_{l=1}^{i-1} \psi_i \psi_l |\{[(S_{li}^{(1)} A_{li})^{\mathrm{T}} + (S_{li}^{(2)} q_{li} k_{li}^* (c_{li}^i)^{\mathrm{T}})^{\mathrm{T}}] P_{li} + P_{li}^{\mathrm{T}} [S_{li}^{(1)} A_{li}$$

$$+ S_{li}^{(2)} q_{li} k_{li}^* (c_{li}^i)^{\mathrm{T}}]\} \sum_{l=i+1}^{s} \psi_i \psi_l \{P_{il} [S_{li}^{(1)} A_{li} + S_{li}^{(2)} q_{li} k_{li}^* (c_{li}^i)^{\mathrm{T}}] + [(S_{li}^{(1)} A_{li})^{\mathrm{T}}$$

$$+ (S_{li}^{(2)} q_{li} k_{li}^* (c_{li}^i)^{\mathrm{T}})^{\mathrm{T}}] P_{il}^{\mathrm{T}}\} + \psi_i^2 [P_{ii} S_{ii}^{(2)} q_{ii} k_{ii}^* (c_{ii}^i)^{\mathrm{T}} + (S_{ii}^{(2)} q_{ii} k_{ii}^* (c_{ii}^i)^{\mathrm{T}})^{\mathrm{T}} P_{ii}]$$

respectively; $\rho_{ij}(S),\ i<j,\ i=1,2,\dots,s,\ j=2,3,\dots,s,$ *are norms of the matrices*

$$\sum_{l=1}^{j-1}\psi_j\psi_l[(S_{li}^{(1)}A_{li}^{\mathrm T})+(S_{li}^{(2)}q_{li}k_{li}^*(c_{li}^i)^{\mathrm T})^{\mathrm T}]P_{lj}+\sum_{l=j+1}^{s}\psi_j\psi_l[(S_{li}^{(1)}A_{li})^{\mathrm T}$$
$$+(S_{li}^{(2)}q_{li}k_{li}^*(c_{li}^i)^{\mathrm T})^{\mathrm T}]P_{jl}^{\mathrm T}+\sum_{l=1}^{i-1}\psi_i\psi_l P_{li}^{\mathrm T}[S_{lj}^{(1)}A_{lj}+S_{lj}^{(2)}q_{lj}k_{lj}^*(c_{lj}^j)^{\mathrm T}]$$
$$+\sum_{l=i+1}^{s}\psi_i\psi_l P_{il}[S_{lj}^{(1)}A_{lj}+S_{lj}^{(2)}q_{lj}k_{lj}^*(c_{lj}^j)^{\mathrm T}]+\frac12\,\psi_i^2\Big\{P_{ii}(S_{ij}^{(1)}A_{ij})$$
$$+(S_{ij}^{(1)}A_{ij})^{\mathrm T}P_{ii}+P_{ii}(S_{ij}^{(2)}q_{ij}k_{ij}^*(c_{ij}^j)^{\mathrm T})+(S_{ij}^{(2)}q_{ij}k_{ij}^*(c_{ij}^j)^{\mathrm T})^{\mathrm T}P_{ii}$$
$$+P_{ii}(S_{ii}^{(2)}q_{ii}k_{ii}^*(c_{ii}^j)^{\mathrm T})+(S_{ii}^{(2)}q_{qq}k_{ii}^*(c_{ii}^j)^{\mathrm T})^{\mathrm T}P_{ii}\Big\}$$
$$+\frac12\,\psi_j^2\Big\{P_{ji}(S_{ji}^{(1)}A_{ji})+(S_{ji}^{(1)}A_{ji})^{\mathrm T}P_{ji}+P_{jj}(S_{ji}^{(2)}q_{ji}k_{ji}^*(c_{ji}^i)^{\mathrm T})$$
$$+(S_{ji}^{(2)}q_{ji}k_{ji}^*(c_{ji}^i)^{\mathrm T})^{\mathrm T}P_{jj}+P_{jj}(S_{jj}^{(2)}q_{jj}k_{jj}^*(c_{jj}^i)^{\mathrm T})+(S_{jj}^{(2)}q_{jj}k_{jj}^*(c_{jj}^i)^{\mathrm T})^{\mathrm T}P_{jj}\Big\}$$

respectively. Here

$$k_{ij}^*=\begin{cases}k_{ij} & \textit{for}\ \ \sigma_{ij}(S_{ij}^{(k)}q_{ij})^{\mathrm T}P_{ij}x_j>0,\quad i,j=1,2,\dots,s;\quad k=1,2;\\ 0 & \textit{in other cases};\end{cases}$$
$$k_{ii}^*=\begin{cases}k_{ii} & \textit{for}\ \ \sigma_{ii}(S_{ii}^{(2)}q_{ii})^{\mathrm T}P_{ii}x_i>0,\quad i=1,2,\dots,s;\\ -k_{ii} & \textit{for}\ \ \sigma_{ii}(S_{ii}^{(2)}q_{ii})^{\mathrm T}P_{ii}x_i<0,\quad i=1,2,\dots,s;\end{cases}$$

$c_{ij}^k\in R^{n_k}$ *is the k-th component of the vector* c_{ij}.

Estimate (a) is proved by transformation of the left-hand part of (a). Namely,

$$\begin{aligned}\psi_i^2(D_{x_i}^+v_{ii})^{\mathrm T}f_i(x^i)&=\psi_i^2\{\dot x_i^{\mathrm T}P_{ii}x_i+x_i^{\mathrm T}P_{ii}\dot x_i\}\\&=\psi_i^2\{[A_{ii}x_i+S_{ii}^{(2)}q_{ii}f_{ii}(\tilde\sigma_{ii})^{\mathrm T}]P_{ii}x_i+x_i^{\mathrm T}P_{ii}[A_{ii}x_i+S_{ii}^{(2)}q_{ii}f_{ii}(\tilde\sigma)]\\&\le x_i^{\mathrm T}\{\psi_i^2[P_{ii}A_{ii}+A_{ii}^{\mathrm T}P_{ii}+P_{ii}S_{ii}^{(2)}q_{ii}k_{ii}^*(c_{ii}^i)^{\mathrm T}\\&+(S_{ii}^{(2)}q_{ii}k_{ii}^*(c_{ii}^i)^{\mathrm T})^{\mathrm T}P_{ii}]\}x_i\le\rho_i^{(1)}(S)\|x_i\|^2\end{aligned}$$

for all $x_i\in\mathcal N_{ix_0},\ i\in[1,s]$.

Estimate (b) is proved in the same way as estimate (a).

Proposition 2.6.3 *Let all conditions of Proposition 2.6.2 be satisfied. Then for expression (2.6.46) the inequality*

$$D^*v(x,\psi)\le u^{\mathrm T}Cu\quad \textit{for all}\ \ (x,S)\in\mathcal N_{x_0}\times\mathcal S, \tag{2.6.47}$$

is satisfied, where $C=[c_{ij}],\ i,j\in[1,s],\ c_{ii}=\rho_i^{(1)}(S^*)+\rho_i^{(2)}(S^*),\ c_{ij}=c_{ji}=\rho_{ij}(S^*),\ (i\ne j)\in[1,s],\ S^*\in\mathcal S$ *is a constant matrix such that* $\rho_i^{(k)}(S)\le\rho_i^{(k)}(S^*),\ \rho_{ij}(S)\le\rho_{ij}(S^*),\ k=1,2.$

Proof When all conditions of Proposition 2.6.3 are satisfied, we have

$$D^+v(x,\psi)\triangleq\psi^{\mathrm T}[D^+v_{ij}(x_i,x_j)]\psi=\sum_{i=1}^{s}\psi_i^2(D^+v_{ii}(x_i))^{\mathrm T}(f_i(x^i)+f_i^*(x,S))$$
$$+2\sum_{i=1}^{s}\sum_{\substack{j=2\\ j>i}}^{s}\psi_i\psi_j(D^*v_{ij}(x_i,x_j))^{\mathrm T}(f_i(x^i)+f_i^*(x,S))$$

$$
\begin{aligned}
&\le \sum_{i=1}^{s}(\rho_i^{(1)}(S)+\rho_i^{(2)}(S))\|x_i\|^2 + 2\sum_{i=1}^{s}\sum_{\substack{j=2\\ j>i}}^{s}\rho_{ij}(S)\|x_i\|\,\|x_j\| \\
&\le \sum_{i=1}^{s}(\rho_i^{(1)}(S^*)+\rho_i^{(2)}(S^*))\|x_i\|^2 + 2\sum_{i=1}^{s}\sum_{\substack{j=2\\ j>i}}^{s}\rho_{ij}(S^*)\|x_i\|\,\|x_j\| \\
&= \sum_{i=1}^{s} c_{ii}\|x_i\|^2 + 2\sum_{i=1}^{s}\sum_{\substack{j=2\\ j>i}}^{s} c_{ij}\|x_i\|\,\|x_j\| = u^{\mathrm{T}}Cu
\end{aligned}
$$

for all $(x,s)\in\mathcal{N}_{x_0}\times\mathcal{S}$. QED.

Basing on the obtained estimates for the matrix-valued function the following result is stated.

Theorem 2.6.4 *Let the motion equations of large-scale Lur'e system (2.6.43) be such that for them the matrix-valued function (2.5.2) is constructed with elements (2.5.3) and for derivative (2.6.46) estimate (2.6.47) is satisfied. If*

(a) *matrices A and B in estimate (2.5.5) are positive definite;*
(b) *matrix C in estimate (2.6.46) is negative definite on $\mathcal{S}$,*

then the equilibrium state $x=0$ of system (2.6.43) is uniformly asymptotically stable on $\mathcal{S}$.

Moreover, if $\mathcal{N}_{ix}=R^{n_i}$, then the equilibrium state $x=0$ of system (2.6.43) is uniformly asymptotically stable in the whole on $\mathcal{S}$.

Proof If estimates (2.5.5) are satisfied, then under condition (a) of Theorem 2.6.4 the function (2.5.4) is positive definite and decreasing on $\mathcal{N}_{ix}$. Estimate (2.6.47) and condition (b) of Theorem 2.6.4 imply that the expression $D^+v(x,\psi)|_{(2.6.43)}$ is negative definite on $\mathcal{N}_{x_0}\times\mathcal{S}$.

These conditions are sufficient for structural uniform asymptotic stability of the equilibrium state $x=0$ of system (2.6.43) on $\mathcal{S}$.

In the case $\mathcal{N}_{ix}=R^{n_i}$ function (2.5.4) is positive definite decreasing and radially unbounded. This together with other conditions of the theorem proves its second assertion.

Remark 2.6.1 Theorem 2.6.4 remains valid, if the matrix A is conditionally positive definite and the matrix C is conditionally negative definite.

Example 2.6.1 Let system (2.6.43) be the fourth-order system of Lur'e type decomposed into two interconnected second order subsystems defined by the vectors and matrices
(2.6.48)

$$
A_{11}=\begin{pmatrix}-3 & 0\\ 0 & -3\end{pmatrix},\quad A_{12}=\begin{pmatrix}-5 & 0\\ -1 & -5\end{pmatrix},\quad A_{21}=\begin{pmatrix}5 & 0\\ 1 & 5\end{pmatrix},
$$

$$
A_{22}=\begin{pmatrix}0.1 & 0\\ 0 & 0.1\end{pmatrix},\quad q_{1l}=\begin{pmatrix}0.1\\ 0\end{pmatrix},\quad q_{2l}=\begin{pmatrix}0\\ 0.1\end{pmatrix},
$$

$$
c_{1l}^{\mathrm{T}}=(0.1;\ 0;\ 0.1;\ 0),\quad c_{2l}^{\mathrm{T}}=(0.1;\ 0;\ 0;\ 0.1),\quad k_{il}=1,\quad i,l=1,2;
$$

$$
S_{ij}^{(1)}=I,\quad S_{ij}^{(2)}=s_{ij}I,\quad i,j=1,2,\quad I=\mathrm{diag}\,(1,1).
$$

Here $s_{ij}\colon[-\infty,+\infty]\to[0,1]$ is a structural parameter.

The structural set $\mathcal{S}$ is determined as follows

$$
\mathcal{S}=\{S\colon\ S_{ij}^{(1)}=I,\ \ 0\le S_{ij}^{(2)}\le I,\quad i,j,=1,2\}
$$

For the elements of matrix-valued function (2.5.2) taken in the form

$$v_{ii}(x_i) = x_i^{\mathrm{T}} I x, \quad i = 1, 2;$$
$$v_{12}(x_1, x_2) = v_{21}(x_1, x_2) = x_1^{\mathrm{T}}\, 0.1\, I x_2,$$

the estimates

$$v_{ii}(x_i) \ge \|x_i\|^2, \quad i = 1, 2;$$
$$v_{12}(x_1, x_2) \ge -0.1\, \|x_1\|\, \|x_2\|$$

are satisfied. Let $\psi^{\mathrm{T}} = (1, 1)$, then the matrix $\tilde{A}$ corresponding to the matrix A in estimate (2.5.5)

$$\tilde{A} = \begin{pmatrix} 1 & -0.1 \\ -0.1 & 1 \end{pmatrix}$$

is positive definite.

For this choice of the elements $v_{ij}(\cdot)$, $i, j = 1, 2$, we have

(1) for $k_i^* = 0$: $\rho_1^{(1)}(S) = -6$, $\rho_2^{(1)}(S) = 0.2$, $\rho_{12}^{(1)}(S) = 1.1 + 0.02\, S_{11}$, $\rho_{21}^{(2)}(S) = -0.9 + 0.02\, S_{22}$, $\rho_{12}(S) = 0.29$;

(2) for $k_i^* = k_i = 1$: $\rho_1^{(1)}(S) = -6 + 0.02\, S_{11}$, $\rho_2^{(1)}(S) = 0.2 + 0.01\, S_{22}$, $\rho_1^{(2)}(S) = 1.1 + 0.02\, S_{11} + 0.001\, S_{21}$, $\rho_2^{(2)}(S) = -0.9 + 0.001\, S_{12} + 0.02\, S_{22}$, $\rho_{12}(S) = 0.29 + 0.011\, S_{11} + 0.01\, S_{12} + 0.005\, S_{21} + 0.007\, S_{22}$.

The matrix $\widetilde{C}$ corresponding to the matrix C in estimate (2.6.47) has the form

$$\widetilde{C} = \begin{cases} \begin{pmatrix} -4.88 & 0.29 \\ 0.29 & -0.68 \end{pmatrix} & \text{for } k_i^* = 0; \\ \begin{pmatrix} -4.859 & 0.323 \\ 0.323 & -0.669 \end{pmatrix} & \text{for } k_i^* = k_i = 1 \end{cases}$$

and is negative definite.

Thus, all conditions of Theorem 2.6.3 are satisfied and the equilibrium state $x = 0$ of system (2.6.43) with vectors and matrices (2.6.48) is structurally asymptotically stable in the whole on $\mathcal{S}$.

2.7 Notes and References

Section 2.1 Mathematical methods of investigation of continuous large scale systems in the absence of structural perturbations are presented in the well-known monographs. The known defects of the stability criteria obtained in terms of the Liapunov vector functions (see Piontkovskii and Rutkovskaya [1], and Martynyuk and Slyn'ko [1]) impelled many investigators to develop other approaches for qualitative analysis of motions of large scale systems. The method of Liapunov matrix-valued functions is one of them (for the details see Martynyuk [13, 17]).

Section 2.2 In the description of the adopted model of nonclassical structural perturbations we proceed from the fact that the architecture of complexity and/or multidimensionness of a real system presumes the evolution from "simple" to "complex" (see Simon [1], Levins [1], Bronowski [1], etc.) with the stable structure on each hierarchical level. In other words, the dynamics of free subsystems (2.2.3) is a prototype of the dynamical properties of the interacting subsystems (2.2.2) and the whole system (C). That is why system (2.2.5) and system (11) from the monograph by Grujić, *et al.* [1], pp. 157 – 160, are similar in their form but different in their content. Namely, system (2.2.5) is a result of mathematical composition of the individual subsystems (2.2.3) for a given model of nonclassical structural perturbations. For Remarks 2.2.1 and 2.2.2 see Grujic *et al.* [1].

Section 2.3 The estimates for the class of Liapunov functions applied in this section were obtained by Martynyuk and Miladzhanov [1, 2]. In this section some results of Krasovskii [1], and Djordjević [1] were used.

Section 2.4 This section is based on the results by Martynyuk [6], Martynyuk and Miladzhanov [1 – 3], Martynyuk and Stavroulakis [1, 2], and Miladzhanov [1, 2]. Definition 2.4.1 is based on some results of Liapunov [1], Chetaev [2], and Siljak [1 – 3] (see and cf. Grujic *et al.* [1], pp. 160, and Martynyuk and Miladzhanov [1, 2]).

Section 2.5 The results of analysis of large scale linear system are new. Alongside the results obtained in Section 2.4 some results by Djordjević [1] are applied.

Section 2.6 The results of Sections 2.6.1 and 2.6.3 are new as referred to the application of matrix-valued function and two measures in the investigation of stability of nonlinear system under nonclassical structural perturbations. Some results in the direction were presented in the paper by Martynyuk [8]. The analysis of stability with respect to two measures of nonlinear systems without structural perturbations in terms of matrix-valued function was carried out by Martynyuk and Chernienko [1]. The investigations of motion stability with respect to two measures by means of scalar Liapunov function were summarized in the monograph by Lakshmikantham, Leela, *et al.* [1], Lakshmikantham and Liu [1].

The mathematical model of energy system studied in Section 2.6.2 was investigated earlier by means of scalar (see Ribbens-Pavella [1, 2]) and vector Liapunov functions (see Grujic *et al.* [1], Voronov and Matrosov [1], etc.). The results obtained via the application of matrix-valued function in the investigation of this model are presented according to Martynyuk and Miladzhanov [7].

The results for large-scale Lur'e-Postnikov systems under nonclassical structural perturbations are new.

3

DISCRETE-TIME LARGE-SCALE SYSTEMS

3.1 Introduction

An important place in the investigation of stability of discrete systems belongs to the Liapunov direct method, which is a natural generalization of the Liapunov direct method for continuous systems. Various generalizations of the Liapunov method for large scale discrete-time systems are based on the use of a scalar, vector or auxiliary matrix-valued function

The aim of Chapter 3 is to establish tests for the stability of an equilibrium state of large scale discrete-time systems under nonclassical structural perturbations by using the matrix-valued Liapunov function. As in the continuous case (see Chapter 2) we consider two general statements of the problem, i.e. Problem D_A and Problem D_B.

The presentation of the main results of the chapter is arranged as follows:

Section 3.2 provides a description of the composition of large scale discrete system for the given connectedness model.

Section 3.3 deals with a description of the classes of matrix-valued functions applied in the investigation of stability-like properties of discrete systems. Also general theorems on stability under nonclassical structural perturbations in terms of existence of auxiliary matrix-valued function are presented here.

In Section 3.4 various sufficient stability conditions are set out for the system of equations under consideration within the framework of solution of Problems D_A and D_B.

In Section 3.5 the efficiency of the applied approaches is analysed; the approaches are based on three classes of auxiliary Liapunov functions in the estimation of robustness boundary of uncertain linear system. It is shown that the application of the heirarchical Liapunov functions yields the widest estimates for the norms of the matrices describing uncertainties in dynamical system.

3.2 Nonclassical Structural Perturbations in Discrete-Time Systems

The object of investigation in this chapter are the models of real systems and/or processes whose mathematical description is made by the systems of the first order ordinary difference equations under nonclassical structural and/or parametric perturbations. For the class of systems (subsystems) under consideration the notations D and D_i are used respectively and the following hypotheses are adopted.

H_1. The time-discrete system D consists of m interacting subsystems D_i, each of the subsystems is described by the subsystem of the first order ordinary difference equations whose order is not changed on the interval of motion investigation of this system.

H_2. Parametric and/or external perturbations of system D are characterized by the matrix $P = (p_1^{\mathrm{T}}, p_2^{\mathrm{T}}, \dots, p_m^{\mathrm{T}})^{\mathrm{T}} \in R^{m\times q}$ as in the continuous case. The set of all the matrices addmissible for the given system is designated by

$$\mathcal{P} = \{P\colon\ P_1 \le P(\tau) \le P_2,\ \ \tau \in N_\tau^+\}. \tag{3.2.1}$$

Here P_1 and P_2 are the prescribed constant matrices.

H_3. The family $\mathcal{F}$ of vector-mappings $(f^1, f^2, \dots, f^N)$ is determined, $f^k\colon N_\tau^+ \times R^n \times R^{s\times q} \to R^n$, where N is a real number, $f_i^k\colon N_\tau^+ \times R^n \times R^{1\times q} \to R^{n_i}$, for all $k = 1, 2, \dots, N$, $i = 1, 2, \dots, m$, $n_1 + n_2 + \dots + n_m = n$.

H_4. The dynamics of the i-th interconnected subsystem D_i in system D is described by the finite-dimentional first order ordinary difference equations

$$x_i(\tau+1) = f_i(\tau, x(\tau), p_i), \quad i = 1, 2, \dots, m, \tag{3.2.2}$$

where $x_i \in R^{n_i}$, $f_i \in \mathcal{F}_i = \{f_i^1, f_i^2, \dots, f_i^N\}$, $x = (x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, \dots, x_i^{\mathrm{T}}, \dots, x_m^{\mathrm{T}})^{\mathrm{T}}$. It is assumed that $f_i(\tau, 0, 0) = 0$ for all $\tau \in N_\tau^+$.

The number N in the definition of the families $\mathcal{F}$ and $\mathcal{F}_i$, $i = 1, 2, \dots, m$, and the variations of the exponent $k = k(\tau)$ on the set $\mathcal{N} = \{1, \dots, N\}$, $k(\tau) \in \mathcal{N}$ for all $\tau \in N_\tau^+$, describe structural changes in system (3.2.2).

H_5. The dynamics of the i-th isolated subsystem D_i in system D is described by the finite-dimensional first order ordinary difference equations

$$x_i(\tau+1) = g_i(\tau, x_i), \quad i = 1, 2, \dots, m, \tag{3.2.3}$$

where $x_i \in R^{n_i}$, and the functions $g_i\colon N_\tau^+ \times R^{n_i} \to R^{n_i}$ are determined by the correlations

$$g_i(\tau, x_i) = f_i(\tau, x^i, 0), \quad i = 1, 2, \dots, m.$$

It is assumed that $g_i(\tau, 0) = 0$ for all $\tau \in \mathcal{N}$.

Similarly to Section 2.2 the dynamics of the whole system D can be described in terms of systems (3.2.2) and (3.2.3) by the systems of the first order ordinary difference equations

$$x_i(\tau+1) = g_i(\tau, x_i) + S_i(\tau) h_i(\tau, x, p_i), \quad i = 1, 2, \dots, m, \tag{3.2.4}$$

where $S_i(\tau)$ is a structural matrix of the i-the subsystem $S_i\colon N_\tau^+ \to R^{n_i \times N_{n_i}}$, $h_i\colon N_\tau^+ \times R^n \times R^{1\times q} \to R^{N_{n_i}}$.

Setting

$$g = (g_1^{\mathrm{T}}, \dots, g_m^{\mathrm{T}})^{\mathrm{T}}, \quad h = (h_1^{\mathrm{T}}, \dots, h_n^{\mathrm{T}})^{\mathrm{T}}$$

and taking into account the designations from Section 2.2, one can present system (3.2.4) in the vector form

$$x(\tau+1) = g(\tau, x(\tau)) + S(\tau) h(\tau, x(\tau), P), \tag{3.2.5}$$

where $x \in R^n$, $g\colon N_\tau^+ \times R^n \to R^n$, $h\colon N_\tau^+ \times R^n \times R^{m\times q} \to R^n$, $S(\tau)$ is the structural matrix of system D.

Further we assume that the system (3.2.5) satisfies the existence and uniqueness conditions for a solution $x(\tau; x_0, t_0)$ of equation (3.2.1) for any $x_0 \in R^n$, $t_0 \geq 0$, and $\tau \in N^+$. Moreover, $x(t_0; x_0, t_0) = x_0$.

Assume that $g(\tau, x(\tau)) = 0$ for all $\tau \in N^+$ and $P \in \mathcal{P}$ if and only if $x(\tau) = x_e$, i.e. the state $x(\tau) = x_e$ is the unique equilibrium state of system (3.2.5).

Remark 3.2.1 The discrete analogue of system of differential inequalities (2.2.10) for system (3.2.5) is the system of difference inequalities of the form

$$T^m(\tau, x(\tau), P_1, S) \leq x(\tau+1) \leq T^M(\tau, x(\tau), P_2, S). \tag{3.2.6}$$

Here $T(\tau, x(\tau), P, S) = g(\tau, x(\tau)) + S(\tau) h(\tau, x(\tau), p)$, $T^m\colon N_\tau^+ \times R^n \times R^{m\times q} \times \mathcal{S} \to R^n$, $T^M\colon N_\tau^+ \times R^n \times R^{m\times q} \times \mathcal{S} \to R^n$, $\tau \in N_\tau^+$.

By definition of the set $\mathcal{P}$ for all $(\tau, x, P, S) \in N_\tau^+ \times R^n \times \mathcal{P} \times \mathcal{S}$ the ineqiality

$$T^m(\tau, x(\tau), P_1, S) \leq T^M(\tau, x(\tau), P_2, S).$$

is fulfilled.

The vector-function $\chi(\tau)$ determined on $N_{\tau_0}^+ = [\tau_0, +\infty) \cap N_\tau^+$ is called the solution of inequality (3.2.6), if

$$T^m(\tau, \chi(\tau), P_1, S) \leq \chi(\tau+1) \leq T^M(\tau, \chi(\tau), P_2, S)$$

for all $\tau \in N_\tau^+$.

Since

$$T^m(\tau, x(\tau), P_1, S) \leq T(\tau, x(\tau), P, S) \leq T^M(\tau, x(\tau), P_2, S)$$

for all $(\tau, x, P, S) \in N_\tau^+ \times R^n \times \mathcal{P} \times \mathcal{S}$, every solution of system (3.2.5) is the solution of inequality (3.2.6).

3.3 Liapunov's Matrix-Valued Functions

The properties of the stability of discrete-time systems under nonclassical structural perturbations can be determined in accordance with the definitions from Sections 1.2 and 2.2.

The solution of the problem concerning the stability of a large scale discrete-time system under nonclassical structural perturbations is based on the natural extension of the theory of the Liapunov direct method on the basis of the matrix-valued function.

Therefore, along with system (3.2.5) we consider the matrix-valued function

$$U(\tau, x(\tau)) = [v_{ij}(\tau, x(\tau))], \quad i, j = 1, 2, \dots, m, \tag{3.3.1}$$

which we use to construct the function of class *SL* (see Section 1.2)

$$v(\tau, x(\tau), \psi) = \psi^{\mathrm{T}} U(\tau, x(\tau))\psi, \tag{3.3.2}$$

where $\psi = (\psi_1, \psi_2, \dots, \psi_m)^{\mathrm{T}}$, $\psi_i \neq 0$, $i = 1, 2, \dots, m$.

We note that if $\psi = J = (1, 1, \dots, 1)^{\mathrm{T}} \in R^m$, the function (3.3.2) becomes

$$v(\tau, x(\tau)) = \sum_{i,j=1}^{m} v_{ij}(\tau, x(\tau)). \tag{3.3.3}$$

For the first differences of the matrix-valued function (3.3.1) and scalar function (3.3.2) along the solutions of system (3.2.5) we introduce the notation

$$\Delta U(\tau, x(\tau)) = [\Delta v_{ij}(\tau, x(\tau))], \quad i, j = 1, 2, \dots, m, \tag{3.3.4}$$

$$\Delta v(\tau, x(\tau)) = \psi^{\mathrm{T}} \Delta U(\tau, x(\tau))\psi, \tag{3.3.5}$$

where the first differences of the functions $v_{ij}(\tau, x(\tau))$ relative to $(\tau, x(\tau)) \in N_\tau^+ \times \mathcal{N}$ are determined in accordance with Section 1.3.

Definition 3.3.1 The matrix-valued function $U\colon N_\tau^+ \times R^n \to R^{m\times m}$ is called

(1) *Liapunov matrix function (LMF) of the $S(\psi)$ type* if
  (a) the MF $U(\tau, x(\tau))$ is positive definite and decreases on $N_\tau^+ \times \mathcal{N}$;
  (b) the MF $\Delta U(\tau, x(\tau))$ is nonpositive on $N_\tau^+ \times \mathcal{N}$, for any $(P, S) \in \mathcal{P} \times \mathcal{S}$ and $\Delta U(\tau, 0) = 0$ for all $\tau \in N_\tau^+$;
(2) *an LMF of the $AS(\psi)$ type* if
  (a) the MF $U(\tau, x(\tau))$ is positive definite and decreases on $N_\tau^+ \times \mathcal{N}$;
  (b) the MF $\Delta U(\tau, x(\tau))$ is positive definite on $N^+ \times \mathcal{N}$, for any $(P, S) \in \mathcal{P} \times \mathcal{S}$ and $\Delta U(\tau, 0) = 0$ for all $\tau \in N_\tau^+$;
(3) *an LMF of the $NS(\psi)$ type* if
  (a) the MF $U(\tau, x(\tau))$ is positive and bounded in the domain $v(\tau, x(\tau), \psi) > 0$;
  (b) the MF $\Delta U(\tau, x(\tau))$ for all $\tau \in N_\tau^+$ is positive definite in the domain $v(\tau, x(\tau), \psi) > 0$ for at least one pair $(P, S) \in \mathcal{P} \times \mathcal{S}$ and $\Delta U(\tau, 0) = 0$ for all $\tau \in N_\tau^+$.

Theorems on the stability of discrete-time systems on the basis of the matrix-valued function (3.3.1) and its first difference (3.3.4) – (3.3.5) are generalized in the following form.

Theorem 3.3.1 *For the state $x(\tau) = 0$ of system (3.2.5) to be stable (uniformly) on $\mathcal{P} \times \mathcal{S}$ it is sufficient that matrix-valued function $U\colon\ N_\tau^+ \times \mathcal{N} \to R^{m\times m}$ of the $S(\psi)$ type to exist for any natural m.*

Proof When the conditions of Theorem 3.3.1 are satisfied, so are the inequalities

$$v(\tau, x(\tau), \psi) > 0 \quad \text{and} \quad \Delta v(\tau, x(\tau), \psi) \le 0$$

for $x \in B_\eta = \{x \in R^n\colon\ \|x\| < \eta\}$ and for $(P, S) \in \mathcal{P} \times \mathcal{S}$.

Suppose that $\varepsilon \in (0, \eta)$ is given. We denote $l = \min\{v(\tau, x(\tau), \psi)\colon \|x\| = \varepsilon\}$. The quantity l is positive, as the minimum of a positive function on a compact set, that function being continuous in discrete time. Suppose that $G = \{x \in R^n\colon\ v(\tau, x(\tau), \psi) < l/2\}$ and G_0 is a bound subset of G, containing $x = 0$. Clearly, the sets G and G_0 are open. If $x_0 \in G_0$, then

$$\Delta v(\tau, x_0, \psi) \le 0 \quad \text{for any} \quad (P, S) \in \mathcal{P} \times \mathcal{S}$$

and

$$v(\tau + 1,\, x(\tau + 1),\, \psi) \le v(\tau, x_0, \psi) \le \frac{l}{2}$$

and, therefore, $x_0 \in G$. Since $v(\tau, x(\tau), \psi)$ is a continuous function, there exists a $\delta > 0$ such that $B_\delta \subset G_0$. Therefore, if $x_0 \in B_\delta$, then $x_0 \in G$ and $x(\tau; t_0, x_0) \in G_0 \subset B_\varepsilon$, for any $(P, S) \in \mathcal{P} \times \mathcal{S}$. The theorem has thus been proved.

Theorem 3.3.2 *For the state $x(\tau) = 0$ of system (3.2.5) to be asymptotically stable (uniformly) on $\mathcal{P} \times \mathcal{S}$ it is sufficient that a matrix-valued function $U\colon\ N_\tau^+ \times \mathcal{N} \to R^{s\times s}$ of the $AS(\psi)$ type to exist for any natural m.*

Proof Since all the conditions of Theorem 3.3.1 are satisfied, the state $x(\tau) = 0$ is stable for any $(P, S) \in \mathcal{P} \times \mathcal{S}$. From the fact that the matrix-valued function $U(\tau, x(\tau))$ is a function of class $AS(\psi)$ it follows that there exist functions $\chi_k \in K$ for every $k = 1, 2, \dots, m$, and matrices $G(P, S) \in \mathcal{P} \times \mathcal{S}$ $(\lambda_{\max}(G) < 0)$ such that

$$\Delta v(\tau, x(\tau), \psi) \le \chi^{\mathrm{T}}(\|x(\tau)\|) G \chi(\|x(\tau)\|),$$

or

$$\Delta v(\tau, x(\tau), \psi) \le -\lambda_{\min}(G) \chi^{\mathrm{T}}(\|x(\tau)\|) \chi(\|x(\tau)\|),$$

for all $(P, S) \in \mathcal{P} \times \mathcal{S}$, where $\chi(\|x(\tau)\|) = (\chi_1(\|x(\tau)\|),\ \dots,\ \chi_m(\|x(\tau)\|))^{\mathrm{T}}$. Since $\chi_k \in K$, there exists a function $\Phi \in K$ such that

$$\Phi(\|x(\tau)\|) \ge \chi^{\mathrm{T}}(\|x(\tau)\|) \chi(\|x(\tau)\|)$$

and, therefore,

$$\Delta v(\tau, x(\tau), \psi) \le -\lambda_{\max}(G) \Phi(\|x(\tau)\|), \quad x(\tau) = 0,$$

for all $x(\tau) \in B_\varepsilon$ and all $(P, S) \in \mathcal{P} \times \mathcal{S}$. Therefore, the equilibrium state $x(\tau) = 0$ is uniformly asymptotically stable on $\mathcal{P} \times \mathcal{S}$.

Theorem 3.3.3 *For the state $x(\tau) = 0$ of system (3.2.5) to be unstable on $\mathcal{P} \times \mathcal{S}$ it is sufficient for a matrix-valued function $U\colon\ N_\tau^+ \times \mathcal{N} \to R^{m\times m}$ of the $NS(\psi)$ type to exist for any natural m.*

Proof Suppose that a matrix-valued function of the $NS(\psi)$ type exists for system (3.2.5). We shall prove that then the zero solution of system (3.2.5) is unstable. We conduct the proof by contradiction. Suppose that the state $x(\tau) = 0$ of system (3.2.5) is stable. We choose the quantity $\varepsilon > 0$ such that for the function

$$\Delta v(\tau, x(\tau), \psi) = \psi^{\mathrm{T}} \Delta U(\tau, x(\tau)) \psi$$

with $x \in B_\varepsilon = \{x \in R^n\colon\ \|x\| < \varepsilon\} \setminus \{0\}$ in the domain $v(\tau, x(\tau), \psi) > 0$ the inequality

$$\Delta v(\tau, x(\tau), \psi) \ge \alpha^{\mathrm{T}}(\|x(\tau)\|) H \alpha(\|x(\tau)\|), \tag{3.3.6}$$

is satisfied; here $\alpha^{\mathrm{T}}(\|x(\tau)\|) = (\alpha_1(\|x(\tau)\|), \dots, \alpha_m(\|x(\tau)\|))$ and $H(P, S) \ge H$, H is $m \times m$ matrix and $\lambda_{\min}(H) \ge 0$ for at least one pair $(P, S) \in \mathcal{P} \times \mathcal{S}$. Here $\alpha_k \in K$ for every $k = 1, 2, \dots, m$.

Since the functions $\alpha_k \in K$, there exists a function $\kappa \in K$:

$$\kappa(\|x(\tau)\|) \le \alpha^{\mathrm{T}}(\|x(\tau)\|) \alpha(\|x(\tau)\|)$$

such that

$$\Delta v(\tau, x(\tau), \psi) \ge \lambda_{\min}(H) \kappa(\|x(\tau)\|), \tag{3.3.7}$$

in the domain $v(\tau, x(\tau), \psi) > 0$.

According to the assumption that the state $x(\tau) = 0$ of system (3.2.5) is stable on $\mathcal{P} \times \mathcal{S}$ for any $\varepsilon > 0$ we can prove that $\delta = \delta(\varepsilon) > 0$ so that $x_0 \in B_\delta = \{x \in R^n \colon \|x\| < \delta\}$, then $x(\tau; t_0, x_0) \in B_\varepsilon$ for all $\tau \in N_\tau^+$ and all $(P, S) \in \mathcal{P} \times \mathcal{S}$. For a matrix-valued function of the $NS(\psi)$ type there exists a $x_0 \in B_\delta$ such that $v(\tau, x_0, \psi) > 0$. Then $x(\tau; t_0, x_0)$ is bounded and remains in B_ε for all $\tau \in N_\tau^+$, then $x(\tau; t_0, x_0)$ tends to the state $(x = 0) \in \{x \in R^n \colon \Delta v(\tau, x(\tau), \psi) = 0\} \cup B_\varepsilon$. It thus follows that

$$v(\tau, x(\tau; t_0, x_0), \psi) \to v(t_0, 0, \psi) = 0, \quad \text{for} \quad (\psi \neq 0).$$

On the other hand, according to (3.3.7)

$$\Delta v(\tau, x(\tau), \psi) \geq 0$$

and

$$v(\tau, x(\tau), \psi) \geq 0.$$

Therefore,

$$0 < v(t_0, x_0, \psi) \leq \cdots \leq v(\tau - 1,\, x(\tau - 1),\, \psi) \leq v(\tau, x(\tau), \psi).$$

The contradiction that has been obtained shows that the statement of Theorem 3.3.3 holds, as was to be proved.

3.4 Tests for Stability Analysis

The Problem $\mathrm{D_A}$ is a natural generalization of the Problem $\mathrm{C_A}$ for the class of the multidimensional discrete-time systems.

3.4.1 The Problem $\mathrm{D_A}$ Assume that the time-discrete system D is obtained in result of decomposition of the isolated subsystems (3.2.3) and the corresponding interacting subsystems (3.2.2). It is required to obtain stability conditions of different types for the state $x = 0$ of system (3.2.5) for the known dynamical properties of subsystems (3.2.3) and qualitative properties of the interconnection functions between the subsystems.

Let us introduce some assumptions about the components v_{ij} of the matrix-valued function $U(\tau, x(\tau))$ and dynamical properties of the subsystem D_i.

Assumption 3.4.1 There exist

(1) neighborhoods $\mathcal{N}_i \subseteq R^{n_i}$, $i = 1, 2, \dots, m$, of equilibrium states $x_i(\tau) = 0$ invariant in discrete time;
(2) functions $\varphi_{ik} \colon \mathcal{N}_i \to R_+$, $i = 1, 2, \dots, m$, $k = 1, 2$, $\varphi_{ik} \in K(KR)$;
(3) constants $\underline{\alpha}_{ij}, \overline{\alpha}_{ij}$, $i, j = 1, 2, \dots, m$, and a matrix-valued function $U(\tau, x(\tau))$ with the elements

$$v_{ii} = v_{ii}(\tau, x_i(\tau)), \quad v_{ij} = v_{ij}(\tau, x_i(\tau), x_j(\tau)), \quad i \neq j,$$
$$v_{ij} = v_{ji}, \quad v_{ii}(\tau, 0) = v_{ij}(\tau, 0, 0) = 0,$$

satisfying the estimates

(a) $\underline{\alpha}_{ii}\varphi_{i1}^2(\|x_i(\tau)\|) \le v_{ii}(\tau, x_i(\tau)) \le \overline{\alpha}_{ii}\varphi_{i2}^2(\|x_i(\tau)\|)$ for all $(\tau, x_i(\tau)) \in N_\tau^+ \times \mathcal{N}_i$, $i = 1, 2, \dots, m$;

(b) $\underline{\alpha}_{ij}\varphi_{i1}(\|x_i(\tau)\|)\varphi_{j1}(\|x_j(\tau)\|) \le v_{ij}(\tau, x_i(\tau), x_j(\tau)) \le \overline{\alpha}_{ij}\varphi_{i2}(\|x_i(\tau)\|)\varphi_{j2}(\|x_j(\tau)\|)$ for all $(\tau, x_i(\tau), x_j(\tau)) \in N_\tau^+ \times \mathcal{N}_i \times \mathcal{N}_j$, and for all $i \ne j$.

Here, $v_{ii}(\tau, x_i(\tau))$ correspond to subsystem (3.2.3), and $v_{ij}(\tau, x_i(\tau), x_j(\tau))$ take into account the relations $h_i(\tau, x(\tau), p_i)$ between the equations in (3.2.4).

Proposition 3.4.1 *Under the conditions of Assumption 3.4.1, function (3.3.2) satisfies the following estimate:*

$$u_1^{\mathrm{T}} H^{\mathrm{T}} A H u_1 \le v(\tau, x(\tau), \psi) \le u_2^{\mathrm{T}} H^{\mathrm{T}} B H u_2 \tag{3.4.1}$$

for all $(\tau, x_i(\tau), x_j(\tau)) \in N_\tau^+ \times \mathcal{N}_i \times \mathcal{N}_j$, $i, j = 1, 2, \dots, m$, *where*

$$\begin{gathered}
u_k^{\mathrm{T}} = (\varphi_{1k}(\|x_1(\tau)\|),\ \varphi_{2k}(\|x_2(\tau)\|),\ \dots,\ \varphi_{mk}(\|x_m(\tau)\|)), \quad k = 1, 2, \\
H = \operatorname{diag}(\psi_1, \psi_2, \dots, \psi_m), \quad A = [\underline{\alpha}_{ij}], \quad \underline{\alpha}_{ij} = \underline{\alpha}_{ji}, \\
B = [\overline{\alpha}_{ij}], \quad \overline{\alpha}_{ij} = \overline{\alpha}_{ji}, \quad i, j = 1, 2, \dots, m, \\
\psi^{\mathrm{T}} = (\psi_1, \psi_2, \dots, \psi_m).
\end{gathered}$$

Assumption 3.4.2 There exist

(1) neighborhoods $\mathcal{N} \subseteq R^{n_i}$ of the states $x_i(\tau) = 0$, $i = 1, 2, \dots, m$, invariant in discrete time;

(2) functions $\varphi_i \colon \mathcal{N} \to R_+$, $i = 1, 2, \dots, m$, $\varphi_i \in K(KR)$ and v_{ij}, $i, j = 1, 2, \dots, m$, defined in Assumption 3.4.1 and such that

 (a) the functions $v_{ij}(\tau, x(\tau))$ are defined either on $N_\tau^+ \times \mathcal{N}_{i0}$ or $N_\tau^+ \times R^{n_i}$;

 (b) the functions $v_{ij}(\tau, x_i(\tau), x_j(\tau))$ are defined either on $N_\tau^+ \times \mathcal{N}_{i0} \times \mathcal{N}_{j0}$ or $N_\tau^+ \times R^{n_i} \times R^{n_j}$, for all $i \neq j$, $\mathcal{N}_{i0} = \{x_i(\tau)\colon x_i(\tau) \in \mathcal{N}_i,\ x_i(\tau) \neq 0\}$;

(3) constants α_{1i}, $\alpha_{2i}(P, S)$, $\alpha_{ij}(P, S)$ $(i \neq j)$, $i, j = 1, 2, \dots, m$, such that the following inequalities are satisfied by virtue of multidimensional discrete systems (3.2.5) and subsystems (3.2.5):

 (a) $\psi_i^2\{v_{ii}(\tau, x_i^i(\tau+1)) - v_{ii}(\tau, x_i^i(\tau))\} \le \alpha_{1i}\varphi_i^2(\|x_i(\tau)\|)$ for all $x_i(\tau) \in \mathcal{N}_{i0}, \quad i = 1, 2, \dots, m$;

 (b)
$$\sum_{i=1}^{m} \psi_i^2\{v_{ii}(\tau, x_i(\tau+1)) - v_{ii}(\tau, x_i^i(\tau+1)) + v_{ii}(\tau, x_i^i(\tau))$$
$$- v_{ii}(\tau, x_i(\tau))\} + 2\sum_{i=1}^{m}\sum_{j=1}^{m} \psi_i\psi_j\{v_{ij}(\tau, x_i(\tau+1), x_j(\tau+1))$$
$$- v_{ij}(\tau, x_i(\tau), x_j(\tau))\} \le \sum_{i=1}^{m} \alpha_{2i}(P, S)\varphi_i^2(\|x_i(\tau)\|)$$
$$+ 2\sum_{i=1}^{m}\sum_{j=1}^{m} \alpha_{ij}(P, S)\varphi_i(\|x_i(\tau)\|)\varphi_j(\|x_j(\tau)\|),$$
 for all $(\tau, x_i, x_j) \in N_\tau^+ \times \mathcal{N}_{i0} \times \mathcal{N}_{j0}$ for all $(P, S) \in \mathcal{P} \times \mathcal{S}$.

Remark 3.4.1 Here, $x_i^i(\cdot)$ means that the difference is taken with regard for subsystems (3.2.5)

Proposition 3.4.2 *If all conditions of Assumption 3.4.2. are satisfied, the following estimate holds:*

$$v(\tau,\, x(\tau+1),\, \psi) - v(\tau, x(\tau), \psi) \le u^{\mathrm{T}} C(P, S) u \tag{3.4.2}$$

for all $(\tau, x) \in N_\tau^+ \times \mathcal{N}_{i0}$, *and for all* $(P, S) \in \mathcal{P} \times \mathcal{S}$, *where*

$$u^{\mathrm{T}} = (\varphi_1(\|x_1\|),\, \varphi_2(\|x_2\|),\, \dots,\, \varphi_{mk}(\|x_m\|)),$$
$$C(P, S) = [C_{ij}(P, S)], \quad i, j = 1, 2, \dots, m,$$
$$C_{ij}(P, S) = \alpha_{ij}(P, S), \quad \textit{for all} \quad i \neq j,$$
$$C_{ii}(P, S) = \alpha_{1i} + \alpha_{2i}(P, S), \quad i = 1, 2, \dots, m.$$

Theorem 3.4.1 *Suppose that the multidimensional discrete system (3.2.5) is such that all conditions of Assumptions 3.4.1 and 3.4.2 are satisfied and, in addition,*

(1) *the matrix A is positive definite;*
(2) *there exists a negative semi-definite matrix $G \in R^{m\times m}$ such that the matrix $C(P,S)$ satisfies the estimate*

$$\frac{1}{2}(C(P,S)+C^{\mathrm{T}}(P,S)) \le G \quad \textit{for all} \quad (P,S)\in\mathcal{P}\times\mathcal{S}.$$

Then the equilibrium state $x(\tau)=0$ of system (3.2.5) is uniformly stable on $\mathcal{P}\times\mathcal{S}$.

Proof If the conditions of Assumption 3.4.1, Proposition 3.4.1, and condition (1) of Theorem 3.4.1 are satisfied, then function (3.3.2) is positive definite on $N_\tau^+\times\mathcal{N}_{i0}$. The conditions of Assumption 3.4.2, Proposition 3.4.2, and condition (2) of Theorem 3.4.1 imply that $v(\tau,x(\tau),\psi)\ge v(\tau,\,x(\tau+1),\,\psi)$ for any $(P,S)\in\mathcal{P}\times\mathcal{S}$. In this case, for every pair $(P,S)\in\mathcal{P}\times\mathcal{S}$, the conditions which are sufficient for the stability of the zero solution of the multidimensional discrete system (3.2.5) on $\mathcal{P}\times\mathcal{S}$ are satisfied (see Section 1.3).

Theorem 3.4.2 *Let the multidimensional discrete system (3.2.5) be such that the conditions of Assumptions 3.4.1 and 3.4.2 are satisfied and, in addition,*

(1) *the matrices A and B are positive definite;*
(2) *there exists a negative definite matrix $G_1 \in R^{m\times m}$ such that the matrix $C(P,S)$ satisfies the estimate*

$$\frac{1}{2}(C(P,S)+C^{\mathrm{T}}(P,S)) \le G_1 \quad \textit{for all} \quad (P,S)\in\mathcal{P}\times\mathcal{S}.$$

Then the equilibrium state $x(\tau)=0$ of system (3.2.5) is uniformly asymptotically stable on $\mathcal{P}\times\mathcal{S}$.

The proof of Theorem 3.4.2 is similar to that of Theorem 3.4.1.

Example 3.4.1 Consider a linear large scale discrete-time system decomposed into two subsystems

$$x_i(\tau+1)=A_{ii}x_i(\tau)+A_{ij}s_{ij}x_j(\tau),\quad i\ne j,\quad i,j=1,2, \tag{3.4.3}$$

where A_{ij} are constant matrices of the corresponding order,

$$S=\{S\colon\ S=\mathrm{diag}\,\{S_1,S_2\},\ \ S_i=[s_{ij}J_i],\ \ 0\le s_{ij}\le 1,\ \ i\ne j,\ \ i,j=1,2\},$$
$$J_i=\mathrm{diag}\,(1,1)\in R^{n_i},\quad x_i\in R^{n_i},\quad i=1,2,\quad \text{and}\quad x=(x_1^{\mathrm{T}},x_2^{\mathrm{T}})^{\mathrm{T}}\in R^n.$$

For system (3.4.3) we construct a matrix-valued function $U(x)$ with the elements

$$v_{11}=x_1^{\mathrm{T}}B_1x_1,\quad v_{22}=x_2^{\mathrm{T}}B_2x_2,\quad v_{12}=v_{21}=x_1^{\mathrm{T}}B_3x_2, \tag{3.4.4}$$

where B_1 and B_2 are symmetric positive definite matrices, and B_3 is a constant matrix.

It is easy to verify that functions (3.4.4) satisfy the following estimates:

$$\lambda_m(B_i)\|x_i\|^2\le\nu_{ii}(x_i(\tau))\le\lambda_M(B_i)\|x_i\|^2,\quad i=1,2,$$
$$-\lambda_M^{1/2}(B_3B_3^{\mathrm{T}})\|x_1\|\,\|x_2\|\le v_{12}(x_1(\tau),x_2(\tau))\le\lambda_M^{1/2}(B_3B_3^{\mathrm{T}})\|x_1\|\,\|x_2\|,$$

where $\lambda_m(B_i)$ are, respectively, the minimum and maximum eigenvalues of the matrices B_i, $i = 1, 2$, and $\lambda_M^{1/2}(B_3 B_3^{\mathrm{T}})$ is the norm of the matrix B_3.

In this case, the matrices A and B from estimate (3.4.1) have the form

$$A = \begin{pmatrix} \lambda_m(B_1) & -\lambda_M^{1/2}(B_3 B_3^{\mathrm{T}}) \\ -\lambda_M^{1/2}(B_3 B_3^{\mathrm{T}}) & \lambda_m(B_2) \end{pmatrix},$$

$$B = \begin{pmatrix} \lambda_M(B_1) & \lambda_M^{1/2}(B_3 B_3^{\mathrm{T}}) \\ \lambda_M^{1/2}(B_3 B_3^{\mathrm{T}}) & \lambda_M(B_2) \end{pmatrix}.$$

In order that these matrices be positive definite, it is sufficient that the condition

$$\lambda_m(B_1)\lambda_m(B_2) > \lambda_M^{1/2}(B_3 B_3^{\mathrm{T}}) \tag{3.4.5}$$

be satisfied. Since the matrices B_1 and B_2 are positive definite, we have $\lambda_M(B_i) > \lambda_m(B_i) > 0$, $i = 1, 2$.

Let $\psi^{\mathrm{T}} = (1, 1)$. Then for the matrix-valued function $U(x)$ indicated above, the elements of the matrix $C(S)$ have the form

$$C_{11}(S) = \lambda_M(C_1) + \lambda_M(C_3(S)),$$
$$C_{22}(S) = \lambda_M(C_2) + \lambda_M(C_4(S)),$$
$$C_{12}(S) = \lambda_M^{1/2}(C_5(S) C_5^{\mathrm{T}}(S)),$$

where

$$
\begin{gathered}
C_1 = A_{11}^{\mathrm T} B_1 A_{11} - B_1, \quad C_2 = A_{22}^{\mathrm T} B_2 A_{22} - B_2,\\
C_3(S) = (A_{21}s_{21})^{\mathrm T} B_2 A_{21}s_{21} + 2A_{11}^{\mathrm T} B_3 A_{21}s_{21} - B_1,\\
C_4(S) = (A_{12}s_{12})^{\mathrm T} B_1 A_{12}s_{12} + 2(A_{12}s_{12})^{\mathrm T} B_3 A_{22} - B_2,\\
C_5(S) = A_{11}^{\mathrm T} B_3 A_{22} + (A_{21}s_{21})^{\mathrm T} B_3^{\mathrm T} A_{12}s_{12} - B_3.
\end{gathered}
$$

Thus, if inequality (3.4.5) is satisfied and there exists a negative semi-definite matrix G such that $\frac{1}{2}(C(S) + C^{\mathrm T}(S)) \le G$ for all $S \in \mathcal{S}$, then all conditions of Theorem 3.4.1 (3.4.2) are satisfied and the zero solution of the large scale discrete-time system (3.4.3) is stable (asymptotically stable).

3.4.2 The Problem $\mathbf{D}_{\mathrm B}$ In this section we propose the solution of the problem of nonlinear dynamics of discrete-time systems.

Problem $D_{\mathrm B}$ Assume that the discrete-time system D is obtained in result of composition of the interconnected subsystems (3.2.2). It is required to determine stability conditions of different types for the equilibrium state $x = 0$ of system (3.2.5) in terms of the dynamical properties of the interconnected subsystems (3.2.4) without additional information on the dynamical properties of the isolated subsystems (3.2.3).

Assumption 3.4.3 Assume that

(1) conditions (1) and (2) of Assumption 3.4.2 are satisfied;
(2) there exist constants $\rho_{ii}(P,S)$ and $\rho_{ij}(P,S)$, $i = 1,2,\dots,m$, $j = 2,3,\dots,m$, $i < j$, such that, by virtue of the multidimensional discrete system (3.2.5), the following inequality holds:

$$
\begin{gathered}
\sum_{i=1}^{m} \psi_i^2 \{\nu_{ii}(\tau, x_i(\tau+1)) - \nu_{ii}(\tau, x_i(\tau))\}\\
+ 2\sum_{i=1}^{m} \sum_{\substack{j=2\\ j>i}}^{m} \psi_i\psi_j \{\nu_{ij}(\tau, x_i(\tau+1), x_j(\tau+1)) - \nu_{ij}(\tau, x_i(\tau), x_j(\tau+1))\}\\
\le \sum_{i=1}^{m} \rho_{ii}(P,S)\varphi_i^2(\|x_i(\tau)\|) + 2\sum_{i=1}^{m} \sum_{\substack{j=2\\ j>i}}^{m} \rho_{ij}(P,S)\varphi_i(\|x_i(\tau)\|)\varphi_j(\|x_j(\tau)\|)\\
\text{for all} \quad (\tau, x_i, x_j) \in N_\tau^+ \times \mathcal{N}_{i0} \times \mathcal{N}_{j0}, \quad \text{for all} \quad (P,S) \in \mathcal{P} \times \mathcal{S}.
\end{gathered}
$$

Proposition 3.4.3 *If the conditions of Assumption 3.4.3 are satisfied, then the first difference of the function $v(\tau, x(\tau))$ satisfies the following estimate:*

$$
\begin{gathered}
v(\tau, x(\tau+1), \psi) - v(\tau, x(\tau), \psi) \le u^{\mathrm T} G(P,S) u\\
\textit{for all} \quad (\tau, x) \in N_\tau^+ \times \mathcal{N}_0, \quad \textit{and for all} \quad (P,S) \in \mathcal{P} \times \mathcal{S},
\end{gathered} \tag{3.4.6}
$$

where

$$
\begin{gathered}
u^{\mathrm T} = (\varphi_1(\|x_1\|), \varphi_2(\|x_2\|), \dots, \varphi_m(\|x_m(\tau)\|)),\\
C(P,S) = [\rho_{ij}(P,S)], \quad i,j = 1,2,\dots,m, \quad \rho_{ij}(P,S) = \rho_{ji}(P,S).
\end{gathered}
$$

Theorem 3.4.3 *Let the multidimensional discrete system (3.2.5) be such that the conditions of Assumptions 3.4.1 and 3.4.3 are satisfied and, in addition,*

(1) *the matrix A is positive definite;*
(2) *there exists a negative semi-definite matrix $\widetilde{G} \in R^{m\times m}$ such that the matrix $C(P,S)$ satisfies the estimate $\frac{1}{2}(C(P,S)+C^{\mathrm{T}}(S)) \le \widetilde{G}$ for all $(P,S) \in \mathcal{P}\times\mathcal{S}$.*

Then the equilibrium state $x(\tau)=0$ of the multidimensional discrete system (3.2.5) is uniformly stable on $\mathcal{P}\times\mathcal{S}$.

The proof of Theorem 3.4.3 is similar to that of Theorem 3.4.1.

Theorem 3.4.4 *Let the multidimensional discrete system (3.2.5) be such that the conditions of Assumptions 3.4.1 and 3.4.3 are satisfied and, in addition,*

(1) *the matrices A and B are positive definite;*
(2) *there exists a negative definite matrix $\widetilde{G}_1 \in R^{m\times m}$ such that the matrix $C(P,S)$ satisfies the estimate $\frac{1}{2}(C(P,S)+C^{\mathrm{T}}(P,S)) \le \widetilde{G}_1$ for all $(P,S) \in \mathcal{P}\times\mathcal{S}$.*

Then the equilibrium state $x(\tau)=0$ of the multidimensional discrete system (3.2.5) is uniformly asymptotically stable on $\mathcal{P}\times\mathcal{S}$.

The proof of Theorem 3.4.4 is similar to that of Theorem 3.4.2.

Example 3.4.2 Consider the linear multidimensional discrete system (3.4.3). Assume that, for system (3.4.3), the matrix function with elements (3.4.4) is constructed and $\psi^{\mathrm{T}}=(1,1)$. Then the elements of the matrix $C(S)$ have the form

$$\rho_{11}(S)=\lambda_m(C_1(S)),\quad \rho_{22}(S)=\lambda_M(C_2(S)),$$
$$\rho_{11}(S)=\lambda_M^{1/2}(C_3(S)C_3^{\mathrm{T}}(S)),$$

where

$$C_1(S)=A_{11}^{\mathrm{T}}B_1A_{11}+(A_{21}s_{21})^{\mathrm{T}}B_2A_{21}s_{21}+2A_{11}^{\mathrm{T}}B_3A_{21}s_{21}-2B_1,$$
$$C_2(S)=A_{22}^{\mathrm{T}}B_2A_{22}+(A_{12}s_{12})^{\mathrm{T}}B_1A_{12}s_{12}+2(A_{12}s_{12})^{\mathrm{T}}B_3A_{22}-2B_2,$$
$$C_3(S)=A_{11}^{\mathrm{T}}B_3A_{22}+(A_{21}s_{21})^{\mathrm{T}}B_3^{\mathrm{T}}A_{12}s_{12}-B_3.$$

Thus, if, for system (3.4.3), the matrix function with elements (3.4.4) is constructed, inequality (3.4.5) is satisfied, and, in addition, there exists a negative semi-definite (negative definite) matrix $\widetilde{G}$ such that $\frac{1}{2}(C(S)+C^{\mathrm{T}}(S)) \le \widetilde{G}$ for all $S\in\mathcal{S}$, then the conditions of Theorem 3.4.3 (3.4.4) are satisfied and the zero solution of the multidimensional discrete system (3.4.3) is stable (asymptotically stable) on $\mathcal{P}\times\mathcal{S}$.

Example 3.4.3 Consider a discrete system of the fourth order which consists of two subsystems of the second order described by the following systems of equations

(3.4.7) $$x_i(\tau+1)=-0.1x_i+0.2x_j+0.5s_{i1}(\tau)x_i+0.3s_{i2}(\tau)x_j,\quad i=1,2,$$

where $x_i = (x_{i1}, x_{i2})^{\mathrm{T}} \in R^2$, $s_{ij}(\tau) \in [0,1]$, $i,j = 1,2$, for all $\tau \in N_\tau^+$, and the structure matrix $S_i(\tau)$ has the form

$$S_i(\tau) = \begin{pmatrix} 1 & 0 & s_{i1}(\tau) & 0 & s_{i2}(\tau) & 0 \\ 0 & 1 & 0 & s_{i1}(\tau) & 0 & s_{i2}(\tau) \end{pmatrix}, \qquad i = 1,2.$$

The structure set of system (3.4.7) is determined as follows

$$S = \Big\{ S(\tau)\colon \; S(\tau) = \begin{pmatrix} S_1(\tau) & 0 \\ 0 & S_1(\tau) \end{pmatrix}, \quad S_i(\tau) = (J_2,\, s_{i1}(\tau)J_2,\, s_{i2}(\tau)J_2),$$
$$s_{ij}(\tau) \in [0,1] \quad \text{for all} \quad \tau \in N_\tau^+, \quad i,j = 1,2 \Big\}.$$

For system (3.4.7), we construct a matrix function $U(x)$ with the elements

$$v_{ii}(x_i) = x_i^2, \quad i = 1,2, \quad v_{12}(x_1,x_2) = 0.5\,, x_1x_2,$$

which satisfy the estimates

$$v_{ii}(x_i) \geq \|x_i\|^2, \quad i = 1,2.$$

The matrices

$$A = \begin{pmatrix} 1 & -0.5 \\ -0.5 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$$

are positive definite.

Let $\psi^{\mathrm{T}} = (1,1)$. Then for the matrix-valued function $U(x)$ introduced above, the elements of the matrix $C(S)$ have the form

$$\rho_{11}(S) = 0.03 + 0.09s_{21} + 0.15s_{11}s_{21} + 0.25s_{11}^2 + 0.09s_{21}^2 - 1 \le -0.39,$$
$$\rho_{22}(S) = 0.03 + 0.09s_{12} + 0.15s_{12}s_{22} + 0.09s_{12}^2 + 0.25s_{22}^2 - 1 \le -0.39,$$
$$\rho_{12}(S) = 0.01 + 0.05s_{11} + 0.03s_{12} + 0.03s_{21} + 0.05s_{22} + 0.15s_{11}s_{12} + 0.25s_{11}s_{22} + 0.09s_{12}s_{22} + 0.15s_{21}s_{22} - 0.5 \le 0.31.$$

It is easy to verify that the matrix

$$\frac{1}{2}(C(S) + C^{\mathrm{T}}(S)) \le \widetilde{G}_1 = \begin{pmatrix} -0.39 & 0.31 \\ 0.31 & -0.39 \end{pmatrix}$$

is negative definite.

Since the conditions of Theorem 3.4.4 are satisfied, the zero solution of system (3.4.7) is asymptotically stable on $\mathcal{S}$.

In conclusion, note that the construction of a scalar Liapunov function for the multidimensional discrete system (3.2.5) is an important and difficult problem in the theory of discrete systems. The application of the matrix-valued function $U(\tau, x(\tau))$ to the construction of the scalar function $v(\tau, x(\tau), \psi)$ simplifies the problem to a certain extent due to weakened requirements on the components v_{ij}, $i, j = 1, 2, \dots, m$. This, in turn, enables one to more adequately take into account the correlation between the independent subsystems (3.2.3).

We emphasize that the suggested method for the analysis of the stability of the multidimensional discrete system (3.2.5) is distinguished by its simplicity and generality, and all established sufficient conditions of stability and asymptotic stability are represented in terms of the property of special matrices to have a fixed sign.

3.5 Certain Trends of Generalizations and Applications

In this section we presents the results of estimating the robust stability bounds for discrete-time system in terms of three approaches based on scalar, vector and hierarchical Liapunov functions. It is shown that the hierarchical Liapunov function allows one to obtain the most wide bounds for the uncertain matrix in the investigation of discrete system.

We consider an uncertain discrete-time system

$$x(\tau+1) = A\,x(\tau) + f(x(\tau), \alpha), \tag{3.5.1}$$

where $x \in R^n$, $\tau \in \mathcal{T}_\tau = \{t_0 + k,\ k = 0, 1, 2, \dots\}$, $t_0 \in R$, A is a constant $n \times n$ matrix, $f\colon R^n \times S \to R^n$, $S \subseteq R^d$ is a compact set. Under specific conditions (we don't cite them here) dynamics of the system (3.5.1) is topologically equivalent to dynamics of the system

$$x(\tau+1) = (A + E)\,x(\tau), \tag{3.5.2}$$

where A is the same matrix, as in system (3.5.1), E is an uncertain $n \times n$ matrix, about which it is known that it lies in some compact set $S_1 \subset R^{n\times n}$. Further we will investigate the system (3.5.2).

Our purpose is to compare the results of estimating the robust bounds of discrete system obtained in terms of three approaches involving scalar, vector and hierarchical Liapunov function. In the section it is shown that the hierarchical Liapunov function provides more wide bounds for estimation of the uncertain matrix.

3.5.1 Scalar approach We assume that for the matrix A the condition $|\sigma_i(A)| < 1$ is realized for all $i = 1, 2, \dots, n$. In this case the Liapunov equation

$$A^{\mathrm{T}} P A - P = -G \tag{3.5.3}$$

has a unique solution $P \in R^{n\times n}$ for arbitrary symmetric and positive definite matrix $G \in R^{n\times n}$. In this case the matrix P is symmetric and positive definite. According to the results of Sezer and Siljak [1], we apply the function

$$v(x) = (x^{\mathrm{T}} P x)^{1/2}. \tag{3.5.4}$$

in robustness analysis of the system (3.5.2). Let us denote by $\sigma_m(P)$ and $\sigma_M(P)$ the maximum and minimum eigenvalues of the matrix P.

Following Sezer and Siljak [1] we have the following.

Theorem 3.5.1 *Let the nominal system*

$$x(\tau + 1) = A x(\tau) \tag{3.5.5}$$

be asymptotically stable. If

$$\|E\| < \mu(G), \tag{3.5.6}$$

where

$$\mu(G) = \frac{\sigma_m(G)}{\sigma_M^{1/2}(P - G)\sigma_M^{1/2}(P) + \sigma_M(P)},$$

then the uncertain system (3.5.2) is asymptotically stable.

Here $\|E\| = \sup\limits_{\|x\|\le 1} \|Ex\|$, $\|x\| = (x^{\mathrm{T}} x)^{1/2}$ is the Euclidean norm of vector x.

It is known (see Sezer and Šiljak [1]), that $\mu(G)$ takes the largest value, if $G = I$. The expression (3.5.6) is a robust bound for the system (3.5.2), obtained in the framework of scalar approach.

3.5.2 Vector approach We decompose system (3.5.2) into two interconnected subsystems

(3.5.7)
$$\widehat{S}_i\colon \quad x_i(\tau + 1) = (A_i + E_i) x_i(\tau) + (B_j + U_j) x_j(\tau), \quad i, j = 1, 2, \quad i \ne j.$$

Here $x_i \in R^{n_i}$, A_i and B_i are submatrices of the known matrix

$$A = \begin{pmatrix} A_1 & B_1 \\ B_2 & A_2 \end{pmatrix}, \tag{3.5.8}$$

E_i and U_i are submatrices of the uncertain matrix

$$E = \begin{pmatrix} E_1 & U_1 \\ U_2 & E_2 \end{pmatrix}, \tag{3.5.9}$$

where $B_1, U_1 \in R^{n_1 \times n_2}$, $B_2, U_2 \in R^{n_2 \times n_1}$, $A_i, E_i \in R^{n_i \times n_i}$, $i = 1, 2$.

Assumption 3.5.1 We assume that:

(1) the nominal subsystems

$$x_i(\tau + 1) = A_i x_i(\tau) \tag{3.5.10}$$

are asymptotically stable, i.e. there exist unique symmetric and positive definite matrices $P_i \in R^{n_i \times n_i}$, which satisfy the Liapunov matrix equations

$$A_i^{\mathrm{T}} P_i A_i - P_i = -G_i, \quad i = 1, 2, \tag{3.5.11}$$

where G_i are arbitrary symmetric and positive definite matrices;

(2) there exists a constant $\gamma \in (0, 1)$ such that

$$\|B_1\| \, \|B_2\| < \gamma^2 \mu_1 \mu_2 \tag{3.5.12}$$

where $\mu_i = (\sigma_M^{1/2}(P_i - I_i)\sigma_M^{1/2}(P_i) + \sigma_M(P_i))^{-1}$, P_i are solutions of the Liapunov matrix equations (3.5.11) for the matrices $G_i = I_{n_i}$, I_{n_i} are $n_i \times n_i$ identity matrices, $i = 1, 2$.

We define the constants

$$a = \sigma_M^{1/2}(P_1)\sigma_M^{1/2}(P_2), \quad b = \sigma_M^{1/2}(P_1)\sigma_M^{1/2}(P_2)\left(\|B_1\| + \|B_2\|\right),$$
$$\mu_i = (\sigma_M^{1/2}(P_i - I_i)\sigma_M^{1/2}(P_i) + \sigma_M(P_i))^{-1}, \quad i = 1,2,$$
$$\alpha_i = \sigma_M^{1/2}(P_i)\mu_i = (\sigma_M^{1/2}(P_i - I_i) + \sigma_M^{1/2}(P_i))^{-1}, \quad i = 1,2,$$
$$c = \gamma^2\alpha_1\alpha_2 - \sigma_M^{1/2}(P_1)\sigma_M^{1/2}(P_2)\|B_1\|\,\|B_2\|,$$
$$\varepsilon = \frac{1}{2a}((b^2 + 4ac)^{1/2} - b),$$

where P_i are solutions of the Liapunov matrix equations (3.5.11) for the matrices $G_i = I_{n_i}$, $i = 1,2$.

Theorem 3.5.2 *Assume that for the uncertain system (3.5.2) the decomposition (3.5.7) – (3.5.9) takes place and all conditions of Assumption 3.1 are satisfied. If the submatrices E_i and U_i satisfy the inequalities*

$$\|E_i\| \le (1-\gamma)\mu_i, \quad \|U_i\| < \varepsilon, \quad i = 1,2, \tag{3.5.13}$$

then the equilibrium $x = 0$ of (3.5.2) is asymptotically stable.

Proof For the nominal subsystems (3.5.10) by (3.5.11) we construct the norm-like functions

$$v_i(x_i) = (x_i^{\mathrm{T}} P_i x_i)^{1/2}, \quad i = 1,2, \tag{3.5.14}$$

and the scalar function

$$v(x) = d_1 v_1(x_1) + d_2 v_2(x_2), \tag{3.5.15}$$

where d_1 and d_2 are some positive constants.

For the first differences $\Delta v_i(x_i)$ of the functions (3.5.14) along the solutions of (3.5.7) we have the estimates:

$$\begin{aligned}\Delta v_i(x_i)\big|_{\hat S_i} &= v_i(A_i x_i) - v_i(x_i) + v_i((A_i + E_i)x_i) - v_i(A_i x_i)\\
&+ v_i((A_i + E_i)x_i + (B_i + U_i)x_j) - v_i((A_i + E_i)x_i)\\
&\le (x_i^{\mathrm{T}} A_i^{\mathrm{T}} P_i A_i x_i)^{1/2} - (x_i^{\mathrm{T}} P_i x_i)^{1/2} + \sigma_M^{1/2}(P_i)\|E_i x_i\|\\
&+ \sigma_M^{1/2}(P_i)\|(B_i + U_i)x_j\| \le \frac{x_i^{\mathrm{T}} A_i^{\mathrm{T}} P_i A_i x_i - x_i^{\mathrm{T}} P_i x_i}{(x_i^{\mathrm{T}} A_i^{\mathrm{T}} P_i A_i x_i)^{1/2} + (x_i^{\mathrm{T}} P_i x_i)^{1/2}}\\
&+ \sigma_M^{1/2}(P_i)\|E_i\|\,\|x_i\| + \sigma_M^{1/2}(P_i)(\|B_i\| + \|U_i\|)\|x_j\|\\
&\le -(\alpha_i - \sigma_M^{1/2}(P_i)\|E_i\|)\,\|x_i\| + \sigma_M^{1/2}(P_i)\left(\|B_i\| + \|U_i\|\right)\|x_j\|,\end{aligned}$$

where $i,j = 1,2,\ i \ne j$.

Here we use the known inequality (see Sezer and Siljak [1])

$$(p^{\mathrm{T}} P p)^{1/2} - (q^{\mathrm{T}} P q)^{1/2} \le \sigma_M^{1/2}(P)\|p - q\|$$

for all $p, q \in R^n$, $P \in R^{n\times n}$ is a symmetric and positive definite matrix.

From here we arrive at the following inequality

$$\Delta v(x)\big|_{(\widehat S_1,\widehat S_2)} \le d_1\Delta v_1(x_1)\big|_{\widehat S_1} + d_2\Delta v_2(x_2)\big|_{\widehat S_2} \le -\tilde d^{\mathrm{T}} W z, \tag{3.5.16}$$

where $\tilde d = (d_1, d_2)^{\mathrm{T}}$, $z = (\|x_1\|, \|x_2\|)^{\mathrm{T}}$, $W = (w_{ij})$ is 2×2 matrix with the elements

$$w_{ij} = \begin{cases} \alpha_i - \sigma_M^{1/2}(P_i)\,\|E_i\| & \text{if} \quad i = j,\\ -\sigma_M^{1/2}(P_i)(\|B_i\| + \|U_i\|) & \text{if} \quad i \ne j.\end{cases}$$

As all conditions of Theorem 3.5.2 are satisfied, it is not difficult to verify that the matrix W is the M-matrix (see Siljak [4]). Really

$$\begin{aligned}
w_{11}w_{22} - w_{12}w_{21} &= [\alpha_1 - \sigma_M^{1/2}(P_1)\|E_1\|][\alpha_2 - \sigma_M^{1/2}(P_2)\|E_2\|] \\
&- \sigma_M^{1/2}(P_1)\sigma_M^{1/2}(P_2)(\|B_1\| + \|U_1\|)(\|B_2\| + \|U_2\|) \\
&> \left[\alpha_1 - \sigma_M^{1/2}(P_1)(1-\gamma)\mu_1\right]\left[\alpha_2 - \sigma_M^{1/2}(P_2)(1-\gamma)\mu_2\right] \\
&- \sigma_M^{1/2}(P_1)\sigma_M^{1/2}(P_2)(\|B_1\| + \varepsilon)(\|B_2\| + \varepsilon) = \gamma^2\alpha_1\alpha_2 \\
&- \sigma_M^{1/2}(P_1)\sigma_M^{1/2}(P_2)(\|B_1\| + \varepsilon)(\|B_2\| + \varepsilon) = -\sigma_M^{1/2}(P_1)\sigma_M^{1/2}(P_2)\varepsilon^2 \\
&- \sigma_M^{1/2}(P_1)\sigma_M^{1/2}(P_2)(\|B_1\| + \|B_2\|)\varepsilon + \gamma^2\alpha_1\alpha_2 \\
&- \sigma_M^{1/2}(P_1)\sigma_M^{1/2}(P_2)\|B_1\|\,\|B_2\| = -a\varepsilon^2 - b\varepsilon + c.
\end{aligned}$$

By condition (2) of Assumption 3.5.1

$$\begin{aligned}
c &= \gamma^2\alpha_1\alpha_2 - \sigma_M^{1/2}(P_1)\sigma_M^{1/2}(P_2)\|B_1\|\,\|B_2\| \\
&= \sigma_M^{1/2}(P_1)\sigma_M^{1/2}(P_2)\big[\gamma^2\mu_1\mu_2 - \|B_1\|\,\|B_2\|\big] > 0
\end{aligned}$$

and therefore $-a\varepsilon^2 - b\varepsilon + c = 0$, and $w_{11}w_{22} - w_{12}w_{21} > 0$.

It is clear that the function (3.5.15) is positive definite and its first difference (3.5.16) is negative definite. These conditions are sufficient for the asymptotic stability of the equilibrium $x = 0$ of (3.5..2). The proof of Theorem 3.5.2 is complete.

Thus the inequalities (3.5.13) are the robust bounds for the system (3.5.2), obtained in terms of the vector approach.

3.5.3 Hierarchical approach As is known (see Ikeda and Siljak [1]), the essence of this method is as follows: starting from the constructing an auxiliary Liapunov function, we take into account a hierarchical structure of the system (3.5.2) or realize a multilevel decomposition of the initial system. Further the second approach is applied precisely.

We decompose each subsystems (3.5.7) into two interconnected components

$$\widetilde{C}_{ij}: \quad x_{ij}(\tau+1) = (A_{ij} + E_{ij})\,x_{ij}(\tau) + (B_{ij} + U_{ij})\,x_{ik}(\tau), \quad i,j,k = 1,2, \quad j \neq k, \tag{3.5.17}$$

where $x_{ij} \in R^{n_{ij}}$, $R^{n_i} = R^{n_{i1}} \times R^{n_{i2}}$, $A_{ij}, E_{ij} \in R^{n_{ij}\times n_{ij}}$, $B_{i1}, U_{i1} \in R^{n_{i1}\times n_{i2}}$, $B_{i2}, U_{i2} \in R^{n_{i2}\times n_{i1}}$,

$$A_i = \begin{pmatrix} A_{i1} & B_{i1} \\ B_{i2} & A_{i2} \end{pmatrix}, \qquad E_i = \begin{pmatrix} E_{i1} & U_{i1} \\ U_{i2} & E_{i2} \end{pmatrix}.$$

Assume that the matrices B_i and U_i have a block structure:

$$B_i = \begin{pmatrix} M_{11}^{(i)} & M_{12}^{(i)} \\ M_{12}^{(i)} & M_{22}^{(i)} \end{pmatrix}, \qquad U_i = \begin{pmatrix} F_{11}^{(i)} & F_{12}^{(i)} \\ F_{12}^{(i)} & F_{22}^{(i)} \end{pmatrix}, \tag{3.5.18}$$

where $M_{jk}^{(i)}, F_{jk}^{(i)} \in R^{n_{ij}\times n_{lk}}$, $i,j,k,l = 1,2$, $i \neq l$.

We extract from (3.5.17) the independent components

$$C_{ij}: \quad x_{ij}(\tau+1) = (A_{ij} + E_{ij})\,x_{ij}(\tau), \quad i,j = 1,2, \tag{3.5.19}$$

with the same designations of variables as in system (3.5.17).

In order to state the robust bounds we require the following assumptions.

Assumption 3.5.2 The nominal components

$$x_{ij}(\tau+1) = A_{ij}x_{ij}(\tau), \quad i,j = 1,2, \tag{3.5.20}$$

are asymptotically stable, i.e. there exist unique symmetric and positive definite matrices P_{ij}, which satisfy the Liapunov matrix equations

$$A_{ij}^{\mathrm{T}}P_{ij}A_{ij} - P_{ij} = -G_{ij}, \qquad i,j = 1,2, \tag{3.5.21}$$

where G_{ij} are arbitrary symmetric and positive definite matrices.

Let P_{ij} be solutions of the Liapunov matrix equations (3.5.21) for the identity matrices $G_{(ij)} = I_{ij}$. We define the constants

$$\begin{aligned}
&\alpha_{ij} = \sigma_M^{1/2}(P_{ij})\mu_{ij} = (\sigma_M^{1/2}(P_{ij} - I_{ij}) + \sigma_M^{1/2}(P_{ij}))^{-1},\\
&\mu_{ij} = (\sigma_M^{1/2}(P_{ij} - I_{ij})\sigma_M^{1/2}(P_{ij}) + \sigma_M(P_{ij}))^{-1},\\
&\varepsilon_i = \frac{1}{2a_i}((b_i^2 + 4a_ic_i)^{1/2} - b_i),
\end{aligned}$$

$$a_i = \sigma_M^{1/2}(P_{i1})\sigma_M^{1/2}(P_{i2}),$$
$$b_i = \sigma_M^{1/2}(P_{i1})\sigma_M^{1/2}(P_{i2})(\|B_{i1}\| + \|B_{i2}\|),$$
$$c_i = \gamma_i^2\alpha_{i1}\alpha_{i2} - \sigma_M^{1/2}(P_{i1})\sigma_M^{1/2}(P_{i2})\|B_{i1}\|\,\|B_{i2}\|, \quad i,j = 1,2.$$

Assumption 3.5.3 There exist constants $\gamma_i \in (0,1)$ such that

(3.5.22) $$\|B_{i1}\|\,\|B_{i2}\| < \gamma_i^2\mu_{i1}\mu_{i2}, \quad i = 1,2.$$

Let us construct an auxiliary function on the base of the functions

(3.5.23) $$v_{ij}(x_{ij}) = (x_{ij}^{\mathrm{T}}P_{ij}x_{ij})^{1/2},$$

by formula
$$v_i(x_i) = d_{i1}v_{i1}(x_{i1}) + d_{i2}v_{i2}(x_{i2}), \quad i = 1,2,$$

where d_{ij} are some positive constants. We introduce 2×2 matrices $W_i = (w_{jk}^{(i)})$ with the elements

$$w_{jk}^{(i)} = \begin{cases} \gamma_i\alpha_{ij} & \text{if} \quad j = k,\\ -\sigma_M^{1/2}(P_{ij})(\|B_{ij}\| + \overline{\varepsilon}_i) & \text{if} \quad j \neq k. \end{cases}$$

Here $0 < \overline{\varepsilon}_i < \varepsilon_i$.

Further we need the following proposition.

Proposition 3.5.1 *We assume that*

(1) *discrete system (3.5.2) is decomposed on the first level to the system (3.5.7) and on the second level to the systems (3.5.17);*
(2) *all conditions of Assumptions 3.5.2 and 3.5.3 are satisfied;*
(3) *for the submatrices E_{ij}, U_{ij} of the matrices E_i, $i = 1,2$, the estimates*
$$\|E_{ij}\| \le (1-\gamma_i)\mu_{ij}, \quad \|U_{ij}\| \le \overline{\varepsilon}_i, \quad i,j = 1,2.$$
are realized.

Then there exist vectors $\hat{d}_1, \hat{d}_2 \in R^2$ with positive components such that the first differences $\Delta v_i(x_i)\big|_{C_{ij}}$ for the functions $v_i(x_i)$ satisfy the estimates

(3.5.24) $$\Delta v_i(x_i)\big|_{C_{ij}} \le -\hat{d}_i^{\mathrm{T}}W_iz_i, \quad i = 1,2$$

and the matrices W_i are the M-matrices.

Here $\hat{d}_i = (d_{i1}, d_{i2})^{\mathrm{T}}$, $z_i = (\|x_{i1}\|, \|x_{i2}\|)^{\mathrm{T}}$.

The proof of Proposition 3.5.1 is analogous to that of Theorem 3.5.1.

Under the hypotheses of Proposition 3.5.1 the matrices W_i are the M-matrices and, according to Siljak [4], the vectors $\hat{d}_i^{\mathrm{T}}W_i = (d_{i1}w_{11}^{(i)} + d_{i2}w_{21}^{(i)},\, d_{i1}w_{12}^{(i)} + d_{i2}w_{22}^{(i)})$ have positive components.

Let us denote

(3.5.25)
$$\pi_i = \min\{d_{i1}w_{11}^{(i)} + d_{i2}w_{21}^{(i)}; d_{i1}w_{12}^{(i)} + d_{i2}w_{22}^{(i)}\}, \quad i = 1,2,$$
$$m = \frac{1}{2}\left(\frac{\pi_1\pi_2}{(d_{11}\sigma_M^{1/2}(P_{11}) + d_{12}\sigma_M^{1/2}(P_{12}))\,(d_{21}\sigma_M^{1/2}(P_{21}) + d_{22}\sigma_M^{1/2}(P_{22}))}\right)^{1/2}$$

and give a method of optimal choice of the constants d_{i1}, d_{i2}, $i = 1,2$.

Proposition 3.5.2 *Let the matrices* W_1 *and* W_2 *be the M-matrices and* $w_{12}^{(i)},\, w_{21}^{(i)} < 0$*, then*

$$\sup_{d\in D} m(d) = m(d_1^*, 1, d_2^*, 1)$$

$$= \frac{1}{2}\Bigg(\frac{w_{11}^{(1)}w_{22}^{(1)} - w_{12}^{(1)}w_{21}^{(1)}}{\sigma_M^{1/2}(P_{11})(w_{22}^{(1)} - w_{21}^{(1)}) + \sigma_M^{1/2}(P_{12})(w_{11}^{(1)} - w_{12}^{(1)})} \times \frac{w_{11}^{(2)}w_{22}^{(2)} - w_{12}^{(2)}w_{21}^{(2)}}{\sigma_M^{1/2}(P_{21})(w_{22}^{(2)} - w_{21}^{(2)}) + \sigma_M^{1/2}(P_{22})(w_{11}^{(2)} - w_{12}^{(2)})}\Bigg)^{1/2}, \tag{3.5.26}$$

where

$$D = \Bigg\{d = (d_{11}, d_{12}, d_{21}, d_{22})^{\mathrm{T}} \in R^4 \colon\ -\frac{w_{21}^{(1)}}{w_{11}^{(1)}} < \frac{d_{11}}{d_{12}} < -\frac{w_{22}^{(1)}}{w_{12}^{(1)}},\ -\frac{w_{21}^{(2)}}{w_{11}^{(2)}} < \frac{d_{21}}{d_{22}} < -\frac{w_{22}^{(2)}}{w_{12}^{(2)}}\Bigg\},$$

$$d_1^* = \frac{w_{22}^{(1)} - w_{21}^{(1)}}{w_{11}^{(1)} - w_{12}^{(1)}}, \qquad d_2^* = \frac{w_{22}^{(2)} - w_{21}^{(2)}}{w_{11}^{(2)} - w_{12}^{(2)}}.$$

Proof As the matrices W_1 and W_2 are the M-matrices, then $w_{11}^{(i)}, w_{22}^{(i)} > 0$, $w_{12}^{(i)},\, w_{21}^{(i)} < 0$ and consequently,

$$-\frac{w_{22}^{(i)}}{w_{12}^{(i)}} > -\frac{w_{21}^{(i)}}{w_{11}^{(i)}} > 0.$$

On computing of the constant π_i and m we can set $d_{12} = d_{22} = 1$, $d_{11} = d_1$, $d_{21} = d_2$ and

$$d_i \in D_i = \Bigg\{d_i \in R\colon\ -\frac{w_{21}^{(i)}}{w_{11}^{(i)}} < d_i < -\frac{w_{22}^{(i)}}{w_{12}^{(i)}}\Bigg\}, \qquad i = 1, 2.$$

Let us denote

$$m_i(d_i) = \frac{\pi_i}{d_i\sigma_M^{\frac{1}{2}}(P_{i1}) + \sigma_M^{\frac{1}{2}}(P_{i2})}, \qquad i = 1, 2, \tag{3.5.27}$$

and note that

$$\sup_{d\in D} m(d) = \frac{1}{2}\big(\sup_{d_1\in D_1} m_1(d_1) \sup_{d_2\in D_2} m_2(d_2)\big). \tag{3.5.28}$$

By (3.5.25) for the function $m_i(d_i)$ we get the expressions

$$m_i(d_i) = \begin{cases} \dfrac{d_i w_{11}^{(i)} + w_{21}^{(i)}}{d_i\sigma_M^{1/2}(P_{i1}) + \sigma_M^{1/2}(P_{i2})}, & \text{if} \quad -\dfrac{w_{21}^{(i)}}{w_{11}^{(i)}} < d_i \le d_i^*, \\[2ex] \dfrac{d_i w_{12}^{(i)} + w_{22}^{(i)}}{d_i\sigma_M^{1/2}(P_{i1}) + \sigma_M^{1/2}(P_{i2})}, & \text{if} \quad d_i^* \le d_i < -\dfrac{w_{22}^{(i)}}{w_{21}^{(i)}}. \end{cases}$$

For the first derivatives $m_i'(d_i)$ we have

(3.5.29)

$$m_i'(d_i) = \begin{cases} \dfrac{w_{11}^{(i)}\sigma_M^{1/2}(P_{i2}) - w_{21}^{(i)}\sigma_M^{1/2}(P_{i1})}{\left(d_i\sigma_M^{1/2}(P_{i1}) + \sigma_M^{1/2}(P_{i2})\right)^2}, & \text{if} \quad -\dfrac{w_{21}^{(i)}}{w_{11}^{(i)}} < d_i < d_i^*, \\ \dfrac{w_{12}^{(i)}\sigma_M^{1/2}(P_{i2}) - w_{22}^{(i)}\sigma_M^{1/2}(P_{i1})}{\left(d_i\sigma_M^{1/2}(P_{i1}) + \sigma_M^{1/2}(P_{i2})\right)^2}, & \text{if} \quad d_i^* < d_i < -\dfrac{w_{22}^{(i)}}{w_{21}^{(i)}}, \end{cases}$$

therefore $m_i'(d_i) > 0$ for $-\dfrac{w_{21}^{(i)}}{w_{11}^{(i)}} < d_i < d_i^*$ and $m_i'(d_i) < 0$ for $d_i^* < d_i < -\dfrac{w_{22}^{(i)}}{w_{21}^{(i)}}$. From here it follows that

$$\sup_{d_i \in D_i} m_i(d_i) = m_i(d_i^*) = \frac{w_{11}^{(i)}w_{22}^{(i)} - w_{12}^{(i)}w_{21}^{(i)}}{\sigma_M^{1/2}(P_{i1})(w_{22}^{(i)} - w_{21}^{(i)}) + \sigma_M^{1/2}(P_{i2})(w_{11}^{(i)} - w_{12}^{(i)})}.$$

Substituting by the values of $m_i(d_i^*)$ into (3.5.28), we get the identity (3.5.26). Proposition 3.5.2 is proved.

Assumption 3.5.4 Let for the submatrices $M_{jk}^{(i)}$ of the matrices B_i the inequalities

$$\overline{m} = \max \|M_{jk}^{(i)}\| < m$$

be realized for all $i, j, k = 1, 2$.

The following proposition is basic in the method of hierarchical Liapunov functions in the robust stability problem of the system (3.5.2).

Theorem 3.5.3 *We assume that for the uncertain system (3.5.2) the two-level decomposition (3.5.7), (3.5.17) is realized and all conditions of Assumptions 3.5.2 – 3.5.4 are satisfied. If the inequalities*

$$\|E_{ij}\| \le (1-\gamma_i)\mu_{ij}, \quad \|U_{ij}\| \le \overline{\varepsilon}_i, \quad \|F^{(i)}_{jk}\| < m - \overline{m}$$

are fulfilled for all $i, j, k = 1, 2$, *then the equilibrium* $x = 0$ *of the system (3.5.2) is asymptotically stable.*

Proof Under the hypotheses of Proposition 3.5.1 there exist constants $d_{ij} > 0$ for which $\hat{d}_i^{\mathrm{T}} W_i z_i > 0$. In view of designations (3.5.25), we get from estimate (3.5.24)

$$\Delta v_i(x_i)\big|_{S_i} \le -\pi_i \left(\|x_{i1}\|^2 + \|x_{i2}\|^2\right)^{1/2} = -\pi_i \|x_i\|, \quad i = 1, 2.$$

Since for $i \ne k$ the estimates

$$\Delta v_{i1}(x_{i1})\big|_{\widehat{S}_i} \le \Delta v_{i1}(x_{i1})\big|_{S_i} + \sigma_M^{1/2}(P_{i1})(2\overline{m} + \|F^{(i)}_{11}\| + \|F^{(i)}_{12}\|)\|x_k\|,$$
$$\Delta v_{i2}(x_{i2})\big|_{\widehat{S}_i} \le \Delta v_{i2}(x_{i2})\big|_{S_i} + \sigma_M^{1/2}(P_{i2})(2\overline{m} + \|F^{(i)}_{21}\| + \|F^{(i)}_{22}\|)\|x_k\|,$$

are true, then

$$\begin{aligned}\Delta v_i(x_i)\big|_{\widehat{S}_i} &= d_{i1}\Delta v_{i1}(x_{i1})\big|_{S_i} + d_{i2}\Delta v_{i2}(x_{i2})\big|_{\widehat{S}_i} \\ &\le -\pi_i\|x_i\| + \big[d_{i1}\sigma_M^{1/2}(P_{i1})\big(2\overline{m} + \|F^{(i)}_{11}\| + \|F^{(i)}_{12}\|\big) \\ &\quad + d_{i2}\sigma_M^{1/2}(P_{i2})\big(2\overline{m} + \|F^{(i)}_{21}\| + \|F^{(i)}_{22}\|\big)\big]\|x_k\|.\end{aligned} \tag{3.5.30}$$

For the function

$$v(x) = d_1 v_1(x_1) + d_2 v_2(x_2)$$

in view of estimates (3.5.30) we get

$$\Delta v(x)\big|_S = d_1 \Delta v_1(x_1)\big|_{\widehat{S}_1} + d_2 \Delta v_2(x_2)\big|_{\widehat{S}_2} \le -\hat{d}^{\mathrm{T}} W z, \tag{3.5.31}$$

where $\hat{d} = (d_1, d_2)^{\mathrm{T}}$, $z = (\|x_1\|, \|x_2\|)^{\mathrm{T}}$ and W is a 2×2 - matrix with the elements

$$w_{jk} = \begin{cases} \pi_j & \text{for} \quad j = k, \\ -d_{j1}\sigma_M^{1/2}(P_{j1})(2\overline{m} + \|F^{(j)}_{11}\|) + \|F^{(j)}_{12}\|) & \\ \qquad - d_{j2}\sigma_M^{1/2}(P_{j2})(2\overline{m} + \|F^{(j)}_{21}\|) + \|F^{(j)}_{22}\|) & \text{for} \quad j \ne k. \end{cases}$$

Under the hypotheses of Theorem 3.5.3 the matrix W in the estimate (3.5.31) is the M-matrix. Thus the matrices W_1, W_2, W are the M-matrices and it is sufficient for asymptotic stability of the system (3.5.2).

3.5.4 Discussion and some examples The hierarchical approach in robust stability problem permits a more complete allowance for the dynamic characteristics of the nominal system on each hierarchical level and thus a more exact definition of robust bounds for the system (3.5.2). We illustrate efficiency of the approach proposed in the Section by a simple example.

Let us assume that in the system (3.5.2) the matrix A has the form

$$A = \begin{pmatrix} 0.5 & 0.01 & 0.03 & 0 \\ 0.01 & 0.125 & 0 & 0.03 \\ 0.03 & 0 & 0.25 & 0.005 \\ 0 & 0.03 & 0.005 & 0.125 \end{pmatrix}. \tag{3.5.32}$$

3.5.4.1 Scalar approach Let us compute the matrices and constants occurring in the framework of the scalar approach (see Theorem 3.5.1):

$$P = \begin{pmatrix} 1.336149 & 0.008512 & 0.032104 & 0.000737 \\ 0.008512 & 1.017019 & 0.000708 & 0.007761 \\ 0.032104 & 0.000708 & 1.068495 & 0.002057 \\ 0.000737 & 0.007761 & 0.002057 & 1.016891 \end{pmatrix};$$

$$\sigma(P) \approx 1.340176; \quad \sigma_M(P - I) \approx 0.340176; \quad \mu \approx 0.496185.$$

Here I is a 4×4- unit matrix. From here the robust bound for the system (3.5.2) with the matrix (3.5.32) is determined by the inequality

$$\|E\| < 0.496185 \tag{3.5.33}$$

for all matrices $E \in S$.

3.5.4.2 Vector approach According to this approach we decompose the matrix (3.5.32) and denote

$$A_1 = \begin{pmatrix} 0.5 & 0.01 \\ 0.01 & 0.125 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.25 & 0.005 \\ 0.005 & 0.125 \end{pmatrix},$$
$$B_1 = B_2 = \begin{pmatrix} 0.03 & 0 \\ 0 & 0.03 \end{pmatrix}.$$

The uncertain matrix E is represented in the form (3.5.9). The matrices and constants occurring in the framework of vector approach are:

$$P_1 \approx \begin{pmatrix} 1.333581 & 0.008469 \\ 0.008469 & 1.016029 \end{pmatrix}, \quad P_2 \approx \begin{pmatrix} 1.066699 & 0.002031, \\ 0.002031 & 1.015902 \end{pmatrix},$$
$$\sigma_M(P_1) \approx 1.333807, \quad \sigma_M(P_2) \approx 1.066780,$$
$$\mu_1 \approx 0.449733, \quad \mu_2 \approx 0.749800.$$

Hence we have the estimates of submatrices norms in the form

$$\|E_1\| \le 0.499733(1 - \gamma), \quad \|E_2\| \le 0.749800(1 - \gamma), \quad \gamma \in (0, 1). \tag{3.5.34}$$

Let $\gamma = 0.25$. Besides $\varepsilon \approx 0.012303$. Finally, for the matrix E represented in the form (3.5.9), we get the estimates:

$$\|E_1\| \le 0.374800, \quad \|E_2\| \le 0.562350, \quad \|U_i\| < 0.012303, \quad i = 1, 2. \tag{3.5.35}$$

For example the matrix

$$\widetilde{E} = \operatorname{diag}\{0.37,\ 0.37,\ 0.56,\ 0.56\}$$

satisfies the inequalities (3.5.35). But $\|\widetilde{E}\| = 0.56$, and consequently, the norm of uncertain matrix $\widetilde{E}$ does not satisfy the inequality (3.5.33).

3.5.4.3 Hierarchical approach According to the proposed algorithm we accomplish the two-level decomposition of system (3.5.2) with the matrix (3.5.32) and as a result we get:

$$A_{11} = 0.5, \quad A_{12} = 0.125, \quad A_{21} = 0.25, \quad A_{22} = 0.125.$$

Let

$$\gamma_1 = 0.5, \quad \gamma_2 = 0.125.$$

Numerical values of the corresponding constants are:

$$\begin{aligned}
&\sigma_M(P_{11}) \approx 1.333333, \quad && \sigma_M(P_{12}) \approx 1.015873, \quad && \mu_{11} = 0.5, \quad && \mu_{12} = 0.875,\\
&\sigma_M(P_{21}) \approx 1.066666, && \sigma_M(P_{22}) \approx 1.015873, && \mu_{21} = 0.75, && \mu_{22} = 0.875,\\
&\varepsilon_1 \approx 0.320718, && \varepsilon_2 \approx 0.096261. && &&
\end{aligned}$$

We shall set $\overline{\varepsilon}_1 = 0.05$, and $\overline{\varepsilon}_2 = 0.006$. In this case for the matrices W_1 and W_2 we get the expressions

$$W_1 \approx \begin{pmatrix} 0.288675 & -0.069282 \\ -0.060474 & 0.440958 \end{pmatrix}, \qquad W_2 \approx \begin{pmatrix} 0.096824 & -0.011360 \\ -0.011086 & 0.110239 \end{pmatrix}.$$

The matrices W_1 and W_2 are the M-matrices as their non-diagonal elements are negative and their principal minors are positive.

The constant m is computed by the formula (3.5.26): $m \approx 0.038392$. Thus, the following restrictions are imposed on submatrices of E:

(3.5.36)

$$\begin{gathered}
\|E_{11}\| \le 0.25, \quad \|E_{12}\| \le 0.4375, \quad \|E_{21}\| \le 0.65625, \quad \|E_{22}\| \le 0.765625,\\
\|U_{1j}\| \le 0.05, \quad \|U_{2j}\| \le 0.006, \quad \|F_{jk}^{(i)}\| < 0.008392.
\end{gathered}$$

For example the matrix

$$\overline{E} = \operatorname{diag}\{0.25,\, 0.43,\, 0.65,\, 0.76\}.$$

satisfies the inequalities (3.5.36). Since $\|\overline{E}\| = 0.76$, the matrix $\overline{E}$ does not satisfy condition (3.5.33). Moreover $\|\,\mathrm{diag}\,\{0.65,\, 0.76\}\| = 0.76 > 0.75$ and it means that for the matrix $\overline{E}$ conditions (3.5.34) are not satisfied for any $\gamma \in (0,1)$.

So, the general conclusion from this example is: the hierarchical Liapunov function allows a more complete use of the potential possibilities of Liapunov direct method in robustness analysis of discrete system (3.5.2).

3.6 Notes and References

Section 3.1 Qualitative theory of discrete systems under nonclassical structural perturbations is of a considerable interest with regard to a number of applied problems of nonlinear dynamics. In the framework of the method of matrix-valued Liapunov functions for the class of dynamical systems some results have been obtained recently. Some of them are presented in this Section.

In a series of monographs (see, for example, Bromberg [1], La-Salle [1], Tsypkin [1], Furasov [1], Lakshmikantham, Leela, *et al.* [1, 2], Michel, *et al.* [1], Hahn [1], etc.) discussed are the various methods of application of the Liapunov direct method for the qualitative analysis of motions of discrete-time systems without structural perturbations.

Section 3.2 The large scale discrete-time system with structural perturbations is composed according to the approach presented in Chapter 2. As in the continuous case this allows the application of the generalized Liapunov direct method based on matrix-valued function in the construction of the corresponding stability conditions.

Section 3.3 In this section the matrix-valued function is applied within the framework of the scalar approach. The section is based on the results by Martynyuk [12].

Section 3.4 The main results of this section are due to Martynyuk and Miladzhanov [1], and Martynyuk, Miladzhanov, and Muminov [1].

Section 3.5 This section is based on the results by Lukyanova and Martynyuk [1]. The comparative analysis of the obtained results is carried out using some results by Sezer and Šiljak [1].

4

IMPULSIVE LARGE-SCALE SYSTEMS

4.1 Introduction

It is well-known that the Liapunov direct method is of great importance for the stability analysis of solutions of continuous nonlinear differential and discrete-time equations (see Chapters 2 − 3). This fruitful technique has been developed during the twentieth century in two main directions. First, the working out of constructive techniques of Liapunov function constructions and second, the expansion of this method for the systems of equations other than ordinary differential equations. One of classes of such systems are impulsive systems.

Therefore our aim in this Chapter is to develop the Liapunov direct matrix-valued functions method for the impulsive systems under nonclassical structural perturbations and to establish a new sufficient conditions for the presence of various dynamical properties of solutions to the equations under consideration.

Chapter 4 is arranged as follows.

Section 4.2 deals with description of impulsive system under nonclassical structural perturbations.

In Section 4.3 the problem on stability of impulsive system under nonclassical structural perturbations is stated.

In Section 4.4 proposed are algorithms for determing the property of having a fixed sign of matrix-valued functions and their decrease (increase) estimations on the trajectories of system under consideration.

In Section 4.5 various sufficient conditions are established for stability, asymptotic stability and instability of large scale impulsive system under nonclassical structural perturbations. Here an example of forth order system with structural perturbations is presented illustrating the proposed technique.

In Section 4.6 the obtained results are discussed as well as the possibilities of the proposed generalization of the Liapunov direct method and some trends of applications.

4.2 Nonclassical Structural Perturbations in Impulsive Systems

We consider an impulsive system under noclassical structural perturbations obtained as a result of composition of complex nonlinear ordinary differential equations with impulsive perturbations.

In this section we study the class of impulsive systems I with subsystems I_i whose description is based on the following assumptions.

H_1. The evolution of system I and its subsystems $I_i,\ i = 1, 2, \dots, m,$ is described by the impulsive systems of equations whose order is not changed on the general interval of existence of solutions to system I.

H_2. Impulsive effect on system I (subsystems I_i) occurs at the moments $\tau_k \in C^1(\Omega, (0, \infty))$, $\Omega \subseteq R$, $k = 1, 2, \dots$, for which $\tau_k(x) < \tau_{k+1}(x)$ for all k and $\tau_k(x) \to \infty$ uniformly on $x \in \Omega$ ($\tau_k^* \in C(\Omega_i, (0, \infty))$, $\Omega_i \subseteq R^{n_i}$, $\tau_k^*(x_i) < \tau_{k+1}^*(x_i)$ for all k and $\tau_k^*(x_i) \to \infty$ uniformly on $x_i \in \Omega_i$, $n_1 + \dots + n_m = n$).

H_3. The matrix $P = (p_1^{\mathrm{T}}, p_2^{\mathrm{T}}, \dots, p_m^{\mathrm{T}})^{\mathrm{T}} \in R^{m \times q}$ describes the internal and/or external parametric effects on system I. The set of all admissible matrices for system I is designated as before by $\mathcal{P} = \{P\colon\ P_1 \le P(t) \le P_2\}$, for all $t \in \mathcal{T}$.

H_4. The family of functions $\mathcal{F}$ described in Assumption H_3 from Section 2.2 is determined.

H_5. The evolution of the interacting subsystems I_i in system I is determined by the equations

$$
\begin{aligned}
\frac{dx_i}{dt} &= f_i(t, x, p_i) \quad \text{for} \quad t \ne \tau_k(x_i), \\
\Delta x_i(t) &= I_{ik}(x_i) + I_{ik}^*(x) \quad \text{for} \quad t = \tau_k(x_i), \quad k = 1, 2, \dots,
\end{aligned}
\tag{4.2.1}
$$

where $x_i \in \Omega_i$, $f_i \in \mathcal{F}_i$, $\mathcal{F}_i = \{f_i^1, \dots, f_i^N\}$, $I_{ik}\colon R^{n_i} \to R^{n_i}$, $I_{ik}^*\colon R^n \to R^{n_i}$, $\Delta x_i(t) = x_i(t^+) - x_i(t^-)$.

H_6. The evolution of the i-th isolated subsystem in system I is described by the equations

$$\begin{aligned}\frac{dx_i}{dt} &= w_i(t,x) \quad \text{for} \quad t \neq \tau_k(x),\\ \Delta x_i(t) &= I_{ik}(x_i) \quad \text{for} \quad t = \tau_k(x_i), \quad k = 1,2,\dots,\end{aligned} \tag{4.2.2}$$

where $w_i\colon \mathcal{T} \times R^{n_i} \to R^{n_i}$, $I_{ik}\colon R^{n_i} \to R^{n_i}$, $k = 1,2,\dots.$ Here the functions $w_i(t,x_i)$ are determined from the correlations

$$w_i(t,x_i) = f_i(t,x^i,0), \quad i = 1,2,\dots,m,$$

where $x^i = (0,\dots,0,x_i^{\mathrm{T}},0,\dots,0)^{\mathrm{T}} \in R^n$.

In view of the designations of Section 2.2 the impulsive system I with structural and parametric perturbations is represented as

$$\begin{aligned}\frac{dx_i}{dt} &= w_i(t,x) + S_i(t)r_i(t,x,p_i) \quad \text{for} \quad t \neq \tau_k(x),\\ \Delta x_i(t) &= I_{ik}(x) + I^*_{ik}(x) \quad \text{for} \quad t = \tau_k(x),\\ k &= 1,2,\dots, \quad i = 1,2,\dots,m,\end{aligned} \tag{4.2.3}$$

or in the vector form

$$\begin{aligned}\frac{dx}{dt} &= w(t,x) + S(t)r(t,x,P), \quad t \neq \tau_k(x),\\ \Delta x &= I_k(x), \quad t = \tau_k(x), \quad k = 1,2,\dots.\end{aligned} \tag{4.2.4}$$

The matrix

$$S(t) = \operatorname{diag}(S_1(t),\, S_2(t),\, \dots,\, S_k(t)) \in \mathcal{S} \tag{4.2.5}$$

describing all structural variations of the system (4.2.4) is called a *structural matrix* of impulsive system (4.2.4).

We assume that $w(t,0) = 0$, $I_k(0) = 0$ for all $k = 1,2,\dots$ and $r(t,0,P) = 0$ if and only if $x = 0$ for all $t \in \mathcal{T}_0$. It is clear that $x = 0$ is an equilibrium.

Further we designate

$$\mathcal{T}_0 \times D(\rho) = [t_0,\infty) \times \{x \in R^n\colon\ \|x\| \le \rho < \rho_0\}, \quad \rho_0 = \text{const} > 0.$$

The functions $\tau_i(x)$ and a number ρ satisfy the condition which excludes the beating of the solutions of system (4.2.4) on the surfaces

$$(S_k)\colon\ t = (\tau_k(x)), \quad k = 1,2,\dots.$$

Throughout this chapter, we will assume that for each $(t_0,x_0) \in \mathcal{T}_0 \times D(\rho)$, there exists at least one solution of (4.2.4) for all $(P,S) \in \mathcal{P} \times \mathcal{S}$ which satisfies the initial condition $x(t_0) = x_0$.

4.3 Definitions of Stability

The possibility of beating of the solutions of the equations on the surfaces $(S_k)\colon\ t = (\tau_k(x)),\ k = 1, 2, \dots,$ causes an essential difficulty when large scale impulsive system (4.2.4) is investigated. Moreover, in the general case there is no continuous dependence of the solutions of large scale impulsive system (4.2.4) on the initial conditions which would be uniform on finite interval. This fact requires the notion of stability in the sense of Liapunov of the solutions of large scale impulsive system (4.2.4) be corrected. Therefore, the notion of stability of solutions of large scale impulsive system (4.2.4) is adopted in the form formulated according to general properties of impulsive systems solution and it is assumed that the beating of the solutions on surfaces $(S_k)\colon\ t = (\tau_k(x)),\ k = 1, 2, \dots,$ is absent. In view of the results from Chapter 2 we introduce the following notions.

Definition 4.3.1 The zero solution $x = 0$ of the system (4.2.4) is

(a) *stable on* $\mathcal{P} \times \mathcal{S}$ if and only if it is stable in the sense of Liapunov for each pair of $(P, S) \in \mathcal{P} \times \mathcal{S}$;
(b) *asymptotically stable on* $\mathcal{P} \times \mathcal{S}$ if and only if it is asymptotically stable in the sense of Liapunov for each pair of $(P, S) \in \mathcal{P} \times \mathcal{S}$;
(c) *unstable on* $\mathcal{P} \times \mathcal{S}$ if and only if there exists at least one pair of $(P, S) \in \mathcal{P} \times \mathcal{S}$ for which the state $x = 0$ is unstable.

Remark 4.3.1 The definitions of the above mentioned types of stability for the given pair (P, S) are the same as for the systems without impulsive perturbations.

4.4 Tests for Stability and Instability Analysis

Problem $\mathrm{I_A}$**.** Assume that the evolution of the system I with impulsive perturbations is obtained in result of composition of evolutions of the isolated subsystems (4.2.2). It is required to establish sufficient conditions for various types of stability (instability) of the equilibrium state $x = 0$ of system (4.2.4) under certain dynamical properties of the isolated subsystems (4.2.2).

4.4.1 Auxiliary estimations for matrix-valued functions Before going over to the stability (instability) conditions for the state $x = 0$ of system (4.2.4) we shall establish some useful estimates for the auxiliary scalar pseudo-quadratic function constructed in terms of matrix-valued function. These estimates are some development of those presented in Chapter 2 for continuous systems.

Assumption 4.4.1 Assume that there exist:

(1) open connected time invariant neighborhoods

$$\mathcal{N}_{jx} = \{x_j \in R^{n_j}\colon\ \|x_j\| < h_{j0}\} \subseteq R^{n_j}$$

of the states $x_j = 0,\ j = 1, 2, \dots, m,\ h_{j0} = \text{const} > 0$;
(2) functions $\varphi_{j1}, \psi_{j1}\colon \mathcal{N}_{jx} \to R_+,\ \varphi_{j1}, \psi_{j1} \in K$;

(3) constants a_{ji}, b_{ji}, $j, i = 1, 2, \dots, m$, and a matrix-valued function $U(t, x) = [v_{ji}(t, \cdot)]$ with the elements

$$v_{jj} = v_{jj}(t, x_j), \quad v_{ji} = v_{ij} = v_{ji}(t, x_j, x_i)$$

for all $j \neq i$, $v_{jj}(t, 0) = v_{ji}(t, 0, 0) = 0$, $j, i = 1, 2, \dots, m$, defined and continuously differentiable in the domain $\mathcal{T}_0 \times D(\rho_0)$, where $\rho_0 = \min h_{j0}$, $j = 1, 2, \dots, m$, and satisfying the estimates

(a) $a_{jj}\varphi_{j1}^2(\|x_j\|) \leq v_{jj}(t, x_j) \leq b_{jj}\psi_{j1}^2(\|x_j\|)$ for all $(t, x_j) \in \mathcal{T}_0 \times \mathcal{N}_{jx}$, $j = 1, 2, \dots, m$;

(b) $a_{ji}\varphi_{j1}(\|x_j\|)\varphi_{i1}(\|x_i\|) \leq v_{ji}(t, x_j, x_i) \leq b_{ji}\psi_{j1}(\|x_j\|)\psi_{i1}(\|x_i\|)$ for all $(t, x_j, x_i) \in \mathcal{T}_0 \times \mathcal{N}_{jx} \times \mathcal{N}_{ix}$, $j, i = 1, 2, \dots, m$, $j \neq i$.

As before we shall introduce the scalar function

(4.4.1) $$v(t, x, \eta) = \eta^{\mathrm{T}} U(t, x)\eta, \quad \eta \in R_+^m, \quad \eta > 0,$$

and its total derivative

(4.4.2) $$Dv(t, x, \eta) = \eta^{\mathrm{T}} DU(t, x)\eta,$$

where

$$DU(t, x) = [Dv_{ji}(t, \cdot)], \quad j, i = 1, 2, \dots, m,$$

due to the system (4.2.4). For the details see Section 1.4.

Proposition 4.4.1 *If all conditions of Assumption 4.4.1 are satisfied, then for the function (4.4.1)*

(4.4.3) $u_1^{\mathrm T} H^{\mathrm T} A H u_1 \le v(t,x,\eta) \le u_2^{\mathrm T} H^{\mathrm T} B H u_2$, *for all* $(t,x) \in \mathcal{T}_0 \times \mathcal{N}_x$,

where

$$
\begin{aligned}
u_1^{\mathrm T} &= (\varphi_{11}(\|x_1\|),\, \varphi_{21}(\|x_2\|),\, \dots,\, \varphi_{m1}(\|x_m\|)),\\
u_2^{\mathrm T} &= (\psi_{11}(\|x_1\|),\, \psi_{21}(\|x_2\|),\, \dots,\, \psi_{m1}(\|x_m\|)),\\
H &= \operatorname{diag}[\eta_1,\eta_2,\dots,\eta_m], \quad A = [a_{ji}], \quad B = [b_{ji}],\\
a_{ji} &= a_{ij}, \quad b_{ji} = b_{ij}, \quad j,i = 1,2,\dots,m,
\end{aligned}
$$

$\mathcal{N}_x \subseteq \mathcal{N}_{1x} \times \mathcal{N}_{2x} \times \ldots \times \mathcal{N}_{mx}$ *is an open connected neighborhood of the state* $x = 0$, *such that*

$$
\mathcal{N}_x = x \in R^n\colon\ \|x\| < \rho_0, \quad \rho_0 = \min_j h_{j0}.
$$

The proof of Proposition 4.4.1 is similar to that of Lemma 1 in Martynyuk and Miladzhanov [1].

Assumption 4.4.2 Assume that there exist

(1) open connected time invariant neighborhoods $\mathcal{N}_{jx} \subseteq R^{nj}$ of the states $x_j = 0$, $j = 1,2,\dots,m$, and open connected neighborhood $\mathcal{N}_x \subseteq \mathcal{N}_{1x} \times \mathcal{N}_{2x} \times \ldots \times \mathcal{N}_{mx}$ of the state $x = 0$;

(2) the functions v_{ji}, $j,i = 1,2,\dots,m$, mentioned in Assumption 4.4.1 and the functions φ_j, $j = 1,2,\dots,m$, φ_m, φ_M such that on the domain $\mathcal{T}_0 \times D(\rho_0)$ the conditions

$$
\varphi_j(0) = \varphi_m(0) = \varphi_M(0) = 0
$$

and

$$
0 < \varphi_m(v(t,x,\eta)) \le \sum_{j=1}^{m} \varphi_j^2(v_{jj}(t,x_j)) \le \varphi_M(v(t,x,\eta))
$$

are satisfied;

(3) constants $\rho_j^{(1)}$, $\rho_j^{(2)}(P,S)$, $\rho_{ji}(P,S)$, $j \ne i = 1,2,\dots,m$, and

(a) $\eta_j^2 D_t v_{jj} + (D_{x_j} v_{jj})^{\mathrm T} f_j(t,x^j) \le \rho_j^{(1)} \varphi_j^2(v_{jj}(t,x_j))$ for all $(t,x_j) \in \mathcal{T}_0 \times \mathcal{N}_{jx0}$, $j = 1,2,\dots,m$;

(b) $\displaystyle\sum_{j=1}^{m} \eta_j^2 (D_{x_j} v_{jj})^{\mathrm T} S_j(t) r_j(t,x,P_j)$

$$
\begin{aligned}
&+ 2\sum_{j=1}^{m} \sum_{\substack{i=2\\ i>j}}^{m} \eta_j \eta_i \{ D_t v_{ji} + (D_{x_j} v_{ji})^{\mathrm T} [f_j(t,x^j) + S_j(t) r_j(t,x,P_j)]\\
&+ (D_{x_i} v_{ji})^{\mathrm T} [f_i(t,x^i) + S_i(t) r_i(t,x,P_i)]\}\\
&\le \sum_{j=1}^{m} \rho_j^{(2)}(P,S) \varphi_j^2(v_{jj}(t,x_j))\\
&+ 2\sum_{j=1}^{m} \sum_{\substack{i=2\\ i>j}}^{m} \rho_{ji}(P,S) \varphi_j(v_{jj}(t,x_j)) \varphi_i(v_{ii}(t,x_i))
\end{aligned}
$$

for all $(t,x_j,x_i,P,S) \in \mathcal{T}_0 \times \mathcal{N}_{jx_0} \times \mathcal{N}_{ix_0} \times \mathcal{P} \times \mathcal{S}$, $j \ne i$, $j = 1,2,\dots,m$.

For the above relations

$$\mathcal{N}_{jx0} = \{x_j \in R^{nj}:\ x_j \in \mathcal{N}_{jx},\ x_j \neq 0\},$$

and

$$t \neq \tau_k(x), \quad k = 1, 2, \dots .$$

Proposition 4.4.2 *If all conditions of Assumption 4.4.2 are satisfied, then for expression (4.4.2)*

$$\begin{aligned} &Dv(t,x,\eta) \le u^{\mathrm{T}} G(P,S) u, \quad t \neq \tau_k(x), \\ &\textit{for all} \quad (t,x,P,S) \in \mathcal{T}_0 \times \mathcal{N}_{x0} \times \mathcal{P} \times \mathcal{S} \end{aligned} \tag{4.4.4}$$

where

$$\begin{aligned} u^{\mathrm{T}} &= (\varphi_1(v_{11}(t,x_1)),\ \varphi_2(v_{22}(t,x_2)),\ \dots\ \varphi_m(v_{mm}(t,x_m))), \\ G(P,S) &= [\sigma_{ji}(P,S)], \quad j,i = 1,2,\dots,m, \\ \sigma_{jj}(P,S) &= \rho_j^{(1)} + \rho_j^{(2)}(P,S), \quad j = 1,2,\dots,m, \\ \sigma_{ji}(P,S) &= \rho_{ji}(P,S), \quad j \neq i, \quad j,i = 1,2,\dots,m. \end{aligned}$$

Proposition 4.4.2 is proved in a similar manner as Lemma 2 in Martynyuk and Miladzhanov [1].

Corollary 4.4.1 Let all conditions of Assumption 4.4.2 be satisfied. If there exists a constant matrix Q such that for the matrix $G(P,S)$ the inequality

$$\frac{1}{2}\left(G(P,S) + G^{\mathrm{T}}(P,S)\right) \le Q, \quad \text{for all} \quad (P,S) \in \mathcal{P} \times \mathcal{S} \tag{4.4.5}$$

holds component-wise and

(1) $\lambda_M(Q) < 0$;
(2) $\lambda_M(Q) < 0$.

Then the estimates

$$Dv(t,x,\eta) \le \lambda_M(Q)\varphi_m(v(t,x,\eta)), \quad \text{for all} \quad (t,x) \in \mathcal{T}_0 \times \mathcal{N}_{x0}; \tag{4.4.6}$$

$$Dv(t,x,\eta) \le \lambda_M(Q)\varphi_M(v(t,x,\eta)), \quad \text{for all} \quad (t,x) \in \mathcal{T}_0 \times \mathcal{N}_{x0}; \tag{4.4.7}$$

are satisfied respectively.

Proof Let all conditions of Assumption 4.4.2 be satisfied and there exist a constant matrix Q satisfying the inequality (4.4.5). In this connection Proposition 4.4.2 yields

$$Dv(t,x,\eta) \le u^{\mathrm{T}} G(P,S) u \le u^{\mathrm{T}} Q u \le \lambda_M(Q)\|u\|^2$$

$$= \lambda_M(Q) \sum_{j=1}^{s} \varphi_j^2(U_{jj}(t,x_j) \le \begin{cases} \lambda_M(Q)\varphi_m(v(t,x,\eta)) & \text{if } \lambda_M(Q) < 0 \\ \lambda_M(Q)\varphi_M(v(t,x,\eta)) & \text{if } \lambda_M(Q) > 0. \end{cases}$$

Assumption 4.4.3 Assume that there exist

(1) the functions u_{ji}, $j, i = 1, 2, \dots, m$, mentioned in Assumption 4.4.1 and functions ψ_j, $j = 1, 2, \dots, m$, ψ_m, ψ_M, $\psi_j(0) = \psi_m(0) = \psi_M(0) = 0$, such that on the domain $\mathcal{T}_0 \times D(\rho_0)$

$$0 < \psi_m(v(\tau_k(x), x)) \leq \sum_{j=1}^{m} \psi_j^2(v_{jj}(\tau_k(x^j), x_j)) \leq \psi_M(v(\tau_k(x), x)), \quad i = 1, 2, \dots; \tag{4.4.8}$$

(2) constants $\alpha_j^{(1)}$, $\alpha_j^{(2)}$, α_{jl} $(j \neq l)$, $j, l = 1, 2, \dots, m$, and

(a) $$\eta_j^2\{v_{jj}(\tau_k(x^j), x_j + J_{ij}(x^j)) - v_{jj}(\tau_k, (x^j), x_j)\} \leq \alpha_j^{(1)} \psi_j^{(2)}(v_{jj}(\tau_k, (x^j), x_j)) \quad \text{for all} \quad x_j \in \mathcal{N}_{jl}, \quad j = 1, 2, \dots, m;$$

(b) $$\begin{aligned}
&\sum_{j=1}^{m} \eta_j^2\{v_{jj}(\tau_k(x),\, x_j + J_{kj}(x)) - v_{jj}(\tau_k(x^j),\, x_j + J_{kj}(x^j))\} \\
&+ v_{jj}(\tau_k(x^j), x_j) - v_{jj}(\tau_k(x), x_j) \\
&+ 2\sum_{j=1}^{m} \sum_{\substack{l=2 \\ l>j}}^{m} \eta_j \eta_l \{v_{jl}(\tau_k(x),\, x_j + J_{kj}(x),\, x_l + J_{kl}(x)) \\
&- v_{jl}(\tau_k(x), x_j, x_l)\} \leq \sum_{j=1}^{m} \alpha_j^{(2)} \psi_j^{(2)}(v_{jj}(\tau_k(x^j), x_j)) \\
&+ 2\sum_{j=1}^{m} \sum_{\substack{l=2 \\ l>j}}^{m} \alpha_{jl} \psi_j(v_{jj}(\tau_k(x^j), x_j)) \psi_l(v_{ll}(\tau_k(x^l), x_l))
\end{aligned}$$

for all $\quad (x_j, x_l) \in \mathcal{N}_{jx} \times \mathcal{N}_{lx}, \quad k = 1, 2, \dots$

Proposition 4.4.3 *If all conditions of Assumption 4.4.3 are satisfied, then*

$$v(\tau_k(x),\, x + J_k(x)) - v(\tau_k(x), x) \le u_k^{\mathrm{T}} C u_k, \tag{4.4.9}$$

where

$$u_k^{\mathrm{T}} = (\psi_1(v_{11}(\tau_k(x^1), x_1)),\, \dots\,,\, \psi_m(v_{mm}(\tau_k(x^m), x_m))), \quad k = 1, 2, \dots\,,$$
$$C = [c_{ji}], \quad j, i = 1, 2, \dots, m, \quad c_{ji} = c_{ij},$$
$$c_{jj} = \alpha_j^{(1)} + \alpha_j^{(2)}, \quad c_{ji} = \alpha_{ji}, \quad j \ne i, \quad j, i = 1, 2, \dots, m.$$

Proof Under all conditions of Assumption 4.4.3 we have

$$\begin{aligned}
v(\tau_k(x),\, x + J_k(x)) - v(\tau_k(x), x) &= \eta^{\mathrm{T}}[v(\tau_k(x),\, x + J_k(x)) - v(\tau_k(x), x)]\eta \\
&= \sum_{j=1}^{m} \eta_j^2 \{v_{jj}(\tau_k(x),\, x_j + J_{kj}(x)) - v_{jj}(\tau_k(x), x_j)\} \\
&+ 2\sum_{j=1}^{m} \sum_{\substack{l=2\\ l>j}}^{m} \eta_j \eta_l \{v_{jl}(\tau_k(x),\, x_j + J_{kj}(x)),\, x_l + J_{kl}(x)) - v_{jl}(\tau_k(x),\, x_j, x_l)\} \\
&= \sum_{j=1}^{m} \eta_j^2 \{v_{jj}(\tau_k(x^j),\, x_j + J_{kj}(x^j)) - v_{jj}(\tau_k(x^j), x_j)\} \\
&+ \sum_{j=1}^{m} \eta_j^2 \{v_{jj}(\tau_k(x),\, x_j + J_{kj}(x)) - v_{jj}(\tau_k(x^j), x_j)\} \\
&+ \sum_{j=1}^{m} \eta_j^2 \{v_{jj}(\tau_k(x),\, x_j + J_{kj}(x)) \\
&+ 2\sum_{j=1}^{m} \sum_{\substack{l=2\\ l>j}}^{m} \eta_j \eta_l \{v_{jl}(\tau_k(x),\, x_j + J_{kj}(x)),\, x_l + J_{kl}(x)) - v_{jl}(\tau_k(x), x_j, x_l)\} \\
&\le \sum_{j=1}^{m} \alpha_j^{(1)} \psi_j^{(2)}(v_{jj}(\tau_k(x^j), x_j)) + \sum_{j=1}^{m} \alpha_j^{(2)} \psi_j^{(2)}(v_{jj}(\tau_k(x^j), x_j) \\
&+ 2\sum_{j=1}^{m} \sum_{\substack{l=2\\ l>j}}^{m} \alpha_{jl} \psi_j(v_{jj}(\tau_k(x^j), x_j)) \psi_l(v_{ll}(\tau_k(x^l), x_l)) \\
&= \sum_{j,l=1}^{m} c_{lj} \psi_j(v_{jj}(\tau_k(x^j), x_j)) \psi_l(v_{ll}(\tau_k(x^l), x_l)) = u_k^{\mathrm{T}} C u_k, \quad k = 1, 2, \dots\,.
\end{aligned}$$

Corollary 4.4.2 If all conditions of Assumption 4.4.3 are satisfied and

(1) $\lambda_M(C) < 0$;
(2) $\lambda_M(C) > 0$,

then for all $x \in \mathcal{N}_{x0}$

(4.4.10)
$$v(\tau_k(x),\, x_j + J_{kj}(x)) - v(\tau_k(x),\, x) \le \lambda_M(C)\, \psi_m(v(\tau_k(x), x));$$
(4.4.11)
$$v(\tau_k(x),\, x_j + J_{kj}(x)) - v(\tau_k(x),\, x) \le \lambda_M(C)\, \psi_M(v(\tau_k(x), x))$$

respectively. Here $\lambda_M(C)$ is a maximal eigenvalue of the matrix C.

Corollary 4.4.2 is proved in the same way as Corollary 4.4.1.

Assumption 4.4.4 Assume that there exist

(1) the functions $u_{ji},\ j, i = 1, 2, \dots, m,$ mentioned in Assumption 4.4.1 and the functions $\psi_j,\ j = 1, 2, \dots, m,\ \psi_m,\ \psi_M$ mentioned in Assumption 4.4.3;

(2) constants $\beta_j^{(1)},\ \beta_j^{(2)},\ \beta_{ji},\ j \ne i,\ j, i = 1, 2, \dots, m,$ and for all $k = 1, 2, \dots$

(a) $\eta_j^2 v_{jj}(\tau_k(x^j),\, x_j + J_{kj}(x^j)) \le \beta_j^{(1)} \psi_j^2(v_{jj}(\tau_k(x^j),\, x_j))$ for all $x_j \in \mathcal{N}_{ji},\quad j = 1, 2, \dots, m;$

(b)
$$\sum_{j=1}^m \eta_j^2 \{v_{jj}(\tau_k(x),\, x_j + J_{kj}(x)) - v_{jj}(\tau_k(x^j),\, x_j + J_{kj}(x^j))\}$$
$$+ 2 \sum_{j=1}^m \sum_{k=2}^m \eta_j \eta_l v_{jl}(\tau_k(x),\, x_j + J_{kj}(x),\, x_l + J_{kl}(x))$$
$$\le \sum_{j=1}^m \beta_j^{(2)} \psi_j^2(v_{jj}(\tau_k(x^j), x_j))$$
$$+ 2 \sum_{j=1}^m \sum_{\substack{l=2 \\ l>j}}^m \beta_{jl} \psi_j(v_{jj}(\tau_k(x^j), x_j)) \psi_l(v_{ll}(\tau_k(x^l), x_l))$$
for all $(x_j, x_l) \in \mathcal{N}_{jx} \times \mathcal{N}_{lx}$.

Proposition 4.4.4 *If all conditions of Assumption 4.4.4 are satisfied, then*

(4.4.12) $v(\tau_k(x),\, x + J_k(x)) \le u_k^{\mathrm{T}} C^* u_k, \quad k = 1, 2, \dots \quad$ *for all* $\quad x \in \mathcal{N}_x,$

where

$$C^* = [c_{ji}^*], \quad j, i = 1, 2, \dots, m, \quad c_{ji}^* = c_{ij}^*,$$
$$c_{jj}^* = \beta_j^{(1)} + \beta_j^{(2)}, \quad c_{ji}^* = \beta_{ij} \quad \textit{for all} \quad j \ne i, \quad j, i = 1, 2, \dots, m.$$

The proof of Proposition 4.4.4 is similar to that of Proposition 4.4.3.

Corollary 4.4.3 If all conditions of Assumption 4.4.4 are satisfied and

(1) $\lambda_M(C^*) < 0;$

(2) $\lambda_M(C^*) > 0,$

then for all $k = 1, 2, \dots$ and for all $x \in \mathcal{N}_{x0}$

$$v(\tau_k(x),\, x + J_k(x)) \le \lambda_M(C^*)\psi_m(v(\tau_k(x), x)), \tag{4.4.13}$$

$$v(\tau_k(x),\, x + J_k(x)) \le \lambda_M(C^*)\psi_M(v(\tau_k(x), x)) \tag{4.4.14}$$

respectively.

Here $\lambda_M(C^*)$ is a maximal eigenvalue of the matrix C^*.

Corollary 4.4.3 is proved in the same way as Corollary 4.4.1.

Assumption 4.4.5 Assume that the conditions (1) and (2) of Assumption 4.4.2 are satisfied and in the inequalities of condition (3) of Assumption 4.4.2 the inequality sign "$\le$" is reversed "$\ge$".

Proposition 4.4.5 *If conditions of Assumption 4.4.5 are satisfied, then for expression (4.4.2):*

$$Dv(t,x,\eta) \ge u^{\mathrm{T}}G(P,S)u \quad t \ne \tau_k(x), \quad k = 1, 2, \dots, \tag{4.4.15}$$

where u and $G(P,S)$ are defined as in Proposition 4.4.2.

Proof Let conditions of Assumption 4.4.5 be satisfied, then (4.4.15) yields:

$$Dv(t,x,\eta) = \eta^{\mathrm{T}} DU(t,x)\eta =$$
$$\sum_{j=1}^{m} \eta_j^{(2)} Dv_{jj}(t,x_j) + 2\sum_{j=1}^{m}\sum_{\substack{l=2\\ l>j}}^{m} \eta_j\eta_l Dv_{jl}(t,x_j,x_l)$$

$$
\begin{aligned}
&= \sum_{j=1}^{m} \eta_j^{(2)} \{D_t v_{jj} + (D_{x_j} v_{jj})^{\mathrm{T}} f_j(t, x^j) + (D_{x_j} v_{jj})^{\mathrm{T}} S_j(t) r_j(t, x, P_j)\} \\
&+ 2 \sum_{j=1}^{m} \sum_{\substack{l=2 \\ l>j}}^{m} \eta_j \eta_l \{D_t v_{jl} + (D_{x_j} v_{jl})^{\mathrm{T}} (f_j(t, x^j) + S_j(t) r_j(t, x, p_j)) \\
&+ (D_{x_l} v_{jl})^{\mathrm{T}} (f_l(t, x^l) + S_l(t) r_l(t, x, p_l))\} \\
&\geq \sum_{j=1}^{m} (\rho_j^{(1)} + \rho_j^{(2)}(P, S)) \varphi_j^2(v_{jj}(t, x_j)) \\
&+ 2 \sum_{j=1}^{m} \sum_{\substack{l=2 \\ l>j}}^{m} \rho_{jl}(P, S) \varphi_j(v_{jj}(t, x_j)) \varphi_l(v_{ll}(t, x_l)) = u^{\mathrm{T}} G(P, S) u
\end{aligned}
$$

for all $t \neq \tau_k(x),\ k = 1, 2, \dots$.

Corollary 4.4.4 Let all conditions of Assumption 4.4.5 be satisfied. If there exists a constant matrix L such that for the matrix $G(P, S)$ the inequality

$$G(P, S) \geq L \quad \text{for all} \quad (P, S) \in \mathcal{P} \times \mathcal{S} \tag{4.4.16}$$

holds component-wise, and

(1) $\lambda_m(L) < 0$;
(2) $\lambda_M(L) > 0$,

then the estimates from below

$$Dv(t, x, \eta) \geq \lambda_m(L)\, \varphi_M(v(t, x, \eta)), \tag{4.4.17}$$

$$Dv(t, x, \eta) \geq \lambda_m(L)\, \varphi_m(v(t, x, \eta)) \quad \text{for all} \quad (t, x) \in \mathcal{T}_0 \times \mathcal{N}_{x0} \tag{4.4.18}$$

hold true respectively.

Here $\lambda_m(L)$ is a minimal eigenvalue of the matrix L.

The proof is similar to that of Corollary 4.4.1.

Assumption 4.4.6 Assume that the condition (1) of Assumption 4.4.3 is satisfied and in condition (2) the inequality sign " $\leq$" is reversed " $\geq$".

Proposition 4.4.6 *If all conditions of Assumption 4.4.6 are satisfied for all* $k = 1, 2, \dots$, *then*

$$v(\tau_k(x),\, x + J_k(x)) - v(\tau_k(x), x) \geq u_k^{\mathrm{T}} C u_k, \quad k = 1, 2, \dots, \quad \textit{for all} \quad x \in \mathcal{N}_{x0}, \tag{4.4.19}$$

where u_k *and* L *are the same as in Proposition 4.4.3.*

The proof is similar to that of Proposition 4.4.3.

Corollary 4.4.5 If in the inequalities (4.4.19) $\lambda_m(C) > 0$, then for all $k = 1, 2, \dots$

$$v(\tau_k(x),\, x + J_k(x)) - v(\tau_k(x), x) \geq \lambda_m(C) \psi_m v(\tau_k(x), x), \quad \text{for all} \quad x \in \mathcal{N}_{x0}, \quad k = 1, 2, \dots . \tag{4.4.20}$$

Assumption 4.4.7 Assume that the condition (1) of Assumption 4.4.4 is satisfied and in condition (2) of Assumption 4.4.4 the inequality sign " $\leq$" is reversed " $\geq$".

Proposition 4.4.7 *If all conditions of Assumption 4.4.7 are satisfied, then for all* $k = 1, 2, \dots$

$$v(\tau_k(x),\, x + J_k(x)) \geq u_k^{\mathrm{T}} C^* u_k, \quad \textit{for all} \quad x \in \mathcal{N}_{x0}, \tag{4.4.21}$$

where u_k *and* C^* *are the same as in Proposition 4.4.4.*

The proof is similar to that of Proposition 4.4.3.

Corollary 4.4.5 If in the inequality (4.4.21) $\lambda_m(C^*) > 0$, then for all $k = 1, 2, \dots$

$$\begin{gathered} v(\tau_k(x),\, x + J_k(x)) \geq \lambda_m(C^*)\psi_m v(\tau_k(x), x), \\ \text{for all} \quad x \in \mathcal{N}_{x0}, \quad k = 1, 2, \dots. \end{gathered} \tag{4.4.22}$$

Assumption 4.4.8 Let

$$\Pi_j = \{(t, x_j) \in \mathcal{T}_0 \times R^{n_j} \colon \; u_{jj}(t, x_j) > 0\}$$

be the positiveness domains for the functions

$$u_{jj}(t, x_j), \quad j = 1, 2, \dots, m,$$

and for every $t \geq t_0$ they have non-zero open intersection with the plane $t = \text{const}$ adjoining to the origin, and in this domain the functions U_{ji}, $j, i = 1, 2, \dots, m$, are bounded.

Proposition 4.4.8 *If all conditions of Assumptions 4.4.1 and 4.4.8 are satisfied and the matrix A in the estimate (4.4.3) is positive definite, i.e.* $\lambda_m(H^{\mathrm{T}} A H) > 0$, *then*

(a) *the domain* $\Pi = \{(t, x) \in \mathcal{T}_0 \times D(\rho) \colon \; v(t, x, \eta) > 0\}$ *of positiveness of function* $v(t, x, \eta)$ *for any* $t \in \mathcal{T}_0$ *has non-zero open intersection with the plane* $t = \text{const}$ *adjoining to the origin;*
(b) *on the domain* Π *the function* $v(t, x, \eta)$ *is bounded.*

Proof If the conditions of Assumption 4.4.1 are satisfied together with the condition $\lambda_m(H^{\mathrm{T}} A H) > 0$, then the positiveness domain of the function $v(t, x, \eta)$ is

$$\widetilde{\Pi} = \{(t, x) \in \mathcal{T}_0 \times D(\rho) \colon \; x \neq 0\}$$

which has an open intersection with the plane $t = \text{const}$ for each $t \in \mathcal{T}_0$. Moreover, the positiveness of the functions $u_{jj}(t, x_j)$, $j = 1, 2, \dots, m$, is a necessary condition for the positiveness of function $v(t, x, \eta)$, therefore $\widetilde{\Pi} \subseteq \bigcap_{j=1}^m \Pi_j$ for every $j = 1, 2, \dots, m$ by the condition of Assumption 4.4.8 has non-zero open intersection with the plane $t = \text{const}$ adjoining to the origin. This proves the assertion (a) of Proposition 4.4.8.

The boundedness of functions u_{ji}, $j, i = 1, 2, \dots, m$, implies that the matrix $U(t, x)$ is bounded, but then the function $v(t, x, \eta)$ constructed by formula (4.4.1) will be bounded as well.

4.4.2 Tests for stability and instability The results presented in Section 4.4.1 enable us to formulate the following theorems on stability and asymptotic stability of the zero solution of large scale impulsive system (4.2.4).

Theorem 4.4.1 *Let large scale impulsive system (4.2.4) be such that*

(1) *in the domain* $\mathcal{T}_0 \times D(\rho)$ *all conditions of Hypotheses 1, 2 and 3 are satisfied;*

(2) *the matrix* A *is positive definite (i.e.* $\lambda_m(H^{\mathrm{T}}AH) > 0$*);*

(3) *there exists a negative semi-definite or equal to zero matrix* Q *such that for the matrix* $G(P,S)$ *the estimate*

$$\frac{1}{2}(G(P,S) + G^{\mathrm{T}}(P,S)) \le Q$$

$$\text{for all} \quad (P,S) \in \mathcal{P} \times \mathcal{S}, \quad t \ne \tau_k(x), \quad k = 1, 2, \dots,$$

is satisfied element-wise;

(4) *the matrix* C *is negative semi-definite or equal to zero (i.e.* $\lambda_M(C) \le 0$*).*

Then the zero solution of large-scale impulsive system (4.2.4) is stable on $\mathcal{P} \times \mathcal{S}$*.*

If condition (4) is modified as follows

(5) *the matrix* C *is negative definite (i.e.* $\lambda_M(C) < 0$*),*

then the zero solution of large scale impulsive system (4.2.4) is asymptotically stable on $\mathcal{P} \times \mathcal{S}$*.*

Proof Under the conditions of Assumptions 4.4.1, Proposition 4.4.1 and condition (1) of Theorem 4.4.1 the function $v(t,x,\eta)$ (see (4.4.1)) is positive definite. The conditions of Assumption 4.4.2, Proposition 4.4.2 and condition (3) of Theorem 4.4.1 imply

$$Dv(t,x,\eta) \le 0 \quad \text{for all} \quad (P,S) \in \mathcal{P} \times \mathcal{S}, \quad t \ne \tau_k(x), \quad k = 1, 2, \dots$$

and the conditions of Assumption 4.4.3, Proposition 4.4.3 and condition (4) of Theorem 4.4.1 yield

$$v(\tau_k(x),\, x + J_k(x)) \le v(\tau_k(x), x), \quad t = \tau_k(x), \quad k = 1, 2, \dots .$$

In this connection for each pair $(P, S) \in \mathcal{P} \times \mathcal{S}$ the conditions being sufficient for the stability of the zero solution of large scale impulsive system (4.2.4) are satisfied on $\mathcal{P} \times \mathcal{S}$.

If instead of condition (4) of Theorem 4.4.1 the condition (4.2.4) of the Theorem is satisfied, then Proposition 3 and Corollary 4.4.2 yield

$$v(\tau_k(x),\, x + J_k(x)) - v(\tau_k(x), x) \le \lambda_M(C)\psi_M(v(\tau_k(x), x)), \quad k = 1, 2, \dots$$

and therefore the conditions being sufficient for asymptotic stability are satisfied for every $(P, S) \in \mathcal{P} \times \mathcal{S}$. Hence, the zero solution of large scale impulsive system (4.2.4) is asymptotically stable on $\mathcal{P} \times \mathcal{S}$.

Theorem 4.4.2 *Let large scale impulsive system (4.2.4) be such that*

(1) *on the domain $\mathcal{T}_0 \times D(\rho)$ Hypotheses 1, 2 and 4 are satisfied;*
(2) *the matrix A is positive definite (i.e. $\lambda_m(H^{\mathrm{T}}AH) > 0$);*
(3) *there exists a negative definite matrix $\widetilde{Q} \in R^{m\times m}$ such that the estimate*

$$\frac{1}{2}\big(G(P, S) + G^{\mathrm{T}}(P, S)\big) \le \widetilde{Q} \quad \text{for all} \quad (P, S) \in \mathcal{P} \times \mathcal{S}$$

is satisfied;
(4) $\lambda_M(C^*) > 0$
(5) *the functions $\tau_k(x)$, $k = 1, 2, \dots$, satisfy the inequality*

$$\sup_k \Big(\min_{x\in D(\rho)} \tau_{k+1}(x) - \max_{x\in D(\rho)} \tau_k(x) \Big) = \theta > 0,$$

where $\rho < \rho_0$.

If there exists a constant $\alpha_0 > 0$ such that for every $\alpha \in (0, \alpha_0]$ the functions $\varphi_m(y)$ and $\psi_M(y)$ satisfy the inequality

$$-\frac{1}{\lambda_M(\tilde{Q})} \int\limits_{\alpha}^{\lambda_M(C^*)\psi_M(\alpha)} \frac{dy}{\varphi_m(y)} \le \theta, \tag{4.4.23}$$

then the zero solution of large scale impulsive system (4.2.4) is stable on $\mathcal{P} \times \mathcal{S}$.

If instead of inequality (4.4.23) for some $\gamma > 0$ the inequality

$$-\frac{1}{\lambda_M(\tilde{Q})} \int\limits_{\alpha}^{\lambda_M(C^*)\psi_M(\alpha)} \frac{dy}{\varphi_m(y)} \le \theta - \gamma \tag{4.4.24}$$

is satisfied, then the zero solution of large scale impulsive system (4.2.4) is asymptotically stable on $\mathcal{P} \times \mathcal{S}$.

The proof of Theorem 4.4.2 is similar to that of Theorem 4.4.1.

Theorem 4.4.3 *Let large scale impulsive system (4.2.4) be such that*

(1) *on the domain* $\mathcal{T}_0 \times D(\rho)$ *Assumptions 4.4.1, 4.4.2 and 4.4.4 hold;*
(2) *the matrix* A *is positive definite (i.e.* $\lambda_m(H^{\mathrm{T}}AH) > 0$*);*
(3) *there exists a matrix* $\widetilde{Q} \in R^{m\times m}$ *for which*
   (a) $\frac{1}{2}(G(P,S) + G^{\mathrm{T}}(P,S)) \le \widetilde{Q}$ *for all* $(P,S) \in \mathcal{P} \times \mathcal{S}$*;*
   (b) $\lambda_M(\tilde{Q}) > 0$*;*
(4) $\lambda_M(C^*) > 0$*;*
(5) *the functions* $\tau_k(x)$, $k = 1, 2, \dots,$ *satisfy for some* $\theta_1 > 0$ *the inequality*

$$\max_{x\in D(\rho)} \tau_k(x) - \min_{x\in D(\rho)} \tau_{k-1}(x) \le \theta_1, \quad \rho \le \rho_0;$$

(6) *there exists a constant* α_0 *such that for every* $\alpha \in (0, \alpha_0]$ *the functions* $\varphi_m(y)$ *and* $\psi_M(y)$ *satisfy the inequality*

$$\frac{1}{\lambda_M(\widetilde{Q})} \int\limits_{\lambda_M(C^*)\psi_M(\alpha)}^{\alpha} \frac{dy}{\varphi_m(y)} \ge \theta_1 + \gamma. \tag{4.4.25}$$

Then the zero solution of large scale impulsive system(4.2.4) is asymptotically stable on $\mathcal{P} \times \mathcal{S}$.

The proof of Theorem 4.4.3 is similar to that of Theorem 4.4.1.

Assume that large scale impulsive system (4.2.4) is decomposed into m interconnected impulsive subsystems (4.2.4), (4.2.5).

Theorem 4.4.4 *Let large scale impulsive system (4.2.4), (4.2.5) be such that*

(1) *Assumptions 4.4.1, 4.4.5, 4.4.6 and 4.4.8 are satisfied on the domain* Π*;*
(2) *the matrix* A *is positive definite (i.e.* $\lambda_m(H^{\mathrm{T}}AH) > 0$*);*
(3) *there exists a positive semi-definite or equal to zero matrix* $L \in R^{m\times m}$ *for which*

$$\frac{1}{2}(G(P,S) + G^{\mathrm{T}}(P,S)) \ge L \quad \text{for all} \quad (P,S) \in \mathcal{P} \times \mathcal{S};$$

(4) *the matrix* C *is positive definite (i.e.* $\lambda_m(C) > 0$*).*

Then the zero solution of large scale impulsive system (4.2.4), (4.2.5) is unstable on $\mathcal{P} \times \mathcal{S}$.

Proof Under the conditions of Assumptions 4.4.1 and 4.4.8, Propositions 4.4.1, 4.4.8 and condition (2) of Theorem 4.4.4 the function $v(t, x, \eta)$ is positive definite and possesses properties (a) and (b) (see Proposition 4.4.8).

Assumption 4.4.5, Proposition 4.4.5 and condition (3) of Theorem 4.4.4 imply that for every pair $(P, S) \in \mathcal{P} \times \mathcal{S}$ the inequality

$$Dv(t, x, \eta) \geq 0, \quad t \neq \tau_k(x), \quad k = 1, 2, \dots,$$

is satisfied.

Under the conditions of Assumption 4.4.6, Proposition 4.4.6, Corollary 4.4.5 and condition (4) of Theorem 4.4.4 one has

$$v(\tau_k(x),\, x + J_k(x)) - v(\tau_k(x), x) \geq \lambda_m(C)\psi_m(v(\tau_k(x), x)), \quad k = 1, 2, \dots.$$

Moreover, for every pair $(P, S) \in \mathcal{P} \times \mathcal{S}$ all hypotheses of Theorem 1.4.7 are satisfied with function $\psi(y) = \lambda_m(C)\psi_m(y)$ i.e. the zero solution of the system (4.2.4), (4.2.5) is unstable on $\mathcal{P} \times \mathcal{S}$.

Theorem 4.4.5 *Let large scale impulsive system (4.2.4), (4.2.5) be such that*

(1) *Assumption 4.4.1, 4.4.5, 4.4.7 and 4.4.8 hold in the domain* Π*;*
(2) *the matrix* A *is positive definite (i.e.* $\lambda_m(H^{\mathrm{T}}AH) > 0$*);*
(3) *there exists a matrix* $\widetilde{L} \in R^{m \times m}$ *such that*

(a) $\frac{1}{2}(G(P, S) + G^{\mathrm{T}}(P, S)) \geq \widetilde{L}$ *for all* $(P, S) \in \mathcal{P} \times \mathcal{S}$*;*

(b) $\lambda_m(\widetilde{L}) < 0$*;*

(4) *the matrix* C^* *is positive definite (i.e.* $\lambda_m(C^*) > 0$*);*
(5) *for some constant* $\theta_1 > 0$ *the functions* $\tau_k(x)$ *satisfy the inequalities*

$$\max_{x \in D(\rho)} \tau_k(x) - \min_{x \in D(\rho)} \tau_{k-1}(x) \le \theta_1, \quad \rho < \rho_0 \quad \textit{for all} \quad k = 1, 2, \dots;$$

(6) *there exists a constant* α_0 *such that for every* $\alpha \in (0, \alpha_0]$ *the functions* $\varphi_M(y)$ *and* $\psi_m(y)$ *satisfy the inequality*

$$-\frac{1}{\lambda_m(\widetilde{L})} \int\limits_{\alpha}^{\lambda_m(C^*)\psi_m(\alpha)} \frac{dy}{\varphi_M(y)} \ge \theta_1 + \gamma. \tag{4.4.26}$$

Then the zero solution of large scale impulsive system (4.2.4), (4.2.5) is unstable on $\mathcal{P} \times \mathcal{S}$.

Theorem 4.4.5 is proved in the same way as Theorem 4.4.4. Besides, for each pair $(P, S) \in \mathcal{P} \times \mathcal{S}$ all conditions of Theorem 1.4.6 with the functions

$$\varphi(y) = -\lambda_m(\widetilde{L})\varphi_M(y) \quad \text{and} \quad \psi(y) = \lambda_m(C^*)\psi_M(y).$$

are satisfied.

Theorem 4.4.6 *Let large scale impulsive system (4.2.4), (4.2.5) be such that*

(1) *Assumption 4.4.1, 4.4.5, 4.4.7 and 4.4.8 hold in the domain* Π*;*
(2) *the matrix* A *is positive definite (i.e.* $\lambda_m(H^{\mathrm{T}}AH) > 0$*);*
(3) *there exists a positive definite matrix* $L^* \in R^{m\times m}$ *for which*

$$\frac{1}{2}\,(G(P,S) + G^{\mathrm{T}}(P,S)) \ge L^* \quad \textit{for all} \quad (P,S) \in \mathcal{P} \times \mathcal{S};$$

(4) $\lambda_m(C^*) > 0$*;*
(5) *for some constant* $\theta > 0$ *the functions* $\tau_k(x)$ *satisfy the inequality*

$$\sup_k \Big(\min_{x \in D(\rho)} \tau_{k-1}(x) - \max_{x \in D(\rho)} \tau_k(x) \Big) = \theta > 0, \quad \rho < \rho_0;$$

(6) *for a constant* $\gamma > 0$ *the functions* $\varphi_m(y)$ *and* $\psi_m(y)$ *satisfy the inequality*

$$-\frac{1}{\lambda_m(L^*)} \int\limits_{\lambda_m(C^*)\psi_m(\alpha)}^{\alpha} \frac{dy}{\varphi_m(y)} \le \theta - \gamma.$$

Then the zero solution of large scale impulsive system (4.2.4), (4.2.5) is unstable on $\mathcal{P} \times \mathcal{S}$.

This Theorem is proved in the same way as Theorem 4.4.4. We note that all hypotheses on instability of Theorem 1.4.6 are satisfied with the functions

$$\varphi(y) = \lambda_m(L^*)\varphi_m(y) \quad \text{and} \quad \psi(y) = \lambda_m(C^*)\psi_m(y),$$

provided all conditions of Theorem 4.4.6 hold.

Example 4.4.1 Consider an impulsive fourth order system consisting of two subsystems of the second order which are described by the systems of equations

$$\frac{dx_j}{dt} = -x_j^3 + 0,5x_i^3 + 0,25S_{j1}x_j^3 + 0,3S_{j2}(t)x_i^3,$$

$$
(4.4.27) \qquad \begin{aligned} &t \neq \tau_k(x_1, x_2),\\ \Delta x_j = &- x_j + \sigma x_i, \quad t = \tau_k(x_1, x_2),\\ &j, i = 1, 2; \quad j \neq i, \end{aligned}
$$

where $x_j = (x_{j1}, x_{j2})^{\mathrm{T}} \in R^2$, $j = 1, 2$.

In this example $\mathcal{P} = \{0\}$ and the structural matrices $S_j(t)$ are of the form

$$
S_j(t) = \begin{pmatrix} 1 & 0 & S_{j1}(t) & 0 & S_{j2}(t) & 0 \\ 0 & 1 & 0 & S_{j1}(t) & 0 & S_{j2}(t) \end{pmatrix}, \qquad j = 1, 2,
$$

$$
S(t) = \mathrm{diag}\,\{S_1(t), S_2(t)\}.
$$

The structural set of the system (4.4.27) is defined as

$$
\mathcal{S} = \Big\{ S(t)\colon\ S(t) = \begin{pmatrix} S_1(t) & 0 \\ 0 & S_2(t) \end{pmatrix}, \quad S_j(t) = (I_2,\ s_{j1} I_2,\ s_{j2}(t) I_2),
$$

$$
s_{ji}(t) \in [0, 1] \ \text{ for all } \ t \in R, \ \ s_{ji}(\tau_k(x)) = 0, \ \ j, i = 1, 2, \ \ k = 1, 2, \dots \Big\}.
$$

For the system (4.4.27) we construct the matrix function $U(x)$ with the elements

$$
v_{jj}(x_j) = x_j^2, \quad j = 1, 2; \quad v_{12}(x_1, x_2) = 0,5 x_1 x_2,
$$

satisfying the estimates

$$
\begin{aligned} v_{jj}(x_j) &\geq \|x_j\|^2, \quad j = 1, 2,\\ v_{12}(x_1, x_2) = v_{21}(x_1, x_2) &\geq -0,5 \|x_1\|\, \|x_2\|. \end{aligned}
$$

The matrix

$$
A = \begin{pmatrix} 1 & -0,5 \\ -0,5 & 1 \end{pmatrix}
$$

is positive definite.

Let $\eta^{\mathrm{T}} = (1, 1)$, then for the above-mentioned matrix-valued function $U(x)$, the elements of matrix $G(S)$ (see (4.4.4)) and matrix C (see (4.4.9)) have the form

$$
\begin{aligned} \sigma_{11}(s) &= -1,5 + 0,5 s_{11}(t) + 0,6 s_{22}(t) \leq -0,4;\\ \sigma_{22}(s) &= -1,5 + 0,6 s_{12}(t) + 0,5 s_{21}(t) \leq -0,4;\\ \sigma_{12}(s) &= 0,05(s_{11}(t) + s_{21}(t)) + 0,15(s_{12}(t) + s_{21}(t)) \leq 0,4, \end{aligned}
$$

$$
C = \begin{pmatrix} \sigma^2 - 1 & \dfrac{|\sigma^2 - 1|}{2} \\ \dfrac{|\sigma^2 - 1|}{2} & \sigma^2 - 1 \end{pmatrix}
$$

It is easy to verify that the matrix Q

$$
G(S) \leq Q = \begin{pmatrix} -0,4 & 0,4 \\ 0,4 & -0,4 \end{pmatrix}
$$

is negative semi-definite, and the matrix C for $\sigma = \pm 1$ is equal to zero, and for $|\sigma| < 1$ it is negative definite (i.e. $\lambda_m(C) < 0$).

Since all conditions of Theorem 4.4.1 are satisfied, the zero solution of the impulsive systems (4.4.27) under nonclassical structural perturbations for $\sigma = \pm 1$ is stable on $\mathcal{S}$ and for $|\sigma| < 1$ is asymptotically stable on $\mathcal{S}$.

4.5 Linear Systems Analysis

We consider the linear large scale impulsive system decomposed into m subsystems

$$\begin{aligned} \frac{dx_i}{dt} &= A_i x_i + \sum_{j=1}^{m} S_{ij} A_{ij} x_j, \quad t \neq \tau_k(x), \\ \Delta x_i &= J_{ki} x_i + \sum_{\substack{j=1 \\ j \neq i}}^{m} J_{kij} x_j, \quad t = \tau_k(x), \\ & i = 1, 2, \dots, m, \quad k = 1, 2, \dots, \end{aligned} \tag{4.5.1}$$

where $x_i \in R^{n_i}$, $\sum\limits_{i=1}^{m} n_i = n$, $x = (x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, \dots, x_m^{\mathrm{T}})^{\mathrm{T}} \in R^n$, A_i, J_{ki}, A_{ij}, J_{kij} are constant matrices of the correspondent dimensions, the set $\mathcal{S}$ and matrices S, S_i, S_{ij} are defined in Appendix 1, the values $\tau_k(x)$, $k = 1, 2, \dots$ are ordered by $\tau_k(x) < \tau_{k+1}(\tau)$ and such that $\tau_k(x) \to +\infty$ as $k \to +\infty$. We shall assume, for simplicity, that the system (4.5.1) satisfies all required conditions so that all solutions $x(t) = x(t, t_0, x_0)$ of (4.5.1) exist for all $t \geq t_0$.

For the system (4.5.1) we construct a matrix-valued function

$$U(x) = [v_{ij}(x_i, x_j)], \quad i,j = 1,2,\dots,m \tag{4.5.2}$$

with the elements

$$v_{ii}(x_i) = x_i^{\mathrm{T}} B_{ii} x_i, \quad i = 1,2,\dots,m \tag{4.5.3}$$

and

$$v_{ij}(x_i, x_j) = x_i^{\mathrm{T}} B_{ij} x_j, \quad i \neq j, \quad i,j = 1,2,\dots,m. \tag{4.5.4}$$

Here B_{ii} are constant positive definite matrices, and B_{ij} are constant matrices.

We introduce the following assumption.

Assumption 4.5.1 Assume that there exist

(1) the matrix-valued function (4.5.2) with the elements (4.5.3) and (4.5.4);

(2) the constants a_{ji}, b_{ji}, $i,j = 1,2,\dots,m$, satisfying the estimates

(a) $a_{ii}\|x_i\|^2 \leq U_{ii}(x_i) \leq b_{ii}\|x_i\|^2$ for all $x_i \in \mathcal{N}_{ix}$, $i = 1,2,\dots,m$,

(b) $a_{ji}\|x_j\|\,\|x_i\| \leq U_{ij}(x_i, x_j) \leq b_{ji}\|x_j\|\,\|x_i\|$ for all $(x_i, x_j) \in \mathcal{N}_{ix} \times \mathcal{N}_{jx}$, $i \neq j$, $i,j = 1,2,\dots,m$.

Proposition 4.5.1 *If all conditions of Assumption 4.5.1 are satisfied, then the function*

$$v(x,\eta) = \eta^{\mathrm{T}} U(x) \eta, \quad \eta \in R_+^m, \quad \eta > 0 \tag{4.5.5}$$

satisfies the bilateral inequality

$$\begin{gathered} u^{\mathrm{T}} H^{\mathrm{T}} A H u \leq v(x,\eta) \leq u^{\mathrm{T}} H^{\mathrm{T}} B H u \\ \textit{for all} \quad x \in \mathcal{N}_x = \mathcal{N}_{1x} \times \mathcal{N}_{2x} \times \ldots \times \mathcal{N}_{mx}. \end{gathered} \tag{4.5.6}$$

Here

$$u^{\mathrm{T}} = (\|x_1\|, \|x_2\|, \dots, \|x_m\|), \quad A = [a_{ij}],$$
$$B = [b_{ij}], \quad H = \operatorname{diag}[\eta_1, \eta_2, \dots, \eta_m].$$

The proof of Proposition 4.5.1 is similar to that of Proposition 4.4.1 (see and cf. Djordjevic [1]).

Together with the function (4.5.5) its total derivative

$$Dv(x,\eta) = \eta^{\mathrm{T}} DU(x) \eta \tag{4.5.7}$$

along the solutions $x(t, t_0, x_0)$ of the system (4.5.1) is constructed

Assumption 4.5.2 Assume that there exist

(1) the matrix-valued function (4.5.2) with the elements (4.5.3) and (4.5.4);

(2) the constants $\widetilde{\rho}_j^{(1)}(S)$, $\widetilde{\rho}_j^{(2)}(S)$, $\widetilde{\rho}_{ij}(S)$, $i \neq j$, $i,j = 1,2,\dots,m$, and

(a) $\eta_j^2\{(D_{x_j} U_{jj}(x_j))^{\mathrm{T}} A_j x_j\} \leq \widetilde{\rho}_j^{(1)}\|x_j\|^2$ for all $x_j \in \mathcal{N}_{jx0}$, $j = 1,2,\dots,m$;

$$
\begin{aligned}
\text{(b)}\quad & \sum_{j=1}^{m} \eta_j^2 (D_{x_j} U_{jj}(x_j))^{\mathrm{T}} \sum_{\substack{i=1\\ i\neq j}}^{m} S_{ij} A_{ij} x_i \\
& + 2\sum_{j=1}^{m}\sum_{i=j+1}^{m} \eta_j\eta_i \Big\{ (D_{x_j} U_{ji}(x_j, x_i))^{\mathrm{T}} \Big(A_j x_j + \sum_{\substack{k=1\\ k\neq j}}^{m} S_{kj} A_{kj} x_k \Big) \\
& + \Big(D_{x_j} U_{ji}(x_j, x_i))^{\mathrm{T}} (A_i x_i + \sum_{\substack{k=1\\ k\neq i}}^{m} S_{ik} A_{ik} x_k \Big) \Big\} \\
& \le \sum_{j=1}^{m} \widetilde{\rho}_j^{(2)}(S) \|x_j\|^2 + 2\sum_{j=1}^{m}\sum_{i=j+1}^{m} \widetilde{\rho}_{ji}(S) \|x_j\|\,\|x_i\|
\end{aligned}
$$

for all $\quad (x_i, x_j) \in \mathcal{N}_{jx0} \times \mathcal{N}_{ix0} \times \mathcal{S}, \quad (i \neq j), \quad i, j = 1, 2, \dots, m.$

Proposition 4.5.2 *If all conditions of Assumption 4.5.2 are satisfied, then for expression (4.5.7) we get*

$$
(4.5.8) \qquad Dv(x,\eta) \le u^{\mathrm{T}} \overline{G}(S) u, \quad \textit{for all} \quad x \in \mathcal{N}_{x0} \times \mathcal{S},
$$

where

$$
\begin{aligned}
u^{\mathrm{T}} &= (\|x_1\|, \|x_2\|, \dots, \|x_m\|), \\
\overline{G}(S) &= [\bar{\sigma}_{ji}(S)], \quad i, j = 1, 2, \dots, m, \\
\bar{\sigma}_{ji}(S) &= \bar{\sigma}_{ij}(S), \\
\bar{\sigma}_{jj}(S) &= \widetilde{\rho}_j^{(1)} + \widetilde{\rho}_j^{(2)}(S), \\
\bar{\sigma}_{ji}(S) &= \widetilde{\rho}_{ji}(S), \quad j \neq i, \quad i, j = 1, 2, \dots, m.
\end{aligned}
$$

The proof of Proposition 4.5.2 is similar to that of Proposition 4.5.1.

It can be easily verified that for $t \neq \tau_k(x)$, $k = 1, 2, \dots,$ the estimate

$$
(4.5.9) \quad Dv(x,\eta) \le \lambda_M(\overline{G}(S)) \|u\|^2, \quad \text{for all} \quad x \in \mathcal{N}_{x0}, \quad \text{for all} \quad S \in \mathcal{S}
$$

is true. Here $\lambda_M(\cdot)$ is the maximal eigenvalues of $(\cdot)$. If $\eta^{\mathrm{T}} = (1, 1, \dots, 1) \in R_+^m$ then from (4.5.6) we get

$$
(4.5.10) \qquad \lambda_m(A)\|u\|^2 \le v(x,\eta) \le \lambda_M(B)\|u\|^2
$$

and for $\lambda_m(A) > 0$ we get

$$
(4.5.11) \qquad \lambda_M^{-1}(B) v(x,\eta) \le \|u\|^2 \le \lambda_m^{-1}(A) v(x,\eta).
$$

Therefore, the estimate (4.5.9) can be represented as

$$
Dv(x,\eta) \le \begin{cases} \lambda_M(\overline{G}(S)) \lambda_m^{-1}(A) v(x,\eta) & \text{for } \lambda_M(\overline{G}(S)) > 0; \\ \lambda_M(\overline{G}(S)) \lambda_M^{-1}(B) v(x,\eta) & \text{for } \lambda_M(\overline{G}(S)) < 0. \end{cases}
$$

Proposition 4.5.3 *If for the system (4.5.1) the condition (1) of Assumption 4.5.1 is satisfied, then for the function (4.5.5) when* $t = \tau_k(x)$, $k = 1, 2, \dots,$ *the inequalities*

$$v(x + J_k(x), \eta) - v(x, \eta) \le u_k^{\mathrm{T}} \overline{C} u_k; \tag{4.5.12}$$

and

$$v(x + J_k(x), \eta) \le u_k^{\mathrm{T}} \overline{C}^* u_k, \tag{4.5.13}$$

are satisfied, where

$$u_k^{\mathrm{T}} = (\|x_1(\tau_k(x))\|, \|x_2(\tau_k(x))\|, \dots, \|x_m(\tau_k(x))\|),$$

$$J_k(x) = J_{ki} x_i + \sum_{\substack{j=1 \\ j \ne i}}^{m} J_{kij} x_j,$$

$$\overline{C} = [\bar{c}_{ij}], \quad \bar{c}_{ij} = \bar{c}_{ji}, \quad i, j = 1, 2, \dots, m,$$

$$\overline{C}^* = [\bar{c}^*_{ij}], \quad \bar{c}^*_{ij} = \bar{c}^*_{ji}, \quad i, j = 1, 2, \dots, m,$$

$$\bar{c}_{ii} = \lambda_M(C_{ii}), \quad \bar{c}_{ij} = \lambda_M^{1/2}(C_{ij} C_{ij}^{\mathrm{T}}), \quad i \ne j, \quad i, j = 1, 2, \dots, m,$$

$$\lambda_M^{1/2}(\cdot) \quad \textit{is a norm of matrix } (\cdot),$$

$$\bar{c}^*{}_{ii} = \lambda_M(C^*_{ii}), \quad \bar{c}^*_{ij} = \lambda_M^{1/2}(C^*_{ij} C^{*T}_{ij}), \quad i \ne j, \quad i, j = 1, 2, \dots, m;$$

and

$$C_{ii} = J_{ki}^{\mathrm{T}} B_{ii} + B_{ii} J_{ki} + J_{ki}^{\mathrm{T}} B_{ii} J_{ki} + \sum_{\substack{j=1 \\ j \ne i}}^{m} J_{kij}^{\mathrm{T}} B_{jj} J_{kji}$$

$$+\sum_{\substack{j=1\\j\neq i}}^{m}(B_{ij}J_{kji}+J_{kji}^{\mathrm{T}}B_{ij})+\sum_{\substack{j=1\\j\neq i}}^{m}(J_{ki}^{\mathrm{T}}B_{ij}J_{kji}+J_{kji}^{\mathrm{T}}B_{ij}J_{ki})$$

$$+\sum_{\substack{l=1\\l\neq i}}^{m}\sum_{\substack{j=1\\j\neq i}}^{m}(J_{kli}^{\mathrm{T}}B_{lj}J_{kji}+J_{kji}^{\mathrm{T}}B_{lj}J_{kli}),\quad i=1,2,\dots,m;$$

$$C_{ij}=B_{ii}J_{kij}+J_{kij}^{\mathrm{T}}B_{ii}+J_{ki}^{\mathrm{T}}B_{ii}J_{kij}+J_{kij}^{\mathrm{T}}B_{ii}+J_{ki}$$

$$+\sum_{\substack{l=1\\l\neq i,j}}^{m}(J_{kli}^{\mathrm{T}}B_{ll}J_{klj}+J_{kli}^{\mathrm{T}}B_{ll}J_{kli})+B_{ij}J_{kj}+J_{ki}^{\mathrm{T}}B_{ij}+J_{ki}^{\mathrm{T}}B_{ij}J_{kj}$$

$$+\sum_{\substack{l=1\\l\neq i,j}}^{m}(B_{il}J_{klj}+J_{kli}^{\mathrm{T}}B_{lj}+J_{ki}^{\mathrm{T}}B_{il}J_{klj}+J_{kli}^{\mathrm{T}}B_{lj}J_{kj})$$

$$+\sum_{l=1}^{m}\sum_{\substack{l=1\\l\neq r}}^{m}J_{kli}^{\mathrm{T}}B_{lr}J_{krj},\quad i\neq j,\quad i,j=1,2,\dots,m;$$

$$C_{ii}^{*}=B_{ii}+C_{ii},\quad C_{ij}^{*}=B_{ij}+C_{ij},\quad i\neq j,\quad i,j=1,2,\dots,m.$$

Proof First we consider the inequality (4.5.12). For all $t=\tau_k(x)$, $k=1,2,\dots,$ for the function (4.5.5) and the system (4.5.1), we have

$$v(x+J_k(x),\,\eta)-v(x,\eta)=\sum_{i=1}^{m}U_{ii}\Bigg(x_i+J_{ki}x_i+\sum_{\substack{j=1\\j\neq i}}^{m}J_{kij}x_j\Bigg)$$

$$+2\sum_{i=1}^{m}\sum_{\substack{j=1\\j>i}}^{m}U_{ij}\Bigg(x_i+J_{ki}x_i+\sum_{\substack{l=1\\l\neq i}}^{m}J_{kil}x_l,\ x_j+J_{kj}x_j+\sum_{\substack{l=1\\l\neq i}}^{m}J_{kjl}x_l\Bigg)$$

$$-\sum_{i=1}^{m}U_{ii}(x_i)-2\sum_{i=1}^{m}\sum_{\substack{j=1\\j>i}}^{m}U_{ij}(x_i,x_j)$$

$$=\sum_{i=1}^{m}\Bigg(x_i+J_{ki}x_i+\sum_{\substack{j=1\\j\neq i}}^{m}J_{kij}x_j\Bigg)^{\mathrm{T}}B_{ii}\Bigg(x_i+J_{ki}x_i+\sum_{\substack{j=1\\j\neq i}}^{m}J_{kij}x_j\Bigg)$$

$$+2\sum_{i=1}^{m}\sum_{\substack{j=1\\j>i}}^{m}\Bigg(x_i+J_{ki}x_i+\sum_{\substack{l=1\\l\neq i}}^{m}J_{kil}x_l\Bigg)^{\mathrm{T}}B_{ij}\Bigg(x_j+J_{kj}x_j+\sum_{\substack{l=1\\l\neq j}}^{m}J_{kjl}x_l\Bigg)$$

$$-\sum_{i=1}^{m}x_i^{\mathrm{T}}B_{ii}x_i-2\sum_{i=1}^{m}\sum_{\substack{j=1\\j>i}}^{m}x_i^{\mathrm{T}}B_{ij}x_j$$

$$=\sum_{i=1}^{m}x_i^{\mathrm{T}}\Bigg(B_{ii}J_{ki}+J_{ki}^{\mathrm{T}}B_{ii}+J_{ki}^{\mathrm{T}}B_{ii}J_{ki}+\sum_{\substack{j=1\\j\neq i}}^{m}J_{kji}^{\mathrm{T}}B_{ii}J_{kji}\Bigg)x_i$$

$$+2\sum_{i=1}^{m}\sum_{j=1}^{m}x_i^{\mathrm{T}}\Bigg(B_{ii}J_{kij}+J_{kij}^{\mathrm{T}}B_{ii}+J_{ki}^{\mathrm{T}}B_{ii}J_{kij}+J_{kij}^{\mathrm{T}}B_{ii}J_{ki}$$

$$+\sum_{\substack{l=1\\ l\neq i,j}}^{m}\big(J_{kli}^{\mathrm T}B_{ll}J_{klj}+J_{klj}^{\mathrm T}B_{ll}J_{kli}\big)\Big)x_j+\sum_{i=1}^{m}x_i^{\mathrm T}\Bigg\{\sum_{\substack{j=1\\ j\neq i}}^{m}\big(B_{ij}+J_{kij}+J_{kji}^{\mathrm T}B_{ij}\big)$$

$$+\sum_{\substack{j=1\\ j\neq i}}^{m}\big(J_{ki}^{\mathrm T}B_{ij}J_{kji}+J_{kji}^{\mathrm T}B_{ij}J_{ki}\big)+\sum_{\substack{l=1\\ l\neq i}}^{m}\sum_{\substack{j=1\\ j\neq i}}^{m}\big(J_{kli}^{\mathrm T}B_{lj}J_{kji}+J_{kji}^{\mathrm T}B_{lj}J_{kli}\big)\Bigg\}x_i$$

$$+2\sum_{i=1}^{m}\sum_{\substack{j=1\\ j>i}}^{m}x_i^{\mathrm T}\Bigg\{B_{ij}J_{kj}+J_{ki}^{\mathrm T}B_{ij}+J_{ki}^{\mathrm T}B_{ij}J_{kj}$$

$$+\sum_{\substack{l=1\\ l\neq i,j}}^{m}\big(B_{il}J_{klj}+J_{kli}^{\mathrm T}B_{lj}+J_{ki}^{\mathrm T}B_{il}J_{klj}+J_{kli}^{\mathrm T}B_{lj}J_{kj}\big)+\sum_{l=1}^{m}\sum_{\substack{r=1\\ l\neq r}}^{m}J_{li}^{\mathrm T}B_{lr}J_{rj}\Bigg\}x_j$$

$$=\sum_{i=1}^{m}x_i^{\mathrm T}C_{ii}x_i+2\sum_{i=1}^{m}\sum_{\substack{j=1\\ j>i}}^{m}x_i^{\mathrm T}C_{ij}x_j$$

$$\leq\sum_{i=1}^{m}\lambda_M(C_{ii})\|x_i\|^2+2\sum_{i=1}^{m}\sum_{\substack{j=1\\ j>i}}^{m}\lambda_M^{1/2}(C_{ij}C_{ij}^{\mathrm T})\|x_i\|\|x_j\|=u_k^{\mathrm T}Cu_k,$$
$$k=1,2,\dots.$$

Inequality (4.5.13) is proved in the same way.

Corollary 4.5.1 If all conditions of Proposition 4.5.3 are satisfied, then for the function (4.5.5) for $t=\tau_k(x),\ k=1,2,\dots,$ the following estimates hold true

(4.5.14) $$v(x+J_k(x),\eta)-v(x,\eta)\leq\Delta v(x,\eta)$$

where

$$\Delta=\begin{cases}\lambda_M(\overline{C})\lambda_M^{-1}(B) & \text{for }\ \lambda_M(\overline{C})<0;\\ \lambda_M(\overline{C})\lambda_m^{-1}(A) & \text{for }\ \lambda_M(\overline{C})>0;\end{cases}$$

and

(4.5.15) $$v(x+J_k(x),\eta)\leq\Delta^* v(x,\eta),$$

where

$$\Delta^*=\begin{cases}\lambda_M(\overline{C}^*)\lambda_M^{-1}(B) & \text{for }\ \lambda_M(\overline{C}^*)<0,\\ \lambda_M(\overline{C}^*)\lambda_m^{-1}(A) & \text{for }\ \lambda_M(\overline{C}^*)>0.\end{cases}$$

The assertions (4.5.14) and (4.5.15) follow from Proposition 4.5.3 and the inequality (4.5.11).

Proposition 4.5.4 *If $t\neq\tau_k(x),\ k=1,2,\dots,$ then for the total derivative (4.5.7) of the function (4.5.5) the estimate*

(4.5.16)

$$Dv(x,\eta)\geq u^{\mathrm T}\underline{G}(S)u\quad\textit{for all}\quad(x\neq 0)\in R^n\quad\textit{and}\quad\textit{for all}\quad S\in\mathcal{S}$$

is true, where

$$\underline{G}(S)=[\underline{\sigma}_{ij}(S)],\quad \underline{\sigma}_{ij}=\underline{\sigma}_{ji},\quad i,j=1,2,\dots,m,$$
$$\underline{\sigma}_{ii}=\underline{\rho}_{1i}+\underline{\rho}_{2i}(S),\quad i=1,2,\dots,m,\quad S\in\mathcal{S}$$
$$\underline{\sigma}_{ij}=\frac{1}{2}\big(\underline{\rho}_{1ij}(S)+\underline{\rho}_{1ji}(S)+\underline{\rho}_{2ij}(S)+\underline{\rho}_{2ji}(S)+\underline{\rho}_{3ij}(S)+\underline{\rho}_{3ji}(S)\big),$$
$$i,j=1,2,\dots,m,\quad i\neq j.$$

$\underline{\rho}_{1i}$ *and* $\underline{\rho}_{2i}(S)$ *are minimal eigenvalues of the matrices*

$$Q_i = \eta_i^2(B_{ii}A_i + A_i^{\mathrm{T}}B_{ii}), \quad i = 1, 2, \dots, m;$$

$$P_i = \sum_{j=1}^{i-1} \eta_i\eta_j(B_{ji}^{\mathrm{T}}S_{ij}A_{ji} + (S_{ji}A_{ji})^{\mathrm{T}}B_{ji})$$

$$+ \sum_{j=i+1}^{m} \eta_i\eta_j(B_{ij}S_{ji} + (S_{ji}A_{ji})^{\mathrm{T}}B_{ij}), \quad i \neq j = 1, 2, \dots, m,$$

the vector u^{T} *is defined as in Proposition 4.5.4, and* $\underline{\rho}_{rij}$, $r = 1, 2, 3$; $i, j = 1, 2, \dots, m$, *are computed.*

The proof of Proposition 4.5.4 is similar to that of Proposition 4.5.2.

Let $\eta^{\mathrm{T}} = (1, 1, \dots, 1) \in R_+^m$. Then in view of (4.5.16) and (4.5.11) the inequality

$$Dv(x, \eta) \geq \lambda_m(\underline{G}(S))\|u\|^2, \quad \text{for all} \quad S \in \mathcal{S} \tag{4.5.17}$$

can be rewritten in the form

$$Dv(x, \eta) \geq \begin{cases} \lambda_m(\underline{G}(S))\lambda_m^{-1}(A)v(x, \eta) & \text{for } \lambda_m(\underline{G}(S)) < 0 \\ \lambda_m(\underline{G}(S))\lambda_M^{-1}(B)v(x, \eta) & \text{for } \lambda_m(\underline{G}(S)) > 0. \end{cases}$$

Proposition 4.5.5 *Let $t = \tau_k(x),\ k = 1, 2, \dots,$ then for the function (4.5.5) and the system (4.5.1)*

(a) $v(x + J_k(x),\, \eta) - v(x, \eta) \geq u_k^{\mathrm{T}} \underline{C} u_k, \quad k = 1, 2, \dots$

and

(b) $v(x + J_k(x),\, \eta) \geq u_k^{\mathrm{T}} \underline{C}^* u_k, \quad k = 1, 2, \dots,$

where

$$\begin{aligned}
\underline{C} = [\underline{c}_{ij}], &\quad \underline{c}_{ij} = \underline{c}_{ji}, \quad i, j = 1, 2, \dots, m, \\
\underline{C}^* = [\underline{c}^*_{ij}], &\quad \underline{c}^*_{ij} = \underline{c}^*_{ji}, \quad i, j = 1, 2, \dots, m, \\
\underline{c}_{ii} = \lambda_m(C_{ii}), &\quad \underline{c}_{ij} = -\bar{c}_{ij}, \quad i \neq j, \quad i, j = 1, 2, \dots, m; \\
\underline{c}^*_{ii} = \lambda_m(C^*_{ii}), &\quad \underline{c}^*_{ij} = -\bar{c}^*_{ij}, \quad i \neq j, \quad i, j = 1, 2, \dots, m;
\end{aligned}$$

and u^{T}, $J_k(x)$, $\underline{c}_{ij}$, $\underline{c}^*_{ij}$, $\underline{c}_{ii}$, $\underline{c}^*_{ii}$ are defined as in Proposition 4.5.3.

Proof The proof of this Proposition is similar to that of Proposition 4.5.3.

Corollary 4.5.1 If all conditions of Proposition 4.5.5 are satisfied, then for the function (4.5.5) and the system (4.5.1) for $t = \tau_k(x),\ k = 1, 2, \dots$

(a) $v(x + J_k(x),\, \eta) - v(x, \eta) \geq \Delta V(x, \eta)$, where

$$\underline{\Delta} = \begin{cases} \lambda_m(\underline{C})\lambda_m^{-1}(A) & \text{for } \lambda_m(\underline{C}) < 0, \\ \lambda_m(\underline{C})\lambda_M^{-1}(B) & \text{for } \lambda_m(\underline{C}) > 0; \end{cases}$$

and

(b) $V(x + J_k(x),\, \eta) \geq \underline{\Delta}^* V(x, \eta)$, where

$$\underline{\Delta}^* = \begin{cases} \lambda_m(\underline{C}^*)\lambda_m^{-1}(A) & \text{for } \lambda_m(\underline{C}^*) < 0, \\ \lambda_m(\underline{C}^*)\lambda_M^{-1}(B) & \text{for } \lambda_m(\underline{C}^*) > 0. \end{cases}$$

Proof The proof follows from Proposition 4.5.5 and (4.5.11).

Theorem 4.5.1 *Let the system (4.5.1) be such that the matrix-valued function (4.5.2) is constructed with the elements (4.5.3) and (4.5.4) and*

(1) *the matrix A in (4.5.6) is positive definite, i.e. $\lambda_m(A) > 0$;*

(2) *there exists a matrix Q such that for the matrix $\overline{G}(S)$ the estimate*
$$\frac{1}{2}(\overline{G}(S) + \overline{G}^{\mathrm{T}}(S)) \leq Q \quad \text{for all} \quad S \in \mathcal{S}$$
is satisfied component-wise;

(3) *the matrix Q is negative semi-definite or equal to zero, i.e. the inequality $\lambda_m(Q) \leq 0$ holds.*

Then the zero solution of the system (4.5.1) is stable in the whole on $\mathcal{S}$.

If instead of the condition (3) the following condition is satisfied

(4) *the matrix $\overline{C}$ in (4.5.12) is negative definite, i.e. $\lambda_M(\overline{C}) < 0$,*

then the zero solution of the system (4.5.1) is asymptotically stable in the whole on $\mathcal{S}$.

The proof of Theorem 4.5.1 is similar to that of Theorem 4.4.1.

Theorem 4.5.2 *Let the system (4.5.1) be such that the matrix-valued function (4.5.2) is constructed with the elements (4.5.3) and (4.5.4) and*

(1) *the matrix A in (4.5.6) is positive definite, i.e. $\lambda_m(A) > 0$;*
(2) *there exists a negative definite matrix $Q^- \in R^{m\times m}$ such that*

$$\frac{1}{2}(\overline{G}(S) + \overline{G}^{\mathrm{T}}(S)) \le Q^- \quad \text{for all} \quad S \in \mathcal{S};$$

(3) $\lambda_M(\overline{C}^*) > 0$;
(4) *the functions $\tau_k(x)$, $k = 1, 2, \dots,$ satisfy the inequality*

$$\tau_{k+1}(x) - \tau_k(x) = \theta, \quad \theta > 0.$$

If

$$-\frac{\lambda_M(B)}{\lambda_M(Q^-)} \ln \frac{\lambda_M(\overline{C}^*)}{\lambda_m(A)} \le \theta, \tag{4.5.18}$$

then the zero solution of the system (4.5.1) is stable in the whole on $\mathcal{S}$.

If instead of (4.5.18) the condition

$$-\frac{\lambda_M(B)}{\lambda_M(Q^-)} \ln \frac{\lambda_M(\overline{C}^*)}{\lambda_m(A)} \le \theta - \gamma \tag{4.5.19}$$

holds for some $\gamma > 0$, then the zero solution of the system (4.5.1) is asymptotically stable in whole on $\mathcal{S}$.

Proof The assertion of Theorem 4.5.2 follows from Theorem 4.4.2.

Theorem 4.5.3 *Let the system (4.5.1) be such that the matrix-valued function (4.5.2) is constructed with the elements (4.5.3) and (4.5.4) and*

(1) *the matrix A in (4.5.6) is positive definite, i.e. $\lambda_m(A) > 0$;*
(2) *there exists a matrix $Q^+ \in R^{m\times m}$ for which*
   (a) $\overline{G}(S) \le Q^+$ *for all* $S \in \mathcal{S}$
   (b) $\lambda_M(Q^+) > 0$;
(3) $\lambda_M(\overline{C}^*) > 0$;
(4) *the functions $\tau_k(x)$ satisfy for some $\theta_1 > 0$ and for all $k = 1, 2, \dots$ the inequality*

$$\tau_k(x) - \tau_{k-1}(x) \le \theta_1.$$

If in addition the condition

$$\frac{\lambda_m(A)}{\lambda_M(Q^+)} \ln \frac{\lambda_m(A)}{\lambda_M(\bar{C}^*)} \ge \theta_1 \tag{4.5.20}$$

is satisfied, then the zero solution of the system (4.5.1) is stable in the whole on $\mathcal{S}$.

If instead of (4.5.20) the inequality

$$\frac{\lambda_m(A)}{\lambda_M(Q^+)} \ln \frac{\lambda_m(A)}{\lambda_M(\bar{C}^*)} \ge \theta_1 + \gamma$$

holds for some $\gamma > 0$, then the zero solution of the system (4.5.1) is asymptotically stable in the whole on $\mathcal{S}$.

Proof The assertion of this theorem follows from Theorem 4.4.3.

Theorem 4.5.4 *Let the system (4.5.1) be such that the matrix-valued function (4.5.2) is constructed with the elements (4.5.3) and (4.5.4) and*

(1) *the matrix A in (4.5.6) is positive definite, i.e. $\lambda_m(A) > 0$;*
(2) *there exists a positive semi-definite or equal to zero matrix $\underline{Q}$ such that for the matrix $\underline{G}(S)$ the estimate*

$$\frac{1}{2}\left(\underline{G}(S) + \underline{G}^{\mathrm{T}}(S)\right) \geq \underline{Q} \quad \text{for all} \quad S \in \mathcal{S}$$

is fulfilled element-wise;
(3) *the matrix $\underline{C}$ is positive definite, i.e. $\lambda_m(\underline{C}) > 0$.*

Then the zero solution of the system (4.5.1) is unstable on $\mathcal{S}$.

Proof The proof of the theorem is similar to that of Theorem 4.4.4.

Theorem 4.5.5 *Let the system (4.5.1) be such that the matrix-valued function (4.5.2) is constructed with the elements (4.5.3) and (4.5.4) and*

(1) *the matrix A in (4.5.6) is positive definite, i.e. $\lambda_m(A) > 0$;*
(2) *there exists a matrix $\underline{Q}^- \in R^{m\times m}$ such that*
   (a) $\frac{1}{2}\left(\underline{G}(S) + \underline{G}^{\mathrm{T}}(S)\right) \geq \underline{Q}^-$ *for all* $S \in \mathcal{S}$;
   (b) $\lambda_m(Q^-) < 0$;
(3) *the matrix $\underline{C}^*$ is positive definite, i.e. $\lambda_m(\underline{C}^*) > 0$;*
(4) *for some constant $\theta_1 > 0$ the values $\tau_k(x)$, $k = 1, 2, \dots$, satisfy the inequality*

$$\tau_k(x) - \tau_{k-1}(x) \leq \theta_1, \quad \text{for all} \quad k = 1, 2, \dots .$$

If for some $\gamma > 0$ *the inequality*

$$-\frac{\lambda_m(A)}{\lambda_M(\underline{Q}^-)} \ln \frac{\lambda_m(\bar{C}^*)}{\lambda_M(B)} \geq \theta_1 + \gamma$$

holds, then the zero solution of the system (4.5.1) is unstable on $\mathcal{S}$.

Proof The validity of this theorem follows from Theorem 4.4.5.

Theorem 4.5.6 *Let the system (4.5.1) be such that the matrix-valued function (4.5.2) be constructed with the elements (4.5.3) and (4.5.4) and*

(1) *the matrix* A *in (4.5.6) is positive definite, i.e.* $\lambda_m(A) > 0$;
(2) *there exists a matrix* $\underline{Q}^+ \in R^{s\times s}$ *such that*
  (a) $\frac{1}{2}\left(\underline{G}(S) + \underline{G}^{\mathrm{T}}(S)\right) \geq \underline{Q}^+$ *for all* $S \in \mathcal{S}$
  (b) $\lambda_m(\underline{Q}^+) > 0$;
(3) $\lambda_m(\underline{C}^*) > 0$ *i.e. the matrix* $\underline{C}^*$ *is positive definite;*
(4) *for some constant* $\theta > 0$ *the values* $\tau_k(x)$, $k = 1, 2, \dots,$ *satisfy the correlation*

$$\tau_{k+1}(x) - \tau_k(x) = \theta > 0.$$

If for some $\gamma > 0$

$$\frac{\lambda_M(B)}{\lambda_m(\underline{Q}^+)} \ln \frac{\lambda_M(B)}{\lambda_m(\bar{C}^*)} \leq \theta - \gamma,$$

then the zero solution of the system (4.5.1) is unstable on $\mathcal{S}$.

Proof The proof of this theorem is similar to the proof of Theorem 4.4.6.

Example 4.5.1 Let the system (4.5.1) be a fourth order system decomposed into two subsystems of the second order which are defined by the matrices:

$$A_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -2 & 1 \\ -1 & -2 \end{pmatrix}, \quad A_{12} = A_{21} = I_2;$$

(4.5.21)
$$J_{ki} = \operatorname{diag}\{-1, -1\}, \quad i = 1, 2; \quad J_{k12} = J_{k21} = 10^{-1} I_2,$$
$$\mathcal{S} = \{S\colon\ S = \operatorname{diag}\{S_1, S_2\}, \quad S_i = [S_{i1}, S_{i2}],$$
$$S_{ii} = I_2, \quad S_{ij} = s_{ij} I_2, \quad 0 \leq s_{ij} \leq 1, \quad i \neq j, \quad i, j = 1, 2\},$$

where $I_2 = \operatorname{diag}\{1, 1\}$.

For this example the elements (4.5.3) and (4.5.4) of the matrix-valued function (4.5.2) are constructed in the form

$$v_{ii}(x_i) = x_i^{\mathrm{T}} I_2 x_i, \quad i = 1, 2;$$
$$v_{12}(x_1, x_2) = v_{21}(x_1, x_2) = x_1^{\mathrm{T}} 10^{-1} I_2 x_2.$$

It is clear that they satisfy the estimates

$$\|x_i\|^2 \leq v_{ii}(x_i) \quad \text{for all} \quad x_i \in R^{n_i}, \quad i = 1, 2,$$
$$-0.1\|x_1\|\,\|x_2\| \leq v_{12}(x_1, x_2) \leq 0.1\|x_1\|\,\|x_2\|.$$

For $\eta^{\mathrm{T}} = (1,1) \in R_+^2$ the matrices

$$A = \begin{pmatrix} 1 & -0.1 \\ -0.1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0.1 \\ 0.1 & 1 \end{pmatrix}$$

are positive definite, i.e.

$$\lambda_m(A) = 0.9 \quad \text{and} \quad \lambda_M(B) = 1.1.$$

For this choice of the elements of the matrix-valued function $U(x)$ we have

$$\begin{aligned}
\bar\sigma_{11}(S) &= -2 + 0.2 s_{21} \le -1.8;\\
\bar\sigma_{22}(S) &= -4 + 0.2 s_{12} \le -3.8;\\
\bar\sigma_{12}(S) &= \frac{1}{2}\big(s_{12} + s_{21} + \sqrt{(-0.2 + s_{21})^2 + 0.01}\\
&\quad + \sqrt{(-0.1 + s_{12})^2 + 0.01} + 0.432\big) \le 2.1,\\
c_{ii} &= -0.968, \quad i = 1,2; \quad c_{12} = 0.099.
\end{aligned}$$

The matrices

$$\overline{G}(S) \le Q = \begin{pmatrix} -1.8 & 2.1 \\ 2.1 & -3.8 \end{pmatrix}$$

and

$$\overline{C} = \begin{pmatrix} -0.968 & 0.099 \\ 0.099 & -0.968 \end{pmatrix}$$

are negative definite which is confirmed by the estimate

$$\lambda_M(Q) = -0.474 < 0; \quad \lambda_M(\overline{C}) = -0.867 < 0.$$

Thus, all conditions of Theorem 4.5.1 are satisfied and the zero solution of the system (4.5.1) with matrices (4.5.21) is structurally asymptotically stable in the whole on $\mathcal{S}$.

4.6 Certain Trends of Generalizations and Applications

This section deals with two problems. In the first problem we establish conditions under which the stability of solutions with respect to two measures in the continuous system under nonclassical structural perturbations implies the same type of stability of solutions to the impulsive system under nonclassical structural perturbations. In the second problem we establish sufficient stability conditions for the system of Lurie-Postnikov type in the presence of impulsive and nonclassical structural perturbations.

4.6.1 Stability with respect to two measures Together with the impulsive system

$$\begin{aligned}
\frac{dx}{dt} &= Q(t,x,P,S), \quad t \ne \tau_k(x),\\
\Delta x &= I_k(x), \quad t = \tau_k(x),\\
x(t_0^+) &= x_0,
\end{aligned} \tag{4.6.1}$$

we shall consider a continuous system under nonclassical structural perturbations

$$\begin{aligned}
\frac{dy}{dt} &= \widetilde{Q}(t,y,P,S),\\
y(t_0) &= x_0.
\end{aligned} \tag{4.6.2}$$

Assume that for the systems (4.6.1) and (4.6.2) all conditions formulated for these classes of systems in Sections 4.2 and 2.2 respectively are satisfied.

Further we shall need the comparison functions defined below (see, Lakshmikantham, *et al.* [1]).

Definition 4.6.1 A function:

(a) η *belongs to the class PC* if $\eta\colon R_+ \to R_+$ is continuous on $(\tau_{k-1}, \tau_k]$, and $\lim\limits_{t\to\tau_k^+} \eta(t) = \eta(\tau_k^+)$;

(b) ζ *belongs to the class PCK* if $\zeta\colon R_+ \to R_+$, $\zeta(\cdot, u) \in PC$ for each $u \in R_+$, and $\zeta(t, \cdot) \in K$ for each $t \in R_+$;

(c) ρ *belongs to the class* $\mathcal{M}$ if $\rho\colon R_+ \times R^n \to R_+$, $\rho(\cdot, x) \in PC$ for each $x \in R^n$ and $\rho(t, \cdot) \in C(R_+ \times R^n, R_+)$ for each $t \in R_+$ and $\inf\limits_{x\in R^n} \rho(t, x) = 0$.

The definitions of (ρ_0, ρ)-stability of the impulsive system (4.6.1) are formulated in view of Definition 1.4.2 and the definition of stability with respect to two measures from Section 2.6.1.

Let $U\colon R_+ \times R^n \to R^{m\times m}$, $m > 1$, be a matrix-valued function. We consider the function

$$v(t, y, \eta) = \eta^{\mathrm{T}} U(t, y)\eta, \quad \eta \in R^m_+. \tag{4.6.3}$$

Further the functions of the class SL_0 are applied (see Definition 1.4.3) which were constructed in terms of matrix-valued functions $U(t, x)$.

For the function (4.6.3) for $(s, x) \in G_0$, $G_0 = \bigcup\limits_{k=1}^{\infty} G_{0k}$, $G_{k0} = \{(t, x) \in R_+ \times R^n\colon \tau_{k-1}(x) < t < \tau_k(x)\}$, $t_0 \le s \le t$, we define the expression

$$D^+ v(s, y(t, s, x), \eta) = \eta^{\mathrm{T}} D^+ U(s, y(t, s, x))\eta. \tag{4.6.4}$$

Here

$$
\begin{aligned}
&D^+U(s, y(t,s,x)) \\
&\qquad = \limsup \{[U(s+\vartheta,\, y(t,\, s+\vartheta,\, x+\vartheta Q(s,x,P,S))) \\
&\qquad - U(s,\, y(t,s,x))]\,\vartheta^{-1}, \quad \vartheta \to 0^+\},
\end{aligned}
$$

where $y(t,s,x)$ is a solution of the continuous system (4.6.2) such that $y(s,s,x) = x$. If $\widetilde{Q}(t,y,P,S) = 0$ then

$$
\begin{gathered}
y(t,s,x) = x, \\
y(t,\, s+\vartheta,\, x+\vartheta Q(s,x,P,S)) \equiv x + \vartheta Q(s,x,P,S),
\end{gathered}
$$

and the expression (4.6.4) becomes

$$
D^+v(s,x,\eta) = \eta^{\mathrm{T}} D^+U(s,x)\eta, \tag{4.6.5}
$$

where

$$
\begin{aligned}
&D^+U(s,x) \\
&\qquad = \limsup \{[U(s+\vartheta,\, x+\vartheta Q(s,x,P,S)) - U(s,x)]\,\vartheta^{-1}, \quad \vartheta \to 0^+\}.
\end{aligned}
$$

In the expressions (4.6.4) and (4.6.5) the total derivative of the function (4.6.3) is computed element-wise.

Further we designate

$$
S(\rho,\sigma) = \{(t,x) \in (\tau_{k-1},\tau_k] \times R^n :\ \rho(t,x) < \sigma,\ \ \sigma = \text{const} > 0\}, \quad k = 1,2,\dots .
$$

We give some sufficient conditions for preserving the (ρ_0,ρ)-stability of impulsive system if corresponding system without impulsive perturbation is stable with respect to two measures.

Theorem 4.6.1 *Assume that for systems (4.6.1) and (4.6.2) the following conditions hold:*

(1) *the measures ρ_0, ρ^*, and ρ are in M, and function $U(t,x,\eta) \in SL_0$;*
(2) *the measure ρ^* is continuous with respect to the measure ρ and $\rho^*(t,x)$ is continuous and nondecreasing in t;*
(3) *the function $v(t,x,\eta)$ is weakly ρ^*-decreasing and for some $\sigma > 0$ the function $v(t,x,\eta)$ is ρ-positive definite on the set $S(\rho,\sigma)$;*
(4) *there exist comparison functions u_i of class K and $m \times m$–matrix $B = B(P,S)$ such that for any $(P,S) \in \mathcal{P} \times \mathcal{S}$ the estimate*

$$
D^+v(s,\, y(t,s,x),\, \eta) \le u^{\mathrm{T}}(\|y\|)B(P,S)\,u(\|y\|) \tag{4.6.6}
$$

holds for any $t > t_0$, provided that $s \in [t_0,t)$ for all $(s,x) \in S(\rho,\sigma) \cap G_0$;
(5) *there exists a constant $\sigma_0 \in (0,\sigma)$ such that $\rho(\tau_k^+,\, x + I_k(x)) < \sigma$ as soon as $\rho(\tau_k,x) < \sigma_0$;*
(6) *for all $(\tau_k,x) \in S_k \cap S(\rho,\sigma)$*

$$
v(\tau_k^+,\, y(t,\tau_k^+,\, x+I_k(x)),\, \eta) \le v(\tau_k,\, y(t,\tau_k,x),\, \eta), \quad k = 1,2,\dots;
$$

(7) *there exists a constant $m \times m$–matrix $\overline{B}$ such that $B(P,S) \le \overline{B}$ for all $(P,S) \in \mathcal{P} \times \mathcal{S}$ and the matrix $B^* = \dfrac{1}{2}(\overline{B} + \overline{B}^{\mathrm{T}})$ is negative semi-definite.*

Then (ρ_0, ρ^*)–stability (asymptotic (ρ_0, ρ^*)–stability) of system (4.6.2) on $\mathcal{P} \times \mathcal{S}$ implies (ρ_0, ρ)–stability (asymptotic (ρ_0, ρ)–stability) of the impulsive system (4.6.1) on $\mathcal{P} \times \mathcal{S}$.

Proof The condition (3) of Theorem 4.6.1 implies that there exist the function $a \in K$ such that

$$a(\rho(t, x)) \le v(t, x, \eta) \quad \text{for all} \quad (t, x) \in S(\rho, \sigma) \tag{4.6.7}$$

and the function $b \in CK$ such that

$$v(t, x, \eta) \le b(t, \rho^*(t, x)) \tag{4.6.8}$$

for all $(t, x) \in S(\rho^*, \delta_0)$ for some value $\delta_0 > 0$.

Further by condition (2) of Theorem 4.6.1 there exist a constant $\delta_1 > 0$ and a comparison function ψ of class CK such that

$$\rho(t, x) \le \psi(t, \rho^*(t, x)) \quad \text{for all} \quad (t, x) \in S(\rho^*, \delta_1), \tag{4.6.9}$$

where the constant δ_1 satisfies the condition

$$\psi(t_0, \delta_1) < \sigma. \tag{4.6.10}$$

Let $\varepsilon \in (0, \sigma)$ and $t_0 \in R_+$. For the given function b of class CK we take a constant $\Delta = \Delta(t_0, \varepsilon) < \min(\sigma, \delta_0, \delta_1)$ so that the condition $u < \Delta$ yields the estimate

$$b(t_0, u) < a(\varepsilon). \tag{4.6.11}$$

Further we assume that the system (4.6.2) is (ρ_0, ρ^*)–stable on $\mathcal{P} \times \mathcal{S}$. Moreover, given Δ, there exists $\delta = \delta(t_0, \Delta) > 0$ $(\delta < \Delta)$ such that the condition $\rho_0(t_0, x_0) < \delta$ implies

$$\rho^*(t, y(t; t_0, x_0)) < \Delta, \quad t \ge t_0, \tag{4.6.12}$$

for all $(P, S) \in \mathcal{P} \times \mathcal{S}$, where $y(t; t_0, x_0)$ is a solution of system (4.6.2) for any values of $(P, S) \in \mathcal{P} \times \mathcal{S}$.

Let $x(t) = x(t; t_0, x_0)$ be a solution of the impulsive system (4.6.1) with

The conditions (4) and (7) of Theorem 4.6.1 imply

$$D^+ v(s, y(t; s, x), \eta) \le \lambda_M(B^*) u^{\mathrm{T}} u \le 0 \tag{4.6.13}$$

for any $t > t_0$, provided that $s \in [t_0, t)$ and $(s, x) \in S(\rho, \sigma) \cap G_0$. Here $\lambda_M(\cdot)$ is the maximal eigenvalue of the matrix B^*, $\lambda_M(B^*) \le 0$, and

$$u^{\mathrm{T}} = (u_1^{1/2}(\|y\|), \ldots, u_m^{1/2}(\|y\|)).$$

Then the conditions (4.6.7) – (4.6.13) yield

$$a(\rho(t_0, x_0)) \le v(t_0, x_0, \eta) \le b(t_0, \rho^*(t_0, x_0)) < a(\varepsilon). \tag{4.6.14}$$

Hence we find that $\rho(t_0, x_0) < \varepsilon$.

Let us show that $\rho(t, x(t)) < \varepsilon$ for all $(P, S) \in \mathcal{P} \times \mathcal{S}$ and for all $t \ge t_0$. If this is not true, then there exists a value $t^* > t_0$ such that $\rho(t^*, x(t^*)) \ge \varepsilon$ for at least one pair $(P, S) \in \mathcal{P} \times \mathcal{S}$. In the case when $t^* < \tau_1(x)$ the proof of the assertion made is not associated with the presence of the impulsive perturbations of the system (4.6.1) and is carried out in a standard manner (see, e.g. Lakshmikantham, Leela, *et al.* [1]). Therefore, this case is omitted here. Assume that $t^* \in (\tau_k, \tau_{k+1}]$ for some k, and moreover, for at least one pair $(P, S) \in \mathcal{P} \times \mathcal{S}$

$$\rho(t^*, x(t^*)) \ge \varepsilon \quad \text{and} \quad \rho(t, x(t)) < \varepsilon \tag{4.6.15}$$

if $t \in [\tau_0, \tau_k]$.

According to the choice of $\varepsilon \in (0, \sigma)$ and by estimates (4.6.15) we have

$$\rho(\tau_k, x(\tau_k)) < \varepsilon < \sigma.$$

The condition (5) of Theorem 4.6.1 implies

$$\rho(\tau_k^+, x(\tau_k^+)) = \rho(\tau_k^+,\, x_k + I_k(x_k)) < \delta, \tag{4.6.16}$$

where $x_k = x(\tau_k)$. Therefore, there exists a value $\tilde{t} \in (\tau_k, t^*]$ such that

$$\varepsilon \le \rho(\tilde{t}, x(\tilde{t})) < \sigma, \quad \rho(t, x(t)) < \sigma \tag{4.6.17}$$

for all $t \in [t_0, \tilde{t})$ and for all $(P, S) \in \mathcal{P} \times \mathcal{S}$.

Designate $m(s) = v(s,\, y(\tilde{t}; s, x(s)),\, \eta)$ for $s \in [t_0, \tilde{t}\,]$ where $y(t; s, x(s))$ is any solution of system 4.6.2) for all $(P, S) \in \mathcal{P} \times \mathcal{S}$. Inequality (4.6.13) implies that $D^+ m(s) \le 0$ for $t_0 \le s < \tilde{t}$, for all $(s, x(s)) \in G_0$ and for all $(P, S) \in \mathcal{P} \times \mathcal{S}$. Hence we find that $m(s) \le m(t_0^+)$, $t_0 \le s < t_1$, where $t_1 = \tau_1(x(t_1))$. If $(s, x(s)) \in G_{01}$ and $s \to t_1$, then $m(t_1) \le m(t_0^+)$. Note that $m(t_1^+) = v(t_1^+,\, y(\tilde{t}; t_1^+,\, x(t_1) + I_1(x(t_1))),\, \eta)$. The condition (6) of Theorem 4.6.1 implies that $m(t_1^+) \le m(t_1) \le m(t_0^+)$.

Thus, for the values $s \in (\tau_k, \tilde{t}\,)$ we get the estimate $m(s) \le m(\tau_k^+) \le m(t_0^+)$. If $(s, x(s)) \in G_{0,k+1}$ and $s \to \tilde{t}^+$, then $m(\tilde{t}\,) \le m(t_0^+)$. The fact that the measure ρ^* is nondecreasing in t and the conditions (4.6.11) and (4.6.12) yield

$$
\begin{aligned}
v(\tilde{t}, x(\tilde{t}\,), \eta) &= v(\tilde{t}, y(\tilde{t}; t, x(\tilde{t}\,)), \eta) = \lim_{s \to \tilde{t}} v(s, y(\tilde{t}; s, x(s)), \eta) \\
&\le v(t_0^+, y(\tilde{t}; t_0, x_0), \eta) \le b(t_0, \rho^*(t_0, y(\tilde{t}; t_0, x_0))) \\
&\le b(\tilde{t}, \rho^*(\tilde{t}, y(\tilde{t}; t_0, x_0))) < a(\varepsilon).
\end{aligned} \tag{4.6.18}
$$

On the other hand, by conditions (4.6.7) and (4.6.17) we have the inequalities

$$
v(\tilde{t}, x(\tilde{t}), \eta) \ge a(\rho(\tilde{t}, x(\tilde{t}\,))) = a(\varepsilon),
$$

which contradict the inequality (4.6.18). Therefore the assertion is true that $\rho(t, x(t)) < \varepsilon$ for all $(P, S) \in \mathcal{P} \times \mathcal{S}$ and for all $t \ge t_0$.

This proves (ρ_0, ρ)–stability of solutions of system (4.6.1) on $\mathcal{P} \times \mathcal{S}$.

Now we shall prove that under conditions of Theorem 4.6.1 the solutions of system (4.6.1) possess the property of (ρ_0, ρ)–attraction on $\mathcal{P} \times \mathcal{S}$. Assume that system (4.6.2) is asymptotically (ρ_0, ρ^*)–stable on $\mathcal{P} \times \mathcal{S}$. Then for any $t_0 \in R_+$ there exists $\delta^* = \delta^*(t_0) > 0$ such that the condition $\rho_0(t_0, x_0) < \delta^*$ implies $\lim\limits_{t \to \infty} \rho^*(t, y(t; t_0, x_0)) = 0$ for all $(P, S) \in \mathcal{P} \times \mathcal{S}$.

The fact that system (4.6.1) is (ρ_0, ρ)–stable implies that for any $t_0 \in R_+$ and $\sigma_0 \in (0, \sigma)$ there exists $\delta_0^* = \delta(t_0, \sigma_0) > 0$, $\delta_0^* < \delta^*$, such that the condition $\rho_0(t_0, x_0) < \delta_0^*$ implies that $\rho(t, x(t)) < \sigma_0$ for all $t \ge t_0$, where $x(t)$ is a solution of system (4.6.1) for $(P, S) \in \mathcal{P} \times \mathcal{S}$.

As noted above, the conditions (4) and (7) of Theorem 4.6.1 yield

$$
D^+ v(t, y(t; s, x(s)), \eta) \le 0, \quad s \in [t_0, t),
$$

for all $(s, x(s)) \in G^0$ and for all $(P, S) \in \mathcal{P} \times \mathcal{S}$.

The arguments similar to the above ones lead to the estimate

$$
0 \le v(t, x(t), \eta) \le v(t_0, y(t; t_0, x_0), \eta),
$$

which is true for all $t \ge t_0$.

For the sufficiently large $t \ge t_0$

$$
v(t_0, y(t; t_0, x_0), \eta) \le b(t_0, \rho^*(t_0, y(t; t_0, x_0))) \le b(t_0, \rho^*(t, y(t; t_0, x_0))).
$$

Hence it follows that

$$
\lim_{t \to \infty} v(t, x(t), \eta) = 0
$$

for all $(P,S) \in \mathcal{P} \times \mathcal{S}$. Since the function v is ρ–positive definite,

$$\lim_{t\to\infty} \rho(t, x(t)) = 0 \quad \text{for all} \quad (P,S) \in \mathcal{P} \times \mathcal{S}.$$

This proves the asymptotic (ρ_0, ρ)–stability of solutions to system (4.6.1) for all $(P,S) \in \mathcal{P} \times \mathcal{S}$.

Actually, when the function (4.6.3) constructed in terms of the matrix-valued function $U(t,y)$ is applied, the condition (6) of Theorem 4.6.1 becomes

$$(6^*) \qquad \begin{aligned} &v(\tau_k^+, y(t;\tau_k^+, x + I_k(x)), \eta) - v(\tau_k, y(t;\tau_k, x), \eta) \\ &\le -\psi^{\mathrm{T}}(v_k(\tau_k, y(t;\tau_k, x), \eta)) C_k \psi(v_k(\tau_k, y(t;\tau_k, x), \eta)), \end{aligned}$$

where $\overline{\lambda}_k(C_k) \ge 0$, $\sum\limits_{k=1}^{\infty} \overline{\lambda}_k(C_k) = \infty$, $\psi \in C(R_+, R_+^m)$, $\psi(0) = 0$, $\psi(s) > 0$ for $s > 0$, $\overline{\lambda}_k(C_k)$ are the maximal eigenvalues of some matrices C_k, $k = 1, 2, \dots$. Then Theorem 4.6.1 is developed as follows.

Theorem 4.6.2 *Assume that conditions (1) – (5) and (7) of Theorem 4.6.1 and condition (6*) are satisfied. Then (ρ_0, ρ^*)–stability of system (4.6.2) implies asymptotic (ρ_0, ρ)–stability of system (4.6.1).*

The proof of this theorem is similar to that of Theorem 4.6.1.

Note, that for the impulsive system under nonclassical structural perturbations it is reasonable to consider the set of measures discussed in Section 2.6.1.

4.6.2 Stability of Lur'e-Postnikov impulsive systems We consider the large scale impulsive system

$$(4.6.20) \qquad \begin{aligned} \frac{dx_i}{dt} &= \sum_{\ell=1}^{m} S_{i\ell}^{(1)} A_{i\ell} x_\ell + \sum_{\ell=1}^{m} S_{i\ell}^{(2)} q_{i\ell} f_{i\ell}(\sigma_{i\ell}), \\ \sigma_{i\ell} &= c_{i\ell}^{\mathrm{T}} x, \quad i = 1, 2, \dots, m, \quad t \ne \tau_k(x), \quad k = 1, 2, \dots \\ \Delta x_i &= \sum_{\ell=1}^{m} J_{ki\ell} x_\ell + \sum_{\ell=1}^{m} b_{i\ell} g_{i\ell}(\sigma_{i\ell}^*), \\ \sigma_{i\ell}^* &= c_{i\ell}^{\mathrm{T}} x(\tau_k(x)), \quad i = 1, 2, \dots, m, \quad t = \tau_k(x), \quad k = 1, 2, \dots \end{aligned}$$

where

$$\begin{aligned} \sigma_{i\ell}^{-1} f_{i\ell}(\sigma_{i\ell}) &\in [0, K_{i\ell}] \subseteq R_+, \\ (\sigma_{i\ell}^*)^{-1} g_{i\ell}(\sigma_{i\ell}^*) &\in [0, \widetilde{K}_{i\ell}] \subseteq R_+, \end{aligned}$$

$A_{i\ell}$, $J_{ki\ell}$ are constant matrices, $x_i \in R^{n_i}$, $n_1 + n_2 + \ldots + n_m = n$, $g_{i\ell}, b_{i\ell}$ are constant vectors and $K_{i\ell}$, $\widetilde{K}_{i\ell}$ are positive constants, all of the appropriate dimensions. The matrices $S_{i\ell}^{(1)}$, $S_{i\ell}^{(2)}$ and the structural set $\mathcal{S}$ are described in Section 1.5. The independent subsystems corresponding to system (4.2.4) are obtained by replacing x in (4.2.4) with x^i, where $x^i = (0, \dots, 0, x_i^{\mathrm{T}}, 0, \dots, 0)^{\mathrm{T}} \in R^{n_i}$:

$$(4.6.21) \qquad \begin{aligned} \frac{dx_i}{dt} &= A_{ii} x_i + q_{ii} f_{ii}(\widetilde{\sigma}_{ii}), \quad t \ne \tau_k(x^i) \\ \Delta x_i &= J_{kii} x_i + b_{ii} g_{ii}(\widetilde{\sigma}_{ii}^*), \quad t = \tau_k(x^i), \end{aligned}$$

where

$$\widetilde{\sigma}_{ii} = c_{ii}^{\mathrm{T}} x_i, \quad \widetilde{\sigma}_{ii}^* = \widetilde{c}_{ii}^{\mathrm{T}} x_i(\tau_k(x^i)), \quad i = 1, 2, \dots, m.$$

In order to simplify system (4.6.20) we introduce the designations

$$f_i(x^i) = A_{ii} x_i + q_{ii} f_{ii}(\widetilde{\sigma}_{ii}), \quad \widetilde{\sigma}_{ii} = c_{ii}^{\mathrm{T}} x_i,$$

$$F_i(x, S) = \sum_{\substack{\ell=1 \\ \ell \neq i}}^{m} S_{i\ell}^{(1)} A_{i\ell} x_\ell + \sum_{\substack{\ell=1 \\ \ell \neq i}}^{m} S_{i\ell}^{(2)} q_{i\ell} f_{i\ell}(\sigma_{i\ell}) + S_{ii}^{(2)} q_{ii} [f_{ii}(\sigma_{ii}) - f_{ii}(\widetilde{\sigma}_{ii})],$$

$$\sigma_{i\ell} = c_{i\ell}^{\mathrm{T}} x_i,$$

$$g_i(x^i) = J_{kii} x_i + b_{ii} g_{ii}(\widetilde{\sigma}_{ii}^*),$$

$$G_i(x) = \sum_{\substack{\ell=1 \\ \ell \neq i}}^{m} J_{ki\ell} x_\ell + \sum_{\substack{\ell=1 \\ \ell \neq i}}^{m} b_{i\ell} g_{i\ell}(\sigma_{i\ell}^*) + b_{ii} [g_{ii}(\widetilde{\sigma}_{ii}^*) - g_{ii}(\widetilde{\sigma}_{ii}^*)],$$

Then system (4.2.4) becomes

$$\begin{aligned} \frac{dx_i}{dt} &= f_i(x^i) + F_i(x, S), \quad t \neq \tau_k(x), \\ &\quad S \in \mathcal{S}, \quad i = 1, 2, \dots, m, \quad k = 1, 2, \dots, \\ \Delta x_i &= g_i(x^i) + G_i(x), \quad t = \tau_k(x), \\ &\quad k = 1, 2, \dots, \quad i = 1, 2, \dots, m. \end{aligned} \tag{4.6.22}$$

Together with system (4.6.21) and subsystems (4.6.22) we consider the matrix-valued function

(4.6.23) $$U(x) = [u_{ij}(x_i, x_j)], \quad u_{ij} = u_{ji}, \quad i, j = 1, 2, \dots, m,$$

the elements of which are determined as

(4.6.24) $$u_{ij}(x_i, x_j) = x_i^{\mathrm{T}} P_{ij} x_j, \quad i, j = 1, 2, \dots, m,$$

where $x_i \in R^{n_i}$, $x_j \in R^{n_j}$, P_{ii} are symmetric, positive definite matrices, and P_{ij} are constant matrices for all $i, j = 1, 2, \dots, m$.

It is known that the functions (4.2.4) satisfy the estimates

(4.6.25)

(a) $$\lambda_m(P_{ii})\|x_i\|^2 \le u_{ii}(x_i) \le \lambda_M(P_{ii})\|x_i\|^2, \quad \text{for all} \quad x_i \in R^{n_i}, \quad i = 1, 2, \dots, m;$$

(b) $$-\lambda_M^{1/2}(P_{ij}P_{ij}^{\mathrm{T}})\|x_i\|\,\|x_j\| \le u_{ij}(x_i, x_j) \le \lambda_M^{1/2}(P_{ij}P_{ij}^{\mathrm{T}})\|x_i\|\,\|x_j\|$$
$$\text{for all} \quad x_i \in R^{n_i}, \quad x_j \in R^{n_j}, \quad \text{for all} \quad (i \ne j) = 1, 2, \dots m,$$

where $\lambda_m(P_{ii})$ are the minimal and $\lambda_M(P_{ii})$ are the maximal eigenvalues of the matrices P_{ii}, and $\lambda_M^{1/2}(P_{ij}P_{ij}^{\mathrm{T}})$ is the norm of the matrices P_{ij}.

Using the matrix-valued function (4.6.23) and the constant vector $\eta = (1, 1, \dots, 1) \in R_+^m$ we construct the function

(4.6.26) $$v(x, \eta) = \eta^{\mathrm{T}} U(x) \eta$$

and consider its total derivative

(4.6.27) $$Dv(x, \eta) = \eta^{\mathrm{T}} DU(x) \eta,$$

where

$$DU(x) = [Du_{ij}(x_i, x_j)], \quad i, j = 1, 2, \dots, m,$$

along the solutions of system (4.2.4).

Proposition 4.6.1 *If the estimates (4.6.25) are satisfied, then for the function (4.6.26) the two-sided inequality*

(4.6.28) $$u^{\mathrm{T}} A u \le v(x, \eta) \le u^{\mathrm{T}} B u \quad \textit{for all} \quad x \in R^n$$

holds true, where

$$u^{\mathrm{T}} = (\|x_1\|, \|x_2\|, \dots, \|x_m\|),$$
$$A = [\underline{\alpha}_{ij}], \quad B = \overline{\alpha}_{ij}, \quad i, j = 1, 2, \dots, m,$$
$$\underline{\alpha}_{ii} = \lambda_m(P_{ii}), \quad \overline{\alpha}_{ii} = \lambda_M(P_{ii})$$
$$\underline{\alpha}_{ij} = \underline{\alpha}_{ji} = -\overline{\alpha}_{ij} = -\overline{\alpha}_{ji} = -\lambda_M^{1/2}(P_{ij}P_{ij}^{\mathrm{T}}).$$

Proof The proof of Proposition 4.6.1 follows from Djordjević [1] (see also Martynyuk and Stavroulakis [1]).

Corollary 4.6.5 If inequality (4.6.28) is satisfied, then

(4.6.29) $$\lambda_m(A)\|u\|^2 \le v(x, \eta) \le \lambda_M(B)\|u\|^2 \quad \text{for all} \quad x \in R^n, \quad \eta = (1, 1, \dots, 1) \in R_+^m,$$

and for

(4.6.30) $$\lambda_m(A) > 0, \quad \lambda_M(B) > 0,$$
$$\lambda_M^{-1}(B) v(x, \eta) \le \|u\|^2 \le \lambda_m^{-1}(A) v(x, \eta).$$

Proposition 4.6.2 *If for system (4.6.20) the matrix-valued function (4.6.23) is constructed with the elements (4.6.24), then for the derivatives of function (4.6.26) along the solutions of (4.6.20) for* $t \neq \tau_k(x),\ k = 1, 2, \dots,$ *the estimates*

$$\text{(a)} \qquad (Dx_i u_{ii})^{\mathrm T} f_i(x^i) \le \rho_i^{(1)} \|x_i\|^2 \quad \textit{for all} \quad x_i \in R^{n_i},\ i = 1, 2, \dots, m;$$

$$\begin{aligned}\text{(b)} \qquad &\sum_{i=1}^{m} (Dx_i u_{ii})^{\mathrm T} F_i(x, S) + 2 \sum_{i=1}^{m} \sum_{\substack{j=2\\ j>i}}^{m} \Big\{ (Dx_i u_{ij})^{\mathrm T} \\ &\times (f_i(x^i) + F_i(x, S)) + (Dx_j u_{ij})^{\mathrm T} (f_i(x^i) + F_j(x, S)) \Big\} \\ &\le \sum_{i=1}^{m} \rho_i^{(2)}(S) \|x_i\|^2 + 2 \sum_{i=1}^{m} \sum_{\substack{j=2\\ j>i}}^{m} \rho_{ij}(S) \|x_i\|\, \|x_j\| \\ &\qquad \textit{for all} \quad (x_i, x_j) \in R^{n_i} \times R^{n_j}, \quad \textit{for all} \quad S \in \mathcal{S},\end{aligned}$$

are satisfied, where $\rho_i^{(1)}$ *and* $\rho_i^{(2)}(S),\ i = 1, 2, \dots, m,$ *are maximal eigenvalues of the matrices*

$$P_{ii} A_{ii} + A_{ii}^{\mathrm T} P_{ii} + P_{ii} q_{ii} k_{ii}^*(c_{ii}^i)^{\mathrm T} + (q_{ii} k_{ii}^*(c_{ii}^i)^{\mathrm T})^{\mathrm T} P_{ii};$$

$$\begin{aligned}&\sum_{\ell=1}^{i-1} \Big\{ \Big[\Big(S_{\ell i}^{(1)} A_{\ell i} \Big)^{\mathrm T} + \Big(S_{\ell i}^{(2)} q_{\ell i} k_{\ell i}^*(c_{\ell i}^i)^{\mathrm T} \Big)^{\mathrm T} \Big] P_{\ell i} \\ &\quad + P_{\ell i}^{\mathrm T} \Big[S_{\ell i}^{(1)} A_{\ell i} + S_{\ell i}^{(2)} q_{\ell i} k_{\ell i}^*(c_{\ell i}^i)^{\mathrm T} \Big] \Big\} \\ &\quad + \sum_{\ell=i+1}^{m} \Big\{ P_{i\ell} \Big[S_{\ell i}^{(1)} A_{\ell i} + S_{\ell i}^{(2)} q_{\ell i} k_{\ell i}^*(c_{\ell i}^i)^{\mathrm T} \Big] \\ &\quad + \Big[\Big(S_{\ell i}^{(1)} A_{\ell i} \Big)^{\mathrm T} + \Big(S_{\ell i}^{(2)} q_{\ell i} k_{\ell i}^*(c_{\ell i}^i)^{\mathrm T} \Big)^{\mathrm T} \Big] P_{i\ell}^{\mathrm T} \Big\} \\ &\quad + \sum_{\substack{j=2\\ j>i}}^{m} \Big\{ P_{ij} S_{ji}^{(2)} q_{ji} k_{ji}^{**}(c_{ji}^i)^{\mathrm T} + \Big(S_{ji}^{(2)} q_{ji} k_{ji}^{**}(c_{ji}^i)^{\mathrm T} \Big)^{\mathrm T} P_{ij} \Big\}\end{aligned}$$

respectively, $\rho_{ij}(S),\ i < j,\ i = 1, 2, \dots, m,\ j = 2, \dots, m,$ *are the norms of the matrices*

$$\begin{aligned}&\sum_{\ell=1}^{j-1} \Big[\Big(S_{\ell i}^{(1)} A_{\ell i} \Big)^{\mathrm T} + \Big(S_{\ell i}^{(2)} q_{\ell i} k_{\ell i}^*(c_{\ell i}^i)^{\mathrm T} \Big)^{\mathrm T} \Big] P_{\ell j} \\ &\quad + \sum_{\ell=j+1}^{m} \Big[\Big(S_{\ell i}^{(1)} A_{\ell i} \Big)^{\mathrm T} + \Big(S_{\ell i}^{(2)} q_{\ell i} k_{\ell i}^*(c_{\ell i}^i)^{\mathrm T} \Big)^{\mathrm T} \Big] P_{\ell j} \\ &\quad + \sum_{\ell=1}^{j-1} P_{\ell i}^{\mathrm T} \Big[S_{\ell j}^{(1)} A_{\ell j} + S_{\ell j}^{(2)} q_{\ell j} k_{\ell j}^*(c_{\ell j}^j)^{\mathrm T} \Big] \\ &\quad + \sum_{\ell=i+1}^{m} P_{i\ell} \Big[S_{\ell j}^{(1)} A_{\ell j} + S_{\ell j}^{(2)} q_{\ell j} k_{\ell j}^*(c_{\ell j}^j)^{\mathrm T} \Big] \\ &\quad + \frac{1}{2} \Big\{ P_{ii} \Big(S_{ij}^{(1)} A_{ij} \Big) + \Big(S_{ij}^{(1)} A_{ij} \Big)^{\mathrm T} P_{ii} + P_{ii} \Big(S_{ij}^{(2)} q_{ij} k_{ij}^*(c_{ij}^j)^{\mathrm T} \Big) \\ &\quad + \Big(S_{ij}^{(2)} q_{ij} k_{ij}^*(c_{ij}^j)^{\mathrm T} \Big) P_{ii} + P_{ii} \Big(q_{ii} k_{ii}^*(c_{ii}^j)^{\mathrm T} \Big) + \Big(q_{ii} k_{ij}^*(c_{ii}^j)^{\mathrm T} \Big)^{\mathrm T} P_{ii} \Big\}\end{aligned}$$

$$+\frac{1}{2}\Big\{P_{ji}\Big(S_{ji}^{(1)}A_{ji}\Big)+\Big(S_{ji}^{(1)}A_{ji}\Big)^{\mathrm{T}}P_{ji}+P_{jj}\Big(S_{ji}^{(2)}q_{ji}k_{ji}^{*}(c_{ji}^{i})^{\mathrm{T}}\Big)$$
$$+\Big(S_{ji}^{(2)}q_{ji}k_{ji}^{*}(c_{ji}^{i})^{\mathrm{T}}\Big)^{\mathrm{T}}P_{jj}+P_{jj}\big(q_{jj}k_{jj}^{*}(c_{jj}^{i})^{\mathrm{T}}\big)$$
$$+\big(q_{jj}k_{jj}^{*}(c_{jj}^{i})^{\mathrm{T}}\big)^{\mathrm{T}}P_{jj}\Big\},\qquad i=1,2,\dots,m,\quad j=1,2,\dots,m,$$

respectively.

Here

$$k_{ij}^{*}=\begin{cases}k_{ij} & \textit{for}\quad \sigma_{ij}\Big(S_{ij}^{(2)}q_{ij}\Big)^{\mathrm{T}}P_{ij}x_j>0,\quad i,j=1,2,\dots,m,\\ 0 & \textit{in other cases;}\end{cases}$$

$$k_{ij}^{**}=\begin{cases}k_{ij} & \textit{for}\quad \sigma_{ii}q_{ii}^{\mathrm{T}}P_{ii}x_i>0,\quad i=1,2,\dots,m,\\ -k_{ij} & \textit{for}\quad \sigma_{ii}q_{ii}^{\mathrm{T}}P_{ii}x_i<0,\quad i=1,2,\dots,m.\end{cases}$$

The proof is carried out in the same way as that of Proposition 4.6.1.

Proposition 4.6.3 *If all conditions of Proposition 4.6.2 are satisfied, then for (4.6.27)*

$$Dv(x,\eta)\le u^{\mathrm{T}}\Theta u,\qquad \textit{for all}\quad (x,S)\in R^n\times\mathcal{S} \tag{4.6.31}$$

where

$$\Theta=[\theta_{ij}],\quad i,j=1,2,\dots,m,\quad \theta_{ii}=\rho_i^{(1)}+\rho_i^{(2)}(S^{*}),$$
$$\theta_{ij}=\theta_{ji}=\rho_{ij}(S^{*}),\quad S^{*}\in\mathcal{S},\quad i\ne j,\quad i,j=1,2,\dots,m,$$

is the constant matrix such that

$$\rho_i^{(2)}(S) \le \rho_i^{(2)}(S^*)$$

and

$$\rho_{ij}(S) < \rho_{ij}(S^*).$$

The proof of this Proposition is similar to that of Lemma 3 in Martynyuk and Stavroulakis [1] (see also Miladzhanov [3]).

Corollary 4.6.6 If inequalities (4.6.29) and (4.6.30) are satisfied, then for (4.6.31)

$$Dv(x,\eta) \le \begin{cases} \lambda_m(\Theta)\lambda_M^{-1}(B)v(x,\eta) & \text{for} \quad \lambda_M(\Theta) < 0, \\ \lambda_m(\Theta)\lambda_m^{-1}(A)v(x,\eta) & \text{for} \quad \lambda_M(\Theta) > 0. \end{cases} \tag{4.6.32}$$

The proof follows from Proposition 4.6.5 and Corollary 4.6.5.

Proposition 4.6.4 *For the function (4.6.26) for $t = \tau_k(x)$, $k = 1, 2, \dots$, due to system (4.6.20) the estimates*

$$v(x + J_k^*(x),\, \eta) - v(x,\eta) \le u_k^{\mathrm{T}} \Lambda u_k, \tag{4.6.33}$$

$$v(x + J_k^*(x),\, \eta) \le u_k^{\mathrm{T}} \Lambda^* u_k, \tag{4.6.34}$$

hold true, where

$$\begin{gathered}
u_k^{\mathrm{T}} = (\|x_1(\tau_k(x))\|,\, \|x_2(\tau_k(x))\|, \dots, \|x_m(\tau_k(x))\|);\\
J_k^*(x) = \sum_{\ell=1}^m J_{ki\ell} x_\ell + \sum_{\ell=1}^m b_{i\ell} g_{i\ell}(\sigma_{i\ell}^1);\\
\Lambda = [\omega_{ij}], \quad \omega_{ij} = \omega_{ji}, \quad i,j = 1,2,\dots,m;\\
\Lambda^* = [\xi_{ij}], \quad \xi_{ij} = \xi_{ji}, \quad i,j = 1,2,\dots,m;\\
\omega_{ii} = \lambda_M(\Omega_{ii}), \quad \omega_{ij} = \lambda_M^{1/2}(\Omega_{ij}\Omega_{ij}^{\mathrm{T}}), \quad i \ne j, \quad i,j = 1,2,\dots,m;\\
\xi_{ii} = \lambda_M(\Psi_{ii}), \quad \xi_{ij} = \lambda_M^{1/2}(\Psi_{ij}\Psi_{ij}^{\mathrm{T}}), \quad i \ne j, \quad i,j = 1,2,\dots,m;
\end{gathered}$$

$$\begin{aligned}
\Omega_{ii} = {} & P_{ii}J_{kii} + J_{kii}^{\mathrm{T}}P_{ii} + \sum_{j=1}^m J_{kji}^{\mathrm{T}} P_{jj} J_{kji} + P_{ii}\Big(\sum_{\ell=1}^m b_{i\ell}\widetilde{k}_{i\ell}^*(\widetilde{c}_{i\ell}^i)^{\mathrm{T}}\Big)\\
& + \Big(\sum_{\ell=1}^m b_{i\ell}\widetilde{k}_{i\ell}^*(\widetilde{c}_{i\ell}^i)^{\mathrm{T}}\Big)^{\mathrm{T}} P_{ii} + \sum_{j=1}^m \Big\{ J_{kji}P_{jj}\Big(\sum_{\ell=1}^m b_{j\ell}\widetilde{k}_{j\ell}^*(\widetilde{c}_{j\ell}^j)^{\mathrm{T}}\Big)\\
& + \Big(\sum_{\ell=1}^m b_{j\ell}\widetilde{k}_{j\ell}^*(\widetilde{c}_{j\ell}^j)^{\mathrm{T}}\Big)^{\mathrm{T}} P_{jj}J_{kji}\Big\}\\
& + \sum_{j=1}^m \Big(\sum_{\ell=1}^m b_{j\ell}\widetilde{k}_{j\ell}^*(\widetilde{c}_{j\ell}^j)^{\mathrm{T}}\Big)^{\mathrm{T}} P_{jj}\Big(\sum_{\ell=1}^m b_{j\ell}\widetilde{k}_{j\ell}^*(\widetilde{c}_{j\ell}^j)^{\mathrm{T}}\Big)\\
& + \sum_{j=1}^m \Big\{ P_{ij}J_{kji} + J_{kji}^{\mathrm{T}}P_{ij} + P_{ij}\Big(\sum_{\ell=1}^m b_{j\ell}\widetilde{k}_{j\ell}^*(\widetilde{c}_{j\ell}^i)^{\mathrm{T}}\Big)\\
& + \Big(\sum_{\ell=1}^m b_{j\ell}\widetilde{k}_{j\ell}^*(\widetilde{c}_{j\ell}^i)^{\mathrm{T}}\Big)^{\mathrm{T}} P_{ij} + \sum_{\ell=1}^m \Big(J_{k\ell i}^{\mathrm{T}}P_{\ell j}J_{kji} + J_{kji}^{\mathrm{T}}P_{\ell j}J_{k\ell i}\Big)\\
& + J_{kji}^{\mathrm{T}}P_{ij}\Big(\sum_{\ell=1}^m b_{j\ell}\widetilde{k}_{j\ell}^*(\widetilde{c}_{j\ell}^i)^{\mathrm{T}}\Big) + \Big(\sum_{\ell=1}^m b_{j\ell}\widetilde{k}_{j\ell}^*(\widetilde{c}_{j\ell}^i)^{\mathrm{T}}\Big)^{\mathrm{T}} P_{ij}J_{kji}
\end{aligned}$$

$$+\Big(\sum_{\ell=1}^{m} b_{i\ell}\widetilde{k}^*_{i\ell}(\widetilde{c}^{\,i}_{i\ell})^{\mathrm T}\Big)^{\mathrm T} P_{ij}\Big(\sum_{\ell=1}^{m} b_{j\ell}\widetilde{k}^*_{j\ell}(\widetilde{c}^{\,i}_{j\ell})^{\mathrm T}\Big)$$
$$+\Big(\sum_{\ell=1}^{m} b_{j\ell}\widetilde{k}^*_{j\ell}(\widetilde{c}^{\,i}_{j\ell})^{\mathrm T}\Big)^{\mathrm T} P_{ij}\Big(\sum_{\ell=1}^{m} b_{i\ell}\widetilde{k}^*_{i\ell}(\widetilde{c}^{\,i}_{i\ell})^{\mathrm T}\Big)\Big\},\quad i=1,2,\dots,m;$$

$$\Omega_{ij}=P_{ii}J_{kij}+J^{\mathrm T}_{kji}P_{jj}+P_{ii}\Big(\sum_{\ell=1}^{m} b_{i\ell}\widetilde{k}^*_{i\ell}(\widetilde{c}^{\,i}_{i\ell})^{\mathrm T}\Big)$$
$$+\Big(\sum_{\ell=1}^{m} b_{j\ell}\widetilde{k}^*_{j\ell}(\widetilde{c}^{\,j}_{j\ell})^{\mathrm T}\Big)^{\mathrm T} P_{jj}+\sum_{\ell=1}^{m}(J^{\mathrm T}_{k\ell i}P_{\ell\ell}J_{k\ell j}+J^{\mathrm T}_{k\ell j}P_{\ell\ell}J_{k\ell i})$$
$$+\sum_{r=1}^{m}\Big\{J^{\mathrm T}_{kri}P_{rr}\Big(\sum_{\ell=1}^{m} b_{r\ell}\widetilde{k}^*_{r\ell}(\widetilde{c}^{\,r}_{r\ell})^{\mathrm T}\Big)+\Big(\sum_{\ell=1}^{m} b_{r\ell}\widetilde{k}^*_{r\ell}(\widetilde{c}^{\,r}_{r\ell})^{\mathrm T}\Big)^{\mathrm T} P_{rr}J_{kri}\Big\}$$
$$+\Big(\sum_{\ell=1}^{m}(P_{i\ell}J_{k\ell j}+J^{\mathrm T}_{k\ell j}B_{\ell j})\Big)+P_{ij}\Big(\sum_{\ell=1}^{m} b_{j\ell}\widetilde{k}^*_{j\ell}(\widetilde{c}^{\,j}_{j\ell})^{\mathrm T}\Big)$$
$$+\Big(\sum_{\ell=1}^{m} b_{i\ell}\widetilde{k}^*_{i\ell}(\widetilde{c}^{\,i}_{i\ell})^{\mathrm T}\Big)^{\mathrm T} P_{ii}+\Big(\sum_{\ell=1}^{m} b_{i\ell}\widetilde{k}^*_{i\ell}(\widetilde{c}^{\,j}_{i\ell})^{\mathrm T}\Big)$$
$$+\Big(\sum_{\ell=1}^{m} b_{i\ell}\widetilde{k}^*_{i\ell}(\widetilde{c}^{\,i}_{i\ell})^{\mathrm T}\Big)P_{ij}+\sum_{\ell=1}^{m}\sum_{\substack{r=1\\ r\neq\ell}}^{m} J^{\mathrm T}_{k\ell j}P_{\ell r}J^{\mathrm T}_{krj}$$
$$+\sum_{\substack{r=1\\ r\neq i,j}}^{m}\Big\{J^{\mathrm T}_{krj}P_{rj}\Big(\sum_{\ell=1}^{m} b_{j\ell}\widetilde{k}^*_{j\ell}(\widetilde{c}^{\,j}_{j\ell})^{\mathrm T}\Big)^{\mathrm T}+\Big(\sum_{\ell=1}^{m} b_{i\ell}\widetilde{k}^*_{i\ell}(\widetilde{c}^{\,i}_{i\ell})^{\mathrm T}\Big)^{\mathrm T} P_{ir}J_{krj}\Big\}$$
$$+\Big(\sum_{\ell=1}^{m} b_{i\ell}\widetilde{k}^*_{i\ell}(\widetilde{c}^{\,i}_{i\ell})^{\mathrm T}\Big)^{\mathrm T} P_{ij}\Big(\sum_{\ell=1}^{m} b_{j\ell}\widetilde{k}^*_{j\ell}(\widetilde{c}^{\,j}_{j\ell})^{\mathrm T}\Big),\quad i\neq j,\quad i,j=1,2,\dots,m;$$
$$\Psi_{ii}=P_{ii}+\Omega_{ii},\quad \Psi_{ij}=P_{ij}+\Omega_{ij},\quad i\neq j=1,2,\dots,m.$$

Here

$$\widetilde{k}^*_{ij}=\begin{cases}\widetilde{k}_{ij} & \textit{if the corresponding multiplier is positive};\\ 0 & \textit{in other cases.}\end{cases}$$

The proof is similar to that of Lemma 4 in Martynyuk and Stavroulakis [1].

Corollary 4.6.7 Under all conditions of Proposition 4.6.4 for function (4.6.26) when $t=\tau_k(x),\ k=1,2,\dots,$ the estimates

$$v(x+J_k(x),\,\eta)-v(x,\eta)\le\gamma v(x,\eta), \tag{4.6.40}$$

where

$$\gamma=\begin{cases}\lambda_M(\Lambda)\lambda_M^{-1}(B) & \text{for}\quad \lambda_M(\Lambda)<0,\\ \lambda_M(\Lambda)\lambda_m^{-1}(A) & \text{for}\quad \lambda_M(\Lambda)>0,\end{cases}$$

and

$$v(x+J_k(x),\,\eta)\le\gamma^* v(x,\eta), \tag{4.6.41}$$

where

$$\gamma^* = \begin{cases} \lambda_M(\Lambda^*)\lambda_M^{-1}(B) & \text{for} \quad \lambda_M(\Lambda^*) < 0, \\ \lambda_M(\Lambda^*)\lambda_m^{-1}(A) & \text{for} \quad \lambda_M(\Lambda^*) > 0, \end{cases}$$

hold true.

The proof follows from Proposition 4.6.4 and Corollary 4.6.6.

For system (4.6.1), the following stability problem is formulated. It is necessary to formulate conditions related to the coefficients which appear in the system and also to introduce structural perturbation, such that the trivial solution of system (4.6.1) is asymptotically stable in the whole on $\mathcal{S}$ for an arbitrary function f of the class under consideration.

We shall introduce the following notions.

Definition 4.6.2 The zero solution $x = 0$ of (4.6.1) is *absolutely stable under nonclassical structural perturbation* (i.e. absolutely stable on $\mathcal{S}$) if it is absolutely stable for each $S \in \mathcal{S}$ in the sense of Lur'e-Postnikov [1].

The above Propositions and Corollaries allow us to establish sufficient conditions for absolute stability of the zero solution of system (4.6.1) on $\mathcal{S}$.

Theorem 4.6.4 *Let system (4.6.1) be such that the matrix-valued function (4.6.23) is constructed with the elements (4.6.24) and*

(1) *the matrix A in (4.6.28) is positive definite, i.e. $\lambda_m(A) > 0$;*
(2) *the matrix Θ in (4.6.32) is negative semi-definite or equals to zero, i.e. $\lambda_M(\Theta) \le 0$;*
(3) *the matrix Λ in (4.6.33) is negative definite.*

Then the zero solution of system (4.6.1) is absolutely stable on $\mathcal{S}$.

Proof Under all conditions of Theorem 4.6.4

(a) the function $v(x,\eta)$ is positive definite;

(b) for the function $v(x,\eta)$ and $t \neq \tau_k(x)$, $k = 1, 2, \dots,$ we have

$$Dv(x,\eta) \le 0 \quad \text{for all} \quad S \in \mathcal{S}, \quad \text{and} \quad x \in R^n;$$

(c) for the function $v(x,\eta)$ and $t = \tau_k(x)$, $k = 1, 2, \dots,$ we have

$$v(x + J_k(x), \eta) - v(x,\eta) \le \lambda_M(\Lambda)\lambda_M^{-1}(B)v(x,\eta) \quad \text{for all} \quad x \in R^n.$$

By Theorem 1.4.4 from Chapter 1 for (a)−(c) the zero solution of system (4.6.1) is asymptotically stable in the whole on $\mathcal{S}$. Since here $\mathcal{N}_{ix} = R^{n_i}$, $i = 1, 2, \dots, m$, and $\mathcal{N}_x = \mathcal{N}_{1x} \times \dots \times \mathcal{N}_{mx} = R^n$.

Theorem 4.6.5 *Let system (4.6.1) be such that the matrix-valued function (4.6.23) is constructed with the elements (4.6.24) and*

(1) *the matrix A in (4.6.28) is positive definite, i.e.* $\lambda_m(A) > 0$;
(2) *the matrix Θ in (4.6.31) is negative definite, i.e.* $\lambda_M(\Theta) < 0$;
(3) *the matrix Λ^* in (4.6.34) is positive definite, i.e.* $\lambda_M(\Lambda^*) > 0$;
(4) *the function* $\tau_k(x), k = 1, 2, \dots,$ *satisfies the inequality*

$$\sup_k \left(\min_{x \in R^n} \tau_{k+1}(x) - \max_{x \in R^n} \tau_k(x) \right) = \theta > 0.$$

If for some $\gamma > 0$ *the inequality*

$$-\frac{\lambda_M(B)}{\lambda_M(\Theta)} \ln \frac{\lambda_M(\Lambda^*)}{\lambda_m(A)} \le \theta - \gamma,$$

is satisfied, then the zero solution of system (4.6.1) is absolutely stable on $\mathcal{S}$.

The proof follows from Propositions 4.6.1−4.6.2 and Theorem 1.4.5.

Theorem 4.6.6 *Let system (4.6.1) be such that the matrix-valued function (4.6.23) is constructed with the elements (4.6.24) and*

(1) *the matrix A in (4.6.28) is positive definite, i.e.* $\lambda_m(A) > 0$;
(2) *the matrix Θ in (4.6.31) is positive definite, i.e.* $\lambda_M(\Theta) > 0$;
(3) *the matrix Λ^* in (4.6.34) is positive definite, i.e.* $\lambda_M(\Lambda^*) > 0$;
(4) *the functions* $\tau_k(x), k = 1, 2, \dots,$ *for some* $\theta_1 > 0$ *satisfy the inequality*

$$\max_{x \in R^n} \tau_k(x) - \min_{x \in R^n} \tau_{k-1}(x) \le \theta_1, \quad k = 1, 2, \dots.$$

If for some $\gamma > 0$ *the inequality*

$$\frac{\lambda_m(A)}{\lambda_M(\Theta)} \ln \frac{\lambda_m(A)}{\lambda_m(\Lambda^*)} \ge \theta_1 + \gamma,$$

is satisfied, then the zero solution of system (4.6.1) is absolutely stable on $\mathcal{S}$.

Proof The statement of Theorem 4.6.6 follows from Propositions 4.6.1−4.6.2 and Theorem 3 in Martynyuk and Stavroulakis [1] (see also Miladzhanov [1]).

Example 4.6.1 Let system (4.6.1) be a fourth-order system of the Lur'e-Postnikov type decomposed into two subsystems determined by the following vectors and matrices:

$$
\begin{aligned}
&A_{ii} = \begin{pmatrix} -4 & 1 \\ 1 & -4 \end{pmatrix}, \quad i = 1, 2, \\
&A_{12} = A_{21} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad q_{i\ell} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\
&c^r_{i\ell} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad k_{i\ell} = 1, \quad i, \ell, r = 1, 2; \\
&J_{kii} = \operatorname{diag}\{-1, 1\}, \quad J_{k12} = J_{k21} = \operatorname{diag}\{0.1, 0.1\}, \\
&b_{i\ell} = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix}, \quad \widetilde{c}^r_{i\ell} = \begin{pmatrix} 0.1 \\ 0 \end{pmatrix}, \quad \widetilde{k}_{i\ell} = 1, \quad i, \ell, r = 1, 2; \\
&S^{(r)}_{ii} = \operatorname{diag}\{1, 1\}, \quad S^{(r)}_{ij} = s^{(r)}_{ij} \operatorname{diag}\{1, 1\}, \\
&0 \le s^{(r)}_{ij} \le 1, \quad i, j, r = 1, 2, \quad i \ne j.
\end{aligned}
\tag{4.6.42}
$$

For this example, the elements of the matrix-valued function (4.6.23) are taken in the form

$$
\begin{aligned}
&u_{ii}(x_i) = x_i^{\mathrm{T}} I_2 \, x_i, \quad i = 1, 2; \\
&u_{12}(x_1, x_2) = u_{21}(x_1, x_2) = x_1^{\mathrm{T}} 0,1 \, I_2 x_2,
\end{aligned}
$$

where $I_2 = \operatorname{diag}\{1, 1\}$.

Let also $\eta^{\mathrm{T}} = (1, 1) \in R^2_+$. It is easy to verify that the matrices

$$
A = \begin{pmatrix} 1 & -0.1 \\ -0.1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0.1 \\ 0.1 & 1 \end{pmatrix}
$$

are positive definite because

$$
\lambda_m(A) = 0.9 \quad \text{and} \quad \lambda_M(B) = 1.1.
$$

For such a choice of the matrix-valued function (4.6.23), we have

$$
\Theta = \begin{pmatrix} 3.75 & 3.35 \\ 3.35 & -3.75 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} -0.917 & 0.502 \\ 0.502 & -0.917 \end{pmatrix},
$$

It is easy to check that matrices Θ and Λ are negative definite. Therefore, all conditions of Theorem 4.6.1 are satisfied and the zero solution of system (4.6.1) specified by vectors and matrices (4.6.42) is absolutely stable on $\mathcal{S}$.

4.7 Notes and References

Section 4.1 The methods of qualitative analysis of nonlinear systems discussed in Chapters 1 – 3 have several analogues for the impulsive systems. Thus yet Krylov and Bogolyubov [1] paid attention to the possibility to apply the methods of nonlinear mechanics in the investigation of systems with impulsive perturbations. Some monographs and textbooks which contain the development of the theory of impulsive systems without structural perturbations have been mentioned in Notes and References to Chapter 1. Though many problems of nonlinear dynamics of systems under impulsive and nonclassical structural perturbations remain still open.

Section 4.2 Dynamics of the impulsive systems for the given model of nonclassical structural perturbations was first studied by Miladzhanov [3, 6]. These investigations were further developed and refined by Martynyuk and Miladzhanov [6] (see also Martynyuk and Stavroulakis [1 – 3]).

Section 4.3 The definitions of stability of impulsive systems under nonclassical structural perturbations take into account some peculiarities of the systems of this class which have been discussed in Section 1.4.

Sections 4.4 – 4.5 These sections are based on the results by Martynyuk and Miladzhanov [6], Martynyuk and Stavroulakis [1– 3] and Miladzhanov [3, 6]. In the investigation of linear impulsive systems some results are used which were obtained in Chapter 2 for linear continuous systems under nonclassical structural perturbations.

Section 4.6 The evolution analysis of the impulsive system with respect to two or more different measures (see Leela [1], Lakshmikantham and Liu [1], Martynyuk [11], etc.) is a possible direction of generalization of the method of matrix-valued Liapunov functions for the system with impulsive and nonclassical structural perturbations. The results of Section 4.6.1 are new (see Martynyuk and Chernetskaya [1], and cf. Kou, *et al.* [1]). In Section 4.6.2 we use general results of this Chapter and the results by Martynyuk and Miladzhanov [6], and Martynyuk and Stavroulakis [1].

5

SINGULARLY PERTURBED LARGE-SCALE SYSTEMS

5.1 Introduction

In this chapter we propose some development of the direct Liapunov method for the given class of systems of equations in terms of auxiliary matrix valued functions. This allows us to weaken the requirements to the dynamical properties of the individual subsystems and to extend the variation limits for the small parameters μ_i for senior derivatives of the systems of perturbed motion equations.

The chapter is arranged as follows.

Section 5.2 sets out the method of composition of large–scale system on the basic of individual subsystems for the given model of connectedness.

Sections 5.3 − 5.4 contain the results of development of a new method of stability and/or instability analysis of large–scale system under nonclassical structural perturbations.

In section 5.5. similar problems are discussed for linear singularly perturbed systems for uniform and nonuniform time scaling.

In final Section 5.6 two problems of practical importance are considered. One of the problems relates to absolute stability under nonclassical structural perturbations and the other deals with gyroscopic stabilization of orbital apparatus.

5.2 Nonclassical Structural Perturbations in Singularly Perturbed Systems

The real systems in which the fast and slow variables can potentially exist are modeled by means of the systems of equations with small parameter at senior variable (singularly perturbed systems). The class of systems of equations under consideration (furtheron designated as F) is described basing on the hypotheses below (cf. Grujić *et al.* [1]).

H_1. System F consists of q subsystems of ordinary differential equations with structural perturbations and r subsystems with structural perturbations and small parameters at senior derivatives. The order of fast and slow components of the system remains unchanged during all the period of system F functionning.

H_2. Dynamics of the i-th interconnected subsystem F_i in system F is described by the equations

$$
\begin{aligned}
\frac{dx_i}{dt} &= f_i(t, x, y, P_i, S_i), \\
\mu_i \frac{dy_i}{dt} &= g_i(t, x, y, M, P_{q+i}, S_{q+i}),
\end{aligned}
\tag{5.2.1}
$$

where $x_i \in R^{n_i}$, $y_i \in R^{m_i}$, f_i and g_i are continuous vector–functions of the corresponding dimensions, μ_i are small positive parameters, $\mu_i \in (0, 1]$ and $M = \operatorname{diag}\{\mu_1, \dots, \mu_n\}$.

H_3. Dynamics of the i-th isolated subsystem $\widehat{F}_i$ in system F is described by the equations

$$\begin{aligned} \frac{dx_i}{dt} &= f_i(t, x^i, y^i, P_i, S_i), \\ \mu_i \frac{dy_i}{dt} &= g_i(t, x^i, y^i, M, P_{q+i}, S_{q+i}), \end{aligned} \tag{5.2.2}$$

where $x^i = (0, 0, \dots, 0, x_i^{\mathrm{T}}, 0, 0, \dots, 0)^{\mathrm{T}} \in R^n$, $n = n_1 + n_2 + \dots + n_q$, $x_i \in R^{n_i}$, $y^i = (0, 0, \dots, 0, y_i^{\mathrm{T}}, 0, 0, \dots, 0)^{\mathrm{T}} \in R^m$, $m = m_1 + m_2 + \dots + m_r$, $y_i \in R^{m_i}$.

In the case when $q = r$, the equations

$$\begin{aligned} \frac{dx_i}{dt} &= f_i(t, x^i, y^i, P_i, S_i), \\ 0 &= g_i(t, x^i, y^i, 0, P_{q+i}, S_{q+i}) \end{aligned} \tag{5.2.3}$$

describe the dynamics of the i -th isolated subsystem $\widehat{F}_{i0}$ of system F, and the equations

$$\frac{dy_i}{dt_i} = g_i(\alpha, b^i, y^i, 0, P_{q+i}, S_{q+i}) \tag{5.2.4}$$

characterize the boundary layer of the fast subsystem $\widehat{F}_{t_i}$ of system F. Here $\alpha \in R$, $b^i = (0, \dots, 0, b_i^{\mathrm{T}}, 0, \dots, 0)^{\mathrm{T}} \in R^n$, $b_i \in R^{n_i}$, $t_i = \frac{t - t_0}{\mu_i}$, $i = 1, 2, \dots, r$.

H_4. Dynamics of the whole system F is described by the equations

$$\begin{aligned} \frac{dx_i}{dt} &= f_i(t, x, y, P_i, S_i), \quad i = 1, 2, \dots, q, \\ \mu_i \frac{dy_i}{dt} &= g_i(t, x, y, M, P_{q+i}, S_{q+i}), \quad i = 1, 2, \dots, r, \end{aligned} \tag{5.2.5}$$

where $x_i \in R^{n_i}$, $\sum\limits_{i=1}^{q} n_i = n$, $y_i \in R^{m_i}$, $\sum\limits_{i=1}^{r} m_i = m$, $q + r = s$, the parametric perturbations P_i, $i = 1, 2, \dots, q$, and the structural matrices S_i, $i = 1, 2, \dots, s$, are determined in the same way as in Section 2.2. Here $\mu_i \in (0, 1]$ and the set of all admissible values of M is designated as

$$\mathcal{M} = \{M| : \ 0 < M \le I\}, \quad I = \operatorname{diag}\{1, 1, \dots, 1\} \in R^{r \times r},$$

Moreover

$$\mathcal{M}_m = \{M : \ 0 < \mu_i < \mu_{im}, \ \forall\, i \in [1, r]\},$$

where μ_{im} is an admissible upper value of μ_i.

If in the system of equations (5.2.5) all μ_i (formally) form a zero set, then the equations

$$\begin{aligned} \frac{dx_i}{dt} &= f_i(t, x, y, P_i, S_i), \quad i = 1, 2, \dots, q, \\ 0 &= g_i(t, x, y, 0, P_{q+i}, S_{q+i}), \quad i = 1, 2, \dots, r, \end{aligned} \tag{5.2.6}$$

describe the dynamics of the interconnected degenerated subsystem F_0 of system F, and the equations

$$\frac{dy_i}{dt_i} = \tau_i g_i(\alpha, b, y, 0, P_{q+i}, S_{q+i}), \quad i = 1, 2, \dots, r, \tag{5.2.7}$$

characterize the behaviour of the interconnected fast subsystem F_t (the boundary layer of system F).

If the small parameters μ_i are not mutually connected, then the system F has r essentially independent time scales t_i:

$$t_i = \frac{t - t_0}{\mu_i}, \quad i = 1, 2, \dots, r. \tag{5.2.8}$$

In this case the time scaling is nonuniform.

The time scales t_i can be interconnected through the values τ_i:

$$\frac{t_i}{t_1} = \tau_i, \quad i = 1, 2, \dots, r, \tag{5.2.9}$$

which are variable within certain limits

$$\tau_i \in [\underline{\tau}_i, \overline{\tau}_u], \quad i = 1, 2, \dots, r, \tag{5.2.10}$$

where $0 < \underline{\tau}_i \le \overline{\tau}_i < \infty$, $\forall\, i \in [1, r]$.

In the case (5.2.9) and (5.2.10) the time scaling is uniform and

$$\tau_i = \frac{\mu_1}{\mu_i}, \quad i = 1, 2, \dots, r. \tag{5.2.11}$$

Obviously, in this case $\underline{\tau}_1 = \tau_1 = \overline{\tau}_1 = 1$.

Everywhere below it is assumed that the correlations

$$0 = g_i(t, x, y, 0, P_{q+i}, S_{q+i}), \quad \forall (t, x, y) \in R \times \mathcal{N}_x \times \mathcal{N}_y,$$

are satisfied for each pair $(P, S) \in \mathcal{P} \times \mathcal{S}$ iff $y = 0$ and

$$0 = g_i(t, x^i, y^i, 0, P_{q+i}, S_{q+i}), \quad \forall (t, x^i, y^i) \in R \times \mathcal{N}_x \times \mathcal{N}_y,$$

are satisfied for each pair $(P, S) \in \mathcal{P} \times \mathcal{S}$ iff $y^i = 0$.

Therefore systems (5.2.6) and (5.2.3) are equivalent to the system

$$\frac{dx_i}{dt} = f_i(t, x^i, 0, P_i, S_i), \tag{5.2.12}$$

$$\frac{dx_i}{dt} = f_i(t, x, 0, P_i, S_i), \quad i = 1, 2, \dots, q, \tag{5.2.13}$$

respectively.

5.3 Tests for Stability Analysis

In the qualitative analysis for the given class of large scale systems the question whether different time scales t_i are interconnected is of importance.

General purpose of our investigation is to obtain conditions under which stability of zero solution of the initial system is implied by stability of some independent degenerated subsystems and stability of the independent subsystems describing boundary layer with allowance for the qualitative properties of interconnections between the subsystems.

5.3.1 Non-uniform time scaling Assume that the correlation $q = r$ is satisfied. Then system (5.2.5) is represented as

$$\begin{aligned} \frac{dx_i}{dt} &= f_i(t, x^i, 0, P_i, S_i) + f_i^* + f_i^{**}, \quad i = 1, 2, \dots, q, \\ \mu_i \frac{dy_i}{dt} &= g_i(\alpha, b^i, y^i, P_{q+i}, S_{q+i}) + g_i^* + g_i^{**}, \quad i = 1, 2, \dots, q, \end{aligned} \tag{5.3.1}$$

where

$$\begin{aligned} f_i^* &= f_i(t, x^i, y^i, P_i, S_i) - f_i(t, x^i, 0, P_i, S_i), \\ g_i^* &= g_i(t, x^i, y^i, M^i, P_{q+i}, S_{q+i}) - g_i(\alpha, b^i, y^i, 0, P_{q+i}, S_{q+i}), \\ f_i^{**} &= f_i(t, x, y, P_i, S_i) - f_i(t, x^i, y^i, P_i, S_i), \\ g_i^{**} &= g_i(t, x, y, M, P_{q+i}, S_{q+i}) - g_i(t, x^i, y^i, M^i, P_{q+i}, S_{q+i}). \end{aligned}$$

Here the functions f_i^* and g_i^* describe the connections between equations of the i-th independent singularly perturbed subsystem (F_i) of the system F, and the functions f_i^{**} and g_i^{**} describe all the rest connections in system F.

In view of results from Martynyuk and Miladzhanov [1−5] we introduce some assumptions.

Assumption 5.3.1 There exist

(1) open connected neighborhoods $\mathcal{N}_{ix} \subseteq R^{n_i}$, $\mathcal{N}_{iy} \subseteq R^{m_i}$ of the states $x_i = 0$ and $y_i = 0$ respectively;

(2) functions φ_{ik}, ψ_{ik} of Hahn class $K\,(KR)$, $k = 1, 2$, $i \in [1, q]$, constants $\underline{\alpha}_{ij}$, $\overline{\alpha}_{ij}$, $\underline{\alpha}_{i,\,q+j}$, $\overline{\alpha}_{i,\,q+j}$, $\underline{\alpha}_{q+i,\,q+j}$, $\overline{\alpha}_{q+i,\,q+j}$, $i, j = 1, 2, \dots, q$, and the matrix–function

$$U(t,x,y,M) = \begin{pmatrix} U_{11}(t,x) & U_{12}(t,x,y,M) \\ U_{12}^{\mathrm{T}}(t,x,y,M) & U_{22}(t,y,M) \end{pmatrix} \tag{5.3.2}$$

where

$$\begin{gathered}
U_{11}(t,x) = [v_{ij}(t,\cdot)], \quad v_{ii} = v_{ii}(t,x_i), \\
v_{ij} = v_{ji} = v_{ij}(t,x_i,x_j), \quad i,j = 1,2,\dots,q; \\
U_{22}(t,y,M) = [v^*_{q+i,q+j}(t,\cdot)], \quad v^*_{q+i,q+i} = \mu_i v_{q+i,q+i}(t,y_i), \\
v^*_{q+i,q+j} = v^*_{q+j,q+i} = \mu_i\mu_j v_{q+i,q+j}(t,y_i,y_j), \quad i,j = 1,2,\dots,q; \\
U_{12}(t,x,y,M) = [\mu_j v_{i,q+j}(t,x_i,y_j)], \quad i,j = 1,2,\dots,q, \quad 2q = s,
\end{gathered}$$

such that

(a) $\underline{\alpha}_{ij}\varphi_{i1}(x_i)\varphi_{j1}(x_j) \le v_{ij}(t,\cdot) \le \overline{\alpha}_{ij}\varphi_{i2}(x_i)\varphi_{j2}(x_j)$, $\forall\,(t,x_i,x_j) \in R \times \mathcal{N}_{ix} \times \mathcal{N}_{jx}$, $i,j = 1,2,\dots,q$, $j \ge i$;

(b) $\underline{\alpha}_{q+i,q+j}\psi_{i1}(y_i)\psi_{j1}(y_j) \le v_{q+i,q+j}(t,\cdot) \le \overline{\alpha}_{q+i,q+j}\psi_{i2}(y_i)\psi_{j2}(y_j)$, $\forall\,(t,y_i,y_j) \in R \times \mathcal{N}_{iy} \times \mathcal{N}_{jy}$, $i,\, j = 1,2,\dots,q$, $j \ge i$;

(c) $\underline{\alpha}_{i,q+j}\varphi_{i1}(x_i)\psi_{j1}(y_j) \le v_{i,q+j}(t,\cdot) \le \overline{\alpha}_{i,q+j}\varphi_{i2}(x_i)\psi_{j2}(y_j)$, $\forall\,(t,x_i,y_j) \in R \times \mathcal{N}_{ix} \times \mathcal{N}_{jy}$, $i,j = 1,2,\dots,q$.

By means of the matrix–function (5.3.2) and the constant vector $\eta \in R^s_+$ we introduce the function

$$v(t,x,y,M) = \eta^{\mathrm{T}} U(t,x,y,M)\eta \tag{5.3.3}$$

and consider the expressions of the upper right Dini derivative

$$\begin{gathered}
D^+v(t,x,y,M) = \eta^{\mathrm{T}} D^+U(t,x,y,M)\eta, \\
D^+U(t,x,y,M) \overset{\mathrm{def}}{=} [D^+v_{rk}(t,\dots)], \quad r,k = 1,2,\dots,s.
\end{gathered} \tag{5.3.4}$$

For the function (5.3.3) the following assertion holds true.

Proposition 5.3.1 *Under conditions of Assumption 5.3.1 for function (5.3.3) the bilateral estimate*

$$\begin{gathered}
u_1^{\mathrm{T}} A(M)u_1 \le v(t,x,y,M) \le u_2^{\mathrm{T}} B(M)u_2, \\
\forall\,(t,x,y,M) \in R \times \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{M},
\end{gathered}$$

is satisfied, where

$$\mathcal{N}_x \subseteq \mathcal{N}_{1x} \times \mathcal{N}_{2x} \times \ldots \times \mathcal{N}_{qx}, \quad \mathcal{N}_y \subseteq \mathcal{N}_{1y} \times \mathcal{N}_{2y} \times \ldots \times \mathcal{N}_{qy},$$

$$u_k^{\mathrm{T}} = (\varphi_{1k}(x_1),\, \dots,\, \varphi_{qk}(x_q),\, \psi_{1k}(y_1),\, \dots\, \psi_{qk}(y_q)), \quad k = 1, 2,$$
$$A(M) = H^{\mathrm{T}} A_1(M) H, \quad B(M) = H^{\mathrm{T}} A_2(M) H,$$
$$H = \operatorname{diag}\{\eta_1, \eta_2, \dots, \eta_s\}, \quad s = 2q,$$
$$A_1(M) = \begin{pmatrix} A_{11} & A_{12}(M) \\ A_{12}^{\mathrm{T}}(M) & A_{22}(M) \end{pmatrix}, \quad A_2(M) = \begin{pmatrix} \overline{A}_{11} & \overline{A}_{12}(M) \\ \overline{A}_{12}^{\mathrm{T}}(M) & \overline{A}_{22}(M) \end{pmatrix},$$
$$A_{11} = [\underline{\alpha}_{ij}], \quad \underline{\alpha}_{ij} = \underline{\alpha}_{ji}, \quad \overline{A}_{11} = [\overline{\alpha}_{ij}], \quad \overline{\alpha}_{ij} = \overline{\alpha}_{ji},$$
$$A_{12}(M) = [\mu_j \underline{\alpha}_{i,q+j}], \quad \overline{A}_{12}(M) = [\mu_j \overline{\alpha}_{i,q+j}],$$
$$A_{22}(M) = [\mu_{ij}^* \underline{\alpha}_{q+i,q+j}] \quad \underline{\alpha}_{q+i,q+j} = \underline{\alpha}_{q+j,q+i},$$
$$\overline{A}_{22}(M) = [\mu_{ij}^*] \overline{\alpha}_{q+i,q+j}, \quad \overline{\alpha}_{q+i,q+j} = \overline{\alpha}_{q+j,q+i},$$
$$\mu_{ij}^* = \begin{cases} \mu_i & \textit{for} \quad i = j, \\ \mu_i \mu_j & \textit{for} \quad i \neq j, \end{cases} \qquad i,\, j = 1, 2, \dots, q.$$

Proof Let all conditions of Assumption 5.3.2 be satisfied. Then for function (5.3.3) we have

$$v(t,x,y,M) = \sum_{i=1}^{q} \eta_i^2 v_{ii}(t,x_i) + 2 \sum_{i=1}^{q} \sum_{j=2}^{q} \eta_i \eta_j v_{ij}(t,x_i,x_j)$$
$$+ \sum_{i=1}^{q} \eta_{q+i}^2 \mu_i v_{q+i,q+i}(t,y_i) + 2 \sum_{i=1}^{q} \sum_{\substack{j=2 \\ j>i}}^{q} \eta_{q+i} \eta_{q+j} \mu_i \mu_j v_{q+i,q+j}$$

$$
\begin{aligned}
&+ 2\sum_{i=1}^{q}\sum_{j=1}^{q}\eta_i\eta_{q+j}\mu_j v_{i,q+j}(t,x_i,y_j) \geq \sum_{i=1}^{q}\eta_i^2\underline{\alpha}_{ii}\varphi_{i1}^2\\
&+ 2\sum_{i=1}^{q}\sum_{\substack{j=2\\ j>i}}^{q}\eta_i\eta_j\underline{\alpha}_{ij}\varphi_{i1}(x_i)\varphi_{j1}(x_j) + \sum_{i=1}^{q}\eta_{q+i}^2\mu_i\underline{\alpha}_{q+i,q+j}\psi_{i1}^2(y_i)\\
&+ 2\sum_{i=1}^{q}\sum_{\substack{j=2\\ j>i}}^{q}\eta_{q+i}\eta_{q+j}\mu_i\mu_j\underline{\alpha}_{q+i,q+j}\psi_{i1}(y_i)\psi_{j1}(y_j)\\
&+ 2\sum_{i=1}^{q}\sum_{j=1}^{q}\eta_i\eta_{q+j}\mu_j\underline{\alpha}_{i,q+j}\varphi_{i1}(x_i)\psi_{j1}(y_j)\\
&= (\varphi_{11}(x_1),\dots,\varphi_{q1}(x_q),\,\psi_{11}(y_1),\,\dots,\,\psi_{q1}(y_q))^{\mathrm{T}}\operatorname{diag}\{\eta_1,\dots,\eta_{2q}\}\\
&\quad\times\begin{pmatrix} A_{11} & A_{12}(M)\\ A_{12}^{\mathrm{T}}(M) & A_{22}(M)\end{pmatrix}\operatorname{diag}\{\eta_1,\eta_2,\dots,\eta_{2q}\}\\
&\times(\varphi_{11}(x_1),\dots,\varphi_{q1}(x_q),\,\psi_{11}(y_1),\,\dots,\,\psi_{q1}(y_q)) = u_1^{\mathrm{T}}A(M)u_1.
\end{aligned}
$$

The upper estimate is proved in the same manner.

Assumption 5.3.2 There exist

(1) functions $\varphi_i,\ \psi_i$ of class $K\,(KR)$, $i = 1,2,\dots,q$;

(2) functions $v_{ij},\ v_{i,q+j},\ v_{q+i,q+j},\ i,\ j = 1,2,\dots,q$ satisfying the conditions mentioned in Assumption 5.3.1, and

  (a) functions $v_{ij}(t,x_i,x_j)$ are continuous on $(R\times\mathcal{N}_{ix_0}\times\mathcal{N}_{jx_0})$ or on $(R\times R^{n_i}\times R^{n_j})$;

  (b) functions $v_{i,q+j}(t,x_i,y_j)$ are continuous on $(R\times\mathcal{N}_{ix_0}\times\mathcal{N}_{jy_0})$ or on $(R\times R^{n_i}\times R^{m_j})$;

  (c) functions $v_{q+i,q+j}(t,y_i,y_j)$ are continuous on $(R\times\mathcal{N}_{iy_0}\times\mathcal{N}_{jy_0})$ or on $(R\times R^{m_i}\times R^{m_j})$;

(3) real numbers $\rho_{\alpha i}(P,S),\ \rho_{\beta ij}(P,S),\ \alpha = 1,2,\dots,13,\ \beta = 1,2,\dots,8,$ $i,\ j = 1,2\dots,q$, and the following conditions are satisfied

  (a) $\eta_i^2D_t^+v_{ii} + \eta_i^2(D_{x_i}^+v_{ii})^{\mathrm{T}}f_i(t,x^i,0,P_i,S_i) \leq \rho_{1i}(P,S)\varphi_i^2(x_i)$,
  $\forall\,(t,x_i,P,S)\in R\times\mathcal{N}_{ix_0}\times\mathcal{P}\times\mathcal{S},\ \ i = 1,2,\dots,q$;

  (b) $\eta_{q+i}^2\mu_iD_t^+v_{q+i,q+i}+\eta_{q+i}^2(D_{y_i}^+v_{q+i,q+i})^{\mathrm{T}}g_i(\alpha,b^i,y^i,0,P_{q+i},S_{q+i})$
  $\leq \rho_{2i}(P,S)\psi_i^2(y_i),\ \ \forall\,(t,y_i,M,P,S)\in R\times\mathcal{N}_{iy_0}\times\mathcal{M}\times\mathcal{P}\times\mathcal{S}$,
  $i = 1,2,\dots,q$;

  (c) $\eta_i^2(D_{x_i}^+v_{ii})^{\mathrm{T}}f_i^* + \eta_{q+i}^2(D_{y_i}^+v_{q+i,q+i})^{\mathrm{T}}g_i^* + 2\eta_i\eta_{q+i}\{\mu_iD_t^+v_{i,q+i}$
  $+ \mu_i(D_{x_i}^+v_{i,q+i})^{\mathrm{T}}f_i(t,x^i,y^i,P_i,S_i)$
  $+ (D_{y_i}^+v_{i,q+i})^{\mathrm{T}}g_i(t,x^i,y^i,M^i,P_{q+i},S_{q+i})\}$
  $\leq (\rho_{3i}(P,S) + \mu_i\rho_{4i}(P,S))\varphi_i^2(x_i) + (\rho_{5i}(P,S)$
  $+ \mu_i\rho_{6i}(P,S))\psi_i^2(y_i) + 2(\rho_{7i}(P,S) + \mu_i\rho_{8i}(P,S))\varphi_i(x_i)\psi_i(y_i)$,
  $\forall\,(t,x_i,y_i,M,P,S)\in R\times\mathcal{N}_{ix_0}\times\mathcal{N}_{iy_0}\times\mathcal{M}\times\mathcal{P}\times\mathcal{S}$,
  $i = 1,2,\dots,q$;

  (d) $\displaystyle\sum_{i=1}^{q}\eta_i^2(D_{x_i}^+v_{ii})^{\mathrm{T}}f_i^{**} + \sum_{i=1}^{q}\eta_{q+i}^2(D_{y_i}^+v_{q+i,q+i})^{\mathrm{T}}g_i^{**}$

$$
\begin{aligned}
&+ \sum_{i=1}^{q} 2\eta_i\eta_{q+i}\Big\{\mu_i(D^+_{x_i}v_{i,q+i})^{\mathrm T} f_i^{**} + (D^+_{y_i}v_{i,q+i})^{\mathrm T} g_i^{**}\Big\}\\
&+ \sum_{i=1}^{q}\sum_{\substack{j=2\\ j>i}}^{q} \eta_i\eta_j\Big\{D^+_t v_{ij} + (D^+_{x_i}v_{ij})^{\mathrm T} f_i(t,x,y,P_i,S_i)\\
&\quad + (D^+_{x_j}v_{ij})^{\mathrm T} f_j(t,x,y,P_j,S_j)\Big\}\\
&+ 2\sum_{i=1}^{q}\sum_{\substack{j=2\\ j>i}}^{q} \eta_{q+i}\eta_{q+j}\Big\{\mu_i\mu_j D^+_t v_{q+i,q+j}\\
&\quad + \mu_j(D^+_{y_i}v_{q+i,q+j})^{\mathrm T} g_i(t,x,y,M,P_{q+i},S_{q+i})\\
&\quad + \mu_i(D^+_{y_j}v_{q+i,q+j})^{\mathrm T} g_j(t,x,y,M,P_{q+j},S_{q+j})\Big\}\\
&+ 2\sum_{i=1}^{q}\sum_{\substack{j=1\\ j\neq i}}^{q} \eta_i\eta_{q+j}\Big\{\mu_j D^+_t v_{i,q+j} + \mu_j(D^+_{x_i}v_{i,q+j})^{\mathrm T}\\
&\times f_i(t,x,y,P_i,S_i) + (D^+_{y_j}v_{i,q+j})^{\mathrm T} g_j(t,x,y,M,P_{q+j},S_{q+j})\Big\}\\
&\le \sum_{i=1}^{q}\Big\{(\rho_{9i}(P,S) + \mu_i\rho_{10i}(P,S))\varphi_i^2(x_i)\\
&\quad + \Big(\rho_{11i}(P,S) + \mu_i\rho_{12i}(P,S) + \mu_i\Big(\sum_{\substack{j=2\\ j>i}}^{q}\mu_j\Big)\rho_{13i}\Big)\psi_i^2(y_i)\Big\}\\
&+ 2\sum_{i=1}^{q}\sum_{\substack{j=2\\ j>i}}^{q}\Big\{(\rho_{1,i,j}(P,S) + \mu_i\rho_{2,i,j}(P,S))\varphi_i(x_i)\varphi_j(x_j)\\
&\quad + (\rho_{3,i,j}(P,S) + \mu_i\rho_{4,i,j}(P,S)\\
&\quad + \mu_i\mu_j\rho_{5,i,j}(P,S))\psi_i(y_i)\psi_j(y_j)\Big\} + 2\sum_{i=1}^{q}\sum_{\substack{j=1\\ j\neq i}}(\rho_{6,i,j}(P,S)\\
&\quad + \mu_i\rho_{7,i,j}(P,S) + \mu_i\mu_j\rho_{8,i,j}(P,S))\varphi_i(x_i)\psi_j(y_j),\\
&\forall\,(t,x,y,M,P,S) \in R\times\mathcal N_{x_0}\times\mathcal N_{y_0}\times\mathcal N\times\mathcal P\times\mathcal S,\ \text{ where}
\end{aligned}
$$

$$
\mathcal N_{ix_0} = \{x_i\colon\ x_i\in\mathcal N_{ix},\ x_i\neq 0\},\quad \mathcal N_{iy_0} = \{y_i\colon\ y_i\in\mathcal N_{iy},\ y_i\neq 0\},\quad i=1,2,\dots,q,\quad 2q=s.
$$

Proposition 5.3.2 *Under conditions of Assumption 5.3.2 the estimate*

$$
D^+v(t,x,y,M) \le u^{\mathrm T}G(M,P,S)u,\quad \forall\,(t,x,y,M,P,S)\in R\times\mathcal N_{x_0}\times\mathcal N_{y_0}\times\mathcal M\times\mathcal P\times\mathcal S,
$$

is true, where

$$
\begin{aligned}
&u^{\mathrm T} = (\varphi_1(1),\dots,\varphi_q(x_q),\psi_1(y_1),\dots,\psi_q(y_q)),\\
&G(M,P,S) = [\sigma_{ij}(M,P,S)],\quad \sigma_{ij}=\sigma_{ji},\quad i,\,j=1,2,\dots,s,\\
&\sigma_{ii}(M,P,S) = \rho_{1i}(P,S) + \rho_{3i}(P,S) + \rho_{9i}(P,S) + \mu_i(\rho_{4i}(P,S)\\
&\qquad + \rho_{10i}(P,S)),\quad i=1,2,\dots,q;\\
&\sigma_{q+i,q+i}(M,P,S) = \rho_{2i}(P,S) + \rho_{5i}(P,S) + \rho_{11i}(P,S)
\end{aligned}
$$

$$+\mu_i\biggl(\rho_{6i}(P,S)+\rho_{12i}(P,S)+\biggl(\sum_{\substack{j=2\\ j>i}}^{q}\mu_j\biggr)\rho_{13i}(P,S)\biggr),$$

$$i=1,2,\dots,q;$$

$$\sigma_{i,q+i}(M,P,S)=\rho_{7i}(P,S)+\mu_i\rho_{8i}(P,S),\quad i=1,2,\dots,q;$$

$$\sigma_{ij}(M,P,S)=\rho_{1ij}(P,S)+\mu_i\rho_{2ij},\quad i=1,2,\dots,q,\quad j=2,3,\dots,q,\quad j>i;$$

$$\sigma_{q+i,q+j}(M,P,S)=\rho_{3ij}(P,S)+\mu_i\rho_{4ij}(P,S)+\mu_i\mu_j\rho_{5ij}(P,S),$$
$$i=1,2,\dots,q,\quad j=2,3,\dots,q,\quad j>i;$$

$$\sigma_{i,q+j}(M,P,S)=\rho_{6ij}(P,S)+\mu_i\rho_{7ij}(P,S)+\mu_i\mu_j\rho_{8ij}(P,S),$$
$$i,\,j=1,2,\dots,q,\quad i\neq j.$$

Proof Let all conditions of Assumption 5.3.2 be satisfied. Then for the expression (5.3.4) we have

$$D^+v(t,x,y,M)=\sum_{i=1}^{q}\Bigl\{\eta_i^2D_t^+v_{ii}+(D_{x_i}^+v_{ii})^{\mathrm T}f_i(t,x^i,0,P_i,S_i)$$
$$+\eta_{q+i}^2\mu_iD_t^+v_{q+i,q+i}+\eta_{q+i}^2(D_{y_i}^+v_{q+i,q+i})^{\mathrm T}g_i(\alpha,b^i,y^i,0,P_{q+i},S_{q+i})$$
$$+\eta_i^2(D_{x_i}^+v_{ii})^{\mathrm T}f_i^*+\eta_{q+i}^2(D_{y_i}^+v_{q+i,q+i})^{\mathrm T}g_i^*+2\eta_i\eta_{q+i}\bigl(\mu_iD_t^+v_{i,q+i}$$
$$+\mu_i(D_{x_i}^+v_{i,q+i})^{\mathrm T}f_i(t,x^i,y^i,P_i,S_i)$$
$$+(D_{y_i}^+v_{i,q+i})^{\mathrm T}g_i(t,x^i,y^i,M^i,P_{q+i},S_{q+i})\bigr)$$

$$
\begin{aligned}
&+ \eta_i^2 (D_{x_i}^+ v_{ii})^{\mathrm T} f_i^{**} + \eta_{q+i}^2 (D_{y_i}^+ v_{q+i,q+i})^{\mathrm T} g_i^{**} \\
&+ 2\eta_i\eta_{q+i}\big(\mu_i (D_{x_i}^+ v_{i,q+i})^{\mathrm T} f_i^{**} + (D_{y_i}^+ v_{i,q+i})^{\mathrm T} g_i^{**}\big)\Big\} \\
&+ 2\sum_{i=1}^{q}\sum_{\substack{j=2\\ j>i}}^{q}\Big\{\eta_i\eta_j\Big(D_t^+ v_{ij} + (D_{x_i}^+ v_{ij})^{\mathrm T} f_i(t,x,y,P_i,S_i) \\
&+ (D_{x_j}^+ v_{ij})^{\mathrm T} f_j(t,x,y,P_j,S_j)\Big) + \eta_{q+i}\eta_{q+j}\Big(\mu_i\mu_j D_t^+ v_{q+i,q+j} \\
&+ \mu_j (D_{y_j}^+ v_{q+i,q+j})^{\mathrm T} g_i(t,x,y,M,P_{q+i},S_{q+i}) \\
&+ \mu_i (D_{y_j}^+ v_{q+i,q+j})^{\mathrm T} g_j(t,x,y,M,P_{q+j},S_{q+j})\Big)\Big\} \\
&+ 2\sum_{i=1}^{q}\sum_{\substack{j=1\\ j\neq i}}^{q}\eta_i\eta_{q+j}\Big\{\mu_j D_t^+ v_{i,q+j} + \mu_j (D_{x_i}^+ v_{i,q+j})^{\mathrm T} f_i(t,x,y,P_i,S_i) \\
&+ (D_{y_j}^+ v_{i,q+j})^{\mathrm T} g_j(t,x,y,M,P_{q+j},S_{q+j})\Big\} \\
&\le \sum_{i=1}^{q}\Big\{\rho_{1i}(P,S) + \rho_{3i}(P,S) + \rho_{9i}(P,S) + \mu_i(\rho_{4i}(P,S) \\
&+ \rho_{10i}(P,S)\Big\}\varphi_i^2(x_i) + \sum_{i=1}^{q}\Big\{\rho_{2i}(P,S) + \rho_{5i}(P,S) + \rho_{11,i}(P,S) \\
&+ \mu_i\left(\rho_{6i}(P,S) + \rho_{12}(P,S) + \left(\sum_{\substack{j=2\\ j>i}}^{q}\mu_j\right)\rho_{13,i}(P,S)\right)\Bigg\}\psi_i^2(y_i) \\
&+ \sum_{i=1}^{q}\{\rho_{7i}(P,S) + \mu_i\rho_{8i}(P,S)\}\varphi_i(x_i)\psi_i(y_i) \\
&+ 2\sum_{i=1}^{q}\sum_{\substack{j=2\\ j>i}}^{q}\{\rho_{1ij}(P,S) + \mu_i\rho_{2ij}(P,S)\}\varphi_i(x_i)\varphi_j(x_j) \\
&+ 2\sum_{i=1}^{q}\sum_{\substack{j=2\\ j>i}}^{q}\{\rho_{3ij}(P,S) + \mu_i\rho_{4ij}(P,S) + \mu_i\mu_j\rho_{5ij}(P,S)\}\psi_i(y_i)\psi_j(y_j) \\
&+ 2\sum_{i=1}^{q}\sum_{\substack{j=1\\ j\neq i}}^{q}\{\rho_{6ij}(P,S) + \mu_i\rho_{7ij}(P,S) + \mu_i\mu_j\rho_{8ij}(P,S)\}\varphi_i(x_i)\psi_j(y_j) \\
&= u^{\mathrm T} G(M,P,S)u.
\end{aligned}
$$

Theorem 5.3.1 *Let the perturbed motion equations (5.2.5) be such that all conditions of Assumptions 5.3.1 and 5.3.2 are satisfied and*

(a) *matrix $A(M)$ is positive definite for any $\mu_i \in (0, \widetilde{\mu}_{i1})$ and for $\mu_i \to 0,\ i = 1, 2, \dots, q$;*

(b) *there exists a matrix $\overline{G}(M)$ which is negative definite for any $\mu_i \in (0, \widetilde{\mu}_{i2})$ and for $\mu_i \to 0,\ i = 1, 2, \dots, q$, such that for the matrix $G(M,P,S)$ determined in Proposition 5.3.2 the estimate*

$$G(M,P,S) \le \overline{G}(M), \quad \forall\,(M,P,S) \in \mathcal{M}\times\mathcal{P}\times\mathcal{S}.$$

is satisfied.

Then the equilibrium state $(x^{\mathrm{T}}, y^{\mathrm{T}})^{\mathrm{T}} = 0$ *of system F is uniformly asymptotically stable for any* $\mu_i \in (0, \widetilde{\mu}_i)$ *and for* $\mu_i \to 0$ *on* $\mathcal{P} \times \mathcal{S}$*, where* $\widetilde{\mu}_i = \min\{1, \widetilde{\mu}_{i1}, \widetilde{\mu}_{i2}\}$*.*

If, moreover, $\mathcal{N}_{i1} \times \mathcal{N}_{iy} = R^{n_i+m_i}$*, the functions* φ_{ik}*,* ψ_{ik}*,* φ_i, ψ_i *are of class KR, then the equilibrium state* $(x^{\mathrm{T}}, y^{\mathrm{T}})^{\mathrm{T}} = 0$ *of system F is uniformly asymptotically stable in the whole for any* $\mu_i \in (0, \widetilde{\mu}_i)$ *and for* $\mu_i \to 0$ *on* $\mathcal{P} \times \mathcal{S}$*.*

Proof Under conditions of Assumption 5.3.1, Proposition 5.3.1 and condition (a) of Theorem 5.3.1 the function $v(t, x, y, M)$ is positive definite for any $\mu_i \in (0, \widetilde{\mu}_{i1})$ and for $\mu_i \to 0$ it is decreasing on $\mathcal{N}_{ix} \times \mathcal{N}_{iy}$. Conditions of Assumption 5.3.2, Proposition 5.3.2 and condition (b) of Theorem 5.3.1 imply that the expression $D^+v(t, x, y, M)$ is negative definite for any $\mu_i \in (0, \widetilde{\mu}_{i2})$ and for $\mu_i \to 0$ for each $(P, S) \in \mathcal{P} \times \mathcal{S}$.

These conditions are sufficient for uniform asymptotic stability of the equilibrium state of system (5.2.5) for any $\mu_i \in (0, \widetilde{\mu}_i)$ and for $\mu_i \to 0$ on $\widetilde{\mathcal{M}} \times \mathcal{P} \times \mathcal{S}$ since all conditions of Theorem 7 from Chapter 1 of the monograph by Grujić, *et al.* [1] are satisfied.

In the case when $\mathcal{N}_{ix} \times \mathcal{N}_{iy} = R^{n_i+m_i}$, the function $v(t, x, y, M)$ is positive definite, decreasing and radially unbounded. This fact together with the other conditions of the theorem prove the second statement.

Remark 5.3.1 From the condition of matrix $A(M)$ positive definiteness and matrix $G(M)$ negative definiteness the values $\widetilde{\mu}_{i1}$ and $\widetilde{\mu}_{i2}$ are determined respectively, since $\widetilde{\mu}_i = \min\{1, \widetilde{\mu}_{i1}, \widetilde{\mu}_{i2}\}$ is the lower estimate of the upper bound of the admissible μ_i so that $\widetilde{\mathcal{M}} = \{M\colon\ 0 < \mu_i < \widetilde{\mu}_i,\ i = 1, 2, \dots, q\}$.

Example 5.3.1 Consider nonlinear and nonstationary 8-th order system consisting of two interconnected 4-th order subsystems described by the equations

$$
\begin{aligned}
&\frac{dx_i}{dt} = (1 + \sin^2 t)(-x_i^3 + 0.1\, y_i^3) + 0.2 s_{i1}(t) y_j^3 \cos^2 t,\\
&\mu_i \frac{dy_i}{dt} = (1 + \sin^2 t)(-y_i^3 + 0.1\, \mu_i x_i^3) + 0.2 s_{2+i,1}(t) x_j^3 \cos^2 t,\\
&\qquad i, j = 1, 2, \quad i \neq j,
\end{aligned}
\tag{5.3.5}
$$

where $x_i = (x_{i1}, x_{i2})^{\mathrm{T}} \in R^2$, $y_i = (y_{i1}, y_{i2})^{\mathrm{T}} \in R^2$, $M = \operatorname{diag}\{\mu_1, \mu_2\}$, $\mathcal{M} = \{M\colon\ 0 < \mu_i \le 1,\ i = 1, 2\}$, $s_{ij}(t) \in [0, 1]$, $i = 1, 2, 3, 4$, $j = 1, 2$ and

$$
S_i = \begin{pmatrix} 1 & 0 & s_{i1}(t) & 0 \\ 0 & 1 & 0 & s_{i1}(t) \end{pmatrix}, \qquad i = 1, 2, 3, 4.
$$

For system (5.3.5) the elements of the matrix–function (5.3.2) are taken as follows

$$
\begin{aligned}
&v_{ii}(x_i) = x_i^2, \quad v_{2+i,2+i}(y_i) = \mu_i y_i^2, \quad v_{ij} = v_{2+i,2+j} = v_{i,2+j} = 0\\
&\qquad v_{i,2+i}(x_i, y_i) = 0.1\, \mu_i x_i y_i, \quad i, j = 1, 2, \quad i \neq j.
\end{aligned}
$$

Let $\eta^{\mathrm{T}} = (1,\ 1,\ 1,\ 1)$. Then the matrix

$$
A(M) = \begin{pmatrix} A_{11} & A_{12}(M) \\ A_{12}(M) & A_{22}(M) \end{pmatrix},
$$

where

$$A_{11} = \mathrm{diag}\,(1,\,1), \quad A_{22}(M) = \mathrm{diag}\,(\mu_1, \mu_2),$$
$$A_{12}(M) = \mathrm{diag}\,(-0.1\,\mu_1,\, -0.1\,\mu_2),$$

is positive definite for any $\mu_i \in (0, 1]$ and for $\mu_i \to 0,\ i = 1, 2.$

The elements of the matrix $\overline{G}(M)$ are of the form

$$\overline{\sigma}_{ii}(M) = -2 + 0.26\,\mu_i, \quad i = 1, 2;$$
$$\overline{\sigma}_{2+i,2+i}(M) = -1.8 + 0.06\,\mu_i, \quad i = 1, 2;$$
$$\overline{\sigma}_{i,2+i}(M) = 0, \quad i = 1, 2; \quad \overline{\sigma}_{ij}(M) = \sigma_{2+i,2+j}(M) = 0.01\mu_i,$$
$$\overline{\sigma}_{i,2+j}(M) = 0.2(1 + \mu_i), \quad i, j = 1, 2, \quad i \neq j.$$

Moreover, the matrix $\overline{G}(M)$ is negative definite for any $\mu_i \in (0,\,1]$ and for $\mu_i \to 0,\ i = 1, 2.$ Therefore, by Theorem 5.3.1 the equilibrium state $(x^{\mathrm{T}}, y^{\mathrm{T}})^{\mathrm{T}} = 0 \in R^8$ of system (5.3.5) is uniformly asymptotically stable in the whole on $\mathcal{M} \times \mathcal{S},$, where $x = (x_1^{\mathrm{T}}, x_2^{\mathrm{T}})^{\mathrm{T}} \in R^4,\ y = (y_1^{\mathrm{T}}, y_2^{\mathrm{T}})^{\mathrm{T}} \in R^4,$

$$\mathcal{S} = \Big\{ S\colon \quad S = \mathrm{diag}\,(S_1, S_2, S_3, S_4),$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \le S_i \le \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad i = 1, 2, 3, 4 \Big\}.$$

5.3.2 Uniform time scaling In the case of uniform time scaling the system (5.2.5) is represented as

$$(5.3.6)\qquad \begin{aligned} \frac{dx_i}{dt} &= f_i(t,x,0,P_i,S_i) + f_i^*, \quad i = 1,2,\dots,q,\\ \mu_1\frac{dy_i}{dt} &= \tau_i g_i(\alpha,b,y,0,P_{q+i},S_{q+i}) + \tau_i g_i^*, \quad i = 1,2,\dots,r, \end{aligned}$$

where

$$\begin{aligned} f_i^* &= f_i(t,x,y,P_i,S_i) - f_i(t,x,0,P_i,S_i),\\ g_i^* &= g_i(t,x,y,M,P_{q+i},S_{q+i}) - g_i(\alpha,b,y,0,P_{q+i},S_{q+i}), \end{aligned}$$

and

$$\tau_i \in [\underline{\tau}_i, \overline{\tau}_i], \quad 0 < \underline{\tau}_i < \overline{\tau}_i < +\infty, \quad \underline{\tau}_1 = \tau_1 = \overline{\tau}_1 = 1.$$

To study system (5.3.6) we make some assumptions.

Assumption 5.3.3 There exist

(1) open connected neighborhoods $\mathcal{N}_{ix} \subseteq R^{n_i}$ and $\mathcal{N}_{jy} \subseteq R^{m_j}$ of the states $x_i = 0$ and $y_j = 0$ respectively;

(2) functions $\varphi_{ik}\colon \mathcal{N}_{ix} \to R_+$, $\psi_{jk}\colon \mathcal{N}_{jy} \to R_+$, $i = 1,2,\dots,q$, $j = 1,2,\dots,r$, $q+r = s$, $k = 1, 2$, φ_{ik}, ψ_{jk} are of class K (KR);

(3) constants $\underline{\alpha}_{ip}$, $\overline{\alpha}_{ip}$, $\underline{\alpha}_{q+j,q+l}$, $\overline{\alpha}_{q+j,q+l}$, $\underline{\alpha}_{i,q+j}$, $\overline{\alpha}_{i,q+j}$, $i,p = 1,2,\dots,q$, $j,l = 1,2,\dots,r$, $q+r = s$, and matrix–function

$$(5.3.7)\qquad U(t,x,y,\mu_1) = \begin{pmatrix} U_{11}(t,x) & \mu_1 U_{12}(t,x,y)\\ \mu_1 U_{12}^{\mathrm{T}}(t,x,y) & \mu_1 U_{22}(t,y) \end{pmatrix},$$

where

$$\begin{aligned} U_{11(t,x)} &= [v_{ip}(t,x_i,x_p)], \quad v_{ip} = v_{pi}, \quad i,\, p = 1,2,\dots,q;\\ U_{22}(t,y) &= [v_{q+i,q+l}(t,y_j,y_l)], \quad v_{q+j,q+l} = v_{q+l,q+j}, \quad j,\, l = 1,2,\dots,r;\\ U_{12}(t,x,y) &= [v_{i,q+j}(t,x_i,y_j)], \quad i = 1,2,\dots,q, \quad j = 1,2,\dots,r, \end{aligned}$$

whose elements satisfy the estimates

(a) $\underline{\alpha}_{ip}\varphi_{i1}(x_i)\varphi_{p1}(x_p) \le v_{ip}(t,x_i,x_p) \le \overline{\alpha}_{ip}\varphi_{i2}(x_i)\varphi_{p2}(x_p)$, $\forall\,(t,x_i,x_p) \in R\times\mathcal{N}_{ix}\times\mathcal{N}_{px}$, $i,\, p = 1,2,\dots,q$, $i \le p$;

(b) $\underline{\alpha}_{q+j,q+l}\psi_{j1}(y_j)\psi_{l1}(y_l) \le v_{q+i,q+l}(t,y_j,y_l) \le \overline{\alpha}_{q+j,q+l}\psi_{j2}(y_j)$ $\times\psi_{l2}(y_l)$, $\forall\,(t,y_j,y_l) \in R\times\mathcal{N}_{jy}\times\mathcal{N}_{ly}$, $(j \le l) \in [1,\, r]$;

(c) $\underline{\alpha}_{i,q+j}\varphi_{i1}(x_i)\psi_{j1}(y_j) \le v_{i,q+j}(t,x_i,y_j) \le \overline{\alpha}_{i,q+j}\varphi_{i2}(x_i)\psi_{j2}(y_j)$ $\forall\,(t,x_i,y_j) \in R\times\mathcal{N}_{ix}\times\mathcal{N}_{jy}$, $i = 1,2,\dots,q$, $j = 1,2,\dots,r$, $q+r = s$.

Matrix–function (5.3.7) and constant vector $\eta \in R^s_+$ allow us to construct an auxiliary function

$$(5.3.8)\qquad v(t,x,y,\mu_1) = \eta^{\mathrm{T}} U(t,x,y,\mu_1)\eta.$$

Alongside function (5.3.8) we consider the expression of the upper right Dini derivative

$$D^+v(t,x,y,\mu_1) = \eta^{\mathrm{T}} D^+U(t,x,y,\mu_1)\eta, \tag{5.3.9}$$

where

$$D^+U(t,x,y,\mu_1) \overset{\mathrm{def}}{=} \begin{pmatrix} D^+U_{11}(t,x) & \mu_1 D^+U_{12}(t,x,y) \\ \mu_1 D^+U_{12}^{\mathrm{T}}(t,x,y) & \mu_1 D^+U_{22}(t,y) \end{pmatrix},$$

$$D^+U_{11} = [D^+v_{ip}(t,\cdot)], \quad D^+U_{12} = [D^+v_{ij}(t,\cdot)],$$

$$D^+U_{22} = [D^+v_{jl}(t,\cdot)], \quad i,p = 1,2,\dots,q; \quad j,l = 1,2,\dots,r; \quad q+r = s.$$

Proposition 5.3.3 *Under conditions of Assumption 5.3.3 the function (5.3.8) satisfies the bilateral estimate*

$$u_1^{\mathrm{T}} A(\mu_1) u_1 \le v(t,x,y,\mu_1) \le u_2^{\mathrm{T}} B(\mu_1) u_2,$$
$$\forall\, (t,x,y,\mu_1) \in R \times \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{M},$$

where

$$u_1^{\mathrm{T}} = (\varphi_{11}(x_1),\, \dots,\, \varphi_{q1}(x_q),\, \psi_{11}(y_1),\, \dots,\, \psi_{r1}(y_r)),$$
$$u_2^{\mathrm{T}} = (\varphi_{12}(x_1),\, \dots,\, \varphi_{q2}(x_q),\, \psi_{12}(y_1),\, \dots,\, \psi_{r2}(y_r)),$$
$$A(\mu_1) = H^{\mathrm{T}} A_1(\mu_1) H, \quad B(\mu_1) = H^{\mathrm{T}} A_2(\mu_1) H, \quad H = \mathrm{diag}\,\{\eta_1,\dots,\eta_s\},$$
$$A_1(\mu_1) = \begin{pmatrix} A_{11} & \mu_1 A_{12} \\ \mu_1 A_{12}^{\mathrm{T}} & \mu_1 A_{22} \end{pmatrix}, \qquad A_2(\mu_1) = \begin{pmatrix} \overline{A}_{11} & \mu_1 \overline{A}_{12} \\ \mu_1 \overline{A}_{12}^{\mathrm{T}} & \mu_1 \overline{A}_{22} \end{pmatrix},$$
$$A_{11} = [\underline{\alpha}_{ip}], \quad \underline{\alpha}_{ip} = \underline{\alpha}_{pi}, \quad \overline{A}_{11} = [\overline{\alpha}_{ip}], \quad \overline{\alpha}_{ip} = \overline{\alpha}_{pi},$$
$$A_{22} = [\underline{\alpha}_{q+j,q+l}], \quad \underline{\alpha}_{q+j,q+l} = \underline{\alpha}_{q+l,q+j},$$
$$\overline{A}_{22} = [\overline{\alpha}_{q+j,q+l}], \quad \overline{\alpha}_{q+j,q+l} = \overline{\alpha}_{q+l,q+j},$$
$$A_{12} = [\underline{\alpha}_{i,q+j}], \quad \overline{A}_{12} = [\overline{\alpha}_{i,q+j}],$$
$$i,p = 1,2,\dots,q, \quad j,l = 1,2,\dots,r, \quad q+r = s.$$

The proof of Proposition 5.3.3 is similar to that of Proposition 5.3.1.

Proposition 5.3.4 *If in Proposition 5.3.3 the matrices A_{11} and A_{22} are positive definite, then the function (5.3.8) is positive definite for any $\mu_1 \in (0,\, \mu_1^*)$ and for $\mu_1 \to 0$, where*

$$\mu_1^* = \min\left\{1,\ \frac{\lambda_m(A_{11}^*)\lambda_m(A_{22}^*)}{\lambda_M(A_{12}^* A_{12}^{*\mathrm{T}})}\right\},$$
$$A_{11}^* = H_1^{\mathrm{T}} A_{11} H_1, \quad A_{22}^* = H_2^{\mathrm{T}} A_{22} H_2, \quad A_{12}^* = H_1 A_{12} H_2,$$
$$H_1 = \mathrm{diag}\,\{\eta_1,\eta_2,\dots,\eta_q\}, \quad H_2 = \mathrm{diag}\,\{\eta_{q+1},\eta_{q+2},\dots,\eta_s\}.$$

Proposition 5.3.4 is proved by the immediate testing.

Assumption 5.3.4 There exist

(1) open connected neighborhoods $\mathcal{N}_{ix} \subseteq R^{n_i}$ and $\mathcal{N}_{jy} \subseteq R^{m_j}$ of the states $x+i=0$ and $y_j=0$ respectively;

(2) functions φ_i, ψ_j of class $K\,(KR)$, $i=1,2,\dots,q$, $j=1,2,\dots,r$;

(3) functions $v_{ip}=v_{pi}$, $v_{q+j,q+l}=v_{q+l,q+j}$, $v_{i,q+j}$, $i,p=1,2,\dots,r$, $j,l=1,2,\dots,r$, which satisfy the conditions of Assumption 5.3.3, and

(a) $v_{ip}(t,x_i,x_p) \in C$ on $(R \times \mathcal{N}_{ix0} \times \mathcal{N}_{px0})$ or on $(R \times R^{n_i} \times R^{n_p})$;

(b) $v_{q+i,q+l}(t,y_j,y_l) \in C$ on $(R \times \mathcal{N}_{jy0} \times \mathcal{N}_{ly0})$ or on $(R \times R^{m_j} \times R^{m_l})$;

(c) $v_{i,q+j}(t,x_i,y_j) \in C$ on $(R \times \mathcal{N}_{ix0} \times \mathcal{N}_{jy0})$ or on $(R \times R^{n_i} \times R^{m_j})$;

(4) real numbers $\rho_{\alpha i}(P,S)$, $\rho_{\alpha ip}(P,S)$, $\rho_{\alpha,q+j}(P,S)$, $\rho_{\alpha,q+j,q+l}(P,S)$, $\rho_{\beta,,i,q+j}(P,S)$, $\alpha=1,2,3$, $\beta=1,2$, $i,p=1,2,\dots,q$, $j,l=1,2,\dots,r$, $q+r=s$ and

(a) $$\eta_i^2 D_t^+ v_{ii} + \eta_i^2 (D_{x_i}^+ v_{ii})^{\mathrm{T}} f_i(t,x,0,P_i,S_i) \le \rho_{1i}(P,S)\varphi_i^2(x_i) + \sum_{\substack{p=1\\p\neq i}}^{q} \rho_{1ip}(P,S)\varphi_i(x_i)\varphi_p(x_p),$$
$$\forall\,(t,x_i,P,S) \in R \times \mathcal{N}_{ix0} \times \mathcal{P} \times \mathcal{S},\quad i=1,2,\dots,q;$$

(b) $\eta_{q+j}^2 \mu_1 D_t^+ v_{q+j,q+j}$

$$+ \eta_{q+j}^2 \tau_j (D_{y_j}^+ v_{q+j,q+j})^{\mathrm{T}} g_j(\alpha, \underset{\sim}{\ } y, 0, P_{q+j}, S_{q+j})$$

$$\leq \rho_{1,q+j}(P,S)\psi_j^2(y_j) + \sum_{\substack{l=1\\ l\neq j}}^{r} \rho_{1,q+j,q+l}(P,S)\psi_j(y_j)\psi_l(y_l),$$

$$\forall\, (t, y_j, \mu_j, P, S) \in R \times \mathcal{N}_{jy0} \times \mathcal{M} \times \mathcal{P} \times \mathcal{S}, \;\; j = 1, 2, \dots, r;$$

(c) $$\sum_{i=1}^{q} \eta_i^2 (D_{x_i}^+ v_{ii})^{\mathrm{T}} f_i^* + \sum_{j=1}^{r} \eta_{q+j}^2 \tau_j (D_{y_j}^+ v_{q+j,q+j})^{\mathrm{T}} g_j^*$$

$$+ 2\sum_{i=1}^{q}\sum_{\substack{p=2\\ p>i}}^{q} \eta_i \eta_p \Big\{ D_t^+ v_{ip} + (D_{x_i}^+ v_{ip})^{\mathrm{T}} f_i(t,x,y,P_i,S_i)$$

$$+ (D_{x_p}^+ v_{ip})^{\mathrm{T}} f_p(t,x,y,P_p,S_p) \Big\}$$

$$+ 2\sum_{j=1}^{r}\sum_{\substack{l=2\\ l>j}}^{r} \eta_{q+j}\eta_{q+l} \Big\{ \mu_1 D_t^+ v_{q+j,q+l}$$

$$+ \tau_j (D_{y_j}^+ v_{q+j,q+l})^{\mathrm{T}} g_j(t,x,y,M,P_{q+j},S_{q+j})$$

$$+ \tau_l (D_{y_l}^+ v_{q+j,q+l})^{\mathrm{T}} g_l(t,x,y,M,P_{q+l},S_{q+l}) \Big\}$$

$$+ 2\sum_{i=1}^{q}\sum_{j=1}^{r} \eta_i \eta_{q+j} \Big\{ \mu_i D_t^+ v_{v,q+j} + \mu_1 (dxv_{i,q+j})^{\mathrm{T}} f_i(t,x,y,P_i,S_i)$$

$$+ \tau_j (D_{y_j}^+ v_{i,q+j})^{\mathrm{T}} g_j(t,x,y,M,P_{q+j},S_{q+j}) \Big\}$$

$$\leq \sum_{i=1}^{q} (\rho_{2i}(P,S) + \mu_1 \rho_{3i}(P,S)) \varphi_i^2(x_i)$$

$$+ \sum_{j=1}^{r} (\rho_{2,q+i}(P,S) + \mu_1 \rho_{3,q+j}(P,S)) \psi_j^2(y_j)$$

$$+ 2\sum_{i=1}^{q}\sum_{\substack{p=2\\ p>i}}^{q} (\rho_{2ip}(P,S) + \mu_1 \rho_{3ip}(P,S)) \varphi_i(x_i)\varphi_p(x_p)$$

$$+ 2\sum_{j=1}^{r}\sum_{\substack{l=2\\ l>j}}^{r} (\rho_{2,q+j,q+l}(P,S) + \mu_1 \rho_{3,q+j,q+l}(P,S)) \psi_j(y_j)\psi_l(y_l)$$

$$+ \sum_{i=1}^{q}\sum_{j=1}^{r} (\rho_{1,i,q+j}(P,S) + \mu_1 \rho_{2,i,q+j}(P,S)) \varphi_i(x_i)\psi_j(y_j),$$

$$\forall\, (t, x_i, y_j, M, P, S) \in R \times \mathcal{N}_{ix_0} \times \mathcal{N}_{jy_0} \times \mathcal{M} \times \mathcal{P} \times \mathcal{S}.$$

Proposition 5.3.5 *Under all conditions of Assumption 5.3.4 for the expression (5.3.9) the estimate*

$$D^+ v(t,x,y,\mu_1) \leq u^{\mathrm{T}} \overline{C} u + \mu_1 u^{\mathrm{T}} \overline{G} u,$$
$$\forall\, (t,x,y,\mu_1,P,S) \in R \times \mathcal{N}_{x_0} \times \mathcal{N}_{y_0} \times \mathcal{M} \times \mathcal{P} \times \mathcal{S}, \quad \forall\, \tau_j \in [\underline{\tau}_j, \overline{\tau}_j],$$

holds, where

$$u^{\mathrm{T}} = (\varphi_1(x_1), \dots, \varphi_q(x_q), \psi_1(y_1), \dots, \psi_r(y_r)),$$
$$\overline{C}[\overline{c}_{ij}], \quad \overline{c}_{ij} = \overline{c}_{ji}, \quad \overline{G} = [\sigma_{ij}], \quad \overline{\sigma}_{ij}\overline{\sigma}_{ji}, \quad i,j \in [1,\, s],$$
$$\overline{c}_{ip} = \rho_{1ip}(\overline{P},\overline{S}) + \rho_{2ip}(\overline{P},\overline{S}), \quad \overline{\sigma}_{ip} = \rho_{3ip}(\overline{P},\overline{S}), \quad i,p \in [1,\, q], \quad p > i,$$
$$\overline{c}_{q+j,q+j} = \rho_{1,q+j}(\overline{P},\overline{S}) + \rho_{2,q+j}(\overline{P},\overline{S}),$$

$$
\begin{gathered}
\overline{\sigma}_{q+j,q+j} = \rho_{3,q+j}(\overline{P},\overline{S}), \quad j=1,2,\dots,r,\\
\overline{c}_{q+j,q+l} = \rho_{1,q+j,q+l}(\overline{P},\overline{S}) + \rho_{2,q+j,q+l}(\overline{P},\overline{S}),\\
\overline{\sigma}_{q+j,q+l} = \rho_{3,q+j,q+l}(\overline{P},\overline{S}), \quad j,l=1,2,\dots,r, \quad j>l,\\
\overline{c}_{i,q+j} = \rho_{1,i,q+j}(\overline{P},\overline{S}), \quad \overline{\sigma}_{i,q+j} = \rho_{2,i,q+j}(\overline{P},\overline{S}),\\
i=1,2,\dots,q, \quad j=\underset{\cdot}{0}r, \quad q+r=s.
\end{gathered}
$$

Here $\overline{P}$, $\overline{S}\in\mathcal{S}$ are constant matrices such that

$$
\begin{gathered}
\rho_{\alpha i}(P,S)\le\rho_{\alpha i}(\overline{P},\overline{S}), \quad \rho_{\alpha ip}(P,S)\le\rho_{\alpha ip}(\overline{P},\overline{S}),\\
\rho_{\alpha,q+j}(P,S)\le\rho_{\alpha,q+j}(\overline{P},\overline{S}), \quad \rho_{\alpha,q+j,q+l}(P,S)\le\rho_{\alpha,q_j,q+l}(\overline{P},\overline{S}),\\
\rho_{\beta,i,q+i}(P,S)\le\rho_{\beta,i,q+i}(\overline{P},\overline{S}), \quad \alpha=1,2,3, \quad \beta=1,2,\\
i,\,p=1,2,\dots,q, \quad j,\,l=1,2,\dots,r, \quad q+r=s.
\end{gathered}
$$

Proof of Proposition 5.3.5 is similar to that of Proposition 5.3.2.

Proposition 5.3.6 *If in Proposition 5.3.5 the matrix $\overline{C}$ is negative-definite and $\lambda_M(\overline{G})>0$, then the expression $D^+v(t,x,y,\mu_1)$ defined by (5.3.9) is negative-definite for any $\mu_1\in(0,\,\mu_1^{**})$ and for $\mu_1\to 0$ where*

$$
\mu_1^{**}=\min\left\{1,\,-\frac{\lambda_M(\overline{C})}{\lambda_M(\overline{G})}\right\}.
$$

The proof of Proposition 5.3.6 follows from the analysis of the inequality

$$
D^+v(t,x,y,\mu_1)\le u^{\mathrm{T}}\overline{C}u+\mu_1u^{\mathrm{T}}\overline{G}u\le(\lambda_M(\overline{C})+\mu_1\lambda_M(\overline{G}))\|u\|^2.
$$

Remark 5.3.2 If in Proposition 5.3.6 $\lambda_M(\overline{G})\le 0$, then expression (5.3.9) is negative definite for any $\mu_1\in(0,\,1]$ and for $\mu_1\to 0$.

Theorem 5.3.2 *Let the perturbed motion equations (5.3.6) be such that all conditions of Assumptions 5.3.3 and 5.3.4 are satisfied and*

(1) *matrices A_{11} and A_{22} are positive definite;*
(2) *matrix $\overline{C}$ is negative definite;*
(3) $\mu_1\in(0,\widetilde{\mu}_1)$, $\mu_i=\mu_1\tau_i^{-1}$, $\tau_i\in[\underline{\tau}_i,\overline{\tau}_i]$, $i\in[1,r]$ *where* $\widetilde{\mu}_1=\min\{\mu_1^*,\mu_1^{**}\}$.

Then the equilibrium state $(x^{\mathrm{T}},y^{\mathrm{T}})^{\mathrm{T}}=0$ of system (5.3.6) is uniformly asymptotically stable on $\widetilde{\mathcal{M}}\times\mathcal{P}\times\mathcal{S}$.

If all conditions of the theorem are satisfied for $\mathcal{N}_{ix}\times\mathcal{N}_{jy}=R^{n_i+m_j}$ and functions φ_i,ψ_j are of class KR, then the equilibrium state $(x^{\mathrm{T}},y^{\mathrm{T}})^{\mathrm{T}}=0$ of system (5.3.6) is uniformly asymptotically stable in the whole on $\widetilde{\mathcal{M}}\times\mathcal{P}\times\mathcal{S}$, where $\widetilde{\mathcal{M}}=\{M\colon 0<\mu_1<\widetilde{\mu}_1,\ \mu_i=\mu_1\tau_i^{-1},\ \tau_i\in[\underline{\tau}_i,\overline{\tau}_i],\ i=1,2,\dots,r\}$.

Proof Under conditions of Assumption 5.3.3, Proposition 5.3.3 and conditions (1) and (3) of Theorem 5.3.2 the function $v(t,x,y,\mu_1)$ is positive definite on $\widetilde{\mathcal{M}}$ and decreasing on $\mathcal{N}_x\times\mathcal{N}_y$. Conditions of Assumption 5.3.4, Proposition 5.3.5 and conditions (2) and (3) of Theorem 5.3.2 imply that the expression $D^+v(t,x,y,\mu_1)$ is negative definite on $\widetilde{\mathcal{M}}\times\mathcal{P}\times\mathcal{S}$.

In the case when $\mathcal{N}_{ix} \times \mathcal{N}_{jy} = R^{n_i+m_j}$ the function $v(t,x,y,\mu_1)$ is positive definite, decreasing and radially unbounded. This fact together with the other conditions of Theorem 5.3.2 prove its second assertion.

Example 5.3.2 Consider nonstationary 4-th order system consisting of two interconnected 2-nd order subsystems

$$\begin{aligned}
\frac{dx_i}{dt} &= \frac{1}{1+\cos^2 t}\left\{-\frac{1-\sin 2t}{2}x_i + 0.02\,S_{i1}y_i + 0.03\,S_{i2}y_j\right\},\\
\mu_i\frac{dy_i}{dt} &= \frac{1}{1+\cos^2 t}\left\{-\frac{4-\mu_j\sin 2t}{2}y_i\right.\\
&\quad \left. + 0.01\mu_i(S_{q+i,1}x_i + S_{q+i,2}x_j)\right\},\\
&\quad i,j = 1,2; \quad i \neq j,
\end{aligned} \tag{5.3.10}$$

where $t, x_i, y_i \in R$, $\mathcal{M} = \{M\colon\ 0 < \mu_i < 1,\ i = 1,2\}$, $M = \operatorname{diag}\{\mu_1,\mu_2\}$, $\underline{\tau}_2 = \frac{1}{2}$, $\overline{\tau}_2 = 1$, so that $\tau_2 \in [\frac{1}{2}, 1]$, $S_{ij} = S_{ij}(t) \in [0,1]$, $i,j = 1,2$.

The elements of the matrix–function (5.3.10) are taken as follows

$$\begin{aligned}
v_{ii}(t,x_i) &= (1+\cos^2 t)x_i^2, \quad i = 1,2,\\
v_{2+i,2+i}(t,y_i) &= (1+\cos^2 t)y_i^2, \quad i = 1,2,\\
v_{ip}(t,x_i,x_p)v_{2+j,2+l}(t,y_j,y_l) &= 0, \quad i,j,p,l = 1,2,\\
v_{i,2+j}(t,x_i,y_j) &= 0.1(1+\cos^2 t)x_iy_j, \quad i,j = 1,2.
\end{aligned}$$

Let $\eta^{\mathrm{T}} = (1,1,1,1),$. Then the matrices $A_{11} = A_{22} = \operatorname{diag}\{1,1\}$ are positive definite and the matrix

$$A(\mu_1) = \begin{pmatrix} A_{11} & \mu_1 A_{12} \\ \mu_1 A_{12}^{\mathrm{T}} & \mu_1 A_{22} \end{pmatrix}, \quad \text{where} \quad A_{12} = \begin{pmatrix} -0.2 & -0.2 \\ -0.2 & -0.2 \end{pmatrix},$$

is also positive definite for any $\mu_1 \in (0,1]$ and for $\mu_1 \to 0,$, since $\mu_1^* = \min\{1, 2.5\} = 1$.

For such choice of the elements of matrix–function (5.3.7) we have

$$\begin{gathered}
\rho_{1i} = -1, \quad i = 1,2; \quad \rho_{13} = -4; \quad \rho_{14} = -1; \quad \rho_{2j} = 0, \quad j = 1,2,3,4; \\
\rho_{31}(S) = 0.01(S_{31} + S_{42}); \quad \rho_{32}(S) = 0.01(S_{32} + S_{42}); \\
\rho_{33}(S) = 0.002S_{11} + 0.003S_{22}; \quad \rho_{34}(S) = 0.003S_{12} + 0.002S_{21}; \\
\rho_{212}(S) = 0; \quad \rho_{312}(S) = 0.01(S_{31} + S_{32} + S_{41} + S_{42}); \\
\rho_{234}(S) = 0; \quad \rho_{334}(S) = 0.002(S_{11} + S_{21}) + 0.003(S_{12} + S_{22}); \\
\rho_{113}(S) = 0.2 + 0.04S_{11}; \quad \rho_{213}(S) = 0.05 + 0.2S_{31}; \\
\rho_{114}(S) = 0.1 + 0.06S_{12}; \quad \rho_{214}(S) = 0.05 + 0.1S_{42}; \\
\rho_{123}(S) = 0.2 + 0.06S_{22}; \quad \rho_{223}(S) = 0.05 + 0.1S_{32}; \\
\rho_{124}(S) = 0.1 + 0.04S_{21}; \quad \rho_{224}(S) = 0.05 + 0.1S_{41}.
\end{gathered}$$

The matrices $\overline{C}$ and $\overline{G}$ consist of the elements

$$\begin{gathered}
\overline{c}_{11} = \overline{c}_{22} = \overline{c}_{44} = -1, \quad \overline{c}_{33} = -4, \quad \overline{c}_{12} = 0, \quad \overline{c}_{34} = 0, \\
\overline{c}_{13} = 0.24, \quad \overline{c}_{14} = 0.16, \quad \overline{c}_{23} = 0.26, \quad \overline{c}_{24} = 0.14; \\
\overline{\sigma}_{11} = 0.02, \quad \overline{\sigma}_{22} = 0.02, \quad \overline{\sigma}_{33} = 0.005, \quad \overline{\sigma}_{44} = 0.005, \\
\overline{\sigma}_{12} = 0.04, \quad \overline{\sigma}_{34} = 0.01, \quad \overline{\sigma}_{13} = 0.25, \quad \overline{\sigma}_{14} = 0.15, \\
\overline{\sigma}_{23} = 0.15, \quad \overline{\sigma}_{24} = 0.15.
\end{gathered}$$

Besides, the matrix $\overline{C}$ is negative definite and

$$\mu_1^{**} = \min\{1,\, 2,\, 1,\, \dots\} = 1.$$

So, all conditions of Theorem 5.3.2 are satisfied, $\widetilde{\mu}_1 = \min\{\mu_1^*, \mu_1^{**}\} = 1$, and therefore the equilibrium state of system (5.3.10) is uniformly asymptotically stable in the whole on $\mathcal{M} \times \mathcal{S}$.

5.4 Tests for Instability Analysis

5.4.1 Non-uniform time scaling Instability of solutions is considered in two cases. First, we shall consider the case of nonuniform time scaling. To this end we need the following assumptions and estimates.

Assumption 5.4.1 The inequalities of Assumption 5.3.2 hold true when the inequality sign is reversed, i.e. "$\le$" becomes "$\ge$".

Proposition 5.4.1 *Under conditions of Assumption 5.4.1 for the expression (5.3.9) the estimate*

$$\begin{gathered}
D^+v(t,x,y,M) \ge u^{\mathrm{T}}G(M,P,S)u, \\
\forall\,(t,x,y,M,P,S) \in R \times \mathcal{N}_{x_0} \times \mathcal{N}_{y_0} \times \mathcal{M} \times \mathcal{P} \times \mathcal{S},
\end{gathered}$$

holds true, where u^{T} *and* $G(M,P,S)$ *are defined in the same way as in Proposition 5.3.5.*

The proof is similar to that of Proposition 5.3.5.

Theorem 5.4.1 *Let the perturbed motion equations (5.2.5) be such that all conditions of Assumptions 5.3.1 and 5.4.1 are satisfied and*

(a) *matrices* $A(M)$ *and* $B(M)$ *are positive definite for any* $\mu_i \in (0,\,\mu_i^*)$ *and for* $\mu_i \to 0$, $i = 1,2,\dots,q$, *where* $\mu_i^* = \min\{\overline{\mu}_{i1}, \overline{\mu}_{i2}\}$*;*
(b) *there exists a matrix* $\underline{G}(M)$ *which is positive definite for any* $\mu_i \in (0,\,\overline{\mu}_{i3})$ *and for* $\mu_i \to 0$, $i = 1,2,\dots,q$, *such that for the matrix* $G(M,P,S)$ *defined by Proposition 5.3.7 the extimate*

$$G(M,P,S) \ge \underline{G}(M), \quad \forall\,(M,P,S) \in \mathcal{M}\times\mathcal{P}\times\mathcal{S}.$$

is satisfied.

Then the equilibrium state $(x^{\mathrm{T}}, y^{\mathrm{T}})^{\mathrm{T}} = 0$ *of system (5.2.5) is unstable for any* $\mu_i \in (0,\,\overline{\mu}_i)$ *and for* $\mu_i \to 0$ *on* $\mathcal{P}\times\mathcal{S}$, *where*

$$\overline{\mu}_i = \min\{1,\,\mu_i^*,\,\overline{\mu}_{i3}\}.$$

Proof We construct the scalar function $v(t,x,y,M)$ in the same way as in Section 5.3.1. Under the conditions of Assumption 5.3.1 and by condition (a) of Theorem 5.3.3 the function $v(t,x,y,M)$ is positive definite for any $\mu_i \in (0,\,\mu_i^*)$ and for $\mu_i \to 0$, $i = 1,2,\dots,q$, and admits infinitely small upper limits on $\mathcal{N}_x \times \mathcal{N}_y$. The conditions of Assumption 5.3.5, Proposition 5.3.7 and condition (b) of Theorem 5.3.3 imply that the expression $D^+v(t,x,y,M)$ is a function being positive definite for any $\mu_i \in (0,\,\overline{\mu}_{i3})$ and for $\mu_i \to 0$, for every $(P,S) \in \mathcal{P}\times\mathcal{S}$. These conditions are known to be sufficient for instability of the equilibrium state of system (5.2.5) for any $\mu_i \in (0,\,\overline{\mu}_i)$ and for $\mu_i \to 0$ on $\mathcal{M}\times\mathcal{P}\times\mathcal{S}$.

Remark 5.4.1 By the condition of positive definiteness of the matrices $A(M)$, $B(M)$ and $\underline{G}(M)$ the values $\overline{\mu}_{i1}$, $\overline{\mu}_{i2}$ and $\overline{\mu}_{i3}$ are determined respectively, since $\overline{\mu}_i = \min\{1,\,\overline{\mu}_{i1},\,\overline{\mu}_{i2},\,\overline{\mu}_{i3}\}$ is the lower estimate of the upper boundary of the admissible μ_i, so that $\overline{\mathcal{M}} = \{M\colon\ 0 < \mu_i < \overline{\mu}_i,\ i = 1,2,\dots,q\}$.

5.4.2 Uniform time scaling

Assumption 5.4.2 The inequalities of Assumption 5.3.4 hold when the inequality sign is reversed, i.e. "$\le$" becomes "$\ge$".

Proposition 5.4.2 *Under all conditions of Assumption 5.3.6 for the expression (5.3.9) the estimate*

$$D^+v(t,x,y,\mu_1) \ge u^{\mathrm{T}}\overline{C}u + \mu_1 u^{\mathrm{T}}\overline{G}u,$$
$$\forall\,(t,x,y,\mu_1,P,S) \in R\times\mathcal{N}_{x_0}\times\mathcal{N}_{y_0}\times\mathcal{M}\times\mathcal{P}\times\mathcal{S}, \quad \forall\,\tau_i \in [\underline{\tau}_i,\,\overline{\tau}_i],$$

takes place, where u^{T}, $\overline{C}$, *and* $\overline{G}$ *are determined as in Proposition 5.3.5.*

The proof is similar to that of Proposition 5.3.2.

Proposition 5.4.3 *If in Proposition 5.4.2 the matrix* $\overline{C}$ *is positive definite and* $\lambda_m(\overline{G}) < 0,$, *then the expression* $D^+v(t,x,y,\mu_1)$ *is positive definite for any* $\mu_1 \in (0,\, \mu_1^{**})$ *and for* $\mu_1 \to 0$, *where*

$$\mu_1^{**} = \min\{1,\, -\lambda_m(\overline{C})\lambda_m^{-1}(G)\}.$$

The proof follows from the analysis of the inequality

$$D^+v(t,x,y,\mu_1) \geq u^{\mathrm{T}}\overline{C}u + \mu_1 u^{\mathrm{T}}\overline{G}u \geq (\lambda_m(\overline{C}) + \mu_1\lambda_m(\overline{G}))\|u\|^2.$$

Remark 5.4.1 If in Proposition 5.3.9 $\lambda_m(\overline{G}) \geq 0$, then the expression $D^+v(t,x,y,\mu_1)$ is positive definite for any $\mu_1 \in (0,\,1]$ and for $\mu_1 \to 0$.

Theorem 5.4.2 *Let the perturbed motion equations (5.2.5) be such that all conditions of Assumptions 5.4.2 and 5.4.3 are satisfied and*

(1) *matrices* A_{11}, A_{22}, $\bar{A}_{11}$, $\bar{A}_{22}$ *and* $\overline{C}$ *are positive definite;*

(2) $\mu_1 \in (0,\, \overline{\mu}_1)$, $\mu_i = \mu_1\tau_i^{-1}$, $\tau_i \in [\underline{\tau}_i,\, \overline{\tau}_i]$, $i \in [1,\, r]$, *where*

$$\begin{gathered}\overline{\mu}_1 = \min\{\mu_1^*,\, \mu_1^{**},\, \lambda_M(\bar{A}_{11}^*)\,\lambda_M(\bar{A}_{22}^*)\,\lambda_M^{-1}(\bar{A}_{12}^*\bar{A}_{12}^{*\mathrm{T}})\},\\ \bar{A}_{11}^* = H_1^{\mathrm{T}}\bar{A}_{11}H, \quad \bar{A}_{22}^* = H_2^{\mathrm{T}}\bar{A}_{22}H_2, \quad \bar{A}_{12}^* = H_1\bar{A}_{12}H_2,\\ H_1 = \mathrm{diag}\{\eta_1, \eta_2, \dots, \eta_q\}, \quad H_2 = \mathrm{diag}\{\eta_{q+1},\, \eta_{q+2},\, \dots,\, \eta_s\}.\end{gathered}$$

Then the equilibrium state $(x^{\mathrm{T}}, y^{\mathrm{T}})^{\mathrm{T}} = 0$ *of system (5.2.5) is unstable on* $\overline{\mathcal{M}} \times \mathcal{P} \times \mathcal{S}$, *where* $\overline{\mathcal{M}} = \{M\colon\; 0 < \mu_1 < \overline{\mu}_1,\; \mu_i = \mu_1\tau_i^{-1},\; i = 1, 2, \dots, q\}$.

The proof is similar to that of Theorem 5.2.2

5.5 Linear Systems

5.5.1 Non-uniform time scaling Consider the linear singularly perturbed system

(5.5.1)
$$\frac{dx_i}{dt} = A_i x_i + \sum_{l=1}^{q} (S_{il}^1 A_{il} x_l + S_{il}^2 A'_{il} y_l), \quad i = 1, 2, \dots, q,$$
$$\mu_i \frac{dy_i}{dt} = B_i y_i + \sum_{l=1}^{q} (\mu_i S_{q+i,l}^1 B_{il} x_l + S_{q+i,l}^2 B'_{il} y_l), \quad i = 1, 2, \dots, q,$$

where A_i, S_i, A_{il}, A'_{il}, B_{il} and B'_{il} are constant matrices, all matrices and vectors are of the corresponding order, and S_{il}^1, S_{il}^2, $S_{q+i,l}^1$ and $S_{q+i,l}^2$ are diagonal matrices, $\mu_i \in (0,\, 1]$, $\forall\, i = 1, 2, \dots, q$. Let

$$S_i = \begin{pmatrix} S_{i1}^1 & S_{i2}^1 & \dots & S_{i,i-1}^1 & 0 & S_{i,i+1}^1 & \dots & S_{iq}^1 \\ S_{i1}^2 & S_{i2}^2 & \dots & S_{i,i-1}^2 & J & S_{i,i+1}^2 & \dots & S_{iq}^2 \\ S_{q+i,1}^1 & S_{q+i,2}^1 & \dots & S_{q+i,i-1}^1 & J & S_{q+i,i+1}^1 & \dots & S_{q+i,q}^1 \\ S_{q+i,1}^2 & S_{q+i,2}^2 & \dots & S_{q+i,i-1}^2 & J & S_{q+i,i+1}^2 & \dots & S_{q+i,q}^2 \end{pmatrix},$$
$$i = 1, 2, \dots, q, \quad S = \operatorname{diag}\{S_1, S_2, \dots, S_q\}.$$

The structural set is defined as

$$\mathcal{S} = \{S\colon\ 0 \le S_{jl}^k \le J,\ \ S_{ii}^1 = S_{q+i,i}^2 = 0,\ \ S_{ii}^2 = S_{q+i,i}^1 = J,$$
$$i, l = 1, 2, \dots, q,\ \ j = 1, 2, \dots, 2q,\ \ k = 1, 2\},$$

where J is an identity matrix of the corresponding dimensions.

The independent singularly perturbed subsystems corresponding to system (5.4.1) are obtained by substitution by x^i and y^i for x and y

(5.5.2)
$$\frac{dx_i}{dt} = A_i x_i + A'_{ii} y_i, \quad \forall\, i = 1, 2, \dots, q,$$
$$\mu_i \frac{dy_i}{dt} = B_i y_i + \mu_i B_{ii} x_i, \quad \forall\, i = 1, 2, \dots, q.$$

Construct matrix $U(t, x)$ for system (5.5.1) with elements

(5.5.3)
$$v_{ij}(x_i, x_j) = v_{ji}(x_i, x_j) = x_i^{\mathrm{T}} P_{ij} x_j, \quad i, j = 1, 2, \dots, q;$$
$$v_{i,q+j}(x_i, y_j) = x_i^{\mathrm{T}} P_{i,q+j} y_j, \quad i, j = 1, 2, \dots, q, \quad 2q = s;$$
$$v_{q+i,q+j}(y_i, y_j) = v_{q+j,q+i}(y_i, y_j) = y_i^{\mathrm{T}} P_{q+i,q+j} y_j, \quad i, j = 1, 2, \dots, q,$$

where P_{ii}, $P_{q+i,q+j}$ $(i \ne j)$, $P_{i,q+j}$ are constant matrices.

For functions (5.5.3) the following estimates are satisfied

(a) $\lambda_m(P_{ii})\|x_i\|^2 \le v_{ii}(x_i) \le \lambda_M(P_{ii})\|x_i\|^2, \quad \forall\, x_i \in \mathcal{N}_{ix_0}, \quad i \in [1,\, q];$

(b) $\lambda_m(P_{q+i,q+i})\|y_i\|^2 \le v_{q+i.q+i}(y_i) \le \lambda_M(P_{q+i,q+i})\|y_i\|^2,$
$\forall\, y_i \in \mathcal{N}_{iy_0}, \quad \forall\, i = 1, 2, \dots, q;$

(c) $-\,l_M^{1/2}(P_{ij}P_{ij}^{\mathrm{T}})\|x_i\|\,\|x_j\| \le v_{ij}(x_i, x_j) \le \lambda_M^{1/2}(P_{ij}P_{ij}^{\mathrm{T}})\|x_i\|\,\|x_j\|,$

(5.4.4)

$\forall\,(x_i,x_j)\in\mathcal{N}_{ix_0}\times\mathcal{N}_{jx_0},\quad \forall\,i,j=1,2,\dots,q,\quad i\neq j;$

(d) $-\lambda_M^{1/2}(P_{q+i,q+j})P^{\mathrm{T}}_{q+i,q+j}\|y_i\|\,\|y_j\|\le v_{q+i,q+j}(y_i,y_j)\le$ $\lambda_M^{1/2}(P_{q+i,q+j})P^{\mathrm{T}}_{q+i,q+j}\|y_i\|\,\|y_j\|,\quad \forall\,(y_i,y_j)\in\mathcal{N}_{iy_0}\times\mathcal{N}_{jy_0},$ $\forall\,i,j=1,2,\dots,q,\quad i\neq j;$

(e) $-\lambda_M^{1/2}(P_{i,q+j}P^{\mathrm{T}}_{i,q+j})\|x_i\|\,\|y_j\|\le v_{i,q+j}(x_i,y_j)\le$ $\lambda_M^{1/2}(P_{i,q+j}P^{\mathrm{T}}_{i,q+j})\|x_i\|\,\|y_j\|,\quad \forall\,(x_i,y_j)\in\mathcal{N}_{ix_0}\times\mathcal{N}_{jy_0},$ $i,j=1,2,\dots,q,$

where $\lambda_m(P_{ii})$ and $\lambda_m(P_{q+i,q+i})$ are minimal eigenvalues, $\lambda_M(P_{ii})$ and $\lambda_M(P_{q+i,q+i})$ are maximal eigenvalues of matrices P_{ii} and $P_{q+i,q+i}$ respectively; $\lambda_M^{1/2}(P_{ij}P^{\mathrm{T}}_{ij})$, $\lambda_M^{1/2}(P_{q+i,q+j}P^{\mathrm{T}}_{q+i,q+j})$ and $\lambda_M^{1/2}(P_{i,q+j}P^{\mathrm{T}}_{i,q+j})$ are norms of matrices P_{ij}, $P_{q+i,q+j}$ and $P_{i,q+j}$ respectively.

When estimates (5.5.4) are satisfied for function (5.3.3) with elements (5.5.3) the bilateral inequality

$$u^{\mathrm{T}}A(M)u\le v(x,y,M)\le u^{\mathrm{T}}B(M)u.$$

takes place.

Here matrices $A(M)$ and $B(M)$ are defined as in Proposition 5.3.1,

$$\begin{gathered}u^{\mathrm{T}}=(\|x_1\|,\|x_2\|,\dots,\|x_q\|,\|y_1\|,\|y_2\|,\dots,\|y_q\|),\\ \underline{\alpha}_{ii}=\lambda_m(P_{ii}),\quad \underline{\alpha}_{q+i,q+i}=\lambda_m(P_{q+i,q+i}),\quad \underline{\alpha}_{ij}=-\lambda_M^{1/2}(P_{ij}P^{\mathrm{T}}_{ij}),\\ \underline{\alpha}_{q+i,q+j}=-\lambda_M^{1/2}(P_{q+i,q+j}P^{\mathrm{T}}_{q+i,q+j}),\quad \underline{\alpha}_{i,q+j}=-\lambda_M^{1/2}(P_{i,q+j}P^{\mathrm{T}}_{i,q+j}),\\ \overline{\alpha}_{ii}=\lambda_M(P_{ii}),\quad \overline{\alpha}_{q+i,q+i}=\lambda_M(P_{q+i,q+i}),\quad \overline{\alpha}_{ij}=-\underline{\alpha}_{ij},\\ \overline{\alpha}_{q+i,q+j}=-\underline{\alpha}_{q+i,q+j},\quad \overline{\alpha}_{i,q+j}=-\underline{\alpha}_{i,q+j},\quad \forall\,i,j=1,2,\dots,q.\end{gathered}$$

Let $\eta^{\mathrm{T}}=(1,1,\dots,1)\in R^s_+,$, then the expression of total derivative of function (5.3.3) with elements (5.5.3) is

$$(5.5.5)\qquad DV(x,y,M)=z^{\mathrm{T}}C(S)z+z^{\mathrm{T}}G(M,S)z,\quad \forall\,(x,y)\in R^q\times R^q,$$

where

$$\begin{gathered}z=(x_1^{\mathrm{T}},x_2^{\mathrm{T}},\dots,x_q^{\mathrm{T}},y_1^{\mathrm{T}},y_2^{\mathrm{T}},\dots,y_q^{\mathrm{T}})^{\mathrm{T}};\\ C(S)=[c_{ij}(S)],\quad i,j=1,2,\dots,s;\\ G(M,S)=[\sigma_{ij}(M,S)],\quad i,j=1,2,\dots,s;\quad s=2q.\end{gathered}$$

The elements of the matrix $C(S)$ are

$$\begin{aligned}c_{ii}(S)&=P_iA_i+A_i^{\mathrm{T}}P_{ii}+\sum_{l=1}^{i-1}\left(P^{\mathrm{T}}_{li}(S^1_{li}A_{li}))+(S^1_{li}A_{li})^{\mathrm{T}}P_{li}\right)\\ &\quad+\sum_{l=1}^{q}\left(P_{il}(S^1_{li}A_{li})+(S^1_{li}A_{li})^{\mathrm{T}}P^{\mathrm{T}}_{il}\right),\quad i=1,2,\dots,q;\end{aligned}$$

$$c_{q+i,q+i}(S)=P_{q+i,q+i}B_i+B_i^{\mathrm{T}}P_{q+i,q+i}+\sum_{l=1}^{i-1}\left(P^{\mathrm{T}}_{q+l,q+i}(S^2_{q+l,i}B'_{li}\right)$$

$$+ (S^2_{q+l,i}B'_{li})^{\mathrm{T}}P_{q+l,q+i}\big) + \sum_{l=i}^{q}\big(P_{q+i,q+l}(S^2_{q+l,i}B'_{li})$$

$$+ (S^2_{q+l,i}B'_{li})^{\mathrm{T}}P^{\mathrm{T}}_{q+i,q+l}\big), \quad i = 1,2,\dots,q;$$

$$c_{ij}(S) = c_{ji}(S) = P_{ij}A_j + A_i^{\mathrm{T}}P_{ij} + \sum_{l=1}^{i-1}\big(P^{\mathrm{T}}_{li}(S^1_{lj}A_{lj})$$

$$+ (S^1_{lj}A_{lj})^{\mathrm{T}}P_{lj}\big) + \sum_{l=i}^{j-1}\big(P_{il}(S^1_{lj}A_{lj}) + (S^1_{li}A_{li})^{\mathrm{T}}P_{lj}\big)$$

$$+ \sum_{l=j}^{q}\big(P_{il}(S^1_{lj}A_{lj}) + (S^1_{li}A_{li})^{\mathrm{T}}P^{\mathrm{T}}_{jl}\big), \quad i,j = 1,2,\dots,q, \quad j > i;$$

$$c_{q+i,q+j}(S) = c_{q+j,q+i}(S) = 0, \quad i,j = 1,2,\dots,q, \quad j > i;$$

$$c_{i,q+j}(S) = P_{i,q+j}B_j + \sum_{l=1}^{i-1}P^{\mathrm{T}}_{li}(S^2_{lj}A'_{lj}) + \sum_{l=i}^{q}P_{il}(S^2_{lj}A'_{lj})$$

$$+ \sum_{l=1}^{q}P_{i,q+l}(S^2_{q+l,j}B'_{lj}), \quad i,j = 1,2,\dots,q.$$

The elements of the matrix $G(M,S)$ are

$$\sigma_{ii}(M,S) = \mu_i\sigma^*_{ii}(S), \quad i = 1,2,\dots,q;$$

$$\sigma^*_{ii}(S) = \sum_{l=1}^{q}\big(P_{i,q+l}(S^1_{q+l,i}B_{li}) + (S^1_{q+l,i}B_{li})^{\mathrm{T}}P^{\mathrm{T}}_{i,q+l}\big), \quad i = 1,2,\dots,q;$$

$$\sigma_{q+i,q+i}(M,S) = \mu_i\sigma^*_{q+i,q+i}(S), \quad i = 1,2,\dots,q;$$

$$\sigma^*_{q+i,q+i}(S) = \sum_{l=1}^{q} \big((S^2_{li}A'_{li})^{\mathrm{T}} P_{l,q+i} + P^{\mathrm{T}}_{l,q+i}(S^2_{li}A'_{li})\big), \quad i = 1, 2, \dots, q;$$

$$\sigma_{ij}(M,S) = \sigma_{ji}(M,S) = \mu_j \sigma^*_{ij}(S), \quad i,j = 1,2,\dots,q, \quad j > i;$$

$$\sigma^*_{ij}(S) = \sum_{l=1}^{q} \big(P_{i,q+l}(S^1_{q+l,j}B_{lj}) + (S^1_{q+l,j}B_{lj})^{\mathrm{T}} P_{i,q+l}\big), \quad j > i = 1,2,\dots,q;$$

$$\sigma_{q+i,q+j}(M,S) = \sigma_{q+j,q+i}(M,S) = \mu_i \sigma^*_{q+i,q+j}(S) + \mu_j \sigma^{**}_{q+i,q+j}(S),$$
$$i,j = 1,2,\dots,q, \quad j > i;$$

$$\sigma^*_{q+i,q+j}(S) = P_{q+i,q+j}B_j + \sum_{l=1}^{i-1} P^{\mathrm{T}}_{q+l,q+i}(S^2_{q+l,j}b'_{lj})$$
$$+ \sum_{l=i}^{q} P_{q+i,q+l}(S^2_{q+l,j}B'_{lj}), \quad i,j = 1,2,\dots,q, \quad j > i;$$

$$\sigma^{**}_{q+i,q+j}(S) = B^{\mathrm{T}}_i P_{q+i,q+j} + \sum_{l=1}^{j-1} (S^2_{q+l,i}B'_{li})^{\mathrm{T}} P_{q+l,q+j}$$
$$+ \sum_{l=j}^{q} (S^2_{q+l,i}B'_{li})^{\mathrm{T}} P^{\mathrm{T}}_{q+j,q+l} + \sum_{l=1}^{q} \big((S^2_{li}A'_{li})^{\mathrm{T}} P_{l,q+j}$$
$$+ P^{\mathrm{T}}_{l,q+j}(S^2_{li}A'_{l;i})\big), \quad j > i = 1,2,\dots,q;$$

$$\sigma_{i,q+j}(M,S) = \mu_j \sigma^*_{i,q+j}(S) + \mu_i\mu_j \sigma^{**}_{i,q+j}(S), \quad i,j = 1,2,\dots,q;$$

$$\sigma^*_{i,q+j}(S) = A^{\mathrm{T}}_i P_{i,q+j} + \sum_{l=1}^{q} (S^1_{li}A_{li})^{\mathrm{T}} P_{l,q+j}, \quad i,j = 1,2,\dots,q;$$

$$\sigma^{**}_{i,q+j}(S) = \sum_{l=1}^{i-1} P^{\mathrm{T}}_{q+l,q+i}(S^1_{q+l,j}B_{lj}) + \sum_{l=i}^{q} P_{q+i,q+l}(S^1_{q+l,j}B_{lj}),$$
$$i,j = 1,2,\dots,q.$$

We designate the upper boundary of expression (5.5.5) by $DV_M(x,y,M)$ and find the estimate

$$DV_M(x,y,M) \le u^{\mathrm{T}} \overline{G}(M) u, \tag{5.5.6}$$

where

$$u^{\mathrm{T}} = (\|x_1\|, \|x_2\|, \dots, \|x_q\|, \|y_1\|, \|y_2\|, \dots, \|y_q\|),$$
$$\overline{G}(M) = [\overline{c}_{ij} + \overline{\sigma}_{ij}(M)], \quad i,j = 1,2,\dots,s, \quad s = 2q.$$

The elements of the matrix $\overline{G}(M)$ are

$$\overline{c}_{ii} = \lambda_M(c_{ii}(S^*)), \quad \overline{c}_{q+i,q+i} = \lambda_M(c_{q+i,q+i}(S^*)), \quad i = 1,2,\dots,q;$$

$$\overline{c}_{ij} = \lambda^{1/2}_M(c_{ij}(S^*)c^{\mathrm{T}}_{ij}(S^*)) = \overline{c}_{ji}, \quad i,j = 1,2,\dots,q, \quad j > i;$$

$$\overline{c}_{q+i,q+j} = \overline{c}_{q+j,q+i} = 0, \quad i,j = 1,2,\dots,q, \quad j > i;$$

$$\overline{c}_{i,q+j} = \lambda^{1/2}_M(c_{i,q+j}(S^*)c^{\mathrm{T}}_{i,q+j}(*S^*)), \quad i,j = 1,2,\dots,q;$$

$$\overline{\sigma}_{ii}(M) = \mu_i \lambda_M(\sigma^*_{ii}(S^*)), \quad i = 1,2,\dots,q;$$

$$\overline{\sigma}_{q+i,q+i}(M) = \mu_i \lambda_M(\sigma^*_{q+i,q+i}(S^*)), \quad i = 1,2,\dots,q;$$

$$\overline{\sigma}_{ij}(M) = \overline{\sigma}_{ji}(M) = \mu_j \lambda_M^{1/2}(\sigma_{ij}^*(S^*)\sigma_{ij}^{*\mathrm{T}}(S^*)), \quad j > i = 1, 2, \dots, q;$$

$$\overline{\sigma}_{q+i,q+j}(M) = \overline{\sigma}_{q+j,q+i}(M) = \mu_i \lambda_M^{1/2}(\sigma_{q+i,q+j}^*(S^*)\sigma_{q+i,q+j}^{*\mathrm{T}}(S^*))$$
$$+ \mu_j \lambda_M^{1/2}(\sigma_{q+i,q+j}^{**}(S^*)\sigma_{q+i,q+j}^{**\mathrm{T}}(S^*)), \quad i, j = 1, 2, \dots, q, \quad j > i;$$

$$\overline{\sigma}_{i,q+j}(M) = \mu_j \lambda_M^{1/2}(\sigma_{i,q+j}^*(S^*)\sigma_{i,q+j}^{*\mathrm{T}}(S^*))$$
$$+ \mu_i \mu_j \lambda_M^{1/2}(\sigma_{i,q+j}^{**}(S^*)\sigma_{i,q_j}^{**\mathrm{T}}(S^*)), \quad i, j = 1, 2, \dots, q.$$

Here $S^* \in \mathcal{S}$ is a constant matrix such that

$$c_{ij}(S) \le c_{ij}(S^*), \quad \sigma_{ij}^*(S) \le \sigma_{ij}^*(s^*), \quad i, j = 1, 2, \dots, s,$$
$$\sigma_{i,q=j}^{**}(S) \le \sigma_{i,q+j}^{**}(S^{**}), \quad \sigma_{q+i,q+j}^{**}(S) \le \sigma_{q+i,q+j}^{**}(S^*), \quad i, j = 1, 2, \dots, q.$$

Theorem 5.5.1 *Let the equations of linear singularly perturbed large-scale system (5.5.1) be such that for this system it is possible to construct matrix-function (5.3.2) with elements (5.5.3) which satisfies estimates (5.5.4) and for the expression (5.5.5) estimate (5.5.6) holds true and*

(1) *matrix $A(M)$ is positive definite for any $\mu_i \in (0, \widetilde{\mu}_{i1})$ and for $\mu_i \to 0$, $i = 1, 2, \dots, q$;*
(2) *matrix $\overline{G}(M)$ is negative definite for any $\mu_i \in (0, \widetilde{\mu}_{i2})$ and for $\mu_i \to 0$, $i = 1, 2, \dots, q$.*

Then the equilibrium state $(x^{\mathrm{T}}, y^{\mathrm{T}})^{\mathrm{T}} = 0$ of system (5.5.1) is structurally uniformly asymptotically stable in the whole for any $\mu_i \in (0, \widetilde{\mu}_i)$ and for $\mu_i \to 0$ on $\mathcal{S}$, where $\widetilde{\mu}_i = \min\{1, \widetilde{\mu}_{i1}, \widetilde{\mu}_{i2}\}$.

Here $\widetilde{\mu}_{i1}$ and $\widetilde{\mu}_{i2}$ are determined by conditions of matrix $A(M)$ positive definiteness and matrix $\overline{G}(M)$ negative definiteness respectively.

This theorem is proved in the same manner as Theorem 5.4.1.

Example 5.5.1 Let system (5.5.1) be the 12-th order system $n = m = 6$ decomposed into three $q = r = 3$ interconnected singularly perturbed subsystems determined by the matrices
(5.5.7)

$$A_1 = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix},$$
$$A_{12} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \quad A_{21} = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix},$$
$$A_{13} = A_{31} = A_{23} = A_{32} = 10^{-1}J,$$
$$A'_{ij} = 10^{-1}J, \quad i, j = 1, 2, 3;$$
$$B_i = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}; \quad B_{ij} = 10^{-1}J, \quad B'_{ij} = 0, \quad i, j = 1, 2, 3;$$
$$S_{jl}^k = \mathrm{diag}\,\{s_{jl}k,\, s_{jl}^k\}, \quad k = 1,\, 2, \quad l = 1, 2, 3, \quad j = 1, 2, 3, 4, 5, 6;$$
$$0 \le s_{jl}^k \le 1, \quad s_{ii}^1 = s_{3+i,i} = 0, \quad s_{ii}^2 = s_{3+i,i}^1 = 1, \quad s_{21}^1 = 1, \quad i = 1, 2, 3.$$

In the matrix-function (5.3.2) the elements are taken as follows
(5.5.8)

$$v_{ii}(x_i) = x_i^{\mathrm{T}} J x_i, \quad v_{ij}(x_i, x_j) = x_i^{\mathrm{T}} 10^{-1} J x_j,$$
$$v_{3+i,3+i}(y_i) = y_i^{\mathrm{T}} 2 J y_i, \quad v_{3+i,3+j}(y_i, y_j) = 0, \quad i, j = 1, 2, 3, \quad i \neq j;$$
$$v_{i,3+j}(x_i, y_j) = x_i^{\mathrm{T}} 10^{-1} J y_j, \quad i, j = 1, 2, 3; \quad j = \operatorname{diag}\{1, 1, 1\}.$$

It is easy to see that for these elements

$$
\begin{aligned}
& v_{ii}(x_i) \geq \|x_i\|^2, \quad i = 1, 2, 3; \\
& v_{ij}(x_i, x_j) \geq -0.1 \, \|x_i\| \, \|x_j\|, \quad i, j = 1, 2, 3, \quad i \neq j; \\
& v_{3+i,3+i}(y_i) \geq 2 \|y_i\|^2, \quad i = 1, 2, 3; \\
& v_{i,3+j}(x_i, y_j) \geq -0.1 \, \|x_i\| \, \|y_j\|, \quad i, j = 1, 2, 3.
\end{aligned} \tag{5.5.9}
$$

Let $\eta^{\mathrm{T}} = (1, 1, 1, 1, 1, 1)$, then the matrix $A(M)$ becomes

$$A(M) = \begin{pmatrix} A_{11} & -A_{12}(M) \\ -A_{12}^{\mathrm{T}}(M) & A_{22}(M) \end{pmatrix},$$

where

$$A_{11} \begin{pmatrix} 1 & -0.1 & -0.1 \\ -0.1 & 1 & -0.1 \\ -0.1 & -0.1 & 1 \end{pmatrix}, \quad A_{12}(M) = 0.1 \begin{pmatrix} \mu_1 & \mu_1 & \mu_3 \\ \mu_1 & \mu_1 & \mu_3 \\ \mu_1 & \mu_1 & \mu_3 \end{pmatrix},$$
$$A_{22}(M) = \operatorname{diag}\{2\mu_1, \, 2\mu_2, \, 2\mu_3\},$$

and is positive definite for any $\mu_i \in (0, \, 1]$ and for $\mu_i \to 0, \; i = 1, 2, 3.$

For such choice of the elements of matrix-function (5.3.2) the elements of the matrix $\overline{G}(M)$ are defined as

$$\overline{c}_{11} = -0.38; \quad \overline{c}_{22} = -7.38; \quad \overline{c}_{33} = -5.96; \quad \overline{c}_{12} = \overline{c}_{21} = 0.17;$$
$$\overline{c}_{13} = \overline{c}_{31} = 0.08; \quad \overline{c}_{23} = \overline{c}_{32} = 0.19; \quad \overline{c}_{3+i,3+i} = -8, \quad i = 1,2,3;$$
$$\overline{c}_{3+i,3+j} = \overline{c}_{3+j,3+i} = 0, \quad i,j = 1,2,3, \quad i \neq j;$$
$$\overline{\sigma}_{ii}(M) = \overline{\sigma}_{3+i,3+i}(M) = 0.6 \cdot 10^{-1}\mu_i, \quad i = 1,2,3;$$
$$\overline{\sigma}_{3+i,3+j}(M) = \overline{\sigma}_{3+j,3+i}(M) = 0.1\mu_i + 0.04\mu_j, \quad i,j = 1,2,3, \quad i \neq j;$$
$$\overline{\sigma}_{ij}(M) = \overline{\sigma}_{ji}(M) = 0.6 \cdot 10^{-1}\mu_j, \quad i,j = 1,2,3, \quad i \neq j;$$
$$\overline{c}_{i,3+j} = 0.8 \cdot 10^{-1}, \quad i,j = 1,2,3;$$
$$\sigma_{i,3+j}(M) = 0.18\mu_j + 0.1\mu_i\mu_j, \quad i = 1,2, \quad j = 1,2,3;$$
$$\sigma_{2,3+j}(M) = 0.9 \cdot 10^{-1}\mu_j + 0.1\mu_2\mu_j, \quad j = 1,2,3.$$

Moreover, the matrix $\overline{G}(M)$ is negative definite for any $\mu_i \in (0,1)$ and for $\mu_i \to 0,\ i = 1,2,3$.

By Theorem 5.4.1 the equilibrium state $(x^{\mathrm{T}}, y^{\mathrm{T}})^{\mathrm{T}} = 0 \in R^{12}$ of the system determined in this example, is uniformly asymptotically stable in the whole on $\mathcal{M} \times \mathcal{S}$, where $\mathcal{M} = \{\mu_i\colon\ 0 < \mu_i \leq 1,\ i = 1,2,3\}$.

Remark 5.5.2 In this example the independent degenerate subsystem

$$\frac{dx_1}{dt} = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix} x_1$$

is unstable and the independent singularly perturbed subsystem

$$\frac{dx_1}{dt} = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix} x_1 + \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix} y_1,$$
$$\mu_1 \frac{dy_1}{dt} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} y_1 + \mu_1 \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix} x_1$$

is not stable for any $\mu_1 \in (0,1]$.

5.5.2 Uniform time scaling In the case of uniform time scaling system (5.5.1) is of the form

$$\frac{dx_i}{dt} = A_i x_i + \sum_{\alpha=1}^{q} S^1_{i\alpha} A_{i\alpha} x_\alpha + \sum_{\beta=1}^{r} S^2_{i\beta} A'_{i\beta} y_\beta, \quad i = 1,2,\dots,q,$$

$$\mu_1 \frac{dy_j}{dt} = \tau_j B_j y_j + \mu + 1 \sum_{\alpha=1}^{q} S^1_{q+j,\alpha} B_{j\alpha} x_\alpha + \tau_j \sum_{\beta=1}^{r} S^2_{q+j,\beta} B'_{j\beta} y_\beta, \quad j = 1,2,\dots,r, \tag{5.5.10}$$

where A_i, B_j,$A_{i\alpha}$, $A'_{i\beta}$, $B_{j\alpha}$ and $B'_{j\beta}$ are constant matrices. All matrices and vectors are of the corresponding order, and $S^1_{i\alpha}$, $S^2_{i\beta}$, $S^1_{q+j,\alpha}$, $S^2_{q+j,\beta} \in \mathcal{S}$ are the diagonal matrices, $\mathcal{S}$ is determined in the same way as Section 5.2, $\mu_1 \in (0,1]$, $q + r = s$, $\tau_j \in [\underline{\tau}_j, \overline{\tau}_j]$.

Assume that $\underline{\tau}_j$ and $\overline{\tau}_j$, $j = 1,2,\dots,r$, are given.

We construct matrix-function (5.3.2) for system (5.5.10) with the elements

$$
\begin{gathered}
v_{ip}(x_i,x_p)=v_{pi}(x_i,x_p)=x_i^{\mathrm T}P_{ip}x_p,\quad i,p=1,2,\dots,q;\\
v_{q+j,q+l}(y_j,y_l)=v_{q+l,q+j}(y_j,y_l)=y_j^{\mathrm T}P_{q+j,q+l}y_l,\\
v_{i,q+j}(x_i,y_j)=x_i^{\mathrm T}P_{i,q+j}y_j,\quad i=1,2,\dots,q,\\
j=1,2,\dots,r,\quad q+r=s,
\end{gathered}
\tag{5.5.11}
$$

where P_{ii}, $P_{q+j,q+j}$ are symmetric positive definite matrices; P_{ip}, $i\neq p$, $P_{q+j,q+l}$, $j\neq l$, $P_{i,q+j}$ are constant matrices.

For function (5.5.11) the following estimates are satisfied

$$
\begin{aligned}
&\text{(a)}\ \ \lambda_m(P_{ii})\|x_i\|^2\le v_{ii}(x_i)\le\lambda_M(P_{ii})\|x_i\|^2,\quad\forall\,x_i\in\mathcal N_{ix_0},\\
&\qquad i=1,2,\dots,q;\\
&\text{(b)}\ \ \lambda_m(P_{q+j,q+j})\|y_j\|^2\le v_{q+j,q+j}(y_j)\le\lambda_M(P_{q+j,q+j})\|y_j\|^2,\\
&\qquad\forall\,y_j\in\mathcal N_{jy_0},\quad j=1,2,\dots,r;\\
&\text{(c)}\ \ -\lambda_M^{1/2}(P_{ip}P_{ip}^{\mathrm T})\|x_i\,\|x_p\|\le v_{ip}(x_i,x_p)\le\lambda_M^{1/2}(P_{ip}P_{ip}^{\mathrm T})\|x_i\,\|x_p\|,\\
&\qquad\forall\,(x_i,x_p)\in\mathcal N_{ix_0}\times\mathcal N_{px_0},\quad i,p=1,2,\dots,q,\quad i\neq p;\\
&\text{(d)}\ \ -\lambda_M^{1/2}(P_{q+j,q+l}P_{q+j,q+l}^{\mathrm T})\|y_j\|\,\|y_l\|\le v_{q+j,q+l}(y_j,y_l)\le\\
&\qquad\lambda_M^{1/2}(P_{q+j,q+l}P_{q+j,q+l}^{\mathrm T})\|y_j\|\,\|y_l\|,\quad\forall\,(y_j,y_l)\in\mathcal N_{jy_0}\times\mathcal N_{ly_0},\\
&\qquad j,l=1,2,\dots,r,\quad j\neq l;\\
&\text{(e)}\ \ -\lambda_M^{1/2}(P_{i,q+j}P_{i,q+j}^{\mathrm T})\|x_i\|\,\|y_j\|\le v_{i,q+j}(x_i,y_j)\le\\
&\qquad\lambda_M^{1/2}(P_{i,q+j}P_{i,q+j}^{\mathrm T})\|x_i\|\,\|y_j\|,\quad\forall\,(x_i,y_j)\in\mathcal N_{ix_0}\times\mathcal N_{jy_0},\\
&\qquad i=1,2,\dots,q,\quad j=1,2,\dots,r,\quad q+r=s,
\end{aligned}
\tag{5.5.12}
$$

where $\lambda_m(\cdot)$ are the minimal eigenvalues, $\lambda_M(\cdot)$ are the maximal eigenvalues, and $\lambda_M^{1/2}(\cdot,\cdot)$ is the matrix norm.

If estimates (5.5.12) are satisfied for function (5.3.2) with elements (5.5.11) the bilateral estimate

$$
\begin{gathered}
u^{\mathrm T}A(\mu_1)u\le v(x,y,\mu_1)\le u^{\mathrm T}B(\mu_1)u,\\
\forall\,(x_i,y_j,\mu_1)\in\mathcal N_{ix_0}\times\mathcal N_{jy_0}\times\mathcal M,
\end{gathered}
$$

holds, where $u^{\mathrm T}=(\|x_1\|,\|x_2\|,\dots,\|x_q\|,\|y_1\|,\|y_2\|,\dots,\|y_r\|)$, the matrices $A(\mu_1)$ and $B(\mu_1)$ are defined as in Proposition 5.3.3 with the elements

$$
\begin{gathered}
\underline{\alpha}_{ii}=\lambda_m(P_{ii});\quad\underline{\alpha}_{ip}=\underline{\alpha}_{pi}=-\lambda_M^{1/2}(P_{ip}P_{ip}^{\mathrm T}),\quad i\neq p=1,2,\dots,q;\\
\overline{\alpha}_{ii}=\lambda_M(P_{ii});\quad\overline{\alpha}_{ip}=\overline{\alpha}_{pi}=\lambda_M^{1/2}(P_{ip}P_{ip}^{\mathrm T}),\quad i\neq p=1,2,\dots,q;\\
\underline{\alpha}_{q+j,q+j}=\lambda_m(P_{q+j,q+j});\\
\underline{\alpha}_{q+j,q+l}=\underline{\alpha}_{q+l,q+j}=-\lambda_M^{1/2}(P_{q+j,q+l}P_{q+j,q+l}^{\mathrm T}),\quad j,l=1,2,\dots,r,\quad j\neq l;\\
\overline{\alpha}_{q+j,q+j}=\lambda_M(P_{q+j,q+j});\\
\overline{\alpha}_{q+j,q+l}=\overline{\alpha}_{q+l,q+j}=\lambda_M^{1/2}(P_{q+j,q+l}P_{q+j,q+l}^{\mathrm T}),\quad j,l=1,2,\dots,r,\quad j\neq l;\\
\underline{\alpha}_{i,q+j}=-\lambda_M^{1/2}(P_{i,q+j}P_{i,q+j}^{\mathrm T}),\quad\overline{\alpha}_{i,q+j}=-\underline{\alpha}_{i,q+j},\quad i\in[1,q],\quad j\in[1,r].
\end{gathered}
$$

It is easy to verify that if the matrices A_{11}^* and A_{22}^* are positive definite, then the function $V(x, y, \mu_1)$ is positive definite for any $\mu_1 \in (0, \mu_1^*)$ and for $\mu_1 \to 0$, where μ_1^* is defined as in Proposition 5.3.4.

Let $\eta^{\mathrm{T}} = (1, 1, \dots, 1) \in R^s$. We designate the upper boundary of the total derivative of function (5.3.8) with elements (5.5.11) by $DV_M(x, y, \mu_1)$, and find

$$-DV_M(x, y, \mu_1) \le u^{\mathrm{T}} \overline{C} u + \mu_1 1 u^{\mathrm{T}} \overline{G} u, \tag{5.5.13}$$

where $\overline{C} = [\overline{c}_{ij}]$, $\overline{c}_{ij} = \overline{c}_{ji}$, $i, j = 1, 2, \dots, s$; $\overline{G} = [\overline{\sigma}_{ij}]$, $\overline{\sigma}_{ij} = \overline{\sigma}_{ji}$, $i, j = 1, 2, \dots, s$, the matrices with elements

$$\overline{c}_{ii} = \rho_{1i}(\overline{S}) + \rho_{2i}(\overline{S}), \quad \overline{\sigma}_{ii} = \rho_{3i}(\overline{S}), \quad i = 1, 2, \dots, q;$$

$$\overline{c}_{ip} = \rho_{1ip}(\overline{S}) + \rho_{2ip}(\overline{S}), \quad \overline{\sigma}_{ip} = \rho_{3ip}(\overline{S}), \quad i, p = 1, 2, \dots, q, \quad p > i;$$

$$\overline{c}_{q+j,q+j} = \rho_{1,q+j}(\overline{S}) + \rho_{2,q+j}(\overline{S}), \quad \overline{\sigma}_{q+j,q+j} = \rho_{3,q+j}(\overline{S}), \quad j = 1, 2, \dots, r;$$

$$\overline{c}_{q+j,q+l} = \rho_{1,q+j,q+l}(\overline{S}) + \rho_{2,q+j,q+l}(\overline{S}), \quad \overline{\sigma}_{q+j,q+l} = \rho_{3,q+j,q+l}(\overline{S}),$$
$$j, l = 1, 2, \dots, r, \quad l > j;$$

$$\overline{c}_{i,q+j} = \rho_{1,i,q+j}(\overline{S}), \quad \overline{\sigma}_{i,q+j} = \rho_{2,i,q+j}(\overline{S}),$$
$$i = 1, 2, \dots, q, \quad j = 1, 2, \dots, r, \quad q + r = s;$$

$$\rho_{1i}(\overline{S}) = \lambda_M(C_{ii}^1(\overline{S})), \quad \rho_{1ip}(\overline{S}) = \lambda_M^{1/2}(C_{ip}^1(\overline{S}) C_{ip}^{1\mathrm{T}}(\overline{S})),$$

$$\rho_{2i}(\overline{S}) = \lambda_M(C_{ii}^2(\overline{S})), \quad \rho_{2ip}(\overline{S}) = \lambda_M^{1/2}(C_{ip}^2(\overline{S}) C_{ip}^{2\mathrm{T}}(\overline{S})),$$

$$\rho_{3i}(\overline{S}) = \lambda_M(\sigma_{ii}(\overline{S})), \quad \rho_{3ip}(\overline{S}) = \lambda_M^{1/2}(\sigma_{ip}(\overline{S}) \sigma_{ip}^{\mathrm{T}}(\overline{S})),$$

$$i, p = 1, 2, \dots, q, \quad p > i;$$

$$\rho_{1,q+j}(\tau_j^*, \overline{S}) = \lambda_M(C^1_{q+j,q+j}(\tau_j^*, \overline{S})),$$

$$\rho_{1,q+j,q+l}(\tau_j^*, \overline{S}) = \lambda_M^{1/2}(C^1_{q+j,q+l}(\tau_j^*, \overline{S})C^{1\mathrm{T}}_{q+j,q+l}(\tau_j^*, \overline{S})),$$

$$\rho_{2,q+j}(\tau_j^*, \overline{S}) = \lambda_M(C^2_{q+j,q+j}(\tau_j^*, \overline{S})),$$

$$\rho_{2,q+j,q+l}(\tau_j^*, \overline{S}) = \lambda_M^{1/2}(C^2_{q+j,q+l}(\tau_j^*, \overline{S})C^{2\mathrm{T}}_{q+j,q+l}(\tau_j^*, \overline{S})),$$

$$\rho_{3,q+j}(\overline{S}) = \lambda_M(\sigma_{q+j,q+j}(\overline{S})),$$

$$\rho_{3,q+j,q+l}(\overline{S}) = \lambda_M^{1/2}(\sigma_{q+j,q+l}(\overline{S})\sigma^{\mathrm{T}}_{q+j,q+l}(\overline{S})), \quad j, l = 1, 2, \dots, r, \quad l > j;$$

$$\rho_{1ij}(\tau_j^*, \overline{S}) = \lambda_M^{1/2}(c_{i,q+j}(\tau_j^*, \overline{S})c^{\mathrm{T}}_{i,q+j}(\tau_j^*, \overline{S})),$$

$$\rho_{2ij}(\overline{S}) = \lambda_M^{1/2}(\sigma_{i,q+j}(\overline{S})\sigma^{\mathrm{T}}_{i,q+j}(\overline{S})),$$

$$i = 1, 2, \dots, q, \quad j = 1, 2, \dots, r, \quad q + r = s;$$

$$c^1_{ii}(A) = P_{ii}A_{ii} + A^{\mathrm{T}}_{ii}P_{ii},$$

$$c^2_{ii}(S) = \sum_{\alpha=1}^{i-1}\left(P^{\mathrm{T}}_{\alpha i}(S^1_{\alpha i}A_{\alpha i}) + (S^1_{\alpha i}A_{\alpha i})^{\mathrm{T}}P_{\alpha i}\right) + \sum_{\alpha=i}^{q}\left(P_{i\alpha}(S^1_{\alpha i}A_{\alpha i}) + (S^1_{\alpha i}A_{\alpha i})^{\mathrm{T}}P^{\mathrm{T}}_{i\alpha}\right);$$

$$\sigma_{ii}(S) = \sum_{\beta=1}^{r}\left(P_{i,q+\beta}(S^1_{q+\beta,i}B_{\beta i}) + (S^1_{q+\beta,i}B_{\beta i})^{\mathrm{T}}P^{\mathrm{T}}_{i,q+\beta}\right), \quad i = 1, 2, \dots, q;$$

$$c^1_{ip}(S) = \sum_{\alpha=1}^{q}\left(P^{\mathrm{T}}_{ii}(S^1_{\alpha p}A_{\alpha p}) + (S^1_{\alpha p}A_{\alpha p})^{\mathrm{T}}P_{ii}\right),$$

$$c^2_{ip}(S) = P_{ip}A_p + A^{\mathrm{T}}_pP_{ip} + \sum_{\alpha=1}^{i-1}\left(P^{\mathrm{T}}_{\alpha i}(S^1_{\alpha p}A_{\alpha p}) + (S^1_{\alpha p}A_{\alpha p})^{\mathrm{T}}P_{\alpha i}\right) + \sum_{\alpha=i+1}^{p-1}\left(P_{i\alpha}(S^1_{\alpha p}A_{alp}) + (S^1_{\alpha p}A_{alp})^{\mathrm{T}}P_{i\alpha}\right) + \sum_{\alpha=p+1}^{q}\left(P_{i\alpha}(S^1_{\alpha p}A_{\alpha p}) + (S^1_{\alpha p}A_{\alpha p})^{\mathrm{T}}P^{\mathrm{T}}_{i\alpha}\right),$$

$$\sigma_{ip}(S) = \sum_{\beta=1}^{r}\left(P_{i,q+\beta}(S^1_{q+\beta,p}B_{\beta p}) + (S^1_{q+\beta,p}B_{\beta p})^{\mathrm{T}}P_{i,q+\beta}\right),$$

$$i, p = 1, 2, \dots, q, \quad p > i;$$

$$c^1_{q+j,q+j}(\tau_j^*, S) = P_{q+j,q+j}\tau_jB_j + \tau_jB^{\mathrm{T}}_jP_{q+j,q+j},$$

$$c^2_{q+j,q+j}(\tau_j, S) = \sum_{\beta=1}^{j-1}\left(P^{\mathrm{T}}_{q+\beta,q+j}\tau_j(S^2_{q+\beta,j}B'_{\beta j}) + \tau_j(S^2_{q+\beta,j}B'_{\beta j})^{\mathrm{T}}P_{q+\beta,q+j}\right) + \sum_{\beta=j}^{r}\left(P_{q+j,q+\beta}\tau_j(S^2_{q+\beta,j}B'_{\beta j}) + \tau_j(S^2_{q+\beta,j}B'_{\beta j})^{\mathrm{T}}P^{\mathrm{T}}_{q+j,q+\beta}\right),$$

$$\sigma_{q+j,q+j}(S) = \sum_{\alpha=1}^{q} \left((S^2_{\alpha j} A'_{\alpha j})^{\mathrm{T}} P_{\alpha,q+j} + P^{\mathrm{T}}_{\alpha,q+j}(S^2_{\alpha j} A'_{\alpha j})\right), \quad j = 1,2,\dots,r;$$

$$c^1_{q+j,q+l}(\tau_l, S) = \sum_{\beta=1}^{r} P^{\mathrm{T}}_{q+j,q+j} \tau_l (S^2_{q+\beta,l} B'_{\beta l}),$$

$$\begin{aligned} c^2_{q+j,q+l}(\tau_j, S) &= P_{q+j,q+l} \tau_l B_l + \tau_j B^{\mathrm{T}}_j P_{q+j,q+l} \\ &+ \sum_{\beta=1}^{j-1} \left(P^{\mathrm{T}}_{q+\beta,q+j} \tau_l (S^2_{q+\beta,l} B'_{\beta l}) + \tau_j (S^2 2_{q+\beta,j} B'_{\beta j})^{\mathrm{T}} P_{q+\beta,q+l}\right) \\ &+ \sum_{\beta=j+1}^{l-1} \left(P_{q+j,q+\beta} \tau_l (S^2_{q+\beta,l} B'_{\beta l}) + \tau_j (S^2_{q+\beta,j} B'_{\beta j})^{\mathrm{T}} P_{q+\beta,q+l}\right) \\ &+ \sum_{\beta=l+1}^{r} \left(P_{q+j,q+\beta} \tau_l (S^2_{q+\beta,l} B'_{\beta l}) + \tau_j (S^2_{q+\beta,j} B'_{\beta j})^{\mathrm{T}} P_{q+l,q+\beta}\right), \end{aligned}$$

$$\sigma_{q+j,q+l}(S) = \sum_{\alpha=1}^{q} \left((S^2_{\alpha j} A'_{\alpha j})^{\mathrm{T}} P_{\alpha,q+l} + P^{\mathrm{T}}_{\alpha,q+l}(S^2_{\alpha j} A'_{\alpha j})\right),$$
$$j, l = 1,2,\dots,r, \quad l > j;$$

$$\begin{aligned} c_{i,q+j}(\tau_j, S) &= P_{i,q+j} + \sum_{\alpha=1}^{i-1} P^{\mathrm{T}}_{\alpha i}(S^2_{\alpha j} A'_{\alpha j}) \\ &+ \sum_{\alpha=i}^{q} P_{i\alpha}(S^2_{\alpha j} A'_{\alpha j}) + \sum_{\beta=1}^{r} P_{i,q+\beta}(S^2_{q+\beta,j} B'_{\beta j}) \tau_j, \end{aligned}$$

$$\begin{aligned} \sigma_{i,q+j}(S) &= A^{\mathrm{T}}_i P_{i,q+j} + \sum_{\alpha=1}^{q} (S^1_{\alpha i} A_{\alpha i}) P_{\alpha,q+j} \\ &+ \sum_{\beta=1}^{j-1} P^{\mathrm{T}}_{q+\beta,q+i}(S^1_{q+\beta,j} B_{\beta j}) + \sum_{\beta=j}^{q} P_{q+i,q+\beta}(S^1_{q+\beta,j} B_{\beta j}), \\ & i = 1,2,\dots,q, \quad j = 1,2,\dots,2, \quad q + r = s. \end{aligned}$$

Here $\overline{S} \in \mathcal{S}$ is a constant matrix such that

$$c^k_{ip}(S) \le c^k_{ip}(\overline{S}), \quad \forall\, S \in \mathcal{S}, \quad i, p = \dot{0}q, \quad p \ge i, \quad k = 1,2;$$
$$c^k_{q+j,q+l}(\tau_j, S) \le c^k_{q+j,q+l}(\tau^*_j, \overline{S}), \quad \forall\, S \in \mathcal{S}, \quad l \ge j = 1,2,\dots,r, \quad k = 1,2;$$
$$\sigma_{ij}(S) \le \sigma_{ij}(\overline{S}), \quad \forall\, S \in \mathcal{S}, \quad i, j = 1,2,\dots,s, \quad s = q + r;$$
$$c_{i,q+j}(\tau_j, S) \le c_{i,q+j}(\tau^*_j, \overline{S}), \quad \forall\, S \in \mathcal{S}, \quad i = 1,2,\dots,q, \quad j = 1,2,\dots,r.$$

The value τ^*_j is defined as

$$\tau^*_j = \begin{cases} \underline{\tau}_j, & \text{if the correponding factors are negative,} \\ \overline{\tau}_j, & \text{if the correponding factors are positive.} \end{cases}$$

Note that if the matrix $\overline{C}$ is negative definite, i.e. $\lambda_M(\overline{C}) < 0$ and $\lambda_M(G) > 0$, then the function $DV_M(x, y, \mu_1)$ is negative definite for any $\mu_1 \in (0, \mu^{**}_1)$ and for $\mu_1 \to 0$, where $\mu^{**}_1 = \min\{1, -\lambda_M(\overline{C})/\lambda_M(\overline{G})\}$.

If $\lambda_M(\overline{C}) < 0$ and $\lambda_M(G) < 0$, then $\mu^{**}_1 = 1$.

Theorem 5.5.2 *Let linear singularly perturbed large-scale system (5.5.10) be such that for this system it is possible to construct the matrix-function (5.3.2) with elements (5.5.11) satisfying estimates (5.5.12) and for function $DV_M(x, y, \mu_1)$ estimate (5.5.13) is fulfilled. Also*

(1) *matrices A^*_{11} and A^*_{22} are positive definite;*
(2) *matrix $\overline{C}$ is negative-definite;*
(3) $\mu_1 \in (0, \widetilde{\mu}_1)$, $\mu_i = \mu_1 \tau_i^{-1}$, $i = 1, 2, \dots, r$, *where*

$$\tau_i \in [\underline{\tau}_i, \overline{\tau}_i], \quad \widetilde{\mu}_1 = \min\{1, \mu_1^*, \mu_1^{**}\}.$$

Then the equilibrium state $(x^{\mathrm{T}}, y^{\mathrm{T}})^{\mathrm{T}} = 0$ of system (5.5.10) is uniformly asymptotically stable in the whole on $\widetilde{\mathcal{M}} \times \mathcal{S}$, where

$$\widetilde{\mathcal{M}} = \{M\colon \; 0 < \mu_1 < \widetilde{\mu}_1, \; \mu_i = \mu_1 \tau_i^{-1}, \; i = 1, 2, \dots, r\}.$$

The proof of this theorem follows from Theorem 5.3.1.

Example 5.5.3 Let system (5.5.10) be the 8-th order system $n = m = 4$, decomposed into two interconnected singularly perturbed subsystems $q = r = 2$ defined by the matrices

$$A_i = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, \quad A_{i\alpha} = A'_{i\beta} = 10^{-2} J;$$
$$B_i = \begin{pmatrix} -4 & 1 \\ 1 & -4 \end{pmatrix}, \quad B_{j\alpha} = B'_{j\beta} = 0.5 \cdot 10^{-2} J;$$
$$J = \operatorname{diag}\{1, 1\}, \quad \underline{\tau}_2 = 0.5, \quad \overline{\tau}_2 = 1, \quad \mu_2 = \mu_1 \tau_2^{-1}.$$

In the matrix-function (5.3.2) the elements $v_{ij}(\cdot)$ are taken as:

$$v_{ii}(x_i) = x_i^{\mathrm{T}} J x_i; \quad v_{2+i,2+i}(y_i) = y_i^{\mathrm{T}} J y_i, \quad i = 1,\, 2;$$
$$v_{12}(x_1, x_2) = x_1^{\mathrm{T}} \cdot 10^{-1} J x_2; \quad v_{34}(y_1, y_2) = y_1^{\mathrm{T}} \cdot 10^{-1} y_2,$$
$$v_{i,2+j}(x_i, y_j) = x_i^{\mathrm{T}} \cdot 10^{-1} J y_j, \quad i, j = 1,\, 2, \quad J = \operatorname{diag}\{1,\, 1\}.$$

Obviously, for these elements the following estimates are true

$$v_{ii}(x_i) \geq \|x_i\|^2, \quad i = 1,\, 2; \quad v_{12}(x_1, x_2) \geq -0.1\|x_1\|\,\|x_2\|;$$
$$v_{2+i,2+i}(y_i) \geq \|y_i\|^2, \quad i = 1,\, 2; \quad v_{34}(y_1, y_2) \geq -0.1\|y_1\|\,\|y_2\|;$$
$$v_{i,2+j}(x_i, y_j) \geq -0.1\|x_i\|\,\|y_j\|, \quad i, j = 1,\, 2.$$

Let $\eta^{\mathrm{T}} = (1, 1, 1, 1),$, then the matrix

$$A(\mu_1) = \begin{pmatrix} A_{11} & \mu_1 A_{12} \\ \mu_1 A_{12}^{\mathrm{T}} & \mu_1 A_{22} \end{pmatrix},$$

where

$$A_{11} = A_{22} = \begin{pmatrix} 1 & -0.1 \\ -0.1 & 1 \end{pmatrix}, \qquad A_{12} = \begin{pmatrix} -0.1 & -0.1 \\ -0.1 & -0.1 \end{pmatrix},$$

is positive definite for any $\mu_i \in (0, 1]$ and for $\mu_1 \to 0$.

For such choice of the elements of matrix (5.3.2) the elements of the matrices $\overline{C}$ and $\overline{G}$ are specified as

$$\overline{c}_{ii} = -1.996, \quad i = 1,\, 2; \quad \overline{c}_{12} = 0.6674; \quad \overline{c}_{2+i,2+i} = -2.996, \quad i = 1,\, 2;$$
$$\overline{c}_{34} = 0.6474; \quad \overline{c}_{1j} = 0,2874; \quad \overline{c}_{2j} = 0.2888, \quad j = 1,\, 2;$$

and

$$\overline{\sigma}_{ii} = 0; \quad \overline{\sigma}_{12} = 0.002, \quad i = 1,\, 2; \quad \overline{\sigma}_{2+i,2+i} = \overline{\sigma}_{34} = 0.004, \quad i = 1,\, 2;$$
$$\overline{\sigma}_{i3} = 0.312438, \quad \overline{\sigma}_{i4} = 0.311178, \quad i = 1,\, 2.$$

For the elements of the matrices $\overline{C}$ and $\overline{G}$ specified in such way we have

$$\lambda_M(\overline{C}) = -1.018975; \quad \lambda_M(\overline{G}) = 0.8819733$$

and

$$\mu_1^{**} = \min\left\{1,\ -\frac{\lambda_M(\overline{C})}{\lambda_M(\overline{G})}\right\} = \min\{1;\ 1.1553354\} = 1.$$

Thus, by Theorem 5.5.2 the equilibrium state $(x^{\mathrm{T}}, y^{\mathrm{T}})^{\mathrm{T}} = 0 \in R^8$ of the system defined in Example 5.5.3 is uniformly asymptotically stable in the whole on $\mathcal{M} \times \mathcal{S}_8$.

5.6 Certain Trends of Generalizations and Applications

In this section general results of this Chapter are applied in two cases. In Subsection 5.6.1 we present the results of stability analysis for singularly perturbed large scale Lur'e-Postnikov systems under nonclassical structural perturbations. A numerical example of the 12-th order system decomposed into three interconnected singularly perturbed subsystems is considered as illustration. In Subsection 5.6.2 we establish the conditions of a spacecraft motion stabilization by means of three gyroframes. Here the possibility of application of the matrix-valued function is shown and the obtained result is compared with those obtained in terms of the vector Liapunov function.

5.6.1 Lur'e-Postnikov systems

5.6.1.1 Non-uniform time scaling Consider a singularly perturbed large-scale system (see Grujic et al.[1])

$$
\begin{aligned}
\frac{dx_i}{dt} &= A_i x_i + \sum_{l=1}^{q} S_{il}^{(1)} A_{il} y_l + q_{i1} f_{i1}(\sigma_{i1}),\\
\sigma_{i1} &= c_{i1}^{\mathrm T} x + c_{i2}^{\mathrm T} y, \quad \forall\, i = 1,2,\dots,q,\\
\mu_i \frac{dy_i}{dt} &= \sum_{l=1}^{q} \mu_i S_{q+i,l}^{(1)} B_{il} x_l + B_i y_i + q_{i2} l_{-2}(\sigma_{i2}) + q_{i3} f_{i3}(\sigma_{i3}),\\
\sigma_{i2} &= \mu_i c_{i3}^{\mathrm T} x_i + c_{i4}^{\mathrm T} y_i,\\
\sigma_{i3} &= \sum_{\substack{l=1\\ l\neq i}}^{q} \left(\mu_i c_{i5}^{\mathrm T} S_{q+i,l}^{(2)} x_l + c_{l6}^{\mathrm T} S_{q+i,l}^{(3)} y_l \right),\\
& i = 1,2,\dots,q,
\end{aligned}
\tag{5.6.1}
$$

where

$$\sigma_{ij}^{-1} f_{ij}(\sigma_{ij}) \in [0, k_{ij}] \subset R_+, \quad i = 1,2,\dots,q, \quad j = 1,2,3,$$

all matrices and vectors are of the appropriate order, and $S_{il}^{(1)}$, $S_{q+i,l}^{(1)}$, $S_{q+i,l}^{(2)}$, $S_{q+i,l}^{(3)}$ are the diagonal matrices. Let

$$
S_i = \begin{pmatrix}
S_{i1}^{(1)} & S_{i2}^{(1)} & \dots & S_{i,i-1}^{(1)} & I & S_{i,i+1}^{(1)} & \dots & S_{iq}^{(1)}\\
S_{q+i,1}^{(1)} & S_{q+i,2}^{(1)} & \dots & S_{q+i,i-1}^{(1)} & 0 & S_{q+i,i+1}^{(1)} & \dots & S_{q+i,q}^{(1)}\\
S_{q+i,1}^{(2)} & S_{q+i,2}^{(2)} & \dots & S_{q+i,i-1}^{(2)} & 0 & S_{q+i,i+1}^{(2)} & \dots & S_{q+i,q}^{(2)}\\
S_{q+i,1}^{(3)} & S_{q+i,2}^{(3)} & \dots & S_{q+i,i-1}^{(3)} & 0 & S_{q+i,i+1}^{(3)} & \dots & S_{q+i,q}^{(3)}
\end{pmatrix},
$$
$$S = \mathrm{diag}\,\{S_1, S_2, \dots, S_q\}.$$

The structural set is determined as

$$
\begin{aligned}
\mathcal{S} = \{S\colon\ 0 \le S_{jl}^{(k)} \le I,\ \ S_{ii}^{(1)} = S_{q+i,i}^{(1)} = I,\ \ S_{q+i,i}^{(2)} = S_{q+i,i}^{(3)} = 0,\\
i,j = 1,2,\dots,q,\quad j = 1,2,\dots,2q,\quad k = 1,2,3\},
\end{aligned}
$$

where I is an identity matrix of the appropriate dimensions.

The independent singularly perturbed subsystems corresponding to system (5.6.1) are obtained as a result of substitution by x^i and y^i for x and y:

$$
\begin{aligned}
&\frac{dx_i}{dt} = A_i x_i + A_{ii} y_i + q_{i1} f_{i1}(\widetilde{\sigma}_{i1}),\\
&\widetilde{\sigma}_{i1} = c_{i1}^{\mathrm{T}} x^i + c_{i2}^{\mathrm{T}} y^i, \quad \forall\, i = 1, 2, \dots, q,\\
&\mu_i \frac{dy_i}{dt} = \mu_i B_{ii} x_i + B_i y_i + q_{i2} f_{i2}(\sigma_{i2}),\\
&\qquad \forall\, i = 1, 2, \dots, q, \quad 2q = s,
\end{aligned}
\tag{5.6.2}
$$

where $x^i = (0^{\mathrm{T}}, \dots, 0^{\mathrm{T}}, x_i^{\mathrm{T}}, 0^{\mathrm{T}}, \dots, 0^{\mathrm{T}})^{\mathrm{T}} \in R^n$, $x_i \in R^{n_i}$, $n_1 + n_2 + \dots + n_q = n$, $y^i = (0^{\mathrm{T}}, \dots, 0^{\mathrm{T}}, y_i^{\mathrm{T}}, 0^{\mathrm{T}}, \dots, 0^{\mathrm{T}})^{\mathrm{T}} \in R^m$, $y_i \in R^{m_i}$, $m_1 + m_2 + \dots + m_q = m$.

We introduce the designations

$$
\begin{aligned}
f_{i0} &= A_i x_i + q_{i1} f_{i1}(\widetilde{\sigma}_{i1}^0), \quad \widetilde{\sigma}_{i1}^0 = c_{i1}^{\mathrm{T}} x^i,\\
g_{i0} &= B_i y_i + q_{i2} f_{i2}(\sigma_{i2}^0), \quad \sigma_{i2}^0 = c_{i2}^{\mathrm{T}} y^i,\\
f_i^* &= A_{ii} y_i + q_{i1}[f_{i1}(\widetilde{\sigma}_{i1}) - f_{i1}(\widetilde{\sigma}_{i1}^0)],\\
g_i^* &= \mu_i B_{ii} x_i + q_{i2}[f_{i2}(\sigma_{i2}) - f_{i2}(\sigma_{i2}^0)],\\
f_i^{**} &= \sum_{\substack{l=1\\ l\neq i}}^{q} S_{il}^{(1)} A_{il} y_l + q_{i1}[f_{i1}(\sigma_{i1}) - f_{i1}(\widetilde{\sigma}_{i1})],
\end{aligned}
$$

$$g_i^{**} = \sum_{\substack{l=1 \\ l\neq i}}^{q} \mu_i S_{q+i,l}^{(1)} B_{il} x_l + q_{i3} f_{i3}(\sigma_{i3}),$$
$$\forall\, i = 1, 2, \ldots, q, \quad 2q = s.$$

The system (5.6.1) becomes

$$(5.6.3) \qquad \begin{aligned} \frac{dx_i}{dt} &= f_{i0} + f_i^* + f_i^{**}, \quad i = 1, 2, \ldots, q, \\ \frac{dy_i}{dt} &= g_{i0} + g_i^* + g_i^{**}, \quad i = 1, 2, \ldots, q. \end{aligned}$$

Here the vector-functions f_{i0} and g_{i0} correspond to the independent degenerated subsystem

$$(5.6.4) \qquad \frac{dx_i}{dt} = A_i x_i + q_{i1} f_{i1}(\widetilde{\sigma}_{i1}^0), \quad \widetilde{\sigma}_{i1}^0 = c_{i1}^{\mathrm{T}} x^i,$$

and subsystem describing the boundary layer

$$(5.6.5) \qquad \mu_i \frac{dy_i}{dt} = B_i y_i + q_{i2} f_{i2}(\sigma_{i2}^0), \quad \sigma_{i2}^0 = c_{i2}^{\mathrm{T}} y_i.$$

The vector-functions $f_{i0} + f_i^*$ and $g_{i0} + g_i^*$ correspond to the independent singularly perturbed subsystems (5.6.2).

Alongside system (5.6.1) and subsystems (5.6.2), (5.6.4), (5.6.5) we shall consider the matrix-function (5.3.2) with the elements (5.5.3) satisfying estimates (5.5.4). As is known, for function (5.3.3) the inequality

$$(5.6.6) \qquad u^{\mathrm{T}} A(M) u \le V(x, y, M) \le u^{\mathrm{T}} B(M) u.$$

is valid. Besides, the matrices $A(M)$, $B(M)$ and vector u are determined as in Proposition 5.3.1.

Proposition 5.6.1 *If for system (5.6.1) the matrix-function (5.3.2) with elements (5.5.3) is constructed, then for the Dini derivatives of functions (5.5.3)*

(a) $\eta_j^2 (D_{x_i}^+ v_{ii})^{\mathrm{T}} f_{i0} \le \rho_{1i} \|x_i\|^2, \quad \forall\, x_i \in \mathcal{N}_{ix0}, \quad \forall\, i \in [1, q];$

(b) $\eta_{q+i}^2 (D_{y_i}^+ v_{q+i,q+i})^{\mathrm{T}} g_{i0} \le \rho_{2i} \|y_i\|^2, \quad \forall\, y_i \in \mathcal{N}_{iy0}, \quad \forall\, i \in [1, q];$

(c)
$$\begin{aligned} &\eta_i^2 (D_{x_i}^+ v_{ii})^{\mathrm{T}} f_i^* + \eta_{q+i}^2 (D_{y_i}^+ v_{q+i,q+i})^{\mathrm{T}} g_i^* \\ &\quad + 2\eta_i \eta_{q+i} \{\mu_i (D_{x_i}^+ v_{i,q+i})^{\mathrm{T}} (f_{i0} + f_i^*) + (D_{y_i}^+ v_{i,q+i})^{\mathrm{T}} (g_{i0} + g_i^*)\} \\ &\quad \le (\rho_{3i} + \mu_i \rho_{4i}) \|x_i\|^2 + (\rho_{5i} + \mu_i \rho_{6i}) \|y_i\|^2 \\ &\quad + 2(\rho_{7i} + \mu_i \rho_{8i}) \|x_i\| \, \|y_i\|, \quad \forall\, (x, y, M) \in \mathcal{N}_{ix0} \times \mathcal{N}_{iy0} \times \mathcal{M}, \\ &\quad i = 1, 2, \ldots, q; \end{aligned}$$

(d)
$$\begin{aligned} &\sum_{i=1}^{q} \eta_i^2 (D_{x_i}^+ v_{ii})^{\mathrm{T}} f_i^{**} + \sum_{i=1}^{q} \eta_{q+i}^2 (D_{y_i}^+ v_{q+i,q+i})^{\mathrm{T}} g_i^{**} \\ &\quad + 2 \sum_{i=1}^{q} \eta_i \eta_{q+i} \{\mu_i (D_{x_i}^+ v_{i,q+i})^{\mathrm{T}} f_i^{**} + (D_{y_i}^+ v_{i,q+i})^{\mathrm{T}} g_i^{**}\} \end{aligned}$$

$$
\begin{aligned}
&+ 2\sum_{i=1}^{q}\sum_{\substack{j=2\\ j>i}}^{q}\eta_i\eta_j\big\{(D_{x_i}^{+}v_{ij})^{\mathrm{T}}(f_{i0}+f_i^{*}+f_i^{**})\\
&\qquad\qquad + (D_{x_j}^{+}v_{ij})^{\mathrm{T}}(f_{j0}+f_j^{*}+f_j^{**})\big\}\\
&+ 2\sum_{i=1}^{q}\sum_{\substack{j=2\\ j>i}}^{q}\eta_{q+i}\eta_{q+j}\big\{\mu_j(D_{y_i}^{+}v_{q+i,q+j})^{\mathrm{T}}(g_{i0}+g_i^{*}+g_i^{**})\\
&\qquad\qquad + \mu_i(D_{y_j}^{+}v_{q+i,q+j})^{\mathrm{T}}(g_{i0}+g_i^{*}+g_i^{**})\big\}\\
&+ 2\sum_{i=1}^{q}\sum_{\substack{j=1\\ (j\neq i)}}^{q}\eta_i\eta_{q+j}\big\{\mu_j(D_{x_i}^{+}v_{i,q+j})^{\mathrm{T}}(f_{i0}+f_i^{*}+f_i^{**})\\
&\qquad\qquad + (D_{y_j}^{+}v_{i,q+j})^{\mathrm{T}}(g_{i0}+g_i^{*}+g_i^{**})\\
&\le \sum_{i=1}^{q}\big\{(\rho_{9i}(s)+\mu_i\rho_{10i}(s))\|x_i\|^2+\rho_{11i}(s)\|y_i\|^2\big\}\\
&+ 2\sum_{i=1}^{q}\sum_{\substack{j=2\\ j>i}}^{q}\big\{(\rho_{1ij}(s)+\mu_i\rho_{2ij}(s))\|x_i\|\,\|x_j\|\\
&\qquad\qquad + (\rho_{3ij}(s)+\mu_i\rho_{4ij}(s)+\mu_j\rho_{5ij}(s))\|y_i\|\,\|y)j\|\big\}\\
&+ 2\sum_{i=1}^{q}\sum_{j=1}^{q}\big(\rho_{6ij}(s)+\mu_i\rho_{7ij}(s)+\mu_i\mu_j\rho_{8ij}(s)\big)\|x_i\|\,\|y_j\|,\\
&\forall\,(x_i,y_i,M,S)\in\mathcal{N}_{ix_0}\times\mathcal{N}_{iy_0}\times\mathcal{M}\times\mathcal{S},
\end{aligned}
$$

where $\rho_{\alpha i}$, $\alpha = 1,2,\dots,6$, $i = 1,2,\dots,q$, $\rho_{\beta i}(s)$, $\beta = 9,10,11$, $i = 1,2,\dots,q$, *are maximal eigenvalues of the matrices*

$$
\begin{aligned}
&\eta_i^2[P_{ii}A_i + A_i^{\mathrm{T}}P_{ii} + P_{ii}k_{i1}^{*}q_{i1}(c_{i1}^{i})^{\mathrm{T}} + (k_{i1}^{*}q_{i1}(c_{i1}^{i})^{\mathrm{T}})^{\mathrm{T}}P_{ii}];\\
&\eta_{q+i}^2[P_{q+i,q+i}B_i + B_i^{\mathrm{T}}P_{q+i,q+i} + P_{q+i,q+i}k_{i2}^{*}q_{i2}(c_{i2}^{i})^{\mathrm{T}}\\
&\qquad + (k_{i2}^{*}q_{i2}(c_{i2}^{i})^{\mathrm{T}})^{\mathrm{T}}P_{q+i,q+i}];\\
&\eta_i^2[P_{ii}k_{i1}^{*}q_{i1}(c_{i1}^{i})^{\mathrm{T}} + (k_{i1}^{*}q_{i1}(c_{i1}^{i})^{\mathrm{T}})^{\mathrm{T}}P_{ii}];\\
&\frac{1}{2}\eta_i\eta_{q+i}(P_{i,q+i}B_{ii} + B_{ii}^{\mathrm{T}}P_{i,q+i}^{\mathrm{T}} + P_{i,q+i}k_{i2}^{*}q_{i2}c_{i3}^{\mathrm{T}} + (k_{i2}^{*}q_{i2}c_{i3}^{\mathrm{T}})^{\mathrm{T}}P_{i,q_i}^{\mathrm{T}});\\
&\eta_{q+i}^2[P_{q+i,q+i}k_{i2}^{*}q_{i2}c_{i4}^{\mathrm{T}} + (k_{i2}^{*}q_{i2}c_{i4}^{\mathrm{T}})^{\mathrm{T}}P_{q+i,q+i}];\\
&\frac{1}{2}\eta_i\eta_{q+i}(P_{i,q+i}A_{ii} + A_{ii}^{\mathrm{T}}P_{i,q+i}^{\mathrm{T}} + P_{i,q+i}k_{i1}^{*}q_{i1}(c_{i2}^{i})^{\mathrm{T}} + (k_{i1}^{*}q_{i1}(c_{i2}^{i})^{\mathrm{T}})^{\mathrm{T}}P_{i,q+i});\\
&\eta_i^2[P_{ii}k_{i1}^{*}q_{i1}(c_{i1}^{i})^{\mathrm{T}} + (k_{i1}^{*}q_{i1}(c_{i1}^{i})^{\mathrm{T}})^{\mathrm{T}}P_{ii}]\\
&\quad + \sum_{\substack{j=2\\ j>i}}^{q}\eta_i\eta_j\{(k_{j1}^{*}q_{j1}(c_{j1}^{i})^{\mathrm{T}})^{\mathrm{T}}P_{ji} + P_{ji}^{\mathrm{T}}k_{j1}^{*}q_{j1}(c_{j1}^{i})^{\mathrm{T}}\};\\
&\eta_i\eta_{q+i}[P_{i,q+i}k_{i3}^{*}q_{i3}c_{i5}^{\mathrm{T}}S_{q+i,i}^{(2)} + (k_{i3}^{*}q_{i3}c_{i5}^{\mathrm{T}}S_{q+i,i}^{(2)})^{\mathrm{T}}P_{i,q+i}^{\mathrm{T}}];\\
&\eta_i\eta_{q+i}[P_{i,q+i}^{\mathrm{T}}k_{i1}^{*}q_{i1}(c_{i2}^{i})^{\mathrm{T}} + (k_{i1}^{*}q_{i1}(c_{i2}^{i})^{\mathrm{T}})^{\mathrm{T}}P_{i,q+i}\\
&\quad + P_{q+i,q+i}^{\mathrm{T}}k_{i3}^{*}q_{i3}c_{i6}^{\mathrm{T}}S_{q+i,i}^{(3)} + (k_{i3}^{*}q_{i3}c_{i6}^{\mathrm{T}}S_{q+i,i}^{(3)})^{\mathrm{T}}P_{q+i,q+i}]
\end{aligned}
$$

respectively, and ρ_{7i}, ρ_{8i}, $\rho_{kij}(S)$, $k = 1, 2, \ldots, 8$, $i, j = 1, 2, \ldots, q$, *are norms of the matrices*

$$\eta_i^2 P_{ii}(A_{ii} + kqi1(c_{i2}^i)^{\mathrm{T}}) + \eta_i \eta_{q+i} P_{i,q+i}(B_i + k_{i2}^* q_{i2} c_{i4}^{\mathrm{T}});$$

$$\eta_{q+i}^2 (B_{ii}^{\mathrm{T}} + (k_{i2}^* q_{i2} c_{i3}^{\mathrm{T}})^{\mathrm{T}}) P_{q+i,q+i} + \eta_i \eta_{q+i}(A_i + (k_{i1}^* q_{i1}(c_{i1}^i)^{\mathrm{T}})^{\mathrm{T}}) P_{i,q+i};$$

$$\eta_i^2 P_{ii} k_{i1}^* q_{i1}(c_{i1}^j)^{\mathrm{T}} + \eta_i \eta_j (P_{ij} A_j + A_i^{\mathrm{T}} P_{ij})$$
$$+ \sum_{l=1}^{i-1} \eta_l \eta_j \{P_{li}^{\mathrm{T}} k_{l1}^* q_{l1}(c_{l1}^j)^{\mathrm{T}} + (k_{l1}^* q_{l1}(c_{l1}^j)^{\mathrm{T}})^{\mathrm{T}} P_{lj}\}$$
$$+ \sum_{l=i}^{j-1} \{\eta_i \eta_l P_{il} k_{l1}^* q_{l1}(c_{l1}^j)^{\mathrm{T}} + \eta_l \eta_j (k_{l1}^* q_{l1}(c_{l1}^j)^{\mathrm{T}})^{\mathrm{T}} P_{lj}\}$$
$$+ \sum_{l=j}^{q} \{\eta_i \eta_l P_{il} k_{l1}^* q_{l1}(c_{l1}^j)^{\mathrm{T}} + \eta_j \eta_l (k_{l1}^* q_{l1}(c_{l1}^j)^{\mathrm{T}})^{\mathrm{T}} P_{jl}^{\mathrm{T}}\};$$

$$\eta_i \eta_{q+i} P_{i,q+i}(S_{q+i,j}^{(1)} B_{ij} + k_{i3}^* q_{i3} c_{j5}^{\mathrm{T}} S_{q+i,j}^{(2)})$$
$$+ \sum_{\substack{l=1 \\ l \neq i}}^{q} \eta_i \eta_{q+l} P_{i,q+l}(S_{q+l,j}^{(1)} B_{lj} + k_{l3}^* q_{l3} c_{j5}^{\mathrm{T}} S_{q+l,j}^{(2)});$$

$$P_{i,q+i}^{\mathrm{T}} k_{i1}^* q_{i1}(c_{i2}^j)^{\mathrm{T}} + \eta_i \eta_{q+i}(k_{i1}^* q_{i1}(c_{i2}^j)^{\mathrm{T}})^{\mathrm{T}} P_{i,q+i};$$

$$\eta_{q+i}\eta_{q+j}P_{q+i,q+j}(B_j + k_{j2}q_{j2}c_{j4}^{\mathrm{T}}) + \eta_i\eta_{q+i}P_{i,q+i}^{\mathrm{T}}(S_{ij}^{(1)}A_{ij}) + \sum_{l=1}^{i-1}\eta_{q+l}\eta_{q+i}P_{q+l,q+i}(k_{l3}^*q_{l3}c_{j6}^{\mathrm{T}}S_{q+l,j}^{(3)}) + \sum_{l=i}^{q}\eta_{q+i}\eta_{q+l}P_{q+i,q+l}(k_{l3}^*q_{l3}c_{j6}^{\mathrm{T}}S_{q+l,j}^{(3)});$$

$$\eta_{q+i}\eta_{q+j}(B_i^{\mathrm{T}} + (k_{i2}^*q_{i2}c_{i4}^{\mathrm{T}})^{\mathrm{T}})P_{q+i,q+j} + \sum_{l=1}^{i-1}\eta_l\eta_i P_{li}^{\mathrm{T}}S_{lj}^{(1)}A_{lj} + \sum_{l=i}^{q}\eta_i\eta_j P_{il}S_{lj}^{(1)}A_{lj} + \sum_{l=1}^{q}\eta_i\eta_{q+l}P_{i,q+l}^{\mathrm{T}}(k_{i1}^*q_{i1}(c_{j2}^{l})^{\mathrm{T}}) + \sum_{l=1}^{i-1}\eta_{q+i}\eta_{q+l}P_{q+l,q+i}^{\mathrm{T}}(k_{l3}^*q_{l3}c_{i6}^{\mathrm{T}}S_{q+l,j}^{(3)}) + \sum_{l=i}^{q}\eta_{q+i}\eta_{q+l}P_{q+i,q+l}(k_{l3}^*q_{l3}c_{j6}^{\mathrm{T}}S_{q+l,j}^{(3)});$$

$$\eta_i^2P_{ii}(S_{ij}^{(1)}A_{ij} + k_{i1}^*q_{i1}(c_{i2}^{j})^{\mathrm{T}}) + \eta_i\eta_{q+i}(k_{i1}^*q_{i1}(c_{i1}^{j})^{\mathrm{T}})^{\mathrm{T}}P_{i,q+i} + \eta_i\eta_{q+j}P_{i,q+j}(B_j + k_{j2}^*q_{j2}c_{i4}^{\mathrm{T}}) + \sum_{l=1}^{i-1}\eta_l\eta_i P_{li}^{\mathrm{T}}(S_{jl}^{(1)}A_{lj} + k_{l1}^*q_{l1}(c_{l2}^{j})^{\mathrm{T}}) + \sum_{l=i+1}^{q}\eta_i\eta_l P_{il}(S_{lj}^{(1)}A_{lj} + k_{l1}^*q_{l1}(c_{l2}^{j})^{\mathrm{T}}) + \sum_{l=1}^{q}\eta_j\eta_{q+l}P_{j,q+l}(k_{l3}^*q_{l3}c_{l6}^{\mathrm{T}}S_{q+l,i}^{(3)});$$

$$\eta_{q+j}^2P_{q+j,q+j}^{\mathrm{T}}(S_{q+j,i}^{(1)}B_{ji} + k_{j3}^*q_{j3}c_{j5}^{\mathrm{T}}S_{q+j,i}^{(2)}) + \eta_i\eta_{q+j}A_i^{\mathrm{T}}P_{i,q+j} + \sum_{l=1}^{q}\eta_l\eta_{q+j}(k_{l1}^*q_{l1}(c_{l1}^{j})^{\mathrm{T}})^{\mathrm{T}}P_{l,q+j};$$

$$\sum_{l=1}^{i-1}\eta_{q+l}\eta_{q+i}P_{q+l,q+i}^{\mathrm{T}}(S_{q+l,j}^{(1)}B_{lj} + k_{l3}^*q_{l3}c_{j5}S_{q+l,j}^{(2)}) + \sum_{l=i}^{q}\eta_{q+i}\eta_{q+l}P_{q+i,q+l}^{\mathrm{T}}(S_{q+l,j}^{(1)}B_{lj} + k_{l3}^*q_{l3}c_{j5}S_{q+l,j}^{(2)}) + \eta_{q+i}\eta_{q+j}[(k_{i2}^*q_{i2}c_{i3}^{\mathrm{T}})^{\mathrm{T}}P_{q+i,q+j} + P_{q+i,q+j}(k_{i2}^*q_{i2}c_{i3}^{\mathrm{T}})]$$

respectively. Here $i \neq j$, k_{ij}^* *and* k_{ii}^* *are determined in the same way as* k_{ir} *in Section 2.4,* $c_{ij}^k \in R^{n_k}$ *is the* k*–th component of the vector* c_{ij}.

The proof of Proposition 5.6.1 follows from direct computations.

Proposition 5.6.2 *Let all conditions of Proposition 5.6.1 be satisfied. Then for the total derivative of function (5.3.3) with elements (5.5.3)*

$$DV(x,y,M) \leq u^{\mathrm{T}}\widetilde{G}(M)u \quad \forall\,(x,y,M,S) \in \mathcal{N}_{x_0} \times \mathcal{N}_{y_0} \times \mathcal{M} \times \mathcal{S}, \tag{5.6.7}$$

where

$$u^{\mathrm{T}} = (\|x_1\|, \|x_2\|, \dots, \|x_q\|, \|y_1\|, \|y_2\|, \dots, \|y_q\|),$$
$$\widetilde{G}(M) = [\widetilde{c}_{ij} + \widetilde{\sigma}_{ij}(M)], \quad i,j = 1,2,\dots,s, \quad s = 2q.$$

The elements of the matrix $\widetilde{G}(M)$ *are*

$$\widetilde{c}_{ii} = \rho_{1i} + \rho_{3i} + \rho_{9i}(S^*); \quad \widetilde{c}_{ij} = \widetilde{c}_{ji} = \rho_{1ij}(S^*), \quad i \neq j;$$
$$\widetilde{c}_{q+i,q+i} = \rho_{2i} + \rho_{5i} + \rho_{11,i}(S^*);$$
$$\widetilde{c}_{q+i,q+j} = \widetilde{c}_{q+j,q+i} = \rho_{3ij}(S^*), \quad i \neq j;$$
$$\widetilde{c}_{i,q+i} = \rho_{7i}; \quad \widetilde{c}_{i,q+j} = \rho_{6ij}(S^*), \quad i \neq j = 1,2,\dots,q;$$
$$\widetilde{\sigma}_{ij}(M) = \mu_i(\rho_{4i} + \rho_{10i}(S^*)); \quad \widetilde{\sigma}_{q+i,q+i}(M) = \mu_i\rho_{6i};$$
$$\widetilde{\sigma}_{ij}(M) = \widetilde{\sigma}_{ji}(M) = \mu_i\rho_{2ij}(S^*), \quad i \neq j;$$
$$\widetilde{\sigma}_{q+i,q+j}(M) = \mu_i\rho_{4ij}(S^*) + \mu_j\rho_{5ij}(S^*), \quad i \neq j;$$
$$\widetilde{\sigma}_{i,q+i}(M) = \mu_i\rho_{8i};$$
$$\widetilde{\sigma}_{i,q+j}(M) = \mu_i\rho_{7ij}(S^*) + \mu_i\mu_j\rho_{8ij}(S^*), \quad i \neq j = 1,2,\dots,q;$$

$S^* \in \mathcal{S}$ *is a constant matrix such that*

$$\rho_{ki}(S) \leq \rho_{ki}(S^*), \quad \rho_{rij}(S) \leq \rho_{rij}(S^*), \quad k = 9,10,11,12,$$
$$r = 1,2,\dots,9, \quad i,j = 1,2,\dots,q, \quad i \neq j.$$

The proof of Proposition 5.6.2 is similar to that of Proposition 5.3.2.

Theorem 5.6.1 *Let the equations of singularly perturbed Lur'e system (5.6.1) be such that the matrix (5.3.2) is constructed with the elements (5.5.3) satisfying estimates (5.5.4) and for the total Dini derivative of function (5.3.3) the estimate (5.6.7) is satisfied and*

(1) *matrix* $A(M)$ *is positive definite for any* $\mu_i \in (0, \widetilde{\mu}_{i1})$ *and for* $\mu_i \to 0$, $i = 1,2,\dots,q$*;*
(2) *matrix* $\widetilde{G}(M)$ *is negative definite for any* $\mu_i \in (0, \widetilde{\mu}^*_{i2})$ *and for* $\mu_i \to 0$, $i = 1,2,\dots,q$*.*

Then the equilibrium state $(x^{\mathrm{T}}, y^{\mathrm{T}})^{\mathrm{T}} = 0$ *of system (5.6.1) is uniformly asymptotically stable for any* $\mu_i \in (0, \widetilde{\mu}^*_i)$ *and for* $\mu_i \to 0$ *on* $\mathcal{S}$*, where* $\mu^*_i = \min\{1, \widetilde{\mu}_{i1}, \widetilde{\mu}^*_{i2}\}$*.*

If, moreover, $\mathcal{N}_{ix} = R^{n_i}$*,* $\mathcal{N}_{iy} = R^{m_i}$*,, then the equilibrium state* $(x^{\mathrm{T}}, y^{\mathrm{T}})^{\mathrm{T}} = 0$ *of system (5.6.1) is uniformly asymptotically stable in the whole for any* $\mu_i \in (0, \widetilde{\mu}^*_i)$ *and for* $\mu_i \to 0$ *on* $\mathcal{S}$*.*

In these relations $\widetilde{\mu}_{i1}$ *and* $\widetilde{\mu}_{i2}$ *are determined by the conditions of matrix* $A(M)$ *positive definiteness and matrix* $\widetilde{G}(M)$ *negative definiteness respectively.*

This theorem is proved by the same method as Theorem 5.3.1.

5.6.1.2 Uniform time scaling In the case of uniform time scaling system (5.6.1) is (see Grujić *et al.* [1])

$$
\begin{aligned}
\frac{dx_i}{dt} &= A_i x_i + \sum_{\alpha=1}^{r} S_{i\alpha}^{(1)} A_{i\alpha} y_\alpha + q_{i1} f_{i1}(\sigma_{i1}),\\
\sigma_{i1} &= \widehat{c}_{i1}^{\mathrm{T}} x + \widehat{c}_{i2}^{\mathrm{T}} y, \quad \forall\, i = 1, 2, \ldots, q;\\
\mu_i \frac{dy_i}{dt} &= \sum_{\beta=1}^{q} S_{q+i}^{(1)} B_{i\beta} x_\beta + \tau_i B_i y_i\\
&\quad + \tau_i q_{i2} f_{i2}(\sigma_{i2}) + \tau_i q_{i3} f_{i3}(\sigma_{i3}),\\
\sigma_{i2} &= \mu_i \widehat{c}_{i3}^{\mathrm{T}} + \widehat{c}_{i4}^{\mathrm{T}} y_i,\\
\sigma_{i3} &= \sum_{\beta=1}^{q} \mu_i \widehat{c}_{\beta 5}^{\mathrm{T}} S_{q+i,\beta}^{(2)} x_\beta + \sum_{\alpha=1}^{r} \widehat{c}_{\alpha 6}^{\mathrm{T}} S_{q+i,\alpha}^{(3)} y_\alpha\\
&\qquad \forall\, i = 1, 2, \ldots, r, \quad q + r = s,
\end{aligned}
\tag{5.6.8}
$$

where

$$
\sigma_{ij}^{-1} f_{ij}(\sigma_{ij}) \in [0,\, k_{ij}] \subset R_+, \qquad \begin{cases} i = 1, 2, \ldots, q \ \text{ when } \ j = 1, \\ i = 1, 2, \ldots, r \ \text{ when } \ j = 2, 3. \end{cases}
$$

The structural matrices $S_{ij}^{(1)}$, $S_{ij}^{(2)}$, $S_{ij}^{(3)}$, and S, and the set $\mathcal{S}$ are determined as in Section 5.5.1, $\tau_i \in [\underline{\tau}_i, \overline{\tau}_i]$, , the numbers $\underline{\tau}_i$ and $\overline{\tau}_i$ are given.

For the analysis of asymptotic stability of large scale system of Lur'e type (5.6.8) we construct matrix-function (5.3.7) with elements (5.5.11) satisfying estimates (5.5.12). For function (5.5.13) the bilateral estimate

$$(5.6.9)\qquad \begin{gathered} u^{\mathrm T}A(\mu_1)u \le V(x,y,\mu_1) \le u^{\mathrm T}B(\mu_1)u,\\ \forall\,(x_i,y_j,\mu_1)\in \mathcal N_{ix_0}\times \mathcal N_{jy_0}\times \mathcal M,\quad \forall\,\tau_i\in[\underline{\tau}_i,\overline{\tau}_i], \end{gathered}$$

is true, where u, $A(\mu_1)$ and $B(\mu_1)$ are determined as in Section 5.3.2.

Assume that $\eta^{\mathrm T}=(1,1,\dots,1,1)\in R^s_+$, $s=q+r$.

Proposition 5.6.3 *If for system (5.6.8) the matrix function (5.3.7) with the elements (5.6.9) is constructed, then for the total derivative of function (5.3.8) by virtue of system (5.6.8)*

$$(5.6.10)\qquad \begin{gathered} DV(x,y,\mu_1)\le u^{\mathrm T}C^*u+\mu_1u^{\mathrm T}G^*u,\\ \forall\,(x,y,\mu_1,S)\in \mathcal N_{x_0}\times \mathcal N_{y_0}\times \mathcal M\times \mathcal S, \end{gathered}$$

where

$$\begin{aligned}
u^{\mathrm T}&=(\|x_1\|,\,\|x_2\|,\,\dots,\,\|x_q\|,\,\|y_1,\,\|y_2\|,\,\dots,\,\|y_r\|);\\
C^*&=[c^*_{ij}],\quad c^*_{ij}=c^*_{ji},\quad i,j=1,2,\dots,s;\\
G^*&=[\sigma^*_{ij}],\quad \sigma^*_{ij}=\sigma^*_{ji},\quad i,j=1,2,\dots,s;\\
c^*_{ii}&=\lambda_M(c_{ii}),\\
c_{ii}&=P_{ii}A_i+A_i^{\mathrm T}P_{ii}+P_{ii}q_{i1}k^*_{i1}(\widehat c^{\,i}_{i1})^{\mathrm T}+(q_{i1}k^*_{i1}(\widehat c^{\,i}_{i1})^{\mathrm T})^{\mathrm T}P_{ii}\\
&\quad+2\sum_{\substack{p=2\\p>i}}^{q}(Pq_{p1}k^*_{p1}(c^p_{p1})^{\mathrm T}(q_{p1}k^*_{p1}(c^p_{p1})^{\mathrm T})^{\mathrm T}P^{\mathrm T}_{ip},\\
&\quad i=1,2,\dots,q;\\
c^*_{q+j,q+j}&=\lambda_M(C_{q+j,q+j}(\tau^*_j,\overline S)),\\
C_{q+j,q+j}(\tau^*_j,\overline S)&=P_{q+j,q+j}\tau^*_jB_j+\tau^*_jB_j^{\mathrm T}P_{q+j,q+j}+P_{q+j,q+j}\tau^*_jq_{j2}k^*_{j2}(\widehat c_{j4})^{\mathrm T}\\
&\quad+(\tau^*_jq_{j2}k^*_{j2}(\widehat c_{j4})^{\mathrm T})^{\mathrm T}P_{q+j,q+j}+P_{q+j,q+j}\tau^*_jq_{j3}k^*_{j3}(\widehat c_{j6})^{\mathrm T}\overline S^{(3)}_{q+j,j}\\
&\quad+(\tau^*_jq_{j3}k^*_{j3}(\widehat c_{j6})^{\mathrm T}\overline S^{(3)}_{q+j,j})^{\mathrm T}P_{q+j,q+j}\\
&\quad+\sum_{\substack{l=2\\l>j}}^{r}\Big\{P_{q+j,q+l}\tau^*_lq_{l3}k^*_{l3}(\widehat c_{j6})^{\mathrm T}\overline S^{(3)}_{q+l,j}\\
&\qquad+(\tau^*_lq_{l3}k^*_{l3}(\widehat c_{j6})^{\mathrm T}\overline S^{(3)}_{q+l,j})^{\mathrm T}P^{\mathrm T}_{q+j,q+l}\\
&\qquad+(\tau^*_jq_{j3}k^*_{j3}(\widehat c_{l6})^{\mathrm T}\overline S^{(3)}_{q+j,l})^{\mathrm T}P_{q+j,q+l}\\
&\qquad+P^{\mathrm T}_{q+j,q+l}\tau^*_jq_{j3}k^*_{j3}(\widehat c_{l6})^{\mathrm T}\overline S^{(3)}_{q+j,l}\Big\},\quad j=1,2,\dots,r;\\
c^*_{ip}&=\lambda_M^{1/2}(C^{\mathrm T}_{ip}C_{ip}),\\
C_{ip}&=P_{ii}q_{i1}k^*_{i1}(\widehat c^{\,p}_{i1})^{\mathrm T}+(q_{p1}k^*_{p1}(\widehat c^{\,i}_{p1})^{\mathrm T})^{\mathrm T}P_{pp}+A_i^{\mathrm T}P_{ip}+P_{ip}A_p\\
&\quad+\sum_{\beta=1}^{i}P^{\mathrm T}_{\beta p}q_{\beta1}k^*_{\beta1}(\widehat c^{\,p}_{\beta1})^{\mathrm T}+\sum_{\substack{\beta=i+1\\\beta\ne p}}^{q}(q_{\beta1}k^*_{\beta1}(\widehat c^{\,p}_{\beta1})^{\mathrm T})^{\mathrm T}P_{\beta p}\\
&\quad+\sum_{\substack{\beta=1\\\beta\ne i}}^{p}P_{i\beta}q_{\beta1}k^*_{\beta1}(\widehat c^{\,i}_{\beta1})^{\mathrm T}+\sum_{\beta=p+1}^{q}(q_{\beta1}k^*_{\beta1}(\widehat c^{\,i}_{\beta1})^{\mathrm T})^{\mathrm T}P^{\mathrm T}_{i\beta},
\end{aligned}$$

$$i,\, p = 1, 2, \ldots, q, \quad p > i;$$

$$c^*_{q+j,q+l} = \lambda_M^{1/2}(C^{\mathrm{T}}_{q+j,q+l}(\tau^*_j, \overline{S})\, C_{q+j,q+l}(\tau^*_j, \overline{S})),$$

$$\begin{aligned} C_{q+j,q+l}(\tau^*_j, \overline{S}) =& P_{q+j,q+l}\tau^*_j q_{j3} k^*_{j3}(\widehat{c}_{l6})^{\mathrm{T}} \overline{S}^{(3)}_{q+j,l} \\ &+ (\tau^*_l q_{l3} k^*_{l3}(\widehat{c}_{j6})^{\mathrm{T}} \overline{S}^{(3)}_{q+l,j})^{\mathrm{T}} P_{q+l,q+j} \\ &+ P_{q+j,q+l}\tau^*_l B_l + \tau^*_j B^{\mathrm{T}}_j P_{q+j,q+l} \\ &+ P_{q+j,q+l}\tau^*_l q_{l2} k^*_{l2}(\widehat{c}_{l4})^{\mathrm{T}} + (\tau^*_j q_{j2} k^*_{j2}(\widehat{c}_{j4})^{\mathrm{T}})^{\mathrm{T}} P_{q+j,q+l} \\ &+ \sum_{\alpha=1}^{j} P^{\mathrm{T}}_{q+\alpha,q+l}\tau^*_\alpha q_{\alpha 3} k^*_{\alpha 3}(\widehat{c}_{j6})^{\mathrm{T}} \overline{S}^{(3)}_{q+\alpha,j} \\ &+ \sum_{\substack{\alpha=j+1 \\ \alpha\neq l}}^{r} (\tau^*_\alpha q_{\alpha 3} k^*_{\alpha 3}(\widehat{c}_{j6})^{\mathrm{T}} \overline{S}^{(3)}_{q+\alpha,j})^{\mathrm{T}} P_{q+\alpha,q+l} \\ &+ \sum_{\substack{\alpha=1 \\ \alpha\neq j}}^{l} P_{q+j,q+\alpha}\tau^*_\alpha q_{\alpha 3} k^*_{\alpha 3}(\widehat{c}_{l6})^{\mathrm{T}} \overline{S}^{(3)}_{q+\alpha,l} \\ &+ \sum_{\alpha=l+1}^{r} (\tau^*_\alpha q_{\alpha 3} k^*_{\alpha 3}(\widehat{c}_{l6})^{\mathrm{T}} \overline{S}^{(3)}_{q+\alpha,l})^{\mathrm{T}} P_{q+j,q+\alpha}, \end{aligned}$$

$$j,\, l = 1, 2, \ldots, r, \quad l > j;$$

$$c^*_{i,q+j} = \lambda_M^{1/2}(C^{\mathrm{T}}_{i,q+j}(\tau^*_j, \overline{S})\, C_{i,q+j}(\tau^*_j, \overline{S}),$$

$$\begin{aligned} C_{i,q+j}(\tau^*_j, \overline{S}) =& P_{ii}\overline{S}^{(1)}_{ij} A_{ij} + P_{ii} q_{i1} k^*_{i1}(\widehat{c}^{\,j}_{i2})^{\mathrm{T}} \\ &+ \sum_{\substack{p=2 \\ p>i}}^{q} \{P_{ip}\overline{S}^{(1)}_{pj} A_{pj} + P_{ip} q_{p1} k^*_{p1}(\widehat{c}^{\,j}_{p2})^{\mathrm{T}}\} \\ &+ P_{i,q+j}\tau^*_j B_j + P_{i,q+j}\tau^*_j q_{j2} k^*_{j2}(\widehat{c}_{j4})^{\mathrm{T}} \\ &+ \sum_{\alpha=1}^{r} P_{i,q+\alpha}\tau^*_\alpha q_{\alpha 3} k^*_{\alpha 3}(\widehat{c}_{j6})^{\mathrm{T}} \overline{S}^{(3)}_{q+\alpha,j}, \end{aligned}$$

$$i = 1, 2, \ldots, q, \quad j = 1, 2, \ldots, r, \quad q + r = s;$$

$$\sigma^*_{ii} = \lambda_M(\sigma_{ii}(\overline{S})),$$

$$\begin{aligned} \sigma_{ii}(\overline{S}) =& P_{i,q+i} q_{i2} k^*_{i2}(\widehat{c}_{i3})^{\mathrm{T}} + (q_{i2} k^*_{i2}(\widehat{c}_{i3})^{\mathrm{T}})^{\mathrm{T}} P_{i,q+i} \\ &+ \sum_{j=1}^{r} \{P_{i,q+i} q_{j3} k^*_{j3}(\widehat{c}_{i5})^{\mathrm{T}} \overline{S}^{(2)}_{q+j,i} + (q_{j3} k^*_{j3}(\widehat{c}_{j5})^{\mathrm{T}} \overline{S}^{(2)}_{q+j,i})^{\mathrm{T}} P^{\mathrm{T}}_{i,q+j}\}, \end{aligned}$$

$$i = 1, 2, \ldots, q;$$

$$\sigma^*_{q+j,q+j} = \lambda_M(\sigma_{q+j,q+j}(\overline{S})),$$

$$\begin{aligned} \sigma_{q+j,q+j}(\overline{S}) = \sum_{i=1}^{q} \Big\{ & (\overline{S}^{(1)}_{ij} A_{ij})^{\mathrm{T}} P_{i,q+j} + P_{i,q+j}\overline{S}^{(1)}_{ij} A_{ij} \\ &+ (q_{i1} k^*_{i1}(\widehat{c}^{\,j}_{i2})^{\mathrm{T}})^{\mathrm{T}} P_{i,q+j} + P^{\mathrm{T}}_{i,q+j} q_{i1} k^*_{i1}(\widehat{c}^{\,j}_{i2})^{\mathrm{T}} \Big\}, \end{aligned}$$

$$j = 1, 2, \ldots, r;$$

$$\sigma_{ip}^* = \lambda_M^{1/2}(\sigma_{ip}^{\mathrm{T}}(\overline{S})\,\sigma_{ip}(\overline{S})),$$

$$\begin{aligned}\sigma_{ip}(\overline{S}) =& P_{i,q+p}q_{p2}k_{p2}^*(\widehat{c}_{p3})^{\mathrm{T}} + (q_{i2}k_{i2}^*(\widehat{c}_{i3})^{\mathrm{T}})^{\mathrm{T}}P_{p,q+i} \\ &+ \sum_{j=1}^{r}\Big\{P_{i,q+j}q_{j3}k_{j3}^*(\widehat{c}_{p5})^{\mathrm{T}}\overline{S}^{(2)}_{q+j,p} + (q_{j3}k_{j3}^*(\widehat{c}_{i5})^{\mathrm{T}}\overline{S}^{(2)}_{q+j,i})^{\mathrm{T}}P_{p,q+j}\Big\},\end{aligned}$$

$$i,\, p = 1, 2, \dots, q, \quad p > i;$$

$$\sigma_{q+j,q+l}^* = \lambda_M^{1/2}(\sigma_{q+j,q+l}^{\mathrm{T}}(\overline{S})\,\sigma_{q+j,q+l}(\overline{S})),$$

$$\begin{aligned}\sigma_{q+j,q+l}(\overline{S}) =& \sum_{i=1}^{q}\Big\{\left(\overline{S}_{ij}^{(1)}A_{ij}\right)^{\mathrm{T}}P_{i,q+l} + P_{i,q+j}^{\mathrm{T}}\left(\overline{S}_{il}^{(1)}A_{il}\right)^{\mathrm{T}} \\ &+ (q_{i1}k_{i1}^*(\widehat{c}_{i2}^{\,j})^{\mathrm{T}})^{\mathrm{T}}P_{i,q+l} + P_{i,q+j}^{\mathrm{T}}q_{i1}k_{i1}^*(\widehat{c}_{i2}^{\,l})^{\mathrm{T}}\Big\},\end{aligned}$$

$$j,\, l = 1, 2, \dots, r, \quad l > j;$$

$$\sigma_{i,q+j}^* = \lambda_M^{1/2}(\sigma_{i,q+j}^{\mathrm{T}}(\overline{S})\,\sigma_{i,q+j}(\overline{S})),$$

$$\begin{aligned}\sigma_{i,q+j}(\overline{S}) =& \left(\overline{S}^{(1)}_{q+j,i}B_{ji}\right)^{\mathrm{T}}P_{q+j,q+j} + \left(q_{j3}k_{j3}^*(\widehat{c}_{i5})^{\mathrm{T}}\overline{S}^{(2)}_{q+j,i}\right)^{\mathrm{T}}P_{q+j,q+j} \\ &+ \sum_{\substack{l=2\\ l>j}}^{r}\Big\{\left(\overline{S}^{(1)}_{q+l,i}B_{li}\right)^{\mathrm{T}}P_{q+j,q+l}^{\mathrm{T}} + \left(\overline{S}^{(1)}_{q+l,i}B_{li}\right)^{\mathrm{T}}P_{q+j,q+l} \\ &+ \left(q_{l3}k_{l3}^*(\widehat{c}_{i5})^{\mathrm{T}}\overline{S}^{(2)}_{q+l,i}\right)^{\mathrm{T}}P_{q+j,q+l}^{\mathrm{T}} + \left(q_{l3}k_{l3}^*(\widehat{c}_{i5})^{\mathrm{T}}\overline{S}^{(2)}_{q+l,i}\right)^{\mathrm{T}}P_{q+j,q+l}\end{aligned}$$

$$+\,\delta^* q_{l2}k^*_{l2}(\widehat{c}_{l3})^{\mathrm T}P^{\mathrm T}_{q+j,q+l}+\delta^*\big(q_{l2}k^*_{l2}(\widehat{c}_{l3})^{\mathrm T}\big)^{\mathrm T}P_{q+j,q+l}\Big\}$$

$$+\,A_i^{\mathrm T}P_{i,q+j}+\sum_{\beta=1}^{q}\big(q_{\beta 1}k^*_{\beta 1}(\widehat{c}^{\,i}_{be1})^{\mathrm T}\big)^{\mathrm T}P_{\beta,q+j},$$

$$i=1,2,\dots,q,\quad j=1,2,\dots,r,\quad q+r=s;$$

$$\delta^*=\begin{cases}1 & for\ \ l=i,\\ 0 & for\ \ l\neq i.\end{cases}$$

Here k^*_{ij}, $i\neq j$, and k^*_{ii} are determined as in Section 5.6.1.2 of this Chapter, τ_j is determined as in Section 5.2.2, c^k_{ij} is the k–th component of vector c_{ij}, $\overline{S}\in\mathcal{S}$ is a constant matrix such that

$$C_{q+j,q+l}(\tau^*_j,S)\le C_{q+j,q+l}(\tau^*_j,\overline{S}),\quad \forall\,S\in\mathcal{S},\quad j,l=1,2,\dots,r;$$

$$C_{i,q+j}(\tau^*_j,S)\le C_{i,q+j}(\tau^*_j,\overline{S}),\quad \forall\,S\in\mathcal{S},\quad i=1,2,\dots,q,\quad j=1,2,\dots,r;$$

$$\sigma_{ij}(S)\le\sigma_{ij}(\overline{S}),\quad \forall\,S\in\mathcal{S},\quad i,j=1,2,\dots,s=q+r.$$

Proposition 5.6.3 is proved by the immediate calculation of total derivatives of functions (5.6.9) by virtue of system (5.6.8) with their subsequent estimation from above.

Corollary 5.6.1 If in Proposition 5.6.3 the matrix C^* is negative definite, i.e. $\lambda_M(C^*)<0$ and

(a) $\lambda_M(G)>0$

or

(b) $\lambda_M(G)\ge 0$,

then the function $DV(x,y,\mu_1)$ is negative definite

(a) for any $\mu_1\in(0,\widehat{\mu}_1)$ and for $\mu_1\to 0$, where

$$\widehat{\mu}_1=\min\{1,\,-\lambda_M(C^*)/\lambda_M(G)\},$$

or

(b) for any $\mu_1\in(0,1]$ and for $\mu_1\to 0$ respectively.

Theorem 5.6.2 *Let the equations of singularly perturbed LSS of Lur'e type (5.6.8) be such that for the system the matrix-function (5.3.7) is constructed with the elements (5.6.9) satisfying estimates (5.6.10) and for the total derivative of function (5.3.3) by virtue of system (5.6.8) the correlation (5.6.12) is satisfied and*

(1) *matrices* A^*_{11} *and* A^*_{22} *are positive definite;*
(2) *matrix* C^* *is negative definite;*
(3) $\mu_1\in(0,\widetilde{\mu}_1)$, $\mu_i=\tau_i^{-1}\mu_1$, $i=1,2,\dots,r$, $\tau_i\in[\underline{\tau}_i,\overline{\tau}_i]$, $\mu_1=\min\{1,\mu^*_1,\widehat{\mu}_1\}$.

Then the equilibrium state $(x^{\mathrm T},y^{\mathrm T})^{\mathrm T}=0$ *of system (5.6.8) is uniformly asymptotically stable on* $\widetilde{\mathcal{M}}\times\mathcal{S}$*, , where*

$$\widetilde{\mathcal{M}}=\{M\colon\ 0<\mu_1<\widetilde{\mu}_1\},\quad \mu_i=\tau_i^{-1}\mu_1,\quad i=1,2,\dots,r.$$

If all conditions of Theorem 5.6.2 are satisfied for $\mathcal{N}_{ix} \times \mathcal{N}_{jy} = R^{n_i \times n_j}$, *then the equilibrium state* $(x^{\mathrm{T}}, y^{\mathrm{T}})^{\mathrm{T}} = 0$ *of system (5.6.8) is uniformly asymptotically stable in the whole on* $\widetilde{\mathcal{M}} \times \mathcal{S}$.

Proof The assertion of this theorem follows from Theorem 5.3.2.

Remark 5.6.1 If $\lambda_M(G) \le 0$, then assertion of Theorem 5.6.2 remains valid for $\widetilde{\mu}_1 = \min\{1, \mu_1^*\}$.

Example 5.6.1 Let system (5.6.1) be the 12-th order system $(n = m = 6)$ of Lur'e type decomposed into three interconnected singularly perturbed subsystems $(q = r = 3)$ determined by the vectors and matrices (see [Grujic et al. [1])

$$A_i = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \quad A_{ii} = J, \quad A_{ij} = \gamma J, \quad i \ne j, \quad \gamma = \frac{1}{2000},$$

$$q_{i1} = \begin{pmatrix} 0 \\ 0.1 \end{pmatrix}, \quad c_{i1}^i = \begin{pmatrix} -0.01 \\ 0 \end{pmatrix}, \quad c_{i2}^i = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$c_{i1}^j = \begin{pmatrix} \gamma \\ 0 \end{pmatrix}, \quad c_{i2}^j = \begin{pmatrix} 0 \\ \gamma \end{pmatrix}, \quad i \ne j, \quad k_{i1} = 2,$$

$$B_i = \begin{pmatrix} -4 & 1 \\ 1 & -4 \end{pmatrix}, \quad B_{ii} = 10^{-3} J, \quad B_{ij} = \gamma J, \quad i \ne j,$$

$$q_{i2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad q_{i3} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad c_{i3} = \begin{pmatrix} 10^{-3} \\ 0 \end{pmatrix}, \quad c_{i4} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$c_{j5} = \begin{pmatrix} 0 \\ \gamma \end{pmatrix}, \quad c_{j6} = \begin{pmatrix} \gamma \\ 0 \end{pmatrix}, \quad k_{i2} = k_{i3} = 1.$$

The elements of the matrix function (5.3.2) are taken as

$$v_{ii}(x_i) = x_i^{\mathrm{T}} \begin{pmatrix} 0.3 & 0.1 \\ 0.1 & 0.3 \end{pmatrix} x_i; \quad v_{3+j,3+j}(y_j, \mu_j) = \mu_j y_j^{\mathrm{T}} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} y_j;$$
$$i, j = 1, 2, 3;$$
$$v_{ip}(x_i, x_p) = x_i^{\mathrm{T}} \begin{pmatrix} 0.01 & 0 \\ 0 & 0.01 \end{pmatrix} x_p, \quad i, p = 1, 2, 3, \quad p > i;$$
$$v_{3+j,3+l}(y_j, y_l, M) = \mu_i \mu_j y_j^{\mathrm{T}} \begin{pmatrix} 0.01 & 0 \\ 0 & 0.01 \end{pmatrix} y_l, \quad j, l = 1, 2, 3, \quad l > j;$$
$$v_{i,q+j}(x_i, y_j, \mu_j) = \mu_j x_i^{\mathrm{T}} \begin{pmatrix} 0.01 & 0 \\ 0 & 0.01 \end{pmatrix} y_j, \quad i, j = 1, 2, 3.$$

For the constructed functions

$$v_{ii}(x_i) \ge 0.2\,\|x_i\|^2, \quad i = 1, 2, 3;$$
$$v_{3+j,3+j}(y_j, \mu_j) \ge 3\mu_j\,\|y_j\|^2, \quad j = 1, 2, 3;$$
$$v_{ip}(x_i, x_p) = v_{pi}(x_i, x_p) \ge -0.01\,\|x_i\|\,\|x_p\|, \quad i, p = 1, 2, 3, \quad p > i;$$
$$v_{3+j,3+l}(y_j, y_l, M) = v_{3+l,3+j}(y_j, y_l, M) \ge -0.01\mu_j\mu_l\,\|y_j\|\,\|y_l\|,$$
$$j, l = 1, 2, 3, \quad l > j;$$
$$v_{i,3+j}(x_i, y_j, \mu_j) \ge -0.01\mu_j\,\|x_i\|\,\|y_j\|, \quad i, j = 1, 2, 3.$$

The matrix

$$A_1(M) = \begin{pmatrix} A_{11} & A_{12}(M) \\ A_{12}^{\mathrm{T}}(M) & A_{22}(M) \end{pmatrix},$$

where

$$A_{11} = \begin{pmatrix} 0.2 & -0.01 & -0.01 \\ -0.01 & 0.2 & -0.01 \\ -0.01 & -0.01 & 0.2 \end{pmatrix},$$

$$A_{12}(M) = \begin{pmatrix} -0.01\mu_1 & -0.01\mu_2 & -0.01\mu_3 \\ -0.01\mu_1 & -0.01\mu_2 & -0.01\mu_3 \\ -0.01\mu_1 & -0.01\mu_2 & -0.01\mu_3 \end{pmatrix},$$

$$A_{22}(M) = \begin{pmatrix} 2\mu_1 & -0.01\mu_1\mu_2 & -0.01\mu_1\mu_3 \\ -0.01\mu_1\mu_2 & 2\mu_2 & -0.01\mu_2\mu_3 \\ -0.01\mu_1\mu_3 & -0.01\mu_2\mu_3 & 2\mu_3 \end{pmatrix},$$

is positive definite for $\mu_j \in (0,\, 1]$ and for $\mu_j \to 0,\ j = 1, 2, 3$.

For such choice of the elements of matrix-function (5.3.2) we have

$$\rho_{1i} = \begin{cases} -0.15383688 & \text{for} \quad k_{i1}^* = 2, \\ -0.15278641 & \text{for} \quad k_{i1}^* = 0, \end{cases}$$

$$\rho_{2i} = \begin{cases} -8.917237 & \text{for} \quad k_{i2}^* = 1, \\ -12 & \text{for} \quad k_{i2}^* = 0, \end{cases}$$

$$\rho_{3i} = \begin{cases} 0.00171 & \text{for} \quad k_{i1}^* = 2, \\ 0 & \text{for} \quad k_{i1}^* = 0, \end{cases}$$

$$\rho_{4i} = \begin{cases} 2.309017 \cdot 10^{-5} & \text{for} \quad k_{i2}^* = 1, \\ 10^{-5} & \text{for} \quad k_{i2}^* = 0, \end{cases}$$

$$\rho_{5i} = \begin{cases} 3.2360679 & \text{for} \quad k_{i2}^* = 1, \\ 0 & \text{for} \quad k_{i2}^* = 0, \end{cases}$$

$$\rho_{6i} = \begin{cases} 1.4828427 \cdot 10^{-2} & \text{for} \quad k_{i1}^* = 2, \\ 10^{-2} & \text{for} \quad k_{i1}^* = 0, \end{cases}$$

$$\rho_{7i} = \begin{cases} 0.46264281 & \text{for} \quad k_{i1}^* = 2, \quad k_{i2}^* = 1, \\ 0.38015581 & \text{for} \quad k_{i1}^* = 0, \quad k_{i2}^* = 1, \\ 0.45199337 & \text{for} \quad k_{i1}^* = 2, \quad k_{i2}^* = 0, \\ 0.37 & \text{for} \quad k_{i1}^* = k_{i2}^* = 0, \end{cases}$$

$$\rho_{8i} = \begin{cases} 0.02407149 & \text{for} \quad k_{i1}^* = 2, \quad k_{i2}^* = 1, \\ 0.02279776 & \text{for} \quad k_{i1}^* = 2, \quad k_{i2}^* = 0, \\ 0.02408377 & \text{for} \quad k_{i1}^* = 0, \quad k_{i2}^* = 1, \\ 0.02280625 & \text{for} \quad k_{i1}^* = k_{i2}^* = 0, \end{cases}$$

$$\rho_{91} = \begin{cases} 0.02497321 & \text{for} \quad k_{11}^* = 2, \\ 0 & \text{for} \quad k_{11}^* = 0, \end{cases}$$

$$\rho_{92} = \begin{cases} 0.0149 & \text{for} \quad k_{21}^* = 2, \\ 0 & \text{for} \quad k_{21}^* = 0, \end{cases}$$

$$\rho_{93} = \begin{cases} 0.00828427 & \text{for} \quad k_{31}^* = 2, \\ 0 & \text{for} \quad k_{31}^* = 0, \end{cases}$$

$$\rho_{10,i}(S^*) = \begin{cases} 4 \cdot 10^{-5} & \text{for} \quad k_{i3}^* = 1, \\ 0 & \text{for} \quad k_{i3}^* = 0, \end{cases}$$

$$\rho_{11,i}(S^*) = \begin{cases} 4.8626044 \cdot 10^{-3} & \text{for} \quad k_{i1}^* = 2, \quad k_{i3}^* = 1, \\ 3.2360679 \cdot 10^{-3} & \text{for} \quad k_{i1}^* = 2, \quad k_{i3}^* = 0, \\ 4 \cdot 10^{-5} & \text{for} \quad k_{i1}^* = 0, \quad k_{i3}^* = 1, \\ 0 & \text{for} \quad k_{i1}^* = k_{i3}^* = 0, \end{cases}$$

$$\rho_{1ij}(S^*) = \begin{cases} 0.020000022 & \text{for} \quad k_{i1}^* = 2, \\ 0 & \text{for} \quad k_{i1}^* = 0, \end{cases}$$

$$\rho_{2ij}(S^*) = \begin{cases} 3 \cdot 10^{-5} & \text{for} \quad k_{i3}^* = 1, \\ 1.5 \cdot 10^{-5} & \text{for} \quad k_{i3}^* = 0, \end{cases}$$

$$\rho_{3ij}(S^*) = \begin{cases} 10^{-6} & \text{for} \quad k_{i1}^* = 2, \\ 0 & \text{for} \quad k_{i1}^* = 0, \end{cases}$$

$$\rho_{4ij}(S^*) = \begin{cases} 0.05121 & \text{for} \quad k_{i2}^* = k_{i3}^* = 1, \\ 0.0499996 & \text{for} \quad k_{i2}^* = 0, \quad k_{i3}^* = 1, \\ 0.051161 & \text{for} \quad k_{i2}^* = 1, \quad k_{i3}^* = 0, \\ 0.049949 & \text{for} \quad k_{i2}^* = k_{i3}^* = 0, \end{cases}$$

$$\rho_{5ij}(S^*) = \begin{cases} 0.064213 & \text{for} \quad k^*_{i1} = k^*_{i2} = k^*_{i3} = 1, \\ 0.0614868 & \text{for} \quad k^*_{i1} = 0, \quad k^*_{i2} = k^*_{i3} = 1, \\ 0.050653 & \text{for} \quad k^*_{i1} = 2, \quad k^*_{i2} = 0, \quad k^*_{i3} = 1, \\ 0.06158 & \text{for} \quad k^*_{i1} = 2, \quad k^*_{i2} = 1, \quad k^*_{i3} = 0, \\ 0.050612 & \text{for} \quad k^*_{i1} = k^*_{i2} = 0, \quad k^*_{i3} = 1, \\ 0.0611818 & \text{for} \quad k^*_{i1} = 0, \quad k^*_{i2} = 1, \quad k^*_{i3} = 0, \\ 0.050414 & \text{for} \quad k^*_{i1} = 2, \quad k^*_{i2} = k^*_{i3} = 0, \\ 0.050015 & \text{for} \quad k^*_{i1} = k^*_{i2} = k^*_{i3} = 0, \end{cases}$$

$$\rho_{6ij}(S^*) = \begin{cases} 0.04989 & \text{for} \quad k^*_{i1} = k^*_{i2} = k^*_{i3} = 0, \\ 0.04995 & \text{for} \quad k^*_{i1} = 2, \quad k^*_{i2} = k^*_{i3} = 0, \\ 0.049960962 & \text{for} \quad k^*_{i1} = 0, \quad k^*_{i2} = 1, \quad k^*_{i3} = 0, \\ 0.049891 & \text{for} \quad k^*_{i1} = k^*_{i2} = 0, \quad k^*_{i3} = 1, \\ 0.04998 & \text{for} \quad k^*_{i1} = 2, \quad k^*_{i2} = 1, \quad k^*_{i3} = 0, \\ 0.04997 & \text{for} \quad k^*_{i1} = 2, \quad k^*_{i2} = 0, \quad k^*_{i3} = 1, \\ 0.049965 & \text{for} \quad k^*_{i1} = 0, \quad k^*_{i2} = k^*_{i3} = 1, \\ 0.04999 & \text{for} \quad k^*_{i1} = 2, \quad k^*_{i2} = k^*_{i3} = 1, \end{cases}$$

$$\rho_{7ij}(S^*) = \begin{cases} 0.021251 & \text{for} \quad k^*_{i1} = 2, \quad k^*_{i3} = 1, \\ 0.020212 & \text{for} \quad k^*_{i1} = 0, \quad k^*_{i3} = 1, \\ 0.023521 & \text{for} \quad k^*_{i1} = 2, \quad k^*_{i3} = 0, \\ 0.023454 & \text{for} \quad k^*_{i1} = k^*_{i3} = 0, \end{cases}$$

$$\rho_{8ij}(S^*) = \begin{cases} 0.0018061 & \text{for} \quad k^*_{i2} = k^*_{i3} = 1, \\ 0.001798 & \text{for} \quad k^*_{i2} = 0, \quad k^*_{i3} = 1, \\ 0.001124 & \text{for} \quad k^*_{i2} = 1, \quad k^*_{i3} = 0, \\ 0.0011 & \text{for} \quad k^*_{i2} = k^*_{i3} = 0. \end{cases}$$

Let $\eta = (1,1,1,1,1,1)$, then the elements of the matrix $\widetilde{G}(M)$ are determined as

$$\widetilde{c}_{ii} = -0.1261032, \quad i = 1,2,3; \quad \widetilde{c}_{ij} = \widetilde{c}_{ji} = 1012, \quad i,j = 1,2,3, \quad i \neq j;$$

$$\widetilde{c}_{3+i,3+i} = -5.676308, \quad i = 1,2,3;$$

$$\widetilde{c}_{3+i,3+j} = \widetilde{c}_{3+j,3+i} = 10^{-6}, \quad i,j = 1,2,3, \quad i \neq j;$$

$$\widetilde{c}_{i,3+i} = 0.4626428, \quad i = 1,2,3; \quad \widetilde{c}_{i,q+j} = 0.04999, \quad i,j = 1,2,3, \quad i \neq j;$$

$$\widetilde{\sigma}_{ii} = 6.309017 \cdot 10^{-5}\mu_i, \quad i = 1,2,3;$$

$$\widetilde{\sigma}_{ij} = \widetilde{\sigma}_{ji} = 3 \cdot 10^{-5}\mu_i, \quad i,j = 1,2,3, \quad i \neq j;$$

$$\widetilde{\sigma}_{3+i,3+i} = 0.0148284\mu_i, \quad i = 1,2,3;$$

$$\widetilde{\sigma}_{3+i,3+j} = \widetilde{\sigma}_{3+j,3+i} = 0.051161\mu_i + 0.064213\mu_j, \quad i,j = 1,2,3, \quad i \neq j;$$

$$\widetilde{\sigma}_{i,q+i} = 0.02408377\mu_i, \quad i = 1,2,3;$$

$$\widetilde{\sigma}_{i,q+j} = 0.023521\mu_i + 0.0018061\mu_i\mu_j, \quad i,j = 1,2,3, \quad i \neq j.$$

For such definition of the elements the matrix $\widetilde{G}(M)$ is negative definite for $\mu_j \in (0,\,1]$ and for $\mu_j \to 0$, $j = 1,2,3$.

By Theorem 5.6.2 the equilibrium state $(x^{\mathrm{T}}, y^{\mathrm{T}})^{\mathrm{T}} = 0 \in R^{12}$ of the system specified in this example, is absolutely stable on $[0, K] \times \widehat{\mathcal{M}} \times \mathcal{S}$, where

$$K = \operatorname{diag}\{2, 1, 1, 2, 1, 1, 2, 1, 1\},$$
$$\widehat{\mathcal{M}} = \{M\colon\ M = \operatorname{diag}\{\mu_1, \mu_2, \mu_3\},\ \ \mu_j \in (0,\, 1],\ \ \forall\, j = 1, 2, 3\}.$$

Remark 5.6.3 In monograph by Grujic et al.[1] it was shown that the equilibrium state $(x^{\mathrm{T}}, y^{\mathrm{T}})^{\mathrm{T}} = 0 \in R^{12}$ of system (5.6.1) is absolutely stable under nonclassical structural perturbations for $\mu_j \in (0,\, 0.447]$, $j = 1, 2, 3$. The application of the matrix-valued function extends the domain of the admissible values of the parameters $\mu_j, j = 1, 2, 3$, for which stability under nonclassical structural perturbations occurs.

5.6.2 Stabilization of an orbital apparatus The objective of the present study is to apply the method of Liapunov's matrix functions to derive new stability conditions for a spacecraft (SC), which is oriented in inertial space by a control system with executive devices in the form of three gyroscopic frames.

5.6.2.1 Mathematical model of the system It is assumed that a spacecraft represents a solid with principal central moments of inertia J_1, J_2 and J_3, the precession axes of the gyroscopic frames are directed along the principal axes of the spacecraft, the elements of the gyrostabilizer (GS) are perfectly rigid, and the gyroscopes of each pair are identical and have constant speeds of self-rotation. Let γ_i be the airborne angles determining the orientation of the spacecraft, p_i be the projections of the angular velocity of the spacecraft onto the body axes, α_i be the precession angle, A'_i be the moment of inertia of each gyroshroud (with a rotor) about the axis of self-rotation, B'_i be the moment of inertia of each gyroscope about the precession axis, C'_i be the equatorial moment of inertia of each gyroshroud (with the rotor), and H'_i be the intrinsic moment of momentum of the gyroscopes of the gyroframe Γ_i. Assuming that $A'_i = C'_i$, we obtain for the SC–GS system the system of equations of motion (for the details see Abdullin *et al.* [1] and the references therein)

$$\begin{aligned} &I_1\dot{p}_1 + (I_3 - I_2)p_2p_3 + H_1\dot{\alpha}_1 \cos\alpha_1 + H_3p_2 \sin\alpha_3 \\ &\quad - H_2p_3 \sin\alpha_2 = M_1 + M_{p1} \qquad (123); \\ &B_i\ddot{\alpha}_1 - H_ip_i \cos\alpha_i + b_i\dot{\alpha}_i = M_{yi} + M_{\alpha i}, \quad i = 1,2,3, \\ &\dot{\gamma}_1 = (p_1 \cos\gamma_3 - p_2 \sin\gamma_3)/\cos\gamma_2, \\ &\dot{\gamma}_2 = p_1 \sin\gamma_3 + p_2 \cos\gamma_3, \\ &\dot{\gamma}_3 = p_3 + (p_2 \sin\gamma_3 - p_1 \cos\gamma_3)\cos\gamma_2. \end{aligned} \tag{5.6.11}$$

Here, the symbol (123) designates cyclic permutation of indices,

$$\begin{aligned} &I_1 = J_1 + B_2 + A_1 + A_3 \qquad (123), \\ &B_i = 2B'_i, \quad A_i = 2A'_i, \quad H_i = 2H'_i, \end{aligned}$$

and M_i is the projection of the disturbing moment acting on the spacecraft on the axis Ox^i, M_π is the moment created by the gyroscope unloading system, b_i is the coefficient of viscous friction in the precession axis, M_{yi} is the control moment created by the torque sensor (TS), and $M_{\alpha i}$ is the disturbing moment along the precession axis.

The above assumptions are not exactly realized in real structures, and this leads to the occurence in system (5.6.11) of additional inertial, gyroscopic, and other moments as disturbing factors. It is assumed that their action is reduced in some way to moments that enter the expressions for M_i and $M_{\alpha i}$ as addends and can be estimated satisfactorily. Apart from the mentioned moments, M_i also includes moments of external forces, reactive moments, and moments due to the debalance of GS rotors, and $M_{\alpha i}$ includes moments of dry friction, moments of dynamic debalance of rotors, and other moments in the TS [1].

The necessity of allowing for GS unloading arises when a prescribed orientation of a spacecraft should be maintained for a long time. It is accepted that unloading is relized by the law

$$M_\pi(\alpha_i(t)) = \begin{cases} 0, & \text{for} \quad t \in [t_k, \tau_k); \\ -M_\pi^0 \operatorname{sign} \alpha_i(\tau_k), & \text{for} \quad t \in [\tau_k, t_{k+1}), \ \ |\alpha_i(\tau_k)| = \alpha_i^0, \end{cases}$$

where α_i^0 is the precession angle at which unloading starts, τ_k are the start-up times $(k = 0, 1, 2, \dots)$, and $t_k = \tau_{k-1} + T_\pi \quad (k \geq 1)$ are the unloading cutoff times, and the constants T_π and M_π^0 are selected so that $|\alpha_i(t_{k+1})|$ is sufficiently small.

The control moment M_{yi} is formed based on information obtained from angle sensors (AS) γ_i and angular-velocity sensors (AVS) p_i and $\dot{\alpha}_i$. A control in the form

$$M_{yi} = f_i(\alpha_i)(K_{1i}\gamma_i + K_{2i}p_i) - K_{3i}\dot{\alpha}_i$$

for ideal TS, AS, and AVS is considered qualitative.

Using the actual characteristics of the sensors, we can determine M_{yi} from the equations

$$\begin{gathered} T_i\dot{M}_{yi} + M_{yi} = F_i(\sigma_i), \\ \sigma_i = f_i(\alpha_i)(F_{1i}(\gamma_i) + F_{2i}(p_i)) - F_{3i}(\dot{\alpha}_i), \end{gathered}$$

where T_i is the time constant of the control circuit of the gyroframes Γ_i and $f_i(\alpha_i) = \sec\alpha_i$ or $f_i(\alpha_i) \equiv 1$; $F_i(\sigma_i)$ and $F_{ji}(x_i)$ are the nonlinear characteristics of the TS, AS, and AVS, which vary in time, are ambiguous and in the domain

$$\begin{gathered} |\gamma_i| \leq \gamma_i^* < \frac{\pi}{2}, \qquad |p_i| \leq p_i^*, \\ |\dot{\alpha}_i| \leq q_i^* = \alpha_i^*, \qquad |\alpha_i| \leq \alpha_i^* \leq \frac{\pi}{2}, \end{gathered} \tag{5.6.12}$$

satisfy the conditions

$$\begin{gathered} |F_{ji}(x_j) - k_{ji}x_j| \leq x_{ji}^0, \\ \min(\sigma_i^*, \sigma_i - \sigma_i^0) \leq F_i(\sigma_i) \leq \max(-\sigma_i^*, \sigma_i + \sigma_i^0). \end{gathered}$$

The values of x_{ji}^0 $(x_{1i}^0 = \gamma_i^0,\ x_{2i}^0 = p_i^0,\ x_{3i}^0 = \dot{\alpha}_i^0 = q_i^0)$ are determined by the dead zones, the noise of both the most sensitive elements and signal amplifiers, quantization, and other nonlinearities of the characteristics. The value of σ_i^0 is determined by the TS hysteresis, the dead zone, and other nonlinearities of the static characteristics of both the sensor itself and amplifiers, noise in the amplifiers, errors and quantization in computational devices, etc. The quantity σ_i^* addresses the saturation of the TS or amplifying devices.

In the monograph by Abdullin *et al.* [1], the following notation was introduced

$$\begin{gathered} x_{1i} = \gamma_i, \quad x_{2i} = p_i, \quad x_{3i} = q_i = \dot{\alpha}_i, \quad x_{4i} = M_{yi}/B_i = u_i; \\ \alpha_i = H_i/I_i; \quad a_{ij} = H_j/I_i, \ (i \neq j); \\ \Delta a_1 = (I_3 - I_2)/I_1 \quad (123); \\ \psi_1 = a_{13}p_2\sin\alpha_3 - a_{12}p_3\sin\alpha_2 + \Delta a_1 p_2 p_3 \quad (123); \\ \theta_1 = [p_1(\cos\gamma_3 - \cos\gamma_2) - p_2\sin\gamma_3]/\cos\gamma_2; \\ \theta_2 = p_2(\cos\gamma_3 - 1) + p_1\sin\gamma_3; \end{gathered}$$

$$\theta_3 = (p_2 \sin\gamma_3 - p_1 \cos\gamma_3)\cos\gamma_2;$$
$$m_i = M_i/I_i; \quad m_\pi = M_\pi/I_i; \quad m_{\alpha_i} = M_{\alpha i}/B_i; \quad n_{ji} = k_{ji}/(B_i T_i);$$
$$g_i = H_i/B_i; \quad h_i = b_i/B_i; \quad d_i = 1/T_i; \quad \nu_i = \sigma_i/(B_i T_i);$$
$$\varphi_i(\nu_i) = F_i(B_i T_i \nu_i)/(B_i T_i) - \nu_i;$$
$$r_i = \nu_i - f_i(\alpha_i)(n_{1i}\gamma_i + n_{2i}p_i) + n_{3i}g_i.$$

In these designations, system (5.6.11) is reduced to the form (see Abdullin *et al.* [1])

$$\begin{aligned} &\dot{x}_{1i} = x_{2i} + \theta_i,\\ &\dot{x}_{2i} = -a_i x_{3i}\cos\alpha_i + m_i + m_\pi - \psi_i,\\ &\dot{x}_{3i} = -h_i x_{3i} + g_i x_{2i}\cos\alpha_i + x_{4i} + m_{\alpha i},\\ &\dot{x}_{4i} = -d_i x_{4i} + f_i(\alpha_i)(n_{1i}x_{1i} + n_{2i}x_{2i}) - n_{3i}x_{3i} + r_i + \varphi_i\nu_i,\\ &\dot{\alpha} = x_{3i}, \quad i = 1,2,3. \end{aligned} \tag{5.6.13}$$

Let us further introduce notations corresponding to the method of study developed in Martynyuk and Miladzhanov [4, 5]

$$x_j = (x_{j1}, x_{j2}, x_{j3})^{\mathrm{T}}, \quad j = 1,2,3,4; \quad \alpha = (\alpha_1, \alpha_2, \alpha_3)^{\mathrm{T}};$$
$$\mu_i = \cos\gamma_i, \quad i = 1,2,3; \quad \mu_k = \cos\alpha_{k-3}, \quad k = 4,5,6;$$
$$m = (m_1, m_2, m_3)^{\mathrm{T}}; \quad m_\alpha = (m_{\alpha 1}, m_{\alpha 2}, m_{\alpha 3})^{\mathrm{T}}; \quad m_p = (m_{p1}, m_{p2}, m_{p3})^{\mathrm{T}};$$
$$A = \mathrm{diag}\,(a_1, a_2, a_3); \quad G = \mathrm{diag}\,(g_1, g_2, g_3); \quad H = \mathrm{diag}\,(h_1, h_2, h_3);$$

$$D = \operatorname{diag}(d_1, d_2, d_3); \quad N_i = \operatorname{diag}(n_{i1}, n_{i2}, n_{i3}), \quad i = 1, 2, 3;$$
$$M = \operatorname{diag}(\mu_4, \mu_5, \mu_6), \quad r = (r_1 + \varphi_1(\nu_1),\, r_2 + \varphi_2(\nu_2),\, r_3 + \varphi_3(\nu_3))^{\mathrm T};$$

$$S_1 = \begin{pmatrix} \mu_3 & -\mu_3' & 0 \\ -\mu_2\mu_3' & \mu_2\mu_3' & 0 \\ -\mu_2^2\mu_3' & \mu_2^2\mu_3' & \mu_2 \end{pmatrix}, \quad \mu_3' = \sqrt{1-\mu_3^2};$$

$$S_2 = \begin{pmatrix} 0 & a_{13}\mu_5' & \Delta a_1 p_2 - a_{12}\mu_6' \\ \Delta a_2 p_3 - a_{23}\mu_6' & 0 & a_{21}\mu_4' \\ a_{32}\mu_5' & \Delta a_3 p_1 - a_{31}\mu_4' & 0 \end{pmatrix},$$

$$\mu_k' = \sqrt{1-\mu_k^2}; \quad k = 4, 5, 6.$$

We transform system (5.6.13) into the form

$$\begin{aligned} \mu_2 \dot x_1 &= S_1 x_2, \\ \dot x_2 &= -S_2 x_2 - MAx_3 + m + m_p, \\ \dot x_3 &= -Hx_3 + MGx_2 + x_4 + m_\alpha, \\ M\dot x_4 &= -MDx_4 + N_1 x_1 + N_2 x_2 - MN_3 x_3 + Mr, \\ \dot\alpha &= x_3. \end{aligned} \tag{5.6.14}$$

Here, the matrices S_1 and S_2 and constants μ_j, $j = 1, \dots, 6$ satisfy the following conditions in domain (5.6.12)

$$\underline{S}_1 \le S_1 \le \overline{S}_1, \quad \underline{S}_2 \le S_2 \le \overline{S}_2, \quad \mu_j \in (0, 1],$$

where

$$\underline{S}_1 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad \overline{S}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

$$\underline{S}_2 = \begin{pmatrix} 0 & 0 & \Delta a_1 p_2^* - a_{12} \\ \Delta a_2 p_3^* - a_{23} & 0 & 0 \\ 0 & \Delta a_3 p_1^* - a_{31} & 0 \end{pmatrix},$$

$$\overline{S}_2 = \begin{pmatrix} 0 & a_{13} & \Delta a_1 p_2^* \\ \Delta a_2 p_3^* & 0 & a_{21} \\ a_{32} & \Delta a_3 p_1^* & 0 \end{pmatrix}.$$

5.6.2.2 Construction of Liapunov's matrix function Let us construct a two-index system of functions for system (5.6.14)

$$\begin{aligned} U(x_1, x_2, x_3, x_4, \alpha, \mu_2, M) &= [\vartheta_{ij}(\cdot)], \\ \vartheta_{ij} = \vartheta_{ji}, &\quad i, j = 1, 2, \dots, 5, \end{aligned} \tag{5.6.15}$$

with the elements

$$\vartheta_{11}(x_1, \mu_2) = \mu_2 x_1^{\mathrm T} B_{11} x_1, \quad \vartheta_{12}(x_1, x_2, \mu_2) = \mu_2 x_1^{\mathrm T} B_{12} x_2,$$
$$\vartheta_{13} = 0; \quad \vartheta_{14}(x_1, x_4, \mu_2, M) = \mu_2 x_1^{\mathrm T} B_{14}(Mx_4), \quad \vartheta_{15} = 0,$$
$$\vartheta_{22}(x_2) = x_2^{\mathrm T} B_{22} x_2, \quad \vartheta_{23}(x_2, x_3) = x_2^{\mathrm T} B_{23} x_3,$$

$$\vartheta_{24}(x_2,x_4,M)=x_2^{\mathrm{T}}B_{24}(Mx_4),\quad \vartheta_{25}(x_2,\alpha)=x_2^{\mathrm{T}}B_{25}\alpha,$$
$$\vartheta_{33}(x_3)=x_3^{\mathrm{T}}B_{33}x_3,\quad \vartheta_{34}(x_3,x_4,M)=x_3^{\mathrm{T}}B_{34}(Mx_4),$$
$$\vartheta_{35}(x_3,\alpha)=x_3^{\mathrm{T}}B_{35}\alpha,\quad \vartheta_{44}(x_4,M)=(Mx_4)^{\mathrm{T}}B_{44}x_4,$$
$$\vartheta_{45}=0,\quad \vartheta_{55}(\alpha)=\alpha^{\mathrm{T}}B_{55}\alpha. \tag{5.6.16}$$

Here, B_{ii}, $i=1,2,3,4,5$, are symmetric, positive-definite matrices and B_{12}, B_{14}, B_{23}, B_{24}, B_{25}, B_{34}, and B_{35} are constant matrices.

The following estimates hold for functions (5.6.16)

$$\mu_2\lambda_m(B_{11})\|x_1\|^2\le\vartheta_{11}(x_1,\mu)\le\mu_2\lambda_M(B_{11})\|x_1\|^2;$$
$$\lambda_m(B_{22})\|x_2\|^2\le\vartheta_{22}(x_2,\mu)\le\lambda_M(B_{22})\|x_2\|^2;$$
$$\lambda_m(B_{33})\|x_3\|^2\le\vartheta_{33}(x_3,\mu)\le\lambda_M(B_{33})\|x_3\|^2;$$
$$\underline{\mu}\lambda_m(B_{44})\|x_4\|^2\le\vartheta_{44}(x_4,M)\le\overline{\mu}\lambda_M(B_{44})\|x_4\|^2;$$
$$-\mu_2\lambda_M^{1/2}(B_{12}B_{12}^{\mathrm{T}})\|x_1\|\,\|x_2\|\le\vartheta_{12}(x_1,x_2,\mu_2)$$
$$\le\mu_2\lambda_M^{1/2}(B_{12}B_{12}^{\mathrm{T}})\|x_1\|\,\|x_2\|;$$
$$-\mu_2\overline{\mu}\lambda_M^{1/2}(B_{14}B_{14}^{\mathrm{T}})\|x_1\|\,\|x_4\|\le\vartheta_{14}(x_1,x_4,\mu_2,M)$$
$$\le\mu_2\overline{\mu}\lambda_M^{1/2}(B_{14}B_{14}^{\mathrm{T}})\|x_1\|\,\|x_4\|;$$
$$-\lambda_M^{1/2}(B_{23}B_{23}^{\mathrm{T}})\|x_2\|\,\|x_3\|\le\vartheta_{23}(x_2,x_3)\le\lambda_M^{1/2}(B_{23}B_{23}^{\mathrm{T}})\|x_2\|\,\|x_3\|;$$
$$-\mu_2\lambda_M^{1/2}(B_{12}B_{12}^{\mathrm{T}})\|x_1\|\,\|x_2\|\le\vartheta_{12}(x_1,x_2,\mu_2)\le \tag{5.6.17}$$
$$\le\mu_2\lambda_M^{1/2}(B_{12}B_{12}^{\mathrm{T}})\|x_1\|\,\|x_2\|;$$
$$-\overline{\mu}_2\lambda_M^{1/2}(B_{24}B_{24}^{\mathrm{T}})\|x_2\|\,\|x_4\|\le\vartheta_{24}(x_2,x_4,M)\le$$
$$\le\overline{\mu}_2\lambda_M^{1/2}(B_{24}B_{24}^{\mathrm{T}})\|x_2\|\,\|x_4\|;$$
$$-\lambda_M^{1/2}(B_{25}B_{25}^{\mathrm{T}})\|x_2\|\,\|\alpha\|\le\vartheta_{25}(x_2,\alpha)\le\lambda_M^{1/2}(B_{25}B_{25}^{\mathrm{T}})\|x_2\|\,\|\alpha\|;$$
$$-\overline{\mu}\lambda_M^{1/2}(B_{34}B_{34}^{\mathrm{T}})\|x_3\|\,\|x_4\|\le\vartheta_{34}(x_3,x_4,M)\le$$
$$\le\overline{\mu}\lambda_M^{1/2}(B_{34}B_{34}^{\mathrm{T}})\|x_3\|\,\|x_4\|;$$
$$-\lambda_M^{1/2}(B_{35}B_{35}^{\mathrm{T}})\|x_3\|\,\|\alpha\|\le\vartheta_{35}(x_3,\alpha)\le\lambda_M^{1/2}(B_{35}B_{35}^{\mathrm{T}})\|x_3\|\,\|\alpha\|;$$
$$-\lambda_m^{1/2}(B_{55}B_{55}^{\mathrm{T}})\|\alpha\|^2\le\vartheta_{55}(\alpha)\le\lambda_M^{1/2}(B_{55}B_{55}^{\mathrm{T}})\|\alpha\|^2,$$

where $\underline{\mu}=\min\{\mu_4,\mu_5,\mu_6\}$, $\overline{\mu}=\max\{\mu_4,\mu_5,\mu_6\}$, $\lambda_m(B_{ii})$ are the minimum eigenvalues, $\lambda_M(B_{ii})$ are the maximum eigenvalues of the matrices B_{ii}, $i=1,2,3,4,5$, and $\lambda_M^{1/2}(B_{ij}B_{ij}^{\mathrm{T}})$ are the norms of the matrices B_{ij} for $i<j$.

Uzing matrix-valued function (5.6.15) and the constant vector $\eta=(1,1,1,1,1)^{\mathrm{T}}$, we introduce the function

$$\Theta(x_1,x_2,x_3,x_4,\alpha,\mu_2,M)=\eta^{\mathrm{T}}U(x_1,x_2,x_3,x_4,\alpha,\mu_2,M)\eta. \tag{5.6.18}$$

It is easy to verify that if the elements of matrix function (5.6.15) satisfy estimates (5.6.17), then function (5.6.18) satisfies the two-sided estimate

$$u^{\mathrm{T}}\underline{B}u\le\Theta(x_1,x_2,x_3,x_4,\alpha,\mu_2,M)^{\mathrm{T}}\le u^{\mathrm{T}}\overline{B}u,$$

where

$$
\begin{gathered}
u = (\|x_1\|, \|x_2\|, \|x_3\|, \|x_4\|, \|\alpha\|)^{\mathrm{T}},\\
\underline{B} = [\underline{b}_{ij}]_{i,j=1}^{5}, \quad \underline{b}_{ij} = \underline{b}_{ji}, \quad \overline{B} = [\overline{b}_{ij}]_{i,j=1}^{5}, \quad \overline{b}_{ij} = \overline{b}_{ji},\\
\underline{b}_{11} = \mu_2\lambda_m(B_{11}), \quad \underline{b}_{22} = \lambda_m(B_{22}), \quad \underline{b}_{33} = \lambda_m(B_{33}), \quad \underline{b}_{44} = \mu\lambda_m(B_{44}),\\
\underline{b}_{55} = \lambda_m(B_{55}), \quad \overline{b}_{11} = \mu_2\lambda_M(B_{11}), \quad \overline{b}_{22} = \lambda_M(B_{22}), \quad \overline{b}_{33} = \lambda_M(B_{33}),\\
\overline{b}_{44} = \mu\lambda_M(B_{44}), \quad \overline{b}_{55} = \lambda_M(B_{55}), \quad \overline{b}_{12} = -\underline{b}_{12} = \mu_2\lambda_M^{1/2}(B_{12}B_{12}^{\mathrm{T}}),\\
\overline{b}_{13} = \underline{b}_{13} = 0, \quad \overline{b}_{14} = -\underline{b}_{14} = \mu_2\overline{\mu}\lambda_M^{1/2}(B_{14}B_{14}^{\mathrm{T}}),\\
\overline{b}_{15} = \underline{b}_{15} = 0, \quad \overline{b}_{23} = -\underline{b}_{23} = \lambda_M^{1/2}(B_{23}B_{23}^{\mathrm{T}}),\\
\overline{b}_{24} = \underline{b}_{24} = \overline{\mu}\lambda_M^{1/2}(B_{24}B_{24}^{\mathrm{T}}), \quad \overline{b}_{25} = \underline{b}_{25} = \lambda_M^{1/2}(B_{25}B_{25}^{\mathrm{T}}),\\
\overline{b}_{34} = \underline{b}_{34} = \overline{\mu}\lambda_M^{1/2}(B_{34}B_{34}^{\mathrm{T}}), \quad \overline{b}_{35} = \underline{b}_{35} = \lambda_M^{1/2}(B_{35}B_{35}^{\mathrm{T}}),\\
\overline{b}_{45} = \underline{b}_{45} = 0.
\end{gathered}
$$

5.6.2.3 Test for stability analysis Further, we obtain an upper estimate of the total derivative of function (5.6.18) along the solution of system (5.6.14) in the form

$$\frac{d}{dt}\,\Theta(x_1,x_2,x_3,x_4,\alpha,\mu_2,M)=\sum_{i=1}^{5}\frac{d\vartheta_{ii}}{dt}+2\sum_{i=1}^{5}\sum_{j=2}^{5}\frac{d\vartheta_{ij}}{dt}\le\sum_{i=1}^{4}x_i^{\mathrm T}K_{ii}x_i$$

$$+\,2\sum_{i=1}^{4}\sum_{j=2}^{4}x_i^{\mathrm T}K_{ij}x_j+\sum_{i=1}^{4}x_i^{\mathrm T}K'_{i1}\alpha+\sum_{i=1}^{4}x_i^{\mathrm T}K'_{i2}(m+m_p)+\sum_{i=1}^{4}x_i^{\mathrm T}K'_{i3}m_\alpha$$

$$+\sum_{i=1}^{4}x_i^{\mathrm T}K'_{i4}r+m_\alpha^{\mathrm T}B_{35}\alpha+(m+m_p)^{\mathrm T}B_{25}\alpha,$$

where

$$K_{11}=\mu_2^*(B_{14}N_1+(B_{14}N_1)^{\mathrm T});$$

$$K_{22}=-\,\widetilde S_2^{\mathrm T}B_{22}-B_{22}\widetilde S_2+B_{12}+B_{12}^{\mathrm T}+B_{23}M^*G+(B_{23}M^*G)^{\mathrm T}$$
$$+\,B_{24}N_2+(B_{24}N_2)^{\mathrm T};$$

$$K_{33}=-\,H^{\mathrm T}B_{33}-B_{33}H-(M^*A)^{\mathrm T}B_{23}-B_{23}M^*A$$
$$-\,B_{34}M^*N_3-(B_{34}M^*N_3)^{\mathrm T};$$

$$K_{44}=-(M^*D)^{\mathrm T}B_{44}M^*D+B_{34}M^*+(B_{34}M^*)^{\mathrm T};$$

$$K_{12}=B_{11}\widetilde S_1+\mu_2^*B_{12}\widetilde S_2+\mu_2^*B_{14}N_2;$$

$$K_{13}=-\mu^*B_{12}M^*A-\mu_2^*B_{14}N_3+N_1^{\mathrm T}B_{34};$$

$$K_{14}=N_1^{\mathrm T}B_{44}-\mu_2^*B_{14}M^*D+N_1^{\mathrm T}B_{24}^{\mathrm T};$$

$$K_{23}=-\,B_{22}M^*A+(M^*G)^{\mathrm T}B_{33}-\widetilde S_2^{\mathrm T}B_{23}-B_{23}H$$
$$-\,B_{24}M^*N_3+N_2^{\mathrm T}B_{34}+B_{25};$$

$$K_{24}=B_{33}+N_2^{\mathrm T}B_{44}+B_{23}-B_{24}M^*D+\mu_2^*\widetilde S_1^{\mathrm T}B_{14}$$
$$-\,\widetilde S_2B_{24}M^*+(M^*G)^{\mathrm T}B_{34};$$

$$K_{34}=-(M^*N_3)^{\mathrm T}B_{44}-(M^*A)^{\mathrm T}B_{24}-B_{34}M^*D-H^{\mathrm T}B_{34}M^*;$$

$$K'_{11}=0;\quad K'_{21}=2(\widetilde S_2^{\mathrm T}B_{25}+(M^*G)^{\mathrm T}B_{35});$$

$$K'_{31}=2(B_{55}-(M^*A)^{\mathrm T}B_{25}-H^{\mathrm T}B_{35});\quad K'_{41}=2B_{35};$$

$$K'_{12}=2\mu_2^*B_{22};\quad K'_{22}=2B_{22};\quad K'_{23}=2B_{23};\quad K'_{42}=2M^*B_{24}^{\mathrm T};$$

$$K'_{13}=0;\quad K'_{23}=2B_{23};\quad K'_{33}=2B_{33};\quad K'_{43}=2M^*B_{34}^{\mathrm T};$$

$$K'_{14}=2\mu_2B_{14}M^*;\quad K'_{24}=2B_{24}M^*;\quad K'_{34}=2B_{34}M^*;\quad K'_{44}=2B_{44}M^*;$$

$$\mu_2^*=\begin{cases}\cos\gamma_2^* & \text{if the corresponding factors are negative,}\\ 1 & \text{if the corresponding factors are positive,}\end{cases}$$

$$M^*=\operatorname{diag}(\mu_4^*,\mu_5^*,\mu_6^*),$$

$$\mu_k^*=\begin{cases}\cos\alpha_{k-3}^* & \text{if the corresponding factors are negative,}\\ 1 & \text{if the corresponding factors are positive,}\end{cases}$$

After simple transformations, we find the estimate

(5.6.19)
$$\frac{d}{dt}\,\Theta(x_1,x_2,x_3,x_4,\alpha,\mu_2,M) \le w^{\mathrm T}Kw + \beta_1\|\alpha\|w + \beta_2\|m+m_p\|w + \beta_3\|m_\alpha\|w + \beta_4\|r\|w + \lambda_M^{1/2}(B_{35}B_{35}^{\mathrm T})\|m_\alpha\|\,\|\alpha\| + \lambda_M^{1/2}(B_{25}B_{25}^{\mathrm T})\|m+m_p\|\,\|\alpha\|,$$

where

$$w = (\|x_1\|,\|x_2\|,\|x_3\|,\|x_4\|)^{\mathrm T},$$
$$K = [\rho_{ij}]_{i,j=1}^4, \qquad \rho_{ij}=\rho_{ji},$$
$$\rho_{ii} = \lambda_M(K_{ii}), \quad \rho_{ij} = \lambda_M(K_{ij}K_{ij}^{\mathrm T}), \quad i,\,j=1,2,3,4, \quad i\ne j,$$
$$\beta_j = (\beta_{1j},\beta_{2j},\beta_{3j},\beta_{4j})^{\mathrm T}, \quad j=1,2,3,4,$$

β_{ij} is the norm of the matrices K'_{ij}.

Assume that $\|m_\alpha\| \le \overline{m}_\alpha$, $\|m+m_p\| \le \overline{m}+\overline{m}_p$, and $\|r\| \le \overline{r}$; then estimate (5.6.19) takes the following form in domain (5.6.12)

$$\frac{d}{dt}\,\Theta(x_1,x_2,x_3,x_4,\alpha,\mu_2,M) \le \lambda_M\|w\|^2 + l\|w\| + f, \tag{5.6.20}$$

where

$$l = \|\beta_1\|\,\|\alpha^*\| + \|\beta_2\|(\overline{m}+\overline{m}_p) + \|\beta_3\|\overline{m}_\alpha + \|\beta_4\|\overline{r},$$
$$f = \lambda_M^{1/2}(B_{35}B_{35}^{\mathrm T})\|m_\alpha\|\,\|\alpha\| + \lambda_M^{1/2}(B_{25}B_{25}^{\mathrm T})\|\overline{m}+\overline{m}_p\|\,\|\alpha^*\|.$$

From (5.6.20), it follows that the expression $\frac{d}{dt}\,\Theta(x_1,x_2,x_3,x_4,\alpha,\mu_2,M)$ is negative definite if and only if the following conditions are satisfied

$$\lambda_M(K) < 0,$$
$$\|w\| > \frac{l+\sqrt{l^2+4f\lambda_M(K)}}{-2\lambda_M(K)}. \tag{5.6.21}$$

However, the inequality below holds in domain (5.6.12)

$$\|w\| \le \Big\{\sum_{i=1}^{3}\big[(\gamma_i^*)^2 + (p_i^*)^2 + (q_i^*)^2 + (U_i^*)^2\big]\Big\}^{1/2}, \tag{5.6.22}$$

where $U_i^* > |U_i|$, $i=1,2,3$.

Theorem 5.6.3 *Let LMF (5.6.15) with elements (5.6.16) be constructed for system (5.6.14) and, for this system, condition (5.6.22) be satisfied in domain (5.6.12). If* $\lambda_m(\underline{B}) > 0$, $\lambda_M(K) < 0$, *and the vector* $w = (\|x_1\|,\|x_2\|,\|x_3\|,\|x_4\|)^{\mathrm T}$ *satisfies inequality (5.6.21), then the motion of system (5.6.14) is asymptotically stable.*

Proof From the condition $\lambda_m(\underline{B}) > 0$, it follows that scalar function (5.6.18) is positive-definite in the sense of Liapunov. If the condition $\lambda_M(K) < 0$ and inequality (5.6.21) are satisfied, then the total derivative of function (5.6.18) will be negative definite by virtue of system (5.6.14). As is known, these conditions are sufficient for the motion of system (5.6.14) to be asymptotically stable.

5.6.2.4 Conclusion remarks The stability conditions for a spacecraft formulated in Theorem 5.6.3 are consistent with the conditions of the Liapunov's like Theorem on asymptotic stability of motion. Namely, in this theorem, auxiliary function (5.6.18) constructed on the basis of matrix-valued function (5.6.15) is applied, and the conditions of its definite negativity are established.

Thus, the method of matrix-valued functions allows us to take into account all features of the system under consideration and the cross links between subsystems and does not require constructing a comparison system, which happens when Liapunov's vector function is used (see Abdullin *et al.* [1], p. 227).

Note that the application of comparison systems to analysis of motion stability is inevitably associated with a certain type of its quasimonotonicity (otherwise, the appropriate comparison theorems are not applicable). It is well known that the quasimonotonicity of a system is not a necessary condition of the stability of its trivial solution. Also, it is well known that the property of quasimonotonicity is not associated with the essence of a stability problem but is due to the key feature of the comparison method used.

The method of Liapunov's matrix functions for the system (5.6.14) allows us to keep all the advantages of Liapunov's direct method without introducing into it side conditions that are not characteristic of this flexible method.

5.7 Notes and References

Section 5.1 The results of stability analysis of solutions for the given class of systems in terms of vector functions are presented in Grujić *et al.* [1]. In this chapter we use some results from the above monograph and set out the results of development of a new method of qualitative analysis of singularly perturbed systems dynamics in terms of auxiliary matrix-valued functions. In the framework of this approach we succeed in reducing the requirement to the individual subsystems of system F and extending the boundaries of the admissible upper values of small parameters as compared to those obtained or/and applied in terms of the vector Liapunov function.

Section 5.2 In this section we use the same models of singularly perturbed system under nonclassical structural perturbations as in Grujić *et al.* [1], but in the mathematical composition of large scale system for the given model of connectedness.

Sections 5.3 – 5.4 These two sections are based on the results by Martynyuk and Miladzhanov [4, 5, 9], and Miladzhanov [4, 5].

Section 5.5 Some results of this section are presented by Martynyuk and Miladzhanov [9].

Section 5.6 Subsection 5.6.1 is based on the papers by Martynyuk and Miladzhanov [9, 10]. Also some estimates from Section 5.5 are used.

Section 5.6.2 is based on the paper by Martynyuk and Miladzhanov [8]. Besides, we employ some results by Voronov and Matrosov (Eds.) [1] who applied the vector Liapunov function in the solution of the problem.

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