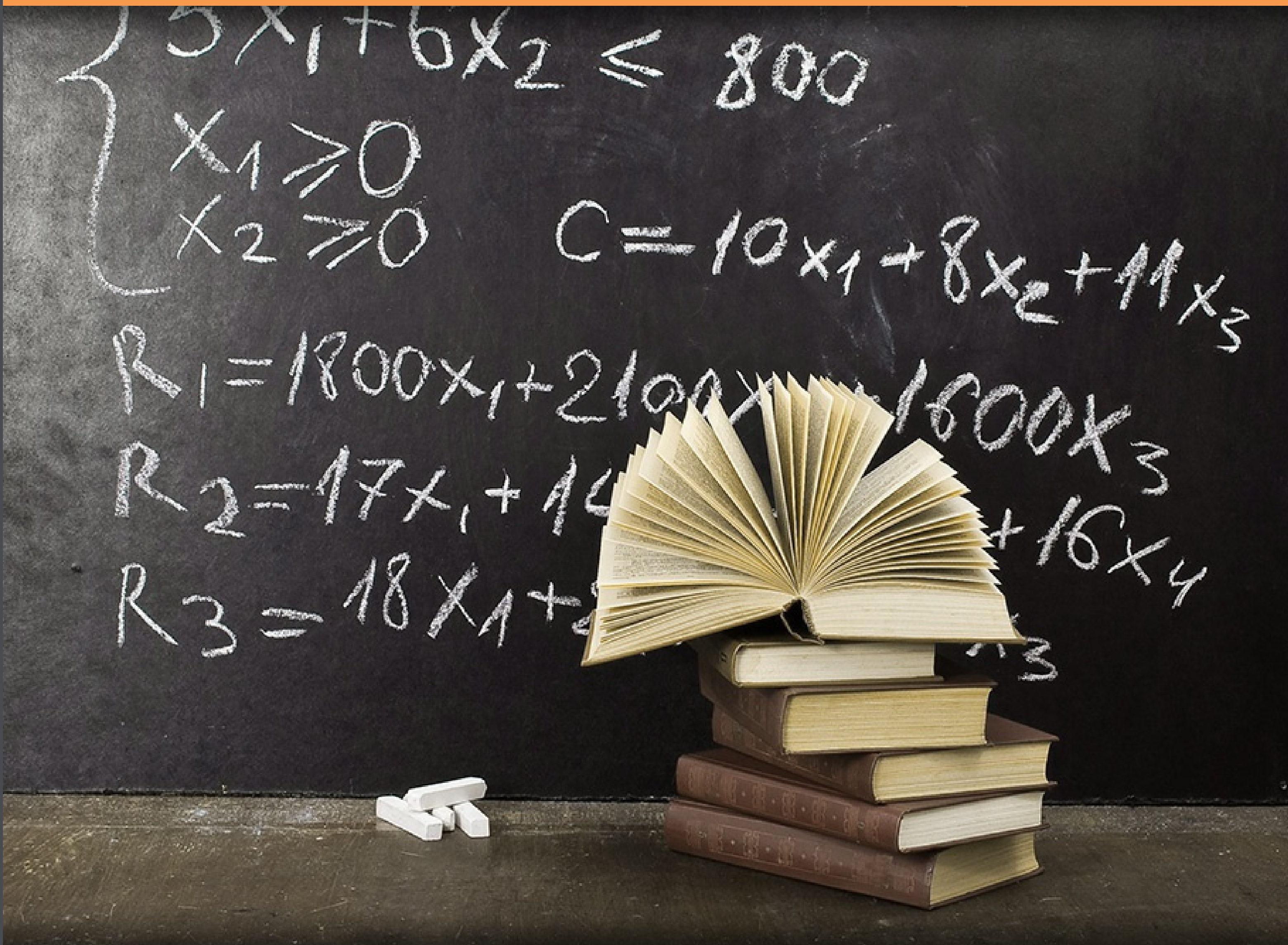
Stability Analysis via Matrix Functions Method

Part II

A. A. Martynyuk



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4

STABILITY ANALYSIS OF STOCHASTIC SYSTEMS

4.1 Introduction

The impact estimation of perturbations, both determined and random ones, is of a great importance for the functioning of real physical systems. Therefore, it is reasonable to consider systems modeled by stochastic differential equations. The present chapter deals with the various types of probability stability for the above mentioned type of equations and develops the method of matrix-valued Liapunov functions with reference to the system of equations of Kats-Krasovskii's form [82] and Ito's form [78]. In the chapter sufficient conditions are formulated for stability and asymptotic stability with respect to probability, global stability with respect to probability, etc.

The notion of averaged derivative of matrix-valued Liapunov function along solutions of the system that has the meaning of infinitesimal operator [34] is crucial in the investigations of this chapter. In a large number of cases this operator defines unequivocally a random Markov process that models the perturbation in the system.

4.2 Stochastic Systems of Differential Equations in General

4.2.1 Notations

For the convenience of readers we collect the following additional nomenclature.

Let R^n be an n-dimensional Euclidean space with norm $\|\cdot\|$, $\nabla_u = \partial/\partial u$, $\nabla_{uv} = \partial^2/\partial u \partial v$, where u and v can be either scalars or vectors. For instance, if $x \in R^n$ and $v \in R^m \to R$, then $\nabla_x v$ denotes the gradient of vector v and $\nabla_{xx} v$ is a matrix with elements $\partial^2 v/\partial x_i \partial x_j$, $i, j \in [1, n]$.

Let $\mathcal{T} = R_+ = [0, +\infty)$ and $(\Omega, \mathcal{A}, P)$ denote a probability space with probability measure P, defined on the σ-algebra $\mathcal{A}$ of ω-sets ($\omega \in \Omega$) in the sample space Ω. Every $\mathcal{A}$ measurable function on Ω is said to be random variable. A sequence of the random variables designated by $\{x(t),\, t \in \mathcal{T}\}$ is called a random process with parameter value t from $\mathcal{T}$. We designate by $R[\mathcal{T}, R[\Omega, R^n]]$ the class of random processes defined on $\mathcal{T}$ with the values in $R[\Omega, R^n]$. Random function $x \in R[[a,b]\colon R[\Omega, R^n]]$ is called measurable on the product, provided that $x(t,\omega)$ is a function measurable on $(\mathcal{A}' \times \mathcal{A})$ and defined on $[a,b] \times \Omega$ with the values in R^n, where A' designates the σ-algebra of measurable in the sense of Lebesque sets on $[a,b]$.

For the set $A \in \mathcal{A}$, $P(A)$ denotes the probability of event A and $P(A/B)$ means the conditional probability of event A under condition $B \in \mathcal{A}$. Function $x(t,\omega)$ is called continuous with respect to $t \in [a,b]$ if

$$P\left\{\bigcup_{t\in[a,b]}\left\{\lim_{\delta\to 0}[\|x(t+\delta)-x(t)\|] \neq 0\right\}\right\} = 0,$$

where $\delta > (< 0)$ when $t = a(b)$.

We designate by $C[[a,b], R[\Omega, R^n]]$ the class of continuous functions defined on $[a,b]$.

Function $x(t)$ admits derivative $x'(t)$ for $t \in [a,b]$ provided

$$P\left\{\bigcup_{t\in[a,b]}\left\{\lim_{\delta\to 0}\left[\left\|\frac{x(t+\delta)-x(t)}{\delta} - x'(t)\right\|\right] \neq 0\right\}\right\} = 0.$$

Let E denote the expectation operator and $\{x_t,\, t \in \mathcal{T}\}$ be a Markov process. Then $E_{x,s}x_t$ denotes the expected value of x_t at $t \in \mathcal{T}$ if it is known that $x_s = x$.

4.2.2 The Motion Equations of Random Parameter Systems

4.2.2.1 Equations of Kats-Krasovskii Form. We consider a system modeled by equations of the form

$$\frac{dx}{dt} = f(t, x, y(t)) \tag{4.2.1}$$

with determined initial conditions

$$x(t_0) = x_0, \tag{4.2.2}$$

$$y(t_0) = y_0. \tag{4.2.3}$$

Here $x \in R^n$, $t \in \mathcal{T}$ (or $t \in \mathcal{T}_\tau = [\tau, +\infty), \tau \geq 0$), $y(t)$ is a perturbation vector that can take the values from $Y \subset R^n$ for every $t \in \mathcal{T}$.

We assume that the vector function f is continuous with respect to every variable and satisfies Lipschitz condition in variable x, i.e.

$$\|f(t, x', y) - f(t, x'', y)\| \leq L\|x' - x''\|$$

in domain $B(\mathcal{T}, \rho, Y)\colon t \in \mathcal{T}$, $\|x\| < \rho$, $y \in Y$ ($\rho = \text{const}$ or $\rho = +\infty$) uniformly in $t \in \mathcal{T}$ and $y \in Y$, and is bounded for all $(t, y) \in \mathcal{T} \times Y$ in every bounded domain $\|x\| < \rho^*$ ($\rho^* = \text{const} > 0$).

Moreover, we assume that

$$f(t, 0, y(t)) = 0 \qquad \forall\,(t, y) \in \mathcal{T} \times Y, \tag{4.2.4}$$

i.e. the unperturbed motion of system (4.2.1) corresponds to the solution $x(t) \equiv 0$.

In system (4.2.1) the random perturbation $y(t)$ is considered to be a random Markov process (see e.g. Doob [31] and Dynkin [34]). Further, two main types of random Markov functions are under consideration.

Case A. The vector $y(t)$ consists of components y_s, $s = 1, 2, \dots, r$ which are independent of each others pure discontinuous Markov processes, the transition functions $P\{y, \tau; \mathcal{A}, t\}$ of which admit the expansion

$$\begin{aligned} &P\{y_s(t + \Delta t) \leq \beta,\ y_s(t + \Delta t) \neq \eta \mid y_s(t) = \eta\} \\ &\qquad\qquad = q_s(t, \eta, \beta)\Delta t + o(\Delta t), \end{aligned} \tag{4.2.5}$$

$$\begin{aligned} &P\{y_s(\tau) \equiv \eta,\ t < \tau \leq t + \Delta t \mid y_s(t) = \eta\} \\ &\qquad\qquad = 1 - \tilde{q}_s(t, \eta)\Delta t + o(\Delta t). \end{aligned} \tag{4.2.6}$$

Here $o(\Delta t)$ is an infinitesimal value of the highest order of smallness relatively Δt, $q_s(t, \eta, \beta)$ and $\tilde{q}_s(t, \eta)$ are some known functions such that

$$q_s(t, \eta, \infty) = \tilde{q}_s(t, \eta), \qquad s = 1, 2, \dots, r.$$

In general we assume almost all realizations $y_s(t, \omega)$ of random process $y(t)$ to be piecewise constant functions continuous from the right.

It should be noted that if the set $Y = \{y_1, \dots, y_k\}$ is one-dimensional and finite, then the representation of functions $q(t, \eta, \beta)$ and $\tilde{q}(t, \eta)$ means the representation of transition matrix

$$p_{ij}(t + \Delta t) = q(t, i, j)\Delta t + o(\Delta t), \qquad i \neq j \tag{4.2.7}$$

where $p_{ij}(t, t+\Delta t)$ is a probability of transition $y_i \to y_j$ during the time from t to $t+\Delta t$.

The process $y(t)$ is called a homogeneous Markov chain with a finite number of states, if $q(t, i, j) = \tilde{q}(i, j)$.

Case B. Vector $y(t)$ is a solution of the generalized differential Ito equation (see e.g. Arnold [5] or Gikhman and Skorokhod [42]).

$$dy(t) = a(t, y(t))dt + b(t, y(t))d\omega(t) + \int c(t, y(t), u)\tilde{\nu}(dt, du) \tag{4.2.8}$$

Besides, $a(t, y)$ and $c(t, y, u)$ are r-component vectors with values in R^r, $y \in R^r$, $u \in R^r$, $b(t, y)$ is a $r \times m$-matrix, $\omega(t)$ is a standard m-dimensional Wienner process with independent coordinates, $\tilde{\gamma}(t, A) = \nu(t, A) - t\lambda(A)$, $\gamma(t, A)$ is a Poisson measure in R^r having a compact carrier, $E\nu(t, A) = t\lambda(A)$, the process $\omega(t)$ and the measure $\nu(t, A)$ are independent of each other.

For the existence conditions with only probability 1 and continuous from the right solution of the equation (4.2.8) see Gikhman and Skorokhod [42].

Following Kats and Krasovskii [82] we shall use the following descriptive interpretation of the solution of (4.2.1). Let almost every realization $y(t, \omega)$ of a random process $y(t)$ and the initial condition (4.2.2), (4.2.3) generate completely continuous realization $x(t, \omega)$ of solutions to the equation

$$\frac{dx}{dt} = f(t, x, y(t, \omega)) \tag{4.2.9}$$

lying in the domain $B(\mathcal{T}, \rho, Y)$ and continuable on $\mathcal{T}_\tau = [\tau, +\infty)$.

Then, the set of these realizations forms an $(n+r)$-dimensional random Markov process $\{x(t), y(t)\}$ that will be referred to as the solution of equations (4.2.1) satisfying conditions (4.2.2) and (4.2.3).

4.2.2.2 Equation of Ito Form. We consider the equation

$$dx = f(t, x)dt + \sigma(t, x)dy(t), \tag{4.2.10}$$

where $t \in \mathcal{T}$, $x_t \in R^n$, $f\colon \mathcal{T} \times R^n \to R^n$, $\sigma\colon \mathcal{T} \times R^n \to R^{n\times m}$ and $\{y(t),\, t \in \mathcal{T}\}$ is a Markov process with independent increments. The system of the equations (4.2.10) is perturbed by two specific types of stochastic processes.

Case C. $\{y(t),\, t \in \mathcal{T}\} \triangleq \{z_t, t \in \mathcal{T}\}$ is a normed m-dimensional Wienner process with independent components.

Case D. $\{y(t),\, t \in \mathcal{T}\} \triangleq \{q_t, t \in \mathcal{T}\}$ is a normed m-dimensional discontinuous Poisson process with independent components.

For the physical interpretation of equation (4.2.10) see e.g. Arnold [5], Kushner [90], et al. Functions f and σ are assumed to be smooth enough and there exists a separable and measurable Markov process $\{x_t,\, t \in \mathcal{T}\}$ satisfying system (4.2.10), that is completely continuous with probability 1.

4.2.3 The Concept of Probability Stability

The notions of probability stability are obtained in terms of Definitions 1.2.1–1.2.3 by replacement of ordinary convergence $x \to 0$, used there, by various types of the probability convergence (convergence with respect to probability, convergence in mean square or almost probable stability). Before we introduce the definitions let us pay attention to the following.

Let the process $y(t)$ be defined by Ito equation (4.2.8). Moreover, equations (4.2.1) and (4.2.8) and initial conditions (4.2.2) and (4.2.3) generate $(n+r)$-dimensional Markov process $\{x_t, y(t)\}$.

If $x(t_0) = 0$, then we have with probability 1 that $x(t) = 0$ for all $t \in \mathcal{T}$ and, therefore, the vector function $\{0, y(t)\}$ is a solution of this system. Let $y(t) \in Y$ for all $t \in \mathcal{T}$, and the set $D = \{0, Y\}$ is a time-invariant set for the process $\{x_t, y(t)\}$ in the sense that

$$P\{\,\{x(t), y(t)\} \in D \mid x(t_0) = x_0,\; y(t_0) = y_0\} = 1$$

for $\{x_0, y_0\} \in D$.

Similar equality is valid for the processes $\{x(t), y(t)\}$ generated by pure discontinuous Markov functions $y(t)$. Therefore, the notion of probability stability discussed herein is based on the stability of an invariant set, for instance $D = \{0, Y\}$.

Definition 4.2.1. The state $x = 0$ of the system (4.2.1) is:

(i) *stable in probability with respect to* $\mathcal{T}_i$ if and only if for every $t_0 \in \mathcal{T}_i$ and every $\varepsilon > 0$, and $1 > p > 0$ there exists $\delta(t_0, \varepsilon) > 0$, such that

$$\|x_0\| < \delta(t_0, \varepsilon) \quad \text{and} \quad y_0 \in Y \tag{4.2.11}$$

implies

$$P\left\{\sup_{t \geq t_0} \|x(t; t_0, x_0, y_0\| < \varepsilon \mid x_0, y_0\right\} > 1 - p \tag{4.2.12}$$

for all $t \in \mathcal{T}_0$;

(ii) *uniformly stable in probability with respect to* $\mathcal{T}_i$ if and only if both (i) holds and for every $\varepsilon > 0$ the corresponding maximal δ_M obeying (i) satisfies

$$\inf\,[\delta_M(t_0, \varepsilon) \colon t_0 \in \mathcal{T}_i] > 0;$$

(iii) *stable in probability in the whole with respect to* $\mathcal{T}_i$ if and only if both (i) holds and

$$\delta_M(t_0, \varepsilon) \to +\infty \quad \text{as} \quad \varepsilon \to +\infty \quad \forall\, t_0 \in \mathcal{T}_i;$$

(iv) *uniformly stable in probability in the whole with respect to* $\mathcal{T}_i$ if and only if both (ii) and (iii) holds.

(v) *unstable in probability with respect to* $\mathcal{T}_i$ if and only if there are $t_0 \in \mathcal{T}_i$, $\varepsilon > 0$, $p > 0$ and $\tau \in \mathcal{T}_0$, $\tau > t_0$ such that for every $\delta > 0$ there is $x_0\colon \|x_0\| < \delta$ and $y_0 \in Y$, for which

$$P\{\, \|x(\tau; t_0, x_0, y_0\| > \varepsilon \mid x_0, y_0\} > 1 - p.$$

The expression "with respect to $\mathcal{T}_i$" is omitted from (i)–(v) if and only if $\mathcal{T}_i = R$.

Definition 4.2.2. The state $x = 0$ of the system (4.2.1) is:

(i) *attractive in probability with respect to* $\mathcal{T}_i$ if and only if for every $t_0 \in \mathcal{T}_i$ there exists $\Delta(t_0) > 0$ and for every $\varsigma > 0$ there exists $\tau(t_0, x_0, y_0, \varsigma) \in [0, +\infty)$ and $p > 0$ such that

$$\|x_0\| < \Delta(t_0) \quad \text{and} \quad y_0 \in Y$$

implies

$$P\left\{ \sup_{t \geq t_0 + \tau} \|x(t; t_0, x_0, y_0\| < \varsigma \mid x_0, y_0 \right\} > 1 - p;$$

(ii) (x_0, y_0)-*uniformly attractive in probability with respect to* $\mathcal{T}_i$ if and only if both (i) is true and for every $t_0 \in \mathcal{T}_i$ there exists $\Delta(t_0) > 0$ and for $\varsigma \in (0, +\infty)$ there exists $\tau_u[t_0, \Delta(t_0), Y, \varsigma] \in [0, +\infty)$ such that

$$\sup\,[\tau_m(t_0, x_0, y_0, \varsigma)\colon x_0 \in B_\Delta(t_0),\; y_0 \in Y] = \tau_u[t_0, \Delta(t_0), Y, \varsigma]$$

(iii) t_0-*uniformly attractive in probability with respect to* $\mathcal{T}_i$ if and only if (i) is true, there is $\Delta > 0$ and for every $(x_0, y_0, \varsigma) \in B_\Delta \times Y \times (0, +\infty)$ there exists $\tau_u(\mathcal{T}_i, x_0, y_0, \varsigma) \in [0, +\infty)$ such that

$$\sup\,[\tau_m(t_0, x_0, y_0, \varsigma)\colon t_0 \in \mathcal{T}_i,\ y_0 \in Y] = \tau_u[\mathcal{T}_i, x_0, y_0, \varsigma];$$

(iv) *uniformly attractive in probability with respect to* $\mathcal{T}_i$ if and only if both (ii) and (iii) hold, that is, that (i) is true, there exists $\Delta > 0$ and for every $\varsigma \in (0, +\infty)$ there is $\tau_u[\mathcal{T}_i, \Delta, Y, \varsigma) \in [0, +\infty)$ such that

$$\sup\,[\tau_m(t_0, x_0, y_0, \varsigma)\colon (t_0, x_0, y_0) \in \mathcal{T}_i \times B_\Delta \times Y] = \tau_u(\mathcal{T}_i, \Delta, Y, \varsigma).$$

(v) The properties (i)–(iv) hold "*in the whole*" if and only if (i) is true for every $\Delta(t_0) \in (0, +\infty)$ and every $t_0 \in \mathcal{T}_i$.

The expression "with respect to $\mathcal{T}_i$" is omitted if and only if $\mathcal{T}_i = R$.

DEFINITION 4.2.3. The state $x = 0$ of the system (4.2.1) is:

(i) *asymptotically stable in probability with respect to* $\mathcal{T}_i$ if and only if it is both stable in probability with respect to $\mathcal{T}_i$ and attractive in probability with respect to $\mathcal{T}_i$;
(ii) *equi-asymptotically stable in probability with respect to* $\mathcal{T}_i$ if and only if it is both stable in probability with respect to $\mathcal{T}_i$ and (x_0, y_0)-uniformly attractive in probability with respect to $\mathcal{T}_i$;
(iii) *quasi-uniformly asymptotically stable in probability with respect to* $\mathcal{T}_i$ if and only if it is both uniformly stable in probability with respect to $\mathcal{T}_i$ and t_0-uniformly attractive in probability with respect to $\mathcal{T}_i$;
(iv) *uniformly asymptotically stable in probability with respect to* $\mathcal{T}_i$ if it is both uniformly stable in probability with respect to $\mathcal{T}_i$ and uniformly attractive in probability with respect to $\mathcal{T}_i$;
(v) the properties (i)–(iv) hold "*in the whole*" if and only if both the corresponding stability in probability of $x = 0$ and the corresponding attraction in probability of $x = 0$ hold in the whole;
(vi) *exponentially stable in probability with respect to* $\mathcal{T}_i$ if and only if there are $\Delta > 0$ and real numbers $\alpha \geq 1$, $\beta > 0$ and $0 < p < 1$ such that $\|x_0\| < \Delta$ and $y_0 \in Y$ implies

$$P\left\{\sup_{t \geq t_0} \|x(t; t_0, x_0, y_0\| < \alpha \|x_0\| \exp[-\beta(t - t_0)] \mid x_0, y_0\right\} > 1 - p.$$

This holds in the whole if and only if it is true for $\Delta = +\infty$.

The expression “with respect to $\mathcal{T}_i$” is omitted if and only if $\mathcal{T}_i = R$.

REMARK 4.2.1. The definitions of stability in probability based on the inequality

$$P\{\|x(t;t_0,x_0,y_0)\| < \varepsilon \mid x(t_0) = x_0,\ y(t_0) = y_0\} > 1 - p \tag{4.2.13}$$

under the condition

$$\|x_0\| < \delta \quad \text{and} \quad y_0 \in Y$$

does not characterize separate realizations of the process $\{x(t), y(t)\}$. I.e. the solution can satisfy the condition (4.2.13), though at the same time almost all realizations may not leave the domain $\|x\| < \varepsilon$ (at various times). Therefore, following Kats and Krasovskii [82] we consider inequality (4.2.12) instead of (4.2.13).

Remark 4.2.2. The probabilities mentioned in Definitions 4.2.1–4.2.3 are not specified in the general case by the finite dimensional distributions of the process $\{x(t), y(t)\}$ and may not exist. However, it is known (see Doob [31]) that a separable modification of the process $\{x(t), y(t)\}$ can be considered, having with probability 1 the realization continuous from the right. In this case all realizations in question have the meaning.

4.2.4 Stochastic Matrix-Valued Liapunov Function

We relate with the system (4.2.1) the stochastic matrix-valued function

$$\Pi(t, x, y(t)) = [v_{kl}(t, x, y(t))], \qquad k,\, l \in [1, s] \tag{4.2.14}$$

where $(t, x, y) \in B$ and $v_{kl}(t, 0, y(t)) \equiv 0\ \forall\, t \in \mathcal{T}$ and $y \in Y$, and, besides, $v_{kl}(t, \cdot) = v_{lk}(t, \cdot)\ \forall\, (k \neq l) \in [1, s],\ v_{kl} \in C(\mathcal{T} \times R^n \times Y, R\,[Y, R])$.

Similar to the determined case (see Chapter 2) the property of having a fixed sign of matrix-valued stochastic function (4.2.14) is of importance in the stability investigation of a stochastic system (4.2.1).

The concept of the property of having a fixed sign must correspond to

(1) the property of having a fixed sign of stochastic matrix;
(2) the property of having a fixed sign of scalar stochastic Liapunov function;
(3) the construction of direct Liapunov method for stochastic systems.

To achieve this we act as follows.

Let $z \in R^s$ and function $V \in C(\mathcal{T} \times R^n \times Y^s \times R^s, R\,[Y, R])$ be defined by the formula

$$V(t, x, y, z) = z^{\mathrm{T}}\Pi(t, x, y(t))z. \tag{4.2.15}$$

In view of Definitions 2.2.1–2.2.2 we present some definitions for stochastic matrix-valued Liapunov function.

Definition 4.2.4. The stochastic matrix-valued function $\Pi\colon R_+ \times B(\rho) \times Y \to R\,[Y, R^{s\times s}]$ is referred to as

(i) *positive (negative) definite,* if and only if there exists a time-invariant connected neighborhood $\mathcal{N}$ of point $x = 0$ ($\mathcal{N} \subseteq R^n$) and positive definite in the sense of Liapunov function $w(x)$ such that
   (a) Π is continuous, i.e. $\Pi \in C(R_+ \times \mathcal{N} \times Y, R\,[Y, R^{s\times s}])$
   (b) $\Pi(t, 0, y) = 0\ \ \forall\, t \in R_+$ and $y \in Y$;
   (c) $\inf V(t, x, y, z) = w(x)\ \ \forall\,(t, y, z) \in R_+ \times Y \times R^s$;
   ($\sup V(t, x, y, z) = -w(x)\ \ \forall\,(t, y, z) \in R_+ \times Y \times R^s$);

(ii) *positive (negative) definite on* $\mathcal{S}$, if and only if all conditions of Definition 4.2.4 (i) are satisfied for $\mathcal{N} = \mathcal{S}$;

(iii) *positive (negative) definite in the whole,* if and only if all conditions of Definition 4.2.4 (i) are satisfied for $\mathcal{N} = R^n$.

Remark 4.2.3. If function Π does not depend on $t \in R_+$, then in Definition 4.2.4 the requirement of function $w(x)$ existence is omitted and conditions (a)–(c) are modified, and condition (c) becomes

(c′) $V(x, y, z) = z^{\mathrm{T}}\Pi(x, y)z > 0\ \ \forall\,(x \neq 0,\ z \neq 0, y) \in \mathcal{N} \times R^s \times Y$,
   $(V(x, y, z) < 0\ \ \forall\,(x \neq 0,\ z \neq 0, y) \in \mathcal{N} \times R^s \times Y)$.

Definition 4.2.5. The stochastic matrix-valued function $\Pi\colon R_+ \times B(\rho) \times Y \to R\,[Y, R^{s\times s}]$ is referred to as

(i) *positive semi-definite,* if and only if there exist a time-invariant connected neighborhood $\mathcal{N}$ of point $x = 0$ ($\mathcal{N} \subseteq R^n$) such that
   (a) Π is continuous in $(t, x) \in R_+ \times \mathcal{N}$;
   (b) Π is non-negative on $\mathcal{N}$: $z^{\mathrm{T}}\Pi(t, x, y)z \geq 0\ \ \forall\,(t, x, y) \in R_+ \times \mathcal{N} \times Y$.
   (c) Π vanishes at the origin $z^{\mathrm{T}}\Pi(t, 0, y)z = 0\ \ \forall\,(z \neq 0,\ y \in Y)$;

(ii) *positive semi-definite on* $R_+ \times \mathcal{S} \times Y$ if and only if (i) holds for $\mathcal{N} = \mathcal{S}$;

(iii) *positive semi-definite in the whole* if and only if (i) holds for $\mathcal{N} = R^n$;

(iv) *negative semi-definite (in the whole)* if and only if $(-\Pi)$ is positive semi-definite (in the whole) respectively.

The following assertion is proved in the same manner as Proposition 2.6.1 from Chapter 2.

Proposition 4.2.1. *The stochastic matrix-valued function* $\Pi\colon R_+ \times B(\rho) \times Y \to R[Y, R^{s\times s}]$ *is positive definite, if and only if there exists a vector* $z \in R^s$ *and a positive definite in the sense of Liapunov function* $a \in K$ *such that*

$$z^{\mathrm{T}}\Pi(t,x,y)z = z^{\mathrm{T}}\Pi_+(t,x,y)z + a(x), \tag{4.2.16}$$

where $\Pi_+(t,x,y)$ *is a stochastic positive semi-definite matrix-valued function.*

Definition 4.2.6. The stochastic matrix-valued function $\Pi\colon R_+ \times B(\rho) \times Y \to R\,[Y, R^{s\times s}]$ is referred to as

(i) *decreasing,* if and only if there exists a time-invariant connected neighborhood $\mathcal{N}$ of point $x = 0$ and a positive definite on $\mathcal{N}$ function $b \in K$ such that

$$V(t,x,y,z) = z^{\mathrm{T}}\Pi(t,x,y)z \le b(x)$$

for all $(t,x,y) \in R_+ \times \mathcal{N} \times Y \times R^s$;

(ii) *decreasing on* $\mathcal{S}$ if and only if (i) holds for $\mathcal{N} = \mathcal{S}$;

(iii) *decreasing in the whole* if and only if (i) holds for $\mathcal{N} = R^n$.

Proposition 4.2.2. *The stochastic matrix-valued function* $\Pi\colon R_+ \times B(\rho) \times Y \to R\,[Y, R^{s\times s}]$ *is decreasing, if and only if there exists a vector* $z \in R^s$ *and a positive definite in the sense of Liapunov function* $c \in K$ *such that*

$$z^{\mathrm{T}}\Pi(t,x,y)z = z^{\mathrm{T}}Q_-(t,x,y)z + c(x), \tag{4.2.17}$$

where $Q_-(t,x,y)$ *is a stochastic negative semi-definite matrix-valued function.*

Definition 4.2.7. The stochastic matrix-valued function $\Pi\colon R_+ \times R^n \times Y \to R\,[Y, R^{s\times s}]$ is referred to as *radially unbounded* if and only if $z^{\mathrm{T}}\Pi(t,x,y)z \to \infty$ as $\|x\| \to +\infty$ and $y \in Y$, $t \in R_+$.

Proposition 4.2.3. *The stochastic matrix-valued function* $\Pi\colon R_+ \times R^n \times Y \to R\,[Y, R^{s\times s}]$ *is radially unbounded, if and only if there exist a vector* $z \in R^s$ *and a function* $\gamma \in KR$ *such that*

$$z^{\mathrm{T}}\Pi(t,x,y)z = z^{\mathrm{T}}Q_+(t,x,y)z + \gamma(\|x\|) \tag{4.2.18}$$

for all $(t,x,y) \in R_+ \times R^n \times Y$, *where* $Q_+(t,x,y)$ *is a positive semi-definite in the whole matrix-valued function.*

We indicate a class of auxiliary stochastic function $v_{kl}(t,x,y(t))$, $k,\, l = 1,2,\dots,s$ using which it is possible to construct the function (4.2.15) satisfying all conditions of Definitions 4.2.4–4.2.7.

State vector $x \in R^n$ of the system (4.2.1) is represented in the form $x = (p^{\mathrm{T}}, q^{\mathrm{T}}, r^{\mathrm{T}})^{\mathrm{T}}$, where $p \in R^{n_1}$, $q \in R^{n_2}$, $r \in R^{n_3}$ and $n_1+n_2+n_3 = n$.

Assumption 4.2.1. There exists time-invariant connected neighborhoods $\mathcal{N}_p \subseteq R^{n_1}$, $\mathcal{N}_q \subseteq R^{n_2}$ and $\mathcal{N}_r \subseteq R^{n_3}$ of the equilibrium states $p = 0$, $q = 0$ and $r = 0$ respectively, functions $\varphi_i(\|p\|)$, $\psi_i(\|q\|)$, $\chi_i(\|r\|)$, $i = 1,2$ of class $K\,(KR)$ and constants $\underline{\alpha}_{jk}$, $\overline{\alpha}_{jk}$, $\forall\,(j,k) \in [1,3]$ and $\overline{\alpha}_{jj}$ and $\underline{\alpha}_{jj} > 0$, $j \in [1,3]$ are such that

(a) $\underline{\alpha}_{11}\varphi_1^2(\|p\|) \le v_{11}(t,x,y) \le \overline{\alpha}_{11}\varphi_2^2(\|p\|)\ \forall\,(t,x,y) \in R_+ \times \mathcal{N}_0 \times Y$,

(b) $\underline{\alpha}_{22}\psi_1^2(\|q\|) \le v_{22}(t,x,y) \le \overline{\alpha}_{22}\psi_2^2(\|q\|)\ \forall\,(t,x,y) \in R_+ \times \mathcal{N}_0 \times Y$;

(c) $\underline{\alpha}_{33}\chi_1^2(\|r\|) \le v_{33}(t,x,y) \le \overline{\alpha}_{33}\chi_2^2(\|r\|)\ \forall\,(t,x,y) \in R_+ \times \mathcal{N}_0 \times Y$;

(d) $\underline{\alpha}_{12}\varphi_1(\|p\|)\psi_1(\|q\|) \le v_{12}(t,x,y) \le \overline{\alpha}_{12}\varphi_2(\|p\|)\psi_2(\|q\|)\ \forall\,(t,x,y) \in R_+ \times \mathcal{N}_0 \times Y$;

(e) $\underline{\alpha}_{13}\varphi_1(\|p\|)\chi_1(\|r\|) \le v_{13}(t,x,y) \le \overline{\alpha}_{13}\varphi_2(\|p\|)\chi_2(\|r\|)\ \forall\,(t,x,y) \in R_+ \times \mathcal{N}_0 \times Y$;

(f) $\underline{\alpha}_{23}\psi_1(\|q\|)\chi_1(\|r\|) \le v_{23}(t,x,y) \le \overline{\alpha}_{23}\psi_2(\|q\|)\chi_2(\|r\|)\ \forall\,(t,x,y) \in R_+ \times \mathcal{N}_0 \times Y$;

(g) $\underline{\alpha}_{21}\psi_1(\|q\|)\varphi_1(\|p\|) \le v_{21}(t,x,y) \le \overline{\alpha}_{21}\psi_2(\|q\|)\varphi_2(\|p\|)\ \forall\,(t,x,y) \in R_+ \times \mathcal{N}_0 \times Y$;

(h) $\underline{\alpha}_{31}\chi_1(\|r\|)\varphi_1(\|p\|) \le v_{31}(t,x,y) \le \overline{\alpha}_{31}\chi_2(\|r\|)\varphi_2(\|p\|)\ \forall\,(t,x,y) \in R_+ \times \mathcal{N}_0 \times Y$;

(i) $\underline{\alpha}_{32}\chi_1(\|r\|)\psi_1(\|q\|) \le v_{32}(t,x,y) \le \overline{\alpha}_{32}\chi_2(\|r\|)\psi_2(\|q\|)\ \forall\,(t,x,y) \in R_+ \times \mathcal{N}_0 \times Y$,

where $\mathcal{N}_0 = \mathcal{N}_{p0} \times \mathcal{N}_{q0} \times \mathcal{N}_{r0}$; $\mathcal{N}_{p0} = \{p \in \mathcal{N}_p,\, p \ne 0\}$, $\mathcal{N}_{q0} = \{q \in \mathcal{N}_q,\, q \ne 0\}$, $\mathcal{N}_{r0} = \{r \in \mathcal{N}_r,\, r \ne 0\}$.

Proposition 4.2.4. *If all conditions of Assumption 4.2.1 are satisfied, then for the function*

$$V(t, x, y, \eta) = \eta^{\mathrm{T}} \Pi(t, x, y) \eta, \tag{4.2.19}$$

with a constant positive vector $\eta \in R^s_+$ *the bilateral estimate*

$$(4.2.20) \qquad u^{\mathrm{T}} H^{\mathrm{T}} A_1 H u \le V(t, x, y, \eta) \le w^{\mathrm{T}} H^{\mathrm{T}} A_2 H w$$

takes place for all $(t, x, y) \in R_+ \times \mathcal{N}_0 \times Y$, *where*

$$\begin{aligned} u^{\mathrm{T}} &= (\varphi_1(\|p\|), \psi_1(\|q\|), \chi_1(\|r\|)), \\ w^{\mathrm{T}} &= (\varphi_2(\|p\|), \psi_2(\|q\|), \chi_2(\|r\|)) \end{aligned}$$

and $A_1 = [\underline{\alpha}_{kl}]$, $A_2 = [\overline{\alpha}_{kl}]$, $H = diag\,(\eta_1, \eta_2, \eta_2)$.

Estimates (4.2.20) are proved by direct substitution by estimates (a)–(i) from Assumption 4.2.1 into the form

$$V(t, x, y, \eta) = \sum_{l,k=1}^{s} \eta_l \eta_k v_{lk}(t, x, y).$$

Estimates (4.2.20) imply

Proposition 4.2.5. *If in the bilateral estimate (4.2.20)*

(1) *the matrix* $H^{\mathrm{T}} A_1 H$ *is positive definite (semi-definite);*
(2) *the matrix* $H^{\mathrm{T}} A_2 H$ *is positive definite;*
(3) *the condition (1) is satisfied and functions* $\varphi_1, \psi_1, \chi_1$ *are of class* KR,

then stochastic function (4.2.19) is

(1) *positive definite (semi-definite);*
(2) *decreasing;*
(3) *radially unbounded*

respectively.

Proof. Assertion (1) of Proposition 4.2.5 follows from the fact that

$$\lambda_m(\tilde{A}_1) u^{\mathrm{T}} u \le u^{\mathrm{T}} H^{\mathrm{T}} A_1 H u, \qquad \lambda_m(\tilde{A}_1) > 0,$$

where $\tilde{A}_1 = H^{\mathrm{T}} A_1 H$. In fact, since $(\varphi_1, \psi_1, \chi_1) \in K$, then a function $\Phi \in K$, $\Phi = \Phi(\|x\|)$ is found such that

$$\Phi(\|x\|) \le \varphi_1^2(\|p\|) + \psi_1^2(\|q\|) + \chi_1^2(\|r\|).$$

Therefore,

$$\lambda_m(\tilde{A}_1)\Phi(\|x\|) \le u^{\mathrm{T}} H^{\mathrm{T}} A_1 H u \le V(t, x, y, \eta)$$

for all $(t, x, y) \in R_+ \times \mathcal{N}_0 \times Y$.

Assertions (2) and (3) of Proposition 4.2.5 are proved similarly.

4.2.5 Structure of the Stochastic Matrix-Valued Function Averaged Derivative

The averaged derivative, that is computed as in determined case without integrating system (2.2.1), is analogous to the total derivative of matrix-valued function for the stochastic system (4.2.1).

Let (τ, x, y) be a point in domain $B(\mathcal{T}, \rho, Y)$.

Definition 4.2.8. Any of the limits

$$\begin{aligned} D^+E[\Pi] &= \limsup \big\{\{E[\Pi(t, x, y) \mid x(\tau) = x,\ y(\tau) = y] \\ &\qquad - \Pi(\tau, x, y)\}(t-\tau)^{-1} \colon t \to \tau + 0\big\}; \\ D_+E[\Pi] &= \liminf \big\{\{E[\Pi(t, x, y) \mid x(\tau) = x,\ y(\tau) = y] \\ &\qquad - \Pi(\tau, x, y)\}(t-\tau)^{-1} \colon t \to \tau + 0\big\}; \end{aligned} \tag{4.2.21}$$

where $E[\,\cdot \mid \cdot\,]$ is a conditional mathematical expectation, is called an *averaged derivative of stochastic matrix-valued function* $\Pi(t, x, y(t))$ *along the solution of system* (4.2.1) at point (τ, x, y). $D^*E[\Pi]$ denotes the case, when $D^+E[\Pi]$ and $D_+E[\Pi]$ are applicable.

The value $D^*E[\Pi]$ is an averaged value of the stochastic matrix-valued function $\Pi(t, x, y)$ derivative along all realizations of process $\{x(t), y(t)\}$ initiating from point (x, y) at time τ. If

$$\begin{aligned} T^+\Pi &= \int P\{\tau, x, y; t, du, dz\}\Pi(t, u, z) \\ &= E[\Pi(t, x(t), y(t)) \mid x(\tau) = x,\ y(\tau) = y], \end{aligned}$$

where $P\{\cdots\}$ is a transition function of solution to system (4.2.1) with the initial conditions $x(\tau) = x$, $y(\tau) = y$, then

$$D^+E[\Pi] = \limsup \big\{[T_\tau^t \Pi - \Pi(\tau, x, y)](t-\tau)^{-1} : t \to \tau + 0\big\}; \tag{4.2.22}$$

$$D_+E[\Pi] = \liminf \left\{ [T_\tau^t \Pi - \Pi(\tau, x, y)](t-\tau)^{-1} : t \to \tau + 0 \right\} \quad (4.2.23)$$

at the point (τ, x, y).

The right-side part of (4.2.22) and (4.2.23) is a weak infinitesimal operator of process $\{x(t), y(t)\}$.

We shall present the formulas for $D^+E[\Pi]$ computation for various realizations of the random process $y(t)$.

1. Let in the system (4.2.1) the process $y(t)$ be pure discontinuous and be described by the relations (4.2.5) and (4.2.6). Then $\dfrac{dE[\Pi]}{dt}$ along solutions of system (4.2.1) at point (τ, x, y) is computed as

$$\frac{dE[\Pi]}{dt} = \nabla_\tau v_{kl}(\tau, x, y) + [\nabla_x v_{kl}(\tau, x, y)]^{\mathrm{T}} f(\tau, x, y(t))$$

$$+\sum_{\mu=1}^{r}\int[v_{kl}(\tau,x,y+\beta_\mu)-v_{kl}(\tau,x,y)]d_\beta q(\tau,y,\beta) \tag{4.2.24}$$

for all $(k,l)\in[1,s]$, where β_μ is a vector, every μ-th component of which equals to β, and the others are zero.

2. Let in the system (4.2.1) $y(t)$ be a simple scalar Markov chain with a finite or countable number of states and transition probabilities satisfying the correlation

$$P\{y(t)=y_j \mid y(\tau)=y_i\}=q_{ij}(t-s)+o(t-s)$$

for all $i\neq j$. We compute $\dfrac{dE[\Pi]}{dt}$ by the formula

$$\begin{aligned}\frac{dE[\Pi]}{dt}&=\nabla_\tau v_{kl}(\tau,x,y)+[\nabla_x v_{kl}(\tau,x,y)]^{\mathrm{T}}f(\tau,x,y(t))\\&+\sum_{j\neq i}[v_{kl}(\tau,x,y_j)-v_{kl}(\tau,x,y_i)]q_{ij}.\end{aligned} \tag{4.2.25}$$

3. Let in the system (4.2.1) $y(t)$ be a Markov process generated by the generalized differential Ito equation (4.2.8). In this case we compute $\dfrac{dE[\Pi]}{dt}$ at point (τ,x,y) by the formula

$$\begin{aligned}\frac{dE[\Pi]}{dt}&=\nabla_\tau v_{kl}(\tau,x,y)+[\nabla_x v_{kl}(\tau,x,y)]^{\mathrm{T}}f(\tau,x,y(t))\\&+[\nabla_y v_{kl}(\tau,x,y)]^{\mathrm{T}}(a(\tau,y)-g(\tau,y))\\&+\int(v_{kl}(\tau,x,y+c(\tau,y,u))-v_{kl}(\tau,x,y))\lambda(du)\\&+\frac{1}{2}\,\mathrm{tr}\,[\nabla_{xx}v_{kl}(\tau,x,y)b(\tau,y)b^{\mathrm{T}}(\tau,y)],\quad\forall\,k,l\in[1,s].\end{aligned} \tag{4.2.26}$$

where $g(\tau, y) = \int c(\tau, y, u)\lambda(du)$.

COROLLARY 4.2.1. If in the formula (4.2.26) $c(t, y, u) \equiv 0$, then $\dfrac{dE[\Pi]}{dt}$ corresponds to the case when $y(t)$ is a diffusion process.

REMARK 4.2.4. Operator $\dfrac{dE[\Pi]}{dt}$ for $c \neq 0$ is local in variable x, but non-local in y.

4. Let in the system (4.2.10) $y(t)$ be a normalized Wienner process with independent components. We compute $\dfrac{dE[\Pi]}{dt}$ at point (τ, x) by the formula

$$\begin{aligned}\frac{dE[\Pi]}{dt} &= \nabla_\tau v_{kl}(\tau, x) + [\nabla_x v_{kl}(\tau, x)]^{\mathrm{T}} f(\tau, x)\\ &+ \frac{1}{2}\,\mathrm{tr}\,[\sigma(t, x)^{\mathrm{T}} \nabla_{xx} v_{kl}(\tau, x)]\sigma(t, x)],\end{aligned} \tag{4.2.27}$$

where $k,\, l \in [1, s]$.

5. Let in the system (4.2.10) $y(t)$ be a normalized jump Poisson process with independent components q_i. Then $\dfrac{dE[\Pi]}{dt}$ at point (τ, x) is computed by the formula

$$\begin{aligned}\frac{dE[\Pi]}{dt} &= \nabla_\tau v_{kl}(\tau, x) + [\nabla_x v_{kl}(\tau, x)]^{\mathrm{T}} f(\tau, x)\\ &+ \sum_{i=1}^{m} \int_{q_i} [v_{kl}(\tau, x + \sigma_i(t, x)q_i) - v_{kl}(\tau, x)] p_i dP_i(dq_i),\end{aligned} \tag{4.2.28}$$

where $k,\, l \in [1, s]$.

Here it is assumed that during the interval Δt the jumps take place with the probability $P_i \Delta t + o(\Delta t)$ and the zero average of the jumps obeys the probability $P_i(\cdot)$.

We establish Liapunov correlation for stochastic matrix-valued function $\Pi(t, x, y(t))$. With this end we construct function (4.2.19) by means of vector $\eta \in R^s_+$. Let $V(t, x, y, \eta)$ be such that for it there exists

$$E\big[V(t, x(t), y(t), \eta) \mid x(\tau) = x,\, y(\tau) = y\big]$$

and

$$\frac{dE[V]}{dt} = H(\tau, x, y) \tag{4.2.29}$$

on the trajectories of the Markov process $\{x(t), y(t)\}$ at point (τ, x, y). Moreover, we assume that

$$\lim_{t \to \tau + 0} E\big[H(t, x(t), y(t)) \mid x(\tau) = x,\ y(\tau) = y\big] = H(\tau, x, y).$$

Then we have

$$\begin{aligned} E\big[V(t, x(t), y(t), \eta) \mid x(\tau) = x,\ y(\tau) = y\big] &= V(\tau, x, y, \eta) \\ + \int_\tau^t E\big[H(u, x(u), y(u)) \mid x(\tau) = x, y(\tau) = y\big]\, du. \end{aligned} \tag{4.2.30}$$

Formula (4.2.30) is valid for the homogeneous Markov processes and functions V independent of time (see Dynkin [34]) and for the processes being considered here (see Kushner [90]).

Let $Q \subset R^n$ be a bounded open set and $U = Q \times Y$ be a set from which the process $\{x(t), y(t)\}$ comes out for the first time at time τ_*. It is easy to notice that $\tau_m(t) = \min\{t, \tau_*\}$ is a Markov momentum, such that $E\tau_m(t) < +\infty$. Therefore, if $\{x(s), y(s)\} \in U$, then

$$E\big[V(\tau_m, x(\tau_m), y(\tau_m), \eta) \mid x(\tau) = x,\ \tau) = y\big] = V(\tau, x, y, \eta) + E\left[\int_\tau^{\tau_m} H(u, x(u), y(u))du \mid x(\tau) = x,\ \tau) = y\right]$$

is valid.

It is also clear that the process $\{x(\tau_m(t)), y(\tau_m(t))\}$ is strictly Markov.

Between $\frac{d}{dt}E[\Pi]$ and $\frac{d}{dt}E[V]$ it is true that

$$\frac{d}{dt}E[V(t, x, y, \eta)] = \eta^{\mathrm{T}} \frac{d}{dt}E[\Pi(t, x, y)]\eta. \tag{4.2.31}$$

We return back to the system (4.2.1) and assume that $y(t)$ is a simple scalar Markov chain with a finite number of states. System (4.2.1) is decomposed into three subsystems

$$\begin{aligned} \frac{dp}{dt} &= X(t, p, 0, 0, y(t)) + F(t, p, q, r, y(t)); \\ \frac{dq}{dt} &= Y(t, 0, q, 0, y(t)) + G(t, p, q, r, y(t)); \\ \frac{dr}{dt} &= Z(t, 0, 0, r, y(t)) + H(t, p, q, r, y(t)); \end{aligned} \tag{4.2.32}$$

where $p \in R^{n_1},\ q \in R^{n_2},\ r \in R^{n_3},\ n_1 + n_2 + n_3 = n,$

$$\begin{array}{ll} X \in C(R_+ \times B_1(\rho),\ R[Y, R^{n_1}]), & Y \in C(R_+ \times B_2(\rho),\ R[Y, R^{n_2}]), \\ Z \in C(R_+ \times B_3(\rho),\ R[Y, R^{n_3}]), & F \in C(R_+ \times B,\ R[Y, R^{n_1}]), \\ G \in C(R_+ \times B,\ R[Y, R^{n_2}]), & H \in C(R_+ \times B,\ R[Y, R^{n_3}]), \end{array}$$

and $B = B_1(\rho) \times B_2(\rho) \times B_3(\rho)$.

Vector-functions X, Y and Z and F, G and H vanish, if and only if $p = q = r = 0$ respectively.

We introduce designation

$$\Delta(v_{kl}) = \sum_{i \neq j} \alpha_{ij}[v_{kl}(t, \cdot, i) - v_{kl}(t, \cdot, j)], \qquad k,\ l = 1, 2, 3.$$

Assumption 4.2.2. There exist the real numbers ρ_{kr}, $k = 1, 2, 3$; $r = 1, 2, \dots, 12$ and comparison functions $\varphi(\|p\|)$, $\psi(\|q\|)$, $\chi(\|r\|)$ of class $K(KR)$ such that

(a) $\nabla_t v_{11} + (\nabla_p v_{11})^{\mathrm{T}} X + \frac{1}{2}\Delta(v_{11}) \le \rho_{11}\varphi^2(\|p\|)\ \forall\,(t, p, y) \in R_+ \times \mathcal{N}_p \times Y$;

(b) $\nabla_t v_{12} + (\nabla_p v_{12})^{\mathrm{T}} X + \frac{1}{4}\Delta(v_{12}) \le \rho_{12}\varphi(\|p\|)\psi(\|q\|)\ \forall\,(t, p, q, y) \in R_+ \times \mathcal{N}_p \times \mathcal{N}_q \times Y$;

(c) $\nabla_t v_{13} + (\nabla_p v_{13})^{\mathrm{T}} X + \frac{1}{4}\Delta(v_{13}) \le \rho_{13}\varphi(\|p\|)\chi(\|r\|)\ \forall\,(t, p, r, y) \in R_+ \times \mathcal{N}_p \times \mathcal{N}_r \times Y$;

(d) $\nabla_t v_{22} + (\nabla_q v_{22})^{\mathrm{T}} Y + \frac{1}{2}\Delta(v_{22}) \le \rho_{21}\psi^2(\|q\|)\ \forall\,(t, q, y) \in R_+ \times \mathcal{N}_q \times Y$;

(e) $\nabla_t v_{21} + (\nabla_q v_{21})^{\mathrm{T}} Y + \frac{1}{4}\Delta(v_{21}) \le \rho_{22}\varphi(\|p\|)\psi(\|q\|)\ \forall\,(t, p, q, y) \in R_+ \times \mathcal{N}_p \times \mathcal{N}_q \times Y$;

(f) $\nabla_t v_{23} + (\nabla_q v_{23})^{\mathrm{T}} Y + \frac{1}{4}\Delta(v_{23}) \le \rho_{23}\psi(\|q\|)\chi(\|r\|)\ \forall\,(t, q, r, y) \in R_+ \times \mathcal{N}_q \times \mathcal{N}_r \times Y$;

(g) $\nabla_t v_{33} + (\nabla_r v_{33})^{\mathrm{T}} Z + \frac{1}{2}\Delta(v_{33}) \le \rho_{31}\chi^2(\|r\|)\ \forall\,(t, r, y) \in R_+ \times \mathcal{N}_r \times Y$;

(h) $\nabla_t v_{31} + (\nabla_r v_{31})^{\mathrm{T}} Z + \frac{1}{4}\Delta(v_{31}) \le \rho_{32}\varphi(\|p\|)\chi(\|r\|)\ \forall\,(t, p, r, y) \in R_+ \times \mathcal{N}_p \times \mathcal{N}_r \times Y$;

(i) $\nabla_t v_{32} + (\nabla_r v_{32})^{\mathrm{T}} Z + \frac{1}{4}\Delta(v_{32}) \le \rho_{33}\psi(\|q\|)\chi(\|r\|)\ \forall\,(t, q, r, y) \in R_+ \times \mathcal{N}_q \times \mathcal{N}_r \times Y$.

and for all $(t,p,q,r,y) \in R_+ \times \mathcal{N}_p \times \mathcal{N}_q \times \mathcal{N}_r \times Y$:

(a′) $(\nabla_p v_{11})^{\mathrm T} F + \frac{1}{2}\Delta(v_{11}) \le \rho_{14}\varphi^2(\|p\|) + \rho_{15}\varphi(\|p\|)\psi(\|q\|) + \rho_{16}\varphi(\|p\|)\chi(\|r\|)$;

(b′) $(\nabla_p v_{12})^{\mathrm T} F + \frac{1}{4}\Delta(v_{12}) \le \rho_{17}\psi^2(\|q\|) + \rho_{18}\varphi(\|p\|)\psi(\|q\|) + \rho_{19}\psi(\|q\|)\chi(\|r\|)$;

(c′) $(\nabla_p v_{13})^{\mathrm T} F + \frac{1}{4}\Delta(v_{13}) \le \rho_{1.10}\chi^2(\|r\|) + \rho_{1.11}\varphi(\|p\|)\chi(\|r\|) + \rho_{1.12}\psi(\|q\|)\chi(\|r\|)$;

(d′) $(\nabla_q v_{22})^{\mathrm T} G + \frac{1}{2}\Delta(v_{22}) \le \rho_{24}\psi^2(\|q\|) + \rho_{25}\varphi(\|p\|)\chi(\|r\|) + \rho_{26}\psi(\|q\|)\chi(\|r\|)$;

(e′) $(\nabla_q v_{21})^{\mathrm T} G + \frac{1}{4}\Delta(v_{21}) \le \rho_{27}\varphi^2(\|p\|) + \rho_{28}\varphi(\|p\|)\psi(\|q\|) + \rho_{29}\varphi(\|p\|)\chi(\|r\|)$;

(f′) $(\nabla_q v_{23})^{\mathrm T} G + \frac{1}{4}\Delta(v_{23}) \le \rho_{2.10}\chi^2(\|r\|) + \rho_{2.11}\varphi(\|p\|)\chi(\|r\|) + \rho_{2.12}\psi(\|q\|)\chi(\|r\|)$;

(g′) $(\nabla_r v_{33})^{\mathrm T} H + \frac{1}{2}\Delta(v_{33}) \le \rho_{34}\chi^2(\|r\|) + \rho_{35}\varphi(\|p\|)\chi(\|r\|) + \rho_{36}\psi(\|q\|)\chi(\|r\|)$;

(h′) $(\nabla_r v_{13})^{\mathrm T} H + \frac{1}{4}\Delta(v_{13}) \le \rho_{37}\varphi^2(\|p\|) + \rho_{38}\varphi(\|p\|)\psi(\|q\|) + \rho_{39}\varphi(\|p\|)\chi(\|r\|)$;

(i′) $(\nabla_r v_{23})^{\mathrm T} H + \frac{1}{4}\Delta(v_{23}) \le \rho_{3.10}\psi^2(\|q\|) + \rho_{3.11}\varphi(\|p\|)\psi(\|q\|) + \rho_{3.12}\psi(\|q\|)\chi(\|r\|)$.

PROPOSITION 4.2.6. *If for the system (4.2.1), decomposed to the form of (4.2.32), there exists a stochastic matrix-valued function* $\Pi(t,x,y)$ *the elements of which satisfy the conditions of Assumption 4.2.1 and all conditions of Assumption 4.2.2 are satisfied, then the structure of stochastic matrix-valued function averaged derivative* $\dfrac{dE[V]}{dt}$ *is defined by the inequality*

$$\frac{dE[V]}{dt} = \eta^{\mathrm T}\frac{dE[\Pi]}{dt}\eta \le u^{\mathrm T} S u \qquad \forall\,(t,x,y) \in R_+ \times \mathcal{N}_0 \times Y \tag{4.2.33}$$

where $s \times s$*-matrix* S *has the elements expressed by formulas*

$$
\begin{aligned}
c_{kl} &= c_{lk}, \qquad (k,l) \in [1,3]: \\
c_{11} &= \eta_1^2(\rho_{11}+\rho_{14}) + 2\eta_1(\eta_2\rho_{27}+\eta_3\rho_{37}), \\
c_{22} &= \eta_2^2(\rho_{21}+\rho_{24}) + 2\eta_2(\eta_1\rho_{17}+\eta_3\rho_{3.10}), \\
c_{33} &= \eta_3^2(\rho_{31}+\rho_{34}) + 2\eta_3(\eta_1\rho_{1.10}+\eta_2\rho_{2.10}), \\
c_{12} &= \frac{1}{2}\eta_1^2\rho_{15} + \frac{1}{2}\eta_2^2\rho_{25} + \eta_1\eta_2(\rho_{12}+\rho_{22}+\rho_{18}+\rho_{28}) \\
&\quad + \eta_3(\eta_1\rho_{38}+\eta_2\rho_{3.11}), \\
c_{13} &= \frac{1}{2}\eta_1^2\rho_{16} + \frac{1}{2}\eta_3^2\rho_{35} + \eta_1\eta_3(\rho_{13}+\rho_{32}+\rho_{1.11}+\rho_{39}) \\
&\quad + \eta_2(\eta_1\rho_{29}+\eta_3\rho_{2.11}), \\
c_{23} &= \frac{1}{2}\eta_2^2\rho_{26} + \frac{1}{2}\eta_3^2\rho_{36} + \eta_2\eta_3(\rho_{23}+\rho_{33}+\rho_{2.12}+\rho_{3.12}) \\
&\quad + \eta_1(\eta_2\rho_{19}+\eta_3\rho_{1.12}).
\end{aligned}
$$

THE PROOF of this proposition is similar to the proof of Proposition 2.7.3.

REMARK 4.2.5. Actually, the structure of the stochastic matrix-valued function $\Pi(t, x, y)$ averaged derivative is established by formula (4.2.33) and is based on the stochastic SL-function (see Martynyuk [120]). The structure of the stochastic matrix-valued function $\Pi(t, x, y)$ averaged derivative is somewhat different provided the stochastic VL-function is applied, i.e.

$$L(t, x, y) = A\Pi(t, x, y)b, \tag{4.2.34}$$

where A is a constant $s \times s$-matrix and b is an s-vector.

4.3 Stability to Systems in Kats–Krasovskii Form

In terms of the stochastic matrix-valued function $\Pi(t, x, y)$ constructed for system (4.2.1), the criteria of stability with respect to probability are in form similar to Theorems 2.3.1–2.3.3.

Theorem 4.3.1. *Let the equations of perturbed motion (4.2.1) are such that:*

(1) *there exists a matrix-valued function* $\Pi\colon R_+ \times B(p) \times Y \to R[Y, R^{s\times s}]$ *in the time-invariant neighborhood* $\mathcal{N} \subseteq R^n$ *of equilibrium state* $x = 0$*;*
(2) *there exists a vector* $\eta \in R^s$ $(\eta \in R^s_+)$*;*
(3) *stochastic scalar function (4.2.19) is positive definite;*
(4) *the averaged derivative (4.2.25) is negative definite or negative semi-definite.*

Then the equilibrium state $x = 0$ *of system (4.2.1) is stable with respect to probability.*

Proof. Let arbitrary numbers $\varepsilon \in (0, \rho)$, $\rho \in (0, 1)$ and $t_0 \in R_+$ be given. Under the conditions (1)–(2) of Theorem 4.3.1 we have the function

$$V(t, x, y, \eta) = \eta^{\mathrm{T}}\Pi(t, x, y)\eta, \quad \eta \in R^s \quad (\eta \in R^s_+),$$

that is positive definite by condition (3) of Theorem 4.3.1. Therefore, a number $\varepsilon_1 > 0$ is found, such that

$$\inf V(t, x, y, \eta) = \varepsilon_1 \quad \text{for} \quad t \in R_+, \quad \|x\| \geq \varepsilon, \quad y \in Y, \quad \eta \in R^s \quad (\eta \in R^s_+).$$

We designate $B(\varepsilon) = \{(x, y) \in R^n \times Y \colon \|x\| < \varepsilon, \, y \in Y\}$. Let τ_ε be the time of trajectory $(x(t), y(t))$ first leaving the domain $B(\varepsilon)$ and let $\tau_\varepsilon(\tau) = \min(\tau, \tau_\varepsilon)$. We have by condition (4)

$$\begin{aligned} E[V(\tau_\varepsilon(\tau), x(\tau_\varepsilon(\tau)), y(\tau_\varepsilon(\tau)), \eta) \mid x(t_0) = x_0, \, y(t_0) = y_0] \\ \leq V(t_0, x_0, y_0, \eta). \end{aligned} \tag{4.3.1}$$

Now we take $\delta > 0$ so that

$$\sup V(t_0, x, y_0) < p\varepsilon_1 \tag{4.3.2}$$

whenever $\|x\| \leq \delta$.

The estimates (4.3.1) and (4.3.2) imply

$$\begin{aligned} p\varepsilon_1 > V(t_0, x_0, y_0, \eta) &\geq E[V(\tau_\varepsilon(\tau), x(\tau_\varepsilon(t)), y(\tau_\varepsilon(\tau)), \eta) \mid x_0, y_0] \\ &\geq \varepsilon_1 P\left\{\sup_{t_0 \leq t \leq \tau} \|x(t)\| \geq \varepsilon \mid x_0, y_0\right\}. \end{aligned}$$

Hence we get for $\tau \to +\infty$

$$P\left\{\sup_{t \geq t_0} \|x(t)\| \geq \varepsilon \mid x_0, y_0\right\} < p.$$

This proves the theorem.

Theorem 4.3.2. *Let the equations of perturbed motion (4.2.1) are such that:*

(1) *hypotheses (1) and (2) of Theorem 4.3.1 are satisfied;*
(2) *the stochastic matrix-valued function* $\Pi(t, x, y)$ *is positive definite and decreasing;*
(3) *the averaged derivative* $\dfrac{dE[V]}{dt}$ *is negative definite.*

Then the equilibrium state $x = 0$ *of the system (4.2.1) is asymptotically stable with probability* $p(H)$*, i.e. if* $\|x_0\| \leq H_0$ *and* $y_0 \in Y$, $t_0 \geq 0$ *then*

$$P\left\{\sup_{t \geq t_0} \|x(t)\| < H \mid x_0, y_0\right\} \geq 1 - p(H), \qquad H_0 < H.$$

PROOF. Let a number $p(H) < 1$ be given. Theorem 4.3.1 implies that under the conditions of Theorem 4.3.2 the equilibrium state $x = 0$ of system (4.2.1) is stable with respect to probability. Therefore, for any $\varepsilon \in (0, \rho)$ and $t_0 \geq 0$ a $\delta = \delta(t_0, \varepsilon) > 0$ can be found such that

$$P\left\{\sup_{t\geq t_0} \|x(t)\| < \varepsilon \mid x_0, y_0\right\} > 1 - p(H), \tag{4.3.3}$$

whenever

$$\|x_0\| < \delta \quad \text{and} \quad y_0 \in Y.$$

Let us show that the number H_0 mentioned in conditions of Theorem 4.3.2 can be taken as $H_0 = \delta$. To this end we define for arbitrary numbers $\gamma \in (0, \varepsilon)$ and $0 < q < +\infty$ the number $\gamma_1 > 0$ from the inequality

(4.3.4)

$$\sup \left[V(t, x, y, \eta) \text{ for } t \in R_+,\ \|x\| < \gamma_1,\ y \in Y,\ \eta \in R^s_+\right]$$
$$< \frac{q}{2} \inf \left[V(t, x, y, \eta) \text{ for } t \in R_+,\ \gamma_1 \leq \|x\| \leq \varepsilon,\ y \in Y \text{ and } \eta \in R^s_+\right].$$

The arguments similar to those used in the proof of Theorem 4.3.1 yield

$$P\left\{\sup_{\tau>t} \|x(\tau)\| < \gamma \mid x(t), y(t)\right\} > 1 - \frac{1}{2}q, \tag{4.3.5}$$

whenever

$$\|x(t)\| \le \gamma_1 \quad \text{and} \quad y(t) \in Y.$$

We claim that there exists a $\tau > t_0$ such that

$$P\{\|x(t_0+\tau)\| < \gamma_1 \mid x_0, y_0\} > 1 - \frac{1}{2}q - p(H). \tag{4.3.6}$$

If this is not true, then for trajectory $\{x(t), y(t)\}$ the inequality

$$P\{\gamma_1 \le \|x(t)\| < \varepsilon,\, t \ge t_0 \mid x_0, y_0\} > \frac{1}{2}q.$$

holds, that yields by condition (3) of Theorem 4.3.2

$$\lim_{t\to\infty} E[V(\tau_\alpha(t), x(\tau_\alpha(t)), y(\tau_\alpha(t)), \eta) \mid x_0, y_0] = -\infty. \tag{4.3.7}$$

Here $\tau_\alpha(t) = \min(\tau^*, t)$, where τ^* is a time of trajectory $(x(t), y(t))$ first leaving the set $B_1 = \{(x,y)\colon \gamma_1 < \|x\| < \varepsilon,\, y \in Y\}$.

Since the function $\Pi(t,x,y)$ is positive definite, the correlation (4.3.7) can not be satisfied. This proves inequality (4.3.6). The estimates (4.3.3), (4.3.5) and (4.3.6) imply that for arbitrary $q > 0$ a $\tau > 0$ is found so that

$$P\left\{\sup_{t\ge t_0+\tau} \|x(t)\| < \gamma \mid x_0, y_0\right\} > 1 - q - p(H),$$

whenever $\|x_0\| < H_0$ and $y_0 \in Y$.

This proves Theorem 4.3.2.

THEOREM 4.3.3. *Let the equations of perturbed motion (4.2.1) are such that:*

(1) *hypotheses (1), (2) and (3) of the Theorem 4.3.1 are satisfied for* $\mathcal{N} = R^n$*;*
(2) *the function* $\Pi(t,x,y)$ *is positive definite in the whole and radially unbounded;*
(3) *the averaged derivative* $\dfrac{dE[V]}{dt}$ *is negative definite in* $B(\mathcal{T}, \infty, Y)$*.*

Then the equilibrium state $x = 0$ *of the system (4.2.1) is stable with respect to probability in the whole.*

A theorem allowing us to find asymptotic stability with respect to probability and stability with respect to probability in the whole on the basis of negative semi-definite averaged derivative is considered.

Let an open domain G containing the origin be definite in space R^n. Function $\psi(t,x,y)\colon T_0\times G\times Y\to R$ is referred to as positive definite on $G\times Y$ if for any numbers $r>\varepsilon>0$ there exists a number $\delta>0$ such that $\psi(t,x,y)\geq\delta$ holds for all $t\geq t_0$, $(x,y)\in(\mathcal{N}\cap\{\varepsilon\leq\|x\|\leq r\}\times Y)$.

Matrix-valued function $\Phi(t,x,y)\colon T_0\times G\times Y\to R^{m\times m}$ satisfies hypotheses A if:

(a) the function Φ is bounded for all $t\geq t_0$ in any finite domain $\|x\|\leq\rho$, $y\in Y$;

(b) averaged derivative $\eta^{\mathrm{T}}\dfrac{dM[\Phi]}{dt}\eta$ is bounded in any finite domain due to system (4.2.1), i.e. there exists a constant K such that

$$\left|\eta^{\mathrm{T}}\frac{dM[\Phi]}{dt}\eta\right|\leq K;$$

(c) the function $\eta^{\mathrm{T}} \dfrac{dM[\Phi]}{dt} \eta$ is positive definite in domain $G \times Y$.

Then the following statement is valid.

THEOREM 4.3.4. *Let the equations of perturbed motion (4.2.1) as definite in domain $B(T_0, \infty, Y)$ and such that:*

(1) *hypotheses (1) and (2) of Theorem 4.3.3 are satisfied;*
(2) *averaged derivative (4.2.13) satisfies hypothesis*

$$\eta^{\mathrm{T}} \frac{dM[\Pi]}{dt} \eta \le H(x) \le 0,$$

where $H(x)$ is continuous in domain G;
(3) *the set $D = \{x\colon x \ne 0,\, H(x) = 0\}$ is non-empty and does not possess mutual points with bound $\partial\mathcal{N}$ in domain $\mathcal{N}$ in the sense that* $\inf \|x_1 - x_2\| > K^2 > 0 \;\; x_1 \in \partial G,\; x_2 \in D \cap \{\varepsilon \le \|x\| \le r\}$;
(4) *there exists a matrix-valued function $\Phi(t, x, y)$ satisfying hypotheses A.*

Then the equilibrium state $x = 0$ of the system (4.2.1) is stable with respect to probability in the whole.

4.4 Stability to Systems in Ito's Form

4.4.1 Decomposition of perturbed motion equations

We consider a system of the equations with random parameters in the form

$$d\omega(t) = f(t,\omega)dt + \sigma(t,\omega)d\xi(t), \tag{4.4.1}$$

where $t \in \mathcal{T}$, $\omega \in R^n$, $f\colon \mathcal{T} \times R^n \to R^n$, $\sigma\colon \mathcal{T} \times R^n \to R^{n\times m}$, and $\{\xi(t),\, t \in \mathcal{T}\}$ is an independent measurable random Markov process.

Assume that the system (4.4.1) allows decomposition into l interconnected subsystems that can be described by equations in the form

$$\begin{aligned} d\omega_i =& f_i(t,\omega_i)dt + \sigma_{ii}(t,\omega_i)d\xi_i \\ &+ g_i(t,\omega)dt + \sum_{j=1}^{l} \sigma_{ij}(t,\omega_j)d\xi_j, \qquad i \in [1,l]. \end{aligned} \tag{4.4.2}$$

Each interconnected subsystem (4.4.2) consists of the independent subsystem

$$d\omega_i = f(t,\omega_i)dt + \sigma_{ii}(t,\omega_i)d\xi_i, \qquad i \in [1,l], \tag{4.4.3}$$

and link functions

$$g_i(t,\omega)dt + \sum_{j=1}^{l} \sigma_{ij}(t,\omega_j)d\xi_j, \qquad i \in [1,l]. \tag{4.4.4}$$

Here $\omega_i \in R^{n_i}$, $\omega \in R^n$, $\omega = (\omega_1^{\mathrm{T}}, \omega_2 T, \dots, \omega_l^{\mathrm{T}})^{\mathrm{T}}$, $\xi_i \in R^{m_i}$, $f_i\colon \mathcal{T}_0 \times R^{n_i} \to R^{n_i}$, $\sigma_{ij}\colon \mathcal{T} \times R^{n_j} \to R^{n_i\times m_j}$, $g_i\colon \mathcal{T} \times R^{n_1} \times \dots \times R^{n_l} \to R^{n_i}$, and $\{\xi_i(t),\, t \in \mathcal{T}\}$ are independent measurable Markov processes.

We assume on function f_i and σ_{ii} that they satisfy the existence condition for solutions to subsystems (4.4.3), and link functions (4.4.4) vanish, if and only if $\omega_j = 0$ and $\omega = 0$. Thus, the points $\omega = 0$ and $\omega_j = 0$, $j \in [1,l]$ are the only equilibrium states of systems (4.4.1), (4.4.2) and (4.4.3) respectively.

The transformation of systems (4.4.1) to (4.4.2) is referred to as the decomposition of stochastic Ito system of the first level. Suppose that from system (4.4.1) couples (i, j) of interconnected subsystems are taken in the form

$$
\begin{aligned}
d\omega_i =& f_{ij}(t,\omega_i,\omega_j)dt + \sigma_{ii}(t,\omega_i)d\xi_i + \sigma_{ij}(t,\omega_j)d\xi_j \\
&+ g_{ij}(t,\omega)dt + \sum_{\substack{k=1\\(k\neq i,j)}}^{l} \sigma_{ik}(t,\omega_k)d\xi_k, \qquad i \in [1,l]. \\
d\omega_j =& f_{ji}(t,\omega_j,\omega_i)dt + \sigma_{jj}(t,\omega_j)d\xi_j + \sigma_{ji}(t,\omega_i)d\xi_i \\
&+ g_{ji}(t,\omega)dt + \sum_{\substack{k=1\\(k\neq i,j)}}^{l} \sigma_{jk}(t,\omega_k)d\xi_k, \qquad (i \neq j) \in [1,l].
\end{aligned} \tag{4.4.5}
$$

Here $f_{ij}\colon \mathcal{T} \times R^{n_i} \times R^{n_j} \to R^{n_i} \times R^{n_j}$, $g_{ij}\colon \mathcal{T} \times R^{n} \to R^{n_i} \times R^{n_j}$. We introduce following designations $\omega_{ij} = (\omega_i^{\mathrm{T}}, \omega_j^{\mathrm{T}})^{\mathrm{T}}$, $\overline{f}_{ij}(t, \omega_{ij}) = (f_{ij}^{\mathrm{T}}, f_{ji}^{\mathrm{T}})^{\mathrm{T}}$; $\overline{g}_{ij}(t, \omega) = (g_{ij}^{\mathrm{T}}, g_{ji}^{\mathrm{T}})^{\mathrm{T}}$, $\sigma_{ij}^k = [\sigma_{ik}^{\mathrm{T}}, \sigma_{jk}^{\mathrm{T}}]^{\mathrm{T}}$, $d\xi_{ij} = (d\xi_i^{\mathrm{T}}, d\xi_j^{\mathrm{T}})^{\mathrm{T}}$, and

$$\overline{\sigma}_{ij} = \begin{pmatrix} \sigma_{ii} & \sigma_{ij} \\ \sigma_{ji} & \sigma_{jj} \end{pmatrix}.$$

Then the (i, j) couple (4.4.5) can be represented as

$$\begin{aligned} d\omega_{ij} =& \overline{f}_{ij}(t, \omega_{ij})dt + \overline{\sigma}_{ij} d\xi_{ij} + \overline{g}_{ij}(t, \omega)dt \\ & + \sum_{\substack{k=1 \\ (k \neq i,j)}}^{l} \sigma_{ij}^k d\xi_k, \qquad (i \neq j) \in [1, l]. \end{aligned} \tag{4.4.6}$$

Besides, the free (i, j) couple has the form

$$d\omega_{ij} = \overline{f}_{ij}(t, \omega_{ij})dt + \overline{\sigma}_{ij} d\xi_{ij} \qquad (i \neq j) \in [1, l]. \tag{4.4.7}$$

and the link functions are represented by the formulas

$$\overline{g}_{ij}(t, \omega)dt + \sum_{\substack{k=1 \\ (k \neq i,j)}}^{l} \sigma_{ij}^k d\xi_k, \qquad (i \neq j) \in [1, l]. \tag{4.4.8}$$

Further we need the following assumptions.

Assumption 4.4.1. There exists a time invariant open connected neighborhood $\mathcal{N}_i \subseteq R^{n_i}$, a function $v_{ii}(t, \omega_i)\colon \mathcal{T} \times \mathcal{N}_i \to R_+$, the comparison functions ψ_{i1}, ψ_{i2} and ψ_{i3} and the positive real numbers ρ_i such that for all $i \in [1, l]$ estimates

(a) $\psi_{i1}(\|\omega_i\|) \le v_{ii}(t, \omega_i) \le \psi_{i2}(\|\omega_i\|)$;

(b) $\dfrac{dE_i[v_{ii}(t, \omega_i)]}{dt} \le p_i \psi_{i3}(\|\omega_i\|)$

are satisfied for any $\omega_i \in \mathcal{N}_i$ and $t \in \mathcal{T}$.

Definition 4.4.1. The isolated subsystems (4.4.3) possesses *property* $A(\mathcal{N}_i)$, provided all conditions of Assumption 4.4.1 are satisfied for each of the subsystems.

Definition 4.4.2. If in Assumption 4.4.1 $\psi_{i1}(\|\omega_i\|) = c_{i1}\|\omega_i\|^2$, $\psi_{i2}(\|\omega_i\|) = c_{i2}\|\omega_i\|^2$ and $\psi_{i3}(\|\omega_i\|) = \dfrac{c_{ii}}{\rho_i}\|\omega_i\|^2$, where c_{i1} and c_{i2} are positive constants, and c_{ii} constants $i \in [1, l]$, then isolated subsystem (4.4.3) is said to possess *property* $B(\mathcal{N}_i)$.

Definition 4.4.3. If in Assumption 4.4.1 $\mathcal{N}_i = R^{n_i}$ for all $i \in [1, l]$ and functions ψ_{11}, $\psi_{12} \in KR$, then isolated subsystems (4.4.3) are said to possesses *property* $B_i(\infty)$.

Assumption 4.4.2. There exist a time-invariant open connected products of neighborhood $\mathcal{N}_i \times \mathcal{N}_j \subseteq R^{n_i} \times R^{n_j}$ of point $\omega_{ij} = 0$, functions $v_{ij}(t, \omega_{ij})\colon \mathcal{T} \times \mathcal{N}_i \times \mathcal{N}_j \to R_+$, a functions ψ^1_{ij}, ψ^2_{ij} and ψ_{i3} of class K and positive real numbers β^1_{ij}, β^2_{ij} and β^3_{ij} such that for all $(i < j) \in [1, l]$ the estimates

(a) $\psi^1_{ij}(\|\omega_{ij}\|) \le v_{ij}(t, \omega_{ij}) \le \psi^2_{ij}(\|\omega_{ij}\|)$;

(b) $\dfrac{dE_{ij}[v_{ij}]}{dt} \le \beta^1_{ij}\psi_{i3}(\|\omega_i\|) + 2\beta^2_{ij}\psi^{1/2}_{i3}(\|\omega_i\|)\psi^{1/2}_{j3}(\|\omega_j\|) + \beta^3_{ij}\psi_{i3}(\|\omega_j\|)$

are satisfied for any $\omega_{ij} \in \mathcal{N}_i \times \mathcal{N}_j$ and $t \in \mathcal{T}$.

Defiinition 4.4.4. Isolated couples (i, j) of subsystems (4.4.7) possesses *property* $A(\mathcal{N}_i \times \mathcal{N}_j)$, if for every of them all conditions of Assumption 4.4.2 are satisfied.

Definition 4.4.5. If in Assumption 4.4.2 $\psi^1_{ij} = c^1_{ij}\|\omega_{ij}\|^2$, $\psi^2_{ij} = c^2_{ij} \times \|\omega_{ij}\|^2$ and $\beta^1_{ij}\psi_{i3}(\|\omega_i\|) + 2\beta^2_{ij}\psi^{1/2}_{i3}(\|\omega_i\|)\psi^{1/2}_{j3}(\|\omega_j\|) + \beta^3_{ij}\psi_{j3}(\|\omega_j\|) =$

$c^3_{ij}\|\omega_{ij}\|^2$, $(i < j) \in [1, l]$, where c^1_{ij}, c^2_{ij}, c^3_{ij} are constant, then the independent couples (i, j) of subsystems (4.4.7) are said to possess *property* $B(\mathcal{N}_i \times \mathcal{N}_j)$.

Definition 4.4.6. If in Assumption 4.4.2 $\mathcal{N}_i = R^{n_i}$ and the functions Ψ^1_{ij}, $\Psi^2_{ij} \in KR$, then the independent couples (i, j) of the subsystems (4.4.7) are said to possess *property* $A_{ij}(\infty)$.

Remark 4.4.1. In Assumptions 4.4.1 and 4.4.2 the constants ρ_i, $i \in [1, l]$ and c^3_{ij} $(i < j) \in [1, l]$ are negative if independent subsystems (4.4.3) and independent couples (i, j) of subsystems (4.4.7) are exponentially stable with respect to probability.

Remark 4.4.2. The matrix B_{ij}, defined by the expression

$$B_{ij} = \begin{pmatrix} \beta^1_{ij} & \beta^2_{ij} \\ \beta^2_{ij} & \beta^3_{i3} \end{pmatrix} \qquad (i < j) \in [1, l]$$

is negative semi-definite (negative definite), if the independent couples (i, j) of subsystems (4.4.7) are stable (asymptotically stable) with respect to probability.

4.4.2 Structure of the Hierarchical Matrix-Valued Function Averaged Derivative

We construct for subsystems (4.4.3) the functions $v_{ii}(t, \omega_i)$, $i \in [1, l]$ and for couples (i, j) of subsystems (4.4.7) the functions $v_{ij}(t, \omega_{ij})$ $(i < j) \in [1, l]$. Let us construct from the above mentioned elements the matrix-valued function.

$$\Pi(t, \omega) = [v_{ij}(t, \cdot)], \tag{4.4.9}$$

where $\Pi\colon \mathcal{T} \times R^{n_i} \times R^{n_j} \times Y \to R[Y, R^{l\times l}]$.

The function (4.4.9) reflects the hierarchy of stochastic subsystems (4.4.3) and (4.4.7) in the large-scale system (4.4.1).

The application of formula (4.2.27) to systems (4.4.2) and (4.4.6) yields the following expressions for $\dfrac{dE[\cdot]}{dt}$

$$
\begin{aligned}
\frac{dE[v_{ii}(t,\omega)]}{dt} &= \sum_{j=1}^{l}[f_j(t,\omega_j)+g_j(t,\omega)]^{\mathrm{T}}\nabla_{\omega_j}v_{ii}(t,\omega_i)\\
&\quad+\frac{1}{2}\mathrm{tr}[\sigma_{ii}^{\mathrm{T}}(t,\omega_i)\nabla_{\omega_i\omega_i}v_{ii}(t,\omega_i)\sigma_{ii}(t,\omega_i)]\\
&\quad+\frac{1}{2}\sum_{\substack{j,k,m=1\\(j\neq i)}}^{l}\mathrm{tr}[\sigma_{kj}^{\mathrm{T}}(t,\omega_j)\nabla_{\omega_k\omega_m}v_{ii}(t,\omega_i)\sigma_{mj}(t,\omega_j)]\\
&\quad+\nabla_t v_{ii}(t,\omega_i)\\
&=\sum_{j=1}^{l}[f_j(t,\omega_j)+g_j(t,\omega)]^{\mathrm{T}}\delta_{ij}\nabla_{\omega_j}v_{ii}(t,\omega_i)\\
&\quad+\frac{1}{2}\mathrm{tr}[\sigma_{ii}^{\mathrm{T}}(t,\omega_i)\nabla_{\omega_i\omega_i}v_{ii}(t,\omega_i)\sigma_{ii}(t,\omega_i)]\\
&\quad+\frac{1}{2}\sum_{\substack{j,k,m=1\\(j\neq i)}}^{l}\mathrm{tr}[\sigma_{kj}^{\mathrm{T}}(t,\omega_j)\delta_{ki}\delta_{mi}\nabla_{\omega_i\omega_i}v_{ii}(t,\omega_i)\sigma_{mj}(t,\omega_j)]\\
&\quad+\nabla_t v_{ii}(t,\omega_i)\\
&=[f_i(t,\omega_i)+g_i(t,\omega)]^{\mathrm{T}}\nabla_{\omega_i}v_{ii}(t,\omega_i)\\
&\quad+\frac{1}{2}\mathrm{tr}[\sigma_{ii}^{\mathrm{T}}(t,\omega_i)\nabla_{\omega_i\omega_i}v_{ii}(t,\omega_i)\sigma_{ii}(t,\omega_i)]\\
&\quad+\frac{1}{2}\sum_{\substack{j=1\\(j\neq i)}}^{l}\mathrm{tr}[\sigma_{ij}^{\mathrm{T}}(t,\omega_j)\nabla_{\omega_i\omega_i}v_{ii}(t,\omega_i)\sigma_{ij}(t,\omega_j)]+\nabla_t v_{ii}(t,\omega_i)\\
&=\frac{dE_i[v_{ii}(t,\omega_i)]}{dt}+g_i^{\mathrm{T}}(t,\omega)\nabla_{\omega_i}v_{ii}(t,\omega_i)\\
&\quad+\frac{1}{2}\sum_{\substack{j=1\\(j\neq i)}}^{l}\mathrm{tr}[\sigma_{ij}^{\mathrm{T}}(t,\omega_j)\nabla_{\omega_i\omega_i}v_{ii}(t,\omega_i)\sigma_{ij}(t,\omega_j)],\\
&\quad(i\neq j)\in[1,l].
\end{aligned}
\tag{4.4.10}
$$

Similarly we have

$$\begin{aligned}
\frac{dE[v_{ij}(t,\omega)]}{dt} &= [\overline{f}_{ij}(t,\omega_{ij}) + \overline{g}_{ij}(t,\omega)]^{\mathrm{T}}\nabla_{\omega_{ij}} v_{ij}(t,\omega_{ij}) \\
&\quad + \frac{1}{2}\mathrm{tr}[\overline{\sigma}_{ij}^{\mathrm{T}}(t,\omega_{ij})\nabla_{\omega_{ij}\omega_{ij}} v_{ij}(t,\omega_{ij})\overline{\sigma}_{ij}(t,\omega_{ij})] \\
&\quad + \frac{1}{2}\sum_{\substack{k=1 \\ (k\neq i,j)}}^{l} \mathrm{tr}[\sigma_{ij}^{k}(t,\omega_k)^{\mathrm{T}}\nabla_{\omega_{ij}\omega_{ij}} v_{ij}(t,\omega_{ij})\sigma_{ij}^{k}(t,\omega_k)] \\
&\quad + \nabla_t v_{ij}(t,\omega_{ij}) \\
&= \frac{dE_{ij}[v_{ij}(t,\omega_{ij})]}{dt} + \overline{g}_{ij}^{\mathrm{T}}(t,\omega)\nabla_{\omega_{ij}} v_{ij}(t,\omega_{ij}) \\
&\quad + \frac{1}{2}\sum_{\substack{k=1 \\ (k\neq i,j)}}^{l} \mathrm{tr}[\sigma_{ij}^{kT}(t,\omega_k)\nabla_{\omega_{ij}\omega_{ij}} v_{ij}(t,\omega_{ij})\sigma_{ij}^{k}(t,\omega_k)], \\
&\quad (i<j)\in[1,l],
\end{aligned} \tag{4.4.11}$$

where $\nabla_u = \dfrac{\partial}{\partial u}$, and δ_{ij} is the Kronecker symbol.

REMARK 4.4.3. If, in particular, $\sigma_{ij}(t,\omega) = 0$ for all $i \neq j$, then (4.4.10) and (4.4.11) become

$$\frac{dE[v_{ii}(t,\omega_i)]}{dt} = \frac{dE_i[v_{ii}(t,\omega_i)]}{dt} + (g_i(t,\omega))^{\mathrm{T}}\nabla_{\omega_i} v_{ii}(t,\omega_i), \quad i \in [1,l]; \tag{4.4.12}$$

$$\frac{dE[v_{ij}(t,\omega)]}{dt} = \frac{dE_{ij}[v_{ij}(t,\omega_{ij})]}{dt} + (\overline{g}_{ij}(t,\omega))^{\mathrm{T}}\nabla_{\omega_{ij}} v_{ij}(t,\omega_{ij}), \quad (i<j) \in [1,l] \tag{4.4.13}$$

Thus, the structure of averaged derivative (4.4.10), (4.4.11) represents adequately the hierarchical dependence of subsystems in large-scale system (4.4.1).

4.4.3 Sufficient Conditions for Stability to Probability of Stochastic Ito System

To formulate sufficient conditions for stability with respect to the probability of system (4.4.1) we make some assumptions on the system.

Assumption 4.4.3. The system (4.4.1) allows first and second level decompositions and

(1) independent subsystems (4.4.3) possess property $A(\mathcal{N}_i)\ \forall\, i \in [1, l]$;
(2) independent couples (i, j) of subsystems (4.4.7) possess property $A(\mathcal{N}_i \times \mathcal{N}_j)\ \forall\, (i \neq j) \in [1, l]$.

Remark 4.4.4. If for the system (4.4.1) there exist p and q $(p < q) \in [1, l]$ for which no free couple (p, q) of (4.4.7) can be found, then we take $v_{pq}(t, x_{pq}) \equiv 0$.

Assumption 4.4.4. There exist time-invariant neighborhoods $\mathcal{N}_i \subseteq R^{n_i}$ and $\mathcal{N}_i \times \mathcal{N}_i \subseteq R^{n_i} \times R^{n_j}$ of states $\omega_i = 0$ and ω_{ij} respectively, constants b_{ij}, d_i, $\gamma_{qp}^{ij} = \gamma_{qp}^{ji}$, α_{ij}, ν_{ij}^{k}, μ_{ij}^{k} and functions $\varphi_{i3} \in K$ such that estimates

(1) $g_i^{\mathrm{T}} \nabla_{\omega_i} v_{ii}(t, \omega_i) \leq \psi_{i3}^{1/2}(\|\omega_i\|) \sum_{k=1}^{l} b_{ik} \psi_{k3}^{1/2}(\|\omega_k\|)$,

(2) $\overline{g}_{ij}^{\mathrm{T}} \nabla_{\omega_{ij}} v_{ij}(t, \omega_{ij}) \leq \sum_{\substack{k=1\\ p=k}}^{l} \gamma_{kp}^{ij} \psi_{k3}^{1/2}(\|\omega_k\|) \psi_{p3}^{1/2}(\|\omega_p\|)$;

(3) $(u^i)^{\mathrm{T}} \nabla_{\omega_i \omega_i} v_{ii}(t, \omega_i) u^i \leq d_i \|u^i\|^2$;

(4) $(u_k^i)^{\mathrm{T}} \nabla_{\omega_{ij} \omega_{ij}} v_{ij}(t, \omega_{ij}) u_k^i \leq \nu_{ij}^{k} \|u_k^i\|^2$;

(5) $\|\sigma_{ij}(t, \omega_j)\|^2 \leq \alpha_{ij} \psi_{j3}(\|\omega_j\|)$;

(6) $\|\sigma_{ij}^{k}(t, \omega_k)\|^2 \leq \mu_{ij}^{k} \psi_{k3}(\|\omega_k\|)$,

are satisfied for all u_k^i, $\omega_i \in R^{n_i}$, $\omega_{ij} \in R^{n_i} \times R^{n_j}$, $t \in \mathcal{T}$, $(i \neq j) \in [1, l]$, p, $k = 1, 2, \ldots, l$.

An important part in the structure of averaged derivative of the function (4.2.15) is played by a symmetric $l \times l$ matrix

$$S = \frac{1}{2}(\overline{S} + \overline{S}^{\mathrm{T}}),$$

where $\overline{S}$ is an upper triangle matrix with elements $\overline{s}_{pq}$ defined as

$$\overline{s}_{pp} = \eta_p^2\Big(\rho_p + b_{pp} + \frac{1}{2} \sum_{\substack{i=1\\ i \neq p}}^{l} d_p \alpha_{pi}\Big) + 2\eta_p \sum_{i=p+1}^{l} \eta_i \beta_{pi}^{1}$$
$$+ 2\eta_p \sum_{i=p+1}^{l} \gamma_{pp}^{pi} \eta_i + 2\eta_p \sum_{i=1}^{l} \beta_{pi}^{3} \eta_i + \frac{1}{2}\eta_p \sum_{k,j=1}^{l} \mu_{pj}^{k} \nu_{pj}^{k} \eta_k;$$

$$\overline{s}_{pq} = \eta_p^2 b_{pq} + 4\beta_{pq}^2 \eta_p \eta_q + \sum_{k=1}^{l} \sum_{j=k+1}^{l} \gamma_{pq}^{kj} \eta_k \eta_j,$$

$$\overline{s}_{qp} = 0, \quad (p < q) \in [1, l], \quad \eta \in R_+^l, \quad \eta > 0.$$

Sufficient conditions for stability with probability of the system (4.4.1) are obtained in terms of the function (4.4.9) being applied in construction of the function

$$V(t, \omega) = \eta^{\mathrm{T}} \Pi(t, \omega) \eta, \quad \eta \in R_+^l, \quad \eta > 0 \tag{4.4.14}$$

Namely, we shall prove the following result.

Theorem 4.4.1. *Let the perturbed motion of the equation (4.4.1) are such that:*

(1) $\{\xi(t), t \in \mathcal{T}\}$ *is a normalized Wienner process and* $\sigma_{ij}(t,\omega) \neq 0$, $\forall\,(i,j) \in [1,l]$;
(2) *all conditions of Assumptions 4.4.1–4.4.4 are satisfied*
(3) *the matrix* S *is*
   (a) *negative semi-definite;*
   (b) *negative definite.*

Then the equilibrium state $\omega = 0$ *of system (4.4.1) is*

(a) *stable in probability;*
(b) *asymptotically stable in probability.*

Proof. We take the functions $v_{ij}(t,\cdot)$ according to Assumption 4.4.1 and a vector $\eta \in R^l_+$, $\eta > 0$. The function (4.4.14) in coordinate form is

$$\begin{aligned} V(t,\omega) &= \sum_{i=1}^{l} \eta_i^2 v_{ii}(t,\omega_i) + \sum_{\substack{i,j=1\\(i\neq l)}}^{l} \eta_i\eta_j v_{ij}(t,\omega_{ij}) \\ &= \sum_{i=1}^{l} \eta_i^2 v_{ii}(t,\omega_i) + 2\sum_{i=1}^{l}\sum_{j=i+1}^{l} \eta_i\eta_j v_{ij}(t,\omega_{ij}) \end{aligned} \tag{4.4.15}$$

Assumption 4.4.1 implies that at the presence of properties $\mathcal{A}(\mathcal{N}_i)$ and $\mathcal{A}(\mathcal{N}_i \times \mathcal{N}_j)$ the bilateral estimate

$$\sum_{i=1}^{l} \eta_i^2 \psi_{i1}(\|\omega_i\|) + \sum_{\substack{i,j=1\\(i\neq j)}}^{l} \eta_i \eta_j \psi_{ij}^1(\|\omega_{ij}\|) \le V(t,\omega)$$
$$\le \sum_{i=1}^{l} \eta_i^2 \psi_{i2}(\|\omega_i\|) + \sum_{\substack{i,j=1\\(i\neq j)}}^{l} \eta_i \eta_j \psi_{ij}^2(\|\omega_{ij}\|)$$

is valid for function (4.4.15) when all $\omega_i \in \mathcal{N}_i$, $\omega_{ij} \in \mathcal{N}_i \times \mathcal{N}_j$ and $t \in \mathcal{T}$.

Since $\psi_{i1}, \psi_{i2} \in K$ and $\psi_{ij}^1, \psi_{ij}^2 \in K$, then the function $V(t,\omega)$ is positive definite and decreasing. Moreover, functions $\Psi_1(\|\omega\|)$ and $\Psi_2(\|\omega\|) \in K$ can be found such that

$$\Psi_1(\|\omega\|) \le V(t,\omega) \le \Psi_2(\|\omega\|) \tag{4.4.16}$$

for all $\omega \in \mathcal{N} = \mathcal{N}_1 \times \cdots \times \mathcal{N}_l$, $t \in \mathcal{T}$.

For the function (4.4.14) the averaged derivative $\dfrac{dE[V]}{dt}$ along the solutions of (4.4.1) is

$$\frac{dE[V(t,\omega)]}{dt} = \eta^{\mathrm{T}} \frac{dE[\Pi(t,\omega)]}{dt} \eta = \sum_{i=1}^{l} \eta_i^2 \left(\frac{dE_i[v_{ii}(t,\omega)]}{dt} + g_i^{\mathrm{T}} \nabla_{\omega_i} v_{ii}(t,\omega_i) \right.$$
$$+ \frac{1}{2} \sum_{\substack{j=1\\(j\neq l)}}^{l} \mathrm{tr}\left[\sigma_{ij}^{\mathrm{T}}(t,\omega_j) \nabla_{\omega_i\omega_i} v_{ii}(t,\omega_i) \sigma_{ij}(t,\omega_j)\right]$$
$$+ 2 \sum_{i=1}^{l} \sum_{j=i+1}^{l} \eta_i \eta_j \left(\frac{dE_{ij}[v_{ij}(t,\omega_{ij})]}{dt} \right.$$
$$\left. + \frac{1}{2} \sum_{\substack{k=1\\(k\neq i,j)}}^{l} \mathrm{tr}\left[(\sigma_{ij}^k)^{\mathrm{T}} \nabla_{\omega_{ij}\omega_{ij}} v_{ij}(t,\omega_{ij}) \sigma_{ij}^k\right] \right).$$

for all $t \in \mathcal{T}$ and $\omega_i \in \mathcal{N}_i$, $\omega_{ij} \in \mathcal{N}_i \times \mathcal{N}_j$.

In view of conditions (c) of Assumptions 4.4.3 and 4.4.4 we get the estimate

$$
\begin{aligned}
\frac{dE[V(t,\omega)]}{dt} \le & \sum_{i=1}^{l} \eta_i^2 [P_i \Psi_{i3}(\|\omega_i\|) \\
& + \Psi_{i3}^{1/2}(\|\omega_i\|) \sum_{k=1}^{l} b_{ik} \Psi_{k3}^{1/2}(\|\omega_k\|) + \frac{1}{2} \sum_{\substack{k=1 \\ (k\neq i)}}^{l} d_k \alpha_{ik} \Psi_{k3}(\|\omega_k\|)] \\
& + 2 \sum_{i=1}^{l} \sum_{j=i+1}^{l} \eta_i \eta_j [\beta_{ij}^1 \Psi_{i3}(\|\omega_i\|) + 2\beta_{ij}^2 \Psi_{i3}^{1/2} \Psi_{j3}^{1/2} \qquad (4.4.17) \\
& + \beta_{ij}^3 \Psi_{j3}(\|\omega_j\|) + \sum_{k=1,p=k}^{l} \gamma_{kp}^{ij} \Psi_{k3}^{1/2}(\|\omega_k\|) \Psi_{p3}^{1/2}(\|\omega_p\|) \\
& + \frac{1}{2} \sum_{\substack{k=1 \\ (k\neq j,i)}}^{l} \nu_{ij}^k \mu_{ij}^k \Psi_{k3}(\|\omega_k\|)]
\end{aligned}
$$

for all $t \in \mathcal{T}$ and $\omega_i \in \mathcal{N}_i$.

With regard to the structure of the matrix S we get from estimate (4.4.12)

$$\frac{dE[V(t,\omega)]}{dt} \leq \Psi^{\mathrm{T}}(\|\omega\|)S\Psi(\|\omega\|) \tag{4.4.18}$$

where $\Psi(\|\omega\|) = \Big(\psi_{13}^{1/2}(\|\omega_1\|), \dots, \psi_{l3}^{1/2}(\|\omega_l\|)\Big)^{\mathrm{T}}$.

Since by condition (3)(a) of Theorem 4.4.1 the matrix S is negative semi-definite, then $\lambda_M(S) \leq 0$ and

$$\frac{dE[V(t,\omega)]}{dt} \leq \lambda_M(s) \sum_{i=1}^{l} \psi_{i3}(\|\omega_i\|)$$

for all $t \in \mathcal{T}$ and $\omega_i \in \mathcal{N}_i$.

Since $\varphi_{i3} \in K$, there exists a comparison function $\Psi_3(\|\omega\|) \in K$ such that

$$\sum_{i=1}^{l} \psi_{i3}(\|\omega_i\|) \leq \Psi_3(\|\omega\|)$$

for all $\omega_i \in \mathcal{N}_i$ and $\omega \in \mathcal{N} = \mathcal{N}_1 \times \dots \mathcal{N}_l$.

Hence

$$\frac{dE[V(t,\omega)]}{dt} \le \lambda_M(S)\Psi_3(\|\omega\|) \tag{4.4.19}$$

is negative semi-definite for all $t \in \mathcal{T}$ and $\omega \in \mathcal{N}$.

Thus all conditions of Theorem 4.3.1 from Section 4.3 are satisfied, and the equilibrium state $\omega = 0$ of system (4.4.1) is stable in probability.

To verify assertion (b) of Theorem 4.4.1 it is sufficient to note that under condition (3)(b) in the estimate (4.4.18) $\lambda_M < 0$. Then according to inequality (4.4.19) all hypotheses of Theorem 4.3.2 are satisfied and the equilibrium state $\omega = 0$ of system (4.4.1) is asymptotically stable in probability.

The Theorem 4.4.1 is proved.

Assumption 4.4.5. The system (4.4.1) allows the first and the second level decompositions and

(1) independent subsystems (4.4.3) possess the property $B_j(\infty)$, $j \in [1, l]$;
(2) independent couples (i, j) of the subsystems (4.4.7) possess the property $A_{ij}(\infty)$, $\forall\,(i \neq j) \in [1, l]$.

Theorem 4.4.2. *Let the perturbed motion of the equations (4.4.1) are such that*

(1) $\{\xi(t), t \in \mathcal{T}\}$ *is a normalized process and* $\sigma_{ij}(t, \omega_j) \neq 0$ $\forall\,(i, j) \in [1, l]$*;*
(2) *all conditions of Assumption 4.4.5 are satisfied;*
(3) *the conditions of Assumption 4.4.4 are satisfied for* $\mathcal{N}_i = R^{n_i}$, $\mathcal{N}_i \times \mathcal{N}_j = R^{n_i} \times R^{n_j}$ *with functions* $\varphi_{i3} \in KR$, $i \in [1, l]$*;*
(4) *the matrix* S *is negative definite.*

Then, the equilibrium state $\omega = 0$ *of system (4.4.1) is asymptotically stable in probability in the whole.*

Proof. Under the conditions of Assumption 4.4.5 the function (4.4.14) satisfies estimates (4.2.20) and its averaged derivative (4.2.27) satisfies inequality (4.4.18) where functions $\psi_{i3} \in KR$. In consequence of condition (4) of Theorem 4.4.2 and estimate (4.4.19), $\dfrac{dE[V(t,\omega)]}{dt}$ is negative definite for all $t \in \mathcal{T}$ and $\omega \in R^n$. Thus, all conditions of Theorem 4.3.3 are satisfied and the equilibrium state $\omega = 0$ of system (4.4.1) is asymptotically stable in probability in the whole.

Remark 4.4.5. If for the perturbed motion the equations (4.4.1) are such that $\sigma_{ij}(t, \omega_j) = 0$ for all $(i, j) \in [1, l]$, i.e. random interconnections between the subsystems are absent, then the structure of matrix S is simplified and its elements are:

$$\overline{s}_{pp} = \eta_p^2(\rho_p + b_{pp}) + 2\eta_p \sum_{i=p+1}^{l} \eta_i \beta_{pi}^1 + 2\eta_p \sum_{i=p+1}^{l} \gamma_{pp}^{pi}\eta_i + 2\eta_p \sum_{i=1}^{l} \beta_{pi}^3 \eta_i,$$

$$\overline{s}_{pq} = \eta_p^2 b_{pq} + 4\beta_{pq}^2 \eta_p \eta_q + \sum_{k=1}^{l} \sum_{j=k+1}^{l} \gamma_{pq}^{kj} \eta_k \eta_j, \qquad \forall\, (p < q) \in [1, l];$$

$$\overline{s}_{qp} = 0, \qquad p < q.$$

Here $\eta_p,\ p \in [1, l]$ are components of vector $\eta \in R_+^l$.

4.5 Applications

In this section general results on stochastic stability are applied in the investigation of some real processes models.

4.5.1 Stochastic Version of the Lefschetz Problem

The following problem is a development of the Lefschetz [100] problem we dealt with in Chapter 2.

Let us decompose system (4.2.1) into two subsystems

$$\begin{aligned} \frac{dp}{dt} &= X(t, p, 0, y(t)) + F(t, p, q, y(t)), \\ \frac{dq}{dt} &= X(t, 0, q, y(t)) + G(t, p, q, y(t)), \end{aligned} \tag{4.5.1}$$

where $p \in R^{n_1}$, $q \in R^{n_2}$, $X \in C[T_0 \times B_p, R[\Omega, R^{n_1}]]$, $Y \in C[T_0 \times B_q, R[\Omega, R^{n_2}]]$, $F \in C[T_0 \times B, R[\Omega, R^{n_1}]]$, $G \in C[T_0 \times B, R[\Omega, R^{n_2}]]$, X, F, Y, G vanish if and only if $p = 0$ and $q = 0$ respectively.

Assumption 4.5.1. There exist time-invariant neighborhoods $\mathcal{N}_p \subseteq R^{n_1}$, $\mathcal{N}_q \subseteq R^{n_2}$ of the equilibrium states $p = 0$, $q = 0$ respectively, and a matrix-valued function $\Pi(t,x,y)$ with elements v_{kl} $k,\, l = 1,2$ such that

$$
\begin{aligned}
&\underline{\alpha}_{11}\zeta_1^2(\|p\|) \le v_{11}(t,p,y) \le \overline{\alpha}_{11}\zeta_2^2(\|p\|) \quad \forall p \in \mathcal{N}_{p0}, \forall y \in Y;\\
&\underline{\alpha}_{22}\zeta_1^2(\|q\|) \le v_{22}(t,q,y) \le \overline{\alpha}_{22}\zeta_2^2(\|q\|) \quad \forall q \in \mathcal{N}_{q0}, \forall y \in Y;\\
&\underline{\alpha}_{12}\zeta_1(\|p\|)\psi_1(\|q\|) \le v_{12}(t,p,q,y) \le \overline{\alpha}_{12}\zeta_2(\|p\|)\psi_2(\|q\|)\\
&\qquad\qquad \forall (p,q,y) \in \mathcal{N}_{p0} \times \mathcal{N}_{q0} \times Y
\end{aligned}
\tag{4.5.2}
$$

where $\mathcal{N}_{p0} = \{p \in \mathcal{N}_p,\ p \ne 0\}$, $\mathcal{N}_{q0} = \{q \in \mathcal{N}_q,\ q \ne 0\}$, $\overline{\alpha}_{kk}, \underline{\alpha}_{kk} = \text{const} > 0$, $\underline{\alpha}_{12}, \overline{\alpha}_{12} = \text{const}$, $k = 1,2$; ζ_k, ψ_k are functions of class K.

If conditions of the Assumption 4.5.1 are satisfied, properties of the function (4.4.14) (property of having a fixed sign, the existence of an infinitely small upper bound; an infinitely large lower bound) are defined by properties of matrices $A = H^{\mathrm{T}} A_1 H$; $B = H^{\mathrm{T}} A_2 H$ where

$$
A_1 = [\underline{\alpha}_{kl}], \quad A_2 = [\overline{\alpha}_{kl}], \quad H = \operatorname{diag}(\eta_1, \eta_2), \qquad k,\, l = 1,2. \tag{4.5.3}
$$

We introduce the designation

$$
\Delta(v_{kl}) = \sum_{i \ne j} \alpha_{ij}[v_{kl}(t,\cdot,i) - v_{kl}(t,\cdot,j)], \qquad k,\, l = 1,2
$$

Assumption 4.5.2. There exist constants ρ_{kr}, $k = 1,2$; $r = 1,2,\ldots,10$ and functions $\zeta(\|p\|)$, $\psi(\|q\|)$ of class $K(KR)$ such that

$$
\begin{aligned}
\nabla_t v_{11} + (\nabla_p v_{11}^{\mathrm{T}})X + \frac{1}{2}\Delta(v_{11}) &\le \rho_{11}\zeta^2 + h_{11}(\zeta,\psi),\\
\nabla_t v_{22} + (\nabla_q v_{22}^{\mathrm{T}})Y + \frac{1}{2}\Delta(v_{22}) &\le \rho_{12}\psi^2 + h_{21}(\zeta,\psi),\\
(\nabla_p v_{11}^{\mathrm{T}})F + \frac{1}{2}\Delta(v_{11}) &\le \rho_{12}\zeta^2 + \rho_{13}\zeta\psi + \rho_{14}\psi^2 + h_{12}(\zeta,\psi),\\
(\nabla_q v_{22}^{\mathrm{T}})G + \frac{1}{2}\Delta(v_{22}) &\le \rho_{22}\zeta^2 + \rho_{23}\zeta\psi + \rho_{24}\psi^2 + h_{22}(\zeta,\psi),\\
\nabla_t v_{12} + (\nabla_p v_{12}^{\mathrm{T}})X + \frac{1}{4}\Delta(v_{12}) &\le \rho_{15}\zeta^2 + \rho_{16}\zeta\psi + \rho_{17}\psi^2 + h_{13}(\zeta,\psi),\\
(\nabla_q v_{12}^{\mathrm{T}})Y + \frac{1}{4}\Delta(v_{12}) &\le \rho_{25}\zeta^2 + \rho_{25}\zeta\psi + \rho_{27}\psi^2 + h_{23}(\zeta,\psi),\\
(\nabla_p v_{12}^{\mathrm{T}})F + \frac{1}{4}\Delta(v_{12}) &\le \rho_{18}\zeta^2 + \rho_{19}\zeta\psi + \rho_{110}\psi^2 + h_{14}(\zeta,\psi),\\
(\nabla_q v_{12}^{\mathrm{T}})G + \frac{1}{4}\Delta(v_{12}) &\le \rho_{28}\zeta^2 + \rho_{29}\zeta\psi + \rho_{210}\psi^2 + h_{24}(\zeta,\psi),
\end{aligned}
$$

where $h_{sk}(\zeta,\psi)$, $k = 1,2$; $s = 1,2,3,4$ are polynomials with respect to ζ, ψ containing additives of power higher than two.

Proposition 4.5.1. *If all hypotheses of Assumption 4.5.2 are satisfied and*

(a) *the matrix* $C = [c_{ij}],\ c_{ij} = c_{ji},\ i \neq j;\ i, j = 1, 2$ *with elements*

$$
\begin{aligned}
c_{11} &= \eta_1^2(\rho_{11} + \rho_{12}) + \eta_2^2\rho_{22} + 2\eta_1\eta_2(\rho_{15} + \rho_{18} + \rho_{25} + \rho_{28}),\\
c_{22} &= \eta_1^2\rho_{14} + \eta_2^2(\rho_{21} + \rho_{24}) + 2\eta_1\eta_2(\rho_{17} + \rho_{110} + \rho_{27} + \rho_{210}),\\
c_{12} &= \frac{1}{2}(\eta_1^2\rho_{13} + \eta_2^2\rho_{23}) + \eta_1\eta_2(\rho_{16} + \rho_{19} + \rho_{26} + \rho_{29}),
\end{aligned}
$$

is negative definite, then due to system (4.5.1) averaged derivative

$$\eta^{\mathrm{T}} \frac{dE[\Pi]}{dt} \eta = \frac{dE[V]}{dt}, \qquad \eta \in R^2_+ \tag{4.5.4}$$

is a negative definite function.

If besides hypothesis (a), hypothesis (b) is satisfied, hypotheses of Assumptions 4.5.1 and 4.5.2 are satisfied

$$\begin{aligned} h(\zeta,\psi) = \eta_1^2(h_{11}+h_{12}) + \eta_2^2(h_{21}+h_{22}) \\ + 2\eta_1\eta_2(h_{13}+h_{14}+h_{23}+h_{24}) \leq 0 \end{aligned} \tag{4.5.5}$$

for $p \in R^{n_1}$, $q \in R^{n_2}$, $n_1 + n_2 = n$ *and for functions* $\zeta(\|p\|) \in KR$, $\psi(\|q\|) \in KR$, *then the averaged derivative (4.5.4) is negative definite in the whole.*

Theorem 4.5.1. *If the system of equations of perturbed motion (4.5.1) is such that all hypotheses of Assumptions 4.5.1 and 4.5.2 (a) are satisfied and matrices* A *and* B *are positive definite and matrix* C *is negative definite, the equilibrium state* $p = 0$, $q = 0$ *of the system (4.5.1) is asymptotically stable with respect to probability.*

If in Assumption 4.5.1 and 4.5.2 $\mathcal{N}_p = R^{n_1}$, $\mathcal{N}_q = R^{n_2}$ *the functions* $\zeta(\|p\|)$ *and* $\psi(\|q\|)$ *are of class* KR, *the equilibrium state* $p = 0$, $q = 0$ *is asymptotically stable with respect to probability in the whole.*

The assertions of Theorem 4.5.1 are implied by estimate

$$\frac{dE[V]}{dt} \leq \xi^{\mathrm{T}} C \xi + h(\xi) \tag{4.5.6}$$

where $\xi = (\zeta,\psi)^{\mathrm{T}}$ and by the fact that if the hypotheses of Theorem 4.5.1 are satisfied, the hypotheses of Theorems 4.3.2 and 4.3.3 are satisfied respectively.

4.5.2 Stability in Probability of Oscillating System

Let us consider for an oscillating system the perturbed motion equations which are of the form

$$\begin{aligned}\frac{dp}{dt} &= A_1(y)p + f_1(p,q,r,y(t)),\\ \frac{dq}{dt} &= A_2(y)q + f_2(p,q,r,y(t)),\\ \frac{dr}{dt} &= A_3(y)r + f_3(p,q,r,y(t)).\end{aligned} \tag{4.5.7}$$

Here $p,\ q,\ r \in R^2,\ f_i \in C(B, R[Y, R^2])$,

$$A_i(y) = \begin{pmatrix} 0 & 1 \\ -b_i(y) & -a_i(y) \end{pmatrix}, \qquad i = 1,2,3$$

.

The functions $a_i(y)$ and $b_i(y)$ are bounded and $y(t)$ is a homogeneous Markov chain with a finite number of states $Y = \{y_1, \dots, y_r\}$ and with transitional probabilities

$$p_{ij}(\tau) = \alpha_{ij}\tau + o(\tau), \qquad \alpha_{ij} = \text{const} \quad (i \neq j) \in [1, r].$$

We designate $b_i(y_k) = b_k^i,\ a_i(y_k) = a_k^i$ and assume that $b_k^i > 0$. Matrix-valued function $\Pi(p,q,r,y(t))$ elements $v_{ik}(\cdot)$ are taken in the form

$$\begin{aligned}
v_{11}(p, y_k) &= p^{\mathrm{T}} \operatorname{diag}\left(1, \frac{1}{b_k^1}\right) p,\\
v_{22}(q, y_k) &= q^{\mathrm{T}} \operatorname{diag}\left(1, \frac{1}{b_k^2}\right) q,\\
v_{33}(r, y_k) &= r^{\mathrm{T}} \operatorname{diag}\left(1, \frac{1}{b_k^3}\right) r,\\
v_{12}(p, q, y_k) &= p^{\mathrm{T}} \operatorname{diag}\left(1, \frac{0,1}{b_k^1}\right) q,\\
v_{13}(p, r, y_k) &= p^{\mathrm{T}} \operatorname{diag}\left(1, \frac{0,1}{b_k^3}\right) r,\\
v_{23}(q, r, y_k) &= q^{\mathrm{T}} \operatorname{diag}\left(1, \frac{0,1}{b_k^2}\right) r,\\
v_{ij}(\cdot) &= v_{ij}(\cdot) \quad \forall\, (i \neq j) \in [1, 3].
\end{aligned} \tag{4.5.8}$$

It is easy to notice that for functions (4.5.8) the following estimates are valid

(a) if $0 < b_k^i \le 1,\ i = 1, 2, 3,$ then

$$
\begin{aligned}
v_{11}(p, y_k) &\ge \|p\|^2; & v_{22}(q, y_k) &\ge \|q\|^2;\\
v_{33}(r, y_k) &\ge \|r\|^2; & v_{12}(p, q, y_k) &\ge -0,1\|p\|\|q\|;\\
v_{13}(p, r, y_k) &\ge -0,1\|p\|\|r\|; & v_{23}(q, r, y_k) &\ge -0,1\|q\|\|r\|;
\end{aligned}
$$

(b) if $b_k^i > 1,\ i = 1, 2, 3,$ then

$$
\begin{aligned}
v_{11}(p, y_k) &\ge \frac{1}{b_k^1}\|p\|^2; & v_{22}(q, y_k) &\ge \frac{1}{b_k^2}\|q\|^2;\\
v_{33}(r, y_k) &\ge \frac{1}{b_k^3}\|r\|^2; & v_{12}(p, q, y_k) &\ge -\frac{0,1}{b_k^1}\|p\|\|q\|;\\
v_{13}(p, r, y_k) &\ge -\frac{0,1}{b_k^3}\|p\|\|r\|; & v_{23}(q, r, y_k) &\ge -\frac{0,1}{b_k^2}\|q\|\|r\|.
\end{aligned}
$$

For the function

$$V(p,q,r,y(t)) = \eta^{\mathrm{T}}\Pi(p,q,r,y(t))\eta, \tag{4.5.9}$$

where $\eta \in R^3_+$ the matrix A_1 in the estimate of (4.2.20) has the form

$$\overline{A}_1(y_k) = \begin{cases} \begin{pmatrix} 1 & -0,1 & -0,1 \\ -0,1 & 1 & -0,1 \\ -0,1 & -0,1 & 1 \end{pmatrix}, & \text{if } 0 < b_k^i \le 1, \quad i = 1,2,3; \\ \begin{pmatrix} \frac{1}{b_k^1} & -\frac{0,1}{b_k^1} & -\frac{0,1}{b_k^3} \\ -\frac{0,1}{b_k^1} & \frac{1}{b_k^2} & -\frac{0,1}{b_k^2} \\ -\frac{0,1}{b_k^1} & -\frac{0,1}{b_k^2} & \frac{1}{b_k^3} \end{pmatrix}, & \text{if } b_k^i > 1, \quad i = 1,2,3. \end{cases}$$

The matrix $\overline{A}_1$ is positive definite, if

$$\frac{b_k^1}{b_k^3} + \frac{b_k^3}{b_k^2} + \frac{b_k^2}{b_k^1} < 99,8, \qquad k = 1,2,\dots,r. \tag{4.5.10}$$

For the averaged derivative $\dfrac{dE[v_{ij}(\cdot)]}{dt}$ of the function $\Pi(p,q,r,y(t))$ with elements (4.5.8) it is easy to establish estimate in the form

$$\frac{dE[V(p,q,r,y(t))]}{dt} \le u^{\mathrm{T}} S u \tag{4.5.11}$$

where $u = (\|p\|, \|q\|, \|r\|)^{\mathrm{T}}$, $\eta = (1,1,1)^{\mathrm{T}}$ and matrix S elements are

(4.5.12)

$$c_{ii}(y_k) = -\Big(\frac{2a_k^i}{b_k^i} - \Delta b^i - \sum_{l=1}^{3} d_{ii}^l\Big), \quad i = 1,2,3; \quad k = 1,2,\dots,r;$$

$$c_{12}(y_k) = \frac{1}{2}\Big(\sum_{l=1}^{6} d_{12}^l + \frac{0,1}{b_k^1}|a_k^1 + a_k^2| + 0,1|\Delta b^1|\Big), \quad k = 1,2,\dots,r;$$

$$c_{13}(y_k) = \frac{1}{2}\Big(\sum_{l=1}^{6} d_{13}^l + \frac{0,1}{b_k^3}|a_k^1 + a_k^3| + 0,1|\Delta b^3|\Big), \quad k = 1,2,\dots,r;$$

$$c_{23}(y_k) = \frac{1}{2}\Big(\sum_{l=1}^{6} d_{23}^l + \frac{0,1}{b_k^2}|a_k^2 + a_k^3| + 0,1|\Delta b^2|\Big), \quad k = 1,2,\dots,r,$$

where

$$\Delta b^i = \sum_{j\neq k}^{r}\Big(\frac{1}{b_j^i} - \frac{1}{b_k^i}\Big)\alpha_{kj}, \quad i = 1,2,3; \quad k = 1,2,\dots,r.$$

Here d_{ij}^k, $i, j \in [1,3]$, $k \in [1,6]$ are constants that are found when estimating $\dfrac{dE[v_{ij}(\cdot)]}{dt}$.

The matrix S with elements (4.5.12) is negative definite if

(a) $$\frac{2a_k^i}{b_k^i} - \Delta b^i > \sum_{l=1}^{3} d_{ii}^l, \quad i = 1,2,3; \quad k = 1,2,\dots,r;$$

(b) $$\prod_{i=1}^{2}\Big(\frac{2a_k^i}{b_k^i} - \Delta b^i - \sum_{l=1}^{3} d_{ii}^l\Big) > \frac{1}{4}\Big(\sum_{l=1}^{6} d_{12}^l + W_k^1(a_k^1, a_k^2, b_k^1, b^1)\Big);$$

(c) $$\prod_{i=1}^{3}\Big(\frac{2a_k^i}{b_k^i} - \Delta b^i - \sum_{l=1}^{3} d_{ii}^l\Big)$$
$$+ \frac{1}{4}\Big(\sum_{l=1}^{6} d_{12}^l + W_k^1(a_k^1, a_k^2, b_k^1, b^1)\Big)\Big(\sum_{l=1}^{6} d_{13}^l + W_k^3(a_k^1, a_k^3, b_k^3, b^3)\Big)$$
$$< \frac{1}{4}\Big(\sum_{l=1}^{6} d_{13}^l + W_k^3(a_k^1, a_k^3, b_k^3, b^3)\Big)\Big(\sum_{l=1}^{3} d_{22}^l + \Delta b^2 - \frac{2a_k^2}{b_k^2}\Big)$$
$$+ \frac{1}{4}\Big(\sum_{l=1}^{6} d_{23}^l + W_k^2(a_k^2, a_k^3, b_k^2, b^2)\Big)\Big(\sum_{l=1}^{3} d_{22}^l + \Delta b^1 - \frac{2a_k^1}{b_k^1}\Big)$$

$$+\frac{1}{4}\Big(\sum_{l=1}^{6} d_{12}^{l} + W_k^1(a_k^1, a_k^2, b_k^1, b^1)\Big)^2\Big(\sum_{l=1}^{3} d_{33}^{l} + \Delta b^3 - \frac{2a_k^3}{b_k^3}\Big),$$
$$k = 1, 2, \dots, r,$$

where

$$W_k^1(a_k^1, a_k^2, b_k^1, b^1) = \frac{0,1}{b_k^1}|a_k^1 + a_k^2| + 0,1|\Delta b^1|;$$
$$(4.5.13)\quad W_k^2(a_k^2, a_k^3, b_k^2, b^2) = \frac{0,1}{b_k^2}|a_k^2 + a_k^3| + 0,1|\Delta b^2|;$$
$$W_k^3(a_k^1, a_k^3, b_k^3, b^3) = \frac{0,1}{b_k^3}|a_k^1 + a_k^3| + 0,1|\Delta b^3|, \qquad k = 1, 2, \dots, r.$$

Thus, under conditions (4.5.10) the function (4.5.9) is positive definite, and when inequalities (a)–(c) are satisfied its averaged derivative (4.5.11) is negative definite.

Applying Theorem 4.3.3 we conclude that conditions (4.5.10) and (a)–(c) are sufficient for stability in probability in the whole of the equilibrium state $p = q = r = 0$ of oscillating system (4.5.7).

4.5.3 Stability in Probability of a Regulation System

We consider an autonomous stochastic regulation system

$$d\omega_i = \sum_{j=1}^{l} A_{ij}\omega_j dt + \sigma_i(\omega_i)dz_i + b_i f_i(\theta_i)dt, \qquad i \in [1, l], \tag{4.5.14}$$

where $\theta_i = \sum\limits_{k=1}^{l} c_{ik}^{\mathrm{T}}\omega_k$, $b_i, \omega_i \in R^{n_i}$, $c_{ik} \in R^{n_k}$, A_{ij} are constant matrices of the corresponding to vector ω_j dimensions, $\{z_i(t),\, t \in \mathcal{T}\}$, is a m_i-dimensional Wienner process. Besides, $f_i(\theta_i) = 0$, if and only if $\theta_i = 0$, $0 \le f_i(\theta_i) < k_i\theta_i^2$ provided $\theta_i \ne 0$.

First level decomposition results in the system

$$d\omega_i = A_{ii}\omega_i dt + \sigma_i(\omega_i)dz_i + \sum_{\substack{j=1 \\ (j\ne i)}}^{l} A_{ij}\omega_j dt + b_i f_i(\theta_i)dt, \quad i \in [1, l], \tag{4.5.15}$$

with the independent subsystems

$$d\omega_i = A_{ii}\omega_i dt + \sigma_i(\omega_i)dz_i \quad i \in [1, l], \tag{4.5.16}$$

and link functions

$$g_i(\omega) = \sum_{\substack{j=1\\(j\neq i)}}^{l} A_{ij}\omega_j dt + b_i f_i(\theta_i)dt, \qquad i \in [1, l]. \tag{4.5.17}$$

The second level decomposition yields

$$\begin{aligned} d\omega_i =& A_{ii}\omega_i dt + A_{ij}\omega_j dt + \sigma_i(\omega_i)dz_i \\ &+ \sum_{\substack{k=1\\(k\neq i,j)}}^{l} A_{ik}\omega_k dt + b_i f_i(\theta_i)dt; \\ d\omega_j =& A_{jj}\omega_j dt + A_{ji}\omega_i dt + \sigma_j(\omega_j)dz_j \\ &+ \sum_{\substack{k=1\\(k\neq i,j)}}^{l} A_{jk}\omega_k dt + b_j f_j(\theta_j)dt. \end{aligned} \tag{4.5.18}$$

Equations (4.5.18) are represented as

$$\begin{gathered} d\omega_{ij} = \overline{A}_{ij}\omega_{ij}dt + \sigma_{ij}dz_{ij} + \sum_{\substack{k=1\\(k\neq i,j)}}^{l} \overline{A}_{ij}^{k}\omega_k dt + B_{ij}dt, \\ (i<j) \in [1, l]. \end{gathered} \tag{4.5.19}$$

Here $\omega_{ij} = (\omega_i^{\mathrm{T}}, \omega_j^{\mathrm{T}})^{\mathrm{T}}$, $\omega_{ij} \in R^{n_i} \times R^{n_j}$ and matrices $\overline{A}_{ij}$, $\overline{A}_{ij}^{k}$, σ_{ij}, B_{ij} with dimensions $(n_i+n_j)\times(n_i+n_j)$, $(n_i+n_j)\times n_k$, $(n_i+n_j)\times(m_i+m_j)$, $(n_i+n_j)\times 1$ $\forall\,(i,j,k) \in [1,l]$ respectively, are defined by formulas

$$\begin{gathered} \overline{A}_{ij}^{k} = (A_{ik}^{\mathrm{T}}, A_{jk}^{\mathrm{T}})^{\mathrm{T}}; \qquad B_{ij} = (b_i^{\mathrm{T}} f_i(\theta_i), b_j^{\mathrm{T}} f_j(\theta_i)); \\ \sigma_{ij} = \operatorname{diag}(\sigma_i(\omega_i), \sigma_j(\omega_j)); \qquad \overline{A}_{ij} = \begin{pmatrix} A_{ii} & A_{ij} \\ A_{ji} & A_{jj} \end{pmatrix}. \end{gathered}$$

Alongside the systems (4.5.15) and (4.5.19) we shall consider the matrix-valued function

$$\Pi(\omega) = [\operatorname{diag}(v_{ii}(\omega_i)) + (v_{ij}(\omega_{ij}))], \quad i<j \in [1,l], \quad i = 1, 2, \dots, l \tag{4.5.20}$$

with the elements

$$v_{ii}(\omega_i) = \omega_i^{\mathrm{T}} P_{ii}\omega_i \qquad i \in [1,l];$$
$$v_{ij}(\omega_{ij}) = \omega_{ij}^{\mathrm{T}} P_{ij}\omega_{ij} \qquad (i<j) \in [1,l].$$

Matrices P_{ii} are found by Liapunov equations

(4.5.21) $$A_{ii}^{\mathrm{T}} P_{ii} + P_{ii}A_{ii} = -G_{ii}, \qquad i \in [1,l]$$

where G_{ii} are symmetric positive definite matrices of dimensions $n_i \times n_i$.

Matrices P_{ij} are also found by Liapunov equations

(4.5.22) $$\overline{A}_{ij}^{\mathrm{T}} P_{ij} + P_{ij}\overline{A}_{ij} = -G_{ij}, \qquad (i<j) \in [1,l]$$

where G_{ij} are symmetric positive definite matrices of dimensions $(n_i + n_j) \times (n_i + n_j)$.

The functions $v_{ii}(\omega_i)$ and $v_{ij}(\omega_{ij})$ are positive definite if matrices A_{ii} and $\overline{A}_{ij}$ are stable. We shall suppose that this condition is satisfied for the systems (4.5.15) and (4.5.19).

Now we introduce symmetric matrices of dimensions $n_p \times n_p$:

$$\begin{aligned}\Xi_{pp} = {}& \eta_p^2 G_{pp} + 2\eta_p^2 c_{pp} b_p^{\mathrm{T}} k_p^* P_{pp} + \eta_p \sum_{\substack{j=1\\(j\neq p)}}^{l} \eta_j (G_{pj}^p + G_{jp}^j)\\ & + 4\eta_p \sum_{j=p+1}^{l} \eta_j \big[c_{pp} b_p^{\mathrm{T}} k_p^* P_{pj}^{\mathrm{T}} + c_{jp} b_j^{\mathrm{T}} k_j^* \overline{P}_{pj}^{\mathrm{T}}\\ & + c_{jp} b_p^{\mathrm{T}} k_p^* \overline{P}_{pj} + c_{jj} b_j^{\mathrm{T}} k_j^* P_{jp}^j\big], \qquad p \in [1,l],\end{aligned}$$

and matrices of dimensions $n_p \times n_q$, $(p<q) \in [1,l]$:

$$\begin{aligned}\Xi_{pq} = {}& \eta_q^2 A_{qp}^{\mathrm{T}} P_{qq} + \eta_p^2 P_{pp} b_p^{\mathrm{T}} k_p^* c_{pq}^{\mathrm{T}} + 2\eta_p\eta_q \overline{G}_{pq}\\ & + \eta_p \sum_{\substack{j=q\\(j\neq p)}}^{l} \eta_j (A_{qp}^{\mathrm{T}} P_{qj}^q + A_{jp}^{\mathrm{T}} \overline{P}_{qj}^{\mathrm{T}}) + \eta_q \sum_{j=q}^{l} \eta_j (A_{jp}^{\mathrm{T}} \overline{P}_{jq} + A_{qp}^{\mathrm{T}} P_{jq}^j)\\ & + \eta_p \sum_{\substack{j=p\\(j\neq q)}}^{l} ({P_{pj}^p}^{\mathrm{T}} A_{pq} + \overline{P}_{pj}^{\mathrm{T}} A_{jq})\eta_j + \eta_p \sum_{\substack{j=p\\(j\neq q)}}^{l} \eta_j (\overline{P}_{jp}^{\mathrm{T}} A_{jq} + {P_{jq}^j}^{\mathrm{T}} A_{pq})\\ & + 2\eta_p \sum_{\substack{k=1\\(k\neq p,q)}}^{l} (c_{pk} b_p^{\mathrm{T}} k_p^* P_{pq}^p + c_{qk} b_q^{\mathrm{T}} k_q^* \overline{P}_{pq}^{\mathrm{T}}\\ & + c_{pk} b_p^{\mathrm{T}} k_p^* \overline{P}_{pq} + c_{qk} b_q^{\mathrm{T}} k_q^* P_{pq}^q)\eta_q, \qquad \eta_p \in R_+, \quad \eta_p > 0.\end{aligned}$$

Here

$$k_i^* = \begin{cases} k_i, & \text{for} \quad \sum_{k=1}^{l} \omega_k^{\mathrm{T}} c_{ik} b_i^{\mathrm{T}} P_{pq}^i \omega_i > 0 \\ & (\text{or} \quad \theta_i b_i^{\mathrm{T}} \overline{P}_{pq} \omega_i > 0, \ \text{or} \quad \theta_i b_i^{\mathrm{T}} P_{ii} \omega_i > 0); \\ 0, & \text{in the other cases.} \end{cases}$$

We designate by $\lambda_M(\Xi_{pp})$ and $\lambda_M^{1/2}(\Xi_{pq}\Xi_{pq})$ the maximal eigenvalue of matrix Ξ_{pp} and the norm of matrix $\Xi_{pq}^{\mathrm{T}}\Xi_{pq}$ respectively. For system (4.5.14) the following result is valid.

THEOREM 4.5.2. *If system of the equations (4.5.14) is such that*

(1) *the first and second level decompositions are described by equations (4.5.15) and (4.5.19) respectively;*
(2) *the matrices A_{ii} and $\overline{A}_{ij}$ in systems (4.5.15) and (4.5.19) are stable;*
(3) *the matrix S with elements*

$$s_{pq} = \begin{cases} \lambda_M(\Xi_{pp}) + \sigma_{pp}, & p = q; \\ \lambda_M^{1/2}(\Xi_{pq}\Xi_{pq}), & p < q; \\ s_{qp}, & p > q, \quad (p,q) \in [1,l], \end{cases}$$

  (a) *negative semi-definite;*
  (b) *negative definite.*

Then, the equilibrium state $\omega = 0$ of system (4.5.14) is

(a) *uniformly stable in probability;*
(b) *uniformly asymptotically stable in probability.*

PROOF. We construct by means of the function (4.5.20) the function

$$V(\omega) = \eta^{\mathrm{T}}\Pi(\omega)\eta, \quad \eta \in R^l_+, \quad \eta > 0. \tag{4.5.23}$$

By condition (2) of Theorem 4.5.2 the function $V(\omega)$ is positive definite and radially unbounded. For the averaged derivative

$$\frac{dE[V(\omega)]}{dt} = \eta^{\mathrm{T}}\frac{dE[\Pi(\omega)]}{dt}\eta, \qquad \eta \in R^l_+$$

it is easy to obtain the estimate

$$\begin{aligned} \frac{dE[V(\omega)]}{dt} &\le \sum_{i=1}^{l}(\lambda_M(\Xi_{ii}) + \sigma_{ii})\|\omega_i\|^2 \\ &+ 2\sum_{i=1}^{l}\sum_{j=i+1}^{l}\lambda_M^{1/2}(\Xi_{ij}^{\mathrm{T}}\Xi_{ij})\|\omega_i\|\|\omega_j\| = u^{\mathrm{T}}Su, \end{aligned} \tag{4.5.24}$$

where $u = (\|\omega_1\|, \dots, \|\omega_l\|)^{\mathrm{T}}$. Under the condition (3) of Theorem 4.5.2 the averaged derivative (4.5.24) is negative semi-definite or negative definite. According to Theorem 4.3.4 the Theorem 4.5.2 is proved.

4.6 Notes

4.1. General outlines on probability theory and theory of stochastic processes can be found in the books by Doob [31], Gikhman and Skorokhod [42], Dynkin [34], etc. The problems of stochastic stability are presented in a number of monographs (see e.g. Kushner [90], Arnold [5], Khasminskii [83], Michel and Miller [143], Ladde and Lakshmikantham [91], etc.). In these investigations the second Liapunov method is further developed with interesting applications.

4.2. Stochastic system in the form of (4.2.1) is called here the Kats–Krasovskii form with reference to Kats and Krasovskii [82] where it was introduced.

Basic definitions of stochastic stability are formulated as the generalization of Definitions 2.2.1—2.2.4 from Chapter 2 for stochastic systems. Stochastic matrix-valued function is introduced according to Martynyuk [115] and averaged derivative is due to Kats and Krasovskii [82] and Martynyuk [115].

4.3. Theorems 4.3.1–4.3.4 are due to Martynyuk [115].

4.4. Theorems 4.4.1, 4.4.2 are taken from Azimov and Martynyuk [8] and Azimov [6].

4.5. Stochastic version of the Lefschetz [100] problem is presented according to Martynyuk [115]. Oscillating system (4.5.7) was investigated by Azimov and Martynyuk [8], and system of automatic control (4.5.14) was considered by Azimov [7].

5

SOME MODELS OF REAL WORLD PHENOMENA

5.1 Introduction

This chapter contains several examples of real world phenomena that illustrate the versatility and applicability of the matrix-valued Liapunov functions in stability investigation of its equilibrium state.

Section 5.2 deals with mathematical models in population. The neighborhood of the non-trivial equilibrium state is investigated in the general case for a predator-prey system and estimates of stability, asymptotic stability and instability domains are found in this section.

In Section 5.3 the model of an orbital astronomical observatory is considered. Conditions are established under which the whole system is stable even though its separate subsystem are unstable.

In Section 5.4 we discuss a power system model consisting of N generators. General conditions are specified for asymptotic stability of the equilibrium state of such a system to be applicable in the case of $3, 5$ and 7 generators to obtain the system parameters such that the system is asymptotically stable, while the method of scalar or vector Liapunov functions have failed to work herein.

Finally, in Section 5.5 the motion in space of winged aircraft is treated.

5.2 Population Models

We shall discuss in this section mathematical models in population dynamics. In particular, we consider mathematical models of population growth of competing as well as predator-prey species as prototype models of our analysis. The models are based on certain simplifying assumptions as stated below.

(1) The density of a species, that is, the number of individuals per unit area, can be represented by a single variable, when differences of age, sex and genotype are ignored.
(2) Crowding affects all population members equally. This is unlikely to be true if the members of the species occur in clumps rather than being evently distributed throughout the available space.
(3) The affects of interactions within and between species are instantaneous. This means that there is no delayed action on the dynamics of the population.
(4) Abiotic environmental factors are sufficiently constant.
(5) Population growth rate is density-dependent even at the lowest densities. It may be more reasonable to suppose that there is some threshold density below which individuals do not interfere with one another.
(6) The females in a sexually reproducing population always find mates, even though the density may be low.

The assumptions relative to density dependency and crowding affects the fact that the growth of any species in a restricted environment must eventually be limited by a shortage of resources.

5.2.1 Competition

For simplicity, let us first consider a two-species community model living together and competing with each other for the same limiting resources. Under assumptions (1)–(6), a mathematical model of population growth of two competing species is described by

$$\begin{aligned} \frac{dx_1}{dt} &= x_1(a_1 - b_{11}x_1 - b_{12}x_2), \\ \frac{dx_2}{dt} &= x_2(a_2 - b_{21}x_1 - b_{22}x_2), \end{aligned} \tag{5.2.1}$$

where x_i is the population density of species i for $i = 1, 2$ and for $i, j = 1, 2$, a_i, b_{ij} are positive constants. These equations are derived from the Verhulst-Pearl logistic equation

$$\frac{dx_i}{dt} = x_i(a_i - b_{ii}x_i), \quad i = 1, 2, \tag{5.2.2}$$

by including the additional terms $-b_{ij}x_j$ for $i, j = 1, 2$ and $i \neq j$ to describe the inhibiting effects of each species on its competior. The logistic equation is best regarded as a purely descriptive equation.

The important features of (5.2.2) are:

(a) The species increase exponentially whenever they are isolated.
(b) They approach their equilibrium without oscillations in the absence of its competitor.

In (5.2.1), for $i = 1, 2$ $a_i x_i$ can be interpreted as the potential rate of increase that the i^{th} species would grow if the resources were unlimited and intra/inter-specific effects are neglected. Here a_i is the intrinsic rate of natural increase of the i^{th} species, $a_i/b_{ii} = k_i$ is referred as the carrying capacity if the i^{th} species. From this (5.2.2) can be written as

(5.2.3) $$\frac{dx_i}{dt} = a_i x_i \left(1 - \frac{x_i}{k_i}\right).$$

We observe that the per capita growth rate $\left(\frac{dx_i}{dt}\right) \Big/ x_i$ will be negative or positive depending on the population density $x_i > k_i$ or $x_i < k_i$. Thus the constants k_i determine the saturation level of population densities.

5.2.2 Predator-Prey

In the community of competing species, each species inhibits the multiplication of the other species. In a community of two species in which one species is a parasite or predator and the other its host or prey, a different form of interaction between these two species takes place. The mathematical models for host-parasite and predator-prey systems are equivalent. Obviously, the more abundant the prey, the more opportunities there are for the predator to breed. However, as the predator population grows, the number of prey eaten by the predator increases. To formulate the mathematical model describing the predator-prey interaction between two species, we assume the following: (a) in the absence of a predator, the prey species satisfies assumptions (1)–(6) and (b) the predator cannot survive without the presence of prey and the rate at which prey are eaten is proportional to the product of the densities of predator and prey. Under these assumptions, a mathematical model describing the predator-prey interaction between a prey and a predator in a given community is given by

$$\begin{aligned} \frac{dx_1}{dt} &= x_1(a_1 - b_{11}x_1 - b_{12}x_2), \\ \frac{dx_2}{dt} &= x_2(-a_2 + b_{21}x_1), \end{aligned} \tag{5.2.4}$$

where x_1 is prey density and x_2 is predator density and a_1, a_2, b_{11}, b_{21} are positive constants.

From the foregoing discussion with regard to the two-species competition model and the predator-prey model, we can readily generalize to n interacting species so that the general model is described by

$$\frac{dx_i}{dt} = x_i\left(a_i + \sum_{j=1}^{n} b_{ij}x_j\right), \qquad x_i(0) = x_{i0} \geq 0, \tag{5.2.5}$$

where x_i is density of the i^{th} species in the community, a_i, $-b_{ii}$ are positive constants and b_{ij}, $i \neq j$, are constants with any sign. Any arbitrary sign for b_{ij}, $i \neq j$, allows us a greater flexibility for the interactions between the i^{th} and j^{th} species in the community. For example, in a competitive model, b_{ij}, b_{ji}, $i \neq j$, are both negative, while for a predator-prey model, b_{ij}, b_{ji}, $i \neq j$, are of opposite signs. In a model for commensalism (symbiosis), b_{ij}, b_{ji}, $i \neq j$, are both positive.

The system (5.2.5) is represented in the vector form

$$\frac{dx}{dt} = X(a + Bx), \qquad x(0) = x_0 \leq 0 \tag{5.2.6}$$

and decomposed into two subsystems

$$\begin{gathered} \frac{dx_s}{dt} = X_s(a_s + A_{s1}x_1 + A_{s2}x_2), \\ x_s(0) = x_s \leq 0, \quad s = 1, 2. \end{gathered} \tag{5.2.7}$$

Here $x = (x_1^{\mathrm{T}}, x_2^{\mathrm{T}})^{\mathrm{T}} \in R_+^n$, $x_s \in R_+^{n_s}$, $(a_1^{\mathrm{T}}, a_2^{\mathrm{T}})^{\mathrm{T}} \in R^n$, $B = [A_{sj}]$, $s, j = 1, 2$; $a_s = (a_{s1}, a_{s2}, \ldots, a_{sn_s})^{\mathrm{T}} \in R^{n_s}$, A_{sj} are constant matrices $n_s \times n_j$, $X = \operatorname{diag}(X_1, X_2)$, $X_s = \operatorname{diag}(x_{s1}, \ldots, x_{sn_s})$, $s = 1, 2$.

Equilibrium population are determined by

$$X(a + Bx) = 0. \tag{5.2.8}$$

From (5.2.8) it is easy to conclude that $x = 0$ is an equilibrium which is not interesting and so, we must assume that $X \neq 0$. In this case (5.2.8) reduces to

$$a + Bx = 0, \tag{5.2.9}$$

where B is an n by n matrix and a is an n-vector.

We assume that there exists an equilibrium population $x^* > 0$ as a positive solution

$$x^* = -B^{-1}a \tag{5.2.10}$$

of (5.2.9). This assumption is consistent with consideration of community stability. In the case when b has all off-diagonal elements non-negative, that is B is a Metzler matrix, then it is known that stability of B implies $x^* > 0$. It is possible to show that for a Metzler matrix B, the quasi-dominant diagonal condition

$$d_j|b_{jj}| > \sum_{\substack{i=1 \\ i\neq j}}^{n} d_i|b_{ij}| \tag{5.2.11}$$

with $d_i > 0$, is equivalent to saying that $-B^{-1}$ is non-negative and since B^{-1} cannot have a row of zeros, positivity of the vector a implies positivity of x^*.

If B is a Metzler matrix, then an elegant solution of the problem on stability of state x^* is obtained by means of the function

$$V(x) = \sum_{i=1}^{n} d_i \left[x_i - x_i^* - x_i^* \ln\left(\frac{x_i}{x_i^*}\right)\right], \qquad d_i > 0.$$

Our aim is to establish stability conditions for system (5.2.6) without assuming matrix B being Metzler. This may be achived by decomposition of system (5.2.6) with further application of the matrix-valued function.

By means of the Liapunov transformation

$$y = x - x^* \tag{5.2.12}$$

we reduce the system (5.2.7) to the form

$$\frac{dy_s}{dt} = X_s^*(A_{s1}y_1 + A_{s2}y_2) + Y_s(A_{s1}Y_1 + A_{s2}y_2) \tag{5.2.13}$$

where

$$X_s^* = \operatorname{diag}\{x_{s1}^*, x_{s2}^*, \dots, x_{sn_s}^*\}, \qquad s = 1, 2,$$
$$Y_s = \operatorname{diag}\{y_{s1}, y_{s2}, \dots, y_{sn_s}\}, \qquad s = 1, 2.$$

For the system (5.2.13) the matrix-valued function

(5.2.14) $$U(y) = [v_{sj}(y_s, y_j)], \qquad s, j = 1, 2,$$

is constructed with the elements

(5.2.15) $$\begin{aligned} v_{ss}(y_s) &= y_s^{\mathrm{T}} P_s y_s, \qquad s = 1, 2, \\ v_{sj}(y_s, y_j) &= v_{js}(y_j . y_s) = y_1^{\mathrm{T}} P_3 y_2. \end{aligned}$$

Here P_s are positive definite symmetric matrices of the dimensions $n_s \times n_s$, $s = 1, 2$, and P_3 is a constant matrix n_1 by n_2.

For the function

(5.2.16) $$V(y, \eta) = \eta^{\mathrm{T}} U(y) \eta, \qquad \eta \in R^2,$$

the following estimates are valid

(5.2.17) $$u^{\mathrm{T}} H^{\mathrm{T}} D_1 H u \le V(y, \eta) \le u^{\mathrm{T}} H^{\mathrm{T}} D_2 H u,$$

where

$$u^{\mathrm{T}} = (\|y_1\|, \|y_2\|), \qquad H = \operatorname{diag}\{\eta_1, \eta_2\},$$

$$D_1 = \begin{pmatrix} \lambda_m(P_1) & -\operatorname{sign}(\eta_1\eta_2)\lambda_M^{1/2}(P_3P_3^{\mathrm{T}}) \\ -\operatorname{sign}(\eta_1\eta_2)\lambda_M^{1/2}(P_3P_3^{\mathrm{T}}) & \lambda_m((P_2) \end{pmatrix},$$

$$D_2 = \begin{pmatrix} \lambda_M(P_1) & -\operatorname{sign}(\eta_1\eta_2)\lambda_M^{1/2}(P_3P_3^{\mathrm{T}}) \\ -\operatorname{sign}(\eta_1\eta_2)\lambda_M^{1/2}(P_3P_3^{\mathrm{T}}) & \lambda_M(P_2) \end{pmatrix}.$$

We have for the function $D^+V(y, \eta) = \eta^{\mathrm{T}} D^+ U(y) \eta$:

(5.2.18) $$\begin{aligned} D^+V(y, \eta) &= \eta^{\mathrm{T}} D^+ U(y) \eta = \eta_1^2 D^+ v_{11}(y_1) + 2\eta_1\eta_2 D^+ v_{12}(y_1, y_2) \\ &+ \eta_2^2 D^+ v_{22}(y_2) = y_1^{\mathrm{T}} [F_{11} + G_{11}] y_1 + 2y_1^{\mathrm{T}} F_{12} y_2 \\ &+ y_2^{\mathrm{T}} [F_{22} + G_{22}] y_2 + 2y_1^{\mathrm{T}} G_{12} y_2. \end{aligned}$$

Here

$$F_{11} = \eta_1^2 \left[P_1 X_1^* A_{11} + \left(X_1^* A_{11} \right)^{\mathrm{T}} P_1 \right] + \eta_1 \eta_2 \left[P_3 X_2^* A_{21} + \left(X_2^* A_{21} \right)^{\mathrm{T}} P_3^{\mathrm{T}} \right];$$

$$F_{12} = \eta_1^2 P_1 X_1^* A_{12} + \eta_2^2 \left(X_2^* A_{21} \right)^{\mathrm{T}} P_2 + \eta_1 \eta_2 \left[\left(X_1^* A_{11} \right)^{\mathrm{T}} P_3 + P_3^{\mathrm{T}} X_2^* A_{22} \right];$$

$$F_{22} = \eta_2^2 \left[P_2 X_2^* A_{22} + \left(X_2^* A_{22} \right)^{\mathrm{T}} P_2 \right] + \eta_1 \eta_2 \left[\left(X_1^* A_{12} \right)^{\mathrm{T}} P_3 + P_3^{\mathrm{T}} X_1^* A_{12} \right];$$

$$G_{11} = \eta_1^2 \left[P_1 Y_1 A_{11} + \left(Y_1 A_{11} \right)^{\mathrm{T}} P_1 \right] + \eta_1 \eta_2 \left[P_3 Y_2 A_{21} + \left(Y_2 A_{21} \right)^{\mathrm{T}} P_3^{\mathrm{T}} \right];$$

$$G_{12} = \eta_1^2 P_1 Y_1 A_{12} + \eta_2^2 \left(Y_2 A_{21} \right)^{\mathrm{T}} P_2 + \eta_1 \eta_2 \left[P_3 Y_2 A_{22} + \left(Y_1 A_{22} \right)^{\mathrm{T}} P_3 \right];$$

$$G_{22} = \eta_2^2 \left[P_2 Y_2 A_{22} + \left(Y_2 A_{22} \right)^{\mathrm{T}} P_2 \right] + \eta_1 \eta_2 \left[P_3^{\mathrm{T}} Y_1 A_{12} + \left(Y_1 A_{12} \right)^{\mathrm{T}} P_3 \right].$$

We have for (5.2.18) the estimate

$$D^+V(y,\eta) \le u^{\mathrm{T}}\,[C + G(y)]\,u, \tag{5.2.19}$$

where

$$u^{\mathrm{T}} = (\|y_1\|, \|y_2\|),$$
$$C = [c_{sj}], \quad s,j = 1,2, \quad c_{12} = c_{21},$$
$$G(y) = [\sigma_{sj}(y)], \quad \sigma_{12}(y) = \sigma_{21}(y).$$

Here c_{11}, c_{22} are maximal eigenvalues of the matrices F_{11}, F_{22}; c_{12} is the norm of matrix F_{12} and $\sigma_{sj}(y)$ is the norm of matrix G_{sj}, $s,j = 1,2$.

It follows from (5.2.18) that

$$D^+V(y,\eta) \ge u^{\mathrm{T}}[C^* - G(y)]u, \tag{5.2.20}$$

where

$$C^* = \begin{pmatrix} c_{11}^* & -c_{12} \\ -c_{21} & c_{22}^* \end{pmatrix}$$

and c_{11}^*, c_{22}^* are minimal eigenvalues of the matrices F_{11}, F_{22} respectively.

Let us introduce the following notations

$$\begin{aligned}
\Pi_1 = &\{y \in R_+^n : \sigma_{11}(y) + c_{11} \le 0, \quad \sigma_{22}(y) + c_{22} \le 0, \\
&(\sigma_{11}(y) + c_{11})(\sigma_{22}(y) + c_{22}) - (\sigma_{12}(y) + c_{12})^2 \ge 0\}; \\
\Pi_2 = &\{y \in R_+^n : \sigma_{11}(y) + c_{11} < 0, \quad \sigma_{22}(y) + c_{22} < 0, \\
&(\sigma_{11}(y) + c_{11})(\sigma_{22}(y) + c_{22}) - (\sigma_{12}(y) + c_{12})^2 > 0\}; \\
\Pi_3 = &\{y \in R_+^n : c_{11}^* - \sigma_{11}(y) > 0, \quad c_{22}^* - \sigma_{22}(y) > 0, \\
&(c_{11}^* - \sigma_{11}(y))(c_{22}^* - \sigma_{22}(y)) - (\sigma_{12}(y) + c_{12})^2 > 0\}.
\end{aligned}$$

Estimates (5.2.17), (5.2.19) and (5.2.20) yield the following assertion.

PROPOSITION 5.2.1. *The equilibrium state x^* of the system (5.2.6) is:*

(1) *Stable (asymptotically) in the domain Π_1 (Π_2) if the matrix D_1 is positive definite and the matrix C is negative definite.*
(2) *Unstable in the domain Π_3 if the matrices D_1 and C^* are positive definite.*

PROOF. The fact that the matrix D_1 is positive definite yields that the function $V(y)$ if positive definite for all $y \in R^n_+$. Since the matrix C is negative definite, then by estimate (5.2.19) the function $D^+V(y)$ is non-positive in the domain Π_1. Hence all conditions of Theorem 2.3.3 are satisfied, and the equilibrium state x^* is stable.

The other assertion of Proposition 5.2.1 follows from Theorem 2.3.7.

5.3 Model of Orbital Astronomic Observatory

According to Geiss, Cohen et al. [40] the orbital astronomic observatory consists of following blocks:

(1) observatory vehicle
(2) observatory body
(3) compensation system
(4) engine
(5) system of data (error) processing.

The subsystems (1)–(4) are physycal and its states are characterized by the variables y_1, y_2, y_3 and y_4 respectively. Under some assumptions the mathematical model of the motion control system for the observatory is described by the equations

$$\begin{aligned}
\frac{dy_1}{dt} &= F_1(y_1)y_2 + F_1(y_1)d_1 + c_1 y_2,\\
\frac{dy_2}{dt} &= Y(y_2)d_2 - \beta_1 f_2(\sigma, y_3) + Y(y_2) f_2(\sigma, y_3) + (\beta_2 I + c_2) y_4,\\
\frac{dy_3}{dt} &= -\beta_3 f_1(\sigma) - \beta_4 y_3,\\
\frac{dy_4}{dt} &= -\beta_1 f_2(\sigma, y_3) - \beta_2 y_4,\\
\sigma &= F_2(y_1)y_1 + c_2 y_1 + F_2(y_1)d_3.
\end{aligned} \tag{5.3.1}$$

Here $y_1 = (y_{11}, y_{12}, y_{13}, y_{14})^{\mathrm{T}}$, $y_i = (y_{i1}, y_{i2}, y_{i3})^{\mathrm{T}}$, $i = 2, 3, 4$

$$\sigma = (\sigma_1, \sigma_2, \sigma_3)^{\mathrm{T}},$$

(5.3.2)

$$F_1(y_1) = \begin{pmatrix} 0 & f_{12} & f_{13} \\ 0 & f_{22} & f_{23} \\ 0 & f_{32} & f_{33} \\ 0 & f_{42} & f_{43} \end{pmatrix}$$

and

$$
\begin{aligned}
f_{12} &= \sin(y_{13}+\alpha_3) - \sin\alpha_3,\\
f_{13} &= \cos(y_{13}+\alpha_3) - \cos\alpha_3,\\
f_{22} &= -\sin(y_{14}+\alpha_4) + \sin\alpha_4,\\
f_{23} &= -\cos(y_{14}+\alpha_4) + \cos\alpha_4,\\
f_{32} &= -\mathrm{tg}\,(y_{11}+\alpha_1)\cos(y_{13}+\alpha_3) + \mathrm{tg}\,\alpha_1\cos\alpha_3,\\
f_{33} &= \mathrm{tg}\,(y_{11}+\alpha_1)\sin(y_{13}+\alpha_3) - \mathrm{tg}\,\alpha_1\sin\alpha_3,\\
f_{42} &= \mathrm{tg}\,(y_{12}+\alpha_2)\cos(y_{14}+\alpha_4) - \mathrm{tg}\,\alpha_2\cos\alpha_4,\\
f_{43} &= -\mathrm{tg}\,(y_{12}+\alpha_2)\sin(y_{14}+\alpha_4) + \mathrm{tg}\,\alpha_2\sin\alpha_4,
\end{aligned}
\tag{5.3.3}
$$

$$
F_2(y_1) = \begin{pmatrix} 0 & 0 & 0 & 0\\ g_{21} & g_{22} & 0 & 0\\ g_{31} & g_{32} & 0 & 0 \end{pmatrix},
$$

where

$$
\begin{aligned}
g_{32} &= -\delta[\sin(y_{13}+\alpha_3) - \sin\alpha_3],\\
g_{21} &= \delta[\cos(y_{14}+\alpha_4) - \cos\alpha_4],\\
g_{22} &= \delta[\cos(y_{13}+\alpha_3) - \cos\alpha_3],\\
g_{31} &= -\delta[\sin(y_{14}+\alpha_4) - \sin\alpha_4].
\end{aligned}
$$

Furthermore

$$
Y(y_2) = J^{-1}\begin{pmatrix} 0 & y_{23} & -y_{22}\\ -y_{23} & 0 & y_{21}\\ y_{22} & -y_{21} & 0 \end{pmatrix},
\tag{5.3.4}
$$

$$
Z_1(\zeta) = \begin{pmatrix} 100\,\mathrm{sign}\,\zeta, & |\zeta| \geq 100\\ \zeta, & |\zeta| \leq 100 \end{pmatrix},
\tag{5.3.5}
$$

(5.3.6) $$Z_2(\zeta) = \begin{pmatrix} 26\,\mathrm{sign}\,\zeta, & |\zeta| \geq 26 \\ \zeta, & |\zeta| \leq 26 \end{pmatrix},$$

(5.3.7) $$f_1(\sigma) = \begin{pmatrix} z_1(\sigma_1 + \alpha_8) - z_1(\alpha_8) \\ z_1(\sigma_2 + \alpha_9) - z_1(\alpha_9) \\ z_1(\sigma_3 + \alpha_{10}) - z_1(\alpha_{10}) \end{pmatrix},$$

(5.3.8)
$$f_2(\sigma, y_3) = \begin{pmatrix} z_2[\beta_1 z_1(\sigma_1 + \alpha_8) + y_{31} + \alpha_{11}] - z_2[\beta_1 z_1(\alpha_8) + \alpha_{11}] \\ z_2[\beta_1 z_1(\sigma_2 + \alpha_9) + y_{32} + \alpha_{12}] - z_2[\beta_1 z_1(\alpha_3) + \alpha_{12}] \\ z_2[\beta_1 z_1(\sigma_3 + \alpha_{10}) + y_{33} + \alpha_{13}] - z_2[\beta_1 z_1(\alpha_{10} + \alpha_{13}] \end{pmatrix},$$

(5.3.9) $$C_1 = \begin{pmatrix} 0 & \sin\alpha_3 & \cos\alpha_3 \\ 0 & -\sin\alpha_4 & -\cos\alpha_4 \\ 1 & -\mathrm{tg}\,\alpha_1\cos\alpha_3 & \mathrm{tg}\,\alpha_1\sin\alpha_3 \\ 1 & \mathrm{tg}\,\alpha_2\cos\alpha_4 & -\mathrm{tg}\,\alpha_2\sin\alpha_4 \end{pmatrix},$$

(5.3.10) $$C_2 = J^{-1}\begin{pmatrix} \alpha_{14} & \alpha_{15} & 1 & 0 \\ \delta\cos\alpha_4 & \delta\cos\alpha_3 & 0 & 0 \\ -\delta\sin\alpha_4 & -\delta\sin\alpha_3 & 0 & 0 \end{pmatrix},$$

(5.3.11) $$d_i = \begin{pmatrix} d_{i1} \\ d_{i2} \\ d_{i3} \end{pmatrix}, \qquad i = 1, 2, 3.$$

In the neighborhood of the equilibrium state

(5,3,12) $$y_i = 0, \quad i = 1, 2, 3, 4, \quad \sigma = 0$$

under some additional assumptions the system (5.3.1) is reduced to the form

(5.3.13) $$\frac{dx_i}{dt} = A_{i1}x_1 + A_{i2}x_2 + A_{i3}x_3 + \nu B_i f(\Sigma), \qquad \Sigma = Cx, \qquad \forall\, i = 1, 2, 3,$$

besides, $x_i,\ i = 1, 2, 3$, is determined as

$$x_1 = \begin{pmatrix} \Delta\varphi \\ \Delta V_\varphi \\ \Delta\omega_\varphi \end{pmatrix}, \qquad x_2 = \begin{pmatrix} \Delta\theta \\ \Delta V_\theta \\ \Delta\omega_\theta \end{pmatrix}, \qquad x_3 = \begin{pmatrix} \Delta\psi \\ \Delta V_\psi \\ \Delta\omega_\psi \end{pmatrix},$$

and (φ, θ, ψ) are Euler anglers specifying the rotating motion of the observatory, $(\omega_\varphi, \omega_\theta, \omega_\psi)$ are the velocities of its changing, $V_\varphi, V_\omega, V_\psi$ are the components of vector V that determines the velocity of plane-parallel motion, x_1, x_2, x_3 specify the observatory deviation from the directed position

$$\begin{array}{lll} \Delta\varphi = \varphi^* - \varphi, & \Delta\theta = \theta^* - \theta, & \Delta\psi = \psi^* - \psi; \\ \Delta V_\varphi = V_\varphi^* - V_\varphi, & \Delta V_\theta = V_\theta^* - V_\theta, & \Delta V_\psi^* - V_\psi; \\ \Delta\omega_\varphi = \omega_\varphi^* - \omega_\varphi, & \Delta\omega_\theta = \omega_\theta^* - \omega_\theta, & \Delta\omega_\psi = \omega_\psi^* - \omega_\psi. \end{array}$$

Here φ^*, θ^*, ψ^*; ω^*_φ, ω^*_θ; ω^*_ψ; $V^*{}_\varphi$, V^*_θ, V^*_ψ are the parameters of the observatory directed position. The matrices A_{ij} and B_i are

$$A_{11} = \begin{pmatrix} 0 & a_1 & 0 \\ a_2 & -a_3 & a_4 \\ -a_5 & 0 & -a_5 \end{pmatrix}, \quad A_{12} = -A_{13} = \begin{pmatrix} 0 & 0 & 0 \\ -a_6 & 0 & 0 \\ -a_7 & 0 & 0 \end{pmatrix},$$

$$B_i = \begin{pmatrix} 0 & 0 & 0 \\ \delta_{i1} & \delta_{i2} & \delta_{i3} \\ 0 & 0 & 0 \end{pmatrix}, \qquad A_{22} = A_{33} = \begin{pmatrix} 0 & -a_1 & 0 \\ 2a_2 & -a_3 & a_4 \\ -2a_5 & 0 & -a_5 \end{pmatrix}$$

δ_{ij} is a Kronecker delta, $A_{ij} = 0,\ i = 2,3;\ j = 1,2,3\ \forall\,(i \neq j)$,

$$C = \begin{pmatrix} r_{11}^{\mathrm{T}} & r_{12}^{\mathrm{T}} & r_{13}^{\mathrm{T}} \\ 0 & r_{22}^{\mathrm{T}} & 0 \\ 0 & 0 & r_{33}^{\mathrm{T}} \end{pmatrix},$$

$$r_{1i}^{\mathrm{T}} = \left(\rho_{1i}^1, \rho_{1i}^2, \rho_{1i}^3\right), \quad i = 1,2,3; \quad r_{ji}^{\mathrm{T}} = (\rho_{j1}, \rho_{j2}, \rho_{j3}), \quad j = 2,3;$$

$$f(\Sigma) = (\varphi_1(\sigma_1), \varphi_2(\sigma_2), \varphi_3(\sigma_3))^{\mathrm{T}}, \quad \Sigma = (\sigma_1, \sigma_2, \sigma_3)^{\mathrm{T}},$$

$$\frac{\varphi_i(\sigma_i)}{\sigma_i} \in [0,1] \quad \forall\,\sigma_i \in R, \quad \varphi_i(\sigma_i) \in C(R,R).$$

The elements $a_s,\ s = 1,2,\dots,7$, of the matrices A_{ij} as well as the values $r_{1i}^k\ (i.k) \in [1,3],\ r_{ik},\ i = 2,3,\ k \in [1,3]$ are known real constants.

System (5.3.13) has a unique equilibrium state $(x = 0) \in R^3$.

The problem is to establish conditions for asymptotic stability in the whole of system (5.3.13).

Let us use the algorithm of constructing the hierarchical Liapunov function (see Martynyuk and Krapivny [124]). The first level decomposition of system (5.3.13) results in the independent subsystems

(5.3.14) $$\frac{dx_i}{dt} = A_{ii}x_i + (1 - \delta_{1i})\nu B_i f(\Sigma), \qquad i = 1,2,3$$

and the relation functions

$$
\begin{aligned}
&g_1(x) = A_{12}x_2 + A_{13}x_3 + \nu B_1 f(\Sigma), \\
&g_i(x) = 0, \qquad i = 2, 3.
\end{aligned}
\tag{5.3.15}
$$

The second level decomposition yields three couples of the independent subsystems

$$
\frac{dx_{ij}}{dt} = \bar{A}_{ij}x_{ij} + \nu B_{ij} f(\Sigma), \qquad (i < j) = 1, 2, 3,
\tag{5.3.16}
$$

where $x_{ij} = \left(x_i^{\mathrm{T}}, x_j^{\mathrm{T}}\right)^{\mathrm{T}}$ and the matrices $\bar{A}_{ij}$ and B_{ij} are

$$
\bar{A}_{ij} = \begin{pmatrix} A_{11} & A_{1j} \\ 0 & A_{jj} \end{pmatrix}, \quad \bar{A}_{23} = \begin{pmatrix} A_{22} & 0 \\ 0 & A_{33} \end{pmatrix}, \quad B_{ij} = \begin{pmatrix} \delta_{2i}B_i \\ B_j \end{pmatrix}.
$$

The relation functions between them are

$$
\begin{aligned}
&\bar{g}_{1j}(x) = A_{1j}^k + \nu \overline{B}_1 f(\Sigma), \qquad (i \neq k) = 2, 3, \\
&\bar{g}_{23}(x) = 0,
\end{aligned}
\tag{5.3.17}
$$

where

$$
A_{1j}^k = \begin{pmatrix} A_{1k} \\ A_{k1} \end{pmatrix} = \begin{pmatrix} A_{1k} \\ 0 \end{pmatrix}, \qquad \overline{B}_1 = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}.
$$

We construct for the subsystem (5.3.14) the function

$$
v_{ii}(x_i) = x_i^{\mathrm{T}} H_{ii} x_i, \qquad i = 1, 2, 3,
\tag{5.3.18}
$$

where $H_{ii} > 0$ satisfy the algebraic Liapunov equations

$$
A_{ii}^{\mathrm{T}} H_{ii} + H_{ii} A_{ii} = G_{ii}, \qquad i = 1, 2, 3,
\tag{5.3.19}
$$

where $G_{ii} < 0$ if and only if the subsystems

$$
\frac{dx_i}{dt} = A_{ii}x_i
$$

are asymptotically stable. For functions (5.3.18) the estimates

$$
\begin{aligned}
&\lambda_m(H_{ii})\|x_i\|^2 \leq v_{ii}(x_i) \leq \lambda_M(H_{ii})\|x_i\|^2 \\
&\forall x_i \in R^{n_i}, \quad i = 1, 2, 3, \quad n_1 = n_2 = n_3 = 3,
\end{aligned}
\tag{5.3.20}
$$

are known.

Assume that for all $x_i \in R^3$ for the functions $v_{ii}(x_i)$ time-derivative along the solutions of subsystems (5.3.14) the estimates

$$\left.\frac{dv_{ii}(x_i)}{dt}\right|_{(5.3.14)} \leq \rho_{ii}^0 \|x_i\|^2, \qquad i = 1, 2, 3 \tag{5.3.21}$$

are satisfied and for (5.3.15)

$$\left(\frac{\partial v_{ii}(x_i)}{\partial x_i}\right)^{\mathrm{T}} g_1(x) \leq \|x_i\|^{1/2} \sum_{k=1}^{3} \mu_{ik} \|x_k\|^{1/2}, \tag{5.3.22}$$

where

$$\begin{aligned} \rho_{ii}^0 &= \lambda_M(G_{ii}) + 2(1 - \delta_{1i})\nu\|H_{ii}\|\|r_i\|, \quad i = 1,2,3; \\ \mu_{11} &= 2\nu\|H_{11}\|\,\|r_{11}\|; \\ \mu_{12} &= 2\|H_{11}\|\;[\|A_{12}\| + \nu\|r_{12}\|]\,; \\ \mu_{13} &= 2\|H_{11}\|\;[\|A_{13}\| + \nu\|r_{13}\|]\,; \\ \mu_{ik} &= 0, \quad i = 2,3; \quad k = 1,2,3. \end{aligned} \tag{5.3.23}$$

We construct for (i,j)-couples of subsystem (5.3.16) the functions

$$v_{ij}(x_{ij}) = x_{ij}^{\mathrm{T}} H_{ij} x_{ij}, \quad (i < j) = 1,2,3, \tag{5.3.24}$$

where the matrices $H_{ij} > 0$ satisfy the algebraic Liapunov equations

$$\bar{A}_{ij}^{\mathrm{T}} H_{ij} + H_{ij}\bar{A}_{ij} = G_{ij}, \quad (i < j) = 1,2,3, \tag{5.3.25}$$

for $G_{ij} < 0$ if and only if (i,j)-couples

$$\frac{dx_{ij}}{dt} = \bar{A}_{ij} x_{ij}, \quad (i < j) = 1,2,3,$$

are asymptotically stable.

For functions $v_{ij}(x_{ij})$ the estimates

$$\begin{gathered} \lambda_m(H_{ij})\|x_{ij}\|^2 \leq v_{ij}(x_{ij}) \leq \lambda_M(H_{ij})\|x_{ij}\|^2 \\ \forall x_{ij} \in R^{n_i \times n_j}, \quad (i < j) = 1,2,3, \end{gathered} \tag{5.3.26}$$

take place.

We assume now that for the functions $v_{ij}(x_{ij})$ time-derivative along the solutions of subsystems (5.3.16) the estimates

$$\frac{dv_{ij}(x_{ij})}{dt}\bigg|_{(5.3.16)} \le \rho^1_{ij}\|x_i\| + 2\rho^2_{ij}\|x_i\|^{1/2}\|x_j\|^{1/2} + \rho^3_{ij}\|x_j\| \tag{5.3.27}$$

are satisfied for all $x_i \in R^3$ and for (5.3.17)

$$\left(\frac{\partial v_{ij}(x_{ij})}{\partial x_{ij}}\right)^{\mathrm{T}} g_{1j}(x) \le \sum_{\substack{k=1\\p=k}}^{3} \nu^{ij}_{kp}\|x_k\|^{1/2}\|x_p\|^{1/2}. \tag{5.3.28}$$

The contstants ρ^1_{ij}, ρ^2_{ij}, ρ^3_{ij} can be determined as follows

$$\begin{aligned} \rho^1_{ij} &= \lambda_M(G_{ij}) + 2\nu\delta_{2i}\|H^j_{22}\|\,\|r_{22}\|, \\ \rho^2_{ij} &= \nu\|\overline{H}_{ij}\|\,\|r_{ij}\| + \nu\delta_{2i}\|\overline{H}_{23}\|\,\|r_{22}\|, \\ \rho^3_{ij} &= \lambda_M(G_{ij}) + 2\nu\|H^i_{ij}\|\,\|r_{jj}\|, \quad (i<j) = 1,2,3 \end{aligned} \tag{5.3.29}$$

and the constants ν^{ij}_{kp} as follows

$$\begin{aligned} \nu^{1j}_{11} &= 2\nu\|H^j_{11}\|\|r_{11}\|, \\ \nu^{1j}_{jj} &= 2\nu\|\overline{H}_{ij}\|(\delta_{2j}\|r_{12}\| + \delta_{31}\|r_{13}\|), \\ \nu^{1j}_{23} &= 2\|\overline{H}_{ij}\|[\|A_{1k} + \nu(\delta_{3j}\|r_{12}\| + \delta_{2j}\|r_{13}\|)]; \\ \nu^{1j}_{1k} &= 2\|H^j_{11}\|\,[(1-\delta_{jk})\|A_{1k}\| + \nu\|r_{1k}\|] \\ &\quad + 2\delta_{kj}\nu\|\overline{H}_{ij}\|\|r_{11}\|, \qquad k,\, j = 2,3, \\ \nu^{12}_{33} &= \nu^{13}_{22} = \nu^{23}_{kp} = 0 \qquad \forall\,(k \le p) = 1,2,3. \end{aligned} \tag{5.3.30}$$

Here the matrices H^j_{ij} and $\overline{H}_{ij}$, $(i<j) = 1,2,3$, of the dimensions 3×3 are the blocks of the matrix H_{ij} so that

$$H_{ij} = \begin{pmatrix} H^j_{ii} & \overline{H}_{ij} \\ \overline{H}^{\mathrm{T}}_{ij} & H^i_{jj} \end{pmatrix}.$$

Using the matrix-valued function $U(x)$ with elements (5.3.18) and (5.3.24), and by virtue of (5.3.21), (5.3.22), (5.3.27) and (5.3.28) we see that

$$\frac{dV(x,\eta)}{dt} \le \varphi^{\mathrm{T}}(\|x\|)S\varphi(\|x\|), \tag{5.3.31}$$

where

$$V(x,\eta) = \eta^{\mathrm{T}} U(x)\eta, \quad \eta \in R^3_+, \quad \eta > 0,$$
$$\varphi(\|x\|) = \left(\|x_1\|^{1/2}, \ldots, \|x_3\|^{1/2}\right).$$

The matrix S in (5.3.31) has the form

$$S = \frac{1}{2}(\Pi + \Pi^{\mathrm{T}}),$$

where Π is the upper triangle matrix with the elements

$$
\begin{aligned}
\pi_{kk} &= \eta_k^2(\rho_{kk}^0 + \mu_{kk}) + 2\eta_k \sum_{i=1}^{k-1} \eta_i \rho_{ik}^3 \\
&\quad + 2\eta_k \sum_{i=k+1}^{3} \eta_i \rho_{ki}^1 + \sum_{\substack{i,j=1 \\ i\neq j}}^{3} \eta_i \eta_j \nu_{kk}^{ij}, \\
\pi_{kp} &= \eta_k^2 \mu_{kp} + 4\eta_k \eta_p \rho_{kp}^2 + 2\sum_{i=1}^{s} \sum_{j=i+1}^{s} \eta_i \eta_j \nu_{kp}^{ij}, \quad k < p, \\
\pi_{pk} &= 0, \qquad k < p.
\end{aligned}
\tag{5.3.32}
$$

The matrix S in the estimate (5.3.31) is negative definite, if

$$
s_{11} < 0, \quad s_{22} < 0, \quad s_{33} < 0 \tag{5.3.33}
$$

and

$$
s_{11}s_{22} - s_{12}^2 > 0, \quad \det S < 0 \tag{5.3.34}
$$

since $s_{ij} > 0 \ \forall\, (i \neq j) \in [1,3]$.

Stability conditions (5.3.33), (5.3.34) are analyzed for two cases, first, for the case when only the first level decomposition is made. This corresponds to the approach based on the vector Liapunov function, applied by Grujić, Martynyuk and Ribbens-Pavella [57].

In this case the elements of matrix Π for system (5.3.13) are in view of (5.3.23)–(5.3.30) and (5.3.32)

$$
\begin{aligned}
\pi_{ii} &= \eta_i^2 \left[\lambda_M(G_{ii}) + 2\nu \|H_{ii}\| \, \|r_{ii}\|\right], && i = 1,2,3; \\
\pi_{1j} &= 2\eta_1^2 \|H_1\| \, \left[\|A_{1j}\| + \nu \|r_{1j}\|\right], && j = 2,3; \\
\pi_{23} &= \pi_{ji} = 0 && \forall\, (j > i) = 1,2,3.
\end{aligned}
$$

We introduce the designations

$$Q = \operatorname{diag}(\eta_1^2, \eta_2^2, \eta_3^2)$$

and the matrix $D = [d_{ij}]$ the elements of which are expressed via the elements of matrix Π as follows

$$d_{ij} = \frac{\pi_{ij}}{\eta_i^2}, \qquad (i, j) = 1, 2, 3.$$

Therefore we have

$$S = \frac{1}{2}(\overline{\Pi} + \overline{\Pi}^{\mathrm{T}}) = \frac{1}{2}(QD + D^{\mathrm{T}}Q). \tag{5.3.35}$$

The matrix S is negative definite if and only if the matrix D is an M-matrix. The matrix D is an upper triangular and $d_{ij} \geq 0$ $(i, j) = 1, 2, 3$, hence, if $d_{ii} < 0$, then D is the M-matrix. Therefore, the conditions for matrix S being negative definite are

$$\lambda_M(G_{ii}) + 2\nu \|H_{ii}\| \, \|r_{ii}\| < 0 \qquad \forall\, i = 1, 2, 3. \tag{5.3.36}$$

These are the well-known conditions for the asymptotic stability in the whole of system (5.3.13).

Let us show conditions (5.3.33), (5.3.34) for the asymptotic stability in the whole of the system (5.3.13) to be more general than the conditions (5.3.36).

The conditions (5.3.36) are satisfied if $\lambda_M(G_{ii}) < 0$. This means that the subsystems

$$\frac{dx_i}{dt} = A_{ii}x_i, \qquad i = 1, 2, 3, \tag{5.3.37}$$

obtained from (5.3.14) must be asymptotically stable.

Therefore, if one of the subsystems (5.3.37) is unstable, the conditions (5.3.36) are not satisfied and the approach based on the vector function does not work.

Assume the 3rd subsystem from (3.5.37) is unstable, i.e. $\lambda_M(G_{33}) > 0$. In view of the second level decomposition one of conditions (3.5.33), namely $s_{33} < 0$ becomes

$$\begin{aligned} \eta_3^2(\lambda_M(G_{33}) + 2\nu\|H_{33}\| \, \|r_{33}\|) + 2\eta_1\eta_3\lambda_M(G_{13}) \\ + 2\eta_2\eta_3(\lambda_M(G_{33}) + 2\nu\|H_{33}^2\| \, \|r_{33}\|) < 0. \end{aligned} \tag{5.3.38}$$

It is clear that, if the 3rd subsystem forms asymptotically stable couples $(2,3)$ and $(1,3)$, then $\lambda_M(G_{13}) < 0$ and $\lambda_M(G_{23}) < 0$. This may prove to be sufficient for inequality (5.3.36) to be satisfied. However this inequality may be derived by means of the matrix-valued function only.

Thus, the application of the matrix-valued function and two-level decomposition yields less strict conditions for the asymptotic stability in the whole of the system (5.3.13) as compared with conditions (5.3.36) established by means of the vector Liapunov function.

5.4 Power System Model

The dynamical and structural complexity combined with the high order of the power system make many methods developed in theory of differential equations inapplicable in the investigation of these systems. The method of Liapunov functions (scalar, vector or matrix) is one of the methods used in the analysis of stability and the estimation of asymptotic stability domains. In this section we shall show the application of the matrix-valued Liapunov function to be advantageous as compared with the results by the vector Liapunov function.

5.4.1 Description of the Power System

Considered is the N-machine power system with uniform mechanical damping λ. The i^{th} machine motion is modeled by the equations

$$M_i\ddot{\delta}_i + D_i\dot{\delta} = P_{mi} - P_{ei}, \qquad i = 1, 2, \dots, N, \tag{5.4.1}$$

where

$$P_{ei} = E_i^2 Y_{ii}\cos\theta_{ii} + \sum_{j\neq i}^{n} E_i E_j Y_{ij}\cos(\delta_{ij} - \theta_{ij}), \tag{5.4.2}$$

and $M_i \in R$ is the inertia coefficient of the i^{th} machine, $D_i \in R$ is the mechanical damping of the i^{th} machine, $P_{mi} \in R$ is the mechanical power delivered by the i^{th} machine, $E_i \in R$ is the modulus of the internal voltage, $Y_{ij} \in R$ is the magnitude of the (i,j)-th element of the reduced admittances matrix Y, $\delta_i \in R$ is the absolute rotor angle: $\delta_{ij} = \delta_i - \delta_j = \delta_{iN} - \delta_{iN}$,

$\delta_{ij}^0 = \delta_i^0 \delta_j^0$, $\theta_{ij} \in R$ is the angle of the (i,j)-th element of the reduced admittances matrix.

Let us take the N^{th} machine as a standard one and introduce $(2N-1)$ state variables

$$\begin{aligned} \sigma_{iN} &= \delta_{iN} - \delta_{iN}^0, \qquad && i \neq N; \\ \omega_i &= \dot{\delta}_i, && i = 1, 2, \dots, N, \end{aligned} \tag{5.4.3}$$

where $\sigma_{ij} \in R$ is a subsidiary variable, $\omega_i \in R$ is the absolute angular speed of the i^{th} machine rotor. Here δ_{iN}^0 are the solutions of the system of equations

$$E_i^2 Y_{ii} \cos\theta_{ii} + \sum_{j \neq i}^{N} E_i E_j Y_{ij} \cos(\delta_{iN}^0 - \delta_{jN}^0 - \theta_{ij}) = P_{mi}, \quad i = 1, 2, \dots, N. \tag{5.4.4}$$

The motion of the whole N-machine system can be described by the equations

$$\begin{aligned} \dot{\sigma}_{iN} &= \omega_{iN}, \\ \dot{\omega}_i &= -\lambda\omega_i - M_i^{-1} \sum_{j \neq i}^{N} A_{ij} f_{ij}(\sigma_{ij}), \qquad i = 1, 2, \dots, N, \end{aligned} \tag{5.4.5}$$

where $A_{ij} = E_j E_i Y_{ij}$, $A_i = A_{iN}$, f_{ij} are non-linear functions

$$f_{ij}(\sigma_{ij}) = \cos(\sigma_{ij} + \sigma_{ij}^0 - \theta_{ij}) - \cos(\delta_{ij}^0 - \theta_{ij}), \tag{5.4.6}$$

satisfying the conditions

$$f_{ij}(0) = 0, \quad 0 \leq \frac{f_{ij}(\sigma_{ij})}{\sigma_{ij}} \leq \xi_{ij}, \quad \sigma_{ij} \neq 0, \tag{5.4.7}$$

as soon as σ_{ij} take the value on compact intervals J_{ij}:

$$J_{ij} = \left\{ \sigma_{ij} : -2(\pi - \theta_{ij} + \delta_{ij}^0) \leq \sigma_{ij} \leq 2(\theta_{ij} - \delta_{ij}^0) \right\}. \tag{5.4.8}$$

The constants ξ_{ij} in (5.4.7) are determined as follows

$$\xi_{ij} = \left. \frac{\partial f_{ij}(\sigma_{ij})}{\partial \sigma_{ij}} \right|_{\sigma_{ij}=0}.$$

5.4.2 Mathematical Decomposition of the Power system model

The state vector of the whole system is designated as

$$\hat{x} = (\sigma_{1N}, \omega_1, \sigma_{2N}, \omega_2, \ldots, \sigma_{N-1,N}, \omega_{N-1}, \omega_N)^{\mathrm{T}},$$

and the subvectors

$$x_i = (\sigma_{iN}, \omega_{iN})^{\mathrm{T}} = (x_{i1}, x_{i2})^{\mathrm{T}}, \qquad i = 1, 2, \ldots, N-1, \tag{5.4.9}$$

are introduced.

System (5.4.5) is represented as

$$\begin{aligned} \frac{dx_i}{dt} &= P_i x_i + B_i F_i(\sigma_i) + h_i(x), \\ \sigma_i &= C_i^{\mathrm{T}} x_i, \qquad i = 1, 2, \ldots, s. \end{aligned} \tag{5.4.10}$$

Each subsystem of (5.4.10) consist of free subsystems

$$\frac{dx_i}{dt} = P_i x_i + B_i F_i(\sigma_i),$$
$$\sigma_i = C_i^{\mathrm{T}} x_i, \qquad i = 1, 2, \dots, s, \tag{5.4.11}$$

and relation functions

$$h_i(x) = \begin{pmatrix} 0 \\ \sum\limits_{j \neq i}^{N-1} (-M_i^{-1} A_{ij} f_{ij}(\sigma_{ij}) + M_N^{-1} A_{Nj} f_{Nj}(\sigma_{Nj})) \end{pmatrix}. \tag{5.4.12}$$

The vector of nonlinearities $F_i(\sigma_i)$ is a decomposition of two nonlinearities

$$f_{i1}(\sigma_{i1}) = \cos(\sigma_{iN} + \delta_{iN}^0 - \theta_{iN}) - cos(\delta_{iN}^0 - \theta_{iN}),$$
$$f_{i2}(\sigma_{i2}) = \cos(\sigma_{Ni} + \delta_{Ni}^0 - \theta_{iN}) - cos(\delta_{Ni}^0 - \theta_{iN}). \tag{5.4.13}$$

The other matrices and functions appearing in the system (5.4.14) are

$$P_i = \begin{pmatrix} 0 & 1 \\ 0 & -\lambda \end{pmatrix},$$

$\lambda = D_i M_i^{-1}$ is a uniform damping, $i - 1, 2, \dots, N$;

$$B_i = \begin{pmatrix} 0 & 0 \\ -M_i^{-1} A_i & M_N^{-1} A_i \end{pmatrix}, \qquad C_i^{\mathrm{T}} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}.$$

5.4.3 Application Algorithm of the Matrix-Valued Function

The elements v_{ij} of the matrix-valued function $U(x)$ are taken as

(5.4.14)
$$v_{ii}(x_i) = x_i^{\mathrm{T}} H_i x_i + \sum_{k=1}^{2} \gamma_{ik} \int_0^{\sigma_{ik}} f_{ik}(\sigma_{ik})\, d\sigma_{ik}, \quad i = 1, 2, \dots, s,$$

(5.4.15)
$$v_{ij}(x_i, x_j) = \alpha_{ij} \int_0^{\sigma_{ij}} f_{ij}(\sigma_{ij})\, d\sigma_{ij} \quad (i \neq j), \quad i, j = 1, 2, \dots, s.$$

Here H_i are 2×2 symmetric positive definite matrices, γ_{ik} and α_{ij} are arbitrary positive numbers.

Let $\eta = (1, \dots, 1)^{\mathrm{T}} \in R_+^s$ and

$$\dot V(x, \eta) = \eta^{\mathrm{T}} \dot U(x) \eta, \qquad \dot U(x) = [\dot v_{ij}(x_i, x_j)].$$

The function v_{ij} time-derivative along the solutions of the i^{th} interconnected subsystem is

(5.4.16)
$$\frac{dv_{ii}}{dt} = \left.\frac{dv_{ii}}{dt}\right|_{(5.4.11)} + \left.\frac{dv_{ii}}{dt}\right|_{(5.4.12)},$$

where

(5.4.17)
$$\left.\frac{dv_{ii}}{dt}\right|_{(5.4.11)} = 2x_i^{\mathrm{T}} H_i [P_i x_i + B_i F_i(\sigma_i)] + \sum_{k=1}^{2} \gamma_{ik} f_{ik}(\sigma_{ik}) \dot\sigma_{ik},$$

(5.4.18)
$$\left.\frac{dv_{ii}}{dt}\right|_{(5.4.12)} = 2x_i^{\mathrm{T}} H_i h_i(x).$$

Further we introduce the following matrices

(5.4.19)
$$r_i = \operatorname{diag}\{\gamma_{i1}, \gamma_{i2}\},$$

(5.4.20)
$$\Phi_i = \operatorname{diag}\left\{ \frac{f_{i1}(\sigma_{i1})}{\sigma_{i1}}, \frac{f_{i1}(\sigma_{i1})}{\sigma_{i1}} \right\} \in [a_i, b_i],$$

where $a_i = \operatorname{diag}\{\varepsilon_{i1}, \varepsilon_{i2}\}$ and $b_i = \{\xi_{i1}, \xi_{i2}\}$ are prescribed valu

The expressions (5.4.18) and (5.4.18) are transformed as

$$\left.\frac{dv_{ii}}{dt}\right|_{(5.4.11)} = -x_i^{\mathrm{T}}[G_i - (aH_iB_i + P_i^{\mathrm{T}}C_ir_i)\Phi_iC_i^{\mathrm{T}}]; \tag{5.4.21}$$

where

$$-G_i = H_iP_i + P_i^{\mathrm{T}}H_i$$

and

$$\left.\frac{dv_{ii}}{dt}\right|_{(5.4.12)} = 2x_i^{\mathrm{T}} H_i D_{ia} x_i + 2x_i^{\mathrm{T}} H_i \sum_{j \neq i}^{s} D_{ib} x_j, \tag{5.4.22}$$

where

$$D_{ia} = \begin{pmatrix} 0 & 0 \\ -M_i^{-1} \sum\limits_{j \neq i}^{s} A_{jj} \Phi_{ij} & 0 \end{pmatrix},$$
$$D_{ib} = \begin{pmatrix} 0 & 0 \\ M_i^{-1} A_{ij} \Phi_{ij} - M_N^{-1} A_N \Phi_N & 0 \end{pmatrix}$$

and

$$\begin{aligned} &\Phi_{ij}(0) = 0, \\ &\Phi_{ij}(\sigma_{ij}) = \frac{f_{ij}(\sigma_{ij})}{\sigma_{ij}}, \quad \sigma_{ij} \neq 0. \end{aligned} \tag{5.4.23}$$

Combining (5.4.21) and (5.4.22) yields

$$\begin{aligned} \frac{dv_{ii}}{dt} &= -x_i^{\mathrm{T}} \left\{ G_i - (2H_i B_i + P_i^{\mathrm{T}} C_i r_i) \Phi_i C_i^{\mathrm{T}} - 2H_i D_{ia} \right\} x_i \\ &\quad + 2x_i^{\mathrm{T}} H_i \sum_{j \neq i}^{s} D_{ib} x_j. \end{aligned} \tag{5.4.24}$$

For functions v_{ij} defined by (5.4.15) we have

$$\begin{aligned} \frac{dv_{ii}}{dt} &= \alpha_{ij} \Phi_{ij} x_i^{\mathrm{T}} dd^{\mathrm{T}} P_i x_i - \alpha_{ij} \Phi_{ij} x_i^{\mathrm{T}} (dd^{\mathrm{T}} P_j + P_j^{\mathrm{T}} dd^{\mathrm{T}}) x_j \\ &\quad + \alpha_{ij} \Phi_{ij} x_j^{\mathrm{T}} dd^{\mathrm{T}} P_j x_i, \qquad i \neq j, \end{aligned} \tag{5.4.25}$$

where $d = (1,0)^{\mathrm{T}}$.

We have for function

$$\dot V(x,\eta) = -\sum_{i=1}^{s} x_i^{\mathrm{T}} D_{ii} x_i + \sum_{i=1}^{s}\sum_{j\neq i}^{s} x_i^{\mathrm{T}} D_{ij} x_j, \tag{5.4.26}$$

where

$$\begin{aligned} D_{ii} &= G_i - (2H_iB_i + P_i^{\mathrm{T}} C_i^{\mathrm{T}} r_i)\Phi_i C_i^{\mathrm{T}} - 2H_i D_{ia} \\ &\quad - \sum_{j\neq i}^{s} (\alpha_{ij}\Phi_{ij} + \alpha_{ij}\Phi_{ji}) dd^{\mathrm{T}} P_i \end{aligned} \tag{5.4.27}$$

and

$$D_{ij} = 2H_i D_{ib} - \alpha_{ij}\Phi_{ij}(dd^{\mathrm{T}} P_j + P_i^{\mathrm{T}} dd^{\mathrm{T}}). \tag{5.4.28}$$

Further we show that the right-hand part of (5.4.26) can be estimated by the expression $w^{\mathrm{T}}(x)Aw(x)$, i.e.

$$\dot V(x,\eta) \le w^{\mathrm{T}}(x)Aw(x), \tag{5.4.29}$$

where $w(x) = (\|x_1\|,\dots,\|x_s\|)^{\mathrm{T}}$, $A = [a_{ij}]$, $i,j = 1,2,\dots,s$. Here a_{ij} is a computed in terms of estimate of the right-hand part of (5.4.26).

If we set $W(x) = \operatorname{diag}\{\|x_1\|,\|x_2\|,\dots,\|x_s\|\}$, then

$$\dot V(x,\eta) \le \eta^{\mathrm{T}} W(x)AW(x)\eta.$$

It should be noted that $\dot U(x)$ is not estimated by the expression $W(x)AW(x)$ in view of (5.4.24)–(5.4.28).

Then the matrices H_i are taken in the form

$$H_i = \begin{pmatrix} \lambda h_{12}^i & h_{12}^i \\ h_{12}^i & \dfrac{1+k_i}{\lambda} h_{12}^i \end{pmatrix}, \tag{5.4.30}$$

where k_i are arbitrary positive constants and matrices G_i are computed

$$G_i = \begin{pmatrix} 0 & 0 \\ 0 & 2k_i h_{12}^i \end{pmatrix}. \tag{5.4.31}$$

We take the constants

$$\gamma_{i1} = 2M_i^{-1}A_i h_{22}^i,$$
$$\gamma_{i2} = 2M_N^{-1}A_i h_{22}^i, \tag{5.4.32}$$
$$\alpha_{ij} = \alpha_{ji} = M_i^{-1}A_{ij}h_{22}$$

and transform the expression $-x_i^{\mathrm{T}} D_{ii} x_i$ as

$$\begin{aligned} -x_i^{\mathrm{T}} D_{ii} x_i = -2h_{i2}^i \Bigg\{ A_i \left(M_i^{-1} \frac{f_{i1}(\sigma_{i1})}{\sigma_{i1}} + M_N^{-1} \frac{f_{i2}(\sigma_{i2})}{\sigma_{i2}} \right) \\ + M_i^{-1} \sum_{j \neq i}^{s} A_{ij} \Phi_{ij} \Bigg\} x_{i1}^2 - 2k_i h_{i2}^i x_{i2}^2 \\ + \sum_{j \neq i}^{s} M_i^{-1} h_{22}^i A_{ij} (\Phi_{ji} + \Phi_{ij}) x_{i1} x_{i2}. \end{aligned} \tag{5.4.33}$$

The right-hand part of (5.4.33) may be estimated by the value $-\lambda_{im}(Q_i)\|x_i\|^2$:

$$-x_i^{\mathrm{T}} D_{ii} x_i \le -\lambda_{im}(Q_i)\|x_i\|^2, \qquad i = 1, 2, \dots, s, \tag{5.4.34}$$

where $\lambda_{im}(Q_i)$ is the minimal eigenvalue of the matrix Q_i, the elements of which are determined as

$$\begin{aligned} q_{11}^i = q_{22}^i &= 2h_{i2}^i \Big\{ A_i(M_i^{-1}\varepsilon_{i1} + M_N^{-1}\varepsilon_{i2}) + M_i^{-1} \sum_{j \ne i}^{s} A_{ij}\varepsilon_{ij} \Big\}, \\ q_{12}^i &= -\frac{1}{2} M_i^{-1} h_{22}^i \sum_{j \ne i}^{s} \max(\xi_{ij}, \xi_{ji}). \end{aligned} \tag{5.4.35}$$

We note that $\varepsilon_{ij} \in (0, \xi_{ij})$ and the constants k_i are taken according to

$$k_i = A_i(M_i^{-1}\varepsilon_{i1} + M_N^{-1}\varepsilon_{i2}) + M_i^{-1} \sum_{j \ne i}^{s} A_{ij}\varepsilon_{ij}. \tag{5.4.36}$$

We have in view of (5.4.28)

(5.4.37)

$$\begin{aligned} x_i^{\mathrm{T}} D_{ij} x_j &= 2h_{i2}(M_i^{-1} A_{ij}\Phi_{ij} - M_N^{-1} A_{Nj}\Phi_{Nj}) x_{i1} x_{j1} \\ &\quad - \alpha_{ij}\Phi_{ij} x_{i1} x_{i2} \\ &\quad + \left\{ 2h_{22}^i (M_i^{-1} A_{ij}\Phi_{ij} - M_N^{-1} A_{Nj}\Phi_{Nj} - \alpha_{ij}\Phi_{ij} \right\} x_{i2} x_{j1}. \end{aligned}$$

To estimate the right-hand part of (5.4.37) the functions $Z_1\colon R^2 \to R$ and $Z_2\colon R^3 \to R$ are introduced by the formulas

$$Z_1(\alpha,\beta) = \min\left\{\sqrt{2}\max(|\alpha|,|\beta|),\ (|\alpha|+|\beta|)\right\},$$

$$Z_2(\alpha,\beta,\gamma) = \min\Big\{\sqrt{2}\max(|\alpha|,|\beta|,|\gamma|),\ (|\alpha|+|\beta|+|\gamma|),$$
$$Z_1(\alpha,\beta)+|\gamma|,\ Z_1(\alpha,\beta)+|\beta|,\ Z_2(\beta,\gamma)+|\alpha|\Big\}.$$

Having noted that the expressions $x_{i1}x_{j1}$, $x_{i1}x_{j2}$, $x_{i2}x_{j1}$ can be treated as the components of the 3-dimensional subspace, where each of the expressions may take either positive, negative or zero value, the estimate of the righ-hand part of (5.4.37) can be obtained in the form

(5.4.38)
$$\begin{aligned} x_i^{\mathrm{T}} D_{ij} x_j \le Z_2 \{&2h_{12}^i \max\left(M_i^{-1}A_{ij}\xi_{ij},\, M_N^{-1}A_{Nj}\xi_{nj}\right), \\ &M_i^{-1}A_{ij}h_{22}^i\xi_{ij},\, h_{22}^i \max\left(M_i^{-1}A_{ij}\xi_{ij},\right. \\ &\left.2M^{-1}A_{Nj}\xi_{Nj}\right)\}\,\|x_i\|\,\|x_j\|. \end{aligned}$$

In view of (5.4.34) and (5.4.38) we get for the elements a_{ij} of matrix A:

(5.4.39)
$$\hat a_{ij} = \begin{cases} -\lambda_{im} \quad (i=j); \\ Z_2\{2h_{12}^i \max(M_i^{-1}A_{ij}\xi_{ij},\, M_N^{-1}A_{Nj}\xi_{Nj}, \\ \quad M_i^{-1}A_{ij}h_{22}^i\xi_{ij},\, h_{22}^i\max(M_i^{-1}A_{ij}\xi_{ij}, \\ \quad 2M_N^{-1}A_{Nj}\xi_{Nj}\} \quad (i\ne j) \end{cases}$$

and

(5.4.40)
$$a_{ij} = \frac{1}{2}(\hat a_{ij} + \hat a_{ji}), \qquad i,j = 1,2,\dots,s.$$

We formulate now the following assertion.

PROPOSITION 5.4.1. *In order for the equilibrium state $x = 0$ of system (5.4.10) to be asymptotically stable it is sufficient that the inequalities*

(5.4.41)
$$(-1)^k \begin{vmatrix} a_{11} & \dots & a_{1s} \\ \dots & \dots & \dots \\ a_{s1} & \dots & a_{ss} \end{vmatrix} > 0, \qquad k = 1,2,\dots,s,$$

be satisfied.

Proof. Let the matrix A in estimate (5.4.29) be constructed according to (5.4.39) and (5.4.40). When inequalities (5.4.41) are satisfied, the matrix A is negative definite, and by (5.4.40) $A = A^{\mathrm{T}}$. The function $V(x, \eta) = \eta^{\mathrm{T}} U(x) \eta$ is positive definite, since $H_i = H_i^{\mathrm{T}}$ is positive definite, $\gamma_{ik} > 0$ and $\alpha_{ij} > 0$ and the integral terms in (5.4.14) and (5.4.15) are non-negative in the neighborhood of $x = 0$. Thus, function $V(x, \eta)$ for system (5.4.10) is positive definite and $\dot{V}(x, \eta)$ is negative definite in the neighborhood of $x = 0$ due to inequalities (5.4.41). By Theorem 2.3.3 the equilibrium state $x = 0$ of system (5.4.10) is asymptotically stable.

5.4.4 Numerical examples

5.4.4.1 Example. The proposed algorithm of the power system stability analysis is applicable to the 3-machine power system considered by Jocić, Ribbens-Pavella and Šiljak [79]. We admit the following parameter values for the system (5.4.10):

$$N=3; \quad E_1=1.017; \quad E_2=1.005; \quad E_3=1.033; \quad \delta_{12}=5^\circ;$$
$$\delta_{13}=2^\circ; \quad \delta_{23}=-3^\circ; \quad Y_{12}=0.98\times 10^{-3}\angle 86^\circ; \quad Y_{13}=0.114\angle 88^\circ;$$
$$Y_{23}=0.106\angle 89^\circ; \quad M_1=M_2=0.01; \quad M_3=2.0.$$

Treating the third machine as a standard one we get two subsystems. Let us take the constants $\lambda=0.3$, $\varepsilon_{11}=\varepsilon_{21}=0.06$ and $\varepsilon_{12}=\varepsilon_{23}=\xi_{12}=\xi_{21}=0.001$. The matrix $\hat{A}=[\hat{a}_{ij}]$, defined by formula (5.4.39) is of the form

$$\hat{A}=\begin{pmatrix} -1.1506 & 1.0814 \\ 1.0671 & -1.0437 \end{pmatrix}.$$

The matrix $2A=\hat{A}+\hat{A}^{\mathrm{T}}$ satisfies conditions (5.4.41) and therefore the equilibrium state $x=0$ is asymptotically stable. It is important to note that in this case Jocić, Ribbens-Pavella and Šiljak [79] established the conditions of asymptotic stability for $\lambda=100$, $\varepsilon=0,99$. In a paper by Shaaban and Grujić [164] the asymptotic stability of the system in question was stated for $\lambda=0.45$, $\varepsilon_{11}=\varepsilon_{21}=0.10$.

The asymptotic stability conditions for the equilibrium state $x=0$ obtained herein are the least value for the parameters λ and ε.

5.4.4.2 Example. Let in system (5.4.10) $N = 4$ and the parameter values are the following (see El-Abiad and Nagappan [35]):

$$E_1 = 1.057/5.7^\circ, \quad E_2 = 1.152/14.4^\circ, \quad E_3 = 1.095/2.3^\circ, \quad E_4 = 1.0/0.1^\circ,$$
$$Y_{11} = 0.88/-88.1^\circ, \quad Y_{22} = 0.873/-83.2^\circ, \quad Y_{33} = 1.014/-75.5^\circ,$$
$$Y_{44} = 2.447/-69,7^\circ, \quad Y_{12} = 0.124/82.1^\circ, \quad Y_{13} = 0.065/82.4^\circ,$$
$$Y_{23} = 0.064/88.2^\circ, \quad Y_{24} = 0.655/96.8^\circ,$$
$$Y_{34} = 0.754/99^\circ, \quad Y_{14} = 0.658/91.1^\circ;$$
$$M_1 = 1130, \quad M_2 = 2260, \quad M_3 = 1508, \quad M_4 = 75\,350.$$

Choosing the fourth machine as a standard one we get three subsystems. For the values $\lambda = 0.8$, $\varepsilon_{11} = \varepsilon_{21} = \varepsilon_{31} = 0.5$ the matrix $\hat{A}$ (see formula (5.4.39)) is

$$\hat{A} = \begin{pmatrix} -4.9087 & 3.7790 & 1.8484 \\ 1.8109 & -2.7037 & 0.9811 \\ 1.4073 & 1.4898 & -4.8370 \end{pmatrix}.$$

The matrix $a = \dfrac{1}{2}\left(\hat{A} + \hat{A}^{\mathrm{T}}\right)$ satisfies the conditions (5.4.41) and therefore, the state $x = 0$ of the system is asymptotically stable. Earlier it has been stated (see Grujić and Shaaban [61]) that the asymptotic stability of the equilibrium state $x = 0$ of the system takes place provided that $\lambda = 1.0$ and $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0.60$.

Therefore, this case as well the proposed algorithm allows us to establish the conditions of asymptotic stability for smaller valies of λ and ε.

5.4.4.3 Example. Let in system (5.4.10) $N = 7$ and the parameter values are taken following Shaaban and Grujić [164]. Taking the seventh machine as a standard one we get six subsystems. For the values $\lambda = 2.0$, $\varepsilon_{i1} = 0.80$, $i = 1, 2, 3$; $\varepsilon_{j1} = 0.85$, $j = 4, 5, 6$, the matrix $\hat{A}$ (see (5.4.39)) is

$$\hat{A} = \begin{pmatrix} -2.0176 & 1.0286 & 0.2408 & 0.2521 & 0.2876 & 0.2730 \\ 1.3301 & -2.3742 & 0.2660 & 0.2785 & 0.3177 & 0.2952 \\ 0.2944 & 0.3111 & -1.8805 & 0.8070 & 0.2744 & 0.2594 \\ 0.2910 & 0,2714 & 0.7547 & -1.9315 & 0.2848 & 0.2577 \\ 0.3022 & 0.2949 & 0.2357 & 0.2505 & -1.9757 & 0.7701 \\ 0.3155 & 0.2941 & 0.2461 & 0.2577 & 0.8847 & -2.1405 \end{pmatrix}$$

and $a = \dfrac{1}{2}\left(\hat{A} + \hat{A}^{\mathrm{T}}\right)$ satisfies the conditions (5.4.41). Then the equilibrium state $x = 0$ of the system is asymptotically stable. In the above mentioned paper by Shaaban and Grujić [164] the asymptotic stability of the equilibrium state was established for $\lambda = 3.0$ and $\varepsilon_{i1} = 0.95$, $i = 1, 2, \dots, 6$. This applies to the smaller values of λ and ε as well as to the asymptotic stability of the equilibrium state $x = 0$.

The application of the approach to three–four and seven-machine system enables us to conclude as follows (see Grujić and Shaaban [61]):

(1) We can decrease the value of the parameter λ for which asymptotic stability of $x = 0$ of the system is assured (value of λ is decreased from 100 to only 0.3 for the three-machine system, and decreased by 33% of that in Shaaban and Grujić [164] for the four and seven machine systems). Noting that the smaller value of λ means that the generator is less damped and that it is more difficult to assure stability, we can deduce that the developed approach is more powerful then those developed so far via vector Liapunov functions.

(2) Smaller value of the parameter ε can be assumed and the asymptotic stability assured by applying the developed approach (value of ε is assumed to be 85% of that in Shaaban and Grujić [164] for the four and seven machine systems, and it is decreased from 0.10 to only 0.06 for the three-machine system). This essentially means that the developed approach can lead to larger asymptotic stability domain estimates.

(3) Using the developed approach, we can decrease the conservativeness of the decomposition-aggregation method.

(4) The matrix-valued Liapunov function methodology leads to more adequate scalar Liapunov functions for power systems and simplifies their construction via the vector Liapunov function concept.

(5) The stability test computation is reduced to only the negative definiteness test of a single elementwise constant aggregation symmetric matrix. Its dimension is reduced to the number $s = N - 1$ of the subsystems of an N-machines power system.

5.5 The Motion in Space of Winged Aircraft

According to Aminov and Sirazetdinov [2] we will consider the case when the aircraft, moving with fixed absolute value of the velocity, performs a manouvre with constant load factor. Thus, to the undistrturbed motion there corresponds constant values of the angles of attack α_0 and of side-slip β_o, and angular velocities of pitch ω_{z0}, yaw ω_{y0} and rotation ω_{x0}. Their deviations from the perturbed values will be called α, β, ω_z, ω_y, ω_x respectively. The deviations of the angular velocities of side-slip, yaw and rotation must not exceed given limits.

We consider the equations of the perturbed motion in the form (see Byushgens and Studnev [18])

$$
\begin{aligned}
\frac{d\alpha}{dt} &= \mu\omega_z - \frac{1}{2}c_y^{\alpha}\alpha - \mu\beta\omega_x - \frac{1}{2}c_y^{\delta_e}\delta_e,\\
\frac{d\omega_z}{dt} &= m_z^{\alpha}\alpha + m_x^{\omega_z}\omega_z - \mu A\omega_x\omega_y + m_z^{\delta_e}\delta_e,\\
\frac{d\beta}{dt} &= \mu\omega_y + \frac{1}{2}c_z^{\beta}\beta + \mu\alpha\omega_x + \frac{1}{2}c_z^{\delta_r}\delta_r,\\
\frac{d\omega_y}{dt} &= m_y^{\beta}\beta + m_y^{\omega_y}\omega_y + \mu B\omega_x\omega_z + m_y^{\delta_r}\delta_r,\\
\frac{d\omega_x}{dt} &= m_x^{\beta}\beta + m_x^{\omega_x}\omega_x - \mu C\omega_y\omega_z + m_x^{\delta_a}\delta_a,
\end{aligned} \tag{5.5.1}
$$

where

$$A = \frac{J_y - J_x}{J_z} > 0, \quad B = \frac{J_z - J_x}{J_y} > 0, \quad C = \frac{J_z - J_y}{J_x} > 0,$$

and μ is the aircraft relative density, c_u are the coefficients of the aerodynamic forces, m_u are the coefficients of the aerodynamic moments, δ_e, δ_r, δ_a are the deviations of the elevator, aileron and rudder, and J_x, J_y, J_z are the aircraft moments of inertia with respect to the connected coordinate system.

We take the law of stabilization in the form

$$
\begin{gathered}
\delta_e = k_e^{\alpha}\alpha + k_e^{z}\omega_z, \qquad \delta_r = k_r^{\beta}\beta + k_r^{y}\omega_y,\\
\delta_a = k_a^{\beta}\beta + k_a^{x}\omega_x.
\end{gathered} \tag{5.5.2 }
$$

We substitute the values (5.5.2) into equations (5.5.1). We use the notations

$$
(5.5.3)\quad
\begin{gathered}
x_1 = \omega_x, \quad x_2 = \omega_y, \quad x_3 = \omega_z, \quad x_4 = \alpha, \quad x_5 = \beta, \\
a_{11} = m_x^\beta + k_a^\beta m_x^{\delta_a}, \qquad a_{15} = m_x^{\omega_x} + k_a^x m_x^{\delta_a}, \\
a_{22} = m_y^\beta + k_r^\beta m_y^{\delta_r}, \qquad a_{25} = m_y^{\omega_y} + k_r^y m_y^{\delta_r}, \\
a_{33} = m_z^\alpha + k_e^\alpha m_z^{\delta_e}, \qquad a_{34} = m_z^{\omega_z} + k_e^z m_z^{\delta_e}, \\
a_{44} = \frac{1}{2}\left(c_y^\alpha + k_e^\alpha c_y^{\delta_e}\right), \quad a_{43} = \mu - \frac{1}{2} k_e^z c_y^{\delta_e}, \\
a_{55} = \frac{1}{2}\left(c_z^\beta + k_r^\beta c_z^{\delta_r}\right), \quad a_{52} = \mu + \frac{1}{2} k_r^y c_z^{\delta_r}, \\
b_1 = -\mu C, \quad b_2 = \mu B, \quad b_3 = -\mu A, \quad b_4 = -\mu, \quad b_5 = \mu.
\end{gathered}
$$

Using this notation we can write system (5.5.1) as

$$
(5.5.4)\quad
\begin{aligned}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{15}x_5 + b_1 x_2 x_3, \\
\frac{dx_2}{dt} &= a_{22}x_2 + a_{25}x_5 + b_2 x_1 x_3. \\
\frac{dx_3}{dt} &= a_{33}x_3 + a_{34}x_4 + b_3 x_1 x_2, \\
\frac{dx_4}{dt} &= a_{43}x_3 + a_{44}x_4 + b_4 x_1 x_5, \\
\frac{dx_5}{dt} &= a_{52}x_2 + a_{55}x_5 + b_5 x_1 x_4.
\end{aligned}
$$

We shall find the conditions connected to the coefficients of the system (5.5.4) under which the solution of the system $x = 0$ is multistability, i.e., asymptotically stable with respect to (x_4, x_5), and stable with respect to (x_1, x_2, x_3).

We use the Theorem 2.6.1. In our example $N = 2$, i.e., there are two groups of variables (x_1, x_2, x_3) and (x_4, x_5). We consider the matrix-valued Liapunov function

$$
U(x) = \frac{1}{2}\,\mathrm{diag}\,[-b_2 b_3 x_1^2,\ 2b_1 b_3 x_2^2,\ -b_1 b_2 x_3^2,\ x_4^2,\ x_5^2],
$$

and $\eta \in R_+^5,\ \eta_i = 1,\ i = 1, 2, \ldots, 5.$

The function

$$
(5.5.5)\quad \eta^{\mathrm{T}} U(x)\eta = V(x,\eta) = \frac{1}{2}\left(-b_2 b_3 x_1^2 + 2b_1 b_3 x_2^2 - b_1 b_2 x_3^2 + x_4^2 + x_5^2\right)
$$

is positive definite, decreasing and radially unbounded. In view of the system (5.5.4) the derivative of the function (5.5.5) is

$$
\begin{aligned}
DV(x,\eta) = &- b_2b_3a_{11}x_1^2 - b_2b_3a_{15}x_1x_5 + 2b_1b_3a_{22}x_2^2 \\
&+ (2b_1b_3a_{25} + a_{52})x_2x_5 - b_1b_2a_{33}x_3^2 \\
&+ (a_{43} - b_1b_2a_{34})x_3x_4 + a_{44}x_4^2 + a_{55}x_5^2.
\end{aligned} \tag{5.5.6}
$$

In order to solve our problem we have to find the conditions whereby function (5.5.6) is non-positive with respect to (x_1, x_2, x_3) and negative definite with respect to (x_4, x_5).

The method of finding these conditions is given by Aminov and Sirazetdinov [3] and is as follows. We equate the derivative $DV(x,\eta)$ of (5.5.6) to the function

$$
\begin{aligned}
W(x) = &- (c_{11}x_1 + c_{15}x_5)^2 - (c_{22}x_2 + c_{25}x_5)^2 \\
&- (c_{33}x_3 + c_{34}x_4)^2 - (c_{4}x_4)^2 - (c_{5}x_5)^2
\end{aligned}
\tag{5.5.7}
$$

and, comparing coefficient of like terms of (5.5.6) and (5.5.7), we find the conditions for the existence of the coefficients of function (5.5.7) which are in fact the required conditions for the function (5.5.6) to be non-positive with respect to (x_1, x_2, x_3) and negative definite with respect to (x_4, x_5). These conditions are

$$
\begin{aligned}
&a_{11} < 0, \quad a_{22} < 0, \quad a_{33} < 0, \quad a_{44} + \frac{(a_{43} - b_1 b_2 a_{34})^2}{b_1 b_2 a_{33}} < 0, \\
&a_{55} + \frac{a_{15}^2 b_2 b_3}{a_{11}} - \frac{(2b_1 b_3 a_{25} + a_{52})^2}{2b_1 b_3 a_{22}} < 0.
\end{aligned}
\tag{5.5.8}
$$

On substituting the values of the coefficients (5.5.3) into inequality (5.5.8) we obtain the sufficient conditions that solve the aircraft space manouvre problem.

5.6 Notes

5.2. The basic result of this section (Proposition 5.2.1) is new. The description of model and the competition discussion is due to Lakshmikantham, Leela and Martynyuk [94]. For the large number of references on this topic see Freedman [36]. The application of the Metzler matrix theory and vector

Liapunov functions in the investigation of thise problems is due to Šiljak [167], Grujić and Burgat [56], etc.

5.3. The description of the model of an orbital astronomical observatory is taken from Geiss, Cohen et al. [40] and Grujić [55]. The results of investigation of this model are cited following Krapivny supervised by A. A. Martynyuk. The comparison of the obtained results with those by Grujić, Martynyuk and Ribbens-Pavella [57] has displayed the advantages of the matrix-valued function application. For other results on the subject see Šiljak [167], Abdullin, Anapolskii et al [1], etc.

5.4. The results of this section are due to Grujić and Shaaban [61]. The scalar Liapunov functions are applied by El-Abiad and Nagappan [35], Michel, Fouad and Vittal [142]. For the application of vector Liapunov functions see Pai and Narayana [151], Grujić, Martynyuk and Ribbens-Pavella [57], Grujić and Ribbens-Pavella [58], [59], Grujić, Ribbens-Pavella and Bouffioux [60], Jocić, Ribbens-Pavella and Šiljak [79], Michel, Nam and Vittal [144], Shaaban and Grujić [164], [165], etc. Matrix-valued Liapunov functions are applied by Miladzhanov [145] including the systems with structural perturbations.

5.5. The results of this section are due to Martynyuk [111] and Aminov and Sirazetdinov [2].

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