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Partial differential equations and operators

Fundamental solutions and semigroups Part II

Jan A. Van Casteren

Part II

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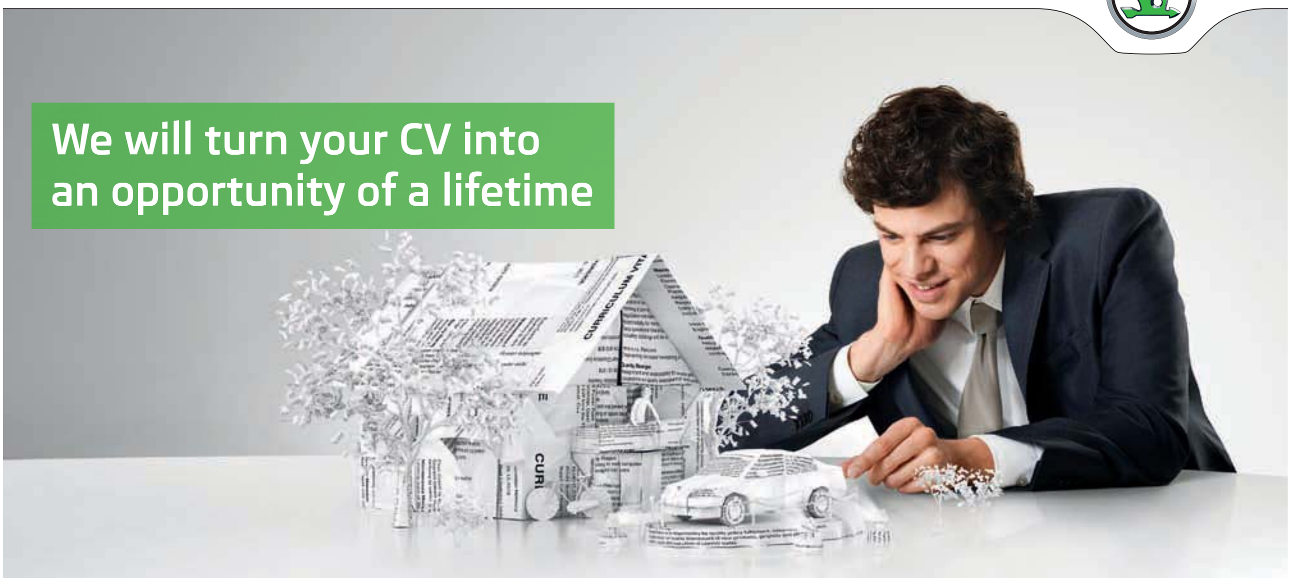

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CHAPTER 5

Operators in Hilbert space

1. Some results in Banach algebras

In this section $(A, \|\cdot\|)$ stands for a complex Banach algebra. A complex Banach algebra is a Banach space over the complex number field $\mathbb{C}$ with a multiplication $(x, y) \mapsto xy$ which is jointly continuous. Moreover we will assume that there is an identity e. This multiplication has the following properties: $x(yz) = (xy)z$, $(x+y)\,z = xz + yz$, $x\,(y+z) = xy + xz$, $\alpha(xy) = (\alpha x)y = x(\alpha y)$ for all x, y, $z \in A$, and for all $\alpha \in \mathbb{C}$. The identity element e satisfies $ex = xe = x$ for all $x \in A$. Moreover, the norm satisfies the multiplicative property $\|xy\| \leqslant \|x\|\,\|y\|$ for all x, $y \in A$. In addition, $\|e\| = 1$. An element $x \in A$ is called invertible if there exists an element $y \in A$ such that $yx = xy = e$. The group of invertible elements of A is denoted by $G(A)$. It is known that $G(A)$ is an open subset of A, and that the application $x \mapsto x^{-1}$ is a homeomorphism from $G(A)$ onto $G(A)$. If $x \in A$ is such that $\|e - x\| < 1$, then x belongs to $G(A)$. Its inverse is given by $y = \lim_{n\to\infty} \sum_{j=0}^{n} (e-x)^j$. Observe that $|\lambda| > \|x\|$ implies that $\lambda e - x = \lambda\,(e - \lambda^{-1}x)$ belongs to $G(A)$. A linear functional $\varphi :\ A \to \mathbb{C}$ which is multiplicative in the sense that $\varphi\,(xy) = \varphi(x)\varphi(y)$ for all x, $y \in A$, is called a complex homomorphism. Most of the time it is assumed that $\varphi(e) \neq 0$, and so $\varphi(e) = 1$. Let φ be a non-zero complex homomorphism. Notice that $1 = \varphi(e) = \varphi(x)\varphi\,(x^{-1})$, $x \in G(A)$, and so $\varphi(x) \neq 0$, Consequently, for $x \in A$ arbitrary and $|\lambda| > \|x\|$, we see that $\varphi(x) \neq \lambda$. In other words $|\varphi(x)| \leqslant \|x\|$. Whence a complex homomorphism is automatically continuous. We also need the following lemma.

5.1. Lemma. *Let $(x_n)_n$ be a sequence in $G(A)$ which converges to $x \in A$. Suppose that $M := \sup_n \|x_n^{-1}\| < \infty$. Then $x \in G(A)$.*

Proof. We estimate

$$\left\|e - x_n^{-1}x\right\| \leqslant \left\|x_n^{-1}\,(x_n - x)\right\| \leqslant \left\|x_n^{-1}\right\|\,\|x_n - x\| \leqslant M\,\|x_n - x\| < 1,$$

for n large enough. It follows that $x_n^{-1}x$ belongs to $G(A)$ for n large enough. But then $\left(x_n^{-1}x\right)^{-1} x_n = x^{-1}$. This completes the proof of Lemma 5.1. □

5.2. Definition. Let $(A, \|\cdot\|)$ be a complex Banach algebra. The symbol $G(A)$ stands for the group of invertible elements. Then $G(A)$ is an open subset of A and the application $x \mapsto x^{-1}$ is a homeomorphism from $G(A)$ to $G(A)$. Let $x \in A$. A complex number λ belongs to the spectrum of x, denoted by $\sigma(x)$, if $\lambda e - x$ does

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not belong to $G(A)$. It follows that $\sigma(x)$ is a closed subset of $\mathbb{C}$, and that $\sigma(x)$ is contained in the disc of radius $\|x\|$. It can be proved that $\sigma(x) \neq \emptyset$. It follows that $\sigma(x)$ is a compact subset of $\mathbb{C}$ contained in the disc $\{\lambda \in \mathbb{C} : |\lambda| \leqslant \|x\|\}$, which is non-empty. The spectral radius $\rho(x)$ of $x \in A$ is defined by $\rho(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}$.

Without a complete proof we mention the following theorem, which is Theorem 10.12 in Rudin [**113**].

5.3. Theorem. *Let x be an element of a Banach algebra. Then $\sigma(x)$ is a non-empty compact subset of $\mathbb{C}$, and the spectral radius $\rho(x)$ satisfies*

$$\rho(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \inf_{n\in\mathbb{N}} \|x^n\|^{1/n}. \tag{5.1}$$

Outline of a proof. Let $0 \neq x \in A$. The fact that $\sigma(x) \neq \emptyset$ follows from the observation that the function $f : \lambda \mapsto (\lambda e - x)^{-1}$ is a holomorphic A-valued map on $\mathbb{C}\backslash\sigma(x)$. If $\sigma(x)$ were empty, then this function would be a bounded holomorphic function. By Liouville's theorem it would be constant, and so $f(\lambda) \equiv 0$. So that $x = xe = xf(\lambda)(\lambda e - x) = 0$, which is a contradiction. The equalities

$$\rho(x) = \limsup_{n\to\infty} \|x^n\|^{1/n} = \inf_n \|x^n\|^{1/n} \tag{5.2}$$

follow from the following considerations. If λ belongs to $\sigma(x)$, then it is easy to see that λ^n belongs to $\sigma(x^n)$, and so $|\lambda| \leqslant \|x^n\|^{1/n}$. Hence

$$\rho(x) \leqslant \inf_n \|x^n\|^{1/n}. \tag{5.3}$$

As above, put $f(\lambda) = (\lambda e - x)^{-1}$, and let Γ_r be the contour $\Gamma_r(\vartheta) = re^{i\vartheta}$, $-\pi \leqslant \varphi \leqslant \pi$. Then, for $r > \rho(x)$,

$$x^n = \frac{1}{2\pi i}\int_{\Gamma_r} \lambda^n f(\lambda)\,d\lambda, \quad n \in \mathbb{N}. \tag{5.4}$$

From (5.4) it follows that

$$\|x^n\| \leqslant r^{n+1} \sup\{\|f(\lambda)\| : |\lambda| = r\},$$

and hence $\limsup_{n\to\infty} \|x^n\|^{1/n} \leqslant r$. Since $r > \rho(x)$ is arbitrary we infer that

$$\limsup_{n\to\infty} \|x^n\|^{1/n} \leqslant \rho(x).$$

This in combination with (5.3) yields the inequalities in (5.2) and completes an outline of the proof of Theorem 5.3. □

5.4. Remark. The second equality in (5.3) can be shown without an appeal to the spectral radius $\rho(x)$. Define the number ρ as $\rho = \inf_n \|x^n\|^{1/n}$, fix $\varepsilon > 0$ and choose $m \in \mathbb{N}$ in such a way that $\|x^m\| \leqslant (\rho + \varepsilon)^m$. Then, for $\ell \geqslant 1$, $\ell \in \mathbb{N}$, and $0 \leqslant j \leqslant m$, we have

$$\left\|x^{\ell m+j}\right\| \leqslant \|x^m\|^{\ell} \left\|x^j\right\| \leqslant (\rho + \varepsilon)^{\ell m} \|xj\|. \tag{5.5}$$

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From (5.5) we obtain

$$\begin{aligned}\limsup_{n\to\infty}\|x^n\|^{1/n} &\leqslant \limsup_{\ell\to\infty}\max_{0\leqslant j\leqslant m-1}\left\|x^{\ell m+j}\right\|^{1/(\ell m+j)}\\ &\leqslant \limsup_{\ell\to\infty}\max_{0\leqslant j\leqslant m-1}(\rho+\varepsilon)^{\ell m/(\ell m+j)}\left\|x^j\right\|^{1/(\ell m+j)} = \rho+\varepsilon. \qquad (5.6)\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the inequality in (5.6) shows the inequality $\limsup_{n\to\infty}\|x^n\|^{1/n} \leqslant \inf_n \|x^n\|^{1/n}$, and therefore

$$\limsup_{n\to\infty}\|x^n\|^{1/n} = \inf_n \|x^n\|^{1/n}.$$

The following theorem says that a complex Banach algebra which is also a division algebra is isometrically isomorphic with the complex number field.

5.5. Theorem (Theorem of Gelfand-Mazur). *Let A be a Banach algebra in which every non-zero element is invertible. Then there exists an algebra isomorphism $\lambda : A \to \mathbb{C}$ which identifies A and $\mathbb{C}$ as algebras.*

Proof of Theorem 5.5. Let $x \in A$, and choose $\lambda \in \sigma(x)$. If $\lambda' \neq \lambda$, then $\lambda' e - x$ is non-zero, and so $\lambda' e - x$ is invertible. In other words $\sigma(x)$ is a singleton, $\{\lambda(x)\}$ say. Then $x - \lambda(x)e = 0$, and the mapping $x \mapsto \lambda(x)$ identifies A with $\mathbb{C}$ as algebras. This completes the proof of Theorem 5.5. $\square$

5.6. Corollary. *Let M be a proper maximal ideal in a commutative Banach algebra A. Then there exists a complex homomorphism $h : A \to \mathbb{C}$ such that $M = N(h) = \{x \in A : h(x) = 0\}$.*

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Proof of Corollary 5.6. Consider the space $A_M := A/M$ with the standard multiplication and standard norm $\|x + M\| = \inf\{\|x + y\| : y \in M\}$. Observe that $\|e + M\| = 1$. Then A_M is a division algebra. For assume that $x \notin M$, then since M is a maximal proper ideal there exists $y \in A$ such that $xy + M = e + M$. It follows that, in the Banach algebra A_M, $(x + M)(y + M) = xy + M = e + M$. Consequently, A/M is a division algebra. By Theorem 5.5 there exists an algebra isomorphism $\lambda : A/M \to \mathbb{C}$. Let $\pi : A \to A/M$ be the mapping $x \mapsto x + M$. Finally put $h(x) = \lambda(\pi(x))$, $x \in A$. Since $\lambda(e + M) = 1$, it follows that h is a complex homomorphism with $h(e) = 1$ and with $N(h) = M$. This completes the proof of Corollary 5.6. □

Proposition 5.7 is a slight improvement of Lemma 10.16 in [**113**]. It is applied there with V and W being groups of invertible elements in a complex Banach algebra, or with V and W being the resolvent sets of elements of a Banach algebra.

5.7. Proposition. *Let V and W be open subsets of a locally connected topological Hausdorff space. Assume that $V \subseteq W$. The following assertions are equivalent (by a component of W a connected component of W is meant):*

(i) *The boundary of V is a subset of the boundary of W,* i.e. *boundary$(V) \subseteq$ boundary(W);*
(ii) $V = \bigcup\{$*component of* W : *component* $(W) \cap V \neq \varnothing\}$.

Proof. (i) ⇒ (ii). Let x be an element of V and let W_x be the connected component of W that contains x. Let $y \in W_x \backslash V$. Then it follows that $y \in W_x \backslash \overline{V}$, because assume that y belongs to $\overline{V}$. Then y belongs to $\overline{V} \backslash V =$ boundary (V). Assertion (i) then implies that y belongs to the boundary of W. Since W is open it then follows that y does not belong to W. This is a contradiction. As a consequence the inclusion $y \in W_x \backslash \overline{V}$ certainly holds. But then it is obvious that $W_x = (W_x \cap V) \cup \left(W_x \backslash \overline{V}\right)$. However, W_x is open and connected, and so since x belongs to $W_x \cap V$ and since $W_x \cap V$ is open we get $W_x = W_x \cap V$, and hence $W_x \subseteq V$. This proves (ii).

(ii) ⇒ (i). Let $x \in \overline{V} \backslash V$. Assume that x belongs to W. Let W_x be the connected component of W that contains x. Then there are two possibilities:

$$W_x \cap V = \varnothing \quad \text{or} \quad W_x \cap V \neq \varnothing.$$

If $W_x \cap V = \varnothing$, then it follows that $W_x \cap \overline{V} = \varnothing$ and thus $x \notin \overline{V}$. But, by hypothesis, $x \in \overline{V} \backslash V =$ boundary (V). Consequently, $W_x \cap V \neq \varnothing$. But from (ii) it then follows that $V \supseteq W_x$ and so $x \in V \backslash V = \varnothing$. This is a contradiction. From $x \in \overline{V} \backslash V$ it apparently follows that $x \in \overline{V} \backslash W \subseteq \overline{W} \backslash W$. Whence

$$\text{boundary}(V) = \overline{V} \backslash V \subseteq \overline{W} \backslash W = \text{boundary}(W),$$

and so the proof of Proposition 5.7 is complete. □

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5.8. Proposition. *Let A and B be complex Banach algebras. Let e_B be the identity of B, and suppose that $e_B \in A$ and that $A \subseteq B$. Then the inclusions $G(A) \subseteq A \cap G(B)$ and $\text{boundary}_A(G(A)) \subseteq \text{boundary}_A(A \cap G(B))$ hold.*

Proof. Let x be an element of $G(A)$. Then there exists $z \in A$ with the property that $xz = zx = e$. So there exists $z \in B$ with $xz = zx = e$. Whence it follows that $G(A) \subseteq G(B) \cap A$.

Let x be an element in the A-boundary of $G(A)$. Then $x \notin G(A)$ and there exists a sequence $(x_n) \subseteq G(A)$ with the property that $\lim_{n\to\infty} x_n = x$. By Lemma 5.1 (see also Lemma 10.17 in [**113**]) we see $\sup_n \|x_n^{-1}\|_A = \infty$. Assume now that x does not belong to the A-boundary of the set $(A \cap G(B))$. Then we get either $x \in \overline{A \cap G(B)}$ or $x \notin \overline{A \cap G(B)}$. But x in the A-boundary of $G(A)$ implies $x \in \overline{G(A)} \subseteq \overline{A \cap G(B)}$. Hence, if x does not belong to the A-boundary of $A \cap G(B)$, then we have $x \in A \cap G(B)$. But then, since $x_n \to x$, we obtain that $x_n^{-1} \to x^{-1}$ in $G(B)$. But then it follows that $\sup_n \|x_n^{-1}\| < \infty$. This is a contradiction.

This completes the proof of Proposition 5.8. $\square$

5.9. Proposition. *Again A and B are Banach algebras with $A \subseteq B$ and with $e = e_B \in A$. Let $x \in A$. Then the following inclusions hold: $\sigma_A(x) \supseteq \sigma_B(x)$ and $\text{boundary}\,(\sigma_A(x)) \subset \text{boundary}\,(\sigma_B(x))$.*

Proof. Since we have

$$\begin{aligned}\mathbb{C}\backslash\sigma_A(x) &= \{\lambda \in \mathbb{C} : \lambda e - x \in G(A)\} \\ &\subseteq \{\lambda \in \mathbb{C} : \lambda e - x \in G(B)\} = \mathbb{C}\backslash\sigma_B(x),\end{aligned}$$

it follows that $\sigma_A(x) \supseteq \sigma_B(x)$. Next let λ be in boundary $(\sigma_A(x))$. Then it follows that $\lambda \in$ boundary $(\mathbb{C}\backslash\sigma_A(x))$. Consequently, there exists a sequence (λ_n) in $\mathbb{C}\backslash\sigma_A(x)$ such that $\lambda_n \to \lambda$, and such that $\lambda e - x \notin G(A)$. But then we get $\lambda_n e - x \in G(A) \subseteq A \cap G(B)$, with $\lambda_n \to \lambda$ and with $\lambda e - x \notin G(A)$. Since $\sup_n \left\|(\lambda_n e - x)^{-1}\right\| = \infty$ it is impossible that $\lambda e - x$ belongs to $G(B)$, and hence λ belongs to boundary $(\sigma_B(x))$.

This completes the proof of Proposition 5.9. $\square$

The following theorem says that if elements x and y in a Banach algebra are close, the their spectra are also close.

5.10. Theorem. *Let Ω be an open subset of $\mathbb{C}$, and let $x \in A$ be such that $\sigma(x) \subset \Omega$. Then there exists a $\delta > 0$ such that $\|y\| < \delta$ implies $\sigma(x+y) \subset \Omega$.*

Proof. The function $\lambda \mapsto \left\|(\lambda e - x)^{-1}\right\|$ is continuous on the set $\mathbb{C}\backslash\Omega$. In addition, it tends to 0 when $|\lambda| \to \infty$. It follows that

$$M = \sup\left\{\left\|(\lambda e - x)^{-1}\right\| : \lambda \in \mathbb{C}\backslash\Omega\right\} < \infty.$$

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If $y \in A$ is such that $\|y\| < 1/M$, then we have $\left\|(\lambda e - x)^{-1} y\right\| < 1$, and consequently, for $\lambda \in \mathbb{C}\backslash\Omega$ we have that the element

$$\lambda e - (x + y) = (\lambda e - x)\left[e - (\lambda e - x)^{-1} y\right]$$

is invertible. This proves 5.10 with $\delta = 1/M$. □

1.1. Symbolic calculus. Let K be compact subset of an open subset Ω in $\mathbb{C}$. Then there exists a concatenation of oriented curves $\Gamma = \gamma_1 * \cdots * \gamma_n$, where $\gamma_j : [\alpha_j, \beta_j] \to \Omega$, $1 \leqslant j \leqslant n$, are continuous differentiable curves, which surrounds K in the sense that

$$\operatorname{Ind}_\Gamma(\zeta) := \frac{1}{2\pi i}\int_\Gamma \frac{d\lambda}{\lambda - \zeta} = \begin{cases} 1, & \text{if } \zeta \in K; \\ 0, & \text{if } \zeta \in \mathbb{C}\backslash\Omega. \end{cases} \tag{5.7}$$

It follows that, for f in $\operatorname{Hol}(\Omega)$, *i.e.* for f holomorphic on Ω, the Cauchy formula

$$f(\zeta) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)\frac{d\lambda}{\lambda - \zeta}, \quad \zeta \in K, \tag{5.8}$$

holds. We say that the contour Γ surrounds K in Ω. If Ω is an open subset of $\mathbb{C}$, the we write $A_\Omega = \{x \in A : \sigma(x) \subset \Omega\}$. Theorem 5.10 says that A_Ω is an open subset of A. The mapping $f \mapsto \widetilde{f}$, $f \in \operatorname{Hol}(\Omega)$ where

$$\widetilde{f}(x) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)\,(\lambda e - x)^{-1}\, d\lambda, \quad x \in A_\Omega, \tag{5.9}$$

is what people call a symbolic calculus. Here Γ surrounds $\sigma(x)$ in Ω. Let $\widetilde{\operatorname{Hol}}(A_\Omega)$ be the collection of all functions $x \mapsto \widetilde{f}(x)$, $x \in A_\Omega$, as given by (5.9). It is noticed that, by Cauchy's theorem, the value of $\widetilde{f}(x)$ does not depend on the choice of Γ as long as Γ surrounds $\sigma(x)$ in Ω. Some properties are collected in the following theorem.

5.11. Theorem. *Let $Hol(\Omega)$ and $\widetilde{Hol}(A_\Omega)$ be as above. The mapping $f \mapsto \widetilde{f}$ is a linear multiplicative isomorphism from $Hol(\Omega)$ onto $\widetilde{Hol}(A_\Omega)$, which is jointly continuous in the following sense. If $(x_n)_n \subset A_\Omega$ is a sequence which converges to $x \in A_\Omega$, and if $(f_n)_n \subset Hol(\Omega)$ which converges uniformly on compact subsets of Ω to $f \in Hol(\Omega)$, then $\widetilde{f}(x) = \lim_{n\to\infty} f_n(x_n)$. Moreover, if $p_n(\lambda) = \lambda^n$, $\lambda \in \mathbb{C}$, then $\widetilde{p}_n(x) = x^n$, $n \in \mathbb{N}$.*

For the convenience of the reader we insert a proof.

Proof. We begin with the multiplication property, *i.e.* $\widetilde{fg}(x) = \widetilde{f}(x)\widetilde{g}(x)$, $x \in A_\Omega$, whenever f and g belong to $\operatorname{Hol}(\Omega)$. To this end we pick $x \in A_\Omega$, and choose concatenations Γ_1 and Γ_2 which surround $\sigma(x)$ in Ω, but Γ_2 is also chosen in such a way that it surrounds the set $\Omega_1 := \{\lambda \in \Omega : \operatorname{Ind}_{\Gamma_1}(\lambda) = 1\}$. Since Γ_1 surrounds $\sigma(x)$ we know that $\sigma(x) \subset \Omega_1$. Then we have, for f, $g \in \operatorname{Hol}(\Omega)$,

$$\widetilde{f}(x)\widetilde{g}(x) = -\frac{1}{4\pi^2}\int_{\Gamma_1} f(\lambda)\int_{\Gamma_2} g(\mu)\,(\lambda e - x)^{-1}\,(\mu e - x)^{-1}\, d\mu\, d\lambda$$

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(resolvent identity $(\lambda e - x)^{-1} - (\mu e - x)^{-1} = (\mu - \lambda)(\lambda e - x)^{-1}(\mu e - x)^{-1}$)

$$
\begin{aligned}
&= -\frac{1}{4\pi^2}\int_{\Gamma_1}\int_{\Gamma_2}\frac{f(\lambda)g(\mu)}{\mu-\lambda}\left\{(\lambda e - x)^{-1} - (\mu e - x)^{-1}\right\} d\mu\, d\lambda \\
&= -\frac{1}{4\pi^2}\int_{\Gamma_1} f(\lambda)\int_{\Gamma_2}\frac{g(\mu)}{\mu-\lambda}\, d\mu\,(\lambda e - x)^{-1}\, d\lambda \\
&\quad + \frac{1}{4\pi^2}\int_{\Gamma_2}\int_{\Gamma_1}\frac{f(\lambda)}{\mu-\lambda}\, d\lambda\, g(\mu)\,(\mu e - x)^{-1}\, d\mu
\end{aligned}
$$

(apply Cauchy's integral formula)

$$
\begin{aligned}
&= \frac{1}{2\pi i}\int_{\Gamma_1} f(\lambda)g(\lambda)\,(\lambda e - x)^{-1}\, d\lambda + \frac{1}{4\pi^2}\int_{\Gamma_2} 0 \times g(\mu)\,(\mu e - x)^{-1}\, d\mu \\
&= \widetilde{fg}(x). \qquad (5.10)
\end{aligned}
$$

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This proves the multiplication property. Next let $(x_n)_n \subset A_\Omega$ which converges in A to $x \in A_\Omega$, and let $(f_n)_n$ be a sequence of holomorphic functions on Ω which converges, uniformly on compact subsets of Ω to a function f. Then, from complex analysis it follows that f belongs to $\mathrm{Hol}(\Omega)$. Since the sequence $(x_n)_n$ converges to $x \in A_\Omega$, it follows that the set K defined by $K = \bigcup_{n=1}^{\infty} \sigma(x_n) \cup \sigma(x)$ is compact. This can be seen as follows. Let $(\alpha_n)_n$ be a sequence in K. We have to show that some subsequence $(\alpha_{n_k})_k$ converges in K. If there exists $k \in \mathbb{N}$ such that $\sigma(x_k)$ contains infinitely many members of the sequence $(\alpha_n)_n$, then we are done, because $\sigma(x_k)$ is compact, and so some subsequence of the sequence $(\alpha_n)_n$ converges (in $\sigma(x_k) \subset K$). If, on the other hand, for every k the spectrum $\sigma(x_k)$ contains at most finitely members of the sequence $(x_n)_n$, then without loss of generality we may assume that $\alpha_n \in \sigma(x_n)$. Then we choose a decreasing sequence of open subset $(U_k)_k$ with the following properties $\sigma(x) \subset U_k \subset \overline{U_k} \subset \Omega$, $\overline{U_k}$ is compact, and $\sigma(x) = \bigcap_k U_k$. Then by Theorem 5.10 the subsets $A_{U_k} = \{y \in A : \sigma(y) \subset U_k\}$, $k \in \mathbb{N}$, are open. It follows that for n_k large enough α_{n_k} belongs to U_k. Since, *e.g.*, $\overline{U_1}$ is compact, the subsequence $(\alpha_{n_k})_k \subset U_1$ has a further subsequence which converges to α in $\overline{U_1}$. Since α_{n_k} belongs to U_k, and $\sigma(x) = \bigcap_k U_k$, it follows that α is a member of $\sigma(x) \subset K$. This proves that the subset K is sequentially compact. But for subsets of $\mathbb{C}$ this is the same as compact. Next let Γ be a concatenation of curves which surrounds K in Ω. Then we have

$$\begin{aligned}
\widetilde{f}_n(x_n) - \widetilde{f}(x) &= \frac{1}{2\pi i}\int_\Gamma \{f_n(\lambda) - f(\lambda)\}(\lambda e - x_n)^{-1}\, d\lambda \\
&\quad + \frac{1}{2\pi i}\int_\Gamma f(\lambda)\left\{(\lambda e - x_n)^{-1} - (\lambda e - x)^{-1}\right\} d\lambda \\
&= \frac{1}{2\pi i}\int_\Gamma \{f_n(\lambda) - f(\lambda)\}(\lambda e - x_n)^{-1}\, d\lambda \\
&\quad + \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda e - x_n)^{-1}(x_n - x)(\lambda e - x)^{-1}\, d\lambda. \qquad (5.11)
\end{aligned}$$

Let Γ^* be the image of Γ in $\mathbb{C}$. Then Γ^* is a compact subset of $\Omega \backslash K$. Since taking inverses is a continuous operation on the group of invertible elements $G(A)$, it then follows that

$$\sup_{\lambda \in \Gamma^*} \sup_{n \in \mathbb{N}} \left\|(\lambda e - x_n)^{-1}\right\| = M < \infty. \qquad (5.12)$$

The equality in (5.11), the property in (5.47) together with the convergence property, *i.e.* $\lim_{n\to\infty} \|x_n - x\| = 0$ results in $\lim_{n\to\infty} \left\|\widetilde{f}_n(x_n) - \widetilde{f}(x)\right\| = 0$. Altogether this completes the proof of Theorem 5.11. □

For an alternative proof, using Runge's theorem, we refer the reader to the literature; for example Theorem 10.27 in Rudin [**113**] is a good source. This is also true for the following theorem.

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5.12. Theorem (Spectral mapping theorem). *Suppose that* $x \in A_\Omega$ *and* $f \in Hol(\Omega)$. *Then* $\widetilde{f}(x)$ *is invertible in* A *if and only if* $f(\lambda) \neq 0$ *for all* $\lambda \in \sigma(x)$. *Moreover,* $\sigma\left(\widetilde{f}(x)\right) = f(\sigma(x))$.

Proof. If $f \in \mathrm{Hol}(\Omega)$ is such that $f(\lambda) \neq 0$ for all $\lambda \in \sigma(x)$, the there exists an open subset Ω_1 of Ω which contains $\sigma(x)$ such that $f(\lambda)g(\lambda) = 1$ for some holomorphic function g defined on Ω_1. By Theorem 5.11 with Ω_1 in place of Ω, we see that $\widetilde{f}(x)\widetilde{g}(x) = e = \widetilde{g}(x)\widetilde{f}(x)$. Hence, $\widetilde{f}(x)$ is invertible. Next suppose that $f(\alpha) = 0$ for some $\alpha \in \sigma(x)$. Then there exists a holomorphic h function on Ω such that $f(\lambda) = (\lambda - \alpha)\, h(\lambda)$, $\lambda \in \Omega$. It follows that $\widetilde{f}(x) = (x - \alpha e)\, \widetilde{h}(x) = \widetilde{h}(x)\,(x - \alpha e)$. Hence, since α belongs to $\sigma(x)$, $\widetilde{f}(x)$ is not invertible. This proves the first part of the theorem.

Next fix $\beta \in \mathbb{C}$. Then, by definition, β belongs to $\sigma\left(\widetilde{f}(x)\right)$ if and only if $\widetilde{f}(x) - \beta e$ is not invertible in A. By the first part, applied to $f - \beta$, this is the case if and only if $f - \beta$ has a zero in $\sigma(x)$, that is, if and only $\beta \in f(\sigma(x))$.

This completes the proof of Theorem 5.12. □

1.2. On square roots in Banach algebras. In this subsection we will discuss the existence of square roots of an element in a Banach algebra. In the proof of assertion (c) of Theorem 5.14 we need the following lemma.

5.13. Lemma. *The following equality holds:*

$$\int_0^\pi \frac{1}{\cos\vartheta} \log\frac{1+\cos\vartheta}{1-\cos\vartheta}\, d\vartheta = \pi^2. \tag{5.13}$$

The method of proof of Lemma 5.13 which is presented is also employed in the proof of assertion (c) in Theorem 5.14.

Proof. Properties of the function $t \mapsto \arctan t$ yield the first one of the following identities:

$$\begin{aligned}\pi^2 &= 4\int_0^\infty \int_0^\infty \frac{1}{t_1^2+1}\frac{1}{t_2^2+1}\, dt_2\, dt_1 \\ &= 4\int_0^\infty \int_0^\infty \frac{1}{t_1^2t_2^2+t_1^2+t_2^2+1}\, dt_2\, dt_1\end{aligned}$$

(employ polar coordinates: $t_1 = r\cos\vartheta$, $t_2 = r\sin\vartheta$, $r > 0$, $0 \leqslant \vartheta \leqslant \frac{1}{2}\pi$)

$$= 4\int_0^\infty r \int_0^{\pi/2} \frac{1}{r^4\cos^2\vartheta\sin^2\vartheta + r^2 + 1}\, d\vartheta\, dr$$

(make the substitutions $\rho = r^{-2}$, and $\varphi = 2\vartheta$)

$$= \int_0^\infty \int_0^\pi \frac{1}{\frac{1}{4}\sin^2\varphi + \rho + \rho^2}\, d\varphi\, d\rho$$

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$$= \int_0^\infty \int_0^\pi \frac{1}{\left(\rho + \frac{1}{2}\right)^2 - \frac{1}{4}\cos^2\varphi}\, d\varphi\, d\rho$$

$$= \int_0^\infty \int_0^\pi \frac{1}{\cos\varphi} \int_{-\frac{1}{2}\cos\varphi}^{\frac{1}{2}\cos\varphi} \frac{1}{\left(\rho + \frac{1}{2} - s\right)^2}\, ds\, d\varphi\, d\rho$$

(apply Fubini's theorem and make the substitution $\rho = \left(\frac{1}{2} - s\right) r$)

$$= \int_0^\pi \frac{1}{\cos\varphi} \int_{-\frac{1}{2}\cos\varphi}^{\frac{1}{2}\cos\varphi} \frac{1}{\frac{1}{2} - s}\, ds \int_0^\infty \frac{1}{(r+1)^2}\, dr\, d\varphi$$

$$= \int_0^\pi \frac{1}{\cos\varphi} \log \frac{1 + \cos\varphi}{1 - \cos\varphi}\, d\varphi,$$

which shows equality (5.13) in Lemma 5.13. $\square$

In assertion (e) of Theorem 5.14 below the space A is a complex Banach algebra with identity e, and with an involution * which is not necessarily continuous. It has the standard properties of an involution: $(\alpha x + \beta y)^* = \overline{\alpha} x^* + \overline{\beta} y^*$, $(xy)^* = y^* x^*$. $x^{**} = x$, α, $\beta \in \mathbb{C}$, x, $y \in A$. We discuss existence and uniqueness of square roots of elements of a Banach algebra (with an involution in assertion (e)).

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5.14. THEOREM. *Let A be a Banach algebra with a not necessarily continuous involution. The following assertions hold true.*

(a) *Existence of square roots. Let x be an element of a Banach algebra $(A, \|\cdot\|)$ with the property that $\sigma(x) \cap (-\infty, 0] = \emptyset$. Then there exists $y \in A$ with the following properties: y, $y^2 = x$, and $\sigma(y) = \left\{\sqrt{\lambda} : \lambda \in \sigma(x)\right\} \subset \{\lambda \in \mathbb{C} : \Re\lambda > 0\}$.*

(b) *Uniqueness of square roots. Suppose that $x \in A$ is such that $\sigma(x) \cap (-\infty, 0) = \emptyset$. There exists only one element $y \in A$ such that $y^2 = x$, which has the property that its spectrum $\sigma(y)$ is contained in the closed right half plane $\{\lambda \in \mathbb{C} : \Re\lambda \geqslant 0\}$, and which satisfies*

$$\sup\left\{|\lambda| \left\|(\lambda e + y)^{-1}\right\| : \Re\lambda > 0\right\} < \infty. \tag{5.14}$$

The element y is given by the (improper) Riemann integral

$$y = \frac{2}{\pi}\int_0^\infty x\left(t^2 e + x\right)^{-1} dt. \tag{5.15}$$

(c) *Let $x \in A$ be such that $\sigma(x) \cap (-\infty, 0) = \emptyset$, and such that*

$$\sup\left\{\lambda \left\|(\lambda e + x)^{-1}\right\| : \lambda > 0\right\} < \infty. \tag{5.16}$$

Define y as in (5.15). Then $y^2 = x$.

(d) *Let $y \in A$ be such that the integral*

$$\int_0^\infty y\left(t^2 e + y^2\right)^{-1} dt = \lim_{\varepsilon \downarrow 0}\lim_{R \to \infty}\int_\varepsilon^R y\left(t^2 e + y^2\right)^{-1} dt$$

exists. Then the limit

$$p = \lim_{\varepsilon} \varepsilon\left(\varepsilon e + y^2\right)^{-1}$$

exists, and

$$\frac{4}{\pi^2}\left(\int_0^\infty y\left(t^2 e + y^2\right)^{-1} dt\right)^2 = e - p. \tag{5.17}$$

Moreover, $p^2 = p$ and $px = xp = 0$.

(e) *Let x be as in assertion (a),* i.e. *x is an element of a Banach algebra $(A, \|\cdot\|)$ with the property that $\sigma(x) \cap (-\infty, 0] = \emptyset$. Then there exists a unique element $y \in A$ with the following properties: $y^2 = x$, $(y^*)^2 = x^*$, and $\sigma(y^*) = \overline{\sigma(y)} = \left\{\sqrt{\lambda} : \lambda \in \sigma(x^*)\right\} \subset \{\lambda \in \mathbb{C} : \Re\lambda > 0\}$. In fact y is given by*

$$y = \frac{2}{\pi}\int_0^\infty x\left(t^2 e + x\right)^{-1} dt,$$

and y^* is given by

$$y^* = \frac{2}{\pi}\int_0^\infty x^*\left(t^2 e + x^*\right)^{-1} dt.$$

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If $x = x^*$, then $y = y^*$, and $\lambda \in \sigma(y)$ if and only if $\overline{\lambda} \in \sigma(y)$. The element x is positive in the sense that $x = x^*$ and $\sigma(x) \subset (0, \infty)$ if and only if y is so.

Proof. (a) Choose a contour Γ which surrounds $\sigma(x)$ in $\mathbb{C}$, and put

$$y = x\frac{1}{2\pi i}\int_\Gamma \frac{1}{\sqrt{\lambda}}\left(\lambda e - x\right)^{-1} d\lambda$$

(deform the curve Γ: $\lambda = te^{i(\pm\pi\mp\varepsilon)}$, and let ε tend to 0 from above)

$$= -x\frac{1}{2\pi i}\int_\infty^0 \frac{1}{\sqrt{t}}\frac{1}{e^{\frac{1}{2}i\pi}}\left(-te - x\right)^{-1} dt - x\frac{1}{2\pi i}\int_0^\infty \frac{1}{\sqrt{t}}\frac{1}{e^{-\frac{1}{2}i\pi}}\left(-te - x\right)^{-1} dt$$

$$= x\frac{1}{2\pi}\int_0^\infty \frac{1}{\sqrt{t}}\left(te + x\right)^{-1} dt + x\frac{1}{2\pi}\int_0^\infty \frac{1}{\sqrt{t}}\left(te + x\right)^{-1} dt$$

$$= x\frac{1}{\pi}\int_0^\infty \frac{1}{\sqrt{t}}\left(te + x\right)^{-1} dt$$

(substitute $t = s^2$)

$$= \frac{2x}{\pi}\int_0^\infty \left(s^2 e + x\right)^{-1} ds = \lim_{n\to\infty}\sum_{k=0}^{n2^n}\frac{2x}{\pi}\left(k^2 2^{-2n}e + x\right)^{-1}\frac{1}{2^n} = \lim_{n\to\infty} y_n,$$

where $y_n := \sum_{k=0}^{n2^n}\frac{2x}{\pi}\left(k^2 2^{-2n}e + x\right)^{-1}\frac{1}{2^n}$. Notice that $y^2 = x$, and that y_n belongs to the commutative sub-Banach algebra A_0 generated by x and e. In fact for $\alpha > 0$ the element $(\alpha e + x)^{-1} \in A_0$, because, by Runge's theorem (in fact by Lemma 4.66), $(\alpha e + x)^{-1} = \lim_{n\to\infty} p_n(\alpha e + x)$, where $(p_n)_{n\in\mathbb{N}}$ is an appropriate sequence of polynomials in one variable. More precisely, for any polynomial p and for an appropriate contour Γ in $\mathbb{C}\backslash\left((-\infty, 0] \cup \sigma(x)\right)$ we have:

$$(\alpha e + x)^{-1} - p\left(\alpha e + x\right) = \frac{1}{2\pi i}\int_\Gamma \left(\frac{1}{\alpha + \lambda} - p\left(\alpha + \lambda\right)\right)\left(\lambda e - x\right)^{-1} d\lambda,$$

and hence

$$\begin{aligned}&\left\|(\alpha e + x)^{-1} - p\left(\alpha e + x\right)\right\| \\ &\leqslant \frac{1}{2\pi}\text{length}\left(\Gamma\right)\sup_{\lambda\in\Gamma^*}\left|\frac{1}{\alpha + \lambda} - p\left(\alpha + \lambda\right)\right|\sup_{\lambda\in\Gamma^*}\left\|(\lambda e - x)^{-1}\right\|.\end{aligned}$$

Since by the spectral mapping theorem $\sigma(x) = \sigma\left(y^2\right) = \left(\sigma(y)\right)^2$, and since $\sigma(x)$ does not contain negative real numbers it follows that the set $\sigma(y) \cap i\mathbb{R}$ is empty. In addition the function $f : \mathbb{C}\backslash(-\infty, 0] \to \mathbb{C}$ defined by

$$f(\lambda) = \frac{2}{\pi}\int_0^\infty \frac{\lambda}{s^2 + \lambda}\, ds = \sqrt{\lambda}, \quad \lambda \in \mathbb{C}\backslash(-\infty, 0],$$

is analytic in an open neighborhood of $\sigma(x)$, and therefore

$$\sigma(y) = \left\{f(\lambda) : \lambda \in \sigma(x)\right\} = \left\{\sqrt{\lambda} : \lambda \in \sigma(x)\right\} \subset \left\{\lambda \in \mathbb{C} : \Re\lambda > 0\right\}.$$

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This proves assertion (a) of Theorem 5.14.

(b) Next, we proceed with proving the uniqueness of "taking square roots" with spectrum in the closed right half plane. Let $y \in A$ be such that $y^2 = x$ and suppose that y satisfies the assumptions made in assertion (b). We will prove that y is represented as in (5.15). First we observe that

$$\sigma(y) \subset \{\lambda \in \mathbb{C} : \Re\lambda > 0\} \cup \{0\}. \tag{5.18}$$

This is because $y^2 = x$ and $\sigma(x) \cap (-\infty, 0) = \varnothing$, so that by the spectral mapping theorem $\sigma(y) \cap i\mathbb{R} \subset \{0\}$. Since, by assumption $\sigma(y)$ is contained in the closed right half plane, the claim in (5.18) follows. Let, for $r > 0$, the semi-circle $\{\lambda \in \mathbb{C} : \Re\lambda \geqslant 0, |\lambda| = r\}$ be parameterized by $\Gamma_r(\vartheta) = re^{i\vartheta}$, $-\frac{1}{2}\pi \leqslant \frac{1}{2}\pi$. By Cauchy's theorem from complex analysis we infer, for $0 < \varepsilon < R < \infty$, the equality:

$$\begin{aligned}
&\frac{1}{\pi i}\int_{\Gamma_\varepsilon} (ze + y)^{-1}\frac{dz}{z} - \frac{1}{\pi i}\int_{\Gamma_R} (ze + y)^{-1}\frac{dz}{z} \\
&= -\frac{1}{\pi i}\int_{-iR}^{-i\varepsilon} (ze + y)^{-1}\frac{dz}{z} - \frac{1}{\pi i}\int_{i\varepsilon}^{iR} (ze + y)^{-1}\frac{dz}{z} \\
&= \frac{2}{\pi}\int_{\varepsilon}^{R} \left(t^2 e + y^2\right)^{-1} dt.
\end{aligned} \tag{5.19}$$

In (5.19) we let $R \to \infty$ to obtain:

$$\frac{2}{\pi}\int_{\varepsilon}^{\infty} \left(t^2 e + y^2\right)^{-1} dt = \frac{1}{\pi i}\int_{\Gamma_\varepsilon} (ze + y)^{-1}\frac{dz}{z}. \tag{5.20}$$

From (5.20) we get:

$$\begin{aligned}
&\frac{2}{\pi}\int_{\varepsilon}^{\infty} y\left(t^2 e + y^2\right)^{-1} dt = \frac{1}{\pi i}\int_{\Gamma_\varepsilon} y\,(ze + y)^{-1}\frac{dz}{z} \\
&= \frac{1}{\pi i}\int_{\Gamma_\varepsilon} \frac{dz}{z} e - \frac{1}{\pi i}\int_{\Gamma_\varepsilon} (ze + y)^{-1} dz = e - \frac{1}{\pi i}\int_{\Gamma_\varepsilon} (ze + y)^{-1} dz.
\end{aligned} \tag{5.21}$$

From (5.21) we infer:

$$\begin{aligned}
&\frac{2}{\pi}\int_{\varepsilon}^{\infty} y^2\left(t^2 e + y^2\right)^{-1} dt = y - \frac{1}{\pi i}\int_{\Gamma_\varepsilon} y\,(ze + y)^{-1} dz \\
&= y - \frac{1}{\pi i}\int_{\Gamma_\varepsilon} 1\,dz\,e + \frac{1}{\pi i}\int_{\Gamma_\varepsilon} z\,(ze + y)^{-1} dz \\
&= y - \frac{2\varepsilon}{\pi} e + \frac{1}{\pi i}\int_{\Gamma_\varepsilon} z\,(ze + y)^{-1} dz.
\end{aligned} \tag{5.22}$$

In (5.22) we let $\varepsilon \downarrow 0$. By employing (5.14) and the equality $y^2 = x$ the equality in (5.15) follows. This proves assertion (b).

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(c) Let x be as in (c) and let y be as in (5.15). Then like in the proof of Lemma 5.13 we have

$$\begin{aligned} y^2 &= \frac{4}{\pi^2}\int_0^\infty \int_0^\infty x^2 \left(t_1^2 e + x\right)^{-1} \left(t_2^2 e + x\right)^{-1} \, dt_2 \, dt_1 \\ &= \frac{4}{\pi^2}\int_0^\infty \int_0^\infty x^2 \left(t_1^2 t_2^2 e + \left(t_1^2 + t_2^2\right) x + x^2\right)^{-1} \, dt_2 \, dt_1 \end{aligned}$$

(employ polar coordinates: $t_1 = r\cos\vartheta$, $t_2 = r\sin\vartheta$, $r > 0$, $0 \leqslant \vartheta \leqslant \frac{1}{2}\pi$)

$$= \frac{4}{\pi^2}\int_0^\infty r \int_0^{\pi/2} x^2 \left(r^4 e \cos^2\vartheta \sin^2\vartheta + r^2 x + x^2\right)^{-1} \, d\vartheta \, dr$$

(make the substitutions $\rho = r^{-2}$, and $\varphi = 2\vartheta$)

$$\begin{aligned} &= \frac{1}{\pi^2}\int_0^\infty \int_0^\pi x^2 \left(\frac{1}{4} e \sin^2\varphi + \rho x + \rho^2 x^2\right)^{-1} \, d\varphi \, d\rho \\ &= \frac{1}{\pi^2}\int_0^\infty \int_0^\pi x^2 \left(\left(\rho x + \frac{1}{2} e\right)^2 - \frac{1}{4} e \cos^2\varphi\right)^{-1} \, d\varphi \, d\rho \\ &= \frac{1}{\pi^2}\int_0^\infty \int_0^\pi \frac{1}{\cos\varphi} \int_{-\frac{1}{2}\cos\varphi}^{\frac{1}{2}\cos\varphi} x^2 \left(\rho x + \left(\frac{1}{2} - s\right) e\right)^{-2} \, ds \, d\varphi \, d\rho \end{aligned}$$

(apply Fubini's theorem and make the substitution $\rho = \left(\frac{1}{2} - s\right) r$)

$$\begin{aligned} &= \frac{1}{\pi^2}\int_0^\pi \frac{1}{\cos\varphi} \int_{-\frac{1}{2}\cos\varphi}^{\frac{1}{2}\cos\varphi} \frac{1}{\frac{1}{2} - s} \, ds \int_0^\infty x^2 \left(rx + e\right)^{-2} \, dr \, d\varphi \\ &= \frac{1}{\pi^2}\int_0^\pi \frac{1}{\cos\varphi} \log \frac{1 + \cos\varphi}{1 - \cos\varphi} \, d\varphi \, x^2 \int_0^\infty \left(x + \rho e\right)^{-2} \, d\rho \end{aligned}$$

(employ Lemma 5.13)

$$= x^2 \int_0^\infty \left(x + \rho e\right)^{-2} \, d\rho. \tag{5.23}$$

From (5.23) we get

$$y^2 = \lim_{R\to\infty} \lim_{\varepsilon\downarrow 0} \left(x^2 \left(\varepsilon e + x\right)^{-1} - x^2 \left(Re + x\right)^{-1}\right) = \lim_{\varepsilon\downarrow 0} \left(x - \varepsilon x + \varepsilon^2 \left(\varepsilon e + x\right)^{-1}\right) = x. \tag{5.24}$$

In the final steps of (5.24) we used the assumption (5.16) on x. This proves assertion (c).

(d) As in the proof of assertion (c) we have

$$\frac{4}{\pi^2}\left(\int_0^\infty y \left(t^2 e + y^2\right)^{-1} \, dt\right)^2 = \int_0^\infty y^2 \left(\rho e + y^2\right)^{-2} \, d\rho. \tag{5.25}$$

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From (5.25) we infer

$$\frac{4}{\pi^2}\left(\int_0^\infty y\left(t^2e+y^2\right)^{-1}dt\right)^2=\lim_{\varepsilon\downarrow 0}\lim_{R\to\infty}y^2\left(\left(\varepsilon e+y^2\right)^{-1}-\left(Re+y^2\right)^{-1}\right)$$
$$=\lim_{\varepsilon\downarrow 0}\left(e-\varepsilon\left(\varepsilon e+y^2\right)^{-1}\right)=e-p. \tag{5.26}$$

Next we write, for $\lambda>0$ and $\varepsilon>0$,

$$\lambda\varepsilon\left(\lambda e+y^2\right)^{-1}\left(\varepsilon e+y^2\right)^{-1}=\frac{\lambda}{\lambda-\varepsilon}\left(\varepsilon\left(\varepsilon e+y^2\right)^{-1}-\varepsilon\left(\lambda e+y^2\right)^{-1}\right),$$
$$=\frac{\varepsilon}{\varepsilon-\lambda}\left(\lambda\left(\lambda e+y^2\right)^{-1}-\lambda\left(\varepsilon e+y^2\right)^{-1}\right). \tag{5.27}$$

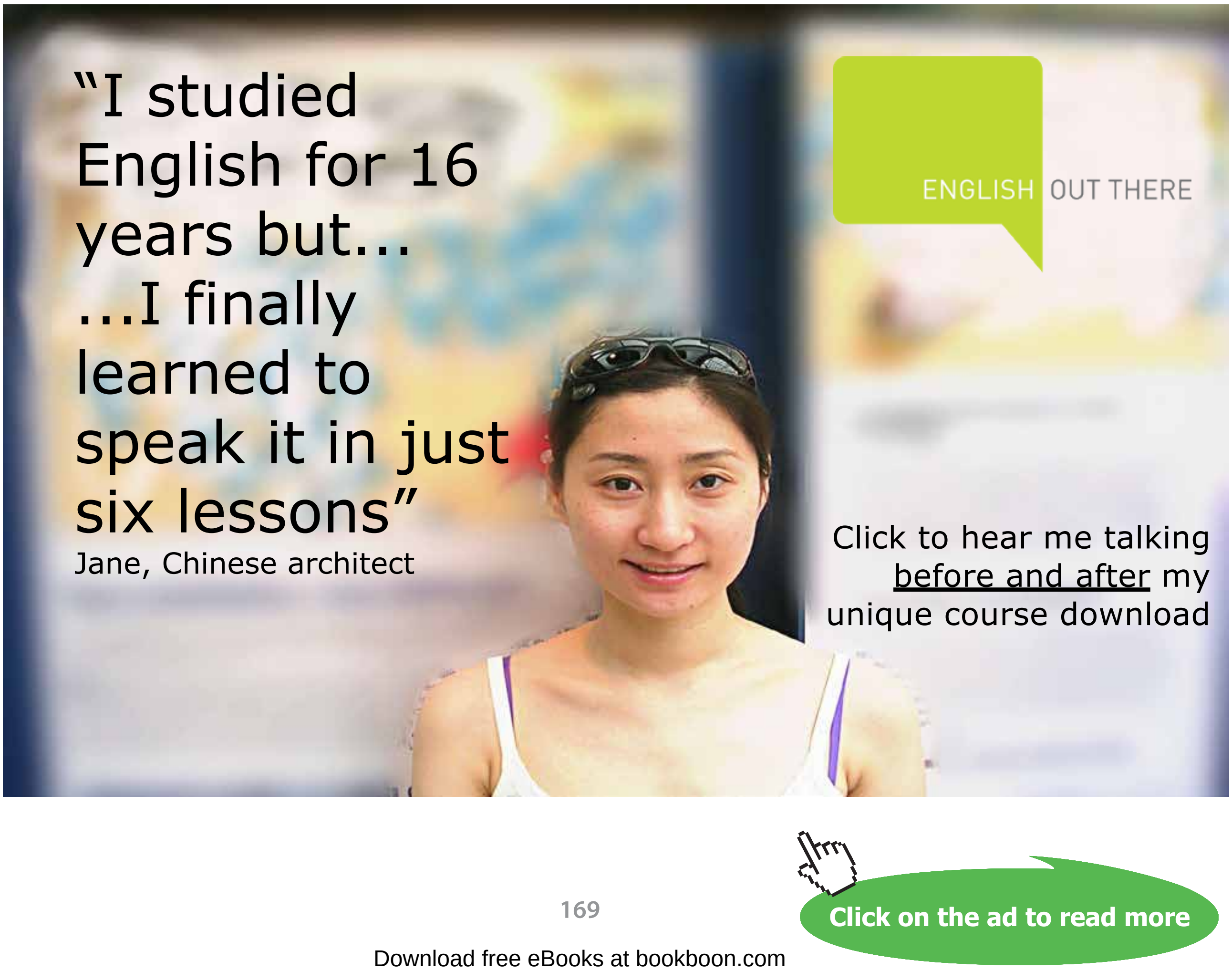
In (5.27) we let ε tend to 0, and we get:

$$\lambda\left(\lambda e+y^2\right)^{-1}p=p,\quad \lambda>0. \tag{5.28}$$

From (5.28) we infer $\lambda p=\left(\lambda e+y^2\right)p$, and so $y^2p=0$. In (5.28) we also let λ tend to 0 to obtain $p^2=p$. If in (5.27) we let λ tend to 0, we obtain

$$\varepsilon p\left(\varepsilon e+y^2\right)^{-1}=p,\quad \varepsilon>0,$$

and hence $py^2=0$. This proves assertion (d).



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(e) This assertion is a consequence of the assertions (a) and (b). Since $\lambda \in \mathbb{C}$ belongs to $\sigma(y)$ if and only if $\overline{\lambda}$ belongs to $\overline{\sigma(y^*)}$, we see that $\sigma(y) = \overline{\sigma(y)}$ provided that $y = y^*$.

Altogether this completes the proof of Theorem 5.14. □

1.3. On C^*-algebras. We need the following property of positive elements in a C^*-algebra A. A C^*-algebra is a Banach algebra with an involution which has the following property $\|x\|^2 = \|x^*x\|$, $x \in A$. An element $u \in A$ is called positive if $u = u^*$ and if $\sigma(u) \subset [0, \infty)$. If $u \in A$ is positive, then the same is true for $\|u\| e - u$.

5.15. Proposition. *If u and v are positive elements in a C^*-algebra A, then $u + v$ is also positive.*

Proof. Put $\alpha = \|u\|$, $\beta = \|v\|$, and $\gamma = \alpha + \beta$. We know that $\sigma(\alpha e - u) \subset [0, \alpha]$ and $\sigma(\beta e - v) \subset [0, \beta]$. Then it follows that $\|\gamma e - w\| \leqslant \gamma$. Since $\sigma(\gamma e - u - v)$ is real it follows that $\sigma(\gamma e - u - v) \subset [-\gamma, \gamma]$, and consequently, $\sigma(u + v) \subset [0, \gamma]$.

This completes the proof of Proposition 5.15. □

5.16. Proposition. *Let y be an element of a C^*-algebra A. Let A_0 be the algebra generated by yy^* and the identity e. Then the spectrum of yy^*, viewed as an element of A_0, is contained in the interval $\left[0, \|y\|^2\right]$. In fact the following identity is true:*

$$yy^* = \frac{2}{\pi} \int_0^\infty (yy^*)^2 \left(t^2 e + (yy^*)^2\right)^{-1} dt. \tag{5.29}$$

Proof. First suppose that $\Im\lambda \neq 0$. Write $\lambda = \alpha + i\beta$, with α and β belonging to $\mathbb{R}$. Choose $t \in \mathbb{R}$ in such a way that $\alpha^2 + 2\beta t + \beta^2 > \|yy^*\|^2$. Then

$$\frac{\|yy^* + ite\|^2}{|\lambda + it|^2} = \frac{\|(yy^* + ite)(yy^* - ite)\|}{\alpha^2 + \beta^2 + 2\beta t + t^2} = \frac{\|yy^*\|^2 + t^2}{\alpha^2 + \beta^2 + 2\beta t + t^2} < 1,$$

and hence $\lambda e - yy^* = (\lambda + it) e - yy^* - ite = (\lambda + it) \left(e - \dfrac{yy^* + ite}{\lambda + it}\right)$ is invertible with inverse $(\lambda + it)^{-1} \displaystyle\sum_{k=0}^{\infty} \frac{(yy^* + ite)^k}{(\lambda + it)^k}$. It follows that $\lambda e - yy^*$ is invertible in A_0 whenever $\Im\lambda \neq 0$. Next we consider the case where λ belongs to $\mathbb{R}$. If $|\lambda| > \|yy^*\| = \|y|^2$, then $\lambda e - yy^*$ is invertible in A_0 via a Neumann series: $(\lambda e - yy^*)^{-1} = \dfrac{1}{\lambda} \displaystyle\sum_{k=0}^{\infty} \frac{1}{\lambda^k} (yy^*)^k$. It follows that $\sigma(yy^*)$ is a subset of $\left[-\|y\|^2, \|y\|^2\right]$.

Put $w = (|yy^*| - yy^*) y = y(|y^*y| - y^*y)$, where $|yy^*|$ is defined as the positive square of $(yy^*)^2$, which can be defined using Gelfand transforms in the algebra generated by yy^*. It can also be defined by employing the integral representation

$$|yy^*| = \frac{2}{\pi} \int_0^\infty (yy^*)^2 \left(s^2 e + (yy^*)^2\right)^{-1} ds$$

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$$= \frac{2}{\pi} \int_0^\infty yy^*y \left(s^2 e + (y^*y)^2\right)^{-1} y^* \, ds. \tag{5.30}$$

Since, for $s > 0$, the element

$$(yy^*)^2 \left((yy^*)^2 + s^2 e\right)^{-1} = yy^* \left(yy^* + ise\right)^{-1} yy^* \left(yy^* - ise\right)^{-1},$$

is the product of two elements in A_0, it itself belongs to A_0. In addition we have

$$\begin{aligned} \left\| \int_0^\varepsilon (yy^*)^2 \left(s^2 e + (yy^*)^2\right)^{-1} ds \right\| &\leqslant \int_0^\varepsilon \left\| (yy^*)^2 \left(s^2 e + (yy^*)^2\right)^{-1} \right\| ds \\ \leqslant \rho \left(\int_0^\varepsilon (yy^*)^2 \left(s^2 e + (yy^*)^2\right)^{-1} ds \right) &\leqslant \sup_{\xi \in \sigma(yy^*)} \int_0^\varepsilon \frac{\xi^2}{s^2 + \xi^2} \, ds \leqslant \varepsilon. \end{aligned}$$

Here $\rho(x)$ represents the spectral radius of the element x. Consequently $|yy^*|$ is a member of A_0. Then

$$\begin{aligned} ww^* &= (|yy^*| - yy^*) \, yy^* \, (|yy^*| - yy^*) = \left(|yy^*| \, yy^* - (yy^*)^2\right) (|yy^*| - yy^*) \\ &= -\left(|yy^*| - yy^*\right)^2 |yy^*| = -\left\{ (|yy^*| - yy^*) \sqrt{|yy^*|} \right\}^2 =: -w_1^2 \end{aligned} \tag{5.31}$$

is negative in the sense that ww^* is self-adjoint and has its spectrum in the closed negative half-axis $(-\infty, 0]$. Since, by the same token,

$$w^*w = -\left\{ (|y^*y| - y^*y) \sqrt{|y^*y|} \right\}^2 =: -w_2^2, \tag{5.32}$$

we infer that w^*w is negative as well, because w_1 as well as w_2 is self-adjoint. Choose self-adjoint elements u and v such that $w = u + iv$. In fact $u = \frac{1}{2}(w + w^*)$, $v = \frac{1}{2i}(w - w^*)$. Then $w^*w + ww^* = 2u^2 + 2v^2$, and hence, by (5.31) and (5.32)

$$2\left(u^2 + v^2\right) = w^*w + ww^* = -w_1^2 - w_2^2. \tag{5.33}$$

From Proposition 5.15 it follows that $w^*w + ww^* = 2u^2 + 2v^2$ is positive in the sense that $w^*w + ww^*$ is self-adjoint and has its spectrum in the closed positive half-axis. On the other hand, by (5.33) we see that $w^*w + ww^*$ is negative in the sense that its spectrum is contained in $(-\infty, 0]$ and that $w^*w + ww^*$ is self-adjoint. But elements whisk are positive as well as negative are 0. Consequently, $w^*w + ww^* = 0$. Proposition 5.17 below shows that the spectrum of w^*w coincides, except for possibly the complex number 0, with the spectrum of ww^*. Hence, w^*w is positive as well as negative; its spectrum is just $\{0\}$. Thus,

$$\|w\|^2 = \|w^*\|^2 = \|w^*w\| = \rho\left(w^*w\right) = 0.$$

Here $\rho\left(w^*w\right)$ denotes the spectral radius of w^*w. However, if $w = 0$, then we get

$$(|yy^*| - yy^*) \, yy^* = 0,$$

and hence,

$$(|yy^*| - yy^*)^2 = 2\left(yy^* - |yy*|\right) yy^* = 0.$$

Consequently, $yy^* = |yy^*|$ is positive. The representation in (5.29) then follows from (5.30).

This completes the proof of Proposition 5.16. □

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In the proof of Proposition 5.16 we used the following result.

5.17. PROPOSITION. *Let x and y be two elements in a Banach algebra. Then $\{0\} \cup \sigma(xy) = \{0\} \cup \sigma(yx)$.*

PROOF. Suppose $\lambda \neq 0$. If z inverts $e - \frac{1}{\lambda}yx$, then $e + \frac{1}{\lambda}xzy$ inverts $e - \frac{1}{\lambda}xy$:

$$\left(e + \frac{1}{\lambda}xzy\right)\left(e - \frac{1}{\lambda}xy\right) = e - \frac{1}{\lambda}xy + \frac{1}{\lambda}xz\left(e - \frac{1}{\lambda}yx\right)y = e - \frac{1}{\lambda}xy + \frac{1}{\lambda}xy = e.$$

This completes the proof of Proposition 5.17. □

5.18. REMARK. In fact in the proof of Proposition 5.16 we could have avoided the use of Proposition 5.17 by the following argument. Let w, u, v, w_1, w_2 be as in the proof of Proposition 5.16. Then we proved that $w^*w + ww^* = -w_1^2 - w_2^2 = 0$. Since w_1 and w_2 are self-adjoint, we see that $w_1 = w_2 = 0$. Since $w_1 = (|yy^*| - yy^*)\sqrt{|yy^*|}$ we get

$$(|yy^*| - yy^*)^2 = 2(|yy^*| - yy^*)\sqrt{|yy^*|}\sqrt{|yy^*|} = 0,$$

and thus $yy^* = |yy^*|$ is positive.

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Next we apply Theorem 5.14 and Proposition 5.16 to obtain the polar decompositions of elements in a C^*-algebra.

5.19. THEOREM. *Let y be an element of a C^*-algebra A with identity e. Define the following elements:*

$$\begin{aligned} u_\varepsilon &= \frac{2}{\pi}\int_\varepsilon^\infty y\left(t^2 e + y^* y\right)^{-1} dt, \quad \varepsilon > 0, \\ |y| &= \frac{2}{\pi}\int_0^\infty y^* y\left(t^2 e + y^* y\right)^{-1} dt, \\ |y^*| &= \frac{2}{\pi}\int_0^\infty y y^*\left(t^2 e + y y^*\right)^{-1} dt, \end{aligned} \tag{5.34}$$

Then the following assertions hold true:

(1) *The elements $|y|$ and $|y^*|$ are positive and satisfy the following equalities: $|y|^2 = y^* y$, $|y^*|^2 = yy^*$. They are the only positive elements in A which satisfy these equalities.*
(2) *The element u_ε^* is given by*

$$u_\varepsilon^* = \frac{2}{\pi}\int_\varepsilon^\infty y^*\left(t^2 e + yy^*\right)^{-1} dt. \tag{5.35}$$

(3) *The following equalities hold:*

$$\lim_{\varepsilon\downarrow 0} u_\varepsilon\,|y| = y \quad \text{and} \quad \lim_{\varepsilon\downarrow 0} u_\varepsilon^*\,|y^*| = y^*. \tag{5.36}$$

(4) *The following equalities hold:*

$$\lim_{\varepsilon\downarrow 0} u_\varepsilon^* u_\varepsilon\,|y| = |y| \quad \text{and} \quad \lim_{\varepsilon\downarrow 0} u_\varepsilon u_\varepsilon^*\,|y^*| = |y^*|\,. \tag{5.37}$$

5.20. REMARK. Since by Proposition 5.16 elements of the form yy^*, and so also of the form y^*y, are positive, we see that $|y|$ is positive. It is called the (positive) square root of the element y^*y, and often written as $|y| = \sqrt{y^*y}$. Heuristically, the equalities in (3) are written as $y = u\,|y|$ (polar decomposition of the element y) and $y^* = u^*\,|y^*|$ (polar decomposition of the element y^*). The equalities in (4) suggest to write $u^*u\,|y| = |y|$ and $uu^*\,|y^*| = |y^*|$ respectively. These equalities say that u and u^* are partial isometries on the range of (the multiplication operators) $|y|$ and $|y^*|$. In the context of bounded or closed linear operators with, domain and range in a Hilbert space, these notions will be justified in the sense that $|T|$ is the unique positive operator S with $S^2 = T^*T$, that $Ux = \frac{2}{\pi}\int_0^\infty T\left(t^2 I + T^*T\right)^{-1} x\,dt$, $x \in H$, is a so-called partial isometry, *i.e.* $\|Ux\| = \|x\|$ for x in the closure of the range of $|T|$, and that U^*U is an orthogonal projection on the closure of the range of $|T|$. Also notice that the closure of the range of $|T|$ coincides with the closure of the range of T^*T. A similar observation goes for the operator T^*.

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PROOF OF THEOREM 5.19. (1) From Proposition 5.16 it follows that the element y^*y is positive. From assertion (b) in Theorem 5.14 it follows that $|y|^2 = y^*y$. A similar argument applies to yy^*. The assertion about the uniqueness follows from assertion (b) in Theorem 5.14.

(2) This assertion follows from the fact that taking an adjoint is a continuous operation, and from the observe that for $t > 0$ the equality

$$\left(t^2 e + y^* y\right)^{-1} y^* = y^* \left(t^2 e + y y^*\right)^{-1}$$

holds.

(3) First we show that

$$\lim_{\varepsilon_2,\varepsilon_1\downarrow 0,\, \varepsilon_2>\varepsilon_1} \frac{2}{\pi}\int_{\varepsilon_1}^{\varepsilon_2} y\left(t^2 e + y^* y\right)^{-1} dt\,|y| = \lim_{\varepsilon_2,\varepsilon_1\downarrow 0,\, \varepsilon_2>\varepsilon_1} \frac{2}{\pi}\int_{\varepsilon_1}^{\varepsilon_2} y\,|y|\left(t^2 e + y^* y\right)^{-1} dt = 0. \tag{5.38}$$

If $0 < \varepsilon_1 < \varepsilon_2$, then

$$\begin{aligned}
&\left\|\int_{\varepsilon_1}^{\varepsilon_2} y\,|y|\left(t^2 e + y^* y\right)^{-1} dt\right\|^2 \\
&= \left\|\int_{\varepsilon_1}^{\varepsilon_2}\int_{\varepsilon_1}^{\varepsilon_2} \left(t_1^2 e + y^* y\right)^{-1} |y|\, y^* y\, |y| \left(t_2^2 e + y^* y\right)^{-1} dt_2\, dt_1\right\| \\
&= \left\|\int_{\varepsilon_1}^{\varepsilon_2}\int_{\varepsilon_1}^{\varepsilon_2} \left(t_1^2 e + y^* y\right)^{-1} \left(y^* y\right)^2 \left(t_2^2 e + y^* y\right)^{-1} dt_2\, dt_1\right\| \\
&\leqslant \sup_{\lambda>0}\int_{\varepsilon_1}^{\varepsilon_2}\int_{\varepsilon_1}^{\varepsilon_2} \frac{\lambda^2}{\left(t_1^2+\lambda\right)\left(t_2^2+\lambda\right)}\, dt_2\, dt_1 \\
&\leqslant \sup_{\lambda>0}\int_{0}^{\varepsilon_2}\int_{0}^{\varepsilon_2} \frac{\lambda^2}{\left(t_1^2+\lambda\right)\left(t_2^2+\lambda\right)}\, dt_2\, dt_1 \\
&= \sup_{\lambda>0}\int_{0}^{\varepsilon_2/\sqrt{\lambda}}\int_{0}^{\varepsilon_2/\sqrt{\lambda}} \frac{\lambda^2}{\left(t_1^2+1\right)\left(t_2^2+1\right)}\, dt_2\, dt_1 \\
&\leqslant \sup_{\lambda>0}\left(\sqrt{\lambda}\arctan\left(\frac{\varepsilon_2}{\sqrt{\lambda}}\right)\right)^2 \leqslant \varepsilon_2^2.
\end{aligned} \tag{5.39}$$

So from (5.39) the equality (5.38) follows. As a consequence we see that

$$\lim_{\varepsilon\downarrow 0} u_\varepsilon\,|y| = \frac{2}{\pi}\int_0^\infty y\,|y|\left(t^2 e + y^* y\right)^{-1} dt \tag{5.40}$$

exists. The element $u_\varepsilon\,|y|$ can be rewritten in the form

$$\begin{aligned}
u_\varepsilon\,|y| &= \frac{4}{\pi^2}\int_\varepsilon^\infty\int_0^\infty y\left(t_1^2 e + y^* y\right)^{-1} y^* y \left(t_2^2 e + y^* y\right)^{-1} dt_1\, dt_2 \\
&= \frac{4}{\pi^2}\int_\varepsilon^\infty\int_0^\infty y y^* y\left(t_1^2 e + y^* y\right)^{-1} \left(t_2^2 e + y^* y\right)^{-1} dt_1\, dt_2.
\end{aligned} \tag{5.41}$$

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From the definition of $|y|$ together with (5.40) it follows that, as $\varepsilon \downarrow 0$, the expression in (5.41) converges to

$$\lim_{\varepsilon\downarrow 0} u_\varepsilon |y| = \frac{4}{\pi^2}\int_0^\infty\int_0^\infty yy^*y\left(t_1^2e + y^*y\right)^{-1}\left(t_2^2e + y^*y\right)^{-1}\,dt_1\,dt_2$$

(like in (5.23))

$$\begin{aligned} &= \int_0^\infty yy^*y\left(\rho e + y^*y\right)^{-2}\,d\rho \\ &= \lim_{\varepsilon\downarrow 0}\lim_{R\to\infty} yy^*y\left\{\left(\varepsilon e + y^*y\right)^{-1} - \left(Re + y^*y\right)^{-1}\right\} \\ &= \lim_{\varepsilon\downarrow 0}\left\{y - \varepsilon y\left(\varepsilon e + y^*y\right)^{-1}\right\} = y. \end{aligned} \tag{5.42}$$

The final equality in (5.42) follows from the estimate:

$$\left\|\varepsilon y\left(\varepsilon e + y^*y\right)^{-1}\right\|^2 = \varepsilon^2\left\|\left(\varepsilon e + y^*y\right)^{-1}y^*y\left(\varepsilon e + y^*y\right)^{-1}\right\| \leqslant \sup_{\lambda>0}\frac{\varepsilon^2\lambda}{\left(\varepsilon+\lambda\right)^2} = \frac{\varepsilon}{4}. \tag{5.43}$$

The proof of the equality $\lim_{\varepsilon\downarrow 0} u_\varepsilon^* |y^*| = y^*$ is exactly the same with the roles of y and y^* interchanged. This proves assertion (3).

(4) We have the equalities

$$\begin{aligned} u_\varepsilon^* u_\varepsilon |y| &= \frac{4}{\pi^2}\int_\varepsilon^\infty\int_\varepsilon^\infty y^*\left(t_1^2e + yy^*\right)^{-1}y\left(t_2^2e + y^*y\right)^{-1}|y|\,dt_1\,dt_2 \\ &= \frac{4}{\pi^2}\int_\varepsilon^\infty\int_\varepsilon^\infty y^*y\left(t_1^2e + y^*y\right)^{-1}\left(t_2^2e + y^*y\right)^{-1}|y|\,dt_1\,dt_2 \\ &= \frac{4}{\pi^2}\int_\varepsilon^\infty\int_\varepsilon^\infty |y|\,y^*y\left(t_1^2e + y^*y\right)^{-1}\left(t_2^2e + y^*y\right)^{-1}\,dt_1\,dt_2. \end{aligned} \tag{5.44}$$

In (5.44) we let $\varepsilon \downarrow 0$ to obtain:

$$\begin{aligned} \lim_{\varepsilon\downarrow 0} u_\varepsilon^* u_\varepsilon |y| &= \frac{4}{\pi^2}\int_0^\infty\int_0^\infty |y|\,y^*y\left(t_1^2e + y^*y\right)^{-1}\left(t_2^2e + y^*y\right)^{-1}\,dt_1\,dt_2 \\ &= \int_0^\infty |y|\,y^*y\left(\rho e + y^*y\right)^{-2}\,d\rho \\ &= \lim_{\varepsilon\downarrow 0}\lim_{R\to\infty} |y|\,y^*y\left\{\left(\varepsilon e + y^*y\right)^{-1} - \left(Re + y^*y\right)^{-1}\right\} = |y|, \end{aligned} \tag{5.45}$$

where in the final we employed the equality:

$$\lim_{\varepsilon\downarrow 0}\varepsilon\,|y|\left(\varepsilon e + y^*y\right)^{-1} = 0. \tag{5.46}$$

The equality in (5.46) follows because

$$\left\|\varepsilon\,|y|\left(\varepsilon e + y^*y\right)^{-1}\right\| \leqslant \sup_{\lambda>0}\frac{\varepsilon\lambda}{\varepsilon+\lambda^2} = \frac{1}{2}\sqrt{\varepsilon}. \tag{5.47}$$

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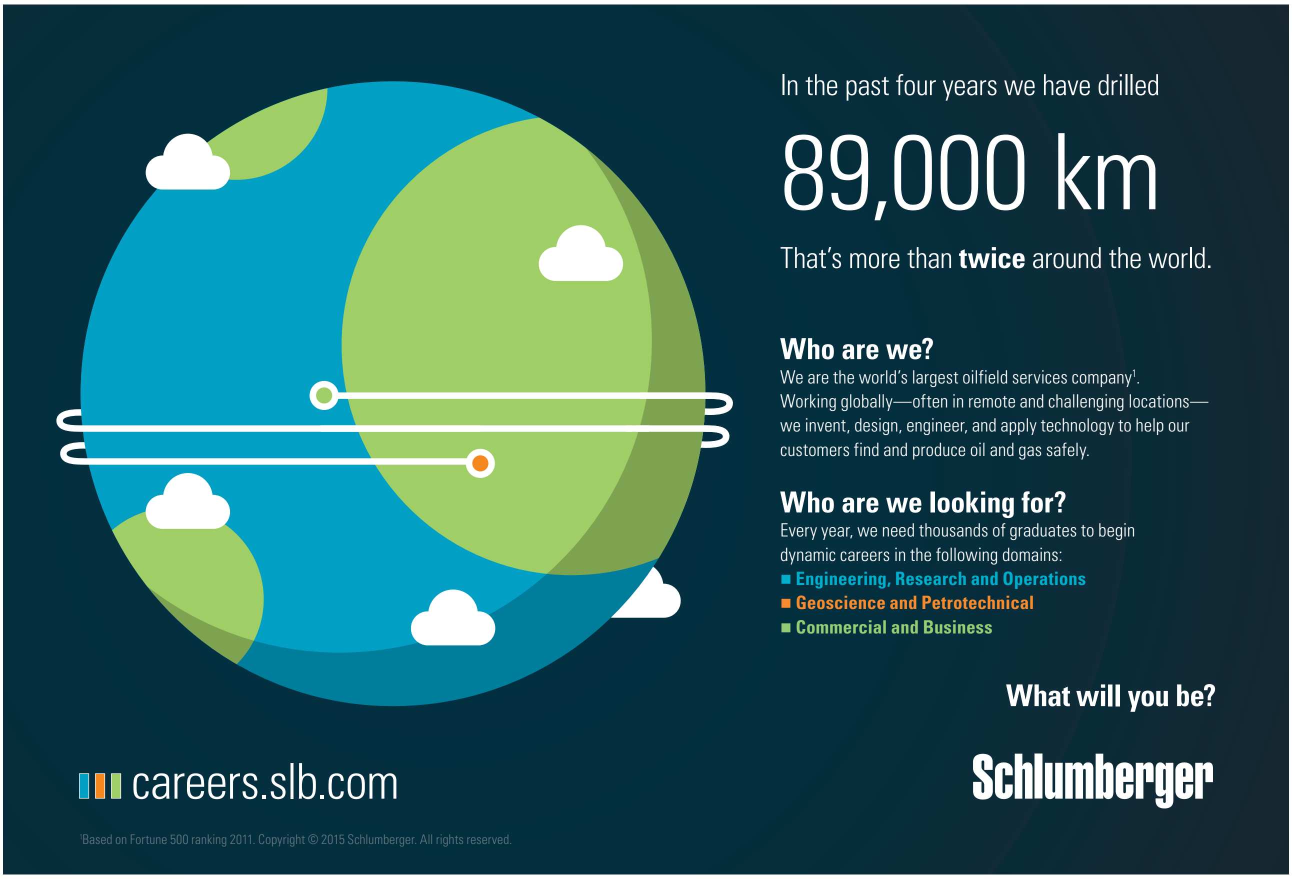
In order to show that the first equality in (5.45) is valid it suffices to prove that

$$\lim_{\varepsilon \downarrow 0} \frac{4}{\pi^2} \int_{\varepsilon}^{\infty} \int_{0}^{\varepsilon} |y|\, y^* y \left(t_1^2 e + y^* y\right)^{-1} \left(t_2^2 e + y^* y\right)^{-1} dt_1\, dt_2 = 0. \tag{5.48}$$

The first equality in (5.45) follows from the following estimates:

$$\begin{aligned} &\left\| \int_{\varepsilon}^{\infty} \int_{0}^{\varepsilon} |y|\, y^* y \left(t_1^2 e + y^* y\right)^{-1} \left(t_2^2 e + y^* y\right)^{-1} dt_1\, dt_2 \right\| \\ &\leqslant \sup_{\lambda > 0} \int_{\varepsilon}^{\infty} \int_{0}^{\varepsilon} \frac{\lambda^3}{\left(t_1^2 + \lambda^2\right)\left(t_2^2 + \lambda^2\right)}\, dt_1\, dt_2 \\ &= \sup_{\lambda > 0} \lambda \arctan\left(\frac{\lambda}{\varepsilon}\right) \arctan\left(\frac{\varepsilon}{\lambda}\right) \leqslant \frac{\pi}{2}\varepsilon. \end{aligned} \tag{5.49}$$

By interchanging the roles of y and y^* the equality $\lim_{\varepsilon \downarrow 0} u_\varepsilon u_\varepsilon^* \, |y^*| = |y^*|$ is obtained. This completes the proof of assertion (4), and also of Theorem 5.19. $\square$



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1.4. On Gelfand transforms. Let A be a commutative Banach algebra with identity e. In the following theorem Δ_A stands for the collection of all non-zero complex homomorphisms. Let $h \in \Delta_A$. Then we know that $|h(x)| \leqslant \|x\|$, $x \in A$: see the beginning of Section 1. In other words Δ_A is a weak*-closed subset of the closed dual unit ball. Equipped with the relative weak*-topology the set Δ_A is a compact Hausdorff space. To every $x \in A$ we can assign a continuous function $\widehat{x} : \Delta_A \to \mathbb{C}$ such that

$$\widehat{x}(h) = h(x), \quad h \in \Delta_A. \tag{5.50}$$

Again, let $h \in \Delta_A$. It also follows that

$$|h\,(x)|^n = |h\,(x^n)| \leqslant \|x^n\|\,,$$

and so $|h(x)| \leqslant \rho(x)$, $x \in A$. In other words

$$\sup_{h\in\Delta_A} |\widehat{x}(h)| = \sup_{h\in\Delta_A} |h(x)| \leqslant \rho(x).$$

Next let $x \in A$, and let $\lambda \in \sigma(x)$. Then the ideal $(\lambda e - x)\,A$ is contained in a proper maximal ideal M. From Corollary 5.6 it follows that there exists a complex homomorphism $h : A \to \mathbb{C}$ such that $h(y) = 0$ for all $y \in M$. Then $\lambda = h(x)$. Conversely, if $x \in A$, and if $h \in \Delta_A$, then $h(x)e - x$ belongs to null-space of h, and hence $h(x| \in \sigma(x)$.

5.21. Definition. The space Δ_A equipped with the (relative) weak*-topology is called the maximal ideal space of the commutative Banach algebra A. The transform $x \mapsto \widehat{x}$ is called the Gelfand transform of $x \in A$.

Some of the results of in the following theorem follow from the previous discussions.

5.22. Theorem. *Let Δ_A be the maximal ideal space of a Banach algebra A. Then the following assertions hold true.*

(a) *Δ_A is a compact Hausdorff space.*
(b) *The Gelfand transform is an algebra homomorphism of A onto a subalgebra $\widehat{A}$ of $C\,(\Delta_A)$. Its kernel is Rad(A), the radical of A,* i.e. *the intersection of all its maximal ideals.*
(c) *For each $x \in A$, the range of $\widehat{x}$ is the spectrum $\sigma(x)$. Hence $\|\widehat{x}\|_\infty = \rho(x) \leqslant \|x\|$.*

Proof. The Banach-Alaoglu theorem implies that the closed unit ball of A^* viewed as a complex Banach space is weak*-compact. Since it is not so difficult to prove that Δ_A is weak*-closed, it follows that Δ_A is compact for the weak*-topology. The remarks preceding Definition 5.21 then essentially prove Theorem 5.22. ☐

The following theorem shows that a commutative C^*-algebra A is *-isometric with $C\,(\Delta_A)$ as C^*-algebra (with complex conjugation as involution).

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5.23. Theorem (Theorem of Gelfand-Naimark). *Let A e a commutative C^*-algebra with maximal ideal space Δ_A. The Gelfand transform is then an isometric isomorphism of A onto $C(\Delta_A)$, with the additional property that*

$$\widehat{x^*}(h) = \overline{\widehat{x}(h)} = \overline{h(x)}, \quad x \in A,\ h \in \Delta_A. \tag{5.51}$$

In addition, if $u \in A$ is positive, then $\widehat{u} \geqslant 0$.

Proof. Let $h \in \Delta_A$ and $u = u^* \in A$. Let $h(u) = \alpha + i\beta$ with $\alpha, \beta \in \mathbb{R}$. Then, since $h(e) = \|h\| = 1$, we have,

$$\alpha^2 + (\beta + t)^2 = |h(u + ite)|^2 \leqslant \|u + ite\|^2 = \left\|u^2 + t^2 e\right\| \leqslant \left\|u^2\right\| + t^2. \tag{5.52}$$

From (5.52) it follows that $\alpha^2 + \beta^2 + 2\beta t \leqslant \|u^2\|$, $t \in \mathbb{R}$, and hence $\beta = 0$. Hence we have $h(u) = \alpha \in \mathbb{R}$. If $x \in A$ is arbitrary, then we write $x = u + iv$ with $u = u^*$ and $v = v^*$. Then $h(x^*) = h(u - iv) = h(u) - ih(v) = \overline{h(u) + ih(v)} = \overline{h(u + iv)} = \overline{h(x)}$. This proves the equalities in (5.51). Next we will show that the Gelfand transform is isometric. To this end we pick $x \in A$ and consider

$$\begin{aligned} \|\widehat{x}\|_\infty^2 &= \sup\left\{\overline{\widehat{x}(h)}\widehat{x}(h) : h \in \Delta_A\right\} = \sup\left\{\widehat{x^* x}(h) : h \in \Delta_A\right\} \\ &= \rho(x^* x) = \lim_{n\to\infty} \left\|(x^* x)^{2^n}\right\|^{2^{-n}} = \|x^* x\| = \|x\|^2. \end{aligned} \tag{5.53}$$

The equalities in (5.53) show $\|\widehat{x}\|_\infty = \|x\|$, $x \in A$. The Stone-Weierstrass theorem entails that the space $\widehat{A} := \{\widehat{x} : x \in A\}$ is dense in $C(\Delta_A)$. Let $f \in C(\Delta_A)$. Then there exists a sequence $(x_n)_n \subset A$ such that $\lim_{n\to\infty} \|f - \widehat{x}_n\|_\infty = 0$. Since $\|x_n - x_m\| = \|\widehat{x}_n - \widehat{x}_m\|_\infty$, it follows that $(x_n)_n$ is a Cauchy sequence in A. The algebra A being complete implies that there exists $x \in A$ such that $x = \lim_{n\to\infty} x_n$. It is the easy to see that $f = \widehat{x}$.

From assertion (c) of Theorem 5.22 it follows that the range of $\widehat{u}$ coincides with $\sigma(u)$. Since, by hypothesis, $\sigma(u)$ is contained in $[0, \infty)$, the final conclusion in Theorem 5.23 follows. □

5.24. Proposition. *Let A be a C^*-algebra generated by x and x^* and the identity. Suppose that x and x^* commute;* i.e. $xx^* = x^*x$. *Define the mapping $\Psi : C(\sigma(x)) \to A$ via the identity $\widehat{\Psi(f)}(h) = f(h(x))$, $h \in \triangle_A$. If f is holomorphic on a neighborhood Ω of $\sigma(x)$, then $\Psi(f) = \widetilde{f}(x)$.*

Proof. Let Γ be a closed curve which surrounds $\sigma(x)$ in Ω. If h belongs to $\triangle_A$, then by Cauchy's theorem we have

$$h(\Psi(f)) = \widehat{\Psi(f)}(h) = f(h(x)) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda - h(x))^{-1}\, d\lambda$$

(h is a continuous complex-valued homomorphis of algebras)

$$= h\left(\frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda e - x)^{-1}\, d\lambda\right) = h\left(\widetilde{f}(x)\right),$$

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and hence the Gelfand transform of the element $\Psi(f) - \widetilde{f}(x)$ is identically 0. The mapping $y \mapsto \widehat{y}$ is a C^*-algebra isomorphism from A onto $C(\triangle)$, and consequently $\Psi(f) - \widetilde{f}(x) = 0$. The proof of Proposition 5.24 is now complete. $\square$

5.25. PROPOSITION. *Let G_1 be the connected component of $G = G(A)$ containing the identity e. Then $G_1 = \bigcup_{n\in\mathbb{N}} \{\exp(x_1)\cdots\exp(x_n) : x_j \in A, \quad 1 \leqslant j \leqslant n\}$.*

PROOF. Put $\Gamma = \bigcup_{n\in\mathbb{N}} \{\exp(x_1)\cdots\exp(x_n) : x_j \in A, \quad 1 \leqslant j \leqslant n\}$. Then

$$\Gamma \supseteq \{\exp(x) : x \in A\} \supseteq \{y \in A : \sigma(y) \subset \mathbb{C}\backslash(-\infty, 0]\}.$$

In other words the subset Γ contains an open neighborhood of the identity e. Next let y be an arbitrary element in Γ. Then the subset

$$\left\{z \in G : \sigma\left(y^{-1}z\right) \subset \mathbb{C}\backslash(-\infty, 0]\right\}$$

is an open subset of Γ. If $\sigma\left(y^{-1}z\right) \subset \mathbb{C}\backslash(-\infty, 0]$, then, by symbolic calculus, $y^{-1}z = \exp(x)$ for some $x \in A$, and hence $z \in \Gamma$. In fact x can be defined by

$$x = \int_0^1 \left((1-\rho)e + \rho y^{-1}z\right)^{-1}\left(y^{-1}z - e\right) d\rho = \int_0^1 \left((1-\rho)y + \rho z\right)^{-1}(z-y)\, d\rho. \tag{5.54}$$

The equality $\exp(x) = y^{-1}z$ follows because, with

$$\int_0^1 \frac{\lambda - 1}{1 - \rho + \rho\lambda}\, d\rho = \int_1^\lambda \frac{1}{z}\, dz = \log\lambda, \quad \lambda \in \mathbb{C}\backslash(-\infty, 0],$$

we have $\lambda = \exp(\log\lambda)$, so that $y^{-1}z = \exp(x)$. Moreover the set

$$\left\{z \in G : \sigma\left(y^{-1}z\right) \subset \mathbb{C}\backslash(-\infty, 0]\right\} = y\left\{w \in G : \sigma(w) \subset \mathbb{C}\backslash(-\infty, 0]\right\}$$

is an open subset of Γ. It follows that Γ is an open subgroup of G. Consequently, $G_1 = \bigcup_{x\in G_1} x\Gamma$, where each coset $x\Gamma$, $x \in G_1$, is open. Since G_1 is open and connected it follows that $G_1 = \Gamma$. This completes the proof of Proposition 5.24. $\square$

5.26. COROLLARY. *Let x belong to $G(A)$, and let $x_1, \dots, x_n$ be elements in A. Since the curve $t \mapsto x\exp(tx_1)\cdots\exp(tx_n)x^{-1}$, $0 \leqslant t \leqslant 1$, connects the element e with $x\exp(x_1)\cdots\exp(x_n)x^{-1}$, it follows that $x\exp(x_1)\cdots\exp(x_n)x^{-1}$ belongs to $G_1 = \Gamma$ and thus can be written as a product of finitely many exponentials.*

1.5. Resolution of the identity. The following definition will be employed with Ω a compact or locally compact Hausdorff space with Borel field. It introduces the reader to the concept of *resolution of the identity*. In case the resolution of the identity pertains to a single self-adjoint or normal operator $T = \int_{\sigma(T)} \lambda\, dE_T(\lambda)$, then we also say that $E_T(\cdot)$ is the spectral decomposition of T.

5.27. DEFINITION. Let $\mathcal{B} = \mathcal{B}_S$ be the Borel field of a topological Hausdorff space S, and let H be a complex Hilbert space with space of bounded linear operators $\mathcal{L}(H)$. A resolution of the identity on $\mathcal{B}$ is a mapping $E : \mathcal{B} \to \mathcal{L}(H)$ with the following properties:

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(a) $E(\varnothing) = 0$, $E(S) = I$;
(b) Each $E(B)$, $B \in \mathcal{B}$, is a self-adjoint projection;
(c) $E(B_1 \cap B_2) = E(B_1)\,E(B_2)$, B_1, $B - 2 \in \mathcal{B}$;
(d) If B_1 and B_2 in $\mathcal{B}$ are such that $B_1 \cap B_2 = \varnothing$, then $E(B_1 \cup B_2) = E(B_1) + E(B_2)$;
(e) For every $x \in H$ and $y \in H$ the set function $B \mapsto E_{x,y}(B) = \langle E(B)x, y\rangle$, $B \in \mathcal{B}$, is a complex Borel measure on $\mathcal{B}$.

Let $B \mapsto E(B)$, $B \in \mathcal{B}$, be a resolution of the identity. It then follows that for every $x \in H$ the set function $B \mapsto E(B)x$ is an H-valued measure, which implies that

$$\lim_{n\to\infty} \sum_{j=1}^{n} E(B_j)\,x = E\left(\cup_{j=1}^{\infty} B_j\right) x,$$

whenever the sequence $(B_j)_j \subset \mathcal{B}$ is mutually disjoint, that is $B_{j_1} \cap B_{j_2} = \varnothing$ if $j_1 \neq j_2$.

5.28. Theorem. *Let A be commutative C^*-algebra of continuous linear operators on a Hilbert space H. Then there exists a (unique) resolution of the identity $\widehat{E}$ on the Borel field of the maximal ideal space Δ_A with the property that*

$$\langle Tx, y\rangle = \int_{\Delta_A} \widehat{T}\, d\widehat{E}_{x,y}, \tag{5.55}$$

where $\widehat{E}_{x,y}(B) = \left\langle \widehat{E}(B)x, y\right\rangle$, x, $y \in H$.

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The equality in (5.55) is often written as $T = \int \widehat{T}\, d\widehat{E}$.

Proof. A proof is based on the Riesz representation theorem. For x, $y \in H$ we consider the linear functional $\Lambda_{x,y} : \widehat{T} \mapsto \langle Tx, y\rangle$, $\widehat{T} \in C(\Delta_A)$. Then

$$\left|\Lambda_{x,y}\left(\widehat{T}\right)\right| \leqslant \left\|\widehat{T}\right\|_{\infty} \|x\| \|y\|, \quad \widehat{T} \in C(\Delta_A). \tag{5.56}$$

Since, by Theorem 5.23, $\left\{\widehat{T} : T \in A\right\} = \widehat{A} = C(\Delta_A)$ the functional $\Lambda_{x,y}$ is everywhere defined, and by (5.56) it is continuous, so that by the Riesz representation theorem there exists a complex measure $\widehat{E}_{x,y}$ on the Borel field of Δ_A such that

$$\langle Tx, y\rangle = \int_{\Delta_A} \widehat{T}\, \widehat{E}_{x,y}, \quad T \in A. \tag{5.57}$$

The representation in (5.57) holds for all x, $y \in H$, and for all $T \in A$. Then it can be proved that there exists a resolution of the identity $\widehat{E}$ such that $\widehat{E}_{x,y}(B) = \left\langle \widehat{E}(B)x, y\right\rangle$, x, $y \in H$, B Borel subset of Δ_A. This completes an outline of the proof of Theorem 5.28. □

5.29. Corollary. *Let A and $\widehat{E}$ be as in Theorem 5.28, and let $T \in A$. Then define the resolution of the identity $B \mapsto E_T(B)$, B a Borel subset of $\mathbb{C}$, by $E_T(B) = \widehat{E}\left[\widehat{T} \in B\right]$, B Borel subset of $\mathbb{C}$. Let $f : \sigma(T) \to \mathbb{C}$ be a bounded Borel function. Then*

$$\int_{\Delta_A} f \circ \widehat{T}\, d\widehat{E} = \int_{\sigma(T)} f(\lambda)\, dE_T(\lambda) \tag{5.58}$$

in the sense that

$$\int_{\Delta_A} f \circ \widehat{T}\, d\widehat{E}_{x,y} = \int_{\sigma(T)} f(\lambda)\, dE_{T,x,y}(\lambda) \tag{5.59}$$

where $E_{T,x,y}(B) = \langle E_T(B)x, y\rangle$, x, $y \in H$. In particular, when $f(\lambda) = \lambda$, $\lambda \in \sigma(T)$, the equality

$$T = \int_{\sigma(T)} \lambda\, dE_T(\lambda) = \int_{\mathbb{C}} \lambda\, dE_T(\lambda)$$

holds.

Let $L^\infty\left(\sigma(T), \mathcal{B}_{\sigma(T)}, E_T\right)$ be the space of all complex bounded Borel functions on $\mathbb{C}$ where two Borel functions f_1, f_2 are identified whenever $E_T\left[f_1 \neq f_2\right] = 0$. Corollary 5.29 yields the existence of a symbolic calculus for bounded normal operators. In other words the mapping $\Phi_T : L^\infty\left(\sigma(T), \mathcal{B}_{\sigma(T)}, E_T\right) \to \mathcal{L}(H)$, defined by

$$\Phi_T(f) = \int_{\sigma(T)} f(\lambda)\, dE_T(\lambda), \quad f \in L^\infty\left(\sigma(T), \mathcal{B}_{\sigma(T)}, E_T\right),$$

defines a symbolic calculus in the sense that $\Phi_T(fg) = \Phi_T(f)\Phi_T(g)$, $\Phi_T(\overline{f}) = \Phi_T(f)^*$, f, $g \in L^\infty\left(\sigma(T), \mathcal{B}_{\sigma(T)}, E_T\right)$. Moreover, $f(\lambda) = \lambda$ yields $\Phi_T(f) = T$. Often $\Phi_T(f)$ is written as $f(T)$.

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If $U = T \in A$ happens to unitary, that is if $U^*U = UU^*$, then from Theorem 5.28 it follows that U can be written in the form $U = \int_{\sigma(U)} \lambda\, dE_U^{(1)}(\lambda$. However, we write

$$U = \int_{-\pi}^{\pi} e^{i\vartheta}\, dE_U(\vartheta). \tag{5.60}$$

Here $E_U(B)$, B Borel subset of $[-\pi, \pi]$, is defined by

$$E_U(B) = E_U^{(1)} \{\lambda \in \mathbb{C} : \arg(\lambda) \in B\}.$$

The argument of a complex number is counted between $-\pi$ (excluded) and π (included).

Let $f : \mathbb{C} \to \mathbb{C}$ be a Borel measurable function. If $(D_j)_j$ be a sequence of open subsets of $\mathbb{C}$ with the property that $E_T[f^{-1}(D_j)] = E_T[f \in D_j] = 0$, for $j \in \mathbb{N}$. Then we have, for $x \in H$ arbitrary,

$$E_{T,x,x}\left[f^{-1}\left(\cup_{j=1}^{\infty} D_j\right)\right] \leqslant \sum_{j=1}^{\infty} E_{T,x,x}\left[f^{-1}(D_j)\right] = 0.$$

It follows that there exists a largest open subset V of $\mathbb{C}$ such that $E_T[f \in V] = E_T[f^{-1}(V)] = 0$. The complement of V is called the E_T-essential range of the function f. The following theorem shows that the spectrum of $f(T)$ is contained in the E_T-essential range of f. A Borel function $g : \mathbb{C} \to \mathbb{C}$ is called E_T-essentially bounded if there exists a finite constant M such that the essential range is contained in a disc with radius M. This is equivalent to saying that, for some finite constant M, $E_T[|f| > M] = 0$.

5.30. Theorem. *Let $T = \int_{\sigma(T)} \lambda\, dE_T(\lambda)$ be a (bounded) normal operator on a Hilbert space H, and let $f : \mathbb{C} \to \mathbb{C}$ be a Borel measurable function. Then the spectrum of $f(T) = \int_{\sigma(T)} f(\lambda)\, dE_T(\lambda)$ is contained in the E_T-essential range of f. If f is continuous, then $\sigma(f(T)) = f(\sigma(T))$.*

Proof. Let α belong to the complement of the E_T-essential range of the function f. Then the function $g : \lambda \mapsto \dfrac{1}{\alpha - f(\lambda)}$, $\lambda \in \mathbb{C}$, is E_T-essentially bounded. It follows that $1 = (\alpha - f(\lambda))\, g(\lambda)$, and so by symbolic calculus

$$\begin{aligned} I &= \int_{\sigma(T)} g(\lambda)\, dE_T(\lambda) \left(\alpha I - \int_{\sigma(T)} f(\lambda)\, dE_T(\lambda)\right) \\ &= \left(\alpha I - \int_{\sigma(T)} f(\lambda)\, dE_T(\lambda)\right) \int_{\sigma(T)} g(\lambda)\, dE_T(\lambda). \end{aligned} \tag{5.61}$$

From (5.61) it follows that the operator $\alpha I - f(T)$ has a bounded inverse $g(T)$. Consequently, α does not belong to the spectrum of $f(T)$. This shows that the complement of the E_T-essential range is contained in the complement of the spectrum of $f(T)$. In other words, the spectrum of $f(T)$ is contained in the E_T-essential range of f. This proves the first part of the theorem. Next let $f : \sigma(T) \to \mathbb{C}$ be continuous,

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and let $\alpha \in \mathbb{C}$. Then, sice $f(\sigma(T))$ is compact, the function $\dfrac{1}{\alpha - f}$ is bounded (on $\sigma(T)$ if and only if $\alpha \notin f(\sigma(T))$. Like above it follows that $\alpha \in \sigma(f(T))$ if and only if $\alpha \in f(\sigma(T))$. This completes the proof of Theorem 5.30. $\square$

5.31. Theorem. *Let $T = T^*$ be a not necessarily bounded linear self-adjoint operator with domain and range in the Hilbert space H. Then there exists a resolution of the identity $E_T(\cdot)$ on the Borel field of $\mathbb{R}$ such that $Tx = \int_{\mathbb{R}} \lambda \, dE_T(\lambda)x$, $x \in D(T)$. In fact $x \in H$ belongs to $D(T)$ if and only if $\int_{\mathbb{R}} \lambda^2 d\langle E_T(\lambda)x, x\rangle < \infty$.*

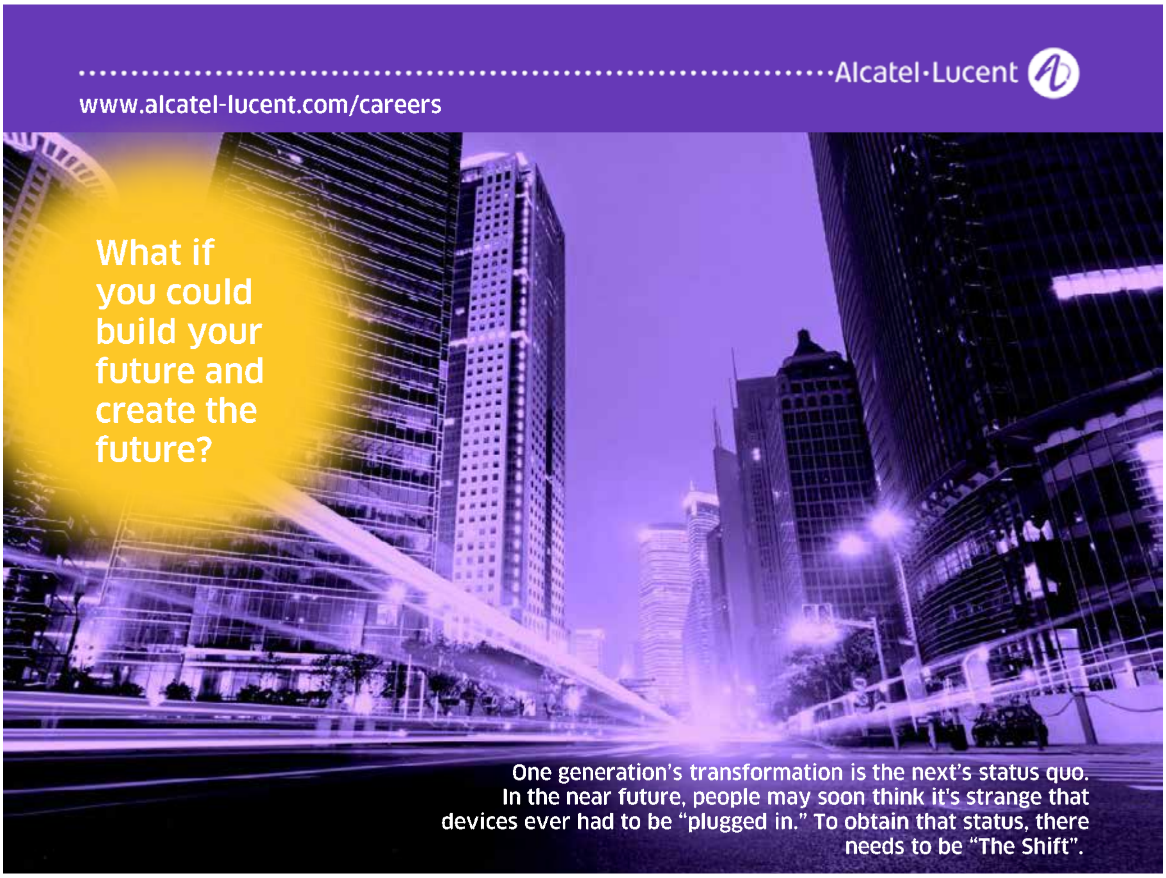
As in the remarks following Corollary 5.29 the equality $Tx = \int_{\mathbb{R}} \lambda \, dE_T(\lambda)x$, $x \in D(T)$, yields a symbolic calculus, by writing $f(T)x = \int_{\mathbb{R}} f(\lambda)\, dE_T(\lambda)x$, $x \in H$, whenever $f : \mathbb{R} \to \mathbb{C}$, is a bounded Borel function. Again we have $(fg)(T) = f(T)g(T)$, and $\overline{f}(T) = f(T)^*$, for all complex bounded Borel functions f and g defined on $\mathbb{R}$.

Outline of a proof. Let U be the unitary operator defined by the Cayley transform:

$$U = (I + iT)(I - iT)^{-1}.$$

Then

$$T = i(I - U)(I + U)^{-1}.$$



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Let $E_U(\cdot)$ be the resolution of the identity, or the spectral decomposition, corresponding to U, *i.e.* $U = \int_{-\pi}^{\pi} e^{i\vartheta} dE_U(\vartheta)$. Define the resolution of the identity $E_T(\cdot)$, which hopefully corresponds to T, by the equality

$$E_T(B) = E_U\left[\vartheta \in (-\pi, \pi] : \tan\frac{1}{2}\vartheta \in B\right], \tag{5.62}$$

where B is a Borel subset of $\mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{C}$ be a Borel measurable function, and let $x \in H$ be such that

$$\int_{-\pi}^{\pi} \left| f\left(\tan\frac{1}{2}\vartheta\right)\right|^2 d\left\langle E_U(\vartheta)x, x\right\rangle < \infty.$$

Then we have

$$\int_{\mathbb{R}} f(\lambda)\, dE_T(\lambda)x = \int_{-\pi}^{\pi} f\left(\tan\frac{1}{2}\vartheta\right) dE_U(\vartheta)x. \tag{5.63}$$

In (5.63) we insert $f(\lambda) = \lambda$ and choose $x \in H$ such that

$$\int_{-\pi}^{\pi} \left| \tan\left(\frac{1}{2}\vartheta\right)\right|^2 d\left\langle E_U(\vartheta)x, x\right\rangle < \infty.$$

Then we deduce

$$\begin{aligned}\int_{\mathbb{R}} \lambda\, dE_T(\lambda)x &= \int_{-\pi}^{\pi} \tan\left(\frac{1}{2}\vartheta\right) dE_U(\vartheta)x = \int_{-\pi}^{\pi} \frac{e^{i\vartheta} - 1}{i\left(e^{i\vartheta}+1\right)}\, dE_U(\vartheta)x \\ &= \int_{-\pi}^{\pi} \frac{i\left(1 - e^{i\vartheta}\right)}{1 + e^{i\vartheta}}\, dE_U(\vartheta)x = i\left(I - U\right)\left(I + U\right)^{-1} x = Tx. \end{aligned} \tag{5.64}$$

The equality in (5.64) shows the equality $Tx = \int_{\mathbb{R}} \lambda\, dE_T(\lambda)x$, $x \in D(T)$. This completes an outline of the proof of Theorem 5.31. □

In the context of self-adjoint operators T we have the following version of the spectral mapping theorem.

5.32. THEOREM. *Let $T = \int_{\sigma(T)} \lambda\, dE_T(\lambda)$ be a self-adjoint operator with domain and range in a Hilbert space H, and let $f : \mathbb{R} \to \mathbb{C}$ be a Borel measurable function. Then the spectrum of $f(T) = \int_{\sigma(T)} f(\lambda)\, dE_T(\lambda)$ is contained in the E_T-essential range of f. If $f : \sigma(T) \to \mathbb{C}$ is continuous, then $\sigma\left(f(T)\right)$ is contained in the closure of $f\left(\sigma(T)\right)$.*

PROOF. The first part of the proof follows in exactly the same manner as in the proof of Theorem 5.30. If $f : \sigma(T) \to \mathbb{C}$ is continuous, and if $\alpha \in \mathbb{C}$ does not belong to the closure of $f\left(\sigma(T)\right)$, then the function $g := \dfrac{1}{\alpha - f}$ is bounded on $\sigma(T)$. By symbolic calculus it follows that the function $g(T)$ is a bounded inverse of $\alpha I - f(T)$. Consequently, such $\alpha \in \mathbb{C}$ does not belong to spectrum of $f(T)$. This proves the second part of the theorem, and completes the proof of Theorem 5.32. □

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A densely defined closed linear operator T is called normal if $D(T) = D(T^*)$ and if $T^*T = TT^*$. The following theorem establishes a spectral decomposition for a normal operator T with domain and range in the Hilbert space H.

5.33. THEOREM. *Let T be a normal operator with domain and range in the Hilbert space H. Then there exists a resolution of the identity $E_T(\cdot)$ pertaining to T such that $T = \int_{\mathbb{C}} \lambda\, dE_T(\lambda)$. In fact a vector $x \in H$ belongs to $D(T) = D(T^*)$ if and only if $\int_{\mathbb{C}} |\lambda|^2\, d\langle E_T(\lambda)x, x\rangle < \infty$.*

OUTLINE OF A PROOF. The operator T admits a polar decomposition of the form $T = U|T|$. Here we may assume that U is unitary, that $|T| \geqslant 0$, and that U and $|T|$ commute: $U|T| = |T|U$. The polar decomposition is explained in Theorem 5.41. In fact the operator U in Theorem 5.41 is only a partial isometry. However, in case T is normal we have $N(T^*) = N(T)$, and we may assume that $Uy = y$ if $Ty = T^*y = 0$. In addition, the closure of the range of T^* is the same as the closure of the range of T. It follows that the partial isometry U which possesses the property that U^*U is the orthogonal projection on the closure of the range of T^* can be considered as a unitary operator. For details see Corollary 5.42. From the construction of U it follows that it commutes with $|T|$. The operator U admits a resolution of the identity $E_U(\cdot)$: see (5.60). So we have $U = \int_{-\pi}^{\pi} e^{i\vartheta}\, dE_U(\vartheta)$. The operator $|T|$ is self-adjoint and positive. So by Theorem 5.31 there exists a resolution of the identity $E_{|T|}(\cdot)$ such that $|T| = \int_0^\infty t\, dE_{|T|}(t)$. The resolutions of the identities $E_U(\cdot)$ and $E_{|T|}(\cdot)$ commute in the sense that $E_U(B_1)E_{|T|}(B_2) = E_{|T|}(B_2)E_U(B_1)$, whenever B_1 is a Borel subset of the interval $[-\pi, \pi]$, and B_2 is a Borel subset of $[0, \infty)$. For the latter see Lemma 5.34. Define the resolution of the identity $E_T(\cdot)$ on the Borel field of $\mathbb{C}$ by

$$E_T(B) = E_U \otimes E_{|T|}\left[(\vartheta, t) \in (-\pi, \pi] \times [0, \infty) : \lambda = te^{i\vartheta} \in B\right].$$

Then $T = \int_{\mathbb{C}} \lambda\, dE_T(\lambda)(t)$, and $x \in D(T)$ if and only if $\int_{\mathbb{C}} |\lambda|^2\, d\langle E_T(\lambda)x, x\rangle < \infty$. This completes an outline of the proof of Theorem 5.33. □

5.34. LEMMA. *Let T be a densely defined normal operator on a Hilbert space H. Let $T = U|T|$ be its polar decomposition where the operator U is supposed to be unitary. Let $E_U(\cdot)$ be the resolution of the identity corresponding to U, and let $E_{|T|}(\cdot)$ be the resolution of the identity corresponding to $|T|$. Let B_1 be a Borel subset of the interval $[-\pi, \pi]$, and let B_2 be a Borel subset of $[0, \infty)$. Then the equality $E_U(B_1)E_{|T|}(B_2) = E_{|T|}(B_2)E_U(B_1)$. In other words the resolutions of the identity $E_U(\cdot)$ and $E_{|T|}(\cdot)$ commute.*

PROOF. From the constructions of U and $|T|$ it follows that $U|T| = |T|U$: see the proof of Theorem 5.41. Then it also follows that $U^*|T| = |T|U^*$: see Corollary 5.42. The operator $|T|$ is closed, and has dense domain. Let $\mu \in \mathbb{C}$ be such that $\Re\mu \geqslant 0$, and let $\lambda > 0$. Then $R(\lambda I + \mu|T|) = H$, and the following inequality holds for all $x \in D(|T|)$:

$$\|\lambda x + \mu|T|(x)\| \geqslant \lambda\|x\|.$$

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From the Lumer-Phillips theorem it follows that $-\mu\,|T|$ generates a contraction semigroup $\left\{e^{-t\mu|T|} : t \geqslant 0\right\}$: see Theorem 6.13. Moreover, from the way Theorem 6.13 is proved we infer that the operators U and $e^{-t\mu|T|}$, $t \geqslant 0$, commute, and that the same is true for U^* and $e^{-t\mu|T|}$, $t \geqslant 0$. Since μ is arbitrary in the closed right-half plane, we deduce that

$$p\left(U^*, U\right) e^{-\mu|T|} = e^{-\mu|T|} p\left(U^*, U\right), \quad \Re\mu \geqslant 0, \tag{5.65}$$

where $p\left(\overline{\lambda}, \lambda\right)$ is a polynomial in two variables. By a standard approximation procedure and using the Stone-Weierstrass theorem the equality in (5.65) implies an equality of the form:

$$\int_{-\pi}^{\pi} f\left(e^{i\vartheta}\right) dE_U(\vartheta) \int_0^\infty g(t)\, dE_{|T|}(t) = \int_0^\infty g(t)\, dE_{|T|}(t) \int_{-\pi}^{\pi} f\left(e^{i\vartheta}\right) dE_U(\vartheta), \tag{5.66}$$

where f is any continuous function on the unit circle in $\mathbb{C}$, and where g is any function in $C_0[0,\infty)$. In fact the equality in (5.66) is first proved for $g(t)$ of the form $g(t) = \int_{\mathbb{R}} e^{-i\xi t}\varphi(\xi)\, d\xi$ where φ is an arbitrary function in $L^1(\mathbb{R})$. By another limiting procedure the equality in (5.66) also holds if f and g are indicator functions of open and compact subsets of the unit circle and the positive half-axis respectively. But then it is also true for indicator functions of Borel subsets. However, the latter is the same as saying that the resolutions of the identity $E_U(\cdot)$ and $E_{|T|}(\cdot)$ commute. This completes the proof of Lemma 5.34. $\square$

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In the context of (unbounded) normal operators T we have the following version of the spectral mapping theorem.

5.35. THEOREM. *Let $T = \int_{\sigma(T)} \lambda \, dE_T(\lambda)$ be a normal operator with domain and range in a Hilbert space H, and let $f : \mathbb{C} \to \mathbb{C}$ be a Borel measurable function. Then the spectrum of $f(T) = \int_{\sigma(T)} f(\lambda) \, dE_T(\lambda)$ is contained in the E_T-essential range of f. If $f : \sigma(T) \to \mathbb{C}$ is continuous, then $\sigma(f(T))$ is contained in the closure of $f(\sigma(T))$.*

The proof of Theorem 5.35 follows exactly the same pattern as that of Theorem 5.32. Therefore it is omitted.

5.36. THEOREM. *Let $T = \int \lambda \, dE(\lambda)$ be a self-adjoint (bounded or unbounded) operator in a Hilbert space H. Then, for $-\infty < a < b < \infty$,*

$$\begin{aligned}
&\lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_a^b \left\{ ((\tau - i\varepsilon) I - T)^{-1} - ((\tau + i\varepsilon) I - T)^{-1} \right\} f d\tau \\
&= E(a,b) f + \frac{1}{2} E\{a\} f + \frac{1}{2} E\{b\} f = E(a,b] f + \frac{1}{2} E\{a\} f - \frac{1}{2} E\{b\} f, \quad f \in H.
\end{aligned}$$

As a corollary to the previous theorem we see that spectral decompositions corresponding to a self-adjoint operator are unique. Observe that $E\{a\} \neq 0$ implies that $E\{a\}$ is the orthogonal projection onto the subspace consisting of those vectors which are eigenvectors of the operator T corresponding to the eigenvalue a.

PROOF. Fix $\varepsilon > 0$ and $f \in H$. Then the following equalities are self-explanatory:

$$\begin{aligned}
&\frac{1}{2\pi i} \int_a^b \left\{ ((\tau - i\varepsilon) I - T)^{-1} - ((\tau + i\varepsilon) I - T)^{-1} \right\} f d\tau \\
&= \frac{1}{2\pi i} \int_a^b 2i\varepsilon \left(\varepsilon^2 I + (\tau I - T)^2 \right)^{-1} f d\tau \\
&= \frac{1}{2\pi i} \int_a^b \int 2i\varepsilon \left(\varepsilon^2 + (\tau - \lambda)^2 \right)^{-1} dE(\lambda) f d\tau
\end{aligned}$$

(apply Fubini's theorem)

$$= \frac{1}{2\pi i} \int \int_a^b 2i\varepsilon \left(\varepsilon^2 + (\tau - \lambda)^2 \right)^{-1} d\tau dE(\lambda) f$$

(substitute $\tau - \lambda = \varepsilon\sigma$)

$$\begin{aligned}
&= \frac{1}{\pi} \int \int_{(a-\lambda)/\varepsilon}^{(b-\lambda)/\varepsilon} \frac{1}{1 + \sigma^2} d\sigma \, dE(\lambda) f \\
&= \frac{1}{\pi} \int \left(\arctan \frac{b - \lambda}{\varepsilon} - \arctan \frac{a - \lambda}{\varepsilon} \right) dE(\lambda) f \\
&= \frac{1}{\pi} \int_{(-\infty, a)} \left(\arctan \frac{b - \lambda}{\varepsilon} - \arctan \frac{a - \lambda}{\varepsilon} \right) dE(\lambda) f
\end{aligned}$$

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$$
\begin{aligned}
&+\frac{1}{\pi}\int_{\{a\}}\left(\arctan\frac{b-\lambda}{\varepsilon}-\arctan\frac{a-\lambda}{\varepsilon}\right)dE(\lambda)\,f\\
&+\frac{1}{\pi}\int_{(a,b)}\left(\arctan\frac{b-\lambda}{\varepsilon}-\arctan\frac{a-\lambda}{\varepsilon}\right)dE(\lambda)\,f\\
&+\frac{1}{\pi}\int_{\{b\}}\left(\arctan\frac{b-\lambda}{\varepsilon}-\arctan\frac{a-\lambda}{\varepsilon}\right)dE(\lambda)\,f\\
&+\frac{1}{\pi}\int_{(b,\infty)}\left(\arctan\frac{b-\lambda}{\varepsilon}-\arctan\frac{a-\lambda}{\varepsilon}\right)dE(\lambda)\,f\\
=&\ \frac{1}{\pi}\int_{(-\infty,a)}\left(\arctan\frac{b-\lambda}{\varepsilon}-\arctan\frac{a-\lambda}{\varepsilon}\right)dE(\lambda)\,f\\
&+\frac{1}{\pi}\left(\arctan\frac{b-a}{\varepsilon}\right)E(\{a\})\,f\\
&+\frac{1}{\pi}\int_{(a,b)}\left(\arctan\frac{b-\lambda}{\varepsilon}+\arctan\frac{\lambda-a}{\varepsilon}\right)dE(\lambda)\,f\\
&+\frac{1}{\pi}\left(\arctan\frac{b-a}{\varepsilon}\right)E(\{b\})\,f\\
&+\frac{1}{\pi}\int_{(b,\infty)}\left(\arctan\frac{\lambda-a}{\varepsilon}-\arctan\frac{\lambda-b}{\varepsilon}\right)dE(\lambda)\,f. \qquad (5.67)
\end{aligned}
$$

In (5.67) we let ε tend to 0 from to obtain the result in Theorem 5.36.

Notice that $\lim_{\varepsilon\downarrow 0}\arctan\left(\frac{c}{\varepsilon}\right)=\frac{\pi}{2}$ whenever $c>0$. □

5.37. THEOREM. *Let $T\in\mathcal{L}(H)$ be a normal operator, and let the C^*-algebra A be generated by the operator T and I. Then A contains T^* and A is a commutative C^*-subalgebra of $\mathcal{L}(H)$. Moreover, there exists a resolution of the identity E defined on the σ-field $\mathcal{B}_{\sigma(T)}$ consisting of all Borel subsets of $\sigma(T)$ such that $\widetilde{f}(T)=\int_{\sigma(T)}f(\lambda)dE(\lambda)$ for all functions f which are holomorphic in a neighborhood of $\sigma(T)$. In particular it follows that $T=\int_{\sigma(T)}\lambda dE(\lambda)$. Moreover an operator $S_0\in\mathcal{L}(H)$ commutes with T if and only it commutes with $E(B)$ for all Borel subsets B of $\sigma(T)$.*

PROOF. Following Theorem 12.22 in Rudin's book there exists a resolution of the identity $\widehat{E}$ defined on the Borel field of the maximal ideal space $\triangle_A$ of A such that $S=\int_{\triangle_A}\widehat{S}d\widehat{E}$ for all $S\in A$. Let $\varphi:\triangle_A\to\sigma(T)$ be the identification of $\triangle_A$ and $\sigma(T)$ given by $\varphi(h)=h(T)=\widehat{T}(h)$, $h\in\triangle_A$. Then φ is a homeomorphism from the compact set $\triangle_A$ onto the compact set $\sigma(T)$. For every x, $y\in H$ we define the image measure $E_{x,y}$ under φ on $\mathcal{B}_{\sigma(T)}$, *i.e.*

$$E_{x,y}(B)=\widehat{E}_{x,y}\left[\varphi\in B\right]=\int 1_B\circ\varphi d\widehat{E}_{x,y},\quad E\in\mathcal{B}_{\sigma(T)},$$

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and define $E(B)$ by the equality $\langle E(B)x, y\rangle = E_{x,y}(B)$, x, $y \in H$. Then $E(\cdot)$ is resolution of the identity defined on $\mathcal{B}_{\sigma(T)}$. Moreover, for f a bounded Borel function defined on $\sigma(T)$ we have

$$\left\langle \left(\int_{\sigma(T)} f(\lambda)E(d\lambda)\right)x, y\right\rangle = \int_{\sigma(T)} f(\lambda)\,\langle E(d\lambda)x, y\rangle$$
$$= \int_{\triangle_A} f\circ\varphi d\widehat{E}_{x,y} = \left\langle \left(\int_{\triangle_A} f\circ\varphi d\widehat{E}\right)x, y\right\rangle,$$

or what is the same $\int_{\sigma(T)} f(\lambda)E(d\lambda) = \int_{\triangle_A} f\circ\varphi d\widehat{E} = \int_{\triangle_A} f\left(\widehat{T}\right) d\widehat{E}$. In addition, let f be a function which is holomorphic in a neighborhood of $\sigma(T)$. Let Γ de a contour that surrounds $\sigma(T)$ in an open neighborhood of $\sigma(T)$ on which f is holomorphic. Then by Cauchy's formula we have

$$\left\langle \int_{\sigma(T)} f(\lambda)E(d\lambda)x, y\right\rangle = \left\langle \int_{\triangle_A} f\left(\widehat{T}\right) d\widehat{E}x, y\right\rangle$$
$$= \int_{\triangle_A} f\left(\widehat{T}\right) d\widehat{E}_{x,y} = \int_{\triangle_A} \frac{1}{2\pi i}\int_\Gamma f(\lambda)\left(\lambda - \widehat{T}\right)^{-1} d\lambda\, d\widehat{E}_{x,y}$$
$$= \int_{\triangle_A} \frac{1}{2\pi i}\int_\Gamma \widehat{f(\lambda)(\lambda I - T)^{-1}}\, d\lambda\, d\widehat{E}_{x,y}$$
$$= \int_{\triangle_A} \widehat{\widetilde{f}}(T)\, d\widehat{E}_{x,y} = \left\langle \widetilde{f}(T)x, y\right\rangle,$$

and so $\int_{\sigma(T)} f(\lambda)E(d\lambda) = \widetilde{f}(T)$.

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Finally, let $S_0 \in \mathcal{L}(H)$ be such that $S_0T = TS_0$. Then by the Theorem of Fuglede-Putnam-Rosenblum we see that $S_0T^* = T^*S_0$, and hence $S_0p(T,T^*) = p(T,T^*)S_0$ for all complex polynomials $p(\lambda, \overline{\lambda})$. Since the polynomials $p(T,T^*)$ are dense in A it follows that $S_0S = SS_0$ for all $S \in A$. Then Theorem 12.22 in Rudin's book [**113**] implies that $S_0E(B) = E(B)S_0$ for all $B \in \mathcal{B}_{\sigma(T)}$. Conversely, if $S_0E(B) = E(B)S_0$ for all $B \in \mathcal{B}_{\sigma(T)}$, then

$$S_0p(T,T^*) = S_0 \int_{\sigma(T)} p(\lambda, \overline{\lambda})\, E(d\lambda) = \int_{\sigma(T)} p(\lambda, \overline{\lambda})\, E(d\lambda)\, S_0 = p(T,T^*)\, S_0.$$

This completes the proof of Theorem 5.37. □

The theorem of Fuglede-Putnam-Rosenblum can be formulated as follows. Recall that an operator M is called normal whenever $M^*M = MM^*$. If an operator M is normal, then the operator $U := e^{M^*}e^{-M} = e^{M^*-M}$ is unitary in the sense that $U^*U = UU^* = I$, and so $\|U\|^2 = \|U^*U\| = 1$.

5.38. Theorem. *Let M and N be bounded normal operators on a Hilbert space H. Let $T : H \to H$ be a bounded linear operator with the intertwining property,* i.e. *$MT = TN$. Then $M^*T = TN^*$.*

Proof. Consider the operator valued analytic function

$$f : \lambda \mapsto e^{\lambda M^*}Te^{-\lambda N^*} = e^{\lambda M^*}e^{-\overline{\lambda}M}Te^{\overline{\lambda}N}e^{-\lambda N^*} = e^{\lambda M^*-\overline{\lambda}M}Te^{\overline{\lambda}N-\lambda N^*} \tag{5.68}$$

where in the first equality we used the intertwining property of the operator T, and in the second one the normality of the operators M and N. As observed above the operators $e^{\lambda M^*-\overline{\lambda}M}$ and $e^{\lambda N^*-\overline{\lambda}N}$, $\lambda \in \mathbb{C}$, are unitary. By (5.68) it follows that the operator norm of the function f can be estimated as follows:

$$\|f(\lambda)\| \leqslant \left\|e^{\lambda M^*-\overline{\lambda}M}\right\| \|T\| \left\|e^{\lambda N^*-\overline{\lambda}N}\right\| = \|T\|. \tag{5.69}$$

From (5.69) we see that the everywhere defined analytic function $\lambda \mapsto f(\lambda)$ is bounded. Liouville's theorem then implies that $f(\lambda) = f(0) = T$, and hence $e^{\lambda M^*}T = Te^{\lambda N}$, $\lambda \in \mathbb{C}$. Consequently, by taking derivatives we obtain $M^*T = TN^*$. This completes the proof of Theorem 5.38. □

2. Closed linear operators

Throughout this section H stands for a complex Hilbert space with inner-product $\langle \cdot, \cdot \rangle$, and norm $\|x\|^2 = \langle x, x \rangle$. Let $T : H \to H$ be a closed linear operator withe dense domain $D(T) \subset H$ and range $R(T) \subset H$. Its graph $G(T)$ is a closed linear subspace of the product Hilbert space $H \times H$, *i.e.* $G(T) = \{(x, Tx) : x \in D(T)\}$. Its adjoint T^* is a linear operator with domain $D(T^*)$ and range $R(T^*)$ in H. Its domain $D(T^*)$ consists of those vectors $y \in H$ for which the linear functional $x \mapsto \langle Tx, y \rangle$, $x \in D(T)$, is a continuous linear function on H. By the Riesz-Fischer representation theorem there exists, for a given vector $y \in D(T^*)$ a vector $z = T^*y \in H$ such that

$$\langle Tx, y \rangle = \langle x, z \rangle = \langle x, T^*y \rangle, \quad \text{for all } x \in D(T).$$

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Since $D(T)$ is dense the vector z is unique, and therefore we are entitled to write $z = T^*y$. Moreover, the mapping $y \mapsto T^*y$, $y \in D(T^*)$, is linear. Its graph $G(T^*) = \{(y, T^*y) : y \in D(T^*)\}$ is a closed linear subspace of $H \times H$. Let the operator $V : H \times H \to H \times H$ by the (unitary) anti-flip operator: $V(x, y) = (-y, x)$, $(x, y) \in H \times H$. In addition write $Q = I + T^*T$, so that $D(Q) = \{x \in D(T) : Tx \in D(T^*)\}$. A densely defined operator T is called symmetric if $T \subset T^*$. The latter means that for all x, $y \in D(T)$ the equality $\langle Tx, y\rangle = \langle x, Ty\rangle$ holds. It also means that $G(T) \subset G(T^*)$. If $T = T^*$, then T is called self-adjoint. A linear operator T with domain and range in H is called positive, denoted by $T \geqslant 0$, if $\langle Tx, x\rangle \geqslant 0$ for all $x \in D(T)$. If T is positive, then $\langle x, Tx\rangle = \langle Tx, x\rangle$, $x \in D(T)$. By the polarization formula we see $\langle Tx, y\rangle = \langle x, Ty\rangle$, x, $y \in D(T)$. Consequently, such operators are symmetric. If, in addition, $D(T)$ is dense in H and closed, then T^* exists, and $T = T^{**} \subset T^*$. The equality $T = T^{**}$ follows from assertion (2) and (4) in Theorem 5.39 below.

In the following theorem we collect some properties of closed densely defined operators.

5.39. Theorem. *Let T be a closed densely defined linear operator with domain and range in the Hilbert space H. The following assertions hold true.*

(1) *The space H admits the orthogonal decomposition: $H = N(T^*) \oplus \overline{R(T)}$.*
(2) *The space $H \times H$ with its natural Hilbert space structure admits the orthogonal decomposition: $H \times H = VG(T) \oplus G(T^*)$, and hence $G(T^*) = VG(T)^{\perp}$.*
(3) *Let the vectors a and b belong to H. Then the system of equations $-Tx + y = a$, $x + T^*y = b$ has a unique solution with $x \in D(T)$ and $y \in D(T^*)$.*
(4) *The domain of T^* is dense, T^{**} exists and coincides with T;*
(5) *The operator Q is a one-to-one mapping from $D(Q)$ onto H, it satisfies $Q \geqslant I$, and there are bounded linear operators B and C that satisfy $\|B\| \leqslant 1$, $\|C\| \leqslant 1$, $C = TB$, and*
$$B(I + T^*T) \subset (I + T^*T)B = I.$$
Moreover, $B \geqslant 0$, and T^*T is densely defined and self-adjoint.
(6) *If T' is the restriction of T to $D(T^*T)$, then the closure of $G(T')$ in $H \times H$ coincides with $G(T)$. In other words $D(T^*T)$ is a core for T.*

5.40. Proposition. *If $T = T^*$ and $T \geqslant 0$, then $\sigma(T) \subset [0, \infty)$.*

Proof. Let $\lambda \in \mathbb{C}$ be such that $\Im\lambda \neq 0$. Then, from the equalities
$$\|\lambda x + Tx\|^2 = |\lambda|^2 + 2\Re\lambda \langle Tx, x\rangle + \|Tx\|^2 = (\Im\lambda)^2 \|x\|^2 + \|\Re\lambda x + Tx\|^2,$$
it follows that
$$\|\lambda x + Tx\| \geqslant |\Im\lambda| \|x\|, \ \lambda \in \mathbb{C}, \text{ and } \|\lambda x + Tx\| \geqslant \Re\lambda \|x\|, \ \Re\lambda > 0, \ x \in D(T). \tag{5.70}$$
From (5.70) it follows that the range of the operator $\lambda I + T$ is closed whenever $\Im\lambda \neq 0$ or $\Re\lambda > 0$. For the same range of values of λ it also follows from (5.70)

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that the null-space of $\lambda I + T$ is the singleton $\{0\}$. By assertion (1) and the fact the $T = T^*$ we infer that $R(\lambda I - T) = H$ and $N(\lambda I - T) = \{0\}$ for $\Im\lambda \neq 0$. Consequently, $\Im\lambda \neq 0$ implies $\lambda \subset \mathbb{C}\backslash\sigma(T)$. It also follows that $R(\lambda I + T) = H$ and $N(\lambda I + T) = \{0\}$ for $\Re\lambda > 0$. So that $\Re\lambda > 0$ implies $-\lambda \in\subset \mathbb{C}\backslash\sigma(T)$. This leads to the conclusion that $\sigma(T) \subset [0, \infty)$, and completes the proof of Proposition 5.40. $\square$

An operator T is called essentially self-adjoint if the closure of its graph is again the graph of an operator $\overline{T}$, and if this closure is self-adjoint. Since a densely operator T is closed if and only if $\overline{T} = T^{**}$, T is essentially self-adjoint if and only if $T^{**} = T^*$.

Proof Theorem 5.39. (1) It is clear that $N(T^*) = R(T)^{\perp}$. Then it follows that the subspace $N(T^*) + \overline{R(T)}$ is closed in H. We shall prove that it is dense. So let $a \in H$ be such that $\langle x, a\rangle = 0$ for all $x \in N(T^*)$, and also such that $\langle Ty, a\rangle = 0$ for all $y \in D(T)$. Then a belongs to $D(T^*)$ and $T^*a = 0$. So a belongs to $N(T^*)$. But then we choose $x = a$ to obtain that $\langle a, a\rangle = 0$, and so $a = 0$. This proves assertion (1) of Theorem 5.39.

(2) Let (x_1, y_1) and (x_2, y_2) be members of $H \times H$. Then their inner-product, or scalar product, $\langle (x_1, y_1), (x_2, y_2)\rangle$ is defined by

$$\langle (x_1, y_1), (x_2, y_2)\rangle = \langle x_1, x_2\rangle + \langle y_1, y_2\rangle .$$

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So it is easy to see that $VG(T)$ and $G(T^*)$ are orthogonal subspaces in $H \times H$. It is also easy to see that $G(T^*)$ is closed in $H \times H$, and the same is obvious for $VG(T)$. It follows that the space $VG(T) + G(T^*)$ is closed in $H \times H$. We shall prove that it is dense. Let the pair $(a, b) \in H \times H$ be orthogonal to both subspaces $V(G(T))$ and $G(T^*)$. It follows that

$$\begin{aligned}
&-\langle Tx, a\rangle + \langle x, b\rangle = \langle(-Tx, x), (a, b)\rangle = 0 \quad \text{for all } x \in D(T), \text{ and}\\
&\langle y, a\rangle + \langle T^*y, b\rangle = \langle(y, T^*y), (a, b)\rangle = 0 \quad \text{for all } y \in D(T).
\end{aligned} \tag{5.71}$$

From the first equality in (5.71) it follows that a belongs to $D(T^*)$ and that $b = T^*a$. Plugging this into the second equality in (5.71) and putting $y = a$ shows $\langle a, a\rangle + \langle T^*a, T^*a\rangle = 0$, and hence $a = 0$. Since $b = T^*a$ we see that $b = 0$ as well. This proves assertion (2) of Theorem 5.39.

(3) This assertion easily follows from the decomposition in assertion (2).

(4) If the vector $a \in H$ is orthogonal to $D(T^*)$, then the vector $(a, 0)$ is orthogonal to the graph $G(T^*)$, and so $(a, 0)$ belongs to $VG(T)$. That is $a = -T0 = 0$. Whence the first part of assertion (4) has been proved. Since $D(T^*)$ is dense its adjoint T^{**} exists. It readily follows that $T \subset T^{**}$. By the decomposition in assertion (2) it follows that $T = T^{**}$. This proves assertion (4) of Theorem 5.39.

(5) Fix $h \in H$ and choose operators vectors $f \in D(T)$ and $g \in D(T^*)$ such that

$$(0, h) = (-Tf, f) + (g, T^*g) \tag{5.72}$$

in the space $H \times H$. The mappings $h \mapsto f$ and $h \mapsto g$ are linear; call them B respectively C. Then $TBh = Ch$ and $h = Bh + T^*Ch = Bh + T^*TBh$. In other words $h = (I + T^*T)\,Bh$. This means that the operator B is a right inverse of $I+T^*T$. Since, for any $h \in H$, Bh belongs to $D(T^*T)$, we also have, for $g \in D(T^*T)$,

$$\begin{aligned}
&(I + T^*T)\,(g - B\,(I + T^*T)\,g) = (I + T^*T)\,g - (I + T^*T)\,B\,(I + T^*T)\,g\\
&= (I + T^*T)\,g - (I + T^*T)\,g = 0,
\end{aligned}$$

we infer that the vector $g - B\,(I + T^*T)\,g$ belongs to the null-space of the operator $Q = I + T^*T$. Since $\langle Qf, f\rangle \geqslant \langle f, f\rangle$, $f \in D(Q) = D(T^*T)$, it follows that $g = B\,(I + T^*T)\,g$. In other words the operator B is also a left-inverse of $I + T^*T$. Since the operator B is everywhere defined and symmetric, it is self-adjoint, and since $N(B) = \{0\}$ it has dense range $R\,(I + T^*T)^{-1} = D(T^*T)$. It follows that its inverse is $I + T^*T$ is self-adjoint, and that the same is true for T^*T. From (5.72) and the definitions of the operators B and C it follows that $B = (I + T^*T)^{-1}$ and $C = T\,(I + T^*T)^{-1}$, and

$$(0, h) = (-Tf, f) + (g, T^*) = (-TBh, Bh) + (Ch, T^*Ch)\,, \tag{5.73}$$

and therefore

$$\begin{aligned}
\|h\|^2 &= \|TBh\|^2 + \|Bh\|^2 + \|Ch\|^2 + \|T^*Ch\|^2\\
&\geqslant \|Bh\|^2 + \|Ch\|^2\,.
\end{aligned}$$

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Whence $\|Bh\|^2 + \|Ch\|^2 \leqslant \|h\|^2$. This proves assertion (5).

(6) From the definition of T' it follows that

$$G(T') = \left\{ \left((I + T^*T)^{-1} x, T(I + T^*T)^{-1} x \right) : x \in H \right\} \subset G(T). \tag{5.74}$$

Assuming that $G(T')$ is not dense in the closed subspace $G(T)$. Then there exists a pair $(a, Ta) \in G(T)$ such that

$$\langle x, a \rangle = \left\langle \left((I + T^*T)^{-1} x, T(I + T^*T)^{-1} x \right), (a, Ta) \right\rangle = 0, \quad x \in H. \tag{5.75}$$

Insert $x = a$ into (5.75) results in $a = 0$. Consequently, $G(T')$ is dense in $G(T)$ which is assertion (6).

This completes the proof of Theorem 5.39. □

The following theorem has its analogue in the context of C^*-algebras. The main result is that a closed linear operator in a Hilbert space can be written in the form $T = U|T|$, where U^*U is an orthogonal projection on the closure of the range of T^*T. The theorem is patterned after Theorem 5.19.

5.41. Theorem. *Let T be a closed densely defined linear operator in a Hilbert space. Define the following operators:*

$$\begin{aligned} Ux &= \frac{2}{\pi} \int_0^\infty T\left(t^2 I + T^*T\right)^{-1} x\, dt, \quad x \in H, \\ |T|(x) &= \frac{2}{\pi} \int_0^\infty T^*T\left(t^2 I + T^*T\right)^{-1} x\, dt, \quad x \in D(T), \\ |T^*|(x) &= \frac{2}{\pi} \int_0^\infty TT^*\left(t^2 I + TT^*\right)^{-1} x\, dt, \quad x \in D(T^*), \end{aligned} \tag{5.76}$$

Then the following assertions hold true:

1. *The operators $|T|$ and $|T^*|$ are well-defined, positive, have the same domain as T and T^* respectively, and satisfy the following equalities: $|T|^2 = T^*T$, $|T^*|^2 = TT^*$. These operators are the only self-adjoint positive operators with these properties.*
2. *The operator U is well-defined, it has norm 1, and its adjoint U^* is given by*
$$U^*x = \frac{2}{\pi} \int_0^\infty T^*\left(t^2 e + TT^*\right)^{-1} x\, dt, \quad x \in H. \tag{5.77}$$
3. *The following equalities hold:*
$$U|T|(x) = Tx, \ x \in D(T), \quad \text{and} \quad U^*|T^*|(x) = T^*(x), \quad x \in D(T^*). \tag{5.78}$$
4. *The operators U^*U and UU^* are orthogonal projection on the closures of the ranges $R(T^*T)$ and $R(TT^*)$ respectively. In fact the following equalities hold:*
$$U^*U|T|(x) = |T|(x), \ x \in D(T), \ \text{and}\ UU^*|T^*|(x) = |T^*|(x), \ x \in D(T^*). \tag{5.79}$$

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PROOF. (1) The assertions about the domains of $|T|$ and $|T^*|$ is a consequence of the equalities in assertions (3) and (4). The proof of the equality $|T|^2(x) = T^*Tx$, $x \in D(T^*T)$ can be patterned after the proof of assertion (1) of Theorem 5.19. The uniqueness of these square roots follows like in the proof of assertion (b) of Theorem 5.14. Let us give more details. Let $0 < \varepsilon < R < \infty$, and put

$$U_{\varepsilon,R}x = \frac{2}{\pi}\int_{\varepsilon}^{R} T\left(t^2 I + T^*T\right)^{-1} x\,dt, \quad x \in H.$$

Then

$$U_{\varepsilon,R}^{*}x = \frac{2}{\pi}\int_{\varepsilon}^{R} T^*\left(t^2 I + TT^*\right)^{-1} x\,dt, \quad x \in H. \tag{5.80}$$

Since

$$U_{\varepsilon,R}^{*}U_{\varepsilon,R}x = \frac{4}{\pi^2}\int_{\varepsilon}^{R}\int_{\varepsilon}^{R} T^*T\left(t_1^2 I + T^*T\right)^{-1}\left(t_2^2 I + T^*T\right)^{-1} x\,dt_2\,dt_1, \quad x \in H, \tag{5.81}$$

we see that

$$\left\|U_{\varepsilon,R}\right\|^2 = \left\|U_{\varepsilon,R}^{*}U_{\varepsilon,R}\right\| \leqslant \sup_{\lambda>0}\frac{4}{\pi^2}\int_0^{\infty}\int_0^{\infty}\frac{\lambda^2}{\left(t_1^2+\lambda^2\right)\left(t_2^2+\lambda^2\right)}\,dt_2\,dt_1 = 1. \tag{5.82}$$

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By the same argument we have $\left\|U_{\varepsilon,R}^*\right\|^2 = \left\|U_{\varepsilon,R}U_{\varepsilon,R}^*\right\|^2 \leqslant 1$. Since, for $x \in D(T)$ we get

$$\frac{2}{\pi}\int_\varepsilon^R T^*T\left(t^2I + T^*T\right)^{-1} x\,dt = \frac{2}{\pi}\int_\varepsilon^R T^*\left(t^2I + TT^*\right)^{-1} Tx\,dt = U_{\varepsilon,R}^*Tx \tag{5.83}$$

we infer

$$\left\|\frac{2}{\pi}\int_\varepsilon^R T^*T\left(t^2I + T^*T\right)^{-1} x\,dt\right\| \leqslant \|Tx\|, \quad x \in D(T). \tag{5.84}$$

Introduce the subspace G' of $G(T)$ defined by

$$G' = \left\{(x, Tx) \in G(T) : \lim_{\varepsilon\downarrow 0, R\to\infty} \frac{2}{\pi}\int_\varepsilon^R T^*T\left(t^2I + T^*T\right)^{-1} x\,dt \text{ exists in } H\right\}. \tag{5.85}$$

Then by the inequality in (5.84) G' is a closed subspace of $G(T)$. By assertion (6) of Theorem 5.39 this space is dense in $G(T)$ (relative to the graph norm). Consequently, $G' = G(T)$. It follows that $D(T) \subset D(|T|)$, and that

$$|T|(x) = \lim_{\varepsilon\downarrow 0, R\to\infty} \frac{2}{\pi}\int_\varepsilon^R T^*T\left(t^2I + T^*T\right)^{-1} x\,dt, \quad \text{exists for } x \in D(T),$$

and, consequently, $|T|$ is well-defined. Next let $x \in D(T^*T)$. Then exactly in the same manner as we proved assertion (1) of Theorem 5.19 we infer (see the proof of (c) in Theorem 5.14 as well):

$$\begin{aligned}
|T|^2(x) &= \frac{4}{\pi^2}\int_\varepsilon^\infty (T^*T)^2 \int_0^\infty \left(t_1^2I + T^*T\right)^{-1}\left(t_2^2I + T^*T\right)^{-1} x\,dt_2\,dt_1 \\
&= \int_0^\infty (T^*T)^2(\rho I + T^*T)^{-2} x\,d\rho \\
&= \lim_{\varepsilon\downarrow 0}\lim_{R\to\infty}\left\{T^*T(\varepsilon I + T^*T)^{-1} - (RI + T^*T)^{-1}\right\}T^*Tx \\
&= \lim_{\varepsilon\downarrow 0}\lim_{R\to\infty}\left\{T^*Tx - \varepsilon x + \varepsilon^2(\varepsilon I + T^*T)^{-1}x - (RI + T^*T)^{-1}T^*Tx\right\} \\
&= T^*Tx.
\end{aligned} \tag{5.86}$$

Let the operator $S \geqslant 0$ be such that $S^2 = T^*T$. Then, as in equality (5.22) in the proof of assertion (b) of Theorem 5.14 we have

$$\frac{2}{\pi}\int_\varepsilon^\infty S^2\left(t^2I + S^2\right)^{-1} x\,dt = Sx - \frac{2\varepsilon}{\pi}x + \frac{1}{\pi i}\int_{\Gamma_\varepsilon} z(zI + S)^{-1} x\,dz, \quad x \in D(S). \tag{5.87}$$

By assumption $S^2 = T^*T$, and so in (5.87) we let ε tend to 0 to obtain $Sx = |T|(x)$, $x \in D(S)$. This shows that the only positive square root of T^*T is given by $|T|$. Similar arguments and conclusions apply to the operator TT^* and T^*. This completes the proof of assertion (1) except that we still have to prove that $D(T) \subset D(|T|)$, and $D(T^*) \subset D(|T^*|)$. For the converse inclusions we first prove that $\||T|(x)\| = \|Tx\|$, $x \in D(|T|)$. This equality is easily established for $x \in D(T^*T) = D\left(|T|^2\right)$. Let x belong to the domain of $|T|$. Since, by construction, the operator $|T|$ is self-adjoint, by assertion (6) applied with $|T|$ there exists a sequence

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$(x_n)_n \subset D\left(|T|^2\right) = D\left(T^*T\right)$ with the following properties $\lim_{n\to\infty} x_n = x$, and $\lim_{n\to\infty} |T|(x_n) = |T|(x)$. Then

$$\|Tx_n - Tx_m\| = \|T(x_n - x_m)\| = \||T|(x_n - x_m)\|,$$

and hence $(Tx_n)_n$ is a Cauchy sequence in H. It follows that $y = \lim_{n\to\infty} Tx_n$ exists. The operator T being closed, and $\lim_{n\to\infty} x_n = x$, we infer $x \in D(T)$ and $y = Tx$. These observations lead to

$$\|Tx\| = \lim_{n\to\infty} \|Tx_n\| = \lim_{n\to\infty} \||T|\, x_n\| = \||T|(x)\|, \quad x \in D(|T|). \tag{5.88}$$

Finally, let $x \in D(T)$. Then there exists a sequence $(x_n)_n \subset D(T^*T)$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = x$. By the equalities in (5.88) it follows that $(|T|(x_n))_n \subset H$ is a Cauchy sequence, and therefore its limit $y := \lim_{n\to\infty} |T|(x_n)$ exists. Since $|T|$ is a closed operator it follows that x belongs to $D(|T|)$, and that $y = |T|(x)$. All this implies that $D(T) = D(|T|)$ and that $\||T|(x)\| = \|Tx\|$ for $x \in D(T)$. The same argumentation shows that $\||T^*|(x)\| = \|T^*x\|$ for $x \in D(T^*)$.

(2) From (5.82) we see that the subspace $L \subset H$ defined by

$$L = \left\{ x \in H : \lim_{\varepsilon\downarrow 0} \lim_{R\to\infty} U_{\varepsilon,R} x \, dt \text{ exists in } H \right\} \tag{5.89}$$

is a closed subspace of H. If $x \in N(T)$, then

$$U_{\varepsilon,R} x = \frac{2}{\pi} \int_\varepsilon^R T\left(t^2 I + T^*T\right)^{-1} x \, dt = \frac{2}{\pi} \int_\varepsilon^R \left(t^2 I + TT^*\right)^{-1} Tx \, dt = 0, \tag{5.90}$$

and so x belongs to L. If x is of the form $x = T^*y$, $y \in D(T^*)$, then we have

$$U_{\varepsilon,R} x = U_{\varepsilon,R} T^* y = \frac{2}{\pi} \int_\varepsilon^R T\left(t^2 I + T^*T\right)^{-1} T^* y \, dt = \frac{2}{\pi} \int_\varepsilon^R TT^* \left(t^2 I + TT^*\right)^{-1} y \, dt. \tag{5.91}$$

From (5.91) and assertion (1) we infer that

$$\lim_{\varepsilon\downarrow 0, R\to\infty} U_{\varepsilon,R} x = \lim_{\varepsilon\downarrow 0, R\to\infty} U_{\varepsilon,R} T^* y = \lim_{\varepsilon\downarrow 0, R\to\infty} \frac{2}{\pi} \int_\varepsilon^R TT^* \left(t^2 I + TT^*\right)^{-1} y \, dt = |T^*|(y). \tag{5.92}$$

From (5.90) and (5.92) it follows that $L \supset N(T) + R(T^*)$. By assertion (1) of Theorem 5.39 we see that the subspace $N(T) + R(T^*)$ is dense in H. Since L is a closed subspace, we deduce that $L = H$. Therefore the operator U is well defined. By the expression for $U^*_{\varepsilon,R}$ in (5.81), it follows, in the same manner as we proved that U is well-defined, that U^* is well-defined as well, and that U^* is given by (5.77). This shows assertion (2).

(3) Let $x \in D(T) = D(|T|)$. Then

$$\begin{aligned} U|T|(x) &= \frac{4}{\pi^2} TT^*T \int_0^\infty \int_0^\infty \left(t_1^2 I + T^*T\right)^{-1} \left(t_2^2 I + T^*T\right)^{-1} x \, dt_2 \, dt_1 \\ &= TT^*T \int_0^\infty (\rho I + T^*T)^{-2} \, x d\rho \end{aligned}$$

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$$
\begin{aligned}
&= \lim_{\varepsilon\downarrow 0, R\to\infty} TT^*T\left\{(\varepsilon I + T^*T)^{-1} - (RI + T^*T)^{-1}\right\} \\
&= \lim_{\varepsilon\downarrow 0, R\to\infty} \left\{Tx - TT^*\left(RI + TT^*\right)^{-1} Tx - \varepsilon T\left(\varepsilon I + T^*T\right)^{-1} x\right\} = Tx.
\end{aligned} \tag{5.93}
$$

In the final step we employed the following equalities:

$$
\lim_{R\to\infty} TT^*\left(RI + TT^*\right)^{-1} y = 0, \quad \text{and} \quad \lim_{\varepsilon\downarrow 0} \varepsilon T\left(\varepsilon I + T^*T\right)^{-1} y = 0, \quad y \in H. \tag{5.94}
$$

The first limit is 0, because this is clear for $y \in D\left(TT^*\right)$. Since $D\left(T^*T\right)$ is dense in H and $\left\|R\left(RI + TT^*\right)^{-1}\right\| \leqslant 1$, $R > 0$, the first limit is 0 for all $y \in H$. The second limit is 0 because

$$
\left\|\varepsilon T\left(\varepsilon I + T^*T\right)^{-1}\right\|^2 = \varepsilon^2\left\|\left(\varepsilon I + T^*T\right)^{-1} T^*T\left(\varepsilon I + T^*T\right)^{-1}\right\| \leqslant \sup_{\lambda>0} \frac{\varepsilon\lambda}{(\varepsilon+\lambda)^2} = \frac{1}{4}\varepsilon.
$$

The proof of the equality $U^*\left|T^*\right| = T^*$ is very similar. This proves assertion (3).

(4) Let $x \in H$. From the properties and definitions of the operators U and U^* we deduce the equalities:

$$
\begin{aligned}
U^*Ux &= \frac{4}{\pi^2}\int_0^\infty\int_0^\infty T^*T\left(t_1^2 I + T^*T\right)^{-1}\left(t_2^2 I + T^*T\right)^{-1} x\, dt_2\, dt_1 \\
&= T^*T\int_0^\infty \left(\rho I + T^*T\right)^{-2} x\, d\rho \\
&= \lim_{\varepsilon\downarrow 0}\lim_{R\to\infty}\left\{T^*T\left(\varepsilon I + T^*T\right)^{-1} x - T^*T\left(RI + T^*T\right)^{-1} x\right\}.
\end{aligned} \tag{5.95}
$$

Like in the proof of assertion (3) the second limit in (5.95) vanishes. The fist limit also vanishes if $T^*Tx = 0$. If $x = T^*Ty$, then the first limit in (5.95) is equal to x. In addition, we have

$$
\left\|T^*T\left(\varepsilon I + T^*T\right)^{-1}\right\| \leqslant 1,
$$

and so U^*U is the orthogonal projection on the closure of $R\left(T^*T\right)$. In particular it follows that $U^*U\left|T\right| = \left|T\right|$. The same argument shows that UU^* is an orthogonal projection on the closure of $R\left(TT^*\right)$. In particular it follows that $UU^*\left|T^*\right| = \left|T^*\right|$. This completes the proof of assertion (4).

Altogether this wraps up the proof of Theorem 5.41. □

5.42. Corollary. *Let T be a densely defined normal operator. This means that T is closed and densely defined, that $D\left(T^*\right) = D\left(T\right)$, and that $T^*T = TT^*$. Then there exists a unitary operator U and a positive operator $\left|T\right|$ such that $T = U\left|T\right|$. Moreover, $U\left|T\right| = \left|T\right| U$.*

Proof. On the range of $\left|T\right|$ define U as in Theorem 5.41. On the null space $N\left(\left|T\right|\right)$ define U as the identity operator. Notice that, since T is normal, $\left|T\right| = \left|T^*\right|$, and that $N\left(\left|T\right|\right) = N\left(T\right) = N\left(T^*\right)$. Then from Theorem 5.41 it also follows that $\left|T\right| U = U\left|T\right|$. This completes the proof of Corollary 5.42. □

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CHAPTER 6

Operator semigroups and Markov processes

We will discuss a number of aspects related to one-parameter operator semigroups. We will present some general theory, give some examples, include a result on initial value problems, and make a link with Markov processes, and give some details on Feynman-Kac semigroups. Unfortunately, not all aspects of this theory can be discussed. In particular, this is true for applications of (generators of) semigroup theory, for semigroups related to population dynamics, and for delay equations. In Chapter 7 we will discuss analytic semigroups and certain aspects of the Crank-Nicolson iteration scheme.

1. Generalities on semigroups

Let $(X, \|\cdot\|)$ be a Banach space and let $\{S(t) : t \geqslant 0\}$ be a family of bounded linear operators from X to X. This family is called *strongly continuous* if it possesses the following properties:

(i) $S(0) = I$, $S(s+t) = S(s) \circ S(t)$, for all s, $t \geqslant 0$;
(ii) $\lim_{t\downarrow 0} \|S(t)f - f\| = 0$ for all $f \in X$.

6.1. Remark. Suppose that the family $\{S(t) : t \geqslant 0\}$ possesses property (i). Then it possesses property (ii) if and only if

(ii′) $\lim_{t\downarrow 0} \langle S(t)f, f^*\rangle = \langle f, f^*\rangle$ for all $f \in X$ and for all $f^* \in X^*$.

Property (i) is called the semigroup property, property (ii) is the strong continuity at $t = 0$, and (ii*) is the weak continuity at $t = 0$.

6.2. Remark. Often a strongly continuous semigroup $\{S(t) : t \geqslant 0\}$ is written in the form $S(t) = \exp(tA)$ or $S(t) = \exp(-tH)$. For symbolic manipulation this notation is very convenient. For example, for $\lambda > 0$ large enough,

$$\int_0^\infty e^{-\lambda t} S(t)\, dt = \int_0^\infty e^{-t(\lambda I - A)} dt = (\lambda I - A)^{-1}.$$

Indeed it can be proved that the collection $\{R(\lambda) : \lambda > \omega\}$ is a resolvent family, where each operator $R(\lambda)$, $\lambda > \omega$, is of the form $R(\lambda) = (\lambda I - A)^{-1}$, for some closed, densely defined linear operator A with domain and range in X. the number ω is chosen in such a way that $\|S(t)\| \leqslant M \exp(\omega t)$, $t \geqslant 0$. Such a number ω exists. If $(X, \|\cdot\|)$ is a Hilbert space, and if each operator $S(t)$ is self-adjoint (*i.e.* $S(t) = S(t)^*$), then M may be taken the constant 1.

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6.3. Theorem. *Let $\{S(t) : t \geqslant 0\}$ be a strongly continuous semigroup and put*

$$A = \text{s-}\lim_{t \downarrow 0} \frac{S(t) - I}{t}.$$

This means that

$$Af = \lim_{t \downarrow 0} \frac{S(t)f - f}{t} \quad \textit{for } f \textit{ belonging to its domain}$$

$$D(A) = \left\{ f \in X : \lim_{t \downarrow 0} \frac{S(t)f - f}{t} \quad \textit{exists in } X \right\}.$$

Then A is a closed densely defined linear operator with the following properties

(i) $(\lambda I - A)R(\lambda)f = f$, *for all* $f \in X$;
(ii) $R(\lambda)(\lambda I - A)f = f$, *for all* $f \in D(A)$;
(iii) $G(A) = \{(R(\lambda)f, \lambda R(\lambda)f - f) : f \in X\}$;
(iv) $f \in D(A)$ *implies* $S(t)f \in D(A)$ *and* $AS(t)f = S(t)Af$;
(v) $f \in X$ *implies* $\int_0^t S(s)f\,ds \in D(A)$ *and* $A\int_0^t S(s)f ds = S(t)f - f$.

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6.4. Proposition. *Let $\{S(t) : t \geqslant 0\}$ be a weakly continuous semigroup in $\mathcal{L}(X)$. Then the following assertions are true:*

(i) *There exist real constants M and ω such that*

$$\|S(t)\| \leqslant M \exp(\omega t), \quad t \geqslant 0.$$

Moreover, for $\lambda > \omega$ and for $f \in X$ the integral $\int_0^\infty e^{-\lambda t} S(t) f \, dt$ can be interpreted as an element of X. Upon writing $R(\lambda) f = \int_0^\infty e^{-\lambda t} S(t) f \, dt$ for $\lambda > \omega$ the resolvent identity follows:

$$R(\lambda_1) f - R(\lambda_2) f = (\lambda_2 - \lambda_1) R(\lambda_2) R(\lambda_1) f, \ \lambda_1, \lambda_2 > \omega, \ f \in X.$$

Integrals of the form $\int_0^\infty \varphi(t) S(t) f \, dt$, where the functions φ are Borel measurable and satisfy $\int_0^\infty |\varphi(t)| e^{\omega t} \, dt < \infty$, are elements of X as well.

(ii) $\lim_{t \to s, t \geqslant 0} \|S(t)f - S(s)f\| = 0$, *$f \in X$;*

(iii) *Let M and ω be as in (i) and put*

$$\|f\|_\omega = \sup \{\exp(-\omega t) \|S(t) f\| : t \geqslant 0\}.$$

Then

(a) $\|f\| \leqslant \|f\|_\omega \leqslant M \|f\|$, *for $f \in X$;*

(b) $\|\exp(-\omega s) S(s) f\|_\omega \leqslant \|S(s) f\|_\omega$ *for $f \in X$.*

6.5. Remark. The topology induced by $\|\cdot\|$ coincides with that of $\|\cdot\|_\omega$, the geometric properties are lost. The semigroup $\{\exp(-\omega s) S(s) : s \geqslant 0\}$ consists of contractive operators in the space $(X, \|\cdot\|)$. The assertion in (ii) says that a weakly continuous semigroup is in fact strongly continuous.

6.6. Remark. Fix $f \in X$. From the proof of the assertions in (i) it follows that the following conditions on the Borel measurable $\varphi : [0, \infty) \to \mathbb{C}$ suffice to guarantee that the integral $\int_0^\infty \varphi(t) S(t) f \, dt$ belongs to X:

(a) For every $f^* \in X^*$ the integral $\int_0^\infty |\varphi(t) \langle S(t) f, f^* \rangle| \, dt$ is finite;

(b) The collection functions

$$\{t \mapsto \Phi_{\varphi, f, f^*}(t) := \varphi(t) \langle S(t) f, f^* \rangle : \ f^* \in X^*, \ \|f^*\| \leqslant 1\}$$

is uniformly integrable in the sense that for every $\varepsilon > 0$ there exists $0 \leqslant g_\varepsilon \in L^1([0, \infty))$ such that, for all $f^* \in X^*$, $\|f^*\| \leqslant 1$, the following inequality holds:

$$\int_{|\Phi_{\varphi, f, f^*}| \geqslant g_\varepsilon} |\Phi_{\varphi, f, f^*}(t)| \, dt \leqslant \varepsilon.$$

The assumption in (a) implies that the integral $\int_0^\infty \varphi(t) S(t) f \, dt$ belongs to X^{**}. A consequence of (b) together with Theorem 8.30 is that the latter integral belongs to X.

Proof of Proposition 6.4. The first part of assertion (i) follows from the Banach-Steinhaus theorem. First it is shown that, for some $\delta > 0$, the supremum $\sup \{\|S(t)\| : 0 \leqslant \delta\}$ is finite. If this were not the case, then there would

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exist a sequence $(t_n : n \in \mathbb{N}\}$ of strictly positive real numbers such that $t_n \downarrow 0$ and $\|S(t_n)\| \uparrow \infty$, if n tends to ∞. However, by assumption (ii$'$) we know that for every $f \in X$ and every $f^* \in X^*$, $\lim_{n\to\infty} \langle S(t_n) f, f^*\rangle - \langle f, f^*\rangle = 0$. Hence $\sup_{n\in\mathbb{N}} |\langle S(t_n) f, f^*\rangle| < \infty$ for all $f \in X$ and for all $f^* \in X^*$. Consider, for $f \in X$ fixed, the sequence of continuous linear functions $\Lambda_n : X^* \to \mathbb{C}$, defined by $\Lambda_n(f^*) = \langle S(t_n) f, f^*\rangle$. Then $\sup_n |\Lambda_n(f^*)| < \infty$. But the Banach-Steinhaus theorem then says that $\sup_{n\in\mathbb{N}} \|S(t_n) f\| < \infty$. Since $f \in X$ is arbitrary, another application of the uniform boundedness principle (or the Banach-Steinhaus theorem) then implies $\sup_n \|S(t_n)\| < \infty$. This is a contradiction. As a consequence we infer that, for some $\delta > 0$, $\sup_{0\leqslant t\leqslant\delta} \|S(t)\| < \infty$. Next we have $S(t) = S(\delta)^n S(t - n\delta)$, where $n\delta \leqslant t < (n+1)\delta$. Thus

$$\|S(t)\| \leqslant \sup_{0\leqslant s\leqslant\delta} \|S(s)\|^{n+1} \leqslant \sup_{0\leqslant s\leqslant\delta} \|S(s)\|^{1+t/\delta} \leqslant M \exp(\omega t),$$

where $M = \sup\{\|S(s)\| : 0 \leqslant s \leqslant \delta\}$, and where $\omega = \dfrac{1}{\delta}\log M$.

Next we want to show that for $\lambda > \omega$ and $f \in X$ the integral $\int_0^\infty e^{-\lambda t} S(t) f\, dt$ can be interpreted as a member of X. We first observe that, for $f \in X$ fixed the subspace X_f which, by definition, is the smallest closed subspace of X which contains all vectors of the form $S(t)f$, $t \geqslant 0$, is separable. From the right-continuity of the functions $t \mapsto \langle S(t)f, f^*\rangle$, $f^* \in X^*$, it follows that the space X_f is separable for the weak topology. But then it is also separable for the norm-topology. By considering the functional

$$\Lambda_f : f^* \mapsto \int_0^\infty e^{-\lambda t} \langle S(t)f, f^*\rangle\, dt, \quad f^* \in X^*,$$

we see by the Lebesgue's dominated convergence theorem that $\lim_{n\to\infty} \Lambda_f(f_n^*) = 0$ whenever $\{f_n^*\}_{n=1}^\infty$ is a sequence in X^* which converges to 0 for the weak*-topology. By the Banach-Steinhaus theorem such a sequence is automatically bounded. Also recall that, by the Hahn-Banach extension theorem, continuous linear functionals on X_f have an extension to all of X while preserving their norm. From Theorem 8.30 it follows that there exists a vector $g \in X_f$ such that

$$\Lambda_f(f^*) = \int_0^\infty e^{-\lambda t} \langle S(t)f, f^*\rangle\, dt = \langle g, f^*\rangle \quad \text{for all } f^* \in X^*.$$

The vector g is written as

$$g = R(\lambda)f = \int_0^\infty e^{-\lambda t} S(t) f\, dt.$$

Let the Borel measurable $\varphi : [0,\infty) \to \mathbb{C}$ and $f \in X$ be such that

$$\int_0^\infty |\varphi(t)|\, |\langle S(t)f, f^*\rangle|\, dt < \infty \ \text{ for all } f^* \in X^*.$$

Then the set B_f defined by

$$B_f = \left\{ f^* \in X^* : \int_0^\infty |\varphi(t)\, \langle S(t)f, f^*\rangle|\, dt \leqslant 1 \right\}$$

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is a closed absolutely convex subset of X^* which is absorbing in the sense that for every $f^* \in X^*$ there exists $t > 0$ such that $f^* \in tB_f$. In other words B_f is a barrel in the Banach space X^*. Consequently, B_f is a neighborhood of the origin, and so there exists $\delta > 0$ such that the ball of radius δ, *i.e.*, $\{f^* \in X^* : \|f\| \leqslant \delta\}$ is contained in B_f. Then it follows that

$$\left|\int_0^\infty \varphi(t) \langle S(t)f, f^*\rangle \, dt\right| \leqslant \int_0^\infty |\varphi(t) \langle S(t)f, f^*\rangle| \, dt \leqslant \frac{1}{\delta} \|f^*\|, \quad f^* \in X^*. \tag{6.1}$$

Define, for $f \in X$, the linear functional $\Lambda_{\varphi,f} : X^* \to \mathbb{C}$ by

$$\Lambda_{\varphi,f}(f^*) = \int_0^\infty \varphi(t) \langle S(t)f, f^*\rangle \, dt, \quad f^* \in X^*. \tag{6.2}$$

Then, by (6.1) it follows that $\Lambda_{\varphi,f}$ is a member of X^{**}, and hence the integral $\int_0^\infty \varphi(t)S(t)f \, dt$ can be interpreted as an element of X^{**}. Since, by hypothesis, the integral $\int_0^\infty |\varphi(t)| \, e^{\omega t} \, dt$ is finite, it follows that $\lim_{k\to\infty} \Lambda_{\varphi,f}(f_k^*) = 0$, whenever $\{\varphi_k^*\}_{k=1}^\infty$ is a sequence in X^* which converges in weak*-sense to 0. By Theorem 8.30 it follows that the integral $\int_0^\infty \varphi(t)S(t)f \, dt$ not only belongs to X^{**}, but that it is member of X.

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Assertion (ii) follows from the assertions in (i) together with strong continuity at $t = 0$. The strong continuity at $t = 0$ of the semigroup $\{S(t) : t \geqslant 0\}$ can be proved as follows. Consider the subspace $L \subset X$ defined by

$$L = \left\{ f \in X : \lim_{t \downarrow 0} S(t)f = f \right\}.$$

Then, by (i), L is a closed subspace of X. In addition, since, for $\lambda > \omega$ and $f \in X$,

$$\lim_{t\downarrow 0} S(t)\lambda R(\lambda)f = \lim_{t\downarrow 0} e^{-\lambda t} S(t)\lambda R(\lambda)f = \lim_{t\downarrow 0} e^{-\lambda t} S(t)\lambda \int_0^\infty e^{-\lambda\rho} S(\rho)f\,d\rho$$
$$= \lim_{t\downarrow 0} \lambda \int_t^\infty e^{-\lambda\rho} S(\rho)f\,d\rho = \lambda \int_0^\infty e^{-\lambda\rho} S(\rho)f\,d\rho, \tag{6.3}$$

it follows that $L \supset \lambda R(\lambda)X$, $\lambda > \omega$. By the resolvent property, which reads as follows

$$R(\lambda_1) - R(\lambda_2) = (\lambda_2 - \lambda_1) R(\lambda_1) R(\lambda_2), \quad \lambda_1, \lambda > \omega,$$

we deduce that the subspaces $\lambda R(\lambda)X$ do not depend on $\lambda > \omega$. Let $f \in X$. Then by assumption (ii') we see that

$$\lim_{\lambda\to\infty} \langle \lambda R(\lambda)f, f^* \rangle = \lim_{\lambda\to\infty} \left\langle \lambda \int_0^\infty e^{-\lambda t} S(t)f\,dt, f^* \right\rangle$$
$$= \lim_{\lambda\to\infty} \int_0^\infty e^{-t} \left\langle S\left(\frac{t}{\lambda}\right) f, f^* \right\rangle dt = \langle f, f^* \rangle. \tag{6.4}$$

From (6.4) it follows that the subspace $\lambda R(\lambda)X$ is weakly dense in X. But a weakly dense subspace is strongly dense. So we conclude that $L = X$. Altogether this proves assertion (ii).

Assertion (iii) follows from the first. This completes the proof of Proposition 6.4. □

6.7. Theorem. *Suppose that A_0 generates the semigroup $\{S_0(t) : t \geqslant 0\}$ and that A_1 generates the semigroup $\{S_1(t) : t \geqslant 0\}$. If A_0 extends A_1,* i.e., *if $G(A_0) \supseteq G(A_1)$, then $S_0(t) = S_1(t)$ and $A_0 = A_1$.*

Proof. For $f \in D(A_1)$ we notice the following Duhamel's formula (variation of constants formula)

$$(S_0(t) - S_1(t))\, f = \int_0^t S_0(u)\,(A_0 - A_1)\, S_1(t-u) f du.$$

This equality follows from

$$\int_0^t S_0(u)\,(A_0 - A_1)\, S_1(t-u) f du = \int_0^t \frac{\partial}{\partial u} S_0(u) S_1(t-u) f du$$
$$= S_0(t)f - S_1(t)f.$$

Here we used the closed graph theorem to be sure that, for f belonging to $D(A_1)$, the function $u \mapsto S_0(u)\,(A_0 - A_1)\, S_1(t-u)f$ is continuous. So that the rule of fundamental calculus is available. This finishes the proof of Theorem 6.7 □

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6.8. Remark. Theorem 6.7 says that a semigroup is uniquely determined by its generator. More information on linear operator semigroup theory can be found in, *e.g.* [**33, 47, 48, 97, 139**]; for non-linear semigroup theory the reader is referred to *e.g.* [**12**].

6.9. Remark. An alternative proof of Theorem 6.7 reads as follows. Fix $x_0 \in D(A_0)$. Choose $\lambda_0 > \omega_0$, and $\lambda_1 > \omega_1$. (We suppose that $\|S_j(t)\| \leqslant M_j \exp(\omega_j t)$, $j = 0, 1$.) Then $(\lambda_0 I - A_0)\, x_0 = (\lambda_0 I - A_1)\, x$, for some $x \in D(A_1)$. Since A_0 extends A_1, we get $(\lambda_0 I - A_0)\, x_0 = (\lambda_0 I - A_1)\, x = (\lambda_0 I - A_0)\, x$. Hence $(\lambda_0 I - A_1)(x_0 - x) = 0$. So that $x_0 = x$. Consequently, $x_0 = x \in D(A)$. Whence $D(A_0) \subseteq D(A)$ and thus $G(A_0) \subseteq G(A)$. So we see $A = A_0$.

A detailed account of the following theorem can be found in Engel and Nagel [**48**].

6.10. Theorem (Hille-Yosida). *Let A be a closed linear operator with a domain that is dense in the Banach space X. The following assertions are equivalent:*

(i) *The operator A generates a strongly continuous semigroup $\{S(t) : t \geqslant 0\}$;*
(ii) *There exist finite constants M and ω such that*

$$\left\|(\lambda I - A)^{-n}\right\| \leqslant M(\lambda - \omega)^{-n}, \quad n = 1,\ 2, \ldots, \lambda > \omega.$$

Proof. Outline of a proof (i) $\Rightarrow$ (ii). Use

$$\Gamma(n)\,(\lambda I - A)^{-n} = \int_0^\infty t^{n-1} \exp(-\lambda t)\, S(t)\, dt.$$

(ii) $\Rightarrow$ (i). Prove that the strong operator limit

$$S(t) = \text{s-}\lim_{\lambda\to\infty} \exp\left(\lambda t\left(\lambda R(\lambda) - I\right)\right)$$

exists and that the family $\{S(t) : t \geqslant 0\}$ is a strongly continuous semigroup with generator A. Here we wrote $R(\lambda) = (\lambda I - A)^{-1}$. In case we deal with contraction semigroups, more details can be found in the proof of the implication (ii) $\Longrightarrow$ (i) of Theorem 6.12 below.

This concludes the outline of the proof of Theorem 6.10. □

6.11. Remark. If A satisfies (ii) of the previous theorem, then $\|S(t)\| \leqslant M \exp(\omega t)$, $t \geqslant 0$.

6.12. Theorem (Lumer-Phillips, Hille-Yosida for contraction semigroups). *Let A be a linear operator with domain $D(A)$ and range $R(A)$ in a Banach space X. The following assertions are equivalent:*

(i) *The operator A generates a strongly continuous semigroup $\{S(t) : t \geqslant 0\}$ for which $\|S(t)\| \leqslant 1$, $t \geqslant 0$;*
(ii) *The operator A has dense domain, there exists $\lambda > 0$ such that $R(\lambda I - A) = X$, and A is dissipative: $\|\lambda f - Af\| \geqslant \lambda \|f\|$, $f \in D(A)$.*

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The property in assertion (ii) of Theorem 6.12 is called the *dissipativity* property. For applications the following somewhat stronger version of Theorem 6.12 is often useful.

PROOF. (i) $\Longrightarrow$ (ii) This implication is not so difficult. Define, for $\lambda > 0$, the operator $R(\lambda)$ is defined by $R(\lambda)f = \int_0^\infty e^{-\lambda t} S(t) f \, dt$, $f \in X$. Then $R(\lambda) = (\Lambda I - A)^{-1}$ in the sense of assertion (i) and (ii) in Theorem 6.7. It follows that the range of $(\lambda I - A)$ coincides with X for all $\lambda > 0$, and that $R(\lambda)f$ belongs to $D(A)$ for all $\lambda > 0$. Moreover, by the strong continuity of the semigroup $\{S(t) : t \geqslant 0\}$ we see

$$\lim_{\lambda \to \infty} \lambda R(\lambda) f = \lim_{\lambda \to \infty} \int_0^\infty e^{-t} S\left(t\lambda^{-1}\right) f \, dt = f, \quad f \in X, \tag{6.5}$$

and so $D(A)$ is dense in X. In addition, we have, for $g \in X$ and $\lambda > 0$,

$$\lambda \left\| R(\lambda) g \right\| \leqslant \lambda \int_0^\infty e^{-\lambda t} \left\| S(t) g \right\| \, dt \leqslant \lambda \int_0^\infty e^{-\lambda t} \left\| g \right\| \, dt = \left\| g \right\|, \tag{6.6}$$

Put $g = (\lambda I - A) f$. Then (6.6) implies $\|\lambda f - A f\| \geqslant \lambda \|f\|$, $f \in D(A)$. Hence, assertion (ii) is a consequence of (i).

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(ii) $\Longrightarrow$ (i) Let $\lambda_0 > 0$ be such that $(\lambda_0 I - A) D(A) = X$. Then we define the operator $R(\lambda_9)$ by

$$R(\lambda_0)(\lambda_0 I - A) f = f, \quad f \in D(A). \tag{6.7}$$

For $0 < \lambda < 2\lambda_0$ we define the operator $R(\lambda)$ by

$$R(\lambda) = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n R(\lambda_0)^{n+1}. \tag{6.8}$$

From the dissipativity property in (ii) it follows that $\lambda_0 \|R(\lambda_0)\| \leqslant 1$. Therefore, the series in (6.8) converges for $|\lambda - \lambda_0| < \lambda_0$, or what amounts to the same for $0 < \lambda < 2\lambda_0$. The equalities

$$(\lambda I - A) R(\lambda) f = f, \quad f \in X, \text{ and } R(\lambda (\lambda I - A) f = f, \quad f \in D(A), \tag{6.9}$$

easily follow for $0 < \lambda < 2\lambda_0$. In other words, $R(\lambda) = (\lambda I - L)^{-1}$, $0 < \lambda < 2\lambda_0$. This procedure can be repeated for any $0 < \lambda < 2\lambda_0$ instead of λ_0. The result will be that the inverse operator $R(\lambda) := (\lambda I - A)^{-1}$ exists for every $0 < \lambda < 4\lambda_0$ and that for such λ the inequality $\lambda \|R(\lambda)\| \leqslant 1$ holds. Again repeating these arguments yields the existence of $R(\lambda) := (\lambda I - A)^{-1}$ for $0 < \lambda < 8\lambda_0$. Again we have $\lambda \|R(\lambda)\| \leqslant 1$, $0 < \lambda < 8\lambda_0$. By repeating these arguments often enough we obtain a resolvent family $R(\lambda) = (\lambda I - A)^{-1}$, $\lambda > 0$, such that $\lambda \|R(\lambda)\| \leqslant 1$ for $\lambda > 0$. We still need to construct a semigroup $\{S(t) : t \geqslant 0\}$ with generator A. To this end we introduce the operators $A(\lambda)$, $\lambda > 0$, by

$$A(\lambda) = \lambda^2 R(\lambda) - \lambda I = \lambda A R(\lambda). \tag{6.10}$$

For $f \in D(A)$, and λ, $\mu > 0$, we have the following equalities:

$$\left(e^{tA(\lambda)} - e^{tA(\mu)}\right) f = \int_0^t e^{(t-s)A(\lambda)} \{A(\mu) - A(\lambda)\} e^{sA(\mu)} f \, ds$$

(the operators $A(\lambda)$ and $A(\mu)$ commute)

$$= \int_0^t e^{(t-s)A(\lambda)} e^{sA(\mu)} \{A(\mu) - A(\lambda)\} f \, ds$$

(employ the identities in (6.10))

$$= \int_0^t e^{(t-s)A(\lambda)} e^{sA(\mu)} A(\mu R(\mu) - \lambda R(\lambda)) f \, ds$$

(the vector f belongs to $D(A)$)

$$= \int_0^t e^{(t-s)A(\lambda)} e^{sA(\mu)} \{\mu R(\mu) - \lambda R(\lambda)\} A f \, ds. \tag{6.11}$$

Observe that $\left\|e^{\rho A(\lambda')}\right\| \leqslant 1$, $\rho \geqslant 0$, $\lambda' > 0$. So from (6.11) we infer

$$\left\|\left(e^{tA(\lambda)} - e^{tA(\mu)}\right) f\right\| \leqslant \int_0^t \left\|e^{(t-s)A(\lambda)} e^{sA(\mu)} \{\mu R(\mu) - \lambda R(\lambda)\} A f\right\| ds$$

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$$\leqslant \int_0^t \left\| e^{(t-s)A(\lambda)} e^{sA(\mu)} \right\| \cdot \left\| \left\{ \mu R(\mu) - \lambda R(\lambda) \right\} Af \right\| \, ds$$
$$\leqslant \int_0^t \left\| \left\{ \mu R(\mu) - \lambda R(\lambda) \right\} Af \right\| \, ds = t \left\| \left\{ \mu R(\mu) - \lambda R(\lambda) \right\} Af \right\|. \tag{6.12}$$

We consider the subspace L_1 defined by

$$L_1 = \left\{ f \in X : \lim_{\lambda \to \infty} \lambda R(\lambda) f = f \right\}.$$

Since $\lambda \|R(\lambda)\| \leqslant 1$ the space L_1 is closed in X. Let $\lambda_0 > 0$ be as in (ii), and pick $g \in X$. Then by the resolvent equation we have, for $\lambda \neq \lambda_0$,

$$(\lambda R(\lambda) - I) R(\lambda_0) g = \frac{\lambda}{\lambda - \lambda_0} \left(R(\lambda_0) - R(\lambda) \right) g - R(\lambda_0) g. \tag{6.13}$$

From (6.13) we get

$$\lim_{\lambda \to \infty} (\lambda R(\lambda) - I) R(\lambda_0) g = 0. \tag{6.14}$$

Since $D(A) = R(\lambda_0) X$, from (6.14) it follows that the space L_1 contains the subspace $D(A)$. Since the subspace L_1 is closed, it contains the closure of $D(A)$. By assumption $D(A)$ is dense, and thus $L_1 = X$.

Next consider the subspace L_2 defined by

$$L_2 = \left\{ f \in X : \lim_{\lambda, \mu \to \infty} \sup_{t \in [0,T]} \left\| \left(e^{tA(\lambda)} - e^{tA(\mu)} \right) f \right\| = 0 \text{ for all } 0 < T < \infty \right\}.$$

Since $\left\| e^{tA(\lambda)} \right\| \leqslant 1$, $t \geqslant 0$, $\lambda > 0$, it follows that the space L_2 is closed. Since the space L_1 coincides with X, the inequality in (6.12) implies that L_2 contains the subspace $D(A)$. The subspace $D(A)$ being dense implies that the subspace L_2 coincides with X. Put $S(t)f = \lim_{\lambda \to \infty} e^{tA(\lambda)} f$, $f \in X$. Since the subspace L_2 coincides with X, it follows that

$$\lim_{\lambda \to \infty} \sup_{0 \leqslant t \leqslant T} \left\| e^{tA(\lambda)} f - S(t) f \right\| = 0 \quad \text{for } f \in X \text{ and for all } 0 < T < \infty. \tag{6.15}$$

From (6.15) we infer that the family of operators $\{S(t) : t \geqslant 0\}$ inherits the semigroup property from the families $\left\{ e^{tA(\lambda)} : t \geqslant 0 \right\}$, $\lambda > 0$. For the same reason it is a strongly continuous semigroup. Since $\left\| e^{tA(\lambda)} \right\| \leqslant 1$ we get $\|S(t)\| \leqslant 1$. By letting $\lambda \to \infty$ in the equality

$$e^{tA(\lambda)} f - f = \int_0^t e^{\rho A(\lambda)} A(\lambda) f \, d\rho = A \left(\lambda R(\lambda) \int_0^t e^{\rho A(\lambda)} f \, d\rho, \quad f \in X, \tag{6.16}$$

we obtain that, for $f \in X$, the integral $\int_0^t S(\rho) f \, d\rho$ belongs to $D(A)$ and that

$$S(t) f - f = A \int_0^t S(\rho) f \, d\rho, \quad t > 0, \quad f \in X. \tag{6.17}$$

Let A_0 be the generator of the semigroup $\{S(t) : t \geqslant 0\}$. If f belongs to $D(A_0)$, then (6.17) implies that f belongs to $D(A)$, and that $Af = A_0 f$. In other words A is an extension of A_0. Let f belong to $D(A)$ and fix $\lambda > 0$. Then the operator

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$\lambda I - A_0$ is surjective. It follows that there exists $f_0 \in D(A_0)$ such that the following equalities hold:

$$(\lambda I - A) f = (\lambda I - A_0) f_0 = (\lambda I - A) f_0. \tag{6.18}$$

From (6.18) we see that the vector $f - f_0$ belongs to the zero space of $\lambda I - A$. Since the operator A is dissipative we infer $f = f_0$, and hence a vector in $D(A)$ belongs to $D(A_0)$. However, all this implies that the operators A and A_0 are the same. So that A is the generator of a strongly continuous semigroup.

The proof of Theorem 6.12 is complete now. $\square$

6.13. Theorem (Lumer-Phillips). *Let A be linear a operator with domain $D(A)$ and range $R(A)$ in a Banach space X. The following assertions are equivalent:*

(i) *The operator A is closable and its closure generates a strongly continuous semigroup $\{S(t) : t \geqslant 0\}$ for which $\|S(t)\| \leqslant 1$, $t \geqslant 0$;*
(ii) *The domain $D(A)$ of A is dense, $\|\lambda f - Af\| \geqslant \lambda \|f\|$, for all $\lambda > 0$, and for all $f \in D(A)$, and there exists $\lambda_0 > 0$ for which $R(\lambda_0 I - A)$ is dense in X.*

6.14. Remark. Usually the range property is the difficult part to verify. Assertion (i) in Theorem 6.13 says that the subspace $D(A)$ is a core for $\overline{A}$, the closure of the operator A. If the operator A satisfies the equivalent conditions in Theorem 6.13, then its closure $\overline{A}$ satisfies the equivalent conditions in Theorem 6.12.

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Proof of Theorem 6.13. (i) $\Longrightarrow$ (ii) From assertion (a) in Proposition 6.17 it follows that the operator A is closable. Let $\overline{A}$ be the closure of A. Then by (i) $\overline{A}$ generates a strongly continuous semigroup $\{S(t) : t \geqslant 0\}$ consisting of contraction operators, and so $D(A)$ is dense. Since $\|S(t)\| \leqslant 1$, $t \geqslant 0$, it follows that $\lambda \|R(\lambda)\| \leqslant 1$, $\lambda > 0$, where

$$R(\lambda)f = \left(\lambda I - \overline{A}\right)^{-1} f = \int_0^\infty e^{-\lambda t} S(t) f\, dt, \quad f \in X.$$

But then it easily follows that $\overline{A}$ is dissipative: see Definition 6.15 below. Since $\left(\lambda I - \overline{A}\right) D\left(\overline{A}\right) = X$, it also follows that, for all $\lambda > 0$, the ranges of $\lambda I - A$ are dense in X.

(ii) $\Longrightarrow$ (i) From assertion (a) in Proposition 6.17 below we see that, under the assumptions in (ii) the operator A is closable. Let $\overline{A}$ be its closure. Then, as is readily verified, the operator $\overline{A}$ possesses the properties described in (ii) of Theorem 6.12. An application of Theorem 6.12 then shows that $\overline{A}$ generates a strongly continuous semigroup consisting of contraction operators.

The proof of Theorem 6.13 is now complete. □

6.15. Definition. Some definitions follow.

(a) As mentioned earlier an operator A with domain $D(A)$ and range $R(A)$ in the Banach space $(X, \|\cdot\|)$ is called *dissipative* if

$$\|\lambda f - Af\| \geqslant \lambda \|f\|, \quad \lambda > 0, \quad f \in D(A).$$

(b) Let E be second countable locally compact Hausdorff space. If in (a) the symbol X denotes the space $C_0(E)$, supplied with the supremum norm, then A is said to satisfy the *maximum principle* if, for every $f \in D(A)$, for which $\sup_{x\in E} \Re f(x) > 0$, there exists $x_0 \in E$ for which $\sup_{x\in E} \Re f(x) = \Re f(x_0)$ and for which $\Re Af(x_0) \leqslant 0$.

(c) If in (b) the space E is compact, then the maximum principle is phrased as follows. For every $f \in D(A)$, there exists $x_0 \in E$ with $\sup_{x\in E} \Re f(x) = \Re f(x_0)$ for which $\Re Af(x_0) \leqslant 0$.

6.16. Remark. An operator A that satisfies the maximum principle can be considered as kind of a generalized second order differential operator. Often this kind of operator is a pseudo-differential operator of order between 0 and 2.

Next we specialize to $X = C_0(E)$, equipped with the supremum norm: $\|f\|_\infty = \sup_{x\in E} |f(x)|$, $f \in C_0(E)$. The space E is supposed to be a second countable (*i.e.* it is a topological space with a countable base for its topology) locally compact Hausdorff space (in particular it is a Polish space). A second-countable locally-compact Hausdorff space is Polish. Let $(U_i)_i$ be a countable basis of open subsets with compact closures, choose for each $i \in \mathbb{N}$, $y_i \in U_i$, together with a continuous

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function $f_i : E \to [0,1]$ such that $f_i(y_i) = 1$ and such that $f_i(y) = 0$ for $y \notin U_i$. Since a locally compact Hausdorff space is completely regular this choice is possible. Put

$$d(x,y) = \sum_{i=1}^{\infty} 2^{-i} |f_i(x) - f_i(y)| + \left| \frac{1}{\sum_{i=1}^{\infty} 2^{-i} f_i(x)} - \frac{1}{\sum_{i=1}^{\infty} 2^{-i} f_i(y)} \right|, \quad x, y \in E.$$

This metric gives the same topology, and it is not too difficult to verify its completeness. For this notice that the sequence $(f_i)_i$ separates the points of E, and therefore the algebraic span (*i.e.* the linear span of the finite products of the functions f_i) is dense in $C_0(E)$ for the topology of uniform convergence. A proof of the fact that a locally compact space is completely regular can be found in Willard [**154**] Theorem 19.3. The connection with Urysohn's metrization theorem is also explained. A related construction can be found in Garrett [**53**]: see Dixmier [**39**] Appendix V as well.

6.17. Proposition. *The following assertions are true.*

(a) *Suppose that the operator A is dissipative and that its range is contained in the closure of its domain. Then the operator A is closable.*
(b) *If the operator A satisfies the maximum principle, then A is dissipative.*

Proof. (a) Let $(f_n) \subset D(A)$ be any sequence with the following properties:

$$\lim_{n\to\infty} f_n = 0, \quad \text{and} \quad g = \lim_{n\to\infty} A f_n$$

exists in $C_0(E)$. Then we consider

$$\left\|(\lambda f_n + g_m) - \lambda^{-1} A(\lambda f_n + g_m)\right\|_\infty \geqslant \|\lambda f_n + g_m\|_\infty,$$

where $(g_m) \subset D(A)$ converges to g. First we let n tend to infinity, then λ, and finally m. The result will be $\lim_{m\to\infty} \|g_m - g\|_\infty \geqslant \lim_{m\to\infty} \|g_m\|_\infty = \|g\|_\infty$. Hence $g = 0$.

(b) Let $f \neq 0$ belong to $D(A)$, choose $\alpha \in \mathbb{R}$ and $x_0 \in E$ in such a way that $0 < \|f\|_\infty = \Re \exp(i\alpha) f(x_0) = \sup_{x\in E} \Re \exp(i\alpha) f(x)$, and that $\Re A(\exp(i\alpha) f)(x_0) \leqslant 0$. Then

$$\begin{aligned} \|\lambda f - Af\|_\infty &\geqslant \Re\left(\exp(i\alpha)(\lambda f - Af)(x_0)\right) \\ &= \lambda \Re\left(\exp(i\alpha) f(x_0)\right) - \Re\left(\exp(i\alpha) Af\right)(x_0) \geqslant \lambda \|f\|_\infty. \end{aligned}$$

This completes the proof of Proposition 6.17. □

6.18. Definition. A strongly continuous semigroup $\{S(t) : t \geqslant 0\}$ in $C_0(E)$ is called a *Feller semigroup* or *Feller-Dynkin semigroup* if it possesses the following positivity property: for all $f \in C_0(E)$, for which $0 \leqslant f \leqslant 1$, and for all $t \geqslant 0$, the inequality $0 \leqslant S(t)f \leqslant 1$ is true. Often a Feller semigroup is called a Feller-Dynkin semigroup, because it leaves the space $C_0(E)$ invariant.

6.19. Remark. From the complex linearity and the assumption that $0 \leqslant f \leqslant 1$, $f \in C_0(E)$, implies $0 \leqslant S(t)f \leqslant 1$, it follows that $\|S(t)f\|_\infty \leqslant \|f\|_\infty$, $f \in C_0(E)$.

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6.20. Remark. It often happens that a semigroup $\{S(t) : t \geqslant 0\}$ is defined on a larger space than $C_0(E)$, *e.g.* on the space of bounded Borel measurable functions. We say that $\{S(t) : t \geqslant 0\}$ is a Feller semigroup, or Feller-Dynkin semigroup if it leaves the space $C_0(E)$ invariant (*i.e.* if $S(t)f$ belongs to $C_0(E)$ whenever f does so and whenever $t \geqslant 0$), and if $0 \leqslant f \leqslant 1$, $f \in C_0(E)$, implies $0 \leqslant S(t)f \leqslant 1$, $t \geqslant 0$.

6.21. Remark. There exists a close relationship between Feller semigroups, strong Markov processes, and well-posed martingale problems: see Theorem 6.36 in Section 3.

6.22. Remark. Let $\{S(t) : t \geqslant 0\}$ be a semigroup of linear operators on $C_0(E)$ with the following property: $0 \leqslant f \leqslant 1$, $f \in C_0(E)$, implies $S(t)f \in C_0(E)$ and $0 \leqslant S(t)f \leqslant 1$, for all $t \geqslant 0$. Then the semigroup $\{S(t) : t \geqslant 0\}$ is strongly continuous if and only if, for all f in a subset of $C_0(E)$ with a dense linear span, and for all $x \in E$, the equality

$$\lim_{t\downarrow 0} S(t)f(x) = f(x) \tag{6.19}$$

holds. It suffices to prove, starting from (6.19), that $\lim_{t\downarrow 0} \|S(t)f - f\|_\infty = 0$ for $f \in C_0(E)$, $0 \leqslant f \leqslant 1$. From (6.19) together with Lebesgue's dominated convergence theorem it follows that

$$\lim_{t\downarrow 0} \int S(t)f(x)\, d\mu(x) = \int f(x)\, d\mu(x), \tag{6.20}$$

for all Borel measures μ on E of bounded variation and for all functions $f \in C_0(E)$, $0 \leqslant f \leqslant 1$. Since, by the Riesz representation theorem, every member of the dual space of $C_0(E)$ can be represented by a complex Borel measure of bounded variation, from (6.20) we may deduce that $\text{w-}\lim_{t\downarrow 0} S(t)f = f$. So from (6.19) it follows that $S(t)f$ converges in the weak sense to f, if t decreases to 0. But a weakly continuous semigroup is strongly continuous. Hence (6.19) implies strong continuity. The converse statement is trivial.

6.23. Proposition. *Suppose that the operator A with domain and range in $C_0(E)$ is such that its range is contained in the closure of its domain. Then the following assertions are true:*

(1) *The operator A satisfies the maximum principle.*
(2) *If $f \in D(A)$, and $\lambda > 0$, then the following inequalities hold:*

$$\begin{aligned} &\inf_{x\in E} \Re\left(\lambda f(x) - Af(x)\right) \leqslant \lambda \inf_{x\in E} \Re f(x) \leqslant \lambda \sup_{x\in E} \Re f(x) \\ &\leqslant \sup_{x\in E} \left(\lambda \Re f(x) - \Re Af(x)\right). \end{aligned} \tag{6.21}$$

(3) *The operator A is closable and its closure satisfies the maximum principle.*
(4) *The operator A is closable, and if $f \in D\left(\overline{A}\right)$, and $\lambda > 0$, then the following inequalities hold:*

$$\begin{aligned} &\inf_{x\in E} \Re\left(\lambda f(x) - \overline{A}f(x)\right) \leqslant \lambda \inf_{x\in E} \Re f(x) \leqslant \lambda \sup_{x\in E} \Re f(x) \\ &\leqslant \sup_{x\in E} \left(\lambda \Re f(x) - \Re \overline{A}f(x)\right). \end{aligned} \tag{6.22}$$

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6.24. Remark. If E is compact, then in Proposition 6.23 it is assumed that the constant function $\mathbf{1}$ belongs to the domain of A and that $A\mathbf{1} = 0$.

Proof of Proposition 6.23. Assume that E is locally compact but not compact. When E is compact the proof follows the same lines, and is left to the reader.

(1) $\Longrightarrow$ (2) Let $f \in D(A)$ be such that $\inf_{x\in E} \Re f(x) < 0$. Then, by assertion (1) there exists $x_0 \in E$ such that $\Re f(x_0) = \inf_{x\in E} \Re f(x) < 0$, and $\Re Af(x_0) \geqslant 0$. It follows that

$$\begin{aligned}\inf_{x\in E} \Re\left(\lambda f(x) - Af(x)\right) &\leqslant \Re\left(\lambda f(x_0) - Af(x_0)\right) \leqslant \lambda \Re f(x_0) \\ &= \lambda \inf_{x\in E} \Re f(x). \end{aligned} \tag{6.23}$$

The inequality obtained in (6.23) proves the first inequality in (6.21) in case $f \in D(A)$ is such that $\inf_{x\in E} \Re f(x) < 0$. If $\inf_{x\in E} \Re f(x) \geqslant 0$, then the first inequality in (6.21) is automatically satisfied because the function $\lambda f - Af$ belongs to $C_0(E)$, and hence it vanishes at "∞". The second inequality in (6.21) is trivial, and the third one follows by applying the previous arguments to the function $-f$ instead of f.

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(2) $\Longrightarrow$ (4) From the proof of assertion (b) in Proposition 6.17 it follows that the operator A is dissipative. By assertion (a) in Proposition 6.17 it is closable. Let f belong to $D\left(\overline{A}\right)$. Then there exists a sequence of functions $(f_n)_{n\in\mathbb{N}} \subset D(A)$ such that

$$\lim_{n\to\infty} \|f_n - f\|_\infty = \lim_{n\to\infty} \left\|Af_n - \overline{A}f\right\|_\infty = 0. \tag{6.24}$$

Since each function f_n satisfies inequalities as in (6.21), the inequalities in (6.22) follow by applying (6.24). Consequently, assertion (4) is proved now.

(4) $\Longrightarrow$ (3) Let $f \in D\left(\overline{A}\right)$ be such that $\sup_{x\in E} \Re f(x) > 0$. Then we have to show that there exists $x_0 \in E$ such that $\sup_{x\in E} \Re f(x) = \Re f(x_0)$ and $\Re\overline{A}f(x_0) \leqslant 0$. From the third inequality in (6.22) it follows that there exist points $x_\lambda \in E$ such that

$$\lambda\Re f(x_\lambda) - \Re\overline{A}f(x_\lambda) \geqslant \lambda \sup_{x\in E} \Re f(x). \tag{6.25}$$

Since E is locally compact there exists a point x_∞ in $E \cup \infty$ which is an adherence point of all families $\{x_\lambda : \lambda \geqslant n\}$, $n \in \mathbb{N}$. Upon dividing the left-hand side and right-hand side of (6.25) by $\lambda > 0$, and letting λ tend to ∞, it follows that

$$\Re f(x_\infty) = \sup_{x\in E} \Re f(x) > 0, \tag{6.26}$$

and, consequently, x_∞ belongs to E. From (6.25) it also follows that

$$-\Re\overline{A}f(x_\lambda) \geqslant \lambda \left(\sup_{x\in E} \Re f(x) - \Re f(x_\lambda)\right) \geqslant 0. \tag{6.27}$$

From (6.26) and by letting λ tend to ∞ in (6.27) it follows that the point $x_\infty \in E$ is such that not only (6.26) is satisfied, but that we also have $\Re\overline{A}f(x_\infty) \leqslant 0$. This proves the implication (4) $\Longrightarrow$ (3).

The implication (3) $\Longrightarrow$ (1) being trivial this completes the proof of Proposition 6.23. $\square$

6.25. Theorem (Lumer-Phillips for Feller semigroups). *The following assertions are equivalent:*

(i) *The operator A is closable and its closure generates a Feller semigroup;*
(ii) *The operator A has dense domain, it verifies the maximum principle, and there exists $\lambda > 0$ such that the range of $\lambda I - A$ is dense in $C_0(E)$.*

If A is closable and if A verifies the maximum principle, then so does its closure: see Proposition 6.23.

Proof of Theorem 6.25. We prove the theorem if E is locally compact, and not compact. The compact case is left as an exercise for the reader.

(i) $\Longrightarrow$ (ii) Let $\{S(t) : t \geqslant 0\}$ be the Feller semigroup generated by $\overline{A}$ the closure of A. Then the domain of $\overline{A}$ is dense, and so is the domain of A. Let $f \in D(A)$ be such

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that $\sup_{x\in E}\Re f(x)>0$, and choose x_0 in such a way that $\Re f(x_0)=\sup_{x\in E}\Re f(x)$. Then

$$\begin{aligned}\Re\left(S(t)f(x_0)-f(x_0)\right)&=S(t)\Re f(x_0)-\Re f(x_0)\\&\leqslant\sup_{x\in E}\Re f(x)-\Re f(x_0)\leqslant 0.\end{aligned}\tag{6.28}$$

In (6.28) we divide by $t>0$ and let t tend to 0 to obtain $\Re Af(x_0)\leqslant 0$. In other words the operator A satisfies the maximum principle. In addition, $R\left(\lambda I-\overline{A}\right)=X$, $\lambda>0$, and consequently, the operators $\lambda I-A$ have dense range. So that assertion (ii) follows from (i).

(ii) $\Longrightarrow$ (i) The operator A satisfies the maximum principle. But then, by assertion (b) in Proposition 6.17, it follows that the operator A is dissipative. By assertion (a) in Proposition 6.17 it follows that the operator A is closable. By Theorem 6.13 we deduce that the operator $\overline{A}$ generates a strongly continuous semigroup $\{S(t):\,t\geqslant 0\}$ consisting of operators $S(t)$ which are contractions: $\|S(t)f\|_\infty\leqslant\|f\|_\infty$, $f\in C_0(E)$. We still need to show that this semigroup has the Feller property, *i.e.*, that $0\leqslant f\leqslant\mathbf{1}$ implies $0\leqslant S(t)f\leqslant\mathbf{1}$. Since the operator A satisfies the maximum principle, its closure does so as well: see Proposition 6.23. Fix $\lambda>0$, and let $f\in D\left(\overline{A}\right)$ be such that $\left(\lambda I-\overline{A}\right)f\geqslant 0$. Then, by assertion (4) in Proposition 6.17 it follows that

$$0\leqslant\inf_{x\in E}\Re\left(\lambda I-\overline{A}\right)(if)(x)\leqslant\lambda\inf_{x\in E}\Re(if)(x).$$

So that $-\Im f(x)\geqslant 0$ for all $x\in E$. So we have $\Im f(x)\leqslant 0$ for all $x\in E$. The same argument applied to $-if$ instead of if yields $\Im f(x)\geqslant 0$ for all $x\in E$. Consequently, $\Im f(x)=0$ for all $x\in E$. In other words the function f is real valued. Another appeal to assertion (4) in Proposition 6.17 then yields

$$0\leqslant\inf_{x\in E}(\lambda I-A)f(x)\leqslant\lambda\inf_{x\in E}f(x),$$

and so $f(x)\geqslant 0$ for all $x\in E$. As a result we have that $\left(\lambda I-\overline{A}\right)f\geqslant 0$ implies $f\geqslant 0$. Put $R(\lambda)=\left(\lambda I-\overline{A}\right)^{-1}$. In other words $g\geqslant 0$ implies $R(\lambda)g\geqslant 0$. Hence the resolvent operators $R(\lambda)$, $\lambda>0$, are positivity preserving. Since

$$S(t)f=\lim_{\lambda\to\infty}e^{-\lambda t}e^{t\lambda^2R(\lambda)}f=\lim_{\lambda\to\infty}e^{-\lambda t}\sum_{n=0}^{\infty}\frac{1}{n!}\left(t\lambda^2R(\lambda)\right)^nf$$

it follows that the operators $S(t)$ are positivity preserving. Hence, $0\leqslant f\leqslant\mathbf{1}$, $f\in C_0(E)$, implies $S(t)f\geqslant 0$, and since $S(t)$ is a contraction, it also follows that $S(t)f\leqslant\mathbf{1}$.

This completes the proof of Theorem 6.25. □

We close Section 1 with a presentation of a result on initial value problems, which is also relevant in system theory. Initial value problems are also called Cauchy problems. The result is due to J. Ball [**9**]. The function f belongs to the space $C([0,\infty),X)$.

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6.26. THEOREM. *Let A be a linear operator with domain and range in a Banach space X. The following assertions are equivalent:*

(i) *The operator A generates a strongly continuous semigroup $\{S(t) : t \geqslant 0\}$;*

(ii) *The operator A has dense domain, it is closed, and for every $x \in D(A)$ there exists a unique function $u_x \in C^1([0,\infty), X)$ such that, for all $t \geqslant 0$, $u_x(t)$ belongs to $D(A)$, the function $(x,t) \mapsto u_x(t)$ is continuous and*

$$u_x'(t) = Au_x(t) + f(t), \quad u_x(0) = x. \tag{6.29}$$

(iii) *The operator A is closed and for every $x \in X$ there exists a unique function $v_x \in C^1([0,\infty), X)$ such that, for all $t \geqslant 0$, $v_x(t)$ belongs to $D(A)$, and*

$$v_x'(t) = x + Av_x(t) + \int_0^t f(s)\,ds, \quad v_x(0) = 0. \tag{6.30}$$

(iv) *The operator A has dense domain and is closed and for every $x \in X$ there exists a unique weakly continuous function w_x, such that, for all $x^* \in D(A^*)$ the equality*

$$\frac{d}{dt}\langle w_x(t), x^*\rangle = \langle x, x^*\rangle + \langle w_x(t), A^*x^*\rangle + \int_0^t \langle f(s), x^*\rangle\,ds, \quad w_x(0) = 0. \tag{6.31}$$

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Outline of a proof of Theorem 6.26. (i) $\Rightarrow$ (ii). Put $u_x(t) = S(t)x + \int_0^t S(t-s)f(s)\,ds$.

(ii) $\Rightarrow$ (iii). Put $v_x(t) = \int_0^t u_x(s)\,ds$.

(iii) $\Rightarrow$ (iv). Put $w_x(t) = v_x(t)$.

(iv) $\Rightarrow$ (i). Put

$$\Phi(t,x,x^*) = \frac{d}{dt}\langle w_x(t), x^*\rangle - \frac{d}{dt}\langle w_0(t), x^*\rangle. \tag{6.32}$$

Then prove that there exists a family $\{S(t) : t \geqslant 0\}$ of continuous linear operators such that

$$\langle S(t)x, x^*\rangle = \Phi(t,x,x^*) = \frac{d}{dt}\langle w_x(t), x^*\rangle - \frac{d}{dt}\langle w_0(t), x^*\rangle. \tag{6.33}$$

Finally prove that the family $\{S(t) : t \geqslant 0\}$ is a strongly continuous semigroup with generator A. All this can be achieved as follows. From our assumptions and definitions, it follows that

$$\langle S(t)x, x^*\rangle - \langle x, x^*\rangle = \left\langle \int_0^t S(s)x ds, A^*x^*\right\rangle. \tag{6.34}$$

From (6.34) we see that the element $\int_0^t S(s)xds$ belongs to $D(A)$, and that

$$A\int_0^t S(s)xds = S(t)x - x.$$

The semigroup property is a consequence of the identity

$$w_x'(t_1+t_2) - w_0'(t_1+t_2) = w'_{w_x'(t_1)-w_0'(t_1)}(t_2) - w_0'(t_2). \tag{6.35}$$

Equality (6.35) can be seen by considering two solutions to the equation in (iv):

$$\begin{aligned} w_{1,x}(t) &= w_x(s+t) - w_x(s) - w_0(s+t) + w_0(s); \\ w_{2,x}(t) &= w_{w_x'(s)-w_0'(s)}(t) - w_0(t). \end{aligned}$$

Put $w(t) = w_{2,x}(t) - w_{1,x}(t)$. Then $w'(t) = Aw(t)$, and $w(0) = 0$. Next consider the following two equations:

$$\begin{aligned} w_x'(t) &= x + Aw_x(t) + \int_0^t f(s)\,ds, \quad w(0) = 0; \\ (w + w_x)'(t) &= x + A(w+w_x)(t) + \int_0^t f(s)\,ds, \quad (w+w_x)(0) = 0. \end{aligned}$$

From the uniqueness in (iv) we get $w + w_x = w_x$ and hence $w = 0$. From the latter we infer $v_{2,x}(t) = v_{1,x}(t)$. As a consequence we obtain the semigroup property: $S(s+t) = S(s)S(t)$. Since $S(t)x - x = w_x'(t) - w_0'(t) - x$ converges weakly to $w_x'(0) - w_0'(0) - x = A(w_x(0) - w_0(0)) = A0 = 0$. It follows that the semigroup $\{S(t); t \geqslant 0\}$ is a weakly continuous semigroup. Such a semigroup is automatically

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strongly continuous. Let $\widetilde{A}$ be its generator, and suppose that x belongs to its domain $D\left(\widetilde{A}\right)$. Then

$$\widetilde{A}x = \lim_{t\downarrow 0} \frac{S(t)x - x}{t} = \lim_{t\downarrow 0} A\frac{\int_0^t S(s)xds}{t} = Ax.$$

As a consequence A is an extension of $\widetilde{A}$. Let x be a member of $D(A)$. Pick λ_0 strictly larger than the growth bound of the semigroup $\{S(t) : t \geqslant 0\}$. Then there exists $x_1 \in D\left(\widetilde{A}\right)$ such that (notice that A extends $\widetilde{A}$)

$$(\lambda_0 I - A)\, x = \left(\lambda_0 I - \widetilde{A}\right) x_1 = (\lambda_0 I - A)\, x_1,$$

and hence $(\lambda_0 I - A)\, x_0 = 0$, where $x_0 = x - x_1$. Put $w_1(t) = \dfrac{e^{\lambda_0 t} - 1}{\lambda_0} x_0$. Then $w_1(0) = 0$ and $w_1'(t) = x_0 + Aw_1(t)$. Since, in addition, $w_0'(t) = Aw_0(t) + \int_0^t f(s)\, ds$, $w_0(0) = 0$, we infer

$$\frac{d}{dt}\left(w_0(t) + w_1(t)\right) = x_0 + A\left(w_0(t) + w_1(t)\right) + \int_0^t f(s)\, ds, \quad w_0(0) + w_1(0) = 0.$$

Since the function $w_{x_0}(t)$ possesses the same property as the function $w_0(t) + w_1(t)$, we infer from (vi) the equality $w_0(t) + w_1(t) = w_{x_0}(t)$ and hence

$$\frac{e^{\lambda_0 t} - 1}{\lambda_0} x_0 = w_{x_0}(t) - w_0(t) \tag{6.36}$$

From (6.36) we obtain, via differentiating with respect to t, $S(t)x_0 = \exp(\lambda_0 t)\, x_0$, $t \geqslant 0$. But then

$$\|x_0\| \exp(\lambda_0 t) \leqslant M \exp(\omega t) \|x_0\|,$$

$t \geqslant 0$, $\omega < \lambda_0$. This can only be possible if $x_0 = 0$, and hence $x = x_1$ belongs to $D\left(\widetilde{A}\right)$. So that, finally, $A = \widetilde{A}$. Hence, the proof of Theorem 6.26 is complete now. □

2. Examples

In this section we present several (interesting) examples of operator semigroups.

2.1. Uniformly continuous semigroups. Let A be a bounded linear operator, and put $S(t) = \sum_{k=0}^{\infty} \dfrac{t^k A^k}{k!}$. The family $\{S(t) : t \geqslant 0\}$ is a strongly continuous semigroup. In fact $\lim_{t\downarrow 0} \|S(t) - I\| = 0$. From the closed graph theorem it follows that a strongly continuous semigroup $\{S(t) : t \geqslant 0\}$ is uniformly continuous (*i.e.* $\lim_{t\downarrow 0} \|S(t) - I\| = 0$) if and only if $S(t) = \exp(tA)$, $t \geqslant 0$, for some bounded linear operator A. The sufficiency is easy to establish. The necessity follows from the closed graph theorem. If $\lim_{t\downarrow 0} \|S(t) - I\| = 0$, then $\lim_{\lambda\to\infty} \|I - \lambda R(\lambda)\| = 0$. Hence, for $\lambda > 0$ sufficiently large we get $\|I - \lambda R(\lambda)\| < 1$. But then the operator

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$\lambda R(\lambda)$ possesses an everywhere defined inverse. As a consequence the generator A of $\{S(t) : t \geqslant 0\}$, which has a closed graph, is everywhere defined. Therefore it is bounded and hence $S(t) = \exp(tA)$.

2.2. Self-adjoint semigroups. Let $H = H^* = \int_{-\omega}^{\infty} \xi E(d\xi)$ be a self-adjoint linear operator in a Hilbert space $\mathcal{H}$, with lower bound $-\omega$. Such an operator H generates the semigroup $\{\exp(-tH) : t \geqslant 0\}$, where $\exp(-tH) = \int_{-\omega}^{\infty} \exp(-t\xi)\, E(d\xi)$. If the operator H possesses a discrete spectrum $\{\lambda_j : j \in \mathbb{N}\}$, then $\exp(-tH) = \sum_{k=1}^{\infty} \exp(-t\lambda_j)\, E\{\lambda_j\}$. The operators $E\{\lambda_j\}$ are the orthogonal projections on the eigenspaces $N(\lambda_j I - H)$. The semigroup $t \mapsto \exp(-tH)$, $t \geqslant 0$, can be extended to $z \to \exp(-zH)$, $\Re z \geqslant 0$. Hence the semigroup $\{\exp(-tH) : t \geqslant 0\}$ extends to an analytic semigroup on $\{z \in \mathbb{C} : \Re z > 0\}$. Of course the mappings $s \mapsto \exp(-isH)$, $s \in \mathbb{R}$, are unitary groups on $\mathcal{H}$.

2.3. Translation group. Let A be the operator $A = \dfrac{d}{dx}$ in $C_0(\mathbb{R})$ or in $L^p(\mathbb{R})$, $1 \leqslant p < \infty$. The corresponding semigroup is given by $\exp(tA)\, f(x) = f(x + t)$, $x \in \mathbb{R}$, $t \geqslant 0$. This semigroup extends to a group in any of the above spaces. It is not strongly continuous in the space $L^\infty(\mathbb{R})$. In fact, a result due to Lotz (see [1]) says that a semigroup $\{S(t) : t \geqslant 0\}$ is strongly continuous in $L^\infty(\mathbb{R})$ if and only if its generator is an everywhere defined bounded linear operator (such semigroups are necessarily uniformly continuous: see Example 2.1). The space $\mathbb{R}$ may be replaced with any locally compact second countable Hausdorff space. Upon replacing the above mentioned spaces with other spaces on which there exists a (semi)group action, the translation (semi)groups serve as a source of examples and counter-examples (compare with the one-sided and two-sided shift in the discrete setting).

2.4. Gaussian semigroup. Let $H_0 = -\frac{1}{2}\Delta$ in $C_0(\mathbb{R}^\nu)$, or in $L^p(\mathbb{R}^\nu)$, $1 \leqslant p < \infty$. Put

$$p_{0,\nu}(t, x, y) = \frac{1}{\left(\sqrt{2\pi t}\right)^\nu} \exp\left(-\frac{|x - y|^2}{2t}\right).$$

(This function is the so-called heat or Gaussian kernel.) Then $-H_0$ generates the semigroup $\{\exp(-tH_0) : t \geqslant 0\}$ given by

$$\exp(-tH_0)\, f(x) = \int p_{0,\nu}(t, x, y) f(y)\, dy.$$

The semigroup property is clear from the equality

$$p_{0,\nu}(s, x, z)p_{0,\nu}(t, z, y) = p_{0,\nu}(s + t, x, y)p_{0,\nu}\left(\frac{st}{s + t}, \frac{sy + tx}{s + t}, z\right).$$

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2.5. Wave operator. In $L^2(0,1)$ we consider the following Cauchy problem or initial value problem:

$$\frac{\partial^2}{\partial t^2}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x), \quad u(t,0) = u(t,1) = 0;$$
$$\frac{\partial}{\partial t}u(0,x) = g(x), \quad u(0,x) = f(x).$$

Put $v_1(t,x) = u(t,x)$, $v_2(t,x) = \dfrac{\partial u}{\partial t}(t,x)$. Then

$$\frac{\partial}{\partial t}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad \begin{pmatrix} v_1(0) \\ v_2(0) \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}. \tag{6.37}$$

We consider this equation in the space $\mathcal{H} = H_0^1(0,1) \times L^2(0,1)$, supplied with the inner-product

$$\left\langle \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\rangle_{\mathcal{H}} = \int_0^1 \frac{\partial v_1(x)}{\partial x}\frac{\overline{\partial w_1(x)}}{\partial x}dx + \int_0^1 v_2(x)\overline{w_2(x)}dx.$$

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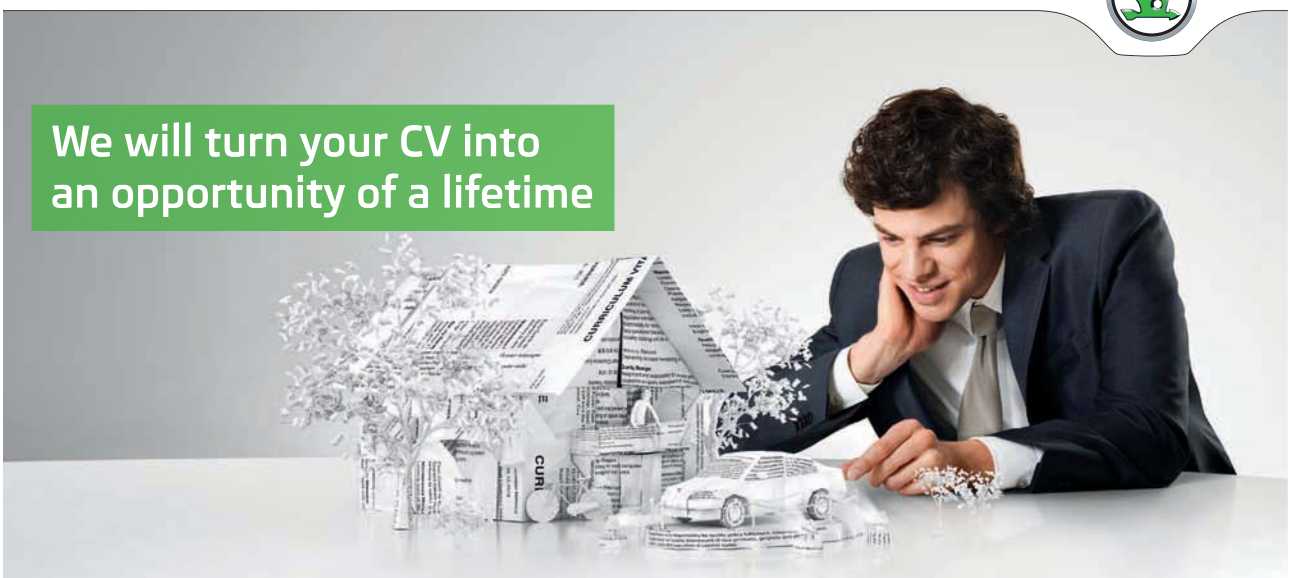

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For the notion of $H_0^1(0,1)$ see Definition 4.38. Put $\mathcal{A} = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$, and put $\varphi_n(x) = \sqrt{2}\sin n\pi x$. A solution to (6.37) is given by the semigroup $\exp(t\mathcal{A})$:

$$\exp(t\mathcal{A})\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \sum_{n=1}^{\infty}\sqrt{2}\left[\langle f,\varphi_n\rangle \cos n\pi t + \frac{1}{n\pi}\langle g,\varphi_n\rangle \sin n\pi t\right]\varphi_n \\ \sum_{n=1}^{\infty}\sqrt{2}\left[-n\pi\langle f,\varphi_n\rangle \sin n\pi t + \langle g,\varphi_n\rangle \cos n\pi t\right]\varphi_n \end{pmatrix}.$$

Moreover we have

$$\left\|\exp(t\mathcal{A})\begin{pmatrix} f \\ g \end{pmatrix}\right\|_{\mathcal{H}} = \left\|\begin{pmatrix} f \\ g \end{pmatrix}\right\|_{\mathcal{H}}, \quad (\exp(t\mathcal{A}))^* = \exp(-t\mathcal{A}).$$

Hence $\{\exp(t\mathcal{A}) : t \geqslant 0\}$ extends to a unitary group on the Hilbert space $\mathcal{H}$.

2.6. Adjoint semigroups. If A is the generator of the strongly continuous semigroup $\{S(t) : t \geqslant 0\}$ in the reflexive Banach space X, then its adjoint A^* generates the strongly continuous semigroup $\{S(t)^* : t \geqslant 0\}$ in the Banach space X^*. If X is not reflexive, then then $\{S(t)^* : t \geqslant 0\}$ need not be strongly continuous, even if $\{S(t) : t \geqslant 0\}$ is. Many semigroups, that are strongly continuous in $L^1(\mathbb{R}^\nu)$, possess adjoints in $L^\infty(\mathbb{R}^\nu)$, which are not strongly continuous. (By Lotz' result, generators of strongly continuous semigroups in $L^\infty(\mathbb{R}^\nu)$ have to be bounded: see [**5**].)

2.7. Dyson-Phillips expansion. If the operator A generates the semigroup

$$\{S(t) : t \geqslant 0\},$$

then $A + B$, where B is bounded linear operator, generates the semigroup:

$$t \mapsto S(t) + \sum_{n=1}^{\infty}\int_{0<s_1<\dots<s_n<t}\int ds_1 \dots ds_n S(s_1)BS(s_2 - s_1)\dots S(s_n - s_{n-1})BS(t - s_n).$$

This is the *Dyson-Phillips expansion* of $\exp(t(A+B))$. This formula is an iteration of the *Duhamel's* or *variation of constants* formula:

$$\exp(t(A+B)) = \exp(tA) + \int_0^t \exp(sA)\, B \exp((t-s)(A+B))\, ds.$$

Extensions to non-necessary bounded operators B are possible.

2.8. Stone's theorem. A family of unitary operators $\{U(t) : t \in \mathbb{R}\}$ on a Hilbert space is a strongly continuous group if and only if there exists a self-adjoint linear operator $H = H^* = \int \xi E(d\xi)$ such that $U(t) = \exp(itH) = \int \exp(it\xi)\, E(d\xi)$.

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2.9. Convolution semigroups of measures. Let $\{\mu_s : s \geqslant 0\}$ be a vaguely continuous semigroup of Borel probability measures on $\mathbb{R}^\nu$. This means the following:

(i) $\mu_0 = \delta_0$ (Dirac measure at 0, the origin of $\mathbb{R}^\nu$) and
$\mu_{s+t}(B) = \iint 1_B(x+y)\, d\mu_s(x)\, d\mu_t(y)$;
(ii) $\lim_{t \downarrow 0} \int f(x)\, d\mu_0(x) = \int f(x)\, d\delta_0(x) = f(0)$, for all $f \in C_0(\mathbb{R}^\nu)$;
(iii) $\mu_t(\mathbb{R}^\nu) = 1$.

Then there exists a continuous negative definite function ψ on $\mathbb{R}^\nu$ ($\mathbb{R}^\nu$ and its dual group are identified) such that

$$\widehat{\mu}_t(\xi) := \int \exp\left(-i \langle x, \xi \rangle\right)\, d\mu_t(x) = \exp\left(-t\psi(\xi)\right), \quad \xi \in \mathbb{R}^\nu, \quad t \geqslant 0.$$

Put $S(t)f(x) = \int f(x-y)\, d\mu_t(y)$. The semigroup $\{S(t) : t \geqslant 0\}$ is a Feller semigroup on $C_0(\mathbb{R}[\nu)$. Every operator $S(t)$, $t \geqslant 0$, commutes with translations on $\mathbb{R}^\nu$: $\tau_x \circ S(t) = S(t) \circ \tau_x$. Here $\tau_x f(y) = f(y-x)$, x, $y \in \mathbb{R}^\nu$. The corresponding Markov processes are the Lévy processes. Particular examples are

(i) $\mu_t(B) = \dfrac{1}{\left(\sqrt{2\pi t}\right)^\nu} \displaystyle\int_B \exp\left(-\frac{|y|^2}{2t}\right) dy, \quad B \subseteq \mathbb{R}^\nu, \quad$ Borel;
(ii) $\mu_t(B) = \dfrac{\Gamma\left((\nu+1)/2\right)}{\pi^{(\nu+1)/2}} \displaystyle\int_B \frac{t}{\left(t^2 + |y|^2\right)^{(\nu+1)/2}}\, dy, \quad B \subseteq \mathbb{R}^\nu, \quad$ Borel.

The first semigroup is called the *heat* or *Gaussian* semigroup, the second one is the *Cauchy* or *Poisson* semigroup. The corresponding negative-definite functions are respectively $\psi(\xi) = \frac{1}{2}|\xi|^2$ (Gaussian semigroup), and $\psi(\xi) = \frac{1}{2}|\xi|$ (Poisson semigroup).

2.10. Semigroups acting on operators. This is a non-commutative version of the example in Subsection 2.9. Again let $\{\mu_t : t \geqslant 0\}$ be a vaguely continuous convolution semigroup of Borel probability measures on $\mathbb{R}$. Let H_0 and H_1 be self-adjoint Hamiltonians in the Hilbert spaces $\mathcal{H}_0$ respectively $\mathcal{H}_1$. Define for $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_0)$, the operator $\exp(-t\mathcal{A})\, T$ via the equality:

$$\exp(-t\mathcal{A})\, T = \int \exp(-i\tau H_0)\, T \exp(i\tau H_1)\, d\mu_t(\tau).$$

On appropriate spaces of linear operators (Hilbert-Schmidt operators, trace class operators, compact operators) the family $\{\exp(-t\mathcal{A}) : t \geqslant 0\}$ is a strongly semigroup. If the spaces $\mathcal{H}_0$ and $\mathcal{H}_1$ are infinite dimensional, it is not strongly continuous on the space of bounded linear operators $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_0)$. This example is related to quantum stochastic processes. In general the operators $\exp(-t\mathcal{A})$ is not completely positive. Only semigroups consisting of completely positive operators correspond to quantum stochastic processes.

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2.11. Quantum dynamical semigroups. A C^*-algebra $\mathcal{M}$ is called a W^*-algebra (or von Neumann algebra) if it is a dual space as a Banach space, *i.e.* if there exists a Banach space $\mathcal{M}_*$ such that $(\mathcal{M}_*)^* = \mathcal{M}$. A Banach space $\mathcal{M}_*$ whose dual is $\mathcal{M}$ is called a predual. Let H be a Hilbert space. Then $\mathcal{L}(H)$, the space of al bounded linear operators on H is a W^*-algebra. Its predual $\mathcal{L}(H)_*$ consists of those linear functionals $f : \mathcal{L}(H) \to \mathbb{C}$ for which there exist two sequences $(x_n)_n$ and $(y_n)_n$ in H such that $\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$, $\sum_{n=1}^{\infty} \|y_n\|^2 < \infty$, and such that

$$f(T) = \sum_{n=1}^{\infty} \langle Tx_n, y_n \rangle, \quad T \in \mathcal{L}(H).$$

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be C^*-algebras, and let $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ be an (algebra) homomorphism, *i.e.*, suppose that $\Phi(\lambda x) = \lambda\Phi(x)$, $\Phi(x + y) = \Phi(x) + \Phi(y)$, $\Phi(xy) = \Phi(x)\Phi(y)$, for $\lambda \in \mathbb{C}$, x, $y \in \mathcal{A}_1$. The homomorphism Φ is called a *-homomorphism, if, in addition, it satisfies $\Phi(x^*) = \Phi(x)^*$, $x \in \mathcal{A}_1$.

6.27. Definition. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be W^*-algebras with preduals $\mathcal{M}_{1*}$ and $\mathcal{M}_{2*}$ respectively, and let $\Phi : \mathcal{M}_1 \to \mathcal{M}_2$ be *-homomorphism. Then Φ is called a W^*-homomorphism provided that it is continuous if $\mathcal{M}_1$ is endowed with the topology $\sigma(\mathcal{M}_1, \mathcal{M}_{1*})$, and if $\mathcal{M}_2$ is endowed with the topology $\sigma(\mathcal{M}_2, \mathcal{M}_{2*})$.

6.28. Proposition. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be W^*-algebras with preduals $\mathcal{M}_{1*}$ and $\mathcal{M}_{2*}$ respectively, and let $\Phi : \mathcal{M}_1 \to \mathcal{M}_2$ be W^*-homomorphism. Then the image $\Phi(\mathcal{M}_1)$ is $\sigma(\mathcal{M}_2, \mathcal{M}_{2*})$-closed, and so $\Phi(\mathcal{M}_1)$ is a W^*-subalgebra of $\mathcal{M}_2$.*

6.29. Definition. Let $\mathcal{A}$ be a C^*-algebra. A C^*-representation of $\mathcal{A}$ is a *-homomorphism π of $\mathcal{A}$ in $\mathcal{L}(H)$ for some Hilbert space H. This C^*-representation is denoted by (π, H).

An important representation theorem for C^*-algebras reads as follows.

6.30. Theorem. *A C^*-algebra $\mathcal{A}$ is C^*-isomorphic and to a uniformly closed self-adjoint subalgebra of $\mathcal{L}(H)$ for some Hilbert space H. Denote this C^*-representation by (π, H). Then $\|\pi(a)\| = \|a\|$, $a \in \mathcal{A}$.*

6.31. Definition. Let $\mathcal{M}$ be a W^*-algebra. A W^*-representation of $\mathcal{M}$ is a W^*-homomorphism π of $\mathcal{A}$ in $\mathcal{L}(H)$ for some Hilbert space H. This W^*-representation is denoted by (π, H).

A representation theorem for W^*-algebras reads as follows.

6.32. Theorem. *Let $\mathcal{M}$ be a W^*-algebra. Then $\mathcal{M}$ has a faithful W^*-representation $\{\pi, H\}$;* i.e. *the representation π is such that $\pi(a) = 0$ if and only if $a = 0$. Therefore $\mathcal{M}$ is W^*-isomorphic to a weakly closed self-adjoint subalgebra of $\mathcal{L}(H)$ for some Hilbert space H. The image $\pi(\mathcal{M})$ is then a W^*-algebra embedded in $\mathcal{L}(H)$, and $\pi(\mathcal{M})$ is W^*-isomorphic to $\mathcal{M}$.*

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Here a subset V of $\mathcal{L}(H)$ is called weakly closed, if it is closed for the topology induced by the semi-norms $T \mapsto |\langle Tx, y\rangle|$, $T \in \mathcal{L}(H)$, x, $y \in H$. The subset V is called self-adjoint, provided $T \in V$ implies $T^* \in V$. For much more details about C^*- and W^*-algebras the reader is referred to Sakai [**116**].

A quantum dynamical semigroup $\{S(t) : t \geqslant 0\}$ is usually defined on a von Neumann algebra $\mathcal{M}$, or a W^*-algebra. It possesses the following properties:

(i) Semigroup property:

$$S(0)(a) = a, \quad S(t+s)(a) = S(t)\big(S(s)(a)\big), \quad \text{for all } a \in \mathcal{M};$$

(ii) The semigroup $\{S(t) : t \geqslant 0\}$ is *completely positive* in the sense that for every $t \geqslant 0$ and for every finite choice of elements belonging to $\mathcal{M}$, $x_1, \dots, x_n$; $y_1, \dots, y_n$, the sum

$$\sum_{j,k=1}^{n} y_j^* S(t)\left(x_j^* x_k\right) y_k$$

is a positive element of $\mathcal{M}$;

(iii) For every $t \geqslant 0$ the operator $S(t)$ is σ-weakly continuous;

(iv) For every $a \in \mathcal{M}$ fixed, the map $t \mapsto S(t)(a)$ is continuous with respect to the σ-weak topology on $\mathcal{M}$.

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The σ-weak continuity is defined via the predual of the W^*-algebra $\mathcal{M}$. Realize the von Neumann algebra $\mathcal{M}$ via the Gelfand-Naimark-Segal construction as a C^*-algebra of operators acting on a Hilbert space $\mathcal{H}$. Let $\mathcal{A}$ be its pre-dual under the duality map:

$$(a, x) \mapsto \operatorname{trace}(x^* a), \quad a \in \mathcal{A}, \quad x \in \mathcal{M}.$$

The trace of a trace class operator $T : \mathcal{H} \to \mathcal{H}$ is defined by

$$\operatorname{trace}(T) = \sum_{j=1}^{\infty} \langle T\xi_j, \xi_j \rangle, \tag{6.38}$$

where the sequence $(\xi_j : j \in \mathbb{N})$ is any complete orthonormal sequence of vectors in $\mathcal{H}$. For trace class operators the sum in (6.38) does not depend on the particular choice of the complete orthonormal sequence $(\xi_j : j \in)$ in $\mathcal{H}$. The operator T is called a *trace class* operator if its trace norm $\|T\|_{\text{trace}}$, defined by

$$\|T\|_{\text{trace}} = \operatorname{trace}(|T|)$$

is finite. Here $|T| = \sqrt{T^*T}$ is the square root of the operator T^*T: see Theorem 5.41. More details on quantum diffusions can be found in: [**16, 67, 92, 96**]. Another book of interest is Alicki and Lendi [**4**].

6.33. Remark. Property (ii) is not shared by Cauchy semigroups. Indeed it should be thought of Cauchy semigroups as an image under a Cauchy (or Poisson) transform. The interesting fact is that this transform associates a generator with the resulting semigroup. As far as we know, the ideas of Poisson and Weierstrass transforms have been studied for the first time by Hille in 1935, but they still enjoy interesting unexplored properties!

6.34. Remark. There are situations, where instead of the logistical law *Cauchy processes* might be more appropriate:

$$\begin{aligned} \exp(-t\mathcal{A})\, T &:= \frac{t}{\pi} \int_{-\infty}^{\infty} \frac{1}{\tau^2 + t^2} V_0(i\tau) T V_1(-i\tau)\, d\tau \\ &= \mathbb{E}^{\text{Cauchy}} \left[V_0(iX(t))\, T V_1(-iX(t)) \right]. \end{aligned} \tag{6.39}$$

The relevant formula is the next one. In [**141**] the central identity was

$$\exp\left(-\frac{t}{2}\mathcal{A}\right) \mathcal{D}(t)(T) = \int_0^t du \exp(-u\mathcal{A})\, V_0(t/2) T V_1(t/2).$$

The basic role of this equality in [**141**] is taken over by the (important) equality in (6.72) in the present book. It is quite well possible, that with the semigroup in Formula (6.39) there can be associated a quantum diffusion. Instead of considering the evolution $\tau \to V_0(i\tau)TV_1(-i\tau)$ one should look at one in the space $\mathcal{H}_0 \times \mathcal{H}_1$ given by $\tau \mapsto V(i\tau)\begin{pmatrix} I & T \\ 0 & I \end{pmatrix} V(-i\tau)$, where $V(i\tau)$, $\tau \in \mathbb{R}$, is the operator matrix $V(i\tau) = \begin{pmatrix} V_0(i\tau) & 0 \\ 0 & V_1(i\tau) \end{pmatrix}$. Here $V_0(i\tau) = e^{-i\tau H_0} = \int e^{-i\tau\xi} E_0(d\xi)$ and $V_1(t) =$

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$e^{-i\tau H_1} = \int e^{-i\tau\eta} E_1(d\eta)$, where $H_0 = \int_{\sigma(H_0)} \lambda\, dE_0(\lambda)$, and $H_1 = \int_{\sigma(H_0)} \lambda\, dE_1(\lambda)$ are the spectral decompositions of the (self-adjoint) Hamiltonians H_0 and H_1.

6.35. Remark (Connections with *double Stieltjes operator integrals*). In [**17, 18, 19, 20**] Birman and Solomyak make a detailed study of operators of the form

$$T \mapsto \int\int \varphi(\xi,\eta) E_0(d\xi) T E_1(d\eta)$$

where φ is an appropriate function. For example we have

$$\exp(-t\mathcal{A})T = \iint \exp\left(-t\left|\xi-\eta\right|\right) E_0(d\xi)TE_1\left(d\eta\right);$$
$$\mathcal{D}(t)T = \iint \frac{\exp(-t\eta)-\exp(-t\xi)}{\xi-\eta} E_0(d\xi)TE_1(d\eta).$$

Here E_0 and E_1 are, not necessarily commuting resolutions of the identity: see 5.27 in Chapter 5. In [**155**], pp 225–228 Yafaev gives some information as well on these so-called *double Stieltjes operator integrals* and so do the authors of [**55**] on page 66.

2.12. Semigroups for system theory. Let $\mathcal{A}_0$ be the generator of the semigroup $\{S_0(t) : t \geqslant 0\}$ in the Banach space X_0 and let $\mathcal{A}_1$ be the generator of the semigroup $\{S_1(t) : t \geqslant 0\}$ in the Banach space X_1. For an operator $B \in \mathcal{L}\left(X_1, X_0\right)$ define the operator $\mathcal{D}(t)B$ by the formula:

$$\mathcal{D}(t)B = \int_0^t S_0(u)BS_1(t-u)\,du.$$

The family $\left\{\begin{pmatrix} S_0(t) & \mathcal{D}(t)B \\ 0 & S_1(t)\end{pmatrix} : t \geqslant 0\right\}$ constitutes a strongly continuous semigroup of continuous linear operators on the space $\mathcal{H}_0 \times \mathcal{H}_1$. Its generator is given by the formula: $\begin{pmatrix} A_0 & B \\ 0 & A_1\end{pmatrix}$. This sort of construction is often used in system theory. See Remark 6.86 as well. For more details see *e.g.* [**30, 31, 32**].

2.13. Semigroups and pseudo-differential operators. A great number of (elliptic) pseudo-differential operators generate strongly continuous semigroups.

Some lower order ($\leqslant 2$) pseudo-differential operators generate Feller semigroups. In fact, let ψ be a non-negative definite function, like $\psi(\xi) = |\xi|^{\alpha}$, $\xi \in \mathbb{R}^{\nu}$, $0 < \alpha \leqslant 2$ fixed. Then the corresponding pseudo-differential operator may be defined by

$$Af(x) = \frac{1}{(2\pi)^{\nu}} \iint \exp\left(i\left\langle x-y,\xi\right\rangle\right)\psi(\xi)f(y)\,dy\,d\xi.$$

Then, some closure of A generates a Lévy process. If the symbol ψ also depends on the position x, then the situations becomes much more complicated: see Jacob [29].

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2.14. Quadratic forms and semigroups. There exist a one-to-one correspondence between the family of lower bounded, closed quadratic forms and strongly continuous semigroups that consist of form positive operators. Certain quadratic forms (closed Dirichlet forms) yield strongly continuous semigroups, consisting of contraction operators, which are form positive and preserve the positivity in a space like $L^2(E,m)$, where m is a Radon measure on the topological space E. With such quadratic one may associate strong Markov processes. In our approach we will start with (generators of) Feller semigroups instead of Dirichlet forms. See Subsection 5.2 for some information on symmetric quadratic forms.

2.15. Ornstein-Uhlenbeck semigroup. Let W be a separable Banach space, supplied with its Borel field $\mathcal{B}(W)$. A probability measure μ on $(W,\mathcal{B}(W))$ is called a *Gaussian measure* if it possesses the following property:

> For every $n \in \mathbb{N}$, and for every finite choice $\ell_1,\dots,\ell_n \in W^*$ (the topological dual of W, there exists $m \in \mathbb{R}^n$, and there exists an $n \times n$ matrix $v = (v_{jk})_{j,k=1}^n$, v symmetric and $v \geqslant 0$, such that
>
> $$\int_W \exp\left(-i\sum_{j=1}^n c_j\ell_j(w)\right) d\mu(w) = \exp\left(i\langle m,c\rangle - \frac{1}{2}\langle vc,c\rangle\right),$$
>
> for all choices $(c_1,\dots,c_n) \in \mathbb{R}^n$.

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In other words the vector $(\ell_1, \dots, \ell_n) \in (W^*)^n$ is a *Gaussian* vector on the probability space $(W, \mathcal{B}(W))$. We also suppose that the support of μ coincides with W. Suppose $m = 0$ (for all choices $\ell_1, \dots, \ell_n$ in W^*). Then there exists a unique Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$, with $\mathcal{H} \subset W$, such that

(i) The embedding $j : \mathcal{H} \hookrightarrow W$ is continuous and $j(\mathcal{H})$ is dense in W;
(ii) For every $\ell \in W^*$ the following equality is valid:

$$\int \exp(-i\ell(w))\, d\mu(w) = \exp\left(-\frac{1}{2}\|\ell\|_{\mathcal{H}}^2\right). \tag{6.40}$$

The equality in (6.40) is equivalent with saying that μ is a Gaussian measure for which $\langle \ell, \ell' \rangle_{\mathcal{H}} = \int \ell(w)\ell'(w)\, d\mu(w)$ for all ℓ and $\ell' \in W^*$. The triple $(W, \mathcal{H}, \mu)$ is called an *abstract Wiener space*. Notice that $W^* \subset \mathcal{H}^* \cong \mathcal{H} \subset W$. A concrete example is given by the r-dimensional Wiener space $W = W_0^r$, given by

$$\begin{aligned} W_0^r &= \{w \in C([0,1], \mathbb{R}^r),\ \ w(0) = 0\},\ \text{supplied with the supremum norm;} \\ H &= \Big\{h \in W :\ h = \left(h^1, \dots, h^r\right),\ \text{for every } 1 \leqslant j \leqslant r \text{ the function } h^j \text{ is} \\ &\qquad \text{absolutely continuous with respect to the Lebesgue measure,} \\ &\qquad \text{with } \frac{dh^j}{dt} = \dot{h}^j \text{ and } \int_0^1 \dot{h}^j(s)^2 ds < \infty\Big\}. \end{aligned}$$

The space $\mathcal{H}$ is then a separable Hilbert space with inner-product

$$\langle h, g \rangle_{\mathcal{H}} = \sum_{j=1}^r \int_0^1 \dot{h}^j(s)\dot{g}^j(s)\, ds, \quad h,\ g \in \mathcal{H}.$$

Notice that $\mathcal{H}$ is isomorphic to $L^2([0,1], \mathbb{R}^r)$. On the spaces $L^p(W, \mathcal{B}(W), \mu) = L^p(W, \mu)$ the Ornstein-Uhlenbeck semigroup $\{S(t) : t \geqslant 0\}$ is defined as follows

$$[S(t)F](w) = \int_W F\left(\exp(-t)\, w + \sqrt{1 - \exp(-2t)}u\right) d\mu(u), \quad F \in L^p(W, \mu).$$

Let J_n, $n \in \mathbb{N}$, be the orthogonal projection in $L^2(W, \mu)$ on the subspace of polynomials of degree exactly equal to n (if P_n denotes the subspace of $L^2(W, \mu)$ consisting of polynomials of degree less than or equal to n, then J_n projects on the subspace $P_n \cap P_{n-1}^{\perp}$). The operator $S(t)$ is also given by

$$S(t)F = \sum_{n=0}^{\infty} \exp(-nt)\, J_n F.$$

(The decomposition $F = \sum_{n=0}^{\infty} J_n F$ in the space $L^2(W, \mu)$ is called the *Itô-Wiener* decomposition.) The generator A of the present Ornstein-Uhlenbeck semigroup in $L^2(W, \mu)$ takes the following form

$$AF = -\sum_{n=0}^{\infty} nJ_nF,\ \text{for } F \in D(A) = \left\{F \in L^2(W, \mu) : \sum_{n=0}^{\infty} n^2 \|J_nF\|_{L^2(W,\mu)}^2 < \infty\right\}.$$

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Let $(W, \mathcal{H}, \mu)$ be a Wiener space and let $(S, \mathcal{B}(S))$ be a measurable space. A measurable mapping $F : W \to S$ is called an S-valued *Wiener* functional. The Wiener functional F is called p-integrable, $1 \leqslant p < \infty$, if S is a Banach space and if the mapping $w \mapsto \|F(w)\|_S$ belongs to $L^p(W, \mu)$. A Winer functional $F : W \to \mathbb{R}$ is called a *polynomial* if the following holds true:

for every $n \in \mathbb{N}$, there exists a polynomial $p = p(x_1, \dots, x_n)$ in n variables
and there exist $\ell_1, \dots, \ell_n \in W^*$,
such that $\quad F(w) = p\left(\ell_1(w), \dots, \ell_n(w)\right),$ for all $w \in W$.

The degree of p is that of F. We may always suppose that the functionals ℓ_j, $1 \leqslant j \leqslant n$, are orthogonal in $\mathcal{H}$: $\langle \ell_j, \ell_k \rangle_{\mathcal{H}} = \delta_{j,k}$. The collection of polynomials of degree $\leqslant n$ is a closed subspace of $L^2(W, \mu)$. The collection of all polynomials is dense in $L^2(W, \mu)$. The Ornstein-Uhlenbeck semigroup plays a fundamental role in Malliavin calculus (or stochastic calculus of variations): [**11, 21, 22, 84, 82, 83, 89, 90, 135, 152, 157**]. For a relatively simple introduction see *e.g.* Friz [**52**].

2.16. Evolutions and semigroups. Let $\{V(r,s) : r \leqslant s\}$ be an evolutionary system on a Banach space X. Basically this means that $V(r,r) = I$, $V(r,s)V(s,t) = V(r,t)$, $r \leqslant s \leqslant t$ (algebraic properties). We also assume the following continuity properties:

$$\text{s-}\lim_{t \downarrow s} V(t,s) = V(s,s) = I = \lim_{r \uparrow s} V(s,r).$$

This system, which is not necessarily time homogeneous, can be made homogeneous in time on spaces like $C_0(\mathbb{R}, X)$. Define the semigroup $\{S(t) : t \geqslant 0\}$ as follows:

$$\left[S(t)f\right](r) = V(r, r+t) f(r+t), \quad f \in C_0(\mathbb{R}, X).$$

3. Markov processes

We begin with a theorem. Some more explanation will follow later.

6.36. Theorem. *The following assertion hold true:*

(a) *(Blumenthal and Getoor [37]) Let $\{S(t) : t \geqslant 0\}$ be a Feller semigroup in $C_0(E)$. Then there exists a strong Markov process (in fact a Hunt process)*

$$\left\{(\Omega, \mathcal{F}, \mathbb{P}_x), (X(t), t \geqslant 0), (\vartheta_t, t \geqslant 0), (E, \mathcal{E})\right\}, \text{ such that}$$
$$\left[S(t)f\right](x) = \mathbb{E}_x\left[f(X(t))\right], \quad f \in C_0(E), \quad t \geqslant 0.$$

*Moreover this Markov process is normal (*i.e. $\mathbb{P}_x\left[X(0) = x\right] = 1$*), is right continuous (*i.e. $\lim_{t \downarrow s} X(t) = X(s)$*, $\mathbb{P}_x$-almost surely), possesses left limits in E on its life time (*i.e. $\lim_{t \uparrow s} X(t)$ *exists in E, whenever $\zeta > s$), and is quasi left continuous (*i.e. *if $(T_n : n \in \mathbb{N})$ is an increasing sequence of $(\mathcal{F}_t)$-stopping times, $X(T_n)$ converges $\mathbb{P}_x$-almost surely to $X(T)$ on the event $\{T < \infty\}$, where $T = \sup_{n \in \mathbb{N}} T_n$).*

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(b) *Conversely, let*

$$\{(\Omega, \mathcal{F}, \mathbb{P}_x), (X(t), t \geqslant 0), (\vartheta_t, t \geqslant 0), (E, \mathcal{E})\}$$

be a strong Markov process which is normal, right continuous, and possesses left limits in E *on its life time. Put* $[S(t)f](x) = \mathbb{E}_x[f(X(t))]$, *for* f *a bounded Borel function,* $t \geqslant 0$, $x \in E$. *Suppose that* $S(t)f$ *belongs to* $C_0(E)$ *for* f *belonging to* $\in C_0(E)$, $t \geqslant 0$. *Then* $\{S(t) : t \geqslant 0\}$ *is a Feller semigroup.*

(c) *Let* A *be the generator of a Feller semigroup in* $C_0(E)$ *and let*

$$\{(\Omega, \mathcal{F}, \mathbb{P}_x), (X(t), t \geqslant 0), (\vartheta_t, t \geqslant 0), (E, \mathcal{E})\}$$

be the corresponding Markov process. For every $f \in D(A)$ *and for every* $x \in E$, *the process*

$$t \mapsto f(X(t)) - f(X(0)) - \int_0^t Af(X(s))\, ds$$

is a $\mathbb{P}_x$*-martingale for the filtration* $(\mathcal{F}_t)_{t \geqslant 0}$, *where each* σ*-field* $\mathcal{F}_t$, $t \geqslant 0$, *is (some closure of)* $\sigma(X(u) : u \leqslant t)$. *In fact the* σ*-field* $\mathcal{F}_t$ *may taken to be* $\mathcal{F}_t = \bigcap_{s>t} \sigma(X(u) : u \leqslant s)$. *It is also possible to complete* $\mathcal{F}_t$ *with respect to* $\mathbb{P}_\mu$, *given by* $\mathbb{P}_\mu(A) = \int \mathbb{P}_x(A)\, d\mu(x)$. *For* $\mathcal{F}_t$ *the following* σ*-field may be chosen:*

$$\mathcal{F}_t = \bigcap_{\mu \in P(E)} \bigcap_{s>t} \{\mathbb{P}_\mu\text{-completion of } \sigma(X(u) : u \leqslant s)\}.$$

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(d) *Conversely, let A be a densely defined linear operator with domain $D(A)$ and range $R(A)$ in $C_0(E)$. Let $(\mathbb{P}_x : x \in E)$ be a unique family of probability measures, on an appropriate measurable space (path space) $(\Omega, \mathcal{F})$ with an appropriate filtration $(\mathcal{F}_t)_{t\geqslant 0}$, such that, for all $x \in E$, $\mathbb{P}_x[X(0) = x] = 1$, and such that for all $f \in D(A)$ the process*

$$t \mapsto f(X(t)) - f(X(0)) - \int_0^t Af(X(s))\,ds$$

is a $\mathbb{P}_x$-martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. Then the operator A possesses a unique extension A_0, which generates a Feller semigroup in $C_0(E)$, provided that $\mathbb{P}_x[X(t) \in E,\, X(s) \in E] = \mathbb{P}_x[X(t) \in E]$ for all $x \in E$, and all $0 \leqslant s < t$. Next, suppose that the path space Ω is the Skorohod space $D\left([0,\infty), E^{\triangle}\right)$ which consists of right-continuous paths, with left limits ω with values in $E^{\triangle}$ with the property that $X(t)(\omega) \in E$, and $0 \leqslant s < t$ implies $X(s)(\omega) \in E$. In addition, suppose that the state variables and translation operators are given by: $X(t)(\omega) = \omega(t)$, $\omega \in D\left([0,\infty), E^{\triangle}\right)$, and $\vartheta_t(\omega)(s) = \omega(s+t)$. The process

$$\left\{(\Omega, \mathcal{F}, \mathbb{P}_x), (X(t), t \geqslant 0), (\vartheta_t, t \geqslant 0), (E, \mathcal{E})\right\}_{x\in E}$$

is then a strong Markov process.

(e) *Suppose that the densely defined linear operator A (with domain and range in $C_0(E)$) possesses the Korovkin property, and suppose that A extends to a generator of a Feller semigroup. Then the martingale problem is well posed for the operator A, and A possesses a unique extension A_0, which generates a Feller semigroup. Moreover the Markov process associated with A_0 solves the martingale problem uniquely for A.*

6.37. Definition. (a) The martingale problem is said to be *well posed* for the operator A (or the martingale problem is said to *uniquely solvable* for the operator A), if for every $x \in E$ there exists a unique probability measure $\mathbb{P}_x$ on the Skorohod space $\Omega = D\left([0,\infty], E^{\triangle}\right)$ (cadlag sample paths), such that for every $f \in D(A)$ the process

$$t \mapsto f(X(t)) - f(X(0)) - \int_0^t Af(X(s))\,ds$$

is $\mathbb{P}_x$-martingale for the filtration $(\mathcal{F}_t)_{t\geqslant 0} := \left(\sigma\left(X(u) : u \leqslant t\right)_{t\geqslant 0}\right)$, and such that $\mathbb{P}_x[X(0) = x] = 1$.

(b) The operator A (with domain and range in $C_0(E)$) is said to possess the *Korovkin* property, if there exists $\lambda_0 > 0$ such that fore every $x_0 \in E$, the space $S(\lambda_0, x_0)$, defined by

$$\begin{aligned} S(\lambda_0, x_0) = \big\{ g \in C_0(E) : \quad & \text{for every } \varepsilon > 0 \text{ the inequality} \\ & \sup\left\{h_1(x_0) : (\lambda_0 I - A)\, h_1 \leqslant \Re\, g + \varepsilon, \quad h_1 \in D(A)\right\} \\ & \geqslant \inf\left\{h_2(x_0) : (\lambda_0 I - A)\, h_2 \geqslant \Re\, g - \varepsilon, \quad h_2 \in D(A)\right\}\big\}, \end{aligned}$$

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coincides with $C_0(E)$. Let D be a subspace of $C_0(E)$ with the property that, for every $x_0 \in E$, the space $S(x_0)$, defined by

$$\begin{aligned}S(x_0) = \{g \in C_0(E) : \quad & \text{for every } \varepsilon > 0 \text{ the inequality} \\ & \sup\{h_1(x_0) : h_1 \leqslant \Re\, g + \varepsilon, \quad h_1 \in D\} \\ & \geqslant \inf\{h_2(x_0) : h_2 \geqslant \Re\, g - \varepsilon, \quad h_2 \in D(A)\}\},\end{aligned}$$

coincides with $C_0(E)$, then such a subspace D could be called a *Korovkin* subspace of $C_0(E)$.

6.38. Remark. For Ω we may take the Skorohod space $\Omega = D\left([0,\infty], E^{\triangle}\right)$. So a sample ω belongs to Ω if it possesses the following properties:

(i) ω is a mapping from $[0,\infty]$ to $E^{\triangle} = E \cup \{\triangle\}$; $\omega(0) \in E$.
(ii) ω is right continuous and possesses left limits in E on the stochastic interval $[0, \zeta(\omega))$, in the sense that $\lim_{t\uparrow s} \omega(s)$ exists in E for

$$s < \zeta(\omega) := \inf\{t > 0 : \omega(t) = \triangle\}.$$

Moreover, if $\omega(s) = \triangle$ and if $t \geqslant s$, then $\omega(t) = \triangle$.
(iii) The set $E^{\triangle}$ is the one-point compactification of E, or, if E is compact, $\triangle$ is an isolated point of $E^{\triangle} = E \cup \{\triangle\}$.

6.39. Remark. The collection $\{\mathcal{F}_t : t \geqslant 0\}$ is a *filtration*: if $s < t$, then $\mathcal{F}_s \subset \mathcal{F}_t$. Every σ-field $\mathcal{F}_t$ is an appropriate completion (extension) of the σ-field $\sigma(X(u) : u \leqslant t)$. The family $\{\mathcal{F}_t : t \geqslant 0\}$ is continuous from the right: $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$. Since we consider more or less the internal history $\{\mathcal{F}_t : t \geqslant 0\}$, $t \geqslant 0$, we suppress the notation $\mathcal{F}_t$, $t \geqslant 0$, in our symbolism of our Markov process:

$$\{(\Omega, \mathcal{F}, \mathbb{P}_x), (X(t), t \geqslant 0), (\vartheta_t, t \geqslant 0), (E, \mathcal{E})\}.$$

Authors often write things like $\left\{(\mathbb{P}_x)_{x\in E},\ (X(t))_{t\geqslant 0}\right\}$, when the other items are clear from the context.

6.40. Remark. The mappings $X(t) : \Omega \to E^{\triangle}$ are called *state variables*; E is referred to as the *state space* (sometimes *stochastic state space*). Put

$$\zeta = \inf\{s > 0 : X(s) = \triangle\}.$$

Then ζ is called the *life time* of the process $\{X(t) : t \geqslant 0\}$.
The motion $\{X(t) : t \geqslant 0\}$ is $\mathbb{P}_x$-almost surely right continuous and possesses left limits in E on its life time:

(i) $\lim_{s\downarrow t} X(s) = X(t)$, (right continuity);
(ii) $s \geqslant t$, $X(t) = \triangle$, implies $X(s) = \triangle$, ($\triangle$ is cemetery);
(iii) $\lim_{s\uparrow t} X(s) = X(t-) \in E$, $t < \zeta$, (left limits in E on its life time).

These assertions hold $\mathbb{P}_x$-almost surely for all $x \in E$. The probability $\mathbb{P}_{\triangle}$ may be defined by $\mathbb{P}_{\triangle}(A) = \delta_{\omega_{\triangle}}(A)$, where $\omega_{\triangle}(s) = \triangle$, $s > 0$.

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6.41. Remark. The *shift* or *translation* operators $\vartheta_s : \Omega \to \Omega$, $s \geqslant 0$, possess the property that $X(t) \circ \vartheta_s = X(t+s)$, $\mathbb{P}_x$-almost surely, for all $x \in E$ and for all s and $t \geqslant 0$. This is an extremely important property. For example $f(X(t)) \circ \vartheta_s = f(X(t+s))$, $f \in C_0(E)$, $s, t \geqslant 0$. If Ω is the Skorohod space $\Omega = D\left([0,\infty], E^{\triangle}\right)$, then $X(t)(\omega) = \omega(t) = X(t,\omega) = \omega(t)$, $\vartheta_t(\omega)(s) = \omega(s+t)$, $\omega \in \Omega$.

6.42. Remark. For every $x \in E$, the measure $\mathbb{P}_x$ is a probability measure on $\mathcal{F}$ with the property that $\mathbb{P}_x\left[X(0) = x\right] = 1$. So the process starts at $X(0) = x$, $\mathbb{P}_x$-almost surely, at $t = 0$. This is the *normality property*.

6.43. Remark. The Markov property can be expressed as follows:

$$\mathbb{E}_x\left[f(X(s+t)) \mid \mathcal{F}_s\right] = \mathbb{E}_x\left[f(X(s+t)) \mid \sigma(X(s))\right] = \mathbb{E}_{X(s)}\left[Y\right], \tag{6.41}$$

$\mathbb{P}_x$-almost surely for all $f \in C_0(E)$ and for all s and $t \geqslant 0$. Of course, the expression $\mathbb{E}\left[Y \mid \mathcal{F}\right]$ denotes conditional expectation. The meaning of $\mathcal{F}_t$ is explained in Remark 6.39. Let $Y : \Omega \to \mathbb{C}$ be a bounded random variable. This means that Y is measurable with respect to the field generated by $\{X(u) : u \geqslant 0\}$. The Markov property is then equivalent to

$$\mathbb{E}_x\left[Y \circ \vartheta_s \mid \mathcal{F}_s\right] = \mathbb{E}_{X(s)}\left[Y\right], \tag{6.42}$$

$\mathbb{P}_x$-almost surely for all random variables Y and for all $s \geqslant 0$. Notice that, intuitively speaking, $\mathcal{F}_s$ is the information from the past, $\sigma\left(X(s)\right)$ is the information at the present, and $Y \circ \vartheta_s$ is measurable with respect to some completion of $\sigma\left\{X(u) : u \geqslant s\right\}$, the information from the future.

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Put $P(t,x,B) = \mathbb{P}_x[X(t) \in B]$. Then $\mathbb{E}_x[f(X(t))] = \int f(y)P(t,x,dy)$, $f \in C_0(E)$. Moreover (6.41) is equivalent to (6.42) and to

$$\begin{aligned} &\mathbb{E}_x\left[\prod_{j=1}^n f_j(X(t_j))\right] \qquad (6.43)\\ &= \iint \dots \int \prod_{j=1}^n f_j(x_j) P(t_1, x, dx_1) P(t_2 - t_1, x_1, dx_2) \dots P(t_n - t_{n-1}, x_{n-1}, dx_n), \end{aligned}$$

for all $0 \leqslant t_1 < t_2 < \cdots < t_n < \infty$ and for all $f_1, \dots, f_n$ in $C_0(E)$.

6.44. Remark. Next we explain the strong Markov property. Since the paths $\{X(t) : t \geqslant 0\}$ are right continuous $\mathbb{P}_x$-almost surely our Markov process is a strong Markov process. Let $S : \Omega \to \infty$ be a *stopping time* meaning that for every $t \geqslant 0$ the event $\{S \leqslant t\}$ belongs to $\mathcal{F}_t$. This is the same as saying that the process $t \mapsto 1_{[S \leqslant t]}$ is adapted. Let $\mathcal{F}_S$ be the natural σ-field associated with the stopping time S, *i.e.*

$$\mathcal{F}_S = \bigcap_{t \geqslant 0} \{A \in \mathcal{F} : A \cap \{S \leqslant t\} \in \mathcal{F}_t\}.$$

Define $\vartheta_S(\omega)$ by $\vartheta_S(\omega) = \vartheta_{S(\omega)}(\omega)$. Consider $\mathcal{F}_S$ as the information from the past, $\sigma\big(X(S)\big)$ as information from the present, and

$$\sigma\{X(t) \circ \vartheta_S : t \geqslant 0\} = \sigma\{X(t+S) : t \geqslant 0\}$$

as the information from the future. The strong Markov property can be expressed as follows:

$$\mathbb{E}_x[Y \circ \vartheta_S | \mathcal{F}_S] = \mathbb{E}_{X(S)}[Y], \ \mathbb{P}_x\text{-almost surely} \qquad (6.44)$$

on the event $\{S < \infty\}$, for all bounded random variables Y, for all stopping times S, and for all $x \in E$. One can prove that under the "cadlag" property events like $\{X(S) \in B, \ S < \infty\}$, B Borel, are $\mathcal{F}_S$-measurable. The passage from (6.44) to (6.42) is easy: put $Y = f(X(t))$ and $S(\omega) = s$, $\omega \in \Omega$. The other way around is much more intricate and uses the cadlag property of the process $\{X(t) : t \geqslant 0\}$. In this procedure the stopping time S is approximated by a decreasing sequence of discrete stopping times $(S_n = 2^{-n}\lceil 2^n S\rceil : n \in \mathbb{N})$. The equality

$$\mathbb{E}_x[Y \circ \vartheta_{S_n} | \mathcal{F}_{S_n}] = \mathbb{E}_{X(S_n)}[Y], \ \mathbb{P}_x\text{-almost surely}, \qquad (6.45)$$

is a consequence of (6.42) for a fixed time. Let n tend to infinity in (6.45) to obtain (6.44). The "strong Markov property" can be extended to the "strong time dependent Markov property":

$$\mathbb{E}_x\left[Y(S + T \circ \vartheta_S, \vartheta_S) \mid \mathcal{F}_S\right](\omega) = \mathbb{E}_{X\left(S(\omega)\right)}\left[\omega' \mapsto Y(S(\omega) + T(\omega'), \omega')\right],$$

$\mathbb{P}_x$-almost surely on the event $\{S < \infty\}$. Here $Y : [0,\infty) \times \Omega \to \mathbb{C}$ is a bounded random variable. The cartesian product $[0,\infty) \times \Omega$ is supplied with the product field $\mathcal{B} \otimes \mathcal{F}$; $\mathcal{B}$ is the Borel field of $[0,\infty)$ and $\mathcal{F}$ is (some extension of) $\sigma(X(u) : u \geqslant 0)$. Important stopping times are "hitting times", or times related to hitting times:

$$T = \inf\left\{s > 0 : X(s) \in E^{\triangle} \backslash U\right\}, \text{ and } S = \inf\left\{s > 0 : \int_0^s 1_{E\backslash U}(X(u))\,du > 0\right\},$$

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where U is some open (or Borel) subset of $E^{\triangle}$. This kind of stopping times have the extra advantage of being *terminal* stopping times, *i.e.* $t+S\circ\vartheta_t = S$ $\mathbb{P}_x$-almost surely on the event $\{S > t\}$. A similar statement holds for the *hitting time* T. The time S is called the *penetration time* of $E\backslash U$. Let $p : E \to [0,\infty)$ be a Borel measurable function. Stopping times of the form

$$S_\xi = \inf\left\{s > 0 : \int_0^s p\big(X(u)\big)\, du > \xi\right\}$$

serve as a *stochastic time change*, because they enjoy the equality:

$$S_\xi + S_\eta \circ \vartheta_{S_\xi} = S_{\xi+\eta}, \quad \mathbb{P}_x\text{-almost surely on the event } \{S_\xi < \infty\}.$$

As a consequence operators of the form $\mathcal{S}(\xi)f(x) := \mathbb{E}_x\left[f\left(X\left(S_\xi\right)\right)\right]$, f a bounded Borel function, possess the semigroup property. Also notice that $S_0 = 0$, provided that the function p is strictly positive.

6.45. Remark. A very important example of a strong Markov process is Brownian motion. Let E be the space $\mathbb{R}^\nu$ and let $\Omega := C\left([0,\infty),\mathbb{R}^\nu\right)$, equipped with the product field $\mathcal{F}$, or even better, with the Borel field coming from the topology of uniform convergence on compact subsets of $[0,\infty)$. Put

$$p_{0,\nu}(t,x,y) = \frac{1}{\left(\sqrt{2\pi t}\right)^\nu}\exp\left(-\frac{|x-y|^2}{2t}\right).$$

Define, for $x_0 \in \mathbb{R}^\nu$, the probability measure $\mathbb{P}_{x_0}$ on $\mathcal{F}$ via the identity

$$\mathbb{E}_{x_0}\left[\prod_{j=1}^n f_j(X(t_j))\right] = \int\dots\int dx_1\dots dx_n \prod_{j=1}^n f_j(x_j)\prod_{j=1}^n p_{0,\nu}(t_j - t_{j-1}, x_{j-1}, x_j), \tag{6.46}$$

where $t_0 = 0$ and $f_1,\dots,f_n$ are bounded Borel measurable functions on $\mathbb{R}^\nu$. The times t_0, $t_1,\dots,t_n$ satisfy $0 = t_0 < t_1 < \dots < t_n < \infty$. Moreover $X(t)(\omega) = \omega(t)$, $\left[\vartheta_s(\omega)\right](t) = \omega(s+t)$, s, $t \geqslant 0$, $\omega \in \Omega$. It is a not so trivial theorem that there exists a genuine probability measure $\mathbb{P}_x$ on Ω such that its finite dimensional distributions are given by (6.46). The corresponding semigroup $\{S(t) : t \geqslant 0\}$ is the classical *Gaussian* or *heat* semigroup:

$$S(t)f(x) = \exp\left(-tH_0\right)f(x) = \int p_{0,\nu}(t,x,y)f(y)\,dy.$$

Its generator is $-H_0 = \frac{1}{2}\Delta$ in $C_0\left(\mathbb{R}^\nu\right)$ or in $L^p\left(\mathbb{R}^\nu\right)$, $1 \leqslant p < \infty$, as the case may be. The family $\{S(t) : t \geqslant 0\}$ is a semigroup in $L^\infty\left(\mathbb{R}^\nu\right)$. However it is not strongly continuous there; it is only weak* continuous. The corresponding Markov process is called ν-dimensional *Brownian motion*. A nice classical application of ν-dimensional Brownian motion is its use in potential theory. A specific example is a description of the solution for the following Dirichlet problem:

$$\begin{cases} \frac{1}{2}\Delta u = 0, & \text{in } U; \\ \lim\limits_{\substack{x\to b\\ x\in U}} u(x) = f(b), & b \in \partial U. \end{cases}$$

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Here ∂U is the boundary of the open set U and $f : \partial U \to \mathbb{R}$ is a bounded continuous function. Put $T = \inf\{t > 0 : X(t) \in \mathbb{R}^\nu \backslash U\}$ and write

$$u(x) = \mathbb{E}_x\left[f(X(T)) : T < \infty\right].$$

Here $\{\mathbb{P}_x, X(t)\}_{x\in\mathbb{R}^\nu}$ is ν-dimensional Brownian motion. Then the function u is harmonic on U. If $\mathbb{P}_b\left[T = 0\right] = 1$, then

$$\begin{aligned}&\lim_{\substack{x\to b\\ x\in U}} \mathbb{E}_x\left[f(X(T)), T < \infty\right] = \mathbb{E}_b\left[f(X(T)), T < \infty\right]\\ &= \mathbb{E}_b\left[f(X(0)), T < \infty\right] = \mathbb{E}_b\left[f(b), T = 0\right] = f(b)\mathbb{P}_b\left[T = 0\right] = f(b).\end{aligned}$$

From Blumenthal's zero-one law, it follows that $\mathbb{P}_b\left[T = 0\right] = 0$ or 1. It equals 1 if b is a regular point of $\mathbb{R}^\nu\backslash U$. The set of points that are irregular constitute a small (a polar) subset of $\mathbb{R}^\nu\backslash U$. In particular if the boundary of U is C^1, then every point of $\mathbb{R}^\nu$ is regular. We say that $\frac{1}{2}\Delta$ generates Brownian motion.

6.46. Remark. The notion of a $\mathbb{C}^m$-valued martingale reads as follows. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{\mathcal{F}_t : t \geqslant 0\}$ be a filtration in $\mathcal{F}$ on Ω. So that $\mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2} \subseteq \mathcal{F}$, for $0 \leqslant t_1 \leqslant t_2$. Let $\{M(t) : t \geqslant 0\}$ be an *adapted* process in $L^1\left((\Omega, \mathcal{F}, \mathbb{P}), \mathbb{C}^m\right)$. This means that, for every $t \geqslant 0$, $M(t)$ is $\mathcal{F}_t$-measurable and, of course, $\mathbb{E}\left(|M(t)|\right) = \mathbb{E}\left(\sqrt{\sum_{j=1}^m |M_j(t)|^2}\right) < \infty$. Here, $M(t) = (M_1(t), \dots, M_m(t))$. If $\mathbb{E}\left[M(t) \mid \mathcal{F}_s\right] = M(s)$, $\mathbb{P}$-almost surely for all $t \geqslant s$, then the family $\{M(t) : t \geqslant 0\}$ is called a *martingale* with respect to $\mathbb{P}$ and the filtration $\{\mathcal{F}_t : t \geqslant 0\}$.

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6.47. Remark. Let A be the generator of the Feller semigroup $\{S(t) : t \geqslant 0\}$ and let $\{(\Omega, \mathcal{F}, \mathbb{P}_x), (X(t), t \geqslant 0), (\vartheta_t, t \geqslant 0), (E, \mathcal{E})\}$ be the corresponding strong Markov process. Fix $f \in D(A)$ and put $M_f(t) = f(x(t)) - f(X(0)) - \int_0^t Af(X(s))\,ds$. Then, for all $x \in E$, the process $\{M_f(t) : t \geqslant 0\}$ is a $\mathbb{P}_x$-martingale. A proof reads as follows $(t > s)$:

$$
\begin{aligned}
&\mathbb{E}_x\left[M_f(t) \mid \mathcal{F}_s\right] - M_f(s) = \mathbb{E}_x\left[M_f(t) - M_f(s) \mid \mathcal{F}_s\right] \\
&= \mathbb{E}_x\left[f(X(t)) - f(X(s))) - \int_s^t Af(X(u))\,du \mid \mathcal{F}_s\right] \\
&= \mathbb{E}_x\left[\left(f(X(t-s)) - f(X(0))) - \int_0^{t-s} Af(X(u))\,du\right) \circ \vartheta_s \mid \mathcal{F}_s\right]
\end{aligned}
$$

(Markov property)

$$
\begin{aligned}
&= \mathbb{E}_{X(s)}\left[f(X(t-s)) - f(X(0))) - \int_0^{t-s} Af(X(u))\,du\right] \\
&= \mathbb{E}_{X(s)}\left[M_f(t-s)\right].
\end{aligned}
$$

So fix $z \in E$. By the fundamental relation between the semigroup $\{S(t) : t \geqslant 0\}$ and the Markov process $\{(\Omega, \mathcal{F}, \mathbb{P}_x), (X(t), t \geqslant 0), (\vartheta_t, t \geqslant 0), (E, \mathcal{E})\}$ we get

$$
\begin{aligned}
\mathbb{E}_z\left[M_f(t-s)\right] &= \mathbb{E}_z\left[f(X(t-s))\right] - \mathbb{E}_z\left[f(X(0))\right] - \int_0^{t-s} \mathbb{E}_z\left[Af(X(u))\right] du \\
&= \left[S(t-s)f\right](z) - f(z) - \int_0^{t-s} \left[S(u)Af\right](z)\,du \\
&= \left[S(t-s)f\right](z) - f(z) - \int_0^{t-s} \frac{\partial}{\partial u}\left[S(u)f\right](z)\,du \\
&= \left[S(t-s)f\right](z) - f(z) - \left(\left[S(t-s)f\right](z) - \left[S(0)f\right](z)\right) = 0.
\end{aligned}
$$

6.48. Remark. In order to define the Markov property we may start with just one probability space

$$\{(\Omega, \mathcal{F}, \mathbb{P}), (X(t), t \geqslant 0), (\vartheta_t, t \geqslant 0), (E, \mathcal{E})\}.$$

The family $\{X(t) : t \geqslant 0\}$ is said to be $\mathbb{P}$-Markovian, if, for all $s \geqslant 0$, and for all bounded random variables $Y : \Omega \to \mathbb{C}$, the equality

$$\mathbb{E}\left[Y \circ \vartheta_s \mid \mathcal{F}_s\right] = \mathbb{E}\left[Y \circ \vartheta_s \mid \sigma(X(s))\right]$$

holds $\mathbb{P}$-almost surely. Then we consider the measures on the Borel field $\mathcal{E}$ given by

$$B \mapsto \mathbb{E}\left[Y \circ \vartheta_s,\ X(t) \in B\right],\ B \in \mathcal{E}, \quad \text{and} \quad B \mapsto \mathbb{P}\left[X(s) \in B\right],\ B \in \mathcal{E}.$$

The first of these two measures is trivially absolutely continuous with respect to the second one. So there exists a function $x \mapsto \mathbb{E}_x[Y]$ such that

$$\frac{\mathbb{E}\left[Y \circ \vartheta_s,\ X(s) \in dx\right]}{\mathbb{P}\left[X(s) \in dx\right]} = \mathbb{E}_x[Y].$$

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Notice that $\mathbb{E}_x[1] = 1$. By the time homogeneity and since the σ-field $\sigma\{X(u),\ u \geqslant 0\}$ is countably determined, the expression $\mathbb{E}_x[Y]$ is well-defined (*i.e.* independent of $s > 0$, and as a function of x Borel measurable). If the state space E is countable, so that the probability measure $B \mapsto \mathbb{P}[X(s) \in B]$ is a discrete measure (a combination of multiples of Dirac measures), then this Radon-Nykodim derivative is an ordinary quotient and we enter the theory of discrete Markov processes. We assume, in the Feller semigroup context, that $x \mapsto \mathbb{E}_x[f(X(t))]$ belongs to $C_0(E)$, whenever f does so and whenever $t \geqslant 0$.

6.49. Remark. Starting from Feller semigroups one may construct the corresponding strong Markov processes. In this construction one first replaces the semigroup $\{S(t) : t \geqslant 0\}$ with a family of (sub-)Markov transition functions $\{P(t,x,B) : t \geqslant 0\}$. Here $B \mapsto P(t,x,B)$ is a (sub-)probability measure on $\mathcal{E}$, with the property that $S(t)f(x) = \int f(y)P(t,x,dy)$, $f \in C_0(E)$, $t \geqslant 0$. From the Riesz representation theorem it follows that such a family of (sub-)probability measures exists. It possesses the following properties:

$$P(0,x,B) = \delta_x(B), \quad P(s+t,x,B) = \int P(s,y,B)P(t,x,dy),$$

s, $t \geqslant 0$, $x \in E$, $B \in \mathcal{E}$. Next put

$$N(t,x,B) = P(t,x,B \cap E) + (1 - P(t,x,E))\, 1_B(\triangle),$$

where now B is a Borel subset of $E^\triangle$. Put $\Omega' = \left(E^\triangle\right)^{[0,\infty]}$, and define the measure $\mathbb{P}_x$ on the product field of $\Omega' = \left(E^\triangle\right)^{[0,\infty]}$ via the equality $(X(t)(\omega) = \omega(t))$:

$$\mathbb{E}_x\left[\prod_{j=1}^n f_j(X(t_j))\right] \tag{6.47}$$
$$\iint \dots \int \prod_{j=1}^n f_j(x_j) N(t_1,x,dx_1)N(t_2-t_1,x_1,dx_2)\dots N(t_n-t_{n-1},x_{n-1},dx_n),$$

where the functions f_j, $1 \leqslant j \leqslant n$ are bounded Borel functions on $E^\triangle$. The hard part is proving that the Skorohod space has full $\mathbb{P}_x$-measure (in fact its outer $\mathbb{P}_x$-measure equals 1). The extension of $\mathbb{P}_x$ to the product field of Ω' is a consequence of the Kolmogorov extension theorem.

6.50. Remark. The fact that the σ-fields $\mathcal{F}_t$, $t \geqslant 0$, may be replaced with larger fields, while still retaining the Markov property (or, more accurately, the strong Markov property) is a consequence of the *cadlag, continue à droite, limitée à gauche* property together with Choquet's theorem on capacitable sets. These larger σ-fields are certain completions of the σ-field generated by the collection $\{X(u) : 0 \leqslant u \leqslant t\}$: see assertion (c) of Theorem 6.36.

6.51. Remark. Since a Feller semigroup possesses a generator, A say, one also says that A generates the associated strong Markov process. For example $\frac{1}{2}\Delta$ generates Brownian motion. This concept yields a direct relation between certain (lower order) pseudo-differential operators and probability theory. The order has to be less than

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or equal to 2. This follows from the theory of Lévy processes and the Lévy-Khinchin formula, which decomposes a continuous negative-definite function into a linear term (probabilistically this corresponds to a deterministic drift), a quadratic term (this corresponds to a diffusion: a continuous Brownian motion-like process), and a term that corresponds to the jumps of the process (compound Poisson process, Lévy measure). Quite a number of problems in classical analysis can be reformulated in probabilistic terms. For instance for certain Dirichlet boundary value problems hitting times are appropriate, for certain initial value problems Markov process theory is relevant. For other problems the martingale approach is more to the point. For example there exists a one-to-one correspondence between the following concepts:

(i) Unique (weak) solutions of stochastic differential equations in $\mathbb{R}^\nu$:
(ii) Unique solutions to the corresponding martingale problem;
(iii) Markovian diffusion semigroups in $\mathbb{R}^\nu$;
(iv) Feller semigroups generated by certain second order differential operators of elliptic type.

(Regular) first order perturbations of second order elliptic differential operators can be studied using the Cameron-Martin-Girsanov transformation. Perturbations of order zero are treated via the Feynman-Kac formula.

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6.52. Remark. In our discussion we started with (generators of) Feller semigroups or, even better, Feller-Dynkin semigroups. Another approach would be to begin with symmetric Dirichlet forms (quadratic form theory) in $L^2(E,m)$, where m is a Radon measure on the Borel field $\mathcal{E}$ of E. (By definition a Radon measure assigns finite values to compact subsets and it is inner and outer regular.) The reader may consult the books by Bouleau and Hirsch [6], by Fukushima, Oshima and Takeda 11], or by Z. Ma and M. Röckner [15]. In the latter reference Ma and Röckner somewhat more general Dirichlet forms are treated. These Dirichlet need not be symmetric, but they obey a certain *cone type* inequality:

$$|\mathcal{E}(f,g)|^2 \leqslant K\mathcal{E}(f,f)\mathcal{E}(g,g), \quad f, \quad g \in D\left(\mathcal{E}\right).$$

Again one says that the Markov process is generated by (or associated to the Dirichlet form $\mathcal{E}$ or to the corresponding closed linear operator: $\mathcal{E}(f,g) = -\langle Af, g\rangle$, $f \in D(A)$, $g \in D\left(\mathcal{E}\right)$. (Notice that only regular Dirichlet forms correspond to Markov processes.) We have taken the approach via $C_0(E)$ instead of $L^2(E,m)$.

6.53. Remark. Examples of (Feller) semigroups can be manufactured by taking a continuous function $\varphi : [0,\infty) \times E \to E$ with the property that

$$\varphi\left(s+t,x\right) = \varphi\left(t,\varphi\left(s,x\right)\right),$$

for all s, $t \geqslant 0$ and $x \in E$. Then the mappings $f \mapsto P(t)f$, with $P(t)f(x) = f\left(\varphi\left(t,x\right)\right)$ defines a semigroup. It is a Feller semigroup, or Feller-Dynkin semigroup, if $\lim_{x\to\triangle}\varphi\left(t,x\right) = \triangle$. An explicit example of such a function, which does not provide a Feller-Dynkin semigroup on $C_0\left(\mathbb{R}\right)$ is given by $\varphi(t,x) = \dfrac{x}{\sqrt{1+\frac{1}{2}tx^2}}$ (example due to V. Kolokoltsov [36]). Here the process $X(t)$ is in fact deterministic: $X(t) = \varphi(t, X(0))$. Put $u(t,x) = P(t)f(x) = f\left(\varphi(t,x)\right)$. Then $\dfrac{\partial u}{\partial t}(t,x) = -\dfrac{x^3}{4}\dfrac{\partial u}{\partial x}(t,x)$. In fact this (counter-)example shows that solutions to the martingale problem do not necessarily give rise to Feller-Dynkin semigroups . These are semigroups which preserve not only the continuity, but also the fact that functions which tend to zero at $\triangle$ are mapped to functions with the same property. However, for Feller semigroups we only require that continuous functions with values in $[0,1]$ are mapped to continuous functions with the same properties. For every $(s,t,x) \in [0,T]^2 \times E$, $0 < s < t$, the equality

$$\mathbb{P}_x\left[X(t) \in E\right] = \mathbb{P}_x\left[X(t) \in E,\, X(s) \in E\right]$$

holds. On the other hand this hypothesis is implicitly assumed, if as sample path space we take the Skorohod space $D\left([0,\infty), E^{\triangle}\right)$. If $X(t) \in E$, then $0 \leqslant s < t$ implies $X(s) \in E$.

The main result, Theorem 2.5, as stated in Van Casteren [**140**] is not correct. That is solutions to the martingale problem can, after having visited $\triangle$, still be alive. Compare this with Remark 2.12 in Van Casteren [**145**].

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To conclude this section we include a simple result on the relation between the generator of a Feller semigroup, or Feller-Dynkin semigroup, and the corresponding Markov process.

6.54. Proposition. *Let the operator A in with domain and range in $C_0(E)$ be the generator of a Feller semigroup $\left\{S(t) = e^{tA} : t \geqslant 0\right\}$ and let*

$$\left\{(\Omega, \mathcal{F}, \mathbb{P}_x), (X(t), t \geqslant 0), (\vartheta_t, t \geqslant 0), (E, \mathcal{E})\right\} \tag{6.48}$$

be the corresponding Markov process. Suppose that the function f belongs to $D(A)$. Then the following equalities hold for $t \geqslant 0$ and $x \in E$:

$$\frac{\partial}{\partial t} S(t)f(x) = \left[AS(t)f\right](x) = \left[S(t)Af\right](x) = \mathbb{E}_x\left[Af\left(X(t)\right)\right] = A\mathbb{E}_{(\cdot)}\left[f\left(X(t)\right)\right](x). \tag{6.49}$$

4. Feynman-Kac semigroups

Suppose that $A = -K_0$ generates a Feller semigroup in $C_0(E)$, and suppose that the corresponding semigroup $\{\exp(-tK_0) : t \geqslant 0\}$ consists of integral operators:

$$\left[\exp(-tK_0) f\right](x) = \int p_0(t,x,y) f(y)\, dm(y), \ \ f \in C_0(E),$$

where m is a Radon measure on the Borel field of E, and where the function $p_0(t,x,y)$ is *symmetric* (*i.e.* $p_0(t,x,y) = p_0(t,y,x)$, x, $y \in E$) and continuous on $(0,\infty)\times E \times E$.

6.55. Remark. If $E = \mathbb{R}^\nu$ with Lebesgue measure and if $K_0 = H_0 = -\frac{1}{2}\Delta$, then $p_0(t,x,y)$ is the classical Gaussian kernel

$$p_0(t,x,y) = p_{0,\nu}(t,x,y) = \frac{1}{\left(\sqrt{2\pi t}\right)^\nu} \exp\left(-\frac{|x-y|^2}{2t}\right).$$

We write $\left[\exp(-tK_0) f\right](x) = \int p_0(t,x,y) f(y)\, dm(y)$ for those functions f for which this integral makes sense for m-almost all $x \in E$. Let $V : E \to [-\infty, \infty]$ be a Kato-Feller potential with respect to K_0. By definition, this means that for every compact subset K of E the following identity is true:

$$\lim_{t \downarrow 0} \sup_{x \in E} \int_0^t \left[\exp(-sK_0)(V_- + V_+)\right](x)\, ds = 0. \tag{6.50}$$

Here $V_+ = \max(V, 0)$, $V_- = \max(-V, 0)$. In case $K_0 = -\frac{1}{2}\Delta$ in $\mathbb{R}^\nu$, many classical potentials from mathematical physics belong to the Kato-Feller class: see Simon [**126**].

For the result in Theorem 6.56 it is only required that (6.50) holds with $V_- + V_+\mathbf{1}_K$ for all compact subsets K instead of $V_- + V_+ = |V|$.

6.56. Theorem. *Let V be Kato-Feller potential, or, even better, suppose that (6.50) holds with $V_- + V_+\mathbf{1}_K$ for all compact subsets K instead of $V_- + V_+ = |V|$.*

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(a) *There exists a closed densely defined linear operator* $K_0 \dot{+} V$ *extending the operator* $K_0 + V$, *which generates a positivity preserving (self-adjoint) semigroup in* $L^2(E, m)$, *denoted by* $\left\{\exp\left(-t\left(K_0 \dot{+} V\right)\right) : t \geqslant 0\right\}$. *This semigroup is given by the Feynman-Kac formula:*

$$\left[\exp\left(-t\left(K_0 \dot{+} V\right)\right) f\right](x) = \mathbb{E}_x\left[\exp\left(-\int_0^t V(X(u))\,du\right) f(X(t))\right], \quad f \in L^2(E, m).$$

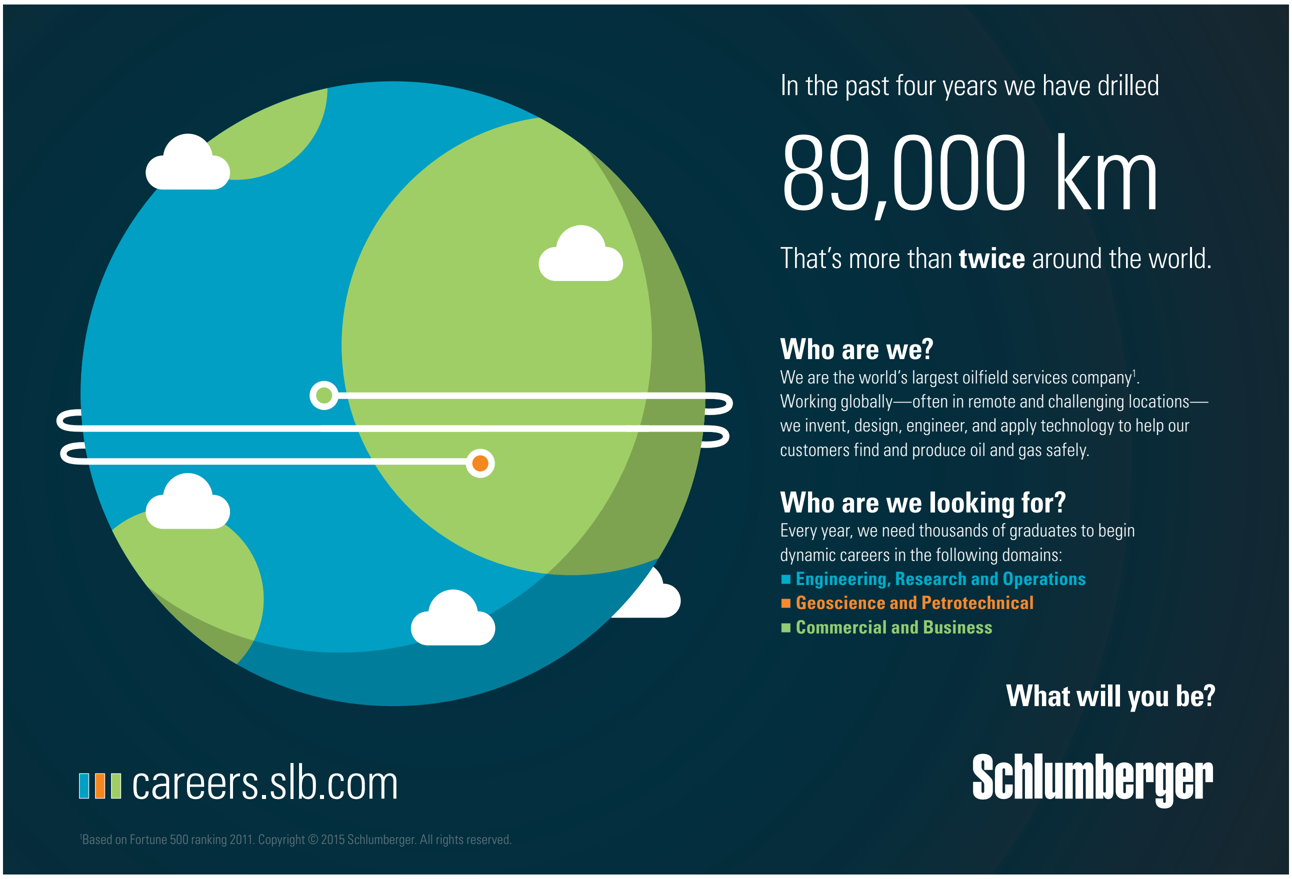
(b) *Every operator* $\exp\left(-t\left(K_0 \dot{+} V\right)\right)$ *is an integral operator with a continuous, symmetric integral kernel* $\exp\left(-t\left(K_0 + V\right)\right)(x, y)$ *given by*

$$\begin{aligned}\exp\left(-t\left(K_0 + V\right)\right)(x, y) &= \lim_{s \uparrow t} \mathbb{E}_x\left[\exp\left(-\int_0^s V(X(u))\,du\right) p_0(t - s, X(s), y)\right] \\ &= \int \exp\left(-\int_0^t V(X(u))\,du\right) d\mu_{0,x}^{t,y}.\end{aligned}$$

The measure $\mu_{0,x}^{t,y}$ *is defined on the* σ*-field* $\sigma\left(X(u) : u < t\right)$, *and as usual can be extended on some completion of this* σ*-field. It is determined by*

$$\mu_{0,x}^{t,y}(A) = \mathbb{E}_x\left[1_A p_0(t - s, X(s), y)\right], \tag{6.51}$$

where the event A *belongs to* $\mathcal{F}_s = \sigma\left(X(u) : u \leqslant s\right)$, *for* $s < t$. *Since the process* $s \mapsto p_0(t - s, X(s), y)$ *is a* $\mathbb{P}_x$*-martingale on the interval* $0 \leqslant s < t$, *it follows that the quantity* $\mu_{0,x}^{t,y}(A)$ *is well-defined: its value does not depend on* s, *as long as* A *belongs to* $\mathcal{F}_s$ *and* $s < t$. *The measure* $\mu_{0,x}^{t,y}$ *could be called the un-normalized Markov bridge kernel.*



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(c) *The quadratic form (generalized Schrödinger form) $\mathcal{E}_V$ associated with the above Feynman-Kac semigroup is given by*

$$\mathcal{E}_V(f,g) = \left\langle \sqrt{K_0} f, \sqrt{K_0} g \right\rangle + \left\langle \sqrt{V_+} f, \sqrt{V_+} g \right\rangle - \left\langle \sqrt{V_-} f, \sqrt{V_-} g \right\rangle,$$

for f, g members of

$$D\left(\sqrt{K_0}\right) \cap \left\{ f \in L^2(E,m), \ \int V_+(x) \left|f(x)\right|^2 dm(x) < \infty \right\}.$$

6.57. Remark. Suppose that Markov process in (6.48) is Brownian motion in $E = \mathbb{R}^d$. In other words, suppose that $K_0 = -\frac{1}{2}\Delta$. Then the measure $\mu_{0,x}^{t,y}$, $t > 0$, x, $y \in \mathbb{R}^d$, defined in (6.51) is called the conditional Brownian bridge measure. It can be normalized through dividing it by the density $p(t,x,y)$.

Indication of a proof. Part of assertion (b) follows from assertion (2) in Theorem 6.64 below with $M(t) = \exp\left(-\int_0^t V(X(s))\, ds\right)$. The proof of the symmetry and continuity of the integral kernel of the Feynman-Kac semigroup

$$\left\{\exp\left(-t\left(K_0 \dot{+} V\right)\right) : t \geqslant 0\right\}$$

is long and tedious, and requires stopping time arguments, and the fact that sets of the form $B \backslash B^r$, where B is a Borel subset of E, and B^r is the collection of regular points of B, are polar sets. For details and for the proof of assertion (c) the reader is referred to [**36**], Chapter 2, 3, and Appendix D. A hint that assertion (a) is true can be seen as follows. Let the function $f \in C_0(E)$ belong to the intersection of the domains of K_0 and V. Suppose that the function $u : (0,\infty) \times E \to \mathbb{C}$ satisfies

$$\frac{\partial u}{\partial t}(t,x) = -\left(K_0 + V\right) u(t,x), \quad \lim_{t \downarrow 0} u(t,x) = f(x). \tag{6.52}$$

Then the function $u(t,x)$ is given by the Feynman-Kac formula:

$$u(t,x) = \mathbb{E}_x\left[\exp\left(-\int_0^t V(X(s))\, ds\right) f\left(X(t)\right)\right], \ t \geqslant 0, \ x \in E. \tag{6.53}$$

A proof of the equality in (6.53) runs as follows. For $t > 0$ and $x \in E$ define the function $v_{t,x} : [0,t) \to \mathbb{C}$ by

$$v_{t,x}(s) = \mathbb{E}_x\left[u\left(t-s, X(s)\right) \exp\left(-\int_0^s V(X(\rho))\, d\rho\right)\right], \ 0 \leqslant s < t. \tag{6.54}$$

Then by Leibniz' rule and Proposition 6.54, with $-K_0$ instead of A, we infer

$$\begin{aligned}\frac{\partial v_{t,x}}{\partial s}(s,x) = \mathbb{E}_x&\left[\frac{\partial}{\partial s} u\left(t-\cdot, X(s)\right)(s) \exp\left(-\int_0^s V(X(\rho))\, d\rho\right)\right] \\ &- \mathbb{E}_x\left[K_0 u\left(t-s,\cdot\right)\left(X(s)\right) \exp\left(-\int_0^s V(X(\rho))\, d\rho\right)\right] \\ &- \mathbb{E}_x\left[V(X(s)) u\left(t-s, X(s)\right) \exp\left(-\int_0^s V(X(\rho))\, d\rho\right)\right]\end{aligned}$$

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$$= -\mathbb{E}_x \left[\left(\frac{\partial}{\partial t} + K_0 + V \right) u\left(t - s, \cdot\right) (X(s)) \exp\left(-\int_0^s V(X(\rho))\, d\rho \right) \right]$$
$$= 0. \tag{6.55}$$

In the final step in (6.55) we employed the first part of (6.52). Consequently, the function $s \mapsto v_{t,x}(s)$ does not depend on s. Hence we may conclude:

$$u(t,x) = v_{t,x}(0) = \lim_{s\uparrow t} v_{t,x}(s) = \lim_{s\uparrow t} \mathbb{E}_x \left[u\left(t - s, X(s)\right) \exp\left(-\int_0^s V(X(\rho))\, ds \right) \right]$$
$$= \mathbb{E}_x \left[\lim_{s\uparrow t} u\left(t - s, X(s)\right) \lim_{s\uparrow t} \exp\left(-\int_0^s V(X(\rho))\, ds \right) \right]$$

(apply the second part of (6.52))

$$= \mathbb{E}_x \left[f\left(X(t-)\right) \exp\left(-\int_0^t V(X(\rho))\, ds \right) \right]$$

(employ the equality $f\left(X(t-)\right) = f\left(X(t)\right)$, $\mathbb{P}_x$-almost surely)

$$= \mathbb{E}_x \left[f\left(X(t)\right) \exp\left(-\int_0^t V(X(\rho))\, ds \right) \right]. \tag{6.56}$$

The equality in (6.56) shows the claim we made above. Put

$$S(t)f(x) = \mathbb{E}_x \left[f\left(X(t)\right) \exp\left(-\int_0^t V(X(\rho))\, ds \right) f(X(t)) \right].$$

Then the Markov property implies that the family $\{S(t) :, t \geqslant 0\}$ has the semigroup property. By the right-continuity of paths it also follows that this semigroup is weakly contiuous, when viewed as a semigroup in $C_0(E)$. But then it turns out to be weakly continuous in $L^2\left(E, m\right)$. For this part to be true one employs Khas'minskii's lemma (see Theorem 6.65) and the density of the space $C_0(E) \cap L^2(E,m)$ in $L^2(E,m)$. Let $-\left(K_0 \dot{+} V\right)$ be the generator of this semigroup. Then $K_0 \dot{+} V$ extends $K_0 + V$. Let f belong to $D\left(K_)\right) \cap D(V)$. Then by Leibniz' rule and the properties of the operator K_0 we have

$$\frac{\partial}{\partial t} \mathbb{E}_x \left[f\left(X(t)\right) \exp\left(-\int_0^t V(X(\rho))\, ds \right) f(X(t)) \right]$$
$$= -\mathbb{E}_x \left[f\left(X(t)\right) \exp\left(-\int_0^t V(X(\rho))\, ds \right) \left(K_0 + V\right) f(X(t)) \right]. \tag{6.57}$$

From (6.57) and the definition of generator we see that $S(t)f$ belongs to $D\left(K_0 \dot{+} V\right)$ and that

$$\left(K_0 \dot{+} V\right) S(t)f = S(t) \left(K_0 + V\right) f. \tag{6.58}$$

By taking $t = 0$ in (6.58) we infer $\left(K_0 \dot{+} V\right) f = S(t) \left(K_0 + V\right)$. As a consequence we see that $\left(K_0 \dot{+} V\right)$ extends $K_0 + V$.

This completes a too brief outline of a proof of Theorem 6.56. □

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6.58. Remark. From our basic assumption it follows that the function V belongs to $L^1_{\mathrm{loc}}(E,m)$. It also follows that the quadratic form $\mathcal{E}_V$ is bounded from below.

A problem we consider is the following. Let V and W be Kato-Feller potentials. Give reasonable conditions on V and W in order that the differences $D(t) := \exp\left(-t\left(K_0 \dot{+} V\right)\right) - \exp\left(-t\left(K_0 \dot{+} W\right)\right)$, $t \geqslant 0$, are compact operators. A nice result we obtained reads as follows. For the existence and properties of the resolution of the identity, see Theorem 5.31 and Definition 5.27.

6.59. Theorem. *Let $\{E_0(\xi) : \xi \in \mathbb{R}\}$ be the spectral decomposition, or resolution of the identity corresponding to $K_0 \dot{+} V$ and let $\{E_1(\xi) : \xi \in \mathbb{R}\}$ be the spectral decomposition (resolution of the identity) corresponding to $K_0 \dot{+} W$. Let*

$$\{(\Omega, \mathcal{F}, \mathbb{P}_x), (X(t), t \geqslant 0), (\vartheta_t, t \geqslant 0), (E, \mathcal{E})\}$$

be the strong Markov process generated by $-K_0$. Suppose that, for some $t_0 > 0$, the function $\exp(-t_0 K_0)|W - V|$ *is bounded, or suppose that*

$$\limsup_{t \downarrow 0} \sup_{x \in E} \mathbb{E}_x \left[\left(\int_0^t (W(X(u)) - V(X(u)))\, du \right)^2 \right] = 0. \tag{6.59}$$

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The following assertions are equivalent:

(i) *For every bounded interval A the operator $E_0(A)(W-V)E_1(A)$ is compact;*
(ii) *For some $t>0$ (for all $t>0$) the operator*

$$\exp\left(-t\left(K_0\dot{+}V\right)\right)(W-V)\exp\left(-t\left(K_0\dot{+}W\right)\right)$$

is compact;
(iii) *For some $t>0$ (for all $t>0$), the operator $D(t)$ is compact.*

6.60. Remark. If $\lim_{t\downarrow 0}\sup_{x\in E}\int_0^t\left[\exp\left(-sK_0\right)|W-V|\right](x)\,ds=0$, then

$$\lim_{t\downarrow 0}\sup_{x\in E}\mathbb{E}_x\left[\left(\int_0^t\left(W(X(u))-V(x(u))\right)\,du\right)^2\right]=0. \tag{6.60}$$

This is a consequence of the Markov property.

6.61. Remark. An equality like (6.50) can probably be used for first order perturbations, where the Cameron-Martin formula is applicable. In such a case we probably have to deal with stochastic integrals instead of the process

$$t\mapsto\int_0^t\left(W(X(u))-V(X(u))\right)\,du.$$

6.62. Remark. Theorem 6.59 is probably not known, even in case we consider $K_0=H_0=-\frac{1}{2}\Delta$. So the corresponding process is Brownian motion.

6.63. Remark. We introduced Brownian motion as a Markov process with a certain transition function. It can also be introduced as a Gaussian process $\{X(t):t\geqslant 0\}$ (assume $\nu=1$) such that $\mathbb{E}\left[X(t)X(s)\right]=\min(s,t)$, or as a Lèvy process with negative definite function $\xi\mapsto\frac{1}{2}|\xi|^2$, or as a martingale with variation process $t\mapsto t$. It can also be seen as a weak limit of symmetric random walks: see, *e.g.*, Bhattacharya and Waymire [**15**].

Proof of Theorem 6.59. (i) $\Rightarrow$ (ii) Fix $t>0$. Operators of the form

$$\exp\left(-t\left(K_0\dot{+}V\right)\right)(W-V)\exp\left(-t\left(K_0\dot{+}W\right)\right)$$

can be approximated (in the uniform operator topology) by operators in the linear span of $\{E_0(A_0)(W-V)E_1(A_1):\ A_0,\ A_1 \text{ bounded interval }\}$.

(ii) $\rightarrow$ (iii) First we assume (6.59) to be satisfied. Fix $t>2\varepsilon>0$ and consider the difference:

$$\begin{aligned}&\int_0^t\exp\left(-u\left(K_0\dot{+}V\right)\right)(W-V)\exp\left(-(t-u)\left(K_0\dot{+}W\right)\right)\,du\\&\quad-\int_\varepsilon^{t-\varepsilon}\exp\left(-u\left(K_0\dot{+}V\right)\right)\exp\left(-(t-u)\left(K_0\dot{+}W\right)\right)\,du\\&=\int_0^\varepsilon\exp\left(-u\left(K_0\dot{+}V\right)\right)(W-V)\exp\left(-(\varepsilon-u)\left(K_0\dot{+}V\right)\right)\,du\\&\quad\exp\left(-(t-\varepsilon)\left(K_0\dot{+}W\right)\right)\end{aligned}$$

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$$
\begin{aligned}
&+\exp\left(-(t-\varepsilon)\left(K_0\dot{+}V\right)\right) \\
&\quad\int_0^{\varepsilon}\exp\left(-u\left(K_0\dot{+}V\right)\right)(W-V)\exp\left(-(\varepsilon-u)\left(K_0\dot{+}W\right)\right)\,du \\
&=\left(\exp\left(-\varepsilon\left(K_0\dot{+}V\right)\right)-\exp\left(-\varepsilon\left(K_0\dot{+}V\right)\right)\right)\exp\left(-(t-\varepsilon)\left(K_0\dot{+}W\right)\right) \\
&\quad+\exp\left(-(t-\varepsilon)\left(K_0\dot{+}V\right)\right)\left(\exp\left(-\varepsilon\left(K_0\dot{+}V\right)\right)-\exp\left(-\varepsilon\left(K_0\dot{+}W\right)\right)\right).
\end{aligned}
$$

Next we fix $f \in L^2(E,m)$ and we consider

$$
\begin{aligned}
&\left[\left(\exp\left(-\varepsilon\left(K_0\dot{+}V\right)\right)-\exp\left(-\varepsilon\left(K_0\dot{+}W\right)\right)\right)\exp\left(-(t-\varepsilon)\left(K_0\dot{+}W\right)\right)f\right](x) \\
&=\mathbb{E}_x\left[\left\{\exp\left(-\int_0^{\varepsilon}V(X(u))\,du\right)-\exp\left(-\int_0^{\varepsilon}W(X(u))\,du\right)\right\}\right. \\
&\quad\left.\times\mathbb{E}_{X(\varepsilon)}\left\{\exp\left(-\int_0^{t-\varepsilon}W(X(u))\,du\right)f(X(t-\varepsilon))\right\}\right] \\
&=\mathbb{E}_x\left[\int_0^1\exp\left(-\int_0^{\varepsilon}((1-s)V(X(u))+sW(X(u)))\,du\right)ds\right. \\
&\quad\times\int_0^{\varepsilon}(W(X(u))-V(X(u)))\,du \\
&\quad\left.\times\mathbb{E}_{X(\varepsilon)}\left\{\exp\left(-\int_0^{t-\varepsilon}W(X(u))\,du\right)f(X(t-\varepsilon))\right\}\right].
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\int\left|\left[\left(\exp\left(-\varepsilon\left(K_0\dot{+}V\right)\right)-\exp\left(-\varepsilon\left(K_0\dot{+}W\right)\right)\right)\exp\left(-(t-\varepsilon)\left(K_0\dot{+}W\right)\right)f\right](x)\right|^2dx \\
&\leqslant\int\mathbb{E}_x\left[\int_0^1\exp\left(-2\int_0^{\varepsilon}((1-s)V(X(u))+sW(X(u)))\,du\right)ds\right. \\
&\quad\left.\times\ \mathbb{E}_{X(\varepsilon)}\left\{\exp\left(-2\int_0^{t-\varepsilon}W(X(u))\,du\right)|f(X(t-\varepsilon))|^2\right\}\right] \\
&\quad\times\ \mathbb{E}_x\left[\left(\int_0^{\varepsilon}(W(X(u))-V(X(u)))\,du\right)^2\right]dx \\
&\leqslant\int_0^1\iiint\exp\left(-\varepsilon\left(K_0+2\left((1-s)V+sW\right)\right)\right)(x,z) \\
&\quad\times\exp\left(-(t-\varepsilon)\left(K_0+2W\right)\right)(z,y)\,|f(y)|^2 \\
&\quad\times\sup_{x\in E}\mathbb{E}_x\left[\left(\int_0^{\varepsilon}(W-V)(X(u))\,du\right)^2\right] \\
&\leqslant\int_0^1\sup_{z\in E}\mathbb{E}_z\left[\exp\left(-2\int_0^{\varepsilon}((1-s)V+sW)(X(u))\,du\right)\right]ds \\
&\quad\times\sup_{y\in E}\mathbb{E}_y\left[\exp\left(-2\int_0^{\varepsilon}W(X(u))\,du\right)\right]\times\sup_{x\in E}\mathbb{E}_x\left[\left(\int_0^{\varepsilon}(W-V)(X(u))\,du\right)^2\right].
\end{aligned}
$$

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From (ii) it follows that the operators

$$\int_{\varepsilon}^{t-\varepsilon} \exp\left(-u\left(K_0\dot{+}V\right)\right)\exp\left(-(t-u)\left(K_0\dot{+}W\right)\right)\,du,$$

$\varepsilon > 0$, are compact. This proves the implication (ii) $\Rightarrow$ (iii) in the presence of (6.59). In the other situation, where we assume that, for some $t_0 > 0$, the function $\exp\left(-t_0K_0\right)|W-V|$ is bounded, we proceed as follows. We shall estimate the L^2-L^2-norm of the operator

$$\exp\left(-t_0\left(K_0\dot{+}V\right)\right)|W-V|\exp\left(-t_0\left(K_0\dot{+}V\right)\right).$$

Therefore, fix $f \geqslant 0$ in $L^2(E,m)$. Then, by the Feynman-Kac formula and Cauchy-Schwartz' inequality, we have

$$\begin{aligned}
&\left(\exp\left(-t_0\left(K_0\dot{+}V\right)\right)f(x)\right)^2 = \left(\mathbb{E}_x\left[\exp\left(-\int_0^{t_0} V(X(s))\,ds\right)f(X(t_0))\right]\right)^2\\
&\leqslant \mathbb{E}_x\left[\exp\left(-2\int_0^{t_0} V(X(s))\,ds\right)\right]\mathbb{E}_x\left[f(X(t_0))^2\right]\\
&\leqslant M_{2V}\exp\left(t_0b_{2V}\right)\left[\exp\left(-t_0K_0\right)f^2\right](x). \qquad (6.61)
\end{aligned}$$

From (6.61) we get

$$\begin{aligned}
&\left\langle\exp\left(-t_0\left(K_0\dot{+}V\right)\right)|W-V|\exp\left(-t_0\left(K_0\dot{+}V\right)\right)f,f\right\rangle\\
&= \left\langle|W-V|\exp\left(-t_0\left(K_0\dot{+}V\right)\right)f,\exp\left(-t_0\left(K_0\dot{+}V\right)\right)f\right\rangle\\
&\leqslant M_{2V}\exp\left(t_0b_{2V}\right)\int|W(x)-V(x)|\left[\exp\left(-t_0K_0\right)f^2\right](x)\,dx\\
&= M_{2V}\exp\left(t_0b_{2V}\right)\left\langle|W-V|,\exp\left(-t_0K_0\right)f^2\right\rangle\\
&= M_{2V}\exp\left(t_0b_{2V}\right)\left\langle\exp\left(-t_0K_0\right)|W-V|,f^2\right\rangle\\
&\leqslant M_{2V}\exp\left(t_0b_{2V}\right)\left\|\exp\left(-t_0K_0\right)|W-V|\right\|_\infty\|f\|_2^2. \qquad (6.62)
\end{aligned}$$

From (6.62) we see that the operator

$$\exp\left(-t_0\left(K_0\dot{+}V\right)\right)|W-V|\exp\left(-t_0\left(K_0\dot{+}V\right)\right) \qquad (6.63)$$

is bounded as an operator from $L^2(E,m)$ to $L^2(E,m)$. By the same token the operator

$$\exp\left(-t_0\left(K_0\dot{+}W\right)\right)|W-V|\exp\left(-t_0\left(K_0\dot{+}W\right)\right)$$

is bounded as well. Fix $\gamma \in \mathbb{R}$ in such a way that, in form sense, $\gamma I + K_0\dot{+}V \geqslant 0$ and $\gamma I + K_0\dot{+}W \geqslant 0$. From (6.63) it follows that operators of the form

$$\exp\left(-t_0\left(K_0\dot{+}V\right)\right)\left(\gamma I + K_0\dot{+}W\right)^{1/2} \text{ and } \left(\gamma I + K_0\dot{+}V\right)^{1/2}\exp\left(-t_0\left(K_0\dot{+}W\right)\right)$$

are bounded. As a consequence, operators of the form

$$E_0(A_0)\left(\gamma I + K_0\dot{+}W\right)^{1/2} \text{ and } \left(\gamma I + K_0\dot{+}V\right)^{1/2}E_1(A_1),$$

where A_0 and A_1 are bounded intervals, are bounded. It follows that

$$\|E_0(A_0)E_1(m,\infty)\|_{2,2}$$

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$$= \left\| E_0(A_0) \left(\gamma I + K_0 \dot{+} W\right)^{1/2} \left(\gamma I + K_0 \dot{+} W\right)^{-1/2} E_1(m, \infty) \right\|_{2,2}$$
$$\leqslant \left\| E_0(A_0) \left(\gamma I + K_0 \dot{+} W\right)^{1/2} \right\|_{2,2} \left\| \left(\gamma I + K_0 \dot{+} W\right)^{-1/2} E_1(m, \infty) \right\|_{2,2}$$

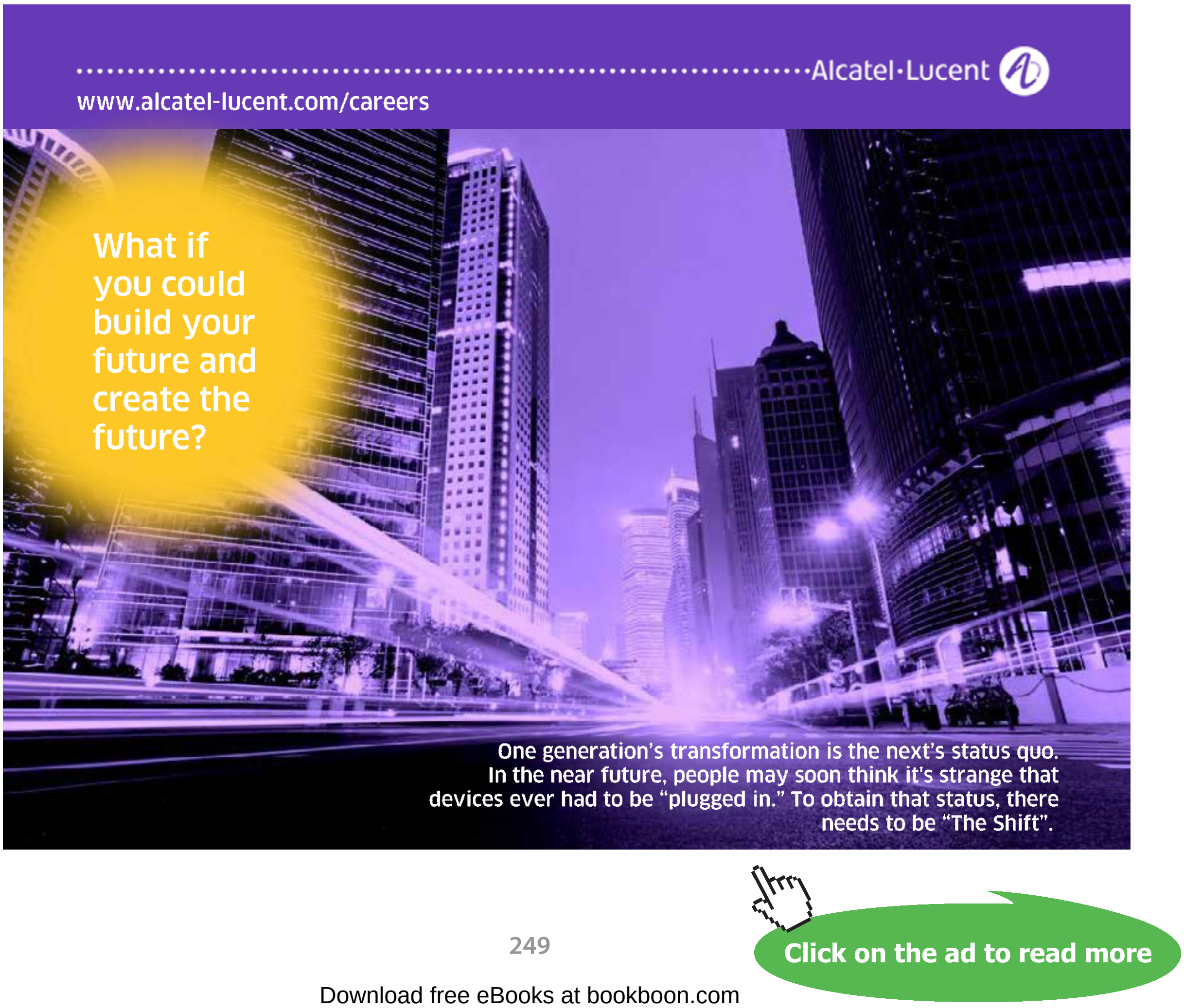
converges to zero, if m tends to ∞ and if A_0 is a bounded interval. The same is true for $\|E_0(m,\infty)E_1(A_1)\|_{2,2}$, if m tends to ∞, and if A_1 is a bounded interval. We may conclude that, for $t > 0$ fixed,

$$\lim_{\varepsilon \downarrow 0} \left\| \exp\left(-t\left(K_0 \dot{+} V\right)\right) \left(I - \exp\left(-\varepsilon\left(K_0 \dot{+} W\right)\right)\right) \right\|_{2,2}$$
$$= \lim_{\varepsilon \downarrow 0} \left\| \left(I - \exp\left(-\varepsilon\left(K_0 \dot{+} V\right)\right)\right) \exp\left(-t\left(K_0 \dot{+} W\right)\right) \right\|_{2,2} = 0.$$

The previous identities yield the following result

$$\lim_{\varepsilon \downarrow 0} \left\| D(t) - \exp\left(-\varepsilon\left(K_0 \dot{+} V\right)\right) D(t) \exp\left(-\varepsilon\left(K_0 \dot{+} W\right)\right) \right\|_{2,2} = 0,$$

where $D(t) = \exp\left(-t\left(K_0 \dot{+} V\right)\right) - \exp\left(-t\left(K_0 \dot{+} W\right)\right)$. Since, by (ii), the operators $\exp\left(-\varepsilon\left(K_0 \dot{+} V\right)\right) D(t) \exp\left(-\varepsilon\left(K_0 \dot{+} W\right)\right)$ $\varepsilon > 0$ are compact, assertion (iii) follows.



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(iii) $\Rightarrow$ (ii). This implication follows from the equality

$$
\begin{aligned}
&t \exp\left(-\frac{t}{2}\left(K_0 \dot{+} V\right)\right)(W-V)\exp\left(-\frac{t}{2}\left(K_0 \dot{+} W\right)\right)\\
&= \frac{\pi}{2t}\int_{-\infty}^{\infty} \frac{1}{\left(\cosh\left(\pi\tau/t\right)\right)^2}\exp\left(i\tau\left(K_0 \dot{+} V\right)\right)\\
&\qquad \int_0^t \exp\left(-u\left(K_0 \dot{+} V\right)\right)(W-V)\exp\left(-(t-u)\left(K_0 \dot{+} W\right)\right)\,du\\
&\qquad \exp\left(-i\tau\left(K_0 \dot{+} W\right)\right)\,d\tau.
\end{aligned}
$$

(ii) $\Rightarrow$ (i) This implication is a consequence of the identity

$$
\begin{aligned}
E_0(A_0)(W-V)E_1(A_1) = &E_0(A_0)\exp\left(t\left(K_0 \dot{+} V\right)\right)\\
&\exp\left(-t\left(K_0 \dot{+} V\right)\right)(W-V)\exp\left(-t\left(K_0 \dot{+} W\right)\right)\\
&\exp\left(t\left(K_0 \dot{+} W\right)\right)E_1(A_1),
\end{aligned}
$$

for A_0 and A_1 bounded intervals. Moreover, for bounded Borel sets A_0 and A_1, the operators $E_0(A_0)\exp\left(t\left(K_0 \dot{+} V\right)\right)$ and $E_1(A_1)\exp\left(t\left(K_0 \dot{+} W\right)\right)$ are bounded. $\square$

The following result is applicable for

$$
M(t) = \exp\left(-\int_0^t V(X(u))\,du\right) \text{ or } M(t) = \exp\left(-\int_0^t V(X(u))\,du\right)1_{\{S>t\}},
$$

where V is a Kato-Feller potential, and where S is a terminal stopping times, *i.e.* $t + S\circ\vartheta_t = S$ $\mathbb{P}_x$-almost surely on the event $\{S > t\}$. Theorem 6.64 shows part (b) of Theorem 6.56.

6.64. Theorem. *Let $\{M(t) : t \geqslant 0\}$ be a multiplicative process taking its values in $[0,\infty)$. This means that for every $t \geqslant 0$, $M(t) : \Omega \to [0,\infty)$ is $\mathcal{F}_t$-measurable and that $M(s+t) = M(s)M(t)\circ\vartheta_s$ for all s and $t \geqslant 0$. We assume*

$$
\lim_{\epsilon\downarrow 0}\int M(t-\epsilon)\,d\mu_{0,x}^{t,y} = \int M(t)\,d\mu_{0,x}^{t,y}.
$$

As above, the defining property of $\mu_{0,x}^{t,y}$ is the equality

$$
\int F d\mu_{0,x}^{t,y} = \mathbb{E}_x\left[F p_0(t-s, X(s), y)\right],
$$

where $F : \Omega \to \mathbb{R}$ is bounded and $\mathcal{F}_s$-measurable ($s < t$). The following assertions are valid:

(1) *The process*

$$
s \mapsto M(s)\int M(t-s)\,d\mu_{0,X(s)}^{t,y}
$$

is a $\mathbb{P}_x$-martingale on the interval $[0,t)$.

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(2) *The following equality is valid:*

$$\mathbb{E}_x\left[M(t)f(X(t))\right] = \iint M(t)\, d\mu_{0,x}^{t,y} f(y)\, dy,$$

where f is greater than or equal to zero and Borel measurable.

(3) *The following Chapman-Kolmogorov identity is valid:*

$$\iint M(t_1)\, d\mu_{0,x}^{t_1,z} \int M(t_2)\, d\mu_{0,z}^{t_2,y} dz = \int M\left(t_1+t_2\right)\, d\mu_{0,x}^{t_1+t_2,y}.$$

As mentioned earlier, the quantity $\mu_{0,t}^{x,y}$ could be called the un-normalized Markov bridge kernel.

Proof. (1) Let $t > s_2 > s_1 \geqslant 0$ and fix $0 < \varepsilon < t - s_2$. Then

$$\mathbb{E}_x\left[M(s_2)\int M(t-s_2-\varepsilon)\, d\mu_{0,X(s_2)}^{t-s_2,y} \mid \mathcal{F}_{s_1}\right]$$

$$= \mathbb{E}_x\left[M(s_1)\left\{M(s_2-s_1)\int M(t-s_2-\varepsilon)\, d\mu_{0,X(s_2-s_1)}^{t-s_2,y}\right\}\circ\vartheta_{s_1} \mid \mathcal{F}_{s_1}\right]$$

(Markov property)

$$= M(s_1)\mathbb{E}_{X(s_1)}\left\{M(s_2-s_1)\int M(t-s_2-\varepsilon)\, d\mu_{0,X(s_2-s_1)}^{t-s_2,y}\right\}$$

(definition of $\mu_{0,z}^{t-s_1,y}$)

$$= M(s_1)\mathbb{E}_{X(s_1)}\left\{M(s_2-s_1)\mathbb{E}_{X(s_2-s_1)}\left[M(t-s_2-\varepsilon)p_0(\varepsilon, X(t-s_2-\varepsilon), y)\right]\right\}$$

(Markov property)

$$= M(s_1)\mathbb{E}_{X(s_1)}\left\{M(s_2-s_1)M(t-s_2-\varepsilon)\circ\vartheta_{s_2-s_1}p_0(\varepsilon, X(t-s_1-\varepsilon), y)\right\}$$

(the process $M(t)$, $t \geqslant 0$, is multiplicative)

$$= M(s_1)\mathbb{E}_{X(s_1)}\left\{M(t-s_1-\varepsilon)p_0(\varepsilon, X(t-s_1-\varepsilon), y)\right\}$$

(definition of $\mu_{0,z}^{t-s_1,y}$)

$$= M(s_1)\int M(t-s_1-\varepsilon)\, d\mu_{0,X(s_1)}^{t-s_1,y}.$$

Finally we let ε tend to zero to obtain (1).

(2) Fix $0 < \varepsilon < t$ and consider

$$\iint M(t-\varepsilon)\, d\mu_{0,x}^{t,y} f(y)\, dy$$

$$\text{(definition } \mu_{0,x}^{t,y}) \quad = \int \mathbb{E}_x\left[M(t-\varepsilon)p_0(\varepsilon, X(t-\varepsilon), y)\right] f(y)\, dy$$

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$$\text{(Fubini)} \quad = \mathbb{E}_x \left[M(t-\varepsilon) \int p_0(\varepsilon, X(t-\varepsilon), y) f(y)\, dy \right]$$

(basic formula: $\mathbb{E}_x \left[f(X(s)) \right] = \left[\exp(-tK_0) f \right](x) = \int p_0(s,x,y) f(y)\, dy$)

$$= \mathbb{E}_x \left[M(t-\varepsilon) \mathbb{E}_{X(t-\varepsilon)} \left\{ f(X(\varepsilon)) \right\} \right]$$

$$\text{(Markov property)} \quad = \mathbb{E}_x \left[M(t-\varepsilon) f(X(t)) \right].$$

Finally, let ε tend to zero to obtain (2).

(3) By assertion (2) we have

$$\begin{aligned} &\int \left(\int M(t_1)\, d\mu_{0,x}^{t_1,z} \int M(t_2)\, d\mu_{0,z}^{t_2,y} \right) dz = \mathbb{E}_x \left[M(t_1) \int M(t_2)\, d\mu_{0,X(t_1)}^{t_2,y} \right] \\ &= \mathbb{E}_x \left[\mathbb{E}_x \left\{ M(t_1) \int M(t_1+t_2-t_1) d\mu_{0,X(t_1)}^{t_1+t_2-t_1,y} \mid \mathcal{F}_0 \right\} \right] \end{aligned}$$

(martingale property: $t_1 \to 0$)

$$\begin{aligned} &= \mathbb{E}_x \left[M(0) \int M(t_1+t_2) d\mu_{0,X(0)}^{t_1+t_2,y} \right] \\ &= \mathbb{E}_x \left[M(0) \right] \int M(t_1+t_2) d\mu_{0,x}^{t_1+t_2,y}. \end{aligned}$$

We notice

$$\begin{aligned} &\int M(t)\, d\mu_{0,x}^{t,z} = \lim_{\varepsilon \downarrow 0} \int M(t-\varepsilon) d\mu_{0,x}^{t,z} \\ &= \lim_{\varepsilon \downarrow 0} \mathbb{E}_x \left[M(t-\varepsilon) p_0(\varepsilon, X(t-\varepsilon), y) \right] \\ &= \lim_{\varepsilon \downarrow 0} \mathbb{E}_x \left[M(0) M(t-\varepsilon) \circ \vartheta_0 p_0(\varepsilon, X(t-\varepsilon), y) \circ \vartheta_0 \right] \\ &= \lim_{\varepsilon \downarrow 0} \mathbb{E}_x \left[M(0) \mathbb{E}_{X(0)} \left\{ M(t-\varepsilon) p_0(\varepsilon, X(t-\varepsilon), y) \right\} \right] \\ &= \lim_{\varepsilon \downarrow 0} \mathbb{E}_x \left[M(0) \right] \mathbb{E}_x \left[M(t-\varepsilon) p_0(\varepsilon, X(t-\varepsilon), y) \right] \\ &= \mathbb{E}_x \left[M(0) \right] \lim_{\varepsilon \downarrow 0} \int M(t-\varepsilon)\, d\mu_{0,x}^{t,z} \\ &= \mathbb{E}_x \left[M(0) \right] \int M(t)\, d\mu_{0,x}^{t,z}. \end{aligned}$$

Hence, the Chapman-Kolmogorov equality

$$\iint M(t_1)\, d\mu_{0,x}^{t_1,z} \int M(t_2)\, d\mu_{0,z}^{t_2,y} dz = \int M\left(t_1+t_2\right)\, d\mu_{0,x}^{t_1+t_2,y}$$

holds indeed. Finally we also notice that

$$\begin{aligned} &\mathbb{E}_x \left[M(0) \right] = \mathbb{E}_x \left[M(0) M(0) \circ \vartheta_0 \right] \\ &= \mathbb{E}_x \left[M(0) \mathbb{E}_{X(0)} \left\{ M(0) \right\} \right] = \left(\mathbb{E}_x \left[M(0) \right] \right)^2 = 1_\Sigma(x), \end{aligned}$$

where $\Sigma = \{ x \in E : \mathbb{E}_x \left[M(0) \right] = 1 \}$.

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The proof of Theorem 6.59 is complete now. $\square$

6.65. THEOREM (Khas'minskii's Lemma: see Simon [**126**]). *Let* $W : E \to [0, \infty]$ *be a Borel measurable function. Put* $\gamma = \limsup_{t \downarrow 0} \sup_{x \in E} \mathbb{E}_x \left[\int_0^t W(X(s))\,ds \right]$*, and suppose* $\gamma < 1$*. The following assertions are true:*

(1) $\gamma = \lim_{a \to \infty} (aI + K_0)^{-1} W(x)$.

(2) *Choose* $t_0 > 0$ *in such a way that* $\alpha := \sup_{x \in E} \mathbb{E}_x \left[\int_0^{t_0} W(X(s))\,ds \right] < 1$*. Then*

$$\sup_{x \in E} \mathbb{E}_x \left[\exp\left(\int_0^{t_0} W(X(s))\,ds \right) \right] \leqslant \frac{1}{1-\alpha}.$$

(3) *Let* t_0 *and* α *be as in (2). Put* $M = \dfrac{1}{1-\alpha}$ *and* $e^b = \left(\dfrac{1}{1-\alpha} \right)^{1/t_0}$*. Then*

$$\mathbb{E}_x \left[\exp\left(\int_0^t W(X(s))\,ds \right) \right] \leqslant M \exp(bt), \quad x \in E, \quad t \geqslant 0.$$

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Proof. (1) Fix $\eta > 0$ and choose $t_0 > 0$ so small that

$$\gamma + \eta > \sup_{x\in E} \mathbb{E}_x \left[\int_0^{t_0} W(X(s))\, ds \right].$$

Then we have

$$\begin{aligned}
&(aI + K_0)^{-1} W(x) = \int_0^\infty e^{-as} \mathbb{E}_x \left[W(X(s)) \right] ds \\
&= a \int_0^\infty e^{-as} \mathbb{E}_x \left[\int_0^t W(X(\sigma))\, d\sigma \right] ds \\
&\leqslant a \int_0^{t_0} e^{-as} \mathbb{E}_x \left[\int_0^{t_0} W(X(\sigma))\, d\sigma \right] ds \\
&\quad + a \sum_{k=2}^\infty \int_{(k-1)t_0}^{kt_0} e^{-a(k-1)t_0} \sum_{j=1}^k \mathbb{E}_x \left[\int_{(j-1)t_0}^{jt_0} W(X(\sigma))\, d\sigma \right] ds \\
&\leqslant a \int_0^{t_0} e^{-as} \mathbb{E}_x \left[\int_0^{t_0} W(X(\sigma))\, d\sigma \right] ds \\
&\quad + at_0 \sum_{k=2}^\infty e^{-a(k-1)t_0} \sum_{j=1}^k \mathbb{E}_x \left[\mathbb{E}_{X((j-1)t_0)} \left\{ \int_0^{t_0} W(X(\sigma))\, d\sigma \right\} \right] \\
&\leqslant \left(1 + at_0 \exp(-at_0) \frac{2 - \exp(-at_0)}{\left(1 - \exp(-at_0)\right)^2} \right) (\gamma + \eta).
\end{aligned}$$

Since $\eta > 0$ is arbitrary, it follows that $\limsup_{a\to\infty} (aI + K_0)^{-1} W(x) \leqslant \gamma$.

In order to prove the reverse inequality we fix $\varepsilon > 0$ and notice the inequality

$$\begin{aligned}
&\mathbb{E}_x \left[\int_0^{\varepsilon/a} W(X(\sigma))\, d\sigma \right] \leqslant ae^\varepsilon \int_0^\infty e^{-as} \mathbb{E}_x \left[\int_0^s W(X(\sigma))\, d\sigma \right] ds \\
&= e^\varepsilon \int_0^\infty e^{-as} \mathbb{E}_x \left[W(X(s)) \right] ds = e^\varepsilon (aI + K_0)^{-1} W(x).
\end{aligned}$$

(2) Upon using the expansion of the exponential and employing the Markov property we see that

$$\begin{aligned}
&\mathbb{E}_x \left[\exp\left(\int_0^{t_0} W(X(s))\, ds \right) \right] \\
&= 1 + \sum_{k=1}^\infty \mathbb{E}_x \left[\int_{0<s_1<\dots<s_{k-1}<t} \int ds_1 \dots ds_{k-1} W(X(s_1)) \dots W(X(s_{k-1})) \right. \\
&\qquad \left. \mathbb{E}_{X(s_{k-1})} \left\{ \int_0^{t_0 - s_{k-1}} W(X(s))\, ds \right\} \right] \\
&\leqslant 1 + \sum_{k=1}^\infty \mathbb{E}_x \left[\int_{0<s_1<\dots<s_{k-1}<t} \int ds_1 \dots ds_{k-1} W(X(s_1)) \dots W(X(s_{k-1})) \right] \alpha
\end{aligned}$$

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$$\leqslant 1 + \sum_{k=1}^{\infty} \alpha^k = \frac{1}{1-\alpha}.$$

This proves assertion 2.

(3) Fix $t > 0$ and choose $k \in \mathbb{N}$ in such a way that $kt_0 \leqslant t < (k+1)t_0$. From the Markov property we infer:

$$\begin{aligned}
&\mathbb{E}_x\left[\exp\left(\int_0^t W(X(s))\,ds\right)\right] \\
&= \mathbb{E}_x\left[\exp\left(\int_0^{kt_0} W(X(s))\,ds\right)\mathbb{E}_{X(kt_0)}\left\{\exp\left(\int_0^{t-kt_0} W(X(s))\,ds\right)\right\}\right] \\
&\leqslant \mathbb{E}_x\left[\exp\left(\int_0^{kt_0} W(X(s))\,ds\right)\right]\frac{1}{1-\alpha} \\
&\leqslant \left(\frac{1}{1-\alpha}\right)^{k+1} \leqslant \frac{1}{1-\alpha}\left(\frac{1}{1-\alpha}\right)^{t/t_0} = Me^{bt}.
\end{aligned}$$

This completes the proof of Theorem 6.65. □

For the convenience of the reader we insert a proof of the Stein and the Riesz-Thorin interpolation theorems. The first theorem is the same as Theorem 4.12.

6.66. Theorem (Theorem of Riesz-Thorin). *Let $(E_0, \mathcal{A}_0, m_0)$ and $(E_1, \mathcal{A}_1, m_1)$ be σ-finite measure spaces, and let*

$$T: L^{p_0}(E_0, \mathcal{A}_0, m_0) + L^{p_1}(E_0, \mathcal{A}_0, m_0) \to L^{q_0}(E_1, \mathcal{A}_1, m_1) + L^{q_1}(E_1, \mathcal{A}_1, m_1)$$

be a linear operator such that

$$T \in \mathcal{L}\left(L^{p_0}(E_0, \mathcal{A}_0, m_0), L^{q_0}(E_1, \mathcal{A}_1, m_1)\right) \cap \mathcal{L}\left(L^{p_1}(E_0, \mathcal{A}_0, m_0), L^{q_1}(E_1, \mathcal{A}_1, m_1)\right).$$

Define, for $0 < t < 1$, p_t and q_t by

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \quad \textit{and} \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}. \tag{6.64}$$

Then $T \in \mathcal{L}\left(L^{p_t}(E_0, \mathcal{A}_0, m_0), L^{q_t}(E_1, \mathcal{A}_1, m_1)\right)$, and setting $M_i = \|T\|_{q_i,p_i}$, $i = 0, 1$, then $\|T\|_{q_t,p_t} \leqslant M_0^{1-t} M_1^t$, $0 \leqslant t \leqslant 1$. In the case that some of the p_i's or the q_i's is ∞ the statement still holds if we set, as usual, $\dfrac{1}{\infty} = 0$.

Recall that the set of the simple functions (= finite linear combinations of indicator functions of measurable sets with finite measure) $a : E_0 \to \mathbb{C}$ is dense in $L^p(E_0, \mathcal{A}_0, m_0)$ or $1 \leqslant p < \infty$, and, for the same reason, the set of the simple functions $b : E_1 \to \mathbb{C}$ is dense in $L^q(E_1, \mathcal{A}_1, m_1)$, for every $q \in [0, \infty)$. Moreover, for each measurable function $f : E_1 \to \mathbb{C}$ we have

$$\|f\|_{L^q} = \sup \frac{1}{\|b\|_{L^{q'}}}\left|\int_{E_1} f(x)b(x)\,dm_1(x)\right|,$$

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where the supremum is taken over all simple function b. Here q' is the conjugate exponent of q: $\dfrac{1}{q} + \dfrac{1}{q'} = 1$. For a concise formulation of the Stein interpolation theorem we introduce the notion of a holomorphic, or analytic, family of operators.

6.67. Definition. Let $S = \{0 \leqslant \Re z \leqslant 1\}$ be the closed unit strip in the complex plane, and let $z \mapsto T(z)$ be family of linear operators defined on the space of simple functions on $(E_0, \mathcal{A}_0, m_0)$. This operator family is called analytic (or holomorphic) if for every pair of simple functions $a : E_0 \to \mathbb{C}$ and $b : E_1 \to \mathbb{C}$, the product $[T(z)a] \cdot b$ is m_1-integrable and the function $z \mapsto \int_{E_1} [T(z)a](x) b(x)\, dm_1(x)$ is continuous and bounded in S, and holomorphic in the interior of S.

Now we formulate the Stein interpolation theorem.

6.68. Theorem (Stein interpolation theorem). *Assume that for every $z \in S$, $T(z)$ is a linear operator defined in the set of the simple functions on E_0, with values in the measurable functions on E_1, such that the function $z \mapsto T(z)$ is holomorphic in the sense of Definition 6.67. Moreover, assume that for some p_j, $q_j \in [1, \infty]$, $j = 0$, 1, the inequalities*

$$\|T(it)a\|_{L^{q_0}} \leqslant M_0 \|a\|_{L^{p_0}}, \quad \text{and} \quad \|T(1+it)a\|_{L^{q_1}} \leqslant M_1 \|a\|_{L^{p_1}}, \quad t \in \mathbb{R} \tag{6.65}$$

hold for every simple function a, and for some finite constants M_0 and M_1. Then for each $t \in (0,1)$, $T(t)$ may be extended to a bounded linear operator, still called $T(t)$), from $L^{p_t}(E_0, \mathcal{A}_0, m_0)$ to $L^{q_t}(E_1, \mathcal{A}_1, m_1)$, with p_t and q_t defined as in (6.64), and $\|T(t)\|_{q_t, p_t} \leqslant M_0^{1-t} M_1^t$.

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Of course, if $T(z) = T$, then, essentially speaking, Theorem 6.68 reduces to Theorem 6.66. A proof of the Stein interpolation theorem may be based on the three line lemma in complex analysis. This lemma reeds as follows. As above, S denotes the closed strip $\{0 \leqslant \Re z \leqslant 1\}$.

6.69. PROPOSITION (Three line lemma). *Let $F : S \to \mathbb{C}$ be a bounded and continuous on S, and let F be analytic on the interior of S. For $0 \leqslant t \leqslant 1$, put*

$$M_t = \sup \{|F(t + iy)| : y \in \mathbb{R}\} .$$

Then the inequality $M_t \leqslant M_0^{1-t} M_1^t$ holds for all $0 \leqslant t \leqslant 1$.

PROOF. The three line lemma follows by applying the maximum modulus theorem to the holomorphic function $F_\varepsilon(z)$, $\varepsilon > 0$, defined by

$$F_\varepsilon(z) = \frac{F(z)}{1 + \varepsilon z} \frac{1}{\alpha^{1-z}\beta^z}, \quad z \in S,$$

where $\alpha > M_0$, $\beta > M_1$. Then $|F_\varepsilon(z)| \leqslant 1$, and hence $|F(z)| \leqslant a^{1-\Re z}\beta^{\Re z}$. By letting α tend to M_0 and β to M_1 we obtain the desired result. $\square$

PROOF OF THEOREM 6.68. For every pair of simple functions $a : E_0 \to \mathbb{C}$, and $b : E_1 \to \mathbb{C}$, we apply the three lines theorem to the function the function $F(z) = \int_{E_1} T(z)f(z)(x)g(z)(x)\, dm_1(x)$, $z \in S$, where f and g are defined by

$$f(z)(x) = \begin{cases} |a(x)|^{p_t\left(\frac{1-z}{p_0} + \frac{z}{p_1}\right)} \frac{a(x)}{|a(x)|}, & \text{if } x \in E_0,\ a(x) \neq 0; \\ 0, & \text{if } x \in E_0,\ a(x) = 0, \end{cases}$$

and

$$g(z)(x) = \begin{cases} |b(x)|^{q_t'\left(\frac{1-z}{q_0'} + \frac{z}{q_1'}\right)} \frac{b(x)}{|b(x)|}, & \text{if } x \in E_1,\ b(x) \neq 0; \\ 0, & \text{if } x \in E_1,\ b(x) = 0. \end{cases} \tag{6.66}$$

Then

$$\begin{aligned} |F(iy)| &\leqslant \int_{E_1} |T(iy)f(iy)(x)g(iy)(x)|\, dm_1(x) \leqslant \|T(iy)f(iy)\|_{L^{q_0}} \|g(iy)\|_{L^{q_0'}} \\ &\leqslant \|T(iy)\|_{q_0,p_0} \|f(iy)\|_{L^{p_0}} \|g(iy)\|_{L^{q_0'}} \leqslant \|T(iy)\|_{q_0,p_0} \|a\|_{L^{p_t}}^{p_t/p_0} \|b\|_{L^{q_t'}}^{q_t'/q_0'}, \end{aligned} \tag{6.67}$$

and, similarly,

$$\begin{aligned} |F(1 + iy)| &\leqslant \int_{E_1} |T(1 + iy)f(1 + iy)(x)g(1 + iy)(x)|\, dm_1(x) \\ &\leqslant \|T(1 + iy)f(1 + iy)\|_{L^{q_1}} \|g(1 + iy)\|_{L^{q_1'}} \\ &\leqslant \|T(1 + iy)\|_{q_1,p_1} \|f(1 + iy)\|_{L^{p_1}} \|g(1 + iy)\|_{L^{q_1'}} \\ &\leqslant \|T(1 + iy)\|_{q_1,p_1} \|a\|_{L^{p_t}}^{p_t/p_1} \|b\|_{L^{q_t'}}^{q_t'/q_1'} . \end{aligned} \tag{6.68}$$

We get

$$|F(t)| = \left| \int_{E_1} [T(t)(a)]\,(x)b(x)\, dm_1(x) \right| \leqslant M_0^{1-t} M_1^{1-t} \|a\|_{L^{p_t}} \|b\|_{L^{q_t'}}, \tag{6.69}$$

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so that

$$\|T(t)a\|_{L^{q_t}} \leqslant M_0^{1-t} M_1^t \|a\|_{L^{p_t}}$$

for every simple a defined in E_0. Since the set of such functions a is dense in L^p the statement in Theorem 6.68 follows. □

As noticed earlier, if $T(z) \equiv T$ in Theorem 6.68, then we get the Riesz-Thorin interpolation theorem 6.66. For a proof of this result the reader is referred to Reed and Simon [**106**], Theorem IX.21 page 40. Other sources of information are Lunardi [**87, 88**]. The paper by Stein [**131**] is the origin of "Stein" interpolation.

6.70. Proposition. *Let $T : L^1(E,m) \to L^1(E,m)$ be a continuous linear map, with norm $\|T\|_{1,1}$. Let it also be a continuous linear map from $L^\infty(E,m)$ to $L^\infty(E,m)$, with norm $\|T\|_{\infty,\infty}$. Then T is a continuous linear map from $L^p(E,m)$ to $L^p(E,m)$ for which $\|T\|_{p,p} \leqslant \|T\|_{1,1}^{1/p} \|T\|_{\infty,\infty}^{1-1/p}$.*

Proof. Apply the Riesz-Thorin interpolation theorem with

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \quad \text{where} \quad p_0 = 1, \quad p_1 = \infty;$$
$$\frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}, \quad \text{where} \quad q_0 = 1, \quad q_1 = \infty.$$

Then

$$\|T\|_{q_t,p_t} \leqslant \|T\|_{q_0,p_0}^{1-t} \|T\|_{q_1,p_1}^{t}.$$

With $t = 1 - 1/p$ we obtain the desired result in Proposition 6.70. □

6.71. Proposition. *Let $T : L^1(E,m) \to L^1(E,m)$ be a continuous linear map, with norm $\|T\|_{1,1}$. Let it also be a continuous linear map from $L^\infty(E,m)$ to $L^\infty(E,m)$, with norm $\|T\|_{\infty,\infty}$. In addition, suppose that it maps $L^1(E,m)$ to $L^\infty(E,m)$, with norm $\|T\|_{\infty,1}$. Then T is a continuous linear mapping from $L^p(E,m)$ to $L^q(E,m)$, where $1 \leqslant p \leqslant q \leqslant \infty$. Its norm $\|T\|_{q,p}$ obeys:*

$$\|T\|_{q,p} \leqslant \|T\|_{1,1}^{1/q} \|T\|_{\infty,1}^{1/p-1/q} \|T\|_{\infty,\infty}^{1-1/p}.$$

Proof. We suppose that $q > p$. Put $t = 1 - \frac{1}{q}$, $r = \frac{p(q-1)}{q-p}$, and $s = 1 - \frac{1}{r}$. Then

$$\frac{1}{p} = \frac{1-t}{1} + \frac{t}{r}, \quad \frac{1}{q} = \frac{1-t}{1} + \frac{t}{\infty};$$
$$\frac{1}{r} = \frac{1-s}{1} + \frac{s}{\infty}, \quad \frac{1}{\infty} = \frac{1-s}{\infty} + \frac{t}{\infty}.$$

Hence, by Riesz-Thorin interpolation (twice),

$$\|T\|_{q,p} \leqslant \|T\|_{1,1}^{1-t} \|T\|_{\infty,r}^{t} \leqslant \|T\|_{1,1}^{1-t} \left(\|T\|_{\infty,1}^{1-s} \|T\|_{\infty,\infty}^{s}\right)^t = \|T\|_{1,1}^{1/q} \|T\|_{\infty,1}^{1/p-1/q} \|T\|_{\infty,\infty}^{1-1/p}.$$

This completes the proof of Proposition 6.71. □

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6.72. Corollary. *Let $T : L^1(E,m) \to L^1(E,m)$ be a continuous linear map, with norm $\|T\|_{1,1}$. Let it also be a continuous linear map from $L^\infty(E,m)$ to $L^\infty(E,m)$, with norm $\|T\|_{\infty,\infty}$. In addition, suppose that it maps $L^1(E,m)$ to $L^\infty(E,m)$, with norm $\|T\|_{\infty,1}$. Then T is a continuous linear mapping from $L^p(E,m)$ to $L^q(E,m)$, where $1 \leqslant p \leqslant q \leqslant \infty$. Moreover suppose that $T = S^2$, where $S = S^*$. Its norm $\|T\|_{q,p}$ obeys:*

$$\|T\|_{q,p} \leqslant \|T\|_{\infty,\infty}^{1-(1/p-1/q)} \|S\|_{\infty,2}^{2(1/p-1/q)}.$$

Proof. Since $T = SS^*$, it follows that

$$\|T\|_{\infty,1} = \|SS^*\|_{\infty,1} \leqslant \|S\|_{\infty,2} \|S^*\|_{2,1} = \|S\|_{\infty,2}^2.$$

From this inequality, together with

$$\|T\|_{1,1} = \|T^*\|_{\infty,\infty} = \|T\|_{\infty,\infty}$$

the result in Corollary 6.72 follows. □

The previous results can be applied to Feynman-Kac semigroups. Put $T(t) = \exp\left(-t\left(K_0 \dot{+} V\right)\right)$. Then $T(t) = T(t)^* = T(t/2)T(t/2)$. We also write $S(z) = V_-^{1-z}\left(aI + K_0\right)^{-1} V_-^z$, $0 \leqslant \Re z \leqslant 1$. Let M and b in $\mathbb{R}$ be such that

$$\|T(t)\|_{\infty,\infty} = \|T(t)1\|_\infty = \sup_{x\in E} \mathbb{E}_x\left[\exp\left(\int_0^t V_-(X(s))\,ds\right)\right] \leqslant Me^{bt}.$$

From Khas'minskii's lemma (Theorem 6.65) it follows that such constants exist.

6.73. Theorem. *Let V be a Kato-Feller potential. The following assertions are valid.*

(1) *The operator $\exp\left(-t\left(K_0 \dot{+} V\right)\right)$ is a mapping from $L^p(E,m)$ to $L^p(E,m)$, $1 \leqslant p \leqslant \infty$. Moreover the following inequality is valid:*

$$\left\|\exp\left(-t\left(K_0 \dot{+} V\right)\right)\right\|_{p,p} \leqslant \left\|\exp\left(-t\left(K_0 \dot{+} V\right)\right)1\right\|_\infty.$$

(2) *The operator $V_-^{1/p}\left(aI + K_0\right)^{-1} V_-^{1/q}$ is a linear mapping from $L^p(E,m)$ to $L^p(E,m)$, $1 \leqslant p \leqslant \infty$. Its norm can be estimated as follows:*

$$\left\|V_-^{1/p}(aI + K_0)^{-1}V_-^{1/q}\right\|_{p,p} \leqslant \left\|(aI + K_0)^{-1}V_-\right\|_\infty.$$

Here $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

(3) *In particular, for a large enough, the operator $V_-^{1/2}\left(aI + K_0\right)^{-1} V_-^{1/2}$ is an operator from $L^2(E,m)$ to $L^2(E.m)$. Its norm can be estimated as follows:*

$$\left\|V_-^{1/2}(aI + K_0)^{-1}V_-^{1/2}\right\|_{2,2} \leqslant \left\|(aI + K_0)^{-1}V_-\right\|_\infty < 1.$$

Moreover, again for a large enough, the operator

$$\left(aI + K_0\right)^{-1/2} V_- \left(aI + K_0\right)^{-1/2} \tag{6.70}$$

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possesses the same L^2*-*L^2*-norm as* $V_-^{1/2}\left(aI+K_0\right)^{-1}V_-^{1/2}$*, which is strictly less than 1.*

(4) *If* $\exp\left(-\frac{1}{2}tK_0\right)$ *is as mapping from* $L^1(E,m)$ *to* $L^\infty(E,m)$*, then the same is true for the operator* $\exp\left(-t\left(K_0\dot{+}V\right)\right)$*. Moreover, for* $1\leqslant p\leqslant q\leqslant\infty$*, the following norm-inequality is valid:*

$$
\begin{aligned}
&\left\|\exp\left(-t\left(K_0\dot{+}V\right)\right)\right\|_{q,p}\\
&\leqslant\left\|\exp\left(-t\left(K_0\dot{+}V\right)\right)1\right\|_\infty^{1-(1/p-1/q)}\\
&\qquad\times\left\|\exp\left(-\frac{t}{2}\left(K_0\dot{+}2V\right)\right)1\right\|_\infty^{1/p-1/q}\left\|\exp\left(-\frac{t}{2}K_0\right)\right\|_{\infty,1}^{1/p-1/q}.
\end{aligned}
$$

Operators of the form (6.70) are called Birman-Schwinger kernels. They are employed to estimate the number of eigenvalues below a certain threshold for Schrödinger type operators: see *e.g.* [**127**].

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PROOF. Assertion (1) is a consequence of Proposition 6.70.

(2) In order to prove this assertion we apply Theorem 6.68 (Stein interpolation) to the holomorphic function $S(z)$. For $\xi \in \mathbb{R}$ the norm $\|S(i\xi)\|_{1,1}$ coincides with

$$\left\|V_- \left(aI + K_0\right)^{-1}\right\|_{1,1} = \left\|\left(aI + K_0\right)^{-1} V_-\right\|_{\infty,\infty} = \left\|\left(aI + K_0\right)^{-1} V_-\right\|_{\infty}.$$

For $\xi \in \mathbb{R}$ the norm $\|S(1 + i\xi)\|_{\infty,\infty}$ coincides with

$$\left\|\left(aI + K_0\right)^{-1} V_-\right\|_{\infty,\infty} = \left\|\left(aI + K_0\right)^{-1} V_-\right\|_{\infty}.$$

Theorem 6.68 with $p_0 = q_0 = 1$, $p_1 = q_1 = \infty$, and $t = 1 - 1/p$, yields the desired result.

(3) In order to prove this assertion we specialize assertion (2) to $p = 2$. Since

$$\begin{aligned}
&\left\|\left(aI + K_0\right)^{-\frac{1}{2}} V_- \left(aI + K_0\right)^{-\frac{1}{2}}\right\|_{2,2} \\
&= \left\|\left(aI + K_0\right)^{-\frac{1}{2}} V_-^{1/2} \left(\left(aI + K_0\right)^{-\frac{1}{2}} V_-^{1/2}\right)^*\right\|_{2,2} \\
&= \left\|\left(\left(aI + K_0\right)^{-\frac{1}{2}} V_-^{1/2}\right)^* \left(aI + K_0\right)^{-\frac{1}{2}} V_-^{1/2}\right\|_{2,2} \\
&= \left\|V_-^{1/2} \left(aI + K_0\right)^{-1} V_-^{1/2}\right\|_{2,2},
\end{aligned}$$

the conclusion in assertion (3) follows.

(4) In order to prove this assertion, we need to estimate the operator $T(t/2)$ as an operator from $L^2(E, m)$ to $L^\infty(E, m)$. Therefore we pick $f \in L^2(E, m)$, and we estimate

$$\begin{aligned}
&\left|\exp\left(-\frac{t}{2}\left(K_0 \dot{+} V\right)\right) f(x)\right|^2 \\
\text{(Feynman-Kac)} \quad &= \left|\mathbb{E}_x\left[\exp\left(-\int_0^{t/2} V(x(s))\, ds\right) f(X(t/2))\right]\right|^2 \\
&\leqslant \mathbb{E}_x\left[\exp\left(-2\int_0^{t/2} V(x(s))\, ds\right)\right] \mathbb{E}_x\left[|f(X(t/2))|^2\right] \\
&\leqslant \left\|\exp\left(-\frac{t}{2}\left(K_0 \dot{+} 2V\right)\right) 1\right\|_\infty \left[\exp\left(-\frac{t}{2} K_0\right) |f|^2\right](x) \\
&\leqslant \left\|\exp\left(-\frac{t}{2}\left(K_0 \dot{+} 2V\right)\right) 1\right\|_\infty \left\|\exp\left(-\frac{t}{2} K_0\right)\right\|_{\infty,1} \|f\|_2^2.
\end{aligned}$$

Combined with Corollary 6.72 this yields the desired result, and completes the proof of 6.73. □

6.74. LEMMA. *Let T and S be closed linear operators in a Hilbert space $\mathcal{H}$. Suppose $T \geqslant S \geqslant \varepsilon I > 0$. Then $T^{-1} \leqslant S^{-1}$. These inequalities are to be understood in form sense.*

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PROOF. Put $A = S^{-\frac{1}{2}}TS^{-\frac{1}{2}}$. Then $T \geqslant S$ if and only if $A \geqslant I$. It follows that

$$\langle f, f\rangle = \left\langle AA^{-\frac{1}{2}}f, A^{-\frac{1}{2}}f\right\rangle \geqslant \left\langle A^{-\frac{1}{2}}f, A^{-\frac{1}{2}}f\right\rangle = \left\langle A^{-1}f, f\right\rangle,$$

for f in the domain of $A^{\frac{1}{2}}$. This proves $A^{-1} \leqslant I$. Hence $T^{-1} \leqslant S^{-1}$, and so the proof of Lemma 6.74 is complete now. □

6.75. REMARK. Another proof is based on the equality

$$\begin{aligned} S^{-1} - T^{-1} &= \int_0^1 \frac{\partial}{\partial\alpha}\left((1-\alpha)T + \alpha S\right)^{-1} d\alpha \\ &= \int_0^1 \left((1-\alpha)T + \alpha S\right)^{-1}(T-S)\left((1-\alpha)T + \alpha S\right)^{-1} d\alpha. \end{aligned}$$

These integrals and derivatives have to be taken in strong sense.

6.76. LEMMA. *Let W and V be Kato-Feller potentials on E. Suppose $W \geqslant V$ (pointwise). Then*

$$\left(aI + K_0\dot{+}V\right)^{-1} \geqslant \left(aI + K_0\dot{+}W\right)^{-1}.$$

This inequality is true in form sense as well as in the sense that $f \geqslant 0$ implies

$$\left(aI + K_0\dot{+}V\right)^{-1} f \geqslant \left(aI + K_0\dot{+}W\right)^{-1} f, \quad \textit{pointwise.}$$

PROOF. Suppose W and V to be bounded. Otherwise replace W and V with respectively $W_{n,m} = \max\left(\min\left(W, m\right), -n\right)$ and $V_{n,m} = \max\left(\min\left(V, m\right), -n\right)$, and let m and n tend to ∞. Since W and are V bounded, we see that in form sense $aI + K_0\dot{+}W = aI + K_0 + W \geqslant aI + K_0 + V = aI + K_0\dot{+}V$. Hence, by virtue of Lemma 6.74, we get $\left(aI + K_0\dot{+}V\right)^{-1} \geqslant \left(aI + K_0\dot{+}W\right)^{-1}$. For the pointwise inequality, one may use the Feynman-Kac representation.

This completes the proof of Lemma 6.76. □

6.77. DEFINITION. (General facts) As above the generator K_0 is perturbed in two ways. The first is a "regular" perturbation, being a *multiplication* operator V. That kind of operator was studied in Theorem 6.56. The other kind of perturbation is the "singular" one, *i.e.* a perturbation by a potential barrier on a closed subset Γ of E. These singular perturbations will be treated presently. Put $\Sigma := E\backslash\Gamma$ and introduce the restriction operator $J = J_\Sigma$ as follows: $Jf = f\restriction_\Sigma$. Then its adjoint $J^* : L^2(\Sigma, m) \to L^2(E, m)$ is given by the canonical extension: $J^*f(x) = f(x)$ for $x \in \Sigma$ and $J^*f(x) = 0$ for $x \in \Gamma$. Moreover, we have $J^*J = 1_\Sigma$ and JJ^* is the identity in $L^2(\Sigma, m)$. By $\left(K_0\dot{+}V\right)_\Sigma$ we denote the generator of the semigroup $\left\{\exp\left(-t\left(K_0\dot{+}V\right)_\Sigma\right) : t \geqslant 0\right\}$. The operator $\exp\left(-t\left(K_0\dot{+}V\right)_\Sigma\right)$ is given by the formula

$$\left[\exp\left(-t\left(K_0\dot{+}V\right)_\Sigma\right) f\right](x) = \mathbb{E}_x\left[\exp\left(-\int_0^t V(X(u))\,du\right) f(X(t)) : S > t\right],$$

where S is the *penetration time* of Γ given by

$$S = \inf\left\{s > 0 : \int_0^s 1_\Gamma(X(\sigma))\,d\sigma > 0\right\}.$$

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Suppose the set of S-regular points coincides with Γ. Then, for $f \in C_0(E)$, the function $g : \Sigma \to \mathbb{C}$, defined by $g(x) = \left[\exp\left(-t\left(K_0 \dotplus V\right)_\Sigma\right) f\right](x)$ possesses a continuous extension to all of E. In fact the canonical extension $J^* f$ is continuous on all of E. This is a consequence of the fact that the "killed" Feynman-Kac or Dirichlet semigroup $\left\{\exp\left(-t\left(K_0 \dotplus V\right)_\Sigma\right) : t \geqslant 0\right\}$ leaves the space $C_\infty(\Sigma)$ invariant: see *e.g.* Doob [**41**], Chapter 1.VIII. For $V = 0$, this is shown in Demuth and Van Casteren [**36**] Appendix D, Theorem D.21. A function in $L^p(E, m)$ can be approximated by functions in $C_0(E)$ in the L^p-norm. So that in the presence of L^1-L^∞-smoothing, the L^p-spaces $L^p(\Sigma, m)$, $1 \leqslant p < \infty$, are mapped into $C_0(\Sigma)$ by the Feynman-Kac semigroups "killed" on Γ. Their canonical extensions then belong to $C_0(E)$: for these results one has to mimic the corresponding proofs of Theorem 2.5 for the singular case. The proof of this theorem was discussed in Chapter 3 of [32]. From formula (2.51) in Corollary 2.32 item (b) of Demuth and Van Casteren [**36**] we see that the operator $\left(K_0 \dotplus V\right)_\Sigma$ extends the operator $J\left(K_0 \dotplus V\right) J^*$. Like in [**36**] we are interested in the *harmonic extension operator* given by the formula (see Definition 2.30 in Chapter 2 of [**36**]):

$$\left[H_\Sigma^{a+V} f\right](x) = \mathbb{E}_x\left[\exp\left(-\int_0^S (a + V(X(u)))\, du\right) f(X(S)) : S < \infty\right], \tag{6.71}$$

for whatever functions f this operator makes sense.

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From the discussions in Proposition 2.31 and its Corollary 2.32 in [**36**], it follows that (6.71) has a meaning for functions $f \in D\left(K_0 \dot{+} V\right)$ or even for functions f in the domain of the generalized Schrödinger form. In Chapter 2 of [**36**] the relation between H_Σ^{a+V} and the resolvents $\left(aI + K_0 \dot{+} V\right)^{-1}$ and $J_\Sigma^* \left(aI + K_0 \dot{+} V\right)_\Sigma^{-1} J_\Sigma$ were discussed. In fact the following version of Dynkin's formula holds (see Proposition 2.31 in [**36**]):

$$H_\Sigma^{a+V} \left(aI + K_0 \dot{+} V\right)^{-1} = \left(aI + K_0 \dot{+} V\right)^{-1} - J_\Sigma^* \left(aI + K_0 \dot{+} V\right)_\Sigma^{-1} J_\Sigma.$$

If $a + V = 0$, then we write H_Σ instead of H_Σ^0, and $T_\Sigma(t)$ instead of $T_\Sigma^{a+V}(t)$.

6.78. Definition. Another family of operators which will play a decisive role consists of the family of projections $\{T_\Sigma(t) : t \geqslant 0\}$, where $T_\Sigma(t) = T_\Sigma^V(t)$ is defined by (see Definition 2.33)

$$\left[T_\Sigma(t) f\right](x) = \mathbb{E}_x \left[\exp\left(-\int_0^S V(X(u))\, du\right) f(X(S)) : S \leqslant t\right].$$

6.79. Theorem. *Let V be a Kato-Feller potentials on E. The following assertions are valid:*

(1) *Suppose $a > b$, where*

$$\mathbb{E}_x \left[\exp\left(\int_0^t V_-(X(s))\, ds\right)\right] \leqslant M \exp(bt), \quad t \geqslant 0.$$

Then the supremum $\sup_{x \in E} \left[H_\Sigma^{a+V} 1\right](x)$ is finite.

(2) *(Dynkin's formula) The following equality is valid:*

$$H_\Sigma^{a+V} \left(aI + K_0 \dot{+} V\right)^{-1} = \left(aI + K_0 \dot{+} V\right)^{-1} - J^* \left(aI + K_0 \dot{+} V\right)_\Sigma^{-1} J.$$

(3) *The following inequality holds in form sense as well as in pointwise sense:*

$$0 \leqslant H_\Sigma^{a+V} \left(aI + K_0 \dot{+} V\right)^{-1} \leqslant \left(aI + K_0 \dot{+} V\right)^{-1}.$$

(4) *The following inequality in form sense is valid:*

$$0 \leqslant \left(aI + K_0 \dot{+} V\right)^{\frac{1}{2}} H_\Sigma^{a+V} \left(aI + K_0 \dot{+} V\right)^{-\frac{1}{2}} \leqslant I.$$

Proof. (1) For this result we refer to [**36**], Proposition 4.20.

(2) For this result we refer also to [**36**], Proposition 2.31.

(3) This assertion follows from the identities:

$$\begin{aligned} H_\Sigma^{a+V} \left(aI + K_0 \dot{+} V\right)^{-1} &= \left(aI + K_0 \dot{+} V\right)^{-1} - J^* \left(aI + K_0 \dot{+} V\right)_\Sigma^{-1} J \\ &= \text{s-}\lim_{\beta \to \infty} \left(\left(aI + K_0 \dot{+} V\right)^{-1} - \left(aI + K_0 \dot{+} V + \beta 1_\Gamma\right)^{-1}\right), \end{aligned}$$

together with Lemma 6.76.

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(4) This assertion follows from assertion (3). In fact, the form sense part of assertion (3) is equivalent to assertion (4).

The proof of Theorem 6.79 is now complete. $\square$

6.80. THEOREM. *Let V be a Kato-Feller potentials on E. The operator*

$$\left(aI + K_0 \dot{+} V\right)^{\frac{1}{2}} H_{\Sigma}^{a+V} \left(aI + K_0 \dot{+} V\right)^{-\frac{1}{2}}$$

is densely defined and extends in a unique fashion to a self-adjoint projection.

PROOF. Put

$$T = \left(aI + K_0 \dot{+} V\right)_{\Sigma}^{-\frac{1}{2}} J \left(aI + K_0 \dot{+} V\right)^{\frac{1}{2}}.$$

In addition we write

$$T_{\beta} = \left(aI + K_0 \dot{+} V + \beta 1_{\Gamma}\right)^{-\frac{1}{2}} \left(aI + K_0 \dot{+} V\right)^{\frac{1}{2}}.$$

Then

$$T_{\beta}^{*} = \left(aI + K_0 \dot{+} V\right)^{\frac{1}{2}} \left(aI + K_0 \dot{+} V + \beta 1_{\Gamma}\right)^{-\frac{1}{2}}, \text{ and}$$
$$T^{*} = \left(aI + K_0 \dot{+} V\right)^{\frac{1}{2}} J^{*} \left(aI + K_0 \dot{+} V\right)_{\Sigma}^{-\frac{1}{2}}.$$

As a consequence we obtain

$$T = \text{s-}\lim_{\beta\to\infty} T_{\beta}.$$

From Dynkin's formula it follows that

$$\begin{aligned}&\left(aI + K_0 \dot{+} V\right)^{\frac{1}{2}} H_{\Sigma}^{a+V} \left(aI + K_0 \dot{+} V\right)^{-\frac{1}{2}}\\ &= I - T^{*}T = \text{s-}\lim_{\beta\to\infty} \left(I - T_{\beta}^{*}T_{\beta}\right).\end{aligned}$$

It follows that T^{*} is bounded and everywhere defined. Hence the operator T is closable, with closure T^{**}. It follows that

$$\left(aI + K_0 \dot{+} V\right)^{\frac{1}{2}} H_{\Sigma}^{a+V} \left(aI + K_0 \dot{+} V\right)^{-\frac{1}{2}} \subseteq I - T^{*}T^{**}.$$

Hence the claim in Theorem 6.79 follows, if we can prove that, for any bounded Borel measurable function f, the equality $\left(H_{\Sigma}^{a+V}\right)^{2} f = H_{\Sigma}^{a+V} f$ holds. For this fact we need the equality $S \circ \vartheta_S = 0$, $\mathbb{P}_x$-almost surely on the event $\{S < \infty\}$. A proof of this equality is indicated in Appendix D of [**36**]: see Theorem D.16 together with Remark 2 on page 403. Next we consider

$$\begin{aligned}&\left[\left(H_{\Sigma}^{a+V}\right)^{2} f\right](x)\\ &= \mathbb{E}_x\left[\exp\left(-\int_0^S (a+V)(X(u))\,du\right)\left[H_{\Sigma}^{a+V} f\right](X(S)), S < \infty\right]\\ &= \mathbb{E}_x\Bigg[\exp\left(-\int_0^S (a+V)(X(u))\,du\right)\\ &\qquad \mathbb{E}_{X(S)}\left\{\exp\left(-\int_0^S (a+V)(X(u))\,du\right) f(X(S)), S < \infty\right\}, S < \infty\Bigg]\end{aligned}$$

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(strong Markov property)

$$= \mathbb{E}_x\left[\exp\left(-\int_0^S (a+V)(X(u))\,du\right)\right.$$

$$\left.\exp\left(-\int_0^{S\circ\vartheta_S} (a+V)(X(u+S))\,du\right) f(X(S+S\circ\vartheta_S)), S\circ\vartheta_S<\infty,\ S<\infty\right]$$

(the equality $S\circ\vartheta_S = 0$ holds $\mathbb{P}_x$-almost surely on the event $\{S<\infty\}$)

$$= \mathbb{E}_x\left[\exp\left(-\int_0^S (a+V)(X(u))\,du\right) f(X(S)),\ S<\infty\right]$$

$$= \left[H_\Sigma^{a+V} f\right](x).$$

This completes the proof of Theorem 6.80. □

6.81. Remark. From Theorem 6.80 it follows that the harmonic extension operator leaves the form domain of the operator $H_0\dot{+}V$ invariant. Its proof uses the fact that the harmonic extension operator H_Σ^{a+V} is an projection operator from $C_b(E)$ to the $a+H_0\dot{+}V$-harmonic function on Σ; it preserves the values of a function $f\in C_b(E)$ on Γ, i.e. $H_\Sigma^{a+V} f\restriction_\Gamma = f\restriction_\Gamma$, $\Gamma = E\backslash\Sigma$.

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In the implication (iii) $\Rightarrow$ (ii) of the proof of Theorem 4.2 equality (6.72) of the following result was used with, $\mathcal{H}_0 = \mathcal{H}_1 = L^2(E,m)$, $H_0 = K_0 \dot{+} V$, $H_1 = K_0 \dot{+} W$, and $T = W - V$.

6.82. THEOREM. *Let, for $j = 0, 1$, $H_j = \int_{-\omega_j}^{\infty} \xi E(d\xi_j)$ be a self-adjoint operator in a Hilbert space $\mathcal{H}_j$ with lower bounds $-\omega_j \in \mathbb{R}$. Let $T : \mathcal{H}_1 \to \mathcal{H}_0$ be an appropriate linear operator. Then the following identity is true:*

$$\begin{aligned} & t\exp\left(-\frac{t}{2}H_0\right) T \exp\left(-\frac{t}{2}H_1\right) \\ &= \frac{\pi}{2}\int_{-\infty}^{\infty} \frac{1}{(\cosh(\pi\tau))^2} \exp(i\tau t H_0)\, \mathcal{D}(t)T \exp(-i\tau t H_1)\, d\tau, \ \text{where} \qquad (6.72) \\ & \mathcal{D}(t)T = \int_0^t \exp(-uH_0)\, T \exp(-(t-u)H_1)\, du. \end{aligned}$$

In the proof double Stieltjes operator integrals are employed. The interested reader is referred to the literature on this subject: Birman and Solomyak [**17, 18, 19**]. Some information on this topic can be found in Yafaev [**155**] as well. It is not very clear under what circumstances these double Stieltjes operator integrals are well defined.

PROOF. We will employ double Stieltjes operator integrals. The main formula in (6.72) is almost trivial from the point of view of double Stieltjes operator integrals (and if one takes the validity of Fubini's theorem for such integrals for granted). Put

$$V_0(t) = e^{-tH_0} = \int_{\sigma(H_0)} e^{-t\xi} E_0(d\xi) \quad \text{and} \quad V_1(t) = e^{-tH_1} = \int_{\sigma(H_1)} e^{-t\eta} E_1(d\eta).$$

A quick proof of the equality in (6.72) runs as follows:

$$\begin{aligned} & \frac{\pi}{2}\int_{-\infty}^{\infty} \frac{1}{(\cosh\pi\tau)^2} V_0(i\tau t_0) \int_0^{t_0} V_0(u) T V_1(t_0-u)\, du V_1(-i\tau t_0)\, d\tau \\ &= \iint \frac{\pi}{2}\int_{-\infty}^{\infty} \frac{\exp(-i\tau t_0(\xi-\eta))}{(\cosh\pi\tau)^2} d\tau \frac{\exp(-t_0\eta) - \exp(-t_0\xi)}{\xi-\eta} E_0(d\xi) T E_1(d\eta) \\ &= \iint \frac{\frac{1}{2}t_0(\xi-\eta)}{\sinh\left(\frac{1}{2}(\xi-\eta)t_0\right)} \frac{\sinh\left(\frac{1}{2}t_0(\xi-\eta)\right)}{\frac{1}{2}(\xi-\eta)} \exp\left(-\frac{1}{2}t_0(\xi+\eta)\right) E_0(d\xi) T E_1(d\eta) \\ &= t_0 \iint \exp\left(-\frac{1}{2}t_0(\xi+\eta)\right) E_0(d\xi) T E_1(d\eta) = t_0 V_0(t_0/2) T V_1(t_0/2). \end{aligned}$$

A proof without double operator integrals will be based on Cauchy's theorem from complex analysis, and on operator valued functions on a horizontal strip in the complex plane. In fact it follows from assertion (iv) of Theorem 6.87 in Section 5 with

$$u(\tau,s) = t_0 V_0(i\tau t_0)\, V_0\left(\left(\frac{1}{2}+s\right)t_0\right) T V_1\left(\left(\frac{1}{2}-s\right)t_0\right) V_1(-i\tau t_0).$$

This completes the proof of Theorem 6.82. $\square$

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6.83. Theorem (Inverse of the equality in (6.72)). *Let f be a rapidly decreasing function. Then*

$$\begin{aligned}
&\int_{-\infty}^{\infty} f(\sigma)V_0\left(i\sigma t_0\right)\mathcal{D}(t_0)TV_1\left(-i\sigma t_0\right)\,d\sigma \qquad (6.73)\\
&\quad -\sum_{j=0}^{n}\binom{n+1}{j+1}(-1)^j\int_{-\infty}^{\infty}\mathbb{E}^{\text{logistic}}\left[f\left(\sigma-\left(U_0+U_1+\cdots+U_j\right)\right)\right]\\
&\quad V_0\left(i\sigma t_0\right)t_0V_0(t_0/2)TV_1(t_0/2)V_1\left(-i\sigma t_0\right)\,d\sigma\\
&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(1-\frac{\xi/2}{\sinh\left(\xi/2\right)}\right)^{n+1}\widehat{f}(\xi)e^{i\sigma\xi}d\xi V_0\left(i\sigma t_0\right)\mathcal{D}(t_0)TV_1\left(-i\sigma t_0\right)\,d\sigma.
\end{aligned}$$

The random variables U_j, $j = 2, 3, \ldots$, are independent copies of the logistically distributed variable U_1, $U_0 \equiv 0$: see Evans, Hastings, and Peacock [**50**]. For a proof of Theorem 6.83 the reader is referred to Proposition 6.88. It is taken from [**142**].

6.84. Remark. Put, for $T : \mathcal{H}_1 \to \mathcal{H}_0$ a (bounded) linear operator,

$$\mathcal{Q}(t_0)T = \mathbb{E}^{\text{logistic}}\left[V_0\left(iU_1t_0\right)TV_1\left(-iU_1t_0\right)\right].$$

Then the formula in (6.72) is the same as saying that

$$t_0V_0\left(t_0/2\right)TV_1\left(t_0/2\right) = \mathcal{Q}(t_0)\mathcal{D}(t_0)T,$$

and the formula in (6.73) is equivalent to the identity:

$$\begin{aligned}
\mathcal{D}(t_0)T &= \int_0^{t_0} V_0(u)TV_1\left(t_0-u\right)\,du\\
&=\sum_{j=0}^{n}\binom{n+1}{j+1}(-1)^j\mathcal{Q}(t_0)^j\left(t_0V_0\left(t_0/2\right)TV_1\left(t_0/2\right)\right)+\left(I-\mathcal{Q}(t_0)\right)^{n+1}\mathcal{D}(t_0)T.
\end{aligned}$$

The question which poses itself is the following. Let $S : \mathcal{H}_1 \to \mathcal{H}_0$ be a bounded linear operator for which the Schatten class norm $\|S\|_p$, $1 \leqslant p \leqslant \infty$ is finite. Does it follow that $\lim_{n\to\infty}\left\|\left(I-\mathcal{Q}(t_0)\right)^{n+1}S\right\|_p = 0$? If $p = 2$ (Hilbert-Schmidt situation), then this result is correct. An argument for this statement runs as follows. First approximate the operator S in Hilbert-Schmidt by an operator-valued integral of the form $S_\varphi := \int\varphi(\sigma)V_0(i\sigma t_0)SV_1(-i\sigma t_0)\,d\sigma$, where φ is a rapidly decreasing function on $\mathbb{R}$. Since $\lim_{n\to\infty}\left\|\left(I-\mathcal{Q}(t_0)\right)^{n+1}S_\varphi\right\|_2 = 0$, and since $\left\|\left(I-\mathcal{Q}(t_0)\right)^{n+1}S\right\|_2 \leqslant \|S\|_2$, we obtain the desired result. In [**143**] this question is answered in more or less full generality. In the proof of Theorem 12 of [**143**] it is shown that

$$\begin{aligned}
&\left\|\int_0^{t_0}V_0(s)TV_1\left(t_0-s\right)\,ds-\sum_{j=0}^{n}\binom{n+1}{j+1}(-1)^j\mathcal{Q}\left(t_0\right)^j\left[t_0V_0\left(t_0/2\right)TV_1\left(t_0/2\right)\right]\right\|\\
&\leqslant\frac{t_0C_0}{\pi}\int_0^{\infty}\int_0^{\infty}\left(1-\widehat{\psi}(\xi)\right)^{\frac{1}{2}n}\frac{\tau}{(1+\tau\xi)^2}d\xi\frac{1}{1+\tau^2}\left\|F(\tau)-F(-\tau)\right\|\,d\tau, \qquad (6.74)
\end{aligned}$$

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where $C_0 < 86$, $\widehat{\psi}(\xi) = \dfrac{\frac{1}{2}\xi}{\sinh\left(\frac{1}{2}\xi\right)}$, and

$$F(\tau) = \exp\left(it_0\tau H_0\right)\left(V_0\left(t_0\right)T - TV_1\left(t_0\right)\right)\exp\left(-it_0\tau H_1\right).$$

The norm in (6.74) can be the usual operator norm, or a Schatten class norm.

4.1. KMS formula. Suppose (τ, s) belongs to $\mathbb{R} \times \left(-\frac{1}{2}, \frac{1}{2}\right)$, and let $T : \mathcal{H}_1 \to \mathcal{H}_0$ be a linear operator with the property that the operators $V_0(t_0)T$ and $V_1(t_0)T^*$ are densely defined. Then, in form sense, the following identity is true:

$$\begin{aligned}
&t_0 V_0\left(i\tau t_0\right) V_0\left(\left(\left(\frac{1}{2}+s\right)t_0\right)TV_1\left(\left(\frac{1}{2}-s\right)t_0\right)V_1\left(-i\tau t_0\right)\right. \\
&= \frac{t_0}{2}\int_{-\infty}^{\infty}\frac{\cos\pi s}{\cosh\pi(\tau-\sigma)-\sin\pi s}V_0\left(t_0\right)V_0\left(i\sigma t_0\right)TV_1\left(-i\sigma t_0\right)\,d\sigma \\
&\quad + \frac{t_0}{2}\int_{-\infty}^{\infty}\frac{\cos\pi s}{\cosh\pi(\tau-\sigma)+\sin\pi s}V_0\left(i\sigma t_0\right)TV_1\left(-i\sigma t_0\right)V_1\left(t_0\right)\,d\sigma.
\end{aligned} \tag{6.75}$$

This formula follows by virtue of the following observation. The function at the right hand side of Formula (6.75) is harmonic on the strip $\mathbb{R} \times \left(-\frac{1}{2}, \frac{1}{2}\right)$ and it possesses boundary values

$$\begin{cases} t_0 V_0\left(t_0\right)V_0\left(i\tau t_0\right)TV_1\left(-i\tau t_0\right), & \text{for } s = \frac{1}{2}, \text{ and} \\ t_0 V_0\left(i\tau t_0\right)TV_1\left(-i\tau t_0\right)V_1\left(t_0\right), & \text{for } s = -\frac{1}{2}. \end{cases}$$

The left hand side is harmonic on the same strip (in fact it is holomorphic there), and has the same boundary values. The uniqueness part on the existence of solutions to the classical Dirichlet problem on a strip, yields the formula in (6.75). For some more details about the KMS-formula see Remark 6.85 below. Upon integrating the identity in (6.75) with respect to s we obtain the next one:

$$\begin{aligned}
\mathcal{D}(t_0)T &= \frac{t_0}{2\pi}\int_{-\infty}^{\infty}\log\left(\frac{\cosh\pi\tau+1}{\cosh\pi\tau-1}\right)V_0\left(i\tau t_0\right)\left\{V_0\left(t_0\right)T + TV_1\left(t_0\right)\right\}V_1\left(-i\tau t_0\right)d\tau \\
&= \frac{t_0}{\pi}\int_{-\infty}^{\infty}\log\left|\coth\left(\frac{1}{2}\pi\tau\right)\right|V_0\left(i\tau t_0\right)\left\{V_0\left(t_0\right)T + TV_1\left(t_0\right)\right\}V_1\left(-i\tau t_0\right)d\tau.
\end{aligned}$$

6.85. Remark. Again we consider the space $\mathcal{H}_0 \times \mathcal{H}_1$ together with

$$V(i\tau) = \begin{pmatrix} V_0(i\tau) & 0 \\ 0 & V_1(i\tau) \end{pmatrix}.$$

Define the flow $\mathcal{A}_\tau$ on $\mathcal{B}\left(\mathcal{H}_0 \times \mathcal{H}_1, \mathcal{H}_0 \times \mathcal{H}_1\right)$ by $\mathcal{A}_\tau(T) = V\left(-i\tau t_0\right)TV\left(i\tau t_0\right)$. Define for $(f, g) \in \mathcal{H}_0 \times \mathcal{H}_1$, and $T \in \mathcal{B}\left(\mathcal{H}_0 \times \mathcal{H}_1, \mathcal{H}_0 \times \mathcal{H}_1\right)$ the function $F_{f,g}(\tau + is)$, $\tau \in \mathbb{R}$, $-\frac{1}{2} \leqslant s \leqslant \frac{1}{2}$ by

$$F_{f,g}(\tau + is) = t_0\left\langle \mathcal{A}_\tau\left(V\left(\left(\frac{1}{2}+s\right)t_0\right)TV\left(\left(\frac{1}{2}-s\right)t_0\right)\right)\begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix}\right\rangle.$$

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The function $F_{f,g}$ is K(ubo)-M(artin)-S(chwinger)-admissible for the operators $V(t_0)$ and T in the sense that it is continuous on the closed strip

$$\left\{\tau + is : \tau \in \mathbb{R},\ -\frac{1}{2} \leqslant s \leqslant \frac{1}{2}\right\},$$

and holomorphic on its interior. Moreover,

$$\begin{aligned} F_{f,g}\left(\tau + \frac{1}{2}i\right) &= t_0 \left\langle V(t_0)\mathcal{A}_\tau(T)\begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix}\right\rangle; \\ F_{f,g}\left(\tau - \frac{1}{2}i\right) &= t_0 \left\langle \mathcal{A}_\tau(T)V(t_0)\begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix}\right\rangle. \end{aligned} \tag{6.76}$$

If $T = T^*$, then $F_{f,g}\left(\tau - \frac{1}{2}i\right) = \overline{F_{f,g}\left(\tau + \frac{1}{2}i\right)}$.

6.86. Remark. Observe that for T a bounded linear operator the family of operators $t \mapsto \begin{pmatrix} V_0(t) & \mathcal{D}(t)T \\ 0 & V_1(t) \end{pmatrix}$ is a strongly continuous semigroup on the space $\mathcal{H}_0 \times \mathcal{H}_1$.

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5. Harmonic functions on a strip

The results in Theorem 6.87 of this section are applicable for real-valued harmonic functions $u(\tau,s)$ with the property that

$$\limsup_{\varepsilon\downarrow 0}\int_{-\infty}^{\infty}\left|\int_{-\frac{1}{2}+\varepsilon}^{\frac{1}{2}-\varepsilon} u(\tau,s)\,ds\right|\,d\tau<\infty,$$

or for the operator valued function $v(\tau,s)$ given by

$$v(\tau,s)=t_0V_0\left(i\tau t_0\right)V_0\left(\left(\frac{1}{2}+s\right)t_0\right)TV_1\left(\left(\frac{1}{2}-s\right)t_0\right)V_1\left(-i\tau t_0\right).$$

Then

$$v(\tau,0)=t_0V_0\left(i\tau t_0\right)V_0\left(\frac{1}{2}t_0\right)TV_1\left(\frac{1}{2}t_0\right)V_1\left(-i\tau t_0\right),\text{ and}$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} v(\tau,s)\,ds=V_0\left(i\tau t_0\right)\mathcal{D}(t_0)TV_1\left(-i\tau t_0\right).$$

If we read the above function $v(\tau,s)$ instead of $u(\tau,s)$, the identities in assertion (iv) of the next theorem yield the basic formula in (6.72) in Theorem 6.82 of Section 4. The author wonders whether there is some relationship between the Stein interpolation theorem, *i.e.* Theorem 6.68, and the results on harmonic functions, including the KMS-function, on the a strip. In fact in both cases (bounded) holomorphic functions are involved.

6.87. Theorem. *Let $(X,\|\cdot\|)$ be a Banach space, and let f_1, and $f_2:\mathbb{R}\to X$ be continuous functions with the property that, for every $\tau\in\mathbb{R}$, the following quantity is finite:*

$$\int_{-\infty}^{\infty}\log\frac{\cosh(\pi(\tau-\sigma))+1}{\cosh(\pi(\tau-\sigma))-1}\left\{\|f_1(\sigma)\|+\|f_2(\sigma)\|\right\}d\sigma.$$

Define the function $u(\tau,s)$, $\tau\in\mathbb{R}$, $-\frac{1}{2}<s<\frac{1}{2}$, by

$$u(\tau,s)$$
$$=\frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos\pi s}{\cosh\pi(\tau-\sigma)-\sin\pi s}f_1(\sigma)\,d\sigma+\frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos\pi s}{\cosh\pi(\tau-\sigma)+\sin\pi s}f_2(\sigma)\,d\sigma.$$

The following assertions are true:

(i) *The function $u(\tau,s)$ is harmonic and*

$$\lim_{s\uparrow\frac{1}{2}}u(\tau,s)=f_1(\tau),\ \text{ and }\ \lim_{s\downarrow-\frac{1}{2}}u(\tau,s)=f_2(\tau);$$

(ii) *If $\int_{-\infty}^{\infty}\|f_1(\sigma)\|\,d\sigma$ and $\int_{-\infty}^{\infty}\|f_2(\sigma)\|\,d\sigma$ are finite, then*

$$\int_{-\infty}^{\infty}u(\tau,s)\,d\tau=(1+2s)\int_{-\infty}^{\infty}f_1(\sigma)\,d\sigma+(1-2s)\int_{-\infty}^{\infty}f_2(\sigma)\,d\sigma$$

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$$= \int_{-\infty}^{\infty} u(\tau, 0)\, d\tau + 2s \int_{-\infty}^{\infty} \left(f_1(\sigma) - f_2(\sigma)\right)\, d\sigma.$$

(iii) *Let the hypotheses be as in (ii). The following equality is valid:*

$$\int_{-\infty}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} u(\tau, s)\, ds\, d\tau = \int_{-\infty}^{\infty} u(\tau, 0)\, d\tau.$$

(iv) *The following identity is true:*

$$\begin{aligned} u(\tau, 0) &= \frac{\pi}{2} \int_{-\infty}^{\infty} \frac{1}{\left(\cosh \pi\,(\tau - \sigma)\right)^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} u(\sigma, s)\, ds\, d\sigma \\ &= \mathbb{E}^{\text{logistic}} \left[\int_{-\frac{1}{2}}^{\frac{1}{2}} u(\tau - U, s)\, ds \right], \end{aligned}$$

where U is a logistically distributed random variable.

Proof of Theorem 6.87. (i) This is a standard result in harmonic analysis about the existence of harmonic functions on a strip with given boundary conditions.

(ii) This result follows from the (elementary) identity

$$\int_{-\infty}^{\infty} \frac{\cos \pi s}{\cosh \pi(\tau - \sigma) - \sin \pi s} d\tau = 1 + 2s, \quad -\frac{1}{2} < s < \frac{1}{2}.$$

(iii) Assertion (iii) follows from (ii) and Fubini's theorem.

(iv) This equality is somewhat more involved. The second equality is a direct consequence of the fact that the random variable U is supposed to be logistically distributed. In order to prove the first equality we notice the following identities:

$$\begin{aligned} \int_{-\frac{1}{2}}^{\frac{1}{2}} u(\sigma, s)\, ds &= \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\cos \pi s}{\cosh \pi(\sigma - \rho) - \sin \pi s} ds \left(f_1(\rho) + f_2(\rho)\right)\, d\rho \\ &= \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \Re \left(\frac{1}{\cosh \pi(\sigma + is - \rho)} \right) ds \left(f_1(\rho) + f_2(\rho)\right)\, d\rho \\ &= \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\infty}^{\infty} \Re \left(\frac{1}{\cosh \pi(\sigma + is - \rho)} \right) \left(f_1(\rho) + f_2(\rho)\right)\, d\rho\, ds. \end{aligned}$$

Here we used the elementary identity

$$\frac{\cos \pi s}{\cosh \pi(\sigma - \rho) - \sin \pi s} + \frac{\cos \pi s}{\cosh \pi(\sigma - \rho) + \sin \pi s} = 2\Re \frac{1}{\cosh \pi(\sigma + is - \rho)},$$

$-\frac{1}{2} < s < \frac{1}{2}$. So, in order establish (iv), it suffices to prove the equality

$$\frac{\pi}{2} \int_{-\infty}^{\infty} \frac{1}{\left(\cosh \pi\,(\tau - \sigma)\right)^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} g(\sigma + is)\, ds\, d\sigma = g(\tau),$$

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where $g(\tau + is)$ is an appropriate analytic function on the strip

$$\left\{\tau + is : \tau \in \mathbb{R},\ -\frac{1}{2} < s < \frac{1}{2}\right\}.$$

For $g(\tau + is)$ we take

$$g(\tau + is) = \int_{-\infty}^{\infty} \frac{1}{\cosh \pi(\tau + is - \rho)} \left(f_1(\rho) + f_2(\rho)\right) d\rho.$$

An application of Fubini's theorem, in conjunction with Cauchy's theorem, yields the following string of identities:

$$\frac{\pi}{2}\int_{-\infty}^{\infty} \frac{1}{\left(\cosh \pi\left(\tau - \sigma\right)\right)^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} g(\sigma + is)\, ds\, d\sigma$$

$$= \lim_{\varepsilon \downarrow 0} \frac{\pi}{2} \int_{-\frac{1}{2}+\varepsilon}^{\frac{1}{2}-\varepsilon} \int_{-\infty}^{\infty} \frac{1}{\left(\cosh \pi\left(\tau - \sigma\right)\right)^2} g(\sigma + is)\, d\sigma\, ds$$

(Cauchy's theorem)

$$= \lim_{\varepsilon \downarrow 0} \frac{\pi}{2} \int_{-\frac{1}{2}+\varepsilon}^{\frac{1}{2}-\varepsilon} \int_{-\infty}^{\infty} \frac{1}{\left(\cosh \pi\left(\tau + is - \sigma\right)\right)^2} g(\sigma)\, d\sigma\, ds$$

(Fubini's theorem once more)

$$= \lim_{\varepsilon \downarrow 0} \frac{\pi}{2} \int_{-\infty}^{\infty} \int_{-\frac{1}{2}+\varepsilon}^{\frac{1}{2}-\varepsilon} \frac{1}{\left(\cosh \pi\left(\tau + is - \sigma\right)\right)^2} ds g(\sigma)\, d\sigma$$

$$= \lim_{\varepsilon \downarrow 0} \frac{\pi}{2} \int_{-\infty}^{\infty} \Re \int_{-\frac{1}{2}+\varepsilon}^{\frac{1}{2}-\varepsilon} \frac{1}{\left(\cosh \pi\left(\tau + is - \sigma\right)\right)^2} ds\, g(\sigma)\, d\sigma$$

$$= \lim_{\varepsilon \downarrow 0} \frac{1}{2} \int_{-\infty}^{\infty} \Re \left(\frac{1}{i} \tanh \pi\left(\tau - \sigma + is\right) \Big|_{s=-\frac{1}{2}+\varepsilon}^{s=\frac{1}{2}-\varepsilon} \right) g(\sigma)\, d\sigma$$

$$= \lim_{\varepsilon \downarrow 0} \int_{-\infty}^{\infty} \Re \left(\frac{1}{i} \frac{-1}{\exp\left(2\pi\left(\tau + is - \sigma\right)\right) + 1} \right) \Big|_{s=-\frac{1}{2}+\varepsilon}^{s=\frac{1}{2}-\varepsilon} g(\sigma)\, d\sigma$$

$$= \lim_{\varepsilon \downarrow 0} \int_{-\infty}^{\infty} \frac{4 \cos \pi\varepsilon \sin \pi\varepsilon}{\left(\exp\left(\pi\sigma\right) - \exp\left(-\pi\sigma\right)\right)^2 + 4 \sin^2 \pi\varepsilon} g(\tau - \sigma)\, d\sigma$$

$$= \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos \pi\varepsilon}{1 + \xi^2} g\left(\tau - \frac{1}{\pi} \log\left(\xi \sin \pi\varepsilon + \sqrt{1 + \xi^2 \sin^2 \pi\varepsilon}\right)\right) \frac{d\xi}{\sqrt{1 + \xi^2 \sin^2 \pi\varepsilon}}$$

$$= g(\tau).$$

The latter proves the first identity in assertion (iv), and completes the proof of Theorem 6.87. □

Upon reading, in Proposition 6.88 below, the above function $v(\tau, s)$ for the harmonic function $u(\tau, s)$, the formula in (6.73) of Theorem 6.83 in Section 4 is obtained.

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6.88. PROPOSITION. *Let f be an n-times differentiable function, belonging to $L^1(\mathbb{R})$. Let the harmonic function $u(\tau, s)$ be as in Theorem 6.87. The following identities are true*

$$
\begin{aligned}
&\int_{-\infty}^{\infty} f(\sigma) \int_{-\frac{1}{2}}^{\frac{1}{2}} u(\sigma, s)\, ds\, d\sigma \\
&= \int_{-\infty}^{\infty} \sum_{j=1}^{n+1} \binom{n+1}{j} (-1)^{j-1} \mathbb{E}^{\text{logistic}} \left[f\left(\sigma - (U_0 + \cdots + U_{j-1})\right)\right] u(\sigma, 0)\, d\sigma \\
&\quad + \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \left(1 - \frac{\frac{1}{2}\xi}{\sinh\left(\frac{1}{2}\xi\right)} \right)^{n+1} \widehat{f}(\xi) e^{i\xi\sigma} d\xi \right] \int_{-\frac{1}{2}}^{\frac{1}{2}} u(\sigma, s)\, ds\, d\sigma \qquad (6.77) \\
&= \int_{-\infty}^{\infty} f(\sigma) u(\sigma, 0)\, d\sigma \\
&+ \int_{-\infty}^{\infty} \mathbb{E}^{\text{logistic}} \left[\prod_{j=1}^{n} U_j \int_0^1 ds_1 \dots \int_0^1 ds_n f^{(n)} \left(\sigma - (s_1 U_1 + \cdots + s_n U_n)\right) \right] u(\sigma, 0)\, d\sigma \\
&+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \left(1 - \frac{\frac{1}{2}\xi}{\sinh\left(\frac{1}{2}\xi\right)} \right)^{n+1} \widehat{f}(\xi) e^{i\xi\sigma} d\xi \right] \int_{-\frac{1}{2}}^{\frac{1}{2}} u(\sigma, s)\, ds\, d\sigma. \qquad (6.78)
\end{aligned}
$$

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The sequence $(U_j)_{j\in\mathbb{N},j\geqslant 1}$ in Proposition 6.88 consists of independent logistically distributed random variables, each distributed according to the same law:

$$\mathbb{P}\left[U_j \in B\right] = \frac{\pi}{2}\int_B \frac{1}{(\cosh \pi\tau)^2} d\tau = \pi^2 \int_0^\infty \frac{\sinh \pi\tau}{(\cosh \pi\tau)^3} \int_{-\tau}^{\tau} 1_B(\sigma)\, d\sigma\, d\tau.$$

The variable U_0 is identically zero.

Proof of Proposition 6.88. The equality of (6.77) and (6.78) is due to the following equality ($n \geqslant 1$, f as in Proposition 6.88):

$$\begin{aligned}&\sum_{j=1}^{n+1}\binom{n+1}{j}(-1)^{j-1}\mathbb{E}^{\text{logistic}}\left[f\left(\sigma-(U_0+\dots+U_{j-1})\right)\right]\\ &= f(\sigma) + \mathbb{E}^{\text{logistic}}\left[\prod_{j=1}^{n} U_j \int_0^1 ds_1 \dots \int_0^1 ds_n f^{(n)}\left(\sigma - (s_1U_1+\dots+s_nU_n)\right)\right]. \qquad (6.79)\end{aligned}$$

Let δ be the Dirac measure at the origin and let φ_0 be the density corresponding to the logistical distribution: $\varphi_0(\tau) = \dfrac{\pi}{2}\dfrac{1}{(\cosh(\pi\tau))^2}$. The identities (the symbol $\left((\delta-\varphi_0)^*\right)^{n+1}$ denotes the $(n+1)$-fold convolution of $\delta - \varphi_0$ with itself)

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\left(1-\frac{\frac{1}{2}\xi}{\sinh\left(\frac{1}{2}\xi\right)}\right)^{n+1}\widehat{f}(\xi)e^{i\xi\sigma}d\xi\right]\int_{-\frac{1}{2}}^{\frac{1}{2}} u(\sigma,s)\,ds\,d\sigma\\ &= \int_{-\infty}^{\infty}\left[\left((\delta-\varphi_0)^*\right)^{n+1} f\right](\sigma)\int_{-\frac{1}{2}}^{\frac{1}{2}} u(\sigma,s)\,ds\,d\sigma\\ &= \int_{-\infty}^{\infty}\sum_{j=0}^{n+1}\binom{n+1}{j}(-1)^j\left[(\delta^*)^{n+1-j} * (\varphi_0^*)^j f\right](\sigma)\int_{-\frac{1}{2}}^{\frac{1}{2}} u(\sigma,s)\,ds\,d\sigma\end{aligned}$$

(assertion (iv) of Theorem 6.87)

$$\begin{aligned}&= \int_{-\infty}^{\infty} f(\sigma)\int_{-\frac{1}{2}}^{\frac{1}{2}} u(\sigma,s)\,ds\,d\sigma + \int_{-\infty}^{\infty}\sum_{j=1}^{n+1}\binom{n+1}{j}(-1)^j\left[(\varphi_0^*)^{j-1} f\right](\sigma)u(\sigma,0)\,d\sigma\\ &= \int_{-\infty}^{\infty} f(\sigma)\int_{-\frac{1}{2}}^{\frac{1}{2}} u(\sigma,s)\,ds\,d\sigma\\ &\quad + \int_{-\infty}^{\infty}\sum_{j=1}^{n+1}\binom{n+1}{j}(-1)^j\mathbb{E}^{\text{logistic}}\left[f\left(\sigma-(U_0+\dots+U_{j-1})\right)\right]u(\sigma,0)\,d\sigma\end{aligned}$$

complete the proof of Proposition 6.88. □

6.89. Theorem. *Suppose that the real-valued harmonic $u(\tau, s)$ possesses the property mentioned in the beginning of the present section:*

$$\limsup_{\varepsilon\downarrow 0}\int_{-\infty}^{\infty}\left|\int_{-\frac{1}{2}+\varepsilon}^{\frac{1}{2}-\varepsilon} u(\tau,s)\,ds\right|d\tau < \infty,$$

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suppose that the function has the property that

$$\int_{-\infty}^{\infty}\left|\left[\int_{-\infty}^{\infty} f(\xi)\frac{\sinh\left(\frac{1}{2}\xi\right)}{\frac{1}{2}\xi}e^{i\tau\xi}d\xi\right]\right| d\tau < \infty,$$

and suppose that vector valued (or operator valued) function $v(\tau,s)$ is bounded in the following sense

$$\sup_{\tau\in\mathbb{R}}\left\|\int_{-\frac{1}{2}}^{\frac{1}{2}} v(\tau,s)\,ds\right\| < \infty.$$

Then the following identities are true:

$$\int_{-\infty}^{\infty} u(\tau,0)\int_{-\frac{1}{2}}^{\frac{1}{2}} v(\tau,s)\,ds\,d\tau = \int_{-\infty}^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}} u(\tau,s)\,ds v(\tau,0)\,d\tau; \tag{6.80}$$

$$\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\xi)\frac{\sinh\left(\frac{1}{2}\xi\right)}{\frac{1}{2}\xi}e^{i\tau\xi}d\xi\right]v(\tau,0)\,d\tau = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\xi)e^{i\tau\xi}d\xi\right]\int_{-\frac{1}{2}}^{\frac{1}{2}} v(\tau,s)\,ds\,d\tau. \tag{6.81}$$

Notice the equalities:

$$\begin{aligned}\int_{-\infty}^{\infty} f(\xi)\exp\left(i\tau\xi\right)\,d\xi &= \frac{\pi}{2}\int_{-\infty}^{\infty}\frac{1}{\left(\cosh\pi(\tau-\sigma)\right)^2}\int_{-\infty}^{\infty} f(\xi)\frac{\sinh\left(\frac{1}{2}\xi\right)}{\frac{1}{2}\xi}\exp\left(i\sigma\xi\right)\,d\xi\,d\sigma \\ &= \mathbb{E}^{\text{logistic}}\left[\int_{-\infty}^{\infty} f(\xi)\frac{\sinh\left(\frac{1}{2}\xi\right)}{\frac{1}{2}\xi}\exp\left(i\left(\tau-U\right)\xi\right)\,d\xi\right].\end{aligned}$$

Proof. The equality in (6.81) follows by inserting the function $u(\tau,s)$ defined through

$$u(\tau,s) = \int_{-\infty}^{\infty} f(\xi)\exp\left(i(\tau+is)\xi\right)\,d\xi,$$

into (6.80). Equality (6.80) follows from (iv) in Theorem 6.87 and Fubini's theorem. In fact we have

$$\int_{-\infty}^{\infty} u(\tau,0)\int_{-\frac{1}{2}}^{\frac{1}{2}} v(\tau,s)\,ds\,d\tau$$

(the equality in assertion (iv) of Theorem 6.87)

$$= \int_{-\infty}^{\infty}\mathbb{E}^{\text{logistic}}\left[\int_{-\frac{1}{2}}^{\frac{1}{2}} u(\tau-U,s_1)\,ds_1\right]\int_{-\frac{1}{2}}^{\frac{1}{2}} v(\tau,s_2)\,ds_2\,d\tau$$

(Fubini twice, translation invariance of Lebesgue measure, and symmetry of the logistic distribution)

$$= \int_{-\infty}^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}} u(\tau,s_1)\,ds_1\,\mathbb{E}^{\text{logistic}}\left[\int_{-\frac{1}{2}}^{\frac{1}{2}} v(\tau-U,s_2)\,ds_2\right]d\tau$$

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(another time we use the equality in assertion (iv) of Theorem 6.87)

$$= \int_{-\infty}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} u(\tau, s)\, ds\, v(\tau, 0)\, d\tau.$$

This concludes the proof of Theorem 6.89. $\square$

6.90. Remark. In particular we have ($\gamma \geqslant 0$):

$$\begin{aligned} &\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\gamma + \frac{1}{2}}{(\tau - \sigma)^2 + \left(\gamma + \frac{1}{2}\right)^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} v(\sigma, s)\, ds\, d\sigma \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\gamma + \frac{1}{2} + s}{(\tau - \sigma)^2 + \left(\gamma + \frac{1}{2} + s\right)^2} ds\ v(\sigma, 0)\, d\sigma \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \log\left(\frac{(\gamma + 1)^2 + (\tau - \sigma)^2}{\gamma^2 + (\tau - \sigma)^2}\right) v(\sigma, 0)\, d\sigma. \end{aligned}$$

Here we employed the harmonic function

$$u(\sigma, s) = \frac{1}{\pi} \frac{\gamma + \frac{1}{2} + s}{(\tau - \sigma)^2 + \left(\gamma + \frac{1}{2} + s\right)^2}, \quad \gamma \geqslant 0.$$

We also introduce the following convex sets of harmonic functions on the strip $\mathbb{R} \times \left(-\frac{1}{2}, \frac{1}{2}\right)$:

$$\begin{aligned}\mathrm{harm}_1 = \Bigg\{ u : \mathbb{R} \times \left(-\frac{1}{2}, \frac{1}{2}\right) \to \mathbb{R} : u \ \text{ harmonic and} \\ \limsup_{\varepsilon \downarrow 0} \int_{-\infty}^{\infty} \left| \int_{-\frac{1}{2}+\varepsilon}^{\frac{1}{2}-\varepsilon} u(\tau, s)\, ds \right| d\tau \leqslant 1 \Bigg\};\end{aligned} \tag{6.82}$$

$$\mathrm{harm}_1^+ = \left\{ u \in \mathrm{harm}_1 : \int_{-a}^{a} u(\tau, s)\, ds \geqslant 0 \text{ for all } 0 \leqslant a < \frac{1}{2} \right\}. \tag{6.83}$$

6.91. Theorem. *Let $(X, \|\cdot\|)$ be a Banach space, and let $v(\tau, s)$ be an X-valued harmonic function on the strip $\mathbb{R} \times \left(-\frac{1}{2}, \frac{1}{2}\right)$. Suppose that* $\sup\left\{\|v(\tau, 0)\| : \tau \in \mathbb{R}\right\}$ *is finite. Then the following equalities are true:*

$$\sup\left\{\|v(\tau, 0)\| : \tau \in \mathbb{R}\right\} \tag{6.84}$$

$$= \sup\left\{ \left\| \int_{-\infty}^{\infty} u(\tau, 0) \int_{-\frac{1}{2}}^{\frac{1}{2}} v(\tau, s)\, ds\, d\tau \right\| : u \in \mathrm{harm}_1 \right\} \tag{6.85}$$

$$= \sup\left\{ \left\| \int_{-\infty}^{\infty} u(\tau, 0) \int_{-\frac{1}{2}}^{\frac{1}{2}} v(\tau, s)\, ds\, d\tau \right\| : u \in \mathrm{harm}_1^+ \right\}. \tag{6.86}$$

Proof. The quantity in (6.86) is trivially dominated by the one in (6.85). Equality (6.80) yields the inequality

$$\left\| \int_{-\infty}^{\infty} u(\tau, 0) \int_{-\frac{1}{2}}^{\frac{1}{2}} v(\tau, s)\, ds\, d\tau \right\| \leqslant \sup\left\{\|v(\tau, 0)\| : \tau \in \mathbb{R}\right\},$$

for u belonging to harm_1. Hence quantity (6.84) dominates (6.85). The equality

$$v(\tau, 0) = \frac{\pi}{2} \int_{-\infty}^{\infty} \frac{1}{\left(\cosh \pi\left(\tau - \sigma\right)\right)^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} v(\sigma, s)\, ds\, d\sigma,$$

together with the identity

$$\frac{\pi}{2} \int_{-\frac{1}{2}+\varepsilon}^{\frac{1}{2}-\varepsilon} \Re \frac{1}{\left(\cosh \pi(\tau + is - \sigma)\right)^2} ds = \frac{\cos \pi\varepsilon \sin \pi\varepsilon}{\left(\cosh \pi(\tau - \sigma)\right)^2 + \sin^2 \pi\varepsilon},$$

$0 < \varepsilon < \frac{1}{2}$, implies that quantity (6.84) is dominated by (6.86). So the proof of Theorem 6.91 is now complete. □

Another corollary is the following one.

6.92. Corollary. *Let $v : \mathbb{R} \times \left(-\frac{1}{2}, \frac{1}{2}\right)$ be an X-valued harmonic function, with boundary values $v\left(\tau, \frac{1}{2}\right)$ and $v\left(\tau, -\frac{1}{2}\right)$ respectively. Suppose*

$$\sup\left\{ \left\| v\left(\tau, \frac{1}{2}\right) + v\left(\tau, -\frac{1}{2}\right) \right\| : \tau \in \mathbb{R} \right\} < \infty.$$

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Then the suprema

$$\sup\left\{\|v(\tau,0)\| : \tau \in \mathbb{R}\right\} \ \ \textit{and} \ \sup\left\{\left\|\int_{-\frac{1}{2}}^{\frac{1}{2}} v(\tau,s)\,ds\right\| : \tau \in \mathbb{R}\right\}$$

are finite. Moreover they obey the following inequalities:

$$\begin{aligned}\sup\left\{\|v(\tau,0)\| : \tau \in \mathbb{R}\right\} &\leqslant \sup\left\{\left\|\int_{-\frac{1}{2}}^{\frac{1}{2}} v(\tau,s)\,ds\right\| : \tau \in \mathbb{R}\right\}\\ &\leqslant \frac{1}{2}\sup\left\{\left\|v\left(\tau,\frac{1}{2}\right)+v\left(\tau,-\frac{1}{2}\right)\right\| : \tau \in \mathbb{R}\right\} < \infty.\end{aligned}$$

PROOF. This result is an easy consequence of the following identities:

$$v(\tau,0) = \frac{\pi}{2}\int_{-\infty}^{\infty}\frac{1}{\left(\cosh\pi\left(\tau-\sigma\right)\right)^2}\int_{-\frac{1}{2}}^{\frac{1}{2}} v(\sigma,s)\,ds\,d\sigma;$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} v(\tau,s)\,ds = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\cos\pi s}{\cosh\pi(\tau-\sigma)-\sin\pi s}ds\left(v\left(\sigma,\frac{1}{2}\right)+v\left(\sigma,-\frac{1}{2}\right)\right)\,d\sigma.$$

The reader should compare this with the proof of assertion (iv) of Theorem 6.87. This completes the proof of Corollary 6.92. □

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CHAPTER 7

Holomorphic semigroups

In this chapter we will discuss certain aspects of holomorphic semigroups. In Theorem 7.3 (generators of) exponentially bounded holomorphic semigroups are described. This is done via so-called sectorial operators. In Section 3 the same is done for (generators of) bounded holomorphic semigroups. These semigroups $\{P(t) : t \geqslant 0\}$ are not only bounded, but, for some $0 < \alpha \leqslant \frac{1}{2}\pi$, they also have an analytic extension in some sector $V_\alpha = \{z \in \mathbb{C} : |\arg z| < \alpha\}$ which is bounded in the sense that $\sup\{\|P(t)\| : t \in V_\alpha\} < \infty$. In section 4 we discuss the relationship between the Crank-Nicolson iteration scheme and generators of bounded analytic semigroups. A certain functional calculus is developed which encompasses the relevant operators; see Theorem 7.13 and its Corollary 7.14, Corollary 7.17. Finally Section 5 is devoted to a discussion on the stability of the Crank-Nicolson iteration scheme.

The author is indebted to Sergey Piskarev, University of Moscow, for interesting discussions on the subject and for some relevant references.

1. Introduction

In this section holomorphic semigroups and their generators are characterized. For a concise formulation for the results we introduce the following notation. Let $0 < \alpha \leqslant \pi$ and put $V_\alpha = \{z \in \mathbb{C} : |\arg z| < \alpha\}$. Let $\{P(t) : t \geqslant 0\}$ be a semigroup of continuous linear operators in a Banach space X. This semigroup is said to be *holomorphic* or *analytic* if there exists $0 < \alpha \leqslant \frac{1}{2}\pi$ and a holomorphic map $\widetilde{P} : V_\alpha \to L(X)$ such that $\widetilde{P}(t) = P(t)$ for $t > 0$. Instead of $\widetilde{P}(t)$ we usually write $P(t)$ for this extension. Again we have $P(z_1 + z_2) = P(z_1) \circ P(z_2)$ for z_1 and z_2 in V_α. We begin with a couple of definitions.

7.1. Definition. The semigroup $\{P(z) : z \in V_\alpha\}$ is said to be *exponentially bounded* if for each $0 < \varphi < \alpha$, there exists constants $M = M_\varphi$ and $\omega = \omega_\varphi$ such that $\|P(z)\| \leqslant M \exp(\omega |z|)$ for all $z \in V_\varphi$.

7.2. Definition. Let $\{P(t) : t \geqslant 0\}$ be a strongly continuous semigroup of operators on a Banach space X. This semigroup is called (uniformly) bounded if

$$\sup\{\|P(t)\| : t \geqslant 0\} < \infty.$$

It is called a (uniformly) bounded analytic semigroup provided that it is (uniformly) bounded and extends to a (uniformly) bounded analytic semigroup on a sector V_α

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where $0 < \alpha \leqslant \pi/2$. This means that, for some $0 < \alpha \leqslant \pi/2$,

$$\sup \{\|P(t)\| : t \in V_\alpha\} < \infty.$$

Usually the adverb "uniformly" is omitted. In Theorem 7.9 below we give a characterization of generators of bounded analytic semigroups in terms of sectorial operators: see Definition 7.5. We also notice that part of this material comes from Chapter 5 in [**139**].

2. Exponentially bounded analytic semigroups

In what follows we collect some of the characterizations of exponentially bounded analytic semigroups.

7.3. Theorem. *Let $\{P(t) : t \geqslant 0\}$ be a strongly continuous semigroup in $L(X)$ with generator A and with resolvent family $\{R(\lambda) : \lambda \in \rho(A)\}$. The following assertions are equivalent:*

(i) *The semigroup $\{P(t) : t \geqslant 0\}$ is analytic and has an exponentially bounded extension.*
(ii) *There exists a complex number ζ, $|\zeta| = 1$, such that*

$$\limsup_{t\downarrow 0} \inf \{\|\zeta x - P(t)x\| : x \in X, \quad \|x\| = 1\} > 0.$$

(iii) *There exists $\pi > \alpha > \frac{1}{2}\pi$ such that $\rho(A) \supset a + \overline{V}_\alpha$ for some $a > 0$ and such that*

$$|\lambda| \|R(\lambda)\| \leqslant C, \quad \lambda \in a + V_\alpha$$

for an appropriate constant C.
(iv) *There exists a constant C and a positive real number b such that $\rho(A) \supset \{\lambda \in \mathbb{C} : \Re\lambda = 0, \quad |\Im\lambda| > b\}$ and such that*

$$|\lambda| \|R(\lambda)\| \leqslant C, \quad \Re\lambda = 0, \quad |\lambda| > b.$$

(v) $\limsup_{t\downarrow 0} \sup \{t \|AP(t)x\| : x \in D(A), \quad \|x\| \leqslant 1\} < \infty$.
(vi) *The operator*

$$\begin{pmatrix} A & 0 \\ A & A \end{pmatrix}$$

generates a strongly continuous semigroup in $X \times X$.
(vii) *There exist a constant C and a positive real number b such that*

$$n \left\| \left(\lambda R(\lambda)\right)^n - \left(\lambda R(\lambda)\right)^{n-1} \right\| \leqslant C, \quad \lambda > bn, \quad n \in \mathbb{N}.$$

(viii) *There exists a polynomial q such that*

$$\limsup_{t\downarrow 0} \|q(P(t))\| < \sup \{|q(z)| : z \in \mathbb{C}, \quad |z| = 1\}.$$

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(ix) *There exists a polynomial q such that*

$$\limsup_{s\downarrow 0, t\downarrow 0, n\to\infty} \left\| \left\{ q\left(P\left(\frac{st}{n}\right)\right)\right\}^n P(t) \right\|^{1/n} < \sup\left\{ |q(z)| : z\in\mathbb{C}, \quad |z|=1\right\}.$$

(x) *For every polynomial q, for which $|q(1)| < \sup\{|q(z)| : |z| = 1\}$, the strict inequality*

$$\limsup_{s\downarrow 0, t\downarrow 0, n\to\infty} \left\| \left\{ q\left(P\left(\frac{st}{n}\right)\right)\right\}^n P(t) \right\|^{1/n} < \sup\left\{ |q(z)| : z\in\mathbb{C}, \quad |z|=1\right\}$$

holds.

7.4. Remark. Let A be the generator of a bounded semigroup $\{P(t) : t \geqslant 0\}$. If in assertion (i) "bounded analytic" replaces "exponentially bounded analytic", in assertion (v) we assume

$$\sup\left\{ t\,\|AP(t)x\| : t > 0,\ x \in D(A), \quad \|x\| \leqslant 1\right\} < \infty,$$

in assertion (vi) the operator $\begin{pmatrix} A & 0 \\ A & A \end{pmatrix}$ generates a strongly continuous bounded semigroup in $X \times X$, and in assertion (vi) we take $b = 0$, then a substantial part of Theorem 7.9 follows.

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7.5. Definition. Operators A which satisfy (iii) in Theorem 7.3 are called *s*ectorial operators.

For more details on sectorial operators see, e.g., Lunardi [**85**] and also Haase [**60**]. The equivalency of (i), (ii) and (iv) is due to Kato: see Kato [**75, 76, 77**]. A proof of the equivalency of (i), (iii), (iv) and (v) can be found in Pazy [**97**], pp. 61-64. In Yosida [**156**], pp. 254–255 the equivalency of (i), (iii) and (v) is proved too. The equivalency of (v), (vi) and (vii) can be found in Crandall, Pazy and Tartar [**28**]. In [**14**] Beurling proves the equivalency of (i), (ix) and (x). The implication (viii) $\Rightarrow$ (ix) is easy. Let q be a polynomial for which the strict inequality in (ix) holds. For appropriately chosen m and n in $\mathbb{N}$ the polynomial $q_0(z) := q(z)^n z^{mn}$ satisfies (viii).

In many textbooks on differential equations applications of analytic semigroup theory can be found. In [**132, 133**] Stewart gives a number of interesting application of the use of analytic semigroups. In [**128**] Sinclair applies holomorphic semigroups to Banach algebra theory. In [**99**] and [**100**] Pisier uses Beurling's characterization of holomorphic semigroups to prove some geometric properties of Banach spaces. Here the following fact is used. If $Q_1, \ldots, Q_n$ are commuting projections in a Banach space, then the mapping $t \mapsto \prod_{k=1}^{n} \{(I - Q_k) + e^{-t}Q_k\}$, $t \geqslant 0$, is a holomorphic semigroup. In [**34**] de Graaf has yet another application of holomorphic semigroups. In [**35**] Delaubenfels introduces the notion of exponentially bounded holomorphic integrated semigroup. He also gives some examples. For regularity properties of solutions of initial value problems, in which sectorial operators play a fundamental role see, e.g., Prüss [**103**], Prüss and Simonett [**104**], Lunardi [**85**] and others.

Before we prove Theorem 7.3 we insert the following proposition. It refines some of its statements. Moreover it yields the equivalence of the assertions (i), (ii), (iii), (iv) and (v) of Theorem 7.3.

7.6. Proposition. *Let A be the generator of a strongly continuous semigroup $\{P(t) : t \geqslant 0\}$ in $L(X)$. The following assertions are equivalent:*

(i) *The semigroup $\{P(t) : t \geqslant 0\}$ has an exponentially bounded holomorphic extension in some angle V_α with $0 < \alpha \leqslant \frac{1}{2}\pi$.*
(ii) *For every $\zeta \in \mathbb{C}$, $|\zeta| = 1$, $\zeta \neq 1$, the inverses $(\zeta I - P(t))^{-1}$ exist, as everywhere defined bounded linear operators, for all sufficiently small positive real numbers t.*
(iii) *There exists $\zeta \in \mathbb{C}$, $|\zeta| = 1$, $\zeta \neq 1$, such that the expression*

$$\liminf_{t \downarrow 0} \inf \{\|\zeta x - P(t)x\| : \ x \in X, \ \|x\| = 1\}$$

is strictly positive.
(iv) *There are constants $M \geqslant 0$, $a \in \mathbb{R}$ and $b \geqslant 0$ such that*

$$M \|(a + i\tau)x - Ax\| \geqslant |\tau| \, \|x\|$$

for all $x \in D(A)$ and for all $\tau \in \mathbb{R}$ with $|\tau| \geqslant b$.

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(v) *There exist constants* $a \in \mathbb{R}$ *and* $\frac{1}{2}\pi < \varphi < \pi$ *such that the resolvent set of* A *contains* $a + V_\varphi$ *and such that*

$$\sup \{|\lambda| \, \|R(\lambda)\| : \lambda \in a + V_\varphi\}$$

is finite.

(vi) *The expression*

$$\limsup_{t \downarrow 0} \sup \{t \, \|AP(t)x\| : \; x \in D(A), \; \|x\| \leqslant 1\}$$

is finite.

Here, as always, $R(\lambda)$ denotes the operator $R(\lambda) = (\lambda I - A)^{-1}$, whenever it exists as a bounded linear operator which is everywhere defined. Sometimes we will use symbolic calculus utilizing operator valued integrals in the complex plane. The contour $\Gamma(a, \varphi)$, $a \in \mathbb{R}$, $0 < \varphi < \pi$, denotes then the boundary of $a + V_\varphi$ oriented in such a way that $a + V_\varphi$ is lying at the right-hand side.

7.7. Remark. Let A be the generator of a bounded semigroup $\{P(t) : t \geqslant 0\}$. If in Proposition 7.6 assertion (i) we replace "exponentially bounded" with "bounded", in (iv) we take $a = 0$, $b = 0$, in (v) we take $a = 0$, and in (vi) we assume

$$\sup \{t \, \|AP(t)x\| : \, t > 0, \, \|x\| \leqslant 1, \, x \in D(A)\} < \infty,$$

then part of Theorem 7.9 follows from the argumentation in the proof just below.

Proof of Proposition 7.6. (i) $\Rightarrow$ (v) Suppose $\|P(t)\| \leqslant M \exp(\omega t)$, for $t \in V_\alpha$, where $0 < \alpha < \frac{1}{2}\pi$ is fixed. Choose $a_1 \in \mathbb{R}$ in such a way that $a_1 \cos\alpha > \omega$ and for $\lambda \in a_1 + V_{\alpha + \frac{1}{2}\pi}$, $\Im\lambda \geqslant 0$, the $L(X)$-valued integrals $R_\alpha(\lambda)$ by

$$R_\alpha(\lambda)x = e^{-i\alpha} \int_0^\infty \exp\left(-\lambda t e^{-i\alpha}\right) P(te^{-i\alpha}) x dt, \quad x \in X.$$

Since

$$\left\|\exp\left(-\lambda t e^{-i\alpha}\right) P\left(te^{-i\alpha}\right)\right\| \leqslant M \exp\left(-(a_1 \cos\alpha - \omega)t\right)$$

for $t \geqslant 0$ and for $\lambda \in \left(a_1 + V_{\alpha + \frac{1}{2}\pi}\right) \bigcap \{\lambda \in \mathbb{C} : \Im\lambda \geqslant 0\}$, these integrals make sense indeed. Next fix $\varphi \in \mathbb{R}$ in such a way that

$$\alpha + \frac{1}{2}\pi > \varphi > \frac{1}{2}\pi, \quad \varphi > 2\alpha,$$

and fix $a \geqslant a_1$ in such a way that

$$a\left(\cos\alpha - \cos(\varphi - \alpha)\right) = 2a \sin\frac{1}{2}\varphi . \sin\left(\frac{1}{2}\varphi - \alpha\right) \geqslant \omega.$$

For $\lambda = a + |\lambda - a| \, e^{i\varphi}$ the following inequlities are valid:

$$\begin{aligned}
\|R_\alpha(\lambda)\| &\leqslant M \int_0^\infty \exp\left(-at\cos\alpha - |\lambda - a| \, t \cos(\varphi - \alpha) + \omega t\right) dt \\
&\leqslant M \left(a \cos\alpha + |\lambda - a| \cos(\varphi - \alpha) - \omega\right)^{-1} \\
&\leqslant M \left(a(\cos\alpha - \cos(\varphi - \alpha)) - \omega + |\lambda| \cos(\varphi - \alpha)\right)^{-1}
\end{aligned}$$

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$$\leqslant M\left(|\lambda|\cos(\varphi-\alpha)\right)^{-1}.$$

Hence

$$|\lambda|\,\|R_\alpha(\lambda)\| \leqslant M\left(\cos(\varphi-\alpha)\right)^{-1}, \quad \lambda\in\Gamma(a,\varphi), \quad \Im\lambda\geqslant 0. \tag{7.1}$$

Similarly the integrals

$$R_{-\alpha}(\lambda)x := e^{i\alpha}\int_0^\infty \exp\left(-\lambda t e^{i\alpha}\right)P(te^{i\alpha})x\,dt, \quad x\in X,$$

make sense for $\lambda\in a_1+V_{\alpha+\frac{1}{2}\pi}$, $\Im\lambda\leqslant 0$. Again we have

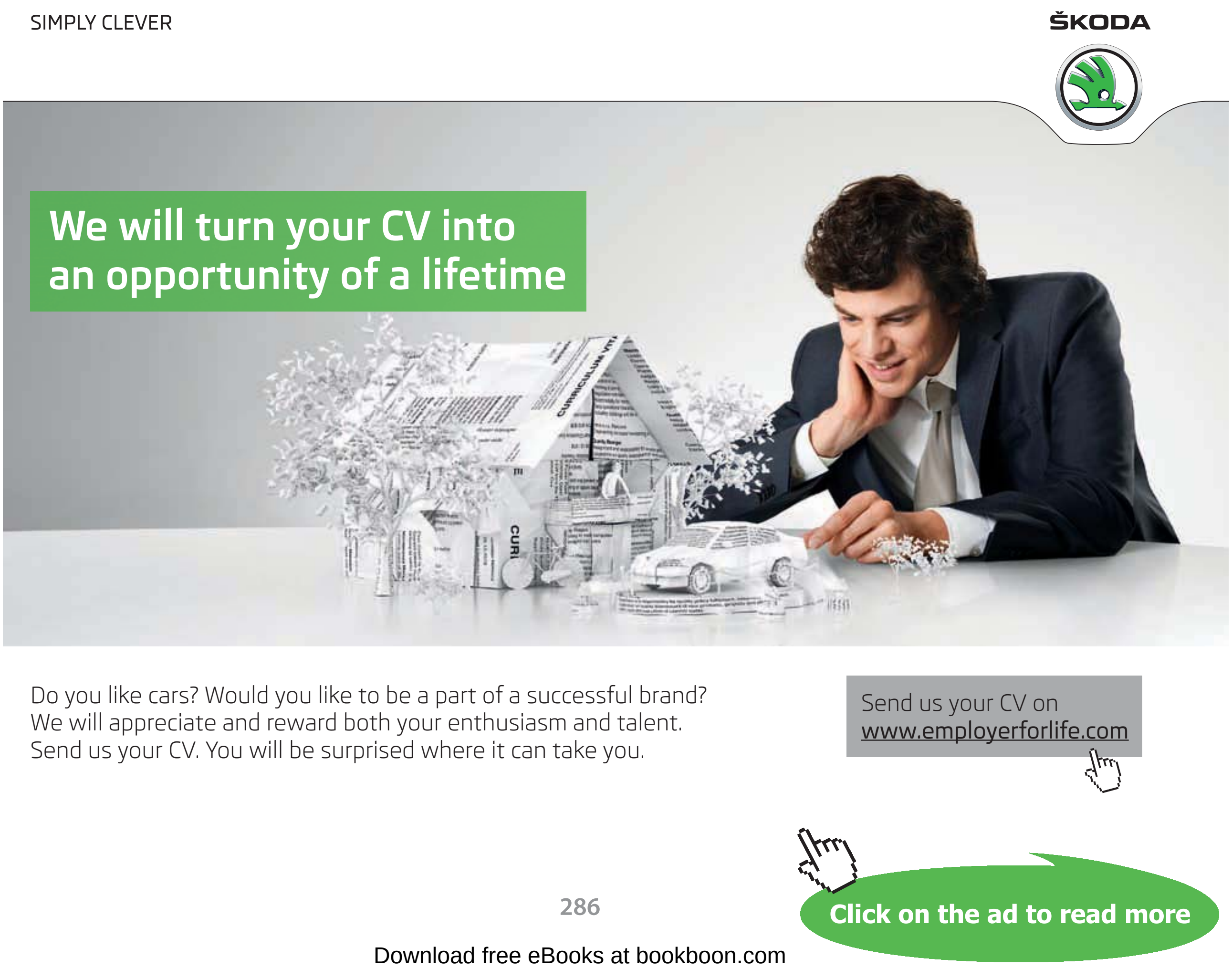
$$|\lambda|\,\|R_{-\alpha}(\lambda)\| \leqslant M\left(\cos(\varphi-\alpha)\right)^{-1}, \quad \lambda\in\Gamma(a,\varphi), \quad \Im\lambda\leqslant 0. \tag{7.2}$$

By Cauchy's theorem we have

$$R(\lambda)=R_\alpha(\lambda), \quad \Re\lambda>a_1, \quad \Im\lambda\geqslant 0, \tag{7.3}$$

and

$$R(\lambda)=R_{-\alpha}(\lambda), \quad \Re\lambda>a_1, \quad \Im\lambda\leqslant 0. \tag{7.4}$$



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In (7.4), as usually,

$$R(\lambda)x = \int_0^\infty \exp(-\lambda t)P(t)xdt = (\lambda I - A)^{-1}x, \quad x \in X,$$

for $\Re\lambda > \omega$. So the holomorphic map $\lambda \mapsto R(\lambda)$, $\Re\lambda > a_1$, extends to an operator valued holomorphic map on the interior of $a_1 + V_{\alpha+\frac{1}{2}\pi}$. This extension is again denoted by $R(\lambda)$ and in fact

$$R(\lambda) = (\lambda I - A)^{-1}, \quad \lambda \in a_1 + V_{\alpha+\frac{1}{2}\pi}. \tag{7.5}$$

From (7.1), (7.2), (7.3), (7.4), (7.5) and the maximum modulus theorem the inequality

$$|\lambda| \|R(\lambda)\| \leqslant M \left(\cos(\varphi - \alpha)\right)^{-1}, \quad \lambda \in a + V_\varphi, \tag{7.6}$$

follows.

(v) $\Rightarrow$ (vi) Let a and φ be as in (v). Then with $\Gamma = \Gamma(a, \varphi)$ we have

$$P(t)x = \frac{1}{2\pi i}\int_\Gamma \exp(\lambda t)R(\lambda)xd\lambda, \quad x \in X, \quad t \geqslant 0. \tag{7.7}$$

The latter follows from the equalities:

$$\begin{aligned}
&\int_0^\infty \exp(-\mu t)\frac{1}{2\pi i}\int_\Gamma \exp(\lambda t)R(\lambda)xd\lambda dt \\
&\quad = \frac{1}{2\pi i}\int_\Gamma \left(\int_0^\infty \exp\left(-(\mu - \lambda)t\right) dtR(\lambda)x\right) d\lambda \\
&\quad = \frac{1}{2\pi i}\int_\Gamma \frac{1}{\mu - \lambda}R(\lambda)d\lambda \\
&\quad = R(\mu)x = \int_0^\infty \exp(-\mu t)P(t)xdt,
\end{aligned}$$

for $\Re\mu > 0$ and $x \in X$ together with the uniqueness of Laplace transforms. Since A is a closed linear operator and since $AR(\lambda) = \lambda R(\lambda) - I$ equality (7.7) implies that for $x \in X$ and $t > 0$ the vector $P(t)x$ belongs to $D(A)$ and that

$$\begin{aligned}
AP(t)x &= \frac{1}{2\pi i}\int_\Gamma \exp(\lambda t)AR(\lambda)xd\lambda \\
&= \frac{1}{2\pi i}\int_\Gamma \exp(\lambda t)\left(\lambda R(\lambda)x - x\right) d\lambda \\
&= \frac{1}{2\pi i}\int_\Gamma \exp(\lambda t)\lambda R(\lambda)xd\lambda.
\end{aligned} \tag{7.8}$$

Consequently, with $M_0 := \sup\left\{|\lambda| \|R(\lambda)\| : \lambda \in \Gamma(a, \varphi)\right\}$, which by (v) is finite, equality (7.8) yields

$$\begin{aligned}
\pi t \|AP(t)x\| &\leqslant t\int_0^\infty \exp\left(at - \rho t\sin\left(\varphi - \frac{1}{2}\pi\right)\right) d\rho.M_0 \|x\| \\
&= \exp(at)\left(\sin\left(\varphi - \frac{1}{2}\pi\right)\right)^{-1}.M_0 \|x\|.
\end{aligned}$$

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This proves (vi).

(vi) $\Rightarrow$ (i) By (vi) there are constants δ and $M_1 > 0$ such that

$$t\,\|AP(t)x\| \leqslant M_1\,\|x\|$$

for all $x \in D(A)$ and for all $0 < t \leqslant \delta$. For $t > 0$ arbitrary select $k \in \mathbb{N}$ in such a way that $k\delta < t \leqslant (k+1)\delta$. With $\omega \geqslant 0$ and $M > 0$ satisfying

$$M = \sup\left\{M_1\left(\frac{t}{\delta}+1\right)\exp(-\omega t)\,\|P(s)\| : 0 < s \leqslant t,\quad t > 0\right\} < \infty,$$

we infer, for $x \in D(A)$,

$$\begin{aligned} t\,\|AP(t)x\| &\leqslant (t-k\delta)\,\|AP(t-k\delta)P(k\delta)x\| + k\delta\,\|AP(\delta)P(t-\delta)x\| \\ &\leqslant M_1\left(\|P(k\delta)x\| + k\,\|P(t-\delta)x\|\right) \\ &\leqslant M_1\left(\frac{t}{\delta}+1\right)\sup\left\{\|P(s)x\| : 0 < s \leqslant t\right\} \\ &\leqslant M\exp(\omega t)\,\|x\|\,. \end{aligned} \tag{7.9}$$

Consequently, since $D(A)$ is dense in X, the operators $AP(t)$, $t > 0$, can be extended to all of X. These extensions will be denoted by $C(t)$, $t > 0$. Moreover (7.9) implies

$$t\,\|C(t)\| \leqslant M\exp(\omega t),\quad t > 0. \tag{7.10}$$

Put $C_0(t) = P(t)$ and $C_k(t) = \left(C(tk^{-1})\right)^k$, $t > 0$, $k \in \mathbb{N}$, $k \geqslant 1$. By induction on n the equality

$$P(t)x = \sum_{k=0}^{n}\frac{(t-t_0)^k}{k!}C_k(t_0)x + \frac{1}{n!}\int_{t_0}^{t}(t-s)^n C_{n+1}(s)x\,ds, \tag{7.11}$$

is readily verified for t, $t_0 > 0$, $n \in \mathbb{N}$, and for $x \in X$. For the time being fix $t_0 > 0$ and consider $t > 0$ with

$$(Me+1)\,|t-t_0| \leqslant t_0. \tag{7.12}$$

By (7.10) and the definitions of the operators $C_n(t)$,

$$t^n\,\|C_n(t)\| \leqslant t^n\left\|C(tn^{-1})\right\|^n \leqslant n^n M^n\exp(\omega t),\quad t > 0,\quad n \in \mathbb{N},$$

and consequently

$$\begin{aligned} &\frac{1}{n!}\left\|\int_{t_0}^{t}(t-s)^n C_{n+1}(s)\,ds\right\| \\ &\quad\leqslant \frac{1}{n!}\left|\int_{t_0}^{t}(t-s)^n\,\|C_{n+1}(s)\|\,ds\right|\,\|x\| \\ &\quad\leqslant \frac{(n+1)^{n+1}}{n!}M^{n+1}\left|\int_{t_0}^{t}\frac{(t-s)^n\exp(\omega s)}{s^{n+1}}\,ds\right|\,\|x\| \\ &\quad\leqslant \frac{(n+1)^{n+1}}{(n+1)!}M^{n+1}\frac{\exp\left(\omega\max(t,t_0)\right)}{\left(\min(t,t_0)\right)^{n+1}}\,|t-t_0|^{n+1}\,\|x\| \end{aligned}$$

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(by Stirling's formula)

$$\leqslant \frac{(Me)^{n+1}}{\sqrt{2\pi(n+1)}} \frac{\exp\left(\omega \max(t,t_0)\right)}{\left(\min(t,t_0)\right)^{n+1}} \left|t-t_0\right|^{n+1} \|x\|$$

(by (7.12))

$$\leqslant \frac{\exp\left(\omega\left(1+(Me+1)^{-1}\right)t_0\right)}{\sqrt{2\pi(n+1)}} \|x\| .$$

Hence in (7.11) the remainder term tends to 0. So

$$P(t) = \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!} C_k(t_0), \quad |t-t_0| \leqslant \frac{t_0}{Me+1}, \quad t>0. \tag{7.13}$$

Next let $t \in \mathbb{C}$ be such that $(Me+1)\,|t-t_0| \leqslant t_0$. Then (7.13) defines $P(t)$ for such t. Moreover, by (7.12) we have for such complex t,

$$\begin{aligned}
\|P(t)\| &\leqslant \|P(t_0)\| + \sum_{k=1}^{\infty} \frac{|t-t_0|^k}{k!} \|C_k(t_0)\| \\
&\leqslant \left(\frac{M}{M_1} + \sum_{k=1}^{\infty} \frac{|t-t_0|^k}{k!} \frac{k^k M^k}{t_0^k}\right) \exp(\omega t_0)
\end{aligned}$$

(by Stirling's formula)

$$\begin{aligned}
&\leqslant \left(\frac{M}{M_1} + \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{\infty} \frac{|t-t_0|^k (Me)^k}{t_0^k}\right) \exp\left(\frac{Me\omega}{Me+1}|t|\right) \\
&\leqslant M\left(\frac{1}{M_1} + \frac{e}{\sqrt{2\pi}}\right) \exp\left(\omega \frac{Me}{Me+1}|t|\right) \\
&\leqslant M_0 \exp(\omega_0 |t|).
\end{aligned}$$

Here $M_0 = M\left(M_1^{-1} + e(2\pi)^{-\frac{1}{2}}\right)$ and $\omega_0 = \omega Me/(Me+1)$. Thus, for $t \in V_\alpha$ with $\alpha = \arcsin \dfrac{1}{Me+1}$, $P(t)$ satisfies

$$\|P(t)\| \leqslant M_0 \exp(\omega_0 |t|).$$

This proves (i).

(v) $\Rightarrow$ (ii) Let a and φ be as in (v), fix $\zeta = \exp(i\vartheta)$, $0 < \vartheta < 2\pi$ and choose $t > 0$ so small that

$$|\arg(-at + i\vartheta + 2k\pi i)| < \varphi, \quad \exists k \in \mathbb{Z}.$$

For such t define the operator $B(t)$ by

$$B(t) = \frac{1}{2\pi i} \int_\Gamma \frac{\exp(\lambda t)}{\exp(\lambda t) - \zeta} R(\lambda) d\lambda,$$

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where $\Gamma = \Gamma(a, \varphi)$. Then by symbolic calculus (see also (7.7)):

$$\begin{aligned}
P(t)B(t) &= B(t)P(t) \\
&= \frac{1}{2\pi i}\int_\Gamma \exp(\lambda t)\frac{\exp(\lambda t)}{\exp(\lambda t)-\zeta}R(\lambda)\,d\lambda \\
&= \frac{1}{2\pi i}\int_\Gamma \{(\exp(\lambda t)-\zeta)+\zeta\}\,\frac{\exp(\lambda t)}{\exp(\lambda t)-\zeta}R(\lambda)\,d\lambda \\
&= \frac{1}{2\pi i}\int_\Gamma \exp(\lambda t)R(\lambda)d\lambda + \frac{\zeta}{2\pi i}\int_\Gamma \frac{\exp(\lambda t)}{\exp(\lambda t)-\zeta}R(\lambda)\,d\lambda \\
&= P(t) + \zeta B(t).
\end{aligned}$$

Hence

$$(\zeta I - P(t))(I - B(t)) = (I - B(t))(\zeta I - P(t)) = \zeta I.$$

This shows (ii).

(ii) $\Rightarrow$ (iii) Trivial.

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(iii) $\Rightarrow$ (iv) Let $\zeta = \exp(i\vartheta)$, $0 < \vartheta < 2\pi$, be as in (iii) and choose $a \in \mathbb{R}$ in such a way that

$$\|P(t)\| \leqslant C\exp(at), \quad t \geqslant 0, \tag{7.14}$$

for some constant C. Then there exists $\eta > 0$ and $t_0 > 0$ such that, for all $x \in D(A)$,

$$\eta\,\|x\| \leqslant \|\zeta x - \exp(-at)P(t)x\|\,, \quad 0 < t < t_0. \tag{7.15}$$

Since, for $x \in D(A)$ and $t > 0$,

$$\zeta x - \exp(-at)P(t)x = \int_0^t \exp\left(i\vartheta\left(1 - \frac{s}{t}\right)\right)P(s)\left(\left(\frac{i\vartheta}{t} + a\right)x - Ax\right)ds.$$

So from 7.14 and 7.15 it follows that

$$\eta\,\|x\| \leqslant Ct\left\|\left(\frac{i\vartheta}{t} + a\right)x - Ax\right\|, \quad 0 < t < t_0, \quad x \in D(A).$$

So, with $M_1 = C\eta^{-1}\vartheta$ and $b_1 = t_0^{-1}\vartheta$,

$$\tau\,\|x\| \leqslant M_1\,\|(i\tau + a)x - Ax\|\,, \tag{7.16}$$

for x in $D(A)$ and $\tau \geqslant b_1$. Similarly, upon replacing ϑ with $\vartheta - 2\pi$ we infer with $M_2 = C\eta^{-1}(2\pi - \vartheta)$ and $b_2 = t_0^{-1}(2\pi - \vartheta)$,

$$|\tau|\,\|x\| \leqslant M_2\,\|(i\tau + a)x - Ax\|\,, \tag{7.17}$$

for $x \in D(A)$ and $\tau < -b_2$. Combining (7.16) and (7.17) yields (iv).

(iv) $\Rightarrow$ (v) Let M, a and b be as in (iv). Let ω be the type of the semigroup $\{P(t) : t \geqslant 0\}$. Fix $\alpha > \omega$, put $b_1 = \max(2M(\alpha - \omega), b)$ and choose $\pi > \varphi > \frac{1}{2}\pi$ in such a way that

$$|\cot\varphi| < \min\left(\frac{1}{2M}, \frac{\alpha - \omega}{b}\right) = \frac{\alpha - \omega}{b_1}. \tag{7.18}$$

For $|\tau| \geqslant b_1$ and $x \in D(A)$ the following inequalities hold true:

$$\begin{aligned}
2M\,\|(a + i\tau)x - Ax\| &= 2M\,\|(a + i\tau)x - Ax + (\alpha - a)x\| \\
&\geqslant 2M\,\|(a + i\tau)x - Ax\| - 2M\,|\alpha - a|\,\|x\| \\
&\geqslant |\tau|\,\|x\| + (|\tau| - 2M\,|\alpha - a|)\,\|x\| \\
&\geqslant |\tau|\,\|x\|\,.
\end{aligned}$$

Hence

$$|\tau|\,\|R(\alpha + i\tau)\| \leqslant 2M, \quad |\tau| \geqslant b_1. \tag{7.19}$$

From (7.18) and (7.19) we see that the series

$$\sum_{k=0}^{\infty}(-\rho\cos\varphi)^k\,(R(\alpha + i\rho\sin\varphi))^{k+1}$$

converges for $\rho\sin\varphi \geqslant b_1$. Moreover

$$R(\alpha + \rho e^{i\varphi}) = \left((\alpha + \rho e^{i\varphi})I - A\right)^{-1} = \sum_{k=0}^{\infty}(-\rho\cos\varphi)^k\,(R(\alpha + i\rho\sin\varphi))^{k+1}$$

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and

$$\left\|R(\alpha+\rho e^{i\varphi})\right\| \leqslant \frac{2M}{\rho\sin\varphi}\frac{1}{1-2M\left|\cot\varphi\right|}$$

for $\rho\sin\varphi \geqslant b_1$. Consequently

$$\begin{aligned}\left|\alpha+\rho e^{i\alpha}\right|\left\|R(\alpha+\rho e^{i\alpha})\right\| &\leqslant 2M\left|\frac{\alpha}{\rho\sin\varphi}+\cot\varphi+i\right|\frac{1}{1-2M\left|\cot\varphi\right|}\\ &\leqslant 2M\left(\frac{|\alpha|}{b_1}+\frac{1}{2M}+1\right)\frac{1}{1-2M\left|\cot\varphi\right|},\end{aligned} \tag{7.20}$$

for $\rho\sin\varphi \geqslant b_1$. Using the fact that $\left|\cot\varphi\right| < \dfrac{b_1}{\alpha-\omega}$ we see that

$$\sup\left\{\left|\alpha+\rho e^{i\varphi}\right|\left\|R(\alpha+\rho e^{i\varphi})\right\| :\ 0\leqslant\rho\leqslant\frac{b_1}{\sin\varphi}\right\}$$

is finite. This together with (7.20) shows that the expression

$$\sup\left\{\left|\alpha+\rho e^{i\varphi}\right|\left\|R(\alpha+\rho e^{i\varphi})\right\| : \rho\geqslant 0\right\} \tag{7.21}$$

is finite. A similar argument shows the finiteness of the expression

$$\sup\left\{\left|\alpha-\rho e^{i\varphi}\right|\left\|R(\alpha-\rho e^{i\varphi})\right\| :\ \rho\geqslant 0\right\}. \tag{7.22}$$

Using (7.21) and (7.22) together with the maximum modulus theorem results in the finiteness of

$$\sup\left\{|\lambda|\left\|R(\lambda)\right\| : \lambda\in\alpha+V_\varphi\right\}.$$

This proves (v). □

Proof of Theorem 7.3. The assertions (i), (ii), (iii), (iv) and (v) are equivalent by Proposition 7.6.

(iii) ⇒ (x) Let C, a and α be as in (iii) and let q be any polynomial with

$$|q(1)| < \sup\left\{|q(z)| : z\in\mathbb{C}, |z|=1\right\}. \tag{7.23}$$

Fix $s_0 > 0$ and fix $t_0 > 0$. Let $0 < s \leqslant s_0$, let $0 < t \leqslant t_0$ and let $\Gamma = \Gamma(at_0,\alpha)$. By symbolic calculus

$$\begin{aligned}&\left\{q\left(P\left(\frac{st}{n}\right)\right)\right\}^n P(t)\\ &= \frac{1}{2\pi i}\int_\Gamma \left\{q\left(\exp\left(\exp\left(\frac{\lambda s}{n}\right)\right)\right)\right\}^n \exp(\lambda t)(\lambda I - tA)^{-1}\,d\lambda.\end{aligned} \tag{7.24}$$

By (iii), (7.24) and some elementary estimates we obtain

$$\left\|\left\{q\left(P\left(\frac{st}{n}\right)\right)\right\}^n P(t)\right\| \leqslant C_1\sup\left\{\left|q\left(\exp\left(\frac{\lambda s}{n}\right)\right)\right|^n : \lambda\in\Gamma(at_0,\alpha)\right\},$$

where

$$C_1 = \frac{C}{\pi}\int_0^\infty \frac{\exp(at_0+\rho\cos\alpha)}{|at_0+\rho\exp(i\alpha)|}\,d\rho.$$

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Hence

$$
\begin{aligned}
&\sup\left\{\left\|\left\{q\left(P\left(\frac{st}{n}\right)\right)\right\}^{n}P(t)\right\| : 0 < s \leqslant s_0, \quad 0 < t \leqslant t_0\right\}\\
&\qquad \leqslant C_1 \sup\left\{\left|q\left(\exp\left(\frac{ast_0}{n}\right)\exp(\lambda)\right)\right|^{n} : 0 < s \leqslant s_0, \quad \lambda \in \Gamma(0,\alpha)\right\}.
\end{aligned}
$$

Consequently, since q is uniformly continuous on compact subsets of $\mathbb{C}$,

$$
\begin{aligned}
&\limsup_{n\to\infty} \sup\left\{\left\|\left\{q\left(P\left(\frac{st}{n}\right)\right)\right\}^{n}P(t)\right\|^{\frac{1}{n}} : 0 < s \leqslant s_0, \quad 0 < t \leqslant t_0\right\}\\
&\qquad \leqslant \sup\left\{\left|q\left(\exp(\lambda)\right)\right| : \lambda \in \Gamma(0,\alpha)\right\}. \qquad (7.25)
\end{aligned}
$$

The set $\{\exp(\lambda) : \lambda \in \Gamma(0,\alpha)\} \bigcup \{0\}$ is a compact subset of $\{z \in \mathbb{C} : |z| \leqslant 1\}$ which touches the circumference of the unit disc in the singleton 1. So the maximum modulus theorem together with (7.23) and (7.25) yields

$$
\begin{aligned}
&\limsup_{n\to\infty} \sup\left\{\left\|\left\{q\left(P\left(\frac{st}{n}\right)\right)\right\}^{n}P(t)\right\|^{\frac{1}{n}} : 0 < s \leqslant s_0, \quad 0 < t \leqslant t_0\right\}\\
&\qquad < \sup\left\{|q(z)| : z \in \mathbb{C}, \quad |z| = 1\right\}.
\end{aligned}
$$

Assertion (x) is an easy consequence of this fact.

(x) $\Rightarrow$ (ix) Yhis implication is trivial.

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(ix) $\Rightarrow$ (viii) Let q be apolynomial that satisfies (ix). Then there are n, $m \in \mathbb{N}$ and $\delta > 0$ such that

$$\sup_{0<t\leqslant\delta} \left\| \left\{ q\left(P\left(\frac{t}{mn} \right) \right) \right\}^{mn} \right\| < \sup\left\{ |q(z)|^n : z \in \mathbb{C}, \quad |z| = 1 \right\}.$$

So with $q_0(z) = q(z)^n z^{mn}$ and $\delta_0 = (mn)^{-1}\delta$ we obtain

$$\sup_{0<t\leqslant\delta_0} \|q_0\left(P(t)\right)\| < \sup\left\{ |q_0(z)| : z \in \mathbb{C}, \quad |z| = 1 \right\}.$$

This proves (viii) with $q = q_0$.

(viii) $\Rightarrow$ (ii) Let the polynomial q be as in (viii) and choose $\zeta \in \mathbb{C}$, $|\zeta| = 1$, in such a way that

$$|q(\zeta)| = \sup\left\{ |q(z)| : z \in \mathbb{C}, \quad |z| = 1 \right\}.$$

Then by (viii), there are $\delta > 0$ and $\eta > 0$ such that

$$\eta \|x\| \leqslant \left(|q(\zeta)| - \|q(P(t))\|\right) \|x\|$$

for $0 < t < \delta$ and for all $x \in X$. So for $0 < t < \delta$ and for $x \in X$ we obtain the inequality:

$$\eta \|x\| \leqslant \|q(\zeta)x - q\left(P(t)\right)x\|. \tag{7.26}$$

Define the polynomial r by the equality $q(\zeta) - q(z) = (\zeta - z)r(z)$, $z \in \mathbb{C}$, and put

$$C = \sup\left\{ \|r\left(P(t)\right)\| : 0 < t < \delta \right\}.$$

Then, because of (7.26),

$$\eta \|x\| \leqslant \|r\left(P(t)\right)\left(\zeta x - P(t)x\right)\| \leqslant C \|\zeta x - P(t)x\|$$

for all $x \in X$ and for all $0 < t < \delta$. So (ii) follows.

(v) $\Rightarrow$ (vi) Let $\delta > 0$ and $C > 0$ be such that $t\|AP(t)x\| \leqslant C\|x\|$ for all $x \in D(A)$ and for all $0 < t < \delta$. As in the proof implication (vi) $\Rightarrow$ (i) of Proposition 7.6 (see (7.9) and (7.10)) there are constants M and ω such that

$$t\|C(t)\| \leqslant M \exp(\omega t), \quad t > 0, \tag{7.27}$$

where $C(t)$ is the canonical extension of $AP(t)$ to all of X. Then the map

$$(x, y) \mapsto \left(P(t)x, tC(t)x + P(t)y\right), \quad x, \quad y \in X,$$

defines a strongly continuous semigroup on $X \times X$ with generator

$$(x, y) \mapsto \left(Ax, Ax + Ay\right), \quad x, \quad y \in D(A).$$

(vi) $\Rightarrow$ (v) Suppose that the map $(x, y) \mapsto (Ax, Ax + Ay)$, x, $y \in D(A)$, generates a strongly continuous semigroup $\{S(t) : t \geqslant 0\}$. Then

$$S(t)(x, y) = \left(P(t)x, tAP(t)x + P(t)y\right), \quad x, \ y \in D(A). \tag{7.28}$$

The latter can be seen as follows. Put $B(x, y) = (Ax, Ax + Ay)$ for x, $y \in D(A)$ and put $D(x, y) = (Ax, Ay)$ for x, $y \in D(A)$. Then D generates the semigroup $\{Q(t) : t \geqslant 0\}$, defined by

$$Q(t)(x, y) = \left(P(t)x, P(t)y\right), \quad x, \quad y \in X.$$

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Moreover for $y \in D(A)$ we have $B(0,y) = (0, Ay)$. Hence $S(t)(0,y) = (0, P(t)y)$, $y \in X$. So, for x and y in $D(A)$ we get

$$\begin{aligned}
S(t)(x,y) - (P(t)x, P(t)y) &= S(t)(x,y) - Q(t)(x,y) \\
&= \int_0^t S(t-s)(B-D)Q(s)(x,y)\, ds \\
&= \int_0^t S(t-s)(B-D)\left(P(s)x, P(s)y\right)\, ds \\
&= \int_0^t S(t-s)\left(0, AP(s)x\right)\, ds \\
&= \int_0^t \left(0, P(t-s)AP(s)x\right)\, ds \\
&= \int_0^t \left(0, AP(t-s)P(s)x\right)\, ds \\
&= \int_0^t \left(0, AP(t)x\right)\, ds = (0, tAP(t)x)\,.
\end{aligned}$$

This shows (7.28). From the strong continuity of the semigroup $\{S(t) : t \geqslant 0\}$ and (7.28) we see that

$$\sup\left\{t\left\|AP(t)x\right\| : x \in D(A),\quad \|x\| \leqslant 1,\quad 0 < t \leqslant 1\right\}$$

is finite. This proves (v).

(v) $\Rightarrow$ (vii) Since $\lambda R(\lambda) - I = AR(\lambda)$, since for λ sufficiently large

$$(n-1)!R(\lambda)^n x = \int_0^\infty s^{n-1}\exp(-\lambda s)P(s)xds, \quad x \in X, \quad n \in \mathbb{N},$$

and since A is a closed linear operator, it follows for $x \in D(A)$ and $n \geqslant 2$,

$$\begin{aligned}
(n-1)!\left\|(\lambda R(\lambda))^n x - (\lambda R(\lambda))^{n-1} x\right\| &= (n-1)!\lambda^{n-1}\left\|AR(\lambda)^n x\right\| \\
= \lambda^{n-1}\left\|A\int_0^\infty s^{n-1}\exp(-\lambda s)P(s)xds\right\| \\
= \lambda^{n-1}\left\|\int_0^\infty s^{n-1}\exp(-\lambda s)AP(s)xds\right\| \\
\leqslant \lambda^{n-1}\int_0^\infty s^{n-1}\exp(-\lambda s)\left\|AP(s)x\right\| ds. \qquad (7.29)
\end{aligned}$$

Since, by (v), for suitable constants $\delta > 0$ and $C > 0$,

$$t\left\|AP(t)x\right\| \leqslant C\left\|x\right\|, \quad 0 < t \leqslant \delta, \quad x \in D(A),$$

we obtain as in the proof of Proposition 7.6 (see (7.9)), again for appropriate constants M and ω,

$$s\left\|AP(s)x\right\| \leqslant M\exp(\omega s)\left\|x\right\|, \quad s \geqslant 0, \quad x \in D(A). \qquad (7.30)$$

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Inserting (7.30) into (7.29) yields

$$
\begin{aligned}
&(n-1)!\left\|(\lambda R(\lambda))^{n} x-(\lambda R(\lambda))^{n-1} x\right\| \\
&\leqslant M \lambda^{n-1} \int_{0}^{\infty} s^{n-2} \exp (-\lambda s) \exp (\omega s) d s .\|x\| \\
&=M\left(\frac{\lambda}{\lambda-\omega}\right)^{n-1}(n-2)!\|x\|,
\end{aligned}
$$

for $x \in D(A)$ and $n \geqslant 2$. Hence, for $\lambda \geqslant 2 \omega n$ we have

$$
n\left\|(\lambda R(\lambda))^{n}-(\lambda R(\lambda))^{n-1} x\right\| \leqslant M_{1}\|x\|, \quad x \in X,
$$

where

$$
M_{1}=M \sup _{n \geqslant 2} \frac{1}{\left(1-\dfrac{1}{2 n}\right)^{n}} \frac{n-\dfrac{1}{2}}{n-1} .
$$

This proves (vii).

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(vii) $\Rightarrow$ (v) From (vii) we get, with $x \in D(A)$ and M, $b > 0$ as in (vii),

$$n \left\| R(\lambda) \left(\lambda R(\lambda)\right)^{n-1} Ax \right\| \leqslant M \left\| x \right\|, \quad \lambda \geqslant bn.$$

So, with $0 < t \leqslant b^{-1}$ and $\lambda = nt^{-1}$,

$$t \left\| \left\{ \frac{n}{t} R\left(\frac{n}{t}\right) \right\}^n Ax \right\| \leqslant M \left\| x \right\|, \quad n \in \mathbb{N}, \quad x \in D(A).$$

Hence the next standard result in semigroup theory

$$P(t)y = \lim_{n\to\infty} \left\{ \frac{n}{t} R\left(\frac{n}{t}\right) \right\}^n y, \quad t \geqslant 0, \quad y \in X,$$

(see e.g. Pazy [**97**] Corollary 5.4) shows

$$t \left\| AP(t)x \right\| \leqslant M \left\| x \right\|, \quad 0 < t \leqslant \frac{1}{b}, \quad x \in D(A).$$

Hence, proof of Theorem 7.3 is now complete. □

We conclude this chapter with the following corollary. It follows from an examination of the proof of the implication (x) $\Rightarrow$ (i) of Theorem 7.3.

7.8. Corollary. *Suppose that the operator A is the generator of a strongly continuous semigroup $\{P(t) : t \geqslant 0\}$. Then $\{P(t) : t \geqslant 0\}$ has an exponentially bounded holomorphic extension if and only if*

$$\limsup_{n\to\infty} \sup_{s,t\in K} \left\| \left\{ q\left(P\left(\frac{st}{n}\right)\right)\right\}^n P(t) \right\| < \sup\left\{ |q(z)| : z \in \mathbb{C}, \quad |z| = 1 \right\}$$

for all compact subsets K of $[0,\infty)$ and for all polynomials q with

$$|q(1)| < \sup\left\{ |q(z)| : z \in \mathbb{C}, \quad |z| = 1 \right\}.$$

3. Bounded analytic semigroups

The following theorem characterizes generators of bounded analytic semigroups as sectorial operators whose resolvent sets contain the right half-plane $\{\lambda \in \mathbb{C} : \Re\lambda > 0\}$.

7.9. Theorem. *Let A be the generator of strongly continuous semigroup. The following assertions are equivalent:*

(i) *The operator A generates a bounded analytic semigroup.*
(ii) *There exists $\frac{1}{2}\pi < \alpha_0 < \pi$ such that $\rho(A) \supset V_{\alpha_0}$ and such that*

$$\sup\left\{ |\lambda| \left\| (\lambda I - A)^{-1} \right\| : \lambda \in V_{\alpha_0} \right\} < \infty. \tag{7.31}$$

(iii) *The resolvent set $\rho(A)$ contains the open half-plane $\{\lambda \in \mathbb{C} : \Re\lambda > 0\}$ and*

$$\sup\left\{ |\lambda| \left\| (\lambda I - A)^{-1} \right\| : \Re\lambda > 0 \right\} < \infty. \tag{7.32}$$

(iv) $\sup\left\{ t \left\| AP(t)x \right\| : t > 0, x \in D(A), \left\| x \right\| \leqslant 1 \right\} < \infty.$

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(v) *The operator*

$$\begin{pmatrix} A & 0 \\ A & A \end{pmatrix}$$

generates a strongly continuous bounded semigroup in $X \times X$.

(vi) *There exist a finite constant* C_1 *such that*

$$n \left\| (\lambda R(\lambda))^n - (\lambda R(\lambda))^{n-1} \right\| \leqslant C_1, \quad \lambda > 0, \quad n \in \mathbb{N}.$$

Let A be the generator of a semigroup $\{P(t) : t \geqslant 0\}$. Informally formulated the operator $\begin{pmatrix} A & 0 \\ A & A \end{pmatrix}$ generates the semigroup $\left\{ \begin{pmatrix} P(t) & 0 \\ tAP(t) & P(t) \end{pmatrix} : t \geqslant 0 \right\}$. This is a special case of the following situation. Let A_1 be the generator of a strongly continuous semigroup $\{P_1(t) : t \geqslant 0\}$ in a Banach space X_1, and let A_2 be the generator of a strongly continuous semigroup $\{P_2(t) : t \geqslant 0\}$ in a Banach space X_2. In addition, let $B : X_1 \to X_2$ be an appropriate, not necessarily bounded, operator. Then, in $X_1 \times X_2$, the operator $\begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}$ generates the semigroup

$$\left\{ \begin{pmatrix} P_1(t) & 0 \\ \int_0^t P_2(s) B P_1(t-s)\, ds & P_2(t) \end{pmatrix} : t \geqslant 0 \right\}.$$

Proof of Theorem 7.9. The reason that the assertions in (iv), (v) and (vi) are equivalent to the assertions in (i), (ii) and (iii) can be found in repeating the proofs of the implications (v) $\Longleftrightarrow$ (vi), (v) $\Longleftrightarrow$ (vii) of Theorem 7.3, and also by repeating the arguments in the implications (i) $\Longrightarrow$ (v) $\Longrightarrow$ (vi) $\Longrightarrow$ (i) of Proposition 7.6. Proofs of the other implications run as follows.

(i) $\Longrightarrow$ (ii) Let $0 < \alpha < \pi/2$ be such that A generates the bounded analytic semigroup $\{P(t) : t \in V_\alpha\}$. From (i) it follows that the sector V_α is such that the constant $C_\alpha := \sup\{\|P(t)\| : t \in V_\alpha\}$ is finite. Choose $0 < \delta < \min(\alpha, \pi/2 - \alpha)$, and put $\alpha_0 = \pi/2 + \alpha - \delta$. Let $\lambda \in V_{\alpha_0}$. Then $\lambda = |\lambda| e^{i\varphi}$ with $|\varphi| \leqslant \alpha_0$. If $\alpha + \delta - \pi/2 \leqslant \varphi \leqslant \alpha - \delta + \pi/2$, the we have

$$(\lambda I - A)^{-1} x = e^{-i\alpha} \int_0^\infty e^{-\lambda t e^{-i\alpha}} P\left(t e^{-i\alpha}\right) x\, dt, \; x \in X. \tag{7.33}$$

From (7.33) we infer:

$$\begin{aligned} \left\| (\lambda I - A)^{-1} x \right\| &\leqslant \int_0^\infty \left\| e^{-\lambda t e^{-i\alpha}} P\left(t e^{-i\alpha}\right) x \right\| dt \\ &\leqslant C_\alpha \int_0^\infty e^{-|\lambda| \cos(\varphi - \alpha) t}\, dt \; \|x\| \\ &\leqslant C_\alpha \int_0^\infty e^{-|\lambda| \cos(\pi/2 - \delta) t}\, dt \; \|x\| \\ &= \frac{C_\alpha}{|\lambda| \sin\delta} \|x\|, \; x \in X. \end{aligned} \tag{7.34}$$

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If $-\alpha+\delta-\pi/2 \leqslant \varphi \leqslant -\alpha-\delta+\pi/2$, the we have

$$(\lambda I - A)^{-1} x = e^{i\alpha} \int_0^\infty e^{-\lambda t e^{i\alpha}} P\left(te^{i\alpha}\right) x\,dt,\ x \in X. \tag{7.35}$$

Observe that $\alpha+\delta-\pi/2 \leqslant \pi/2-\alpha-\delta$, and so that if $\lambda = |\lambda|\, e^{i\varphi}$ with $-\pi < \varphi < \pi$ belongs to V_{α_0}, then φ satisfies $\alpha+\delta-\pi/2 \leqslant \varphi \leqslant \alpha-\delta+\pi/2$ or $-\alpha+\delta-\pi/2 \leqslant \varphi \leqslant -\alpha-\delta+\pi/2$. As in (7.34) from (7.35) we again have

$$\left\|(\lambda I - A)^{-1} x\right\| \leqslant \frac{C_\alpha}{|\lambda| \sin\delta} \|x\|,\ x \in X. \tag{7.36}$$

(ii) $\Longrightarrow$ (iii) This implication is trivial.

(iii) $\Longrightarrow$ (i) Let $\{P(t) : t \geqslant 0\}$ be the semigroup generated by A. Define the family of operators $\{Q(t) : t \geqslant 0\}$ such that

$$tQ(t)x = \frac{1}{2\pi i} \int_{\omega-i\infty}^{\omega+i\infty} e^{\lambda t} (\lambda I - A)^{-2} x\,dt, \quad x \in X. \tag{7.37}$$

In (7.37) ω is strictly, but we may choose it as we please. Let $\mu \in \mathbb{C}$ be such that $\Re\mu > \omega$. Then we have, for $x \in X$,

$$\begin{aligned}\int_0^\infty e^{-\mu t} tQ(t)x\,dt &= \frac{1}{2\pi i} \int_{\omega-i\infty}^{\omega+i\infty} \int_0^\infty e^{-(\mu-\lambda)t}\,dt\,(\lambda I - A)^{-2} x\,d\lambda \\ &= \frac{1}{2\pi i} \int_{\omega-i\infty}^{\omega+i\infty} \frac{1}{\mu-\lambda} (\lambda I - A)^{-2} x\,d\lambda\end{aligned}$$

(Cauchy's theorem, $\omega' > \Re\mu$)

$$= (\mu I - A)^{-2} x + \frac{1}{2\pi i} \int_{\omega'-i\infty}^{\omega'+i\infty} \frac{1}{\mu-\lambda} (\lambda I - A)^{-2} x\,d\lambda$$

(let ω' tend to ∞)

$$= (\mu I - A)^{-2} x = \int_0^\infty e^{-\mu t} tP(t)x\,dt. \tag{7.38}$$

In the final step of (7.38) we employed the fact that the operator A is the generator of the semigroup $\{P(t) : t \geqslant 0\}$. From (7.38) it follows that $Q(t)x = P(t)x$ for $t \geqslant 0$. We still have to prove that $t \mapsto P(t)x$, $t > 0$, extends to an analytic semigroup which is bounded on sector V_α with $0 < \alpha < \pi/2$. To achieve this we apply integration by parts to obtain, for $x \in X$,

$$t^2 P(t)x = t^2 Q(t)x = \frac{1}{\pi i} \int_{\omega-i\infty}^{\omega+i\infty} e^{\lambda t} (\lambda I - A)^{-3} x\,d\lambda. \tag{7.39}$$

Fix $x \in X$ and $t > 0$. From (7.39) we readily infer that $P(t)x$ belongs to $D(A)$, and that

$$t^2 AP(t)x = t^2 AQ(t)x = \frac{1}{\pi i} \int_{\omega-i\infty}^{\omega+i\infty} e^{\lambda t} (\lambda I - A)^{-2} \left(\lambda (\lambda I - A)^{-1} - I\right) x\,d\lambda. \tag{7.40}$$

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Again employing Cauchy's theorem and using (7.40) entails

$$
\begin{aligned}
t^2 AP(t)x &= t^2 AQ(t)x \\
&= \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}\left(e^{\lambda t}+e^{-\lambda t}-2\right)(\lambda I-A)^{-2}\left(\lambda(\lambda I-A)^{-1}-I\right)x\,d\lambda. \qquad (7.41)
\end{aligned}
$$

Put

$$
C=\sup\left\{|\lambda|\left\|(\lambda I-A)^{-1}\right\|:\Re\lambda>0\right\}. \qquad (7.42)
$$

Then by (iii) $C<\infty$. Employing (7.37) we get by Cauchy's theorem

$$
tP(t)x=tQ(t)x=\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}\left(e^{\lambda t}+e^{-\lambda t}-2\right)(\lambda I-A)^{-2}x\,dt, \quad x\in X. \qquad (7.43)
$$

From (7.43) we deduce, by letting $\omega\downarrow 0$, that

$$
\|P(t)x\|\leqslant\frac{1}{2\pi t}\int_{-\infty}^{\infty}\frac{2(1-\cos t\xi)}{\xi^2}\,d\xi\,C^2\|x\|=C^2\|x\|,\ x\in X,\ t>0. \qquad (7.44)
$$

Similarly, from (7.41) we deduce, by letting $\omega\downarrow 0$, that

$$
\begin{aligned}
t^2\|AP(t)x\| &\leqslant \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{2(1-\cos t\xi)}{\xi^2}\,d\xi\,C^2(C+1)\|x\| \\
&= 2tC^2(C+1)\|x\|. \qquad (7.45)
\end{aligned}
$$

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From (7.45) we see

$$t \|AP(t)x\| \leqslant 2C^2 (C+1) \|x\|, \quad t \geqslant 0, \ x \in X. \tag{7.46}$$

Fix $t_0 > 0$, and choose $t \in \mathbb{C}$ such that $2|t-t_0| eC^2 (C+1) < t_0$. Then we define the operator $\widetilde{P}(t)$ by the power series:

$$\begin{aligned} \widetilde{P}(t)x &= P(t_0)x + \sum_{k=1}^{\infty} \frac{(t-t_0)^k}{k!} A^k P(t_0)x \\ &= P(t_0)x + \sum_{k=1}^{\infty} \frac{k^k}{k!} \left(\frac{t}{t_0} - 1\right)^k \left(\frac{t_0}{k} AP\left(\frac{t_0}{k}\right)\right)^k x, \ x \in X. \end{aligned} \tag{7.47}$$

Hence, from (7.44), (7.45), and (7.47) we deduce

$$\begin{aligned} \left\|\widetilde{P}(t)x\right\| &\leqslant C^2 \|x\| + \sum_{k=1}^{\infty} \frac{k^k}{k!} \left|\frac{t}{t_0} - 1\right|^k \left(C^2 (C+1)\right)^k \|x\| \\ &\leqslant C^2 \|x\| + \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \left|\frac{t}{t_0} - 1\right|^k \left(eC^2 (C+1)\right)^k \|x\|, \ x \in X. \end{aligned} \tag{7.48}$$

In the final inequality in (7.48) we applied Stirling's formula which implies

$$\sqrt{2\pi k} \frac{k^k}{e^k} \leqslant k! \leqslant \sqrt{2\pi k} \frac{k^k}{e^k} \exp\left(\frac{1}{12k}\right), \quad k \geqslant 1. \tag{7.49}$$

From (7.48) it follows that $t \mapsto \left\|\widetilde{P}(t)x\right\|$, $\|x\| \leqslant 1$, is bounded as long as there exists $1 > \delta > 0$ such that $|t-t_0| eC^2 (C+1) \leqslant t_0 (1-\delta)$. Next we observe that the operator $\widetilde{P}(t)$ does not really depend on $t_0 > 0$ in the sense that if t_1, $t_2 > 0$, and if $t \in \mathbb{C}$ is such that $|t-t_j| eC^2 (C+1) < t_j$ for $j = 1, 2$, then

$$\sum_{k=0}^{\infty} \frac{(t-t_1)^k}{k!} A^k P(t_1)x = \sum_{k=0}^{\infty} \frac{(t-t_2)^k}{k!} A^k P(t_2)x, \ x \in X. \tag{7.50}$$

This can be achieved by differentiating the functions

$$s \mapsto \sum_{k=0}^{\infty} \frac{(t-s)^k}{k!} A^k P(s)x, \ x \in X. \tag{7.51}$$

Differentiating the function in (7.51) (for fixed $x \in X$) yields:

$$-\sum_{k=1}^{\infty} \frac{(t-s)^{k-1}}{(k-1)!} A^k P(s)x + \sum_{k=0}^{\infty} \frac{(t-s)^k}{k!} A^{k+1} P(s)x = 0,$$

and so the definition of $\widetilde{P}(t)$ does not really depend on t_0. In particular, by taking $t_0 = t$, it follows that $\widetilde{P}(t)x = P(t)x$ whenever $t > 0$. Also notice that for t_0 we may choose $|t|$. Consequently, the operators $\widetilde{P}(t)$ are defined in the sector determined by the inequality

$$|t - |t|| \, eC^2 (C+1) < |t|, \ t \in \mathbb{C}.$$

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In addition $\sup\left\{\left\|\widetilde{P}(t)\right\| : \arg(t) \leqslant \alpha\right\}$ is finite whenever the angle $0 < \alpha < \pi/2$ is such that

$$\left|e^{i\alpha} - 1\right| eC^2 (C+1) \leqslant 1 - \delta. \tag{7.52}$$

It is not so difficult to prove the semigroup property of the family $t \mapsto \widetilde{P}(t)$, $t \in V_\alpha$, where $0 < \alpha < \pi/2$ satisfies (7.52). In fact we have

$$\begin{aligned}
\widetilde{P}(t_1)\widetilde{P}(t_2) &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \frac{(t_1 - |t_1|)^{k_1}}{k_1!}\frac{(t_2 - |t_2|)^{k_2}}{k_2!} A^{k_1+k_2} P(|t_1| + |t_2|)\,x \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(t_1 - |t_1|)^k (t_2 - |t_2|)^{n-k} A^n P(|t_1| + |t_2|)\,x \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}(t_1 + t_2 - |t_1| - |t_2|)^n A^n P(|t_1| + |t_2|)\,x \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}(t_1 + t_2 - |t_1 + t_2|)^n A^n P(|t_1 + t_2|)\,x \\
&= \widetilde{P}(t_1 + t_2)\,x, \quad x \in X.
\end{aligned} \tag{7.53}$$

In other words the family $\left\{\widetilde{P}(t) : t \in V_\alpha\right\}$ represents a uniformly bounded analytic semigroup. The strong continuity at $t = 0$ of the the semigroup $\widetilde{P}(t)$ can easily be proved via the representation

$$\widetilde{P}(t)\,x = P(t_0)\,x + \sum_{k=1}^{\infty}\frac{(t - t_0)^k}{k!}A^k P(t_0)\,x$$

with $t_0 = |t|$ and $t \to 0$, $t \in V_\alpha$. This proves assertion (i).

Altogether this completes the proof of Theorem 7.9. □

Theorem 7.9 can be strengthened somewhat. The idea is that a densely defined closed linear operator A which satisfies (iii) in Theorem 7.9 is in fact the generator of bounded analytic semigroup.

7.10. Theorem. *Let A be a closed linear operator with a domain which is dense in a Banach space X. The following assertions are equivalent:*

(i) *The operator A generates a bounded analytic semigroup.*

(ii) *There exists $\frac{1}{2}\pi < \alpha_0 < \pi$ such that $\rho(A) \supset V_{\alpha_0}$ and such that*

$$\sup\left\{|\lambda|\left\|(\lambda I - A)^{-1}\right\| : \lambda \in V_{\alpha_0}\right\} < \infty. \tag{7.54}$$

(iii) *The resolvent set $\rho(A)$ contains the open half-plane $\{\lambda \in \mathbb{C} : \Re\lambda > 0\}$ and*

$$\sup\left\{|\lambda|\left\|(\lambda I - A)^{-1}\right\| : \Re\lambda > 0\right\} < \infty. \tag{7.55}$$

Our proof requires the following lemma.

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7.11. Lemma. *Let the family $\{Q(t) : t \geqslant 0\}$ be a uniformly bounded family of linear operator with the property that for every $x \in X$ the mapping $t \mapsto Q(t)x$ is Borel measurable. Put $R(\lambda)x = \int_0^\infty e^{-\lambda t} Q(t)x\, dt$. Then the following assertions are equivalent:*

(i) *The family $\{Q(t) : t \geqslant 0\}$ has the semigroup property, i.e. for all $x \in X$ the equality $Q(s+t)x = Q(s)Q(t)x$ holds for almost all $(s,t) \in (0,\infty) \times (0,\infty)$.*
(ii) *The family $\{R(\lambda) : \lambda > 0\}$ has the resolvent property,* i.e.,

$$(\mu - \lambda) R(\lambda) R(\mu) = R(\lambda) - R(\mu)$$

for all $(\lambda, \mu) \in (0,\infty) \times (0,\infty)$.

Proof. Fix $x \in X$, and λ, $\mu > 0$, $\lambda \neq \mu$. First we calculate

$$\begin{aligned}\int_0^\infty \int_0^\infty e^{-\lambda s - \mu t} Q(s+t)x\, dt\, ds &= \int_0^\infty \int_0^\infty e^{-(\lambda-\mu)s} \int_0^\infty e^{-\mu(s+t)} Q(s+t)x\, dt\, ds \\ &= \int_0^\infty e^{-(\lambda-\mu)s} \int_s^\infty e^{-\mu t} Q(t)x\, dt\, ds\end{aligned}$$

(Fubini-Tonelli's theorem)

$$\begin{aligned} &= \int_0^\infty e^{-\mu t} \int_0^t e^{-(\lambda-\mu)s}\, ds Q(t)x\, dt \\ &= \int_0^\infty \frac{e^{-\lambda t} - e^{-\mu t}}{\mu - \lambda} Q(t)x\, dt = \frac{R(\lambda)x - R(\mu)x}{\mu - \lambda}. \end{aligned} \tag{7.56}$$

It is clear that

$$\int_0^\infty \int_0^\infty e^{-\lambda s - \mu t} Q(s) Q(t)x\, dt\, ds = R(\lambda) R(\mu) x. \tag{7.57}$$

The implication (i) $\Longrightarrow$ (ii) follows from (7.56) and (7.57). The other implication, (ii) $\Longrightarrow$ (i) also follows from these identities in conjunction with uniqueness of Laplace transforms.

The proof of Lemma 7.11 is complete now. □

Proof of Theorem 7.10. The proofs of the implications (i) $\Longrightarrow$ (ii) and (ii) $\Longrightarrow$ (iii) are exactly the same as in the proof of Theorem 7.9. In the proof of the implication (iii) $\Longrightarrow$ (i) we have to show that (iii) implies that the operator A generates a strongly continuous semigroup. The proof of this implication in Theorem 7.9 supplies us with a candidate semigroup $\{Q(t) : t > 0\}$ determined by the equality in (7.37)

$$tQ(t)x = \frac{1}{2\pi i} \int_{\omega - i\infty}^{\omega + i\infty} e^{\lambda t} (\lambda I - A)^{-2} x\, dt = \frac{t}{2\pi i} \int_{\omega - i\infty}^{\omega + i\infty} e^{\lambda} (\lambda I - tA)^{-2} x\, dt, \tag{7.58}$$

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for $x \in X$. The second equality follows from the substitution $t\lambda = \lambda'$, and then replacing λ' with λ. As in the proof of the implication (iii) $\Longrightarrow$ (i) of Theorem 7.9 (see (7.38)) we see

$$\int_0^\infty e^{-\lambda t} tQ(t)x\,dt = (\lambda I - A)^{-2}\,x,\ x \in X. \tag{7.59}$$

From (7.59) we obtain, for $x \in X$,

$$-\frac{d}{d\lambda}\int_0^\infty e^{-\lambda t}Q(t)x\,dt = \int_0^\infty e^{-\lambda t}tQ(t)x\,dt = (\lambda I - A)^{-2}\,x = -\frac{d}{d\lambda}(\lambda I - A)^{-1}\,x,$$

and hence, for $x \in X$, the vector-valued function

$$\lambda \mapsto \int_0^\infty e^{-\lambda t}Q(t)x\,dt - (\lambda I - A)^{-1}\,x,\ \lambda > 0,$$

is constant. However, since the function

$$\lambda \mapsto \lambda\int_0^\infty e^{-\lambda t}Q(t)x\,dt - \lambda\,(\lambda I - A)^{-1}\,x,\ \lambda > 0,$$

is bounded, it follows that this constant vanishes. Consequently, we get

$$\int_0^\infty e^{-\lambda t}Q(t)x\,dt = (\lambda I - A)^{-1}\,x,\ x \in X,\ \lambda > 0. \tag{7.60}$$

Since the family $\left\{(\lambda I - A)^{-1} : \lambda > 0\right\}$ has the resolvent property Lemma 7.11 together with the equality in (7.60) yields the semigroup property of the family $\{Q(t) : t > 0\}$.

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It remains to show that the semigroup $\{Q(t) : t \geqslant 0\}$, with $Q(0) = I$, is strongly continuous. Because, once we have established this strong continuity, the proof of the implication (iii) $\Longrightarrow$ (i) can then be finished in the same way as we proved this implication in Theorem 7.9. In order to prove the strong continuity we invoke the equality of the ultimate expressions in (7.58). This equality applies to the effect that

$$\begin{aligned} Q(t)x - x &= \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} e^{\lambda}\left((\lambda I - tA)^{-2}x - \lambda^{-2}x\right)\,dt \\ &= \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} \frac{e^{\lambda}}{\lambda^2}\left(\lambda(\lambda I + tA)^{-1} + I\right)\left(\lambda(\lambda I - tA)^{-1}x - x\right)\,dt \\ &= \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} \frac{e^{\lambda}}{\lambda^2}\left(\lambda(\lambda I + tA)^{-1} + I\right)(tA)(\lambda I - tA)^{-1}x\,dt, \end{aligned} \tag{7.61}$$

for $x \in X$. It is not so difficult to show that, for $\lambda \in \mathbb{C}$ with $\Re\lambda > 0$ fixed, and $x \in X$, we have

$$\lim_{t\downarrow 0}\left(\lambda(\lambda I - tA)^{-1}x - x\right) = \lim_{t\downarrow 0} tA(\lambda I - tA)^{-1}x = 0. \tag{7.62}$$

First this equality is proved for $x \in D(A)$, and then the equality in (7.62) extends to all $x \in X$ because $D(A)$ is dense in X and the function $t \mapsto (\lambda I - tA)^{-1}x$, $t > 0$, is bounded. The fact that $\lim_{t\downarrow 0} Q(t)x = x$ then follows from (7.61) together with Lebesgue's dominated convergence theorem.

Altogether this completes the proof of Theorem 7.10. $\square$

4. Bounded analytic semigroups and the Crank-Nicolson iteration scheme

In the present section we discuss the (implicit) Crank-Nicolson iteration scheme in which a generator A of a bounded analytic semigroup plays a central role. The present material is partly taken from [**146**] and [**144**]. Let $(\tau_j)_{j\in\mathbb{N}}$ be a sequence of strictly positive real numbers, and let A be the generator of a bounded analytic semigroup in a Banach space X. Fix $x_0 \in D(A)$, and define the sequence $(x_n)_{n\in\mathbb{N}} \subset D(A)$ by the (implicit) Crank-Nicolson scheme:

$$\left(I - \frac{1}{2}\tau_{n+1}A\right)x_{n+1} = \left(I + \frac{1}{2}\tau_n A\right)x_n.$$

Put $A_n = \prod_{j=1}^{n}\left(I + \frac{1}{2}\tau_j A\right)\left(I - \frac{1}{2}\tau_j A\right)^{-1}$. Then the sequence $(x_n)_{n\in\mathbb{N}} \subset X$ determined by the Crank-Nicolson scheme is given by $x_n = A_n x_0$. In this paper it is investigated under what conditions on x_0 and the sequence of step-sizes $(\tau_j)_{j\in\mathbb{N}}$ the Crank-Nicolson scheme is stable in the sense that $\sup_{n\in\mathbb{N}}\|A_n x_0\| < \infty$. Put $f_k(\xi) = 2\sum_{j=1}^{k}\arctan\left(\frac{1}{2}\tau_j\xi\right)$, $\xi \in \mathbb{R}$, and $I_k = \displaystyle\int_0^\infty \frac{1}{\xi^2}\left|\int_0^\xi \sin f_k(\eta)\,d\eta\right|\,d\xi$.

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7.12. Theorem. *The Crank-Nicolson scheme is stable provided one of the following conditions is satisfied: (a)* $\sup_k I_{2k} < \infty$ *and* $x \in X$ *is arbitrary; (b)* $\sum_{j=1}^{\infty} \tau_j < \infty$ *and* x *belongs to* $D(A)$*, the domain of* A*; (c)* $\sum_{j=1}^{\infty} \tau_j^{-1} < \infty$ *and* x *belongs to* $R(A)$*, the range of* A*; (d) the sequence of positive step-sizes* $(\tau_j)_{j\in\mathbb{N}}$ *is arbitrary, and* x *belongs to the intersection* $D(A) \cap R(A)$*.*

Suppose that the operator A is the generator of a bounded analytic semigroup in a complex Banach space X. As is well-known this is the case if and only if $D(A)$ is dense and there exists a constant C such that

$$|\lambda| \, \|R(\lambda)\| \leqslant C, \quad \Re\lambda > 0. \tag{7.63}$$

Here $R(\lambda) = (\lambda I - A)^{-1}$. For more details see theorems 7.9 and 7.10. Let $\Omega = \{z \in \mathbb{C} : \Re z < 0\}$ be the open left half-plane in $\mathbb{C}$, and let $f : \Omega \to \mathbb{C}$ be a bounded holomorphic function which has a continuous extension up to the boundary which we call again f. Then we have the following result.

7.13. Theorem. *Let* A *be the generator of a bounded analytic semigroup on the Banach space* $(X, \|\cdot\|)$*, and let* $f : \overline{\Omega} \to \mathbb{C}$ *be a bounded continuous function on* $\overline{\Omega}$ *which is holomorphic on* Ω*. In addition, let the finite constant* C *be as in (7.63). Then the operator* $f(A)$ *has the following representations (*$x \in X$*):*

$$
\begin{aligned}
& f(A)x - f(0)x \\
&= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \int_0^{\lambda} \left(f(\zeta) - f(-\zeta)\right) d\zeta \left\{ (\lambda I - A)^{-2} - \lambda^{-2} I \right\} x \, d\lambda && (7.64) \\
&= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \int_0^{\lambda} \left(f(\zeta) + f(-\zeta) - 2f(0)\right) d\zeta \left\{ (\lambda I - A)^{-2} - \lambda^{-2} I \right\} x \, d\lambda && (7.65) \\
&= \frac{1}{\pi i} \int_{-i\infty}^{i\infty} \int_0^{\lambda} \left(f(\zeta) + f(-\zeta) - 2f(0)\right) d\zeta \int_0^1 A \left(\lambda I - sA\right)^{-3} x \, ds \, d\lambda && (7.66) \\
&= \frac{-1}{\pi i} \int_0^{\infty} \int_0^{\xi} \left(f(i\eta) - f(-i\eta)\right) d\eta \left\{ \left(A^2 - \xi^2 I\right) \left(\xi^2 I + A^2\right)^{-2} + \xi^{-2} I \right\} x \, d\xi && (7.67) \\
&= \frac{1}{\pi i} \int_0^{\infty} \int_0^{\xi} \left(f(i\eta) - f(-i\eta)\right) d\eta \frac{\partial}{\partial \xi} \left\{ \xi^{-1} A^2 \left(\xi^2 I + A^2\right)^{-1} \right\} x \, d\xi && (7.68) \\
&= -\frac{1}{\pi i} \int_0^{\infty} \frac{f(i\xi) - f(-i\xi)}{\xi} A^2 \left(\xi^2 I + A^2\right)^{-1} x \, d\xi && (7.69) \\
&= \lim_{R\to\infty} \left\{ \frac{1}{\pi i} \int_0^{R} \left(f(i\xi) - (-i\xi)\right) \xi \left(\xi^2 I + A^2\right)^{-1} x \, d\xi \right. \\
&\qquad \left. + \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} f\left(-Re^{i\vartheta}\right) x \, d\vartheta - f(0)x \right\} && (7.70) \\
&= \lim_{R\to\infty} \frac{1}{2\pi i} \int_{-iR}^{iR} \left(f(\lambda) - f(-\lambda)\right) \left\{ (\lambda I - A)^{-1} x - \lambda^{-1} x \right\} d\lambda && (7.71)
\end{aligned}
$$

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$$= \frac{1}{\pi}\int_0^\infty \left(2f(0) - f(i\xi) - f(-i\xi)\right) A\left(\xi^2 I + A^2\right)^{-1} x\, d\xi. \tag{7.72}$$

In addition, the following equalities hold for $x \in X$:

$$\begin{aligned} f(A)x &= \lim_{R\to\infty}\left\{\frac{1}{2\pi i}\int_{-iR}^{iR}\int_0^\lambda \left(f(\zeta) - f(-\zeta)\right)\, d\zeta\, (\lambda I - A)^{-2}\, x\, d\lambda \right.\\ &\qquad \left. + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\int_0^1 f\left(-Re^{i\vartheta}\rho\right) x\, d\rho\, d\vartheta\right\} \\ &= \lim_{R\to\infty}\left\{\frac{1}{\pi i}\int_0^R \int_0^\xi \left(f(i\eta) - f(-i\eta)\right)\, d\eta \left(\xi^2 I - A^2\right)\left(\xi^2 I + A^2\right)^{-2} x\, d\xi \right.\\ &\qquad \left. + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\int_0^1 f\left(-Re^{i\vartheta}\rho\right) x\, d\rho\, d\vartheta\right\}. \end{aligned} \tag{7.73}$$

Moreover, let the bounded function f be such that the integral

$$\int_0^\infty \frac{\left|\int_0^\xi \left(f(i\eta) + f(-i\eta) - 2f(0)\right)\, d\eta\right|}{\xi^3}\, d\xi \tag{7.74}$$

is finite. The following inequalities hold as well:

$$\|f(A)\| \leqslant \frac{C^2\left(1 + 2C^2\right)}{\pi}\int_0^\infty \frac{\left|\int_0^\xi \left(f(i\eta) - f(-i\eta)\right)\, d\eta\right|}{\xi^2}\, d\xi + \sup_{\Re\lambda > 0}\left|f(-\lambda)\right|, \tag{7.75}$$

and for $x \in D(A)$,

$$\begin{aligned} \|f(A)x - f(0)x\| &\leqslant \frac{(1+C)^2}{\pi}\int_0^1 \frac{|f(i\xi) - f(-i\xi)|}{\xi}\, d\xi\, \|x\| \\ &\quad + \frac{2C(1+C)}{\pi}\sup_{\xi\in\mathbb{R}}|f(i\xi)|\, \|Ax\|, \end{aligned} \tag{7.76}$$

$$\|f(A)x - f(0)x\| \leqslant \frac{2C^3}{\pi}\int_0^\infty \frac{\left|\int_0^\xi \left(f(i\eta) + f(-i\eta) - 2f(0)\right)\, d\eta\right|}{\xi^3}\, d\xi\, \|Ax\|. \tag{7.77}$$

The integrals in (7.64) through (7.72) have to be interpreted as improper strong Riemann integrals.

PROOF OF THEOREM 7.13. Let $z \in \mathbb{C}$ be such that $\Re z < 0$. First we prove the equality:

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\int_0^\lambda \left(f(\zeta) - f(0)\right)\, d\zeta \left\{\frac{1}{(\lambda - z)^2} - \frac{1}{\lambda^2}\right\} d\lambda = f(z) - f(0). \tag{7.78}$$

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Let γ_R be the curve $\vartheta \mapsto -Re^{-i\vartheta}$, $-\frac{1}{2}\pi \leqslant \vartheta \leqslant \frac{1}{2}\pi$, and let

$$g(\lambda, z) = \int_0^\lambda (f(\zeta) - f(0))\, d\zeta \left\{ \frac{1}{(\lambda - z)^2} - \frac{1}{\lambda^2} \right\}$$

be the integrand in the left-hand side of (7.78). From residue calculus it follows that the left-hand side of (7.78) is equal to

$$f(z) - f(0) + \lim_{R\to\infty} \frac{1}{2\pi i} \int_{\gamma_R} g(\lambda, z)\, d\lambda = f(z) - f(0). \tag{7.79}$$

The reason that the limit in (7.79) vanishes is due to the fact that by assumption the function f is bounded. A similar argument shows that

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \int_0^\lambda (f(0) - f(-\zeta))\, d\zeta \left\{ \frac{1}{(\lambda - z)^2} - \frac{1}{\lambda^2} \right\} d\lambda = 0. \tag{7.80}$$

Adding the equalities in (7.78) and (7.80) yields the equality:

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \int_0^\lambda (f(\zeta) - f(-\zeta))\, d\zeta \left\{ \frac{1}{(\lambda - z)^2} - \frac{1}{\lambda^2} \right\} d\lambda = f(z) - f(0). \tag{7.81}$$

Subtracting the equalities in (7.78) and (7.80) yields the equality:

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \int_0^\lambda (f(\zeta) + f(-\zeta) - 2f(0))\, d\zeta \left\{ \frac{1}{(\lambda - z)^2} - \frac{1}{\lambda^2} \right\} d\lambda = f(z) - f(0). \tag{7.82}$$

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An integration by parts and parametrization in (7.82) yields:

$$\begin{aligned}
f(z)-f(0) &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\int_0^{\lambda}\left(f(\zeta)+f(-\zeta)-2f(0)\right)d\zeta\left\{\frac{1}{(\lambda-z)^2}-\frac{1}{\lambda^2}\right\}d\lambda\\
&= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\left(f(\lambda)+f(-\lambda)-2f(0)\right)\frac{z}{\lambda-z}\frac{d\lambda}{\lambda}\\
&= \frac{1}{2\pi i}\int_{\infty}^{0}\left(f(-i\xi)+f(i\xi)-2f(0)\right)\frac{z}{-i\xi-z}\frac{d\xi}{\xi}\\
&\quad+\frac{1}{2\pi i}\int_0^{\infty}\left(f(i\xi)+f(-i\xi)-2f(0)\right)\frac{z}{i\xi-z}\frac{d\xi}{\xi}\\
&= \frac{1}{\pi}\int_0^{\infty}\left(2f(0)-f(i\xi)-f(-i\xi)\right)\frac{z}{\xi^2+z^2}\,d\xi. \qquad (7.83)
\end{aligned}$$

The equality in (7.72) is a direct consequence of the identity in (7.83). The equalities in (7.81) and (7.82) apply to the effect that the the two equalities in (7.64) and (7.65) hold true. The third equality, i.e. (7.66), follows easily from the second one, i.e. (7.65). The equalities in (7.67) through (7.69) are consequences of the equality in (7.64). The equality in (7.70) is an easy consequence of (7.69) and (7.71) follows from (7.70). The inequality in (7.75) is obtained from the equality in (7.73), the one in (7.76) follows from the equality in (7.69), and the one in (7.77) follows from the equality in (7.66). The constant C is as in (7.63).

This completes the proof of Theorem 7.13. □

By choosing in (7.72) the function

$$f(\lambda)=\prod_{j=1}^{k}\frac{1+\frac{1}{2}\tau_j\lambda}{1-\frac{1}{2}\tau_j\lambda}. \qquad (7.84)$$

we obtain the following corollary.

7.14. Corollary. *The following identities hold for $x\in X$:*

$$\begin{aligned}
&\prod_{j=1}^{k}\left(I+\frac{1}{2}\tau_jA\right)\left(I-\frac{1}{2}\tau_jA\right)^{-1}x-x \qquad (7.85)\\
&=\frac{1}{\pi}\int_0^{\infty}\left|1-\prod_{j=1}^{k}\frac{1+\frac{1}{2}i\tau_j\xi}{1-\frac{1}{2}i\tau_j\xi}\right|^2 A\left(\xi^2I+A^2\right)^{-1}x\,d\xi\\
&=\frac{2}{\pi}\int_0^{\infty}\left(1-\cos f_k(\xi)\right)A\left(\xi^2I+A^2\right)^{-1}x\,d\xi\\
&=\frac{2}{\pi}\int_0^{\infty}\left(1-\cos f_k\left(\eta^{-1}\right)\right)A\left(I+\eta^2A^2\right)^{-1}x\,d\eta, \qquad (7.86)
\end{aligned}$$

where $f_k(\xi)=2\sum_{j=1}^{k}\arctan\left(\frac{1}{2}\tau_j\xi\right)$.

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7.15. Theorem. *Let $(\tau_j)_{j\in\mathbb{N}}$ be a sequence of strictly positive real numbers. Let A be the generator of a bounded analytic semigroup in the Banach space X, and define for any compact subset K of the open set $(0,\infty)$ the operator $x \mapsto \nu(K)x$, $x \in X$, by*

$$\nu(K)x := \frac{2}{\pi}\int_K (-A)\left(I+\xi^2A^2\right)^{-1} x\, d\xi. \tag{7.87}$$

Then, for every $k \in \mathbb{N}$, the vector

$$x - \prod_{j=1}^{k}\left(I+\frac{1}{2}\tau_jA\right)\left(I-\frac{1}{2}\tau_jA\right)^{-1} x \tag{7.88}$$

belongs to the closed convex hull of

$$\left\{2\nu(K)x : K \text{ compact subset of the open semi-axis } (0,\infty)\right\}. \tag{7.89}$$

Moreover, the collection mentioned in (7.89) is contained in $D(A)\cap R(A)$.

Proof. The vector mentioned in (7.88) can be written in the form

$$\begin{aligned}
&x - \prod_{j=1}^{k}\left(I+\frac{1}{2}\tau_jA\right)\left(I-\frac{1}{2}\tau_jA\right)^{-1} x\\
&= \int_0^\infty \left(1-\cos f_k\left(\xi^{-1}\right)\right) d\nu(\xi)x = \lim_{\delta\downarrow 0, R\uparrow\infty}\int_\delta^R \left(1-\cos f_k\left(\xi^{-1}\right)\right) d\nu(\xi)x\\
&= 2\lim_{\delta\downarrow 0, R\uparrow\infty}\int_0^1 \nu\left\{\xi\in[\delta,R], 1-\cos f_k(\xi)\geqslant 2\rho\right\} x\, d\rho\\
&= \lim_{\delta\downarrow 0}\lim_{R\to\infty}\lim_{n\to\infty}\frac{1}{2^{n-1}}\sum_{\ell=1}^{2^n}\nu\left\{\left\{1-\cos f_k\geqslant\frac{\ell}{2^{n-1}}\right\}\cap[\delta,R]\right\}x.
\end{aligned} \tag{7.90}$$

The equalities in (7.90) follow from (7.86) in Corollary 7.14. Since sets of the form $\{1-\cos f_k \geqslant \ell 2^{-n+1}\}\cap[\delta,R]$, $\ell\geqslant 1$, $0<\delta<R<\infty$, are compact the first assertion in Proposition 7.15 follows. Let the constant C be as in (7.63), let K be a compact subset of the open interval $(0,\infty)$, and pick $x\in X$. The second assertion in Proposition 7.15 is a consequence of the following observations:

(1) The inequality
$$\left\|A\left(\xi^2I+A^2\right)^{-1}x\right\| \leqslant \frac{2C}{\xi}\left\|x\right\|$$
implies that $\nu(K)x$ is a vector in X indeed;

(2) The inequality
$$\left\|A^2\left(\xi^2I+A^2\right)^{-1}x\right\| \leqslant \left(1+\frac{C^2}{\xi^2}\right)\left\|x\right\|$$
yields the claim that $\nu(K)x$ belongs to $D(A)$.

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(3) Finally, the inequality

$$\left\|\left(\xi^2 I + A^2\right)^{-1} x\right\| \leqslant \frac{C^2}{\xi^2} \|x\|$$

entails that $\nu(K)x$ is a member of $R(A)$.

The assertions in (1), (2) and (3) imply the second statement in Theorem 7.15, and complete its proof. $\square$

Let $g_k(\rho)$ be the inverse function of $f_k(\xi) = 2\sum_{j=1}^{n} \arctan\left(\frac{1}{2}\tau_j\xi\right)$, $0 \leqslant \xi < \infty$, $f_k(\infty) = k\pi$. That is $f_k\left(g_k(\rho)\right) = \rho$, $0 \leqslant \rho \leqslant k\pi$. Let $h_k(\rho) = \dfrac{1}{g_k(\rho)}$. An alternative way of looking at the vector in (7.88) is to rewrite the expression in (7.86) as follows:

$$\int_0^\infty \left(1 - \cos f_k(\xi)\right)(-A)\left(\xi^2 I + A^2\right)^{-1} x\, d\xi$$

(integration by parts)

$$= \int_0^\infty \sin f_k(\xi) f_k'(\xi) \int_\xi^\infty (-A)\left(\eta^2 I + A^2\right)^{-1} x\, d\eta\, d\xi$$

(make the substitution $\rho = f_k(\xi)$, $\xi = g_k(\rho)$, $0 < \rho < k\pi$)

$$= \int_0^{k\pi} \sin\rho \int_{g_k(\rho)}^\infty (-A)\left(\eta^2 I + A^2\right)^{-1} x\, d\eta\, d\rho$$

(distinguish cases: $k = 2\ell$, $k = 2\ell + 1$, $g_{2\ell+1}\left(\rho + (2\ell + 1\pi)\right) = \infty$, $\rho \geqslant 0$)

$$= \begin{cases} \displaystyle\int_0^\pi \sin\rho \sum_{j=0}^{\ell-1} \int_{g_{2\ell}(\rho+2j\pi)}^{g_{2\ell}(\rho+(2j+1)\pi)} (-A)\left(\eta^2 I + A^2\right)^{-1} x\, d\eta\, d\rho, & k = 2\ell, \\ \displaystyle\int_0^\pi \sin\rho \sum_{j=0}^{\ell} \int_{g_{2\ell+1}(\rho+2j\pi)}^{g_{2\ell+1}(\rho+(2j+1)\pi)} (-A)\left(\eta^2 I + A^2\right)^{-1} x\, d\eta, & k = 2\ell + 1, \end{cases}$$

$\left(h_{2\ell+1}\left(\rho + (2\ell + 1)\right) = 0,\ \rho \geqslant 0\right)$

$$= \begin{cases} \displaystyle\int_0^\pi \sin\rho \sum_{j=0}^{\ell-1} \int_{h_{2\ell}(\rho+(2j+1)\pi)}^{h_{2\ell}(\rho+2j\pi)} (-A)\left(I + \eta^2 A^2\right)^{-1} x\, d\eta\, d\rho, & k = 2\ell, \\ \displaystyle\int_0^\pi \sin\rho \sum_{j=0}^{\ell} \int_{h_{2\ell+1}(\rho+(2j+1)\pi)}^{h_{2\ell+1}(\rho+2j\pi)} (-A)\left(I + \eta^2 A^2\right)^{-1} x\, d\eta, & k = 2\ell + 1. \end{cases} \tag{7.91}$$

Put, for $0 < \rho < \pi$,

$$O_{2\ell}(\rho) = \bigcup_{j=0}^{\ell-1} \left(h_{2\ell}\left(\rho + (2j+1)\pi\right), h_{2\ell}\left(\rho + 2j\pi\right)\right), \quad \text{and}$$

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$$O_{2\ell+1}(\rho) = \bigcup_{j=0}^{\ell-1} \left(h_{2\ell}\left(\rho + (2j+1)\pi\right), h_{2\ell}\left(\rho + 2j\pi\right)\right) \cup \left(0, h_{2\ell+1}\left(\rho + 2\ell\pi\right)\right). \tag{7.92}$$

Then, from (7.86), (7.91) and (7.92) it follows that

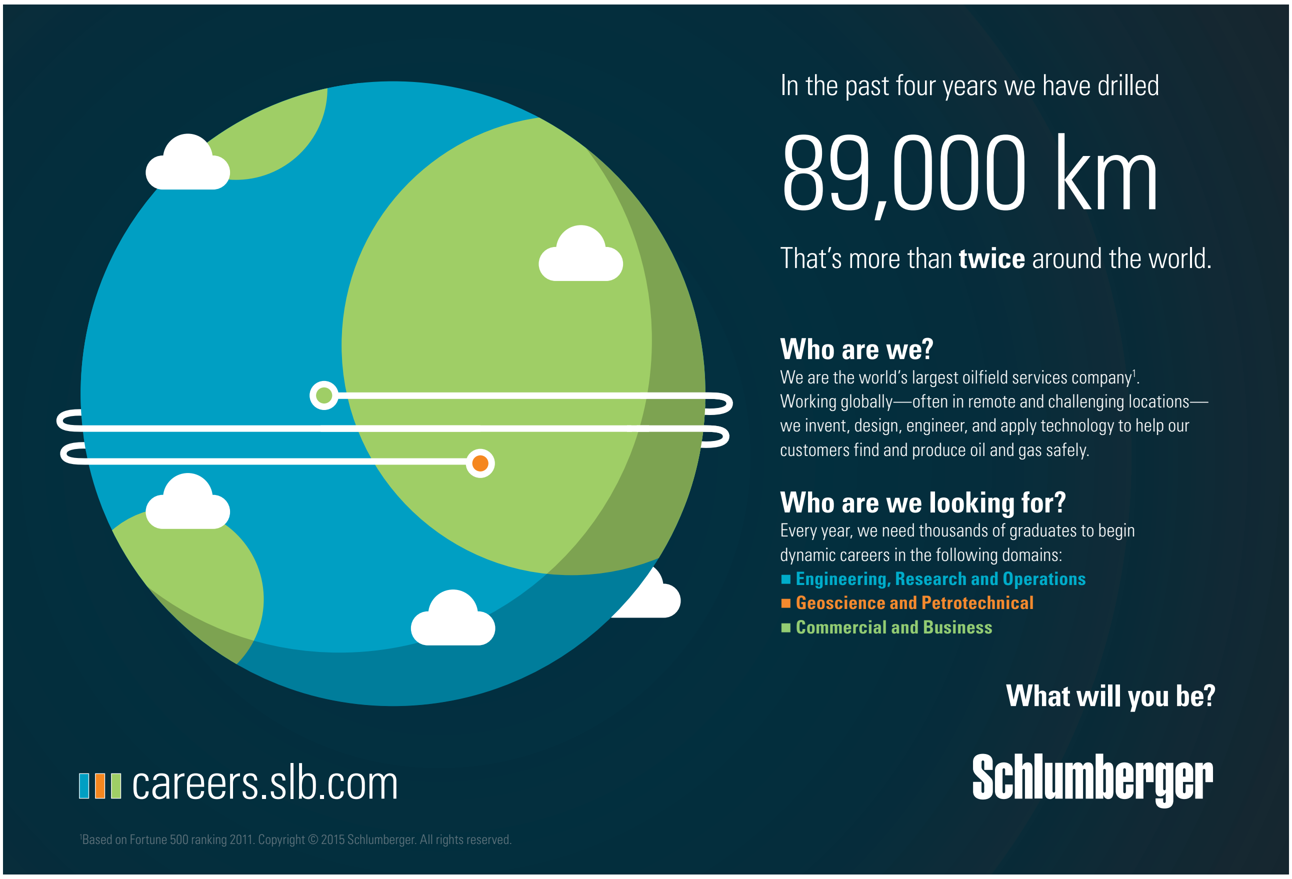
$$x - \prod_{j=1}^{k} \left(I + \frac{1}{2}\tau_j A\right) \left(I - \frac{1}{2}\tau_j A\right)^{-1} x = \int_0^{\pi} \sin\rho\, \nu\left(O_k(\rho)\right) x\, d\rho, \tag{7.93}$$

where ν is determined by (7.87). Observe that, for $x \in X$,

$$\nu(\alpha, \beta)x = \frac{2}{\pi}\left(\arctan\left(-\beta A\right) - \arctan\left(-\alpha A\right)\right) x, \ \ 0 \leqslant \alpha < \beta < \infty.$$

From Corollary 7.14 in conjunction with the theory of vector measures we infer the result in Theorem 7.16. The Orlicz-Pettis theorem says that a weakly-continuous X-valued measure, defined on a σ-field $\mathcal{F}$, is in fact an X-valued bounded vector measure ν: see e.g. Diestel and Uhl [**37**]. A result by Bartle, Dunford and Schwartz says that a bounded countably additive vector measure on a σ-field possesses a relatively weakly compact range: see [**37**] and also Section 1.2 in [**147**]. For some proofs see, e.g., [**71**] and also [**38**]. As a consequence it follows that the collection

$$\left\{\int f\, d\nu = \int_0^1 \nu\left\{f > \rho\right\}\, d\rho : 0 \leqslant f \leqslant \mathbf{1},\ f \in L^{\infty}(\lambda)\right\}$$



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is contained in the closed convex hull of $\{\nu(B) : B \in \mathcal{F}\}$. The measure λ has to be such that, for every $x^* \in X^*$, the complex measure $B \mapsto \langle \nu(B)x, x^* \rangle$ is absolutely continuous relative to the measure λ. Rybakov's theorem says that λ can be taken of the form $B \mapsto \langle \nu(B)x, x_0^* \rangle$ for some $x_0^* \in X^*$: see [**114**]. For closely related material see, e.g., [**73**]. Because of the nature of $\nu(B)x$, in our case for λ we may choose the Lebesgue mesure on $\mathbb{R}_+$. Let X^{**} be the topological bi-dual of the Banach space X.

7.16. Theorem. *Let $(\tau_j)_{j\in\mathbb{N}}$ be a sequence of strictly positive real numbers. Let A be the generator of a bounded analytic semigroup in the Banach space X, and let $x \in X$ be such that the mapping*

$$B \mapsto \langle \nu(B)x, x^* \rangle := \frac{2}{\pi} \int_B \left\langle (-A) \left(I + \xi^2 A^2\right)^{-1} x, x^* \right\rangle d\xi, \quad B \in \mathcal{B}_{\mathbb{R}_+}, \tag{7.94}$$

represents a σ-additive $\mathbb{C}$-valued measure for every $x^* \in X^*$. Then, for every $k \in \mathbb{N}$, the vector

$$x - \prod_{j=1}^{k} \left(I + \frac{1}{2}\tau_j A\right) \left(I - \frac{1}{2}\tau_j A\right)^{-1} x \tag{7.95}$$

belongs to the closed convex hull of the set $\{2\nu(K)x : K \subset (0,\infty),\ K$ compact$\}$, which is bounded in X, and hence the Crank-Nicolson scheme is stable for such $x \in X$.

Notice that the assumption (7.87) in Theorem 7.16 is equivalent to the fact that $x \in X$ is such that for every $x^* \in X^*$ the integral $\int_0^\infty \left|\left\langle A\left(\xi^2 I + A^2\right)^{-1} x, x^* \right\rangle\right| d\xi$ is finite.

7.17. Corollary. *Let the notation and hypotheses be as in Theorem 7.16. In particular, let $(\tau_j)_{j\in\mathbb{N}}$ be any sequence of strictly positive real numbers. Then the Crank-Nicolson iteration scheme with step sizes τ_j, $j \in \mathbb{N}$, is stable provided the initial vector x belongs to $D(A) \cap R(A)$.*

Proof. Let $x = Ay \in R(A)$ be a member of $D(A)$. Then the mapping $B \mapsto \nu(B)x$, $B \in \mathcal{R}_+$, represents a bounded vector measure indeed. The result in Corollary 7.17 then follows from Theorem 7.16. $\square$

For a concise formulation of the main result of this paper we introduce the following definition.

7.18. Definition. Let A be the generator of a bounded analytic semigroup, let $(\tau_j)_{j\in\mathbb{N}}$ be a sequence of strictly positive real numbers, and let $x \in X$. The Crank-Nicolson iteration scheme is said to be stable for x and the step-sizes $(\tau_j)_{j\in\mathbb{N}}$ if the sequence

$$\left\{ \prod_{j=1}^{k} \left(I + \frac{1}{2}\tau_j A\right) \left(I - \frac{1}{2}\tau_j A\right)^{-1} x : k \in \mathbb{N} \right\}$$

is bounded in X.

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7.19. Theorem. *Let A be the generator of a bounded analytic semigroup, and let $(\tau_j)_{j\in\mathbb{N}}$ be a sequence of strictly positive real numbers. The following assertion hold true:*

(a) *If* $\sup_{k\in\mathbb{N}} \int_0^\infty \frac{1}{\xi^2} \left| \int_0^\xi \sin f_k(\eta)\, d\eta \right| d\xi < \infty$, *then the Crank-Nicolson scheme is stable for all $x \in X$. Here $f_k(\xi) = 2\sum_{j=1}^k \arctan\left(\frac{1}{2}\tau_j\xi\right)$.*
(b) *If $\sum_{j=1}^\infty \tau_j < \infty$, then the Crank-Nicolson scheme is stable for all $x \in D(A)$.*
(c) *If $\sum_{j=1}^\infty \tau_j^{-1} < \infty$, then the Crank-Nicolson scheme is stable for all $x \in R(A)$.*
(d) *The Crank-Nicolson scheme is stable for all $x \in D(A) \cap R(A)$ and all sequences of strictly positive step sizes $(\tau_j)_{j\in\mathbb{N}}$.*

7.20. Definition. Let $x \in X$ and let $(\tau_j)_{j\in\mathbb{N}}$. The Crank-Nicolson scheme is called consistent for x and $(\tau_j)_{j\in\mathbb{N}}$, provided that the limit

$$\lim_{n\to\infty} \prod_{j=1}^n \left(I + \frac{1}{2}\tau_j A\right)\left(I - \frac{1}{2}\tau_j A\right)^{-1} x \tag{7.96}$$

exists in X. The Crank-Nicolson scheme is called two step consistent for x and $(\tau_j)_{j\in\mathbb{N}}$, provided that the limit

$$\lim_{n\to\infty} \prod_{j=1}^{2n} \left(I + \frac{1}{2}\tau_j A\right)\left(I - \frac{1}{2}\tau_j A\right)^{-1} x \tag{7.97}$$

exists.

In Theorem 7.24 we shall prove that in assertion (b) and (c) of Theorem 7.19 the conclusion may be strengthened to "consistent" and "two step consistent" respectively instead of just "stable".

Proof of Theorem 7.19. (a) Let de function $f(\lambda)$ be as in (7.84). Then

$$\frac{f(i\eta) - f(-i\eta)}{2i} = \Im \prod_{j=1}^k \frac{1 + \frac{1}{2}i\tau_j\eta}{1 - \frac{1}{2}i\tau_j\eta} = \sin f_k(\eta), \tag{7.98}$$

where $f_k(\eta)$ is as in Corollary 7.14. The inequality in (7.75) then entails the claim in assertion (a).

(b) Let the function $f_k(\xi)$ be as in Corollary 7.14: see (a). Then the equality in (7.86) implies, for $x \in D(A)$,

$$\begin{aligned}
&\prod_{j=1}^k \left(I + \frac{1}{2}\tau_j A\right)\left(I - \frac{1}{2}\tau_j A\right)^{-1} x - x \\
&= \frac{2}{\pi}\int_0^\infty \left(1 - \cos f_k(\xi)\right) A \left(\xi^2 I + A^2\right)^{-1} x\, d\xi \\
&= \frac{4}{\pi}\int_0^\infty \sin^2\left(\frac{1}{2} f_k(\xi)\right)\left(\xi^2 I + A^2\right)^{-1} Ax\, d\xi.
\end{aligned} \tag{7.99}$$

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Let C be the constant from (7.63). Then the equality in (7.99) implies

$$\left\| \prod_{j=1}^{k} \left(I + \frac{1}{2}\tau_j A \right) \left(I - \frac{1}{2}\tau_j A \right)^{-1} x - x \right\| \leqslant \frac{4C^2}{\pi} \int_0^\infty \frac{\sin^2\left(\frac{1}{2} f_k(\xi)\right)}{\xi^2}\, d\xi \, \|Ax\| \,. \tag{7.100}$$

Since

$$\left| \sin\left(\frac{1}{2} f_k(\xi) \right) \right| \leqslant \frac{1}{2} f_k(\xi) = \sum_{j=1}^{k} \arctan\left(\frac{1}{2}\tau_j \xi \right) \leqslant \frac{1}{2}\xi \sum_{j=1}^{k} \tau_j, \tag{7.101}$$

we infer

$$\frac{4}{\pi} \int_0^\infty \frac{\left(\sin\left(\frac{1}{2} f_k(\xi)\right)\right)^2}{\xi^2}\, d\xi \leqslant \frac{1}{\pi} \int_0^\infty \frac{\left(\min\left(2, \xi \sum_{j=1}^{k} \tau_j\right)\right)^2}{\xi^2}\, d\xi = \frac{4}{\pi} \sum_{j=1}^{k} \tau_j. \tag{7.102}$$

The assertion in (b) then follows from (7.100) and (7.102).

(c) The proof of assertion (c) is similar to the one of (b). Without loss of generality assume that k is an even positive integer. Otherwise replace x with the vector $\left(I + \frac{1}{2}\tau_1 A\right)\left(I - \frac{1}{2}\tau_1 A\right)^{-1} x$. Let $x = Ay$ belong to the range of the operator A.

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Then as in (7.99) we have:

$$\begin{aligned}
&\prod_{j=1}^{k}\left(I+\frac{1}{2}\tau_j A\right)\left(I-\frac{1}{2}\tau_j A\right)^{-1}x-x\\
&=\frac{2}{\pi}\int_0^\infty\left(1-\cos f_k(\xi)\right)A^2\left(\xi^2 I+A^2\right)^{-1}y\,d\xi\\
&=\frac{4}{\pi}\int_0^\infty\sin^2\left(\frac{1}{2}f_k(\xi)\right)\left\{y-\xi^2\left(\xi^2 I+A^2\right)^{-1}y\right\}\,d\xi. \qquad (7.103)
\end{aligned}$$

And hence, Let C be the constant from (7.63). Then the equality in (7.99) implies

$$\begin{aligned}
&\left\|\prod_{j=1}^{k}\left(I+\frac{1}{2}\tau_j A\right)\left(I-\frac{1}{2}\tau_j A\right)^{-1}x-x\right\|\\
&\leqslant\frac{4\left(1+C^2\right)}{\pi}\int_0^\infty\sin^2\left(\frac{1}{2}f_k(\xi)\right)\,d\xi\,\|y\|. \qquad (7.104)
\end{aligned}$$

Since k is even we have, by the equality $\arctan\xi+\arctan\dfrac{1}{\xi}=\dfrac{1}{2}\pi$,

$$\begin{aligned}
\left|\sin\left(\frac{1}{2}f_k(\xi)\right)\right| &= \left|\sin\left(\sum_{j=1}^{k}\arctan\left(\frac{1}{2}\tau_j\xi\right)\right)\right|\\
&=\left|\sin\left(\sum_{j=1}^{k}\left(\frac{1}{2}\pi-\arctan\left(\frac{2}{\tau_j\xi}\right)\right)\right)\right|\\
&=\left|\sin\left(\sum_{j=1}^{k}\arctan\left(\frac{2}{\tau_j\xi}\right)\right)\right|\leqslant\frac{2}{\xi}\sum_{j=1}^{k}\frac{1}{\tau_j}. \qquad (7.105)
\end{aligned}$$

From (7.104) and (7.105) it then follows that, with $x=Ay$,

$$\left\|\prod_{j=1}^{k}\left(I+\frac{1}{2}\tau_j A\right)\left(I-\frac{1}{2}\tau_j A\right)^{-1}x-x\right\|\leqslant\frac{16\left(1+C^2\right)}{\pi}\sum_{j=1}^{k}\frac{1}{\tau_j}\,\|y\|. \qquad (7.106)$$

Assertion (c) follows from (7.106).

Assertion (d) being contained in Corollary 7.17 this completes the proof of Theorem 7.19. □

7.21. Remark. Let $x\in X$, and let K be a compact subset of the open interval $(0,\infty)$. A typical element in $D(A)\cap R(A)$ is given by

$$\nu(K)\,x=\frac{2}{\pi}(-A)\int_K\left(I+\xi^2A^2\right)^{-1}x\,d\xi. \qquad (7.107)$$

Moreover, the family $\{\nu(\delta,R)x:\ 0<\delta<R<\infty\}$ is bounded in X, and, for all $k\in\mathbb{N}$,

$$\prod_{j=1}^{k}\left(I+\frac{1}{2}\tau_j A\right)\left(I-\frac{1}{2}\tau_j A\right)^{-1}x-x$$

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$$= \lim_{\delta\downarrow 0,\, R\uparrow\infty} \left\{ \prod_{j=1}^{k} \left(I + \frac{1}{2}\tau_j A\right)\left(I - \frac{1}{2}\tau_j A\right)^{-1} - I \right\} \nu(\delta, R)x. \tag{7.108}$$

7.22. LEMMA. *Let A be the generator of a bounded analytic semigroup, let $(\tau_j)_{j\in\mathbb{N}}$ be a sequence of strictly positive real numbers, and let $x \in X$. The following identity holds:*

$$\begin{aligned}
&\sum_{\ell=1}^{k} \tau_\ell \prod_{j=1}^{\ell-1} \left(I + \frac{1}{2}\tau_j A\right)\left(I - \frac{1}{2}\tau_j A\right)^{-1}\left(I - \frac{1}{2}\tau_\ell A\right)^{-1} x \\
&= \frac{2}{\pi}\int_0^\infty (1 - \cos f_k(\xi))\left(\xi^2 I + A^2\right)^{-1} x \, d\xi.
\end{aligned} \tag{7.109}$$

In general the vector in (7.109) belongs to $D(A)$, and

$$\begin{aligned}
&\sum_{\ell=1}^{k} \tau_\ell A \prod_{j=1}^{\ell-1} \left(I + \frac{1}{2}\tau_j A\right)\left(I - \frac{1}{2}\tau_j A\right)^{-1}\left(I - \frac{1}{2}\tau_\ell A\right)^{-1} x \\
&= \prod_{j=1}^{k} \left(I + \frac{1}{2}\tau_j A\right)\left(I - \frac{1}{2}\tau_j A\right)^{-1} x - x \\
&= \frac{2}{\pi}\int_0^\infty (1 - \cos f_k(\xi))\, A\left(\xi^2 I + A^2\right)^{-1} x \, d\xi.
\end{aligned} \tag{7.110}$$

If k is even, then the vector in (7.110) belongs to $D(A)\cap R(A)$. In fact, the following equalities hold:

$$\begin{aligned}
&\sum_{\ell=1}^{2k} \tau_\ell A \prod_{j=1}^{\ell-1} \left(I + \frac{1}{2}\tau_j A\right)\left(I - \frac{1}{2}\tau_j A\right)^{-1}\left(I - \frac{1}{2}\tau_\ell A\right)^{-1} x \\
&= \prod_{j=1}^{2k} \left(I + \frac{1}{2}\tau_j A\right)\left(I - \frac{1}{2}\tau_j A\right)^{-1} x - x \\
&= \sum_{\ell=1}^{k} \left(\tau_{2\ell-1} + \tau_{2\ell}\right) A \prod_{j=1}^{2(\ell-1)} \left(I + \frac{1}{2}\tau_j A\right)\left(I - \frac{1}{2}\tau_j A\right)^{-1} \\
&\qquad \left(I - \frac{1}{2}\tau_{2\ell-1} A\right)^{-1}\left(I - \frac{1}{2}\tau_{2\ell} A\right)^{-1} x
\end{aligned}$$

$$= \frac{2}{\pi}\int_0^\infty (1 - \cos f_{2k}(\xi))\, A\left(\xi^2 I + A^2\right)^{-1} x \, d\xi \tag{7.111}$$

$$= \frac{2}{\pi}\int_0^\infty \sin f_{2k}(\xi)\, f_{2k}'(\xi) \int_\xi^\infty A\left(\eta^2 I + A^2\right)^{-1} x \, d\eta \, d\xi \tag{7.112}$$

$$= \frac{1}{\pi}\int_0^\infty \sin f_{2k}(\xi)\, f_{2k}'(\xi) \int_{-\pi/2}^{\pi/2} \xi e^{i\vartheta}\left(\xi e^{i\vartheta} I - A\right)^{-1} x \, d\vartheta \, d\xi \tag{7.113}$$

$$= \frac{1}{\pi}\int_0^\infty \sin f_{2k}(\xi)\, f_{2k}'(\xi) \int_{-\pi/2}^{\pi/2} A\left(\xi e^{i\vartheta} I - A\right)^{-1} x \, d\vartheta \, d\xi. \tag{7.114}$$

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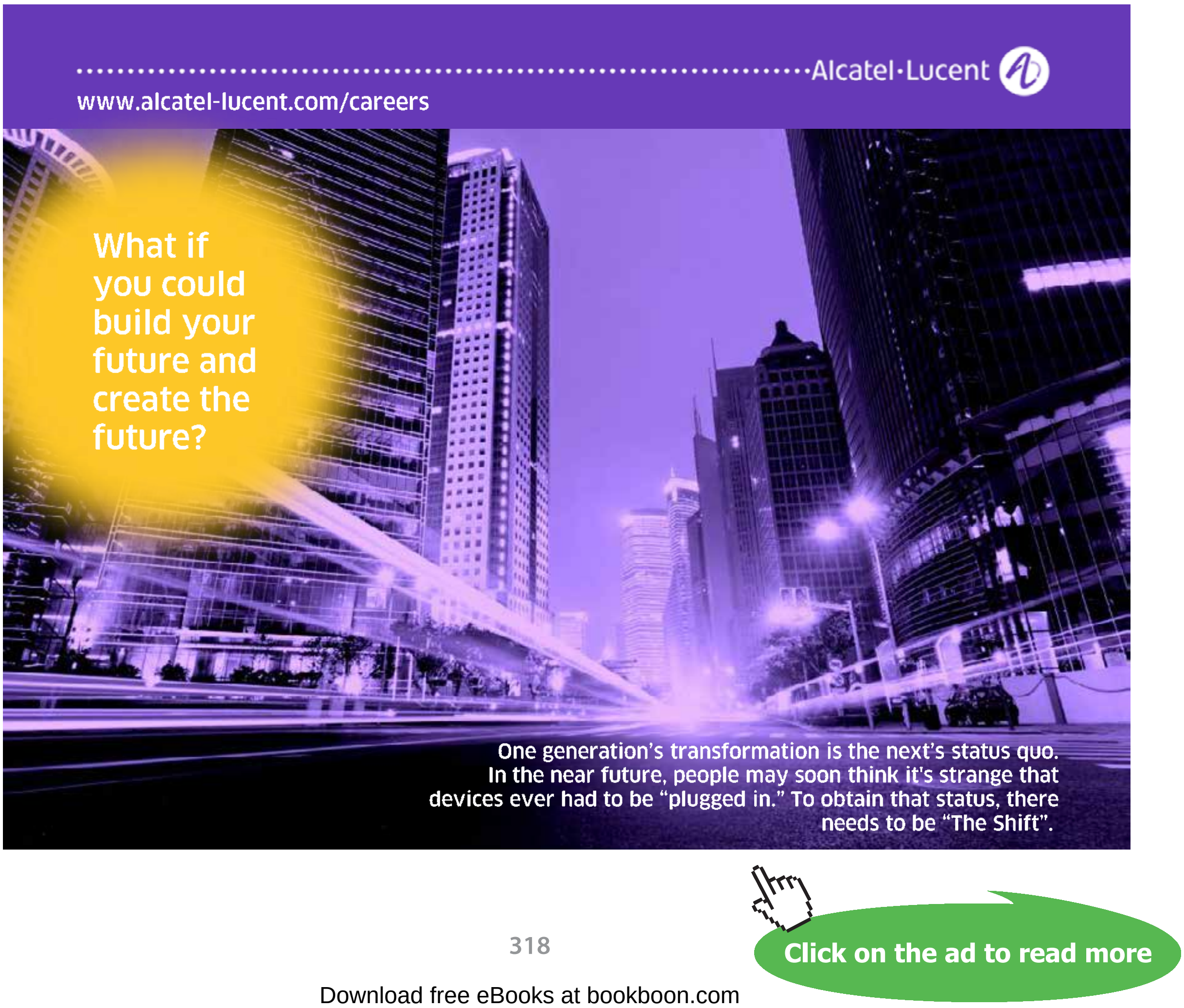
Proof. The second equality in (7.110) is a consequence of equality (7.85) in Corollary 7.14. The first equality in (7.110) and the first two equalities in (7.111) follow from an appropriate choice of commuting operators $(A_j)_{1\leqslant j\leqslant k}$ and $(B_j)_{1\leqslant j\leqslant k}$ in the equality

$$\prod_{j=1}^{k} A_j - \prod_{j=1}^{k} B_j = \sum_{\ell=1}^{k}\prod_{j=1}^{\ell-1} A_j\left(A_\ell - B_\ell\right)\prod_{j=\ell+1}^{k} B_j.$$

Products over a void index set are to be interpreted as I: e.g., $\prod_{j=1}^{0} A_j = I$. The final equality in (7.111) then follows from (7.85) in Corollary 7.14. The equality of the expression in (7.111) and the one in (7.112) follows from integration by parts. The equality of (7.112) and (7.114) is a consequence of the following identity

$$\begin{aligned}\frac{2}{\pi}\int_{\xi}^{R} A\left(\eta^2 I + A^2\right)^{-1} x\,d\eta &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \xi e^{i\vartheta}\left(\xi e^{i\vartheta} I - A\right)^{-1} x\,d\vartheta \\ &\quad - \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} Re^{i\vartheta}\left(Re^{i\vartheta} I - A\right)^{-1} x\,d\vartheta, \qquad (7.115)\end{aligned}$$

for $0 < \xi < R < \infty$ and $x \in X$. The equality in (7.115) follows from Cauchy's theorem applied to the analytic function $\lambda \mapsto (\lambda I - A)^{-1} x$, $\lambda \in V_\alpha$ for some $\alpha > \pi/2$ with a contour bordered by two semi-circles, one of radius ξ and the other of radius R, and two intervals on the imaginary axis, one with endpoints $-iR$ and $-i\xi$, and the other one with endpoints $i\xi$ and iR.



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If we let R tend to ∞ in (7.115) we obtain:

$$\begin{aligned}\frac{2}{\pi}\int_{\xi}^{\infty} A\left(\eta^2 I + A^2\right)^{-1} x\, d\eta &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \xi e^{i\vartheta}\left(\xi e^{i\vartheta} I - A\right)^{-1} x\, d\vartheta \\ &\quad - \lim_{R\to\infty}\frac{1}{\pi}\int_{-\pi/2}^{\pi/2} Re^{i\vartheta}\left(Re^{i\vartheta} I - A\right)^{-1} x\, d\vartheta \\ &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\left\{\xi e^{i\vartheta}\left(\xi e^{i\vartheta} I - A\right)^{-1} x - x\right\} d\vartheta \\ &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} A\left(\xi e^{i\vartheta} I - A\right)^{-1} x\, d\vartheta \end{aligned} \tag{7.116}$$

for $\xi > 0$ and $x \in X$. The equality in (7.116) yields the equality of (7.112) and (7.114). The equality of (7.113) and (7.114) follows from the equality

$$\int_0^{\infty} \sin f_{2k}(\xi) f'_{2k}(\xi)\, d\xi = 0.$$

In order to finish the proof of Lemma 7.22 we still need to show the identity in (7.109). This equality is a consequence of the equalities in (7.110). If the operator A is invertible, then this implication is immediate, otherwise we replace A with $A - \omega I$, $\omega > 0$, and let $\omega \downarrow 0$.

This conclude the proof of Lemma 7.22. □

From the equalities in Lemma 7.22 we obtain the following abstract results in numerical analysis.

7.23. Lemma. *For $k \in \mathbb{N}$ the following equalities hold:*

$$\sum_{\ell=1}^{k}\frac{\tau_\ell}{2} = \frac{1}{\pi}\int_0^{\infty}\frac{1-\cos f_k(\xi)}{\xi^2}\, d\xi, \quad \textit{and} \quad \sum_{\ell=1}^{2k}\frac{2}{\tau_\ell} = \frac{1}{\pi}\int_0^{\infty}\left(1-\cos f_{2k}(\xi)\right) d\xi. \tag{7.117}$$

Proof. The equalities in Lemma 7.22 also hold for complex numbers A with negative real part. By putting $A = 0$ in (7.109) we get the first equality in (7.117). Taking A a negative real number, and multiplying the equalities in (7.111) by A, and letting A tend to $-\infty$ shows the second equality in (7.117). □

7.24. Theorem. *Let A be the generator of a bounded analytic semigroup in a Banach space x. Let $(\tau_j)_{j\in\mathbb{N}}$ be a sequence of strictly positive step sizes in the Crank-Nicolson iteration scheme:*

$$x_{n+1} = \prod_{j=1}^{n}\left(I + \frac{1}{2}\tau_j A\right)\left(I - \frac{1}{2}\tau_j A\right)^{-1} x. \tag{7.118}$$

The following assertion hold:

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(i) *if x belongs to $D(A)$ and* $\sum_{j=1}^{\infty} \tau_j < \infty$, *then the Crank-Nicolson iteration scheme is consistent;*

(ii) *if x belongs to $R(A)$ and* $\sum_{j=1}^{\infty} \frac{1}{\tau_j} < \infty$, *then the Crank-Nicolson iteration scheme is two step consistent.*

PROOF. (i) Assume $x \in D(A)$ and $\sum_{j=1}^{\infty} \tau_j < \infty$. Put

$$f_\infty(\xi) = 2\sum_{j=1}^{\infty} \arctan\frac{\tau_j \xi}{2} = \lim_{k\to\infty} 2\sum_{j=1}^{k} \arctan\frac{\tau_j \xi}{2}.$$

Then the equality in (7.109) implies

$$\begin{aligned}
&\lim_{k\to\infty}\sum_{\ell=1}^{k}\tau_\ell\prod_{j=1}^{\ell-1}\left(I+\frac{1}{2}\tau_j A\right)\left(I-\frac{1}{2}\tau_j A\right)^{-1}\left(I-\frac{1}{2}\tau_\ell A\right)^{-1}x \\
&= \lim_{k\to\infty}\frac{2}{\pi}\int_0^\infty \left(1-\cos f_k(\xi)\right)\left(\xi^2 I + A^2\right)^{-1} x\, d\xi \\
&= \frac{2}{\pi}\int_0^\infty \left(1-\cos f_\infty(\xi)\right)\left(\xi^2 I + A^2\right)^{-1} x\, d\xi. \qquad (7.119)
\end{aligned}$$

If $x \in D(A)$, then (7.119) together with (7.110) implies:

$$\begin{aligned}
&\lim_{k\to\infty}\sum_{\ell=1}^{k}\tau_\ell A\prod_{j=1}^{\ell-1}\left(I+\frac{1}{2}\tau_j A\right)\left(I-\frac{1}{2}\tau_j A\right)^{-1}\left(I-\frac{1}{2}\tau_\ell A\right)^{-1}x \\
&= \lim_{k\to\infty}\prod_{j=1}^{k}\left(I+\frac{1}{2}\tau_j A\right)\left(I-\frac{1}{2}\tau_j A\right)^{-1}x - x \\
&= \frac{2}{\pi}\int_0^\infty \left(1-\cos f_\infty(\xi)\right) A\left(\xi^2 I + A^2\right)^{-1} x\, d\xi. \qquad (7.120)
\end{aligned}$$

This proves assertion (i).

(ii) Assume $x = Ay \in R(A)$ and $\sum_{j=1}^{\infty}\frac{1}{\tau_j} < \infty$. Put $g_k(\xi) = 2\sum_{j=1}^{k}\arctan\frac{2}{\tau_j\xi}$. Then an elementary calculation shows the equality:

$$\cos g_k(\xi) = (-1)^k \cos f_k(\xi).$$

In addition we write

$$g_\infty(\xi) = \lim_{k\to\infty} 2\sum_{j=1}^{k}\arctan\frac{2}{\tau_j\xi}.$$

Then as a consequence of (7.111) we infer:

$$\lim_{k\to\infty}\prod_{j=1}^{2k}\left(I+\frac{1}{2}\tau_j A\right)\left(I-\frac{1}{2}\tau_j A\right)^{-1}x - x$$

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$$
\begin{aligned}
&= \lim_{k\to\infty} \sum_{\ell=1}^{k} \left(\tau_{2\ell-1} + \tau_{2\ell}\right) A \prod_{j=1}^{2(\ell-1)} \left(I + \frac{1}{2}\tau_j A\right)\left(I - \frac{1}{2}\tau_j A\right)^{-1} \\
&\qquad \left(I - \frac{1}{2}\tau_{2\ell-1} A\right)^{-1}\left(I - \frac{1}{2}\tau_{2\ell} A\right)^{-1} x \\
&= \lim_{k\to\infty} \frac{2}{\pi}\int_0^\infty \left(1 - \cos g_{2k}(\xi)\right) A \left(\xi^2 I + A^2\right)^{-1} x \, d\xi \\
&= \frac{2}{\pi}\int_0^\infty \left(1 - \cos g_{\infty}(\xi)\right) A \left(\xi^2 I + A^2\right)^{-1} x \, d\xi. \qquad (7.121)
\end{aligned}
$$

In order to interchange the integral and the limit in the final step in (7.121) we used the finiteness of the integral

$$
\int_0^\infty \sup_k \left\{1 - \cos g_{2k}(\xi)\right\} \left\| A \left(\xi^2 I + A^2\right)^{-1} x \right\| d\xi. \qquad (7.122)
$$

If $x = Ay$, $y \in D(A)$, then

$$
\left\| A \left(\xi^2 I + A^2\right)^{-1} x \right\| \leqslant \left(1 + C^2\right) \|y\| , \qquad (7.123)
$$

where the constant C is as in (7.63).

This completes the proof of Theorem 7.24. □

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Let $x \in X$ and let K be a compact subset of $(0, \infty)$. As in (7.87) the symbol $\nu(K)x$ stands for:

$$\nu(K)x = \frac{2}{\pi} \int_K (-A) \left(I + \eta^2 A^2\right)^{-1} x \, d\eta.$$

7.25. Theorem. *Define the subspace X_A of X by*

$$X_A = \{x \in X : \sup \{\|\nu(K)x\| : \ K \text{ compact subset of } (0, \infty)\} < \infty\} . \tag{7.124}$$

The space X_A coincides with the space

$$\begin{aligned} X_{\nu,\mathrm{weak}} &= \{x \in X : \text{ the mapping } K \mapsto \langle \nu(K)x, x^*\rangle \, , \ K \subset (0, \infty), \ K \text{ compact}, \\ &\qquad\qquad \text{extends to a complex measure on } \mathcal{B}_{\mathbb{R}_+} \text{ for all } x^* \in X^*\} \qquad (7.125) \\ &= \left\{ x \in X : \int_0^\infty \left| \left\langle A \left(I + \eta^2 A^2\right)^{-1} x, x^* \right\rangle \right| d\eta < \infty \text{ for all } x^* \in X^* \right\}. \end{aligned}$$

In addition, the following inequalities hold:

$$\sup_K |\langle \nu(K)x, x^*\rangle| \leqslant \frac{2}{\pi} \int_0^\infty \left| \left\langle (-A) \left(I + \eta^2 A^2\right)^{-1} x, x^* \right\rangle \right| d\eta \leqslant 4 \sup_K |\langle \nu(K)x, x^*\rangle| \tag{7.126}$$

and hence

$$\sup_K \|\nu(K)x\| \leqslant \sup_{\|x^*\| \leqslant 1} \frac{2}{\pi} \int_0^\infty \left| \left\langle (-A) \left(I + \eta^2 A^2\right)^{-1} x, x^* \right\rangle \right| d\eta \leqslant 4 \sup_K \|\nu(K)x\| . \tag{7.127}$$

In (7.126) and (7.127) the suprema are taken over all compact subsets K of $(0, \infty)$.

Proof. The fact that the two spaces mentioned in (7.125) coincide is a standard result in complex measure theory. The theorem of Hahn-Banach shows that the inequalities in (7.127) follow from those in (7.126). The first inequality in (7.126) is trivial. The second one can be proved as follows. Fix $x \in X$ and $x^* \in X^*$. Define, for $j = 1, 2, 3, 4$ the open subsets $B_{x,x^*,j}$ of $(0, \infty)$ as follows:

$$\begin{aligned} B_{x,x^*,1} &= \left\{ \eta > 0 : \ \Re \left\langle -A \left(I + \eta^2 A^2\right)^{-1} x, x^* \right\rangle > 0 \right\}, \\ B_{x,x^*,2} &= \left\{ \eta > 0 : \ \Re \left\langle -A \left(I + \eta^2 A^2\right)^{-1} x, x^* \right\rangle < 0 \right\}, \\ B_{x,x^*,3} &= \left\{ \eta > 0 : \ \Im \left\langle -A \left(I + \eta^2 A^2\right)^{-1} x, x^* \right\rangle > 0 \right\}, \\ B_{x,x^*,4} &= \left\{ \eta > 0 : \ \Im \left\langle -A \left(I + \eta^2 A^2\right)^{-1} x, x^* \right\rangle < 0 \right\}. \end{aligned}$$

Then we have

$$\begin{aligned} &\int_0^\infty \left| \left\langle (-A) \left(I + \eta^2 A^2\right)^{-1} x, x^* \right\rangle \right| d\eta \\ &\leqslant \left| \int_{B_{x,x^*,1}} \Re \left\langle (-A) \left(I + \eta^2 A^2\right)^{-1} x, x^* \right\rangle d\eta \right| \end{aligned}$$

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$$+\left|\int_{B_{x,x^*,2}} \Re\left\langle (-A)\left(I+\eta^2A^2\right)^{-1}x, x^*\right\rangle\, d\eta\right|$$
$$+\left|\int_{B_{x,x^*,3}} \Im\left\langle (-A)\left(I+\eta^2A^2\right)^{-1}x, x^*\right\rangle\, d\eta\right|$$
$$+\left|\int_{B_{x,x^*,4}} \Im\left\langle (-A)\left(I+\eta^2A^2\right)^{-1}x, x^*\right\rangle\, d\eta\right|$$
$$\leqslant 4\sup_K\left|\int_K \left\langle (-A)\left(I+\eta^2A^2\right)^{-1}x, x^*\right\rangle\, d\eta\right| \tag{7.128}$$

The second inequality in (7.126) readily follows from (7.128). In order to complete the proof of Theorem 7.25 it suffices to prove that, if a vector $x \in X$ belongs to $X_{\nu,\mathrm{weak}}$, then $\sup_K \|\nu(K)x\| < \infty$. To this end we consider the following subset of X^*:

$$B_{x,\nu} := \{x^* \in X^* : |\langle \nu(K)x, x^*\rangle| \leqslant 1 \text{ for all compact subsets } K \text{ of } (0,\infty).\} \tag{7.129}$$

Then the subset $B_{x,\nu}$ is a closed, absolutely convex and absorbing subset of the Banach space X^*. In other words it is a barrel in X^*. Since barrels in Banach spaces contain neighborhoods of the origin, it follows that there exists $\delta(x) > 0$ such that the ball of radius $\delta(x)$ is contained in $B_{x,\nu}$. That is to say, if $\|x^*\| \leqslant \delta(x)$, then $|\langle \nu(K)x, x^*\rangle| \leqslant 1$. Or in other words: for all compact subsets $K \subset (0,\infty)$ we have

$$\|x^*\| \leqslant 1 \implies |\langle \nu(K)x, x^*\rangle| \leqslant \frac{1}{\delta(x)}.$$

And hence another application of the theorem of Hahn-Banach shows $\|\nu(K)x\| \leqslant \delta(x)^{-1}$. This concludes the proof of Theorem 7.25. □

Theorem 7.26 shows that on the subspace X_A of X we have stability of the Crank-Nicolson iteration scheme. As mentioned in (7.125) the subspace $X_A = X_{\nu,\mathrm{weak}}$ consists of those $x \in X$ for which $\int_0^\infty \left|\left\langle A\left(I+\eta^2A^2\right)^{-1}x, x^*\right\rangle\right| d\eta < \infty$ for all $x^* \in X^*$. It is observed that $N(A) + R(A) \cap D(A)$ is contained in X_A.

7.26. Theorem. *Let $x \in X_\nu$ and let $(\tau_j)_{j\in\mathbb{N}}$ be any sequence of strictly positive step sizes in the Crank-Nicolson iteration scheme. Then*

$$\sup_{\|x^*\|\leqslant 1}\int_0^\infty \left|\left\langle A\left(I+\eta^2A^2\right)^{-1}x, x^*\right\rangle\right| d\eta < \infty,$$

and

$$\sup_{k\in\mathbb{N}}\left\|\prod_{j=1}^k\left(I+\frac{1}{2}\tau_jA\right)\left(I-\frac{1}{2}\tau_jA\right)^{-1}x - x\right\|$$
$$\leqslant \frac{4}{\pi}\sup_{\|x^*\|\leqslant 1}\int_0^\infty \left|\left\langle A\left(I+\eta^2A^2\right)^{-1}x, x^*\right\rangle\right| d\eta$$

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$$\leqslant \frac{16}{\pi} \sup_K \left\| \int_K A \left(I + \eta^2 A^2\right)^{-1} x \, d\eta \right\|, \tag{7.130}$$

where in the last step the supremum is taken over all compact subsets K of the open half-axis $(0, \infty)$.

In the following theorem we collect some properties of the operator

$$\begin{aligned} x \mapsto P_{R(A)}x &:= \frac{-2}{\pi} \int_0^\infty A \left(I + \eta^2 A^2\right)^{-1} x \, d\eta \\ &= \lim_{\varepsilon \downarrow 0,\, R \uparrow \infty} \frac{-2}{\pi} \int_\varepsilon^R A \left(I + \eta^2 A^2\right)^{-1} x \, d\eta \\ &= \lim_{\varepsilon \downarrow 0,\, R \uparrow \infty} \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \left\{ \left(I - \varepsilon e^{i\vartheta} A\right)^{-1} - \left(I - Re^{i\vartheta} A\right)^{-1} \right\} x \, d\vartheta \\ &= x - \lim_{R \uparrow \infty} \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \left(I - Re^{i\vartheta} A\right)^{-1} x \, d\vartheta, \end{aligned} \tag{7.131}$$

for those $x \in X$ for which this limit exists. The limit in (7.131) exists if and only if x belongs to closure of $N(A) + R(A) \cap D(A)$. Let X_0 be this closure. Then $X_0 = N(A) + \overline{R(A) \cap D(A)} = N(A) + \overline{R(A)}$.

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We also like to mention the following identities ($0 \leqslant \alpha < \beta < \infty$, $x \in X$):

$$\frac{2}{\pi}\int_{\alpha}^{\beta}\left(I+\eta^2A^2\right)^{-1}x\,d\eta = \frac{1}{\pi}\int_{\pi/2}^{\pi/2}(\beta-\alpha)\,e^{i\vartheta}\left(I-\alpha e^{i\vartheta}A\right)^{-1}\left(I-\beta e^{i\vartheta}A\right)^{-1}x\,d\vartheta$$
$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}e^{i\vartheta}\int_{\alpha}^{\beta}\left(I-\eta e^{i\vartheta}A\right)^{-2}x\,d\eta\,d\vartheta, \tag{7.132}$$

and

$$\frac{2}{\pi}\int_{\alpha}^{\beta}(-A)\left(I+\eta^2A^2\right)^{-1}x\,d\eta$$
$$= \frac{1}{\pi}\int_{\pi/2}^{\pi/2}(\beta-\alpha)\,e^{i\vartheta}(-A)\left(I-\alpha e^{i\vartheta}A\right)^{-1}\left(I-\beta e^{i\vartheta}A\right)^{-1}x\,d\vartheta$$
$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\left\{-\beta e^{i\vartheta}A\left(I-\beta e^{i\vartheta}A\right)^{-1}+\alpha e^{i\vartheta}A\left(I-\alpha e^{i\vartheta}A\right)^{-1}\right\}x\,d\vartheta$$
$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\left\{\left(I-\alpha e^{i\vartheta}A\right)^{-1}-\left(I-\beta e^{i\vartheta}A\right)^{-1}\right\}x\,d\vartheta$$
$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\left(-e^{i\vartheta}A\right)\int_{\alpha}^{\beta}\left(I-\eta e^{i\vartheta}A\right)^{-2}x\,d\eta\,d\vartheta. \tag{7.133}$$

The equalities in (7.133) can be understood by applying Fubini's theorem to the final double integral together with some simple manipulations. The equalities in (7.132) follow by first applying (7.133) to an operator of the form $A-\omega I$ instead of A, and then letting ω tend to 0.

7.27. Theorem. *The following assertions hold true:*

(i) *The operator $P_{R(A)}$ is a projection operator from X_0 onto $\overline{R(A)}$, the closure of $R(A)$.*
(ii) $X_0 = N(A) + P_{R(A)}X_0 = N(A) + \overline{R(A)}$.
(iii) $\left\|P_{R(A)}x\right\| \leqslant (1+C)\,\|x\|$, *for* $x \in X_0$.
(iv) *If the space X is reflexive, then $X_0 = X$.*

7.28. Remark. Theorem 7.25 shows that it is useful to investigate the following subspace X_A of X:

$$X_A = \{x \in X : \sup\{\|\nu(K)x\| : \ K \text{ compact subset of } (0,\infty)\} < \infty\}. \tag{7.134}$$

As observed earlier the space X_A contains the subspace $N(A) + R(A) \cap D(A)$. If X_A is a closed subspace, then X_A contains the closure of $N(A) + R(A) \cap D(A)$. If, in addition, X is reflexive, then $X = N(A) + \overline{R(A)}$, and X is the closure of $N(A) + R(A) \cap D(A)$. The space X_A coincides with the space

$$X_{\nu,\text{weak}} = \big\{x \in X : \ B \mapsto \langle \nu(B)x, x^*\rangle\,,\ B \in \mathcal{B}_{(0,\infty)},$$
$$\text{is a complex measure for all } x^* \in X^*\} \tag{7.135}$$

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$$= \left\{ x \in X : \int_0^\infty \left| \left\langle A\left(\xi^2 I + A^2\right)^{-1} x, x^* \right\rangle \right| d\xi < \infty \text{ for all } x^* \in X^* \right\}.$$

The coincidence of the spaces mentioned in (7.135) is closely related to the theory of vector-valued Pettis integrals and Gelfand integrals: see, e.g., Diestel and Uhl [**37**], and also [**94**]. Pettis' theorem says that a set function μ on a σ-field $\mathcal{F}$ with values in a Banach space X and has the property that $B \mapsto \langle \mu(B), x^* \rangle$ for all $x^* \in X$ is in fact an X-valued measure in the sense that if $B = \cup_{j=1}^{\infty} B_j$, $B_j \in \mathcal{F}$, $B_j \cap B_k = \varnothing$, $j \neq k$, then $\lim_{n\to\infty} \sum_{j=1}^{n} \mu(B_j) = \mu(B)$. Define the operators $\arctan(-\xi A)$, $\xi > 0$, by the improper vector valued Riemann integrals:

$$\arctan(-\xi A)\, x = \int_0^\xi (-A)\left(I + \eta^2 A^2\right)^{-1} x\, d\eta = \lim_{\delta \downarrow 0} \int_\delta^\xi (-A)\left(I + \eta^2 A^2\right)^{-1} x\, d\eta, \tag{7.136}$$

for $x \in X$. Then the space $X_A = X_{\nu,\mathrm{weak}}$ also coincides with the space

$$\left\{ x \in X : \sup \left\| \sum_{j=1}^{n} \left(\arctan(-\beta_j A) - \arctan(-\alpha_j A)\right) x \right\| < \infty \right\}, \tag{7.137}$$

where the supremum is taken over all finite number of pairs $(\alpha_j, \beta_j)_{j=1}^n$ such that $0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_n < \beta_n$. In fact (7.137) says that the compact subsets in the definition of the space X_ν (see (7.134)) may be taken of the form $K = \cup_{j=1}^{n} [\alpha_j, \beta_j]$ where $0 < \alpha_j < \beta_j$, $1 \leqslant j \leqslant n$.

7.29. Remark. Consider for $f \in L^\infty(\mathbb{R}_+)$ the function $Tf \in \mathrm{Hol}(-V_0)$ defined by

$$Tf(\mu) = \frac{2}{\pi} \int_0^\infty f(\eta) \frac{-\mu}{1 + \eta^2 \mu^2}\, d\eta. \tag{7.138}$$

Here the space $L^\infty(\mathbb{R}_+)$ consists of all bounded complex-valued Borel-measurable functions f defined on $\mathbb{R}_+ = (0, \infty)$, and $\mathrm{Hol}(\Omega)$ consists of all holomorphic complex-valued function on the open subset Ω of $\mathbb{C}$. The subset $V_\alpha \subset \mathbb{C}$, $0 < \alpha < \pi$, is defined as the sector $V_\alpha = \{\mu \in \mathbb{C} : |\arg(z)| < \alpha\}$. If $|f(\eta)| \leqslant 1$, $\eta > 0$, then $|Tf(\mu)| \leqslant \dfrac{1}{\cos\alpha}$ provided that $\mu \in -V_\alpha$. The latter inequality is a consequence of the fact that

$$\left|1 + \eta^2 e^{2i\alpha}\right| \geqslant \cos\alpha \left(1 + \eta^2\right), \quad \eta \in \mathbb{R}.$$

In fact for f we may choose indicator functions $f = \mathbf{1}_B$ of Borel subsets B of $(0, \infty)$. Then the mapping $B \mapsto T\mathbf{1}_B(\mu)$ yields a $\mathrm{Hol}(-V_0)$-valued measure. These functions have upper-bound $\dfrac{1}{\cos\alpha}$ if μ belongs to sector $-V_\alpha$. In other words the operators

$$B \mapsto T\mathbf{1}_B(A)x := \frac{2}{\pi} \int_B (-A)\left(I + \eta^2 A^2\right)^{-1} x\, d\eta,\ x \in X, \tag{7.139}$$

lead to possible vector measures, like we suggested above. But it also gives rise to an approach by using an H^∞-calculus for sectorial operators. For more details, see, e.g., Haase [**60**]. If in (7.138) we choose $f(\eta) = f_m(\eta) = m(0, \eta]$ where m is a Borel

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measure on $\mathbb{R}_+$ of bounded variation, then, for $x \in X_0 = N(A) + \overline{R(A)}$, we have

$$\begin{aligned}Tf_m(A)x &= \frac{2}{\pi}\int_0^\infty m(0,\eta]\,(-A)\left(I+\eta^2A^2\right)^{-1}x\,d\eta\\ &= \lim_{R\to\infty}\frac{2}{\pi}\int_0^R\int_\rho^R(-A)\left(I+\eta^2A^2\right)^{-1}x\,d\eta\,dm(\rho)\end{aligned}$$

(see (7.133))

$$\begin{aligned}&= \lim_{R\to\infty}\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\int_0^R\left\{\left(I-\rho e^{i\vartheta}A\right)^{-1}-\left(I-Re^{i\vartheta}A\right)^{-1}\right\}x\,dm(\rho)\,d\vartheta \qquad (7.140)\\ &= \lim_{R\to\infty}\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\int_0^R\left(-(R-\rho)e^{i\vartheta}A\right)\left(I-\rho e^{i\vartheta}A\right)^{-1}\left(I-Re^{i\vartheta}A\right)^{-1}x\,dm(\rho)\,d\vartheta\end{aligned}$$

(apply Theorem 7.27)

$$\begin{aligned}&= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\int_0^\infty\left(I-\rho e^{i\vartheta}A\right)^{-1}P_{R(A)}x\,dm(\rho)\,d\vartheta\\ &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\int_0^\infty\left\{\left(I-\rho e^{i\vartheta}A\right)^{-1}-I\right\}P_{R(A)}x\,dm(\rho)\,d\vartheta+m(0,\infty)P_{R(A)}x\end{aligned}$$

(another application of (7.133) with $\alpha = 0$ and $\beta = \rho$)

$$\begin{aligned}&= -\frac{2}{\pi}\int_0^\infty\int_0^\rho(-A)\left(I+\eta^2A^2\right)^{-1}x\,d\eta\,dm(\rho)+m(0,\infty)P_{R(A)}x\\ &= -\frac{2}{\pi}\int_0^\infty\arctan(-\rho A)\,dm(\rho)+m(0,\infty)P_{R(A)}x. \qquad (7.141)\end{aligned}$$

Then from (7.140) we infer

$$\|Tf_m(A)x\| \leqslant \frac{2C}{\pi}\,|m|\,(\mathbb{R}_+)\,\|x\|,\quad x \in X. \qquad (7.142)$$

In (7.142) $B \mapsto |m|\,(B) = \sup_{B_1,\dots,B_n}\sum_{j=1}^n|m(B_j)|$, $B = \cup_{j=1}^n B_j \in \mathcal{B}_{\mathbb{R}_+}$, $B_{j_1}\cap B_{j_2} = \emptyset$, $1 \leqslant j_1,\ j_2 \leqslant$, is the variation measure associated to m. The inequality in (7.142) follows from (7.133) together with (7.42). From (7.141) we also deduce:

$$\frac{2}{\pi}\int_0^\infty m(\eta,\infty)(-A)\left(I+\eta^2A^2\right)^{-1}x\,d\eta = \frac{2}{\pi}\int_0^\infty\arctan(-\rho A)\,x\,dm(\rho),\quad x \in X. \qquad (7.143)$$

We still have another way of checking that under the given conditions of the operator A integrals of the form as in the definition of $X_{\nu,\mathrm{weak}}$ (see (7.135)) are finite.

7.30. Lemma. *Let* $(x,x^*) \in X \times X^*$, *and let the function* $v(\eta) = v_{x,x^*}(\eta)$ *be such that*

$$v(\eta)\left\langle -A\left(I+\eta^2A^2\right)^{-1}x,x^*\right\rangle = \left|\left\langle -A\left(I+\eta^2A^2\right)^{-1}x,x^*\right\rangle\right|. \qquad (7.144)$$

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Suppose that

$$p(v) := \sup_{0<\alpha<1} \inf_{|\varphi|\leqslant\pi} \int_\alpha^1 \frac{|v(\eta) - e^{i\varphi}|}{\eta}\, d\eta + \sup_{1\leqslant R<\infty} \inf_{|\varphi|\leqslant\pi} \int_1^R \frac{|v(\eta) - e^{i\varphi}|}{\eta}\, d\eta < \infty. \tag{7.145}$$

Then the integral

$$\int_0^\infty \left|\left\langle (-A)\left(I + \eta^2 A^2\right)^{-1} x, x^*\right\rangle\right| d\eta$$

is finite. In fact the following inequality holds:

$$\int_0^\infty \left|\left\langle (-A)\left(I + \eta^2 A^2\right)^{-1} x, x^*\right\rangle\right| d\eta \leqslant \left\{C(1+C)p(v) + \frac{\pi}{2}(1+3C)\right\} \|x\| \cdot \|x^*\|. \tag{7.146}$$

Here the constant C is chosen as in (7.63).

Notice that the condition in (7.145) only is a requirement on the behavior of $v(\eta)$ for η small or large.

Proof. Fix $0 < \alpha < 1 < R < \infty$, and choose the function $v_1 \in L^\infty(0,\infty)$ in such a way that $|v_1(\alpha)| = 1 = |v_1(R)|$, and such that

$$\begin{aligned} \inf_{|\varphi|\leqslant\pi} \int_\alpha^1 \frac{|v(\eta) - e^{i\varphi}|}{\eta}\, d\eta &= \int_\alpha^1 \frac{|v(\eta) - v_1(\alpha)|}{\eta}\, d\eta,\ 0 < \alpha < 1, \text{ and} \\ \inf_{|\varphi|\leqslant\pi} \int_1^R \frac{|v(\eta) - e^{i\varphi}|}{\eta}\, d\eta &= \int_1^R \frac{|v(\eta) - v_1(R)|}{\eta}\, d\eta,\ 1 < R < \infty. \end{aligned} \tag{7.147}$$

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Then the following equalities are self-explanatory:

$$\begin{aligned}
&\int_\alpha^R \left|\left\langle -A\left(I+\eta^2A^2\right)^{-1}x, x^*\right\rangle\right| d\eta \\
&= \int_\alpha^1 \frac{v(\eta)-v_1(\alpha)}{\eta}\left\langle -\eta A\left(I+\eta^2A^2\right)^{-1}x, x^*\right\rangle d\eta \\
&\quad + v_1(\alpha)\int_\alpha^1 \left\langle -A\left(I+\eta^2A^2\right)^{-1}x, x^*\right\rangle d\eta \\
&\quad + \int_1^R \frac{v(\eta)-v_1(R)}{\eta}\left\langle -\eta A\left(I+\eta^2A^2\right)^{-1}x, x^*\right\rangle d\eta \\
&\quad + v_1(R)\int_1^R \left\langle -A\left(I+\eta^2A^2\right)^{-1}x, x^*\right\rangle d\eta
\end{aligned}$$

(employ (7.132)

$$\begin{aligned}
&= \int_\alpha^1 \frac{v(\eta)-v_1(\alpha)}{i\eta}\left\langle \left(I-(I-i\eta A)^{-1}\right)(I+i\eta A)^{-1}x, x^*\right\rangle d\eta \\
&\quad + \frac{v_1(\alpha)}{2}\int_{-\pi/2}^{\pi/2} \left\langle\left\{\left(I-\alpha e^{i\vartheta}A\right)^{-1} - \left(I-e^{i\vartheta}A\right)^{-1}\right\}x, x^*\right\rangle d\vartheta \\
&\quad + \int_1^R \frac{v(\eta)-v_1(R)}{i\eta}\left\langle \left(I-(I-i\eta A)^{-1}\right)(I+i\eta A)^{-1}x, x^*\right\rangle d\eta \\
&\quad + \frac{v_1(R)}{2}\int_{-\pi/2}^{\pi/2} \left\langle\left\{\left(I-e^{i\vartheta}A\right)^{-1} - \left(I-Re^{i\vartheta}A\right)^{-1}\right\}x, x^*\right\rangle d\vartheta. \qquad (7.148)
\end{aligned}$$

The inequality in (7.146) follows from (7.147) and (7.148) together with the choice of C in (7.63). This completes the proof of Lemma 7.30. □

7.31. Remark. Let $v : (0,\infty) \to \mathbb{C}$ be cádlág function of bounded variation and, for the time being fix $0 < \alpha < \beta < \infty$. The variation measure $|dv|$ satisfies:

$$\int_\alpha^\beta |dv|(\rho) = \sup\left\{\sum_{j=1}^n |v(\rho_j) - v(\rho_{j-1})| : \alpha = \rho_0 < \rho_1 < \cdots < \rho_n = \beta\right\}. \qquad (7.149)$$

Then, by (7.132) and (7.133) the following equalities hold:

$$\begin{aligned}
&\frac{2}{\pi}\int_\alpha^\beta v(\eta)(-A)\left(I+\eta^2A^2\right)^{-1}x\,d\eta \\
&= v(\beta)\frac{2}{\pi}\int_\alpha^\beta (-A)\left(I+\eta^2A^2\right)^{-1}x\,d\eta - \frac{2}{\pi}\int_\alpha^\beta\int_\alpha^\rho (-A)\left(I+\eta^2A^2\right)^{-1}x\,d\eta\,dv(\rho) \\
&= v(\beta)\frac{2}{\pi}\left(\arctan(-\beta A) - \arctan(-\alpha A)\right)x \\
&\quad - \frac{2}{\pi}\int_\alpha^\beta \left(\arctan(-\rho A) - \arctan(-\alpha A)\right)x\,dv(\rho) \qquad (7.150)
\end{aligned}$$

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Here, for $\rho > \alpha$, we wrote

$$\begin{aligned}
&(\arctan(-\rho A) - \arctan(-\alpha A))\, x = \int_\alpha^\rho (-A)\left(I + \eta^2 A^2\right)^{-1} x\, d\eta \\
&= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left\{\left(I - \alpha e^{i\vartheta} A\right)^{-1} - \left(I - \rho e^{i\vartheta} A\right)^{-1}\right\} x\, d\vartheta \\
&= \frac{1}{2}\int_{-\pi/2}^{\pi/2} -(\rho - \alpha)\, e^{i\vartheta} A \left(I - \rho e^{i\vartheta} A\right)^{-1}\left(I - \alpha e^{i\vartheta} A\right)^{-1} x\, d\vartheta. \qquad (7.151)
\end{aligned}$$

With C as in (7.63) the equalities in (7.151) imply:

$$\|\{\arctan(-\rho A) - \arctan(-\alpha A)\}\, x\| \leqslant \pi C\, \|x\|\,. \qquad (7.152)$$

A combination of (7.150) and (7.152) yields:

$$\frac{2}{\pi}\left\|\int_\alpha^\beta v(\eta)\,(-A)\left(I + \eta^2 A^2\right)^{-1} x\, d\eta\right\| \leqslant 2C\left\{|v(\beta)| + \int_\alpha^\beta |dv|\,(\rho)\right\}\|x\|\,. \qquad (7.153)$$

From (7.153) we infer:

$$\frac{2}{\pi}\sup_{0<\alpha<\beta<\infty}\left\|\int_\alpha^\beta v(\eta)\,(-A)\left(I + \eta^2 A^2\right)^{-1} x\, d\eta\right\| \leqslant 2C\left\{\|v\|_\infty + \int_0^\infty |dv|\,(\rho)\right\}\|x\|\,. \qquad (7.154)$$

5. Stability of the Crank-Nicolson iteration scheme

In this section we return to the problem of the stability of the Crank-Nicolson iteration scheme. The equality in (7.73) of Theorem 7.13 yields:

$$\begin{aligned}
&\prod_{j=1}^{k}\left(I + \frac{1}{2}\tau_j A\right)\left(I - \frac{1}{2}\tau_j A\right)^{-1} x - (-1)^k x \\
&= \frac{2}{\pi}\int_0^\infty \int_0^\xi \sin f_k(\eta)\, d\eta\ \left(\xi^2 I + A^2\right)^{-1}\left(2\xi^2\left(\xi^2 I + A^2\right)^{-1} - I\right) x\, d\xi. \qquad (7.155)
\end{aligned}$$

In (7.73) we chose $f(\lambda) = \prod_{j=1}^{k}\frac{1 + \frac{1}{2}\tau_j\lambda}{1 - \frac{1}{2}\tau_j\lambda}$. Then, as observed in (7.98),

$$\frac{f(i\eta) - f(-i\eta)}{2i} = \Im\prod_{j=1}^{k}\frac{1 + \frac{1}{2}i\tau_j\eta}{1 - \frac{1}{2}i\tau_j\eta} = \sin f_k(\eta) = \sin\left(2\sum_{j=1}^{k}\arctan\left(\frac{1}{2}\tau_j\eta\right)\right), \qquad (7.156)$$

and so (7.155) follows. By putting $A = 0$ in (7.155) we get:

$$\frac{2}{\pi}\int_0^\infty \frac{1}{\xi^2}\int_0^\xi \sin f_k(\eta)\, d\eta\ d\xi = 1 - (-1)^k. \qquad (7.157)$$

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7.32. Theorem. *Let A be the generator of a bounded analytic semigroup, and let $(\tau_j)_{j\in\mathbb{N}}$ be a sequence of strictly positive numbers. Let the constant C be as in (7.63). Then (7.155) implies:*

$$\begin{aligned}&\left\|\prod_{j=1}^{k}\left(I+\frac{1}{2}\tau_j A\right)\left(I-\frac{1}{2}\tau_j A\right)^{-1}x-(-1)^k x\right\|\\ &\leqslant \frac{2C^2\left(2C^2+1\right)}{\pi}\int_0^\infty \frac{1}{\xi^2}\left|\int_0^\xi \sin f_k(\eta)\,d\eta\right|\,d\xi\,\|x\|.\end{aligned} \tag{7.158}$$

Proof. The inequality in (7.158) is a consequence of (7.156) which in turn follows from (7.73) in Theorem 7.13. □

For the convenience of the reader we insert the following lemma.

7.33. Lemma. *As before, put $f_k(\eta)=2\sum_{j=1}^{k}\arctan\left(\frac{1}{2}\tau_j\eta\right)$. Define the quantity $b_{2k}(\eta)$ by*

$$b_{2k}(\eta)=\frac{\sin\left(\dfrac{2}{2k}\displaystyle\sum_{j=1}^{2k}\arctan\left(\frac{1}{2}\tau_j\eta\right)\right)}{\dfrac{1}{2k}\displaystyle\sum_{j=1}^{2k}\dfrac{\tau_j\eta}{1+\frac{1}{4}\tau_j^2\eta^2}}. \tag{7.159}$$

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Then (7.159) implies

$$\int_0^\infty \frac{1}{\xi^2}\left|\int_0^\xi \sin f_{2k}(\eta)\,d\eta\right| d\xi \leqslant 2\int_0^\infty \frac{1-\cos f_{2k}(\eta)}{\eta^2 f_{2k}'(\eta)}\,d\eta \leqslant 2\pi \sup_{\eta>0} b_{2k}(\eta)^2. \tag{7.160}$$

PROOF OF LEMMA 7.33. Integration by parts shows:

$$\frac{1-\cos f_{2k}(\xi)}{f_{2k}'(\xi)} - \int_0^\xi \sin f_{2k}(\eta)\,d\eta = \frac{1}{2}\sum_{j=1}^{2k}\int_0^\xi \frac{1-\cos f_{2k}(\eta)}{f_{2k}'(\eta)} \sum_{j=1}^{2k} \frac{\tau_j^3\eta}{\left(1+\frac{1}{4}\tau_j^2\eta^2\right)^2}\,d\eta \geqslant 0. \tag{7.161}$$

From (7.161) we deduce

$$\begin{aligned}\left|\int_0^\xi \sin f_{2k}(\eta)\,d\eta\right| &= \left|\frac{1-\cos f_{2k}(\xi)}{f_{2k}'(\xi)} - \left(\frac{1-\cos f_{2k}(\xi)}{f_{2k}'(\xi)} - \int_0^\xi \sin f_{2k}(\eta)\,d\eta\right)\right| \\ &\leqslant 2\times \frac{1-\cos f_{2k}(\xi)}{f_{2k}'(\xi)} + \int_0^\xi \sin f_{2k}(\eta)\,d\eta. \end{aligned} \tag{7.162}$$

Hence, from (7.157) and (7.162) we infer

$$\int_0^\infty \frac{1}{\xi^2}\left|\int_0^\xi \sin f_{2k}(\eta)\,d\eta\right| d\xi \leqslant 2\int_0^\infty \frac{1-\cos f_{2k}(\eta)}{\eta^2 f_{2k}'(\eta)}\,d\eta. \tag{7.163}$$

The inequality in (7.163) proves the first inequality in (7.160). In order to show the second inequality in (7.160) we proceed as follows. The equality $b_{2k}(\eta) = \dfrac{2k\sin\dfrac{f_{2k}(\eta)}{2k}}{\sum_{j=1}^{2k}\sin\varphi_j(\eta)}$ also follows, and upon writing $\psi_{2k}(\eta) = \dfrac{f_{2k}(\eta)}{2k}$ we obtain

$$\begin{aligned} &\int_0^\infty \frac{1-\cos\left(f_{2k}(\eta)\right)}{\eta^2}\frac{1}{f_{2k}'(\eta)}\,d\eta = \int_0^\infty \frac{1-\cos\left(f_{2k}(\eta)\right)}{\left(\eta f_{2k}'(\eta)\right)^2} f_{2k}'(\eta)\,d\eta \\ &= \int_0^\infty \frac{1-\cos\left(f_{2k}(\eta)\right)}{\left(\sum_{j=1}^{2k}\sin\varphi_j(\eta)\right)^2} f_{2k}'(\eta)\,d\eta \\ &= \int_0^\infty \frac{1-\cos\left(2k\psi_{2k}(\eta)\right)}{2k\sin^2\psi_{2k}(\eta)}\left(\frac{2k\sin\psi_{2k}(\eta)}{\sum_{j=1}^{2k}\sin\varphi_j(\eta)}\right)^2 \psi_{2k}'(\eta)\,d\eta \\ &= \int_0^\infty \frac{1-\cos\left(2k\psi_{2k}(\eta)\right)}{2k\sin^2\psi_{2k}(\eta)}\left(b_{2k}(\eta)\right)^2 \psi_{2k}'(\eta)\,d\eta \\ &\leqslant \left(\sup_{\eta>0} b_{2k}(\eta)\right)^2 \int_0^\infty \frac{1-\cos\left(2k\psi_{2k}(\eta)\right)}{2k\sin^2\psi_{2k}(\eta)}\psi_{2k}'(\eta)\,d\eta \\ &= \left(\sup_{\eta>0} b_{2k}(\eta)\right)^2 \int_0^\pi \frac{1-\cos(2k\psi)}{2k\sin^2\psi}d\psi = \pi\left(\sup_{\eta>0} b_{2k}(\eta)\right)^2. \end{aligned} \tag{7.164}$$

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In the final step in (7.164) we used integration by parts and the trigonometric identity

$$\cot\psi \sin(2k\psi) = 1 - \cos(2k\psi) + 2\sum_{j=1}^{k} \cos(2j\psi), \quad k \geqslant 1,$$

to obtain

$$\begin{aligned}\int_0^\pi \frac{1-\cos(2k\psi)}{2k\sin^2\psi} d\psi &= \int_0^\pi \cot\psi \sin(2k\psi)\, d\psi \\ &= \int_0^\pi \left(1 - \cos(2k\psi) + 2\sum_{j=1}^{k} \cos(2j\psi)\right) d\psi = \pi.\end{aligned}$$

The inequality in (7.164) yields the second inequality in (7.160) and completes the proof of Lemma 7.33. $\square$

The following result is an immediate consequence of Theorem 7.32 and Lemma 7.33.

7.34. Theorem. *Let the notation and hypotheses be as in Theorem 7.32 and Lemma (7.33). Then the following inequality holds:*

$$\left\| \prod_{j=1}^{2k} \left(I + \frac{1}{2}\tau_j A\right)\left(I - \frac{1}{2}\tau_j A\right)^{-1} x - x \right\| \leqslant 4C^2\left(2C^2+1\right)\left(\sup_{\eta>0} b_{2k}(\eta)\right)^2 \|x\|. \tag{7.165}$$

From Theorem 7.34 we see that that for all $x \in X$ the Crank-Nicolson iteration scheme is stable provided that the sequence of functions $\eta \mapsto b_{2k}(\eta)$, $k \in \mathbb{N}$, is uniformly bounded. Also notice that, by the inequality

$$\left\| \left(I + \frac{1}{2}\tau A\right)\left(I - \frac{1}{2}\tau A\right)^{-1} \right\| = \left\| 2 - \left(I - \frac{1}{2}\tau A\right)^{-1} \right\| \leqslant 2 + C,$$

with C as in (7.63), the one-step Crank-Nicolson scheme is stable if and only if the two-step Nicolson iteration scheme is stable.

7.35. Theorem. *Then the following assertions are equivalent:*

(1) *There exists a constant C_1 such that*

$$\#\left\{1 \leqslant j \leqslant 2k : \varphi_j > \frac{\pi}{2}\right\} \wedge \#\left\{1 \leqslant j \leqslant 2k : \varphi_j \leqslant \frac{\pi}{2}\right\} \leqslant C_1 \sum_{j=1}^{2k} \varphi_j \wedge (\pi - \varphi_j). \tag{7.166}$$

(2) *There exists a constant C_2 such that*

$$\left(\sum_{j=1}^{2k} \varphi_j\right) \wedge \left(\sum_{j=1}^{2k} (\pi - \varphi_j)\right) \leqslant C_2 \sum_{j=1}^{2k} \varphi_j \wedge (\pi - \varphi_j). \tag{7.167}$$

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In fact, if C_1 is such that (7.166) holds, then $C_2 = \pi C_1 + 2$ satisfies (7.166). Conversely, if C_2 is a constant for which (7.167) holds, then (7.166) is true with $C_1 = \dfrac{2C_2}{\pi}$.

PROOF. The proof of Theorem 7.35 is essentially speaking contained in the proof of Proposition 7.38. □

By choosing $\varphi_j(\eta) = 2\arctan\left(\frac{1}{2}\tau_j\eta\right)$, and thus $\sin\varphi_j(\eta) = \dfrac{\tau_j\eta}{1+\frac{1}{4}\tau_j^2\eta^2}$, the following result follows from Theorem 7.35.

7.36. THEOREM. *The following assertions are equivalent:*

(1) *There exists a constant C_1 such that for all $\eta > 0$ and $k \in \mathbb{N}$*

$$\#\{1 \leqslant j \leqslant 2k : \tau_j\eta > 2\} \wedge \#\{1 \leqslant j \leqslant 2k : \tau_j\eta \leqslant 2\} \leqslant C_1 \sum_{j=1}^{2k} \frac{\tau_j\eta}{1+\frac{1}{4}\tau_j^2\eta^2}. \tag{7.168}$$

(2) *There exists a constant C_2 such that for all $\eta > 0$ and for all $k \in \mathbb{N}$*

$$\left(\sum_{j=1}^{2k}\varphi_j(\eta)\right) \wedge \left(\sum_{j=1}^{2k}(\pi - \varphi_j(\eta))\right) \leqslant C_2 \sum_{j=1}^{2k} \frac{\tau_j\eta}{1+\frac{1}{4}\tau_j^2\eta^2}. \tag{7.169}$$

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In order to further investigate conditions imposed on the step size $(\tau_j)_{j\in\mathbb{N}}$ we insert the following proposition, which has also some relevance on its own.

7.37. Corollary. *Let the sequence of positive real numbers $\{\tau_j : j \in \mathbb{N}\}$ satisfy (7.168) in Theorem 7.36, and let A be the generator of a bounded analytic semigroup in a Banach space $(X, \|\cdot\|)$ with domain $D(A)$. Then the Crank-Nicolson iteration scheme*

$$\left(I - \frac{1}{2}\tau_{n+1}A\right) x_{n+1} = \left(I + \frac{1}{2}\tau_{n+1}A\right) x_n, \quad x_0 \in D(A), \tag{7.170}$$

is stable in the sense that $\sup_{n\in\mathbb{N}} \|x_n\| < \infty$.

Proof. Corollary 7.37 is a consequence of Theorem 7.36 in conjunction with Theorem 7.34. □

7.38. Proposition. *Put $\varphi_j(\eta) = 2\arctan\left(\frac{1}{2}\tau_j\eta\right)$ and $f_{2k}(\eta) = 2\sum_{j=1}^{2k} \arctan\left(\frac{1}{2}\tau_j\eta\right)$, $\eta > 0$. Then a calculation gives $\eta\varphi_j'(\eta) = \sin\varphi_j(\eta)$. In addition, the following inequalities hold:*

$$\begin{aligned}
&\frac{\pi}{2}\left(\#\left\{1 \leqslant j \leqslant 2k : \varphi_j > \frac{\pi}{2}\right\} \wedge \#\left\{1 \leqslant j \leqslant 2k : \varphi_j \leqslant \frac{\pi}{2}\right\}\right) \\
&\leqslant \left(\sum_{j=1}^{2k} \varphi_j\right) \wedge \left(\sum_{j=1}^{2k} (\pi - \varphi_j)\right) \\
&\leqslant \pi\left(\#\left\{1 \leqslant j \leqslant 2k : \varphi_j > \frac{\pi}{2}\right\} \wedge \#\left\{1 \leqslant j \leqslant 2k : \varphi_j \leqslant \frac{\pi}{2}\right\}\right) \\
&\qquad + 2\sum_{j=1}^{2k} \varphi_j \wedge (\pi - \varphi_j).
\end{aligned} \tag{7.171}$$

In addition the following inequality is true:

$$\begin{aligned}
&\left(\sum_{j=1}^{2k} \varphi_j\right) \wedge \left(\sum_{j=1}^{2k} (\pi - \varphi_j)\right) \\
&\leqslant \left\{ 2 + \frac{\pi}{\dfrac{\sum_{j=1}^{2k} \varphi_j \mathbf{1}_{\{\varphi_j \leqslant \pi/2\}}}{\sum_{j=1}^{2k} \mathbf{1}_{\{\varphi_j \leqslant \pi/2\}}} + \dfrac{\sum_{j=1}^{2k} (\pi - \varphi_j)\mathbf{1}_{\{\varphi_j > \pi/2\}}}{\sum_{j=1}^{2k} \mathbf{1}_{\{\varphi_j > \pi/2\}}}} \right\} \times \sum_{j=1}^{2k} \varphi_j \wedge (\pi - \varphi_j).
\end{aligned} \tag{7.172}$$

Proof. The proofs of the inequalities in (7.171) can be seen from the following more or less self-explanatory arguments:

$$\begin{aligned}
&\frac{\pi}{2}\left(\#\left\{1 \leqslant j \leqslant 2k : \varphi_j > \frac{\pi}{2}\right\} \wedge \#\left\{1 \leqslant j \leqslant 2k : \varphi_j \leqslant \frac{\pi}{2}\right\}\right) \\
&\leqslant \left(\sum_{j=1}^{2k} \varphi_j\right) \wedge \left(\sum_{j=1}^{2k} (\pi - \varphi_j)\right) \leqslant 2\left(\sum_{j=1}^{2k} \varphi_j \wedge \frac{\pi}{2}\right) \wedge \left(\sum_{j=1}^{2k} (\pi - \varphi_j) \wedge \frac{\pi}{2}\right)
\end{aligned}$$

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$$
\begin{aligned}
&= \left(\pi \# \left\{1 \leqslant j \leqslant 2k:\ \varphi_j > \frac{\pi}{2}\right\} + 2 \sum_{j=1,\, \varphi_j \leqslant \frac{1}{2}\pi}^{2k} \varphi_j\right) \\
&\quad \wedge \left(\pi \# \left\{1 \leqslant j \leqslant 2k:\ \varphi_j \leqslant \frac{\pi}{2}\right\} + 2 \sum_{j=1,\, \varphi_j > \frac{1}{2}\pi}^{2k} \left(\pi - \varphi_j\right)\right) \\
&= \left(\pi \# \left\{1 \leqslant j \leqslant 2k:\ \varphi_j > \frac{\pi}{2}\right\} + 2 \sum_{j=1,\, \varphi_j \leqslant \frac{1}{2}\pi}^{2k} \varphi_j \wedge \left(\pi - \varphi_j\right)\right) \\
&\quad \wedge \left(\pi \# \left\{1 \leqslant j \leqslant 2k:\ \varphi_j \leqslant \frac{\pi}{2}\right\} + 2 \sum_{j=1,\, \varphi_j > \frac{1}{2}\pi}^{2k} \varphi_j \wedge \left(\pi - \varphi_j\right)\right) \\
&\leqslant \left(\pi \# \left\{1 \leqslant j \leqslant 2k:\ \varphi_j > \frac{\pi}{2}\right\} + 2 \sum_{j=1}^{2k} \varphi_j \wedge \left(\pi - \varphi_j\right)\right) \\
&\quad \wedge \left(\pi \# \left\{1 \leqslant j \leqslant 2k:\ \varphi_j \leqslant \frac{\pi}{2}\right\} + 2 \sum_{j=1}^{2k} \varphi_j \wedge \left(\pi - \varphi_j\right)\right) \\
&= \pi \left(\# \left\{1 \leqslant j \leqslant 2k:\ \varphi_j > \frac{\pi}{2}\right\} \wedge \# \left\{1 \leqslant j \leqslant 2k:\ \varphi_j \leqslant \frac{\pi}{2}\right\}\right) \\
&\quad + 2 \sum_{j=1}^{2k} \varphi_j \wedge \left(\pi - \varphi_j\right). \qquad (7.173)
\end{aligned}
$$

Next we have the following inequality:

$$
\begin{aligned}
&\left\{ \frac{\sum_{j=1}^{2k} \varphi_j \mathbf{1}_{\{\varphi_j \leqslant \pi/2\}}}{\sum_{j=1}^{2k} \mathbf{1}_{\{\varphi_j \leqslant \pi/2\}}} + \frac{\sum_{j=1}^{2k} \left(\pi - \varphi_j\right) \mathbf{1}_{\{\varphi_j > \pi/2\}}}{\sum_{j=1}^{2k} \mathbf{1}_{\{\varphi_j > \pi/2\}}} \right\} \\
&\quad \times \# \left\{1 \leqslant j \leqslant 2k:\ \varphi_j > \frac{\pi}{2}\right\} \wedge \# \left\{1 \leqslant j \leqslant 2k:\ \varphi_j \leqslant \frac{\pi}{2}\right\} \\
&\leqslant \sum_{j=1}^{2k} \varphi_j \mathbf{1}_{\{\varphi_j \leqslant \pi/2\}} + \sum_{j=1}^{2k} \left(\pi - \varphi_j\right) \mathbf{1}_{\{\varphi_j > \pi/2\}} \\
&= \sum_{j=1}^{2k} \varphi_j \wedge \left(\pi - \varphi_j\right). \qquad (7.174)
\end{aligned}
$$

From (7.174) we see that the right-hand side of (7.172) satisfies:

$$
\left\{ 2 + \frac{\pi}{\frac{\sum_{j=1}^{2k} \varphi_j \mathbf{1}_{\{\varphi_j \leqslant \pi/2\}}}{\sum_{j=1}^{2k} \mathbf{1}_{\{\varphi_j \leqslant \pi/2\}}} + \frac{\sum_{j=1}^{2k} \left(\pi - \varphi_j\right) \mathbf{1}_{\{\varphi_j > \pi/2\}}}{\sum_{j=1}^{2k} \mathbf{1}_{\{\varphi_j > \pi/2\}}}} \right\} \times \sum_{j=1}^{2k} \varphi_j \wedge \left(\pi - \varphi_j\right)
$$

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$$\geqslant \left\{ 2 + \frac{\pi \# \left\{1 \leqslant j \leqslant 2k : \varphi_j > \frac{\pi}{2}\right\} \wedge \# \left\{1 \leqslant j \leqslant 2k : \varphi_j \leqslant \frac{\pi}{2}\right\}}{\sum_{j=1}^{2k} \varphi_j \wedge (\pi - \varphi_j)} \right\}$$

$$\times \sum_{j=1}^{2k} \varphi_j \wedge (\pi - \varphi_j)$$

$$= 2 \sum_{j=1}^{2k} \varphi_j \wedge (\pi - \varphi_j) + \# \left\{1 \leqslant j \leqslant 2k : \varphi_j > \frac{\pi}{2}\right\} \wedge \# \left\{1 \leqslant j \leqslant 2k : \varphi_j \leqslant \frac{\pi}{2}\right\}$$

$$\geqslant \left(\sum_{j=1}^{2k} \varphi_j \right) \wedge \left(\sum_{j=1}^{2k} (\pi - \varphi_j) \right). \tag{7.175}$$

In the final step of (7.175) we applied the second inequality of (7.171). This shows the inequality in (7.172) and completes the proof of Proposition 7.38. □

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The following corollary follows from (7.172) by using standard goniometric inequalities like

$$\sin\varphi \leqslant \varphi \wedge (\pi-\varphi) \leqslant \frac{\pi}{2}\sin\varphi, \quad 0 \leqslant \varphi \leqslant \pi, \text{ and}$$
$$\frac{\pi}{4}\eta \leqslant \arctan\eta \leqslant \eta, \quad 0 \leqslant \eta \leqslant 1. \tag{7.176}$$

7.39. Corollary. *Let the function $b_{2k}(\eta)$ be defined as in (7.159). Then*

$$b_{2k}(\eta) \leqslant \pi \left\{ 1 + \frac{1}{\frac{\sum_{j=1}^{2k} \frac{1}{2}\tau_j\eta \mathbf{1}_{[0,1]}\left(\frac{1}{2}\tau_j\eta\right)}{\sum_{j=1}^{2k} \mathbf{1}_{[0,1]}\left(\frac{1}{2}\tau_j\eta\right)} + \frac{\sum_{j=1}^{2k} \frac{2}{\tau_j\eta} \mathbf{1}_{(1,\infty)}\left(\frac{1}{2}\tau_j\eta\right)}{\sum_{j=1}^{2k} \mathbf{1}_{(1,\infty)}\left(\frac{1}{2}\tau_j\eta\right)}} \right\}, \tag{7.177}$$

where

$$\min_{1\leqslant j\leqslant 2k} \tau_j < \frac{2}{\eta} < \max_{1\leqslant j\leqslant 2k} \tau_j.$$

If $\dfrac{2}{\eta}$ is outside the interval $\left[\min_{1\leqslant j\leqslant 2k}\tau_j, \max_{1\leqslant j\leqslant 2k}\tau_j\right]$, then, as is easily seen, $b_{2k}(\eta) \leqslant \frac{1}{2}\pi$.

7.40. Corollary. *Suppose that*

$$M\left(\left(\tau_j\right)_j\right) := \inf_k \min_\eta \left\{ \frac{\displaystyle\sum_{j=1}^{2k} \frac{\tau_j}{\eta}\mathbf{1}_{[0,1]}\left(\frac{\tau_j}{\eta}\right)}{\displaystyle\sum_{j=1}^{2k} \mathbf{1}_{[0,1]}\left(\frac{\tau_j}{\eta}\right)} + \frac{\displaystyle\sum_{j=1}^{2k} \frac{\eta}{\tau_j}\mathbf{1}_{[0,1]}\left(\frac{\eta}{\tau_j}\right)}{\displaystyle\sum_{j=1}^{2k} \mathbf{1}_{[0,1]}\left(\frac{\eta}{\tau_j}\right)} \right\} > 0, \tag{7.178}$$

where the minimum is taken over all η with the property that

$$\min_{1\leqslant j\leqslant 2k} \tau_j \leqslant \eta \leqslant \max_{1\leqslant j\leqslant 2k} \tau_j.$$

Then the Crank-Nicolson iteration scheme is stable.

Proof. This result follows from Corollary 7.39 by applying it to the functions $\varphi_j(\eta) = 2\arctan\left(\frac{1}{2}\tau_j\eta\right)$ in which η is replaced with $2/\eta$. In addition, the elementary equality $\arctan\eta + \arctan(1/\eta) = \frac{1}{2}\pi$, $\eta > 0$, and the inequality

$$\arctan\eta \geqslant \frac{4}{\pi}\eta, \quad 0 \leqslant \eta \leqslant 1,$$

are used to see that Corollary 7.40 is a consequence of Corollary 7.39. □

The following corollary is a consequence of Theorem 7.34) of Corollary 7.39 and of Corollary 7.40. It shows that the Crank-Nicolson iteration scheme is stable provided that the quantity $M\left(\left(\tau_j\right)_j\right)$ as defined in (7.178) is strictly positive.

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7.41. Corollary. *Let the sequence of positive real numbers* $\{\tau_j : j \in \mathbb{N}\}$ *satisfy* (7.178) *in Corollary 7.40, and let* A *be the generator of a bounded analytic semigroup in a Banach space* $(X, \|\cdot\|)$ *with domain* $D(A)$*. Then the Crank-Nicolson iteration scheme in (7.170) is stable in the sense of Corollary 7.37.*

7.42. Corollary. *Let the sequence* τ_j, $j \in \mathbb{N}$, *be such that* $0 < \inf_j \tau_j \leqslant \sup_j \tau_j < \infty$. *Then Crank-Nicolson iteration scheme, as described in Corollary 7.37 is stable.*

Proof. It is not so difficult to see that

$$M\left((\tau_j)_j\right) \geqslant 2 \inf_k \sqrt{\frac{\min_{1\leqslant j\leqslant 2k} \tau_j}{\max_{1\leqslant j\leqslant 2k} \tau_j}} = 2\sqrt{\frac{\inf_j \tau_j}{\sup_j \tau_j}},$$

and hence, the conclusion in Corollary 7.42 follows from Corollary 7.41. □

7.43. Corollary. *Let* $\tau_j = R(j) > 0$, *where the function* $\eta \mapsto R(\eta)$ *is a rational function taking its values in* $(0, \infty)$. *Then the Crank-Nicolson iteration scheme, as described in Corollary 7.37, is stable.*

Proof. A rational function possesses either one of the following properties:

(1) it ultimately decreases to 0;
(2) it ultimately increases to ∞;
(3) it possesses a strictly positive finite limit.

If $\lim_{j\to\infty} R(j)$ exists and is finite and strictly positive, then the result in Corollary 7.43 follows from Corollary 7.42. Since $M\left((\tau_j)_j\right) = M\left((\tau_j^{-1})_j\right)$, it suffices to consider the situation that, ultimately, $R(\xi)$ increases to ∞. Since stability is only affected for j large, without loss of generality we may assume that the function $\xi \mapsto R(\xi)$ is increasing for $\xi \geqslant \tau_1$. In fact we let the Crank-Nicolson scheme start after m steps, and replace τ_m with τ_1. Then we write $\eta = R(\xi)$, $\tau_j = R(j)$, to obtain:

$$\begin{aligned}
\frac{\sum_{j=1}^{2k} \frac{\tau_j}{\eta} \mathbf{1}_{[0,1]}\left(\frac{\tau_j}{\eta}\right)}{\sum_{j=1}^{2k} \mathbf{1}_{[0,1]}\left(\frac{\tau_j}{\eta}\right)} &= \frac{\sum_{j=1}^{2k} \frac{R(j)}{R(\xi)} \mathbf{1}_{[0,1]}\left(\frac{j}{\xi}\right)}{\sum_{j=1}^{2k} \mathbf{1}_{[0,1]}\left(\frac{j}{\xi}\right)} \\
&= \frac{1}{\lfloor\xi\rfloor} \sum_{j=1}^{\lfloor\xi\rfloor} \frac{R(j)}{R(\xi)} = \int_0^1 \frac{R\left(\lceil \lfloor\xi\rfloor s\rceil\right)}{R(\xi)}\, ds. \qquad (7.179)
\end{aligned}$$

Let $R(\xi)$ be of the form $R(\xi) = \dfrac{P(\xi)}{Q(\xi)}$, where $P(\xi)$ is a polynomial of degree n, and $Q(\xi)$ is a polynomial of degree m. From our assumption on the rational function $R(\xi)$ (its limit is ∞ as $\xi \to \infty$), it follows that $n - m \geqslant 1$. Moreover,

$$\lim_{\xi\to\infty} \int_0^1 \frac{R\left(\lceil \lfloor\xi\rfloor s\rceil\right)}{R(\xi)}\, ds = \int_0^1 s^{n-m}\, ds = \frac{1}{n-m+1}. \qquad (7.180)$$

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Consequently, from (7.179) and (7.180) it follows that $M\left((\tau_j)_j\right) > 0$. The conclusion in Corollary 7.43 then follows form Corollary 7.40. Altogether this completes the proof of Corollary 7.43. $\square$

Notice the $M\left((\tau_j)_j\right) = 0$ when $\tau_j = e^j$. So exponential step sizes may result in non stable Crank-Nicolson schemes.

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CHAPTER 8

Elements of functional analysis

In this chapter we discuss and prove some results which are useful to understand the main part of the book. Among other things the reader will find formulations of the Banach-Steinhaus theorem for Fréchet spaces, the closed graph theorem and results related the Hahn-Banach theorem like Mazur's theorem. The results and proofs are taken from Rudin [**113**], Gohberg and Goldberg [**56**], and from Waelbroeck [**150**]. For elementary proofs of the uniform boundedness principle in Banach spaces, not using a Baire category argument, but kind of a *gliding hump* technique, see *e.g.* Hennefeld [**61**] or Sokal [**129**].

1. Theorem of Hahn-Banach

Let X be a real or complex vector space. A functional $p : X \to \mathbb{R}$ is called sub-additive if $p(x+y) \leqslant p(x) + p(y)$ for all $x, y \in X$. It is called positive homogeneous provided $p(\lambda x) = \lambda p(x)$ for all $\lambda \geqslant 0$ and for all $x \in X$. The functional p is called a semi-norm, if it attains its values in $[0, \infty)$, is sub-additive, and if $p(\lambda x) = |\lambda|\, p(x)$ for all $\lambda \in \mathbb{R}$, or $\lambda \in \mathbb{C}$, and $x \in X$.

8.1. Theorem (Hahn-Banach, analytic version in a real vector space). *Let X be a vector over $\mathbb{R}$, let $p : X \to \mathbb{R}$ be a sub-additive, positive homogeneous functional on X, let M be a real linear subspace of X, and let $f : M \to \mathbb{R}$ be a real-valued linear functional on M with the property that $f(x) \leqslant p(x)$ for all $x \in M$. Then there exists a linear functional $f_0 : X \to \mathbb{R}$ which extends f, i.e. $f_0(x) = f(x)$, $x \in M$, and which is such that $-p(-x) \leqslant f(x) \leqslant p(x)$ for all $x \in X$.*

Proof. Suppose $M \neq X$, and choose $x_1 \notin M$. Put $M_1 = M + \mathbb{R}x_1$. Then M_1 is a vector subspace of X which contains M, and

$$f(x) - p(x - x_1) \leqslant p(y + x_1) - f(y), \quad x, y \in M. \tag{8.1}$$

Choose $\alpha \in \mathbb{R}$ in such a way that

$$\sup_{x \in M} \{f(x) - p(x - x_1)\} \leqslant \alpha \leqslant \inf_{y \in M} \{(y + x_1) - f(y)\}.$$

By (8.1) such a choice is possible. Define the functional $f_1 : M_1 \to \mathbb{R}$ by

$$f_1(x + tx_1) = f(x) + t\alpha, \ x \in M, \ t \in \mathbb{R}.$$

Then $f_1(x) = f(x)$, $x \in M$. Moreover, $-p(-y) \leqslant f_1(y) \leqslant p(y)$, $y \in M_1$. Let $\mathcal{P}$ be the collection of all ordered pairs (M', f') with the following properties: M' is a linear

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subspace of X containing M, and f' is a linear functional on M' which extends f, and which is such that $f'(y) \leqslant p(y)$, $y \in M'$. Partially order this collection by declaring $(M', f') \leqslant (M'', f'')$ to mean that $M' \subset M''$ and $f'(y) = f''(y)$, $y \in M'$. By Hausdorff's maximality theorem there exists a maximal totally ordered subcollection Ω of $\mathcal{P}$. Put $\widetilde{M} = \cup \{M' : (M', f') \in \Omega\}$, and define $\widetilde{f} : \widetilde{M} \to \mathbb{R}$ by $\widetilde{f}(y) = f'(y)$ if $y \in M'$. Then $\widetilde{f}$ is well-defined, and $\widetilde{f}(y) \leqslant p(y)$, $y \in \widetilde{M}$. By the first part of he proof it follows that $\widetilde{M} = X$. It also follows that $\widetilde{f}$ can be taken as f in the theorem. This completes the proof of Theorem 8.1. ☐

8.2. Theorem (Hahn-Banach, analytic version in a complex vector space). *Let X be a vector over $\mathbb{C}$, let $p : X \to [0, \infty)$ be a semi-norm on X, let M be a linear subspace of X, and let $f : M \to \mathbb{C}$ be a complex-valued linear functional on M with the property that $\Re f(x) \leqslant p(x)$ for all $x \in M$. Then there exists a linear functional $f_0 : X \to \mathbb{C}$ which extends f, i.e. $f_0(x) = f(x)$, $x \in M$, and which is such that $|f(x)| \leqslant p(x)$ for all $x \in X$.*

Proof. The proof of the complex version can be recovered from the real version of the Hahn-Banach theorem, by putting $u(x) = \Re f(x)$, $x \in M$. Then the real-valued functional u satisfies the conditions of the (real) Hahn-Banach theorem. Here $f : M \to \mathbb{C}$ is as in the theorem. Let $u_0 : X \to \mathbb{R}$ be the real extension of u to all of X which is such that $u_0(x) \leqslant p(x)$, $x \in M$. The mapping $f_0(x) = u_0(x) - iu_0(ix)$, $x \in X$, then has the required properties. The proof of Theorem 8.2 is complete now. ☐

8.3. Theorem (Hahn-Banach, geometric version). *Let A and B be disjoint convex subsets of a locally convex vector space X.*

(a) *If B open is (and $A \cap B = \varnothing$), then there exists a real number γ and a contiunuous linear functional $\Lambda : X \to \mathbb{C}$ with the property that, for all vectors $b \in B$ and for all vectors $a \in A$, the following inequality is true:*

$$\Re\Lambda(b) < \gamma \leqslant \Re\Lambda(a).$$

(b) *If B is closed and if A is compact (and as above $A \cap B = \varnothing$), then there exist real numbers γ_1 and γ_2 and a continuous linear functional $\Lambda : X \to \mathbb{C}$ with the property that, for all vectors $b \in B$ and for all vectors $a \in A$, the following inequality is true:*

$$\Re\Lambda(b) \leqslant \gamma_1 < \gamma_2 \leqslant \Re\Lambda(a).$$

Proof. (a) Fix a vector $a_0 \in A$ and fix a vector $b_0 \in B$ and consider the neighborhood of the zero-vector V defined by $V = B - A + a_0 - b_0$. Let p_V be its Minkowski functional. Since the vector $a_0 - b_0$ does not belong to V, it follows that $p_V(a_0 - b_0) \geqslant 1$. Define the real linear functional $f : \mathbb{R}(a_0 - b_0) \to \mathbb{R}$ by $f(\lambda(b_0 - a_0)) = \lambda p_V(a_0 - b_0)$, $\lambda \in \mathbb{R}$. Then we have $f(y) \leqslant p(y)$ for all $y \in \mathbb{R}(a_0 - b_0)$. By virtue of the analytic version of the Hahn-Banach theorem there exists a real linear functional $u : X \to \mathbb{R}$ with the properties that $u(x) \leqslant p_V(x)$ for all $x \in X$

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that $u(a_0 - b_0) = p_V(a_0 - b_0) \geqslant 1$. Put $\Lambda(x) = u(x) - iu(ix)$, $x \in X$, and notice that $\Lambda(\lambda x) = \lambda\Lambda(x)$ for $\lambda \in \mathbb{C}$ and $x \in X$. Note that the functional Λ is continuous. Because, let V_0 be an absolutely convex closed neighborhood of the origin contained in V, and let p_{V_0} be its Minkowski functional. Then $|\Lambda(x)| \leqslant p_{V_0}(x)$ for all $x \in X$. If b belongs to B and if a belongs to A, then $\Re\Lambda\left((b - a + a_0 - b_0\right) < 1 \leqslant \Re\Lambda(a_0 - b_0)$. Define the constant γ by $\gamma = \inf\left\{\Re\Lambda(a) : a \in A\right\}$. This constant γ then verifies the required conditions.

(b) Select an open absolutely convex neighborhood of the origin U in such a way that $A \cap (B + U) = \varnothing$. Since A is a compact set and X is locally convex such a neighborhood of the origin exists. Define the zero-neighborhood V by $V = B - A + U + a_0 - b_0$, where a_0 is chosen in A and where b_0 is chosen in B. The vector $a_0 - b_0$ does not belong to V and hence $p_V(a_0 - b_0) \geqslant 1$. Again there exists a functional $\Lambda : X \to \mathbb{C}$ with the property that $\Re\Lambda(a_0 - b_0) = p_V(a_0 - b_0) \geqslant 1$ and for which $\Re\Lambda(x) \leqslant p_V(x)$ for all $x \in X$. Define γ_1 by $\gamma_1 = \sup_{b\in B} \Re\Lambda(b)$ and define γ_2 via the formula $\gamma_2 = \inf_{a\in A} \Re\Lambda(a)$. The inequality $\Re\Lambda(y) \leqslant \Re\Lambda(a) - \Re\Lambda(b)$ follows for all $y \in U$, for all $a \in A$ for all $b \in B$. Consequently

$$\gamma_2 - \gamma_1 \geqslant \sup_{y\in U} \Re\Lambda(y) = \sup_{y\in U} |\Lambda(y)| > 0$$

and also

$$\Re\Lambda(b) \leqslant \gamma_1 < \gamma_2 \leqslant \Re\Lambda(a).$$

This completes the proof of Theorem 8.3. $\square$

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The analytic version of the Hahn-Banach theorem can also be deduced from the geometric version. Putting it differently, let $p : X \to \mathbb{R}$ be positive homogeneous and sub-additive continuous linear functional, defined on a locally convex space X and let $f : M \to \mathbb{C}$ be a linear functional, defined on the linear subspace M and that verifies $\Re f(x) \leqslant p(x)$ for all $x \in M$ and that possesses the property that $\Re f(x_0) = p(x_0) = 1$ for some $x_0 \in M$. Prove that there exists a functional $f_0 : X \to \mathbb{C}$ with the following properties:

(a) $f_0(x) = f(x)$ for $x \in M$ en
(b) $\Re f_0(x) \leqslant p(x)$ for $x \in X$.

For a proof we consider the following two convex subsets of X that are disjoint: $U := \{x \in X : p(x) < 1\}$ and $C := \{x \in M : \Re f(x) \geqslant 1\}$. We notice that the vector x_0 belongs to C and hence C is non-empty. From the geometric version of the Hahn-Banach theorem it follows that there exists a complex linear functional Λ and a constant γ such that the following inequalities are satisfied:

$$\Re\Lambda(u) < \gamma \leqslant \Re\Lambda(v)$$

for all vectors $u \in U$ and for all vectors v for which $\Re f(v) \geqslant 1$. Since the zero-vector belongs to U, it follows that $\gamma > 0$. So we may consider $f_0 := \dfrac{\Lambda}{\gamma}$. Then $\Re f_0(x) < 1$ if $p(x) < 1$ and if $x \in M$ is such that $\Re f(x) \geqslant 1$, then $\Re f_0(x) \geqslant 1$. It follows that $\Re f_0(x) \leqslant p(x)$ for all $x \in X$ and the following assertion follows as well. If $x \in M$ is such that $\Re f(x) \geqslant \eta$, then $\Re f_0(x) \geqslant \eta$ and this is true for any $\eta > 0$. Consequently we see that $\Re f_0(x) \geqslant \Re f(x)$ for all $x \in M$. Since M is a linear subspace, it follows $\Re f(x) = \Re f_0(x)$ for all $x \in M$. This proves the statement.

8.4. Corollary. *Let X be a locally convex vector space and let V be a convex neighborhood of the origin. Let $p_V(x) := \inf\{t > 0 : x \in tV\}$, $x \in X$, be its Minkowski functional, and let*

$$V^0 = \bigcap_{x \in V} \{x^* \in X^* : \Re\langle x, x^*\rangle \leqslant 1\}$$

be its polar set. Then $p_V(x) = \sup\{\Re\langle x, x^*\rangle : x^* \in V^0\}$ for all $x \in X$. In fact, the proof will show that, for $x \in X$ given, there exists a continuous linear functional $x^* \in V^0$ such that $p_V(x) = \Re\langle x, x^*\rangle$.

Proof. Fix $x_0 \in X$ and define $f : \mathbb{R}x_0 \to \mathbb{R}$ by $f(\lambda x_0) = \lambda p_V(x_0)$. Then $f(y) \leqslant p_V(y)$ for all y in the real subspace spanned by x_0. By the Hahn-Banach extension theorem there exists a real linear functional $f_0 : X \to \mathbb{R}$ such that $f_0(x) \leqslant p_V(x)$, $x \in X$, and such that $f(y) = f_0(y)$ for all $y \in \mathbb{R}x_0$. Define the complex linear functional x_0^* by $\langle x, x_0^*\rangle = f_0(x) - if_0(ix)$, for $x \in X$. It follows that $\Re\langle x, x_0^*\rangle = f_0(x) \leqslant p_V(x)$ for all $x \in X$. If x belongs to V, then $p_V(x) \leqslant 1$ and so, for such x, $\Re\langle x, x_0^*\rangle \leqslant 1$. Consequently x_0^* belongs to V^0 and since $\Re\langle x_0, x_0^*\rangle = p_V(x_0)$, we infer that $p_V(x_0) = \Re\langle x_0, x_0^*\rangle$, with $x_0^* \in V^0$. This proves the claim in Corollary 8.4. □

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The following result implies that a convex subset of a locally convex topological vector space $(X, \mathcal{T})$ is $\mathcal{T}$-closed if and only if it is weakly closed. This is Mazur's theorem.

8.5. Proposition. *Let C be a closed convex subset of a locally convex topological vector space $(X, \mathcal{T})$. Then*

$$C = \bigcap \{\Re\Lambda \leqslant \alpha : \ C \subseteq \{\Re\Lambda \leqslant \alpha\}\}.$$

Proof. Pick

$$x_0 \in \bigcap \{\Re\Lambda \leqslant \alpha : C \subseteq \{\Re\Lambda \leqslant \alpha\}\} \tag{8.2}$$

and assume that x_0 does not belong to the $\mathcal{T}$-closed convex subset C. From the geometric version of the Hahn-Banach theorem, it follows that there exists a continuous linear functional $\Lambda : \ X \to \mathbb{C}$ and a constant γ, such that $\Re\Lambda(x_0) > \gamma \geqslant \Re\Lambda(x)$ for all $x \in C$. This contradicts (8.2). Whence

$$\bigcap \{\Re\Lambda \leqslant \alpha : C \subseteq \{\Re\Lambda \leqslant \alpha\}\} \subseteq C.$$

The other inclusion being trivial, this proves Proposition 8.5. □

8.6. Corollary. *Let C be a convex subset of a locally convex topological vector space $(X, \mathcal{T})$. Then C is $\mathcal{T}$-closed if and only if*

$$C = \bigcap \{\Re\Lambda \leqslant \alpha : C \subseteq \{\Re\Lambda \leqslant \alpha\}\}.$$

8.7. Theorem (Alaoglu-Bourbaki). *Let E^* be the topological dual space of a locally convex topological vector space E, and let B be an equi-continuous family of linear functionals in E^*. Then B is relatively compact for the weak* topology. In particular it follows that the polar set U° of a zero-neighborhood U is $\sigma(E^*, E)$-compact.*

Proof. Let p_W be the Minkowski-functional of the convex zero-neighborhood W. Since B is equi-continuous there exists an absolutely convex, closed zero-neighborhood V with the property that

$$\begin{aligned}
B \subseteq V^\circ &:= \bigcap_{x \in V} \{x^* \in E^* : \mathrm{Re}\ x^*(x) \leqslant 1\} \\
&= \bigcap_{x \in V} \{x^* \in E^* : |x^*(x)| \leqslant 1\} \\
&= \bigcap_{x \in V} \{x^* : E \mapsto \mathbb{C} : |x^*(x)| \leqslant p_V(x), x^* \quad \text{linear}\} \\
&= \bigcap_{x \in V, \alpha, \beta \in \mathbb{C}, u, v \in E} \{x^* : E \mapsto \mathbb{C} : |x^*(x)| \leqslant p_V(x), \\
&\qquad\qquad x^*(\alpha u + \beta v) = \alpha x^*(u) + \beta x^*(v)\} \\
&= \bigcap_{x \in V, \alpha, \beta \in \mathbb{C}, u, v \in E} \left\{(\lambda_y)_{y \in E} \in \mathbb{C}^E : |\lambda_x| \leqslant p_V(x), \lambda_{\alpha u + \beta v} = \alpha\lambda_u + \beta\lambda_v\right\}.
\end{aligned}$$

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It follows that V° can be identified with the closed subset $(*)$ of the compact set (Tychonov) $\prod_{x \in E} \{\lambda \in \mathbb{C} : |\lambda| \leqslant p_V(x)\}$. By Tychonov's theorem for infinite cartesian products, it follows that V° is compact. This completes the proof of Theorem 8.7. $\square$

1.1. Baire category. In Theorem 8.10 we need the notion of *Baire category*. The precise definition reads as follows.

8.8. Definition. Let $(S, \mathcal{T})$ be a topological space. A subset $E \subset S$ is said to be *nowhere dense* in S if its closure has empty interior. The sets of *first category* in S are those that are countable unions of nowhere dense subsets. Any subset of S that is not of the first category is said to be of the *second category* in S.

Sometimes subsets of the first category are called *meager*, and subsets of the second category *non-meager*. Let $(S_1, \mathcal{T}_1)$ and $(S_2, \mathcal{T}_2)$ be topological Hausdorff spaces, and let $h : S_1 \to S_2$ be a surjective homeomorphism. Let E be a subset of S_1. Then E and $h(E_1)$ are of the same category in $(S_1, \mathcal{T}_1)$ respectively $(S_2, \mathcal{T}_2)$. Subsets of sets of the first category are of the first category. Countable unions of sets of the first category are of the first category. Closed subsets with empty interior are of the first category. The following theorem implies that complete metric spaces, and locally compact Hausdorff spaces are of the second category in themselves.

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8.9. THEOREM. *If S is either*

(a) *a complete metric space, or*
(b) *a locally compact Hausdorff space,*

then every countable intersection of dense open subsets of S is dense in S.

Although this result is well-known, we include a proof for completeness.

PROOF. Suppose that $(V_j)_{j\in\mathbb{N}}$ is a sequence of dense open subsets of S, and let B_0 be an arbitrary nonempty open subset of S. If $n \geqslant 1$ and an open subset $B_{n-1} \neq \varnothing$ has been chosen, then, because V_n is dense in S, there exists an open subset $B_n \neq \varnothing$ with $\overline{B_n} \subset V_n \cap B_{n-1}$. In case (a), B_n may be taken to be ball of radius $< 1/n$; in case (b) this choice is made in such a way that $\overline{B_n}$ is compact. Put $K = \bigcap_{n\in\mathbb{N}} \overline{B_n}$. In case (a) the centers of the nested balls B_n form a Cauchy sequence which converge to some point of K, and so $K \neq \varnothing$. In case (b), $K \neq \varnothing$ by compactness. The construction shows that $K \subset B_0$ and $K \subset V_n$ for each n. Hence B_0 intersects $\bigcap_{n\in\mathbb{N}} V_n$. This completes the proof of Theorem 8.9. □

8.10. THEOREM. *Let B be a weakly bounded subset of a locally convex topological vector space $(E, \mathcal{T})$. Then B is $\mathcal{T}$-bounded.*

PROOF. Let V be an arbitrary closed $\mathcal{T}$-zero neighborhood, which is absolutely convex. It suffices to prove that the set B is contained in a certain scalar multiple of V. Put

$$K = V^{\circ} = \bigcap_{x\in V} \{x^* \in E^* : |x^*(x)| \leqslant 1\}.$$

Then it follows that

$$V = \bigcap_{x^*\in K} \{x \in E : |x^*(x)| \leqslant 1\}.$$

Assume that x_0 does not belong to V. By the Hahn-Banach theorem there exists a linear functional $x_0^* \in E^*$ in such a way that $\operatorname{Re} x_0^*(x_0) > 1 \geqslant |x_0^*(x)|$ for all $x \in V$. Hence, x_0 is not a member of $\bigcap_{x^*\in K} \{x \in E : |x^*(x)| \leqslant 1\}$. So we obtain

$$\bigcap_{x^*\in K} \{x \in E : |x^*(x)| \leqslant 1\} \subseteq V \subseteq \bigcap_{x^*\in K} \{x \in E : |x^*(x)| \leqslant 1\}.$$

The ultimate inclusion is a trivial consequence of the definition of K. Because B is weakly bounded it follows that

$$K = \bigcup_{n\in\mathbb{N}} \bigcap_{x\in B} \{x^* \in K : |x^*(x)| \leqslant n\}.$$

The theorem of Alaoglu-Bourbaki yields that the set K is $\sigma(E^*, E)$-compact. But a compact space is a Baire space. So there exist $n \in \mathbb{N}$, $\delta > 0$, $x_0^* \in K$, and $x_1, \dots, x_m$ in E such that

$$K_n := \bigcap_{x\in B} \{x^* \in K : |x^*(x)| \leqslant n\} \supseteq \bigcap_{i=1}^{m} \{x^* \in K : |x^*(x_i) - x_0^*(x_i)| \leqslant \delta\}$$

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$$= x_0^* + W \cap (K - x_0^*),$$

where

$$W = \bigcap_{i=1}^{m} \{x^* \in E^* : |x^*(x_i)| \leqslant \delta\}.$$

Since K_n is absolutely convex it follows that

$$K_n \supseteq \frac{1}{2}\left((x_0^* + W) \cap K - (x_0^* + W) \cap K\right) \supseteq \beta\,(W \cap K),$$

where

$$\beta = \frac{\delta}{\delta + \max_{1 \leqslant j \leqslant m} |x_0^*(x_j)|}.$$

For x^* belonging to $W \cap K$ it holds that

$$\beta x^* = \frac{1}{2}\left(x_0^* + \beta(x^* - x_0^*)\right) - \frac{1}{2}\left(x_0^* - \beta(x^* + x_0^*)\right).$$

Because $|\beta\,(x^*(x_j) \pm x_0^*(x_j))| \leqslant \delta$, $1 \leqslant j \leqslant m$, en because K is absolutely convex it follows that the vectors $x_0^* + \beta(x^* - x_0^*) = (1-\beta)x_0^* + \beta x^*$ and $x_0^* - \beta(x^* + x_0^*) = (1-\beta)x_0^* + \beta(-x^*)$ belong to the set $(x_0^* + W) \cap K$. From this we see

$$\beta\,(W \cap K) \subseteq K_n.$$

Next let $y^* \in K$ and consider the vector

$$\frac{\delta}{\delta + \max_{1 \leqslant j \leqslant m} |y^*(x_j)|} y^* + \left(1 - \frac{\delta}{\delta + \max_{1 \leqslant j \leqslant m} |y^*(x_j)|}\right) 0.$$

By the convexity of K and since 0 belongs to K, this vector belongs to K. This vector is a member of W as well. Thus, if $y^* \in K$, the vector $\beta \dfrac{\delta}{\delta + \max |y^*(x_j)|} y^*$ belongs to K_n. Consequently,

$$\begin{aligned}
|y^*(x)| &\leqslant \frac{1}{\beta}\left(1 + \frac{\max_{1 \leqslant j \leqslant m} |y^*(x_j)|}{\delta}\right) n \\
&\leqslant \left(1 + \frac{\max_{1 \leqslant j \leqslant m} |x_0^*(x_j)|}{\delta}\right)\left(1 + \frac{\max_{1 \leqslant j \leqslant m} |y^*(x_j)|}{\delta}\right) n \\
&\leqslant \left(1 + \frac{\max_{1 \leqslant j \leqslant m} p_V(x_j)}{\delta}\right)^2 n
\end{aligned}$$

for $y^* \in K$ and for $x \in B$. Put $M = \left(1 + \dfrac{\max_{1 \leqslant j \leqslant m} p_V(x_j)}{\delta}\right)^2 n$. Then, apparently,

$$|y^*(x)| \leqslant M, \quad x \in B, \quad y^* \in K.$$

Hence, we see that, for $x \in B$, the vector x/M belongs to the bipolar set $(V^\circ)^\circ = K^\circ = V$. From this we see that B is a subset of MV, and completes the proof of Theorem 8.10. □

Although the following theorem is not used in the main text we include it, because it is one of the central results in Functional analysis.

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8.11. Theorem (Krein-Milman). *Let C be a compact convex subset of a locally convex topological space X. Then C coincides with the closed convex hull of the extreme points of C.*

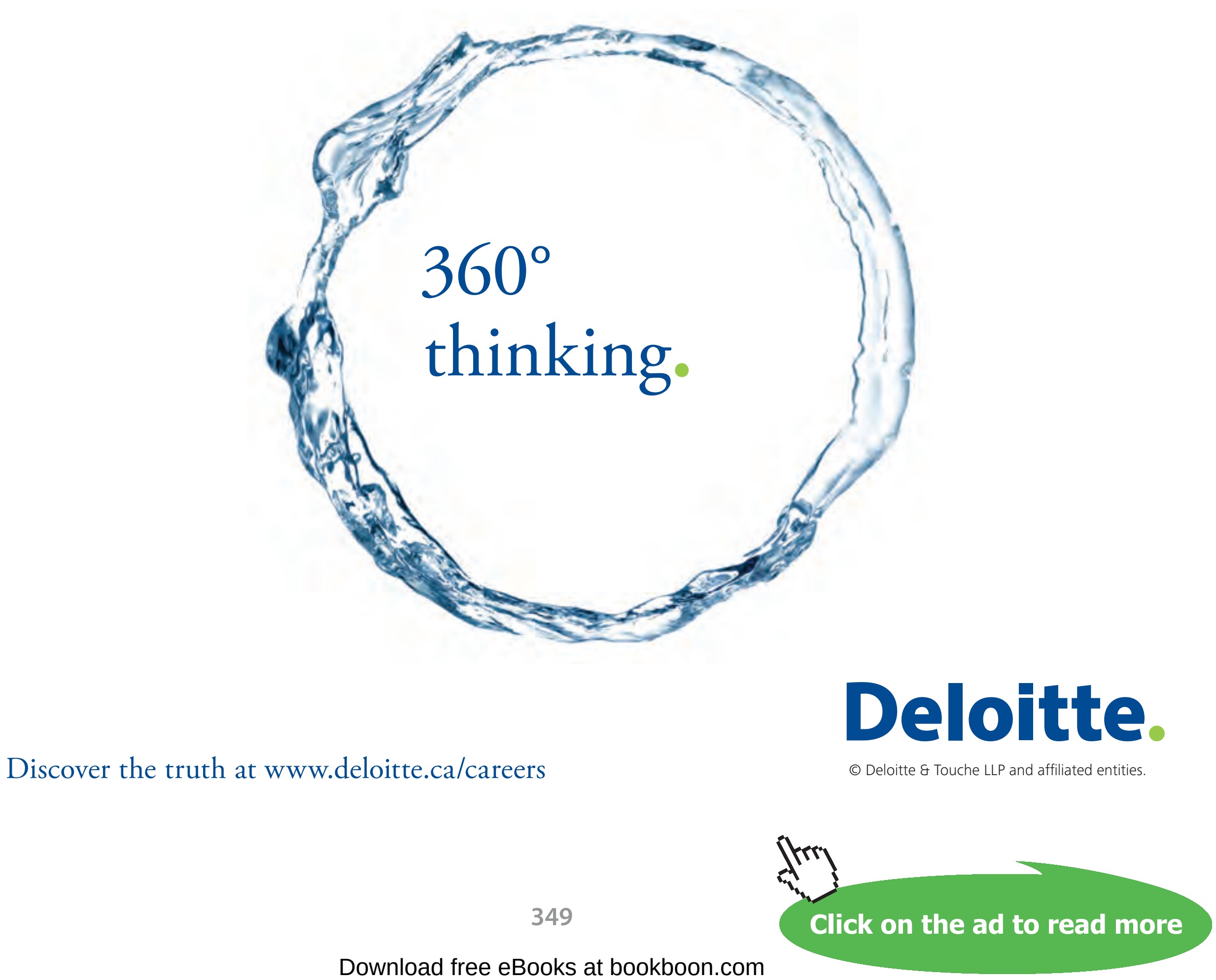
2. Banach-Steinhaus theorems: barreled spaces

A Banach space version of the Banach-Steinhaus theorem, or the uniform boundedness principle reads as follows.

8.12. Theorem (Banach-Steinhaus). *Let X and Y be Banach spaces and let $\mathcal{F}$ be a family of continuous linear operators of X to Y. Suppose that the family $\mathcal{F}$ is pointwise bounded in the sense that for every $x \in X$ the expression $\sup\{\|Tx\| : T \in \mathcal{F}\}$ is finite. Then $\sup\{\|T\| : T \in \mathcal{F}\}$ is finite.*

The closed graph theorem reads as follows.

8.13. Theorem (Closed graph theorem). *Again let X and Y be Banach spaces and let $T : X \to Y$ be an everywhere defined linear operator with the property that its graph $G(T)$, defined by $G(T) = \{(x, Tx) : x \in X\}$, is a closed linear subspace of the cartesian product $X \times Y$. Then the operator T is continuous.*



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The open mapping theorem for Banach spaces reads as follows: see Theorem 8.24 as well.

8.14. Theorem (Open mapping theorem). *Let X be a Banach space and let Y be a normed linear space. Suppose that $T : X \to Y$ is a continuous linear operator that is surjective. So $T(X) = Y$. Let B_X be the open unit ball of X: $B_X = \{x \in X : \| x \| < 1\}$. Then TB_X is an open subset of Y and Y is also a Banach space.*

The following result says that the foregoing theorems (for Banach spaces X and Y) are equivalent. Theorem 8.19 below characterizes those locally convex topological vector spaces X for which an adapted version of the Banach-Steinhaus result holds. For the same spaces the closed graph theorem is valid. The uniform boundedness in Theorem 8.12 is replaced with the equi-continuity of a family of continuous operators. It turns out that the class of spaces X for which the closed graph theorem or the Banach-Steinhaus theorem hold for all Banach spaces Y coincides with the class of the so-called *barreled* spaces: see Definitions 8.16 and 8.20. Corollary 8.22 shows that Fréchet spaces are barreled.

8.15. Theorem. *The following assertions are equivalent for arbitrary Banach spaces X and Y.*

(a) *Let $\mathcal{F}$ be a pointwise bounded family of continuous linear operators from X to Y. Then $\sup_{T \in \mathcal{F}} \|T\| < \infty$. In other words, every pointwise bounded family of continuous linear operators from X to Y is uniformly bounded.*
(b) *Every everywhere defined closed linear operator $T : X \to Y$ is continuous.*
(c) *Every surjective continuous linear operator $T : X \to Y$ is an open mapping.*

Proof. (c) $\Rightarrow$ (b). Define the projection $\Pi : X \times Y \to X$ by $\Pi(x, Tx) = x$. The restriction of Π to $G(T)$ is surjective (and injective). Let Π_G be this restriction. From the open mapping theorem it follows that there exists a $\delta > 0$ with the property that the subset:

$$\Pi_G\left(G(T) \bigcap \{(x,y) \in X \times Y : \|x\| \leqslant 1, \ \|y\| < 1\}\right)$$

contains the ball $\{x \in X : \|x\| \leqslant \delta\}$. Consequently: $\|x\| \leqslant \delta \Rightarrow \|Tx\| \leqslant 1$. For x arbitrary, $x \neq 0$, we obtain

$$\left\| T\left(\delta \frac{x}{\|x\|}\right) \right\| \leqslant 1.$$

Hence $\|Tx\| \leqslant \dfrac{1}{\delta}\|x\|$. This means that T is continuous.

(b) $\Rightarrow$ (a). Suppose that Y is complete (this can always be achieved by taking the completion of Y instead of Y itself. Let $\mathcal{B}(\mathcal{F}, Y)$ be the vector space of all functions $f : \mathcal{F} \to Y$ with the property that $\|f\| := \sup\{\|f(T)\| : T \in \mathcal{F}\} < \infty$. Define the linear operator $A : X \to \mathcal{B}(\mathcal{F}, Y)$ by $[Ax](T) = Tx$, $x \in X$. The operator A is linear, its graph is closed in $X \times \mathcal{B}(\mathcal{F}, Y)$. The operator A is everywhere defined.

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Hence, by the closed graph theorem, A is continuous. This is the same as saying that there exists a finite constant c with the property that

$$\sup_{T\in\mathcal{F}} \|[Ax](T)\| \leqslant c\,\|x\|$$

for all $x \in X$. Hence $\|Tx\| \leqslant c\,\|x\|$, $x \in X$. Whence $\|T\| \leqslant c$, $T \in \mathcal{F}$.

(b) $\Rightarrow$ (c). Consider the mapping $S : Y \to X/N(T)$, defined by $S : Tx \mapsto x + N(T)$, $x \in X$. The operator S is everywhere defined on the Banach space Y. Moreover S has a closed graph. This is so because we have the following. Let $(x_n : n \in \mathbb{N})$ be a sequence in X with the property that, in the cartesian product $Y \times X/N(T)$, (Tx_n, STx_n) converges to $(y, x + N(T))$. Then $\lim_{n\to\infty} x_n + N(T) = x + N(T)$ and $\lim_{n\to\infty} Tx_n = y$. This means that there exists a sequence (z_n) in the zero-space of T such that $\lim_{n\to\infty} \|x_n - x + z_n\| = 0$ and such that $\lim_{n\to\infty} \|T(x_n + z_n) - y\| = 0$. Since T is continuous it follows that $y = Tx$ and so T is closed. Consequently the operator S is closed. So, by the closed graph theorem, it is continuous. Hence there exists a constant c with the property that

$$\inf\{\|x + z\| : Tz = 0\} \leqslant c\,\|Tx\|,$$

for $x \in X$. But the we have

$$T\{\|x\| < 1\} \supseteq \frac{1}{c}\{y \in Y : \|y\| < 1\}.$$

An easy exercise then shows that T is an open mapping in the sense that open subsets of X are mapped onto open subsets of Y.

Altogether this completes the proof of Theorem 8.15, except that the implication (a) $\Rightarrow$ (b) has not been established yet. This is part of Theorem 8.19. $\square$

The fact that the Banach-Steinhaus theorem implies the closed graph theorem is part of the following result. However for a concise formulation we need two definitions.

8.16. Definition. Let X be locally convex topological vector space. A subset W of X is said to be a *barrel* if it is closed, balanced, convex and absorbing.

8.17. Definition. Let X and Y be topological vector spaces. A linear operator $T : X \to Y$ is said to be *almost continuous* if for every zero-neighborhood V in Y the closure of $T^{-1}V$ contains a zero-neighborhood in X.

8.18. Lemma. *Let $\{(X_U, \|\cdot\|) : U \in \mathcal{U}\}$ be a family of normed spaces. The space $\ell^\infty(X_U : U \in \mathcal{U})$, defined by*

$$\ell^\infty(X_U : U \in \mathcal{U}) = \left\{(x_U)_{U\in\mathcal{U}} \in \prod_{U\in\mathcal{U}} X_U : \sup_{U\in\mathcal{U}} \|x_U\|_U < \infty\right\},$$

is a normed vector space. If every space $(X_U, \|\cdot\|_U)$, $U \in \mathcal{U}$ is a Banach space, then so is $\ell^\infty(X_U, U \in \mathcal{U})$.

Proof. The proof of this lemma is elementary. It is left to the reader as an exercise. $\square$

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8.19. Theorem. *Let X be a locally convex vector space. The following assertions are equivalent:*

(a) *Every barrel in X is a zero-neighborhood.*
(b) *A pointwise bounded family of continuous linear operators from X to any locally convex topological vector space is equicontinuous.*
(c) *An everywhere defined operator T defined on all of X with values in a locally convex topological vector space is almost continuous.*
(d) *A closed linear operator, that is everywhere defined on X and with values in a Fréchet space, is continuous.*
(e) *A pointwise bounded family of continuous linear operators defined on X with values in a Banach-space is equicontinuous.*

Proof. (a) $\Rightarrow$ (b). Let Y be any locally convex space and let $\mathcal{F}$ be a pointwise bounded family of continuous linear operators defined on X and with values in Y. Let V be a closed absolutely convex (= balanced and convex) neighborhood of the origin in Y. Put $W = \bigcap_{T\in\mathcal{F}} T^{-1}V$. The subset W is closed and absolutely convex. Since the family $\mathcal{F}$ is pointwise bounded, W is also absorbing. So, by definition, W is a barrel in X. So by (a) W is a neighborhood of the origin in X. This means that the family $\mathcal{F}$ is equicontinuous.

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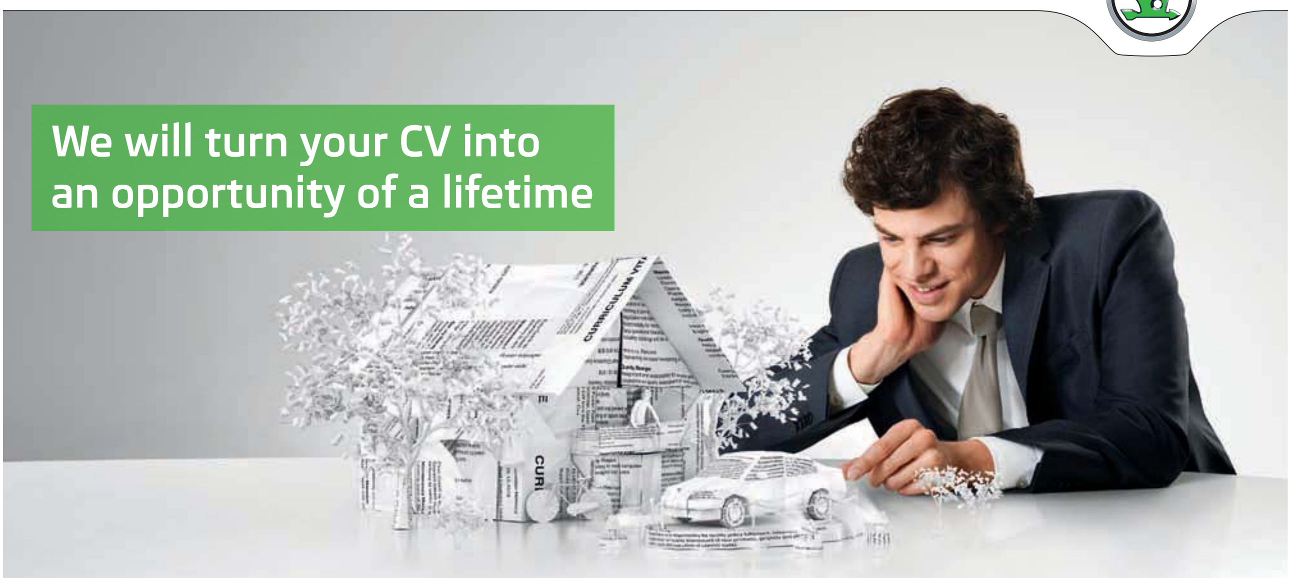

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(b) $\Rightarrow$ (c). Let Y be any locally convex vector space and let $T : X \to Y$ be an everywhere defined linear operator. Let V be a absolutely convex closed zero-neighborhood in Y and let W be the closure of $T^{-1}V$. Then W is a barrel in X. Denote by $\mathcal{U}_X(0)$ the collection of all absolutely convex closed neighborhoods of the origin in X. Fix $U \in \mathcal{U}_X(0)$ and let $G(U+W)$ be the largest subspace contained in the set $U+W$. So that $G(U+W) = \bigcap_{k=1}^{\infty} 2^{-k}(U+W)$. Define the vector space X_{U+W} as follows: $X_{U+W} = \{\lambda x : \lambda \geqslant 0,\ x \in U+W\}/G(U+W)$. The space X_{U+W} is equipped with the Minkowski functional of the union of cosets $U+W+G(U+W) = \bigcup_{x\in U+W}(x+G(U+W))$:

$$\|x+G(U+W)\|_{U+W} = \inf\{\lambda > 0 : x \in \lambda(U+W)\}.$$

This functional renders X_{U+W} into a normed linear space. (Verify this precisely.) Then let $X_{\mathcal{U}+W}$ be the vector space defined by

$$X_{\mathcal{U}+W} = \bigcup_{m=1}^{\infty} m \prod_{U\in\mathcal{U}_X(0)} (U+W+G(U+W)).$$

So a vector $x_{\mathcal{U}+W} = (x_U + G(U+W))_{U\in\mathcal{U}_X(0)}$ belongs to $X_{\mathcal{U}+W}$ if and only if there exists a natural number m with the property that x_U belongs to $m(U+W)$ for for all $U \in \mathcal{U}_X(0)$. It is a matter of routine to verify that $X_{\mathcal{U}+W}$ is a vector space and that the norm $\|x_{\mathcal{U}+W}\|_{\mathcal{U}+W}$, defined by

$$\|x_{\mathcal{U}+W}\|_{\mathcal{U}+W} = \inf\{\lambda > 0 : x_U \in \lambda(U+W),\ U \in \mathcal{U}_X(0)\}$$

turns $\left(X_{\mathcal{U}+W}, \|\cdot\|_{\mathcal{U}+W}\right)$ into a normed vector space. Here, as above,

$$x_{\mathcal{U}+W} = ((x_U + G(U+W)))_{U\in\mathcal{U}_X(0)}$$

is supposed to be a member of $X_{\mathcal{U}+W}$. In fact the normed space $\left(X_{\mathcal{U}+W}, \|\cdot\|_{\mathcal{U}+W}\right)$ coincides with the ℓ^∞-sum of the spaces

$$\left(X_{U+W}, \|\cdot\|_{U+W}\right), \quad U \in \mathcal{U}_X(0).$$

To be precise:

$$\left(X_{\mathcal{U}+W}, \|\cdot\|_{\mathcal{U}+W}\right) = \ell^\infty\left(\left(X_{U+W}, \|\cdot\|_{U+W}\right) : U \in \mathcal{U}_X(0)\right).$$

Define for $U \in \mathcal{U}_X(0)$ the operator $T_U : X \to X_{\mathcal{U}+W}$ as follows:

$$T_U(x) = (\dots, 0, \dots, 0, x+G(U+W), 0, \dots, 0, \dots), \quad x \in X.$$

So only "site U is occupied" by the vector $x+G(U+W)$. Since the unit ball of $X_{\mathcal{U}+W}$ is given by $B_{X_{\mathcal{U}+W}} = \prod_{U\in\mathcal{U}_X(0)}(U+W+G(U+W))$, it follows that the set $T_U^{-1}B_{X_{\mathcal{U}+W}}$ contains the set U. So every operator T_U is continuous. Next fix $x \in X$. Since the operator T is everywhere defined, there exists a strictly positive number $\lambda = \lambda(x) > 0$ with the property that the vector Tx belongs to λV. Hence the vector x belongs to $\lambda T^{-1}V \subseteq \lambda W$. Consequently the vectors $T_U(x)$, $U \in \mathcal{U}_X(0)$, belong to $\lambda B_{X_{\mathcal{U}+W}}$. This means that the family $\{T_U :\ U \in \mathcal{U}_X(0)\}$ is pointwise bounded.

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From (b) it follows that the family $\{T_U : U \in \mathcal{U}_X(0)\}$ is equicontinuous. This means that the intersection

$$\bigcap_{U\in\mathcal{U}_X(0)} \left\{x \in X : T_U x \in B_{X_{\mathcal{U}+W}}\right\} = \bigcap_{U\in\mathcal{U}_X(0)} \{x \in X : x \in U + W\} = W$$

belongs to $\mathcal{U}_X(0)$. This proves (c).

(c) $\Rightarrow$ (d). Let Y be a Fréchet space and let $T : X \to Y$ be an almost continuous linear operator with a closed graph. Let $(V_k : k = 0, 1, 2, \ldots)$ be a sequence of open absolutely convex neighborhoods of 0 in Y with the property that $V_{k+1} + V_{k+1}$ is contained in V_k, $k = 0, 1, 2, \ldots$. Also suppose that the sequence $(V_k : k \in \mathbb{N})$ constitutes a local basis. We shall prove that the closure of $T^{-1}V_1$ is contained in $T^{-1}V_0$. Since, by (c), the operator T is almost continuous, this shows that the set $T^{-1}V_0$ contains the zero-neighborhood $\overline{T^{-1}V_1}$. Hence it will follow that T is continuous. Pick x in the closure of $T^{-1}V_1$. Since the operator T is almost continuous, it follows that the closure of $T^{-1}V_1$ is contained in $T^{-1}V_1 + \cdots + T^{-1}V_\ell + \overline{T^{-1}V_{\ell+1}}$. So, for every $\ell \in \mathbb{N}$ there exist vectors x_j, $1 \leqslant j \leqslant \ell + 1$, such that Tx_j belongs to V_j, $1 \leqslant j \leqslant \ell$, and such that $x - \sum_{j=1}^{\ell+1} x_j$ belongs to the closure of $T^{-1}V_{\ell+1}$. It also follows that the sequence of partial sums $\left(\sum_{j=1}^{\ell} Tx_j : \ell \in \mathbb{N}\right)$ is a Cauchy sequence. Let y be its limit: $y = \sum_{j=1}^{\infty} Tx_j$. Next let U be any neighborhood in $\mathcal{U}_X(0)$. Then the vector $x - \sum_{j=1}^{\ell+1} x_j$ belongs to $T^{-1}V_{\ell+1} + U$. Choose $u \in U$ with the property that $x + u - \sum_{j+1}^{\ell+1} x_j$ belongs to $T^{-1}V_{\ell+1}$. It readily follows that the vector

$$\big(x + u, T(x + u)\big) - (x, y) = \left(u, T(x + u) - T\sum_{j=1}^{\ell+1} x_j\right) + \left(0, \sum_{j=\ell+2}^{\infty} Tx_j\right)$$

belongs to $U \times V_{\ell+1} + \{0\} \times V_{\ell+1}$. Consequently, we have

$$\big(x + u, T(x + u)\big) - (x, y) \in U \times (V_{\ell+1} + V_{\ell+1}) \subseteq U \times V_\ell.$$

This proves that the vector (x, y) belongs to the closure of the graph of T. By assumption the operator T is closed and hence the vector (x, y) belongs to the graph of T. So that $y = Tx$. Since y belongs to V_0, this proves that x belongs to $T^{-1}V_0$. Whence $\overline{T^{-1}V_1} \subseteq T^{-1}V_0$. Another application of the fact that the operator T is almost continuous proves that the set $T^{-1}V_0$ contains a neighborhood of the origin in X. Since V_0 was an arbitrary absolutely convex neighborhood in Y, this proves that the operator T is continuous.

(d) $\Rightarrow$ (e). Let Y be a Banach space and let $\mathcal{F}$ be pointwise bounded family of continuous linear operators T defined in X with taking values in Y. Let $\mathcal{B}(\mathcal{F}, Y)$ be the vector space of all functions $f : \mathcal{F} \to Y$ with the property that its norm $\|f\|$, defined by $\|f\|_{\mathcal{B}} = \sup_{T\in\mathcal{F}} \|f(T)\|$, is finite. Supplied with this norm the vector space $\mathcal{B}(\mathcal{F}, Y)$ becomes a Banach space. Define the operator $A : X \to \mathcal{B}(\mathcal{F}, Y)$ by $Ax(T) = Tx$, $x \in X$, $T \in \mathcal{F}$. The operator A is a closed linear operator from X to $\mathcal{B}(\mathcal{F}, Y)$. Its domain is all of X, because $\mathcal{F}$ is pointwise bounded. Assertion (d) implies that the operator A is continuous. This means that there exists a neighborhood of the

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origin U such that $x \in U$ implies $\|Ax\|_{\mathcal{B}} \leqslant 1$. So $x \in U$ together with $T \in \mathcal{F}$ implies $\|Tx\| \leqslant 1$. This proves the implication (d) $\Rightarrow$ (e).

(e) $\Rightarrow$ (a). Let W be a barrel in X and construct the normed linear space $X_{\mathcal{U}+W}$ as in the proof of the implication (b) $\Rightarrow$ (c). Also construct the family of continuous linear operators $\{T_U : U \in \mathcal{U}_X(0)\}$ as in the proof above. This family is pointwise bounded and so by (e) it follows that it is equicontinuous. As in the proof of the implication (b) $\Rightarrow$ (c) it follows that W is a neighborhood of the origin in X. This proves assertion (a).

This completes the proof of Theorem 8.19. □

8.20. Definition. A locally convex vector space with the property that every barrel in it is a neighborhood of the origin is called a *barreled* space.

From the previous theorem it follows that in a barreled space the closed graph theorem holds and also that the Banach-Steinhaus theorem is valid. Next we are going to prove that Fréchet spaces are barreled. The result will be based on the following proposition.

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8.21. Proposition. *Let X be a Fréchet space and let $\mathcal{F}$ be a pointwise bounded family of continuous linear operators from X to a topological vector space Y. Let $(x_n : n \in \mathbb{N})$ be sequence in X that converges to 0. Then the collection*

$$\{Tx_n : T \in \mathcal{F},\ n \in \mathbb{N}\}$$

is a bounded subset of Y.

Proof. Assume, to arrive at a contradiction, that the set

$$\{Tx_n : T \in \mathcal{F},\ n \in \mathbb{N}\}$$

is not bounded in Y. Then there exists a balanced neighborhood V of zero in Y such that for every $k \in \mathbb{N}$ there exists $T \in \mathcal{F}$ together with $n_k \in \mathbb{N}$ with the property that

$$Tx_{n_k} \notin kV. \tag{8.3}$$

Put $J = \{n_k : k \in \mathbb{N}\}$. Since, for every $n \in \mathbb{N}$, the set $\{Tx_n : T \in \mathcal{F}\}$ is bounded, the set J is an infinite subset of $\mathbb{N}$. Since X is a Fréchet space, there exists an infinite subset J' of J with the property that the sum $\sum_{j\in J''} x_j = \lim_{n\to\infty} \sum_{j\in J''\cap[1,n]} x_j$ converges for every subset J'' of J'. This can be achieved in the following fashion. The sequence $\{n_k : k \in \mathbb{N}\}$ is infinite and so the sequence $(x_{n_k} : k \in \mathbb{N})$ contains a subsequence that converges to 0. This is so because the original sequence $(x_n : n \in \mathbb{N})$ converges to 0. Since X is a Fréchet space there exists a countable local basis $(U_k : k \in \mathbb{N})$ with the property that $U_{k+1} + U_{k+1} \subseteq U_k$. Choose a further subsequence $\left(x_{n_{k_j}} : j \in \mathbb{N}\right)$ with the property that $x_{n_{k_j}}$ belongs to U_j. The set $J' = \left\{n_{k_j} : j \in \mathbb{N}\right\}$ possesses the required property. Since the set V is balanced, it follows from (8.3) that for each $t \in (0,\infty)$ there exists $T \in \mathcal{F}$ and $n \in J'$ with the property that

$$Tx_n \notin tV. \tag{8.4}$$

From these observations we shall derive a contradiction. First choose a balanced open zero-neighborhood V_0 in Y with the property that

$$\overline{V_0} + V_0 + V_0 \subseteq V. \tag{8.5}$$

Let $\widetilde{x}_0$ be the zero-vector in X, put $m_0 = 0$, put $\epsilon_0 = 1$ and let $T_0 : X \to Y$ be the zero-map. We shall construct a sequence of positive real numbers $(\epsilon_n : n \in \mathbb{N})$, with $0 < \epsilon_n \leqslant \dfrac{1}{n}$, $n \in \mathbb{N}$, a sequence of vectors $(\widetilde{x}_n : n \in \mathbb{N})$ in X, a strictly increasing sequence of indexes $(m_n : n \in \mathbb{N}) \subseteq J'$ together with a sequence $(T_n : n \in)$ in $\mathcal{F} \cup \{0\}$, such that for $n \geqslant 1$ the following conditions are verified:

$$\begin{aligned}
\epsilon_n T_n\left(\widetilde{x}_{n-1}\right) &\in V_0, && \text{(i)}\\
\epsilon_k T_k\left(\widetilde{x}_n - \widetilde{x}_k\right) &\in V_0, \quad 0 \leqslant k \leqslant n-1, && \text{(ii)}\\
\widetilde{x}_n - \widetilde{x}_{n-1} &= x_{m_n}, && \text{(iii)}\\
\epsilon_n T_n\left(\widetilde{x}_n - \widetilde{x}_{n-1}\right) &\notin V. && \text{(iv)}
\end{aligned}$$

First we consider the case $n = 1$. Since $\widetilde{x}_0 = 0$ and $T_0 = 0$, (i) and (ii) are always satisfied. By (8.4) there exists an operator $T_1 \in \mathcal{F}$ and there exists $m_1 \in J'$ such

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that $T_1(x_{m_1}) \notin V$. With $\widetilde{x}_1 = x_{m_1}$ and with $\epsilon_1 = 1$ we have

$$\epsilon_1 T_1(\widetilde{x}_1 - \widetilde{x}_0) = T_1(x_{m_1}) \notin V.$$

So the construction of ϵ_1, $\widetilde{x}_1$, m_1 and T_1 has been carried out. Next suppose that $(\epsilon_1, \dots, \epsilon_n)$, $(x_1, \dots, x_n)$, $(m_1, \dots, m_n)$ and $(T_1, \dots, T_n)$ have been chosen in such a way that (i), (ii), (iii) and (iv) are satisfied. Then we choose $0 < \epsilon_{n+1} \leqslant \dfrac{1}{n+1}$ in such a way that

$$\epsilon_{n+1} T(\widetilde{x}_n) \in V_0, \quad T \in \mathcal{F}. \tag{8.6}$$

Since the family $\mathcal{F}$ is pointwise bounded such a choice of ϵ_{n+1} is possible. From (ii) it follows that the set

$$\bigcap_{k=1}^{n} (V_0 - \epsilon_k T_k(\widetilde{x}_n - \widetilde{x}_k))$$

is a zero-neighborhood in Y. Since $\lim_{j\to\infty, j\in J'} x_j = 0$ and since each operator T_k, $1 \leqslant k \leqslant n$, is continuous, it follows that there $m'_n > m_n$, $m'_n \in \mathbb{N}$, with the property that

$$\epsilon_k T_k(x_m) \in V_0 - \epsilon_k T_k(\widetilde{x}_n - \widetilde{x}_k), \quad m \geqslant m'_n, \quad m \in J', \quad 1 \leqslant k \leqslant n. \tag{8.7}$$

By (8.4) there exists a number $m_{n+1} \geqslant m'_n$, $m_{n+1} \in J'$, and $T_{n+1} \in \mathcal{F}$ such that

$$\epsilon_{n+1} T_{n+1}\left(x_{m_{n+1}}\right) \notin V. \tag{8.8}$$

For assume that $\epsilon_{n+1} T_{n+1}(x_m)$ belongs to V for all $m \geqslant m'_n$, $m \in J'$, and let $t_m \in (0, \infty)$, $1 \leqslant m \leqslant m'_n - 1$, $m \in J'$, be such that Tx_m belongs to $t_m V$ for every $T \in \mathcal{F}$. With

$$t = \max\left(\frac{1}{\epsilon_{n+1}}, \max\left\{t_m : 1 \leqslant m \leqslant m'_{n-1},\ m \in J'\right\}\right),$$

it follows that the set $\{Tx_m : T \in \mathcal{F},\ m \in J'\}$ is a subset of tV. This contradicts (8.4). Finally put

$$\widetilde{x}_{n+1} = x_{m_{n+1}} + \widetilde{x}_n. \tag{8.9}$$

Since $m_{n+1} \geqslant m'_n$ it follows from (8.7) that

$$\epsilon_k T_k(\widetilde{x}_{n+1} - \widetilde{x}_n) \text{ belongs to } V_0 - \epsilon_k T_k(\widetilde{x}_n - \widetilde{x}_k), \text{ for } 1 \leqslant k \leqslant n.$$

So

$$\epsilon_k T_k(\widetilde{x}_{n+1} - \widetilde{x}_k) \text{ belongs to } V_0, \text{ for } 1 \leqslant k \leqslant n. \tag{8.10}$$

From (8.6), (8.9), (8.8) and (8.10) it follows that the $(n+1)$-tuples $(\epsilon_1, \dots, \epsilon_{n+1})$, $(x_1, \dots, x_{n+1})$, $(m_1, \dots, m_{n+1})$ and $(T_1, \dots, T_{n+1})$ satisfy (i), (ii), (iii) and (iv) with n replaced with $n+1$. From (iii) we see that $\widetilde{x}_n = \sum_{k=1}^{n} x_{m_k}$. Since $(m_k : k \in \mathbb{N})$ is a subset of J' we conclude that the vector $\widetilde{x} := \lim\limits_{n\to\infty} \widetilde{x}_n$ exists, because the space X is complete. From (ii) together with the continuity of each operator T_k, $k \in \mathbb{N}$, it follows that

$$\epsilon_k T_k(\widetilde{x} - \widetilde{x}_k) \text{ belongs to } \overline{V_0}, \quad k \in \mathbb{N}. \tag{8.11}$$

Since $V_0 = -V_0$ and since

$$\epsilon_n T_n(\widetilde{x}_n - \widetilde{x}_{n-1}) = \epsilon_n T_n(\widetilde{x}) - \epsilon_n T_n(\widetilde{x} - \widetilde{x}_n) - \epsilon_n T_n(\widetilde{x}_{n-1}), \quad n \in \mathbb{N},$$

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we conclude from (i) and (8.11) that

$$\epsilon_n T_n\left(\widetilde{x}_n - \widetilde{x}_{n-1}\right) \quad \text{belongs to} \quad \epsilon_n T_n\left(\widetilde{x}\right) + \overline{V_0} + V_0, \quad n \in \mathbb{N}. \tag{8.12}$$

Since the sequence $(T_n(\widetilde{x}) : n \in \mathbb{N})$ is bounded and since $\lim_{n\to\infty} \epsilon_n = 0$, it follows from (8.12) that the vector $\epsilon_n T_n(\widetilde{x}_n - \widetilde{x}_{n-1})$ belongs to $V_0 + \overline{V_0} + V_0$ for n sufficiently large. Hence, by (8.5),

$$\epsilon_n T_n\left(\widetilde{x}_n - \widetilde{x}_{n-1}\right) \quad \text{belongs to} \quad V \tag{8.13}$$

for n sufficiently large. However (8.13) contradicts (iv). Consequently the assumption that the set $\{Tx_n : T \in \mathcal{F},\ n \in \mathbb{N}\}$ be not bounded is false. This proves Proposition 8.21. □

The following corollary shows that a Fréchet space is barreled. Consequently, the Banach-Steinhaus theorem holds in Fréchet spaces.

8.22. Corollary. *A Fréchet space is barreled.*

Proof. Let X be Fréchet space and let $(U_n : n \in \mathbb{N})$ be a local basis of zero-neighborhoods. Let $\mathcal{F}$ be a pointwise bounded family of continuous linear operators defined on X and attaining values in a locally convex space Y. We have to prove that the family $\mathcal{F}$ is equicontinuous. Suppose not. Then there exists a balanced convex neighborhood V of the origin in Y with the property that for no $n \in \mathbb{N}$ the inclusion $TU_n \subseteq V$ is valid for all $T \in \mathcal{F}$. So for every $n \in \mathbb{N}$ there exists an operator $T_n \in \mathcal{F}$ and a vector $x_n \in U_n$ such that the vector $T_n x_n$ does not belong to nV. Then $\lim_{n\to\infty} x_n = 0$ and so by the previous proposition the set $\{Tx_n : T \in \mathcal{F},\ n \in \mathbb{N}\}$ is bounded in Y. Hence there exists $t > 0$ such that the following inclusion is valid:

$$\{Tx_n : T \in \mathcal{F},\ n \in \mathbb{N}\} \subseteq tV.$$

Since the set V is balanced we have, for $n > t$, $T_n x_n \in tV \subset nV$. But on the other hand $T_n x_n$ does not belong to nV. This is a contradiction. So our assumption that the family $\mathcal{F}$ is not equicontinuous is false. Consequently a pointwise bounded family of continuous linear operators defined on a Fréchet space and attaining values in a locally convex space is equi-continuous. From the main theorem, Theorem 8.19, it then follows that a Fréchet space is barreled. □

8.23. Example. Next we will give an example of a locally convex topological vector space which is barreled, but which is not a Fréchet space. Let Ω be an open subset of $\mathbb{R}^n$, and let $\mathcal{D}(\Omega)$ be the space of all C^∞-functions whose support is a compact subset of Ω. Let $K \subset \Omega$ be a compact. As in the Chapters 1 and 4 we let $\mathcal{D}_K$ be space of all C^∞-functions in Ω whose support is contained in K. Define the semi-norms $p_{m,K} : \mathcal{D}_K \to [0,\infty)$, $m \in \mathbb{N}$, by

$$p_{m,K}(\varphi) = \max_{\alpha\in\mathbb{N}^n,\, |\alpha|\leqslant m} \sup_{x\in K} \left|D^\alpha \varphi(x)\right|, \quad \varphi \in \mathcal{D}_K.$$

The metric

$$d(\varphi,\psi) = \sum_{m=0}^{\infty} \frac{1}{2^{m+1}} \frac{p_{m,K}(\varphi-\psi)}{1 + p_{m,K}(\varphi-\psi)}, \quad \varphi,\ \psi \in \mathcal{D}_K,$$

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turns $\mathcal{D}_K$ into a complete metric space. Let $\mathcal{T}_K$ be the corresponding compatible locally convex topology. That is to say the topological space $(\mathcal{D}_K, \mathcal{T}_K)$ is a complete metrizable locally convex Hausdorff space. In other words it is a Frěchet space. A subset W of $\mathcal{D}(\Omega)$ is called $\mathcal{T}_\Omega$-open if $W \cap \mathcal{D}_K$ is $\mathcal{T}_K$-open for all compact subsets K of Ω. This topology $\mathcal{T}_\Omega$ renders $\mathcal{D}(\Omega)$ into a locally convex topological space in which a sequence $(\varphi_k)_{k\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ converges to $\varphi \in \mathcal{D}(\Omega)$ if and only this sequence is contained in $\mathcal{D}_K$ for some compact subset K of Ω, and converges in $\mathcal{D}_K$ to φ. Then the space $(\mathcal{D}(\Omega), \mathcal{T}_\Omega)$ is barreled. However, it is not complete metrizable. In fact the topology $\mathcal{T}_\Omega$ is the strongest locally convex topology $\mathcal{T}$ on $\mathcal{D}(\Omega)$ with the property that all inclusions $(\mathcal{D}_K, \mathcal{T}_K) \hookrightarrow (\mathcal{D}(\Omega), \mathcal{T})$, $K \subset \Omega$, K compact, are continuous. Instead of taking all compact subsets K of Ω, it suffices to take a sequence of compact subsets K_m, $m \in \mathbb{N}$, such that K_m is contained in the interior of K_{m+1}, and such that $\Omega = \cup_m K_m$. The corresponding topology is called the (strict) inductive limit of the family $(\mathcal{D}_K, \mathcal{T}_K)$. It is often denoted by

$$(\mathcal{D}(\Omega), \mathcal{T}_\Omega) = \varinjlim_{K\subset\Omega,\, K\text{compact}} (\mathcal{D}_K, \mathcal{T}_K) = \lim_{m\longrightarrow\infty} (\mathcal{D}_{K_m}, \mathcal{T}_{K_m}).$$

2.1. The open mapping theorem. The following version of the open mapping theorem is partly taken from Rudin [**113**].

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8.24. Theorem. *Suppose*

(a) X *is a complete metrizable topological vector space;*
(b) Y *is a topological vector space;*
(c) $u : X \to Y$ *is a continuous linear mapping;*
(d) $u(X)$ *is of the second category in* Y.

Then the following assertions hold:

(i) $u(X) = Y$;
(ii) $u(U)$ *is open for every open subset* U *of* X;
(iii) Y *is complete metrizable.*

The same conclusion holds provided that:

(a′) X *is a complete metrizable locally convex topological vector space (i.e. a Fréchet space);*
(b′) Y *is a barreled locally convex topological vector space;*
(c′) $u : X \to Y$ *is a continuous linear mapping;*
(d′) $u(X) = Y$.

Proof. (ii) $\Rightarrow$ (i). The open linear subspace $u(X)$ coincides with the whole space.

(ii) $\Rightarrow$ (iii). Put $\widetilde{X} = X/\mathrm{Ker}(u)$; *i.e.* $\widetilde{X}$ is the quotient space of X modulo the zero space of u: $\mathrm{Ker}(u) = \{x \in X : u(x) = 0\}$. The quotient space is also a complete metrizable space and the mapping $\widetilde{u} : \widetilde{X} \to Y$, defined by $\widetilde{u}\,(x + \mathrm{Ker}(u)) = u(x)$ is surjective and open. Consequently, it is a homeomorphism. If $y_n = u(x_n)$, $n \in \mathbb{N}$, is a Cauchy sequence in Y, then $x_n + \mathrm{Ker}(u)$, $n \in \mathbb{N}$, is a Cauchy sequence in $\widetilde{X}$. However, the quotient space $\widetilde{X}$ inherits its completeness from X. Thus $\lim_{n\to} x_n + \mathrm{Ker}(u)$ converges to $x + \mathrm{Ker}(u)$ for some $x \in X$. It follows that $u(y_n) = \widetilde{u}\,(x_n + \mathrm{Ker}(u))$ converges to $y = u(x)$. Consequently Y is complete. Let d be a translation invariant distance on X, then $d_Y(y_1, y_2) := \inf_{z\in\mathrm{Ker}(u)} d\,(x_1 + z, x_2)$, with $y_1 = u(x_1)$, $y_2 = u(x_2)$, defines a distance on Y. This distance is compatible with the topology.

(ii) Let V be a neighborhood of the origin in X. (If X is a Fréchet space we may and do assume that V is absolutely convex and closed.) We still have to prove that $u(V)$ contains a neighborhood of the origin in Y. Let d be an invariant metric on X, and choose $r > 0$ so small that V_0 defined by $V_0 = \{x \in X : d(x, 0) < r\}$ is contained in V. Put $V_n = \{x \in X : d(x, 0) < 2^{-n}r\}$. Since $u(X) = \bigcup_{k=1}^{\infty} ku\,(V_n)$, $n \in \mathbb{N}$, and since $u(X)$ is of the second category in Y it follows that the closure of $u\,(V_n)$ is a neighborhood of the origin. In case $u(X) = Y$ and Y is barreled the closure of $u\,(V_n)$ contains a barrel, because V_n contains the closure of an absolutely convex neighborhood of the origin in X. Consequently, $\overline{u\,(V_n)}$ is a neighborhood of the origin. We will show that there exists a neighborhood W of the origin in Y such

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that

$$W \subseteq \overline{u(V_1)} \subseteq u(V_0) \subseteq u(V). \tag{8.14}$$

First we have

$$\overline{u(V_1)} \supseteq \overline{u(V_2) - u(V_2)} \supseteq \overline{u(V_2)} - \overline{u(V_2)} \supseteq W,$$

where W is a neighborhood of the origin, because $\overline{u(V_2)}$ has non-empty interior. This proves the first inclusion in (8.14). In order to prove the second inclusion we pick $y_1 \in \overline{u(V_1)}$. Then $y_1 - \overline{u(V_2)}$ is a neighborhood of y_1. Consequently, it has non-empty intersection with $u(V_1)$, because y_1 belongs to the closure of $u(V_1)$. Hence, there exists $x_1 \in V_1$ such that $u(x_1) \in y_1 - \overline{u(V_2)}$. Put $y_2 = y_1 - u(x_1)$. Then $y_2 \in \overline{u(V_2)}$. By induction we find $y_n \in \overline{u(V_n)}$ and $x_n \in V_n$ such that $y_{n+1} := y_n - u(x_n) \in \overline{u(V_{n+1})}$. The latter is true because $y_n - \overline{u(V_{n+1})}$ is a neighborhood of y_n and y_n belongs to the closure of $u(V_n)$. Thus $y_n - \overline{u(V_{n+1})}$ has non-empty intersection with $u(V_n)$. Then the sequence of partial sums $\sum_{j=1}^n x_j$, $n \in \mathbb{N}$, is a Cauchy sequence in X. It converges to x say. Then, since $d(x_1 + \cdots + x_n, 0) \leqslant \sum_{j=1}^n d(x_j, 0) < \sum_{j=1}^n 2^{-j} r \leqslant r$, we see that $d(x,0) = \lim_{n\to\infty} d\left(\sum_{j=1}^n x_j, 0\right) < \sum_{j=1}^\infty 2^{-j} r = r$. It follows that x belongs to $V_0 \subseteq V$. Moreover,

$$\sum_{j=1}^n u(x_j) = \sum_{j=1}^n (y_j - y_{j+1}) = y_1 - y_{n+1}. \tag{8.15}$$

Since y_n belongs to the closure of $u(V_n)$, and since u is continuous and the sequence V_n, $n \in \mathbb{N}$, is a basis of neighborhoods of the origin in X, we see that y_{n+1} converges to 0 in Y. (Let U be a closed neighborhood of the origin in Y. Choose n so large that $u(V_n) \subseteq U$. Since U is closed, the closure of $u(V_n)$ is also contained in U.) It follows that

$$y_1 = \lim_{n\to\infty} (y_1 - y_{n+1}) = \lim_{n\to\infty} \sum_{j=1}^n u(x_j) = u(x),$$

where x belongs to V_0.

So the proof of Theorem 8.24 is complete now. □

2.2. Krein-Smulian and the Eberlein-Smulian theorem. In this subsection we will discuss two interesting results in Banach space theory. Similar results also exist for Frecehet spaces and even for locally convex spaces, *e.g.*, see Schaefer. We next go over the proofs of two fundamental results in Banach space theory, elucidating the weak*-topology and the weak topology, respectively. We follow Section 1.2, Some facts from functional analysis, in [**1**]. The text of Aaserub in turn is based on parts from Conway's book [**27**], and on Robert Whitley's paper [**153**]. We will also quote some results from [**118**]. We begin with the result of Krein-Smulian, the proof of which requires the use of the following two lemmas. We will use the notation $B^\circ = \{x^* \in X^* : |\langle b, x^*\rangle| \leqslant 1 \text{ for all } b \in B\}$, for $B \subset X$, and $X_s = \{x \in X : \|x\| \leqslant s\} = sX_1$, for $s > 0$, with $(X^*)_s$ defined similarly. Here B° is called the polar of B.

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8.25. Lemma. *Let X be a Banach space, $r > 0$ a real number. Let $\mathfrak{F}_r$ be the collection of all finite subsets of $X_{r^{-1}}$. Then $\bigcap_{F \in \mathfrak{F}_r} F^\circ = (X^*)_r$.*

Proof. Let E denote the intersection on the left-hand side. Clearly, $(X^*)_r \subset E$. If $x^* \in E \setminus (X^*)_r$, then $|\langle x, x^* \rangle| > r$ for some $x \in X$ with $\|x\| = 1$ so that $|\langle (x/r), x^* \rangle| > 1$ with $\{x/r\} \in \mathfrak{F}_r$, contradicting the fact that $f \in \{x/r\}^\circ$. This completes the proof of Lemma 8.25. □

8.26. Lemma. *Let X be a Banach space and $A \subset X^*$ a convex set such that $A \cap (X^*)_r$ is weak*-closed for every $r > 0$. If $A \cap (X^*)_1 = \varnothing$, then there exists some $x \in X$ such that $\Re \langle x, x^* \rangle \geqslant 1$ for all $x^* \in A$.*

Proof. We will construct, recursively, a sequence of finite sets $F_0, F_1, \ldots \subset X$ such that, for each $n \in \mathbb{N}$, we have (i) $F_n \subset X_{1/n}$ and (ii) $(X^*)_{n+1} \cap \bigcap_{k=0}^{n} F_k^\circ \cap A = \varnothing$. Put $F_0 = \{0\}$. Assuming that $F_0, \ldots, F_{n-1}$ have been selected such that (i) and (ii) are satisfied, we must find a finite set $F_n \subset X_{1/n}$ such that

$$(X^*)_{n+1} \cap \bigcap_{k=0}^{n} F_k^\circ \cap A = \varnothing.$$

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Put $Q = (X^*)_{n+1} \cap \bigcap_{k=0}^{n-1} F_k^\circ \cap A$. As Q is clearly weak*-closed and is contained in the weak*-compact set $(n+1)(X^*)_1$, Q is weak*-compact. Suppose now for contradiction that $Q \cap F^\circ \neq \varnothing$ for all finite sets $F \subset X_{1/n}$. Note that the collection $\{Q \cap F^\circ : F \in \mathfrak{F}_n\}$ consists of non-empty weak*-closed sets. We claim that it has the finite intersection property, i.e., that any finite subcollection has non-empty intersection. Indeed, if we take $Q \cap G_1^\circ, \ldots, Q \cap G_N^\circ$ in the collection, we get easily that

$$\cap_{k=1}^N Q \cap G_k^\circ = Q \cap \left(\cap_{k=1}^N G_k^\circ\right) = Q \cap \left(\cup_{k=1}^N G_k\right)^\circ \neq \varnothing,$$

by assumption. As Q is weak*-compact, it follows that $\varnothing \neq \bigcap_{F \in \mathfrak{F}_n} Q \cap F^\circ = Q \cap (X^*)_n$ by the previous lemma. This contradicts the assumption on $F_1, \ldots, F_{n-1}$. Thus we can take a finite set $F_n \subset X_{1/n}$ such that $Q \cap F_n^\circ = \varnothing$. Note that $\bigcup_{n=1}^\infty F_n$ is a countable set, which we enumerate as $\{x_n\}_{n=1}^\infty$. It is immediate from (i) that $x_n \to 0$ in norm when $n \to \infty$. Thus we may define a linear map $T : X^* \to c_0$ by $x^* \mapsto (\langle x_n, x^* \rangle)_{n=1}^\infty$, where c_0 is the Banach space of complex sequences that converge to 0, equipped with the supremum-norm.

Note next that $A \cap \bigcap_{n=1}^\infty F_n^\circ = \varnothing$, as otherwise we could pick X^* in this set and $N \in \mathbb{N}$ such that $N \geqslant \|x^*\|$ in which case x^* belongs to $A \cap (X^*)_N \cap \bigcap_{k=0}^{N-1} F_k^\circ$, contradicting (ii). Thus $\|T(x^*)\| = \sup_n |\langle x_n, x^* \rangle| > 1$ for all $x^* \in A$. It follows that the convex sets $T(A)$ and D, the open unit ball of c_0, are disjoint. Thus the Hahn-Banach separation theorem implies that there is some $f \in \ell^1 = (c_0)^*$ and $\alpha \in \mathbb{R}$ such that $\Re f(\varphi) < \alpha \leqslant \Re f(T(x^*))$ for all $x^* \in A$ and all $\varphi \in D$. Without loss of generality, we may assume that $\|f\|_1 = 1$. If $\varphi \in D$, then $|f(\varphi)| = \Re f(\omega\varphi) \leqslant \alpha$ for some $\omega \in \mathbb{C}$ of modulus 1. Thus $1 = \|f\|_1 \leqslant \alpha$. Hence, $1 \leqslant \alpha \leqslant \Re f(T(x^*)) = \Re \sum_{n=1}^\infty f(e_n) \langle x_n, x^* \rangle$ for all $x^* \in A$. It follows that $x = \sum_{n=1}^\infty f(e_n) x_n$ does the trick. Here e_n is the nth unit vector in ℓ^1And so the proof of Lemma 8.26 is complete now. □

We have now essentially proved the Krein-Smulian theorem in Banach spaces.

8.27. Theorem. *(Krein-Smulian) Let X be a Banach space and $A \subset X^*$ a convex set such that $A \cap (X^*)_r$ is weak*-closed for every $r > 0$. Then A is weak*-closed.*

Proof. An analogue of the theorem for norm-closure is trivially true (as every norm-convergent sequence is bounded). In particular, A is norm-closed. We will show that $X^* \backslash A \subset X^* \backslash B$, where B is the weak*-closure of A. Let $x_0^* \in X^* \backslash A$ be given. As A is norm-closed, we can find $r > 0$ such that $B_r(x_0^*) \cap A = \varnothing$. Thus $(1/r)(A - x_0^*) \cap (X^*)_1 = \varnothing$, because translations and dilations are bijections of X^*. By the previous lemma, there exists $x \in X$ such that $(1/r)\Re(x^* - x_0^*)(x) \geqslant 1$ for all $x^* \in A$. Thus x_0^* is not in the weak*-closure of A, which completes the proof of Theorem 8.27. □

We next prove the theorem of Eberlein-Smulian, concerning weak compactness. We will say that a set $C \subset (X^*)_1$ is total if $\cap_{f \in C} \ker(f) = \{0\}$.

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8.28. LEMMA. *Let X be a Banach space such that $(X^*)_1$ contains a countable total set $C = \{f_n\}_{n=1}^{\infty}$. Then the assignment*

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} \left|f_n(x-y)\right|,$$

for x, $y \in X$ defines a metric on X such that the weak topology on any weakly compact subset of X is generated by d.

PROOF. It is clear that d is a metric on X. Let K be weakly compact. Then, as each $f \in X^*$ is weakly continuous, $\sup_{x\in K} |f(x)| < \infty$ for each $f \in X^*$ by the compactness of K. Thus the Uniform Boundedness Principle implies that $C = \sup_{x\in K} \|x\| < \infty$, i.e., that K is norm-bounded. We claim that the identity map on K is continuous when the domain is equipped with the weak topology and the range is equipped with the topology generated by d. If this is true, the identity map is automatically a homeomorphism, which proves what we want. Let $x_\alpha \to x$ weakly in K and let $\varepsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that $\sum_{n=N+1}^{\infty} 2^{-n} < \varepsilon/(4C)$. Now for α so large that $|f_n(x_\alpha - x)| < \varepsilon/(2N)$ for $n = 1, \ldots, N$, we get that

$$d(x_\alpha, x) = \sum_{n=1}^{N} 2^{-n} \left|f_n(x_\alpha - x)\right| + \sum_{n=N+1}^{\infty} 2^{-n} \left|f_n(x_\alpha - x)\right| \leqslant \frac{\varepsilon}{2} + \sum_{n=N+1}^{\infty} 2^{-n} 2C < \varepsilon,$$

proving what we want. So the proof of lemma 8.28 is complete now. $\square$

8.29. LEMMA. *Let X be a separable Banach space. Then there exists a countable total set $C \subset (X^*)_1$.*

PROOF. Let $D = \{x_n\}_{n=1}^{\infty}$ be a countable dense subset of X. For each $n \in \mathbb{N}$, choose via Hahn-Banach extension $f_n \in X^*$ of unit norm such that $f_n(x_n) = \|x_n\|$. Put $C = \{f_n\}_{n=1}^{\infty}$. Let $x \in X$ be such that $f_n(x) = 0$ for all n. Choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x$ in norm. Then

$$\|x\| = \lim_{k\to\infty} \|x_{n_k}\| = \lim_{k\to\infty} \left|f_{n_k}(x_{n_k})\right| = \lim_{k\to\infty} \left|f_{n_k}(x)\right| = 0.$$

The proof of Lemma 8.29 is complete now. $\square$

The following theorem is a consequence of Theorem 8.27. The theorem of Krein-Smulian (see Theorem 6.4 Corollary in [**119**]), or Grothendieck (see Corollary 2 to Theorem 6.2 in [**119**]) plays a dominant role in the proof of Theorem 8.30. Let $(X, \|\cdot\|)$ be Banach space. By definition a sequence $(x_n^*)_{n\in\mathbb{N}} \subset X^*$ belongs to $c_0(\mathbb{N}, X^*)$ if $\lim_{n\to\infty} \langle x, x_n^*\rangle = 0$ for every $x \in X$.

8.30. THEOREM. *Let X be a separable Banach space, and let $f : X^* \to \mathbb{C}$ be a linear functional. Then the following assertions are equivalent:*

(a) *There exists $x \in X$ such that $f(x^*) = \langle x, x^*\rangle$ for all $x^* \in X^*$;*
(b) *For every sequence $(x_n^*)_{n\in\mathbb{N}} \in c_0(\mathbb{N}, X^*)$ the following inequalities hold:*

$$0 \leqslant \sup_{n\in\mathbb{N}} \Re f(x_n^*) < \infty.$$

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(c) *For every sequence* $(x_n^*)_{n\in\mathbb{N}} \in c_0(\mathbb{N}, X^*)$ *the following inequalities hold:*

$$0 \leqslant \limsup_{n\to\infty} \Re f(x_n^*) < \infty.$$

PROOF OF THEOREM 8.30. (a) $\Longrightarrow$ (b) A sequence in $c_0(\mathbb{N}, X^*)$ is bounded with respect to the norm in X^*; this is a consequence of e.g. the Banach-Steinhaus theorem. It is also a consequence of a Baire-category argument applied to the dual unit ball. Hence assertion (b) follows from (a).

(b) $\Longrightarrow$ (c) Let $(x_n^*)_{n\in\mathbb{N}}$ be any sequence in $c_0(\mathbb{N}, X^*)$. Then $(x_k^*)_{k\in\mathbb{N},k\geqslant n}$ is a sequence in $c_0(\mathbb{N}, X^*)$, and so, by (b), $0 \leqslant \sup_{k\geqslant n} \Re f(x_k^*) < \infty$, from which assertion (c) readily follows.

(c) $\Longrightarrow$ (a) In this implication we will employ the Krein-Smulian theorem, or Grothendieck's completeness result. So suppose that (c) holds, and let $(y_n^*)_{n\in\mathbb{N}}$ be any sequence in X^* which converges in weak*-sense to $y^* \in X^*$. By (c) we see $0 \leqslant \limsup_{n\to\infty} \Re f(y_n^* - y^*) < \infty$, and hence

$$\Re f(y^*) \leqslant \limsup_{n\to\infty} \Re f(y_n^*) < \infty. \tag{8.16}$$

From (8.16) it follows that for every $M \in \mathbb{N}$ and every $\alpha \in \mathbb{R}$ the subset

$$\{x^* \in X^* : \|x^*\| \leqslant M, \Re f(x^*) \leqslant \alpha\} \tag{8.17}$$

is sequentially weak*-closed. Since X is separable, and the set in (8.17) is equi-continuous, it follows that sets of the form (8.17) are weak*-closed, not just sequentially weak*-closed. From Krein-Smulian's theorem it follows that for every $\alpha \in \mathbb{R}$ the half-space $\{x^* \in X^* : \Re f(x^*) \leqslant \alpha\}$ is weak*-closed. It then follows that the real hyper-plane $\{x^* \in X^* : \Re f(x^*) = 0\}$ is weak*-closed. Consequently, since $f : X^* \to \mathbb{C}$ is complex linear, there exists a vector $x \in X$ such that $f(x^*) = \langle x, x^*\rangle$, $x^* \in X^*$.

We can also use Grothendieck's theorem. Then we proceed as follows. Instead of considering a set of the form (8.17) we look at the subset $H_{M,\alpha}$ defined by

$$H_{M,\alpha} = \{x^* \in X^* : \|x^*\| \leqslant M, f(x^*) = \alpha\}. \tag{8.18}$$

Then the set in (8.18) is sequentially weak*-closed. Let $(x_n^*)_{n\in\mathbb{N}}$ be a sequence in $H_{M,\alpha}$ which converges to $x^* \in X^*$ in weak*-sense. Then, by (c),

$$\Re f(x^*) \leqslant \limsup_{n\to\infty} \Re f(x_n^*) = \limsup_{n\to\infty} \Re\alpha = \Re\alpha. \tag{8.19}$$

Applying the same argument to the sequence $(-x_n^*)_{n\in\mathbb{N}}$ which converges in weak*-sense to $-x^*$ shows $f(-x^*) \leqslant -\alpha$. This in combination with (8.19) yields $\Re f(x^*) = \alpha$. The same argument can applied to the sequences $(ix_n^*)_{n\in\mathbb{N}}$ and to $(-ix_n^*)_{n\in\mathbb{N}}$. Consequently the subset $H_{M,\alpha}$ is sequentially weak*-closed. Since the space is separable and the set $H_{M,\alpha}$ is equi-continuous it follows that $H_{M,\alpha}$ is weak*-closed. Grothendieck's theorem then implies that the hyper-plane $\{x^* \in X^* : f(x^*) = \alpha\}$ is weak*-closed. Again it follows that there exists $x \in X$ such that $f(x^*) = \langle x, x^*\rangle$, $x^* \in X^*$.

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This completes the proof of Theorem 8.30. □

8.31. THEOREM. *(Eberlein-Smulian) Let A be a subset of a Banach space X. Then the following assertions are equivalent:*

(1) *A is relatively weakly compact;*
(2) *Any sequence in A has a weakly convergent subsequence;*
(3) *Any sequence in A has a weak cluster point.*

PROOF. (1) $\Rightarrow$ (2): Let $\{a_n\}_{n=1}^{\infty}$ be a sequence in A. Denote by X_0 the norm-closure of the span of the a_n. It is easy to see that X_0 is separable and that $A \cap X_0$ is a relatively weakly closed subset of X_0. Indeed, the weak topology on X_0 coincides with the restriction to X_0 of the weak topology on X. (Alternatively, one can note that X_0 is actually a weakly closed subspace of X. In either case, we apply the Hahn-Banach theorem.) By the preceding lemmas, the weak topology on $A \cap X_0$ is metrizable. Thus any sequence in $A \cap X_0$ has a weakly convergent subsequence. In particular, so does $\{a_n\}_{n=1}^{\infty}$.

(2) $\Rightarrow$ (3) This implication is trivial.

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(3) $\Rightarrow$ (1) Assume that A satisfies (3). We claim that A must be norm-bounded. Indeed, for each $x^* \in X^*$, the set $\{\langle x, x^* \rangle : x \in A\}$ is bounded in $\mathbb{C}$. This is a consequence of (3). As above, it follows from the uniform boundedness principle that A is norm-bounded. Denote by J the canonical embedding $X \to X^{**}$. It suffices to show that the weak*-closure B of $J(A)$, which is weak*-compact by the Banach-Alaoglu theorem and the previous paragraph, is contained in $J(X)$. Let now $x^{**} \in B$ be given. We will use compactness to construct a sequence $\{a_n\}_{n=1}^{\infty}$ in A such that, if $x \in X$ is any weak cluster point of this sequence, $x^{**} = J(x)$. This will then complete the proof.

We need the following remark. Suppose Y is a Banach space and F is a finite-dimensional subspace of Y^*. Then the unit sphere of F is compact in F equipped with the norm inherited from Y^*. Thus we can find a 1/4-net $y_1^*, \ldots, y_n^*$ in the unit sphere of F, i.e., a set such that for every $y^* \in F$ with $\|y^*\| = 1$ there is a $1 \leqslant j \leqslant n$ such that $\left\|y^* - y_j^*\right\| < 1/4$. We can choose $y_1, \ldots, y_n \in X$ of unit norm such that $\left|\left\langle y_j, y_j^* \right\rangle\right| \geqslant 3/4$ for $1 \leqslant j \leqslant n$. Then, for any $y^* \in F$, we obtain $\max\{|\langle y_j, y^* \rangle| : 1 \leqslant j \leqslant n\} \geqslant (1/2)\|y^*\|$ by the triangle inequality.

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We will now construct the promised sequence $\{a_k\}_{k=1}^{\infty}$ in A as well as a sequence $\{x_m^*\}_{m=1}^{\infty}$ in X^*. They will be constructed recursively such that, for some strictly increasing sequence $\{n(k)\}_{k=1}^{\infty}$ of integers,

(i) $|[x^{**} - Ja_k](x_m^*)| < 1/k$ for $m \leqslant n(k)$, and
(ii) $\max\{|y^{**}(x_m^*)| : n(k-1) < m \leqslant n(k)\} \geqslant (1/2)\|y^{**}\|$ for all $y^{**} \in \operatorname{span}\{x^{**}, x^{**} - Ja_1, \ldots, x^{**} - Ja_{k-1}\}$.

Fix $x_1^* \in X^*$ of norm 1 such that $|x^{**}(x_1^*)| \geqslant (1/2)\|x^{**}\|$ and put $n(1) = 1$. As $x^{**} \in B$ we can find $a_1 \in A$ such that $|[x^{**} - J(a_1)](x_1^*)| < 1$. Thus (i) and (ii) hold when $k = 1$. Assume now that we have chosen $a_1, \ldots, a_{k-1}, x_1^*, \ldots, x_{n(k-1)}^*$, and $n(0), n(1), \ldots, n(k-1)$ in such a way that (i) and (ii) hold (where $k \geqslant 2$). By the preceding remark, i.e., the previous paragraph, we can find $n(k)$ and $x_{n(k-1)+1}^*, \ldots, x_{n(k)}^*$ such that, for any $y^{**} \in \operatorname{span}\{x^{**}, x^{**} - J(a_1), \ldots, x^{**} - J(a_{k-1})\}$,

$$\max\{|y^{**}(x_m^*)| : n(k-1) < m \leqslant n(k)\} \geqslant (1/2)\|y^{**}\|.$$

Choose next, using the fact that $x^{**} \in B$, $a_k \in A$ such that $|[x^{**} - Ja_k](x_m^*)| < 1/k$ whenever $m \leqslant n(k)$. Let now x be a weak cluster point of $\{a_n\}_{n=1}^{\infty}$. By the Hahn-Banach theorem, x is contained in the norm-closed convex linear span of the sequence $\{a_n\}_{n\in\mathbb{N}}$. As J is an isometry (by the Hahn-Banach theorem), it follows that $x^{**} - J(x)$ belongs to the norm-closed linear span of the vectors $x^{**} - Ja_n$, $n = 1, 2, \ldots$. It follows from (ii) above that $\sup\{|y^{**}(x_m^*)| : m \in \mathbb{N}\} \geqslant (1/2)\|y^{**}\|$ for all $y^{**} \in \operatorname{span}\{x^{**}, x^{**} - Ja_1, x^{**} - Ja_2, \ldots, \}$. Hence the triangle inequality implies that

$$\sup\{|y^{**}(x_m^*)| : m \in \mathbb{N}\} \geqslant (1/2)\|y^{**}\|$$

for all $y^{**} \in \operatorname{span}\{x^{**}, x^{**} - Ja_1, x^{**} - Ja_2, \ldots\}$. In particular, we may take $y^{**} = x^{**} - Jx$. Fix m. Given $N > m$ there exists $n > n(N) \geqslant N > m$ such that $|x_m^*(a_n - x)| < 1/N$. It follows by (i) that

$$|[x^{**} - Jx](x_m^*)| \leqslant |[x^{**} - Ja_n](x_m^*)| + |x_m^*(a_n - x)| < 2/N.$$

Letting N tend to ∞, we get that $|[x^{**} - Jx](x_m^*)| = 0$ for all m, whence

$$\|x^{**} - Jx\| \leqslant 2\sup_{m\in\mathbb{N}}|[x^{**} - Jx](x_m^*)| = 0.$$

Altogether, this completes the proof of Theorem 8.31. □

The following theorem is known as Grothendieck's completeness theorem.

8.32. Theorem. *Let X be a locally convex vector space. The following assertions are equiavalent:*

(a) *The space X is complete;*
(b) *Every linear form on X^*, the topological dual of X, which is $\sigma(X^*, X)$-continuous on every equi-continuous subset of X^* is $\sigma(X^*, X)$-continuous on X^*.*
(c) *Every hyperplane H in X^* for which $H \cap A$ is $(\sigma(X^*, X))$-closed in A for every equi-continuous subset A of X^* is itself $(\sigma(X^*, X))$-closed.*

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Here the topology $\sigma(X^*, X)$ is the weakest locally convex topology on X^* which makes all functionals of the form $x^* \mapsto \langle x, x^* \rangle$, $x^* \in X^*$, where x varies over X, continuous. Of course, this topology is called the weak*-topology. The following theorem is a version of the Krein-Smulian theorem for metrizable locally convex spaces.

8.33. Theorem. *A metrizable locally convex space X is complete if and only a convex set $M \subset X^*$ is $\sigma(X^*, X)$-closed whenever $M \cap U^\circ$ is $\sigma(X^*, X)$-closed for every 0-neighborhood U in X.*

For the proofs of Theorems 8.32 and 8.33 the reader is referred to the literature, *e.g.*, Schaefer [**119**] or Schaefer and Wolff [**118**].

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Subjects for further research and presentations

The following topics may be of interest for a presentation and/or further research:

(1) Detailed treatment of the wave equation. A text can be found in Chapter 3. The wave equation is based on the operator $\dfrac{\partial^2}{\partial t^2} - \Delta$. For a connection with unitary semigroups see Subsection 2.5.

(2) Pseudo-differential operators of general order. For details see, *e.g.*, Trèves [**137**]. An operator of the form

$$P(x, D)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} P(x, \xi) u(y)\, dy\, d\xi,$$

where the function $P(x, \xi)$ is an appropriate function, is called a pseudo-differential operator. The integrand belongs to a certain symbol class. For instance, if $P(x, \xi)$ is an infinitely differentiable function on $\mathbb{R}^n \times \mathbb{R}^n$ with the property that

$$\left| D_\xi^\alpha D_x^\beta P(x, \xi) \right| \leqslant C_{\alpha,\beta} \, (1 + |\xi|)^{m-|\alpha|}$$

for all x, $\xi \in \mathbb{R}^n$, all multi-indices α, β. some constants $C_{\alpha,\beta}$ and some real number m, then P belongs to the symbol class $\mathcal{S}^m_{1,0}$ of Hörmander. The corresponding operator $P(x, D)$ is called a pseudo-differential operator of order m and belongs to the class $\Psi^m_{1,0}$.

(3) Certain pseudo-differential operators of order less than or equal to 2 can be put into correspondence with space-homogeneous or non-space-homogeneous Markov processes. A detailed exposition can be found in Jacob [**68, 69, 70**].

(4) Non-linear partial differential operators: the Hamilton-Jacobi-Bellmann equation, the Hamilton-Jacobi equation, the Euler-Lagrange equation, the Korteweg-Devries equation. A good reference for some of these topics is Evans [**49**]. Some of these equations are (closely) related to optimization problems: see *e.g.* [**13**].

(5) Viscosity solutions to partial differential equations. The standard reference for this subject is Crandall, Ishii, and Lions [**29**]. This topic can also be treated in the context of Backward Stochastic Differential Equations (BSDEs): see, *e.g.*, Pardoux [**95**].

(6) Stationary phase methods for Fourier integral operators. For this topic the reader is referred to Simon [**105**]. The books [**109**], [**108**], and [**107**] by the same authors are also quite interesting. An important related topic is the notion of wavefront set in connection with the singular support of a distribution. The text in [**130**] authored by Hansen, Hilgert, and Paravicini contains relevant material.

(7) General differential operators of elliptic type. An important role is played by Sobolev theory. Some of these operators generate analytic semigroups. The reader may consult Chazarain and Piriou [**25**], Folland [**51**], Hörmander

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[**63, 64, 65, 66**], Strichartz [**134**]. For connection with regularity properties and analytic semigroups the reader is referred to, *e.g.*, Prüss [**103**], Prüss and Simonett [**104**], or Lunardi [**86**].

(8) Elliptic differential operators of second order (and Markov processes); see, *e.g.*, Øksendael [**138**].

(9) Parabolic differential operators (of second order and Markov processes). An interesting article in this context is [**23**]. The abstract of this paper reads: "We present the main concepts of the theory of Markov processes: transition semigroups, Feller processes, infinitesimal generator, Kolmogorov's backward and forward equations, and Feller diffusion. We also give several classical examples including stochastic differential equations (SDEs) and backward stochastic differential equations (BSDEs) and describe the links between Markov processes and parabolic partial differential equations (PDEs). In particular, we state the Feynman-Kac formula for linear PDEs and BSDEs, and we give some examples of the correspondence between stochastic control problems and Hamilton-Jacobi-Bellman (HJB) equations and between optimal stopping problems and variational inequalities. Several examples of financial applications are given to illustrate each of these results, including European options, Asian options, and American put options."

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(10) Differential operators and boundary value problems. A recent book on this topic is Pinsky [**98**]. Several books on partial differential equations contain a chapter on boundary value problems, *e.g.*, see [**110**]. For a more applied version of this topic see, *e.g.*, [**6**]. A modern book with excellent critics is [**123**] written by Shaurer and Levy.

(11) Operator semigroups and differential operators in Banach space; see Chapter 6 in this book. Other texts can be found in [**33**], [**48**], [**139**].

(12) Solutions to stochastic differential equations and the corresponding second order differential equation (of parabolic type) satisfied by the one-dimensional distributions.

(13) Backward stochastic differential equations and their viscosity solutions; see, *e.g.* Pardoux [**95**].

(14) The equation of Rudin-Osher. A relevant book on this topic is [**93**] written by Jean-Michel Morel and Sergio Solimini. A related equation, the equation of Perona-Malik, is discussed in Otmar Scherzer [**120**]. From the description of this book we quote "The Handbook of Mathematical Methods in Imaging provides a comprehensive treatment of the mathematical techniques used in imaging science. The material is grouped into two central themes, namely, Inverse Problems (Algorithmic Reconstruction) and Signal and Image Processing. Each section within the themes covers applications (modeling), mathematics, numerical methods (using a case example) and open questions. Written by experts in the area, the presentation is mathematically rigorous. The entries are cross-referenced for easy navigation through connected topics. Available in both print and electronic forms, the handbook is enhanced by more than 150 illustrations and an extended bibliography. It will benefit students, scientists and researchers in applied mathematics. Engineers and computer scientists working in imaging will also find this handbook useful." Other related work is Grasmair and Lenzen [**58**]

(15) Heat equation on a Riemannian manifold. A relevant book in this context is [**59**]. For connections with stochastic differential equations on manifolds see, *e.g.*, Elworthy [**45, 46**].

(16) Interpolation theorems: Riesz-Thorin, Stein, Marcinkiewicz, and others. An interesting book is [**78**]. In the abstract, the author Mark Kim writes "This expository thesis contains a study of four interpolation theorems, the requisite background material, and a few applications. The materials introduced in the first three sections of Chapter 1 are used to motivate and prove the Riesz-Thorin interpolation theorem and its extension by Stein, both of which are presented in the fourth section. Chapter 2 revolves around Calderón's complex method of interpolation and the interpolation theorem of Fefferman and Stein, with the material in between providing the necessary examples and tools. The two theorems are then applied to a brief study of linear partial differential equations, Sobolev spaces, and Fourier integral operators, presented in the last section of the second chapter." A

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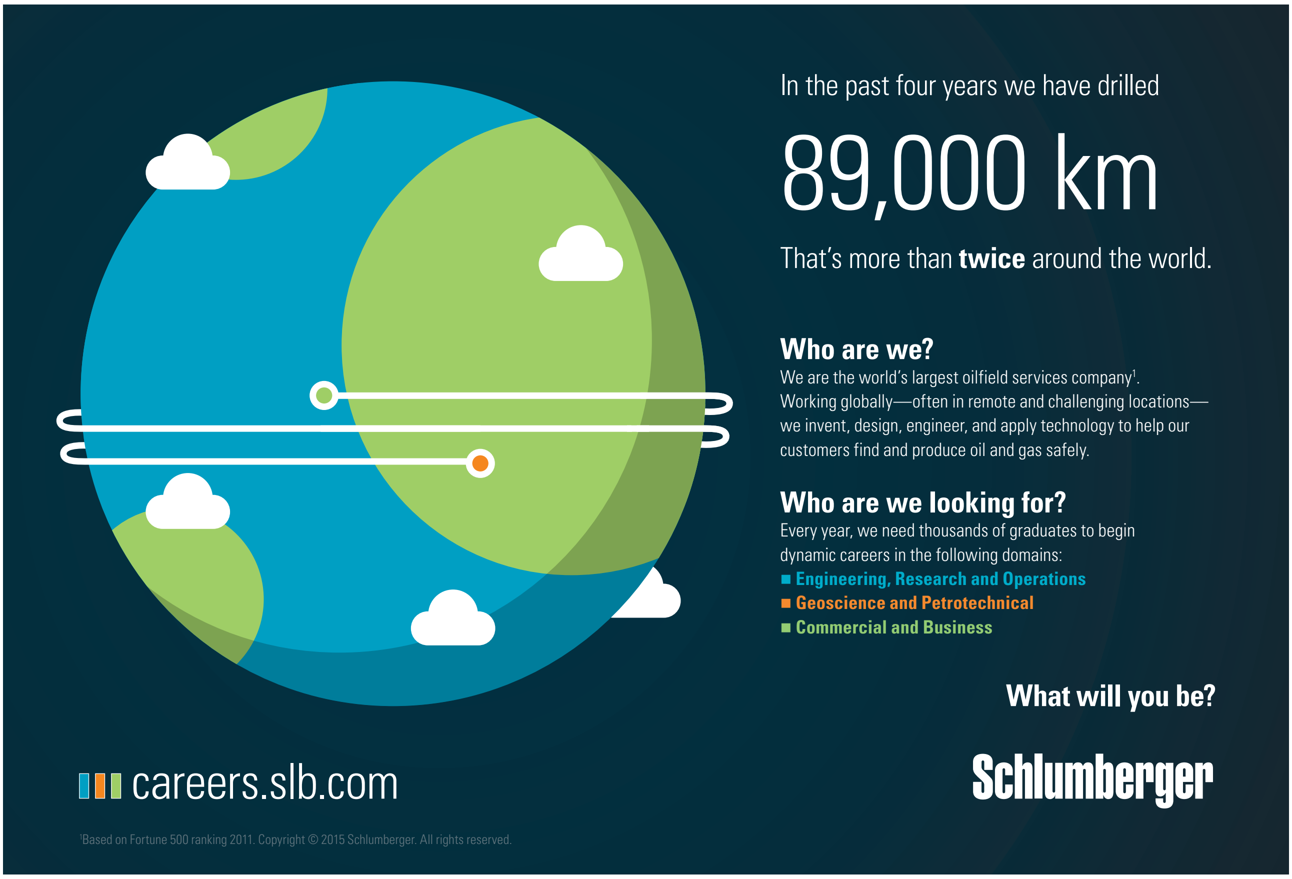
rather recent book on interpolation is [**88**]. For abstract interpolation results see *e.g.*, Voigt [**148, 149**]. Another text containing material about interpolation is Lunardi [**87**].

(17) Oscillatory integrals and related path integrals. There is a lot of literature on this subject. Nice papers on this topic are [**2, 3**]. Interesting books are, *e.g.*, Mazzucchi [**91**], Johnson and Lapidus [**72**], and Kleinert [**79**].

(18) Eigenvalue problems and spectral theory. A possible reference for this topic is Gilbarg and Trudinger [**54**]. There are several other references for this kind of subject.



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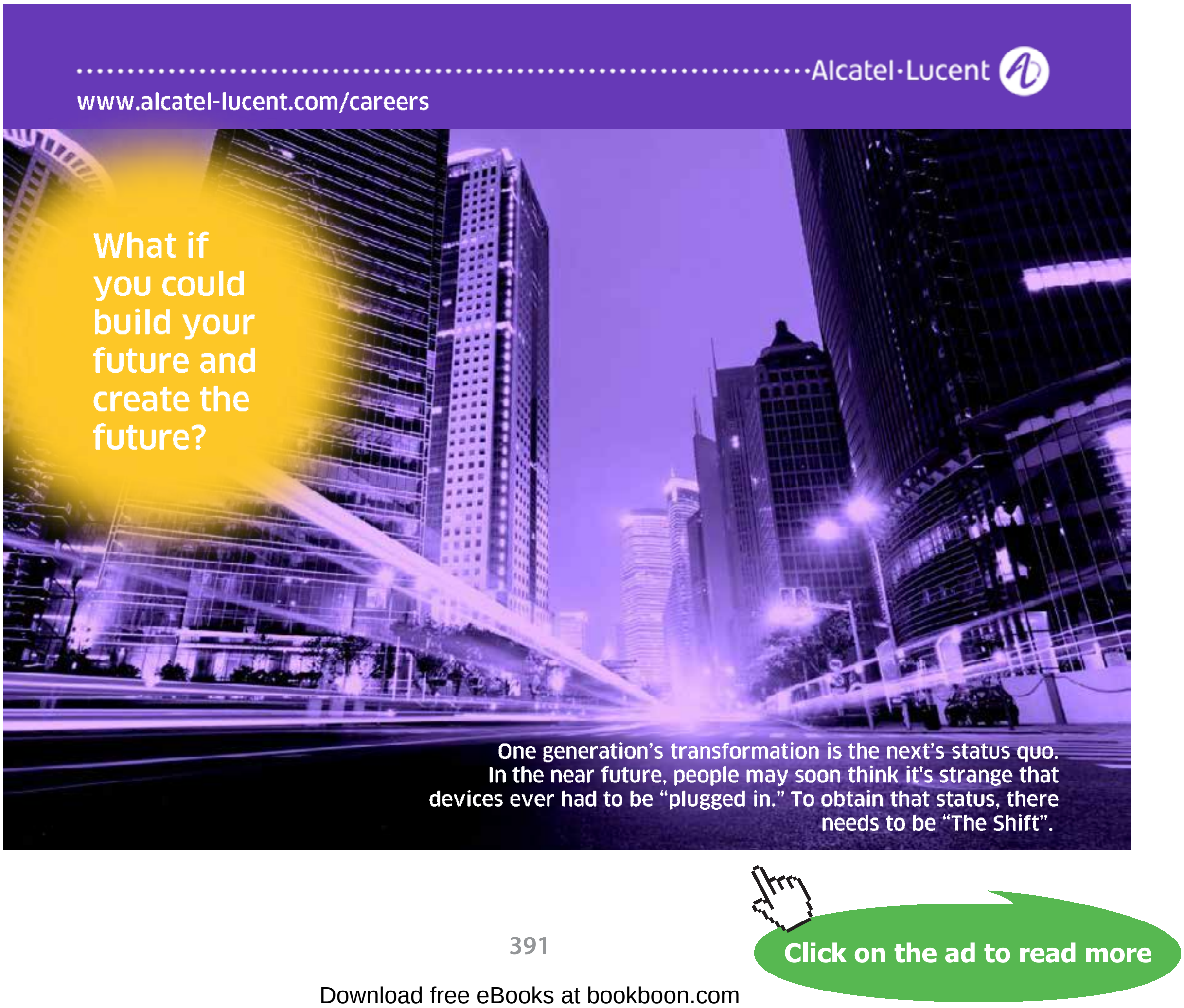

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