# Integral Operators

## Functional Analysis Examples c-5

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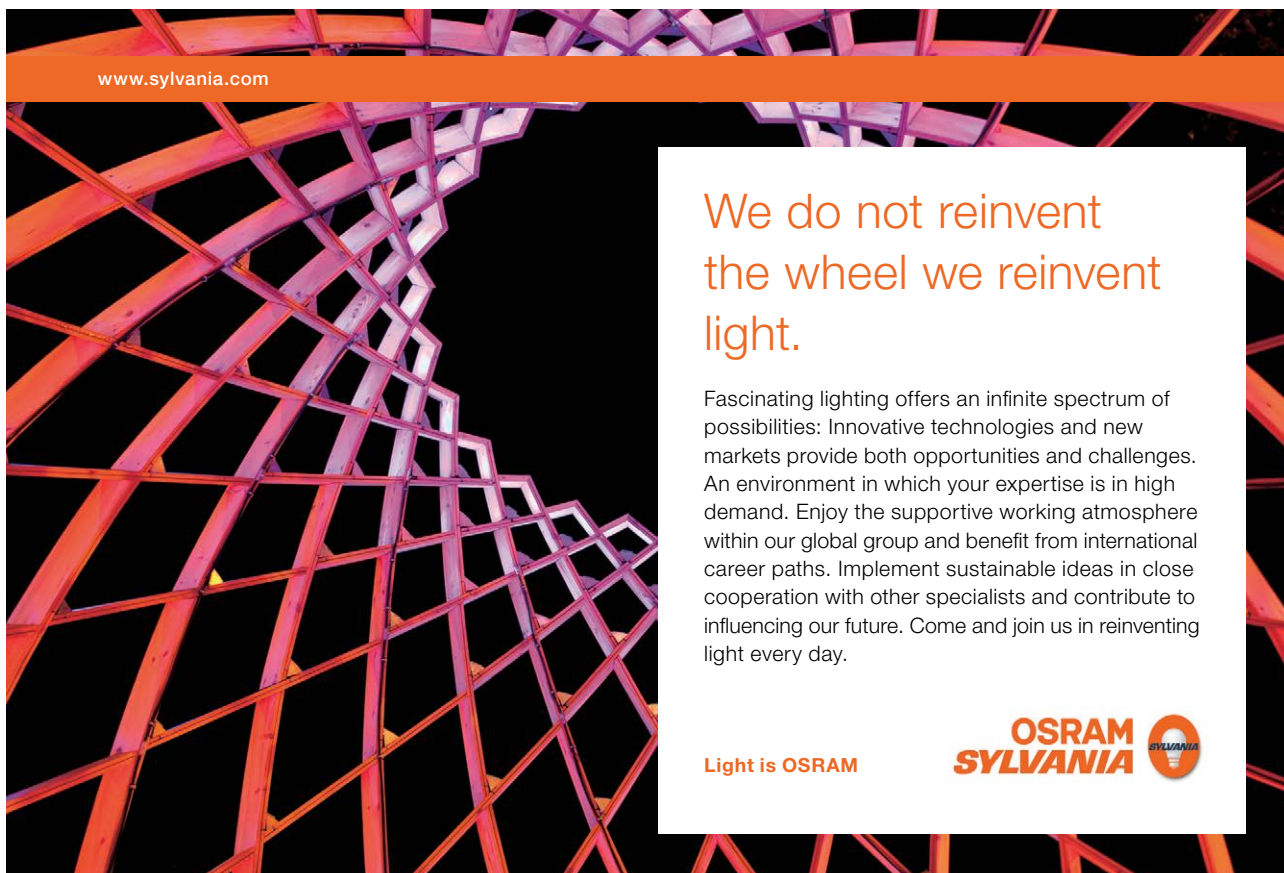
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# Contents

# 1  Hilbert-Schmidt operators

**Example 1.1** *Let $(e_k)$ denote an orthonormal basis in a Hilbert space $H$, and assume that the operator $T$ has the matrix representation $(t_{jk})$ with respect to the basis $(e_k)$. Show that*

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|t_{jk}|^2 < \infty$$

*implies that $T$ is compact.*
*Let $(f_k)$ denote another orthonormal basis in $H$, and let*

$$s_{jk} = (Tf_j, f_k)$$

*so that $(s_{jk})$ is the matrix representation of $T$ with respect to the basis $(f_k)$.*
*Show that*

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|t_{jk}|^2 = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|s_{jk}|^2.$$

*An operator satisfying*

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|t_{jk}|^2 < \infty$$

*is called a* general Hilbert-Schmidt operator.

Write $t_{jk} = (Te_j, e_j)$. It follows from VENTUS, HILBERT SPACES, ETC., EXAMPLE 2.7 that

$$Tx = T\left(\sum_{j=1}^{+\infty} x_j e_j\right) = \sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty} x_j t_{jk} e_k.$$

Define the sequence $(T_n)$ of operators by

$$T_n x = T_n\left(\sum_{j=1}^{+\infty} x_j e_j\right) = \sum_{j=1}^{+\infty}\sum_{k=1}^{n} x_j t_{jk} e_k.$$

The range of $T_n$ is finite dimensional, so $T_n$ is compact. Then we conclude from

$$\|(T - T_n)\,x\|^2 = \left\|\sum_{j=1}^{+\infty}\sum_{n=1}^{+\infty} x_j t_{jk} e_k\right\|^2 = \sum_{k=n+1}^{+\infty}\left|\sum_{j=1}^{+\infty} x_j t_{jk}\right|^2,$$

where

$$\left|\sum_{j=1}^{+\infty} x_j t_{jk}\right|^2 \leq \left\{\sum_{j=1}^{+\infty}|x_j|^2\right\}\cdot\left\{\sum_{j=1}^{+\infty}|t_{jk}|^2\right\},$$

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that

$$\|(T - T_n) x\|^2 \leq \left\{ \sum_{k=n+1}^{+\infty} \sum_{j=1}^{+\infty} |t_{jk}|^2 \right\} \cdot \|x\|^2.$$

It follows that

$$\|T - T_n\|^2 \leq \sum_{k=n+1}^{+\infty} \sum_{j=1}^{+\infty} |t_{jk}|^2.$$

Putting

$$a_k = \sum_{j=1}^{+\infty} |t_{jk}|^2 \geq 0,$$

it follows from the assumption that

$$\sum_{k=1}^{+\infty} a_k = \sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} |t_{jk}|^2 < +\infty.$$

Hence, to every $\varepsilon > 0$ there is an $n \in \mathbb{N}$, such that

$$\sum_{k=n+1}^{+\infty} a_k < \varepsilon^2,$$

from which

$$\|T - T_n\|^2 \leq \sum_{k=n+1}^{+\infty} \sum_{j=1}^{+\infty} |t_{jk}|^2 = \sum_{k=n+1}^{+\infty} a_k < \varepsilon^2,$$

thus $\|T - T_n\| < \varepsilon$, and we have proved that $T_n \to T$. Because all the $T_n$ are compact, we conclude that $T$ is also compact.

Given another orthonormal basis $(f_k)$ of $H$, and let $s_{jk} = (Tf_j, f_k)$. Then an application of Parseval's equation gives that

$$\sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} |(Te_k, f_j)|^2 = \sum_{k=1}^{+\infty} \|Te_j\|^2 = \sum_{k=1}^{+\infty} \sum_{j=1}^{+\infty} |(Te_k, e_j)|^2 = \sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} |t_{kj}|^2$$

and

$$\begin{aligned}
\sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} |(Te_k, f_j)|^2 &= \sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} |(e_k, T^\star f_j)|^2 = \sum_{j=1}^{+\infty} \|T^\star f_j\|^2 = \sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} |(T^\star f_j, f_k)|^2 \\
&= \sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} |(f_j, Tf_k)|^2 = \sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} |(Tf_j, f_k)|^2 = \sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} |s_{jk}|^2,
\end{aligned}$$

hence,

$$\sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} |t_{jk}|^2 = \sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} |t_{kj}|^2 = \sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} |s_{jk}|^2.$$

**Example 1.2** *For a general Hilbert-Schmidt operator we define the Hilbert-Schmidt norm $\|\cdot\|_{\mathrm{HS}}$ by*

$$\|T\|_{\mathrm{HS}} = \left\{ \sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} |t_{jk}|^2 \right\}^{\frac{1}{2}}.$$

*Show that this is a norm, and show that*

$$\|T\| \leq \|T\|_{\mathrm{HS}}$$

*for a general Hilbert-Schmidt operator $T$.*

Write $t_{jk} = (Te_j, e_k)$, and let

$$\|T\|_{\mathrm{HS}} = \left\{ \sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} |t_{jk}|^2 \right\}^{\frac{1}{2}}.$$

Then $\|T\|_{\mathrm{HS}} \geq 0$, and if $\|T\|_{\mathrm{HS}} = 0$, then $t_{jk} = (Te_j, e_k) = 0$ for all $j$, $k \in \mathbb{N}$, thus

$$Te_j = \sum_{k=1}^{+\infty} (Te_j, e_k)\, e_k = \sum_{k=1}^{+\infty} t_{jk} e_k = 0 \qquad \text{for every } j \in \mathbb{N}.$$

It follows that $T = 0$ as required.

We infer from $(\alpha\, T e_j, e_k) = \alpha\, (T e_j, e_k) = \alpha\, t_{jk}$ that

$$\|\alpha\, T\|_{\mathrm{HS}} = \left\{ |\alpha|^2 \sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} |t_{jk}|^2 \right\}^{\frac{1}{2}} = |\alpha| \cdot \|T\|_{\mathrm{HS}}.$$

Finally, if $\mathbf{S} = (s_{jk})$ and $\mathbf{T} = (t_{jk})$, then

$$
\begin{aligned}
\|S + T\|_{\mathrm{HS}}^2 &= \sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} |+ s_{jk} t_{jk}|^2 \leq \sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} \left\{ |s_{jk}|^2 + 2\, |s_{jk}| \cdot |t_{jk}| + |t_{jk}|^2 \right\} \\
&= \|S\|_{\mathrm{HS}}^2 + \|T\|_{\mathrm{HS}}^2 + 2 \sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} |s_{jk}| \cdot |t_{jk}| \\
&\leq \|S\|_{\mathrm{HS}}^2 + \|T\|_{\mathrm{HS}}^2 + 2 \left\{ \sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} |s_{jk}|^2 \right\}^{\frac{1}{2}} \cdot \left\{ \sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} |t_{jk}|^2 \right\}^{\frac{1}{2}} \\
&= \|S\|_{\mathrm{HS}}^2 + \|T\|_{\mathrm{HS}}^2 + 2\, \|S\|_{\mathrm{HS}} \cdot \|T\|_{\mathrm{HS}} = \left\{ \|S\|_{\mathrm{HS}} + \|T\|_{\mathrm{HS}} \right\}^2 ,
\end{aligned}
$$

and we have proved the triangle inequality,

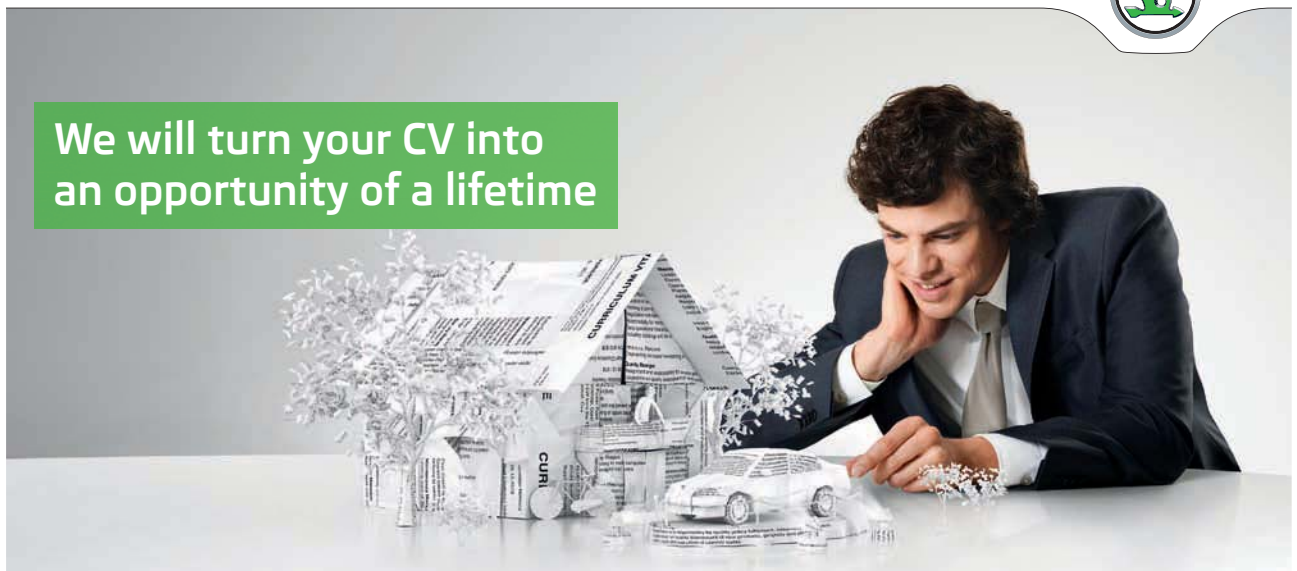$$\|S + T\|_{\mathrm{HS}} \leq \|S\|_{\mathrm{HS}} + \|T\|_{\mathrm{HS}}.$$

We have proved that $\|\cdot\|_{\mathrm{HS}}$ is a norm.

Finally,

$$
\begin{aligned}
\|Tx\|^2 &= \left\| \sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} x_j t_{jk} e_k \right\|^2 = \sum_{k=1}^{+\infty} \left| \sum_{j=1}^{+\infty} x_j t_{jk} \right|^2 \le \sum_{k=1}^{+\infty} \sum_{j=1}^{+\infty} \sum_{\ell=1}^{+\infty} |x_j| \cdot |t_{jk}| \cdot |x_\ell| \cdot |t_{\ell k}| \\
&= \sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} \sum_{\ell=1}^{+\infty} \{ |x_j| \cdot |t_{\ell k}| \} \cdot \{ |x_\ell| \cdot |t_{jk}| \} \\
&\le \left\{ \sum_{j,k,\ell=1}^{+\infty} |x_j|^2 |t_{\ell j}|^2 \right\}^{\frac{1}{2}} \cdot \left\{ \sum_{j,k,\ell=1}^{+\infty} |x_\ell|^2 |t_{jk}|^2 \right\}^{\frac{1}{2}} = \|T\|_{\mathrm{HS}}^2 \cdot \|x\|^2,
\end{aligned}
$$

hence $\|Tx\| \le \|T\|_{\mathrm{HS}} \cdot \|x\|$ for every $x$, and we find that $\|T\| \le \|T\|_{\mathrm{HS}}$.

**Example 1.3** *Define for $f \in L^2(\mathbb{R})$, the operator $K$ by*

$$
Kf(x) = \int_{-\infty}^{\infty} \frac{1}{2} \exp(-|x-t|) f(t) \, dt.
$$

*Show that $Kf \in L^2(\mathbb{R})$ and that $K$ is linear and bounded, with norm $\le 1$.*
*Show that the function $\frac{1}{2} \exp(-|x-t|)$ does not belong to $L^2(\mathbb{R}^2)$, so that $K$ is not a Hilbert-Schmidt operator.*

First we see that

$$
\begin{aligned}
Kf(x) &= \int_{-\infty}^{+\infty} \frac{1}{2} \exp(-|x-t|) f(t) \, dt = \int_{-\infty}^{x} \frac{1}{2} e^{-x} e^{t} f(t) \, dt + \int_{x}^{+\infty} \frac{1}{2} e^{x} e^{-t} f(t) \, dt \\
&= \frac{1}{2} e^{-x} \int_{-\infty}^{x} e^{t} f(t) \, dt + \frac{1}{2} e^{x} \int_{x}^{+\infty} e^{-t} f(t) \, dt.
\end{aligned}
$$

Then

$$
\begin{aligned}
|Kf(x)|^2 &= \left\{ \int_{-\infty}^{+\infty} \frac{1}{2} \exp(-|x-t|) f(t) \, dt \right\}^2 \\
&\le \int_{-\infty}^{+\infty} \frac{1}{2} \exp(-|x-t|) |f(t)| \, dt \cdot \int_{-\infty}^{+\infty} \frac{1}{2} \exp(-|x-u|) |f(u)| \, du \\
&= \frac{1}{4} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp -|x-t| \exp(-|x-u|) \cdot |f(t)| \cdot |f(u)| \, dt \, du \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{4} \exp(-|x-t| - |x-u|) \cdot |f(t)| \cdot |f(u)| \, dt \, du.
\end{aligned}
$$

Hence

$$
\int_{-\infty}^{+\infty} |Kf(x)|^2 dx \le \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} \frac{1}{4} \exp(-|x-t| - |x-u|) \, dx \right\} |f(t)| \cdot |f(u)| \, dt \, du.
$$

If $t \le u$, then

$$
|x-t| + |x-u| = \begin{cases} t - x + u - x = t + u - 2x, & \text{for } x \le t, \\ x - t + u - x = u - t, & \text{for } t \le x \le u, \\ x - t + x - u = 2x - t - u, & \text{for } x \ge u. \end{cases}
$$

This gives the inspiration to the following rearrangement

$$\int_{-\infty}^{+\infty} |Kf(x)|^2 dx \le 2 \int_{-\infty}^{+\infty} \left( \int_t^{+\infty} \left\{ \int_{-\infty}^{+\infty} \frac{1}{4} \exp(-|x-t|-|x-u|) \, dx \right\} |f(u)| du \right) |f(t)| dt,$$

where

$$
\begin{aligned}
\int_{-\infty}^{+\infty} e^{-|x-t|-|x-u|} \, dx &= \int_{-\infty}^t e^{2x-t-u} \, dx + \int_t^u e^{-u+t} \, dx + \int_u^{+\infty} e^{-2x+t+u} \, dx \\
&= \left[ \frac{1}{2} e^{2x-t-u} \right]_{x=-\infty}^t + (u-t)e^{-u+t} + \left[ -\frac{1}{2} e^{-2x+t+u} \right]_{x=u}^{+\infty} \\
&= \frac{1}{2} e^{t-u} + (u-t)e^{t-u} + \frac{1}{2} e^{t-u} = (u-t+1)e^{t-u},
\end{aligned}
$$

and where we have assumed that $t \le u$.

By insertion,

$$\int_{-\infty}^{+\infty} |Kf(x)|^2 \, dx \le \frac{1}{2} \int_{-\infty}^{+\infty} \left\{ \int_t^{+\infty} (u-t+1)e^{t-u} |f(u)| \, du \right\} |f(t)| \, dt.$$

Then we change variables $y = u - t$ and $z = t + u$, thus

$$t = \frac{y+z}{2} \qquad \text{og} \qquad u = \frac{y-z}{2},$$

where $y \in [0, +\infty[$ and $z \in \mathbb{R}$. We get

$$
\begin{aligned}
\int_{-\infty}^{+\infty} |Kf(x)|^2 \, dx &\le \frac{1}{4} \int_{-\infty}^{+\infty} \int_0^{+\infty} (y+1)e^{-y} \left| f\left( \frac{y-z}{2} \right) \right| \cdot \left| f\left( \frac{y+z}{2} \right) \right| \, dy \, dz \\
&= \frac{1}{4} \int_0^{+\infty} \left\{ \int_{-\infty}^{+\infty} \left| f\left( \frac{y-z}{2} \right) \right| \cdot \left| f\left( \frac{y+z}{2} \right) \right| \, dz \right\} (y+1)e^{-y} \, dy.
\end{aligned}
$$

Then for every fixed $y$ it follows by the Cauchy-Schwarz inequality,

$$
\int_{-\infty}^{+\infty} \left| f\left( \frac{y-z}{2} \right) \right| \cdot \left| f\left( \frac{y+z}{2} \right) \right| \, dz
$$

$$
\le \left\{ \int_{-\infty}^{+\infty} \left| f\left( \frac{y-z}{2} \right) \right|^2 \, dz \right\}^{\frac{1}{2}} \cdot \left\{ \int_{-\infty}^{+\infty} \left| f\left( \frac{y+z}{2} \right) \right|^2 \, dz \right\}^{\frac{1}{2}}
$$

$$
\left\{ 2 \int_{-\infty}^{+\infty} \left| f\left( \frac{y-z}{2} \right) \right|^2 \, d\left( \frac{y-z}{2} \right) \right\}^{\frac{1}{2}} \cdot \left\{ 2 \int_{-\infty}^{+\infty} \left| f\left( \frac{y-z}{2} \right) \right|^2 \, d\left( \frac{y+z}{2} \right) \right\}^{\frac{1}{2}}
$$

$$
= 2\|f\|_2 \cdot \|f\|_2 = 2\|f\|_2^2,
$$

and we get by insertion the estimate

$$
\begin{aligned}
\int_{-\infty}^{+\infty} |Kf(x)|^2 \, dx &\le \frac{1}{2} \int_0^{+\infty} (y+1)e^{-y} \, dy \cdot \|f\|_2^2 \\
&= \frac{1}{2} \left[ -e^{-y}(y+1) + \int e^{-y} \, dy \right]_0^{+\infty} \cdot \|f\|_2^2 \\
&= \frac{1}{2} \left[ -e^{-y}(y+2) \right]_0^{+\infty} \cdot \|f\|_2^2 = \|f\|_2^2,
\end{aligned}
$$

so we have proved that $Kf \in L^2(\mathbb{R})$ and that

$$\|Kf\|_2 \le \|f\|_2 \qquad \text{for every } f \in L^2(\mathbb{R}),$$

hence $\|K\| \le 1$.

On the other hand, the kernel $\dfrac{1}{2}\,e^{-|x-t|}$ does not belong to $L^2(\mathbb{R})$, because we get by a formal computation that

$$
\begin{aligned}
\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{4}\,e^{-2|x-t|}\,dx\,dt
&= \frac{1}{4}\int_{-\infty}^{+\infty}\left\{ 2\int_{t}^{+\infty} e^{-2(x-t)}\,dx\right\}dt \\
&= \frac{1}{4}\int_{-\infty}^{+\infty}\left\{\int_{0}^{+\infty} e^{-x}\,dx\right\}dt = \frac{1}{4}\int_{-\infty}^{+\infty} 1\,dt = +\infty.
\end{aligned}
$$

**Example 1.4** *Let $K$ denote the Hilbert-Schmidt operator with kernel*

$$k(x,y) = \sin(x)\,\cos(t), \qquad 0 \le x,\, t \le 2\pi.$$

*Show that the only eigenvalue for $K$ is 0.*
*Find an orthonormal basis for* $\ker(K)$.

First notice that

$$Kf(x) = \int_0^{2\pi} k(x,t)\,f(t)\,dt = \sin(x) \cdot \int_0^{2\pi} \cos(t) \cdot f(t)\,dt,$$

hence $Kf(x) = a(f) \cdot \sin(x)$, where

$$a(f) = \int_0^{2\pi} \cos(t) \cdot f(t)\,dt \in \mathbb{C}.$$

If $\lambda \in \sigma_p(K)$, then the corresponding eigenfunction must be $f(x) = \sin(x)$. Then by insertion,

$$(K\sin)(x) = \sin(x)\int_0^{2\pi} \cos(t) \cdot \sin(t)\,dt = 0,$$

proving that $\lambda = 0$ is the only eigenvalue.

Now,

$$\frac{1}{\sqrt{2\pi}},\ \frac{1}{\sqrt{\pi}}\,\cos(x),\ \frac{1}{\sqrt{\pi}}\,\sin(x),\ \dots,\ \frac{1}{\sqrt{\pi}}\,\cos(nx),\ \frac{1}{\sqrt{\pi}}\,\sin(nx),\ \dots,$$

is an ortonormalbasis for $L^2([0, 2\pi])$, so $\ker(K)$ is spanned by all these with the exception of $\dfrac{1}{\sqrt{\pi}}\,\cos(x)$, in which case

$$
\begin{aligned}
K\left(\frac{1}{\sqrt{\pi}}\,\cos\right)(x) &= \sqrt{\pi}\int_0^{2\pi} \frac{1}{\sqrt{\pi}}\,\cos(t) \cdot \frac{1}{\sqrt{\pi}}\,\cos(t)\,dt \cdot \sin(x) \\
&= \sqrt{\pi} \cdot \sin(x) = \pi \cdot \frac{1}{\sqrt{\pi}}\,\sin(x),
\end{aligned}
$$

and we get in particular, $K^2 \equiv 0$.

Note that

$$
\begin{aligned}
k_2(x,t) &= \int_0^{2\pi} k(x,s)k(s,t)\,ds = \int_0^{2\pi} \sin(x)\cdot\cos(s)\cdot\sin(s)\cdot\cos(t)\,ds \\
&= \sin(x)\cdot\cos(t)\cdot\int_0^{2\pi} \sin(s)\cdot\cos(s)\,ds = 0,
\end{aligned}
$$

which agrees with $K^2 \equiv 0$.

**Example 1.5** *Let $K$ denote the Hilbert-Schmidt operator with continuous kernel $k$ on $L^2(I)$, where $I$ is a closed and bounded interval. Show that all the iterated kernels $K_n$ are continuous on $I^2$ and show that*

$$\|k_n\|_2 \leq \|k\|_2^n.$$

*Show that if $|\lambda|\,\|k\|_2 < 1$, then the series*

$$\sum_{n=1}^{\infty} \lambda^n\, k_n$$

*is convergent in $L^2(I)$.*

Write $I = [a, b]$. It is well-known that

$$k_n(x, t) = \int_a^b f(x, s)\, k_{n-1}(s, t)\, ds.$$

The first claim is proved by induction. Assume that both $k(x, s)$ and $k_{n-1}(s, t)$ are continuous. By subtracting something and then adding it again we get

$$
\begin{aligned}
k_n(x, t) - k_n(x_0, t_0) &= \int_a^b \{k(x, s)k_{n-1}(s, t) - k(x_0, s)k_{n-1}(s, t)\}\, ds \\
&\quad + \int_a^b \{k(x_0, s)k_{n-1}(s, t) - k(x_0, s)k_{n-1}(s, t_0)\}\, ds \\
&= \int_a^b \{k(x, s) - k(x_0, s)\}\, k_{n-1}(s, t)\, ds \\
&\quad + \int_a^b k(x_0, s) \cdot \{k_{n-1}(s, t) - k_{n-1}(s, t_0)\}\, ds.
\end{aligned}
$$

To every $\varepsilon > 0$ there is a $\delta > 0$, such that

$$|k(x, s) - k(x_0, s)| < \varepsilon \qquad \text{for } |x - x_0| < \delta \text{ and all } s \in [a, b],$$

and

$$|k_{n-1}(s, t) - k_{n-1}(s, t_0)| < \varepsilon \qquad \text{for } |t - t_0| < \delta \text{ and all } s \in [a, b].$$

If therefore $|x - x_0| < \delta$ and $|t - t_0| < \delta$, then we get the following estimate,

$$
\begin{aligned}
|k_n(x, t) - k_n(x_0, t_0)| &\leq \int_a^b \varepsilon \cdot \|k_{n-1}\|_\infty\, dx + \int_a^b \|k\|_\infty \cdot \varepsilon\, ds \\
&= (b - a)\, \{\|k\|_\infty + \|k_{n-1}\|_\infty\}\, \varepsilon,
\end{aligned}
$$

and we conclude that $k_n(x, t)$ is continuous, and the claim follows by induction.

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Furthermore,

$$
\begin{aligned}
\|k_n\|_2^2 &= \int_a^b \int_a^b |k_n(x,t)|^2 \, dx \, dt \\
&= \int_a^b \int_a^b \left| \int_a^b k(x,s) k_{n-1}(s,t) \, ds \right| \cdot \left| \int_a^b k(x,r) k_{n-1}(r,t) \, dr \right| \, dx \, dt \\
&\leq \int_a^b \int_a^b \int_a^b \int_a^b |k(x,s)| \cdot |k_{n-1}(s,t)| \cdot |k(x,r)| \cdot |k_{n-1}(r,t)| \, ds \, dr \, dx \, dt \\
&\leq \frac{1}{2} \int_a^b \int_a^b \int_a^b \int_a^b \left\{ |k(x,s)|^2 |k_{n-1}(r,t)|^2 + |k_{n-1}(s,t)|^2 |k(x,r)|^2 \right\} \, ds \, dr \, dx \, dt \\
&= \frac{1}{2} \left\{ \|k\|_2^2 \|k_{n-1}\|_2^2 + \|k_{n-1}\|_2^2 \|k\|_2^2 \right\} = \|k\|_2^2 \|k_{n-1}\|_2^2,
\end{aligned}
$$

and we have proved that

$$\|k_n\|_2 \leq \|k\|_2 \|k_{n-1}\|_2.$$

Hence we get for $n = 2$ that $\|k_2\|_2 \leq \|k\|_2^2$.

Assume that $\|k_{n-1}\|_2 \leq \|k\|_2^{n-1}$. Then

$$\|k_n\|_2 \leq \|k\|_2 \|k_{n-1}\|_2 \leq \|k\|_2 \cdot \|k\|_2^{n-1} = \|k\|_2^n,$$

and the claim follows by induction.

The remaining claim is now trivial, because

$$\left\| \sum_{n=1}^{+\infty} \lambda^n\, k_n(x,t) \right\|_2 \leq \sum_{n=1}^{+\infty} |\lambda|^n \|k_n\|_2 \leq \sum_{n=1}^{+\infty} |\lambda|^n \|k\|_2^n = \sum_{n=1}^{+\infty} \{|\lambda| \cdot \|k\|_2\}^n = \frac{1}{1 - |\lambda| \cdot \|k\|_2},$$

where we have used that the geometric series is convergent for $|\lambda| \cdot \|k\|_2 < 1$.

**Example 1.6** *Let $K$ and $L$ denote the Hilbert-Schmidt operators with continuous kernels $k$ and $\ell$ on $L^2(I)$, where $I$ is a closed and bounded interval. We define the* trace *of $K$, $\mathrm{tr}(K)$ by*

$$\mathrm{tr}(K) = \int_I k(x,x)\, dx,$$

*and similarly for $K$.*
*Show that*

$$|\mathrm{tr}(KL)| \leq \|K\|_{\mathrm{HS}} \|L\|_{\mathrm{HS}},$$

*and*

$$|\mathrm{tr}\,(K^n)| \leq \|K\|_{\mathrm{HS}}^n, \qquad n \geq 2.$$

*Moreover, if $(K_n)$, $(L_n)$ denote sequences of Hilbert-Schmidt operators like above, where*

$$\|K_n - K\|_{\mathrm{HS}} \to 0 \quad and \quad \|L_n - L\|_{\mathrm{HS}} \to 0,$$

*then*

$$\mathrm{tr}\,(K_n L_n) \to \mathrm{tr}(KL).$$

**Remark 1.1** We first show that the claim is not true, if we replace the Hilbert-Schmidt norm $\|\cdot\|_{\mathrm{HS}}$ by the operator norm.

Let

$$k(x,t) = \ell(x,t) = x + t$$

be the kernel of self adjoint Hilbert-Schmidt operators $K$ and $L$ on $L^2([0,1])$. It follows from Example 1.7 below that $\dfrac{1}{2} \pm \dfrac{1}{\sqrt{3}}$ are the two eigenvalues different from zero of both $K$ and $L$, and the norm of $K$ (and $L$) is given by the absolute value of the numerically largest eigenvalue,

$$\|K\| = \|L\| = \frac{1}{2} + \frac{1}{\sqrt{3}}.$$

Furthermore,l

$$\begin{aligned}
\|k\|_2^2 &= \|\ell\|_2^2 = \int_0^1 \int_0^1 (x+t)^2\, dx\, dt = \int_0^1 \int_0^1 \left(x^2 + 2xt + t^2\right) dx\, dt = \int_0^1 \left[\frac{x^3}{3} + x^2 t + x t^2\right]_{x=0}^1 dt \\
&= \int_0^1 \left\{\frac{1}{3} + t + t^2\right\} dt = \frac{1}{3} + \frac{1}{2} + \frac{1}{3} = \frac{7}{6}.
\end{aligned}$$

Finally,

$$\text{tr}(KL) = \int_0^1 \left\{ \int_0^1 (x+s)(s+x)\,ds \right\} dx = \int_0^1 \left\{ \int_0^1 (x+s)^2\,ds \right\} dx = \|k\|_2^2 = \frac{7}{6}.$$

Thus, in this example,

$$\text{tr}(KT) = \frac{7}{6} = \|k\|_2^k > \|K\|^2 = \|K\| \cdot \|L\| = \left\{ \frac{1}{2} + \frac{1}{\sqrt{3}} \right\}^2 = \frac{1}{4} + \frac{1}{3} + \frac{\sqrt{3}}{3},$$

which either can be shown numerically, or of course must follow from the theory, because we always have that $\|K\| \le \|k\|_2$. Here we cannot have equality, if $\sigma_p(K)$ contains at least two different points $\ne 0$. $\Diamond$

Then we turn to the example itself.

Write $I = [a, b]$, and let

$$Ku(x) = \int_a^b k(x,t)u(t)\,dt \qquad \text{and} \qquad Lu(x) = \int_a^b \ell(x,t)u(t)\,dt$$

for $u \in L^2([a,b])$. Then

$$((KL)u)(x) \;=\; K(Lu)(x) = \int_a^b k(x,t)\,Lu(t)\,dt = \int_a^b k(x,t) \left\{ \int_a^b \ell(t,s)u(s)\,ds \right\} dt$$

$$=\; \int_a^b \left\{ \int_a^b k(x,t)\ell(t,s)\,dt \right\} u(s)\,ds,$$

and it follows that the composition $KL$ has the kernel

$$m(x,t) = \int_a^b k(x,s)\ell(s,t)\,ds.$$

Then

$$|\text{tr}(KL)| \;=\; \left| \int_a^b m(x,x)\,dx \right| = \left| \int_a^b \left\{ \int_a^b k(x,t)\ell(t,x)\,dt \right\} dx \right|$$

$$\le\; \int_a^b \left\{ \int_a^b |k(x,t)|^2 dt \right\}^{\frac{1}{2}} \cdot \left\{ \int_a^b |\ell(t,x)|^2 dt \right\}^{\frac{1}{2}} dx.$$

Putting

$$k_1(x) = \left\{ \int_a^b |k(x,t)|^2 dt \right\}^{\frac{1}{2}} \qquad \text{og} \qquad \ell_1(x) = \left\{ \int_a^b |\ell(t,x)|^2 dt \right\}^{\frac{1}{2}},$$

we get $k_1, \ell_1 \in L^2([a,b])$, and it follows from the Cauchy-Schwarz inequality that

$$|\text{tr}(KL)| \;\le\; \int_a^b k_1(x)\ell_1(x)\,dx \le \left\{ k_1(x)^2\,dx \right\}^{\frac{1}{2}} \left\{ \int_a^b \ell_1(x)^2 dx \right\}^{\frac{1}{2}}$$

$$=\; \left\{ \int_a^b \left( \int_a^b |k(x,t)|^2 dt \right) dx \right\}^{\frac{1}{2}} \left\{ \int_a^b \left( \int_a^b |\ell(t,x)|^2 dt \right) dx \right\}^{\frac{1}{2}}$$

$$=\; \|k\|_2 \cdot \|\ell\|_2 = \|K\|_{\text{HS}} \cdot \|L\|_{\text{HS}},$$

and the first claim is proved.

We note that since $KL$ has the kernel

$$m(x,t) = \int_a^b k(x,s)\ell(s,t)\,ds,$$

we have

$$
\begin{aligned}
\|KL\|_{\mathrm{HS}}^2 \;\leq\; & \int_a^b \int_a^b |m(x,t)|^2 dx\,dt = \int_a^b \left\{ \int_a^b \left| \int_a^b k(x,s)\ell(s,t)\,ds \right|^2 dx \right\} dt \\
\leq\; & \int_a^b \left( \int_a^b \left\{ \left( \int_a^b |k(x,s)|^2 ds \right)^{\frac{1}{2}} \left( \int_a^b |\ell(s,t)|^2 ds \right)^{\frac{1}{2}} \right\}^2 dx \right) dt \\
=\; & \int_a^b \left( \int_a^b \left\{ \left( \int_a^b |k(x,s)|^2\,ds \right) \cdot \left( \int_a^b |\ell(s,t)|^2 ds \right) \right\} dx \right) dt \\
=\; & \int_a^b \int_a^b |k(x,s)|^2 ds\,dx \cdot \int_a^b \int_a^b |\ell(s,t)|^2 ds\,dt = \|k\|_2^2 \cdot \|\ell\|_2^2 = \|K\|_{\mathrm{HS}}^2 \cdot \|L\|_{\mathrm{HS}}^2.
\end{aligned}
$$

This proves that we always have

(1)  $\|KL\|_{\mathrm{HS}} \leq \|K\|_{\mathrm{HS}} \cdot \|L\|_{\mathrm{HS}}$.

Recall for $n = 1$ that

$$\mathrm{tr}(K) = \int_a^b k(x,x)\,dx.$$

Choosing $k(x,x) = 1$ and $k(x,t)$ continuous, such that $\|k\|_2 < \varepsilon$, we get

$$\mathrm{tr}(K) = b - a \qquad \text{and} \qquad \|K\|_{\mathrm{HS}}^2 < \varepsilon,$$

which shows that the formula is not true for $n = 1$.

On the other hand, if $n \geq 2$, then it follows from the first question and (1) that

$$\left| \mathrm{tr}\left(K^n\right) \right| = \left| \mathrm{tr}\left(K\,K^{n-1}\right) \right| \leq \|K\|_{\mathrm{HS}} \|K^{n-1}\|_{\mathrm{HS}} \leq \|K\|_{\mathrm{HS}} \|K\|_{\mathrm{HS}}^{n-1} = \|K\|_{\mathrm{HS}}^n.$$

Finally, we note that for any scalar $\lambda$ and any Hilbert-Schmidt operators,

$$\mathrm{tr}(K + \lambda\,L) = \int_a^b \{k(x,x) + \lambda\,\ell(x,x)\}\,dx = \mathrm{tr}(K) + \lambda\,\mathrm{tr}(L),$$

proving that the *trace* is linear on the vector space of all Hilbert-Schmidt operators. Then we get

$$
\begin{aligned}
\mathrm{tr}(KL) - \mathrm{tr}\left(K_n L_n\right) \;=\; & \mathrm{tr}\left(KL - K_n L_n\right) = \mathrm{tr}\left(KL - KL_n + KL_n - K_n L_n\right) \\
=\; & \mathrm{tr}\left(K\left(L - L_n\right)\right) + \mathrm{tr}\left(\left(K - K_n\right)L_n\right) \\
=\; & \mathrm{tr}\left(K\left(L - L_n\right)\right) + \mathrm{tr}\left(\left(K - K_n\right)\left(L_n - L\right)\right) + \mathrm{tr}\left(\left(K - K_n\right)L\right),
\end{aligned}
$$

and it follows from the assumptions and the first part of the example that

$$
\begin{aligned}
& |\mathrm{tr}(KL) - \mathrm{tr}\left(K_n L_n\right)| \\
& \quad \leq \|K\|_{\mathrm{HS}} \|L - L_n\|_{\mathrm{HS}} + \|K - K_n\|_{\mathrm{HS}} \|L - L_n\|_{\mathrm{HS}} + \|K - K_n\|_{\mathrm{HS}} \|L\|_{\mathrm{HS}} \to 0 \qquad \text{for } n \to +\infty.
\end{aligned}
$$

**Example 1.7** *Let $K$ denote the Hilbert-Schmidt operator on $L^2([0,1])$ with kernel*

$$k(x,t) = x + t.$$

*Find all eigenvalues and eigenfunctions for $K$.*
*Solve the equation*

$$Ku = \mu\, u + f, \qquad f \in L^2([0,1]),$$

*when $\mu$ is not in the spectrum for $K$.*

It follows from

$$(2) \quad Kf(x) = x \int_0^1 f(t)\,dt + \int_0^1 t \cdot f(t)\,dt,$$

that every eigenfunction corresponding to an eigenvalue $\lambda \neq 0$ must have the form $f(x) = ax + b$. By insertion into (2) we get

$$Kf(x) = x \int_0^1 (at + b)\,dt + \int_0^1 \left(at^2 + bt\right)\,dt = \left\{\frac{a}{2} + b\right\} x + \left\{\frac{a}{3} + \frac{k}{2}\right\}.$$

This expression is equal to $\lambda(ax+b)$, if and only if $(a,b)$ and $\left(\dfrac{a}{2}+b, \dfrac{a}{3}+\dfrac{b}{2}\right)$ are proportion, thus if and only if

$$0 = \left| \begin{array}{cc} \frac{a}{2}+b & \frac{a}{3}+\frac{b}{2} \\ a & b \end{array} \right| = \frac{ab}{2} + b^2 - \frac{a^3}{3} - \frac{ab}{2} = b^2 - \frac{a^2}{3},$$

hence if and only if $b = \pm \dfrac{1}{\sqrt{3}}\, a$. Since

$$\lambda\, a = \frac{a}{2} + b = \left\{ \frac{1}{2} \pm \frac{1}{\sqrt{3}} \right\} a,$$

the corresponding eigenvalues are $\lambda = \dfrac{1}{2} \pm \dfrac{1}{\sqrt{3}}$.

For $\lambda_1 = \dfrac{1}{2} + \dfrac{1}{\sqrt{3}}$ we get the eigenfunction $f_1(x) = x + \dfrac{1}{\sqrt{3}}$.

For $\lambda_2 = \dfrac{1}{2} - \dfrac{1}{\sqrt{3}}$ we get the eigenfunction $f_2(x) = x - \dfrac{1}{\sqrt{3}}$.

Finally, $K$ is trivially self adjoint, thus $\lambda = 0$ is an eigenvalue for every function

$$f \in \left\{ \mathrm{span}\left( x + \frac{1}{\sqrt{3}}, x - \frac{1}{\sqrt{3}} \right) \right\}^{\perp} = \{\mathrm{span}(1, x)\}^{\perp},$$

hence for every function $f \in L^2([0,1])$, for which

$$\int_0^1 f(t)\, dt = 0 \qquad \text{og} \qquad \int_0^1 t\, f(t)\, dt = 0.$$

Now, $k(x,t) = \overline{k(t,x)}$, so $K$ is self adjoint. Therefore, if we put

$$\varphi_1(x) = \frac{f_1}{\|f_1\|_2} \qquad \text{and} \qquad \varphi_2 = \frac{f_2}{\|f_2\|_2},$$

then the operator $K$ is described by

(3) $\quad K u = \lambda_1 \, (u, \varphi_1)\, \varphi_1 + \lambda_2\, (u, \varphi_2)\, \varphi_2.$

If $(f, \varphi_1) = (f, \varphi_2) = 0$, then it follows by a simple check that the solution of the equation

$$K u = \mu\, u + f, \qquad \text{hvor } \mu \notin \left\{ 0, \frac{1}{2} + \frac{1}{\sqrt{3}}, \frac{1}{2} - \frac{1}{\sqrt{3}} \right\},$$

is given by $u = -\dfrac{1}{\mu}\, f$.

Then assume that $f = a\, \varphi_1 + b\, \varphi_2$. The equation $K u = \mu\, u + f$ can now be written in the form

$$\lambda_1\, (u, \varphi_1)\, \varphi_1 + \lambda_2\, (u, \varphi_2)\, \varphi_2 = \mu \sum_{n=1}^{\infty} (u, \varphi_n) + a\, \varphi_1 + b\, \varphi_2,$$

which implies that

$$u = c_1\,\varphi_1 + c_2\,\varphi_2,$$

where

$$c_1 = (u, \varphi_1) = \frac{a}{\lambda_1 - \mu} = \frac{1}{\lambda_1 - \mu}\,(f, \varphi_1)\,,$$

and

$$c_2 = (u, \varphi_2) = \frac{b}{\lambda_2 - \mu} = \frac{1}{\lambda_2 - \mu}\,(f, \varphi_2)\,.$$

The equation being linear, it follows in general from the rewriting

$$Ku - \mu\,u = f = (f, \varphi_1)\,\varphi_1 + (f, \varphi_2)\,\varphi_2 + \{f - (f, \varphi_1)\,\varphi_1 - (f, \varphi_2)\,\varphi_2\}\,,$$

that

$$\begin{aligned}
u &= \frac{1}{\lambda_1 - \mu}\,(f, \varphi_1)\,\varphi_1 + \frac{1}{\lambda_2 - \mu}\,(f, \varphi_2)\,\varphi_2 - \frac{1}{\mu}\,f + \frac{1}{\mu}\,(f, \varphi_1)\,\varphi_1 + \frac{1}{\mu}\,(f, \varphi_2)\,\varphi_2 \\
&= \frac{\lambda_1}{\mu(\lambda_1 - \mu)}\,(f, \varphi_1)\,\varphi_1 + \frac{\lambda_2}{\mu(\lambda_2 - \mu)}\,(f, \varphi_2)\,\varphi_2 - \frac{1}{\mu}\,f = A\,\varphi_1 + B\,\varphi_2 - \frac{1}{\mu}\,f,
\end{aligned}$$

which *in principle* can be written explicitly by means of the functions $f_i(x)$, $i = 1, 2$. We shall, however, not waste our time on that, because the result will look extremely nasty.

**Example 1.8** *Lad $K$ denote the Hilbert-Schmidt operator on $L^2\left(\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$ with kernel*

$$k(x, t) = \cos(x - t).$$

*Find all eigenvalues and eigenfunctions for $K$.*
*Solve the equation*

$$Ku = \mu\,u + f, \qquad f \in L^2\left(\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right),$$

*when $\mu$ is not in the spectrum for $K$.*

Obviously, $K$ is self adjoint.

It follows in general from

$$\cos(x - t) = \cos(x) \cdot \cos(t) + \sin(x) \cdot \sin(t),$$

that

$$(4)\quad Kf(x) = \cos(x) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(t)\,\cos(t)\,dt + \sin(x) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(t)\,\sin(t)\,dt.$$

Then any eigenfunction corresponding to some eigenvalue $\lambda \neq 0$ must be of the structure

$$f(x) = a \cdot \cos(x) + b \cdot \sin(x).$$

By insertion into (4),

$$
\begin{aligned}
Kf(x) &= \cos(x) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ a \cdot \cos^2 t + b \cdot \sin t \cos t \, dt \right\} + \sin(x) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ a \cdot \sin t \cos t + b \cdot \sin^2 t \right\} \, dt \\
&= \left\{ \frac{a\pi}{2} + 0 \right\} \cos(x) + \left\{ 0 + \frac{b\pi}{2} \right\} \sin(x) = \frac{\pi}{2} \left\{ a \, \cos(x) + b \, \sin(x) \right\} = \frac{\pi}{2} \, f(x),
\end{aligned}
$$

hence $f(x) = a \cdot \cos(x) + b \cdot \sin(x)$ is for every pair $(a, b) \neq (0, 0)$ an eigenfunction corresponding to the eigenvalue $\lambda = \dfrac{\pi}{2}$.

For $\lambda = 0$ we get the eigenspace $\{\cos(x), \sin(x)\}^{\perp}$ i $L^2\left(\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]\right)$.

ALTERNATIVELY, we see that

$$
\cos(x - t) = \frac{1}{2} \, e^{ix} e^{-it} + \frac{1}{2} \, e^{-ix} e^{it}.
$$

We get from

$$
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| e^{\pm ix} \right|^2 \, dx = \pi,
$$

the normed functions

$$
\varphi_1(x) = \frac{1}{\sqrt{\pi}} \, e^{ix} \qquad \text{and} \qquad \varphi_{-1} = \frac{1}{\sqrt{\pi}} \, e^{-ix},
$$

where

$$
(\varphi_1, \varphi_{-1}) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \varphi_1(x) \, \overline{\varphi_{-1}(x)} \, dx = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{2ix} \, dx = \frac{1}{2i\pi} \left\{ e^{i\pi} - e^{-i\pi} \right\} = 0,
$$

hence

$$
k(x, t) = \cos(x - t) = \frac{\pi}{2} \, \varphi_1(x) \, \overline{\varphi_1(t)} + \frac{\pi}{2} \, \varphi_{-1}(x) \, \overline{\varphi_{-1}(t)}.
$$

We obtain directly that $\lambda = \dfrac{\pi}{2}$ is the only eigenvalue $\neq 0$, thus $\|K\| = \dfrac{\pi}{2}$, and the eigenfunctions are $\varphi_1$ and $\varphi_{-1}$.

**Remark 1.2** A basis for $L^2\left(\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]\right)$ is e.g.

$$
\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos 2x, \frac{1}{\sqrt{\pi}} \sin 2x, \frac{1}{\sqrt{\pi}} \cos 4x, \frac{1}{\sqrt{\pi}} \sin 4x, \ldots,
$$

from which it follows that $\{\cos(x), \sin(x)\}^{\perp}$ may be difficult to describe. $\Diamond$

It follows from $\overline{k(t, x)} = k(x, t)$ that $K$ is self adjoint, which also was noted previously. We may therefore apply the standard method where we expand after the eigenfunctions.

First choose $f$, such that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(t) \cos t \, dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(t) \sin t \, dt = 0.$$

Then $Kf = 0$, and we conclude that $u = -\dfrac{1}{\mu} f$ is the only solution.

We get in the general case that

$$
\begin{aligned}
u &= \sum_{n=1}^{+\infty} (u, \varphi_n) \, \varphi_n = \frac{1}{\frac{\pi}{2} - \mu} \left\{ (f, \varphi_1) \, \varphi_1 + (f, \varphi_2) \, \varphi_2 \right\} - \frac{1}{\mu} f + \frac{1}{\mu} (f, \varphi_1) \, \varphi_1 + \frac{1}{\pi} (f, \varphi_2) \, \varphi_2 \\
&= \frac{\frac{\pi}{2}}{\mu \left( \frac{\pi}{2} - \mu \right)} \left\{ (f, \varphi_1) \, \varphi_1 + (f, \varphi_2) \, \varphi_2 \right\} - \frac{1}{\mu} f.
\end{aligned}
$$

Now,

$$\varphi_i = \frac{f_i}{\|f_i\|_2}, \qquad i = 1,\, 2,$$

where $f_1(x) = \cos x$ and $f_2(x) = \sin x$, and $\|f_1\|_2^2 = \|f_2\|_2^2 = \dfrac{\pi}{2}$, hence

$$
\begin{aligned}
u &= \frac{\frac{\pi}{2}}{\mu\left(\frac{\pi}{2} - \mu\right)} \cdot \frac{1}{\frac{\pi}{2}} \left\{ (f, \cos t)\cos(x) + (f, \sin t)\sin(x) \right\} - \frac{1}{\mu}\, f \\
&= \frac{1}{\mu\left(\frac{\pi}{2} - \mu\right)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(t)\,\cos t\, dt \cdot \cos(x) + \frac{1}{\mu\left(\frac{\pi}{2} - \mu\right)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(t)\,\sin t\, dt \cdot \sin(x) - \frac{1}{\mu}\, f(x).
\end{aligned}
$$

Notice that this expression can be written as

$$u = \frac{1}{\mu\left(\frac{\pi}{2} - \mu\right)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x-t)\, f(t)\, dt - \frac{1}{\mu}\, f(x) = \frac{1}{\mu\left(\frac{\pi}{2} - \mu\right)}\, Kf - \frac{1}{\mu}\, f.$$

We have assumed that

$$\mu \notin \sigma(K) = \sigma_p(K) = \left\{ 0, \frac{\pi}{2} \right\}.$$

**Example 1.9** *Let $K$ denote the Hilbert-Schmidt operator on $L^2([-\pi, \pi])$ with kernel*

$$k(x,t) = \{\cos(x) + \cos(t)\}^2.$$

*Find all eigenvalues and eigenfunctions for $K$, and find an orthonormal basis for $\ker(K)$.*

By a simple computation,

$$
\begin{aligned}
k(x,t) &= (\cos x + \cos t)^2 = \cos^2 x + 2\,\cos x\,\cos t + \cos^2 t \\
&= \frac{1}{2}\,\cos 2x + 2\,\cos x\,\cos t + \frac{1}{2}\,\cos 2t + \frac{1}{2} \\
&= \frac{1}{2}\,\cos 2x + 2\,\cos x\,\cos t + \left\{ 1 + \frac{1}{2}\,\cos 2t \right\} \cdot 1.
\end{aligned}
$$

Hence

$$
\begin{aligned}
(5)\quad Kf(x) &= \cos 2x \int_{-\pi}^{\pi} \frac{1}{2}\, f(t)\, dt + \cos x \int_{-\pi}^{\pi} 2\, f(t)\,\cos t\, dt \\
&\quad + \int_{-\pi}^{\pi} f(t)\, dt + \int_{-\pi}^{\pi} \frac{1}{2}\, f(t)\,\cos 2t\, dt.
\end{aligned}
$$

Therefore, any eigenfunction corresponding to an eigenvalue $\lambda \neq 0$ must be of the form

$$f(x) = a \cdot \cos 2x + b \cdot \cos x + c,$$

where we shall find the constants $a$, $b$ and $c$. We get by insertion into (5) that

$$
\begin{aligned}
Kf(x) &= \cos 2x \int_{-\pi}^{\pi} \frac{1}{2} \left( a \cdot \cos 2t + b \cdot \cos t + c \right) dt + \cos x \int_{-\pi}^{\pi} 2(a \cos 2t + b \cos t + c) \cos t \, dt \\
&\quad + \int_{-\pi}^{\pi} (a \cdot \cos 2t + b \cdot \cos t + c) \, dt \\
&\quad + \int_{-\pi}^{\pi} \frac{1}{2} \left( a \cdot \cos 2t + b \cdot \cos t + c \right) \cdot \cos 2t \, dt \\
&= c\pi \cdot \cos 2x + 2b\pi \, \cos x + 2\pi \, c + \frac{a\pi}{2}.
\end{aligned}
$$

This expression is equal to $\lambda a \cdot \cos 2x + \lambda b \cdot \cos x + \lambda c$, if and only if

$$
\lambda \, a = c\pi, \qquad \lambda \, b = 2\pi b, \qquad \lambda \, c = 2\pi \, c + \frac{a\pi}{2}.
$$

We immediately get the eigenvalue $\lambda = 2\pi$ with its corresponding eigenfunction $\cos x$.

The other eigenfunctions are found in the following way: The vectors $(a, c)$ and $\left( c\pi, 2c\pi + \dfrac{a\pi}{2} \right)$ must be proportional, so

$$
0 = \begin{vmatrix} c & 2c + \frac{a}{2} \\ a & c \end{vmatrix} = c^2 - 2ac - \frac{a^2}{2} = (c - a)^2 - \frac{3}{2} a^2,
$$

hence

$$
c = a \pm \sqrt{\frac{3}{2}} \, a = \left\{ 1 \pm \sqrt{\frac{3}{2}} \right\} a,
$$

corresponding to

$$
\lambda = \frac{c\pi}{a} = \left\{ 1 \pm \sqrt{\frac{3}{2}} \right\} \pi.
$$

For $\lambda_1 = \left\{ 1 + \sqrt{\dfrac{3}{2}} \right\} \pi$ we get the eigenfunction

$$
f_1(x) = \cos 2x + 1 + \sqrt{\frac{3}{2}} \qquad \left[ = 2\cos^2 x + \sqrt{\frac{3}{2}} \right].
$$

For $\lambda_2 = \left\{ 1 - \sqrt{\dfrac{3}{2}} \right\} \pi$ we get the eigenfunction

$$
f_2(x) = \cos 2x + 1 - \sqrt{\frac{3}{2}} \qquad \left[ = 2\cos^2 x - \sqrt{\frac{3}{2}} \right].
$$

For $\lambda = 2\pi$ we get the eigenfunction $f_3(x) = \cos x$.

There is no reason here to norm these eigenfunctions. We only notice that they span the same subspace of $L^2([-\pi, \pi])$ as $1$, $\cos x$, and $\cos 2x$ do.

It follows from $\overline{k(t,x)} = k(x,t)$ that $K$ is self adjoint, so the null-space is simply the orthogonal complement of the subspace mentioned above. Thus we conclude that $\ker(K)$ is spanned by

$$\sin x, \ \sin 2x, \ \cos 3x, \ \sin 3x, \ \cos 4x, \ \sin 4x, \ \ldots,$$

i.e. of the usual trigonometric basis with the exception of $1$, $\cos x$ and $\cos 2x$.

**Example 1.10** *Let $K$ denote a self adjoint Hilbert-Schmidt operator on $L^2(I)$ with kernel $k$. Show that $\|K\| = \|k\|_2$ if and only if the spectrum for $K$ consists of at most two points.*

It follows from $K$ being self adjoint that $\overline{k(t,x)} = k(x,t)$ and there exist an ortonormal sequence $(\varphi_n)$ in $L^2(I)$ and a sequence $(\lambda_n)$ of real numbers with $|\lambda_1| \geq |\lambda_2| \geq \cdots$, where either $\lambda_n = 0$ eventually, or $\lambda_n \to 0$, such that

$$(6) \ Ku = \sum_{n=1}^{+\infty} \lambda_n \, (u, \varphi_n) \, \varphi_n \qquad \text{for } u \in L^2(I),$$

where every $\varphi_n$ is an eigenfunction of the corresponding $\lambda_n \in \sigma_p(K)$, and where $0$ is either an eigenvalue or belongs to the continuous spectrum $\sigma_c(K)$, and where

$$\sigma(K) = \{0\} \cup \sigma_p(K).$$

We shall prove that $\|K\| = \|k\|_2$, if and only if $\sigma(K)$ contains at most two points.

1) If $\sigma(K)$ only consists of one point, then $\sigma(K) = \{0\}$, and $Ku \equiv 0$, thus $k(x,t) = 0$ almost everywhere, and it follows trivially that $\|K\| = \|k\|_1 = 0$.

2) If $\sigma(K)$ contains two points, then it follows from the introducing argument that we necessarily must have

$$\sigma(M) = \{0, \lambda\},$$

so the operator is described by

$$Ku = (u, \varphi)\,\varphi = \lambda \int_a^b \varphi(x) \, \overline{\varphi(t)} \, u(t) \, dt,$$

from which we derive that

$$k(x,t) = \lambda \, \varphi(t)\varphi(x).$$

Clearly, $\|K\| = \lambda$. Because $\|\varphi\|_2 = 1$, we get

$$\|k\|_2^2 = \int_a^b \int_a^b |k(x,t)|^2 dx\,dt = |\lambda|^2 \int_a^b \int_a^b |\varphi(x)|^2 |\varphi(t)|^2 dx\,dt = |\lambda|^2.$$

Hence $\|k\|_2 = |\lambda| = \|K\|$ in this case.

3) If $\sigma(K)$ contains more than two points, then

$$\|K\| = \max |\lambda_n| = |\lambda_1|.$$

Furthermore, we get by the computation

$$Ku(x) = \int_I k(x,y)\, u(t)\, dt = \sum_{n=1}^{+\infty} \lambda_n\, (u, \varphi_n)\, \varphi_n(x) = \int_I \sum_{n=1}^{+\infty} \lambda_n\, \varphi_n(x)\, \overline{\varphi_n(t)}\, u(t)\, dt,$$

that

$$\|k\|_2^2 = \sum_{n=1}^{+\infty} \lambda_n^2 > \lambda_1^2 = \|K\|^2,$$

and the claim is proved.

**Example 1.11** *Let* $\{e_1, e_2, \ldots, e_p\}$ *denote a finite orthonormal set in* $L^2(I)$*, and let the Hilbert-Schmidt operator* $K$ *be given by the kernel*

$$k(x,y) = \sum_{i=1}^{p} \sum_{j=1}^{p} k_{ij}\, e_i(x)\, e_j(t).$$

*Find the trace* $\operatorname{tr}(K)$.
*We say that the operator* $K$ *has a* canonical kernel of finite rank.

This example is trivial,

$$\operatorname{tr}(K) = \int_I k(x,x)\, dx = \int_I \sum_{i=1}^{p} \sum_{j=1}^{p} k_{ij}\, e_i(x)\, e_j(x)\, dx = \sum_{i=1}^{p} \sum_{j=1}^{p} k_{ij}\, \delta_{ij} = \sum_{i=1}^{p} k_{ii}.$$

Note that this corresponds to the trace of matrix $(k_{ij})$.

**Example 1.12** *Denote by* $K$ *a self adjoint Hilbert-Schmidt operator on* $L^2(I)$ *of kernel* $k$.
*Prove that* $K$ *is a general Hilbert-Schmidt operator (cf. the definition in* EXAMPLE 1.1*), and find the Hilbert-Schmidt norm* $\|K\|_{\mathrm{HS}}$.

Put

$$Ku = \sum_{n=1}^{+\infty} \lambda_n\, (u, \varphi_n)\, \varphi_n.$$

It follows from VENTUS, HILBERT SPACES ETC., EXAMPLE 2.7 that

$$t_{jk} = (K\varphi_j, \varphi_k) = \left( \sum_{n=1}^{+\infty} \lambda_n\, (\varphi_j, \varphi_n)\, \varphi_n, \varphi_k \right) = (\lambda_j\, \varphi_j, \varphi_k) = \lambda_j\, \delta_{jk},$$

thus $t_{jj} = \lambda_j$ and $t_{jk} = 0$ for $j \neq 0$.

Then by EXAMPLE 1.1, $K$ is a general Hilbert-Schmidt operator, if

$$\sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} |t_{jk}|^2 < +\infty,$$

because it was proved that this number is independent of the choice of orthonormal basis. Furthermore, it follows from EXAMPLE 1.2 that

$$\|K\|_{\mathrm{HS}} = \left\{ \sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} |t_{jk}|^2 \right\}^{\frac{1}{2}}.$$

In the present case we get

$$\|K\|_{\mathrm{HS}} = \left\{ \sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} |\lambda_j|^2\, \delta_{jk} \right\}^{\frac{1}{2}} = \left\{ \sum_{j=1}^{+\infty} |\lambda_j|^2 \right\}^{\frac{1}{2}} = \|k\|_2.$$

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**Example 1.13** *Let*

$$k(x,t) = \{\sin(x) + \sin(t)\}^2 - \frac{1}{8}$$

*be the kernel for a Hilbert-Schmidt operator $K$ on the complex Hilbert space $L^2([-\pi, \pi])$.*
*Show that $K$ is self adjoint and express the range $K\left(L^2([-\pi, \pi])\right)$ of $K$ with the help of the non-normalized basis*

$$1, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots$$

*Find all non-zero eigenvalues and corresponding eigenfunctions for $K$, and determine $\sigma(K)$.*
*Solve the equation $Ku = \pi u - \dfrac{5\pi}{4}$ in $L^2([-\pi, \pi])$.*

1)  Clearly, $k(x,t) \in L^2([-\pi, \pi] \times [-\pi, \pi])$, and

$$\overline{k(t,x)} = (\sin t + \sin x)^2 - \frac{1}{8} = k(x,t),$$

thus $k(x,t)$ is Hermitian, and $K$ is a self adjoint Hilbert-Schmidt-operator. It follows from

$$
\begin{aligned}
k(x,t) &= (\sin x + \sin t)^2 - \frac{1}{8} = \sin^2 x + 2\sin x \cdot \sin t + \sin^2 t - \frac{1}{8} \\
&= -\frac{1}{2}\cos 2x + 2\sin x \cdot \sin t - \frac{1}{2}\cos 2t + \frac{7}{8},
\end{aligned}
$$

that

$$
\begin{aligned}
(7)\ Kf(x) &= \left\{ -\frac{1}{2}\int_{-\pi}^{\pi} f(t)\,dt \right\}\cos 2x + \left\{ 2\int_{-\pi}^{\pi} f(t)\,\sin t\,dt \right\}\sin x \\
&\quad + \left\{ -\frac{1}{2}\int_{-\pi}^{\pi} f(t)\,\cos 2t\,dt + \frac{7}{8}\int_{-\pi}^{\pi} f(t)\,dt \right\}\cdot 1,
\end{aligned}
$$

and we conclude that the range $K\left(L^2([-\pi, \pi])\right)$ is spanned by $1$, $\sin x$ and $\cos 2x$.
(Choose e.g. suitable linear combinations of these three functions in order to conclude that the dimension is 3).

2)  An eigenfunction $f$ corresponding to an eigenvalue $\lambda \neq 0$ must necessarily lie in the range, thus it is of the form

$$f(x) = a \cdot \cos 2x + b \cdot \sin x + c, \qquad a,\,b,\,c \in \mathbb{C}.$$

When we insert this expression into (7) and then apply that $1$, $\sin x$ and $\cos 2x$ are mutually orthogonal, we get

$$
\begin{aligned}
Kf(x) &= \left\{ -\frac{1}{2}c \cdot 2\pi \right\}\cos 2x + \left\{ 2b \cdot \frac{2\pi}{2} \right\}\sin x + \left\{ -\frac{1}{2}a \cdot \frac{2\pi}{2} + \frac{7}{8}c \cdot 2\pi \right\}\cdot 1 \\
&= -c\pi \cdot \cos 2x + 2b\pi \cdot \sin x + \left\{ \frac{7\pi}{4}c - \frac{\pi}{2}a \right\}\cdot 1.
\end{aligned}
$$

We have for comparison,

$$\lambda\, f(x) = \lambda\, a \cdot \cos 2x + \lambda\, b \cdot \sin x + \lambda\, c \cdot 1.$$

The coefficient $b$ occurs only in connection with $\sin x$, hence we conclude that $\sin x$ is an eigenfunction corresponding to the eigenvalue $\lambda = 2\pi$.

Assume that $b = 0$. If $a \cdot \cos 2x + c$ is an eigenfunction, then the vectors

$$\left( -c\pi, \frac{7\pi}{4}\, c - \frac{\pi}{2}\, a \right) = \pi \left( -c, \frac{7}{4}\, c - \frac{1}{2}\, a \right) \quad \text{og} \quad (a, c)$$

must be proportional with the eigenvalue $\lambda = -\dfrac{c}{a}\,\pi$ as the factor of proportion. Thus we get the condition

$$\begin{vmatrix} a & -c \\ c & \frac{7}{4}c - \frac{1}{2}a \end{vmatrix} = c^2 + \frac{7}{4}\, ac - \frac{1}{2}\, a^2 = 0.$$

By solving this equation with respect to $c$ we get

$$c = -\frac{7}{8}\, a \pm \sqrt{\frac{49}{64}\, a^2 + \frac{1}{2}\, a^2} = -\frac{7}{8}\, a \pm \sqrt{\frac{81}{64}\, a^2} = -\frac{7}{8}\, a \pm \frac{9}{8}\, a.$$

We have now two possibilities:

a) For $c = -\dfrac{7}{8}\, a - \dfrac{9}{8}\, a = -2a$ we get $\lambda = -\dfrac{c}{a}\,\pi = 2\pi$, corresponding to the eigenfunction $\cos 2x - 2$.

b) For $c = -\dfrac{7}{8}\, a + \dfrac{9}{8}\, a = \dfrac{1}{4}\, a$ we get $\lambda = -\dfrac{c}{a}\,\pi = -\dfrac{\pi}{4}$, corresponding to the eigenfunction $\cos 2x + \dfrac{1}{4}$.

Summing up,

$$\begin{aligned} \lambda_1 &= 2\pi, & \varphi_1(x) &= \sin x, \\ \lambda_2 &= 2\pi, & \varphi_2(x) &= \cos 2x - 2, \\ \lambda_3 &= -\frac{\pi}{4}, & \varphi_3(x) &= \cos 2x + \frac{1}{4}. \end{aligned}$$

Notice that $\lambda_1 = \lambda_2$, and that the eigenfunctions are not normed.

It follows e.g. from $(K\cos)(x) = 0$ that $\ker(K) \neq \emptyset$, thus

$$\sigma(K) = \sigma_p = \left\{ 0, -\frac{\pi}{2}, 2\pi \right\}.$$

3) The equation $Ku = \pi\, u - \dfrac{5\pi}{4}$ can be solved in several ways:

**First method.** The coefficient $\pi$ of $u$ on the right hand side of the equation does not belong to the spectrum, $\pi \notin \sigma(K)$, hence the solution is unique. Because

$$-\frac{5\pi}{4} = \frac{5\pi}{9}\, (\cos 2x - 2) - \frac{5\pi}{9}\, \left( \cos 2x + \frac{1}{4} \right),$$

we see that $-\dfrac{5\pi}{4}$ lies in the subspace spanned by the eigenvectors

$$\varphi_2(x) = \cos 2x - 2 \qquad \text{and} \qquad \varphi_3(x) = \cos 2x + \frac{1}{4}.$$

Thus we *guess* a solution of the structure

$$u(x) = a \cdot (\cos 2x - 2) + b \cdot \left(\cos 2x + \frac{1}{4}\right).$$

We get by insertion of this structure that

$$
\begin{aligned}
Ku(x) - \pi\, u(x) &= 2\pi a \cdot (\cos 2x - 2) - \frac{\pi}{4}\, b \cdot \left(\cos 2x + \frac{1}{4}\right) \\
&\qquad - \pi a(\cos 2x - 2) - \pi b \left(\cos 2x + \frac{1}{4}\right) \\
&= \pi a(\cos 2x - 2) - \frac{5\pi}{4}\, b \left(\cos 2x + \frac{1}{4}\right) \\
&= \pi \left(a - \frac{5}{4}\, b\right) \cos 2x - \pi \left(2a + \frac{5}{16} - b\right).
\end{aligned}
$$

This expression is equal to $-\dfrac{5\pi}{4}$, if

$$a = \frac{5}{4}\, b \qquad \text{and} \qquad 2 \cdot \frac{5}{4}\, b + \frac{1}{4} \cdot \frac{5}{4}\, b = \frac{5}{4},$$

hence $\dfrac{9}{4}\, b = 1$ and $b = \dfrac{4}{9}$, $a = \dfrac{5}{9}$. Finally, we get by insertion,

$$u(x) = \frac{5}{9}\left(\cos 2x - 2\right) + \frac{4}{9}\left(\cos 2x + \frac{1}{4}\right) = \cos 2x - 1 = -2\sin^2 x.$$

**Method 1a.** A variant of the FIRST METHOD is to guess a solution of the form

$$u(x) = a \cdot \cos 2x + c.$$

Then apply the previous computation from (2) to get

$$Ku(x) = -c\pi \cdot \cos 2x + \left\{\frac{7\pi}{4}\, c - \frac{\pi}{2}\, a\right\},$$

and

$$-\pi\, u(x) = -a\pi \cdot \cos 2x - c\pi,$$

hence

$$Ku(x) - \pi\, u(x) = -(a + c)\cos 2x + \frac{3\pi}{4}\, c - \frac{\pi}{2}\, a.$$

This expression is equal to $-\dfrac{5\pi}{4}$, if and only if

$$c = -a \qquad \text{and} \qquad -\frac{5\pi}{4} = \frac{3\pi}{4}\, c - \frac{\pi}{2}\, a = -\frac{5\pi}{4}\, a,$$

thus $a = 1$ and $c = -1$, and the unique solution is given by

$$u(x) = \cos 2x - 1 = -2\sin^2 x.$$

**Second method.** It is also possible to apply the standard method. A straightforward computation where we explicitly use the previously found eigenfunctions (these should then be normed), would demand a lot of energy, although one at different stages could apply one of the two variants above.

We shall show below how this might be carried out. First put

$$\varphi_1(x) = \sin x, \quad \varphi_2(x) = \cos 2x - 2, \quad \varphi_3(x) = \cos 2x + \frac{1}{4}.$$

Let $\{\varphi_n \mid n \geq 4\}$ denote an orthonormal basis of the null-space $\ker(K)$. Then a solution of the equation

$$Ku = \pi\, u - \frac{5\pi}{4},$$

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has the structure

$$u = \sum_{n=1}^{+\infty} a_n \, \varphi_n, \qquad \text{where } \sum_{n=4}^{+\infty} |a_n|^2 < +\infty.$$

Put $f(x) = -\dfrac{5\pi}{4}$. It follows from

$$(f, \varphi_n) = \left(-\frac{5\pi}{4}, \varphi_n\right) = 0 \qquad \text{for } n \in \mathbb{N} \setminus \{2, 3\},$$

and

$$f(x) = -\frac{5\pi}{4} = c_2(\cos 2x - 2) + c_3 \left(\cos 2x + \frac{1}{4}\right) = (c_2 + c_3)\cos 2x - \left(2c_2 - \frac{1}{4}c_3\right),$$

that $c_3 = -c_2$, and

$$2c_2 - \frac{1}{4}c_3 = 2c_2 + \frac{1}{4}c_2 = \frac{9}{4}c_2 = \frac{5\pi}{4},$$

thus

$$c_2 = \frac{5\pi}{9} \qquad \text{and} \qquad c_3 = -\frac{5\pi}{9}.$$

Then we get by insertion into the equation

$$Ku - \pi\,u = -\frac{5\pi}{4}$$

that

$$\begin{aligned}
Ku - \pi\,u &= \lambda_1 a_1 \varphi_1 + \lambda_2 a_2 \varphi_2 + \lambda_3 a_3 \varphi_3 - \sum_{n=1}^{+\infty} a_n \, \varphi_n \\
&= (2\pi - \pi)a_1\varphi_1 + (2\pi - \pi)a_2\varphi_2 - \left(\frac{\pi}{4} + \pi\right) a_3\varphi_3 - \pi\sum_{n=4}^{+\infty} a_n\varphi_n \\
&= \pi a_1\varphi_1 + \pi a_2\varphi_2 - \frac{5\pi}{4} a_3\varphi_3 - \pi\sum_{nm=4}^{+\infty} a_n\varphi_n \\
&= -\frac{5\pi}{4} = c_2\varphi_2 + c_3\varphi_3,
\end{aligned}$$

and we derive that

$$a_1 = 0, \quad a_2 = \frac{1}{\pi}c_2 = \frac{5}{9}, \quad a_3 = -\frac{4}{5\pi}c_3 = \frac{4}{9}, \quad a_n = 0 \text{ for } n \geq 4,$$

hence

$$u(x) = \frac{5}{9}(\cos 2x - 2) + \frac{4}{9}\left(\cos 2x + \frac{1}{4}\right) = \cos 2x - 1 = -2\sin^2 x.$$

**Example 1.14** *Let $k(x,t) = x + t + 2xt$ be the kernel for the Hilbert-Schmidt operator $K$ on the Hilbert space $H = L^2([-1,1])$.*
*Show that $K$ is self adjoint and determine the range $K(H)$.*
*Find all non-zero eigenvalues and corresponding eigenfunctions for $K$, and determine $\sigma(K)$ as well as $\|K\|$.*
*Express $Kf$, $f \in H$, with the help of the Legendre polynomials $(P_n)$.*
*Let $f(x) = \cosh(1)\cosh(x) - \cosh(2x)$. Show that $(f, P_0) = (f, P_1) = 0$ and solve the equation*

$$Ku(x) + u(x) = f(x).$$

1) It follows from

$$\overline{k(t,x)} = \overline{t + x + 2tx} = x + t + 2xt = k(x,t),$$

that the kernel is Hermitian, thus $K$ is self adjoint. We conclude from

$$Kf(x) = \int_{-1}^{1}(x + t + 2xt)f(t)\, dt = x\int_{-1}^{1}(1 + 2t)f(t)\, dt + \int_{-1}^{1}t\, f(t)\, dt,$$

that the range is $K\left(L^2([-1,1])\right) = \operatorname{span}\{1, x\}$.

2) The only possible eigenfunctions must be of the form $f(x) = ax + b$. We get by insertion the condition

$$\lambda f(x) = \lambda ax + \lambda b = Kf(x) = x\int_{-1}^{1}(1 + 2t)(at + b)\, dt + \int_{-1}^{1}t(qt + b) = dt,$$

hence

$$\lambda a = \int_{-1}^{1}(1 + 2t)(at + b)\, dt = \int_{-1}^{1}\left\{2at^2 + (a + 2b)t + b\right\}\, dt = \frac{4}{3}\,a + 2b$$

and

$$\lambda b = \int_{-1}^{1}\left(at^2 + bt\right)\, dt = \frac{2a}{3}.$$

Hence,

$$\lambda^2 a = \frac{4}{3}\,a\lambda + 2\lambda b = \frac{4}{3}\,\lambda a + \frac{4}{3}\,a.$$

If $a = 0$, then $2b = \left(\lambda - \frac{4}{3}\right)a = 0$, which leads to nothing, so we may assume that $a \neq 0$, e.g. $a = 1$. Then

$$\lambda^2 - \frac{4}{3}\,\lambda - \frac{4}{3} = 0,$$

i.e.

$$\lambda = \frac{2}{3} \pm \sqrt{\frac{4}{9} + \frac{4}{3}} = \frac{2}{3} \pm \sqrt{\frac{16}{9}} = \frac{2}{3} \pm \frac{4}{3} = \begin{cases} 2, \\ \\ -\dfrac{2}{3}. \end{cases}$$

If $\lambda_1 = 2$ and $a = 1$, then $b = \dfrac{1}{\lambda_1} \cdot \dfrac{2a}{3} = \dfrac{1}{3}$, and the corresponding eigenfunction is

$$\varphi_1(x) = x + \frac{1}{3}, \qquad \lambda_1 = 2.$$

If $\lambda_2 = -\dfrac{2}{3}$ and $a = 1$, then $b = \dfrac{1}{\lambda_2} \cdot \dfrac{2a}{3} = -\dfrac{3}{2} \cdot \dfrac{2}{3} = -1$, and the corresponding eigenfunction is

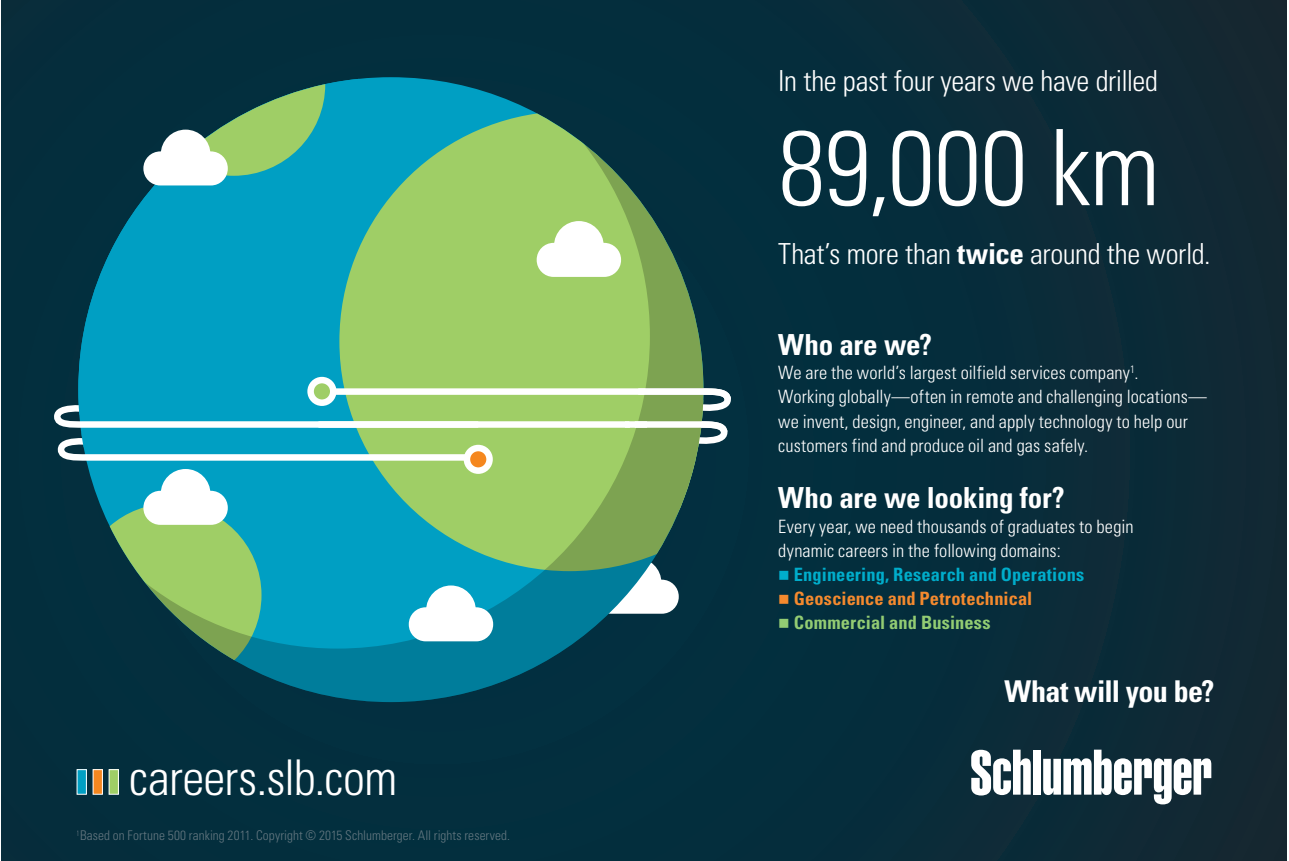$$\varphi_2(x) = x - 1, \qquad \lambda_2 = -\frac{2}{3}.$$

Since $K$ is self adjoint and of Hilbert-Schmidt-type, $\|K\|$ is the absolute value of the eigenvalue of largest absolute value,

$$\|K\| = 2.$$

Finally,

$$\sigma(K) = \sigma_p(K) = \left\{ -\frac{2}{3}, 0, 2 \right\},$$

and every function, which is orthogonal on both $\varphi_1$ and $\varphi_2$, i.e. on both 1 and $x$ by a change of basis, must lie in the eigenspace corresponding to $\lambda = 0$.

3) It is well-known that the Legendre polynomials form an orthogonal system on $L^2([-1,1])$. We have in particular,

$$P_0(t) = 1 \qquad \text{and} \qquad P_1(t) = t,$$

and since $\text{span}\{P_0, P_1\} = K\left(L^2([-1,1])\right)$, we infer that

$$KP_n = 0 \qquad \text{for every } n \geq 2.$$

It follows that if $f = \sum_{n=0}^{+\infty} a_n P_n$, then

$$
\begin{aligned}
Kf(x) &= K\left(\sum_{n=0}^{+\infty} a_n P_n\right)(x) = K\left(\sum_{n=0}^{1} a_n P_n\right)(x) \\
&= K\left(a_0 + a_1 t\right)(x) = \int_{-1}^{1} (a_0 + a_1 t)\,(x + t + 2xt)\,dt \\
&= \int_{-1}^{1} \left\{a_0 x + a_0 t + 2a_0 x \cdot t + a_1 x \cdot t + a_1(1+2x)t^2\right\}\,dt \\
&= 2a_0 x + \frac{2}{3}\,a_1(1+2x) = \left(2a_0 + \frac{4}{3}\,a_1\right)x + \frac{2}{3}\,a_1 \\
&= \left(2a_0 + \frac{4}{3}\,a_1\right) P_1(x) + \frac{2}{3}\,a_1 P_0(x).
\end{aligned}
$$

4) Let $f(x) = \cosh 1 \cdot \cosh x - \cosh 2x$. Then

$$
\begin{aligned}
(f, P_0) &= \int_{-1}^{1} \{\cosh 1 \cdot \cosh x - \cosh 2x\}dx = \cosh 1 \cdot [\sinh x]_{-1}^{1} - \left[\frac{1}{2}\,\sinh 2x\right]_{-1}^{1} \\
&= \cosh 1 \cdot 2\sinh 1 - \frac{1}{2}\cdot 2\sinh 2 = \sinh 2 - \sinh 2 = 0,
\end{aligned}
$$

and

$$(f, P_1) = \int_{-1}^{1} \{\cosh 1 \cdot \cosh x - \cosh 2x\} \cdot x\,dx = 0,$$

because the integrand is an odd function, and because we integrate over a finite symmetric interval.

Finally, we shall solve the equation

$$Ku(x) + u(x) = \cosh 1 \cdot \cosh x - \cosh 2x.$$

If

$$u = \sum_{n=0}^{+\infty} a_n P_n \quad \text{and} \quad \cosh 1 \cdot \cosh x - \cosh 2x = \sum_{n=2}^{+\infty} b_n P_n,$$

then it follows from the above that

$$
\frac{2}{3}\,a_1 P_0 + \left(2a_0 + \frac{4}{3}\,a_1\right) P_1 + a_0 P_0 + a_1 P_1 + \sum_{n=2}^{+\infty} a_n P_n
$$

$$
= \sum_{n=2}^{+\infty} b_n P_n = \cosh 1 \cdot \cosh x - \cosh 2x,
$$

and we conclude that $a_n = b_n$ for $n \geq 2$ and that

$$\begin{cases} a_0 + \dfrac{2}{3}\, a_1 = 0, \\[2mm] 2a_0 + \dfrac{7}{3}\, a_1 = 0, \end{cases} \qquad \text{hence } a_0 = a_1 = 0,$$

and whence

$$u = \sum_{n=2}^{+\infty} a_n P_n = \sum_{n=2}^{+\infty} b_n P_n = \cosh 1 \cdot \cosh x - \cosh 2x.$$

**Example 1.15** *In $L^2([-\pi,\pi])$ we consider the orthonormal basis $(e_n)$, $n \in \mathbb{Z}$, where*

$$e_n(t) = \frac{1}{\sqrt{2\pi}}\, e^{int}.$$

**1.** *Let $\varphi : \mathbb{R} \to \mathbb{C}$ denote a continuous function with period $2\pi$, and assume that $\varphi(-x) = \overline{\varphi(x)}$ for all $x \in \mathbb{R}$. Show that*

$$Ku(x) = \int_{-\pi}^{\pi} \varphi(x-t)\, u(t)\, dt$$

*defines a selfadjoint Hilbert-Schmidt operator on $L^2([-\pi,\pi])$.*

**2.** *Show that all $e_n$ are eigenfunctions for $K$.*

*From now on we assume that $\varphi$ is the periodic extension from $[-\pi,\pi]$ to $\mathbb{R}$ of the function*

$$\varphi(x) = 1 - \frac{|x|}{\pi}.$$

**3.** *Calculate the spectrum of $K$.*

**4.** *Solve the equation*

$$Ku = \frac{2}{\pi}\, u + f \qquad in\ L^2([-\pi,\pi]),$$

*where $f(x) = \sin^2(x) + \sin(x)$.*

**5.** *Solve the equation*

$$Ku = \frac{4}{\pi}\, u + 1 \qquad in\ L^2([-\pi,\pi]).$$

1) The kernel is

$$k(x,t) = \varphi(x-t), \qquad x,\, t \in [-\pi,\pi],$$

where

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |\varphi(x-t)|^2\, dt\, dx = \int_{-\pi}^{\pi}\left\{\int_{-\pi-t}^{\pi-t} |\varphi(u)|^2\, du\right\} dx = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |\varphi(u)|^2\, du\, dx$$
$$= 2\pi\, \|\varphi\|_2^2 < +\infty,$$

proving that $K$ is a Hilbert-Schmidt operator.

ALTERNATIVELY, $\varphi$ is continuous on a compact set, hence $|\varphi(x)| \leq c$ for $x \in [-\pi,\pi]$. Then apply the periodicity to get the estimate

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |\varphi(x-t)|^2\, dt\, dx \leq c^2(2\pi)^2 = 4\pi^2 c^2 < +\infty. \quad \lozenge$$

From $\varphi(-x) = \overline{\varphi(x)}$ follows that

$$\overline{k(t,x)} = \overline{\varphi(t-x)} = \varphi(x-t) = k(x,t),$$

which shows that the kernel is Hermitian, thus $K$ is self adjoint.

2) By insertion of $e_n(x)$ follows by a change of variable,

$$
\begin{aligned}
K e_n(x) &= \int_{-\pi}^{\pi} \varphi(x-t)\, e_n(t)\, dt = \int_{x-\pi}^{x+\pi} \varphi(u)\, e_n(x-u)\, du \\
&= \int_{x-\pi}^{x+\pi} \varphi(u) \cdot e^{-inu}\, du \cdot \frac{1}{\sqrt{2\pi}}\, e^{inx} = \int_{-\pi}^{\pi} \varphi(u)\, e^{-inu}\, du \cdot e_n(x),
\end{aligned}
$$

from which follows that every $e_n(x)$, $n \in \mathbb{Z}$, is an eigenfunction for $K$.

Conversely, if $\psi$ is an eigenfunction, then $\psi = \sum c_n e_n$, hence $\psi$ must lie in the subspace corresponding to the $e_n$, which have the same eigenvalue. This means that the eigenvalues are

$$
\int_{-\pi}^{\pi} \varphi(u)\, e^{-inu}\, du, \qquad n \in \mathbb{Z},
$$

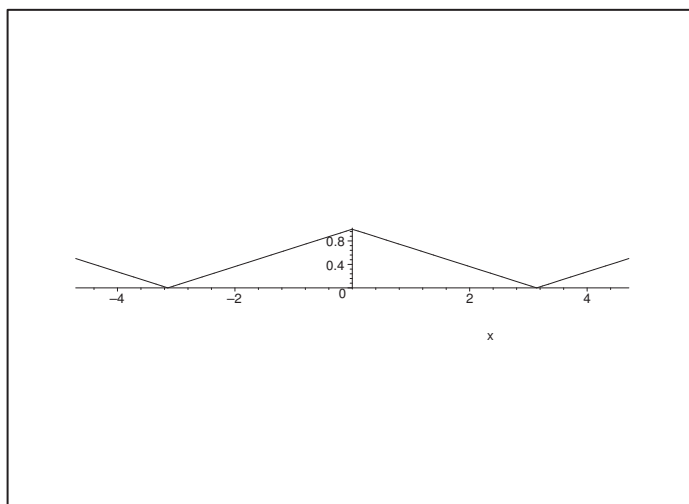and it suffices only to look at the eigenfunctions $e_n(x)$, $n \in \mathbb{Z}$, in the following.



Figure 1: The graph of the function $\varphi$.

3) If $\varphi(x) = 1 - \dfrac{|x|}{\pi}$ for $x \in [-\pi, \pi]$, then we have in particular that $\varphi(-x) = \overline{\varphi(x)}$, and that $\varphi$ is continuous – also after a periodic extension. Therefore, we are again in the situation above. If $n \neq 0$, then the eigenvalues are given by

$$
\begin{aligned}
\int_{-\pi}^{\pi} \left( 1 - \frac{|x|}{\pi} \right) e^{-inx}\, dx &= -\int_{-\pi}^{\pi} \frac{|x|}{\pi}\, e^{-inx}\, dx = -\frac{2}{x} \int_0^{\pi} x \cos(nx)\, dx \\
&= 0 + \frac{2}{n\pi} \int_0^{\pi} \sin(nx)\, dx = \frac{2\{1-(-1)^n\}}{\pi\, n^2}.
\end{aligned}
$$

For $n = 0$ we instead get by considering an area on the figure,

$$
\int_{-\pi}^{\pi} \left( -\frac{|x|}{\pi} \right) dx = \pi.
$$

ALTERNATIVELY,

$$\int_{-\pi}^{\pi} \left( a - \frac{|x|}{\pi} \right) dx = 2\pi - \frac{2}{\pi} \int_0^{\pi} x \, dx = 2\pi - \frac{2\pi^2}{2\pi} = \pi.$$

Summing up,

$$\lambda_0 = \pi, \qquad \begin{cases} \lambda_{2n} = 0, & n \in \mathbb{Z} \setminus \{0\}, \\[2mm] \lambda_{2n+1} = \dfrac{4}{\pi(2n+1)^2}, & n \in \mathbb{Z}, \end{cases}$$

and we conclude that the spectrum is

$$\sigma(K) = \sigma_p(K) = \{0, \pi\} \cup \left\{ \frac{4}{\pi(2n+1)^2} \ \middle| \ n \in \mathbb{N}_0 \right\}.$$

Notice that the eigenspace corresponding to each eigenvalue of the form $\dfrac{4}{\pi(2n+1)^2}$ is of dimension 2, while the eigenspace corresponding to $\lambda_0 = \pi$ is only of dimension 1.

4) Let

$$u = \sum c_n e_n = c_0 e_0 0 \sum_{n \neq 0} c_{2n} e_{2n} + \sum_{n \in \mathbb{Z}} c_{2n+1} e_{2n+1}.$$

Then

$$
\begin{aligned}
f(x) &= \sin^2 x + \sin x = \frac{1 - \cos 2x}{2} + \sin 2 = \frac{1}{2} + \frac{e^{ix} - e^{-ix}}{2i} - \frac{e^{2ix} + e^{-2ix}}{4} \\[2mm]
&= \frac{\sqrt{2\pi}}{2} e_0(x) + i \frac{\sqrt{2\pi}}{2} e_{-1}(x) - i \frac{\sqrt{2\pi}}{2} e_1(x) - \frac{\sqrt{2\pi}}{4} e_2(x) - \frac{\sqrt{2\pi}}{4} e_{-2}(x) \\[2mm]
&= Ku - \frac{2}{\pi} u \\[2mm]
&= \left( \pi - \frac{2}{\pi} \right) c_0 e_0(x) + \sum_{n \in \mathbb{Z} \setminus \{0\}} \left( -\frac{2}{\pi} \right) c_{2n} e_{2n}(x) + \sum_{n \in \mathbb{Z}} \left\{ \frac{4}{(2n+1)^2 \pi} - \frac{2}{\pi} \right\} c_{2n+1} e_{2n+1}(x).
\end{aligned}
$$

It follows from $\dfrac{2}{\pi} \notin \sigma_p(K) = \sigma(K)$ by identification that

$$c_0 = \frac{\sqrt{2\pi}}{2} \cdot \frac{1}{\pi - \dfrac{2}{\pi}} = \sqrt{2\pi} \cdot \frac{\pi}{2(\pi^2 - 2)},$$

and

$$c_{-1} = i \frac{\sqrt{2\pi}}{2} \cdot \frac{1}{\dfrac{4}{\pi} - \dfrac{2}{\pi}} = i\sqrt{2\pi} \cdot \frac{\pi}{4}, \qquad c_1 = \overline{c_{-1}} = -i\sqrt{2\pi} \cdot \frac{\pi}{4},$$

and

$$c_{-2} = c_2 = -\frac{\sqrt{2\pi}}{4} \cdot \frac{1}{-\dfrac{2}{\pi}} = \sqrt{2\pi} \cdot \frac{\pi}{8}, \qquad \text{and } c_n = 0 \quad \text{otherwise.}$$

This implies that

$$
\begin{aligned}
u(x) &= \frac{\pi}{2\left(\pi^2 - 2\right)}\,\sqrt{2\pi}\,e_0(x) + \frac{\pi}{2}\cdot\frac{\sqrt{2\pi}}{2i}\left\{e_1(x) - e_{-1}(x)\right\} + \frac{\pi}{4}\,\frac{\sqrt{2\pi}}{2}\left\{e_2(x) + e_{-2}(x)\right\} \\
&= \frac{\pi}{2\left(\pi^2 - 2\right)} + \frac{\pi}{2}\,\sin x + \frac{\pi}{4}\,\cos 2x.
\end{aligned}
$$

5) In this case, $\dfrac{4}{\pi}$ is an eigenvalue corresponding to the eigenvectors $e_1(x)$ and $e_{-1}(x)$. Since $1 = \sqrt{2\pi}\,e_0$ is orthogonal to $e_1$ and $e_{-1}$, we get

$$
u = c_{-1}e_{-1} + c_1 e_1 + c_0 e_0,
$$

where $c_{-1}$ and $c_1$ are arbitrary constants, and

$$
1 = K\left(c_0 e_0\right) - \frac{4}{\pi}\,c_0 e_0 = \left(\pi - \frac{4}{\pi}\right)c_0 e_0 = \left(\pi - \frac{4}{\pi}\right)c_0 \cdot \frac{1}{\sqrt{2\pi}},
$$

hence

$$
c_0 = \frac{\sqrt{2\pi}}{\pi - \dfrac{4}{\pi}} = \frac{\pi\sqrt{2\pi}}{\pi^2 - 4},
$$

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and we get the solutions

$$u(x) = \frac{\pi\sqrt{2\pi}}{\pi^2 - 4} + \tilde{c}_1\, e^{ix} + \tilde{c}_{-1}e^{-ix},$$

where $\tilde{c}_1$ and $\tilde{c}_{-1} \in \mathbb{C}$ are arbitrary constants.

**Example 1.16** *Let $H$ denote the Hilbert space $L^2([0, 2\pi])$ with the subspace $F = C([0, 2\pi])$, and let $K$ denote the integral operator on $H$ with the kernel*

$$k(x,t) = \begin{cases} \dfrac{i}{2}\,\exp\left(\dfrac{i}{2}\,(x-t)\right), & \text{if } 0 \le t < x \le 2\pi, \\[2mm] \qquad\quad 0 & \text{if } \;\; 0 \le t = x \le 2\pi, \\[2mm] -\dfrac{i}{2}\,\exp\left(\dfrac{i}{2}\,(x-t)\right), & \text{if } \;\; 0 \le x < t \le 2\pi. \end{cases}$$

*1) Show that $K$ is a self adjoint Hilbert-Schmidt operator.*

*2) Assume that $F$ is equipped with the* sup-*norm. Show that $K : H \to F$ is continuous.*

*3) Now let $S$ denote the restriction of $K$ to $F$ (considered as a subspace of $H$). Show that $S$ is injective and that $S^{-1}$ is given by*

$$D\left(S^{-1}\right) = \{g \in C^1([0, 2\pi]) \mid g(0) = g(2\pi)\},$$

*and*

$$S^{-1}g = -i\, g' - \frac{1}{2}\, g \qquad \text{for } g \in D\left(S^{-1}\right).$$

*4) Find all normalized eigenfunctions and associated eigenvalues for $S^{-1}$. Show that all eigenvalues are simple and that the set of normalized eigenfunctions is an orthonormal system in $H$.*

*5) Show that the eigenfunctions for $S^{-1}$ are also eigenfunctions for $K$ and find the associated eigenvalues. Justify that all eigenfunctions for $K$ are given this way, and write the kernel for $K$ using the normalized eigenfunctions.*

*6) Let $f \in H$ be given by the Fourier expansion*

$$f = \sum_{n=-\infty}^{\infty} c_n\, e^{inx}.$$

*Expand $Kf$ using the Fourier coefficients $c_n$ instead of $f$.*

1) The kernel $k(x,t)$ is bounded and continuous for $t \ne x$ in the compact set $[0, 2\pi]^2$, hence $k \in L^2\left([0, 2\pi]^2\right)$ with

$$\|k\|_2^2 = \int_0^{2\pi} \left\{\int_0^{2\pi} |k(x,t)|^2\, dt\right\} dx = \frac{1}{4} \cdot (2\pi)^2 = \pi^2,$$

i.e. $\|k\|_2 = \pi$. This shows that $K$ is a Hilbert-Schmidt operator.

We see from

$$\overline{k(t,x)} \;=\; \begin{cases} -\dfrac{i}{2}\,\exp\left(-\dfrac{i}{2}(t-x)\right), & \text{for } 0 \le x < t \le 2\pi, \\[2ex] \qquad 0 & \text{for } 0 \le x = t \le 2\pi, \\[2ex] \dfrac{i}{2}\,\exp\left(-\dfrac{i}{2}\,(t-x)\right), & \text{for } 0 \le t < x \le 2\pi, \end{cases}$$

$$=\; \begin{cases} \dfrac{i}{2}\,\exp\left(\dfrac{i}{2}\,(x-t)\right), & \text{for } 0 \le t < x \le 2\pi, \\[2ex] \qquad 0 & \text{for } 0 \le t = x \le 2\pi, \\[2ex] -\dfrac{i}{2}\,\exp\left(\dfrac{i}{2}\,(x-t)\right), & \text{for } 0 \le x < t \le 2\pi, \end{cases}$$

$$=\; k(x,t),$$

that $k(x,t)$ is Hermitian,, thus $K$ is a self adjoint Hilbert-Schmidt operator.

2) The operator $K$ is described by

$$\begin{aligned} Kf(x) &= \int_0^{2\pi} k(x,t)\,f(t)\,dt = \frac{i}{2}\int_0^x \exp\left(\frac{i}{2}\,(x-t)\right) f(t)\,dt - \frac{i}{2}\int_x^{2\pi} \exp\left(\frac{i}{2}\,(x-t)\right) f(t)\,dt \\[1ex] &= \frac{i}{2}\,\exp\left(i\,\frac{x}{2}\right)\int_0^x \exp\left(-i\,\frac{t}{2}\right) f(t)\,dt - \frac{i}{2}\,\exp\left(i\,\frac{x}{2}\right)\int_x^{2\pi} \exp\left(-i\,\frac{t}{2}\right) f(t)\,dt \\[1ex] &= \frac{i}{2}\,\exp\left(i\,\frac{x}{2}\right)\left\{\int_0^x \exp\left(-i\,\frac{t}{2}\right) f(t)\,dt + \int_{2\pi}^x \exp\left(-i\,\frac{t}{2}\right) f(t)\,dt\right\}. \end{aligned}$$

Applying the Cauchy-Schwarz inequality over $[x, x+\Delta x]$ we get

$$\left|\int_x^{x+\Delta x} \exp\left(-i\,\frac{t}{2}\right) f(t)\,dt\right| \le \|f\|_2 \cdot \sqrt{\Delta x},$$

where obviously the latter factor in the expression for $Kf(x)$ is continuous. The former factor is also continuous, so $K : H \to F$ is a mapping of $H$ into $F$.

Then we get the estimate

$$\begin{aligned} |Kf(x)| &\le \frac{1}{2}\cdot 1 \cdot\left\{\int_0^x 1\cdot|f(t)|\,dt + \int_x^{2\pi} 1\cdot|f(t)|\,dt\right\} \\[1ex] &\le \frac{1}{2}\,\|f\|_2\left\{\sqrt{x} + \sqrt{2\pi - x}\right\} \le \frac{1}{2}\,\|f\|_2\cdot\left\{\sqrt{\pi} + \sqrt{\pi}\right\} = \sqrt{\pi}\cdot\|f\|_2, \end{aligned}$$

because $\sqrt{x} + \sqrt{2\pi - x}$ has its maximum in the interval $[0, 2\pi]$ at $x = \pi$. Then

$$\|Kf\|_\infty \leq \sqrt{\pi} \cdot \|f\|_2, \qquad \text{hence} \qquad \|K\| \leq \sqrt{\pi},$$

and the linear operator $K : H \to F$ is continuous.

3) Assume that $f \in F$ with $Kf \equiv 0$. Then by (2),

$$\int_0^x \exp\left(-i\,\frac{t}{2}\right) f(t)\, dt + \int_{2\pi}^x \exp\left(-i\,\frac{t}{2}\right) f(t)\, dt = 0,$$

for all $x \in [0, 2\pi]$. Both integrands are continuous, and the sum of the integrals are $C^1$ and constant, hence by differentiation,

$$0 = \exp\left(-i\,\frac{x}{2}\right) f(x) + \exp\left(-i\,\frac{x}{2}\right) f(x) = 2\exp\left(-i\,\frac{x}{2}\right) f(x),$$

and we get $f \equiv 0$, so $S = K_{|F}$ is injective.

It was mentioned above that $Kf \in C^1$, if $f \in C$. Furthermore,

$$Kf(0) = \frac{i}{2} \cdot 1 \left\{ 0 - \int_0^{2\pi} \exp\left(-i\,\frac{t}{2}\right) f(t)\, dt \right\} = -\frac{i}{2} \int_0^{2\pi} \exp\left(-i\,\frac{t}{2}\right) f(t)\, dt,$$

and

$$\begin{aligned} Kf(2\pi) &= \frac{i}{2} \exp\left(i \cdot \frac{2\pi}{2}\right) \left\{ \int_0^{2\pi} \exp\left(-i\,\frac{t}{2}\right) f(t)\, dt + 0 \right\} \\ &= -\frac{i}{2} \int_0^{2\pi} \exp\left(-i\,\frac{t}{2}\right) f(t)\, dt = Kf(0), \end{aligned}$$

so we infer that

$$D\left(S^{-1}\right) = KF \subseteqq \{ g \in C^1([0, 2\pi]) \mid g(0) = g(2\pi) \}.$$

If on the other hand $g \in C^1([0, 2\pi])$ satisfies $g(0) = g(2\pi)$, then we shall check if the equation

$$Kf(x) = \frac{i}{2} \exp\left(i\,\frac{x}{2}\right) \left\{ \int_0^x \exp\left(-i\,\frac{t}{2}\right) f(t)\, dt + \int_{2\pi}^x \exp\left(-i\,\frac{t}{2}\right) f(t)\, dt \right\} = g(x)$$

has a solution $f \in F$. This equation is equivalent to

$$(8) \quad \int_0^x \exp\left(-i\,\frac{t}{2}\right) f(t)\, dt + \int_{2\pi}^x \exp\left(-i\,\frac{t}{2}\right) f(t)\, dt = -2i \exp\left(-i\,\frac{x}{2}\right) g(x),$$

so we get by differentiation,

$$(9) \quad 2\exp\left(-i\,\frac{x}{2}\right) f(x) = -2i \exp\left(-i\,\frac{x}{2}\right) \left\{ -\frac{i}{2}\, g(x) + g'(x) \right\},$$

where (9) is equivalent to that the candidate $f(x)$ must have the structure

$$f(x) = -\frac{1}{2}\, g(x) - i\, g'(x).$$

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It is obvious that $f$ given in this way is continuous, when $g \in C^1$. The proof will be concluded, if we can prove that the additional condition $g(0) = g(2\pi)$ combined with (9) implies (8). The trick is that we write

$$2 \exp\left(-i\frac{x}{2}\right) f(x) = \exp\left(-i\frac{x}{2}\right) f(x) + \exp\left(-i\frac{x}{2}\right) f(x),$$

where we integrate the former term on the right hand side from $0$ to $x$, and the latter from $2\pi$ to $x$. This construction is guaranteed by the assumption $g(0) = g(2\pi)$.

ALTERNATIVELY one may compute explicitly,

$$K f(x) = -i\, K(g')(x) - \frac{1}{2}\, K(g)(x),$$

and then convince oneself by some partial integration that the result is $g(x)$. $\Diamond$

4) The equation $S^{-1} g(x) = \lambda\, g(x)$ for $g \in D\left(S^{-1}\right)$ is rewritten as

$$-i\, g'(x) - \frac{1}{2}\, g(x) = \lambda\, g(x), \qquad g(0) = g(2\pi), \quad g \in C^1([0, 2\pi]),$$

i.e.

$$g'(x) = i\left\{\lambda + \frac{1}{2}\right\} g(x), \qquad g(0) = g(2\pi).$$

The complete solution without the boundary condition is

$$g(x) = c \cdot \exp\left(i\left(\lambda + \frac{1}{2}\right)x\right).$$

Choosing $c = 1$ and inserting into the boundary condition, we get

$$\exp\left(i\left(\lambda + \frac{1}{2}\right)0\right) = 1 = \exp\left(i\left(\lambda + \frac{1}{2}\right)\cdot 2\pi\right),$$

the solutions of which are $\lambda_n + \dfrac{1}{2} = n \in \mathbb{Z}$.

The eigenvalues are

$$\sigma_p\left(S^{-1}\right) = \left\{\lambda_n = n - \frac{1}{2} \ \middle| \ n \in \mathbb{Z}\right\},$$

with the corresponding normalized eigenfunctions

$$e_n(x) = \frac{1}{\sqrt{2\pi}}\, e^{in\pi}, \qquad n \in \mathbb{Z}.$$

5) It follows from $S^{-1}e_n(x) = \lambda_n e_n(x)$ that

$$\lambda_n K e_n(x) = e_n(x), \qquad \text{thus} \qquad K e_n(x) = \frac{1}{\lambda_n}\, e_n(x),$$

and $K$ has the same eigenfunctions as $S^{-1}$, and the corresponding eigenvalues are

$$\left\{\frac{1}{\lambda_n} = \frac{1}{n - \frac{1}{2}} = \frac{2}{2n - 1} \ \middle| \ n \in \mathbb{Z}\right\} \subseteqq \sigma_p(K).$$

Using that $K$ is a self adjoint Hilbert-Schmidt operator, we get that the spectrum is given by

$$\sigma(K) = \{0\} \cup \left\{\frac{2}{2n - 1} \ \middle| \ n \in \mathbb{Z}\right\},$$

where each $\dfrac{2}{2n - 1}$ is an eigenvalue. Now, $K$ is injective according to (3), so 0 is not an eigenvalue, thus

$$\sigma_c(K) = \{0\} \quad \text{and} \quad \sigma_p(K) = \left\{\frac{2}{2n - 1} \ \middle| \ n \in \mathbb{Z}\right\}.$$

Finally,

$$k(x, t) = \sum_{n=-\infty}^{+\infty} \frac{1}{\lambda_n}\, e_n(x)\cdot\overline{e_n(t)} = \frac{1}{\pi}\sum_{n=-\infty}^{+\infty} \frac{1}{2n - 1}\, e^{in(x - t)}.$$

6)  Let $f \in H$ be given by the Fourier expansion

$$f = \sum_{n=-\infty}^{+\infty} c_n e^{inx}.$$

Since $e^{inx}$ is an eigenfunction for $K$ corresponding to the eigenvalue $\dfrac{1}{\lambda_n} = \dfrac{2}{2n-1}$, it follows by a termwise application of $K$ that

$$Kf = \sum_{-\infty}^{+\infty} c_n K\left(e^{in\star}\right) = \sum_{n=-\infty}^{+\infty} \frac{2}{2n-1}\, c_n e^{inx}.$$

# 2  Other types of integral operators

**Example 2.1** *We shall consider $H = L^2([0,1])$ as a real Hilbert space, and define $T : H \to H$ by*

$$Tf(x) = \int_0^x f(t)\, dt.$$

*Show that*

$$|Tf(x)| \le \sqrt{x}\, \|f\|_2,$$

*and use this to show that $\|T\| < 1$.*
*Show that*

$$T^n f(x) = \int_0^x \frac{(x-t)^{n-1}}{(n-1)!}\, f(t)\, dt.$$

*Show that $\log(I+T)$ is a well-defined operator of Volterra type, and find an explicit expression for the kernel of this operator, using only known functions, that is, find $k$ such that*

$$\log(I+T)f(x) = \int_0^x k(x,t)\, f(t)\, dt.$$

1) It follows form the Cauchy-Schwarz inequality that

$$
\begin{aligned}
|Tf(x)| &= \left| \int_0^x f(t)\, dt \right| = \left| \int_0^1 1_{[0,x]}(t)\, f(t)\, dt \right| \le \left\| 1_{[0,x]} \right\|_2 \|f\|_2 \\
&= \left( \int_0^1 \left\{ 1_{[0,x]}(t) \right\}^2 dt \right)^{\frac{1}{2}} \|f\|_2 = \left\{ \int_0^x dt \right\}^{\frac{1}{2}} \|f\|_2 = \sqrt{x} \cdot \|f\|_2.
\end{aligned}
$$

(There are more variants of this computation).

2) It follows from the estimate above that

$$\|Tf\|_2^2 = \int_0^1 |Tf(x)|^2 dx \le \int_0^1 x\, \|f\|_2^2\, dx = \left[ \frac{x^2}{2} \right]_0^1 \|f\|_2^2 = \frac{1}{2}\, \|f\|_2^2,$$

and we conclude that

$$\|T\| \le \frac{1}{\sqrt{2}} < 1.$$

3) The formula clearly holds for $n = 1$. Assume that for some $n \in \mathbb{N}$,

$$T^n f(x) = \int_0^x \frac{(x-t)^{n-1}}{(n-1)!}\, f(t)\, dt, \qquad f \in L^2([0,1]).$$

Interchanging the order of integration in the computation below we get

$$
\begin{aligned}
T^{n+1} f(x) &= T^n(Tf)(x) = \int_0^x \frac{(x-t)^{n-1}}{(n-1)!}\, Tf(t)\, dt = \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} \int_0^t f(s)\, ds\, dt \\
&= \int_0^x \left\{ \int_s^x \frac{(x-t)^{n-1}}{(n-1)!}\, dt \right\} f(s)\, ds = \int_0^x \left[ -\frac{(x-t)^n}{n!} \right]_{t=s}^{t=x} f(s)\, ds \\
&= \int_0^x \frac{(x-s)^n}{n!}\, f(s)\, ds,
\end{aligned}
$$

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and it follows that the formula also holds, when $n$ is replaced by $n + 1$. Then the claim follows by induction.

4) Now,

$$\varphi(\lambda) = \log(1 + \lambda) = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{1}{n} \lambda^n, \qquad \text{for } |\lambda| < 1,$$

and $T \in B(L^2([0,1])$ with $\|T\| \leq \dfrac{1}{\sqrt{2}} < 1$, so the operator $\log(I + T)$ is indeed defined by

$$\varphi(T) = \log(I + T) = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{1}{n} T^n.$$

Each of the $T^n$ is of Volterra type, and $\varphi(T)$ contains only $T^n$ for $n \geq 1$, hence $\varphi(T)$ is also of Volterra type.

5) When we insert the expression for $T^n f$ from (3), we get by purely formal computations that

$$\log(I + T)f(x) = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{1}{n} \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} f(t)\, dt = \sum_{n=1}^{+\infty} \int_0^x \frac{(t-x)^{n-1}}{n!} f(t)\, dt.$$

However, the series $\sum_{n=1}^{+\infty} \dfrac{(t-x)^{n-1}}{n!}$ is *uniformly* convergent for $0 \leq t \leq x \leq 1$. (Notice that we get the sum 1 for $t = x$). Therefore it is indeed legal to interchange summation and integration. The we get for $0 \leq t < x$ the sum

$$\sum_{n=1}^{+\infty} \frac{(t-x)^{n-1}}{n!} = \frac{1}{t-x} \left\{ \sum_{n=0}^{+\infty} \frac{(t-x)^n}{n!} - 1 \right\} = \frac{e^{t-x} - 1}{t-x} = e^{-x} \cdot \frac{e^x - e^t}{x - t}.$$

Note that we for $t \to x$ get the limit $e^{-x} \cdot e^x = 1$.

We get by interchanging summation and integration,

$$\log(I + T)f(x) = \int_0^x e^{-x} \cdot \frac{e^x - e^t}{x - t} f(t)\, dt,$$

so the kernel of the Volterra operator $\log(I + T)$ is given by

$$k(x, t) = \begin{cases} e^{-x} \cdot \dfrac{e^x - e^t}{x - t} & \text{for } 0 \leq t < x \leq 1, \\[3mm] 1 & \text{for } 0 \leq t = x \leq 1, \\[3mm] 0 & \text{otherwise.} \end{cases}$$

**Example 2.2** *In this example it is allowed to change the order of integrations without justification. Consider the operator*

$$Af(x) = \frac{1}{\sqrt{\pi}} \int_0^x \frac{f(t)}{\sqrt{x-t}} \, dt, \qquad x \in [0, 1],$$

*whenever this expression gives sense.*

1) *Show that* $Af \in L^{\infty}([0, 1])$ *if* $f \in L^p([0, 1])$, $p > 2$.

2) *Find the operator* $B = A^2$, *that is find the kernel* $k(x, t)$ *such that*

$$Bf(x) = A^2 f(x) = \int_0^x k(x, t) \, f(t) \, dt$$

*for* $f \in L^p([0, 1])$, $p > 2$.

3) *Show that* $B : L^p([0, 1]) \to L^{\infty}([0, 1])$, $1 \le p \le \infty$ *is bounded.*

4) *Solve the equation*

$$(I - A)f(x) = 1$$

*formally by a Neumann series, and express* $f$ *as*

$$f(x) = g(x) + Ah(x),$$

*where* $g$ *and* $h$ *are known functions. (Here it is not possible to express* $Ah(x)$ *as a known function.) Insert and show that this formal solution* is *a solution.*

**Remark 2.1** First note that the kernel does *not* belong to $L^2([0, 1]^2)$. In fact, it follows from

$$k(x, t) = \begin{cases} \dfrac{1}{\sqrt{x-t}} & \text{for } 0 \le t < x \le 1, \\ 0 & \text{otherwise,} \end{cases}$$

that

$$\int_0^1 \int_0^1 |k(x, t)|^2 dt \, dx = \int_0^1 \left\{ \int_0^x \frac{dt}{x-t} \right\} dx = \int_0^1 [-\ln(x-t)]_{t=0}^x \, dx = +\infty,$$

so we cannot apply the theory of the Hilbert-Schmidt operators. Part of the example is to use other methods. ◊

1) Given $f \in L^p([0, 1])$, where $p > 2$, thus $1 < q < 2$, where $q$ is the conjugated number of $p$, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Then by the *Hölder inequality*

$$
\begin{aligned}
|Af(x)| &\le \frac{1}{\sqrt{\pi}} \int_0^x \frac{|f(t)|}{\sqrt{x-t}} \, dt \le \frac{1}{\sqrt{\pi}} \left\{ \int_0^x |f(t)|^p \, dt \right\}^{\frac{1}{p}} \left\{ \int_0^x \frac{dt}{(x-t)^{q/2}} \right\}^{\frac{1}{q}} \\
&\le \frac{1}{\sqrt{\pi}} \|f\|_p \left\{ \frac{-1}{1-\frac{q}{2}} \left[ (x-t)^{1-\frac{q}{2}} \right]_{t=0}^x \right\}^{\frac{1}{q}} = \frac{1}{\sqrt{\pi}} \|f\|_p \left\{ \frac{1}{1-\frac{q}{2}} x^{1-\frac{q}{2}} \right\}^{\frac{1}{q}} \\
&\le \frac{1}{\sqrt{\pi}} \cdot \left\{ 1 - \frac{q}{2} \right\}^{-\frac{1}{q}} \|f\|_p,
\end{aligned}
$$

where we have used that $1 - \dfrac{q}{2} > 0$, because $p > 2$. This holds for all $x \in [0,1]$, so

$$\|Af\|_\infty \leq \frac{1}{\sqrt{\pi}} \cdot \left\{1 - \frac{q}{2}\right\}^{-\frac{1}{q}} \|f\|_p,$$

and $Af \in L^\infty([0,1])$ for $f \in L^p([0,1])$, when $2 < p < +\infty$.

If instead $p = +\infty$, then we get the following estimate,

$$
\begin{aligned}
|Af(x)| &\leq \frac{1}{\sqrt{\pi}} \int_0^x \frac{|f(t)|}{\sqrt{x-t}}\, dt = \frac{1}{\sqrt{\pi}} \|f\|_\infty \int_0^x \frac{dt}{\sqrt{x-t}} \\
&= \frac{1}{\sqrt{\pi}} \|f\|_\infty \cdot \left[\frac{-1}{1-\dfrac{1}{2}} \sqrt{x-t}\right]_0^x = \frac{2}{\sqrt{\pi}} \sqrt{x} \cdot \|f\|_\infty \leq \frac{2}{\sqrt{\pi}} \|f\|_\infty,
\end{aligned}
$$

and we get in this case that

$$\|Af\|_\infty \leq \frac{2}{\sqrt{\pi}} \|f\|_\infty,$$

hence $Af \in L^\infty([0,1])$ for $f \in L^\infty([0,1])$.

2) Assume again that $f \in L^p([0,1])$, where $p > 2$. Then $Af \in L^\infty([0,1])$ according to (1). From $p_1 = \infty > 2$ follows by another application of (1) that $A^2 f \in L^\infty([0,1])$.

Compute

$$Bf(x) = A^2 f(x) = \frac{1}{\sqrt{\pi}} \int_0^x \frac{1}{\sqrt{x-t}} \, Af(t) \, dt = \frac{1}{\sqrt{\pi}} \int_0^x \frac{1}{\sqrt{x-t}} \left\{ \frac{1}{\sqrt{\pi}} \int_0^t \frac{f(u)}{\sqrt{t-u}} \, du \right\} dt.$$

From $0 \le u \le t \le x \le 1$ we infer by an interchange of the integrals fås follows by the change of variable $xs = t - u$ that

$$\begin{aligned}
Bf(x) &= \frac{1}{\pi} \int_0^x \left\{ \int_u^x \frac{dt}{\sqrt{(x-t)(t-u)}} \right\} f(u) \, du = \frac{1}{\pi} \int_0^x \left\{ \int_0^{x-u} \frac{ds}{\sqrt{\{(x-u)-s)s\}}} \right\} f(u) \, du \\
&= \frac{1}{\pi} \int_0^x \pi \, f(u) \, du = \int_0^x f(t) \, dt,
\end{aligned}$$

where we have used that

$$\int_0^a \frac{ds}{\sqrt{(a-s)s}} = \pi \qquad \text{for } a = x - u > 0.$$

**Remark 2.2** We prove for completeness this formula. We get by the monotonous substitution $s = a \sin^2 \theta$, $\theta \in \left[ 0, \frac{\pi}{2} \right]$,

$$\begin{aligned}
\int_0^a \frac{ds}{\sqrt{(a-s)s}} &= \int_0^{\frac{\pi}{2}} \frac{1 \cdot 2 \sin\theta \cos\theta}{\sqrt{(a - a \sin^2\theta) \cdot a \sin^2\theta}} \, d\theta = 2a \int_0^{\frac{\pi}{2}} \frac{\sin\theta \cos\theta}{\sqrt{a^2(1 - \sin^2\theta) \sin^2\theta}} \, d\theta \\
&= \frac{2a}{|a|} \int_0^{\frac{\pi}{2}} \frac{\cos\theta \sin\theta}{|\cos\theta \sin\theta|} \, d\theta = 2 \int_0^{\frac{\pi}{2}} d\theta = \pi. \qquad \diamond
\end{aligned}$$

The operator is therefore a well-known integral operator, and $A$ corresponds to "integrating one half time from 0". The kernel is explicitly given by

$$k(x,t) = \begin{cases} 1 & \text{for } 0 \le t \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

3) This follows easily from the Hölder inequality,

$$|Bf(x)| \le \int_0^x |f(t)| \, dt \le \int_0^1 |f(t)| \cdot 1 \, dt \le 1 \cdot \|f\|_p,$$

hence $\|Bf\|_\infty \le \|f\|_p$, and $\|B\| \le 1$.

4) The Neumann series is given by

$$(I - A)^{-1} = \sum_{n=0}^{+\infty} A^n,$$

so the formal solution is

$$\begin{aligned}
f(x) &= \sum_{n=0}^{+\infty} A^n 1(x) = \sum_{n=0}^{+\infty} A^{2n} 1(x) + \sum_{n=0}^{+\infty} A^{2n+1} 1(x) \\
&= \sum_{n=0}^{+\infty} B^n 1(x) + A \sum_{n=0}^{+\infty} B^n 1(x) = g(x) + Ag(x),
\end{aligned}$$

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hence

$$h(x) = g(x) \quad = \quad \sum_{n=0}^{+\infty} B^n 1(x) = 1 + \sum_{n=1}^{+\infty} B^n 1(x) = 1 + \sum_{n=1}^{+\infty} \int_0^x \frac{t^{n-1}}{(n-1)!} \cdot 1 \, dt$$

$$= \quad 1 + \sum_{n=1}^{+\infty} \frac{x^n}{n!} = e^x,$$

and the formal solution is

$$f(x) = e^x + A e^x.$$

Then we get by insertion

$$(I - A)f(f) \quad = \quad f(x) - Af(f) = e^x + Ae^x - Ae^x - A^2 e^x$$

$$= \quad e^x - Be^x = e^x - \int_0^x e^t \, dt = e^x - \left[ e^t \right]_0^x = e^x - (e^x - 1) = 1,$$

and we have proved that we have found a solution.

ALTERNATIVELY (and more elegantly),

$$(I - A)(I + A) = (I + A)(I - A) = I - A^2 = I - B.$$

Since $B$ is a Volterra operator, we have that $(I - B)^{-1} = \sum_{n=0}^{+\infty} B^n$ is bounded. Clearly, $A$ and $B = A^2$ commutes, so

$$(I - A)\left\{(I + A)(I - B)^{-1}\right\} = \left\{(I + A)(I - B)^{-1}\right\}(I - A) = I,$$

proving that

$$(I - A)^{-1} = (I + A)(I - B)^{-1}.$$

Hence the equation $(I - A)f = 1$ is equivalent to

$$f(x) = (I - A)^{-1}A(x) = (I + A)\sum_{n=0}^{+\infty} B^n 1(x) = (I + A)e^x = e^x + Ae^x,$$

where we have applied the computation above.

**Example 2.3** *Let $H = L^2([0,1])$ and consider the integral operator*

$$Bf(x) = \int_0^x f(t)\, dt, \qquad \text{for } f \in H.$$

*1) Show that*

$$k(x,t) = \min\{x, t\}, \qquad 0 \le x,\, t \le 1,$$

*is the kernel for the self adjoint Hilbert-Schmidt operator $K = BB^\star$.*

*2) Let $\varphi$ be an eigenfunction for $K$ associated with a non-zero eigenvalue $\lambda$. Justify that $\varphi$ can be taken as a $C^\infty$-function.*
*Next, show that $\varphi$ must satisfy the equation*

$$\lambda\, \varphi''(x) = -\varphi(x),$$

*and use this to find all non-zero eigenvalues for $K$ and all the associated eigenfunctions.*

*3) Assuming the $\|BB^\star\| = \|B^\star\|^2$, show that $\|K\| = \|B\|^2$, and find both $\|K\|$ and $\|B\|$.*

1) The operator $B$ has the kernel

$$b(x,t) = \begin{cases} 1 & \text{for } 0 \le t \le x \le 1, \\ 0 & \text{otherwise,} \end{cases}$$

so

$$b^\star(x,t) = \overline{b(t,x)} = b(t,x) = \begin{cases} 1 & \text{for } 0 \le x \le t \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then the kernel $k(x,t)$ for $K = BB^\star$ is given by

$$\begin{aligned}
k(x,t) &= \int_0^1 b(x,s)b^\star(s,t)\, ds = \int_0^1 b(x,s)b(t,s)\, ds \\
&= \int_0^1 b(\min\{x,t\}, s)\, ds = \min\{x,t\}, \qquad x,\, t \in [0,1].
\end{aligned}$$

2) Since $k(x,t)$ is continuous, we can choose the eigenfunctions continuous. Hence, if $\varphi(x)$ is an eigenfunction corresponding to an eigenvalue $\lambda \neq 0$, then

(10)  $\lambda \lambda \varphi(x) = \displaystyle\int_0^1 k(x,t)\,\varphi(t)\,dt = \int_0^x t\,\varphi(t)\,dt + x \int_x^1 \varphi(t)\,dt.$

If $\varphi$ is continuous, then the right hand side of (10) is differentiable. If $\varphi$ is of class $C^n$, then the right hand side of (10) is of class $C^{n+1}$, hence $\varphi$ is also of class $C^{n+1}$. Then the claim follows by induction, hence $\varphi \in C^\infty$.

When we differentiate (10), we get

$$\lambda \varphi'(x) = x\,\varphi(x) + \int_x^1 \varphi(t)\,dt - x\,\varphi(x) = \int_x^1 \varphi(t)\,dt,$$

hence by another differentiation,

(11)  $\lambda \varphi''(x) = -\varphi(x),$

and the claim is proved.

3) Let $\alpha \in \mathbb{C} \setminus \{0\}$ satisfy the condition $\alpha^2 = \dfrac{1}{\lambda}$. Then the equation (11) has the complete solution

(12)  $\varphi(x) = C_1 e^{i\alpha x} + C_2 e^{-i\alpha x}.$

When (12) is put into (10), and we apply that $\dfrac{1}{\alpha^2} = \lambda$, then

$$\begin{aligned}
\lambda \varphi(x) &= \lambda \left\{ C_1 e^{i\alpha x} + C_2 e^{-i\alpha x} \right\} \\
&= \int_0^x t \left\{ C_1 e^{i\alpha t} + C_2 e^{-i\alpha t} \right\} dt + x \int_x^1 \left\{ C_1 e^{i\alpha t} + C_2 e^{-i\alpha t} \right\} dt \\
&= \left[ t \left\{ \frac{C_1}{i\alpha} e^{i\alpha t} - \frac{C_2}{i\alpha} e^{-i\alpha t} \right\} \right]_0^x - \int_0^x \left\{ \frac{C_1}{i\alpha} e^{i\alpha t} - \frac{C_2}{i\alpha} e^{-i\alpha t} \right\} dt \\
&\quad + x \left[ \frac{C_1}{i\alpha} e^{i\alpha t} - \frac{C_2}{i\alpha} e^{-i\alpha t} \right]_x^1 \\
&= x \left\{ \frac{C_1}{i\alpha} e^{i\alpha x} - \frac{C_2}{i\alpha} e^{-i\alpha x} \right\} - \left[ \frac{C_1}{i^2 \alpha^2} e^{i\alpha t} + \frac{C_2}{i^2 \alpha^2} e^{-i\alpha t} \right]_0^x \\
&\quad + x \left\{ \frac{C_1}{i\alpha} e^{i\alpha} - \frac{C_2}{i\alpha} e^{-i\alpha} \right\} - x \left\{ \frac{C_1}{i\alpha} e^{i\alpha x} - \frac{C_2}{i\alpha} e^{-i\alpha x} \right\} \\
&= \frac{1}{\alpha^2} \left\{ C_1 e^{i\alpha x} + C_2 e^{-i\alpha x} \right\} - \frac{1}{\alpha^2} \left\{ C_1 + C_2 \right\} + \frac{x}{i\alpha} \left\{ C_1 e^{i\alpha} - C_2 e^{-i\alpha} \right\} \\
&= \lambda \varphi(x) - \lambda \left\{ C_1 + C_2 \right\} + \frac{x}{i\alpha} \left\{ C_1 e^{i\alpha} - C_2 e^{-i\alpha} \right\}.
\end{aligned}$$

This equation holds for every $x$, and $\lambda \neq 0$ and $\alpha \neq 0$, so we conclude that

$$C_1 + C_2 = 0 \qquad \text{and} \qquad C_1 e^{i\alpha} - C_2 e^{-i\alpha} = 0,$$

hence $C_2 = -C_1$, and $C_1 \left\{ e^{i\alpha} + e^{-i\alpha} \right\} = 2 C_1 \cos \alpha = 0$, thus

$$\alpha = \frac{\pi}{2} + n\,\pi, \qquad n \in \mathbb{Z}.$$

It follows from

$$\varphi(x) = C_1 e^{i\alpha x} + C_2 e^{-i\alpha x} = C_1\left\{e^{i\alpha x} - e^{-i\alpha x}\right\} = 2i\,C_1 \sin\alpha x,$$

that the eigenfunctions for $K$ corresponding to a $\lambda \in \sigma_p(K) \setminus \{0\}$ are some constant times

$$\varphi_n(x) = \sin\left(\left(n - \frac{1}{2}\right)\pi x\right), \qquad n \in \mathbb{N},$$

corresponding to the eigenvalue

$$\lambda_n = \frac{1}{\alpha_n^2} = \frac{4}{\pi^2}\cdot\frac{1}{(2n+1)^2}, \qquad n \in \mathbb{N}.$$

4) Now, $\|K\|$ is the absolute value of the numerically largest eigenvalue $|\lambda_1|$, so

$$\|K\| = \|BB\star\| = \lambda_1 = \frac{4}{\pi^2}\cdot\frac{1}{(2-1)^2} = \left(\frac{2}{\pi}\right)^2.$$

On the other hand, $BB^\star$ is self adjoint, hence

$$\begin{aligned}
|BB\star\| &= \sup\{|(BB^\star f, f)| \mid f \in L^2([0,1]), \|f\|_2 = 1\}\\
&= \sup\{(B^\star f, B^\star f) \mid f \in L^2([0,1]), \|f\|_2 = 1\}\\
&= \sup\{\|B^\star f\|^2 \mid f \in L^2([0,1]), \|f\|_2 = 1\} = \|B^\star\|^2.
\end{aligned}$$

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Finally, $B \in B(H)$, hence also $B^\star \in B(H)$ with $\|B^\star\| = \|B\|$, and whence

$$\|K\| = \|BB^\star\| = \|B^\star\|^2 = \|B\|^2 = \left(\frac{2}{\pi}\right)^2.$$

Then

$$\|B\| = \frac{2}{\pi},$$

where

$$Bf(x) = \int_0^x f(t)\, dt, \qquad f \in L^2([0,1]).$$

**Example 2.4** *Let $H = L^2([0,1])$ and consider the operator $K$ with domain $D(K) = C([0,1])$ given by*

$$Kf(x) = x \int_0^x f(t)\, dt + \int_x^1 t\, f(t)\, dt, \qquad f \in D(K).$$

*1) Show that $K : D(K) \to C^2([0,1])$, and that*

$$(Kf)'(0) = 0 \qquad and \qquad (Kf)'(1) = (Kf)(1).$$

*2) Show that $K$ is injective and that $K^{-1}$ has the domain*

$$D\left(K^{-1}\right) = \{u \in C^2([0,1]) \mid u'(0) = 0,\ u(1) = u'(1)\},$$

*and the action $K^{-1}u = u''$.*

*3) Show that $K$ is an integral operator with continuous and symmetric kernel and find this kernel.*

*4) Let $\varphi$ and $\psi$ denote eigenfunctions for $K$ associated to the same eigenvalue $\lambda$. Define the function $f$ by*

$$f(x) = \psi(0)\, \varphi(x) - \varphi(0)\, \psi(x),$$

*and use the existence and uniqueness theorem for ordinary differential equations to argue that $f = 0$.*
*Next show that all eigenspaces for $K$ are of dimension one.*

*5) Let $\sigma_p(K) = (\lambda_n)$ denote the sequence of eigenvalues for $K$. Find*

$$\sum_{n=1}^\infty \lambda_n^2.$$

*6) Let $\lambda$ be a positive eigenvalue and let $\mu = \dfrac{1}{\sqrt{\lambda}}$. Express the associated eigenfunction with $\mu$ a transcendent equation for $\mu$.*
*Use a graph argument to show that $K$ has at most one positive eigenvalue.*

1) If $f \in C([0,1])$, then we get immediately that $Kf$ is of class $C^1([0,1])$ and

$$(Kf)'(x) = \int_0^x f(t)\, dt + x\, f(x) - x\, f(x) = \int_0^x f(t)\, dt.$$

This shows that we even have $(Kf)' \in C^1([0,1])$, hence $Kf \in C^2([0,1])$, and

(13) $(Kf)''(x) = f(x)$.

Furthermore,

$$(Kf)'(0) = \int_0^0 f(t)\, dt = 0,$$

and

$$(Kf)(1) = 1 \cdot \int_0^1 f(t)\, dt + \int_1^1 t\, f(t)\, dt = \int_0^1 f(t)\, dt = (Kf)'(1).$$

2) Now, $K$ is linear, hence $K$ is injective, If $Kf(x) \equiv 0$ implies that $f = 0$. This follows from (13) in (1), because

$$f(x) = (Kf)''(x) = 0.$$

Assume that $u \in C^2([0,1])$ satisfies $u'(0) = 0$ and $u(1) = u'(1)$. We shall prove that there is an $f \in C([0,1])$, for which $u = Kf$. According to (13) the only possibility is $f = u''$, which we now check. Using that $u'' \in C([0,1])$, we get

$$\begin{aligned}
Ku''(x) &= x \int_0^x u''(t)\, dt + \int_x^1 t\, u''(t)\, dt = x\,\{u'(x) - u'(0)\} + [t\, u'(t)]_x^1 - \int_x^1 1 \cdot u'(t)\, dt \\
&= x\, u'(x) + u'(1) - x\, u'(x) - [u(t)]_x^1 = u'(1) - u(1) + u(x) = u(x),
\end{aligned}$$

and the claim is proved.

3) We get from the expression for $Kf$ that

$$Kf(x) = \int_0^1 k(x,t)\, f(t)\, dt = \int_0^x f(t)\, dt + \int_x^1 t\, f(t)\, dt = \int_0^1 \max\{x,t\}\, f(t)\, dt,$$

thus

$$k(x,t) = \max\{x,t\} \qquad \text{for } x,\, t \in [0,1],$$

and $k(x,t)$ is clearly continuous in $[0,1]^2$, hence of class $L^2\left([0,1]^2\right)$.

We note that $k(x,t) = \overline{k(t,x)}$, hence the kernel is Hermitian and $K$ is a self adjoint Hilbert-Schmidt operator.

4) This is trivial. We know that $K$ is injective, so $0 \notin \sigma_p(K)$, and if $\lambda \in \sigma_p(K)$, $\lambda \neq 0$, and $K\varphi = \lambda\varphi$, it follows by an application of $K^{-1}$ that

$$\varphi = \lambda\, K^{-1}\varphi, \qquad \text{i.e.} \qquad K^{-1}\varphi = \frac{1}{\lambda}\, \varphi.$$

5) Assume that $\varphi$ and $\psi$ are eigenvectors for $K$ with the same eigenvalue $\lambda$. Then

$$f(x) = \psi(0)\,\varphi(x) - \varphi(0)\,\psi(x)$$

is also an eigenfunction corresponding to $\lambda$, hence $f$ is according to (4) an eigenvector corresponding to the operator $K^{-1} = \dfrac{d^2}{dx^2}$ with the eigenvalue $\dfrac{1}{\lambda}$, so

$$f''(x) = \frac{1}{\lambda}\,f(x).$$

Now, $(K\varphi)'(0) = 0 = \lambda\varphi'(0)$, and analogously for $\psi$, so we conclude from (1) that

$$f(0) = \psi(0)\,\varphi(0) - \varphi(0)\,\psi(0) = 0$$

and

$$f'(0) = \psi(0)\,\varphi - (0) - \varphi(0)\,\psi'(0) = 0.$$

It follows from the existence and uniqueness theorem for linear second order differential equations that

(14) $\quad \dfrac{d^2 f}{dx^2} - \dfrac{1}{\lambda}\,f(x) = 0, \qquad f(0) = 0, \quad f'(0) = 0,$

does only have the solution $f(x) \equiv 0$, hence

(15) $\quad \psi(0)\,\varphi(x) = \varphi(0)\,\psi(x).$

Then assume that $\varphi(0) = 0$ for every eigenfunction. Then also $\varphi'(0) = 0$, cf. the above, so $\varphi$ is a solution of (14), and $\varphi \equiv 0$. This means that $\varphi$ is not an eigenfunction, contradicting the assumption. Therefore, we conclude that $\varphi(0) \neq 0$ for every eigenfunction. Then it follows from (15) that all eigenfunctions of the same eigenvalue are mutually proportional, hence every eigenspace for $K$ has dimension 1.

6) When we use that $K$ is self adjoint and of Hilbert-Schmidt type, cf. (3), we get that all eigenvalues are real, and

$$\sum_{n=1}^{+\infty} \lambda_n^2 = \|k\|_2^2,$$

where we have used (5) that every eigenspace has dimension 1. Then

$$
\begin{aligned}
\sum_{n=1}^{+\infty} \lambda_n^2 &= \|k\|_2^2 = \int_0^1 \int_0^1 \max\{x,t\}^2 \, dt\, dx = \int_0^1 \left\{ \int_0^x x^2 \, dt + \int_x^1 t^2 \, dt \right\} dt \\
&= \int_0^1 \left\{ x^3 + \left[ \frac{t^3}{3} \right]_x^1 \right\} dx = \int_0^1 \left\{ x^3 + \frac{1}{3} - \frac{x^3}{3} \right\} dx = \frac{1}{3} \int_0^1 \left( 2x^3 + 1 \right) dx \\
&= \frac{1}{3} \left[ \frac{x^4}{2} + x \right]_0^1 = \frac{1}{3} \left\{ \frac{1}{2} + 1 \right\} = \frac{1}{2}.
\end{aligned}
$$

7) It follows from (4) that if $\lambda > 0$ and $\mu = \dfrac{1}{\sqrt{\lambda}}$, then

$$\varphi''(x) = \frac{1}{\lambda}\,\varphi(x) = \mu^2\,\varphi(x),$$

the complete solution of which is

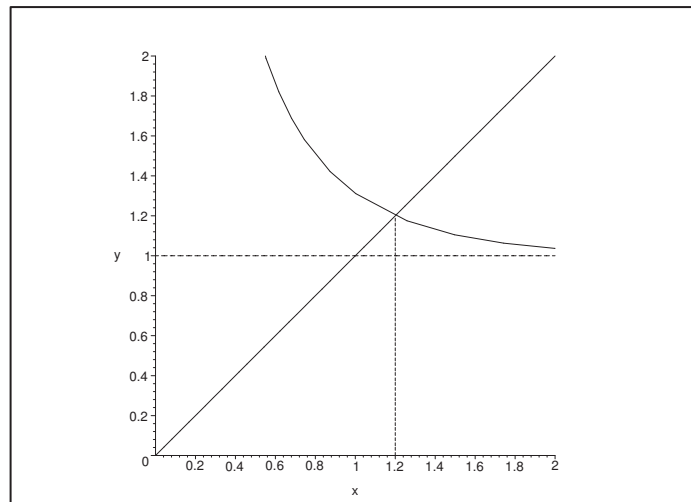$$\varphi(x) = C_1 e^{\mu x} + C_2 e^{-\mu x}.$$



Figure 2: The graphs of $x = \mu$ and $x = \coth \mu$ intersect at $\mu \approx 1.199\,678\,640$.

We shall find the values of $C_1$, $C_2$ and $\mu$, for which $\varphi \in D\left(K^{-1}\right)$. We compute

$$\varphi'(x) = \mu \left\{ C_1 e^{\mu x} - C_2 e^{-\mu x} \right\},$$

and get the conditions (because $\mu > 0$)

$$\varphi'(0) = \mu \left\{ C_1 - C_2 \right\} = 0, \qquad \text{i.e. } C_1 = C_2 = C,$$

and

$$\varphi(1) = C \left\{ e^\mu + e^{-\mu} \right\} = C\mu \left\{ e^\mu - e^{-\mu} \right\} = \varphi'(1),$$

so $\mu$ is a solution of the equation

$$\cosh \mu = \mu \sinh \mu,$$

which we write as

$$\coth \mu = \mu.$$

Considering the graphs we see that this equation has only one solution $\mu > 0$.

**Remark 2.3** Using the iteration

$$\mu_{n+1} = \frac{1}{\tan \mu_n}$$

we get on a pocket calculator that

$$\mu \approx 1.199\,678\,640.$$

Note that

$$\lambda_1^2 = \frac{1}{\mu^4} \approx 0.482\,770\,022 < 0, .5,$$

so

$$\sum_{n=2}^{+\infty} \lambda_n^2 = 0.017\,229\,978 \ll \lambda_1^2.$$

The norm of $K$ is approximately

$$\|K\| = \lambda_1 \approx 0.694\,82.$$

We have for any other eigenvalue $\lambda \in \mathbb{R}$ that $\lambda < 0$, so $\mu = \dfrac{1}{\sqrt{\lambda}}$ is purely imaginary. $\lozenge$

**Example 2.5** *Let $K \in B(H)$, where $H = L^2([0,1])$, be given by*

$$Kf(x) = \int_{1-x}^{1} f(t)\, dt.$$

*1) Show that $K$ is actually bounded.*

*2) Show that the kernel $k(x,t)$ for $K$ is Hermitian, and calculate*

$$\|k\|^2 = \int_0^1 \int_0^1 |k(x,t)|^2\, dt\, dx.$$

*3) Show that the kernel $k_2(x,t)$ for $K^2$ is $\min\{x,t\}$.*

*4) Show that an eigenfunction for $K$ is an eigenfunction for $K^2$.*
*Now, let $f$ denote an eigenfunction for $K$ associated with the eigenvalue $\lambda$. Calculate $\left(K^2 f\right)''$, justify that it belongs to $H$ and show that $f$ is a solution to the equation*

$$\lambda^2 f'' + f = 0.$$

*5) Find all eigenvalues and associated eigenfunctions for $K$.*

*6) Determine $\|K\|$.*

1) Apply the Cauchy-Schwarz inequality in $L^2([1-x,1])$ for $f \in H$. This gives

$$\|Kf\|_2^2 = \int_0^1 \left| \int_{1-x}^{1} 1 \cdot f(t)\, dt \right|^2 dx \leq \int_0^1 \left\{ \sqrt{x} \cdot \|f\|_2 \right\}^2 dx = \|f\|_2^2 \int_0^1 x\, dx = \frac{1}{2}\, \|f\|_2^2,$$

and we conclude that $\|K\| \leq \dfrac{1}{\sqrt{2}}$, thus $K$ is bounded.

2) It follows from

$$Kf(x) = \int_0^1 k(x,t)\, f(t)\, dt = \int_{1-x}^{2} f(t)\, dt = \int_0^1 1_{[1-x,1]}(t)\, f(t)\, dt,$$

that

$$k(x,t) = 1_{[1-x,1]}(t) = \begin{cases} 1 & \text{for } 1-x \leq t \leq 1, \quad x \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Hence, $k(x,t) = 1$, if and only if $x + t \geq 1$, $x$, $t \in [0,1]$, and 0 otherwise, i.e. if and only if

$$(x,t) \in B = \{(x,t) \in [0,1]^2 \mid x + t \geq 1\},$$

so we get (cf. the figure)

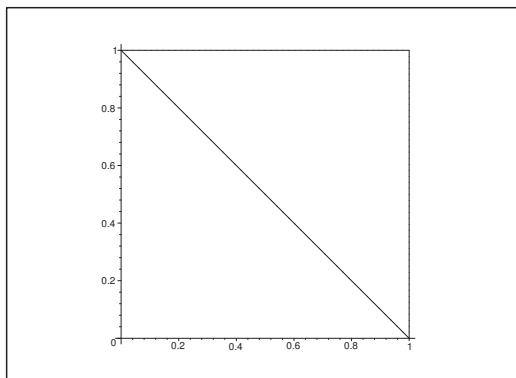$$k(x,t) = 1_B(x,t) = \overline{1_B(t,x)} = \overline{k(t,x)},$$

Figure 3: The domain $B$, where $k(x,t) = 1$, is the upper triangle.

which shows that the kernel is Hermitian.

Then we get

$$\|k\|_2^2 = \int_0^1 \int_0^1 |k(x,t)|^2 \, dt \, dx = \int_0^1 \int_0^1 k(x,t) \, dt \, dx = \text{ area}(B) = \frac{1}{2},$$

possibly in the variant

$$\|k\|_2^2 = \int_0^1 \int_0^1 k(x,t) \, dt \, dx = \int_0^1 (K1)(x) \, dx = \int_0^1 \left\{ \int_{1-x}^1 dt \right\} dx = \int_0^1 x \, dx = \frac{1}{2}.$$

3) The kernel for $K^2$ is given by

$$k_2(x,t) = \int_0^1 k(x,s)k(s,t) \, ds,$$

where the integrand is $\neq 0$, if and only if

$$1 - x \leq s \leq 1 \qquad \text{and} \qquad 1 - s \leq t \leq 1.$$

This provides us with the bounds

$$1 - x \leq s \leq 1 \qquad \text{and} \qquad 1 - t \leq s \leq 1,$$

hence $s \leq 1$ and

$$s \geq \max\{1 - x, 1 - t\} = 1 - \min\{x, t\}.$$

Then by insertion

$$
\begin{aligned}
k_2(x,t) &= \int_0^1 k(x,s)k(s,t) \, ds = \int_{1-\min\{x,t\}}^1 k(x,s)k(s,t) \, ds \\
&= \int_{1-\min\{x,t\}}^1 ds = \min\{x,t\},
\end{aligned}
$$

i.e.

$$k_2(x,t) = \min\{x,t\}, \qquad (x,t) \in [0,1]^2.$$

4) If $Kf = \lambda f$, then of course

$$K^2 f = \lambda K f = \lambda^2 f,$$

so if $f$ is an eigenfunction for $K$ corresponding to the eigenvalue $\lambda$, then $f$ is an eigenfunction for $K^2$ corresponding to the eigenvalue $\lambda^2$.

We get, the kernel for $K^2$ being $k_2$,

$$K^2 f(x) = \int_0^1 \min\{x, t\} \, f(t) \, dt = \int_0^x t \, f(t) \, dt + x \int_x^1 f(t) \, dt.$$

Obviously, $K^2 f$ is differentiable in the weak sense, and we get

$$\left(K^2 f\right)'(x) = x \, f(x) + \int_x^1 f(t) \, dt - x \, f(x) = \int_x^1 f(t) \, dt.$$

This shows that $\left(K^2 f\right)'$ also is weakly differentiable, so

$$\left(K^2 f\right)''(x) = -f(x).$$

If $f$ is an eigenvalue for $K$ corresponding to the eigenvalue $\lambda$, i.e. $Kf = \lambda f$, then it follows from the above that

$$\left(K^2 f\right)(x) = \lambda^2 f(x)$$

and $f$ is differentiable. It follows by induction that $f$ is infinitely often differentiable, so we get from the above that

$$\lambda^2 f''(x) = \left(K^2 f\right)''(x) = -f(x),$$

hence by a rearrangement,

(16) $\lambda^2 f''(x) + f(x) = 0.$

Therefore, if $f$ is an eigenfunction for $K$ with eigenvalue $\lambda$, then $f$ must also fulfil (16). In particular, $\lambda \neq 0$, if $f$ is an eigenfunction. It is well-known that the solutions of (16) are

$$f(x) = c_1 \exp\left(\frac{i}{\lambda}x\right) + c_2 \exp\left(-\frac{i}{\lambda}x\right).$$

From $K^2 f(0) = 0 = \lambda^2 f(0)$ follows that $f(0) = 0$, so we conclude that $c_1 + c_2 = 0$. Putting $c_1 = \dfrac{c}{2i}$, we get $c_2 = -\dfrac{c}{2i}$, and the only possibility of an eigenfunction is

$$f(x) = \frac{c}{2i}\left\{\exp\left(\frac{i}{\lambda}x\right) - \exp\left(-\frac{i}{\lambda}x\right)\right\} = c \cdot \sin\left(\frac{x}{\lambda}\right).$$

5) It remains to find the possible eigenvalues $\lambda$.

Put $c = 1$ and $\alpha = \dfrac{1}{\lambda}$. It follows from $Kf(x) = \lambda f(x)$ that

$$f(x) = \sin\left(\frac{x}{\lambda}\right) = \sin(\alpha x) = \frac{1}{\lambda}Kf(x) = \alpha \cdot K\sin(\alpha \cdot)(x),$$

hence by insertion into the definition of $K$,

$$\begin{aligned}
\sin(\alpha x) &= \alpha \int_{1-x}^{1} \sin(\alpha t)\,dt = [-\cos(\alpha t)]_{1-x}^{1} = \cos(\alpha(1-x)) - \cos\alpha \\
&= \cos\alpha \cdot \cos\alpha x + \sin\alpha \cdot \sin\alpha x - \cos\alpha,
\end{aligned}$$

so

$$(1 - \sin\alpha)\sin\alpha x = \cos\alpha \cdot (\cos\alpha x - 1).$$

This equation is fulfilled for all $x$, if either $\alpha = 0$, which is not possible because $\alpha = \dfrac{1}{\lambda}$, or if $\cos\alpha = 0$ and $\sin\alpha = 1$, hence

$$\alpha_p = \frac{\pi}{2} + 2p\pi, \qquad p \in \mathbb{Z},$$

and we get

$$\lambda_p = \frac{1}{\alpha_p} = \frac{1}{\frac{\pi}{2} + 2p\pi} = \frac{1}{\pi} \cdot \frac{1}{4p+1}, \qquad p \in \mathbb{Z}.$$

Then we derive the point spectrum and the continuous spectrum,

$$\sigma_p(K) = \left\{ \frac{2}{\pi} \cdot \frac{1}{4p+1} \;\middle|\; p \in \mathbb{Z} \right\} \quad \text{and} \quad \sigma_c(K) = \{0\}.$$

The eigenfunction corresponding to

$$\lambda_p = \frac{2}{\pi} \cdot \frac{1}{4p+1}, \qquad p \in \mathbb{Z},$$

is

$$f_p(x) = \sin\left(\left(\frac{\pi}{2} + 2p\pi\right) x\right), \qquad x \in [0,1]; \quad p \in \mathbb{Z}.$$

6) The numerically largest eigenvalue is $\lambda_0 = \dfrac{2}{\pi} > 0$, hence

$$\|K\| = \max\{|\lambda_p| \mid p \in \mathbb{Z}\} = \frac{2}{\pi}.$$

CHECK. As a *check* we use that we should have

$$\frac{1}{2} = \|k\|_2^2 = \sum_{p \in \mathbb{Z}} |\lambda_p|^2.$$

We get

$$\sum_{p \in \mathbb{Z}} |\lambda_p|^2 = \frac{4}{\pi^2} \sum_{p=-\infty}^{+\infty} \frac{1}{(4p+1)^2} = \frac{4}{\pi^2} \sum_{p=0}^{+\infty} \frac{1}{(2p+1)^2} = \frac{4}{\pi^2} \cdot \frac{\pi^2}{8} = \frac{1}{2} = \|k\|_2^2,$$

because it follows from

$$
\begin{aligned}
\frac{\pi^2}{6} &= \sum_{n=1}^{+\infty} \frac{1}{n^2} = \left\{ 1 + \frac{1}{2^2} + \frac{1}{2^4} + \cdots \right\} \sum_{p=0}^{+\infty} \frac{1}{(2p+1)^2} = \sum_{n=0}^{+\infty} \frac{1}{4^n} \sum_{p=0}^{+\infty} \frac{1}{(2p+1)^2} \\
&= \frac{4}{3} \sum_{p=0}^{+\infty} \frac{1}{(2p+1)^2},
\end{aligned}
$$

that

$$\sum_{p=0}^{+\infty} \frac{1}{(2p+1)^2} = \frac{\pi^2}{8}.$$

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