Hilbert Spaces and Operators on Hilbert Spaces

Functional Analysis Examples c-3

Leif Mejlbro
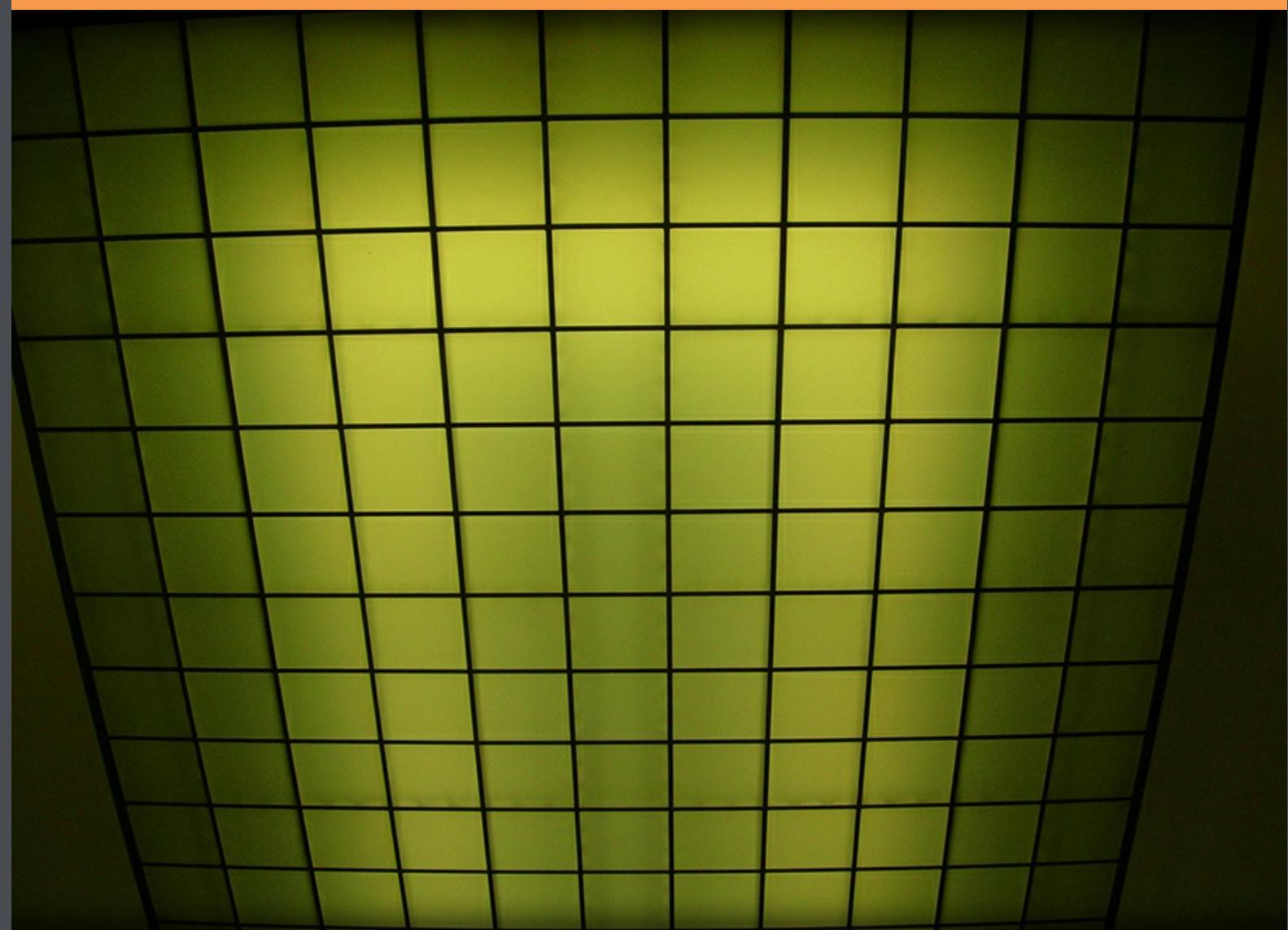
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Example 1.1 *Prove that in a* real *vector space with inner product we have*

$$(x, y) = \frac{1}{4} \left(\|x + y\|^2 - \|x - y\|^2 \right),$$

and in a complex *vector space with inner product we have*

$$(x, y) = \frac{1}{4} \left(\|x + y\|^2 - \|x - y\|^2 + i \|x + iy\|^2 + i \|x - iy\|^2 \right).$$

These are the so-called polarization identities. *They tell us that in a Hilbert space, the inner product is determined by the norm.*

Let V be a real vector space with an inner product. It follows straightforward that

$$\frac{1}{4} \left\{ \|x + y\|^2 - \|x - y\|^2 \right\} = \frac{1}{4} \left\{ (x + y, x + y) - (x - y . x - y) \right\}$$

$$= \frac{1}{4} \left\{ (x, x) + (y, x) + (x, y) + (y, y) - (x, x) + (y, x) + (x, y) - (y, y) \right\}$$

$$= \frac{1}{4} \left\{ 2(y, x) + 2(x, y) \right\} = \frac{1}{2} \left\{ (x, y) + (y, x) \right\} = (x, y),$$

and we have proved the claim concerning real vector spaces.

Let V be a complex vector space with an inner product. Then we get analogously,

$$\frac{1}{4}\left\{\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right\}$$

$$= \frac{1}{4}\left\{(x+y, x+y) - (x-y, x-y) + i(x+iy, x+iy) - i(x-iy, x-iy)\right\}$$

$$= \frac{1}{4}\left\{[(x,x) + (x,y) + (y,x) + (y,y)] - [(x,x) - (x,y) - (y,x) + (y,y)]\right.$$
$$\left. +i[(x,x) + (x,iy) + (iy,x) + (iy,iy)] - i[(x,x) - (x,iy) - (iy,x) + (iy,iy)]\right\}$$

$$= \frac{1}{4}\left\{2(x,y) + 2(y,x) + 2i(x,iy) + 2i(iy,x)\right\}$$

$$= \frac{1}{2}\left\{(x,y) + (y,x) + i\cdot(-i)(x,y) + i\cdot i(y,x)\right\}$$

$$= \frac{1}{2}\left\{(x,y) + (y,x)0(x,y) - (y,x)\right\}$$

$$= \frac{1}{2}\cdot 2(x,y) = (x,y),$$

and the claim is proved in the complex case.

Example 1.2 *Let V be a* real *normed vector space, and assume that the norm satisfies*

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right) \qquad \text{for all } x, y \in V.$$

Show that

$$(x,y) = \frac{1}{4}\left\{\|x+y\|^2 - \|x-y\|^2\right\}$$

defines an inner product in V and that the norm is induced by this inner product.

The task is to prove that in general

1) $(x_1 + x_2, y) = (x_1, y) + (x_2, y),$

2) $(\alpha x, y) = \alpha(x, y).$

In order to prove (1) we start by proving

(1) $(x_1, y) + (x_2, y) = 2\left(\frac{1}{2}(x_1 + x_2), y\right).$

This is done below.

In order to prove (2) we start by assuming that $\alpha \in \mathbb{Z}$ is an integer, then we assume that $\alpha \in \mathbb{Q}$ is rational, and finally we let $\alpha \in \mathbb{R}$ be real. This task is also postponed.

Define the bilinear form $(\cdot, \cdot)$ by

$$(x,y) = \frac{1}{4}\left\{\|x+y\|^2 - \|x-y\|^2\right\}.$$

The vector space is real, so the condition of symmetry

$$(x,y) = (y-x)$$

is trivial. Furthermore,

(2) $(x, x) = \dfrac{1}{4} \left\{ \|x + x\|^2 - \|x - x\|^2 \right\} = \dfrac{1}{4} \|2x\|^2 = \|x\|^2 \geq 0.$

We conclude from (2) that

$(x, x) = \|x\|^{=} 0, \qquad$ if and only if $\qquad x = 0,$

and *if* $(\cdot, \cdot)$ is an inner product, *then* the norm is defined by this inner product.

We see that the claim will be proved if we can prove (1) and (2) above.

1) As mentioned above we first prove (1), i.e.

$$(x_1, y) + (x_2, y) = 2 \left(\dfrac{1}{2}(x_1 + x_2), y \right).$$

We get by computing the left hand side,

$$
\begin{aligned}
(x_1, y) + (x_2, y) &= \dfrac{1}{4} \left\{ \|x_1 + y\|^2 - \|x_1 - y\|^2 + \|x_2 + y\|^2 - \|x_2 - y\|^2 \right\} \\
&= \dfrac{1}{4} \left\{ \|x_1 + y\|^2 + \|x_2 + y\|^2 \right\} - \dfrac{1}{4} \left\{ \|x_1 - y\|^2 + \|x_2 - y\|^2 \right\}.
\end{aligned}
$$

The latter two terms are treated separately by means of the law of parallelograms. (We shall somewhere use the assumption).

If we put $u + v = x_1 + y$ and $u - v = x_2 + y$, then

$$u = \dfrac{1}{2}(x_1 + x_2) + y \qquad \text{og} \qquad v = \dfrac{1}{2}(x_1 - x_x).$$

Then apply the law of parallelograms to get

$$
\begin{aligned}
\|x_1 + y\|^2 + \|x_2 + y\|^2 &= \|u + v\|^2 + \|u - v\|^2 = 2 \left(\|u\|^2 + \|v\|^2 \right) \\
&= 2 \left\{ \left\| \dfrac{1}{2}(x_1 + x_2) + y \right\|^2 + \left\| \dfrac{1}{2}(x_1 - x_2) \right\|^2 \right\}.
\end{aligned}
$$

When y is replaced by $-y$, we get analogously

$$\|x_1 - y\|^2 + \|x_2 - y\|^2 = 2 \left\{ \left\| \dfrac{1}{2}(x_1 + x_2) - y \right\|^2 + \left\| \dfrac{1}{2}(x_1 - x_2) \right\|^2 \right\}.$$

These two expressions are then inserted into the expression above for $(x_1, y) + (x_2, y)$. Thus, by the definition of $(\cdot, \cdot)$,

$$
\begin{aligned}
(x_1, y) + (x_2, y) &= \dfrac{1}{4} \cdot 2 \left\{ \left\| \dfrac{1}{2}(x_1 + x_2) + y \right\|^2 + \left\| \dfrac{1}{2}(x_1 - x_2) \right\|^2 \right\} \\
&\quad - \dfrac{1}{4} \cdot 2 \left\{ \left\| \dfrac{1}{2}(x_1 + x_2) - y \right\|^2 + \left\| \dfrac{1}{2}(x_1 - x_2) \right\|^2 \right\} \\
&= 2 \cdot \dfrac{1}{4} \left\{ \left\| \dfrac{1}{2}(x_1 + x_2) + y \right\|^2 - \left\| \dfrac{1}{2}(x_1 + x_2) - y \right\|^2 \right\} \\
&= 2 \left(\dfrac{1}{2}(x_1 + x_2), y \right),
\end{aligned}
$$

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and the auxiliary result is proved.

We use again the law of parallelograms in the following computation,

$$
\begin{aligned}
(2z, y) &= \frac{1}{2}\left\{\|2z + y\|^2 - \|2z - y\|^2\right\} \\
&= \frac{1}{4}\left\{\|z + (z + y)\|^2 - \|z + (z - y)\|^2\right\} \\
&= \frac{1}{4}\left\{\|z + (z + y)\|^2 + \underline{\|z - (z + y)\|^2}\right\} - \frac{1}{4}\left\{\|z + (z - y)\|^2 + \underline{\|z - (z - y)\|^2}\right\} \\
&= \frac{1}{4} \cdot 2\left\{\|z\|^2 + \|z + y\|^2\right\} - \frac{1}{4} \cdot 2\left\{\|z\|^2 + \|z - y\|^2\right\} \\
&= 2 \cdot \frac{1}{4}\left\{\|z + y\|^2 - \|z - y\|^2\right\} = 2(z, y),
\end{aligned}
$$

where we get for the underlined terms that

$$
\|z - (z + y)\|^2 - \|z - (z - y)\|^2 = \| - y\|^2 - \|y\|^2 = 0.
$$

Finally, use the result above with $z = \frac{1}{2}(x_1 + x_2)$, to obtain

$$
(3)\ \ (x_1, y) + (x_2, y) = 2\left(\frac{1}{2}(x_1 + x_2), y\right) = (x_1 + x_2, y).
$$

2) Let $n \in \mathbb{N}$. Then we get by repeatedly applying (3),

$$
(nx, y) = (x + \cdots + x, y) = (x, y) + \cdots + (x, y) = n(x, y),
$$

thus (2) holds for $\alpha = n \in \mathbb{N}$.

If $\alpha = -1$, then by the definition of $(\cdot, \cdot)$,

$$
(-x, y) = \frac{1}{4}\left\{\| - x + y\|^2 - \| - x - y\|^2\right\} = -\frac{1}{4}\left\{\|x + y\|^2 - \|x - y\|^2\right\} = -(x, y).
$$

Composing this result with the previous one for $n \in \mathbb{N}$ we get

$$
(-nx, y) = -(nx, y) = -n(x, y),
$$

and the formula holds for $\alpha \in \mathbb{Z}$.

Then let $\alpha = \frac{1}{n}$, $n \in \mathbb{N}$. We shall prove that

$$
\left(\frac{1}{n}x, y\right) = \frac{1}{n}(x, y),
$$

or, equivalently,

$$
n\left(\frac{1}{n}x, y\right) = (x, y).
$$

Put $z = \dfrac{x}{n}$. Then it follows by the previous results that

$$n\left(\frac{1}{n}x, y\right) = n(z, y) = (nz, y) = \left(n \cdot \frac{1}{n}x, y\right) = (x, y),$$

and we have extended the formula.

If $\alpha = \dfrac{p}{q} \in \mathbb{Q}$, $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, then it follows from the previous results that

$$(\alpha x, y) = \left(p \cdot \frac{1}{q}x, y\right) = p\left(\frac{1}{q}x, y\right) = p \cdot \frac{1}{q}(x, y) = \alpha(x, y),$$

and the formula is proved for $\alpha \in \mathbb{Q}$.

Now, $\mathbb{Q}$ is dense in $\mathbb{R}$, so we shall only show that the mapping $(\cdot, y_0)$ is continuous in x for every fixed y_0. First notice that

$$|(x - x_0, y_0)| = \frac{1}{4}\left|\,\|x - x_0 + y_0\|^2 - \|x - x_0 - y_0\|^2\,\right|$$

$$= \frac{1}{4}\left\{\|y_0 + (x - x_0)\| + \|y_0 - (x - x_0)\|\right\} \cdot \left|\,\|y_0 + (x - x_0)\| - \|y_0 - (x - x_0)\|\,\right|.$$

If $\|x - x_0\| < \delta$, then it follows by the triangle inequality that

$$\|y_0 + (x - x_0)\| + \|y_0 - (x - x_0)\| \le 2 \{\|y_0\| + \delta\},$$

and

$$|\|y_0 + (x - x_0)\| - \|y_0 - (x - x_0)\|| \le 2\|x - x_0\| \le 2\delta,$$

hence by insertion

$$|(x - x_0, y_0)| \le \frac{1}{4} \cdot 2(\|y_0\| + \delta) \cdot 2\delta = \{\|y_0\| + \delta\} \cdot \delta < \varepsilon$$

for sufficiently small δ. Since $(\cdot, y_0)$ is continuous, the formula for $\mathbb{Q}$ is extended by the continuity to all of $\mathbb{R}$, and the claim is proved.

Remark 1.1 The most difficult part of the example is that the notation $(\cdot, \cdot)$ is a little confusing. One shall always be aware of distinguishing between what is given and what shall be proved. $\diamond$

Remark 1.2 If also the complex law of parallelograms holds in a normed complex vector space, then it is possible to prove the analogous result. Only the computations become much larger. $\diamond$

Example 1.3 *Show that the sup-norm on $C([a, b])$ is not induced by an inner product.*

We know already that *if* a norm is defined by an inner product, *then* we the law of parallelograms holds,

$$\|f + g\|^2 + \|f - g\|^2 = 2 \{\|f\|^2 + \|g\|^2\}.$$

Hence, it suffices to prove that the law of parallelograms does *not* hold for

$$\|f\| = \sup\{|f(t)| \mid t \in [a, b]\} \qquad \text{in } C([a, b]).$$

We may assume that $[a, b] = [0, 1]$.

Choose $f(t) = 1$ and $g(t) = t$ for $t \in [0, 1]$. Then $\|f\| = 1$ and $\|g\| = 1$, and

$$\|f + g\| = \sup_{t \in [0,1]} |1 + t| = 2, \qquad \|f - g\| = \sup_{t \in [0,1]} |1 - t| = 1.$$

Hence

$$\|f + g\|^2 + \|f - g\|^2 = 4 + 1 = 5$$

and

$$2 \left(\|f\|^2 + \|g\|^2\right) = 2(1 + 1) = 4.$$
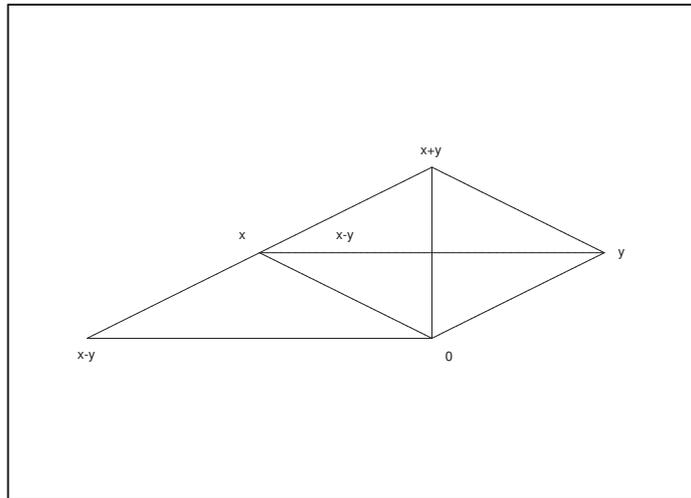
It follows from

$$\|f + g\|^2 + \|f - g\|^2 = 5 \ne 4 = 2 \left(\|f\|^2 + \|g\|^2\right),$$

that the law of parallelograms is not satisfied, so the sup-norm is not defined by an inner product.

Example 1.4 *Prove that in a* real *vector space with inner product we have that* $\|x\| = \|y\|$ *implies that*

$(x + y, x - y) = 0.$

In the case $V = \mathbb{R}^2$ *this is a well-known geometric statement – which one?*

If $\|x\| = \|y\|$, then

$(x + y, x - y) = (x, x) + (y, x) - (x, y) - (y, y) = \|x\|^2 - \|y\|^2 = 0.$

If we sketch the parallelogram of forces, then we have just proved that in a rhomb the diagonals are perpendicular to each other.

Example 1.5 *Let* V_i, $I = 1, \ldots, k$ *be vector spaces equipped with inner products* $(\cdot, \cdot)_i$, *respectively. We define the product space* $\bigotimes_{i=1}^{k} V_i$ *as in* Example 2.16 *in* Ventus, Functional Analysis, Topological and metric spaces, Banach spaces and bounded operators.
Show that we can define an inner product in $\bigotimes_{i=1}^{n} V_i$ *by*

$$((x_1, x_2, \ldots, x_k), (y_1, y_2, \ldots, y_k)) = \sum_{i=1}^{k} (x_i, y_i)_i,$$

and that $\bigotimes_{i=1}^{k} V_i$ *with this inner product is a Hilbert space if* V_i, $i = 1, \ldots, k$ *are Hilbert spaces.*

Due to the structure of the suggested inner product it suffices to prove the claim for $k = 2$, because the general result then follows by induction. In this way we avoid the usual mess of notation.

Consider two vector spaces V_1, $i = 1, 2$, each one with an inner product $(\cdot, \cdot)_i$. We shall prove that

$((x_1, x_2), (y_1, y_2)) := (x_1, y_1)_1 + (x_2, y_2)_2$

defines an inner product on $V \times V$.

First note that

$$((y_1, y_2), (x_1, x_2)) = (y_1, x_1)_1 + (y_2, x_2)_2 = \overline{(x_1, y_1)_1 + (x_2, y_2)_2} = \overline{((x_1, x_2), (y_1, y_2))}.$$

Then

$$
\begin{aligned}
&(\alpha(x_1, x_2) + (y_1, y_2), (z_1, z_2)) \\
&= ((\alpha x_1 + y_1, \alpha x_2 + y_2), (z_1, z_2)) \\
&= (\alpha x_1 + y_1, z_1)_1 + (\alpha x_2 + y_2, z_2)_2 \\
&= \alpha(x_1, z_1)_1 + (y_1, z_1)_1 + \alpha(x_2, z_2)_2 + (y_2, z_2)_2 \\
&= \alpha \{(x_1, z_1)_1 + (x_2, z_2)_2\} + \{(y_1, z_1)_1 + (y_2, z_2)_2\} \\
&= \alpha \left((x_1, x_2), (z_1, z_2)\right) + ((y_1, y_2), (z_1, z_2)).
\end{aligned}
$$

Finally,

$$((x_1, x_2), (x_1, x_2)) = (x_1, x_1)_1 + (x_2, x_2)_2 \geq 0,$$

where we have equality, if and only if both $\|x\|_1 = 0$ and $\|x\|_2 = 0$, thus if and only if $x_1 = 0$ and $x_2 = 0$, which again is the same as saying that $(x_1, x_2) = (0, 0)$.

We have now proved that $(\cdot, \cdot)$ is an inner product on $V_1 \times V_2$.

Then assume that $\bigotimes_{i=1}^{k} V_i$ for some $k \in \mathbb{N}$ is given an inner product by

$$((x_1, x_2, \ldots, x_k), (y_1, y_2, \ldots, x_k)) = \sum_{i=1}^{k} (x_i, y_i)_i,$$

whenever each V_i is equipped with an inner product $(\cdot, \cdot)_i$, $i = 1, \ldots, k$. We proved above that this is true for $k = 2$.

Consider $k + 1$ vector spaces $(U_i, \langle \cdot, \cdot \rangle_i)$, $i = 1, \ldots, k + 1$, each equipped with an inner product. We define

$$V_1 = \otimes_{i=1}^{k} U_i \quad \text{and} \quad ((x_1, \ldots, x_k), (y_1, \ldots, y_k))_1 = \sum_{i=1}^{k} \langle x_i, y_i \rangle_i,$$

and

$$V_2 = U_{k+1} \quad \text{and} \quad (x_{k+1}, y_{k+1})_2 = \langle x_{k+1}, y_{k+1} \rangle_{k+1}.$$

It follows by using the assumption of induction that both $(V_1, (\cdot, \cdot)_1)$ and $(V_2, (\cdot, \cdot)_2)$ are vector spaces with inner products. Then it follows from the result above that

$$V_1 \times V_2 = \bigotimes_{i=1}^{k+1} U_i$$

is also a vector space, equipped with the inner product

$$
\begin{aligned}
((x_1, \ldots, x_k, x_{k+1}), (y_1, \ldots, y_k, y_{k+1})) &= ((x_1, \ldots, x_k), (y_1, \ldots, y_k))_1 + (x_{k+1}, y_{k+1})_2 \\
&= \sum_{i=1}^{k} \langle x_i, y_i \rangle_i + \langle x_{k+1}, y_{k+1} \rangle_{k+1} = \sum_{i=1}^{k+1} \langle x_i, y_i \rangle_i,
\end{aligned}
$$

and the first claim is proved.

Then assume (for $k = 2$) that $(V_i, (\cdot, \cdot)_i)$, $i = 1, 2$, are both Hilbert spaces. We shall prove that $V_1 \times V_2$ becomes a Hilbert space when it is given the inner product which has been constructed her, in other words, we shall prove that $V_1 \times V_2$ is complete.

Let $((x_1^n, x_2^n))_{n \in \mathbb{N}}$ be a Cauchy sequence in $V_1 \times V_2$, which means that to every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$, such that for all $m, n \geq N$,

$$\|(x_1^m, x_2^m) - (x_1^n, x_2^n)\| = \|(x_1^m - x_1^n, x_2^m - x_2^n)\| < \varepsilon.$$

This can also be written

$$(x_1^m - x_1^n, x_1^m - x_1^n)_1 + (x_2^m - x_2^n, x_2^m - x_2^n)_2 < \varepsilon^2.$$

Both terms on the left hand side are ≥ 0, so

$$|x_1^m - x_1^n\| < \varepsilon \qquad \text{og} \qquad \|x_2^m - x_2^n\| < \varepsilon,$$

which shows that (x_1^n) is a Cauchy sequence in V_1, and that (x_2^n) is a Cauchy sequence in V_2. They are both convergent by the assumption,

$$x_1^n \to x_1 \in V_1 \qquad \text{and} \qquad x_2^n \to x_2 \in V_2.$$

It follows that if $(x_1, x_2) \in V_1 \times V_2$, then

$$
\begin{aligned}
\|(x_1, x_2) - (x_1^n, x_2^n)\|^2 &= \|(x_1 - x_1^n, x_2 - x_2^n)\|^2 \\
&= \|x_1 - x_1^n\|^2 + \|x_2 - x_2^n\|^2 \to 0 \qquad \text{for } n \to \infty.
\end{aligned}
$$

We have proved that every Cauchy sequence in $V_1 \times V_2$ is convergent, hence $V_1 \times V_2$ is a Hilbert space.

Finally, if all $(V_1, (\cdot, \cdot)_i)$, $i = 1, \ldots, k$, are Hilbert spaces, then it follows in the same way as in the previous proof of the inner product that $\bigotimes_{i=1}^{k} V_i$ becomes a Hilbert space, when it is equipped with the here constructed inner product.

Example 1.6 *Let x and y be vectors in a vector space with an inner product. Show that $(x, y) = 0$ if and only if*

$$\|x + \alpha y\| = \|x - \alpha y\| \qquad \text{for all scalars.}$$

Moreover, show that $(x, y) = 0$, if and only if

$$\|x + \alpha y\| \geq \|x\| \qquad \text{for all scalars.}$$
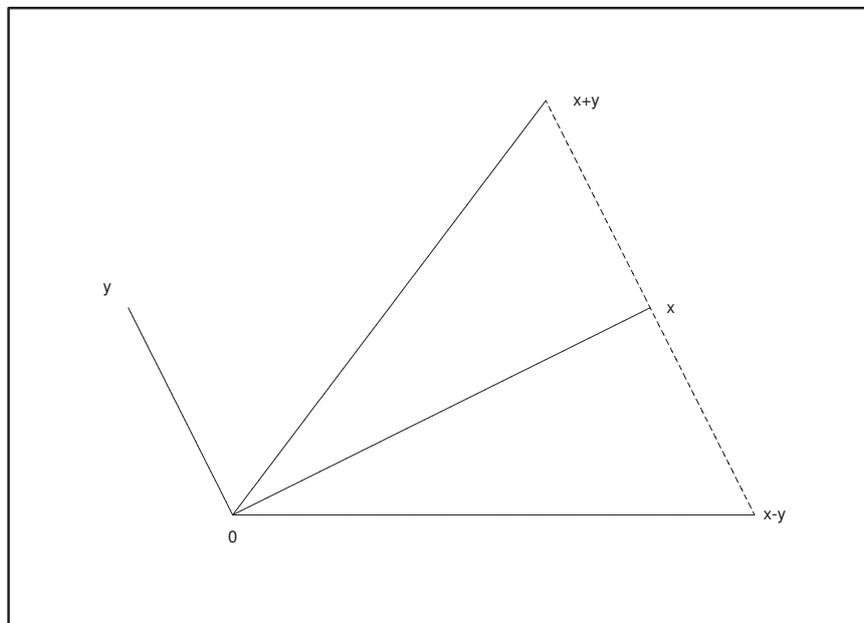
Figure 1: The case $\alpha = 1$.

1) First compute the difference $\|x + \alpha y\|^2 - \|x - \alpha y\|^2$. We get

$$
\begin{aligned}
&\|x + \alpha y\|^2 - \|x - \alpha y\|^2 \\
&= (x + \alpha y, x + \alpha y) - (x - \alpha y, x - \alpha y) \\
&= (x, x) + \overline{\alpha}(x, y) + \alpha(y, x) + |\alpha|^2(y, y) - (x, x) + \overline{\alpha}(x, y) + \alpha(y, x) - |\alpha|^2(y, y) \\
&= 2\left\{ \overline{\alpha}(x, y) + \alpha \overline{(x, y)} \right\} = 2\,\mathrm{Re}\left\{ \overline{\alpha}(x, y) \right\}.
\end{aligned}
$$

a) If $(x, y) = 0$, then we immediately get that

$$\|x + \alpha y\|^2 = \|x - \alpha y\|^2 \qquad \text{for all } \alpha.$$

b) Assume that $(x, y) \neq 0$. Choosing $\alpha = (x, y)$, we get

$$\|x + (x, y)y\|^2 - \|x - (x, y)y\|^2 = 4|(x, y)|^2 > 0,$$

and it follows that $\|x + \alpha y\| \neq \|x - \alpha y$ for $\alpha = (x, y)$.

2) Since $\|x + \alpha y\| \geq \|x\|$ and $\|x + \alpha y\|^2 \geq \|x\|^2$ are equivalent, we first compute $\|x + \alpha y\|^2 - \|x\|^2$. This gives

$$
\begin{aligned}
\|x + \alpha y\|^2 - \|x\|^2 &= (x + \alpha y, x + \alpha y) - (x, x) \\
&= (x, x) + \alpha(y, x) + \overline{\alpha}(x, y) + |\alpha|^2(y, y) - (x, x) \\
&= |\alpha|^2 \|y\|^2 + 2\operatorname{Re}\left\{\overline{\alpha}(x, y)\right\}.
\end{aligned}
$$

a) If $(x, y) = 0$, then

$$\|x + \alpha y\|^2 - \|x\|^2 = |\alpha|^2 \|y\|^2 \geq 0.$$

b) If $(x, y) \neq 0$, if in particular $y \neq 0$, then choose

$$\alpha = -\frac{1}{\|y\|^2}(x, y).$$

We get

$$
\begin{aligned}
\|x - \alpha y\|^2 - \|x\|^2 &= \left\| x - \frac{1}{\|y\|^2}(x, y)y \right\|^2 - \|x\|^2 \\
&= \frac{1}{\|y\|^4}|(x, y)|^2 \|y\|^2 + 2\operatorname{Re}\left\{-\frac{1}{\|y\|^2}|(x, y)|^2\right\} \\
&= \frac{1}{\|y\|^2}|(x, y)|^2 - 2 \cdot \frac{1}{\|y\|^2}|(x, y)|^2 \\
&= -\frac{1}{\|y\|^2}|(x, y)|^2 < 0,
\end{aligned}
$$

hence $\|x + \alpha y\| < \|x\|$, and the claim is proved.

Remark 1.3 Hilbert spaces are the natural generalization of Euclidean spaces. This means that if we only consider a subspace of a Hilbert space spanned by two vectors (like in the present situation), then we might as well give a geometric proof in the usual 2-dimensional plane, because a 2-dimensional Hilbert space is isomorphic to E_2. Analogously in 3-dimensional subspaces. However, only very few have a geometrical understanding of what is going on in Euclidean spaces of higher dimensions. I only remember the late professor Fabricius Bjerre once told me that he himself had some sense of E_4 as being described as "shadows". $\Diamond$

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Example 1.7 *Let x and y be vectors in a* complex *vector space with an inner product, and assume that*

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

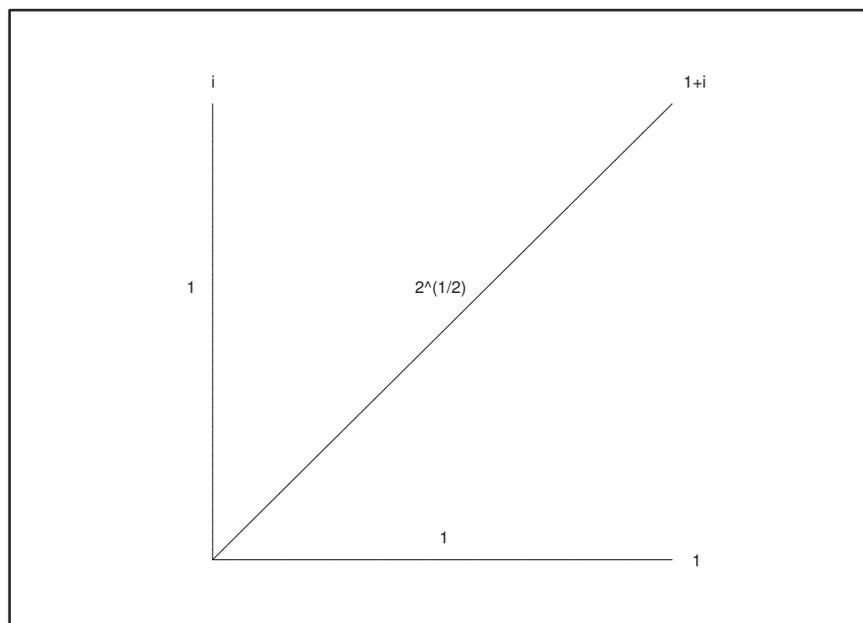*Does this imply that $(x, y) = 0$?*

Figure 2: Vectors in $(\mathbb{C}, +, \cdot, \mathbb{C})$.

The answer is "no"! In fact, choose $V = (\mathbb{C}, +, \cdot, \mathbb{C})$, and

$$x = 1 \qquad \text{and} \qquad y = i.$$

Then

$$|x + y|^2 = |1 + i|^2 = 1^2 + 1^2 = |1|^2 + |i|^2 = |x|^2 + |y|^2 \qquad (= 2),$$

and

$$(x, y) = (1, i) = -i(1, 1) = -i|1|^2 = -i \neq 0.$$

Example 1.8 *Let V be a complex vector space with an inner product and assume that $T \in B(V)$.*
Show that $(Ty, x) = 0$ for all x, $y \in V$ if and only if T is the zero operator.
Show next that $(Tx, x) = 0$ for all $x \in V$ if and only if T is the zero operator.
If the vector space is assumed to be real, do these results hold?

1) Assume that $(Tx, y) = 0$ for all x, $y \in V$. Choosing $y = Tx \in V$ we get

$$\|Tx\|^2 = (Tx, Tx) = 0, \qquad \text{thus } Tx = 0.$$

Since this is true for every x, we infer that T is the zero operator.

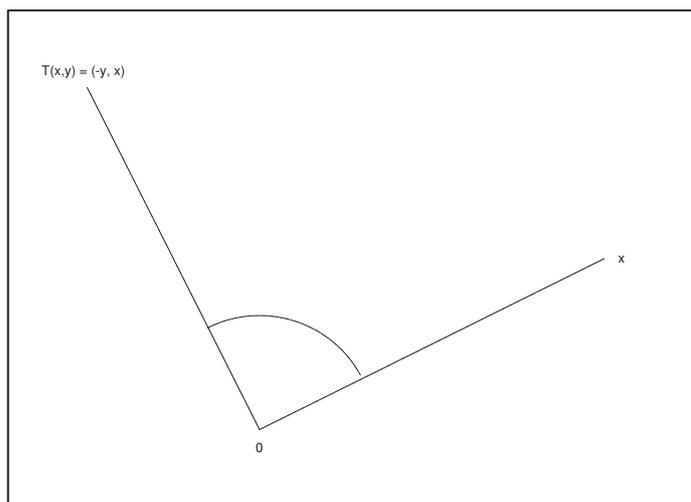Of course this argument also holds if the vector space is real.

Figure 3: Rotation T of $\dfrac{\pi}{2}$ around $(0,0)$ in the plane E_2, cf. (3).

2) Assume that $(Tx, x) = 0$ for every $x \in V$. Then

$$
\begin{aligned}
0 &= (T(x + \alpha y), x + \alpha y) = (Tx + \alpha Ty, x + \alpha y) \\
&= (Tx, x) + (Tx, \alpha y) + \alpha(Ty, x) + |\alpha|^2 (Ty, y) \\
&= \overline{\alpha}(Tx, y) + \alpha(Ty, x).
\end{aligned}
$$

We infer that

$$\overline{\alpha}(Tx, t) = -\alpha(Ty, x) \qquad \text{for alle } x, y \in V \text{ og } \alpha \in \mathbb{C}.$$

Choosing $\alpha = 1$ we get

(4) $(Tx, y) = -(Ty, x)$.

If we instead choose $\alpha = i$, then

$$-i(Tx, y) = -i(Ty, x),$$

hence

(5) $(Tx, y) = +(Ty, x)$.

By combining (4) and (5) we get

$$(Tx, y) = 0 \qquad \text{for all } x, y \in V.$$

Then it follows from (1) that T is identically 0.

3) The last result is not true for *real* vector spaces.
Let $V = E_2$ be the Euclidean plane, and let T denote the rotation around $(0, 0)$ of the angle $\dfrac{\pi}{2}$. Expressed in coordinates we have

$$T(x, y) = (-y, x) \qquad \text{for } (x, y) \in E_2.$$

Clearly, the mapping T is linear, and $\|T\| = 1 \neq 0$, hence $T \neq 0$.

Finally,

$$(T(x, y), (x, y)) = ((-y, x), (x, y)) = -yx + xy = 0.$$

Remark 1.4 We see that it is essential for the proof that we in the derivation of (5) can choose $\alpha = i$. $\Diamond$

Example 1.9 *Prove the law of parallelograms.*

We shall prove that in a vector space of inner product,

$$\|x + y\|^2 + \|x - y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right).$$

A direct computation of the left hand side gives

$$\begin{aligned}
\|x + y\|^2 + \|x - y\|^2 &= (x + y, x + y) + (x - y, x - y) \\
&= (x,x)+(x,y)+(y,x)+(y,y)+(x,x)-(x,y)-(y,x)+(y,y) \\
&= 2(x,x) + 2(y,y) = 2\left\{\|x\|^2 + \|y\|^2\right\},
\end{aligned}$$

and the claim is proved.

Example 1.10 *Denote by M the set of all random variables on a probability space $(\Omega, \mathcal{A}, P)$ of finite second moment. Define*

$$(X, Y) = \text{Cov}(X, Y), \qquad X, Y \in M.$$

Does this define an inner product on M?

It is well-known that the covariance is defined by

$$\text{Cov}(X, Y) = E\left\{(X - \mu_1)(Y - \mu_2)\right\},$$

where

$$E\{X\} = \mu_1, \quad V(X) = \sigma_1^2, \quad \text{og} \quad E\{Y\} = \mu_2, \quad V(Y) = \sigma_2^2.$$

We shall check if we get an inner product.

We have

$$(X, Y) = \text{Cov}(X, Y) = \text{Cov}(Y, X) = (Y, X),$$

and

$$(\alpha X, Y) = \text{Cov}(\alpha X, Y) = \alpha\, \text{Cov}(X, Y) = \alpha\, (X, Y),$$

and

$$(X + Y, Z) = \text{Cov}(X + Y, Z) = \text{Cov}(X, Z) + \text{Cov}(Y, Z) = (X, Z) + (Y, Z).$$

It follows from

$$1 = \varrho(X, X) = \frac{\text{Cov}(X, X)}{\sigma_1^2} = \frac{(X, X)}{\sigma_1^2}, \qquad \text{for } \sigma_1 \neq 0,$$

and $(X, X) = \sigma_1^2$ in general, that $(X, X) = \sigma_1^2 \geq 0$ for every X, and that $(X, X) = 0$, if and only if $V(X) = \sigma_1^2 = 0$, i.e. if and only if X is causally distributed.

We conclude that $(X, Y) = \text{Cov}(X, Y)$ is not an inner product in M, because we cannot distinguish between causal distributions and O.

1.2 Hilbert spaces

Example 1.11 *Let $[a, b]$ be a finite interval.*
Show that $L^2([a, b]) \subset L^1([a, b])$.

The interval $[a, b]$ is bounded, so the constant $1 \in L^2([a, b])$. In fact,

$$\|1\|_2^2 = \int_a^b 1^2 dt = b - a < +\infty.$$

Let $f \in L^2([a, b])$. Then we get by the Cauchy-Schwarz inequality

$$\int_a^b |f(t)|\, dt = \int_a^b |f(t)| \cdot 1\, dt \leq \|f\|_2 \cdot \|1\|_2 = \sqrt{b - a} \cdot \|f\|_2 < +\infty,$$

proving that $f \in L^1([a, b])$, and thus

$$L^2([a, b]) \subseteq L^1([a, b]) \qquad \text{with} \qquad \|f\|_1 \leq \sqrt{b - a} \cdot \|f\|_2.$$

Remark 1.5 We can find $f \in L^1([a, b])$, which does not lie in $L^2([a, b])$. An example is

$$f(x) = \begin{cases} \dfrac{1}{\sqrt{x}} & \text{for } x \in]0, 1], \\[2mm] 0 & \text{for } x = 0. \end{cases}$$

In fact,

$$\left\| \frac{1}{\sqrt{x}} \right\|_1 = \int_0^1 \frac{1}{\sqrt{x}}\, dx = \left[2\sqrt{x} \right]_0^1 = 2,$$

hence $f \in L^1([0, 1])$.
On the other hand,

$$\int_0^1 \left\{ \frac{1}{\sqrt{x}} \right\}^2 dx = \int_0^1 \frac{1}{x}\, dx = \infty,$$

hence $f \notin L^2([0, 1])$.

Example 1.12 *Let T be a linear operator $T : L^2(\mathbb{R}) \to L^2(\mathbb{R}^2)$ satisfying that $f \geq 0$ implies that $Tf \geq 0$.*
Show that

$$\|T(|f|)\| \geq \|T(f)\|,$$

for all $f \in L^2(\mathbb{R})$.
Show that T is bounded.

1) We first assume that f is real. Then

$$|f| - f \geq 0 \qquad \text{and} \qquad |f| + f \geq 0,$$

so

$$T|f| - Tf \geq 0 \qquad \text{and} \qquad T|f| + Tf \geq 0$$

according to the assumption. We conclude that $|Tf| \leq T|f|$ for *real* $f \in L^2(\mathbb{R})$. This estimate implies that

$$\|Tf\|_2^2 = \int_{-\infty}^{+\infty} |Tf|^2 dx \leq \int_{-\infty}^{+\infty} (T|f|)^2 dx = \|T(|f|)\|_2^2,$$

and we conclude that for every *reelle* $f \in L^2(\mathbb{R})$,

$$\|Tf\|_2 \leq \|T(|f|)\|_2.$$

Then assume that f is *complex, $f = g + i\,h$.* This case is far more difficult than one would imagine. We present below one proof among several ones. First apply the result above on the real and the imaginary part separately,

$$
\begin{aligned}
\|Tf\|_2^2 &= \int_{-\infty}^{+\infty} |Tg + i\,Th|^2 dx = \int_{-\infty}^{+\infty} \left\{ |Tg|^2 + |Th|^2 \right\} dx \\
&\leq \int_{-\infty}^{+\infty} (T|g|)^2 dx + \int_{-\infty}^{+\infty} (T|h|)^2 dx.
\end{aligned}
$$

The trick of the present proof is that we choose constants α and $\beta \in \mathbb{R}$, such that $0 < \alpha < \beta$, and then consider the set

$$\Omega_{\alpha,\beta} = \{ x \in \mathbb{R} \mid \alpha|f(x)| \leq |g(x)| < \beta|f(x)| \},$$

where α and β are to be fixed later.
If $x \in \Omega_{\alpha,\beta}$, it follows from the above that

$$\alpha T|f| \leq T|g| \leq \beta T|f| \quad \text{and} \quad \sqrt{1-\beta^2}\, T|f| \leq T|h| \leq \sqrt{1-\alpha^2}\, T|f|,$$

so by a squaring and an addition, (still for $x \in \Omega_{\alpha,\beta}$),

$$\left(1 - \beta^2 + \alpha^2\right)(T|f|)^2 \leq (T|g|)^2 + (T|h|)^2 \leq \left(1 + \beta^2 - \alpha\right)(T|f|)^2.$$

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Choose any $\varepsilon > 0$. Defining

$$\omega_n := \Omega_{\sqrt{n\varepsilon}, \sqrt{(n+1)\varepsilon}} = \{x \in \mathbb{R} \mid \sqrt{n\varepsilon}\, |f(x)| \leq |g(x)| < \sqrt{(n+1)\varepsilon}\, |f(x)|\}$$

it follows that $\mathbb{R} = \bigcup_{n=0}^{\infty} \omega_n$, where the union is disjoint. Then

$$
\begin{aligned}
\|Tf\|_2^2 &\leq \int_{-\infty}^{+\infty} \left\{ (T|g|)^2 + (T|h|)^2 \right\} dx = \sum_{n=0}^{\infty} \int_{\omega_n} \left\{ (T|g|)^2 + (T|h|)^2 \right\} dx \\
&\leq \sum_{n=0}^{\infty} \int_{\omega_n} \left\{ 1 + (n+1)\varepsilon - n\varepsilon \right\} \cdot (T|f|(^2 dx \\
&= (1+\varepsilon) \int_{\bigcup_{n=0}^{\infty} \omega_n} (T|f|)^2 dx = (1+\varepsilon)\, \|T(|f|)\|_2^2.
\end{aligned}
$$

This holds for every $\varepsilon > 0$, so we conclude by taking the limit $\varepsilon \to 0+$ that

$$\|Tf\|_2 \leq \|T(|f|)\|_2,$$

even if $f \in L^2(\mathbb{R})$ is a function of complex values.

2) Then we prove that T is bounded. It follows from (1) that it suffices to consider non-negative functions.

Assume that T is *not* bounded. Then there exists a sequence $f_n \in L^2(\mathbb{R})$, where $f_n \geq 0$ and $\|f_n\|_2 = 1$, such that

$$\|Tf_n\|_2 \geq n^4 \|f_n\|_2 = n^4 \qquad \text{for all } n \in \mathbb{N}.$$

If we put

$$f = \sum_{n=1}^{+\infty} \frac{1}{n^2} f_n,$$

then $f \geq 0$ and

$$\|f\|_2 \leq \sum_{n=1}^{+\infty} \frac{1}{n^2} \|f_n\|_2 = \sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < +\infty,$$

hence $f \in L^2(\mathbb{R})$, i.e. $Tf \in L^2(\mathbb{R})$ by the assumption. In particular,

$$\|Tf\|_2 < +\infty.$$

On the other hand, for every $n \in \mathbb{N}$,

$$Tf \geq T\left(\frac{1}{n^2} f_n\right) = \frac{1}{n^2} Tf_n \geq 0,$$

and it follows that

$$\|Tf\|_2 \geq \frac{1}{n^2} \|Tf_n\|_2 \geq \frac{1}{n^2} n^4 |f_n\|_2 = n^2.$$

This is true for every $n \in \mathbb{N}$, hence $\|Tf\|_2 = +\infty$ contradicting that $Tf \in L^2(\mathbb{R})$. This means that our assumption that T is not bounded, is wrong, so we conclude that T is bounded.

Example 1.13 *Let (e_n) be an orthonormal basis for the Hilbert space H.*
Show that

$$T\left(\sum_{i=1}^{\infty} a_i e_i\right) = (a_1, a_2, \dots)$$

defines an isomorphism from H onto ℓ^2, satisfying $(Tx, Ty) = (x, y)$ for all x, $y \in H$.

Let $x = \sum_n a_n e_n$ and $y = \sum_m b_m e_m$. Using that (e_n) is an orthonormal basis we get

$$(x, y) = \sum_{n.m} a_n \overline{b}_m (e_n, e_m) = \sum_n a_n \overline{b}_n = (Tx, Ty),$$

where we have put $Tx = (a_n) \in \varepsilon^l l^2$ and $Ty = (b_n) \in \ell^2$.

It is obvious that T is linear.
Choosing $y = x$ in the above we get $\|Tx\|^2 = \|x\|^2$, thus $\|Tx\| = \|x\|$, and T is isometric.
If in particular $Tx = 0$, then we infer that $\|x\| = \|Tx\| = 0$, so $x = 0$, and it follows that T is injective.
Finally, T is also surjective. In fact, if $(a_n) \in \ell^2$, then $\sum_n |a_n|^2 < +\infty$. If we therefore put

$$x = \sum_{n=1}^{\infty} a_n e_n,$$

then

$$\|x\|^2 = \sum_{n=1}^{\infty} |a_n|^2 < +\infty,$$

and $x \in H$, and it is obvious that $Tx = (a_n)$.

Example 1.14 *Let M be a subset of a Hilbert space H. Show that $M^{\perp}$ is a closed subspace of H.*
Show that $M \subset \left(M^{\perp}\right)^{\perp}$, and show that $\left(M^{\perp}\right)^{\perp}$ is the smallest closed subspace containing M. It is called the orthogonal complement of M.

By the definition,

$$M^{\perp} = \{y \in H \mid \forall\, x \in M : (x, y) = 0\} = \bigcap_{x \in M} \{y \in H \mid (x, y) = 0\}.$$

The mapping $\varphi_x(y) = (x, y)$ is continuous for every fixed x, hence

$$\{y \in H \mid (x, y) = 0\} = \varphi_x^{\circ -1}(\{0\})$$

is closed, i.e.

$$M^{\perp} = \bigcap_{x \in M} \varphi_x^{\circ -1}(\{0\})$$

is closed as the intersection of a class of closed sets.

If x, $y \in M^{\perp}$, and α is a scalar, then

$$(x + \alpha y, z) = (x, z) + \alpha(y, z) = 0 + \alpha \cdot 0 = 0$$

for every $z \in M$. Then $x + \alpha y \in M^{\perp}$, and it follows that $M^{\perp}$ is a subspace.

Let $x \in M$ and $y \in M^{\perp}$. Then of course $(x, y) = 0$, so we conclude that

$$M \subseteqq \{x \in H \mid \forall\, y \in M^{\perp} : (x, y) = 0\} = \left(M^{\perp}\right)^{\perp}.$$

Clearly, $\left(M^{\perp}\right)^{\perp}$ is a closed subspace. We shall prove that $\left(M^{\perp}\right)^{\perp}$ is the smallest closed subspace which contains M.

Any closed subspace V of H containing M will of course also contain $\overline{M}$, so we may assume that M is closed, $\overline{M} = M$. Then it suffices to prove that if $x \in \left(M^{\perp}\right)^{\perp} \setminus \{0\}$, then there exists a $\lambda \neq 0$, such that $\lambda x \in M$. Then by the definition, $x \in \left(M^{\perp}\right)^{\perp}$, if and only if

$$\forall\, y \in M^{\perp} : (x, y) = 0.$$

By the definition, $y \in M^{\perp}$, if and only if

$$\forall\, z \in M : (y, z) = 0.$$

Assuming that $x \neq 0$ satisfies all conditions, that $\lambda x \notin M$ for all $\lambda \neq 0$, and that

$$(x, y) = (\lambda x, y) \qquad \text{for all } y \in M^{\perp},$$

we infer that there exist a $\lambda \neq 0$ and a sequence $(x_n) \subseteq M$, such that $x_n \to \lambda x$. Since M was assumed to be closed, we infer that $\lambda x \in M$, which contradicts the assumption.

We conclude that there exists $y \in M^\perp$, such that $(x, y) \neq 0$. This shows precisely that $x \notin (M^\perp)^\perp$. Hence, if V is any closed subspace of H, containing M, then $V \supseteq (M^\perp)^\perp$, and $(M^\perp)^\perp$ is therefore the smallest closed subspace with this property.

Example 1.15 *Let (e_n) be an orthogonal sequence in a Hilbert space H, satisfying*

$$\sum_{n=1}^{\infty} \|x_n\|^2 < \infty.$$

Show that the series $\sum_{n=1}^{\infty} x_n$ is convergent in H.
Is this still true if we drop the orthogonality assumption?

Because (x_n) is an orthogonal sequence and H is complete, it suffices to prove that the sectional sequence

$$s_n = \sum_{j=1}^{n} x_n$$

is a Cauchy sequence, i.e.

$$\forall \varepsilon > 0 \, \exists N \, \forall m, n \geq N : \|s_m - s_n\| < \varepsilon.$$

Using that (x_n) is an orthogonal sequence, we get for $m > n$ that

$$\|s_m - s_n\|^2 = \left\| \sum_{j=n+1}^{m} z_j \right\|^2 = \sum_{j=n+1}^{m} \|x_j\|^2.$$

By the assumption, the series $\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$. Thus there exists an N, such that

$$\sum_{j=n+1}^{\infty} \|x_j\|^2 < \varepsilon^2 \qquad \text{for every } n \geq N.$$

If $m > n \geq N$, then

$$\|s_m - s_n\|^2 = \sum_{j=n+1}^{m} \|x_j\|^2 \leq \sum_{j=n+1}^{\infty} \|x_j\|^2 < \varepsilon^2,$$

and we have proved that (s_n) is a Cauchy sequence. The first claim then follows from that H is complete and (s_n) is convergent.

We now construct a simple example, which shows that if (x_n) is just a sequence of vectors in H satisfying that $\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$, then the series $\sum_{n=1}^{\infty} x_n$ is not necessarily convergent in H.

Choose $H = \mathbb{R}$, and let $x_n = \dfrac{1}{n}$. It is well-known that the harmonic series

$$\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

is divergent. It is also well-known that

$$\sum_{n=1}^{\infty} \|x_n\|^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < \infty$$

is convergent.

Example 1.16 *Let H be a Hilbert space (infinite dimensional). Show that there is a sequence of vectors (x_n) such that $\|x_n\| = 1$ for all n and $(x_n, x) \to 0$ for all $x \in H$.*

When H is infinite dimensional, then there exists an orthonormal sequence (x_n), thus $\|x_n\| = 1$, and $(x_m, x_n) = 0$ for $m \neq n$.

Let $x \in H$. It follows from Bessel's inequality that $\sum_{n=1}^{\infty}(x, x_n)x_n$ is convergent in H with

$$\sum_{n=1}^{\infty} |(x, x_n)|^2 \leq \|x\|^2.$$

The series on the left hand side is convergent, hence $|(x, x_n)|^2 \to 0$ for $n \to +\infty$ according to the necessary condition of convergence of a series. This shows that

$$(x, x_n) \to 0 \qquad \text{for } n \to +\infty$$

for every $x \in H$.

Example 1.17 *Let H be a Hilbert space. Show that*

$$\|x - z\| = \|x - y\| + \|y - z\|$$

if and only if $y = \alpha x + (1 - \alpha)z$ for some $\alpha \in [0, 1]$.
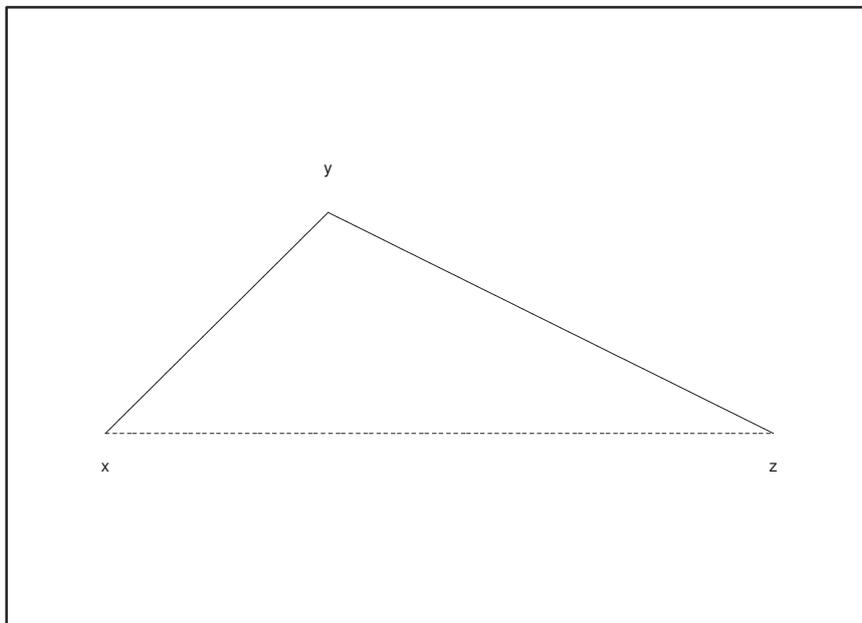

Figure 4: "The smallest detour".

First assume that $y = \alpha x + (1 - \alpha)z$ for some $\alpha \in \mathbb{C}$. Then

$$
\begin{aligned}
\|x - y\| + \|y - z\| &= \|x - \alpha x - (1 - \alpha)z\| + \|\alpha x + (1 - \alpha)z - z\| \\
&= \|(1 - \alpha)(x - z)\| + \|\alpha(x - z)\| = (|1 - \alpha| + |\alpha|)\|x - z\|.
\end{aligned}
$$

If $\alpha \in [0, 1]$, then

$$|1 - \alpha| + |\alpha| = 1 - \alpha + \alpha = 1,$$

hence the equation is fulfilled in this case

If $\alpha \in \mathbb{C} \setminus [0, 1]$, then $|1 - \alpha| + |\alpha| > 1$, and the equation is not fulfilled.

Then assume that y cannot be written in this way. If $x = z$, there is nothing to prove, because $\|x - z\| = 0$, and the right hand side is > 0. Therefore, assume that $x \neq z$. Then

$$y = \alpha(x - z) + w, \qquad \text{where } (w, x - z) = 0.$$

Using a translation we see that we can put $z = 0$, thus

$$y = \alpha x + w, \qquad \text{where } (w, x) = 0 \text{ and } w \neq 0.$$

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We shall prove that

(6) $\|x\| \neq \|x - y\| + \|y\|$.

In fact,

$$
\begin{aligned}
\|x - y\| + \|y\| &= \|(1 - \alpha)x - w\| + \|\alpha x + w\| \\
&= \sqrt{|1 - \alpha|^2 \|x\|^2 + \|w\|^2} + \sqrt{|\alpha|^2 \|x\|^2 + \|w\|^2} \\
&> |1 - \alpha| \cdot \|x\| + |\alpha| \cdot \|x\| \\
&\geq (1 - \alpha + \alpha)\|x\| = \|x\|,
\end{aligned}
$$

and (6) is proved.

Example 1.18 *It is well-known that any continuous function defined on $[0, \pi]$ can be approximated uniformly by linear combinations of cosines. Is this also true if we instead approximate with sines?*

The answer is "no", because every function $\sin jt$ has the value 0 at $t = 0$ and $t = \pi$, so no continuous function f with either $f(0) \neq 0$ or $f(\pi) \neq 0$ can be uniformly approximated by linear combinations of the form

$$
\sum_{j=1}^{k} b_j \sin jt.
$$

Example 1.19 *Let (e_n) be an orthonormal basis for $L^2([0, 1])$. Construct from this an orthonormal basis for $L^2(I)$, where I is a finite interval.*

Let

$$
s = \varphi(t) = \frac{t - a}{b - a}, \qquad t = \varphi^{\circ -1}(s) = a + s(b - a).
$$

Then $[a, b]$ is mapped linearly and bijectively into $[0, 1]$.

Then put

$$
f_k(t) = \frac{1}{\sqrt{b - a}} \, e_k \circ \varphi(t) = \frac{1}{\sqrt{b - a}} \, e_k\left(\frac{t - a}{b - a}\right),
$$

It follows bye the change og variable $s = \dfrac{t - a}{b - a}$ that

$$
\begin{aligned}
(f_m, f_n) &= \int_a^b f_m(t) \, \overline{f_n(t)} \, dt = \int_a^b e_m\left(\frac{t - a}{b - a}\right) \overline{e_m\left(\frac{t - a}{b - a}\right)} \, dt \\
&= \int_0^1 e_m(s) \, \overline{e_b(s)} \, ds = (e_m, e_n) = \delta_{mn},
\end{aligned}
$$

proving that (f_k) is an orthonormal sequence in $L^2([a, b])$.

Now, $L^2([0,1])$ is in a bijective correspondence with $L^2([a,b])$ by

$$g(t) = \left(g \circ \varphi^{\circ -1}\right)(s), \quad \text{for } g \in L^2([a,b]),$$
$$h(s) = (h \circ \varphi)(t), \qquad \text{for } h \in L^2([0,1]),$$

and because (e_k) is a basis for $L^2([0,1])$, we conclude that (f_k) is a basis for $L^2([a,b])$.

Example 1.20 *Let (e_n) be an orthonormal sequence in $L^2(I)$, where I is a finite interval, with the property that for any continuous $f \in L^2(I)$ and any $\varepsilon > 0$ we can find $N \in \mathbb{N}$ and constants a_1, a_2, ..., a_N such that*

$$\left\| f - \sum_{k=1}^{N} a_k e_k \right\| < \varepsilon.$$

Show that (e_n) is an orthonormal basis for $L^2(I)$.

It is well-known that there is an orthonormal basis (φ_k) in $L^2([a,b])$ consisting of *continuous* functions (e.g. some trigonometric system).

Let $\varepsilon > 0$ be given, and consider any $f \in L^2([a,b])$, There exist constants $b_1, \ldots, b_p$, such that

$$\left\| f - \sum_{k=1}^{p} b_k \varphi_k \right\| < \frac{\varepsilon}{2}.$$

We may assume that the φ_k have been arranged such that $b_k \neq 0$ for $k = 1, \ldots, p$. It follows from the assumption that there exist $N_k \in \mathbb{N}$ and constants $a_{k,1}, \ldots, a_{k,N_k}$, such that

$$\left\| \varphi_k - \sum_{j=1}^{N_k} a_{k,j} e_j \right\| < \frac{\varepsilon}{2p|b_k|}, \qquad k = 1, \ldots, p.$$

Then

$$\left\| f - \sum_{k=1}^{p} b_k \sum_{j=1}^{N_k} a_{k,j} e_j \right\| = \left\| f - \sum_{k=1}^{p} b_k \varphi_k + \sum_{k=1}^{p} b_k \left(\varphi - \sum_{j=1}^{N_k} a_{k,j} e_j \right) \right\|$$

$$\leq \left\| f - \sum_{k=1}^{p} b_k \varphi_k \right\| + \sum_{k=1}^{p} |b_k| \cdot \left\| \varphi_k - \sum_{j=1}^{N_k} a_{k,j} e_j \right\|$$

$$< \frac{\varepsilon}{2} + \sum_{k=1}^{p} |b_k| \cdot \frac{\varepsilon}{2p|b_k|} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

The sum $\sum_{k=1}^{p} b_k \sum_{j=1}^{N_k} a_{k,j} e_j$ is a finite linear combination of the e_j for every $\varepsilon > 0$, hence the claim is proved.

Example 1.21 *Let S, $T \in B(H)$, where H is a complex Hilbert space. Prove that if $(Sx, x) = (Tx, x)$ for every $x \in H$, then $S = T$. This result is* not *true for real Hilbert spaces. Give an example.*

We first produce an example which shows that the claim is not true in *real* Hilbert spaces. Let $H = (\mathbb{R}^2, +, \cdot, \mathbb{R})$, and let

$$S(x_1, x_2) = (-x_2, x_1),$$

denote a rotation of the angle $\dfrac{\pi}{2}$ around O, and let $T = 0$. Then

$$(S(x_1, x_2), (x_1, x_2)) = ((-x_2, x_1), (x_1, x_2)) = 0 = (T(x_1, x_2), (x_1, x_2))$$

for all $(x_1, x_2) \in \mathbb{R}^2$, and it is obvious that $S \neq T$.

We have in a complex Hilbertrum (cf. the *polarization identity*)

$$
\begin{aligned}
&\frac{1}{4}\left(S(x+y), x+y\right) - \frac{1}{4}\left(S(x-y), x-y\right) \\
&\quad + \;\frac{i}{4}\left(S(ix+y), ix+y\right) - \frac{i}{4}\left(S(ix-y), ix-y\right) \\
&= \;\frac{1}{4}\left\{(Sx,x)+(Sx,y)+(Sy,x)+(Sy,y)-(Sx,x)+(Sx,y)+(Sy,x)-(Sy,y)\right\} \\
&\quad + \frac{i}{4}\left\{(Sx,x)-(Sx,y)+(Sy,x)+(Sy,y)-(Sx,x)-(Sx,y)+(Sy,x)-(Sy,y)\right\} \\
&= \;(Sy, x).
\end{aligned}
$$
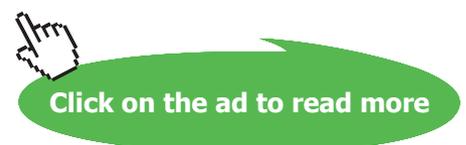
Then it follows from the assumption that

$$
\begin{aligned}
(Sy, x) &= \frac{1}{4}\left(S(x+y), x+y\right) - \frac{1}{4}\left(S(x-y), x-y\right) \\
&\quad + \frac{i}{4}\left(S(ix+y), ix+y\right) - \frac{i}{4}\left(S(ix-y), ix-y\right) \\
&= \frac{1}{4}\left(T(x+y), x+y\right) - \frac{1}{4}\left(T(x-y), x-y\right) \\
&\quad + \frac{i}{4}\left(T(ix+y), ix+y\right) - \frac{i}{4}\left(T(ix-y), ix-y\right) \\
&= (Ty, x),
\end{aligned}
$$

hence

$$(Sy - Ty, x) = ((S-T)y.x) = 0 \qquad \text{for all } x, y \in H.$$

Choosing $x = (S-T)y$, we get $\|(S-T)x\|^2 = 0$, thus $Sx = Tx$ for all $x \in H$, and hence $S = T$.

1.3   Fourier series

Example 1.22 *Find the Fourier series (with respect to the usual trigonometric orthogonal system) for the function*

$$f(x) = |x|, \qquad x \in [-\pi, \pi[,$$

and then find the sum of the series $\sum_{n=1}^{+\infty} \dfrac{1}{n^4}$.

The function $f(x) = |x|$ is even, so we have in the sense of L^2 that

$$|x| = \frac{1}{2} a_0 + \sum_{n=1}^{+\infty} a_n \cos nx,$$

where

$$a_0 = \frac{2}{\pi} \int_0^\pi x \, dx = \frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^\pi = \pi,$$

and where for $n \in \mathbb{N}$,

$$
\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^\pi |x| \cos nx \, dx = \frac{2}{\pi} \int_0^\pi x \cos nx \, dx = \frac{2}{\pi} \left\{ \left[x \cdot \frac{1}{n} \sin nx \right]_0^\pi - \int_0^\pi \frac{1}{n} \sin nx \, dx \right\} \\
&= \frac{2}{\pi} \cdot \frac{1}{n^2} \left[\cos nx \right]_0^\pi = \frac{2}{\pi} \cdot \frac{1}{n^2} \cdot \left\{ (-1)^n - 1 \right\}.
\end{aligned}
$$

We conclude that $a_{2n} = 0$ and that $a_{2n+1} = -\dfrac{4}{\pi(2n+1)^2}$, thus

$$|x| = \frac{\pi}{2} - \sum_{n=0}^{+\infty} \frac{4}{\pi} \cdot \frac{1}{(2n+1)^2} \cos((2n+1)x).$$

Remark 1.6 This holds in the sense of L^2. However, the periodic continuation of f is continuous and piecewise C^1, so one can prove that the result also holds pointwise. $\Diamond$

Using Parseval's equation we get

$$\frac{1}{2} |a_0|^2 + \sum_{n=0}^{+\infty} |a_{2n+1}|^2 = \frac{\pi^2}{2} + \frac{16}{\pi^2} \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^4} = \frac{1}{\pi} \int_{-\pi}^\pi |x|^2 \, dx = \frac{1}{\pi} \left[\frac{x^3}{3} \right]_{-\pi}^\pi = \frac{2}{3} \pi^2,$$

hence by a rearrangement*oraf ved omordning*,

$$\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^4} = \frac{\pi^2}{16} \left\{ \frac{2}{3} \pi^2 - \frac{1}{2} \pi^2 \right\} = \frac{\pi^4}{16} \cdot \frac{1}{6} = \frac{\pi^4}{96}.$$

Then finally,

$$\sum_{n=1}^{+\infty} \frac{1}{n^4} = \left\{ 1 + \frac{1}{2^4} + \frac{1}{(2^4)^2} + \cdots \right\} \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^4} = \frac{1}{1 - \dfrac{1}{16}} \cdot \frac{\pi^4}{96} = \frac{16}{15} \cdot \frac{\pi^4}{96} = \frac{\pi^4}{90}.$$

Example 1.23 *Find the Fourier series (with respect to the usual trigonometric orthogonal system)*
for the function

$$f(x) = \begin{cases} 0, & -\pi \le x < 0, \\ \cos x, & 0 \le x < \pi, \end{cases}$$

and then find the sum of the series $\sum_{n=1}^{+\infty} \dfrac{n^2}{(4n^2 - 1)^2}$.

Let $k \ne 1$. Then

$$\begin{aligned} a_k & = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx = \frac{1}{\pi} \int_0^{\pi} \cos x \cdot \cos kx \, dx \\ & = \frac{1}{2\pi} \int_0^{\pi} \{\cos(k+1)x + \cos(k-1)x\} \, dx \\ & = \frac{1}{2\pi} \left[\frac{1}{k+1} \sin(k+1)x + \frac{1}{k-1} \sin(k-1)x \right]_0^{\pi} = 0, \end{aligned}$$

which for $k = 1$ is supplied with

$$a_1 = \frac{1}{2\pi} \int_0^{\pi} \{\cos 2x + 1\} \, dx = \frac{1}{2}.$$

Furthermore, for $k \ne 1$,

$$\begin{aligned} b_k & = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx = \frac{1}{\pi} \int_0^{\pi} \cos x \cdot \sin kx \, dx \\ & = \frac{1}{2\pi} \int_0^{\pi} \{\sin(k+1)x + \sin(k-1)x\} \, dx \\ & = \frac{1}{2\pi} \left[-\frac{1}{k+1} \cos(k+1)x - \frac{1}{k-1} \cos(k-1)x \right]_0^{\pi} \\ & = \frac{1}{2\pi} \left\{ \frac{(-1) \cdot (-1)^{k+1}}{k+1} + \frac{(-1) \cdot (-1)^{k-1}}{k-1} + \frac{1}{k+1} + \frac{1}{k-1} \right\} \\ & = \frac{1}{2\pi} \left\{ \frac{1 + (-1)^k}{k+1} + \frac{1 + (-1)^k}{k-1} \right\}. \end{aligned}$$

If $k \ne 1$ is odd, then $b_k = 0$. If $k = 2n$ is even, then

$$b_{2n} = \frac{1}{2\pi} \left\{ \frac{2}{2n+1} + \frac{2}{2n-1} \right\} = \frac{1}{\pi} \cdot \frac{4n}{4n^2 - 1}.$$

Finally,

$$b_1 = \frac{1}{2\pi} \int_0^{\pi} \sin 2x \, dx = -\frac{1}{4\pi} [\cos 2x]_0^{\pi} = 0.$$

We have proved that $f(x)$ has the Fourier series

$$f(x) = \frac{1}{2} a_0 + \sum_{k=1}^{+\infty} \{a_k \cos kx + b_k \sin kx\} = \frac{1}{2} \cos x + \sum_{n=1}^{\infty} \frac{4}{\pi} \cdot \frac{n}{4n^2 - 1} \cdot \sin 2nx,$$

where the equalities are in the sense of L^2.

We deduce from Parseval's equation

$$\frac{1}{2}\,a_0^2 + \sum_{k=1}^{+\infty} \left\{ a_k^2 + b_k^2 \right\} = \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx$$

that

$$\frac{1}{4} + \frac{16}{\pi^2} \sum_{n=1}^{+\infty} \frac{n^2}{(4n^2-1)^2} = \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \frac{1}{\pi} \int_0^{\pi} \cos^2 x \, dx = a_1 = \frac{1}{2},$$

hence by a rearrangement,

$$\sum_{n=1}^{+\infty} \frac{n^2}{(4n^2-1)^2} = \frac{\pi^2}{16} \left\{ \frac{1}{2} - \frac{1}{4} \right\} = \frac{\pi^2}{64}.$$

Example 1.24 *Consider the Hilbert space $H = \pi])$ with the orthonormal basis (e_n) defined by*

$$e_n(t) = \frac{1}{\sqrt{2\pi}}\, e^{int}, \qquad n \in \mathbb{Z}.$$

For $f \in H$ we define $\hat{f}$ by

(7) $\hat{f}(x) = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} f(t)\, e^{-ixt}\, dt.$

Show that $\hat{f}$ exists for all $x \in \mathbb{R}$.
Use the function $\hat{f}$ to express the Fourier expansion of $f \in H$ in terms of the orthogonal basis $\left(\sqrt{2\pi}\, e_n\right)$.
Let $\gamma \in \mathbb{R}$ and $f \in H$ be given and define the function g by

$$g(t) = f(t)\, e^{-i\gamma t}.$$

Find $\hat{g}(x)$.
Show that for any $\gamma \in \mathbb{R}$ and $f \in H$ we have

$$\sum_{n=-\infty}^{+\infty} |\hat{f}(n+\gamma)|^2 = \frac{1}{2\pi}\, \|f\|_2^2.$$

Take $f = 1$ and $\gamma = \dfrac{\theta}{\pi}$, $\theta \notin \{p\pi \mid p \in \mathbb{Z}\}$, and show that

$$\frac{1}{\sin^2 \theta} = \sum_{n=-\infty}^{+\infty} \frac{1}{(n\pi + \theta)^2}.$$

1) It follows from the estimate

$$|\hat{f}(x)| \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)| \cdot 1\, dt \le \frac{1}{2\pi} \|1\|_2 \cdot \|f\|_2 = \frac{1}{\sqrt{2\pi}} \|f\|_2,$$

that (7) is convergent for every $x \in \mathbb{R}$, thus $\hat{f}(x)$ exists according to Lebesgue's majorizing theorem.

2) The Fourier series is given by

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx}, \qquad \text{where } c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e^{-ikt}\, dt = \hat{f}(k),$$

hence the Fourier series is

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)\, e^{inx} \qquad \text{for } f \in L^2([-\pi, \pi]).$$

3) Then we compute

$$\hat{g}(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) e^{-ixt}\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e^{-i\gamma t} e^{-ixt}\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e^{-i(x+\gamma)t}\, dt = \hat{f}(x+\gamma).$$

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4) We shall only apply Parseval's equation,

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n+\gamma)|^2 = \sum_{n=-\infty}^{\infty} |\hat{g}(n)|^2 = \frac{1}{2\pi} \|g\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| f(t)\, e^{-i\gamma t} \right|^2 dt = \frac{1}{2\pi} \|f\|_2^2.$$

5) Let $f \equiv 1 \in L^2([-\pi, \pi])$ and $\gamma = \dfrac{\theta}{\pi}$, where $\theta \notin \{p\pi \mid p \in \mathbb{Z}\}$. If $x \neq 0$, then

$$\hat{f}(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} 1 \cdot e^{-ixt}\, dt = \frac{1}{2\pi} \left[-\frac{1}{ix}\, e^{-ixt} \right]_{-\pi}^{\pi} = \frac{1}{\pi x} \cdot \frac{1}{2i} \left\{ e^{ix\pi} - e^{-ix\pi} \right\} = \frac{\sin \pi x}{\pi x}.$$

We note that if $x = 0$, then trivially,

$$\hat{f}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} 1 \cdot 1\, dt = 1 = \lim_{x \to 0} \frac{\sin \pi x}{\pi x}.$$

We shall, however, not use this result.

It follows from (4) that

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \left| \hat{f}\left(n + \frac{\theta}{\pi} \right) \right|^2 &= \sum_{n=-\infty}^{\infty} \left| \frac{\sin\left(\pi \left(n + \dfrac{\theta}{\pi} \right) \right)}{\pi \left(n + \dfrac{\theta}{\pi} \right)} \right| = \sum_{n=-\infty}^{\infty} \left| \frac{(-1)^n \sin \theta}{n\pi + \theta} \right|^2 \\
&= \sum_{n=-\infty}^{\infty} \frac{\sin^2 \theta}{(n\pi + \theta)^2} = \frac{1}{2\pi} \|1\|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} 1^2\, dx = 1,
\end{aligned}$$

hence by a rearrangement, because $n\pi + \theta \neq 0$ for every $n \in \mathbb{Z}$, and because the series trivially is convergent,

$$\sin^2 \theta \sum_{n=-\infty}^{\infty} \frac{1}{(n\pi + \theta)^2} = 1.$$

Because $\sin \theta \neq 0$ for every $\theta \in \mathbb{R} \setminus \{p\pi \mid p \in \mathbb{Z}\}$, we finally get that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\pi n + \theta)^2} = \frac{1}{\sin^2 \theta}.$$

Example 1.25 *Let H denote a Hilbert space, (e_n) an orthonormal basis and $0 < \lambda < 1$ a constant. We define a sequence (g_n) in H by*

$$g_n = \sqrt{1 - \lambda^2}\, e_n, \qquad n \in \mathbb{N}.$$

Find $\lim_n \|f + g_n\|$ for $f \in H$.
Define the unction ξ by

$$\xi(f) = 1 \qquad \text{for } \|f\| < 1,$$

and

$$\xi(f) = 0 \qquad \text{for } \|f\| \ge 1.$$

Show that $\lim_n \xi(f + g_n) = 1$ if $\|f\| < \lambda$ and $\lim_n \xi(f + g_n) = 0$ if $\|f\| > \lambda$.
(The case $\|f\| = \lambda$ is omitted.)

If we put $f = \sum_{k=1}^{+\infty} f_k e_k$, then $f + g_n = \sum_{k=1}^{+\infty} f_k e_k + \sqrt{1 - \lambda^2}\, e_n$, thus

$$\|f + g_n\|^2 = \sum_{k \ne n} |f_k|^2 + \left| f_n + \sqrt{1 - \lambda^2} \right|^2 = \sum_{k \ne n} |f_k|^2 + |f_n|^2 + 1 - \lambda^2 + 2\sqrt{1 - \lambda^2}\, \mathrm{Re}\, f_n.$$

It follows from $f_n \to 0$ for $n \to +\infty$, that

$$\lim_{n \to +\infty} \|f + g_n\|^2 = \|f\|^2 + 1 - \lambda^2.$$

If $\|f\| < \lambda$, then there exists $N(\lambda, f) \in \mathbb{N}$, such that we for every $n \ge N(\lambda, f)$ have $\|f + g_n\| < 1$, i.e.

$$\xi(f + g_n) = 1 \qquad \text{for } n \ge N(\lambda, f),$$

hence

$$\lim_{n \to +\infty} \xi(f + g_n) = 1 \qquad \text{for } \|f\| < \lambda.$$

If $\|f\| > \lambda$, then we find quite analogously $N(\lambda, f) \in \mathbb{N}$, such that $\|f + g_n\| > 1$ for all $n \ge N(\lambda, f)$, hence

$$\xi(f + g_n) = 0 \qquad \text{for } n \ge N(\lambda.f),$$

from which follows that

$$\lim_{n \to +\infty} \xi(f + g_n) = 1 \qquad \text{for } \|f\| > \lambda.$$

1.4 Construction of Hilbert spaces

Example 1.26 *Prove that*

$$\|P_n\|^2 = \frac{2}{2n+1}, \qquad for\ n = 0,\ 1,\ 2,\ \ldots,$$

where P_n are the Legendre polynomials and the space is $L^2([-1,1])$.

It is well-known that

$$P_n(t) = \frac{1}{2^n n!} \frac{d^n}{dt^n}\left((t^2-1)^n\right).$$

Using that we have the factorization $(t^2-1)^n = (t+1)^n(t-1)^n$ we get

$$\frac{d^k}{dt^k}\left((t^2-1)^n\right) = 0 \quad \text{at } t = \pm 1 \quad \text{for } k = 0,\ 1,\ \ldots,\ n-1.$$

Then by partial integration,

$$
\begin{aligned}
\int_{-1}^{1} \{P_n(t)\}^2 \, dt &= \int_{-1}^{1} \frac{1}{2^n n!} \frac{d^n}{dt^n} (t^2-1)^n \cdot \frac{1}{2^n n!} \frac{d^n}{dt^n} (t^2-1)^n \, dt \\
&= \frac{1}{\{2^n n!\}^2} \left\{ \left[\frac{d^n}{dt^n} (t^2-1)^n \cdot \frac{d^{n-1}}{dt^{n-1}} (t^2-1)^n \right]_{t=-1}^{1} \right. \\
&\qquad \left. - \int_{-1}^{1} \frac{d^{n+1}}{dt^{n+1}} (t^2-1)^n \cdot \frac{d^{n-1}}{dt^{n-1}} (t^2-1)^n \, dt \right\} \\
&= \cdots = \frac{(-1)^n}{\{2^n n!\}^2} \int_{-1}^{1} (t^2-1)^n \cdot \frac{d^{2n}}{dt^{2n}} (t^2-1)^n \, dt \\
&= \frac{(-1)^n}{\{2^n n!\}^2} \int_{-1}^{1} (t^2-1)^n \cdot (2n)! \, dt \\
&= \frac{(2n)!}{\{2^n n!\}^2} \int_{-1}^{1} (1-t^2)^n \, dt \\
&= 2 \cdot \frac{(2n)!}{\{2^n n!\}^2} \int_{0}^{1} (1-t^2)^n \, dt.
\end{aligned}
$$

Then apply the monotonous substitution $t = \sin u$ to get

$$
\begin{aligned}
\int_{0}^{1} (1-t^2)^n \, dt &= \int_{0}^{\frac{\pi}{2}} \cos^{2n+1} u \, du = \int_{0}^{\frac{\pi}{1}} \cos^{2n} u \cdot \cos u \, du \\
&= \left[\cos^{2n} u \cdot \sin u \right]_0^{\frac{\pi}{2}} + \int_{0}^{\frac{pi}{2}} 2n \cdot \cos^{2n-1} u \cdot \sin^2 u \, du \\
&= 2n \int_{0}^{\frac{\pi}{2}} \cos^{2n-1} u \cdot (1 - \cos^2 u) \, du \\
&= 2n \int_{0}^{\frac{\pi}{2}} \cos^{2n-1} u \, du - 2n \int_{0}^{\frac{\pi}{2}} \cos^{2n+1} u \, du,
\end{aligned}
$$

hence by a rearrangement and recursion,

$$
\begin{aligned}
\int_{0}^{\frac{\pi}{2}} \cos^{2n+1} u \, du &= \frac{2n}{2n+1} \int_{0}^{\frac{\pi}{2}} \cos^{2n-1} u \, du = \cdots \\
&= \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdots \frac{2}{3} \int_{0}^{\frac{\pi}{2}} \cos u \, du = \frac{\{2^n n!\}^n}{(2n+1)!}.
\end{aligned}
$$

Finally, we get by insertion,

$$
\int_{-1}^{1} \{P_n(t)\}^2 \, dt = 2 \cdot \frac{(2n)!}{\{2^n n!\}^2} \int_{0}^{\frac{\pi}{2}} \cos^{2n+1} u \, du = 2 \cdot \frac{(2n)!}{\{2^n n!\}^2} \cdot \frac{\{2^n \, n!\}^2}{(2n+1)!} = \frac{2}{2n+1}.
$$

Example 1.27 *Show that the Legendre polynomials are orthogonal in $L^2([-1,1])$, and show that* even *normalized Legendre functions (p_n), $n = 0, 2, 4, \ldots$ is an orthonormal basis for the closed subspace of even functions in $L^2([-1,1])$.*
By the way, why is this subspace closed?

We note that

$$P_m(t) = \frac{1}{2^m m!} \frac{d^m}{dt^m} \left((t^2 - 1)^m\right)$$

is a polynomial of degree m,

$$P_m(t) = a_0 + a_1 t + \cdots + a_m t^m.$$

Then clearly the Legendre polynomials are orthogonal, if we can prove that

$$\int_{-1}^1 t^k P_n(t) \, dt = 0, \qquad \text{for } k = 0, 1, \ldots, m \text{ and } m < n.$$

We get by partial integration for $k \le m < n$,

$$
\begin{aligned}
\int_{-1}^1 t^k P_n(t) \, dt &= \frac{1}{2^n n!} \int_{-1}^1 t^k \frac{d^n}{dt^n} \left((t^2 - 1)\right) \, dt \\
&= \left[\frac{1}{2^n n!} t^k \frac{d^{n-1}}{dt^{n-1}} \left((t^2-1)^2\right)\right]_{-1}^1 - \frac{k}{2^n n!} \int_{-1}^1 t^{k-1} \frac{d^{n-1}}{dt^{n-1}} \left((t^2-1)^n\right) \, dt \\
&= \cdots = (-1)^k \frac{k!}{2^n n!} \int_{-1}^1 1 \cdot \frac{d^{n-k}}{dt^{n-k}} \left((t^2-1)^n\right) \, dt \\
&= (-1)^k \cdot \frac{k!}{2^n n!} \left[\frac{d^{n-k-1}}{dt^{n-k-1}} \left((t^2-1)^n\right)\right]_{-1}^1 = 0.
\end{aligned}
$$

In fact, from $k < n$, follows that $n - k - 1 \ge 0$, so

$$\frac{d^{n-k-1}}{dt^{n-k-1}} \left((t^2 - 1)^n\right)$$

is a polynomial, which at least contains the factor $t^2 - 1$, hence the boundary values are 0.

Combining this result with Example 1.26 we obtain that the Legendre polynomials form an orthogonal system.

Denote by $U \subsetneqq L^2([-1,1])$ the closed subspace of all even functions.

We have proved above that $\sqrt{\dfrac{2n+1}{2}} P_n(t)$ is an orthonormal sequence, and since we get them from 1, t, t^2, $\ldots$ by Gram-Schmidt's orthogonalizing method, they form an orthonormal basis for all of $L^2([-1,1])$.

Every function from $L^2([-1,1])$ can uniquely be written as a sum of an even and an odd function. Thus the next claim will be solved if we can prove that $P_n(t)$ is an even function, when n is even, and an odd function for n odd.

Clearly, $(t^2 - 1)^n$ is always an even polynomial, no matter the choice of $n \in \mathbb{N}_0$. By n differentiations we either lower the degree of each term by n, or, the term totally disappears, because its degree is smaller than n. Therefore, if n is even, then all terms of $P_n(t)$ is of even degree, and if n is odd, then all terms of $P_n(t)$ are of odd degree, and the claim follows,

Finally, we notice that it follows from

$$U = \left\{ t^{2n+1} \mid n \in \mathbb{N}_0 \right\}^{\perp},$$

that U is closed.

Example 1.28 *Show that the Hermite polynomial $H_n(t)$ can be written in the form*

$$H_n(t) = n! \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \, \frac{2^{n-2k}}{k!(n-2k)!} \, t^{n-2k},$$

where $[x]$ denotes the integer part of x.
Show that

$$H_{n+1}(t) = 2t \, H_n(t) - H_n'(t),$$

and that

$$H_n'(t) = 2n \, H_{n-1}(t).$$

Use these results to show that H_n is a solution to the Hermite differential equation:

$$\frac{d^2 x}{dt^2} - 2t \, \frac{dx}{dt} + 2nx = 0, \qquad t \in \mathbb{R}.$$

By the definition,

$$H_n(t) = (-1)^n \, e^{t^2} \, \frac{d^n}{dt^n} \left(e^{-t^2} \right),$$

thus

$$H_n'(t) = (-1)^n \, \frac{d}{dt} \left(e^{t^2} \right) \cdot \frac{d^n}{dt^n} \left(e^{-t^2} \right) + (-1)^n e^{t^2} \frac{d^{n+1}}{dt^{n+1}} \left(e^{-t^2} \right) = 2t \cdot H_n(t) - H_{n+1}(t),$$

and hence by a rearrangement,

(8) $\quad H_{n+1}(t) = 2t \cdot H_n(t) - H_n'(t).$

Putting

$$\varphi_n(t) = n! \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \cdot \frac{2^{n-2k}}{k!(n-2k)!} \, t^{n-2k},$$

it follows that

$$\varphi_0(t) = 1 = H_0(t),$$

and

$$\varphi_1(t) = 1!(-1)^0 \cdot \frac{2^{1-2\cdot 0}}{0!(1-0)!}\, t^{1-2\cdot 0} = 2t = H_1(t).$$

If we can prove that φ_n fulfills (8), then it follows by induction that $\varphi_n = H_n$, and the claim is proved.

First we get

$$2t \cdot \varphi_n(t) - \varphi_n'(t) = n! \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \cdot \frac{2^{n+1-2k}}{k!(n-2k)!}\, t^{n+1-2k} - n! \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \cdot \frac{2^{n-2k}}{k!(n-2k)!} \cdot (n-2k) \cdot t^{n-1-2k}.$$

Then we have to split the investigation into the cases of n being even or odd.

If $n = 2p$, then

$$2t\varphi_{2p}(t) - \varphi'_{2p}(t)$$

$$= (2p)! \sum_{k=0}^{p} (-1)^k \cdot \frac{2^{2p+1-2k}}{k!(2p-2k)!} \, t^{2p+1-2k}$$

$$-(2p)! \sum_{k=0}^{p-1} (-1)^k \cdot \frac{2^{2p-2k}}{k!(2p-1-2k)!} \, t^{2p-1-2k}$$

$$= (2p)! \sum_{k=0}^{p} (-1)^k \cdot \frac{2^{2p+1-2k}(2p+1-2k)}{k!(2p+1-2k)!} \cdot t^{2p+1-2k}$$

$$-(2p)! \sum_{k=1}^{p} (-1)^{k-1} \cdot \frac{2^{2p+2-2k}}{(k-1)!(2p+1-2k)!} \, t^{2p+1-2k} \cdot \frac{k}{k}$$

$$= (2p)! \sum_{k=0}^{p} (-1)^k \cdot \frac{2^{2p+1-2k}}{k!(2p+1-2k)!} \cdot (2p+1-2k+2k) t^{2p+1-2k}$$

$$= (2p+1)! \sum_{k=0}^{p} (-1)^k \cdot \frac{2^{2p+1-2k}}{k!(2p+1-2k)!} \cdot t^{2p+1-2k} = \varphi_{2p+1}(t).$$

If $n = 2p + 1$, then analogously,

$$2t \cdot \varphi_{2p+1}(t) - \varphi'_{2p+1}(t)$$

$$= (2p+1)! \sum_{k=0}^{p} (-1)^k \cdot \frac{2^{2p+2-2k}}{k!(2p+1-2k)!} \, t^{tp+2-2k}$$

$$-(2p+1)! \sum_{k=0}^{p} (-1)^k \cdot \frac{2^{2p+1-2k}}{k!(2p-2k)!} \, t^{2p-2k}$$

$$= (2p+1)! \sum_{k=0}^{p} (-1)^k \cdot \frac{2^{2p+2-2k} \cdot (2p+2-2k)}{k!(2p+2-2k)!} \, t^{2p+2-2k}$$

$$-(2p+1)! \sum_{k=1}^{p+1} (-1)^{k-1} \cdot \frac{2^{2p+3-2k}}{(k-1)!(2p+2-2k)!} \cdot \frac{k}{k} \cdot t^{2p+2-2k}$$

$$= (2p+1)! \sum_{k=0}^{p+1} (-1)^k \cdot \frac{2^{2p+2-2k}}{k!(2p+2-2k)!} \cdot (2p+2-2k+2k) \cdot t^{2p+2-2k}$$

$$= (2p+2)! \sum_{k=0}^{p+1} (-1)^k \cdot \frac{2^{2p+2-2k}}{k!(2p+2-2k)!} \cdot t^{2p+2-2k} = \varphi_{2p+2}(t).$$

Then it follows by induction that $\varphi_n(t) = H_n(t)$.

We have already proved above that

$$H_{n+1}(t) = 2t \, H_n(t) - H'_n(t).$$

ALTERNATIVELY,

$$(-1)^n \frac{d^n}{dt^n}\left(e^{-t^2}\right) = e^{-t^2} H_n(t),$$

hence

$$e^{-t^2} H_{n+1}(t) = (-1)^{n+1} \frac{d^{n+1}}{dt^{n+1}}\left(e^{-t^2}\right) = -\frac{d}{dt}\left\{e^{-t^2} H_n(t)\right\} = 2t\, e^{-t^2} H_n(t) - e^{-t^2} H_n'(t).$$

If we multiply by e^{t^2}, we get precisely

$$H_{n+1}(t) = 2t\, H_n(t) - H_n'(t).$$

Using the previous result on the series expansion, we get for $n = 2p$ even that

$$H_n(t) = H_{2p}(t) = (2p)! \sum_{k=0}^{p} (-1)^k \cdot \frac{2^{2p-2k}}{k!(2p-2k)!}\, t^{2p-2k},$$

hence

$$\begin{aligned}
H_n'(t) &= H_{2p}'(t) = (2p)! \sum_{k=0}^{p-1} (-1)^k \cdot \frac{2^{2p-2k}}{k!(2p-1-2k)!}\, t^{2p-1-2k} \\
&= n \cdot (n-1)! \sum_{k=0}^{[\frac{n+1}{2}]} (-1)^k \cdot \frac{2 \cdot 2^{n-1-2k}}{k!(n-1-2k)!}\, t^{n-1-2k} = 2n\, H_{n-1}(t).
\end{aligned}$$

If instead $n = 2p + 1$ is odd, then

$$H_n(t) = H_{2p+1}(t) = (2p+1)! \sum_{k=0}^{p} (-1)^k \cdot \frac{2^{2p+1-2k}}{k!(2p+1-2k)!}\, t^{2p+1-2k},$$

hence

$$\begin{aligned}
H_n'(t) &= H_{2p+1}'(t) = n \cdot (n-1)! \sum_{k=0}^{[\frac{n}{2}]} (-1(^k \cdot \frac{2 \cdot 2^{n-1-.2k}}{k!(n-1-2k)!}\, t^{n-1-2k} \\
&= 2m\, H_{n-1}(t),
\end{aligned}$$

and the claim is proved.

It follows from the above that

$$H_n'(t) = 2t\, H_n(t) - H_{n+1}(t), \qquad H_n'(t) = 2n\, H_{n-1}(t).$$

When we differentiate the former equation, we get

$$H_n''(t) = 2t\, H_n'(t) + 2\, H_n(t) - H_{n+1}'(t),$$

and we have by the latter equation that

$$H_{n+1}'(t) = 2(n+1)H_n(t) = (2n+2)H_n(t),$$

hence by insertion

$$H_n''(t) = 2t\, H_n'(t) - 2n\, H_n(t), \qquad t \in \mathbb{R},$$

and we get by a rearrangement,

$$H_n''(t) - 2t\, H_n'(t) + 2n\, H_n(t) = 0.$$

Example 1.29 *Consider in* $L^2([0,1])$ *the sequence of* Rademacher functions*:*

$$e_n(t) = \sum_{j=0}^{2^n-1} (-1)^j 1_{]\frac{j}{2^n}, \frac{j+1}{2^n}[}(t), \qquad n \in \mathbb{N}.$$

1) Draw the graphs for e_1, e_2, e_3 *and* e_4.

2) Show that (e_n) *is an orthonormal sequence in* $L^2([0,1])$.

3) Show that (e_n) *is not an orthonormal basis.*

1) For some strange reason I have not been able to let MAPLE sketch the graph of $e_4(t)$ – at least not within 4 minutes. The former three graphs look as follows:
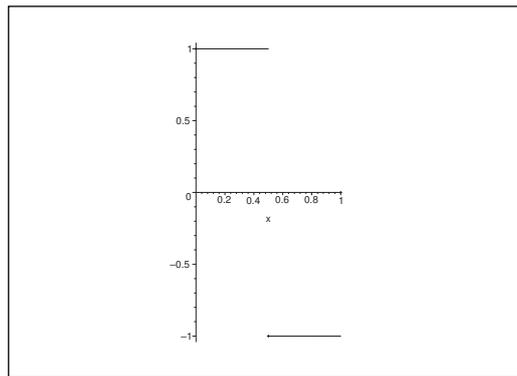
Figure 5: The graph of $e_1(t)$.

2) It follows from $|e_n(t)| = 1$ for almost every $t \in [0,1]$ that

$$(e_n, e_n) = \int_0^1 e_n(t)^2 dt = \int_0^1 dt = 1.$$

If $m > n$, then

$$(e_m, e_n) = \sum_{j=0}^{2^n-1} (-1)^j \left\{ \sum_{k=0}^{2^m-1} \int_{\frac{j}{2^n}}^{\frac{j+1}{2^n}} (-1)^k \cdot 1_{]\frac{k}{2^m}, \frac{k+1}{2^m}[}(t)\, dt \right\} = 0,$$

because a simple consideration of the graph shows that the expression inside $\{\cdots\}$ is 0, when $m > n$, hence $\{e_n\}$ is an orthonormal sequence.

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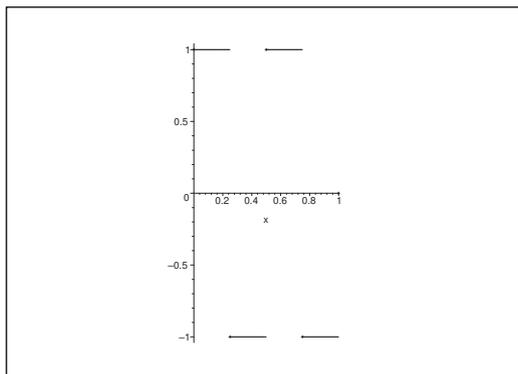
Figure 6: The graph of $e_2(t)$.
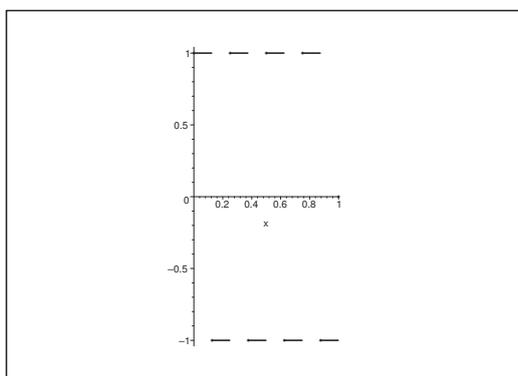
Figure 7: The graph of $e_2(t)$.

3) We prove that $\{e_n\}$ is not total. If $e(t) = 1_{[0,1]}(t) \neq 0$, then

$$(e_n, e) = 0 \qquad \text{for every } n \in \mathbb{N},$$

thus $\{e_n\}$ is not total, and hence not an orthonormal basis either. We see by a parallel translation to $x = \dfrac{1}{2}$ that with respect to this coordinate all functions $e_n(t)$ are odd, explaining the phenomenon.

Example 1.30 *Consider in $L^2([0,1])$ the sequence of* Haar functions*:*

$$h_1(t) = 1,$$

and

$$h_{2^m+k}(t) = \sqrt{2^m} \qquad for \ \frac{k-1}{2^m} \le t \le \frac{2k-1}{2^{2m+1}},$$

$$h_{2^m+k}(t) = -\sqrt{2^m} \quad for \ \frac{2k-1}{2^{m+1}} \le k \le \frac{k}{2^m},$$

$$h_{2^m+k}(t) = 0 \qquad else,$$

where $k = 1, 2, \ldots, 2^m$ and $m = 0, 1, 2, \ldots$.

1) Sketch the graphs of $h_1, h_2, \ldots, h_8$.

2) Show that (h_n) is an orthonormal sequence in $L^2([0,1])$.

3) Show that (h_n) is an orthonormal basis in $L^2([0,1])$.

1)
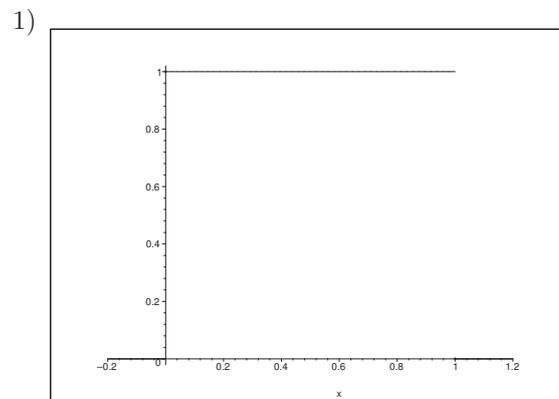
Figure 8: The graph of $h_1(t)$.

2) Let $n \ne m$, e.g. $n > m$. If $h_n(t) \cdot h_m(t) \equiv 0$, there is nothing to prove. If instead $h_n \cdot h_m \ne 0$, then $h_n(t) \cdot h_m(t)$ is piecewise constant, and $h_n(t)$, $n > m$ must be an odd function with respect to the midpoint of $\{t \mid h_n(t) \cdot h_m(t) \ne 0\}$, where we ignore the values at the midpoint and at the endpoints. It follows that

$$(h_n, h_m) = \int_0^1 h_n(t) \cdot h_m(t) \, dt = 0, \qquad \text{for } n > m.$$

Finally,

$$\int_0^1 h_1(t)^2 \, dt = 1,$$

and

$$\int_0^1 \{h_{2^m+k}(t)\}^2 \, dt = \int_{\frac{k-1}{2^m}}^{\frac{k}{2^m}} \left\{\pm\sqrt{2^m}\right\}^2 \, dt = \int_{\frac{k-1}{2^m}}^{\frac{k}{2^m}} 2^m \, dt = 2^m \cdot \frac{1}{2^m} = 1.$$

Hence we have proved that (h_n) is an orthonormal sequence.
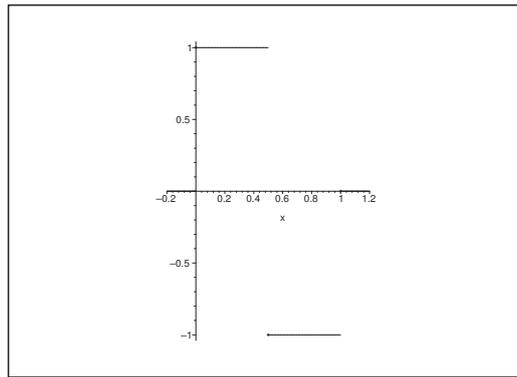
Figure 9: The graph of $h_2(t)$.
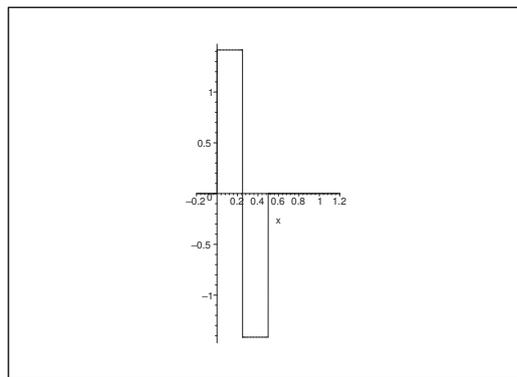
Figure 10: The graph of $h_3(t)$.
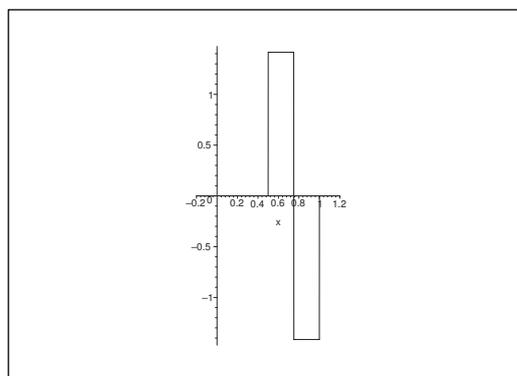
Figure 11: The graph of $h_4(t)$.

3) Assume that

$$(f, h_n) = 0 \qquad \text{for every } n \in \mathbb{N}.$$
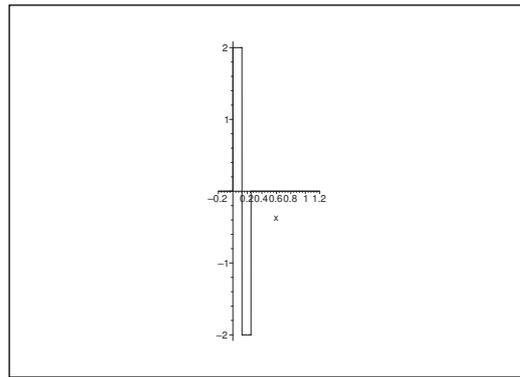
Figure 12: The graph of $h_5(t)$.
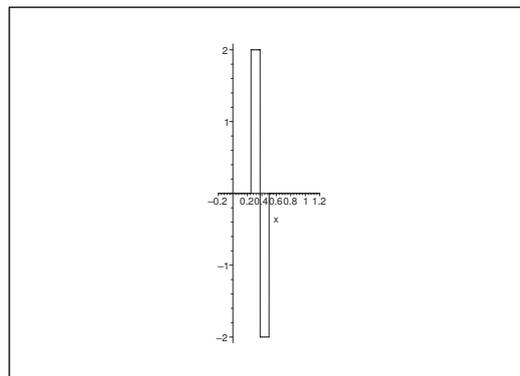
Figure 13: The graph of $h_6(t)$.
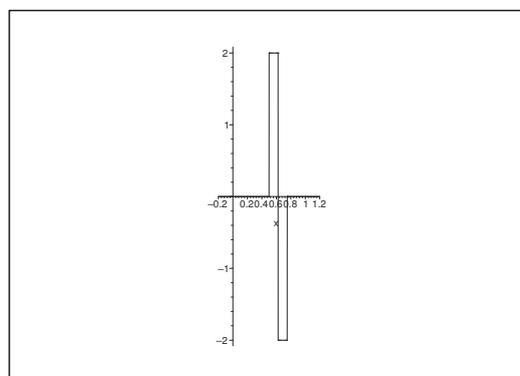
Figure 14: The graph of $h_7(t)$.

We shall prove that $f = 0$ almost everywhere. Put

$$F(t) = \int_0^t f(u)\, du.$$
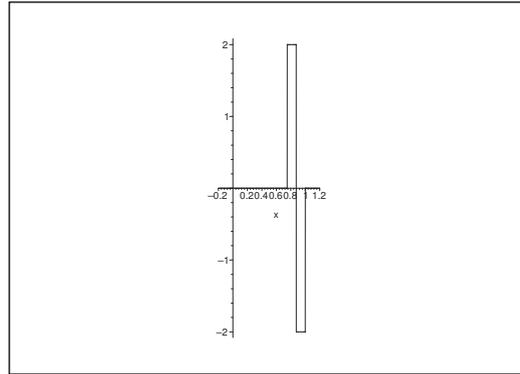
Figure 15: The graph of $h_8(t)$.

Since $f \in L^2([0,1]) \subseteqq L^1([0,1])$, we infer that $F(t)$ is continuous, and that $F(0) = 0$. It follows from

$$(f, h_1) = \int_0^1 f(t)\, dt = F(1) = 0,$$

that $F(1) = 0$. Furthermore,

$$(f, h_2) = \int_0^{\frac{1}{2}} f(t)\, dt - \int_{\frac{1}{2}}^1 f(t)\, dt = 0,$$

thus

$$F\left(\frac{1}{2}\right) = \int_0^{\frac{1}{2}} f(t)\, dt = \int_{\frac{1}{2}}^1 f(t)\, dt,$$

and hence

$$F(t) = 0 = \int_0^{\frac{1}{2}} f(t)\, dt + \int_{\frac{1}{2}}^1 f(t)\, dt = 2\, F\left(\frac{1}{2}\right),$$

proving that $F\left(\frac{1}{2}\right) = 0$. It follows from

$$(f, h_3) = \sqrt{2}\left\{\int_0^{\frac{1}{4}} f(t)\, dt - \int_{\frac{1}{4}}^{\frac{1}{2}} f(t)\, dt\right\} = 0,$$

that

$$F\left(\frac{1}{4}\right) = \int_0^{\frac{1}{4}} f(t)\, dt = \int_{\frac{1}{4}}^{\frac{1}{2}} f(t)\, dt,$$

hence

$$2\, F\left(\frac{1}{4}\right) = \int_0^{\frac{1}{4}} f(t)\, dt + \int_{\frac{1}{4}}^{\frac{1}{2}} f(t)\, dt = F\left(\frac{1}{2}\right) = 0.$$

We continue in this way, so we conclude that

$$F\left(\frac{k}{2^m}\right) = 0 \qquad \text{for all } k = 0, 1, \ldots, 2^m \text{ and all } m \in \mathbb{N}_0.$$

The function $F(t)$ is continuous, and the set

$$\left\{\frac{k}{2^m} \;\middle|\; k = 0, 1, \ldots, 2^m; m \in \mathbb{N}_0\right\}$$

is dense in $[0, 1]$, hence $F(t) \equiv 0$. This implies that $f = 0$ almost everywhere, and we have proved that the (h_n) form an orthonormal basis.

Example 1.31 *Prove that the k-th Laguerre polynomial can be written*

$$L_k(t) = \sum_{j=0}^{k} \frac{(-1)^j}{j!} \left(\begin{array}{c} k \\ j \end{array} \right) t^j,$$

and then prove that

$$\sum_{k=0}^{+\infty} L_k(t)\, w^k = \frac{1}{1-w} \exp\left(-\frac{tw}{1-w} \right) \qquad \text{for } |w| < 1, \quad t \geq 0.$$

By the definition of the Laguerre polynomials,

$$L_k(t) = \frac{1}{k!}\, e^t\, \frac{d^k}{dt^k} \left(t^k\, e^{-t} \right).$$

We first prove *Leibniz's formula* as an auxiliary result. First note that if f, $g \in C^\infty$ (possibly only of class C^n for n sufficiently large), then

$$\frac{d}{dt}(f \cdot g) = \frac{df}{dt} \cdot g + f \cdot \frac{dg}{dt},$$

$$\frac{d^2}{dt^2}(f \cdot g) = \frac{d^2 f}{dt^2} \cdot g + 2\frac{df}{dt} \cdot \frac{dg}{dt} + f \cdot \frac{d^2 g}{dt^2} = \sum_{j=0}^{2} \left(\begin{array}{c} 2 \\ j \end{array} \right) f^{(j)}(t) \cdot g^{(2-j)}(t).$$

We now prove *Leibniz's formula* for differentiation of a product,

$$\frac{d^n}{dt^n}(f \cdot g) = \sum_{j=0}^{n} \left(\begin{array}{c} n \\ j \end{array} \right) f^{(j)}(t) \cdot g^{(n-j)}(t).$$

According to the above this holds at least for $n = 0, 1, 2$.
Assume that the formula is true for some $n \in \mathbb{N}$. Then

$$\begin{aligned}
\frac{d^{n+1}}{dt^{n+1}}(f \cdot g) &= \frac{d}{dt} \sum_{j=0}^{n} \left(\begin{array}{c} n \\ j \end{array} \right) f^{(j)}(t) \cdot g^{(n-j)}(t) \\[2mm]
&= \sum_{j=1}^{n} \left(\begin{array}{c} n \\ j \end{array} \right) f^{(j+1)}(t) g^{(n-j)}(t) + \sum_{j=0}^{n} \left(\begin{array}{c} n \\ j \end{array} \right) f^{(j)}(t) g^{n+1-j}(t) \\[2mm]
&= \sum_{j=1}^{n+1} \left(\begin{array}{c} n \\ j-1 \end{array} \right) f^{(j)}(t) g^{(n+1-j)}(t) + \sum_{j=0}^{n} \left(\begin{array}{c} n \\ j \end{array} \right) f^{(j)}(t) g^{(n+1-j)}(t) \\[2mm]
&= \left(\begin{array}{c} n \\ n \end{array} \right) f^{(n+1)}(t) g(t) + \left(\begin{array}{c} n \\ 0 \end{array} \right) f(t) g^{(n+1)}(t) \\[2mm]
&\quad + \sum_{j=1}^{n} \left\{ \left(\begin{array}{c} n \\ j-1 \end{array} \right) + \left(\begin{array}{c} n \\ j \end{array} \right) \right\} f^{(j)}(t) g^{(n+1-j)}(t) \\[2mm]
&= \left(\begin{array}{c} n+1 \\ n+1 \end{array} \right) f^{(n+1)}(t) g(t) + \left(\begin{array}{c} n+1 \\ 0 \end{array} \right) f(t) g^{(n+1)}(t) \\[2mm]
&\quad + \sum_{j=1}^{n} \left(\begin{array}{c} n+1 \\ j \end{array} \right) f^{(j)}(t) g^{(n+1-j)}(t) \\[2mm]
&= \sum_{j=0}^{n+1} \left(\begin{array}{c} n+1 \\ j \end{array} \right) f^{(j)}(t) g^{(n+1-j)}(t),
\end{aligned}$$

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because

$$\begin{pmatrix} n \\ j-1 \end{pmatrix} + \begin{pmatrix} n \\ j \end{pmatrix} = \frac{n!}{(j-1)!(n+1-j)!} + \frac{n!}{j!(n-j)!}$$

$$= \frac{n!}{j!(n+1-j)!}(j+n+1-n) = \begin{pmatrix} n+1 \\ j \end{pmatrix}.$$

Then Leibniz's formula follows by induction.

When we apply Leibniz's formula on $L_k(t)$, we get

$$L_k(t) = \frac{1}{k!} e^t \frac{d^k}{dt^k} \left(t^k e^{-t} \right) = \frac{1}{k!} e^t \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} \frac{d^{k-j}}{dt^{k-j}} \left(t^k \right) \cdot \frac{d^j}{dt^j} \left(e^{-t} \right)$$

$$= \frac{1}{k!} e^t \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} k \cdot (k-1) \cdots (j+1) \cdot t^{k-(k-j)} \cdot (-1)^j e^{-t}$$

$$= \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} \frac{k(k-1)\cdots(j+1)}{k!} t^j \cdot (-1)^j e^t e^{-t} = \sum_{j=0}^{k} \frac{(-1)^j}{j!} \begin{pmatrix} k \\ j \end{pmatrix} t^j.$$

Then by formal computations,

$$
\begin{aligned}
\sum_{k=0}^{+\infty} L_k(t)\, w^k &= \sum_{k=0}^{+\infty} \sum_{j=0}^{k} \frac{(-1)^j}{j!} \binom{k}{j} t^j w^k = \sum_{j=0}^{+\infty} \sum_{k=j}^{+\infty} \frac{(-1)^j}{j!} \binom{k}{j} t^j w^k \\
&= \sum_{j=0}^{+\infty} \frac{(-1)^j}{j!} \sum_{k=j}^{+\infty} \binom{k}{j} w^k = \sum_{j=0}^{+\infty} \frac{(-1)^j}{j!} \sum_{k=0}^{+\infty} \binom{k+j}{j} w^{k+j} \\
&= \sum_{j=0}^{+\infty} \frac{(-tw)^j}{j!} \sum_{k=0}^{+\infty} \binom{k+j}{j} w^k.
\end{aligned}
$$

We have for $|w| < 1$,

$$
\begin{aligned}
\frac{1}{(1-w)^{j+1}} &= \frac{1}{j!} \frac{d^j}{dw^j} (1-w)^{-1} = \frac{1}{j!} \frac{d^j}{dw^j} \sum_{k=0}^{+\infty} w^k = \frac{1}{j!} \sum_{k=j}^{+\infty} k(k-1)\cdots(k+1-j) w^{k-j} \\
&= \frac{1}{j!} \sum_{k=j}^{+\infty} \frac{k!}{(k-j)!} w^{k-j} = \sum_{j=j}^{+\infty} \binom{k}{j} w^{k-j} = \sum_{j=0}^{+\infty} \binom{k+j}{j} w^k,
\end{aligned}
$$

hence *formally*,

$$
\sum_{k=0}^{+\infty} L_j(t) w^k = \sum_{j=0}^{+\infty} \frac{(-tw)^j}{j!} \cdot \left(\frac{1}{1-w}\right)^{j+1} = \frac{1}{1-w} \sum_{j=0}^{+\infty} \frac{1}{j!} \left(-\frac{tw}{1-w}\right)^j = \frac{1}{1-w} \exp\left(-\frac{tw}{1-w}\right).
$$

These computations are, however, legal, if $|w| < 1$ (an assumption which was used above), and if furthermore it is allowed to interchange the summations. It follows from

$$
|L_k(t)| = \left| \sum_{j=0}^{k} \frac{(-1)^j}{j!} \binom{k}{j} t^j \right| \le \sum_{j=0}^{k} \binom{k}{j} |t|^j = (1+|t|)^j,
$$

that $\sum_{k=0}^{+\infty} L_k(t) w^k$ is convergent for all $|w| < \dfrac{1}{1+|t|}$, and we conclude that all the formal computations above are indeed valid in every closed subset of $|w| < 1$. Hence

$$
\sum_{k=0}^{+\infty} L_k(t) w^k = \frac{1}{1-w} \exp\left(-\frac{tw}{1-w}\right) \qquad \text{for } |w| < \frac{1}{1+|t|}.
$$

Now, for every fixed t the right hand side is analytic in all of $|w| < 1$, thus

$$
\frac{1}{1-w} \exp\left(-\frac{tw}{1-w}\right) = \sum_{k=0}^{+\infty} a_j(t) w^k \qquad \text{for } |w| < 1.
$$

Then it follows from the uniqueness theorem that $a_k(t) = L_k(t)$, hence the formula holds for every $t \ge 0$ and every $w \in \mathbb{C}$ satisfying $|w| < 1$.

Example 1.32 *Prove that*

$$\int_0^{+\infty} e^{-t} L_m(t)\, L_n(t)\, dt = 0 \qquad for\ 0 \le m < n.$$

The Laguerre polynomials $L_k(t)$ are given by

$$L_k(t) = \frac{1}{k!}\, e^t\, \frac{d^k}{dt^k}\left(t^k e^{-t}\right), \qquad k \in \mathbb{N}_0.$$

Using a combinatorial argument on the differentiation of the product it follows that $L_m(t)$ is a polynomial of precisely degree m. Then clearly the condition

$$\int_0^{+\infty} e^{-t}\, t^k\, L_n(t)\, dt = 0 \qquad for\ k = 0,\, 1,\, \ldots,\, m,$$

implies that $\int_0^{+\infty} e^{-t} L_m(t)\, L_n(t)\, dt = 0$, because $L_m(t)$ can be written as a finite linear combination of $1,\, t,\, t^2,\, \ldots,\, t^m$.

We get by partial integration (note that the exponential function secures that the contribution of the "boundary" at $+\infty$ becomes 0),

$$\int_0^{+\infty} e^{-t}\, t^k\, L_n(t)\, dt$$

$$= \int_0^{+\infty} e^{-t}\, t^k \cdot \frac{1}{n!}\, e^t\, \frac{d^n}{dt^n}\left(t^n\, e^{-t}\right) dt = \frac{1}{n!} \int_0^{+\infty} t^k\, \frac{d^n}{dt^n}\left(t^n\, e^{-t}\right) dt$$

$$= \frac{1}{n!}\left[t^k\, \frac{d^{n-1}}{dt^{n-1}}\left(t^n\, e^{-t}\right)\right]_0^{+\infty} - \frac{k}{n!} \int_0^{+\infty} t^{k-1} \cdot \frac{d^{n-1}}{dt^{n-1}}\left(t^n\, e^{-t}\right) dt = \cdots$$

$$= (-1)^k\, \frac{k!}{n!} \int_0^{+\infty} 1 \cdot \frac{d^{n-k}}{dt^{n-k}}\left(t^n\, e^{-t}\right) dt = (-1)^k \cdot \frac{k!}{n!}\left[\frac{d^{n-k-1}}{dt^{n-k-1}}\left(t^n\, e^{-t}\right)\right]_0^{+\infty} = 0,$$

and the claim is proved. Here we have applied that $\dfrac{d^{n-k-1}}{dt^{n-k-1}}\left(t^n\, e^{-t}\right)$

1) is defined (because $k < n$),

2) contains at least one factor t, (thus the contribution of the boundary is zero at 0),

3) has the form $P(t)e^{-t}$, where $P(t)$ is a polynomial, hence the boundary contribution at $+\infty$ is 0 due to the difference of magnitude.

Example 1.33 *Prove that*

$$\int_0^{+\infty} e^{-t} \left\{L_n(t)\right\}^2 \, dt = 1, \qquad n \in \mathbb{N}_0.$$

It follows from the fact that

$$L_n(t) = \frac{1}{n!} \, e^t \, \frac{d^n}{dt^n} \left(t^n e^{-t}\right)$$

is a polynomial of degree n that the integral is convergent. We get by insertion followed by partial integration,

$$
\begin{aligned}
\int_0^{+\infty} e^{-t} \left\{L_n(t)\right\}^2 \, dt &= \frac{1}{(n!)^2} \int_0^{+\infty} e^t \, \frac{d^n}{dt^n} \left(t^n \, e^{-t}\right) \cdot \frac{d^n}{dt^n} \left(t^n \, e^{-t}\right) \, dt \\
&= \frac{1}{(n!)^2} \left[e^t \, \frac{d^n}{dt^n} \left(t^n \, e^{-t}\right) \cdot \frac{d^{n-1}}{dt^{n-1}} \left(t^n \, e^{-t}\right) \right]_0^{+\infty} \\
&\quad - \frac{1}{(n!)^2} \int_0^{+\infty} \frac{d}{dt} \left\{ e^t \, \frac{d^n}{dt^n} \left(t^n \, e^{-t}\right) \right\} \frac{d^{n-1}}{dt^{n-1}} \left(t^n \, e^{-t}\right) \, dt.
\end{aligned}
$$

From $\dfrac{d^{n-1}}{dt^{n-1}} \left(t^n \, e^{-t}\right) = t \cdot \{\cdots\}$ follows that the lower boundary contribution is 0, and since the boundary term is of the form $e^{-t} P(t)$, where $P(t)$ is a polynomial, the upper boundary contribution is also 0. Hence

$$\int_0^{+\infty} e^{-t} \left\{L_n(t)\right\}^2 \, dt = -\frac{1}{(n!)^2} \int_0^{+\infty} \frac{d}{dt} \left\{ et \, \frac{d^n}{dt^n} \left(t^n \, e^{-t}\right) \right\} \cdot \frac{d^{n-1}}{dt^{n-1}} \left(t^n \, e^{-t}\right) \, dt.$$

Repeating the argument above we get after n steps that

$$\int_0^{+\infty} e^{-t} \left\{L_n(t)\right\}^2 \, dt = \frac{(-1)^n}{(n!)^2} \int_0^{+\infty} \frac{d^n}{dt^n} \left\{e^t \frac{d^n}{dt^n} \left(t^n \, e^{-t}\right)\right\} \cdot t^n e^{-t} \, dt.$$

Then notice that $e^t \dfrac{d^n}{dt^n} \left(t^n \, e^{-t}\right)$ is a polynomial of degree n. When we differentiate n times, all terms of lower degree have disappeared, and the only contribution comes from the term of highest degree, which is given by

$$e^t \, t^n \, \frac{d^n}{dt^n} \left(e^{-t}\right) = (-1)^n e^t \, t^n \, e^{-t} = (-1)^n \, t^n.$$

When this is differentiated n times, we get $(-1)^n n!$, thus

$$\frac{d^n}{dt^n} \left\{e^t \frac{d^n}{dt^n} \left(t^n \, e^{-t}\right)\right\} = (-1)^n \, n!.$$

Then by insertion into the above,

$$\int_0^{+\infty} e^{-t} \left\{L_n(t)\right\}^2 dt = \frac{(-1)^n}{(n!)^2} \int_0^{+\infty} (-1)^n \cdot n! \, t^n \, e^{-t} \, dt = \frac{1}{n!} \int_0^{+\infty} t^n \, e^{-t} \, dt = \frac{1}{n!} \cdot n! = 1,$$

where we have applied that

$$\Gamma(n+1) = \int_0^{+\infty} t^n \, e^{-t} \, dt = n!.$$

Example 1.34 *1) Prove for $|w| < 1$, $t \geq 0$ that*

$$\sum_{k=0}^{+\infty} w^k \exp\left(-\frac{t}{2}\right) L_k(t) = \frac{1}{1-w} \exp\left(-\frac{1}{2}\frac{1+w}{1-w}\,t\right) = g(t,w).$$

2) Prove for every $w \in \,]-1,1[$ that

$$\int_0^{+\infty} \left\{ g(t,w) - \sum_{k=0}^{N} w^k \exp\left(-\frac{t}{2}\right) L_k(t) \right\}^2 dt = \frac{1}{1-w^2} - \sum_{k=0}^{N} w^{2k}.$$

3) Prove that the image of $\alpha = \dfrac{1}{2} \cdot \dfrac{1+w}{1-w}$ is $\,]0,+\infty[$ for the range $w \in \,]-1,1[$.

4) Prove that every function $e^{-\alpha t}$ (where $\alpha > 0$) can be approximated in the sense of L^2 over $\,]0,+\infty[$ by a finite linear combination of the form

$$\sum_{k=0}^{N} a_k \exp\left(-\frac{t}{2}\right) L_k(t).$$

1) Let $|w| < 1$ and $t \geq 0$. Then it follows from EXAMPLE 1.31 that

$$L_k(t) = \sum_{j=0}^{k} \frac{(-1)^j}{j!} \binom{k}{j} t^j.$$

and we get the following crude estimate,

$$\left| \exp\left(-\frac{t}{2}\right) L_k(t) \right| \leq \exp\left(-\frac{t}{2}\right) \sum_{j=0}^{k} \binom{k}{j} t^j = \exp\left(-\frac{t}{2}\right) (1+t)^k \leq (1+t)^k,$$

from which we conclude that the series

$$\sum_{k=0}^{+\infty} w^k \exp\left(-\frac{t}{2}\right) L_k(t)$$

is at least convergent for $|w| < \dfrac{1}{1+t}$. We restrict ourselves in the following to a closed disc

contained in $|w| < \dfrac{1}{1+t}$. Then

$$\sum_{k=0}^{+\infty} w^k \exp\left(-\frac{t}{2}\right) L_k(t)$$

$$= \sum_{k=0}^{+\infty} w^k \exp\left(-\frac{t}{2}\right) \sum_{j=0}^{k} \frac{(-1)^j}{j!} \left(\begin{array}{c} k \\ j \end{array}\right) t^j = \exp\left(-\frac{t}{2}\right) \sum_{k=0}^{+\infty} \sum_{j=0}^{k} \frac{(-1)^j}{j!} \left(\begin{array}{c} k \\ j \end{array}\right) t^j \, w^k$$

$$= \exp\left(-\frac{t}{2}\right) \sum_{j=0}^{+\infty} \sum_{k=j}^{+\infty} \frac{(-1)^j}{j!} \left(\begin{array}{c} k \\ j \end{array}\right) t^j \, w^k = \exp\left(-\frac{t}{2}\right) \sum_{j=0}^{+\infty} \frac{1}{j!} (-t)^j \sum_{k=0}^{+\infty} \left(\begin{array}{c} k+j \\ j \end{array}\right) w^{k+j}$$

$$= \exp\left(-\frac{t}{2}\right) \sum_{j=0}^{+\infty} \frac{(-tw)^j}{j!} \sum_{k=0}^{+\infty} \left(\begin{array}{c} k+j \\ j \end{array}\right) w^k.$$

We get from EXAMPLE 1.31 that

$$\sum_{k=0}^{+\infty} \left(\begin{array}{c} k+j \\ j \end{array}\right) w^k = \frac{1}{(1-w)^{j+1}} \qquad \text{for } |w| < 1,$$

so if $|w| < \dfrac{1}{1+t}$, $t \geq 0$, then

$$
\begin{aligned}
\sum_{k=0}^{+\infty} w^k \exp\left(-\frac{t}{2}\right) L_k(t) &= \exp\left(-\frac{t}{2}\right) \sum_{j=0}^{+\infty} \frac{1}{j!} (-tw)^j \cdot \frac{1}{(1-w)^{j+1}} \\
&= \frac{1}{1-w} \exp\left(-\frac{t}{2}\right) \sum_{j=0}^{+\infty} \frac{1}{j!} \left(-\frac{tw}{1-w}\right)^j \\
&= \frac{1}{1-w} \exp\left(-\frac{t}{2}\right) \exp\left(-t \cdot \frac{w}{1-w}\right) \\
&= \frac{1}{1-w} \exp\left(-t \left\{\frac{1}{2} + \frac{w}{1-w}\right\}\right) \\
&= \frac{1}{1-w} \exp\left(-\frac{t}{2} \left\{\frac{1-w}{1-w} + \frac{2w}{1-w}\right\}\right) \\
&= \frac{1}{1-w} \exp\left(-\frac{t}{2} \cdot \frac{1+w}{1-w}\right).
\end{aligned}
$$

This is true for all $|w| < \dfrac{1}{1+t}$. However, the right hand side is an analytic function for $|w| < 1$, and since the series expansion exists and is unique in this domain, the formula holds for $|w| < 1$ and $t \geq 0$.

2) Now $\exp\left(-\dfrac{t}{2}\right) L_k(t)$, $k \in \mathbb{N}_0$, so an orthonormal system on $L^2([0, +\infty[)$ for $w \in]-1, 1[$ is given by

$$\int_0^{+\infty} \left\{ g(t, w) - \sum_{k=0}^{N} w^k \exp\left(-\frac{t}{2}\right) L_k(t) \right\}^2 dt$$

$$= \left\| g(t, w) - \sum_{k=0}^{N} w^k \exp\left(-\frac{t}{2}\right) L_k(t) \right\|^2 = \left\| \sum_{k=N+1}^{+\infty} w^k \exp\left(-\frac{t}{2}\right) L_k(t) \right\|^2$$

$$= \sum_{k=N+1}^{+\infty} w^{2k} = \sum_{k=0}^{+\infty} \left(w^2 \right)^k - \sum_{k=0}^{N} \left(w^2 \right)^k = \frac{1}{1 - w^2} - \sum_{k=0}^{N} w^{2k}.$$

3) It follows from

$$\frac{d\alpha}{dw} = \frac{1}{2} \cdot \frac{1}{(1-w)^2} \cdot (1 - w + 1 + w) = \frac{1}{(1-w)^2} > 0 \qquad \text{for } w \in]-1, 1[,$$

that $\alpha(w)$ is increasing, and $\alpha(-1) = 0$, and $\alpha(w) \to +\infty$ for $w \to 1-$. Hence α runs through $]0, +\infty[$, when w runs through $]-1, 1[$.

4) Putting $\alpha = \dfrac{1}{2} \cdot \dfrac{1+w}{1-w}$ we get

$$\frac{w+1}{w-1} = -2\alpha, \qquad \text{thus} \qquad w = \frac{-2\alpha + 1}{-2\alpha - 1} = \frac{2\alpha - 1}{2\alpha + 1},$$

and

$$1 - w = 1 - \frac{2\alpha - 1}{2\alpha + 1} = \frac{2\alpha + 1 - 2\alpha + 1}{2\alpha + 1} = \frac{2}{2\alpha + 1}.$$

It follows from the above that

$$e^{-\alpha t} = \exp\left(-\frac{1}{2} \cdot \frac{1+w}{1-w} t\right) = \sum_{k=0}^{+\infty} (1-w) w^k \exp\left(-\frac{t}{2}\right) L_k(t)$$

$$= \sum_{k=0}^{+\infty} \frac{2}{2\alpha + 1} \cdot \left(\frac{2\alpha - 1}{2\alpha + 1}\right)^k \exp\left(-\frac{t}{2}\right) L_k(t).$$

Then by (2),

$$\int_0^{+\infty} \left\{ e^{-\alpha t} - \sum_{k=0}^{N} \frac{2}{2\alpha + 1} \left(\frac{2\alpha - 1}{2\alpha + 1}\right)^k \exp\left(-\frac{t}{2}\right) L_k(t) \right\}^2 dt$$

$$= \sum_{k=N+1}^{+\infty} w^{2k} = \frac{w^{2N+2}}{1 - w^2} = \frac{1}{1 - \left(\dfrac{2\alpha - 1}{2\alpha + 1}\right)^2} \cdot \left(\frac{2\alpha - 1}{2\alpha + 1}\right)^{2N+2}$$

$$= \frac{(2\alpha + 1)^2 (2\alpha - 1)^2}{(2\alpha + 1)^2 - (2\alpha - 1)^2} \cdot \left(\frac{2\alpha - 1}{2\alpha + 1}\right)^{2N} = \frac{(4\alpha^2 - 1)^2}{4\alpha} \cdot \left(\frac{2\alpha - 1}{2\alpha + 1}\right)^{2N}.$$

Now $\left(\dfrac{2\alpha - 1}{2\alpha + 1}\right)^2 < 1$ for every $\alpha > 0$, so to every $\varepsilon > 0$ there exists an N, such that

$$\frac{(4\alpha^2 - 1)^2}{4\alpha} \cdot \left(\frac{2\alpha - 1}{2\alpha + 1}\right)^{2N} < \varepsilon,$$

and the claim is proved.

Example 1.35 *Prove that the functions* $\exp\left(-\dfrac{t}{2}\right) L_k(t), \ k \in \mathbb{N}_0$, *form an orthonormal basis for* $L^2([0, +\infty[)$.

Given $\varepsilon > 0$ and $f \in L^2([0, +\infty[)$, i.e.

$$\|f\|_2^2 = \int_0^{+\infty} |f(x)|^2 \, dx < +\infty.$$

The set $C([0, +\infty[)$ is dense in $L^2([0, +\infty[)$, so there exists a $\varphi \in C([0, +\infty[)$, such that

$$\|f - \varphi\|_2 < \frac{\varepsilon}{3}.$$

From $\varphi \in C([0, +\infty[)$ follows that there exists an $A \in \mathbb{R}_+$, such that $\varphi(x) = 0$ for every $x \geq A$. Define

$$\psi(t) = \begin{cases} \dfrac{1}{t} \, \varphi\left(\ln \dfrac{1}{t}\right) & \text{for } t \in \,]0, 1], \\[3mm] 0 & \text{for } t = 0. \end{cases}$$

Then $\psi(t) = 0$ for $t \in \left[0, e^{-A}\right]$, and $\psi(t)$ is continuous in $[0, 1]$.

Given any $\delta > 0$. (We shall later choose δ depending on ε and A). It follows from Weierstraß's Approximation Theorem that there exists a polynomial $\sum_{k=0}^{n} a_k \, t^k$, such that

$$\left| \psi(t) - \sum_{k=0}^{n} a_k \, t^k \right| < \delta = \frac{\varepsilon}{3}.$$

It follows from the above that $\varphi\left(\ln \dfrac{1}{t}\right) = t \, \psi(t)$, thus

$$\left| \varphi\left(\ln \frac{1}{t}\right) - t \sum_{k=0}^{n} a_k \, t^k \right| = \left| \varphi\left(\ln \frac{1}{t}\right) - \sum_{k=0}^{n} a_k \, t^{k+1} \right| < \frac{\varepsilon}{3}.$$

Then we get by the change of variable $x = \ln \dfrac{1}{t}$, i.e. $t = e^{-x}$ and $dx = -\dfrac{dt}{t}$,

$$
\begin{aligned}
\left\| \varphi - \sum_{k=0}^{n} a_k \, e^{-(k+1)x} \right\|_2^2 &= \int_0^{+\infty} \left| \varphi(x) - \sum_{k=1}^{n} a_k \, e^{-(k+1)x} \right|^2 dx = \int_0^1 \left| t\,\psi(t) - t \sum_{k=0}^{n} a_k \, t^k \right|^2 \frac{dt}{t} \\
&= \int_0^1 t \left| \psi(t) - \sum_{k=0}^{n} a_k t^k \right|^2 dt < \left(\frac{\varepsilon}{3}\right)^2 \int_0^1 t \, dt < \left(\frac{\varepsilon}{3}\right)^2.
\end{aligned}
$$

Using the result of EXAMPLE 1.34 it follows that to every $k = 0, \dots, n$, there exist constants $b_{k,j}$, $j = 0, 1, \dots, N_k$, such that

$$|a_k| \cdot \left\| e^{-(k+1)x} - \sum_{j=0}^{N_k} b_{k,j} \, \exp\left(-\frac{x}{2}\right) L_j(x) \right\|_2 < \frac{\varepsilon}{3} \cdot \frac{1}{n+1}.$$

Putting

$$\sum_{k=0}^{n} a_k \sum_{j=0}^{N_k} b_{k,j} \, \exp\left(-\frac{x}{2}\right) L_j(x) = \sum_{j=0}^{N} c_j \, \exp\left(-\frac{x}{2}\right) L_j(x),$$

where $N = \max\{N_0, N_1, \dots, N_n\}$, we get

$$
\begin{aligned}
&\left\| f(x) - \sum_{j=0}^{N} c_j \, \exp\left(-\frac{x}{2}\right) L_j(x) \right\|_2 \\
&\leq \quad \|f - \varphi\|_2 + \left\| \varphi(x) - \sum_{k=0}^{n} a_k \, e^{-(k+1)x} \right\|_2 + \left\| \sum_{k=0}^{n} a_k \, e^{-(k+1)x} - \sum_{k=0}^{N_k} b_{k,j} \, \exp\left(-\frac{x}{2}\right) L_j(x) \right\|_2 \\
&< \quad \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \sum_{k=0}^{n} |a_j| \cdot \left\| e^{-(k+1)x} - \sum_{j=0}^{N_k} b_{k,j} \, \exp\left(-\frac{x}{2}\right) L_j(x) \right\|_2 \\
&< \quad \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \sum_{k=0}^{n} \frac{\varepsilon}{3} \cdot \frac{1}{n+1} = \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\end{aligned}
$$

and we have proved that there to every $f \in L^2([0, +\infty[)$ and every $\varepsilon > 0$ exists a finite linear combination

$$\sum_{j=0}^{N} c_j \exp\left(-\frac{x}{2}\right) L_j(x),$$

such that

$$\left\| f(x) = -\sum_{j=0}^{N} c_j \exp\left(-\frac{x}{2}\right) L_j(x) \right\|_2 < \varepsilon.$$

Because $\left\{ \exp\left(-\frac{x}{2}\right) L_j(x) \right\}_{j \in \mathbb{N}_0}$ is also an orthonormal sequence, we conclude that it is an ortonormal basis.

Example 1.36 *Prove for the n-th Chebyshev polynomial $T_n(x)$ that the coefficient of x^n is 2^{n-1}, $n \in \mathbb{N}$.*

We know that $T_n(x)$ is a polynomial of degree n, hence it follows from the recursion formula,

$$T_{n+1}(x) = 2x\, T_n(x) - T_{n-1}(x), \qquad n \in \mathbb{N},$$

that the coefficient of x^{n+1} in $T_{n+1}(x)$ is equal twice the coefficient of x^n in $T_n(x)$. In fact, the term $-T_{n-1}(x)$ is only of degree $n - 1$.

It follows by induction that if the claim holds for $T_n(x)$, then it is also true for $T_{n+1}(x)$.

Clearly, the claim is true for $T_1 = 1 = 2^{1-1}$, where $n = 1$, so it follows in general by induction.

Example 1.37 *Prove that the n-th Chebyshev polynomial $T_n(x)$ has its n roots lying in the interval $[-1, 1]$.*

The polynomial $T_n(x)$ is of degree n, so it has *at most n* roots in $[-1, 1]$. According to the Fundamental Theorem of Algebra the polynomial has n complex roots. The problem is that some of them might lie outside the given real interval.

In order to find the roots we apply the definition

$$T_n(x) = \cos(n \cdot \operatorname{Arccos} x).$$

From $\cos t = 0$, if and only if $t = \frac{\pi}{2} + p\pi$, $p \in \mathbb{Z}$, we get the condition

$$\operatorname{Arccos} x = \frac{\pi}{n}\left\{ \frac{1}{2} + p \right\}, \qquad p \in \mathbb{Z}.$$

Now $\operatorname{Arccos} x \in [0, \pi]$, so $\frac{\pi}{n}\left\{ \frac{1}{2} + p \right\} \in [0, \pi]$. For the relevant values of $p \in \mathbb{Z}$ we get $\frac{1}{2} + p \in [0, n]$, thus $p = 0, 1, \ldots, n - 1$. These values correspond to

$$x = \cos\left(\frac{\pi}{n}\left\{ \frac{1}{2} + p \right\} \right), \qquad p = 0, 1, \ldots, n - 1.$$

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On the other hand, it is easily seen that these n numbers, which are mutually different and all lie in the in the $[-1, 1]$, indeed are roots in $T_n(x)$.

Example 1.38 *Prove that T_n is a solution of the differential equation*

$$\left(1 - t^2\right) \frac{d^2 x}{dt^2} - t \frac{dx}{dt} + n^2 x = 0, \qquad t \in\,]-1, 1[.$$

From the definition of T_n,

$$x(t) = T_n(t) = \cos(n \cdot \operatorname{Arccos} t),$$

follows by direct computation,

$$
\begin{aligned}
x &= \cos(n \cdot \operatorname{Arccos} t), \\
\frac{dx}{dt} &= -\sin(n \cdot \operatorname{Arccos} t) \cdot n \cdot \left(-\frac{1}{\sqrt{1 - t^2}}\right) = n \cdot \frac{1}{\sqrt{1 - t^2}} \cdot \sin(n \cdot \operatorname{Arccos} t), \\
\frac{d^2 x}{dt^2} &= n \cdot \left\{ -\frac{1}{2} \cdot \frac{-2t}{\left(\sqrt{1 - t^2}\right)^3} \cdot \sin(n \cdot \operatorname{Arccos} t) \right\} \\
&\qquad + n \cdot \frac{1}{\sqrt{1 - t^2}} \cdot \cos(n \cdot \operatorname{Arccos} t) \cdot \left(-\frac{n}{\sqrt{1 - t^2}}\right) \\
&= \frac{nt}{(1 - t^2)\sqrt{1 - t^2}} \cdot \sin(n \cdot \operatorname{Arccos} t) - \frac{n^2}{1 - t^2} \cdot \cos(n \cdot \cos(n \cdot \operatorname{Arccos} t).
\end{aligned}
$$

Then by insertion

$$
\begin{aligned}
&\left(1 - t^2\right) \frac{d^2 x}{dt^2} - t \frac{dx}{dt} + n^2 x \\
&= \frac{nt}{\sqrt{1 - t^2}} \cdot \sin(n \cdot \operatorname{Arccos} t) - n^2 \cdot \cos(n \cdot \operatorname{Arccos} t) \\
&\quad - t \cdot \frac{n}{\sqrt{1 - t^2}} \cdot \sin(n \cdot \operatorname{Arccos} t) + n^2 \cdot \cos(n \cdot \operatorname{Arccos} t) \\
&= 0,
\end{aligned}
$$

and the claim is proved.

Example 1.39 *Prove that*

$$\max_{t \in [-1,1]} \left| \frac{1}{2^{n-1}} T_n(t) \right| = \frac{1}{2^{n-1}}.$$

Let P_b be any other polynomial of degree n and of coefficient 1 to t^n. Prove that

$$\max_{t \in [-1,1]} |P_n(t)| > \frac{1}{2^{n-1}}.$$

From $T_n(t) = \cos(n \cdot \operatorname{Arccos} t)$ follows that $|T_n(t)| \leq 1$. For $t = 1$ we get

$$T_n(1) = \cos(n \cdot \operatorname{Arccos} 1) = \cos(n \cdot 0) = 1,$$

thus

$$\max_{t \in [-1,1]} \left| \frac{1}{2^{n-1}} T_n(t) \right| = \frac{1}{2^{n-1}}.$$

Let P_n be a polynomial of degree n and of coefficient 1 of t^n. Then

$$T_{n+1}(t) = 2t\, T_n(t) - T_{n-1}(t), \qquad n \in \mathbb{N},$$

and $T_0(t) = 1$, $T_1(t) = t$. Since $T_{n-1}(t)$ only has degree $n-1$, it follows by induction that the coefficient of t^n in $T_n(t)$ is 2^{n-1}. This means that

$$Q_{n-1}(t) := \frac{1}{2^{n-1}} T_n(t) - P_n(t)$$

is a polynomial of at most degree $n-1$.

The latter claim is now proved INDIRECTLY. Thus assume that $P_n(t)$ has been chosen, such that

$$\max_{t \in [-1,1]} |P_n(t)| \leq \frac{1}{2^{n-1}}.$$

Now $T_n(t) = \cos(n \cdot \text{Arccos } t) = 1$ for $n \cdot \text{Arccos } t = 2k\pi$, i.e. for $t = \cos\left(\dfrac{2k\pi}{n}\right)$, and $T_n(t) = -1$ for $t = \cos\left(\dfrac{2k+1}{n}\,\pi\right)$. We therefore conclude that

$$Q_{n-1}(t) > 0 \text{ for } t = \cos\left(\frac{2k\pi}{n}\right), \quad \text{and} \quad Q_{n-1}(t) < 0 \text{ for } t = \cos\left(\frac{2k+1}{n}\,\pi\right).$$

Because $|T_n(0)| = |T_n(1)| = 1$, it follows from the continuity that $Q_{n-1}(t)$ has at least as many zeros in $[0, 1]$ as T_n. Now, $T_n(t) = 0$ for $n \cdot \mathrm{Arccos}\, t = \left(k + \dfrac{1}{2}\right)\pi$, i.e. for n different values $t_k = \cos\left(\dfrac{2k+1}{2n}\,\pi\right) \in [0, 1]$, hence $Q_{n-1}(t)$ must have at least n roots in $[0, 1]$. However, $Q_{n-1}(t)$ is at most of degree grad $n - 1$, hence $Q_{n-1}(t) \equiv 0$. We conclude that

$$P_n(t) = \frac{1}{2^{n-1}}\, T_n(t).$$

It follows that if the coefficient of t^n in $P_n(t)$ is 1, and $P_n(t) \neq T_n(t)$, then

$$\max_{t \in [-1,1]} |P_n(t)| > \frac{1}{2^{n-1}}.$$

Remark 1.7 It is easily seen for a general polynomial

$$P_n(t) = \sum_{k=0}^{n} a_k\, t^k, \qquad a_n \neq 0,$$

that

$$\max_{t \in [-1,1]} \left| \sum_{k=0}^{n} a_k\, t^k \right| \geq \frac{|a_k|}{2^{n-1}}. \qquad \Diamond$$

1.5 Orthogonal projections and complement

Example 1.40 *Let H be a Hilbert space and let P and Q denote the orthogonal projections on the closed subspaces M and N, respectively. Show that if $M \perp N$, then $P + Q$ is the orthogonal projection on $M \oplus N$.*

When $M \perp N$, then

$$M \oplus N = \{z \in H \mid z = x + y,\ x \in M,\ y \in N\}.$$

Then

$$((P + Q)x, y) = (Px, y) + (Qx, y) = (x, Py) + (x, Qy) = (x, (P + Q)y)$$

for $x,\ y \in H$, implies the first condition for that $P + Q$ is a projection.

Now, $PQx = 0$ for all $x \in H$, because $Qx \in N \perp M$, and analogously $QPx = 0$, because $Px \in M \perp M$. This shows that $PQ = QP = 0$. We shall apply this result by the reduction below of $(P + Q)^2$. In fact,

$$(P + Q)^2 = (P + Q)(P + Q) = P^2 + PQ + QP + Q^2 = P + 0 + 0 + Q = P + Q.$$

We have now proved that $P + Q$ is the orthogonal projection onto the closed subspace

$$\{z \in H \mid (P + Q)z = Pz + Qz = z\} = \{z \in H \mid z = x + y,\ x \in M,\ y \in N\} = M \oplus N.$$

Example 1.41 *Let P and Q denote the orthogonal projections in a Hilbert space, and assume that $PQ = QP$. Show that $P + Q - PQ$ is an orthogonal projection and find the image of $P + Q - PQ$.*

Using that P and Q are projections, we get

$$(Px, y) = (x, Py) \qquad \text{and} \qquad (Qx, y) = (x, Qy),$$

and $P^2 = P$ and $Q^2 = Q$. We shall prove that assuming that $PQ = QP$ then the same conditions hold for $R = P + Q - PQ$. By a computation,

$$
\begin{aligned}
(Rx, y) &= ((P + Q - PQ)x, y) = (Px, y) + (Qx, y) - (PQx, y) \\
&= (x, Py) + (x, Qy) - (Qx, Py) = (x, Py + Qy) - (x, QPy) \\
&= (x, Py + Qy) - (x, PQy) = (x, (P + Q - PQ)y) = (x, Ry),
\end{aligned}
$$

and the first condition is proved.

Then by another simple computation,

$$
\begin{aligned}
R^2 &= (P + Q - PQ)(P + Q - PQ) \\
&= (P^2 + PQ - P^2Q) + (QP + Q^2 - QPQ) + (-PQP - PQ^2 + PQPQ) \\
&= P + PQ - PQ + PQ + Q - pQ^2 - P^2Q - PQ + P^2Q \\
&= P + PQ + Q - PQ - PQ - PQ + PQ = P + Q - PQ = R,
\end{aligned}
$$

and we have proved that $R = P + Q - PQ$ is a projection.

Remark 1.8 We note that

$$R = I - (I - P)(I - Q).$$

It follows from the assumption $PQ = QP$ that also

$$(I - P)(I - Q) = (I - Q)(I - P).$$

With more theory at hand *it is possible to prove* that if

$$P(H) = M \qquad \text{and} \qquad Q(H) = N,$$

then

$$(I - P)(H) = M^\perp \qquad \text{and} \qquad (I - Q)(H) = N^\perp,$$

and

$$(I - P)(I - Q)(H) = M^\perp \cap N^\perp.$$

Then we get

$$R(H) = \{I - (I - P)(I - Q)\}(H) = \left(M^\perp \cap N^\perp\right)^\perp = M^{\perp\perp} + N^{\perp\perp} = M + N,$$

where

$$M + N = \{z \in H \mid \exists\, x \in M \,\exists\, y \in N : z = x + y\}.$$

Note that the splitting $z = x + y$ is not necessarily unique. $\Diamond$

The proof of the remark above requires a little more of the reader than can be expected. Therefore, we give below an alternative proof.

Let $P(H) = M$ and $Q(H) = N$. It follows from $PQ = QP$ that for every $x \in H$,

$$PQx \in M \cap N.$$

In fact, $PQx = P(Qx) \in M$, and $PQx = QPx = Q(Px) \in N$. Conversely, if $x \in M \cap N$, then clearly

$$PQx = P(Qx)x = Px = x,$$

and we infer from

$$(PQ)^2 = PQPQ = P(QP)Q = P(PQ)Q = P^2Q^2 = PQ,$$

that when $PQ = QP$, then PQ is the orthogonal projection onto $M \cap N$.

We shall prove that the image of $R = P + Q - PQ$ is

$$R(H) = M + N := \{z \in H \mid \exists\, x \in M \,\exists\, y \in N : z = x + y\}.$$

Clearly, $R(H) \subseteqq M + N$. In fact, $Px \in M$, $Qx \in N$ and $PQx \in M \cap N$, from which we conclude that

$$Rx = Px + Qx - PQx \in M + N.$$

Now, M and N are closed subspaces of the Hilbert space H, so they can also be considered as Hilbert spaces. They both contain the closed subspace $M \cap N$, hence there exist a subspace M_1 of M and a subspace N_1 of N, such that

$$M = M_1 \oplus (M \cap N) \qquad \text{and} \qquad N = N_1 \oplus (M \cap N),$$

and we conclude that

(9) $M + N = M_1 \oplus N_1 \oplus (M \cap N)$.

Let $z \in M + N$. We shall only prove that $Rz = z$. By (9) we can write z uniquely in the form

$$z = x_1 + y_1 + z_1, \qquad \text{where } x_1 \in M_1, \quad y_1 \in N_1, \quad z_1 \in M \cap N.$$

In particular,

$$x_1 = Px_1, \qquad y_1 = Qy_1, \qquad Pz_1 = Qz_1 = PQz_1 = QPz_1 = z_1,$$

and we get by insertion that

$$\begin{aligned}
Rz &= (P + Q - PQ)(x_1 + y_1 + z_1) \\
&= (Px_1 + Py_1 + Pz_1) + (Qx_1 + Qy_1 + Qz_1) - (PQx_1 + PQy_1 + PQz_1) \\
&= (x_1 + PQy_1 + z_1) + (QPx_1 + y_1 + z_1) - (QPx_1 + PQy_1 + z_1) \\
&= x_1 + y_1 + z_1 = z.
\end{aligned}$$

We have proved that R is the orthogonal projection onto $M + N$.

Example 1.42 *Consider $C([a,b])$ with the* sup-*norm, (here we take only real functions and consider it as a real vector space) and consider the functionals*

$$\alpha(f) = \max_{t \in [a,b]} f(t), \qquad \beta(f) = \min_{t \in [a,b]} f(t).$$

Are these functionals linear and/or bounded?

None of the functionals is linear. It follows from $\max\{-f(t)\} = -\min f(t)$ that it suffices to consider

$$\alpha(f) = \max_{t \in [0,1]} f(t),$$

even in the interval $[a,b] = [0,1]$.

Choosing $f(t) = t$ and $g(t) = 1 - t$, $t \in [0,1]$ we get

$$\alpha(f) = \alpha(g) = 1.$$

From $f + g \equiv 1$ follows that

$$\alpha(f + g) = 1 \neq 2 = \alpha(f) + \alpha(g),$$

and we have proved that α is not linear.

We have in general

$$|\alpha(f)| = \left| \max_{t \in [a,b]} f(t) \right| \leq \max_{t \in [a,b]} |f(t)| = \|f\|_\infty,$$

and analogously for $\beta(f)$. Hence, both functionals are bounded in $(C([a,b]), \|\cdot\|_\infty)$.

Example 1.43 *Let φ denote a linear function on a vector space V, and assume that $\ker(\varphi) \neq V$. Let $x_0 \in V \backslash \ker(\varphi)$. Show that any vector $x \in V$ can be written in the form $x = ax_0 + y$, where $y \in \ker(\varphi)$.*
Is this expansion unique?

ANALYSIS: *If $x = ax_0 + y$ has the right form, where $y \in \ker(\varphi)$, then*

$$\varphi(x) = a\,\varphi(x_0) + \varphi(y) = a\,\varphi(x_0),$$

because $\varphi(y) = 0$ for $y \in \ker(\varphi)$.

From $x_0 \notin \ker(\varphi)$ follows that $\varphi(x_0) \neq 0$. Thus, the *only possibility* is

$$a = \frac{\varphi(x)}{\varphi(x_0)}.$$

Now let

$$y = x - \frac{\varphi(x)}{\varphi(x_0)}\,x_0.$$

Then

$$\varphi(y) = \varphi(x) - \frac{\varphi(x)}{\varphi(x_0)} \cdot \varphi(x_0) = \varphi(x) - \varphi(x) = 0,$$

hence $y \in \ker(\varphi)$.

Thus, for any given x and x_0,

$$x = a\,x_0 + y = \frac{\varphi(x)}{\varphi(x_0)} \cdot x_0 + \left\{ x - \frac{\varphi(x)}{\varphi(x_0)}\,x_0 \right\}$$

is of the wanted type with

$$x - \frac{\varphi(x)}{\varphi(x_0)}\,x_0 \in \ \ker(\varphi).$$

Here $a = \dfrac{\varphi(x)}{\varphi(x_0)}$ is uniquely determined by the above, so the expansion is unique.

Example 1.44 *Let φ and ψ denote linear functionals on a vector space V, and assume that $\ker(varphi) = \ker(\psi)$. Show that there is a constant $\alpha \in \mathbb{C}$ such that $\alpha\,\varphi = \psi$.*

If $\ker(\varphi) = \ker(\psi) = V$, then $\varphi = \psi = 0$, and there is nothing to prove.

Assume that $\ker(\varphi) = \ker(\psi) \neq V$, and let $x \in V \backslash \ker(\varphi)$. Then $\varphi(x)$ and $\psi(x) \in \mathbb{C} \setminus \{0\}$. Hence, there exists ant $a \in \mathbb{C} \setminus \{0\}$, such that

$$a\,\varphi(x) = \psi(x),$$

for this particular $x \in V \backslash \ker(\varphi)$.

If $y \in V \backslash \ker(\varphi)$, $y \neq x$, is any other element, then there exists analogously a $b \in \mathbb{C} \setminus \{0\}$, such that

$$b\,\varphi(y) = \psi(y).$$

The task is to prove that $a = b$.

Putting

$$\alpha = \psi(x) \quad [= a\,\varphi(x)] \in \mathbb{C} \setminus \{0\} \quad \text{and} \quad \beta = \psi(y) \quad [= b\,\varphi(y)] \in \mathbb{C} \setminus \{0\},$$

we get

$$\psi(\beta\,x - \alpha\,y) = \beta \cdot a\,\varphi(x) - \alpha \cdot b\,\varphi(y) = ab \cdot \varphi(x)\varphi(y) - ab \cdot \varphi(x)\varphi(y) = 0,$$

thus $\beta\,x - \alpha\,y \in \ker(\psi) = \ker(\varphi)$, where the latter follows from the assumption. This means that also

$$0 = \varphi(\beta\,x - \alpha\,y) = b\,\varphi(y)\varphi(x) - a\,\varphi(x)\varphi(y) = (b - a)\varphi(x)\varphi(y).$$

From $\varphi(x) \neq 0$ and $\varphi(y) \neq 0$ follows that $b - a = 0$, hence $a = b$.

The claim follows from that x and y are arbitrarily chosen in $V \backslash \ker(\varphi)$, and that there is nothing to prove in $\ker(\varphi = \ker(\psi)$.

Example 1.45 *Let V denote a normed vector space with norm $\|\cdot\|$. Recall that $V^\star$ denotes the vector space of bounded, linear functionals on V. On $V^\star$ we define the* dual norm $\|\varphi\|^\star$, *which is just the operator norm of φ as an element of $B(V, \mathbb{C})$.*
Let $x \in V$. Show that

$$g_x(\varphi) = \varphi(x), \qquad \varphi \in V^\star,$$

determines an element $g_x \in V^{\star\star}$.
Show that the mapping $x \mapsto g_x$ is a linear and injective mapping from V to $V^{\star\star}$, and that $\|g_x\|^{\star\star} = \|x\|$.
If $x \mapsto g_x$ is also surjective, V is said to be reflexive.
Show that a Hilbert space is reflexive.

Let $g_x(\varphi) = \varphi(x)$, $\varphi \in V^\star$. We shall prove that g_x is linear and bounded. It follows from the definition of g_x that

$$g_x(\varphi + \lambda\psi) = (\varphi + \lambda\psi)(x) = \varphi(x) + \lambda\psi(x) = g_x(\varphi) + \lambda\,g_x(\psi),$$

which shows that g_x is linear. Furthermore,

$$(10) \quad \|g_x\|^{\star\star} = \sup_{\|\varphi\|^\star \leq 1} |g_x(\varphi)| = \sup_{\|\varphi\|^\star \leq 1} |\varphi(x)| = \sup_{\|\varphi\|^\star \leq 1} \|x\| \cdot \left| \varphi\left(\frac{x}{\|x\|}\right) \right| \leq \|x\| \cdot \sup_{\|f\|^\star \leq 1} \|f\|^\star = \|x\|,$$

hence $\|g_x\|^{\star\star} \leq \|x\|$, and g_x is bounded. We have proved that $g_x \in V^{\star\star}$.

Then we prove that the mapping $x \mapsto g_x$ is linear: $V \to V^{\star\star}$,
Assuming that $x + \lambda y \mapsto g_{x+\lambda y}$, we shall prove that $g_{x+\lambda y} = g_x + \lambda \cdot g_y$.
For every $\varphi \in V^\star$,

$$g_{x+\lambda y}(\varphi) = \varphi(x + \lambda y) = \varphi(x) + \lambda\,\varphi(y) = g_x(\varphi) + \lambda\,g_y(\varphi),$$

and the linearity follows.

Next we prove that the mapping $x \mapsto g_x$ is injective. Therefore, let $x \neq y$.

1) If x and y are linearly dependent, then there exists $\lambda \neq 1$, such that $y = \lambda x$. Choose $\varphi \in V^\star$, such that $\varphi(x) = 1$. Then $\varphi(y) = \lambda \neq 1$, and

$$g_x(\varphi) = \varphi(x) \neq \varphi(y) = g_y(\varphi), \qquad \text{thus } g_x \neq g_y.$$

2) If x and y are linearly independent, we choose $\varphi \in V^\star$, such that $\varphi(x) = 1$ and $\varphi(y) = 0$. Then

$$g_x(\varphi) = \varphi(x) = 1 \neq 0 = \varphi(y) = g_y(\varphi).$$

Summing up, it follows that $x \mapsto g_x$ is injective.

It only remains to prove that $\|g_x\|^{\star\star} = \|x\|$. According to (10) we have already proved that $\|g_x\|^{\star\star} \leq \|x\|$. Therefore, it suffices to prove that we can choose $\varphi \in V^\star$, such that

$$\|\varphi\|^\star = 1 \qquad \text{and} \qquad \left| \varphi\left(\frac{x}{\|x\|}\right) \right| = 1.$$

Choose any closed subspace U of V of codimension 1, such that $x \notin U$.

Remark 1.9 The *codimension* is 1, if $\dim V/U = 1$, where V/U denotes the space of classes in V modulo U, i.e. if x_1, $x_2 \in V/U$, then $x_1 = x_2$ in V/U, if $x_1 + U = x_2 + U$, i.e. if there exists an $u \in U$, such that $x_1 - x_2 = u \in U$. $\Diamond$

From $\dim V/U = 1$ follows that every element $y \in V$ has a unique expansion in the form

$$y = \lambda x + u, \qquad \text{where } \lambda \in \mathbb{C} \quad \text{and} \quad u \in U.$$

Define $\varphi \in V^\star$ by

$$\varphi(y) = \lambda \, \|x\|, \qquad \text{i.e.} \ \ker(\varphi) = U.$$

Then $\varphi(x) = \|x\|$, i.e. $\varphi\left(\dfrac{x}{\|x\|}\right) = 1$, and

$$(11) \ \ \varphi(y) = \lambda \, \|x\| = \|y\| \cdot \frac{\lambda \, \|x\|}{\|\lambda x + u\|}.$$

It follows that the subspace U must be chosen, such that $\mathbf{0} \in U$ is the closed point in U to x, because in that case if follows from (11) that $|\varphi(y)| \leq \|y\|$, where we obtain equality, if $y = x$, and the claim is proved.

Finally, let $V = H$ be a Hilbert space.
According to Riesz's Representation Theorem, every continuous linear functional φ on H can be written

$$\varphi(x) = (x, f),$$

where $f \in H$ is uniquely determined by $\varphi \in H^\star$.
This means that $H^\star$ is mapped bijectively onto H by $\varphi \mapsto f$. Consequently, H and $H^\star$ are isomorphic, and we identify $f = \varphi$, hence

$$\varphi(x) = (x, \varphi).$$

By repeating this argument we get that $H^\star$ and $H^{\star\star}$ are isomorphic, and we may consider $H^{\star\star}$ as it was H itself.

Finally, if $g \in H^{\star\star}$ and $\varphi \in H^\star$, then

$$g(\varphi) = (\varphi, g) = \overline{(g, \varphi)} = \overline{\varphi(g)},$$

hence $x \mapsto g_x$ is surjective, and every Hilbert space H is reflexive.

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Example 1.46 *Let $(V, \| \cdot \|)$ be a normed space of $\dim V = n$. Show that $\dim V' = n$, and that V is reflexive.*

Let $e_1, \ldots, e_n$ form a basis for V. Define $f_j \in V'$ by

$$(12) \quad f_j \left(\sum_{i=1}^{n} x_i e_i \right) = x_j, \qquad j = 1, \ldots, n.$$

We claim that $f_1, \ldots, f_n$ are linearly independent. Hence, assume that $\lambda_1 f_1 + \cdots + \lambda_n f_n = 0$. Then in particular,

$$(\lambda_1 f_1 + \cdots + \lambda_n f_n)(e_j) = \lambda_j = 0, \qquad j = 0, \ldots, n,$$

so we conclude that all coefficients are 0, and the claim is proved.

Then we claim that $f_1, \ldots, f_n$ form a basis for V'. Let $f \in V'$ be any element. Then by (12),

$$f(x) = f \left(\sum_{j=1}^{n} x_j e_j \right) = \sum_{j=1}^{n} x_j f(e_j) = \sum_{j=1}^{n} f(e_j) \, f_j \left(\sum_{i=1}^{n} x_i e_i \right) = \sum_{j=1}^{n} \lambda_j f_j(x),$$

where we have put $\lambda_j = f(e_j)$. This is true for every $x \in V$, so

$$f = \sum_{j=1}^{n} \lambda j f_j,$$

and we have proved that f_1, ..., f_n form a basis. We therefore infer that $\dim V' = n$. Then, by repeating the argument, we also get that $\dim V'' = n$.

It follows from EXAMPLE 1.45 that the mapping $x \mapsto g_x$ is injective, so g_{e_1}, ..., g_{e_n} are linearly independent in V'', because e_1, ..., e_n are linearly independent in V. Consequently, g_{e_1}, ..., g_{e_n} form a basis for V'', and the mapping $x \mapsto g_x$ is also surjective, and we have proved that V is a reflexive space.

Example 1.47 *We consider the space of sequences ℓ^p, where $p \geq 1$. Let $y \in \ell^q$, where $\dfrac{1}{p} + \dfrac{1}{q} = 1$.*
(If $p = 1$ then $y \in \ell^\infty$, the space of bounded sequences).
Show that

$$x \mapsto \sum_{i=1}^{\infty} x_i \overline{y}_i$$

defines an element $y^\star \in (\ell^p)^\star$ with norm $\|y^{\star|}\|^\star = \|y\|_q$.

If $y \in \ell^q$, where $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then by Hölder's inequality

$$\left| \sum_{i=1}^{+\infty} x_i \overline{y}_i \right| \leq \|x\|_p \cdot \|y\|_q$$

for every $x \in \ell^p$, proving that the linear mapping

$$y^\star(x) = \sum_{i=1}^{+\infty} x_i \overline{y}_i$$

is bounded, $y^\star \in (y^\star)$, and that $\|y^\star\|^\star \leq \|y\|_q$.

Then choosing $x \in \ell^p$ by $x_i = \operatorname{sign} y_i \cdot |y_i|^{\frac{q}{p}}$, we get

$$y^\star(x) = \sum_{i=1}^{+\infty} |y_i|^{1+\frac{q}{p}} = \sum_{i=1}^{+\infty} |y_i|^{(\frac{1}{q}+\frac{1}{p})q} = \sum_{i=1}^{+\infty} |y_i|^q = \|y\|_q^q = \|y\|_q^{q(\frac{1}{p}+\frac{1}{q})} = \|y\|_q \cdot \|y\|_q^{\frac{q}{p}}.$$

Notice that

$$\|x\|_p = \left\{ \sum_{i=1}^{+\infty} |x_i|^p \right\}^{\frac{1}{p}} = \left\{ \sum_{i=1}^{+\infty} |y_i|^q \right\}^{\frac{1}{p}} = \left\{ \|y\|_q^q \right\}^{\frac{1}{p}} = \|y\|_q^{\frac{q}{p}},$$

from which follows that

$$y^\star(x) = |y^\star(x)| = \|y\|_q \cdot \|x\|_p,$$

and we conclude that $\|y^\star\|^\star \geq \|y\|_q$. When this is combined with the previous estimate, then $\|y^\star\|^\star = \|y\|_q$, as required.

Example 1.48 *Let φ denote a bounded, linear functional on a Hilbert space H, and assume that the domain $D(\varphi)$ is a proper subspace of H. Show that there is exactly one extension φ_1 of φ to H with the property that $\|\varphi_1\| = \|\varphi\|$.*

Since φ is bounded, it can by continuity be extended to the closure $\overline{D(\varphi)}$ of its domain. Hence we may from the beginning assume that $D(\varphi)$ is closed.

The orthogonal complement $D(\varphi)^\perp$ makes sense in the Hilbert space H, i.e. it is defined. Hence, every element $z \in H$ has a unique expansion of the form

$$z = x + y, \qquad \text{where} \quad x \in D(\varphi) \qquad \text{and} \quad y \in D(\varphi)^\perp.$$

Therefore, any bounded extension φ_1 of φ is given by

$$\varphi_1(z) = \varphi(x) + \psi(y),$$

where ψ is a bounded and linear functional with $D(\psi) = D(\varphi)^\perp$.

Both $D(\psi)$ and $D(\varphi)$ are closed subspaces of a Hilbert space. We may therefore consider them as Hilbert spaces. According to Riesz's Representation Theorem there exist $f \in D(\varphi)$ and $g \in D(\varphi)^\perp$ (hence also $h \in H$), such that

$$\varphi(x) = (x, f) \text{ for all } x \in D(\varphi), \qquad \psi(y) = (y, g) \text{ for all } y \in D(\varphi)^\perp,$$

hence

$$\varphi_1(z) = (z, h) = (x + y, h) = (x, f) + (y, g).$$

The spaces $D(\varphi)$ and $D(\varphi)^\perp$ being orthogonal, it follows that

$$\|\varphi_1\|^2 = \|h\|^2 = \|f\|^2 + \|g\|^2 \geq \|f\|^2 = \|\varphi\|^2,$$

so the equality holds, if and only if $\|g\|^2 = 0$, i.e. if and only if $g = 0$. This shows that the requirement $\varphi_1\| = \|\varphi\|$ uniquely fixes the extension to a Hilbert space, and that

$$\varphi_1(z) = \varphi_1(x + y) = \varphi(x),$$

where $z = x + y$ is the unique splitting given by $x \in D(\varphi)$ and $y \in D(\varphi)^\perp$.

Example 1.49 *Let H be a Hilbert space. A mapping $h : H \times H \to \mathbb{C}$ is called* sesquilinear *if, for all x, x_1, $x_2 \in H$ and $\alpha \in \mathbb{C}$ we have*

$$\begin{aligned}
h(x_1 + x_2, x) &= h(x_1, x) + h(x_2, x), \\
h(x, x_1 + x_2) &= h(x, x_1) + h(x, x_2), \\
h(\alpha\, x_1, x_2) &= \alpha\, h(x_1, x_2), \\
h(x_1, \alpha\, x_2) &= \overline{\alpha}\, h(x_1, x_2).
\end{aligned}$$

We say that h is bounded *if there is a constant $c \geq 0$ such that*

$$|h(x_1, x_2)| \leq c \, \|x_1\| \cdot \|x_2\|,$$

for all x_1, $x_2 \in H$. The norm $\|h\|$ is defined as the smallest possible c.
Show that there is an $S \in B(H)$ such that

$$h(x_1, x_2) = (Sx_1, x_2),$$

and that this representation is unique. Show also that $\|h\| = \|S\|$.
A sesquilinear form is called Hermitian *if*

$$h(x, y) = \overline{h(y, x)},$$

for all x, $y \in H$. If, moreover, $h(x, x) \geq 0$ the form is called positive semidefinite.
Show that in this case we have the Schwarz's inequality:

$$|h(x, y)|^2 \leq h(x, x)\, h(y, y),$$

for all x, $y \in H$.

For every fixed x_1 the mapping

$$x_2 \mapsto \overline{h(x_1, x_2)}$$

is a linear bounded functional. Then by Riesz's Representation Theorem there exists an element $x_3 \in H$, such that

$$\overline{h(x_1, x_2)} = (x_2, x_3),$$

i.e.

$$h(x_1, x_2) = (x_3, x_2).$$

Now, x_3 is uniquely determined, so we can define a mapping $S : H \to H$ by $Sx_1 = x_3$, i.e.

$$h(x_1, x_2) = (x_3, x_2) = (Sx_1, x_x).$$

We shall prove that S is linear. It follows from the definition of a sesquilinear form that

$$\begin{aligned}
(S(x_1 + \lambda\, x_2), x) &= h(x_1 + \lambda\, x_2, x) = H(x_1, x) + \lambda\, h(x_2, x) \\
&= (Sx_1, x) + \lambda\, (Sx_2, x) = (Sx_1 + \lambda\, Sx_2, x).
\end{aligned}$$

This is true for all x, so

$$S(x_1 + \lambda\, x_2) = Sx_1 + \lambda\, Sx_2,$$

and S er lineær.

It follows from

$$|(Sx_1, x_2)| = |h(x_1, x_2)| \leq \|h\| \cdot \|x_1\| \cdot \|x_2\|,$$

and $Sx_1 \in H$ for $x_2 = Sx_1$ that

$$\|Sx_1\|^2 \leq \|h\| \cdot \|x_1\| \cdot \|Sx_1\|,$$

so either $\|Sx_1\| = 0$ or $\|Sx_1\| \leq \|h\| \cdot \|x_1\|$. Now, $\|S\|$ is the smallest constant, for which $\|Sx_1\| \leq \|S\| \cdot \|x_1\|$ for all $x_1 \in H$. It therefore follows that $\|S\| \leq \|h\|$.

On the other hand,

$$|h(x_1, x_2)| = |(Sx_1, x_2)| \leq \|Sx_1\| \cdot \|x_2\| \leq \|S\| \cdot \|x_1\| \cdot \|x_2\|,$$

and because $\|h\|$ is the smallest possible constant in such an estimate, we conclude that $\|h\| \leq \|S\|$, hence $\|h\| = \|S\|$.

Assume that h is sesquilinear, Hermitian and positive semidefinite. Then

$$
\begin{aligned}
0 &\leq h(x + \lambda y, x + \lambda y) \\
&= h(x,x) + \lambda h(y,x) + \overline{\lambda} h(x,y) + |\lambda|^2 h(y,y) \\
&= h(x,x) + 2\operatorname{Re}\{\lambda h(y,x)\} + |\lambda|^2 h(y,y).
\end{aligned}
$$

Let $\mu \in \mathbb{R}$ be a real constant. Then choosing

$$\lambda = \pm\operatorname{sign}(\overline{h(y,x)}) \cdot \mu = \pm\operatorname{sign}(h(x,y)) \cdot \mu,$$

we infer from the above for every $\mu \in \mathbb{R}$ that

$$0 \leq \mu^2 \cdot h(y,y) + 2\mu \cdot |h(y,x)| + h(x,x).$$

The right hand side is a polynomial of second degree in $\mu \in \mathbb{R}$ of real coefficients. It then follows from the usual condition of the discriminant that

$$0 \geq B^2 - 4AC = 4 \cdot |h(x,y)|^2 - 4 \cdot h(y,y) \cdot h(x,x),$$

hence by a rearrangement

$$|h(x,y)|^2 \leq h(x,x) \cdot h(y,y).$$

Remark 1.10 The example is a natural generalization of the inner product in a Hilbert space. This can be seen by choosing $h = (\cdot, \cdot)$. $\Diamond$

1.6 Weak convergence

Example 1.50 *Let V and W be Hilbert spaces and let $T \in B(V, W)$. Show that the image of a weakly convergent sequence in V is a weakly convergent sequence in W.*

Let $(x_n) \subset V$ be a weakly convergent sequence in V, thus there exists an x, such that for every $y \in V$,

$$(x_n, y) \to (x, y) \qquad \text{for } n \to +\infty.$$

Let $(x_n) \subset V$ and $T \in B(V, W)$. Then we have for the adjoint operator that $T^\star \in B(W, V)$, so for every $z \in W$,

$$(Tx_n, z) = (x_n, T^\star z) \to (x, T^\star z) = (Tx, z),$$

and (Tx_n) converges weakly towards Tx in W.

2 Operators on Hilbert spaces

2.1 Operators on Hilbert spaces, general

Example 2.1 *In the Hilbert space ℓ^2 we define an operator $T : D(T) \to \ell^2$ by*

$$T((x_n)) = (a_n x_b),$$

where (a_n) is a complex sequence.
Find the maximal possible $D(T)$ and show that T is linear.
Show that $D(T)$ is dense in ℓ^2.
Show that if (a_n) is bounded, then $D(T) = \ell^2$ and T is bounded.

It is obvious that

$$D(T) = \left\{ x \in \ell^2 \;\middle|\; \sum_{n=1}^{+\infty} |a_n|^2 \cdot |x_n|^2 < +\infty \right\}.$$

We shall show that $D(T)$ is a subspace. If $x, y \in D(T)$, then

$$|x_n + y_n|^2 = |x_n|^2 + |y_n|^2 + 2|x_n| \cdot |y_n| \le 2 \left(|x_n|^2 + |y_n|^2 \right),$$

hence

$$\sum_{n=1}^{+\infty} |a_n|^2 |x_n + y_n|^2 \le 2 \left\{ \sum_{n=1}^{+\infty} |a_n|^2 |x_n|^2 + \sum_{n=1}^{+\infty} |a_n|^2 |y_n|^2 \right\} < +\infty,$$

from which follows that $x + y \in D(T)$.
If $x \in D(T)$ and $\lambda \in \mathbb{C}$, then trivially $\lambda x \in D(T)$, and we have proved that $D(T)$ is a subspace of ℓ^2.
It is now obvious that T is linear.

Next we prove that $D(T)$ is dense in ℓ^2. Let $x \in \ell^2$. Then

$$\sum_{n=1}^{+\infty} |x_n|^2 < +\infty.$$

Hence, to every $\varepsilon > 0$ there is an $N \in \mathbb{N}$, such that

$$\sum_{n=N+1}^{+\infty} |x_n|^2 < \varepsilon^2.$$

Putting $x^N = (x_1, x_2, \ldots, 0, 0, \ldots)$, we get $\|x - x^N\|_2 < \varepsilon$. It only remains to prove that $x^N \in D(T)$ for every $N \in \mathbb{N}$. This condition is the same as saying that

$$\sum_{n=1}^{N} |a_n|^2 |x_n|^2 < +\infty,$$

which clearly is fulfilled, because the sum only contains a finite number of terms.

Finally, assume that (a_n) is a bounded sequence, thus there exists an $M > 0$, such that $|a_n| \leq M$ for every $n \in \mathbb{N}$. Let $x \in \ell^2$. Then

$$(13) \quad \sum_{n=1}^{+\infty} |a_n|^2 |x_n|^2 \leq M^2 \sum_{n=1}^{+\infty} |x_n|^2 = M^2 \|x\|_2^2 < +\infty,$$

which shows that every $x \in \ell^2$ is also an element of $D(T)$.

It follows from (13) that

$$\|Tx\|_2^2 = \sum_{n=1}^{+\infty} |a_n|^2 |x_n|^2 \leq M^2 \|x\|_2^2,$$

hence

$$\|Tx\|_2 \leq M \, \|x\|_2,$$

and we have proved that T is bounded.

Example 2.2 *Consider in $L^2(\mathbb{R})$ the operator Q defined by*

$$Qf(x) = x \, f(x),$$

with

$$D(Q) = \left\{ f \in L^2(\mathbb{R}) \mid Qf \in L^2(\mathbb{R}) \right\}.$$

Show that Q is linear but not bounded. Show that $D(Q)$ is dense in $L^2(\mathbb{R})$.
In quantum mechanics Q is called the position operator.

Let $\lambda \in \mathbb{C}$ and $f \in D(Q)$. Then trivially, $\lambda \, f \in D(Q)$.
If $f, \, g \in D(Q)$, then it follows from the inequality

$$|f(x) + g(x)|^2 \leq 2 \left\{ 2|f(x)|^2 + |g(x)|^2 \right\},$$

that

$$\int_{-\infty}^{+\infty} x^2 |f(x) + g(x)|^2 dx \leq 2 \int_{-\infty}^{+\infty} x^2 |f(x)|^2 dx + 2 \int_{-\infty}^{+\infty} x^2 |g(x)|^2 dx < +\infty,$$

and we conclude that $f + g \in D(Q)$, thus $D(Q)$ is a subspace of $L^2(\mathbb{R})$.

If $f, \, g \in L^2(\mathbb{R})$ and $\lambda \in \mathbb{C}$, then

$$Q(f + \lambda \, g) = x \cdot \{f(x) + \lambda \, g(x)\} = x \cdot f(x) + \lambda \cdot x \cdot g(x) = Qf(x) + \lambda \, Qg(x),$$

thus

$$Q(f + \lambda \, g) = Qf + \lambda \, Qg,$$

and we have proved that Q is linear.

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Let $f_n = 1_{[0,n]}$. Then

$$\|f_n\|_2^2 = \int_0^n 1 \, dx = n, \qquad \text{thus} \qquad \|f_n\|_2 = \sqrt{n},$$

and

$$\|Qf_n\|_2^2 = \int_0^n x^2 dx = \frac{n^3}{3}, \quad \text{thus} \quad \|Qf_n\|_2 = \frac{n}{\sqrt{3}} \cdot \sqrt{n} = \frac{n}{\sqrt{3}} \cdot \|f_n\|_2.$$

We conclude that (1) $f_n \in D(Q)$ for every $n \in \mathbb{N}$, and (2) $\|Q\| \geq \dfrac{n}{\sqrt{3}}$ for every $n \in \mathbb{N}$, hence Q is unbounded.

Finally, let $f \in L^2(\mathbb{R})$, i.e. $\int_{-\infty}^{+\infty} |f(x)|^2 dx < +\infty$. Then to every $\varepsilon > 0$ there is an $N \in \mathbb{N}$, such that
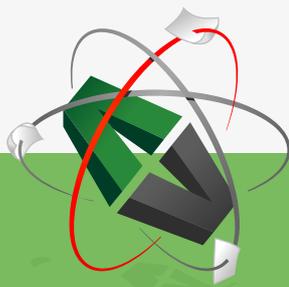
$$\int_{-\infty}^{-N} |f(x)|^2 dx + \int_N^{+\infty} |f(x)|^2 dx < \varepsilon^2.$$

If we put

$$f_N(x) = \begin{cases} f(x) & \text{for } |x| \leq N, \\ 0 & \text{for } |x| > N, \end{cases}$$

then it follows that $\|f - f_n\|_2 < \varepsilon$.

It only remains to prove that $f_N \in D(Q)$. This follows from

$$\|Qf_N\|_2^2 = \int_{-N}^{N} x^2 |f(x)|^2 dx \leq N^2 \int_{-N}^{N} |f(x)|^2 dx \leq N^2 \int_{-\infty}^{+\infty} |f(x)|^2 dx = N^2 \|f\|_2^2,$$

hence

$$\|Qf_N\|_2 \leq N \|f\|_2 < +\infty,$$

and the last claim is proved.

Example 2.3 *Consider in $L^2(\mathbb{R})$ the operator P defined by*

$$Pf = -i \frac{df}{dx},$$

with

$$D(P) = \left\{ f \in L^2(\mathbb{R}) \mid Pf \in L^2(\mathbb{R}) \right\}.$$

Show that P is linear but not bounded.
Show that $D(P)$ is dense in $L^2(\mathbb{R})$.
In quantum mechanics P is called the momentum operator.

Let $f \in D(P)$ and $\lambda \in \mathbb{C}$. Then clearly, $\lambda f \in D(P)$.
If f, $g \in D(P)$, then f and g are differentiable almost everywhere, hence $f + g$ is also differentiable almost everywhere. From f', $g' \in L^2(\mathbb{R})$, follows that also $f' + g' \in L^2(\mathbb{R})$, so $D(P)$ is a vector space.
Then clearly, $Pf(x) = -i \dfrac{df}{dx}(x)$ is linear.

Then we shall show that P is not bounded. Let

$$f_n(x) = \begin{cases} \sin nx & \text{for } x \in [0, 2\pi], \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\|f_n\|_2^2 = \int_0^{2\pi} \sin^2 nx \, dx = \int_0^{2\pi} \cos^2 nx \, dx = \frac{1}{2} \int_0^{2\pi} \left\{ \sin^2 nx + \cos^2 nx \right\} dx = \pi,$$

thus $\|f_n\|_2 = \sqrt{\pi}$ for every $n \in \mathbb{N}$.
It follows that

$$f_n'(x) = \begin{cases} n \cdot \cos nx & \text{for } x \in]0, 2\pi[, \\ \text{not defined} & \text{for } x \in \{0, 2\pi\}, \\ 0 & \text{otherwise.} \end{cases}$$

where $\{0, 2\pi\}$ clearly is a null-set. Then

$$\|Pf_n\|_2^2 = \| -i f_n'(x) \|_2^2 = \int_0^{2\pi} n^2 \cdot \cos^2 nx \, dx = \cdots = n^2 \pi = n^2 \|f_n\|_2^2,$$

so $\|Pf_n\|_2 = n \|f_n\|_2$, and it follows that P is not bounded.

Finally, we shall show that $D(P)$ is dense in $L^2(\mathbb{R})$. Choose any $f \in L^2(\mathbb{R})$, thus $\int_{-\infty}^{+\infty} |f(x)|^2 dx < +\infty$. There exists to every $\varepsilon > 0$ an $N \in \mathbb{N}$, such that

$$\int_{-\infty}^{-N} |f(x)|^2 dx + \int_N^{+\infty} |f(x)|^2 dx < \left(\frac{\varepsilon}{3}\right)^2.$$

If we therefore put

$$f_N(x) = \begin{cases} f(x) & \text{for x} \leq N, \\ 0 & \text{otherwise,} \end{cases}$$

then $f \in L^2(\mathbb{R})$ and $\|f - f_N\|_2 < \dfrac{\varepsilon}{3}$.

Furthermore, there exists a continuous function g on $\mathbb{R}$, such that $g(x) = 0$ for $|x| \geq N$, and such that $\|f_N - g\|_2 < \dfrac{\varepsilon}{3}$.

It follows from Weierstraß's Approximation Theorem that there exists a polynomial $P(x)$ with $P(-N) = P(N) = 0$, such that

$$\max_{x \in [-N,N]} |g(x) - P(x)| < \frac{\varepsilon}{3} \cdot \frac{1}{\sqrt{2N}}.$$

If we put

$$h(x) = \begin{cases} P(x) & \text{for } |x| \leq N, \\ 0 & \text{otherwise.} \end{cases}$$

then

$$\|g - h\|_2^2 = \int_{-N}^N |g(x) - P(x)|^2 dx < \left(\frac{\varepsilon}{3}\right)^2 \cdot \frac{1}{2N} \int_{-N}^N dx = \left(\frac{\varepsilon}{3}\right)^2,$$

and we infer that

$$\|f - h\|_2 \leq \|f - f_N\|_2 + \|f_N - g\|_2 + \|g - h\|_2 < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

The function h is differentiable, if only $x \neq \pm N$, and since h is continuous at $\pm N$, we conclude that

$$h'(x) = \begin{cases} P'(x) & for |x| < N, \\ \text{not defined} & \text{for } x = \pm N, \\ 0 & \text{otherwise,} \end{cases}$$

which of course belongs to $L^2(\mathbb{R})$, because $\{-N, N\}$ is a null-set- This proves that $h \in D(P)$, and $D(P)$ is therefore dense in $L^2(\mathbb{R})$.

Example 2.4 *Let V be a normed vector space. Show that no pair of operators S, $T \in B(V)$ satisfies the* canonical commutator relation:

$$[S,T] = ST - TS = I.$$

HINT*: Show by induction that $ST^n - T^n S = nT^{n-1}$, $n \in \mathbb{N}$, and use this to estimate $\|S\|$ and $\|T\|$.*

Indirect proof. Assume that S and T are two operators from $B(V)$, which satisfy the canonical commutator relation

(14) $ST - TS = I.$

We first prove that these assumptions imply that

$$ST^n - T^n S = nT^{n-1}, \qquad n \in \mathbb{N}.$$

First method. It is actually not necessary to use induction, because adding and subtracting some suitable terms will de the job. We find

$$
\begin{aligned}
ST^n - T^n S &= ST^n - TST^{n-1} + TST^{n-1} - T^2 ST^{n-2} + T^2 ST^{n-2} - \cdots + T^{n-1}ST - T^n S \\
&= \{ST - TS\}T^{n-1} + T\{ST - TS\}T^{n-2} + \cdots + T^{n-1}\{ST - TS\} \\
&= IT^{n-1} + TIT^{n-2} + \cdots + T^{n-1}I = nT^{n-1},
\end{aligned}
$$

and the claim is proved.

Second method. INDUCTION. The claim is trivial for $n = 1$.
Assume that $ST^n - T^n X = nT^{n-1}$, thus by a rearrangement, $ST^n = T^n S + nT^{n-1}$. Then we get for the successor that

$$
\begin{aligned}
ST^{n+1} - T^{n+1}S &= (ST^n)T - T^{n+1}S = T^n ST + nT^n - T^{n+1}S \\
&= nT^n + T^n\{ST - TS\} = (n+1)T^n,
\end{aligned}
$$

which is the same formula, only with n replaced by $n+1$. The claim then follows by induction.

We have assumed that $S, T \in B(V)$, so $\|S\|$ and $\|T\| < +\infty$. Hence it follows from the formula

$$nT^{n-1} = ST^n - T^n S, \qquad n \in \mathbb{N},$$

that

$$
\begin{aligned}
\left\| T^{n-1} \right\| &= \frac{1}{n} \left\| ST^n - T^n S \right\| \leq \frac{1}{n} \left\{ \|S\| \cdot \|T\| \cdot \left\| T^{n-1} \right\| + \left\| T^{n-1} \right\| \cdot \|T\| \cdot \|S\| \right\} \\
&= \frac{2}{n} \|S\| \cdot \|T\| \cdot \left\| T^{n-1} \right\|.
\end{aligned}
$$

This inequality implies that either

$$\exists n \in \mathbb{N} : \left\| T^{n-1} \right\| = 0,$$

or

$$\forall n \in \mathbb{N} : \|S\| \cdot \|T\| \geq \frac{n}{2}.$$

Because the proof is given INDIRECTLY, we shall prove that none of these two possibilities can be fulfilled.

Because we have assumed that $\|S\| \cdot \|T\| < +\infty$, the latter possibility is excluded.

Then assume that $\left\|T^{n-1}\right\| = 0$ for some $n \geq 2$. Then $T^{n-1} = 0$, which implies that

$$(n-1)T^{n-2} = ST^{n-1} - T^{n-1}S = SO - OS = 0,$$

hence $T^{n-2} = O$. We get by recursion after a finite number of steps that $T^1 = T = O$, from which follows that

$$I = ST - ST = O.$$

This relation is only satisfied in the not so interesting case of $V = \{0\}$.

No matter which one we choose of our two possibilities, we obtain a contradiction. Therefore, our (indirect) assumption is wrong, and the original claim is proved.

Example 2.5 *Let V be a normed space and assume that $T \in B(V)$ is bijective. Show that if T^{-1} is bounded, then*

$$\left\|T^{-1}\right\| \geq \|T\|^{-1}.$$

We have tacitly assumed that $V \neq \{0\}$.

If $T \in B)(V)$ is bijective, then T^{-1} exists and $D\left(T^{-1}\right) = V$.

If V is a *Banach space*, we prove below that also $T^{-1} \in B(V)$. If V, however, is just a normed space, then T^{-1} needs *not* being bounded, a statement we shall also give an example of below.

If T^{-1} is unbounded, then it follows from the definition of the norm that $\left\|T^{-1}\right\| = +\infty$, and the inequality above is trivial. Therefore, we now assume that $T^{-1} \in B(V)$. It follows from $I = TT^{-1}$ for every $x \in V$ that

$$\|x\| = \|Ix\| = \left\|TT^{-1}x\right\| \leq \|T\| \cdot \left\|T^{-1}x\right\| \leq \|T\| \cdot \left\|T^{-1}\right\| \cdot \|x\|.$$

We have assumed that $V \neq \{0\}$, so we conclude that

$$\|T\| \cdot \left\|T^{-1}\right\| \geq 1,$$

and we get the wanted inequality.

If V is a Banach space, then it follows from the theorem of a bounded inverse that $T^{-1} \in B(V)$.

If V is not a Banach space, then T^{-1} needs not be bounded. Let $\{\varphi_n \mid n \in \mathbb{N}\}$ be an orthonormal basis in a Hilbert space H, and let

$$V = \operatorname{span}\{\varphi_n \mid n \in \mathbb{N}\}.$$

Then V is dense in H, though V itself is not complete.
We define a bijective and linear operator $T \in B(V)$ by

$$T\left(\sum_{j=1}^{n} a_j \varphi_j\right) = \sum_{j=1}^{n} \frac{1}{j} a_j \varphi_j, \qquad \text{with } \|T\| = 1.$$

Then T is bijective, and clearly

$$T^{-1}\left(\sum_{j=1}^{n} a_j \varphi_j\right) = \sum_{j=1}^{n} j \, a_j \varphi_j$$

is unbounded. Since every element of V can be written as a *finite* linear combination of φ_j, $j \in \mathbb{N}$, we get $D\left(T^{-1}\right) = V$.

Example 2.6 *Let (e_n) denote an orthonormal basis in a Hilbert space H, and define the operator T by*

$$T\left(\sum_{k=1}^{\infty} x_k e_k\right) = \sum_{k=1}^{\infty} x_k e_{k+1}.$$

Show that $T \in B(H)$ and find $\|T\|$.
Show that T is injective and find T^{-1}.

From (e_n) being an orthonormal basis we get

$$\|x\|^2 = \left\|\sum_{k=1}^{\infty} x_k e_k\right\|^2 = \sum_{k=1}^{\infty} |x_k|^2.$$

It follows that T is defined on all of H, and that

$$\|Tx\|^2 = \left\|T\left(\sum_{k=1}^{\infty} x_k e_k\right)\right\|^2 = \left\|\sum_{k=1}^{\infty} x_k e_{k+1}\right\|^2 = \sum_{k=1}^{\infty} |x_k|^2 = \|x\|^2.$$

We infer that $\|T\| = 1$ and that T is linear and defined everywhere in H, hence $T \in B(H)$.

If

$$T\left(\sum_{k=1}^{\infty} x_k e_k\right) = \sum_{k=1}^{\infty} x_k e_{k+1} = 0,$$

then $x_k = 0$ for every $k \in \mathbb{N}$, hence $\sum_{k=1}^{\infty} x_k e_k = 0$, and T is injective.

Then the inverse operator T^{-1} exists and its domain is

$$D\left(T^{-1}\right) = T D(T) = \{x \in H \mid x_1 = (x, e_1) = 0\}.$$

When $x \in D\left(T^{-1}\right)$, i.e. the first coordinate is $x_1 = 0$, then

$$T^{-1}x = T^{-1}\left(\sum_{k=2}^{\infty} x_k e_k\right) = \sum_{k=2}^{\infty} x_k e_{k-1} = \sum_{k=1}^{\infty} x_{k+1} e_k.$$

This can also be written alternatively in the following way:

$$T^{-1}\left(0, x_2, x_3, x_4, \cdots\right) = \left(x_2, x_3, x_4, \cdots\right),$$

i.e. we just delete the zero and move the rest of the sequence one step to the left.

Example 2.7 *Let (e_k) be an orthonormal basis in a Hilbert space H, and let $T \in B(H)$. Define for $j, k \in \mathbb{N}$ the numbers*

$$t_{jk} = (Te_j, e_k).$$

Show that

$$Te_j = \sum_{k=1}^{\infty} t_{jk} e_k,$$

and that

$$\sum_{k=1}^{\infty} |t_{jk}|^2 < \infty \qquad \text{for } j \in \mathbb{N}.$$

The matrix (t_{jk}) is called the matrix form *for T with respect to the orthonormal basis (e_k).*
Let $A, B \in B(H)$ have the forms (a_{jk}) and (b_{jk}) respectively. Find the forms for $A + B$ and AB.

From $t_{jk} = (Te_j, e_k)$ follows that

$$Te_j = \sum_{k=1}^{+\infty} (Te_j, e_k) \, e_k = \sum_{k=1}^{+\infty} t_{jk} e_k \qquad \text{for every } j \in \mathbb{N}.$$

Hence for every $j \in \mathbb{N}$,

$$\|Te_j\|^2 = \left(\sum_{k=1}^{+\infty} t_{jk} e_k, \sum_{k=1}^{+\infty} t_{jk} e_k \right) = \sum_{k=1}^{+\infty} |t_{jk}|^2 < +\infty.$$

From

$$x = \sum_{j=1}^{+\infty} (x, e_j) e_j,$$

follows that

$$Tx = \sum_{j=1}^{+\infty} (x, e_j) Te_j \sum_{j=1}^{+\infty} (x, e_j) \sum_{k=1}^{+\infty} t_{jk} e_k = \sum_{k=1}^{+\infty} \left\{ \sum_{j=1}^{+\infty} t_{jk} (x, e_j) \right\} e_k.$$

Let $A = (a_{jk})$ and $B = (b_{jk})$ be the infinite matrices, which are constructed this way corresponding to the two linear mappings $A, B \in B(H)$. Then

$$(A + B)x = \sum_{k=1}^{+\infty} \left\{ \sum_{j=1}^{+\infty} \{a_{jk} + b_{jk}\} (x, e_j) \right\} e_k,$$

and we conclude that $(a_{jk} + b_{jk})$ is the matrix corresponding to $A + B$.

Furthermore,

$$
\begin{aligned}
ABx &= A \left(\sum_{k=1}^{+\infty} \left\{ \sum_{j=1}^{+\infty} b_{jk}(x, e_j) \right\} e_k \right) = \sum_{\ell=1}^{+\infty} \left(\sum_{k=1}^{+\infty} a_{k\ell} \left\{ \sum_{j=1}^{+\infty} b_{jk}(x, e_j) \right\} \right) e_\ell \\
&= \sum_{\ell=1}^{+\infty} \left(\sum_{j=1}^{+\infty} \left\{ \sum_{k=1}^{+\infty} b_{jk} a_{k\ell} \right\} (x, e_j) \right) e_\ell,
\end{aligned}
$$

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from which we derive that

$$\left(\sum_{k=1}^{+\infty} b_{jk} a_{k\ell}\right)_{j,\ell}$$

is the infinite matrix corresponding to the composite mapping AB.

Example 2.8 *Let (e_k) be an orthonormal basis in a Hilbert space H, and let $T : D(T) \to K$ be a linear operator from the Hilbert space H into the Hilbert space K. Show that if $e_k \in D(T)$ for all $k \in \mathbb{N}$, then $D(T)$ is dense in H.*

This example is trivial. In fact, (e_k) is an orthonormal *basis*, so the set of finite linear combinations of (e_k) is dense H, and because this set is contained in the subspace $D(T)$, we conclude that $D(T)$ is also dense in H.

Example 2.9 *Let $[a, b]$ be a bounded interval. We equip the vector space $C([a, b])$ with the two norms*

$$\|f\|_p = \left\{\int_a^b |f(t)|^p dt\right\}^{\frac{1}{p}}, \qquad \|f\|_q = \left\{\int_a^b |f(t)|^q dt\right\}^{\frac{1}{q}},$$

where $1 \le p < q < +\infty$.
Let $V_p = (C([a, b]), \|\cdot\|_p)$ and $V_1 = (C([a, b]), \|\cdot\|_q)$. Prove that

1) the identity operator from V_q to V_p is bounded,

2) the identity operator from V_p to V_q is not bounded

Remark 2.1 If the interval $[a, b]$ is not bounded, then one can prove that both "identity operators" are unbounded. $\Diamond$

1) First note that

$$\|f\|_p^p = \int_a^b |f(t)|^p dt = \int_a^b |f(t)|^p \cdot 1 \, dt.$$

From $q > p \ge 1$ follows that $p_1 = \dfrac{q}{p} > 1$. Let q_1 denote the conjugate of p_1, given by

$$q_1 = \frac{p_1}{p_1 - 1} = \frac{\frac{q}{p}}{\frac{q}{p} - 1} = \frac{q}{q - p}.$$

Then apply Hölder's inequality with p_1 and q_1 on the integral above,

$$
\begin{aligned}
\|f\|_p^p &= \int_a^b |f(t)|^p 1 \, dt \le \| |f|^p \|_{p_1} \cdot \|1\|_{q_1} = \left\{\int_a^b |f(t)|^{p \cdot \frac{q}{p}} dt\right\}^{\frac{p}{q}} \cdot \left\{\int_a^b 1^{q_1} \, dt\right\}^{\frac{q-p}{q}} \\
&= \left(\left\{\int_a^b |f(t)|^q dt\right\}^{\frac{1}{q}}\right)^p \cdot \left\{(b-a)^{\frac{q-p}{pq}}\right\}^p = \left\{(b-a)^{\frac{1}{p} - \frac{1}{q}} \cdot \|f\|_q\right\}^p.
\end{aligned}
$$

When we extract the "p-th root", then for every $f \in C([a,b])$,

$$\|If\|_p = \|f\|_p \leq (b-a)^{\frac{1}{p}-\frac{1}{q}} \cdot \|f\|_q,$$

which shows that $I : V_q \to V_p$ is bounded with

$$\|I\| \leq (b-a)^{\frac{1}{p}-\frac{1}{q}}.$$

Remark 2.2 By a continuous extensions we get

$$L^q([a,b]) \subset L^p([a,b]), \qquad \text{for } 1 \leq p < q < +\infty. \quad \Diamond$$

2) In order to prove that $I : V_p \to V_q$ is not bounded we only have to choose a sequence (f_n) of functions from $C([a,b])$, such that $(\|f_n\|_p)$ is bounded, while $\|f_n\|_q \to +\infty$. Using that $1 \leq p < q$, we can choose a constant $\alpha \in \,]0,1[$, such that $\alpha\,p < 1 < \alpha q$. Then choose

$$f_n(t) = \frac{1}{\left(t-a+\dfrac{1}{n}\right)^\alpha}, \qquad t \in [a,b].$$

We get that $f_n \in C([a,b])$, and (f_n) is an increasing sequence. Hence

$$\|f_n\|_p^p = \int_a^b \frac{dt}{\left(t-a+\dfrac{1}{n}\right)^{\alpha p}} \leq \int_0^{b-a} \frac{dt}{t^{\alpha p}} = \frac{1}{1-\alpha p}\,(b-a)^{1-\alpha p}.$$

On the other hand, it follows from $\alpha q > 1$ that

$$\|f_n\|_q^q = \int_a^b \frac{dt}{\left(t - a + \dfrac{1}{n}\right)^{\alpha q}} \to +\infty \qquad \text{for } n \to +\infty.$$

We conclude that $I : V_p \to V_q$ is unbounded.

Remark 2.3 The example shows that continuity depends essentially on the structures we give the spaces as normed spaces. $\diamondsuit$

Example 2.10 *Let k be a continuous function in $[a, b] \times [a, b]$, and define the operator K by*

$$Kf(x) = \int_a^b k(x, t)\, f(t)\, dt, \qquad f \in C([a, b]).$$

Prove that $Kf \in C([a, b])$ and that K is a bounded and linear operator of $(C([a, b]), \|\cdot\|_\infty)$ into itself. Prove that

$$\|K\| \le (b - a) \max_{a \le x,\, t \le b} |k(x, t)|.$$

First note that if $f = 0$, then $Kf = 0$. Then let $f \ne 0$, thus $\|f\|_\infty > 0$. It follows that

$$Kf(x + y) - Kf(x) = \int_a^b \{k(x + y, t) - k(x, y)\} f(t)\, dt,$$

hence

$$|Kf(x + y) - Kf(x)| \le \max_{t \in [a,b]} |k(x + y, t) - k(x, t)| \cdot \|f\|_\infty \cdot (b - a).$$

The function k is continuous in a closed and bounded, i.e. compact, domain, so k is uniformly continuous. Then we can find a $\delta > 0$, such that

$$k(x, y) - k(y, u) < \frac{\varepsilon}{\|f\|_\infty \cdot (b - a)} \quad \text{for } \|(x, y) - (y, u)\| < \delta.$$

Therefore, if $|y| < \delta$, then $|Kf(x + y) - Kf(x)| < \varepsilon$, proving that $Kf \in C([a, b])$.

Clearly

$$\begin{aligned}
K(f + \lambda g)(x) &= \int_a^b k(x, t) \cdot \{f(t) + \lambda g(t)\}\, dt = \int_a^b k(x, t)\, f(t)\, dt + \lambda \int_a^b k(x, t)\, g(t)\, dt \\
&= Kf(x) + \lambda\, Kg(x),
\end{aligned}$$

shows that K is linear.

We have

$$\begin{aligned}
|Kf(x)| &= \left| \int_a^b k(x, t) f(t)\, dt \right| \le \int_a^b |k(x, t)| \cdot |f(t)|\, dt \le \max_{a \le x,\, t \le b} |k(x, t)| \cdot \|f\|_\infty \int_a^b dt \\
&= (b - a) \cdot \max_{a \le x,\, t \le b} |k(x, t)| \cdot \|f\|_\infty,
\end{aligned}$$

where the right hand side is independent of x. Thus we conclude that

$$\|Kf\|_\infty \leq (b-a) \cdot \max_{a \leq x,\, t \leq b} |k(x,t)| \cdot \|f\|_\infty,$$

and K is bounded. Because $\|K\|$ is defined as the smallest possible constant in such an estimate, we conclude that

$$\|K\| \leq (b-a) \cdot \max_{a \leq x,\, t \leq b} |k(x,t)|.$$

Example 2.11 *Let H denote a Hilbert space, and let $T \in B(H)$. Show that*

$$\|T\| = \sup\{|(Tx,y)| \mid \|x\| \leq 1,\ \|y\| \leq 1\}.$$

We infer from

$$|(Tx,y)| \leq \|Tx\| \cdot \|y\| \leq \|T\| \cdot \|x\| \cdot \|y\|,$$

that

$$\sup\{|(Tx,y)| \mid \|x\| \leq 1,\ \|y\| \leq 1\} \leq \|T\|.$$

If $T = O$, there is nothing to prove. Thus, let $T \neq O$, and consider the set of $x \in H$, for which $Tx \neq 0$. Putting $y = \dfrac{Tx}{\|Tx\|}$ we get

$$\|Tx\| = \left(Tx, \frac{Tx}{\|Tx\|}\right) \leq \sup\{|(Tx,y)| \mid \|y\| \leq 1\},$$

hence

$$
\begin{aligned}
\|T\| &= \sup\{\|Tx\| \mid \|x\| \leq 1\} \\
&= \sup\{|(Tx,y)| \mid \|x\| \leq 1,\ \|y\| \leq 1\} \leq \|T\|,
\end{aligned}
$$

and we infer that we have equality as wanted.

Example 2.12 *Let (e_k) be an orthonormal basis of a Hilbert space H, and let $T : D(T) \to H$ be a closed, linear operator in H, for which $e_k \in D(T)$ for all $k \in \mathbb{N}$.*
Let $x = \sum_{k=1}^{+\infty} \alpha_k e_k \in H$, and let $x_n = \sum_{k=1}^{n} \alpha_k e_k$, $n \in \mathbb{N}$. Prove that if (Tx_n) is convergent, then $x \in D(T)$ and $Tx = \sum_{k=1}^{+\infty} \alpha_k Te_k$.

It follows from the assumptions that $x_n \to x$. Therefore, if $(Tx_n)_n = \left(\sum_{k=1}^{n} \alpha_k Te_k\right)_n$ is convergent, we infer that $x \in D(T)$ and

$$Tx = \lim_{n \to +\infty} \sum_{k=1}^{n} \alpha_k Te_k = \sum_{k=1}^{+\infty} \alpha_k Te_k.$$

Example 2.13 *Let H denote a Hilbert space and define for $a, b \in H$ the mapping $T_{a,b} : H \to H$ by*

$$T_{a,b}x = (x, b)a.$$

1) Show that $T_{a,b} \in B(H)$ and find $\dim(T_{a,b}(H))$ (when $a, b \neq 0$) as well as the norm of $T_{a,b}$.

2) Find $T_{a,b}^{\star}$.

3) Let $T \in B(H)$ have a one dimensional range. Show that there exist $a, b \in H$ such that $T = T_{a,b}$.

1) If $T_{a,b}x = (x, b)\, a$, then

$$\|T_{a,b}x\| = |(x, b)|\, \|a\| \leq \|a\| \cdot \|b\| \cdot \|x\|,$$

hence $\|T_{a,b}\| \leq \|a\| \cdot \|b\|$, and $T_{a,b} \in B(H)$.

Choose in particular $x = b \neq 0$. Then

$$\|T_{a,b}x\| = \|b\|^2 \|a\| = \|a\| \cdot \|b\| \cdot \|x\|,$$

and we get equality in the estimate above, thus $\|T_{a,b}\| = \|a\| \cdot \|b\|$.

Finally, $T_{a,b}(H)$ is generated by the vector $a \neq 0$. Since also $b \neq 0$, we get

$$\dim(T_{a,b}(H)) = 1.$$

2) By a direct computation,

$$(T_{a,b}x, y) = ((x, b)a, y) = (x, b)(a, y) = \left(x, \overline{(a, y)}\, b\right) = (x, (y, a)b),$$

hence

$$T_{a,b}^{\star}y = (y, a)b.$$

3) Let $\dim(T(H)) = 1$, thus $T(H)$ is generated by a vector $a \neq 0$,

$$T(H) = \{\lambda a \mid \lambda \in \mathbb{C}\},$$

and Tx is determined by a scalar function $\lambda(x)$ by the relation

$$Tx = \lambda(x)\, a.$$

Then

$$\lambda(x) = \left(\lambda(x)\, a, \frac{a}{\|a\|^2}\right) = \frac{1}{\|a\|^2}\,(Tx, a) = \frac{1}{\|a\|^2}\,(x, T^{\star}a),$$

so $T = T_{a,b}$, where $a \neq 0$ generates $T(H)$, and b is given by

$$b = \frac{1}{\|a\|^2}\, T^{\star}a.$$

Example 2.14 *Let $H = L^2([a, b])$, a, b finite and T the operator*

$$Tf(x) = \frac{1}{b-a} \int_a^b f(t)\, dt, \qquad \text{for } x \in [a, b] \quad \text{and} \quad f \in H.$$

1) Show that $T \in B(H)$.

2) Show that T is a projection.

1) The constants a and b are finite, so $1_{[a,b]} \in H$. Let $f \in H$. An application of Cauchy-Schwarz's inequality gives

$$|Tf(x)| = \frac{1}{b-a} \left| \int_a^b 1 \cdot f(t)\, dt \right| \leq \frac{1}{b-a} \sqrt{a-b} \cdot \|f\|_2 < +\infty,$$

showing that T is indeed defined. Since $Tf(x) = c$ is constant for all $x \in [a, b]$, we get

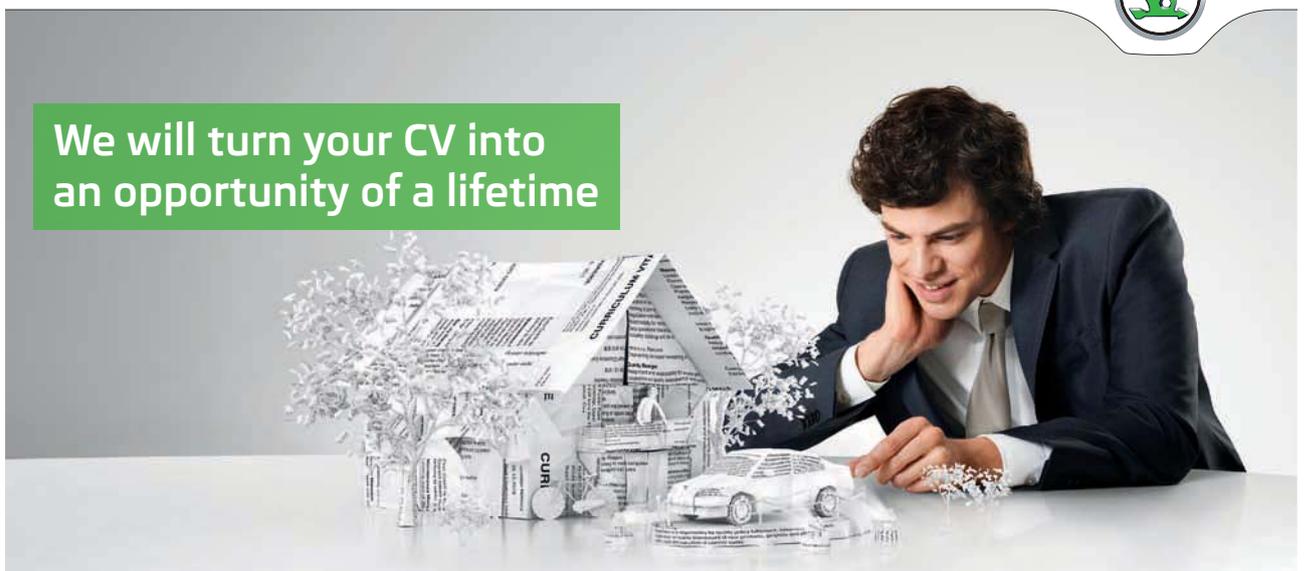$$\|Tf\|_2 = |c| \cdot \|1\|_2 = |c| \cdot \sqrt{b-a} < +\infty,$$

thus $Tf \in H$, and it follows that $T \in B(H)$.

2) We shall only check the two conditions,

$$(Tf, g) = (f, Tg) \qquad \text{for all } f, g \in H,$$

and

$$T^2 f = Tf \qquad \text{for alle } f \in H.$$

Now, $Tf(x)$ and $Tg(t)$ are constant, hence

$$
\begin{aligned}
(Tf, g) &= \int_a^b \frac{1}{b-a} \int_a^b f(t)\, dt \cdot \overline{g(x)}\, dx = \int_a^b f(t)\, dt \cdot \overline{\frac{1}{b-a} \int_a^b g(x)\, dx} \\
&= \int_a^b f(t)\, dt \cdot \overline{Tg(t)} = \int_a^b f(t) \cdot \overline{Tg(t)}\, dt = (f, Tg).
\end{aligned}
$$

Furthermore,

$$T^2 f = T(Tf) = T\left(\frac{1}{b-a} \int_a^b f(t)\, dt \right) = T(f) \cdot T(1) = Tf \cdot \frac{1}{b-a} \int_a^b dt = Tf.$$

Because T is linear, we conclude that T is a projection on the closed subspace

$$M = \{f \in H \mid Tf = f\}.$$

Finally, it follows from $Tf = c$ constant, and

$$T(c) = c \cdot T(1) = c,$$

that M is the subspace consisting of all constants.

Example 2.15 *Let $H = L^2([-\pi, \pi])$ be the Hilbert space with the usual orthonormal basis*

$$e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}, \qquad n \in \mathbb{Z}.$$

Let λ_n, $n \in \mathbb{N}$ be complex numbers satisfying

$$M = \sup_{n \in \mathbb{Z}} |\lambda_n - n| < \frac{\log 2}{\pi},$$

and let

$$f_n(x) = \frac{1}{\sqrt{2\pi}} e^{i\lambda_n x}, \qquad n \in \mathbb{Z}.$$

1. *Show that*

$$f_n(x) - e_n(x) = e_n(x) \sum_{k=1}^{\infty} \frac{\{i(\lambda_n - n)\}^k}{k!} x^k, \qquad n \in \mathbb{Z}.$$

2. *Show that for any $k \in \mathbb{Z}$ we have that*

$$\left\| x^k g \right\| \le \pi^k \left\| g \right\|.$$

Now, let $D : \ell^2(\mathbb{Z}) \to H$ be the linear operator given by

$$S\left((a_n)\right) = \sum_{n=-\infty}^{\infty} a_n \left(f_n - e_n\right), \qquad (a_n) \in \ell^2(\mathbb{Z}).$$

3. *Assuming that the order of summation can be reversed, prove that*

$$\|S\| \le e^{M\pi} - 1.$$

Let the linear operator $K : H \to H$ be given by

$$Kf = S\left((f, e_n)\right) = \sum_{n=-\infty}^{\infty} (f, e_n) \{f_n - e_n\}, \qquad f \in H.$$

4. *Show that $\|K\| < 1$ and deduce that $T = I + K \in B(H)$ has an inverse $T^{-1} \in B(H)$. Calculate Te_n and infer that*

$$\operatorname{span} \{f_n\}^{\perp} = \{0\}.$$

5. *Define*

$$g_n = \left(T^{-1}\right)^\star e_n, \qquad n \in \mathbb{Z},$$

and show that $(f_m, g_n) = \delta_{mn}$ *(Kronecker delta).*

6. *Finally, use the decomposition*

$$T^{-1}f = \sum_{n=-\infty}^{\infty} \left(T^{-1}f, e_n\right) e_n$$

to show that

$$f = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} (f, g_n)\, e^{i\lambda_n x}, \qquad \text{for } f \in H.$$

1) First put $\varphi_n(x) = \dfrac{1}{\sqrt{2\pi}}$ outside as a factor. By a simple series expansion we get for every $n \in \mathbb{Z}$,

$$f_n(x) - \varphi_n(x) = \frac{1}{\sqrt{2\pi}}\, e^{i\lambda_n x} - \frac{1}{\sqrt{2\pi}}\, e^{inx} = \frac{1}{\sqrt{2\pi}}\, e^{inx}\left\{e^{i(\lambda_n - n)x} - 1\right\} = \varphi_n(x) \sum_{k=1}^{+\infty} \frac{\left\{(\lambda_n - n)\right\}^k}{k!}\, x^k.$$

2) This follows from a simple estimate

$$\left\|x^k g\right\|_2^2 = \int_{-\pi}^{\pi} x^{2k} |g(x)|^2 dx \le \pi^{2k} \int_{-\pi}^{\pi} |g(x)|^2 dx = \left\{\pi^k \|g\|_2\right\}^2,$$

hence

$$\left\|x^k g\right\|_2 \le \pi^k \|g\|_2.$$

3) This is not an easy question. We get

$$(15) \qquad \|S((a_n))\|_{L^2} = \left\| \sum_{n=-\infty}^{+\infty} a_n \{f_n - \varphi\} \right\|_{L^2}$$

$$(16) \qquad = \left\| \sum_{n=-\infty}^{+\infty} a_n \varphi_n(x) \sum_{k=1}^{+\infty} \frac{\{i(\lambda_n - n)\}^k}{k!} x^k \right\|_{L^2}$$

$$(17) \qquad = \left\| \sum_{k=1}^{+\infty} x^k \left\{ \sum_{n=-\infty}^{+\infty} a_n \frac{\{i(\lambda_n - n)\}^k}{k!} \varphi_n(x) \right\} \right\|_{L^2}$$

$$(18) \qquad \le \sum_{k=1}^{+\infty} \left\| x^k \left\{ \sum_{n=-\infty}^{+\infty} a_n \frac{\{i(\lambda_n - n)\}^k}{k!} \varphi_n(x) \right\} \right\|_{L^2}$$

$$(19) \qquad \le \sum_{k=1}^{+\infty} \pi^k \left\| \sum_{n=-\infty}^{+\infty} a_n \frac{\{i(\lambda_n - n)\}^k}{k!} \varphi_n(x) \right\|_{L^2}$$

$$(20) \qquad \le \sum_{k=1}^{+\infty} \pi^k \left\{ \sum_{n=-\infty}^{+\infty} |a_n|^n \cdot \frac{|i(\lambda_n - n)|^{2k}}{(k!)^2} \right\}^{\frac{1}{2}}$$

$$(21) \qquad \le \sum_{k=1}^{+\infty} \pi^k \cdot \frac{M^k}{k!} \left\{ \sum_{n=-\infty}^{+\infty} |a_n|^2 \right\}^{\frac{1}{2}}$$

$$(22) \qquad = \left\{ e^{M\pi} - 1 \right\} \|(a_n)\|_{\ell^2} ..$$

The complicated estimate above needs some comments:

- In (15) we apply the definition of S.
- In (16) we apply the result from (1).
- In (17) we reverse the summations.
- In (18) we apply Minkowski's inequality.
- In (19) we apply the result of (2).
- In (20) we apply Parseval's equation.
- In (21) we use the definition of M and then take the square root.
- In (22) we use the exponential series.

Remark 2.4 We note for later use that

$$e^{M\pi} - 1 < 1,$$

This follows from

$$e^{M\pi} - 1 < \exp\left(\frac{\ln 2}{\pi} \cdot \pi\right) - 1 = e^{\ln 2} - 1 = 2 - 1 = 1. \qquad \lozenge$$

4) Then

$$
\begin{align}
(23) \qquad \|Kf\|_{L^2} &= \|S\left((f, \varphi_n)\right)\|_{L^2} \\
(24) \qquad &\leq \left\{e^{M\pi} - 1\right\} \|((f, \varphi_n))\|_{\ell^2} \\
(25) \qquad &= \left\{e^{M\pi} - 1\right\} \|f\|_{L^2}, ,
\end{align}
$$

where we

- in (23) use the definition,
- in (24) use the result of (3),
- in (25) apply Parseval's equation.

It follows from the remark above that

$$\|K\|e^{M\pi} - 1 < 1.$$

We have $K \in B(H)$, so $T = I + K$ has an inverse in $B(H)$, given by the Neumann series

$$T^{-1} = (I + K)^{-1} = \sum_{n=0}^{+\infty} (-1)^n K^n \qquad [\in B(H)].$$

Then

$$T\varphi_n = \varphi_n + K\varphi_n = \varphi_n + \sum_{k=-\infty}^{+\infty} (\varphi_n, \varphi_k)\{f_k - \varphi_k\} = \varphi_n + 1 \cdot \{f_n - \varphi_n\} = f_n,$$

hence

$$T\varphi_n = f_n.$$

Let $g \perp \mathrm{span}\{f_n\}$, i.e.

$$(g, f_n) = (g, T\varphi_n) = (T^\star g, \varphi_n) = 0 \qquad \text{for all } n \in \mathbb{Z}.$$

Because $\{\varphi_n \mid n \in \mathbb{Z}\}$ is an orthonormal system, we must have $T^\star g = 0$.

On the other hand, $\left(T^{-1}\right)^\star$ exists and is equal to $(T^\star)^{-1}$. In particular, $T^\star$ is injective. This means that $T^\star g = 0$ implies that $g = 0$, and we have proved that

$$(\mathrm{span}\,\{f_n\})^\perp = \{0\}.$$

5) By a simple computation,

$$(f_m, g_n) = \left(T\varphi_m, \left(T^{-1}\right)^\star \varphi_n\right) = \left(T^{-1}T\varphi_m, \varphi_n\right) = \delta_{mn}.$$

6) From

$$T^{-1}f = \sum_{n=-\infty}^{+\infty} \left(T^{-1}f, \varphi_n\right)\varphi_n,$$

and T continuous we get by termwise to apply T and by using that $T\varphi_n = f_n$ that for $f \in H$,

$$f = T\left(T^{-1}f\right) = \sum_{n=-\infty}^{+\infty} \left(f, \left(T^{-1}\right)^\star \varphi_n\right) T\varphi_n = \sum_{n=-\infty}^{+\infty} (f, g_n)\, f_n = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} (f, g_n)\, e^{i\lambda_n x}.$$

Remark 2.5 This example shows that if

$$|\lambda_n - n| \le M < \frac{\ln 2}{\pi} \approx 0.2206, \qquad \text{for all } n \in \mathbb{Z},$$

where λ_n is allowed to be complex, then we still have a uniquely determined series expansion

$$(26) \quad f = \sum_{n=-\infty}^{+\infty} a_n\, e^{i\lambda_n x}, \qquad \text{i } L^2([-\pi, \pi]),$$

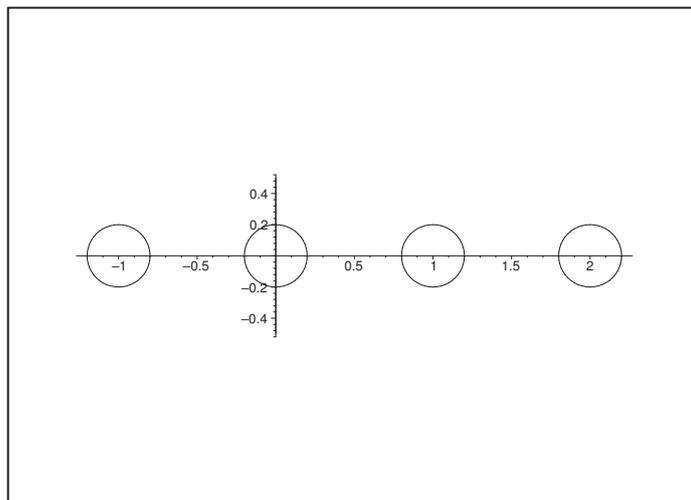where we put $a_n = \dfrac{1}{\sqrt{2\pi}}\,(f, g_n)$.

Figure 16: Illustration of λ_n inside the circle of centre n and radius $M = 0.2 < \dfrac{\ln 2}{\pi}$.

If every $\lambda_n \in \mathbb{R}$, then the series expansion (26) gives a very realistic model from a technical point of view, where we include some "noise", i.e. oscillations which are not commensurable with the usual

harmonic eigen-oscillations. One explicit example is the description of the oscillating string of variable density.

It is very natural to call (26) a *non-harmonic Fourier series*.

It is possible to prove (*"Kadeč's $\frac{1}{4}$ Theorem"*), which states that the best possible result is obtained, when

$$|\lambda_n - n| \le M < \frac{1}{4} \qquad \text{for all } n \in \mathbb{Z}.$$

The proof of this Theorem is far more difficult, and Kadeč's first proof was even wrong.

Finally, it is possible to prove that there exists a sequence (λ_n) such that

$$\sup_{n \in \mathbb{Z}} |\lambda_n - n| = \frac{1}{4},$$

and such that the result (26) does *not* hold for the sequence (λ_n). $\Diamond$

2.2   Closed operators

Example 2.16 *Consider in ℓ^2 the operator T, defined by*

$$T(x_n) = (n\,x_n).$$

Prove that T is a closed and densely defined operator, and prove that

$$T(D(T)) = \ell^2.$$

We shall start by first proving the latter claim. Then notice that

$$Tx = (1x_1, 2x_2, 3x_3, \cdots) = (0, 0, 0, \cdots),$$

if and only if $x = 0$, proving that T is injective, thus the inverse T^{-1} exist. Then it immediately follows that

$$T^{-1}(x_1, x_2, x_3, \cdots) = \left(x_1, \frac{1}{2}\,x_2, \frac{1}{3}\,x_3, \cdots\right),$$

hence

$$\left\|T^{-1}x\right\|^2 = \sum_{n=1}^{+\infty} \frac{1}{n^2}\,|x_n|^2 \leq \sum_{n=1}^{+\infty} |x_n|^2 = \|x\|^2,$$

and T^{-1} is bounded, $\left\|T^{-1}\right\| \leq 1$. We even conclude from

$$T^{-1}(1, 0, 0, \cdots) = (1, 0, 0, \cdots),$$

that $\left\|T^{-1}\right\| = 1$, even if this is not essential for the example.

It is, however, very important that $T^{-1} \in B\left(\ell^2\right)$, so T^{-1} is in particular a closed operator. This means that the graph $\mathcal{G}\left(T^{-1}\right)$ is a closed set. However, $\mathcal{G}(T)$ is obtained from $\mathcal{G}\left(t^{-1}\right)$ by interchanging the two coordinates. Hence it follows that the graph $\mathcal{G}(T)$ for T is also closed, and we have proved that T is a closed operator.

Note, however, that T is an unbounded operator. We have e.g.

$$\|Te_n\| = n \qquad \text{and} \qquad \|e_n\| = 1.$$

Even if the task have been totally solved above, we continue by giving a *direct* proof of that T is closed.

The domain $D(T)$ contains the subspace consisting of all sequences from ℓ^2, which are 0 eventually. The latter subspace is already dense in ℓ^2, so $D(T)$ is also dense in ℓ^2.

Choose $x^{(n)} \in D(T)$ such that $x^{(n)} \to 0$ and $Tx^{(n)} \to y$. If we can prove that $y = 0$, then it follows that T is closed.

Choose any $z \in D(T)$. Then $x \mapsto (x, z)$ is continuous in ℓ^2, and we infer that

$$\left(Tx^{(n)}, z\right) \to (y, z) \qquad \text{for } n \to +\infty.$$

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On the other hand,

$$\left(Tx^{(n)}, z\right) = \sum_{j=1}^{+\infty} j\, x_j^{(n)} \overline{z}_j = \sum_{j=1}^{+\infty} x_j^{(n)} \overline{\{j\, z_j\}} = \left(x^{(n)}, Tz\right) \to 0 \qquad \text{for } n \to \infty,$$

because $x \mapsto (x, Tz)$ also is continuous for every fixed $z \in D(T)$. We conclude for every $z \in D(T)$ (which we have shown above is dense in ℓ^2) that

$$(z, y) = \overline{(y, z)} = \overline{0} = 0.$$

Then it follows from Riesz's Representation Theorem that this is only possible, if $y = 0$.

Example 2.17 *Let T be a closed linear operator $T : D(T) \subset V \to V$, where V is a Banach space, and let $A \in B(V)$.*
Show that $A + T$ and TA are closed, linear operators.

By the definition, $T : D(T) \to V$ is closed, if the graph

$$\mathcal{G}(T) = \{(x, y) \in V \times V \mid x \in D(T), \, y = Tx\}$$

is a closed set.
Let $A \in B(V)$. We shall prove that

$$\mathcal{G}(T + A) = \{(x, y) \in V \times V \mid x \in D(T), \, y = Tx + Ax\}$$

is closed. It suffices to prove that if $(x_n) \subseteqq D(T + A)$ and $x_n \to x$ and $Tx_n + Ax_n \to y$, then $x \in D(T + A)$ and $(T + A)x = y$.

Let $(x_n) \subseteqq D(T + A) = D(T)$ and $x_n \to x$ and $Tx_n + Ax_n \to y$. Then

$$Tx_n \to y - \lim Ax_n = y - Ax.$$

Since T is closed. $x \in D(T)$, and $Tx = y - Ax$, hence by a rearrangement, $Tx + Ax = y$, and it follows that $T + A$ is closed.

Finally, choose a sequence $(x_n) \subseteqq D(TA)$, such that

$$x_n \to x \qquad \text{and} \qquad TAx_n \to y.$$

Then $(Ax_n) \subseteqq D(T)$, and

$$Ax_n \to Ax \qquad \text{and} \qquad T(Ax_n) \to y.$$

Using that T is closed, we get $Ax \in D(T)$ and $TAx = y$. However, $Ax \in D(T)$ means that $x \in D(TA)$, and we have proved that TA is closed.

Example 2.18 *Consider in ℓ^2 the operator T defined as*

$$T(x_1, x_2, \dots) = (x_2, 2x_3, 3x_4, \dots).$$

Show that T is closed, densely defined operator.

Clearly, T is linear, $T : D(T) \to \ell^2$, where $D(T) \subseteq \ell^2$ is dense in ℓ^2. (All sequences which are zero eventually lie in $D(T)$, and this subspace is dense in ℓ^2).

Let $\left(x^{(n)}\right) \subseteq D(T)$, and assume that $x^{(n)} \to x \in \ell^2$ and $Tx^{(n)} \to y \in \ell^2$. If we write

$$x^{(n)} = (x_1^n, x_2^n, x_3^n, \cdots) \qquad \text{and} \qquad y = (y_1, y_2, y_2, \cdots),$$

then

$$y - Tx^{(n)} = (y_1 - x_2^n, y_2 - 2x_3^n, y_3 - 3x_4^n, \cdots) \to 0 \qquad \text{in } \ell^2.$$

In particular,

$$x_2^n \to y_1, \quad x_3^n \to \frac{1}{2}\, y_2, \quad \dots, \quad x_{k+1}^n \to \frac{1}{k}\, y_k, \quad \dots, \qquad \text{for } n \to +\infty,$$

hence the only possible limit value is

$$x = \left(\lim_{n \to +\infty} x_1^n, y_1, \frac{1}{2}\, y_2, \dots, \frac{1}{k}\, y_k, \dots \right).$$

Because $y \in \ell^2$ it is obvious that $x \in \ell^2$. Finally, it is trivial that $Tx = y$, so $x \in D(T)$, and we have proved that T is closed.

Example 2.19 *Let X, Y be Banach spaces, and let*

$$T : D(T) \subset X \to Y$$

be a closed linear operator. Assume that the sequences (u_n), $(v_n) \subset D(T)$ satisfy the condition

$$\lim_n u_n = \lim_n v_n.$$

Show that if (Tu_n) and (Tv_n) are both convergent, then $\lim_n Tu_n = \lim_n Tv_n$.

From the assumption that T is a *closed* operator and

$$x = \lim_{n \to +\infty} u_n, \quad u_n \in D(T), \quad \text{and} \quad \lim_{n \to +\infty} Tu_n = z,$$

then $x \in D(T)$, and $Tx = z$.

Similarly it follows from

$$x = \lim_{n \to +\infty} v_n, \quad v_n \in D(T), \quad \text{and} \quad \lim_{n \to +\infty} Tv_n = q,$$

and from T being closed that also $Tx = q$. Now, $x \in D(T)$, and the mapping is uniquely defined, so

$$\lim_{n \to +\infty} Tu_n = z = q = \lim_{n \to +\infty} Tv_n.$$

Example 2.20 *Let $T : X \to Y$ be a closed linear operator between two normed spaces and let $A \subset X$ be compact. Show that $T(A)$ is closed.*

Let $(y_n) \subseteq T(A)$, and assume that $y_n \to y \in Y$. We shall prove that $y \in T(A)$.

Since $y_n \in T(A)$, there exists at least one $x_n \in A$ (possibly more), such that $Tx_n = y_n$. Choose for each such y_n one $x_n \in A$ of this property. Then (x_n) defines a sequence on a compact set A. It therefore contains a convergent subsequence (z_n), where $z_n \to x \in A$. We have now proved that

$$(z_n) \subseteq A \subset D(T), \quad z_n \to x \in A \quad \text{and} \quad Tz_n \to y \in Y.$$

Now, T is a *closed* operator, so it follows that $x \in D(T)$ and $y = Tx$. Since $x \in A$, we have $y \in T(A)$. This proves that $T(A)$ is closed.

Example 2.21 *Let $T : D(T) \subset X \to Y$ be a closed linear operator between two normed spaces. Show that $\ker(T)$ is a closed subspace of X.*

It is obvious that

$$\ker(T) = \{x \in D(T) \mid Tx = 0\}$$

is a subspace, so we shall only prove that it is closed.

Let $(x_n) \subseteq \ker(T)$ be a sequence, such that $x_n \to x \in X$. We shall prove that $x \in \ker(T)$.

We infer from $Tx_n = 0 \to 0$, and T being closed that $x \in D(T)$ and

$$Tx = \lim_{n \to +\infty} Tx_n = 0,$$

thus $x \in \ker(T)$, and the claim is proved.

Example 2.22 *Let V and W be two Banach spaces (over the same scalar field), and let $T : D(T) \to W$ be a bounded linear operator, where $D(T)$ is a subspace of V.*
Prove that there exists precisely one bounded extension T_1 of T, such that $D(T_1) = \overline{D(T)}$, and prove that $\|T\|_1 = \|T\|$.

If $D(T)$ is closed, there is of course nothing to prove. We assume that $D(T)$ is *not* closed. Let $x \in \overline{D(T)}$. Then there exists a sequence $(x_n) \subseteq D(T)$, such that $x_n \to x$. Because T is bounded, we see that (Tx_n) is a Cauchy sequence on W. In fact,

$$\|Tx_m - Tx_n\| = \|T(x_m - x_n)\| \le \|T\| \cdot \|x_m - x_n\| \to 0 \quad \text{for } m, n \to +\infty.$$

Now, W is a Banach space, so (Tx_n) is convergent with some limit value, which we call w, thus $Tx_n \to w$ for $n \to +\infty$.

If $(y_n) \subseteq D(T)$ is another sequence, for which $y_n \to x$, then $x_n - y_n \in D(T)$, and $x_n - y_n \to x - x = 0$ for $n \to +\infty$. Furthermore, T is bounded, so $Tx_n - Ty_n \ rightarrow 0$, and we get that (Tx_n) and (Ty_n) have the same limit value w, whenever (x_n) and (y_n) converge towards the same x.

Then put $w = \overline{T}x$, i.e.

$$\overline{T}x = \lim_{n \to +\infty} Tx \qquad \text{for } x = \lim_{n \to +\infty} x_n, \quad x_n \in D(T).$$

Then clearly $\overline{T}$ is an extension of T to $\overline{D(T)}$.

Then we prove that $\overline{T}$ is linear. Let $x, y \in \overline{D(T)}$, and let (x_n) and (y_n) be sequences from $D(t)$, for which $x_n \to x$ and $y_n \to y$. If $\alpha, \beta \in L$, then $\alpha x_n + \beta y_n \in D(T)$, and

$$\alpha x_n + \beta y_n \to \alpha x + \beta y \in \overline{D(T)}.$$

Using the linearity of T on $D(T)$ we get

$$\begin{aligned}
\overline{T}(\alpha x + \beta y) &= \lim_{n \to +\infty} T(\alpha x_n + \beta y_n) = \lim_{n \to +\infty} \{\alpha Tx_n + \beta Ty_n\} \\
&= \alpha \lim_{n \to +\infty} Tx_n + \beta \lim_{n \to +\infty} Ty_n = \alpha \overline{T}x + \beta \overline{T}y,
\end{aligned}$$

and it follows that $\overline{T}$ is linear.

Finally, we shall prove that $\overline{T}$ is bounded. Let $x \in \overline{D(T)}$, and let $(x_n) \subseteq D(T)$, $x_n \to x$. Then we have the estimates

$$\|\overline{T}x\| \le \|\overline{T} - Tx_n\| + \|Tx_n\| \le \|\overline{T}x - Tx_n\| + \|T\| \cdot \|x_n\|.$$

From $\overline{T}x = \lim_{n \to +\infty} Tx_n$, follows that $\|\overline{T} - Tx_n\| \to 0$ and $\|x_n\| \to \|x\|$ for $n \to +\infty$. We infer that

$$\|\overline{T}x\| \le \|T\| \cdot \|x\|,$$

which shows that $\|\overline{T}\| \leq \|T\|$. On the other hand, $\overline{T}$ is an extension of T, so trivially $\|\overline{T}\| \geq \|T\|$. Summing up, $\|\overline{T}\| = \|T\|$.

If $T_2 : \overline{D(T)} \to W$ were another bounded extension, then $T_2 - T_1 : \overline{D(T)} \to W$ would be a bounded operator, and

$$(T_2 - T_1)(D(T)) = O(D(T)).$$

However, the only continuous extension of the zero operator on $D(T)$ to the closure $\overline{D(T)}$ is of course the zero operator, hence we conclude that $T_2 = T_1$. Thus the uniqueness is also proved.

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