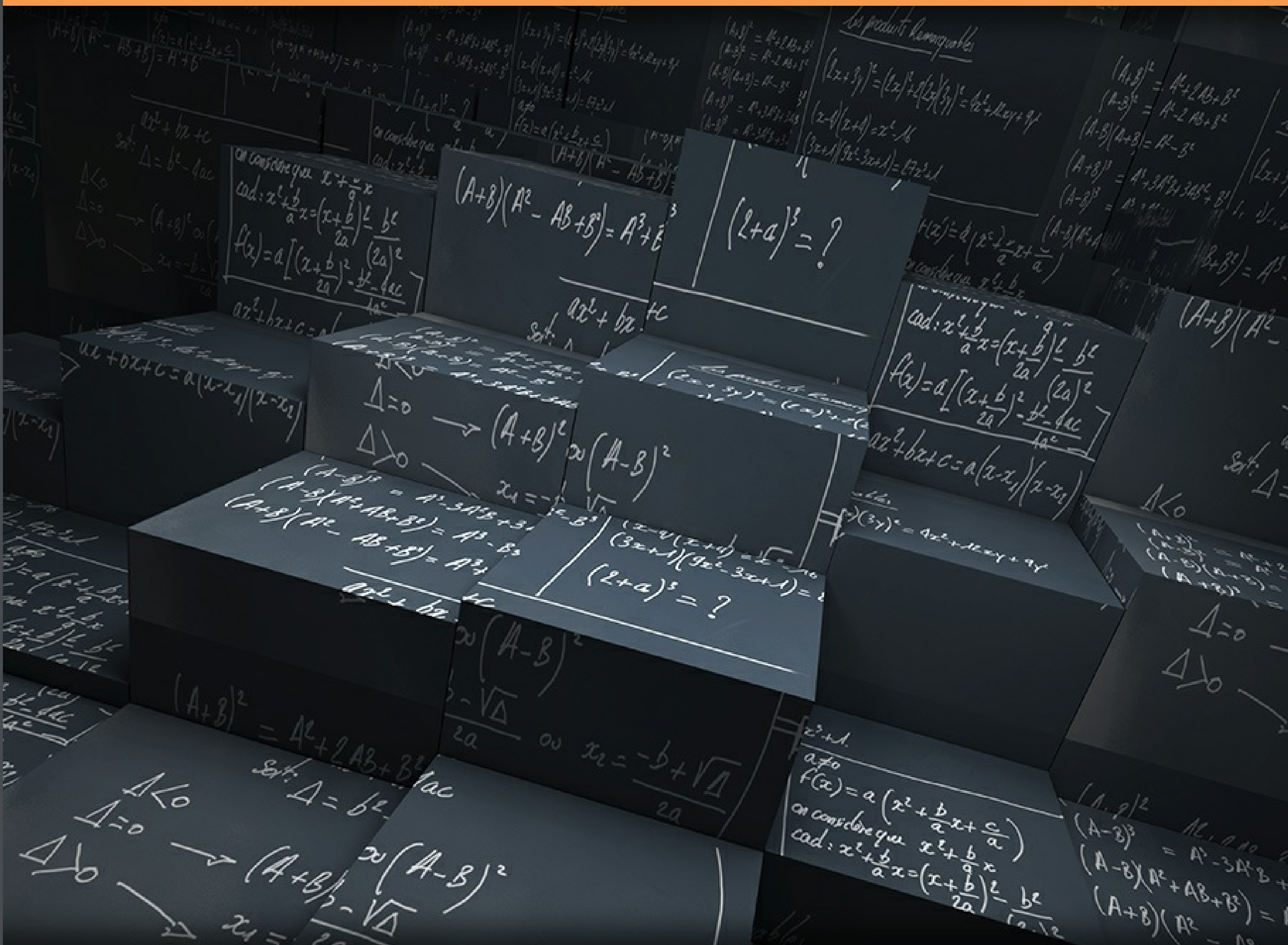
A Short Course in Predicate Logic

Jeff Paris



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Introduction

In our everyday lives we often employ arguments to draw conclusions. In turn we expect others to follow our line of reasoning and thence agree with our conclusions. This is especially true in mathematics where we call such arguments 'proofs' But why are these arguments or proofs so convincing, why should we agree with their conclusions? What is it that makes them 'valid'? In this course we will attempt to formalize what we mean by these notions within a context/language which is adequate to express almost everything we do in mathematics, and much of everyday communication as well.

The presentation given here derives from a lecture course given in the School of Mathematics at Manchester University between 2010 and 2013. Previous to that courses covering similar topics had run for many years with ever diminishing student numbers, the students seemingly finding the notation bewildering and the level of rigor and nit picking detail excessive. As a result they often gave up before the point of realizing how easy, self-evident and downright interesting the subject really is. The primary aim of this current version then was to adopt an approach which avoided as far as possible those initial barriers, and which reached some of the 'good stuff' before any risk of disheartenment setting in.

That is not to say that the approach given here lacks rigor or is at some points 'not quite right'. Far from it. But we will on occasions implicitly accept as obvious or self-evident facts which, looking back later, you might question. If so then that is the time to check for yourself that what has been taken for granted in the text is indeed perfectly correct.

In terms of the choice of material in the course the intention is that it will provide a firm grounding in Predicate Logic such as is necessary for further fields in Mathematical Logic, for example Proof Theory, Model Theory, Set Theory, as well as Philosophical Logic and the diverse applications in Computer Science. In addition, with its presentation of the Completeness Theorem, it aims to provide a broad picture and understanding of relationship between proof and truth and the nature of mathematics in general.

These notes can be studied at two levels, in UK terms Bachelors and Masters. The more demanding material and exercises, primarily aimed at the Master level is marked with an asterisk, *. Unmarked material is intended for both levels and is self contained, requiring nothing from the upper level.

Motivation

Consider the following examples of 'reasoning':

1(a) $$\frac{\textit{10 is a number which is the sum of 4 squares}}{\therefore \textit{ There is a number which is the sum of 4 squares}}$$

2(a) $$\frac{\begin{array}{c}\textit{Every student at this University pays fees}\\ \textit{Monica is a student at this University}\end{array}}{\therefore \textit{Monica pays fees}}$$

In each case the conclusion seems to 'follow' from the assumptions/premises. But in what sense? What do we mean by 'follows'? Since such arguments are common in our everyday lives, especially when as mathematicians we produce proofs of theorems, it would seem worthwhile to understand and answer this question, and that's what logic is all about, it's the study of 'valid reasoning or argument'.

In both the above examples the reasoning seems to be 'valid' (which right now just equates with 'OK'), but what does this mean? A first guess here is that it means: The conclusion is true given that the premises are true. This is close, but we have to be careful here. Consider for example the argument:

3(a) $$\frac{\textit{There is a number which is the sum of 4 squares}}{\therefore \textit{ Every number is the sum of 4 squares}}$$

This does not seem to be 'valid' in the sense of the first two examples, despite the fact that the assumption and conclusion are actually true.

The reason the first two arguments are valid and the last is not is that they do not actually depend on the meaning of 'sum of 4 squares', 'Monica', '10', 'student at this university', 'pays fees' nor what universe of objects (natural numbers in the first and last, people, say, in the second) we are referring to, whereas in the last the **meaning** of 'is the sum of 4 squares' does matter. For example if we change 'sum of 4 squares' to 'sum of 3 squares' then the premiss is true but the conclusion false.

To see this let's write

$\forall$ for 'for all'
$\exists$ for 'there exists'
c for 10
$P(x)$ for 'x is the sum of 4 squares'

Then our first and last examples become:

1(b) $$\frac{P(c)}{\therefore \exists x\, P(x)}$$

3(b) $$\frac{\exists x\ P(x)}{\therefore \forall x\ P(x)}$$

Clearly the conclusion in the first of these 'follows' no matter what universe the x ranges over, no matter what element of that universe c stands for and no matter what property of x $P(x)$ stands for. In other words no matter what they stand for if the premises are true then so is the conclusion. For example if we take this universe to be the set of all buses along Oxford Road, c to stand for the number 43 bus and $P(x)$ to mean that bus x goes to the airport then the first argument would become

1(c) $$\frac{\textit{The 43 bus goes to the airport}}{\therefore \textit{There is a bus on Oxford Road which goes to the airport}}$$

which we would surely accept as 'OK'.

However in the second case we obtain

3 (c) $$\frac{\textit{There is a bus on Oxford Road which goes to the airport}}{\therefore \textit{All buses along Oxford Road go to the airport}}$$

and now the conclusion is false, whilst the premiss is true, so this is clearly not an OK argument.

Similarly in the Monica example if we let

m stand for Monica
$S(x)$ stand for '*x* is a student at this university'
$F(x)$ stand for '*x* pays fees'
$\rightarrow$ stand for 'if … then', equivalently 'implies',

then the example becomes

2 (b) $$\begin{array}{l} \forall x\,(S(x) \rightarrow F(x)) \\ S(m) \\ \hline \therefore F(m) \end{array}$$

and again this looks an OK argument no matter what universe of objects the variable x ranges over, no matter what element of this universe m stands for and no matter what properties of such x, $S(x)$ and $F(x)$ stand for.

In other words, no matter what meaning (or *interpretation*) we give to this universe, m and $S(x)$, $F(x)$, **if** the premises are true **then** so is the conclusion. The validity of the Monica example 2 derives from this fact. The non-validity of our 'all numbers are the sum of 4 squares' example 3 is a consequence of this failing in this case, despite the fact that in this interpretation the conclusion of 3(a) is true.

What we have learnt here is that to understand and investigate 'valid' arguments we need to study formal examples like the one above where all meaning has been stripped away, where we have been left with just the essential bare bones.

Before doing that however it will be useful to give two more examples which introduce another (small) point. Consider the following, where 'number' means 'natural number':

4 (a) $$\frac{\textit{There is a number which is less or equal any number}}{\therefore\ \textit{For every number there is a number which is less or equal to it}}$$

5 (a) $$\frac{\textit{For every number there is a number which is less or equal to it}}{\therefore\ \textit{There is a number which is less or equal any number}}$$

In these cases both the premiss and conclusion are true. However it is only in the first that the conclusion seems to be valid, in other words to 'follow' from the premise. Again if we let x, y range over natural numbers and let $Q(x, y)$ stand for x is less or equal y then they become respectively:

4 (b) $$\frac{\exists x\, \forall y\, Q(x,y)}{\therefore \forall y\, \exists x\, Q(x,y)}$$

5 (b) $$\frac{\forall y\, \exists x\, Q(x,y)}{\therefore \exists x\, \forall y\, Q(x,y)}$$

The validity of the former is (quite) easy to see. For clearly *no matter what universe the x, y range over and no matter what binary (or 2-ary) relation on the universe Q stands for*, ***if** the premise is true **then** so is the conclusion*. This holds simply because of the **forms** of the premise and conclusion, not because of how we interpreted them here.

On the other hand this 'logical' connection between the premise and the conclusion does not hold in the second case. If we interpret the variables x, y as ranging over the universe $\mathbb{N}$ of natural numbers[1] but interpret Q as the 'greater or equal than' relation then the argument interprets as:

5 (c) $$\frac{\textit{For every number there is a number which is greater or equal to it.}}{\therefore \textit{There is a number which is greater or equal any other number}}$$

so the premise is true whilst the so-called conclusion is false.

As our final example consider:

6 (a) $$\frac{x^5 = 2x-1}{\therefore \exists w\, w^5 = 2w-1}$$

One's first thought maybe is that the variable x here is supposed to be a real number, and that the conclusion follows (trivially even) from the premiss. However the conclusion obviously follows whether we're thinking here of x being a real, or a complex number, or a 4×4 matrix or indeed an element of any algebraic structure in which the functions $x \mapsto x^5$ and $x \mapsto 2x-1$ have some meaning.

To sum up then we could say that in examples 1, 2, 4, 6 the conclusion follows *logically* from the premise(s) whereas in examples 3, 5, it does not. It is this notion of 'logical consequence' that this course, and Logic in general, is interested in.Our above considerations lead us to propose a rough definition of an assertion ϕ being a *logical consequence* of assumptions/premises $\theta_1, \theta_2, \ldots, \theta_n$. Namely this holds if no matter how we interpret the range of the variables, the relations, the constants etc. if $\theta_1, \theta_2, \ldots, \theta_n$ are all true then ϕ will be true. To make this a precise definition we need to say what $\theta_1, \ldots, \theta_n, \phi$ can be, what we mean by an 'interpretation' and even what we mean by 'true'. We start with the former.

Formal Languages, Formulae and Sentences

We have seen in the last section that to study valid reasoning we are led to consider formalized, abstract, assertions such as $P(c)$, $\exists x\, P(x)$, $\forall x\,(S(x) \to F(x))$, $\exists x\, \forall y\, Q(x, y)$, $\forall y\, \exists x\, Q(x, y)$, $x^5 = 2x + 1$ appearing in 1(b), 2(b) and 5(b). Expressions which can arise in this way will be called formulae of a language. Formally they are simply *words* built up from the symbols[2] listed below in specified, 'well-formed', ways (so as to make sense):

Symbol	**Standing for**
Relation symbols e.g. P, S, Q etc	unary, binary, etc. relations
Constant symbols, e.g. c, m etc.	constants
Function symbols, e.g. $+$ etc.	unary, binary, etc. functions
Equality symbol, $=$	the binary relation of equality
Variables, x, w etc.	variable elements of the universe on which the quantifiers, relations, functions operate
Connectives: $\to$	implication, 'implies' or 'if $\cdots$ then$\cdots$'
$\wedge$	conjunction, 'and'
$\vee$	disjunction, 'or'
$\neg$	negation, 'not'
Quantifiers: $\forall w$	for all w (Universal quantification)
$\exists w$	there exists w (Existential quantification)
Parenthesis (,)	punctuation

The available relation, function, constant, and if present equality symbols[3], are said[4] to comprise the *language* of which such expressions are formulae. The language we are working in will vary whilst the remaining symbols are the same in all cases.

Definition A language L is a set consisting of some relation symbols (possibly including $=$) and possibly some constant, function symbols. Each relation and function symbol in L has an *arity* (e.g. unary, binary, ternary, etc.).[5]

For example we could have $L = \{P, Q, c, f\}$ where P is a 1-ary or unary relation symbol, Q is a 2-ary or binary relation symbol, f is a unary function symbol and c is a constant symbol.

We use L, L', L_1, L_2, etc. to denote languages.

To make things ultimately simpler (though it might not seem like that at first) we will use $x_1, x_2, x_3, \ldots$ for *free variables*, that is variables which are not linked to a quantifier, and $w_1, w_2, w_3, \ldots$ for *bound variables*, that is variables which are linked with a quantifier.

In order to avoid a flood of notation too early on we shall start by limiting ourselves to *relational languages*, that is languages which have no function, constant symbols, nor equality.

Definition For L a (relational) language the *formulae* of L are defined as follows:

Ll If R is an n-ary relation symbol of L and $x_{i_1}, x_{i_2}, \ldots, x_{i_n}$ (not necessarily distinct) come from the set of free variables $\{x_1, x_2, x_3, \ldots\}$ then $R(x_{i_1}, x_{i_2}, \ldots, x_{i_n})$ is a formula of L.

L2 If θ, ϕ are formulae of L then so are $(\theta \to \phi)$, $(\theta \wedge \phi)$, $(\theta \vee \phi)$, $\neg\theta$.

L3 If ϕ is a formula of L which does not mention w_j and $\phi(w_j / x_i)$ is the result of replacing the free variable x_i everywhere in ϕ by the bound variable w_j then $\exists w_j\, \phi(w_j / x_i)$, $\forall w_j\, \phi(w_j / x_i)$ are formulae of L.

L4 ϕ is a formulae of L just if this follows in a finite number of steps from Ll-3.

We denote the set of all formulae of L by FL. We use θ, ϕ, ψ, χ etc. to denote formulae and Γ, Δ, Ω etc. to denote sets of formulae, possibly empty. Notice that in L3 since we have infinitely many bound variables available and any one formula only mentions finitely many bound (or free) variables we can always pick one which doesn't appear already.

Example

In this example let the language $L = \{P, R\}$ where P is a unary relation symbol and R a ternary relation symbol Then.

1. $R(x_3, x_3, x_1)$ is a formula of L, equivalently $R(x_3, x_3, x_1) \in FL$, by Ll with $i_1 = i_2 = 3, i_3 = 1$. Similarly $P(x_1) \in FL$.
2. From 1 and L3, $\exists w_1\, R(w_1, w_1, x_1) \in FL$.
3. From 1, 2 and L2 $(\exists w_1\, R(w_1, w_1, x_1) \to P(x_1)) \in FL$.
4. From 3 and L3, $\forall w_2(\exists w_1\, R(w_1, w_1, w_2) \to P(w_2)) \in FL$.

Generally to show that some expression/word is a formula of L you need to demonstrate that it can be *constructed* from the relation symbols of L using Ll-3.

To show that some expression is *not* a formula of L the following observation is valuable (and will find many more applications as we proceed):

Every formula θ is actually just a finite string of symbols so we can talk about its *length*, $|\theta|$, meaning the number of symbols in θ where x_i, w_i, $\wedge$, $\neg$, $\to$, $\vee$, $\exists$, $\forall$, (,), R (for R a relation symbol of L) all count as single symbols (commas don't count). So for example

$$|(\exists w_1 R(w_1, w_1, x_1) \to P(x_1))| = 15.$$

A common way of proving that *all* formulae have some property $\mathcal{P}$ is to prove it *by induction in the length of formulae*. That is we show that if all formulae of length less than n have property $\mathcal{P}$ then all formulae of length n also have $\mathcal{P}$, and hence all formulae of length less than $n+1$ have $\mathcal{P}$. (Notice that the 'base case', that all formulae of length less than 0 have $\mathcal{P}$ is trivial true – they all do because there aren't any!) If we can show this then by induction 'for all n all formulae of length less than n have $\mathcal{P}$', so all formulae have $\mathcal{P}$. In fact in practice we do not even need to make n explicit as the following example shows.

Example For L as in the examples $(P(x_1)$ is not a formula of L.
To see this let $\mathcal{P}$ be the property of having the same number of left parentheses '(' as right parentheses ')'. Suppose $\theta \in FL$, and every formulae of length less than $|\theta|$ has $\mathcal{P}$. There are 7 cases:

θ is $R(\vec{x})$ for some relation symbol R of L.

θ is one of $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \to \psi)$ for some $\phi, \psi \in FL$.

θ is one of $\exists w_j\, \chi(w_j/x_i)$, $\forall w_j\, \chi(w_j/x_i)$ for some $\chi \in FL$.

By Inductive Hypothesis the ϕ, ψ, χ (being shorter than θ) contain the same number of right as left round brackets so clearly this also must hold for θ in all 7 cases.

Hence by induction on the length of formulae it must be true for all formulae. But it is not true for $(P(x_1)$ so this cannot be a formula of L.

Reading formulae

We 'read' formulae in the obvious way, for example

$\neg(P(x_1) \wedge P(x_2))$	Not (pause) *P* of x_1 and *P* of x_2
$(\neg P(x_1) \wedge P(x_2))$	Not *P* of x_1 (pause) and *P* of x_2
$\forall w_2(\exists w_1\, R(w_1, w_1, w_2) \to P(w_2))$	For every w_2, if there exists w_1 such that *R* of w_1, w_1, w_2 then *P* of w_2
$\forall w_2 \exists w_1 (R(w_1, w_1, w_2) \to P(w_2))$	For every w_2 there exists w_1 such that if *R* of w_1, w_1, w_2 then *P* of w_2

Notice the difference in the first two formulae above. In the first we first take the conjunction then negate it. In the second we first negate $P(x_1)$ and then take its conjunction with $P(x_2)$. It is the parentheses which enable us to make such expressions unambiguous. For example without it $\neg P(x_1) \wedge P(x_2)$ could have two different readings.

That the use of brackets as we have them really does succeed in avoiding any ambiguity in reading formulae is confirmed by the following theorem.

The Unique Readability Theorem 1 *Let $\theta \in FL$. Then exactly one of the following hold and furthermore in each case the $R, \vec{x}, \phi, \psi, w_j, \eta(w_j/x_i)$ etc. are themselves unique*:

1) $\theta = R(x_{i_1}, x_{i_2}, \ldots, x_{i_r})$ *for some r-ary-relation symbol R of L,*
2) $\theta = \neg\phi$ *for some* $\phi \in FL$,
3) $\theta = (\phi \wedge \psi)$ *for some* $\phi, \psi \in FL$,
4) $\theta = (\phi \vee \psi)$ *for some* $\phi, \psi \in FL$,
5) $\theta = (\phi \to \psi)$ *for some* $\phi, \psi \in FL$,
6) $\theta = \exists w_j\, \eta(w_j/x_i)$ *for some w_j and $\eta \in FL$ with w_j not occurring in η,*
7) $\theta = \forall w_j\, \eta(w_j/x_i)$ *for some w_j and $\eta \in FL$ with w_j not occurring in η.*

Proof* The proof is by induction on the length of $\theta \in FL$. Assume the result (and uniqueness) for all formulae of length less than $|\theta|$.

Since $\theta \in FL$, θ must be of at least[6] one of the forms

i) $R(x_{i_1}, x_{i_2}, \ldots, x_{i_r})$ for some *r*-ary-relation symbol of L,
ii) $\neg\phi, (\phi \wedge \psi), (\phi \vee \psi), (\phi \to \psi)$ for some $\phi, \psi \in FL$,
iii) $\exists w_j\, \eta(w_j/x_i)$, $\forall w_j\, \eta(w_j/x_i)$ for some w_j and $\eta \in FL$, with w_j not occurring in η.

If θ (as a sequence or symbols, i.e. word) starts with a relation symbol R then we must be in case (i) and the R, and after that the $x_{i_1}, x_{i_2}, \ldots, x_{i_r}$ (in that order), are uniquely determined by θ.

If θ starts with $\neg$ the only possibility is that $\theta = \neg\eta$ with $\eta \in FL$ and again θ uniquely determines η. Similarly in cases (iii).

So suppose that θ starts with '('. By induction on the length of formulae we can show that any formula which starts with '(' ends with ')' and is of the form $(\zeta \star \eta)$ for $\star \in \{\wedge, \vee, \to\}$ and $\zeta, \eta \in FL$ and what we have to prove is that θ cannot be written like this in two different ways. So suppose it could, say

$$\theta = (\gamma \star \delta) = (\lambda \dagger \tau)$$

where $\gamma, \delta, \lambda, \tau \in FL, \star, \dagger \in \{\wedge, \vee, \to\}$ and $\gamma = \lambda$. Notice that if $|\gamma| = |\lambda|$ then $\gamma = \lambda$ and hence also $\star = \dagger$ and $\delta = \tau$. So without loss of generality assume that $|\gamma| < |\lambda|$. Then the explicitly exhibited connective $\star$ must occur as a symbol in λ, say that $\lambda = \sigma \star \beta$ where σ, β are words. Clearly we must have $\sigma = \gamma$, so $\lambda = \gamma \star \beta$.

We now obtain our desired contradiction by establishing two properties of formulae by induction on the length. This first, which has already been proved in the notes in fact, is that if $\phi \in FL$ then the number l_ϕ of left parentheses in ϕ is the same as the number r_ϕ of right parentheses in ϕ. In particular then $l_\lambda = r_\lambda$. The second property is that if $\phi \in FL$ and we consider a particular occurrence of a connective, $\diamond$ say, in ϕ, so $\phi = \nu \diamond \varepsilon$ for some strings of symbols ν, ε, then $l_\nu > r_\nu$. [You are left to establish this fact.] Hence since $\lambda \in FL$ and $\lambda = \gamma \star \beta, l_\gamma > r_\gamma$, contradicting $l_\lambda = r_\lambda$. ■

The Unique Readability Theorem provides a rather more sophisticated (and in fact foolproof) method for showing that a particular word, i.e. finite string of symbols, from L is not a formula of L. To illustrate this consider the word

$$(R(x_1, x_1) \to (R(x_1, x_1)) \to R(x_1, x_1)) \tag{1}$$

If this was a formula of L then by case (5) of Unique Readability the only possibility is that either $R(x_1, x_1)$, $(R(x_1, x_1)) \to R(x_1, x_1)$ are both in FL or $R(x_1, x_1) \to (R(x_1, x_1))$ and $R(x_1, x_1)$ are in FL. But $R(x_1, x_1) \to (R(x_1, x_1))$ does not fall under any of the cases of Unique Readability, so it would have to be the case that $(R(x_1, x_1)) \to R(x_1, x_1) \in FL$. But the only case (5) could apply again and $R(x_1, x_1$ would have to be a formula, which it is not since it does not fall under any of the Readability cases. It follows that (1) cannot be in FL.

In fact this method of repeatedly breaking down a word provides a foolproof test of formulahood in that if it does not demonstrate that the word is not a formula then reversing the analysis yields a construction of the word which confirms that it is a formula.

It may appear at this point that we have been excessively fussy about the precise structures to which formulae need to conform and that this doesn't really have much to do with logic. In response we would point out that at this stage it is most important to be able to write correct formulae, and recognize incorrect 'formulae', in order to avoid any possibility of ambiguity. In the logic you meet beyond this course you may be able to take liberties but, like the driving test, you need to start off by knowing and abiding by the rules.

Having emphasized the importance of parentheses we now mention a common abbreviation: In dealing with formulae $(\theta \to \phi)$, $(\theta \vee \phi)$, $(\theta \wedge \phi)$ in we may temporarily drop the *outermost* parentheses, so writing instead $\theta \to \phi$, $\theta \vee \phi$, $\theta \wedge \phi$, where this can cause no confusion.

Notation If ϕ is a formula of L and the free variables appearing in ϕ are amongst[7] $x_{i_1}, x_{i_2}, \ldots, x_{i_n}$ then we may write $\phi(x_{i_1}, x_{i_2}, \ldots, x_{i_n})$ (or $\phi(\vec{x})$) for ϕ(where $\vec{x} = x_{i_1}, x_{i_2}, \ldots, x_{i_n}$). In this case $\phi(t_1, t_2, \ldots, t_n)$ is to be the result of (simultaneously) replacing each x_{i_j} in ϕ by t_j.[8] So for example if ϕ is

$$\forall w_2 (R(x_1, x_3, w_2) \wedge \neg P(x_3))$$

then we might write ϕ as $\phi(x_1, x_3)$, in which case $\phi(t_1, t_2)$ would be

$$\forall w_2 (R(t_1, t_2, w_2) \wedge \neg P(t_2)) .$$

Notice then that with this notation L3 can be written as:

> If $\phi(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$ is a formula of L which does not mention w_j then $\exists w_j\, \phi(x_1, x_2, \ldots, x_{i-1}, w_j, x_{i+1}, \ldots, x_n)$, $\forall w_j\, \phi(x_1, x_2, \ldots, x_{i-1}, w_j, x_{i+1}, \ldots, x_n)$ are formulae of L.

Convention If we quantify a formula $\theta(x_1, \vec{x})$ to get, say, $\exists w_j\, \theta(w_j, \vec{x})$ you should take it as read that w_j does not already appear in $\theta(x_1, \vec{x})$ – so $\exists w_j\, \theta(w_j, \vec{x})$ is again a formula.[9] [For emphasis however we may sometimes still mention this assumed non-occurrence.]

Referring back to the question at the end of the previous section, we now know what the $\theta_1, \ldots, \theta_n, \phi$ are, namely formulae of a language L. We now come to clarify the second part of that question.

Truth

Let L be a relational language. We have seen from the introductory motivation section that, for example, we can give a meaning, or *semantics*, to a formula such as $\exists w_1 \, \forall w_2 \, Q(w_1, w_2)$ by interpreting the bound variables w_1, w_2 as ranging over some *universe* (such as the set of natural numbers $\mathbb{N}$), interpreting the free variables x_i as elements of this universe, interpreting the binary relation symbol Q as a binary relation (such as 'greater than') on this universe, and interpreting the quantifiers and connectives in the obvious way appropriate to their names. We can then talk about a formula being *true in this interpretation*.

For example, with this interpretation of Q etc. and interpreting x_1 as the number $2 \in \mathbb{N}$,

$$\exists w_2 \, Q(w_2, x_1)$$

is true since there does exist a number $w_2 \in \mathbb{N}$ such that w_2 is greater than 2.

However with this same interpretation

$$\forall w_1 \, \exists w_2 \, Q(w_1, w_2)$$

is false since it is not the case that for every $w_1 \in \mathbb{N}$ there is a $w_2 \in \mathbb{N}$ such that w_1 is greater than w_2 (because this fails for $w_1 = 0$).

We now want to make precise what we mean by an 'interpretation' To do that we first need to say what we mean by a 'relation' on a non-empty set A.

In the example given above we have interpreted Q as the binary relation of 'greater than' between natural numbers. Now clearly we could identify

$$\left.\begin{array}{l}\text{'greater than' between}\\ \text{natural numbers}\end{array}\right\} \equiv \left\{\begin{array}{l}\text{the set}\\ \{\langle n, m\rangle \in \mathbb{N}\times\mathbb{N} \mid n > m\}\end{array}\right.$$

In other words we can think of the relation of 'greater than' as a specific subset of $\mathbb{N}^2$. But this is a quite general phenomenon, we can identify any n-ary relation $\mathcal{R}$ on A with a subset of A^n, namely the subset

$$\{\langle a_1, a_2, \ldots, a_n\rangle \in A^n \mid \mathcal{R}(a_1, a_2, \ldots, a_n)\}.$$

Conversely any subset S of A^n determines an n -ary relation on A , namely the relation $\mathcal{S}$ such that

$$\mathcal{S}(a_1, a_2, \ldots, a_n)\,\text{holds} \quad \Leftrightarrow \quad \langle a_1, a_2, \ldots, a_n\rangle \in S.$$

The upshot of all this is that we now see that effectively n -ary relations on A and subsets of A^n *are the same thing*. Realizing this our definition of an interpretation becomes much easier to state.

It turns out (for reasons which hopefully will be clear later) that it is best to split this notion of an interpretation into two parts, the interpretation of the universe and the relations of L and the interpretation of the free variables. The former we call a *structure for* L:

Definition

A *structure* M for a relational language L consists of:

- a non-empty set[10] $|M|$, called the *universe* (or *domain*) of M ,
- for each n -ary relation symbol R of L a subset R^M of $|M|^n$ (equivalently an n -ary relation on $|M|$)[11]

In this case we sometimes write

$$M = \langle |M|, R_1^M, R_2^M, \ldots\rangle$$

where $R_1, R_2, \ldots$ are the relation symbols in L .

Examples

Let $L = \{P, Q\}$ with P 1-ary and Q 2-ary.

Then examples of structures for L are:

a) Universe of M is $\mathbb{N}$, i.e. $|M| = \mathbb{N}$,

$$Q^M = \{\langle n, m\rangle \in \mathbb{N}^2 \mid n > m\},$$

$$P^M = \{\, n \in \mathbb{N} \mid n \,\text{is prime}\,\}.$$

b) Universe of M is $\mathbb{R}$,

$$Q^M = \left\{ \langle s, t \rangle \in \mathbb{R}^2 \mid s^2 = t + 5 \right\},$$

$$P^M = \left\{ s \in \mathbb{R} \mid s \text{ is rational} \right\} = \mathbb{Q}.$$

c) Universe of M is {1, 2, 3},

$$Q^M = \{\langle 1,1 \rangle, \langle 1,2 \rangle, \langle 3,2 \rangle, \langle 2,3 \rangle\},$$

$$P^M = \emptyset.$$

The second part of the 'interpretation', the interpretation of the free variables x_i as elements of the universe of the structure M, we shall refer to as an *assignment*, possibly writing $x_i \mapsto a_i$ to indicate that the variable x_i is being *assigned* the value $a_i \in |M|$, or being *interpreted* as $a_i \in |M|$.

We are now ready to clarify the third 'unknown' in the last paragraph of the initial 'motivation' section, what it means for a formula to be true in an interpretation.

Recall that for a relational language L we have split an 'interpretation' into two parts: a structure for L and an assignment of elements in the universe of that structure to the free variables. Given a formula $\phi(x_1, x_2, \ldots, x_n)$ of L we now wish to define

$$\left.\begin{array}{l}\phi(x_1, x_2, \ldots, x_n) \text{ is true in the} \\ \text{structure } M \text{ for } L \text{ when the} \\ x_1, x_2, \ldots, x_n \text{ are assigned values} \\ a_1, a_2, \ldots, a_n \text{ resp. from the} \\ \text{universe } |M| \text{ of } M\end{array}\right\} \text{ written } M \vDash \phi(a_1, a_2, \ldots, a_n).$$

[Recall that when we write ϕ as $\phi(x_1, x_2, \ldots, x_n)$ it is implicit that all the free variables mentioned in ϕ are amongst $x_1, x_2, \ldots, x_n$ though they do not necessarily all need to actually occur in ϕ.]

For a fixed structure M for L, with universe $|M|$, and any choice of assignment $x_i \mapsto a_i$ to the free variables, we define

$$M \vDash \eta(a_1, a_2, \ldots, a_n)$$

by induction on the length of $\eta(\vec{x}) \in FL$ (for all assignments simultaneously) in the obvious way:

T1 For $R(x_{i_1}, x_{i_2}, \ldots, x_{i_n}) \in FL$, where R is an n-ary relation symbol in L,

$$\begin{aligned} M \vDash R(a_{i_1}, a_{i_2}, \ldots, a_{i_n}) \quad &\Leftrightarrow \quad \langle a_{i_1}, a_{i_2}, \ldots, a_{i_n} \rangle \in R^M \\ &\Leftrightarrow \quad \text{the relation interpreting } R \text{ in } M \\ &\qquad \text{holds for } a_{i_1}, a_{i_2}, \ldots, a_{i_n}. \end{aligned}$$

T2 For formulae $\theta(x_1, x_2, \ldots, x_n)$, $\phi(x_1, x_2, \ldots, x_n)$ etc. of L and $\vec{a} = a_1, \ldots, a_n \in |M|$,

$$\begin{array}{lcl} M \vDash \neg\phi(\vec{a}) & \Leftrightarrow & \text{not } M \vDash \phi(\vec{a}), \text{ i.e. } M \nvDash \phi(\vec{a}) \\ M \vDash \theta(\vec{a}) \wedge \phi(\vec{a}) & \Leftrightarrow & M \vDash \theta(\vec{a}) \text{ and } M \vDash \phi(\vec{a}) \\ M \vDash \theta(\vec{a}) \vee \phi(\vec{a}) & \Leftrightarrow & M \vDash \theta(\vec{a}) \text{ or } M \vDash \phi(\vec{a}) \\ M \vDash \theta(\vec{a}) \rightarrow \phi(\vec{a}) & \Leftrightarrow & M \nvDash \theta(\vec{a}) \text{ or } M \vDash \phi(\vec{a}). \end{array}$$

T3

$$M \vDash \forall w_j\, \psi(w_j, \vec{a}) \quad \Leftrightarrow \qquad \text{For all } b \in |M|,\ M \vDash \psi(b, \vec{a}).$$
$$M \vDash \exists w_j\, \psi(w_j, \vec{a}) \quad \Leftrightarrow \quad \text{For some } b \in |M|,\ M \vDash \psi(b, \vec{a}).$$

Notice that by Theorem 1 (Unique Readability) for a given formula $\eta(\vec{x})$ exactly one of the above cases applies and hence whether or not $M \vDash \eta(\vec{a})$ is unambiguously determined by **Tl-3**.

Notation If $M \vDash \phi(a_1, a_2, \ldots, a_n)$ we say that $\phi(a_1, a_2, \ldots, a_n)$ is *true in* M or that $\phi(x_1, x_2, \ldots, x_n)$ is *satisffied* by $a_1, a_2, \ldots, a_n$ in M .

Examples

1. Let M be as in (a) above, so the universe of $|M|$ is $\mathbb{N}$, P^M is the set of primes and

$$Q^M = \{\langle n, m\rangle \in \mathbb{N}^2 \mid n > m\}.$$

Then using Tl, $M \vDash P(7)$ since $7 \in P^M$, i.e. 7 is a prime. Also $M \nvDash Q(4,7)$ since $\langle 4, 7\rangle \notin Q^M$, i.e. not $(4 > 7)$, so by T2, $M \vDash \neg Q(4,7)$.

Hence by T2,

$$M \vDash P(7) \wedge \neg Q(4,7)$$

and[12] by T3,

$$M \vDash \exists w_2 \left(P(w_2) \wedge \neg Q(4, w_2)\right).$$

In the above example we have moved from simple to more complicated formulae. However in practice when checking if a formula is true in an interpretation we usually start at the complicated end and successively break it down using T1-T3 until we (hopefully) reach a stage where we can 'see' whether or not it is true. For example

$$\begin{aligned} & M \vDash \forall w_1 \exists w_2 (Q(w_2, w_1) \wedge P(w_2)) \\ & \quad \Leftrightarrow \text{ for all } m \in \mathbb{N},\ M \vDash \exists w_2 (Q(w_2, m) \wedge P(w_2)), \text{by T} \\ & \quad \Leftrightarrow \text{ for all } m \in \mathbb{N}, \text{there is some } n \in \mathbb{N} \text{ such that} \\ & \qquad M \vDash Q(n, m) \wedge P(n), T3, \end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow \quad \text{for all } m \in \mathbb{N}, \text{there is some } n \in \mathbb{N} \text{ such} \\
&\qquad \text{that } M \vDash Q(n, m) \text{ and } M \vDash P(n), \text{by T2} \\
&\Leftrightarrow \quad \text{for all } m \in \mathbb{N}, \text{there is some } n \in \mathbb{N} \text{ such} \\
&\qquad \text{that } \langle n, m \rangle \in Q^M \text{ and } n \in P^M, \text{by T1,} \\
&\Leftrightarrow \quad \text{for all } m \in \mathbb{N}, \text{there is some } n \in \mathbb{N} \text{ such} \\
&\qquad \text{that } n > m \text{ and } n \text{ is prime,}
\end{aligned}$$

– which is true, there are unboundedly many primes.

2. Let M be as in (c) above, so M is {1, 2, 3}, $P^M = \emptyset$.

$$Q^M = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 3,2\rangle, \langle 2,3\rangle\}.$$

Then $M \vDash Q(3,2)$ since $\langle 3, 2\rangle \in Q^M$ but $M \nvDash Q(2,1)$ (so $M \vDash \neg Q(2,1)$) since $\langle 2, 1\rangle \notin Q^M$. Hence by T3,

$$M \vDash \exists w_1\, Q(3, w_1). \tag{2}$$

Similarly since $\langle 1, 2\rangle, \langle 2, 3\rangle \in Q^M$, $M \vDash Q(1,2)$ and $M \vDash Q(2,3)$, and hence

$$M \vDash \exists w_1\, Q(1, w_1) \text{ and } M \vDash \exists w_1\, Q(2, w_1). \tag{3}$$

Finally, since 1, 2, 3 are all the elements in the universe of M, from these we obtain from (2) and (3),

$$M \vDash \forall w_2 \exists w_1\, Q(w_2, w_1).$$

We are now ready to put these three features, formulae, interpretation, truth, together to capture our initial intuitions about 'logical consequence'.

Logical Consequence

Definition Let L be a relational language, Γ a set (possibly empty) of formulae of L (*i.e.* $\Gamma \subseteq FL$) and $\theta \in FL$. Then θ is a *logical consequence* of Γ (equivalently Γ *logically implies* θ), denoted $\Gamma \vDash \theta$, if for any structure M for L and any assignment of elements of $|M|$ to the free variables $x_1, x_2, \ldots$ appearing in the formulae in Γ or θ, **if** every formula in Γ is true in that interpretation **then** θ is true in that interpretation.[13]

So, for example if

$$\Gamma = \{\phi_1(x_1, x_2, \ldots, x_{\mathrm{n}}), \phi_2(x_1, x_2, \ldots, x_{\mathrm{n}}), \ldots, \phi_{\mathrm{m}}(x_1, x_2, \ldots, x_{\mathrm{n}})\}$$

then

$$\Gamma \vDash \theta(x_1, x_2, \ldots, x_n) \quad \Leftrightarrow \quad \begin{array}{l} \text{for all structures } M \text{ for } L \text{ and for all} \\ a_1, a_2, \ldots a_n \text{ in the universe of } M, \text{if} \\ M \vDash \phi_i(a_1, a_2, \ldots, a_n) \text{ for } i = 1, 2, \ldots, m \\ \text{then } M \vDash \theta(a_1, a_2, \ldots, a_n). \end{array}$$

In the case $\Gamma = \emptyset$ we usually write $\vDash \theta$ instead of $\emptyset \vDash \theta$. Notice that in this case since every formula in the empty set is true in any interpretation (because there aren't any!) $\vDash \theta(x_1, \ldots, x_n)$ holds just if for every structure M for L and $a_1, \ldots, a_n \in |M|$, $M \vDash \theta(a_1, \ldots, a_n)$.[14] A formula θ with this property is known as a *tautology*. A formula which is false in all interpretations (equivalently its negation is a tautology) is referred as a *contradiction*. An example of a tautology is $(\phi \vee \neg\phi)$, and an example of a contradiction is $(\phi \wedge \neg\phi)$, for $\phi \in FL$.

Examples In what follows take it as read that ϕ, θ etc. are formula from a relational language L and Γ is a set of formulae from L, equivalently $\Gamma \subseteq FL$.

1. $\phi(x_1, x_2, x_3, \ldots, x_n) \vDash \exists w_1\, \phi(w_1, x_2, x_3, \ldots, x_n)$.[15]

Proof Let M be a structure for L with universe $|M|$ and $a_1, a_2, \ldots, a_n \in |M|$. Suppose that

$$M \vDash \phi(a_1, a_2, \ldots, a_n).$$

Then certainly for some $b \in |M|$,

$$M \vDash \phi(b, a_2, a_3, \ldots, a_n),$$

namely $b = a_1$ will do, so by T3

$$M \vDash \exists w_1\, \phi(w_1, a_2, a_3, \ldots, a_n).$$

Since M was an arbitrary structure for L and $a_1, a_2, \ldots, a_n$ arbitrary elements of the universe of M the required result follows.

2. $\forall w_1\, \phi(w_1, x_2, x_3, \ldots, x_n) \vDash \phi(x_1, x_2, x_3, \ldots, x_n)$.

Proof Let M be a structure for L with universe $|M|$ and $a_1, a_2, \ldots, a_n \in |M|$. Suppose that

$$M \vDash \forall w_1\, \phi(w_1, a_2, \ldots, a_n)\,.$$

Then from T3, for all $b \in |M|$,

$$M \vDash \phi(b, a_2, a_3, \ldots, a_n)\,.$$

In particular

$$M \vDash \phi(a_1, a_2, a_3, \ldots, a_n)\,,$$

from which the required result follows.

3. $\exists w_1 \phi(w_1, \vec{x}),\ \forall w_1 (\phi(w_1, \vec{x}) \to \theta(w_1, \vec{x})) \vDash \exists w_1\, \theta(w_1, \vec{x})$,

where $\vec{x} = x_1, x_2, x_3, \ldots, x_n$.

Proof Let M be a structure for L with universe $|M|$ and $\vec{a} = a_1, a_2, \ldots, a_n \in |M|$. Suppose that

$$M \vDash \exists w_1\, \phi(w_1, \vec{a}), \tag{4}$$

$$M \vDash \forall w_1 (\phi(w_1, \vec{a}) \to \theta(w_1, \vec{a})). \tag{5}$$

Then from (4) and T3, for some $b \in |M|$,

$$M \vDash \phi(b, \vec{a}). \tag{6}$$

From (5) and T3,

$$M \vDash \phi(b, \vec{a}) \to \theta(b, \vec{a}).$$

From T2,

$$M \nvDash \phi(b, \vec{a}) \text{ or } M \vDash \theta(b, \vec{a}).$$

By (6) the first of these doesn't hold so it must be the case that

$$M \vDash \theta(b, \vec{a}).$$

T3 now gives that

$$M \vDash \exists w_1\, \theta(w_1, \vec{a}),$$

as required.

Note that there was nothing special about the choice of variable w_1 here, we could in general have been using w_j.

Another Example

$$\Gamma \vDash \theta(\vec{x}) \to \phi(\vec{x}) \Leftrightarrow \Gamma, \theta(\vec{x}) \vDash \phi(\vec{x})$$ [16]

Proof Assume that

$$\Gamma \vDash \theta(\vec{x}) \to \phi(\vec{x}), \tag{7}$$

so we want first to show that

$$\Gamma, \theta(\vec{x}) \vDash \phi(\vec{x}).$$

To this end let M be a structure for L with universe $|M|$ and suppose we have some assignment to the free variable such that $\vec{x} \mapsto \vec{a}$ and under this interpretation every formula in Γ is true and $\theta(\vec{a})$ is true. Then

$$M \vDash \theta(\vec{a}) \tag{8}$$

and from (7), since even formula in Γ is true in this interpretation,

$$M \vDash \theta(\vec{a}) \to \phi(\vec{a}).$$

By T2 then,

$$M \nvDash \theta(\vec{a}) \text{ or } M \vDash \phi(\vec{a}).$$

Using (8) we must have $M \vDash \phi(\vec{a})$.

In summary then we have shown that if all the formulae in Γ and $\theta(\vec{x})$ are true in an interpretation then so is $\phi(\vec{x})$. Hence

$$\Gamma, \theta(\vec{x}) \vDash \phi(\vec{x}).$$

Conversely assume that

$$\Gamma, \theta(\vec{x}) \vDash \phi(\vec{x}). \tag{9}$$

We wish to show that

$$\Gamma \vDash \theta(\vec{x}) \to \phi(\vec{x}).$$

So suppose we have a structure M and an assignment to the free variables (where $\vec{x} \mapsto \vec{a}$) under which every formula in Γ is true. There are now two cases.

Case 1: $M \vDash \theta(\vec{a})$.

In this case every formula in Γ along with $\theta(\vec{x})$ is true under this interpretation so from (9),

$$M \vDash \phi(\vec{a}).$$

Hence (trivially)

$$M \nvDash \theta(\vec{a}) \text{ or } M \vDash \phi(\vec{a})$$

so from T2

$$M \vDash \theta(\vec{a}) \rightarrow \phi(\vec{a}).$$

Case 2: $M \nvDash \theta(\vec{a})$.

In this case again (trivially)

$$M \nvDash \theta(\vec{a}) \text{ or } M \vDash \phi(\vec{a})$$

so from T2

$$M \vDash \theta(\vec{a}) \to \phi(\vec{a}).$$

Either way then

$$M \vDash \theta(\vec{a}) \to \phi(\vec{a}).$$

In summary what we have shown then is that under assumption (9) if we have a structure and an assignment to the free variables in which every formula in Γ is true then $\theta(\vec{x}) \to \phi(\vec{x})$ is also true under that interpretation, i.e.

$$\Gamma \vDash \theta(\vec{x}) \to \phi(\vec{x}),$$

as required.

We have now given several examples of demonstrating that some logical implication does hold. Conversely to show that $\Gamma \vDash \theta$ does not hold, denoted $\Gamma \nvDash \theta$, it is enough to find a structure and an assignment to the free variables as elements of the universe of that structure in which every formula in Γ is true but θ is false.

Example

$$\exists w_1 \exists w_2\, R(w_1, w_2) \nvDash \exists w_1\, R(w_1, w_1).$$

Proof Let M be a structure for L with universe $\{0,1\}$ and let $R^M = \{\langle 0,1\rangle\}$ (we don't need to bother here about any assignment to the free variables –because there aren't any!).

Then $M \vDash R(0,1)$ so $M \vDash \exists w_1 \exists w_2\, R(w_1, w_2)$. However if $M \vDash \exists w_1\, R(w_1, w_1)$ we would have to have

$$M \vDash R(0,0) \text{ or } M \vDash R(1,1),$$

equivalently

$$\langle 0,0\rangle \in R^M \text{ or } \langle 1, 1\rangle \in R^M$$

both of which are false. Hence

$$M \nvDash \exists w_1\, R(w_1, w_1),$$

giving the required counter-example to

$$\exists w_1 \exists w_2\, R(w_1, w_2) \vDash \exists w_1\, R(w_1, w_1).$$

Sentences

Notice that in this last example we did not need to bother about the assignment to free variables because there were none involved.

A formula of L without free variables is called a *sentence* of L. So for example $\forall w_2(\exists w_1\, R(w_1, w_1, w_2) \to P(w_2))$ is a sentence whereas $(\exists w_1\, R(w_1, w_1, x_1) \to P(x_1))$ is a formula but not a sentence (because a free variable, x_1 in this case, occurs in it).

We denote the set of sentences of L by SL.

In most applications of logic we deal with sentences, in which case the assignment of values to free variables doesn't figure and we only need talk about truth in a structure.[17] So if $\theta \in SL$ it makes sense to write $M \vDash \theta$ without specifying any assignment of values to the (non-existent!) free variables. In this case we say that M is a *model* of θ. Similarly if $\Gamma \subseteq SL$ and $M \vDash \theta$ for every $\theta \in \Gamma$ we say that M is a model of Γ and write $M \vDash \Gamma$.

Very often a proof given for sentences trivially generalizes to formulae, as we shall now see.

Example

If $\Gamma, \Delta \subseteq SL$ and $\theta, \phi, \psi \in SL$ and $\Gamma, \theta \vDash \psi$ and $\Delta, \phi \vDash \psi$ then[18] $\Gamma, \Delta, \theta \vee \phi \vDash \psi$.

Proof Let M be a structure for L such that $M \vDash \Gamma \cup \Delta \cup \{\theta \vee \phi\}$, meaning that $M \vDash \eta$ for every sentence $\eta \in \Gamma \cup \Delta \cup \{\theta \vee \phi\}$. Then $M \vDash \Gamma, M \vDash \Delta$ and $M \vDash \theta \vee \phi$, so from T2 either $M \vDash \theta$ or $M \vDash \phi$. Without loss of generality assume $M \vDash \theta$ (since there is complete symmetry here between Γ, θ and Δ, ϕ). Then $M \vDash \Gamma \cup \{\theta\}$ so since $\Gamma, \theta \vDash \psi, M \vDash \psi$. Hence

$$\Gamma, \Delta, \theta \vee \phi \vDash \psi.$$

Logical Equivalence

Definition Formulae $\theta(\vec{x}), \phi(\vec{x}) \in FL$, are *logically equivalent*, written $\theta(\vec{x}) \equiv \phi(\vec{x})$, if for all structures M for L and $\vec{a}$ from $|M|$,

$$M \vDash \theta(\vec{a}) \Leftrightarrow M \vDash \phi(\vec{a}).$$

Notice that

$$
\begin{aligned}
\theta(\vec{x}) \equiv \phi(\vec{x}) \quad &\Leftrightarrow \quad \forall M, \vec{a},\ [M \vDash \theta(\vec{a}) \Rightarrow M \vDash \phi(\vec{a})] \\
&\qquad \text{and } [M \vDash \phi(\vec{a}) \Rightarrow M \vDash \theta(\vec{a})] \\
&\Leftrightarrow \quad \forall M, \vec{a}, [M \vDash (\theta(\vec{a}) \to \phi(\vec{a}))] \\
&\qquad \text{and } [M \vDash (\phi(\vec{a}) \to \theta(\vec{a}))] \\
&\Leftrightarrow \quad \forall M, \vec{a}, [M \vDash (\theta(\vec{a}) \leftrightarrow \phi(\vec{a}))] \\
&\Leftrightarrow \quad \vDash (\theta(\vec{x}) \leftrightarrow \phi(\vec{x})) \\
&\Leftrightarrow \quad \vDash (\theta(\vec{x}) \to \phi(\vec{x}))\ \ \&\ \vDash (\phi(\vec{x}) \to \theta(\vec{x})) \\
&\Leftrightarrow \quad \theta(\vec{x}) \vDash \phi(\vec{x})\ \ \&\ \ \phi(\vec{x}) \vDash \theta(\vec{x})
\end{aligned}
$$

where $(\theta \leftrightarrow \phi)$ is shorthand for $((\theta \to \phi) \wedge (\phi \to \theta))$.

Clearly $\equiv$ is an *equivalence relation*, that is it is:

Reflexive, i.e. it satisfies $\theta \equiv \theta$ for all $\theta \in FL$

Symmetric, i.e. it satisfies $\theta \equiv \phi \Rightarrow \phi \equiv \theta$ for all $\theta, \phi \in FL$,

Transitive, i.e. it satisfies $(\theta \equiv \phi \,\&\, \phi \equiv \psi) \Rightarrow \theta \equiv \psi$ for all $\theta, \phi, \psi \in FL$.

If two formulae are logically equivalent they 'say the same thing' or 'have the same meaning' in the sense that one is true just if the other is. Very often in logic this is the important relationship between formulae, rather than equality. For that reason it is important to be able to recognize some simple logically equivalent formulae:

Some useful logical equivalents

$$(\theta \wedge \phi) \equiv (\phi \wedge \theta) \qquad (\theta \vee \phi) \equiv (\phi \vee \theta)$$
$$\neg\neg\theta \equiv \theta \qquad (\theta \to \phi) \equiv (\neg\theta \vee \phi)$$
$$\neg(\theta \wedge \phi) \equiv (\neg\theta \vee \neg\phi) \qquad \neg(\theta \vee \phi) \equiv (\neg\theta \wedge \neg\phi)$$
$$\neg(\theta \to \phi) \equiv (\theta \wedge \neg\phi) \qquad (\theta \to \phi) \equiv (\neg\phi \to \neg\theta)$$
$$\theta \equiv (\theta \wedge \phi) \vee (\theta \wedge \neg\phi) \qquad \theta \equiv (\theta \vee \phi) \wedge (\theta \vee \neg\phi)$$
$$\theta \vee (\phi \wedge \psi) \equiv (\theta \vee \phi) \wedge (\theta \vee \psi) \qquad \theta \wedge (\phi \vee \psi) \equiv (\theta \wedge \phi) \vee (\theta \wedge \psi)$$
$$\neg\exists w_j \theta(w_j, \vec{x}) \equiv \forall w_j \neg\theta(w_j, \vec{x}) \qquad \neg\forall w_j \theta(w_j, \vec{x}) \equiv \exists w_j \neg\theta(w_j, \vec{x})$$
$$\exists w_j \theta(w_j, \vec{x}) \equiv \exists w_k \theta(w_k, \vec{x}) \qquad \forall w_j \theta(w_j, \vec{x}) \equiv \forall w_k \theta(w_k, \vec{x})$$

$$\exists w_j \exists w_k\, \theta(w_j, w_k, \vec{x}) \equiv \exists w_k \exists w_j\, \theta(w_j, w_k, \vec{x})$$

$$\forall w_j \forall w_k\, \theta(w_j, w_k, \vec{x}) \equiv \forall w_k \forall w_j\, \theta(w_j, w_k, \vec{x})$$

$$\exists w_j\, (\psi(\vec{x}) \wedge \theta(w_j, \vec{x})) \equiv \psi(\vec{x}) \wedge \exists w_j\, \theta(w_j, \vec{x})$$

$$\forall w_j\, (\psi(\vec{x}) \wedge \theta(w_j, \vec{x})) \equiv \psi(\vec{x}) \wedge \forall w_j\, \theta(w_j, \vec{x})$$

$$\exists w_j\, (\psi(\vec{x}) \vee \theta(w_j, \vec{x})) \equiv \psi(\vec{x}) \vee \exists w_j\, \theta(w_j, \vec{x})$$

$$\forall w_j\, (\psi(\vec{x}) \vee \theta(w_j, \vec{x})) \equiv \psi(\vec{x}) \vee \forall w_j\, \theta(w_j, \vec{x})$$

$$\exists w_j\, (\psi(\vec{x}) \to \theta(w_j, \vec{x})) \equiv \psi(\vec{x}) \to \exists w_j\, \theta(w_j, \vec{x})$$

$$\forall w_j\, (\psi(\vec{x}) \to \theta(w_j, \vec{x})) \equiv \psi(\vec{x}) \to \forall w_j\, \theta(w_j, \vec{x})$$

$$\exists w_j\, (\theta(w_j, \vec{x}) \to \psi(\vec{x})) \equiv \forall w_j\, \theta(w_j, \vec{x}) \to \psi(\vec{x})$$

$$\forall w_j\, (\theta(w_j, \vec{x}) \to \psi(\vec{x})) \equiv \exists w_j\, \theta(w_j, \vec{x}) \to \psi(\vec{x})$$

where throughout w_j does not occur in $\psi(\vec{x})$ (and of course w_k does not occur in $\exists w_j\, \theta(w_j, \vec{x})$) .

These can be checked directly from the definition of $\equiv$. We give a couple of examples. Throughout let M be an arbitrary structure for L with $\vec{a}$ from $|M|$.

Then

$$
\begin{aligned}
M \vDash \neg(\theta(\vec{a}) \wedge \phi(\vec{a})) \quad &\Leftrightarrow \quad \text{not } M \vDash (\theta(\vec{a}) \wedge \phi(\vec{a})) \\
&\Leftrightarrow \quad \text{not}[M \vDash \theta(\vec{a}) \text{ and } M \vDash \phi(\vec{a})] \\
&\Leftrightarrow \quad \text{not}\, M \vDash \theta(\vec{a}) \text{ or not } M \vDash \phi(\vec{a} \\
&\Leftrightarrow \quad M \vDash \neg\theta(\vec{a}) \text{ or } M \vDash \neg\phi(\vec{a}) \\
&\Leftrightarrow \quad M \vDash (\neg\theta(\vec{a})\neg\phi(\vec{a}))\,.
\end{aligned}
$$

$$\therefore \quad \neg(\theta(\vec{x}) \vee \phi(\vec{x})) \equiv (\neg\theta(\vec{x}) \vee \neg\phi(\vec{x})).$$

$$M \vDash \exists w_j(\theta(w_j, \vec{a}) \to \psi(\vec{x}))$$

$$
\begin{aligned}
&\Leftrightarrow \quad \exists b \in |M|, (M \vDash (\theta(b, \vec{a}) \to \psi(\vec{a})) \\
&\Leftrightarrow \quad \exists b \in |M|, (M \nvDash \theta(b, \vec{a}) \text{ or } M \vDash \psi(\vec{a})) \\
&\Leftrightarrow \quad [\exists b \in |M|, M \nvDash \theta(b, \vec{a})] \text{ or } M \vDash \psi(\vec{a}) \\
&\Leftrightarrow \quad [\text{not } \forall b \in |M|, M \vDash \theta(b, \vec{a})] \text{ or } M \vDash \psi(\vec{a}) \\
&\Leftrightarrow \quad M \nvDash \forall w_j\, \theta(w_j, \vec{a}) \text{ or } M \vDash \psi(\vec{a}) \\
&\Leftrightarrow \quad M \vDash (\forall w_j\, \theta(w_j, \vec{a}) \to \psi(\vec{a})).
\end{aligned}
$$

$$\therefore \quad \exists w_j\,(\theta(w_j, \vec{x}) \to \psi(\vec{x})) \equiv (\forall w_j\, \theta(w_j, \vec{x}) \to \psi(\vec{x})).$$

Lemma 2

If $\theta_1 \equiv \theta_2$, $\phi_1 \equiv \phi_2$ *and* $\psi_1(x_i, \vec{x}) \equiv \psi_2(x_i, \vec{x})$ *then*[19]*:*

$$(\theta_1 \wedge \phi_1) \equiv (\theta_2 \wedge \phi_2), \qquad (\theta_1 \vee \phi_1) \equiv (\theta_2 \vee \phi_2),$$

$$(\theta_1 \to \phi_1) \equiv (\theta_2 \to \phi_2), \qquad \neg\theta_1 \equiv \neg\theta_2$$

$$\exists w_j\, \psi_1(w_j, \vec{x}) \equiv \exists w_j\, \psi_2(w_j, \vec{x}), \qquad \forall w_j\, \psi_1(w_j, \vec{x}) \equiv \forall w_j\, \psi_2(w_j, \vec{x})$$

Proof Let $\theta_1 = \theta_1(\vec{x})$ etc., M a structure for L and $\vec{a} \in |M|$. Then when $\theta_1 \equiv \theta_2$, $\phi_1 \equiv \phi_2$,

$$
\begin{aligned}
M \vDash \theta_1(\vec{a}) \wedge \phi_1(\vec{a}) \quad &\Leftrightarrow \quad M \vDash \theta_1(\vec{a}) \text{ and } M \vDash \phi_1(\vec{a}) \text{ by T2} \\
&\Leftrightarrow \quad M \vDash \theta_2(\vec{a}) \text{ and } M \vDash \phi_2(\vec{a}) \\
&\Leftrightarrow \quad M \vDash \theta_1(\vec{a}) \wedge \phi_1(\vec{a})\,,
\end{aligned}
$$

and hence $(\theta_1 \wedge \phi_1) \equiv (\theta_2 \wedge \phi_2)$. The cases for the other connectives are exactly similar.

Now suppose that $\psi_1(x_i, \vec{x}) \equiv \psi_2(x_i, \vec{x})$. Then if $M \vDash \exists w_j \psi_1(w_j, \vec{a})$ there is some $b \in |M|$ such that $M \vDash \psi_1(b, \vec{a})$. By the assumed logical equivalence, for this same b, $M \vDash \psi_2(b, \vec{a})$. Hence $M \vDash \exists w_j \psi_2(w_j, \vec{a})$. Obviously the same proof works in the other direction, giving the required result that $\exists w_j\, \psi_1(w_j, \vec{x}) \equiv \exists w_j\, \psi_2(w_j, \vec{x})$. The case for $\forall$ is exactly similar. ■

The Prenex Normal Form Theorem

The next theorem turns out to be a very useful representation result in many areas of logic.[20]

The Prenex Normal Form Theorem, 3

Every formula $\theta(\vec{x})$ *of* L *is logically equivalent to a formula in Prenex Normal Form (PNF), that is of the form*

$$Q_1 w_{j_1} Q_2 w_{j_2} \ldots Q_k w_{j_k} \psi(w_{j_1}, w_{j_2}, \ldots, w_{j_k}, \vec{x})$$

where the $Q_i = \forall$ *or* $\exists$, $i = 1, 2, \ldots, k$ *and there are no quantifiers appearing in* ψ .

Proof* The proof is by induction on the length of θ. Assume the result for formulae of length less than $|\theta|$. As usual there are various cases.

Case 1: $\theta = R(\vec{x})$ where R is a relation symbol of L.

In this case θ is already in PNF.

Case 2: $\theta = \neg\phi$.

By the Inductive Hypothesis we have that

$$\phi \equiv Q_1 w_{j_1}\, Q_2 w_{j_2} \ldots Q_k w_{j_k}\, \psi(w_{j_1}, w_{j_2}, \ldots, w_{j_k}, \vec{x})$$

for some quantifier free ψ. In this case, by Lemma 2

$$\theta = \neg\phi \equiv \neg Q_1 w_{j_1}\, Q_2 w_{j_2} \ldots Q_k w_{j_k}\, \psi(\vec{w}, \vec{x}).$$

We now prove by induction on k that this right hand side is logically equivalent to a formula in PNF (which does it for this case of course). Clearly this is true if $k = 0$ since such a formula would already be in PNF. So assume it's true for $k-1$. Then by the 'useful logical equivalents'

$$\neg Q_1 w_{j_1} Q_2 w_{j_2} \ldots Q_k w_{j_k}\, \psi(\vec{w}, \vec{x}) \equiv Q_1' w_{j_1} \neg Q_2 w_{j_2} \ldots Q_k w_{j_k} \psi(\vec{w}, \vec{x}).$$

where

$$Q_1' = \begin{cases} \exists & \text{if } Q_1 = \forall, \\ \forall & \text{if } Q_1 = \exists. \end{cases}$$

Also, by the Inductive Hypothesis

$$\neg Q_2 w_{j_2} Q_3 w_{j_3} \ldots Q_k w_{j_k} \psi(x_{i_1}, w_{j_2}, w_{j_3}, \ldots, w_{j_k}, \vec{x})$$

is logically equivalent to a formula $\chi(x_{i_1}, \vec{x})$ in PNF. (Here x_{i_1} is some variable which has not already occurred.) So by Lemma 2

$$\begin{aligned}
&\neg Q_1 w_{j_1}\, Q_2 w_{j_2} \ldots Q_k w_{j_k}\, \psi(w_{j_1}, w_{j_2}, \ldots, w_{j_k}, \vec{x}) \\
&\quad \equiv Q_1' w_{j_1} \neg Q_2 w_{j_2} \ldots Q_k w_{j_k}\, \psi(w_{j_1}, w_{j_2}, \ldots, w_{j_k}, \vec{x}) \\
&\quad \equiv Q_1' w_h \neg Q_2 w_{j_2} \ldots Q_k w_{j_k}\, \psi(w_h, w_{j_2}, \ldots, w_{j_k} \vec{x}), \text{ by Lemma 2,} \\
&\qquad \text{where } w_h \text{ does not occur in } \chi(x_{i_1}, \vec{x}),\ \psi(w_{j_1}, \ldots, w_{j_k}, \vec{x}), \\
&\quad \equiv Q_1' w_h\, \chi(w_h, \vec{x}), \text{ by using Lemma 2,}
\end{aligned}$$

and this last formula is in PNF.

Case 3: $\theta = (\phi_1 \wedge \phi_2)$.

By the Inductive Hypothesis we have that

$$\phi_1 \equiv Q_1^1 w_{j_1} Q_2^1 w_{j_2} \ldots Q_k^1 w_{j_k} \psi(w_{j_1}, w_{j_2}, \ldots, w_{j_k}, \vec{x}),$$

$$\phi_2 \equiv Q_1^2 w_{s_1} Q_2^2 w_{s_2} \ldots Q_u^2 w_{s_u}\, \eta(w_{s_1}, w_{s_2}, \ldots, w_{s_u}, \vec{x})$$

for some such right hand side PNF formulae. By Lemma 2 $(\phi_1 \wedge \phi_2)$ is logically equivalent to

$$\begin{aligned} &Q_1^1 w_{j_1} Q_2^1 w_{j_2} \ldots Q_k^1 w_{j_k} \psi(w_{j_1}, w_{j_2}, \ldots, w_{j_k}, \vec{x}) \wedge \\ &Q_1^2 w_{s_1} Q_2^2 w_{s_2} \ldots Q_u^2 w_{s_u} \eta(w_{s_1}, w_{s_2}, \ldots, w_{s_u}, \vec{x}) \end{aligned} \tag{10}$$

so it is enough to show that such a conjunction is logically equivalent to a formula in PNF. This we now prove by induction on $k + u$.

If $k + u = 0$ then the conjunction (10) is already in PNF. Suppose the result holds for $k' + u' < k + u$. Without loss of generality we may suppose that $u > 0$, otherwise we can suppose that $k > 0$ and transpose the conjuncts (which is logically equivalent). By the Inductive Hypothesis let $\chi(x_{i_1}, \vec{x})$ be a formula in PNF logically equivalent to

$$\begin{aligned} &Q_1^1 w_{j_1} Q_2^1 w_{j_2} \ldots Q_k^1 w_{j_k}\, \psi(w_{j_1}, w_{j_2}, \ldots, w_{j_k}, \vec{x}) \wedge \\ &\qquad Q_2^2 w_{s_2} \ldots Q_u^2 w_{s_u}\, \eta(x_{i_1}, w_{s_2}, \ldots, w_{s_u}, \vec{x}) \end{aligned} \tag{11}$$

(where x_{i_1} is a previously unmentioned free variable). Pick h such that w_h does not occur in (10) or $\chi(x_{i_1}, \vec{x})$. Then by the 'useful logical equivalences' and Lemma 2 the PNF formula $Q_1^2 w_h\, \chi(w_h, \vec{x})$ is logically equivalent to each of

$$\begin{aligned} &Q_1^2 w_h\, (Q_1^1 w_{j_1}\, Q_2^1 w_{j_2}\, \ldots\, Q_k^1 w_{j_k} \psi(w_{j_1}, w_{j_2}, \ldots, w_{j_k}, \vec{x}) \wedge \\ &\qquad\qquad Q_2^2 w_{s_2} \ldots Q_u^2 w_{s_u}\, \eta(w_h, w_{s_2}, \ldots, w_{s_u}, \vec{x})) \\ &Q_1^1 w_{j_1}\, Q_2^1 w_{j_2} \ldots Q_k^1 w_{j_k}\, \psi(w_{j_1}, w_{j_2}, \ldots, w_{j_k}, \vec{x}) \wedge \\ &\qquad\qquad Q_1^2 w_h\, Q_2^2 w_{s_2} \ldots Q_u^2 w_{s_u}\, \eta(w_h, w_{s_2}, \ldots, w_{s_u}, \vec{x})) \\ &Q_1^1 w_{j_1}\, Q_2^1 w_{j_2} \ldots Q_k^1 w_{j_k}\, \psi(w_{j_1}, w_{j_2}, \ldots, w_{j_k}, \vec{x}) \wedge \\ &\qquad\qquad Q_1^2 w_{s_1}\, Q_2^2 w_{s_2} \ldots Q_u^2 w_{s_u}\, \eta(w_{s_1}, w_{s_2}, \ldots, w_{s_u}, \vec{x}) \end{aligned}$$

and hence finally to $\phi_1 \wedge \phi_2$ and θ. The proofs for the cases for $\theta = (\phi_1 \vee \phi_2)$ and $\theta = (\phi_1 \to \phi_2)$ are similar and are left as amusing exercises.

Case 4: $\theta = \exists w_j\, \phi(w_j/x_i)$.

This case is easy. Since $|\phi| < |\theta|$ by the Inductive Hypothesis there is a formula χ in PNF logically equivalent to ϕ. Let h be such that w_h does not occur in χ or ϕ. Then

$$\theta = \exists w_j\, \phi(w_j/x_i) \equiv \exists w_h\, \phi(w_h/x_i) \equiv \exists w_h\, \chi(w_h/x_i)$$

and $\exists w_h\, \chi(w_h/x_i)$ is in PNF, as required.

The case for $\theta = \forall w_j\, \phi(w_j/x_i)$ is exactly similar. ■

Example

Find a formula in PNF logically equivalent to $\neg(\forall w_1\, R(w_1) \wedge \exists w_1\, P(w_1))$:

$$\begin{aligned} \neg(\forall w_1\, R(w_1) \wedge \exists w_1\, P(w_1)) &\equiv \neg\forall w_1\, R(w_1) \vee \neg\exists w_1\, P(w_1) \\ &\equiv \exists w_1 \neg R(w_1) \vee \forall w_1\, \neg P(w_1) \end{aligned}$$

by Lemma 2 and the 'Useful Equivalents', UEs, $\neg(\theta \wedge \phi) \equiv (\neg\theta \vee \neg\phi)$

$$\neg\forall w_1\, R(w_1) \equiv \exists w_1\, \neg R(w_1),\ \neg\exists w_1\, P(w_1) \equiv \forall w_1\, \neg P(w_1),$$

$$\equiv \exists w_1\, \neg R(w_1) \vee \forall w_2\, \neg P(w_2)$$

by Lemma 2, reflexivity of $\equiv$ and the UE $\forall w_1\, \neg P(w_1) \equiv \forall w_2\, \neg P(w_2)$,

$$\equiv \forall w_2\, (\exists w_1\, \neg R(w_1) \vee \neg P(w_2)), \tag{12}$$

by the UEs. Also by the UEs,

$$\begin{aligned}(\exists w_1\, \neg R(x_1) \vee \neg P(x_2)) \quad &\equiv \quad (\neg P(x_2) \vee \exists w_1 \neg R(w_1)) \\ &\equiv \quad \exists w_1\, (\neg P(x_2) \vee \neg R(w_1))\end{aligned}$$

so by Lemma 2,

$$\forall w_2(\exists w_1\, \neg R(w_1) \vee \neg P(w_2)) \equiv \forall w_2 \exists w_1(\neg P(w_2) \vee \neg R(w_1))$$

and from this, (12) and transitivity of $\equiv$,

$$\neg(\forall w_1\, R(w_1) \wedge \exists w_1\, P(w_1)) \equiv \forall w_2 \exists w_1\, (\neg P(w_2) \vee \neg R(w_1)),$$

a PNF equivalent (it's not unique, obviously).

Generally the more quantifiers (or the more alternations of blocks of universal and existential quantifiers) there are in a formula in Prenex Normal Form the more it can express, in the sense for example of not being logically equivalent to a formula in Prenex Normal Form with few quantifiers (or alternating blocks of quantifiers). Indeed in several areas of logic this is used as a measure of the complexity of sets defined by formulae.

An exception to this pattern however is when the formula only contains unary relation symbols.[21] In this case having more than one alternation of quantifiers does not give you anything new, as we shall shortly demonstrate. Firstly however we need a little notation.

Given $\phi_1, \phi_2, \ldots, \phi_m \in FL$ we write

$$\phi_1 \wedge \phi_2 \wedge \ldots \wedge \phi_m \qquad \text{or} \qquad \bigwedge_{i=1}^{n} \phi_i$$

for the formula

$$((\ldots(((\phi_1 \wedge \phi_2) \wedge \phi_3) \wedge \phi_4) \wedge \ldots \wedge \phi_{m-1}) \wedge \phi_m).$$

More precisely we define by induction

$$\bigwedge_{i=1}^{1} \phi_i = \phi_1, \qquad \bigwedge_{i=1}^{n+1} \phi_i = \left(\bigwedge_{i=1}^{m} \phi_i \wedge \phi_{n+1} \right).$$

So what we are doing here is is repeatedly taking conjunctions, starting from the left.

It is now rather clear, and certainly straightforward to prove by induction on n, that $\bigwedge_{i=1}^{n} \phi_i$ is true in an interpretation just if each conjunct ϕ_i for $i = 1,\ldots,n$ is true in that interpretation. This is an valuable observation because it means that if we change the order of the ϕ_i, or insert or remove repeats, in this big conjunction then the formula we obtain is logically equivalent to the one we started with. Since much of the time in logic we are only interested in formulae up to logical equivalence this can allow us a useful freedom.

For example for a finite set S of formulae we might simply write $\bigwedge S$ for a conjunction of the formulae in S without specifying the precise order in which this conjunction is supposed to be taken since up to logical equivalence this does not matter.

It is also convenient to identify the conjunction of the set of formulae in the empty set, i.e.

$$\bigwedge \emptyset \text{ or } \bigwedge_{i=1}^{0} \phi_i$$

with some tautology, the precise tautology chosen being irrelevant when we are only interested in formulae up to logical equivalence. Notice that with this convention we still have that that $\bigwedge\emptyset$ is true in an interpretation just if every formula $\phi \in \emptyset$ is true in that interpretation, since they all are,[22] and $\bigwedge\emptyset$ must also be true because it is a tautology.

Exactly similarly given $\phi_1, \phi_2, \ldots, \phi_m \in FL$ we write

$$\phi_1 \vee \phi_2 \vee \ldots \vee \phi_m \quad \text{or} \quad \bigvee_{i=1}^{n} \phi_i$$

for the formula

$$((\ldots(((\phi_1 \vee \phi_2) \vee \phi_3) \vee \phi_4) \vee \ldots \vee \phi_{m-1}) \vee \phi_m).$$

In this case we take the disjunction of the formulae in the empty set to be a contradiction and we have, even for $n = 0$, that $\bigvee_{i=1}^{n} \phi$ is false in an interpretation just if each ϕ_i is false in that interpretation.

Having got that bit of useful notation out of the way let $\theta(x_1,\dots,x_n) \in FL$ and suppose that the relation symbols occurring in θ are $R_1, R_2,\dots,R_m$ and these are all unary. Writing ϕ^1 for ϕ and ϕ^0 for $\neg\phi$ we call a formula of L an *atom* (for $R_1, R_2,\dots, R_m$) if it has the form

$$R_1^{\varepsilon_1}(x_1) \wedge R_2^{\varepsilon_2}(x_1) \wedge \dots \wedge R_m^{\varepsilon_m}(x_1)$$

for some $\varepsilon_1, \varepsilon_2,\dots,\varepsilon_m \in \{0,1\}$.

For example

$$R_1(x_1) \wedge \neg R_2(x_1) \wedge \neg R_3(x_1) \wedge \dots \wedge R_{m-1}^{\varepsilon_{m-1}}(x_1) \wedge \neg R_m^{\varepsilon_m}(x_1)$$

is the atom for $R_1, R_2,\dots,R_m$ with

$$\varepsilon_1 = 1, \varepsilon_2 = 0, \varepsilon_3 = 0,\dots, \varepsilon_{m-1} = 1, \varepsilon_m = 0.$$

Since there are 2^m choices for the finite sequence $\varepsilon_1, \varepsilon_2,\dots,\varepsilon_m \in \{0,1\}$ there are 2^m such atoms, which we shall denote by $\alpha_1(x_1), \alpha_2(x_1),\dots,\alpha_{2^m}(x_1)$.

Now suppose that we are given an interpretation for L. Let $\theta(x_1,\ldots,x_n)$ be as above and let $1 \le i \le m$. Then exactly one of $R_1(x_i)$, $\neg R_1(x_i)$ is true in this interpretation. In other words there is a unique $\varepsilon_1 \in \{0,1\}$ such that $R_1^{\varepsilon_1}(x_i)$ is true in this interpretation. Similarly there is a unique $\varepsilon_2 \in \{0,1\}$ such that $R_2^{\varepsilon_2}(x_i)$ is true in this interpretation. Continuing in this way we see that there is a unique atom α_{h_i} such that $\alpha_{h_i}(x_i)$ is true in this interpretation.

Similarly for each $1 \le k \le 2^m$ exactly one of $\exists w_1\, \alpha_k(w_1)$ and $\neg\exists w_1\, \alpha_k(w_1)$ is true in this interpretation. In other words there is a unique $\delta_k \in \{0,1\}$ such that $(\exists w_1\, \alpha_k(w_1))^{\delta_k}$ is true in this interpretation. Putting these observations together then there are unique finite sequences $j_1, j_2, \ldots, j_n \in \{1,2,\ldots,2^m\}$ and $\delta_1, \delta_2, \ldots, \delta_{2^m} \in \{0,1\}$ such that the formula

$$\bigwedge_{i=1}^{n} \alpha_{j_i}(x_i) \wedge \bigwedge_{j=1}^{2^m} (\exists w_1\, \alpha_j(w_1))^{\delta_j} \tag{13}$$

is true in this interpretation. Call a formula of this form for some $j_1, j_2, \ldots, j_n \in \{1,2,\ldots,2^m\}$ and $\delta_1, \delta_2, \ldots, \delta_{2^m} \in \{0,1\}$ a *diagram* for $x_1,\ldots,x_n$.

Theorem 4* *Let $\theta(x_1,\ldots,x_n) \in FL$ and suppose that the relation symbols occurring in θ are $R_1, R_2, \ldots, R_m$ and these are all unary. Then $\theta(x_1,\ldots,x_n)$ is logically equivalent to a disjunction of diagrams for $x_1,\ldots,x_n$.*

Before we give the proof it is worth noticing that not all diagrams are satisfiable since a diagram might for example have conjuncts $\alpha_1(x_1)$ and $\neg\exists w_1\, \alpha_1(w_1)$. Clearly we could, without loss, drop these unsatisfiable diagrams from the representation given in this theorem.

Proof* The proof is by induction on $|\theta(x_1,\ldots,x_n)|$, where n can vary but the $R_1,\ldots,R_m$ are fixed.

In the case that $\theta = R_r(x_i)$, with, say, $1 \le i \le n$, we have from the above discussion that in any interpretation

$$\begin{aligned} \theta \text{ is true } &\Leftrightarrow R_r(x_i) \text{ is true} \\ &\Leftrightarrow \text{some atom } \textstyle\bigwedge_{k=1}^{m} R_k^{\varepsilon_k}(x_i) \text{ with } \varepsilon_r = 1 \text{ is true} \\ &\Leftrightarrow \text{some diagram } (13) \text{ where } \alpha_{j_i} = \textstyle\bigwedge_{k=1}^{m} R_k^{\varepsilon_k}(x_j) \\ &\qquad \text{with } \varepsilon_r = 1 \text{ is true.} \end{aligned}$$

In other words $R_r(x_i)$ is logically equivalent to the disjunction of all such diagrams.

Now suppose that $\theta(x_1,\ldots,x_n) = (\phi(x_1,\ldots,x_n) \wedge \psi(x_1,\ldots,x_n))$. By inductive hypothesis ϕ is logically equivalent to a disjunction of diagrams for $\vec{x} = x_1, x_2, \ldots, x_n$ so given an interpretation ϕ is true in that interpretation just if the unique diagram which is true in that interpretation is one of these disjuncts. Similarly for ψ.

Hence θ is true in an interpretation just if the diagram true in that interpretation is a disjunct for both ϕ and ψ. Or to put it another way θ is logically equivalent to the disjunction of diagrams which appear in the corresponding forms for both ϕ and ψ. The cases for the other connectives are exactly analogous.

The tricky cases concern the quantifiers. So now suppose that $\theta(x_1,\ldots,x_n) = \exists w_j\, \phi(x_1,\ldots,x_n,w_j)$. By inductive hypothesis then there are diagrams for $x_1,\ldots,x_n,x_{n+1}$, say $\xi_g(x_1,\ldots,x_n,x_{n+1})$ for $g = 1,\ldots,u$, such that

$$\phi(x_1,\ldots,x_{n+1}) \equiv \bigvee_{g=1}^{u} \xi_g(x_1,\ldots,x_n,x_{n+1}).$$

Then from the 'Useful Logical Equivalents', ULE's,

$$\begin{aligned} \exists w_j \phi(x_1,\ldots,x_n,w_j) &\equiv \exists w_2 \left(\bigvee_{g=1}^{u} \xi_g\, (x_1,\ldots,x_n,w_2)\right) \\ &\equiv \left(\bigvee_{g=1}^{u} \exists w_2\, \xi_g(x_1,\ldots,x_n,w_2)\right). \end{aligned} \tag{14}$$

Since each $\xi_g(x_1,\ldots,x_n,w_2)$ is a conjunction of expressions only one of which actually mentions w_2, and that one has the form $\alpha_v(w_2)$ for some atom $\alpha_v(x_1)$, the ULE's give that this $\exists w_2\, \xi_g(x_1,\ldots,x_n,w_2)$ is logically equivalent to a formula of the form

$$\exists w_2\, \alpha_v(w_2) \wedge \zeta(x_1,\ldots,x_n) \tag{15}$$

where ζ is a diagram for $x_1,\ldots,x_n$. If $(\exists w_1\, \alpha_v(w_2))^1$ already appears in ζ then (15) is logically equivalent to ζ. On the other hand if $(\exists w_1\, \alpha_v(w_1))^1$ does not already appear in ζ then $(\exists w_1\, \alpha_v(w_1))^0$, i.e. $\neg\exists w_1\, \alpha_v(w_1)$, must appear in ζ and in that case (15) is not satisfiable.

From the ULE's it now follows that $\exists w_j\, \phi(x_1,\ldots,x_n,w_j)$ is logically equivalent to the disjunction of the (distinct) diagrams for which the (15) yielded a satisfiable ζ, giving the required result.

Finally in the case $\theta(x_1,\ldots,x_n) = \forall w_j\, \phi(x_1,\ldots,x_n,w_j)$ we have by the ULE's that $\theta(x_1,\ldots,x_n) = \neg\exists w_j \neg\phi(x_1,\ldots,x_n,w_j)$. To treat this formula it is simplest to use three of the cases already covered, namely going from $\phi(x_1,\ldots,x_n,x_{n+1})$ (where we can use the Inductive Hypothesis) to $\neg\phi(x_1,\ldots,x_n,x_{n+1})$ (for which we then have the Inductive Hypothesis), thence to $\exists w_j \neg\phi(x_1,\ldots,x_n,w_j)$, and finally to $\neg\exists w_j \neg\phi(x_1,\ldots,x_n,w_j)$. ■

Formal Proofs

We have now given a formulation of what it means for, say, a formula ϕ to *follow logically* from a set Γ of formulae by introducing a semantics, a notion of interpretation (or meaning) and truth, and saying that this 'following' happens just if whenever every $\theta \in \Gamma$ is true then so is ϕ. This seems to have worked out very well, all our initial intuitions have been proved to be spot on.

But there is another way that we might have tried to capture this notion of 'follows'. Namely we could have just written down the *properties* we think 'follows' should have and once we have what appears to be an exhaustive list say that ϕ *follows from* Γ *just if this can be shown purely on the basis of these properties*. In other words we try to pin down 'follows' solely in terms of syntactic rules. [This may not make much sense to you right now but it will later.]

These 'rules' will be of the form

$$\frac{\Gamma_1 \mid \theta_1, \quad \Gamma_2 \mid \theta_2, \ldots, \Gamma_s \mid \theta_s}{\Gamma \mid \theta}$$

where the $\Gamma_1, \Gamma_2, \ldots, \Gamma_s, \Gamma$ are sets of formulae, possibly empty. The 'idea' behind these rules is that they represent situations where one feels that:

If I thought that θ_i follows from Γ_i for $i = 1, 2, \ldots, s$ then I should think that θ follows from Γ.

While that might be the motivation however these rules can be viewed as purely formal, syntactic objects. In particular the $|$ need have no meaning, it'sjust a device for separating the two sides. [Expressions like $\Gamma \mid \theta$ are called *sequents*.] We now give a list of such rules.[23] In these rules the Γ, Δ stand for sets of formulae, and the θ, ϕ, ψ stand for formulae of some relational language L.

The Rules of Proof for the Predicate Calculus

And In (AND) $$\frac{\Gamma \mid \theta, \quad \Delta \mid \phi}{\Gamma \cup \Delta \mid \theta \wedge \phi}$$

And Out (AO) $$\frac{\Gamma \mid \theta \wedge \phi}{\Gamma \mid \theta} \qquad \frac{\Gamma \mid \theta \wedge \phi}{\Gamma \mid \phi}$$

Or In (ORR) $$\frac{\Gamma \mid \theta}{\Gamma \mid \theta \vee \phi} \qquad \frac{\Gamma \mid \theta}{\Gamma \mid \phi \vee \theta}$$

Disjunction (DIS) $$\frac{\Gamma, \theta \mid \psi, \qquad \Delta, \phi \mid \psi}{\Gamma \cup \Delta, (\theta \vee \phi) \mid \psi}$$

Implies In (IMR) $$\frac{\Gamma, \theta \mid \phi}{\Gamma \mid \theta \to \phi}$$

Modus Ponens (MP) $$\frac{\Gamma \mid \theta, \quad \Delta \mid \theta \to \phi}{\Gamma \cup \Delta \mid \phi}$$

Not In (NIN) $$\frac{\Gamma, \theta \mid \phi, \quad \Delta, \theta \mid \neg\phi}{\Gamma \cup \Delta \mid \neg\theta}$$

Not Not Out (NNO) $$\frac{\Gamma \mid \neg\neg\theta}{\Gamma \mid \theta}$$

Monotonicity (MON) $$\frac{\Gamma \mid \theta}{\Gamma \cup \Delta \mid \theta}$$

All In ($\forall$I) $\dfrac{\Gamma \,|\, \theta}{\Gamma \,|\, \forall w_j\, \theta(w_j/x_i)}$ where x_i does not occur in any formula in Γ and w_j does not occur in θ

All Out ($\forall$O) $\dfrac{\Gamma \,|\, \forall w_j\, \theta(w_j, \vec{x})}{\Gamma \,|\, \theta(x_i, \vec{x})}$

Exists In ($\exists$I) $\dfrac{\Gamma \,|\, \theta}{\Gamma \,|\, \exists\, w_j\, \theta'}$ where θ' is the result of replacing any number of occurences of x_i in θ by w_j and w_j does not occur in θ.

Exists Out ($\exists$O) $\dfrac{\Gamma,\, \phi \,|\, \theta}{\Gamma, \exists w_j\, \phi(w_j/x_i) \,|\, \theta}$ where x_i does not occur in θ nor any formula in Γ and w_j does not occur in ϕ.

Finally we have a rule, or *axiom*, which requires no assumptions:

REF $\Gamma \,|\, \theta$ whenever $\theta \in \Gamma$.

We can now give a second formulation of what we mean by 'θ follows from Γ', namely that we can *derive* $\Gamma \,|\, \theta$ *using just* REF *and the rules* AND-$\exists$O, and investigate its relation to logical consequence, $\Gamma \vDash \theta$.

First however we need to make precise what we mean by 'derive using just REF and the rules AND-$\exists$O'.

Definition A (formal) *proof* is a sequence of sequents

$$\Gamma_1 \,|\, \phi_1, \Gamma_2 \,|\, \phi_2 \ldots, \Gamma_m \,|\, \phi_m$$

where the Γ_i are *finite* subsets of FL, the $\phi_i \in FL$ and for $i = 1,2, \ldots,$ m, either $\Gamma_i \,|\, \phi_i$ is an instance of REF or there are some $j_1, j_2, \ldots, j_s < i$ such that

$$\frac{\Gamma_{j_1} \,|\, \phi_{j_1}, \Gamma_{j_2} \,|\, \phi_{j_2}, \ldots, \Gamma_{j_s} \,|\, \phi_{j_s}}{\Gamma_i \,|\, \phi_i}$$

is an instance of one of the rules of proof.

So in order to be a *proof* every sequent $\Gamma_i \mid \phi_i$ appearing in it must be *justified*, either by being an instance of the axiom REF or because it follows from some of the earlier (and so already justified) $\Gamma_{j_k} \mid \phi_{j_k}$. We require the Γ_i to be finite because we want proofs to be simply finite objects whose correctness can be checked *mechanically* in a finite time.

We can now formalize the above version of 'follows':

Definition For $\Gamma \subseteq FL$ and $\theta \in FL$,

$$\begin{aligned}\Gamma \vdash \theta \quad \Leftrightarrow \quad & \exists \text{ a proof } \Gamma_1 \mid \phi_1, \ldots, \Gamma_m \mid \phi_m \\ & \text{such that } \Gamma_m \subseteq \Gamma, \theta = \phi_m.\end{aligned}$$

In this case we say that $\Gamma_1 \mid \phi_1, \ldots, \Gamma_m \mid \phi_m$ *is a proof of* θ *from* Γ. We say Γ 'proves' θ, or, 'there is a proof of θ from Γ', for $\Gamma \vdash \theta$. Notice that in this definition Γ can be infinite (but the Γ_i must be finite, we require that proofs are finite objects that we can physically write down). As with $\vDash$ the left hand side of $\mid$ or $\vdash$ is supposed to be a set of formulae but again we abbreviate $\Gamma \cup \{\psi\}$ to Γ, ψ etc.

Example To show that $\forall w_1\, \psi(w_1, x_1) \vdash \exists w_1\, \psi(w_1, x_1)$

A suitable proof is given by the middle column below:

1.	$\forall w_1\, \psi(w_1, x_1) \mid \forall w_1\, \psi(w_1, x_1)$	REF
2.	$\forall w_1\, \psi(w_1, x_1) \mid \psi(x_2, x_1)$	$\forall$O from 1
3.	$\forall w_1\, \psi(w_1, x_1) \mid \exists w_1\, \psi(w_1, x_1)$	$\exists$I from 2

Notice

1. In this case the left hand side of the final sequent,

   $$\forall w_1\, \psi(w_1, x_1) \mid \exists w_1\, \psi(w_1, x_1)$$

   is the left hand side of $\forall w_1\, \psi(w_1, x_1) \vdash \exists w_1\, \psi(w_1, x_1)$ though we actually only require it to be a subset of it.

2. Recall our convention that if we write a formula ψ as $\psi(\vec{x})$ then all the variable occurring in ψ are amongst $\vec{x}$. Hence on line 2 in this proof x_2 does not already occur in $\psi(w_1, x_1)$ and as a result subsequently replacing x_2 by w_1 in $\psi(x_2, x_1)$ gets us back to the original $\psi(w_1, x_1)$. [Notice also in this step that w_1 cannot appear in $\psi(x_2, x_1)$, as required by the $\exists$I rule.]
3. Formally we don't need columns 1 and 3 above. However for ease of marking (!) you should include them when I ask you for a (formal) proof. [The word 'formal' here is only include when there is a danger of confusing this sort of proof with the sort of 'proof' you give of, say, a theorem.]
4. When writing out proofs such as the one above we may, to save repetition, replace the occurrences of $\forall w_1\, \psi(w_1, x_1)$ on lines 2 & 3 by simply ditto marks (or a vertical line) below the occurrence of this formula on line 1.
5. In this course we shall, for simplicity and to avoid any confusion, continue to use the x_i for free variables and the w_i for bound variables. However once you have got used to this system you will have the confidence to use $x, w, y, z, t, \ldots$ for both free and bound variables, and indeed you will commonly meet this more relaxed usage in the other logic courses such as Model Theory and Gödel's Theorems.

Another Example

The following is a proof of $\neg\exists w_1\, \theta(w_1, x_1) \vdash \forall w_1 \neg\theta(w_1, x_1)$:

1.	$\theta(x_2, x_1)$,	$\neg\exists w_1\, \theta(w_1, x_1) \mid \neg\exists w_1\, \theta(w_1, x_1)$	REF
2.	$\theta(x_2, x_1)$,	$\neg\exists w_1\, \theta(w_1, x_1) \mid \theta(x_2, x_1)$	REF
3.	$\theta(x_2, x_1)$,	$\neg\exists w_1\, \theta(w_1, x_1) \mid \exists w_1\, \theta(w_1, x_1)$	$\exists$I , 2
4.		$\neg\exists w_1\, \theta(w_1, x_1) \mid \neg\theta(x_2, x_1)$	NIN, 1,3
5.		$\neg\exists w_1\, \theta(w_1, x_1) \mid \forall w_1 \neg\theta(w_1, x_1)$	$\forall$I , 4

Notice

1. On line 3 w_1 cannot already appear in $\theta(x_2, x_1)$ since we replaced it everywhere in $\theta(w_1, x_1)$ in forming $\theta(x_2, x_1)$. When you do examples you need not mention that such conditions are fulfilled when they are as clear as it is here.
2. By our convention x_2 does not appear in $\neg\exists w_1\, \theta(w_1, x_1)$ so the places where x_2 appears in $\theta(x_2, x_1)$ are just those that w_1 occupied in $\neg\exists w_1\, \theta(w_1, x_1)$. Again when you write out a formal proof you need not mention such 'obvious' facts.
3. Again by our convention x_2 does not appear in the left hand side formula on line 4 so the $\forall$I rule is being correctly applied.

Strategies for finding proofs

A good strategy if you are stuck trying to find a (formal) proof is to ask yourself 'why do I think that the right hand side follows (in an informal sense) from the left hand side?' In this case you might say: 'Well, if there doesn't exist a w_1 such that $\theta(w_1, x_1)$ then I couldn't have $\theta(x_2, x_1)$, that would be a contradiction. So I must have $\neg\theta(x_2, x_1)$. But I've shown this for *any* x_2 so it must be true for all of them'. Once you've got that far you essentially have your formal proof, all you need to do is match the steps in your informal demonstration with the formal rules of proof of the Predicate Calculus.

Another hint if you are asked to find a proof of $\theta_1,\ldots,\theta_m \vdash \phi$ is consider what you expect to be the final sequent in your proof, namely $\theta_1,\ldots,\theta_m \mid \phi$, and consider what the line above that might be, and so on. In other words working backwards.

Again in this situation it seems reasonable to take as the first m lines of your proof the sequents $\theta_1,\ldots,\theta_m \mid \theta_i$ (alternatively $\theta_i \mid \theta_i$) each justified by REF and see what can be obtained from these by an application of a rule, and so on. Hopefully applying these two processes you will see how to join up the two streams.

Yet another trick worth being aware of here is that if, in this case, you obtain a proof ending in $\theta_1,\ldots,\theta_m \mid \psi$ and you can also see a proof of $\psi \mid \phi$, so (probably) ending in $\psi \mid \phi$, then, by IMR, we can append $\mid (\psi \rightarrow \phi)$ to this proof and concatenating it with the first proof allows you to add the final sequent $\theta_1,\ldots,\theta_m \mid \phi$, justified by MP, to give the required proof.

Finally, if it is any consolation, the fact of the matter is that formal proofs are not easy to find, you'll see just why if you ever study Gödel's Theorems. It's not simply because we human beings are actually pretty slow, overall even the fastest computers cannot do any better.

The Completeness and Compactness Theorems

So now we have two formulations of what it means for θ to *follow* from Γ, namely $\Gamma \vDash \theta$ and $\Gamma \vdash \theta$. The main part of this course involves determining the relationship between them. Before that however it will prove very useful to establish the following result:

Lemma 5

Let $\Gamma_1, \ldots, \Gamma_s, \Gamma \subseteq FL$ (possibly infinite) and $\theta_1, \ldots, \theta_s, \theta \in FL$. Then

(*i*) *If* $\theta \in \Gamma$ *then* $\Gamma \vdash \theta$.

(*ii*) *If* $\Gamma_i \vdash \theta_i$ *for* $i = 1, \ldots, s$ *and*

$$\frac{\Gamma_1 \mid \theta_1, \ldots, \Gamma_s \mid \theta_s}{\Gamma \mid \theta}$$

is an instance of a rule of proof then $\Gamma \vdash \theta$.

Proof For (i) a suitable proof of $\Gamma \vdash \theta$ is just the single sequent $\theta \mid \theta$ since it is justified by REF and

Secondly notice that $\Delta_m - \{\phi\} \subseteq \Delta$ so x_i does not appear in any formula in $\Delta_m - \{\phi\}$ nor in θ so the last sequent in (16) is justified by $\exists$O from its immediate predecessor.

Finally

$$(\Delta_m - \{\phi\}) \cup \{\exists w_j \phi(w_j/x_i)\} \subseteq \Delta \cup \{\exists w_j \phi(w_j/x_i)\} = \Gamma,$$

as required.

The arguments for the remaining rules, some of which appear in the Exercises Section, are similar (and much easier in general). ■

In what follows Lemma 5 will turn out to be very useful because it frequently enables us to avoid talking about actual formal proofs and instead talk directly about the provability relation $\vdash$. For that reason it is well worth making sure you really have grasped what it is saying.

We now set about establishing some connections between $\vDash$ and $\vdash$.

Lemma 6

Let $\Gamma_1, \ldots, \Gamma_s, \Gamma \subseteq FL$ be finite[24] *and $\theta_1, \ldots, \theta_s, \theta \in FL$. Then*

(*i*) *If* $\theta \in \Gamma$ *then* $\Gamma \vDash \theta$.

(*ii*) *If* $\Gamma_i \vDash \theta_i$ *for* $i = 1, \ldots, s$ *and*

$$\frac{\Gamma_1 \mid \theta_1, \ldots, \Gamma_s \mid \theta_s}{\Gamma \mid \theta}$$

is an instance of a rule of proof then $\Gamma \vDash \theta$.

Proof First notice that if Γ is finite then there can only be finitely many free variables which are mentioned in formulae in Γ. In that case we might write $\Gamma(\vec{x})$, where all these variables occur in $\vec{x}$, and $\Gamma(\vec{a})$ for the result of replacing each x_j in $\vec{x}$ in the formulae in Γ by a_j. With this notation then

$$\Gamma(\vec{x}) \vDash \theta(\vec{x}) \Leftrightarrow \text{For all structures } M \text{ for } L \text{ and } \vec{a} \in |M|$$

$$M \vDash \Gamma(\vec{a}) \Rightarrow M \vDash \theta(\vec{a}) \tag{17}$$

where $M \vDash \Gamma(\vec{a})$ is short for $M \vDash \phi(\vec{a})$ for all $\phi(\vec{x}) \in \Gamma$.

Turning to the proof of (i) of the lemma then if $\theta \in \Gamma$ then trivially the above right hand side holds.

To show (ii) we need to check it for each of the rules $\mathrm{AND} - \exists\mathrm{O}$. We'll check it for $\forall$I and leave the rest as exercises (some already appear in the Exercises). Without loss of generality in this case the instance of the rule looks like

$$\frac{\Gamma(x_2,\ldots,x_n) \mid \phi(x_1,x_2,\ldots,x_n)}{\Gamma(x_2,\ldots,x_n) \mid \forall w_j\phi(w_j,x_2,\ldots,x_n)}$$

where x_1 does not occur in any formula in Γ and w_j does not already appear in ϕ. We are told that

$$\Gamma(x_2,\ldots,x_n) \vDash \phi(x_1, x_2,\ldots,x_n). \tag{18}$$

Let M be any structure for L and $a_2, a_3,\ldots,a_n$ elements of the universe of M. Suppose that $M \vDash \Gamma(a_2,\ldots,a_n)$. Then from (17) and (18):

for *any* a_1 from the universe of M, $M \vDash \phi(a_1, a_2,\ldots,a_n)$.

Hence

$$M \vDash \forall w_j\phi(w_j, a_2,\ldots, a_n).$$

Since the structure M for L and $a_2,\ldots,a_n$ from the universe of M were arbitrary we see that we have shown that

$$\Gamma(x_2,\ldots, x_n) \vDash \forall w_j\phi(w_j, x_2,\ldots, x_n),$$

as required. ■

Lemma 6 provides us with a useful means of checking that a strategy we might have for producing a certain formal proof is at least not just wishful thinking. For if we ever get to, or hope to get to as an intermediate step, a sequent $\Gamma \mid \theta$ where we do not have $\Gamma \vDash \theta$ then this cannot be part of a correct proof. This is a practically useful check because it is often quite easy to see whether or not $\Gamma \vDash \theta$.

The Correctness Theorem (for Relational L), 7

Let $\Gamma \subseteq FL$ (possibly infinite) and $\zeta \in FL$. Then

$$\Gamma \vdash \zeta \Rightarrow \Gamma \vDash \zeta.$$

Proof We use a proof technique called 'induction on the length of proof'. Assume that that $\Gamma \vdash \zeta$, say $\Gamma_1 \mid \theta_1, \ldots, \Gamma_m \mid \theta_m$ is a proof of this. So the Γ_i are finite and $\Gamma_m \subseteq \Gamma$, $\theta_m = \zeta$. We prove by induction on i for $i = 1, 2, \ldots, m$ that $\Gamma_i \vDash \theta_i$.

Suppose that we have this already for all $k < i$ where $1 \leq i \leq m$. Notice that in the base case, when $i = 1$, this is vacuously true.

If $\Gamma_i \mid \theta_i$ is justified in this proof because it is an instance of REF then $\theta_i \in \Gamma_i$ so $\Gamma_i \vDash \theta_i$ by Lemma 6(i). Otherwise $\Gamma_i \mid \theta_i$ follows by one of the rules of proof from some earlier $\Gamma_{j_1} \mid \theta_{j_1}, \ldots, \Gamma_{j_s} \mid \theta_{j_s}$, so $j_1, \ldots, j_s < i$ and

$$\Gamma_{j_1} \vDash \theta_{j_1}, \ldots, \Gamma_{j_s} \vDash \theta_{j_s}$$

by inductive hypothesis. By now by Lemma 6(ii), $\Gamma_i \vDash \theta_i$.

From this then we conclude that we must have $\Gamma_m \vDash \theta_m$. Let M be a structure for L and suppose that we have an assignment of elements of the universe of M to the free variables appearing in the formulae in Γ under which every formula in Γ was true in M. Then the same must be true of Γ_m since $\Gamma_m \subseteq \Gamma$. Hence $\zeta = \theta_m$ must be true according to this interpretation, because $\Gamma_m \vDash \theta_m$. We have shown that $\Gamma \vDash \zeta$, as required. ■

The Correctness Theorem is valuable in that it gives us a way of showing that something is *not* provable. Specifically to show that $\Gamma \nvdash \theta$ it is enough to show that $\Gamma \nvDash \theta$ and to do this we only have to exhibit a suitable structure and an assignment to the free variables under which everything in Γ is true but θ is false.

Example

To show that

$$\forall w_1 \exists w_2 (R(w_1, w_2) \wedge R(w_2, w_2)) \nvdash \forall w_1 R(w_1, w_1)$$

let M be the structure for $L = \{R\}$ (R a binary relation symbol) with universe $\{0,1\}$ and

$$R^M = \{\langle 0,1 \rangle, \langle 1,1 \rangle\}.$$

Then

$$M \vDash (R(0, 1) \wedge R(1,1)), \quad M \vDash (R(1,1) \wedge R(1,1)),$$

so

$$M \vDash \exists w_2 (R(0, w_2) \wedge R(w_2, w_2)),\ M \vDash \exists w_2 (R(1, w_2) \wedge R(w_2, w_2)),$$

and hence

$$M \vDash \forall w_1 \exists w_2 (R(w_1, w_2) \wedge R(w_2, w_2)).$$

However $M \nvDash R(0, 0)$ so $M \nvDash \forall w_1 R(w_1, w_1)$. Hence

$$\forall w_1 \exists w_2 (R(w_1, w_2) \wedge R(w_2, w_2)) \nvDash \forall w_1 R(w_1, w_1)$$

so by the Correctness Theorem

$$\forall w_1 \exists w_2 (R(w_1, w_2) \wedge R(w_2, w_2)) \nvdash \forall w_1 R(w_1, w_1).$$

From this Correctness (also sometimes called 'Soundness') Theorem for Predicate Logic (also called the Predicate Calculus) it follows that the notion $\vdash$ of 'follows' is at least as strong as that formalized by $\vDash$. But is it stronger? Given the Correctness Theorem we might suspect that it is not stronger, that in fact these two notions of follows are equivalent. This is indeed the case, and an amazing result it is too as will later be explained. This was first proved by Kurt Gödel in 1929, as 'Gödel's Completeness Theorem', not to be confused with his 'Incompleteness Theorems', though what they do have in common is that they are amongst the most philosophically important theorems in the whole of mathematics.

To show the other direction of the Correctness Theorem, that

$$\Gamma \vDash \zeta \Rightarrow \Gamma \vdash \zeta$$

we start by assuming that $\Gamma \vdash \zeta$ fails, i.e. there is no proof of ζ from Γ and then go on to show that $\Gamma \nvDash \zeta$, that is that there is a structure for L and an assignment to the free variables in which all the formulae in Γ come out to be true but ζ comes out to be false. So what we need to do, starting from the fact that $\Gamma \nvdash \zeta$, is somehow *construct* the required M and assignment to the free variables.

The first step is to rephrase the assumed $\Gamma \nvdash \zeta$, as a statement about *consistency* – for which we will need some definitions and lemmas.

Definition $\Gamma \subseteq FL$ is *inconsistent* if $\Gamma \vdash \phi \wedge \neg\phi$ for some $\phi \in FL$. Γ is *consistent* if it is not inconsistent.

Lemma 8

For $\Gamma \subseteq FL$ the following are equivalent:

i) Γ *is inconsistent.*
ii) $\Gamma \vdash \phi$ *and* $\Gamma \vdash \neg\phi$ *for some* $\phi \in FL$.
iii) $\Gamma \vdash \theta$ *for any* $\theta \in FL$.

Proof (i) $\Rightarrow$ (ii) Assume that Γ is inconsistent, say $\Gamma \vdash \phi \wedge \neg\phi$. Then since

$$\frac{\Gamma \mid \phi \wedge \neg\phi}{\Gamma \mid \phi} \qquad \frac{\Gamma \mid \phi \wedge \neg\phi}{\Gamma \mid \neg\phi}$$

are instances of the AO rule, by Lemma 5(ii), $\Gamma \vdash \phi$ and $\Gamma \vdash \neg\phi$.

(ii) $\Rightarrow$ (iii) Suppose that $\Gamma \vdash \phi$ and $\Gamma \vdash \neg\phi$. Then by Lemma 5(ii) and the MON rule,

$$\Gamma, \neg\theta \vdash \phi, \Gamma, \neg\theta \vdash \neg\phi.$$

Now by Lemma 5(ii) and the NIN rule,

$$\Gamma \vdash \neg\neg\theta$$

and by this same Lemma again and the NNO rule, $\Gamma \vdash \theta$.

(iii) $\Rightarrow$ (i) Exercise!! ■

We shall be dealing with consistent/inconsistent sets of formulae a lot in what follows and will be swapping between the equivalent formulations in Lemma 8 according to which is the most suitable at the time. We shall also be using Lemma 5 frequently in what follows and from now on we will not mention it explicitly, only the rule of proof involved.

The next lemma reveals the relationship between consistency and non-provability hinted at earlier.

Lemma 9

Let $\Gamma \subseteq FL$, $\theta \in FL$. *Then*

$$\Gamma \nvdash \theta \Leftrightarrow \Gamma \cup \{\neg\theta\} \textit{ is consistent.}$$

Proof We prove the contra-positive. If $\Gamma \vdash \theta$ then by MON

$\Gamma \cup \{\neg\theta\} \vdash \theta$ and by REF $\Gamma \cup \{\neg\theta\} \vdash \neg\theta$ so $\Gamma \cup \{\neg\theta\}$ is inconsistent.

Conversely if $\Gamma \cup \{\neg\theta\}$ is inconsistent, say $\Gamma \cup \{\neg\theta\} \vdash \phi$ and

$\Gamma \cup \{\neg\theta\} \vdash \neg\phi$ then by NIN, $\Gamma \vdash \neg\neg\theta$ so $\Gamma \vdash \theta$ by NNO. ■

So if $\Gamma \nvdash \zeta$ then $\Gamma \cup \{\neg\zeta\}$ is consistent and to complete the proof of the Completeness Theorem it is enough to show that whenever $\Delta \subseteq FL$ is consistent then Δ is *satisfiable*, that is there is a structure M for L and an assignment to the free variables according to which every formula in Δ is true. So what we want to do is somehow use Δ to construct such a structure M and assignment to the free variables.

The next few lemmas provide key steps in this construction.

Lemma 10

Let $\Gamma \subseteq FL$ be consistent.

i) *For $\theta \in FL$ at least one of $\Gamma \cup \{\theta\}, \Gamma \cup \{\neg\theta\}$ is consistent.*
ii) *If $\exists w_j \phi(w_j, \vec{x}) \in \Gamma$ and x_i does not occur in any formula in Γ then $\Gamma \cup \{\phi(x_i, \vec{x})\}$ is consistent.*

Proof i) Suppose both were inconsistent. Then for some ϕ_1, ϕ_2

$$\Gamma, \theta \vdash \phi_1,\ \Gamma, \theta \vdash \neg\phi_1,\ \Gamma, \neg\theta \vdash \phi_2,\ \Gamma, \neg\theta \vdash \neg\phi_2.$$

Then by NIN,

$$\Gamma \vdash \neg\theta,\ \Gamma \vdash \neg\neg\theta$$

so Γ is inconsistent, contradiction.

ii) Suppose that $\Gamma \cup \{\phi(x_i, \vec{x})\}$ was inconsistent. Then by Lemma 8(ii)

$$\Gamma, \phi(x_i, \vec{x}) \vdash \theta \wedge \neg\theta$$

where θ is any *sentence* of L. By the $\exists$O rule[25]

$$\Gamma, \exists w_j \phi(w_j, \vec{x}) \vdash \theta \wedge \neg\theta$$

so $\Gamma \cup \{\exists w_j \phi(w_j, \vec{x})\}$ is inconsistent. But this *is* Γ since $\exists w_j \phi(w_j, \vec{x})$ is already a member of Γ, contradiction. ■

Lemma 11

Suppose that $\Gamma_0, \Gamma_1, \Gamma_2, \ldots$ *are consistent subsets of* FL *such that*

$$\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq, \ldots. \tag{19}$$

Then their union

$$\bigcup_{n \in \mathbb{N}} \Gamma_n = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \ldots$$

is consistent.

Proof Suppose on the contrary that $\bigcup_{n \in \mathbb{N}} \Gamma_n$ was inconsistent, say,

$$\bigcup_{n \in \mathbb{N}} \Gamma_n \vdash \phi \wedge \neg\phi.$$

Let $\Delta_1 \mid \theta_1, \ldots, \Delta_{\mathrm{m}} \mid \theta_m$ be a proof of this, so

$$\Delta_{\mathrm{m}} \subseteq \bigcup_{n \in \mathbb{N}} \Gamma_n \tag{20}$$

and $\theta_m = \phi \wedge \neg\phi$. Now by definition of a proof Δ_m is finite, say,

$$\Delta_m = \{\eta_1, \eta_2, \ldots, \eta_r\}.$$

From (20) each $\eta_i \in \Gamma_{k_i}$ for some $k_i \in \mathbb{N}$. Let k be the largest of these k_i. [This is where we need the finiteness of Δ_m, since an infinite set of natural numbers need not have a largest member.] By (19) the Γ_i are increasing so for each $i = 1, 2, \ldots, r$,

$$\eta_i \in \Gamma_{k_i} \subseteq \Gamma_k.$$

But that means that

$$\Delta_m = \{\eta_1, \eta_2, \ldots, \eta_r\} \subseteq \Gamma_k$$

so

$$\Delta_1 \mid \theta_1, \ldots, \Delta_m \mid \theta_m$$

is also a proof of $\theta_m = (\phi \wedge \neg\phi)$ from Γ_k, contradicting the assumed consistency of Γ_k. The result follows. ■

At this point we are going to make an assumption about L which will simplify the proof.[26] We shall assume that we can list, or enumerate, the formulae of L as

$$\eta_1, \eta_2, \eta_3, \ldots, \eta_i, \ldots \quad \text{for } 0 < i \in \mathbb{N}.$$

With this assumption in place we now prove the following:

Lemma 12

Let $\Delta \subseteq FL$ be consistent and suh that there are infinitely many free variables which do not occur in any formula in Δ. Then there is a consistent $\Delta \subseteq \Omega \subseteq FL$ such that

i) *For any $\theta \in FL$ either $\theta \in \Omega$ or $\neg\theta \in \Omega$.*
ii) *If $\exists w_j \phi(w_j, \vec{x}) \in \Omega$ then $\phi(x_r, \vec{x}) \in \Omega$ for some r.*

Proof Let $\eta_1, \eta_2, \eta_3, \ldots$ enumerate FL and define Δ_i for $i \in \mathbb{N}$ inductively as follows.

For $i = 0$ set $\Delta_0 = \Delta$.

Now suppose that $i > 0$ and Δ_{i-1} has been defined and is consistent and has the property that there are infinitely many free variables not occurring in any formula in Δ_{i-1}. Proceed as follows:

If $\{\eta_i\} \cup \Delta_{i-1}$ is consistent and $\eta_i = \exists w_j \phi(w_j, \vec{x})$ for some ϕ, w_j pick an x_r not appearing in any formula in $\{\eta_i\} \cup \Delta_{i-1}$ (possible because there are infinitely many not occurring in any formula in Δ_{i-1} and at most finitely many of them have been ruled out because of occurring in η_i) and set $\Delta_i = \{\eta_i, \phi(x_r, \vec{x})\} \cup \Delta_{i-1}$. By Lemma 10(ii) Δ_i is consistent. Also there are still infinitely many free variables not occurring in any formula in Δ_i since all those for Δ_{i-1} except the finitely many introduced by adding $\eta_i, \phi(x_r, \vec{x})$ are still available.

If $\{\eta_i\} \cup \Delta_{i-1}$ is consistent and η is not of the form $\exists w_j \phi(w_j, \vec{x})$ for any ϕ then put $\Delta_i = \{\eta_i\} \cup \Delta_{i-1}$. Again infinitely many free variables do not occur in any formula in Δ_i and Δ_i is consistent.

Finally if $\{\eta_i\} \cup \Delta_{i-1}$ is not consistent put $\Delta_i = \{\neg\eta_i\} \cup \Delta_{i-1}$. By Lemma 10(i) Δ_i is consistent and again infinitely many free variables do not occur in any formula in Δ_i.

Clearly by induction all the Δ_i get defined and are consistent and satisfy

$$\Delta_0 \subseteq \Delta_1 \subseteq \Delta_2 \subseteq \ldots.$$

Now put

$$\Omega = \bigcup_{i \in \mathbb{N}} \Delta_i.$$

Clearly $\Delta = \Delta_0 \subseteq \Omega$. By Lemma 11 Ω is consistent. To see that Ω has the other required properties let $\theta \in FL$. Then since the η_i enumerate FL, $\theta = \eta_i$ for some i. But then by the construction one of $\eta_i, \neg\eta_i$ (i.e. one of θ, $\neg\theta$) gets into Δ_i and hence into Ω since $\Delta_i \subseteq \Omega$. This shows that Ω has property (i).

To show that Ω also satisfies (ii) suppose that $\theta = \exists w_j \phi(w_j, \vec{x}) = \eta_i \in \Omega$. If in the construction of Δ_i we put in η_i then by the construction, for some r,

$$\phi(x_r, \vec{x}) \in \Delta_i \subseteq \Omega$$

as required. On the other hand if we put $\neg\eta_i$ into Δ_i at this stage we would have both $\eta_i, \neg\eta_i \in \Omega$ so by Lemma 5(i) $\Omega \vdash \eta_i$ and $\Omega \vdash \neg\eta$ so Ω would not be consistent by Lemma 8(ii), contradiction.

It turns out that the Ω constructed in the above lemma has some very nice properties, as we now demonstrate. ∎

Lemma 13

Let Ω be as constructed in Lemma 12. Then for θ, ϕ, $\exists w_j \psi(w_j, \vec{x}) \in FL$:

(a) $\Omega \vdash \theta \Leftrightarrow \theta \in \Omega.$

(b) $\theta \in \Omega \Leftrightarrow \neg\theta \not\in \Omega.$

(c) $(\theta \wedge \phi) \in \Omega \Leftrightarrow \theta \in \Omega$ *and* $\phi \in \Omega.$

(d) $(\theta \vee \phi) \in \Omega \Leftrightarrow \theta \in \Omega$ *or* $\phi \in \Omega.$

(e) $(\theta \to \phi) \in \Omega \Leftrightarrow \theta \not\in \Omega$ *or* $\phi \in \Omega.$

(f) $\exists w_j \psi(w_j, \vec{x}) \in \Omega \Leftrightarrow \psi(x_i, \vec{x}) \in \Omega$ *for some free variable* x_i.

(g) $\forall w_j \psi(w_j, \vec{x}) \in \Omega \Leftrightarrow \psi(x_i, \vec{x}) \in \Omega$ *for all free variables* x_i.

Proof

(a) $\theta \in \Omega \Rightarrow \Omega \vdash \theta$ by REF. Conversely $\theta \not\in \Omega$ implies that $\neg\theta \in \Omega$ by Lemma 12(i) so $\Omega \vdash \neg\theta$ and $\Omega \vdash \theta$ is impossible since otherwise Ω would be inconsistent.

(b) $\theta \in \Omega \Rightarrow \Omega \vdash \theta$ by (a), so $\Omega \nvdash \neg\theta$ otherwise Ω would be inconsistent. $\neg\theta \not\in \Omega$ by (a). Conversely $\theta \not\in \Omega \Rightarrow \neg\theta \in \Omega$ by Lemma 12(i).

(e) Suppose $\theta \not\in \Omega$. Then by (a), (b), $\Omega \vdash \neg\theta$. Therefore, since $\vdash \neg\theta \to (\theta \to \phi)$ (see the Exercises), $\Omega \vdash (\theta \to \phi)$ by MP and $(\theta \to \phi) \in \Omega$ by (a). Similarly if $\phi \in \Omega$ then since $\vdash \phi \to (\theta \to \phi)$ (see the Exercises), we get by MP $\Omega \vdash (\theta \to \phi)$ and the required conclusion follows by (a). This proves the $\Leftarrow$ direction.

To show the converse suppose that neither $\theta \not\in \Omega$ nor $\phi \in \Omega$ hold. Then from (a) and (b) $\Omega \vdash \theta$ and $\Omega \vdash \neg\phi$ and by AND $\Omega \vdash \theta \wedge \neg\phi$. Since $\vdash (\theta \wedge \neg\phi) \to \neg(\theta \to \phi)$ (see the Exercises) by MP $\Omega \vdash \neg(\theta \to \phi)$ and hence by (a), (b), $(\theta \to \phi) \not\in \Omega$, as required.

(c),(d) –see the Exercises.

(f) If $\exists w_j \psi(w_j, \vec{x}) \in \Omega$ then by Lemma 12(ii), $\psi(x_i, \vec{x}) \in \Omega$ for some free variable x_i. Conversely if $\psi(x_i, \vec{x}) \in \Omega$ then by (a) $\Omega \vdash \psi(x_i, \vec{x})$ and by $\exists\mathrm{I}\,\Omega \vdash \exists w_j \psi(w_j, \vec{x})$ so $\exists w_j \psi(w_j, \vec{x}) \in \Omega$ by (a).

(g) If $\forall w_j \psi(w_j, \vec{x}) \in \Omega$ then by (a) $\Omega \vdash \forall w_j \psi(w_j, \vec{x})$ so by $\forall\mathrm{O}$ $\Omega \vdash \psi(x_i, \vec{x})$, and by (a) $\psi(x_i, \vec{x}) \in \Omega$, for any free variable x_i. Conversely suppose $\forall w_j \psi(w_j, \vec{x}) \not\in \Omega$, so by (a), (b), $\Omega \vdash \neg\forall w_j \psi(w_j, \vec{x})$. Since (see the Exercises)

$$\vdash \neg\forall w_j \psi(w_j, \vec{x}) \to \exists w_j \neg\psi(w_j, \vec{x})$$

so by MP, $\Omega \vdash \exists w_j \neg\psi(w_j, \vec{x})$. By (a) and (f) this gives $\neg\psi(x_i, \vec{x}) \in \Omega$ for some free variable x_i so, as required, for this x_i we cannot have $\psi(x_i, \vec{x}) \in \Omega$ otherwise Ω would be inconsistent. ■

We are now ready to prove the big theorem from which the Completeness Theorem will follow as a corollary.

Theorem 14

Let $\Delta \subseteq FL$. Then Δ is consistent if and only if Δ is satisfiable.

Proof Right to left is easy: Suppose Δ is satisfied, say in the structure M for some assignment to the free variables. If Δ was inconsistent we would have

$\Delta \vdash \phi$ and $\Delta \vdash \neg\phi$ for some $\phi \in FL$. But then by the Correctness Theorem ϕ and $\neg\phi$ would both have to be true in this interpretation, contradiction!

In the other direction suppose that Δ is consistent, and for the present that there are infinitely many free variables not mentioned in any formula in Δ. We need to construct a structure M and an assignment to the free variables in M in which every formula in Δ is true (or satisfied).

The construction of M is rather surprising, as we shall now see. Let $\Omega \supseteq \Delta$ be as in Lemmas 12 and 13. Set

$$|M| = \{x_1, x_2, x_3, \ldots\},$$

so the universe of M is the set of free variables (!), and for R an r-ary relation symbol of L set

$$\langle x_{i_1}, x_{i_2}, \ldots, x_{i_r} \rangle \in R^M \Leftrightarrow R(x_{i_1}, x_{i_2}, \ldots, x_{i_r}) \in \Omega,$$

equivalently,

$$M \vDash R(x_{i_1}, x_{i_2}, \ldots, x_{i_r}) \Leftrightarrow R(x_{i_1}, x_{i_2}, \ldots, x_{i_r}) \in \Omega.$$

[Notice that the $x_{i_1}, x_{i_2}, \ldots, x_{i_r}$ here on the left hand side are elements of the universe of M whilst on the right hand side they are free variables.]

Claim

For all $\theta(\vec{x}) \in FL$,

$$M \vDash \theta(\vec{x}) \Leftrightarrow \theta(\vec{x}) \in \Omega.$$

Again it is important to appreciate that the $\vec{x}$ appearing here are serving different roles. The $\vec{x}$ appearing on the left is a vector of *elements of the universe of* M whereas on the right it is a vector of free variables. So on the left it says that the formula $\theta(\vec{x})$ (here $\vec{x}$ is a vector of free variables) is satisfied by, or true of, the elements $\vec{x}$ from the universe of M.

Proof of Claim

The proof is by induction on the length of formulae. Assume the result is true for all formulae of length less than $|\theta|$. There are the usual 7 cases.

If θ is $R(\vec{x})$ for R a relation symbol of L the result is true by definition.

If $\theta(\vec{x}) = \phi(\vec{x}) \to \psi(\vec{x})$ then, since $|\phi(\vec{x})|$, $|\psi(\vec{x})| < |\theta(\vec{x})|$, so by inductive hypothesis

$$M \vDash \phi(\vec{x}) \Leftrightarrow \phi(\vec{x}) \in \Omega, \tag{21}$$

$$M \vDash \psi(\vec{x}) \Leftrightarrow \psi(\vec{x}) \in \Omega. \tag{22}$$

Then

$$
\begin{aligned}
M \vDash \theta(\vec{x}) &\Leftrightarrow M \vDash \phi(\vec{x}) \to \psi(\vec{x}) \\
&\Leftrightarrow M \nvDash \phi(\vec{x}) \text{ or } M \vDash \psi(\vec{x}) \\
&\Leftrightarrow \phi(\vec{x}) \notin \Omega \text{ or } \psi(\vec{x}) \in \Omega \text{ by (21), (22)} \\
&\Leftrightarrow (\phi(\vec{x}) \to \psi(\vec{x})) \in \Omega \text{ by Lemma 13(e)} \\
&\Leftrightarrow \theta(\vec{x}) \in \Omega.
\end{aligned}
$$

The cases for the other connectives are similar.

If $\theta(\vec{x}) = \exists w_j \chi(w_j, \vec{x})$ then since for any x_k, $|\chi(x_k, \vec{x})| < |\theta(\vec{x})|$, by inductive hypothesis

$$M \vDash \chi(x_k, \vec{x}) \Leftrightarrow \chi(x_k, \vec{x}) \in \Omega. \tag{23}$$

Hence

$$
\begin{aligned}
M \vDash \theta(\vec{x}) &\Leftrightarrow M \vDash \exists w_j \chi(w_j, \vec{x}) \\
&\Leftrightarrow M \vDash \chi(x_i, \vec{x}) \text{ for some } x_i \in |M| \\
&\Leftrightarrow \chi(x_i, \vec{x}) \in \Omega \text{ by (23) for some } x_i \\
&\Leftrightarrow \exists w_j \chi(w_j, \vec{x}) \in \Omega \text{ by Lemma 13(f)} \\
&\Leftrightarrow \theta(\vec{x}) \in \Omega.
\end{aligned}
$$

The case for $\theta = \forall w_j \chi(w_j, \vec{x})$ is similar and this completes the proof of the Claim.

But now we have that if $\theta(\vec{x}) \in \Delta$ then $\theta(\vec{x}) \in \Omega$ (since by construction $\Delta \subseteq \Omega$) and in turn $M \vDash \theta(\vec{x})$. In other words the formula $\theta(\vec{x})$ is satisfied in M by the elements $\vec{x}$ of the universe of M.

We are done, well almost! There is a small problem that we assumed at the start of all this that there were infinitely many free variables not occurring in any formula in Δ. So what if that's not the case? Well we first form Δ' by replacing every free variable x_i appearing in a formula in Δ by x_{2i}. Δ' is still consistent (see the Exercises) and now clearly there are infinitely many free variables not occurring in any formula in Δ' (certainly all the x_i with i odd). As above then we can construct M to satisfy Δ'. But then

$$
\begin{aligned}
\theta(x_1, x_2, \ldots, x_n) \in \Delta &\Rightarrow \theta(x_2, x_4, \ldots, x_{2n}) \in \Delta' \\
&\Rightarrow M \vDash \theta(x_2, x_4, \ldots, x_{2n})
\end{aligned}
$$

so Δ is satisfied in M (but now by assigning to the free variable x_i the element x_{2_i} of the universe of M).

Now we're really done! ■

The Completeness Theorem (for Relational L), 15

For $\Gamma \subseteq FL, \theta \in FL$,

$$\Gamma \vDash \zeta \Leftrightarrow \Gamma \vdash \zeta .$$

Proof By the Correctness Theorem in the $\Leftarrow$ direction and by Theorem 14 and the remarks following Lemma 9 in the $\Rightarrow$ direction. ■

The Completeness Theorem is one of the most important results in, or about, mathematics. For taking $\Gamma = \emptyset$ it tells us that

$$\vdash \zeta \Leftrightarrow \vDash \zeta ,$$

informally then, if something *must be true* then we can *prove* it, and conversely. So if this theorem did not hold in the $\Leftarrow$ direction we would be in the position that there would be mathematical truths which could never actually be proved whilst if it failed in the $\Rightarrow$ direction we would be able to prove statements which weren't necessarily true.

This result also clarifies an earlier doubt we might have had about the 'completeness' of the rules of proof that we wrote down. For at the time it seemed entirely possible that we could, and perhaps should, have added further rules to $A\text{O} - \exists\text{O}$. But we can now see that any extra rule we might add will either enable us to prove nothing beyond what we could get from $A\text{O} - \exists\text{O}$ alone, and so effectively be redundant, or will enable us to derive some new '$\Gamma \vdash \theta$' But in that latter case since it was not previously derivable, by the Completeness Theorem, we could not have $\Gamma \vDash \theta$ so we would have a 'proof' of θ from Γ even though there was an interpretation in which every formula in Γ was true whilst θ was false. In other words our 'proofs' would no longer preserve truth.

It is useful to bear the Completeness Theorem in mind when devising strategies for producing formal proofs because it can help one to set intermediate goals. To give an example suppose that you are looking for a proof of some assertion of the form $\theta \vee \phi \vdash \psi$. Now if there is some such formal proof it must be the case, by the Completeness Theorem, that $\theta \vee \phi \vDash \psi$. But then clearly $\theta \vDash \psi$ and $\phi \vDash \psi$, so by Completeness there must be proofs of $\theta \vdash \psi$ and $\phi \vdash \psi$, and if you can find such proofs you can put them together with DIS and obtain the proof you are looking for. The point here is that you have found two intermediate goals for which you know there must be proofs, and the tasks of finding them promises to be simpler that the one you were initially confronted with.

Apart from identifying proof and truth the Completeness Theorem is also remarkable for another reason. The assertion '$\Gamma \vDash \zeta$' is a 'FOR ALL' statement, it says that 'for all the infinitely many structures M if…'. However the assertion '$\Gamma \vdash \zeta$' is a 'THERE EXISTS' statement, it says 'there exists a (finite in fact) proof such that…'. To have a 'FOR ALL' statement equivalent to a 'THERE EXISTS' statement is very rare in mathematics[27] and when it happens it hints at something profound.

Finally, of course, the Completeness Theorem shows that our two, superficially different, formulations of 'follows' are actually one and the same.

The fact that proofs are just finite objects enables us to prove a very useful corollary of the Completeness Theorem:

The Compactness Theorem (for relational *L*) 16

Let $\Gamma \subseteq FL$. *Then* Γ *is satisfiable if and only if every finite subset of* Γ *is satisfiable.*

Proof Clearly if Γ is satisfiable, say in a structure M with some assignment to the free variables, then this same M and assignment also satisfies any subset of Γ, finite or not.

Conversely suppose that Γ is not satisfiable. Then by Theorem 14 Γ is not consistent, say $\Gamma \vdash (\phi \wedge \neg\phi)$. Let

$$\Gamma_1 \mid \theta_1, \Gamma_2 \mid \theta_2, \ldots, \Gamma_m \mid \theta_m$$

be a proof of this, so $\theta_m = (\phi \wedge \neg\phi)$ and $\Gamma_m \subseteq \Gamma$, and, being a left hand side in a proof, Γ_m is finite. But then this proof is also a proof of $\Gamma_{\mathrm{m}} \vdash (\phi \wedge \neg\phi)$, so Γ_m is a finite inconsistent subset of Γ and hence by Theorem 14 a finite unsatisfiable subset of Γ. ■

An Application of the Compactness Theorem

Let L have a single binary relation symbol R and let M be a structure for L. We say that M is *finitely colourable* if there are some finitely many disjoint subsets of $|M|$, say $A_1, A_2, \ldots, A_k$, with union $|M|$ (i.e. a finite partition of $|M|$) such that whenever $b, c \in |M|$ and $M \models R(b, c)$ then b, c are in different A_i. (Thinking of the A_i as colours then this says that if there is a directed edge from b to c (*i.e.* $\langle b, c\rangle \in R^M$), then b and c have different colours.)

Using the Compactness Theorem for Relational Languages we can show that there can be no sentence ψ of L such that, for any structure M for L,

$$M \models \psi \Leftrightarrow M \text{ is finitely colourable} \tag{24}$$

For suppose there was such a $\psi \in SL$ and consider the set of formulae

$$\Gamma = \{R(x_i, x_j) \mid 1 \leq i < j\} \cup \{\psi\}.$$

We shall show that Γ is satisfiable. Let $\Delta \subset \Gamma$ be finite, say m is maximal such that the free variable x_m occurs in some formula in Δ (or $m = 1$ if no free variables occur in formulae in Δ). Then

$$\Delta \subseteq \{R(x_i, x_j) \mid 1 \leq i < j \leq m\} \cup \{\psi\}$$

and this set of formulae is satisfied by $x_i \mapsto i$ in the structure M_m for L given by

$$|M_m| = \{1, 2, \ldots, m\},\ R^{M_m} = \{\langle i, j\rangle \mid 1 \leq i < j \leq m\},$$

– notice that $M_m \vDash \psi$ by (24) and the fact that the partition {1}, {2},..., $\{m\}$ provides a finite colouring of M_m. Hence Δ is satisfiable and hence by Compactness Γ is also satisfiable.

Let M be a structure for L in which Γ is satisfied, by $x_i \mapsto a_i \in |M|$ say. Since $M \vDash \psi$, by (24) M has a finite colouring, $A_1, A_2, \ldots, A_k$ say. Also since $R(x_i, x_j) \in \Gamma$ for $i < j$, $M \vDash R(a_i, a_j)$ so a_i and a_j must get different colours, i.e. be in different A_n. But there are infinitely many a_i and only k colours so this is impossible! We conclude that no such ψ could exist.

Adding Constants and Functions

Up to now we have, to avoid a lot of notation early on, limited ourselves to relational languages. However as we saw from the motivating examples at the start of the course in practice we often also include constants and functions in our reasoning. The plan now is to extend our languages to also include symbols for these (but not yet equality, =). Fortunately the main challenge this will involve is 'getting one's head round the notation' – the same theorems will go through with almost no extra effort.

Our earlier definition gives that with this addition

> *A language L (without equality) is a set consisting of some relation symbols and possibly some constant, function symbols. Each relation and function symbol in L has an arity (e.g. unary, binary, ternary, etc.).*

For this section let L be such a language. The addition of constants $c_1, c_2, \ldots$ and functions $f_1, f_2, f_3, \cdots$ to our language means that not only do we have free variables acting as elements of the universe but also new 'objects' such as c_1, $f_1(c_1, x_2)$, $f_1(f_2(x_1), f_1(c_1, x_2)), \ldots$ etc. for binary f_1, unary f_2 etc. The 'old' free variables together with these new objects are called the *terms of the language L*.

Precisely:

Definition For L a language the *terms* of L are defined as follows:

Te1 The free variables $x_1, x_2, x_3, \ldots,$ are terms of L.

Te2 If c is a constant symbol in L then c is a term of L.

Te3 If f is an n-ary function symbol of L and $t_1, t_2, \ldots, t_n$ are terms of L then $f(t_1, t_2, \ldots, t_n)$ is a term of L.

Te4 t is a term of L just if this follows in a finite number of steps from Tel-3.

We denote the set of all terms of L by TL. Analogously to Theorem 1 we can prove a unique readability results for terms (and for the soon to be introduced formulae of this language).

Example Let L have a binary relation symbol R, a binary function symbol f and a constant symbol c. Then

$$c, x_1, x_2 \in TL \text{, by Tel, Te2}$$

$$f(x_2, x_1),\ f(c, c),\ f(c, x_2) \in TL \text{, by Te3}$$

$$f(f(c, x_2), x_2) \in TL \text{ by Te3}$$

Clearly the definition of the terms of L closely parallels that of the formulae of L and we employ similar conventions. For example if we denote a term t by $t(x_{i_1}, x_{i_2}, \ldots, x_{i_n})$ then it will be implicit that all the free variables occurring in t are amongst $x_{i_1}, x_{i_2}, \ldots, x_{i_n}$ (though they don't all *have* to occur in *t*) and $t(b_1, b_2, \ldots, b_r)$ is the result of simultaneously replacing each x_{i_j} in t by b_j etc..

Generally we will use $t, t_1, t_2, s, s_1, \ldots$ for terms.

As with the formulae we can define the *length* of a term t, denoted $|t|$, as the number of symbols in t where each free variable, constant symbol, function symbol has length 1. So for example $|f_1(f_2(x_1), f_1(c_1, x_2))| = 12$ (again we don't count commas). Again as with formulae we can prove results about terms by *induction on the length of terms*.

For example we can show that, as with formulae, every term contains as many left parentheses '(' as right parentheses ')'

Notice that if L is a relational language (i.e. has no constants or function symbols) then $TL = \{x_1, x_2, x_3, \ldots\}$ is just the set of free variables.

The presence of terms in the language L (in addition to the free variables) forces us to make a minor change to the definition of 'formula of L':

Definition For L a language the *formulae* of L are defined as follows:

L1 If R is an n-ary relation symbol of L and $t_1, t_2, \ldots, t_n$ are terms of L then $R(t_1, t_2, \ldots, t_n)$ is a formula of L.

L2 If θ, ϕ are formulae of L then so are $(\theta \to \phi)$, $(\theta \wedge \phi)$, $(\theta \vee \phi)$, $\neg\theta$.

L3 If ϕ is a formula of L which does not mention w_j and $\phi(w_j/x_i)$ is the result of replacing the free variable x_i in ϕ by the bound variable w_j then $\exists w_j \phi(w_j/x_i)$, $\forall w_j \phi(w_j/x_i)$ are formulae of L.

L4 ϕ is a formulae of L just if this follows in a finite number of steps from Ll-3.

We continue to denote the set of formulae of L by FL (etc.). Continuing with the example of the language L above:

$$R(c, x_2),\ R(c, f(x_2, x_1)) \in FL, \quad \text{by L1}$$
$$(R(c, f(x_2, x_1)) \to R(c, x_2)) \in FL, \quad \text{by L2}$$
$$\forall w_3(R(c, f(w_3, x_1)) \to R(c, w_3)) \in FL, \quad \text{by L3.}$$

Interpretations

The examples at the start of this course already demonstrated how we interpret, or give a semantics to, the function and constants symbols. Namely a constant symbol is interpreted as a fixed element of the universe and an r-ary function symbol is interpreted as a function from r-tuples of element of the universe into the universe.

To give an example for L above if we set the universe to be $\mathbb{N} = \{0,1,2,\dots\}$, interpret c as 3, assign x_1 value 4, interpret f as multiplication and R as 'divides' then

$$\forall w_3(R(c, f(w_3, x_1)) \rightarrow R(c, w_3))$$

becomes

For all natural numbers n, *if 3 divides* $n \times 4$ *then 3 divides* n

– which is true, though if we had instead assigned x_1 the value 9 it would have been false.

As before we split an 'interpretation' into two parts, a structure, which interprets the relation, constant and function symbols of L, and an assignment to the free variables.

Definition

A *structure* M for a language L consists of:

- a non-empty set $|M|$, called the *universe* of M,
- for each n-ary relation symbol R of L a subset R^M of $|M|^n$ (equivalently an n-ary relation on $|M|$),
- for each constant symbol c of L a fixed element c^M of $|M|$,
- for each n-ary function symbol f of L a function $f^M : |M|^n \rightarrow |M|$.

In this case we often write

$$M = \langle |M|, R_1^M, R_2^M, \dots, c_1^M, c_2^M, \dots, f_1^M, f_2^M, \dots, \rangle$$

where $R_1, R_2, \dots, c_1, c_2, \dots,$ and $f_1, f_2, \dots$ are respectively the relation/constant/function symbols of L.

Examples

Let $L = \{R, c, f\}$ as above, so R and f are both binary. Then some structures for L are:

(a) Universe of M is $\mathbb{N}$, i.e. $|M| = \mathbb{N}$,

$$R^M = \{\langle n, m\rangle \in \mathbb{N}^2 \mid n \text{ divides } m\},$$
$$c^M = 3,$$
$$f^M(n, m) = n \times m.$$

(b) Universe of M is $\mathbb{R}$,

$$R^M = \{\langle s, t\rangle \in \mathbb{R}^2 \mid t \neq 0 \;\&\; s / t \in \mathbb{Q}\},$$

$$c^M = 0,$$

$$f^M(r, s) = \begin{cases} 1 & if\ r < s, \\ s & \text{otherwise.} \end{cases}$$

(c) Universe of M is {1, 2, 3},

$$R^M = \{\langle 2,1\rangle, \langle 1,2\rangle, \langle 3,1\rangle, \langle 3,3\rangle\},$$

$$c^M = 3,$$

$$f^M : \{1,2,3\}^2 \to \{1,2,3\} \text{ by } f^M(1,1) = 2,$$

$$f^M(1,2) = 2,\ f^M(1,3) = 3,\ f^M(2,1) = 3,\ f^M(2,2) = 2,$$

$$f^M(2,3) = 1,\ f^M(3,1) = 1,\ f^M(3,2) = 2,\ f^M(3,3) = 3$$

or as an easier to read table:

f^M	1	2	3
1	2	2	3
2	3	2	1
3	1	2	3

Truth

In order to now talk about the truth of a formula in an interpretation we need to first talk about the *value of a term* in an interpretation. So let $t(x_1, x_2, \ldots, x_n) \in TL$ and let M be a structure for L. Then we define that value of $t(\vec{x})$ in M when x_i is assigned value $a_i \in |M|$, written $t^M(a_1, a_2, \ldots, a_n)$, by induction on $|\, t(\vec{x}) \,|$ as follows:

V1 For $t(\vec{x}) = x_i$, $t^M(\vec{a}) = a_i$.

V2 For $t(\vec{x}) = c$, where c is a constant symbol of L, $t^M(\vec{a}) = c^M$.

V3 For $t(\vec{x}) = f(t_1(\vec{x}), t_2(\vec{x}), \ldots, t_r(\vec{x}))$, where f is an r-ary function symbol of L and $t_1(\vec{x})$, $t_2(\vec{x}), \ldots, t_r(\vec{x}) \in TL$,

$$t^M(\vec{a}) = f^M(t_1^M(\vec{a}), t_2^M(\vec{a}), \ldots, t_r^M(\vec{a})).$$

This may look rather complicated but all it really says is: To find $t^M(\vec{a})$ replace the x_i by a_i, the c by c^M, the f by f^M and evaluate. So for example in the last example above if $t(x_1, x_2) = f(f(c, x_1), x_2)$ and $a_1 = 1, a_2 = 3$ then

$$\begin{aligned} t^M(a_1, a_2) &= f^M(f^M(c^M, a_1), a_2) \\ &= f^M(f^M(3,1), 3) \text{ since } a_1 = 1, a_2 = 3, c^M = 3, \\ &= f^M(1,3) \text{ since } f^M(3,1) = 1 \\ &= 3 \text{ since } f^M(1,3) = 3 \end{aligned}$$

Having got the evaluation of terms out of the way we can now define the truth of a formula in a structure for an assignment to the free variables by a minor generalization of the definition for relational languages.

For $\eta(x_1, x_2, \ldots, x_n) \in FL$, M a structure for L and any assignment $x_1 \mapsto a_i \in | M |$ to the free variables, we define

$$M \vDash \eta(a_1, a_2, \ldots, a_n),$$

said '$\eta(a_1, a_2, \ldots, a_n)$ is true in M, or $\eta(x_1, x_2, \ldots, x_n)$ is satisfied in M by $a_1, a_2, \ldots, a_n$', by induction on the length of $\eta(\vec{x}) \in FL$ in the obvious way:

T1 For $R(t_1(\vec{x}), t_2(\vec{x}), \ldots, t_n(\vec{x})) \in FL$, where R is an n-ary relation symbol in L and $t_1(\vec{x})$, $t_2(\vec{x}), \ldots, t_n(\vec{x})$ are terms of L,

$$\begin{aligned} M \vDash R(t_1(\vec{a}), t_2(\vec{a}), \ldots, t_n(\vec{a})) &\Leftrightarrow \langle t_1^M(\vec{a}), t_2^M(\vec{a}), \ldots, t_n^M(\vec{a})\rangle \in R^M \\ &\Leftrightarrow \text{the relation interpreting } R \text{ in } M \\ &\quad\;\; \text{holds for } t_1^M(\vec{a}),\ t_2^M(\vec{a}), \ldots, t_n^M(\vec{a}). \end{aligned}$$

T2 For formulae $\theta(x_1, x_2, \ldots, x_n)$, $\phi(x_1, x_2, \ldots, x_n)$ etc. of L and $a_1, a_2, \ldots, a_n \in |M|$,

$$\begin{aligned} M \vDash \neg\phi(\vec{a}) &\quad\Leftrightarrow\quad \text{not } M \vDash \phi(\vec{a}),\ \text{i.e. } M \nvDash \phi(\vec{a}) \\ M \vDash \theta(\vec{a}) \wedge \phi(\vec{a}) &\quad\Leftrightarrow\quad M \vDash \theta(\vec{a}) \text{ and } M \vDash \phi(\vec{a}) \\ M \vDash \theta(\vec{a}) \vee \phi(\vec{a}) &\quad\Leftrightarrow\quad M \vDash \theta(\vec{a}) \text{ or } M \vDash \phi(\vec{a}) \\ M \vDash \theta(\vec{a}) \to \phi(\vec{a}) &\quad\Leftrightarrow\quad M \nvDash \theta(\vec{a}) \text{ or } M \vDash \phi(\vec{a}). \end{aligned}$$

T3 $M \vDash \forall w_j \psi(w_j, \vec{a}) \Leftrightarrow$ For all $b \in |M|$, $M \vDash \psi(b, \vec{a})$.

$M \vDash \exists w_j \psi(w_j, \vec{a}) \Leftrightarrow$ For some $b \in |M|$, $M \vDash \psi(b, \vec{a})$.

Example

Let L have a constant symbol c, binary function symbol f, unary function symbol g and binary relation symbol E. Let M be the structure for L such that $|M| = \mathbb{N}$, $c^M = 0$, $g^M(n) = n+1$, $f^M(n, m) = n + m$ and E^M is just the equality relation. Then

$$\forall w_1 \forall w_2 E(f(g(w_1), w_2), g(f(w_1, w_2))) \in FL$$

and[28]

$$M \vDash \forall w_1 \forall w_2 E(f(g(w_1), w_2), g(f(w_1, w_2))$$

$\Leftrightarrow$ For all $n, m \in |M|$ $(= \mathbb{N})$, $M \vDash E(f(g(n), m), g(f(n, m)))$

$\Leftrightarrow$ $\forall n, m \in \mathbb{N}, \langle (f(g(n), m))^M, (g(f(n, m)))^M \rangle \in E^M$,

$\Leftrightarrow$ $\forall n, m \in \mathbb{N}, \langle f^M(g^M(n), m),\ g^M(f^M(n, m)) \rangle \in E^M$,

$\Leftrightarrow$ $\forall n, m \in \mathbb{N}, f^M(g^M(n), m) = g^M(f^M(n, m))$,

since E^M is equality,

$\Leftrightarrow$ $\forall n, m \in \mathbb{N}, (n+1) + m = (n+m) + 1$,

since $g^M(k) = k+1$ and $f^M(n, m) = n + m$,

which we know is true.

Note In examples like this we often in practice use more descriptive symbols than E, g, f, c typically using the symbols $\_ = \_$ in place of $E(\_,\_)$, $\_ + \_$ in place of $f(\_,\_)$, the symbol 0 in place of c etc. We also often abbreviate $\forall w_1 \forall w_2$ by $\forall w_1, w_2$, as well as using x, w, y, z, etc., for both free and bound variables. As your confidence grows you will easily adopt these standard practices (!)

Another Example

Let M be as in the example (c) above, so $|M| = \{1,2,3\}$,

$$R^M = \{\langle 2,1\rangle, \langle 1,2\rangle, \langle 3,1\rangle, \langle 3,3\rangle\},$$

$$c^M = 3,$$

f^M	1	2	3
1	2	2	3
2	3	2	1
3	1	2	3

Then $\exists w_1 \forall w_2 R(f(w_1, w_2), x_1)$ is true in M when x_1 is assigned value 1 (equivalently is satisfied by 1 in M) since

$M \vDash R((f(1,1), 1)$, because $f^M(1,1) = 2, \langle 2,1\rangle \in R^M$,
$M \vDash R(f(1,2), 1)$, because $f^M(1,2) = 2, \langle 2,1\rangle \in R^M$,
$M \vDash R(f(1,3), 1)$, because $f^M(1,3) = 3, \langle 3.1\rangle \in R^M$,

so

$$M \vDash \forall w_2 R(f(1, w_2), 1)$$

and hence

$$M \vDash \exists w_1 \forall w_2 R(f(w_1, w_2), 1).$$

We can now define logical consequence by directly generalizing the previous version, viz:

Definition Let L be a language, Γ a set (possibly empty) of formulae of L (*i.e.* $\Gamma \subseteq FL$) and $\theta \in FL$. Then θ is a *logical consequence* of Γ (equivalently Γ *logically implies* θ), denoted $\Gamma \vDash \theta$, if for any structure M for L and any assignment to the free variables $x_1, x_2, \ldots$ appearing in the formulae in Γ or θ, *if* every formula in Γ is true in that interpretation *then* θ is true in that interpretation.[29]

If $\Gamma \subseteq SL$, $\theta \in SL$ (i.e. θ and every formula in Γ is actually a sentence), the usual situation in fact when logic is being applied, then we can drop mention of the assignment part of the interpretation to obtain: Γ logically implies θ, $\Gamma \vDash \theta$, if for every structure M for L, if $M \vDash \phi$ for each $\phi \in \Gamma$[30] then $M \vDash \theta$.

Example

Let $L = \{R, f, \ldots\}$ with R a binary relation symbol and f a unary function symbol. Then

$$\forall w_1 R(w_1, f(w_1)) \vDash \forall w_1 \exists w_2 R(w_1, w_2) \tag{25}$$

Proof Let M be a structure for L such that[31]

$$M \vDash \forall w_1 R(w_1, f(w_1)).$$

Then by (T3), for all $a \in |M|$,

$$M \vDash R(a, f(a)), \;\; so \; \langle a, f^M(a) \rangle \in R^M$$

by (T1). Hence

$$M \vDash R(a, f^M(a)), \; so \; M \vDash \exists w_2 R(a, w_2).$$

Finally since $a \in |M|$ was arbitrary,

$$M \vDash \forall w_1 \exists w_2 R(w_1, w_2),$$

which completes the proof of (25).

Notice that in the above example we have gone from $M \vDash R(a, f(a))$ to $M \vDash R(a, f^M(a))$. And we could equally have gone in the other direction. In fact this facility of 'replacing a term' by its value is quite general, as the next two lemmas show.

Lemma 17

Let $s(x_1, x_2, \ldots, x_n) \in TL$ and $t_1(\vec{x}), t_2(\vec{x}), \ldots, t_n(\vec{x}) \in TL$. Then $s(t_1(\vec{x}), t_2(\vec{x}), \ldots, t_n(\vec{x})) \in TL$ and for any structure M for L and $\vec{a} \in |M|$,

$$(s(t_1(\vec{a}), t_2(\vec{a}), \ldots, t_n(\vec{a})))^M = s^M(t_1^M(\vec{a}), t_2^M(\vec{a}), \ldots, t_n^M(\vec{a})).$$

Proof* The proof is by induction on the length $|s|$ of s. Assume the result holds for terms of length less than $|s|$. There are 3 cases.

Case 1: $s = x_i$, a free variable.

In this case

$$s(t_1(\vec{x}), t_2(\vec{x}),\ldots, t_n(\vec{x})) = t_i(\vec{x}) \in TL$$

and by **Vl**

$$(s(t_1(\vec{a}), t_2(\vec{a}),\ldots, t_n(\vec{a})))^M = (t_i(\vec{a}))^M = t_i^M(\vec{a}) = s^M(t_1^M(\vec{a}), t_2^M(\vec{a}),\ldots, t_n^M(\vec{a})), \text{ as required.}$$

Case 2: $s = c$, a constant symbol.

In this case

$$s(t_1(\vec{x}), t_2(\vec{x}),\ldots, t_n(\vec{x})) = c \in TL$$

and by **V2**

$$(s(t_1(\vec{a}), t_2(\vec{a}),\ldots, t_n(\vec{a})))^M = c^M = s^M(t_1^M(\vec{a}), t_2^M(\vec{a}),\ldots, t_n^M(\vec{a})),$$

as required.

Case 3: $s = f(s_1\ (x_1,\ldots,x_n),\ldots,s_r(x_1,\ldots,x_n))$ where $s_1,\ldots,s_r \in TL$ and f is an r-ary function symbol of L.

In this case,

$$s(t_1(\vec{x}), t_2(\vec{x}),\ldots, t_n(\vec{x})) = f(s_1(t_1(\vec{x}),\ldots,t_n(\vec{x})),\ldots,s_r(t_1(\vec{x}),\ldots,t_n(\vec{x}))).$$

Since the $|s_i| < |s|$ the result already holds for them, so the $s_i(t_1(\vec{x}),\ldots,t_n(\vec{x})) \in TL$ by inductive hypothesis, and hence

$$f(s_1\ (t_1(\vec{x}),\ldots,t_n(\vec{x})),\ldots,s_r(t_1(\vec{x}),\ldots,t_n(\vec{x}))) \in TL$$

by **Te3**. Also, using **V3**,

$$
\begin{aligned}
(s(t_1(\vec{a}), t_2(\vec{a}),\ldots,t_n(\vec{a})))^M &= \\
&= (f(s_1(t_1(\vec{a}),\ldots,t_n(\vec{a})),\ldots,s_r(t_1(\vec{a}),\ldots,t_n(\vec{a}))))^M \\
&= f^M((s_1(t_1(\vec{a}),\ldots,t_n(\vec{a})))^M,\ldots,(s_r(t_1(\vec{a}),\ldots,t_n(\vec{a})))^M) \\
&= f^M(s_1^M(t_1^M(\vec{a}),\ldots,t_n^M(\vec{a})),\ldots,s_r^M(t_1^M(\vec{a}),\ldots,t_n^M(\vec{a}))) \\
&\qquad \text{– by inductive hypothesis,} \\
&= s^M(t_1^M(\vec{a}),\ldots,t_n^M(\vec{a})), \text{ as required.}
\end{aligned}
$$

■

Lemma 18

Let $\theta(x_1, x_2,\ldots,x_n) \in FL$ *and* $t_1(\vec{x})$, $t_2(\vec{x}),\ldots,t_n(\vec{x}) \in TL$.[32] *Then* $\theta(t_1(\vec{x}), t_2(\vec{x}),\ldots,t_n(\vec{x})) \in FL$ *and for any structure* M *for* L *and* $\vec{a} \in |M|$,

$$M \vDash \theta(t_1(\vec{a}), t_2(\vec{a}),\ldots,t_n(\vec{a})) \Leftrightarrow M \vDash \theta(t_1^M(\vec{a}), t_2^M(\vec{a}),\ldots,t_n^M(\vec{a}))$$

Proof* The proof is by induction on the length of $\theta(x_1, x_2,\ldots,x_n)$. Assume true for all formulae of length less than $|\theta|$. There are various cases.

Case 1: $\theta = R(s_1\ (x_1,\ldots,x_n),\ldots,s_r(x_1,\ldots,x_n))$ where the $s_1,\ s_2,\ldots,s_r \in TL$, and R is an r-ary relation symbol of L.

Then the $s_i(t_1(\vec{x}),\ldots,t_n(\vec{x})) \in TL$ as shown in Lemma 17 so

$$\theta(t_1(\vec{x}),\ldots,t_n(\vec{x})) = R(s_1(t_1(\vec{x}),\ldots,t_n(\vec{x})),\ldots,s_r(t_1(\vec{x}),\ldots,t_n(\vec{x}))) \in FL$$

by **L1** and

$$M \vDash \theta(t_1(\vec{a}),\ldots,\ t_n(\vec{a})) \Leftrightarrow$$

$$\Leftrightarrow M \vDash R(s_1(t_1(\vec{a}),\ldots,t_n(\vec{a})),\ldots,s_r(t_1(\vec{a}),\ldots,t_n(\vec{a})))$$

$$\Leftrightarrow \langle(s_1\ (t_1(\vec{a}),\ldots,t_n(\vec{a})))^M,\ldots,(s_r(t_1(\vec{a}),\ldots,t_n(\vec{a}))))^M\rangle \in R^M \quad \text{by } \mathbf{T1},$$

$$\Leftrightarrow \langle s_1^M(t_1^M(\vec{a}),\ldots,t_n^M(\vec{a})),\ldots,s_r^M(t_1^M(\vec{a}),\ldots,t_n^M(\vec{a})))\rangle \in R^M \quad \text{by Lemma 17},$$

$$\Leftrightarrow M \vDash R(s_1(t_1^M(\vec{a}),\ldots,t_n^M(\vec{a})),\ldots,s_r(t_1^M(\vec{a}),\ldots,t_n^M(\vec{a}))) \quad \text{by } \mathbf{T1},$$

$$\Leftrightarrow M \vDash \theta(t_1^M(\vec{a}),\ldots,t_n^M(\vec{a})) \text{ by } \mathbf{T1},$$

as required.

Case 2: $\theta(x_1,\ldots,x_n) = \neg\phi(x_1,\ldots,x_n)$.

In this case since $|\phi| < |\theta|$, $\phi(t_1(\vec{x}),\ldots,t_n(\vec{x})) \in FL$ by inductive hypothesis so

$$\theta(t_1(\vec{x}),\ldots,t_n(\vec{x})) = \neg\phi(t_1(\vec{x}),\ldots,t_n(\vec{x})) \in FL \text{ by } \mathbf{L2}.$$

Also

$$\begin{aligned} M \vDash \theta(t_1(\vec{a}),\ldots,t_n(\vec{a})) &\Leftrightarrow M \nvDash \phi(t_1(\vec{a}),\ldots,t_n(\vec{a})) \\ &\Leftrightarrow M \nvDash \phi(t_1^M(\vec{a}),\ldots,t_n^M(\vec{a})) \text{ by ind. hyp.} \\ &\Leftrightarrow M \vDash \theta(t_1^M(\vec{a}),\ldots,t_n^M(\vec{a})) \text{ by } \mathbf{T2}, \end{aligned}$$

as required. The cases for the other connectives are similar.

Case 3: $\theta(x_1,\ldots,x_n) = \exists w_j\phi(x_1,\ldots,x_n,\ w_j)$ where $\phi(x_1,\ldots,x_n,\ x_{n+1}) \in FL$. [33]

Let x_k not appear in $\vec{x}$ or $x_1,\ x_2,\ldots,x_n$. Then since

$$|\phi\ (x_1,\ldots,x_n,\ x_{n+1})| < |\theta\ (x_1,\ldots,\ x_n)|,$$

by inductive hypothesis

$$\phi(t_1(\vec{x}),\ldots,t_n(\vec{x}),x_k) \in FL$$

and so by **L3**

$$\exists w_j \phi(t_1(\vec{x}),\ldots,t_n(\vec{x}),w_j) = \theta(t_1(\vec{x}),\ldots,t_n(\vec{x})) \in FL.$$

Also

$$\begin{aligned} & M \vDash \theta(t_1(\vec{a}),\ldots,t_n(\vec{a})) \\ & \quad \Leftrightarrow M \vDash \exists w_j \phi(t_1(\vec{a}),\ldots,t_n(\vec{a}),w_j) \\ & \quad \Leftrightarrow \exists b \in |M|, M \vDash \phi(t_1(\vec{a}),\ldots,t_n(\vec{a}),b) \\ & \quad \Leftrightarrow \exists b \in |M|, M \vDash \phi(t_1^M(\vec{a}),\ldots,t_n^M(\vec{a}),b) \quad \text{by ind. hyp.} \\ & \quad \Leftrightarrow M \vDash \exists w_j \phi(t_1^M(\vec{a}),\ldots,t_n^M(\vec{a}),w_j) \\ & \quad \Leftrightarrow M \vDash \theta(t_1^M(\vec{a}),\ldots,t_n^M(\vec{a})), \end{aligned}$$

as required. ■

The case for $\forall$ is similar.

The following corollary to Lemma 18 will prove useful later on.

Corollary 19 *Let M be a structure for L, $t(\vec{x}) \in TL$, $\psi(x_{n+1},\vec{x}) \in FL$ and $\vec{a} \in |M|$, where $\vec{x} = x_1,\ldots,x_n$ etc. Then*

(a) *If $M \vDash \forall w_i \psi(w_i,\vec{a})$ then $M \vDash \psi(t(\vec{a}),\vec{a})$.*

(b) *If $M \vDash \psi(t(\vec{a}),\vec{a})$ then $M \vDash \exists w_i \psi(w_i,\vec{a})$.*

Before we commence with the proof notice that this corollary is not quite as obvious as it might appear at first glance. In (a) for example it says that if the formula $\forall w_i \psi(w_i,\vec{x})$ is satisfied in M by the assignment $x_i \mapsto a_i$ then the formula $\psi(t(\vec{x}),\vec{x})$ is also satisfied in M by this assignment.

Proof For (a), if $M \vDash \forall w_i \psi(w_i,\vec{a})$ then $M \vDash \psi(t^M(\vec{a}),\vec{a})$ by **T3**. Hence by Lemma 18, $M \vDash \psi(t(\vec{a}),\vec{a})$.

Part (b) follows similarly, if $M \vDash \psi(t(\vec{a}),\vec{a})$ then $M \vDash \psi(t^M(\vec{a}),\vec{a})$ by Lemma 18 and $M \vDash \exists w_i \psi(w_i,\vec{a})$ follows by **T3**. ■

Formal Proofs with Constants and Functions

In the case where the language has constant and/or function symbols the rules of proof are the same except that ∀O and ∃I generalize from a free variable substitution (in the case of a relational language where the only terms we have are the free variables) to a general term as follows:

The Rules of Proof for the Predicate Calculus (possibly with constant and function symbols)

And In (AND) $\dfrac{\Gamma \mid \theta,\ \ \Delta \mid \phi}{\Gamma \cup \Delta \mid \theta \wedge \phi}$

And Out (AO) $\dfrac{\Gamma \mid \theta \wedge \phi}{\Gamma \mid \theta}$ $\dfrac{\Gamma \mid \theta \wedge \phi}{\Gamma \mid \phi}$

Or In (ORR) $\dfrac{\Gamma \mid \theta}{\Gamma \mid \theta \vee \phi}$ $\dfrac{\Gamma \mid \theta}{\Gamma \mid \phi \vee \theta}$

Disjunction (DIS) $\dfrac{\Gamma, \theta \mid \psi,\ \ \Delta, \phi \mid \psi}{\Gamma \cup \Delta,\ \ \theta \vee \phi \mid \psi}$

Implies In (IMR) $\dfrac{\Gamma, \theta \mid \phi}{\Gamma \mid \theta \rightarrow \phi}$

Modus Ponens (MP) $\Gamma \mid \theta, \quad \Delta \mid \theta \rightarrow \phi$

Exists In $(\exists \mathrm{I})$ $\dfrac{\Gamma \mid \theta}{\Gamma \mid \exists w_j \theta'}$ where θ' is the result of replacing any number of occurences of the term $t(\vec{x})$ in θ by w_j in θ by w_j and w_j does not occur in θ.

Exists Out $(\exists \mathrm{O})$ $\dfrac{\Gamma, \phi \mid \theta}{\Gamma, \exists w_j \phi(w_j \,/\, x_i) \mid \theta}$ where x_i does not occur in θ nor any formula in Γ and w_j does not occur in ϕ.

REF $\Gamma \mid \theta$ whenever $\theta \in \Gamma$.

We now define (formal) proofs as before but now with these enhanced rules.

Example

A formal proof of $\forall w_1 R(w_1, f(w_1)) \vdash \forall w_1 \exists w_2 R(w_1, w_2)$.

1. $\forall w_1 R(w_1, f(w_1)) \mid \forall w_1 R(w_1, f(w_1))$, REF
2. $\forall w_1 R(w_1, f(w_1)) \mid R(x_1, f(x_1))$, $\forall$O, 1
3. $\forall w_1 R(w_1, f(w_1)) \mid \exists w_2 R(x_{1)} w_2)$, $\exists$I, 2
4. $\forall w_1 R(w_1, f(w_1)) \mid \forall w_1 \exists w_2 R(w_{1)} w_2)$, $\forall$I, 3

Within this enlarged context Lemmas 5, 6 go through just as before except that for the latter we need to quote Lemma 18 for the two enhanced rules.

In more detail suppose the instance of the $\forall \mathrm{O}$ rule is:

$$\frac{\Gamma \mid \forall w_j \psi(w_j, \vec{x})}{\Gamma \mid \psi(t(\vec{x}), \vec{x})}$$

where $t(\vec{x}) \in TL$ and

$$\Gamma \vDash \forall w_j \psi(w_j, \vec{x}). \tag{26}$$

Let M be any structure for L and $x_i \mapsto a_i \in |M|$ an assignment to the free variables such that every formula in Γ is true in this interpretation. Then from (26), since $t^M(\vec{a}) \in |M|$,

$$M \vDash \psi(t^M(\vec{a}), \vec{a}).$$

Therefore by Lemma 18,

$$M \vDash \psi(t(\vec{a}), \vec{a}).$$

This shows that

$$\Gamma \vDash \psi(t(\vec{x}), \vec{x}),$$

as required.

The demonstration for the enhanced $\exists$I rule follows similarly. This then gives the Correctness Theorem:

The Correctness Theorem for L, 20 *Let $\Gamma \subseteq FL$ (possibly infinite) and $\zeta \in FL$. Then*

$$\Gamma \vdash \zeta \quad \Rightarrow \quad \Gamma \vDash \zeta.$$

Defining consistency and satisfiability as before Lemmas 8, 9, 10, 11, 12 go through without alteration. We can now follow the same route to the Completeness Theorem as previously by reducing it to showing that any consistent $\Delta \subseteq FL$ not mentioning infinitely many of the free variables has a maximal consistent extension[34] Ω satisfying $(\mathrm{a})-(\mathrm{g})$ of Lemma 13. Indeed a simple use of the new $\exists$I and $\forall$O rules now allows us to slightly improve parts (f), (g) of that lemma to now give:

Lemma 21

Let $\Delta \subseteq FL$ be consistent and not mentioning infinitely many of the free variables. Then there is a consistent $\Delta \subseteq \Omega \subseteq FL$ such that for $\theta, \phi, \exists w_j \psi(w_j, \vec{x}) \in FL$:

$$
\begin{array}{lll}
(a)\ \ \Omega \vdash \theta & \Leftrightarrow & \theta \in \Omega. \\
(b)\ \ \theta \in \Omega & \Leftrightarrow & \neg\theta \notin \Omega. \\
(c)\ \ (\theta \wedge \phi) \in \Omega & \Leftrightarrow & \theta \in \Omega \text{ and } \phi \in \Omega. \\
(d)\ \ (\theta \vee \phi) \in \Omega & \Leftrightarrow & \theta \in \Omega \text{ or } \phi \in \Omega. \\
(e)\ \ (\theta \to \phi) \in \Omega & \Leftrightarrow & \theta \notin \Omega \text{ or } \phi \in \Omega. \\
(f)\ \ \exists w_j \psi(w_j, \vec{x}) \in \Omega & \Leftrightarrow & \psi(x_i, \vec{x}) \in \Omega \text{ for some free vbl } x_i, \\
 & \Leftrightarrow & \psi(t(\vec{x}), \vec{x}) \in \Omega \text{ for some term } t(\vec{x}). \\
(g)\ \ \forall w_j \psi(w_j, \vec{x}) \in \Omega & \Leftrightarrow & \psi(x_i, \vec{x}) \in \Omega \text{ for all free vbl } x_i, \\
 & \Leftrightarrow & \psi(t(\vec{x}), \vec{x}) \in \Omega \text{ for all terms } t(\vec{x}).
\end{array}
$$

Proof To show the enhanced version of (g) suppose that $\psi(t(\vec{x}), \vec{x}) \in \Omega$ for all terms $t(\vec{x})$. Then certainly $\psi(x_i, \vec{x}) \in \Omega$ for all free variables x_i since the x_i are terms. Now by the old version of Lemma 13(g), $\forall w_j \psi(w_j, \vec{x}) \in \Omega$. By part (a) $\Omega \vdash \forall w_j \psi(w_j, \vec{x})$ and by the enhanced $\forall$O rule, $\Omega \vdash \psi(t(\vec{x}), \vec{x})$ for (any) $t(\vec{x}) \in TL$. Hence by part (a) again $\psi(t(\vec{x}), \vec{x}) \in \Omega$ for any $t(\vec{x}) \in TL$, which takes us full circle.

The proof for (f) follows similar lines. ■

Now recall that at this point in the case of a purely relational language we constructed a structure M by setting

$$\begin{aligned} &| M | = \{x_1, x_2, x_3, \} - \text{ the set of free variables} \\ &\langle x_{i_1}, x_{i_2}, \ldots, x_{i_r} \rangle \in R^M \Leftrightarrow R(x_{i_1}, x_{i_2}, \ldots, x_{i_r}) \in \Omega, \end{aligned}$$

for R an r-ary relation symbol of L, equivalently,

$$M \vDash R(x_{i_1}, x_{i_2}, \ldots, x_{i_r}) \Leftrightarrow R(x_{i_1}, x_{i_2}, \ldots, x_{i_r}) \in \Omega.$$

Now however we may have constant and function symbols in L -so how to interpret them in M? The answer is staring us in the face!

Set:

$$\begin{aligned} &|M| = TL \text{ –the set of terms of } L \\ &c^M = c \in TL \text{ for } c \text{ a constant symbol of } L, \end{aligned}$$

and for $s_1, s_2, \ldots, s_r \in |M| = TL$, f an r-ary function symbol of L and R an r-ary relation symbol of L set[35]

$$f^M(s_1, s_2, \ldots, s_r) = f(s_1, s_2, \ldots, s_r) \in TL$$
$$\langle s_1, s_2, \ldots, s_r \rangle \in R^M \Leftrightarrow R(s_1, s_2, \ldots, s_r) \in \Omega,$$

equivalently,

$$M \vDash R(s_1, s_2, \ldots, s_r) \Leftrightarrow R(s_1, s_2, \ldots, s_r) \in \Omega.$$

Proposition 22 *With* M *defined in this way,*

a) *For* $t(x_1, x_2, \ldots, x_n) \in TL$ *and* $s_1, s_2, \ldots, s_n \in |M|\,(= TL)$,

$$t^M(s_1, s_2, \ldots, s_n) = t(s_1, s_2, \ldots, s_n) \in TL = |M|. \tag{27}$$

b) *For* $\theta(x_1, x_2, \ldots, x_n) \in FL$ *and* $s_1, s_2, \ldots, s_n \in |M|\,(= TL)$,

$$M \vDash \theta(s_1, s_2, \ldots, s_n) \Leftrightarrow \theta(s_1, s_2, \ldots, s_n) \in \Omega. \tag{28}$$

Proof* (a) We show this by induction on $|t(x_1, x_2, \ldots, x_n)|$. If $t(\vec{x}) = x_i$

$$t^M(s_1, s_2, \ldots, s_n) = s_i = t(s_1, s_2, \ldots, s_n) \text{ by } \mathbf{V1}$$

If $t(\vec{x}) = c$ for c a constant symbol of L then by **V2**

$$t^M(s_1, s_2, \ldots, s_n) = c^M = c \text{ (by defn. of } c^M) = t(s_1, s_2, \ldots, s_n)$$

Finally if $t(\vec{x}) = f(t_1(\vec{x}), t_2(\vec{x}), \ldots, t_r(\vec{x}))$ where f is an r-ary function symbol of L and $t_1(\vec{x}), \ldots, t_r(\vec{x})$ are terms of L (and necessarily shorter than $t(\vec{x})$) then (abbreviating 'Inductive Hypothesis' by IH),

$$\begin{aligned} t^M(s_1, s_2, \ldots, s_n) &= f^M(t_1^M(\vec{s}), t_2^M(\vec{s}), \ldots, t_r^M(\vec{s})) \text{ by } \mathbf{V3} \\ &= f^M(t_1(\vec{s}), t_2(\vec{s}), \ldots, t_r(\vec{s})) \text{ by IH} \\ &= f(t_1(\vec{s}), t_2(\vec{s}), \ldots, t_r(\vec{s})) \text{ by defn. of } f^M \\ &= t(\vec{s}), \text{ as required.} \end{aligned}$$

(b) We show this by induction on $|\,\theta(x_1, x_2, \ldots, x_n)\,|$.

In the case $\theta(\vec{x}) = R(t_1(\vec{x}), t_2(\vec{x}), \ldots, t_r(\vec{x})$ for R an r-ary relation symbol of L and $t_1(\vec{x}), t_2(\vec{x}), \ldots, t_r(\vec{x}) \in TL$,

$$
\begin{aligned}
M \vDash \theta(s_1, s_2, \ldots, s_n) & \\
& \Leftrightarrow \quad M \vDash R(t_1(\vec{s}),\, t_2(\vec{s})), \ldots,\, t_r(\vec{s})) \\
& \Leftrightarrow \quad \langle t_1^M(\vec{s}), t_2^M(\vec{s}), \ldots,\, t_r^M(\vec{s}) \rangle \in R^M \\
& \Leftrightarrow \quad \langle t_1(\vec{s}), t_2(\vec{s}), \ldots, t_r(\vec{s}) \rangle \in R^M \quad \text{by } (27) \\
& \Leftrightarrow \quad R(t_1(\vec{s}),\, t_2(\vec{s}), \ldots, t_r(\vec{s})) \in \Omega \quad \text{by defn. of } R^M \\
& \Leftrightarrow \quad \theta(s_1,\, s_2, \ldots, s_n) \in \Omega, \text{ as required}
\end{aligned}
$$

The remaining cases now go through just as before in Theorem 14 but using Lemma 21 in place Lemma 13 and the enhanced (f),(g) in the cases of the quantifiers. To illustrate this last suppose that $\theta(\vec{x}) = \forall w_j \psi(w_j, \vec{x})$. Then

$$
\begin{aligned}
M \vDash \theta(\vec{s}) & \Leftrightarrow M \vDash \forall w_j \phi(w_j,\, \vec{s}) \\
& \Leftrightarrow \forall t \in |M|,\, M \vDash \phi(t, \vec{s}) \\
& \Leftrightarrow \forall t \in TL,\, \phi(t, \vec{s}) \in \Omega, \quad \text{by IH}, \\
& \Leftrightarrow \forall w_j \phi(w_j, \vec{s}) \in \Omega \qquad \text{by Lemma 21(g)} \\
& \Leftrightarrow \theta(\vec{s}) \in \Omega, \quad \text{as required.}
\end{aligned}
$$

■

From (28) it follows that if $\theta(\vec{x}) \in \Delta$ then $M \vDash \theta(\vec{x})$, since $\Delta \subseteq \Omega$. In other words Δ is satisfied in the interpretation with structure M by assignment $x_i \mapsto x_i \in |M|$, as required.

By the same trick as previously we can now dispense with the requirement that there are infinitely many free variables not mentioned in Δ and the Completeness and Compactness Theorems then follow exactly as before (but now for a language possibly containing constant and function symbols):

The Completeness Theorem for L, 23

For $\Gamma \subseteq FL, \zeta \in FL$,

$$\Gamma \vDash \zeta \Leftrightarrow \Gamma \vdash \zeta.$$

The Compactness Theorem for L, 24

Let $\Gamma \subseteq FL$. *Then* Γ *is satisfiable if and only if every finite subset of* Γ *is satisfiable.*

Herbrand's Theorem

A natural if somewhat vague conjecture which might posit one at this juncture is that if a simple formula is provable then it should have a relatively simple proof. This of course depends what one means by 'simple' and there are measures of simplicity for which it is true and others for which it is false. One such positive result however which is of some practical importance is when a formula ϕ being 'simple' means that ϕ is quantifier free, i.e. $\forall$ and $\exists$ do not occur anywhere in ϕ.

Theorem 25 *Suppose there is a proof of the quantiffier free formula ψ. Then there is a proof of ψ which only mentions quantifier free formulae, so does not use any of the rules $\forall I, \forall O, \exists I, \exists O$, only the rules AND-MON.*

Proof We first derive a somewhat stronger result which will shortly have another application. For Γ a set of quantifier free formulae and θ quantifier free write $\Gamma \vdash^{QF} \theta$ to mean that there is a proof of θ from Γ only mentioning quantifier free formulae (so using just the rules AND-MON) and say that Γ is QF-inconsistent if $\Gamma \vdash^{QF} \psi \wedge \neg\psi$ for some (necessarily quantifier free) formula ψ. Say that Γ is QF-consistent if not QF-inconsistent etc.

Let Γ be a QF-consistent set of quantifier free formulae. Just as in the proof of the Completeness Theorem on page 90, but without the need to consider quantified formulae at all, we can extend Γ to an QF-consistent set Ω of quantifier free formulae which is maximally consistent in the sense that if ϕ is quantifier free and $\phi \notin \Omega$ then $\Omega \cup \{\phi\}$ is QF-inconsistent. Similarly, as in the proof of Lemma 13 (or Lemma 21), Ω can be shown to satisfy that for quantifier free θ, ϕ:

(a) $\Omega \vdash^{QF} \theta \quad \Leftrightarrow \quad \theta \in \Omega.$

(b) $\theta \in \Omega \;\; \Leftrightarrow \;\; \neg\theta \notin \Omega.$

(c) $(\theta \wedge \phi) \in \Omega \;\; \Leftrightarrow \;\; \theta \in \Omega$ and $\phi \in \Omega.$

(d) $(\theta \vee \phi) \in \Omega \;\; \Leftrightarrow \;\; \theta \in \Omega$ or $\phi \in \Omega.$

(e) $(\theta \rightarrow \phi) \in \Omega \;\; \Leftrightarrow \;\; \theta \notin \Omega$ or $\phi \in \Omega.$

Analogously to the account on page 92 define a structure M by

$|M| = TL$ – the set of terms of L,
$c^M = c \in TL$ for c a constant symbol of L,

$f^M(s_1, s_2, \ldots, s_r) = f(s_1, s_2, \ldots, s_r) \in TL$ for f an r-ary function symbol of L and $s_1, s_2, \ldots, s_r \in |M| = TL$, and

$$\langle s_1, s_2, \ldots, s_r \rangle \in R^M \Leftrightarrow R(s_1, s_2, \ldots, s_r) \in \Omega,$$

equivalently,

$$M \vDash R(s_1, s_2, \ldots, s_r) \Leftrightarrow R(s_1, s_2, \ldots, s_r) \in \Omega,$$

for R an r-ary relation symbol of L and $s_1, s_2, \ldots, s_r \in |M| = TL.$

Just as in Proposition 22 we can now show that

a) For $t(x_1, x_2, \ldots, x_n) \in TL$ and $s_1, s_2, \ldots, s_n \in TL(=|M|)$,

$$t^M(s_1, s_2, \ldots, s_n) = t(s_1, s_2, \ldots, s_n) \in TL. \tag{29}$$

b) For $\theta(x_1, x_2, \ldots, x_n) \in FL$ quantifier free and $s_1, s_2, \ldots, s_n \in TL(=|M|)$,

$$M \vDash \theta(s_1, s_2, \ldots, s_n) \Leftrightarrow \theta(s_1, s_2, \ldots, s_n) \in \Omega. \tag{30}$$

In particular then (b) gives that Γ is satisfiable, and hence consistent by the Correctness Theorem on page 90.

Returning to the main statement of the theorem to be proved, suppose that $\nvdash^{QF} \psi$. Then analogously to the proof of Lemma 8 $\{\neg\psi\}$ is QF-consistent. But that means by the above that $\{\neg\psi\}$ is consistent, so $\nvdash \psi$, as required. ■

Earlier in these notes it was mentioned that a good idea if you are stuck on trying to concoct a formal proof is to ask yourself why you expect the conclusion to follow from, or more accurately be a logical consequence of, the assumptions and try to use that as a basis for constructing the required proof. Since this often seems to be a successful strategy one might be led to wonder just how fool proof it is. More pointedly, is it the case that if there is a proof then there *must* be a proof along the lines of ones semantic argument?

Without wishing to spend time clarifying and discussing this question in general one situation where it seems particularly pertinent is when we are asked to find a (formal) proof that

$$\vdash \exists w_1, w_2, \ldots, w_m \theta(w_1, w_2, \ldots, w_m, x_1, x_2, \ldots, x_n).$$

It this case you might feel that the obvious way to prove the existence of $w_1, w_2, \ldots, w_m$ satisfying $\theta(w_1, w_2, \ldots, w_m, x_1, x_2, \ldots, x_n)$ would be to actually exhibit some $w_1, w_2, \ldots, w_m$ with this property, necessarily some terms of the language since these are all we have available.

Whilst this intuition is not completely sound in general[36] it nearly holds in the case when $\theta(\vec{x})$ is *quantifier free*, as the following theorem shows

Theorem 26 *Suppose that* $\theta(x_1, x_2, \ldots, x_{n+m})$ *is quantifier free and*

$$\vdash \exists w_1, w_2, \ldots, w_m \theta(x_1, x_2, \ldots, x_n, w_1, w_2, \ldots, w_m). \tag{31}$$

Then for some r *and terms* $t_{i,j} \in TL$, *where* $i = 1, 2, \ldots, r$ *and* $j = 1, 2, \ldots, m$,

$$\vdash^{QF} \bigvee_{i=1}^{r} \theta(x_1, x_2, \ldots, x_n, t_{i,1}, t_{i,2}, \ldots,$$

Proof Suppose on the contrary that for any choice of $t_{i,j} \in TL$,

$$\nvdash^{QF} \bigvee_{i=1}^{r} \theta(x_1, x_2, \ldots, x_n, t_{i,1}, t_{i,2}, \ldots,$$

equivalently by Theorem 25,

$$\nvdash \bigvee_{i=1}^{r} \theta(x_1, x_2, \ldots, x_n, t_{i,1}, t_{i,2}, \ldots, t_{i,m}). \tag{32}$$

Then the set Γ of all formulae of the form

$$\bigwedge_{i=1}^{r} \neg\theta(x_1, x_2, \ldots, x_n, t_{i,1}, t_{i,2}, \ldots, t_{i,m})$$

is satisfiable.

To see this suppose not. Then by the Compactness Theorem there would be a finite unsatisfiable subset of Γ. Furthermore since Γ is closed under conjunction, meaning that whenever $\phi, \psi \in \Gamma$ then $(\phi \wedge \psi) \in \Gamma$, it follows by taking the conjunction of the sentences in this finite subset that there is a single sentence in Γ which is unsatisfiable. Let

$$\bigwedge_{i=1}^{r} \neg\theta(x_1, x_2, \ldots, x_n, t_{i,1}, t_{i,2}, \ldots, t_{i,m})$$

be such a sentence. Then its negation must be a tautology and hence, by logical equivalence,

$$\bigvee_{i=1}^{r} \theta(x_1, x_2, \ldots, x_n, t_{i,1}, t_{i,2}, \ldots, t_{i,m})$$

must also be a tautology. By the Completeness Theorem then

$$\vdash \bigvee_{i=1}^{r} \theta(x_1, x_2, \ldots, x_n, t_{i,1}, t_{i,2}, \ldots, t_{i,m})$$

which contradicts (32).

To sum up then, under this assumption Γ must be consistent. By the construction in the proof of Theorem 25 there is a structure M with $|M| = TL$ and satisfying (30) such that $M \vDash \Gamma$. From (31), the Completeness Theorem and Lemma 18 (with the assignment $x_i \mapsto x_i$),

$$M \vDash \exists w_1, w_2, \ldots, w_m \theta(x_1, x_2, \ldots, x_n, w_1, w_2, \ldots, w_m).$$

Hence by (30) and Lemma 18 for some $t_1, t_2, \ldots, t_m \in |M| = TL$,

$$M \vDash \theta(x_1, x_2, \ldots, x_n, t_1, t_2, \ldots, t_m).$$

But this is a contradiction since $\neg\theta(x_1, x_2, \ldots, x_n, t_1, t_2, \ldots, t_m) \in \Gamma$ so

$$M \vDash \neg\theta(x_1, x_2, \ldots, x_n, t_1, t_2, \ldots, t_m).$$

We conclude that our initial assumption in this proof is false and hence that the required result follows. ■

Theorem 26 is a special case of Herbrand's Theorem which gives an analogous result without the restriction that θ be quantifier free. To go a little deeper into this suppose that $\phi(x_1, x_2, x_3, x_4) \in FL$ is quantifier free and $\forall w_1 \forall w_2 \exists w_3 \phi(x_1, w_1, w_2, w_3)$ is satisfiable, say

$$M \vDash \forall w_1 \forall w_2 \exists w_3 \phi(x_1, w_1, w_2, w_3) \tag{33}$$

for some structure M for L and $a \in |M$. Now let L^+ be the language L augmented with a new binary function symbol f and extend M to L^+ by picking f^M to be some function such that for $c, d \in |M|$,

$$M \vDash \phi(a, c, d, f^M(c, d)).$$

Notice that from (33) for every $c, d \in |M|$ there indeed is *some* $b \in |M|$ such that $\phi(a, c, d, b)$ holds in M. By Lemma 18 then for all $c, d \in |M|$,

$$M \vDash \phi(a, c, d, f(c, d))$$

so[37]

$$M \vDash \forall w_1 \forall w_2 \phi(a, w_1, w_2, f(w_1, w_2)). \tag{34}$$

This shows that if $\forall w_1 \forall w_2 \exists w_3 \phi(x_1, w_1, w_2, w_3)$ is satisfiable then so is $\forall w_1 \forall w_2 \phi(x_1, w_1, w_2, f(w_1, w_2))$, and clearly the converse also holds.

Using the Completeness Theorem and Theorem 26 we now have that for $\phi = \neg\theta$,

$$\begin{aligned}
&\vdash \exists w_1 \exists w_2 \forall w_3 \theta(x_1, w_1, w_2, w_3) \\
&\quad \Leftrightarrow \vDash \exists w_1 \exists w_2 \forall w_3 \theta(x_1, w_1, w_2, w_3) \\
&\quad \Leftrightarrow \forall w_1 \forall w_2 \exists w_3 \phi(x_1, w_1, w_2, w_3) \text{ is not satisfiable} \\
&\quad \Leftrightarrow \forall w_1 \forall w_2 \phi(x_1, w_1, w_2, f(w_1, w_2)) \text{ is not satisfiable} \\
&\quad \Leftrightarrow \vDash \exists w_1 \exists w_2 \theta(x_1, w_1, w_2, f(w_1, w_2)) \\
&\quad \Leftrightarrow \vdash \exists w_1 \exists w_2 \theta(x_1, w_1, w_2, f(w_1, w_2)) \\
&\quad \Leftrightarrow \vdash \bigvee_{i=1}^{m} \theta(x_1, t_{2i-1}, t_{2i}, f(t_{2i-1}, t_{2i})) \text{ for some } m \\
&\qquad \text{and } t_1, t_2, \ldots, t_{2m} \in TL^+.
\end{aligned}$$

To sum up then we have shown that the provability of

$$\exists w_1 \exists w_2 \forall w_3 \theta(x_1, w_1, w_2, w_3)$$

is equivalent to the provability, and hence QF-provability, of some quantifier free formula. Furthermore the method we have use here, introducing Skolem Functions and using Theorem 26, can be iterated so as to apply to any formula in Prenex Normal Form, and indeed any formula since the provability of a formula is equivalent to the provability of a Prenex Normal Form version of the formula. This result it is known as Herbrand's Theorem.

The practical value of this resides in the fact that QF-provability essentially lands us in Propositional Logic where there are well developed techniques for constructing proofs.

Equality

Many structures that we deal with in mathematics have relations, constant, functions *and the binary relation of equality*, for example groups, rings, vector spaces. Such structures are said to be *normal*:

Definition A structure M for a language containing the binary relation symbol $=$ is *normal* if the interpretation $=^M$ of the equality symbol is equality, i.e.

$$=^M \quad \text{is } \{\langle x, y\rangle \in |M|^2 \;\; | x = y\},$$

equivalently, for $a_1, a_2 \in |M|$,[38]

$$(M \vDash a_1 = a_2) \Leftrightarrow a_1 = a_2.$$

In particular then a group is a normal structure for the language with the equality symbol, a constant symbol e and a binary function symbol which satisfies the *Axioms of Group Theory, GPAx:*[39]

$$\begin{aligned}
&\forall w_1 \cdot e w_1 = w_1 \\
&\forall w_1 \exists w_2\, w_2 \cdot w_1 = e \\
&\forall w_1 \forall w_2 \forall w_3 (w_1 \cdot w_2) \cdot w_3 = w_1 \cdot (w_2 \cdot w_3)
\end{aligned} \tag{35}$$

Unfortunately as they currently stand our Completeness and Compactness Theorems do not 'work' if we try to limit ourselves to normal structures.

Initially that might cause you some surprise, after all why not include $=$ as one of the relation symbols of L, isn't that enough?

Well, there's no harm at all in including it as a relation symbol – *the trouble is that in general there is no reason why* $=^M$ *should look anything like equality*! For example there's nothing to stop us landing up with

$$M \vDash a \neq a, \quad \text{or} \quad M \vDash a = b \wedge b \neq a.$$

The point is that equality has a number of special properties and we certainly have to build these in if we want $=^M$ to look anything like equality.

To get a feel for these properties let L be a language with equality, i.e. containing (possibly amongst other relation symbols) the binary relation symbol $=$. Then the following should be true in M if the symbol $=$ is to be interpreted in M as genuine equality:[40]

Eq1 $\forall w_1 w_1 = w_1$

Eq2 $\forall w_1, w_2 (w_1 = w_2 \rightarrow w_2 = w_1)$

Eq3 $\forall w_1, w_2, w_3 ((w_1 = w_2 \wedge w_2 = w_3) \rightarrow w_1 = w_3)$

Eq4 $\forall w_1, \ldots, w_{2r} \left(\left(\bigwedge_{i=1}^{r} w_i = w_{r+i} \right) \rightarrow (R(w_1, w_2, \ldots, w_r) \leftrightarrow R(w_{r+1}, w_{r+2}, \ldots, w_{2r})) \right)$

for R an r-ary relation symbol of L (other than equality).

Eq5 $\forall w_1, \ldots, w_{2r} \left(\left(\bigwedge_{i=1}^{r} w_i = w_{r+i} \right) \rightarrow f(w_1, w_2, \ldots, w_r) = f(w_{r+1}, w_{r+2}, \ldots, w_{2r}) \right)$

for f an r-ary function symbol of L.

Let EqL stand for the sentences Eq1-5. Notice that if L is finite then so is EqL.

The next lemma is so obvious it would be a waste of paper bothering to write down a proof.

Lemma 27

Let L contain equality and let M be a normal structure for L. Then $M \vDash EqL$, i.e. $M \vDash \phi$ for each $\phi \in EqL$.

Lemma 28

Let M be a structure (not necessarily normal) for the language L with equality and such that Eq1-5 are true in M. Then the following are true in M for $t(x_1,\ldots,x_n) \in TL$ and $\theta(x_1,\ldots,x_n) \in FL$:

Eq6 $\forall w_1,\ldots, w_{2n}\left(\left(\bigwedge_{i=1}^{n} w_i = w_{n+i}\right) \to t(w_1, w_2,\ldots, w_n) = t(w_{n+1}, w_{n+2},\ldots, w_{2n})\right)$

Eq7 $\forall w_1,\ldots, w_{2n}\left(\left(\bigwedge_{i=1}^{n} w_i = w_{n+i}\right) \to (\theta(w_1, w_2,\ldots, w_n) \leftrightarrow \theta(w_{n+1}, w_{n+2},\ldots, w_{2n}))\right)$

Proof* **Eq6**: By induction on $|t|$. Assume that Eq6 holds for all $s(\vec{x}) \in TL$ of shorter length.

If $t = c$, a constant symbol, then

$$t(w_1, w_2,\ldots, w_n) = t(w_{n+1}, w_{n+2},\ldots, w_{2n})$$

amounts to $c = c$ which holds in M by Eql. So the required version of Eq6 in this case is

$$\forall w_1,\ldots, w_{2n}\left(\left(\bigwedge_{i=1}^{n} w_i = w_{n+i}\right) \to c = c\right)$$

which also holds in *M*.

If $t = x_i$ then Eq6 is just

$$\forall w_1,\ldots, w_{2n}\left(\left(\bigwedge_{i=1}^{n} w_i = w_{n+i}\right) \to w_i = w_{n+i}\right)$$

which is in fact a tautology (i.e. always true in any structure for L).

If $t(x_1,\ldots,x_n) = f(s_1(x_1,\ldots,x_n),\ldots,s_r(x_1,\ldots,x_n))$ for $s_1,\ldots, s_r \in TL$ and f an r-ary function symbol of L then by inductive hypothesis

$$M \vDash \forall w_1,\ldots, w_{2n}\left(\left(\bigwedge_{i=1}^{n} w_i = w_{n+i}\right) \to s_i(w_1,\ldots, w_n) = s_i(w_{n+1},\ldots, w_{2n})\right) \tag{36}$$

for $i = 1,2,\ldots,r$. Let $a_1, a_2,\ldots, a_{2n} \in |M|$ be such that

c) $M \vDash \bigwedge_{i=1}^{n} a_i = a_{n+i}$.

Then from (36),

$$M \vDash s_i(a_1, a_2, \ldots, a_n) = s_i(a_{n+1}, a_{n+2}, \ldots, a_{2n}),$$

and hence

$$M \vDash \bigwedge_{i=1}^{r} s_i(a_1, a_2, \ldots, a_n) = s_i(a_{n+1}, a_{n+2}, \ldots, a_{2n})$$

and from Eq5 and Corollary 19 (which henceforth we shall use without mention)

$$\begin{aligned} M \vDash\ & f(s_1(a_1, \ldots, a_n), \ldots, s_r(a_1, \ldots, a_n)) \\ & = f(s_1\ (a_{n+1}, \ldots, a_{2n}), \ldots, s_r(a_{n+1}, \ldots, a_{2n})), \end{aligned}$$

equivalently

$$M \vDash t(a_1, a_2, \ldots, a_n) = t(a_{n+1}, a_{n+2}, \ldots, a_{2n}).$$

We have now shown that

$$M \vDash \left(\bigwedge_{i=1}^{n} a_i = a_{n+i} \right) \to t(a_1, a_2, \ldots, a_n) = t(a_{n+1}, a_{n+2}, \ldots, a_{2n})$$

for *any* $a_1, \ldots, a_{2n} \in |M|$ and Eq6 now follows.

Eq7: The proof is by induction on the length of θ. Assume the result is true for formulae shorter than θ.

Suppose that $\theta(x_1, \ldots, x_n) = R(t_1(x_1, \ldots, x_n), \ldots, t_r(x_1, \ldots, x_n))$ for some r-ary relation symbol R in L (R not $=$) and terms $t_1(x_1, \ldots, x_n), \ldots,\ t_r(x_1, \ldots, x_n)$ of L and

$$M \vDash \bigwedge_{i=1}^{n} a_i = a_{n+i}.$$

Then by Eq6

$$M \vDash t_j(a_1, \ldots, a_n) = t_j(a_{n+1}, \ldots, a_{2n}) \text{ for } j = 1, 2, \ldots, r,$$

hence

$$M \vDash \bigwedge_{j=1}^{n} t_j(a_1, \ldots, a_n) = t_j(a_{n+1}, \ldots, a_{2n})$$

and by Eq4

$$\begin{aligned} M \vDash\ & R(t_1(a_1, \ldots, a_n), \ldots, t_r(a_1, \ldots, a_n)) \leftrightarrow \\ & R(t_1(a_{n+1}, \ldots, a_{2n}), \ldots, t_r(a_{n+1}, \ldots, a_{2n})), \end{aligned}$$

equivalently

$$M \vDash \theta(a_1, \ldots, a_n) \leftrightarrow \theta(a_{n+1}, \ldots, a_{2n}),$$

as required.

If $\theta(x_1, \ldots, x_n)$ is $t(x_1, \ldots, x_n) = s(x_1, \ldots, x_n)$ then the required version of Eq7 is:

$$\begin{aligned} & M \vDash \forall w_1, \ldots, w_{2r} \left(\left(\bigwedge_{i=1}^{n} w_i = w_{n+i} \right) \to \right. \\ & (t(w_1, \ldots, w_n) = s(w_1, \ldots, w_n) \\ & \qquad \left. \leftrightarrow t(w_{n+1}, \ldots, w_{2n}) = s(w_{n+1}, \ldots, w_{2n})) \right) \end{aligned} \tag{37}$$

Let $a_1, \ldots, a_{2n} \in |M|$ and suppose that

$$M \vDash \bigwedge_{i=1}^{n} a_i = a_{n+i} \tag{38}$$

and

$$M \vDash t(a_1, \ldots, a_n) = s(a_1, \ldots, a_n). \tag{39}$$

Then by Eq6

$$M \vDash t(a_1, \ldots, a_n) = t(a_{n+1}, \ldots, a_{2n}), \tag{40}$$

$$M \vDash s(a_1, \ldots, a_n) = s(a_{n+1}, \ldots, a_{2n}). \tag{41}$$

From Eq2 and (40) we obtain

$$M \vDash t(a_{n+1}, \ldots, a_{2n}) = t(a_1, \ldots, a_n), \tag{42}$$

and Eq3 and (39), (42) now give

$$M \vDash t(a_{n+1}, \ldots, a_{2n}) = s(a_1, \ldots, a_n). \tag{43}$$

Another application of Eq3 with (41),(43) gives

$$M \vDash t(a_{n+1}, \ldots, a_{2n}) = s(a_{n+1}, \ldots, a_{2n}).$$

A similar argument starting with

$$M \vDash t(a_{n+1}, \ldots, a_{2n}) = s(a_{n+1}, \ldots, a_{2n})$$

in place of (39) yields

$$M \vDash t(a_1, \ldots, a_n) = s(a_1, \ldots, a_n).$$

In summary then from (38) we have concluded

$$M \vDash t(a_1, \ldots, a_n) = s(a_1, \ldots, a_n) \leftrightarrow t(a_{n+1}, \ldots, a_{2n}) = s(a_{n+1}, \ldots, a_{2n}).$$

Since $a_1, \ldots, a_{2n}$ were arbitrary elements of $|M|$, (37) follows.

If $\theta(x_1,\dots,x_r) = \neg\phi(x_1,\dots,x_r)$ then by inductive hypothesis,

$$M \vDash \forall w_1,\dots, w_{2r}\Big(\big(\bigwedge\nolimits_{i=1}^{r} w_i = w_{r+i}\big)$$
$$\rightarrow (\phi(w_1,\dots, w_r) \leftrightarrow \phi(w_{r+1},\dots, w_{2r}))\Big) \tag{44}$$

Let $a_1,\dots,a_{2r} \in |M|$ and assume that

$$M \vDash \bigwedge_{i=1}^{r} a_i = a_{r+i}.$$

Then from (44),

$$M \vDash \phi(a_1, a_2,\dots,a_r) \leftrightarrow \phi(a_{r+1}, a_{r+2},\dots,a_{2r}) \tag{45}$$

equivalently

$$M \vDash \phi(a_1, a_2,\dots,a_r) \Leftrightarrow M \vDash \phi(a_{r+1}, a_{r+2},\dots,a_{2r}).$$

But from this

$$M \nvDash \phi(a_1, a_2, \ldots, a_r) \Leftrightarrow M \nvDash \phi(a_{r+1}, a_{r+2}, \ldots, a_{2r})$$

so

$$M \vDash \neg\phi(a_1, a_2, \ldots, a_r) \Leftrightarrow M \vDash \neg\phi(a_{r+1}, a_{r+2}, \ldots, a_{2r}).$$

Since $\theta = \neg\phi$ working back gives the required version of Eq7 for θ.

The cases for the other connectives are similar.

Now suppose that $\theta(x_1, \ldots, x_r) = \exists w_j \phi(x_1, \ldots, x_r, w_j)$. By inductive hypothesis

$$M \vDash \forall w_1, \ldots, w_{2(r+1)}\Bigg(\left(\bigwedge_{i=1}^{r+1} w_i = w_{r+1+i}\right) \rightarrow$$

$$(\phi(w_1, \ldots, w_{r+1}) \leftrightarrow \phi(w_{r+2}, \ldots, w_{2(r+1)}))\Bigg). \tag{46}$$

Let $a_1, \ldots, a_{2r} \in |M|$ and suppose that

$$M \vDash \bigwedge_{i=1}^{r} a_i = a_{r+i} \tag{47}$$

and

$$M \vDash \theta(a_1, a_2, \ldots, a_r). \tag{48}$$

Then for some $b \in |M|$,

$$M \vDash \phi(a_1, a_2, \ldots, a_r, b). \tag{49}$$

By Eql $M \vDash b = b$, and using this with (47) and (46) we obtain that

$$M \vDash \phi(a_1, a_2, \ldots, a_r, b) \leftrightarrow \phi(a_{r+1}, a_{r+2}, \ldots, a_{2r}, b).$$

Using this and (49) we obtain that

$$M \vDash \phi(a_{r+1}, a_{r+2}, \ldots, a_{2r}, b)$$

and hence

$$M \vDash \theta(a_{r+1}, a_{r+2}, \ldots, a_{2r}).$$

Similarly if we assumed this instead of (48) (and (46), (47)) we would have been able to show (48). Overall then we have shown that

$$M \vDash \left(\bigwedge_{i=1}^{r} a_i = a_{r+i} \right) \to (\theta(a_1, a_2, \ldots, a_r) \leftrightarrow \theta(a_{r+1}, a_{r+2}, \ldots, a_{2r})$$

without any assumptions on the $a_1, \ldots, a_{2r}$ and hence the required version of Eq7 follows.

The case for $\theta(x_1, \ldots, x_r) = \forall w_j \phi(x_1, \ldots, x_r, w_j)$ is similar. ■

Lemma 28 has shown that

$$EqL \vDash Eq6 + Eq7$$

and hence by the Completeness Theorem[41, 42]

Corollary 29

$$EqL \vdash Eq6 + Eq7.$$

Convention* When writing out formal proofs with EqL on the left we will adopt the convention of omitting mention of subsets of EqL on the left of sequents and introduce instances of these axioms (plus Eq6, Eq7) on the right of sequents by quoting as justification which one of Eql, Eq2, … , Eq7 they fall under rather than introducing the instant on both sides of the sequent and quoting REF as the justification –or splicing in a proof of instances of Eq6, Eq7 from EqL. [This will be clear from the following example.]

Example* A formal proof[43] of

$$EqL, \exists w_1(\theta(w_1) \wedge \neg\theta(c)) \vdash \exists w_1 \neg w_1 = c:$$

1	$x_1 = c, \theta(x_1) \wedge \neg\theta(c) \mid \theta(x_1) \wedge \neg\theta(c)$	REF,
2	$x_1 = c, \theta(x_1) \wedge \neg\theta(c) \mid \theta(x_1)$	AO, 1
3	$x_1 = c, \theta(x_1) \wedge \neg\theta(c) \mid \neg\theta(c)$	AO, 1
4	$\mid \forall w_1, w_2(w_1 = w_2 \to (\theta(w_1) \leftrightarrow \theta(w_2)))$	Eq7,
5	$\mid \forall w_2(x_1 = w_2 \to (\theta(x_1) \leftrightarrow \theta(w_2)))$	$\forall$O, 4
6	$\mid x_1 = c \to (\theta(x_1) \leftrightarrow \theta(c))$	$\forall$O, 5
7	$x_1 = c, \theta(x_1) \wedge \neg\theta(c) \mid x_1 = c$	REF,
8	$x_1 = c, \theta(x_1) \wedge \neg\theta(c) \mid (\theta(x_1) \leftrightarrow \theta(c))$	MP, 6, 7
9	$x_1 = c, \theta(x_1) \wedge \neg\theta(c) \mid \theta(x_1) \to \theta(c)$	AO, 8
10	$x_1 = c, \theta(x_1) \wedge \neg\theta(c) \mid \theta(c)$	MP, 2, 9
11	$\theta(x_1) \wedge \neg\theta(c) \mid \neg x_1 = c$	NIN, 3, 10
12	$\theta(x_1) \wedge \neg\theta(c) \mid \exists w_1 \neg w_1 = c$	$\exists$I, 11
13	$\exists w_1(\theta(w_1) \wedge \neg\theta(c)) \mid \exists w_1 \neg w_1 = c$	$\exists$O, 12

We now have in place the syntactic, or proof theoretic, part of the Completeness Theorem for Normal Structures that we are seeking. The appropriate semantic notion is:

Definition For L a language with equality, $\Gamma \subseteq FL$ and $\zeta \in FL$, ζ is a *normal logical consequence of* Γ, denoted $\Gamma \models^{=} \zeta$, if for all *normal* structures M for L and assignments to the free variables by elements of $|M|$, if every formula in Γ is true in M then ζ is true in M.

In other words $\Gamma \models^{=} \zeta$ is the same as $\Gamma \models \zeta$ except that we restrict ourselves entirely to normal structures, that is structures that interpret $=$ as actual equality.

Many results in mathematics actually amount to showing that $\Gamma \models^{=} \zeta$ for some Γ, ζ. For example when we show that in any group the left identity e is also a right identity we are actually showing (recall(35)) that

$$GPAx \models^{=} \forall w_1 w_1 \cdot e = w_1$$

Our next result then provides a valuable link between the Predicate Calculus and mainstream Pure Mathematics:

The Completeness Theorem for Normal Structures, 30

Let L be a language with equality, $\Gamma \subseteq FL$ and $\zeta \in FL$. Then

$$\Gamma \vDash^{=} \zeta \Leftrightarrow \Gamma + EqL \vdash \zeta \Leftrightarrow \Gamma + EqL \vDash \zeta .$$

Proof *

$\Leftarrow$: Suppose that $\Gamma + EqL \vdash \zeta$. By the already proven version of the Completeness Theorem we have that

$$\Gamma + EqL \vDash \zeta , \tag{50}$$

and conversely. Now let M be a normal structure for L such that for some assignment to the free variables

$$M \vDash \Gamma. \tag{51}$$

Then since M is normal, by Lemma 27, $M \vDash EqL$ so with (50) and (51), $M \vDash \zeta$. We have shown that $\Gamma \vDash^{=} \zeta$.

$\Rightarrow$: Suppose that

$$\Gamma + EqL \nvdash \zeta. \tag{52}$$

Analogously to the proof of the previous Completeness Theorem we will show that $\Gamma \nvDash^{=} \zeta$ by constructing a normal structure M and an assignment to the free variables for which $M \vDash \Gamma$ but $M \nvDash \zeta$.

The first step is to apply the previous Completeness Theorem to conclude from (52) that there is a structure N and an assignment to the free variables such that

$$N \vDash \Gamma + EqL \text{ but } N \nvDash \zeta. \tag{53}$$

Unfortunately this N may not be normal. We need to 'factor' N in a similar way to factoring a group G by a normal subgroup K to get the group G/K.

To this end define a binary relation $\sim$ between elements of $|N|$ by

$$a \sim b \Leftrightarrow N \vDash a = b.$$

Since $N \vDash EqL$, N is a model of

$$\begin{gathered}\forall w_1 w_1 = w_1,\\ \forall w_1, w_2(w_1 = w_2 \to w_2 = w_1),\\ \forall w_1, w_2, w_3((w_1 = w_2 \wedge w_2 = w_3) \to w_1 = w_3).\end{gathered}$$

Consequently for any $a, b, c \in |N|$,

$$a \sim a, \qquad a \sim b \Rightarrow b \sim a, \qquad (a \sim b \,\&\, b \sim c) \Rightarrow a \sim c.$$

In other words $\sim$ is an equivalence relation on $|N|$.

For $a \in |N|$ let $[a]$ be the equivalence class of a with respect to $\sim$, *i.e*[44].

$$[a] = \{b \in |N| \,|\, a \sim b\}.$$

Now define a structure M for L by:

$$|M| = \{[a] \,|\, a \in |N|\},$$

for R an r-ary relation symbol of L, including the binary relation symbol $=$, set

$$\begin{aligned} R^M &= \{\langle [a_1], [a_2], \ldots, [a_r] \rangle \mid \langle a_1, a_2, \ldots, a_r \rangle \in R^N\} \\ &= \{\langle [a_1], [a_2], \ldots, [a_r] \rangle \mid N \vDash R(a_1, a_2, \ldots, a_r)\}. \end{aligned} \tag{54}$$

In particular

$$\begin{aligned} [a] =^M [b] &\Leftrightarrow a =^N b \Leftrightarrow (N \vDash a = b) \\ &\Leftrightarrow a \sim b \Leftrightarrow [a] = [b] \end{aligned}$$

so M is normal. For c a constant symbol of L set

$$c^M = [c^N],$$

and for f an r-ary function symbol from L set

$$f^M([a_1], [a_2], \ldots, [a_r]) = [f^N(a_1, a_2, \ldots, a_r)].$$

M will be the normal structure in which will satisfy Γ and $\neg\zeta$.

However first of all we need to show that M is well defined. To see the problem here suppose we had $a_1, a_2, \ldots, a_r, b_1, b_2, \ldots b_r \in |N|$ with $[a_i] = [b_i]$ (equivalently $a_i \sim b_i$, or $N \vDash a_i = b_i$) for $i = 1, 2, \ldots, r$ and

$$N \vDash R(a_1, a_2, \ldots, a_r), \qquad N \nvDash R(b_1, b_2, \ldots, b_r). \tag{55}$$

In that case according to (54) we'd have to set

$$\langle [a_1], [a_2], \ldots, [a_r] \rangle \in R^M \text{ and } \langle [b_1], [b_2], \ldots, [b_r] \rangle \notin R^M.$$

But $\langle [a_1], [a_2], \ldots, [a_r] \rangle$ and $\langle [b_1], [b_2], \ldots, [b_r] \rangle$ are the same thing!

Fortunately (55) cannot happen. For if R is not $=$ then since $N \vDash EqL$, by Eq4

$$N \vDash \forall w_1, \ldots, w_{2r} \Bigg(\bigg(\bigwedge_{i=1}^{r} w_i = w_{r+i} \bigg) \\ \to (R(w_1, \ldots, w_r) \leftrightarrow R(w_{r+1}, \ldots, w_{r+r})) \Bigg).$$

Hence, since $N \vDash a_i = b_i$ for $i = 1, 2, \ldots, r$,

$$N \vDash R(a_1, a_2, \ldots, a_r) \leftrightarrow R(b_1, b_2, \ldots, b_r)$$

so

$$\langle [a_1], [a_2], \ldots, [a_r] \rangle \in R^M \Leftrightarrow \langle [b_1], [b_2], \ldots, [b_r] \rangle \in R^M.$$

In the case R is $=$, if $[a_1] = [b_1], [a_2] = [b_2]$ and $N \vDash a_1 = a_2$ then $a_1 \sim b_1$, $a_2 \sim b_2$, and $a_1 \sim a_2$ so since $\sim$ is an equivalence relation $b_1 \sim b_2$, i.e. $N \vDash b_1 = b_2$ as required.

A similar situation also pertains for the definition of f^M, again it initially seems possible that this might not be well defined since we could have $[a_i] = [b_i]$ (*i.e.* $N \vDash a_i = b_i$) for $i = 1, 2, \ldots, r$ but

$$f^M([a_1], [a_2], \ldots, [a_r]) = [f^N(a_1, a_2, \ldots, a_r)] \\ [f^N(b_1, b_2, \ldots, b_r)] = f^M([b_1], [b_2], \ldots, [b_r]).$$

However again this cannot happen because, since $N \vDash Eq5$,

$$N \vDash \forall w_1, \ldots, w_{2r} \Bigg(\bigg(\bigwedge_{i=1}^{r} w_i = w_{r+i} \bigg) \\ \to f(w_1, w_2, \ldots, w_r) = f(w_{r+1}, w_{r+2}, \ldots, w_{2r}) \Bigg)$$

we get

$$N \vDash f(a_1, a_2, \ldots, a_r) = f(b_1, b_2, \ldots, b_r),$$

so by Lemma 18

$$N \vDash f^N(a_1, a_2, \ldots, a_r) = f^N(b_1, b_2, \ldots, b_r),$$

equivalently

$$[f^N(a_1, a_2, \ldots, a_r)] = [f^N(b_1, b_2, \ldots, b_r)].$$

Having disposed of that possible problem we can now go on to show that M is a normal structure in which we can satisfy Γ and $\neg\zeta$. We show this via two claims:

Claim 1: For any term $t(x_1, x_2, \ldots, x_n) \in TL$ and $a_1, a_2, \ldots, a_n \in |N|$,

$$t^M([a_1], [a_2], \ldots, [a_n]) = [t^N(a_1, a_2, \ldots, a_n)]. \tag{56}$$

We prove this claim by induction on the length of t. If $t = x_i$ then both sides of (56) are $[a_i]$. If t is a constant symbol c then both sides are $[c^N]$. So assume that

$$t(x_1, \ldots, x_n) = f(s_1\ (x_1, \ldots, x_n), \ldots, s_r(x_1, \ldots, x_n))$$

for some terms $s_1, \ldots, s_r$ (so shorter than t) and r-ary function symbol f of L. Then

$$\begin{aligned} t^M([a_1], \ldots, [a_n]) &= f^M(s_1^M([a_1], \ldots, [a_n]), \ldots, s_r^M([a_1], \ldots, [a_r])) \\ &= f^M([s_1^N(a_1, \ldots, a_n)], \ldots, [s_r^N(a_1, \ldots, a_n)]) \quad \text{by IH} \\ &= [f^N(s_1^N(a_1, \ldots, a_n), \ldots, s_r^N(a_1, \ldots, a_n))] \quad \text{by definition} \\ &= [t^N(a_1, \ldots, a_n) \end{aligned}$$

as required.

Claim 2: For any formula $\theta(x_1, x_2, \ldots, x_n) \in FL$ and $a_1, a_2, \ldots, a_n \in |N|$

$$M \vDash \theta([a_1], [a_2], \ldots, [a_n]) \Leftrightarrow N \vDash \theta(a_1, a_2, \ldots, a_n).$$

We prove the claim by induction on the length of θ.

If $\theta(\vec{x}) = R(t_1(\vec{x}),\ldots,t_r(\vec{x}))$ for some r-ary relation symbol R of L (possibly R is $=$) and $t_1(\vec{x}),\ldots,t_r(\vec{x}) \in TL$ then

$$\begin{aligned}
&M \models R(t_1([a_1],\ldots,[a_n]),\ldots,t_r([a_1],\ldots,[a_n])) \\
&\quad \Leftrightarrow \langle t_1^M([a_1],\ldots,[a_n]),\ldots,t_r^M([a_1],\ldots,[a_n])\rangle \in R^M \\
&\quad \Leftrightarrow \langle [t_1^N(a_1,\ldots,a_n)],\ldots,[t_r^N(a_1,\ldots,a_n)]\rangle \in R^M, \quad \text{by Claim 1,} \\
&\quad \Leftrightarrow \langle t_1^N(a_1,\ldots,a_n),\ldots,t_r^N(a_1,\ldots,a_n)\rangle \in R^N, \quad \text{by definition,} \\
&\quad \Leftrightarrow N \models R(t_1\,(a_1,\ldots,a_n),\ldots,t_r(a_1,\ldots,a_n)), \quad \text{by T1,}
\end{aligned}$$

as required.

Now suppose $\theta(\vec{x}) = \neg\phi(\vec{x})$ (so ϕ is shorter than θ). Then

$$\begin{aligned}
M \models \theta([a_1],[a_2],\ldots,[a_n]) &\Leftrightarrow M \not\models \phi([a_1],[a_2],\ldots,[a_n]) \\
&\Leftrightarrow N \not\models \phi(a_1,a_2,\ldots,a_n) \quad \text{by IH} \\
&\Leftrightarrow N \models \theta(a_1,a_2,\ldots,a_n),
\end{aligned}$$

as required.

The cases for the other connectives are similar.

Finally in the case $\theta(\vec{x}) = \exists w_j \phi(x_1, \ldots, x_n, w_j)$,

$$\begin{aligned} M \models \theta([a_1], [a_2], \ldots, [a_n]) & \\ &\Leftrightarrow \exists [b] \in \mid M \mid, M \models \phi([a_1], [a_2], \ldots, [a_n], [b]) \\ &\Leftrightarrow \exists b \in \mid N \mid, N \models \phi(a_1, a_2, \ldots, a_n, b) \quad \text{by IH} \\ &\Leftrightarrow N \models \theta(a_1, a_2, \ldots, a_n), \end{aligned}$$

as required.

The case for $\forall$ is similar and this concludes the proof of Claim 2.

Since there is some assignment to the free variables, say $x_i \mapsto a_i$, for which in N all the formulae in Γ are satisfied but ζ is not it follows from Claim 2 that for the assignment $x_i \mapsto [a_i]$ all the formulae in Γ are satisfied in M but ζ is not. Finally since M is normal this gives, as required,

$$\Gamma \not\models^{=} \zeta. \qquad \blacksquare$$

Corollary 31

Let $EqL \subseteq \Gamma \subseteq FL$ and $\zeta \in FL$. Then

$$\Gamma \models^{=} \zeta \Leftrightarrow \Gamma \models \zeta.$$

Proof Since $EqL \subseteq \Gamma$ by the two Completeness Theorems both sides of this equivalence are equivalent to $\Gamma \vdash \zeta$. $\blacksquare$

Note that in most areas of logic where the Predicate Calculus is applied, for example Model Theory and Gödel's Incompleteness Theorems, we are only interested in normal structures. As a result most of the time logicians will omit mention of 'normal' and just take it as implicit that the structures under consideration are normal, writing $\models$ and $\Gamma \vdash \theta$ for what in this course we would write as $\models^{=}$ and $\Gamma + EqL \vdash \theta$.

As a second corollary to the Completeness Theorem for Normal Structures we are able to give an extension of Herbrand'sTheorem 26 to languages with equality. For suppose that

$$EqL \vdash \exists w_1, w_2, \ldots, w_m \theta(x_1, x_2, \ldots, x_n, w_1, w_2, \ldots, w_m) \tag{57}$$

with $\theta(x_1, x_2, \ldots, x_{n+m})$ quantifier free. Then

$$\models \exists w_1, w_2, \ldots, w_m \theta(x_1, x_2, \ldots, x_n, w_1, w_2, \ldots, w_m)$$

and in this assertion it is clearly enough (see Exercise 10 on page 133) to consider only normal structures for the finite language consisting just of those relation, function and constant symbols actually appearing in θ. Without loss of generality then we may assume that L is this finite language. In this case the conjunction of Eql-5 for L is logically equivalent to a sentence of the form

$$\forall w_1, w_2, \ldots, w_k \psi(w_1, w_2, \ldots, w_k)$$

with ψ quantifier free and using the fact that Eq6-7 are derivable from Eql-5 we obtain from (57) that

$$\forall w_1, w_2, \ldots, w_k \psi(w_1, w_2, \ldots, w_k) \\ \vdash \exists w_1, w_2, \ldots, w_m \theta(x_1, x_2, \ldots, x_n, w_1, w_2, \ldots, w_m).$$

In turn by IMR and the 'Useful Logical Equivalents',

$$\vdash \exists w_1, w_2, \ldots, w_{m+k}(\psi(w_{m+1}, w_{m+2}, \ldots, w_{m+k}) \\ \to \theta(x_1, x_2, \ldots, x_n, w_1, w_2, \ldots, w_m)). \tag{58}$$

We are now in a position to apply the original Theorem 26 which gives us that for some r and terms $t_{i,j} \in TL$, where $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, m$, there is a quantifier free proof of

$$\bigvee_{i=1}^{r} \theta(x_1, x_2, \ldots, x_n, t_{i,1}, t_{i,2}, \ldots, t_{i,m})$$

from some finite set of quantifier free formulae $\xi(s_1, s_2, \ldots, s_g)$ where the $s_i \in TL$ and $\forall w_1, w_2, \ldots, w_g \xi(w_1, w_2, \ldots, w_g)$ is one of the axioms Eql-5 for L.

Just as previously this result can be extended to the provability of general formulae by the introduction of Skolem Functions.

The Completeness Theorem above for Normal Structures gives us as usual a Compactness Theorem:

The Compactness Theorem for Normal Structures, 32

For L a language with equality and $\Gamma \subseteq FL$, Γ is satisfiable in a normal structure if and only if every finite subset of Γ is satisfiable in a normal structure.

Proof From left to right is clear. In the other direction suppose that Γ cannot be satisfied in a normal structure. Then $\Gamma \vDash^{=} \phi \wedge \neg\phi$ for some/any ϕ. Hence from the Completeness Theorem for Normal Structures, $\Gamma + EqL \vdash \phi \wedge \neg\phi$, so $\Gamma + EqL$ is inconsistent. As in the proof of the 'usual' Compactness Theorem there must be a finite subset Δ of Γ for which $\Delta + EqL$ is inconsistent, and hence not satisfiable. But then since EqL will automatically be satisfied in any normal structure this must mean that it is the Δ which cannot be satisfied in a normal structure. ■

An application of the Compactness Theorem

Let L be a language with equality. Then there can be no sentence $\theta \in SL$ such that for M a normal structure for L,

$$M \vDash \theta \quad \Leftrightarrow \quad |M| \text{ is finite}$$

Proof Suppose that there was such a sentence θ and consider the set of formulae

$$\Gamma = \{\theta\} \cup \{\neg x_i = x_j \mid 1 \leq i < j,\ i,\ j \in \mathbb{N}\}.$$

Let Δ be a finite subset of Γ. Then there is a bound, $k \in \mathbb{N}$ say, on the i, j such that $\neg x_i = x_j$ is in Γ. Let

$$\Gamma_k(x_1, x_2, \ldots, x_k) = \{\theta\} \cup \{\neg x_i = x_j \mid 1 < i < j \quad k\} \supseteq \Delta.$$

Let M be any normal structure for L with universe having exactly k elements, say

$$|M| = \{a_1, a_2, \ldots, a_k\}$$

– clearly we can easily make such a structure. Then since $|M|$ is finite $M \vDash \theta$ and $M \vDash \neg a_i = a_j$ for $1 \leq i < j \leq k$ so $\Gamma_k(x_1, x_2, \ldots, x_k)$ is satisfied in $M(by\ x_i \mapsto a_i,\ i = 1, 2, \ldots, k)$.

By the Compactness Theorem then Γ is satisfiable, say in a normal structure K for L by $b_1, b_2, b_3 \ldots \in |K|$. Then $|K|$ must, by our assumption on θ, be finite since $K \vDash \theta$. But also $K \vDash \neg b_i = b_j$ for $1 \leq i < j$, so $b_i \neq b_j$, and $|K|$ has infinitely many elements, contradiction!! ■

It is easy to see that even if we replaced the single sentence θ by a, possibly infinite, set of sentences Λ we would still obtain the same result, that we cannot define 'finiteness' within Predicate Logic.

Several other examples of the use of the Compactness Theorem are given in the Exercises.

In most areas of logic where the Predicate Calculus is applied, for example Model Theory and Gödel's Incompleteness Theorems, we are only interested in normal structures. As a result most of the time logicians will omit mention of 'normal' and just take it as implicit that the structures under consideration are normal, writing $\vDash$ and $\Gamma \vdash \theta$ for what in this course we would write as $\vDash^=$ and $\Gamma + EqL \vdash \theta$.

Exercises

These questions are numbered in the form $X(pY)$. The Y here refers to the page in the notes that you should be up to in order to be fully equipped to tackle the question. If the X is starred it means that the answer to this question relies starred material from the course notes.

It is important to attempt these questions, firstly because 'hands on' is very much the way to master the ideas (and notation.) in this course and secondly because the solutions to parts of these questions are quite often assumed later on in the course notes.

1(p8) Which of the following 'arguments' do you think the conclusion follows from the premises? Try to justify your answers.

(a) $$\frac{\begin{array}{l}\textit{If it rained last night the road would be wet}\\ \textit{The road is wet}\end{array}}{\therefore \textit{ It rained last night}}$$

(b) $$\frac{\begin{array}{l}\textit{Socrates is a man}\\ \textit{All men are mortal}\end{array}}{\therefore \textit{ Socrates is mortal}}$$

(c) $$\frac{\begin{array}{l}\textit{311 is prime}\\ \textit{311 is not prime}\end{array}}{\therefore \textit{ 311 is an odd number}}$$

(d) $$\frac{\textit{Montevideo is the capital of Uruguay}}{\therefore \textit{If you've gotta go you've gotta go}}$$

2(pl2) Let the language L have a binary relation symbol R and a unary relation symbol P. Which of the following are formulae of L? You should justify your answers.

a) $\forall w_3(R(w_3, x_2) \to P(w_3))$

b) $(\exists w_1 R(w_1, w_1) \to \forall w_1 P(w_1))$

c) $P(w_3)$

d) $((((P(x_1) \wedge P(x_2)) \wedge P(x_3)) \wedge (R(x_1, x_2) \wedge R(x_2, x_3)))$

e) $\forall x_3(R(x_3, x_1) \to P(x_3))$

f)* $\exists w_1(R(w_1, w_1) \to \forall w_1 P(w_1))$

3 (pl2) Show by induction on the length of formulae that if $\theta \in FL$, $s, t \in \mathbb{N}^+$ and $\theta(x_t/x_s)$ is the result of replacing the variable x_s everywhere in θ by x_t then $\theta(x_t/x_s) \in FL$.

4* (p12) Suppose that S is a relation symbol of L of arity s and let $\xi(x_1, x_2, \ldots, x_s) \in FL$. For $\phi \in FL$ let $\phi^\star$ be the result of replacing each occurrence of $S(t_1, t_2, \ldots, t_s)$ in ϕ by $\xi(t_1, t_2, \ldots, t_s)$(where the t_i are variables, free or bound). Show that if ϕ and $\xi(x_1, x_2, \ldots, x_s)$ have no free or bound variables in common then $\phi^\star$ is also a formula of L.

5 (pl5) Use Theorem 1 (The Unique Readability Theorem) to show that the following words from the language L with a binary relation symbol R are not formulae of L:

i) $(\exists w_1 R(w_1, x_1) \to R(x_1, w_1))$.
ii) $\exists w_1 (R(w_1, x_1) \wedge \forall w_1 R(x_1, x_1))$.

6 (pl5) Show that if $\exists w_j \phi \in FL$ then $\phi(x_i/w_j) \in FL$.

7 (p24) Let the language L have just a binary relation symbol R. Let M be the structure for L such that $|M| = \{1, 2, 3\}$ and

$$R^M = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 2,3\rangle, \langle 3,3\rangle\}.$$

Which of the following hold?

a) $M \models R(1,2)$

b) $M \models R(1,3) \to \neg R(1,1)$

c) $M \models \exists w_1(R(w_1,2) \wedge R(w_1, w_1))$

d) $M \models \forall w_2 R(1, w_2)$

e) $M \models \forall w_1 \forall w_2((R(w_1, w_2) \wedge R(w_2,2)) \to R(w_1,2))$

f) $M \models \forall w_2 \exists w_1 \neg R(w_1, w_2)$

g) $M \models \forall w_1(\exists w_2 R(w_1, w_2) \to R(w_1, w_1))$

h) $M \models \forall w_1 \exists w_2 \forall w_3(R(w_1, w_2) \to R(w_2, w_3))$

8 (p24) Let the language L have binary relation symbols R, S and a unary relation symbol P. Let M be the structure for L such that $|M| = \mathbb{N}^+ = \{1, 2, 3, \ldots\}$, let P^M be the set of primes and let

$$R^M = \{\langle n, m\rangle \in \mathbb{N}^2 \ | n < m\},\ S^M = \{\langle n, m\rangle \in \mathbb{N}^2 |\ m = n + 2\}.$$

Which of the following are true in *M*?

a) $\forall w_1 P(w_1)$

b) $\forall w_1 \exists w_2(R(w_1, w_2) \wedge P(w_2))$

c) $\forall w_1 \forall w_2((P(w_1) \wedge S(w_1, w_2)) \to P(w_2))$

d) $\forall w_1 \forall w_2(S(w_1, w_2) \to R(w_1, w_2))$

e) $\forall w_1 \forall w_2(R(w_1, w_2) \to \neg R(w_2, w_1))$

f) $\exists w_1 R(w_1, w_1) \to \forall w_1 P(w_1)$

g) $\forall w_1 \exists w_2 \exists w_3(((R(w_1, w_2) \wedge S(w_2, w_3)) \wedge P(w_2)) \wedge P(w_3))$.

9 (p24) Let L be as in question **6** and let M be the structure for L with

$|M| = \mathbb{N}^+ = \{1,2,3,\ldots\}$, $R^M = \{\langle n, m\rangle \in \mathbb{N}^+ \times \mathbb{N}^+ | \, n \text{ divides } m\}$.

Which of the following hold?

i) $M \vDash \forall w_3(R(w_3, 3) \to R(w_3, 9))$,

ii) $M \vDash \forall w_3(R(w_3, 4) \to R(w_3, 6))$,

iii) $M \vDash \exists w_3(R(w_3, 12) \wedge R(w_3, 18)) \wedge \neg R(3, w_3))$.

Is the following sentence true in M?

$$\forall w_1 \forall w_2 \exists w_3 \Big((R(w_3, w_1) \wedge R(w_3, w_2)) \\ \wedge \forall w_4((R(w_4, w_1) \wedge R(w_4, w_2)) \to R(w_4, w_3)) \Big).$$

Find formulae $\phi_1(x_1, x_2)$, $\phi_2(x_1)$, $\phi_3(x_1, x_2)$, $\phi_4(x_1)$ of L such that for $n, m \in |M|$,

$$n = m \Leftrightarrow M \vDash \phi_1(n, m),$$

$$n = 1 \Leftrightarrow M \vDash \phi_2(n),$$

$$gcd\{n, m\} = 1 \Leftrightarrow M \vDash \phi_3(n, m),$$

$$n \text{ is a power of a prime } \Leftrightarrow M \vDash \phi_4(n).$$

Is it possible to find a formula $\chi(x_1, x_2)$ of L such that

$$n < m \Leftrightarrow M \vDash \chi(n, m) \text{ ? } (^*\text{–ed, harder})$$

Let K be the structure for L with $|K| = \mathbb{N} = \{0,1,2,3,\ldots\}$ and $R^K = \{\langle n, m\rangle \in \mathbb{N} \times \mathbb{N} \mid n \le m\}$. Find a sentence η of L such that $M \vDash \eta$ and $K \vDash \neg\eta$.

10 (p24) Let L, L' be languages and let M, M' be structures for L, L' respectively such that $|M| = |M'|$ and for each relation symbol R of $L \cap L'$, $R^M = R^{M'}$. Show that for $\theta(\vec{x}) \in FL \cap FL'$ and $\vec{a} \in |M|$,

$$M \vDash \theta(\vec{a}) \Leftrightarrow M' \vDash \theta(\vec{a}).$$

Hence show that the notion of logical consequence is language independent in the sense that if $\Gamma \subseteq FL \cap FL'$ and $\theta(\vec{x}) \in FL \cap FL'$ then $\theta(\vec{x})$ is true in every interpretation for L in which every formula in Γ is true just if is true in every interpretation for L' in which every formula in Γ is true.

11 (p30) For $\Gamma, \Delta \subseteq SL$ and $\theta, \phi \in SL$ show that

i) $\Gamma, \theta \vDash \phi \Leftrightarrow \Gamma \vDash (\theta \to \phi)$

ii) $\Gamma \vDash \phi \ \& \ \Delta \vDash \theta \Rightarrow \Gamma, \Delta \vDash (\theta \wedge \phi)$

iii) $\Gamma \vDash \theta \ \& \ \Delta \vDash (\theta \to \phi) \Rightarrow \Gamma, \Delta \vDash \phi$

[Here, as usual, Γ, θ is an abbreviation for $\Gamma \cup \{\theta\}$ and Γ, Δ is an abbreviation for $\Gamma \cup \Delta$. Note that exactly the same results hold for $\Gamma, \Delta \subseteq FL$ and $\theta, \phi \in FL$, it's just that we need to argue not just about structures but also about interpretations of the free variables in those structures. In such cases we will, purely for notational simplicity, often prove a result for sentences since the generalization to formulae uses just the same ideas.]

12 (p30) For the language L with a single binary relation symbol R show that no two of the following sentences logically imply the third:

i) $\forall w_1 \forall w_2 \forall w_3 ((R(w_1, w_2) \wedge R(w_2, w_3)) \to R(w_1, w_3))$,

ii) $\forall w_1 \forall w_2 ((R(w_1, w_2) \vee R(w_2, w_1))$,

iii) $\exists w_1 \forall w_2 R(w_1, w_2)$.

13[*] (p30) Let ξ, $\phi(\vec{x})$, $\phi^\star(\vec{x})$ etc. be as in question 4 with ϕ, ξ having no variables, free or bound, in common. Given a structure M for L let $M^\star$ be the structure for L such that $|M^\star| = |M|$ and for R an r-ary relation symbol of L,

$$R^{M^\star} = \begin{cases} R^M & if\ R \neq S, \\ \{\langle a_1, a_2, \ldots, a_s\rangle \ | M \vDash \xi(a_1, a_2, \ldots, a_s)\} & if\ R = S. \end{cases}$$

Show that for $\vec{a} \in |M|$,

$$M^\star \vDash \phi(\vec{a}) \Leftrightarrow M \vDash \phi^\star(\vec{a}).$$

Hence show that if ϕ is a tautology then so is $\phi^\star$

14 (p35) Show the following from the list of 'useful logical equivalences' (to simplify the notation you may assume that all the displayed formulae are actually sentences):

a) $\theta \vee \phi \equiv \phi \vee \theta,$

b) $\forall w_1 \psi(w_1) \equiv \forall w_2 \psi(w_2),$

c) $(\forall w_1 \psi(w_1) \wedge \theta) \equiv \forall w_1(\psi(w_1) \wedge \theta),$

d) $(\exists w_1 \psi(w_1) \to \theta) \equiv \forall w_1(\psi(w_1) \to \theta).$

where in (b), (c) w_1 does not occur in θ.

15 (p35) Which of the following hold (for arbitrary θ, ϕ) ? In each case justify your answer, either by giving a (informal!) proof that it holds or by providing a counter-example:

a) $\neg(\theta \to \phi) \equiv (\theta \to \neg\phi)$

b) $\neg\exists w_1 \theta(w_1) \equiv \forall w_1 \neg\theta(w_1)$

c) $\forall w_1(\theta(w_1) \wedge \phi(w_1)) \equiv (\forall w_1 \theta(w_1) \wedge \forall w_1 \phi(w_1))$

d) $\exists w_1(\theta(w_1) \wedge \phi(w_1)) \equiv (\exists w_1 \theta(w_1) \wedge \exists w_1 \phi(w_1))$

e) $\forall w_1(\theta(w_1) \to \phi(w_1)) \equiv (\forall w_1 \theta(w_1) \to \forall w_1 \phi(w_1))$

f)[*] $\exists w_1(\theta(w_1) \to \phi(w_1)) \equiv (\forall w_1 \theta(w_1) \to \exists w_1 \phi(w_1))$

16(p35) For $\theta_i(\vec{x}) \in FL$ we define $\bigwedge_{i=1}^{n} \theta_i(\vec{x})$ and $\bigvee_{i=1}^{n} \theta_i(\vec{x})$ inductively by

$$\bigwedge_{i=1}^{1} \theta_i(\vec{x}) = \theta_1(\vec{x}), \quad \bigwedge_{i=1}^{n+1} \theta_i(\vec{x}) = \left(\bigwedge_{i=1}^{n} \theta_i(\vec{x}) \right) \wedge \theta_{n+1}(\vec{x}))$$

$$\bigvee_{i=1}^{1} \theta_i(\vec{x}) = \theta_1(\vec{x}), \quad \bigvee_{i=1}^{n+1} \theta_i(\vec{x}) = \left(\bigvee_{i=1}^{n+1} \theta_i(\vec{x}) \right) \vee \theta_{n+1}(\vec{x}).$$

Show that for M a structure for L and $\vec{a} \in |M|$,

$$M \vDash \bigwedge_{i+1}^{n} \theta_i(\vec{a}) \Leftrightarrow M \vDash \theta_i(\vec{a}) \quad \text{for all } 1 \leq i \leq n,$$

$$M \vDash \bigvee_{i=1}^{n} \theta_i(\vec{a}) n \Leftrightarrow M \vDash \theta_i(\vec{a}) \quad \text{for some } 1 \leq i \leq n \quad .$$

17* (p37) Write down formulae in Prenex Normal Form logically equivalent to:

a) $\neg \exists w_1 \forall w_2 R(w_1, w_2)$,

b) $\forall w_1 R(w_1, x_1) \wedge \exists w_1 R(x_2, w_1)$,

c) $\forall w_1 R(w_1, x_1) \to \exists w_2 R(x_2, w_2)$.

18 (p47) Fill-in justifications for the steps in the following formal proof:

1. $\forall w_1 P(w_1) \mid \forall w_1 P(w_1)$
2. $\forall w_1 P(w_1) \mid P(x_1)$
3. $\quad P(x_1) \mid P(x_1)$
4. $\quad P(x_1) \mid (P(x_1) \wedge P(x_1))$
5. $\forall w_1 P(w_1) \mid (P(x_1) \wedge P(x_1))$
6. $\forall w_1 P(w_1) \mid \forall w_1 (P(w_1) \wedge P(w_1))$

If we to append to this proof the sequents

7. $\exists w_1 P(w_1) \mid (P(x_1) \wedge P(x_1))$
8. $\exists w_1 P(w_1) \mid \exists w_1 (P(w_1) \wedge P(w_1))$

would it still be a correct proof? If not how might it be corrected to give the same final sequent?

19 (p50) Give formal proofs of the following:

a) $\vdash (\theta \to \theta)$

b) $\vdash (\phi \to (\theta \to \phi))$

c) $\vdash (\theta \wedge \neg\phi) \to \neg(\theta \to \phi)$

d) $\vdash (\neg\theta \to (\theta \to \phi))$

e) $\neg(\theta \wedge \phi) \vdash (\neg\theta \vee \neg\phi)$

f) $\forall w_1\theta(w_1) \vdash \forall w_2\theta(w_2)$

g) $\exists w_1\theta(w_1) \vdash \exists w_2\theta(w_2)$

h) $\exists w_1\neg\theta(w_1) \vdash \neg\forall w_1\theta(w_1)$

i) $\forall w_1\neg\theta(w_1) \vdash \neg\exists w_1\theta(w_1)$

j) $\exists w_1(\theta(w_1) \vee \phi(w_1)) \vdash (\exists w_1\theta(w_1) \vee \exists w_1\phi(w_1))$

k) $\forall w_1(\theta(w_1) \to \phi(w_1)), \exists w_1\theta(w_1) \vdash \exists w_1\phi(w_1)$

l) $\forall w_1(\theta(w_1) \vee \phi(w_1)) \vdash \forall w_1\theta(w_1) \vee \exists w_1\phi(w_1)$

20* (p53) Show that if $\phi(x_1)$, $\theta(\vec{x}) \in FL$, w_1 does not occur in $\phi(x_1)$ and $\phi(x_i) \vdash \theta(\vec{x})$ for all $i \in \mathbb{N}^+$ then $\exists w_1\phi(w_1) \vdash \theta(\vec{x})$.

21 (p53) Prove Lemma 5(ii) in the case where the rule is (a) AND, (b) $\forall$I, (c) DIS.

22 (p55) Prove Lemma 6(ii) in the case where the rule is (a) ORR, (b) $\forall$O, (c) $\exists$O.

23 (p64) Let Ω be as in Lemma 13. Show that:

c) $(\theta \wedge \phi) \in \Omega \Leftrightarrow \theta \in \Omega$ and $\phi \in \Omega$.

d) $(\theta \vee \phi) \in \Omega \Leftrightarrow \theta \in \Omega$ or $\phi \in \Omega$.

24* (p67) Let L have a single binary relation symbol R. Show that if $\Gamma \subseteq SL$ is satisfiable then Γ is satisfiable in a structure M for L with $|M|$ infinite. Is it necessarily true that Γ must also be satisfiable in a structure with finite universe?

25 (p70) Suppose that $\theta_n \in SL, n \in \mathbb{N}$, are such that for every structure M for L there is some $n \in \mathbb{N}$ such that $M \vDash \theta_n$. Show that for some m

$$\neg\theta_0, \neg\theta_1, \ldots, \neg\theta_{m-1} \vDash \theta_m.$$

26 (p70) Suppose that $\Gamma, \Delta \subseteq SL$ are such that for any structure M for L,

$$M \vDash \Gamma \Leftrightarrow M \nvDash \Delta$$

Show that there are finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that for any structure M for L,

$$M \vDash \Gamma' \Leftrightarrow M \nvDash \Delta'.$$

27 (p70) Let L be the language with unary relation symbols R_n for $n \in \mathbb{N}^+$ and let

$$\Gamma = \{R_n(x_1) \mid n \in \mathbb{N}^+\}.$$

Using the Compactness Theorem for Relational Languages show that there can be no sentence $\psi \in SL$ such that, for any structure M for L,

$$M \vDash \psi \Leftrightarrow \Gamma \text{ is satisfiable in } M.$$

28 (p70) Let L be the language with a single binary relation symbol R. Say that a structure M for L is *connected* if for any $g, h \in |M|$ there are some $a_1, a_2, \ldots, a_n \in |M|$ such that $a_1 = g, a_n = h$ and

$$M \vDash \bigwedge_{i=1}^{n-1} R(a_i, a_{i+1}).$$

Show that there is no sentence θ of L such that for a (normal) structure M for L,

$$M \vDash \theta \Leftrightarrow M \text{ is connected.}$$

[Hint: Assume that such a sentence θ did exist and consider the set of formulae

$$\{\neg\exists w_1, \ldots, w_n((R(x_1, w_1) \wedge R(w_n, x_2)) \wedge \bigwedge_{i=1}^{n-1} R(w_i, w_{i-1})) \mid n \in \mathbb{N}^+\} \cup \{\theta, \neg R(x_1, x_2)\}.]$$

29 (p73) Let the language L have a binary function symbol f, a unary function symbol g and a constant symbol c. Which of the following are terms of L? Justify your answers.

(i) $f(g(f(x_1, x_1)), c)$, (ii) $gg(c)$, (iii) $f(f(x_1, w_1), g(x_1))$,

(iv) $f(f(g(f(c, f(f(g(f(x_1, f(g(x_2), g(g(x_3))))))), c)), x_2)$.

30 (p78) Let L be as in the previous question and let M be a structure for L with $|M| = \mathbb{Z}$, $f^M(x, y) = x - y$, $g^M(x) = x^2$, $c^M = 4$. Evaluate $t^M(2, -5)$ when $t(x_1, x_2)$ is

(i) $f(g(x_1), x_2)$, (ii) $f(f(g(c), x_1), x_2)$, (iii) $g(f(f(x_1, c), g(x_2)))$.

31 (p88) Give formal proofs of the following where R is a unary relation symbol, f is a unary function symbol:

a) $\forall w_1 R(w_1) \vdash \forall w_1 R(f(w_1))$

b) $\exists w_1 R(f(w_1)) \vdash \exists w_1 R(w_1)$

32 (p88) Let M, K be structures for a language L and $t(\vec{x}) \in TL, \phi(\vec{x}) \in FL$. Suppose that $|M| = |K|$ and $R^M = R^K$, $c^M = c^K$, $f^M = f^K$ for every relation, constant, function symbol R, c, f occurring in $t(\vec{x})$ or $\phi(\vec{x})$. Show that for $\vec{a} \in |M|$, $t^M(\vec{a}) = t^K(\vec{a})$ and

$$M \vDash \phi(\vec{a}) \Leftrightarrow K \vDash \phi(\vec{a}).$$

Suppose c is a constant symbol of L and let $\theta(x_1) \in FL$ be such that c does not occur in θ. Use the above result to show that

i)* If $\vDash \theta(c)$ then $\vDash \forall w_j \theta(w_j)$.

Show directly (so without appealing to the Completeness Theorem) that:

ii)* If $\vdash \theta(c)$ then $\vdash \forall w_j \theta(w_j)$.

33 * (p88) Let c_1, c_2 be constant symbols of L and let $\theta(x_1, x_2)$ be a formula of L which does not mention c_1 or c_2. Show that if $\{\theta(c_1, c_2)\}$ is inconsistent then so is $\{\theta(c_1, c_1)\}$.

Is the converse true, that if $\{\theta(c_1, c_1)\}$ is inconsistent then so is $\{\theta(c_1, c_2)\}$?

34 (p95) Let L be the language with constant symbols c_n for $n \in \mathbb{N}$, binary function symbols $f_+, f_\times$ and binary relation symbol $R_<$ and let $L(\varepsilon)$ be L augmented with a new constant symbol ε. Let $\mathcal{R}$ be the structure for L with

$$|\mathcal{R}| = \mathbb{R},\ c_n^{\mathcal{R}} = n,\ R_<^{\mathcal{R}} = \{\langle r, s\rangle \in \mathbb{R} \times \mathbb{R} \mid r < s\}$$

$$f_+^{\mathcal{R}}(r, s) = r + s, \qquad f_\times^{\mathcal{R}}(r, s) = rs.$$

Show that there is a model[45] M of

$$\Omega = \{\theta \in SL \;|\mathcal{R} \models \theta\} \cup \{R_<(c_0, \varepsilon)\} \cup \{R_<(f_\times(c_n, \varepsilon), c_1)|\; n \in \mathbb{N}\}.$$

35 (pl05) Let L be the language with equality, a binary relation symbol R, a binary function symbol f, unary function symbol g and constant symbol c. Which of the following are formulae of L? Justify your answers.

(i) $\forall w_1(x_1 = w_1)$, (ii) $\forall w_1(x_1 = w_1 \vee x_1 = w_1)$, (iii) $\exists w_3 f(w_3, x_1)$, (iv) $\forall w_1(R(x_1, w_1) \to w_1 = x_2)$.

Let M be the (normal) structure for L with $|M| = \mathbb{N}^+ =$

$\{1,2,3, \ldots\}$, $f^M(x, y) = x + y$, $g^M(x) = x^2$, $c^M = 2$,

$R^M = \{\langle n, m\rangle \in (\mathbb{N}^+)^2 \;|n|\,m$ i.e. n divides $m\}$.

Which of the following are true in M?

1) $\forall w_1 f(w_1, w_1) = c$,
2) $\exists w_1 c = g(w_1)$,
3) $\forall w_1 \forall w_2 (R(w_1, w_2) \to R(w_1, g(w_2)))$,
4) $\exists w_1 \forall w_2 \forall w_3 (R(w_2, f(w_1, w_3)) \to R(w_2, w_3))$.

Find $\theta_1(x_1)$, $\theta_2(x_1)$, $\theta_3(x_1)$, $\theta_4(x_1, x_2, x_3)$, $\theta_5(x_1)$, $\theta_6(x_1, x_2, x_3) \in FL$ such that for $n, m, k \in |M|$,

$M \models \theta_1(n) \Leftrightarrow n = 4$,

$M \models \theta_2(n) \Leftrightarrow n = 3$,

$M \models \theta_3(n) \Leftrightarrow n$ is the sum of two squares (of elements of $\mathbb{N}^+$),

$M \models \theta_4(n, m, k) \Leftrightarrow n = \gcd(m, k)$,

$M \models \theta_5(n) \Leftrightarrow n$ is prime,

$M \models \theta_6(n, m, k) \Leftrightarrow n = mk$.

Let K be the (normal) structure for L with

$|K| = \mathbb{Q}^+ = \{q \in \mathbb{Q} \mid q > 0\}$,

$f^K(x, y) = x + y(x, y \in \mathbb{Q}^+$ of course$)\, g^K(x) = x^2$, $c^K = 2$,

$$R^K = \{\langle q, s\rangle \in (\mathbb{Q}^+)^2 \mid q < s\}.$$

Find $\phi \in SL$ such that $M \vDash \phi$, $K \nvDash \phi$.

36 (pl05) Write down sentences $\theta_1, \theta_2, \theta_3$ of L such that for a normal structure M for L,

$$M \vDash \theta_1 \Leftrightarrow |M| \text{ has at most 3 elements,}$$

$$M \vDash \theta_2 \Leftrightarrow |M| \text{ has at least 3 elements,}$$

$$M \vDash \theta_3 \Leftrightarrow |M| \text{ has exactly 3 elements.}$$

Suppose that f is a unary function symbol of L. Show that

$$\forall w_1 \forall w_2 (f(w_1) = f(w_2) \to w_1 = w_2) \wedge \exists w_1 \forall w_2 \neg f(w_2) = w_1$$

is satisfied in some normal structure for L but is not satisfied in any finite normal structure for L.

37 (pl05) In a certain football league every team plays every other team exactly once and either wins, loses or draws. Let M be the structure for the language L with equality and a binary relation symbol R such that $|M|$ is the set of teams in the league and

$$R^M = \{\langle b, c\rangle \in |M| |\ R \neq S, \text{ and team } b \text{ beats team } c\}.$$

Write down formulae $\theta_1(x_1, x_2)$, $\theta_2(x_1)$, θ_3, θ_4, of L such that for $b, c \in |M|$, $R \neq S$,

$M \vDash \theta_1(b, c) \Leftrightarrow$ the match between team b and team c is drawn,

$M \vDash \theta_2(b) \Leftrightarrow$ team b loses all its matches,

$M \vDash \theta_3 \Leftrightarrow$ no team wins all its matches,

$M \vDash \theta_4 \Leftrightarrow$ some team wins all its matches except one.

38 (pl06) Show that

$$\forall w_1(\theta(w_1) \to w_1 = c), \neg\theta(c) \vDash \forall w_1 \neg\theta(w_1).$$

[Purely to simplify the notation you may assume that these are all *sentences* of L.]

39* (p115) Let the language L have a binary relation symbol R, a unary function symbol f and a constant symbol c. Give formal proofs of the following:

a) $EqL, t = s \vdash f(t) = f(s)$, where $t, s \in TL$,

b) $EqL, x_1 = c \vdash (R(x_1, c) \to R(c, c))$,

c) $EqL, \exists w_1 \theta(w_1), \neg\theta(c) \vdash \exists w_1 (\theta(w_1) \wedge \neg w_1 = c)$,

d) $EqL, \forall w_1 (\theta(w_1) \to w_1 = c), \neg\theta(c) \vdash \forall w_1 \neg\theta(w_1)$.

40 (pl24) For L as in the previous question show that

(a) $EqL, f(x_1) = f(x_2) \nvdash x_1 = x_2$,

(b) $EqL, \exists w_1 (\neg w_1 = c \wedge R(w_1, w_1)) \nvdash \neg R(c, c)$

(c) $Eq1, Eq3 \nvdash Eq2$.

41*(p127) The language of arithmetic, LA, has equality, binary function symbols $\pm, \underline{\cdot}$ and constants $\underline{0}, \underline{1}$. Let N be the structure for LA with $|N| = \mathbb{N}$, $\underline{0}^N = 0$, $\underline{1}^N = 1$, and $\pm^N, \underline{\cdot}^N$ the usual addition and multiplication resp. on $\mathbb{N}$. For $1 \leq n \in \mathbb{N}$ what is

$$\pm(\underline{1}, \pm(\underline{1}, \pm(, \ldots \pm(\underline{1}, \pm(\underline{1}, \pm(\underline{1}, \underline{1}))) \ldots))^N$$

when there are n copies of $\pm$?

Let

$$TA = \{\theta \in SLA \mid N \vDash \theta\} = \text{ 'True arithmetic'.}$$

N is called the *standard model of true arithmetic*. By using the Compactness Theorem show that there are 'non-standard models of true arithmetic', that is (normal) models which are not isomorphic to N (i.e. not just N with the elements of $|N|$ renamed).

42*(p127) By using the Compactness Theorem for Normal Structures prove König's Lemma:

Let H be a set of finite strings $a_0 a_1 a_2 a_3 \ldots a_k$ of 0's and 1's such that

1. *If $a_0 a_1 a_2 a_3 \ldots a_k \in H$ and $n \leq k$ then $a_0 a_1 a_2 \ldots a_n \in$*
2. *For each $n \in \mathbb{N}$ there is a string $a_0 a_1 a_2 \ldots a_n \in H$ ($i.e.$ a string in H of length $n+1$).*

Then there is an $infinite$ string $b_0 b_1 b_2 \ldots$ of 0's and 1's such that for all $n \in \mathbb{N}$, $b_0 b_1 b_2 \ldots b_n \in H$.

[Only for those who think this course is too easy.]

Solutions to the Exercises

1 (a) This does not follow. For suppose we put P for 'it rained last night' and Q for 'the road is wet'. Then the argument becomes:

$$\begin{array}{l} \textit{If } P \textit{ then } Q \, (\textit{i.e. } P \to Q) \\ Q \\ \hline \therefore P \end{array}$$

But clearly this $\mathrm{isn't}$ correct in general, for example let P stand for 'the moon is made of green cheese' and Q stand for '5 is prime'. Then both $P \to Q$ and Q are true but P is not true.

(b) This does follow. For let $M(x)$ stand for 'x is a man', let $E(x)$ stand for 'x is mortal', let s stand for Socrates and let the variables range over, say, objective things. Then the argument becomes

$$\begin{array}{l} M(s) \\ \forall x(M(x) \to E(x)) \\ \hline \therefore E(s) \end{array}$$

But clearly this conclusion must be true whenever the premises are both true no matter what properties M and E stand for, no matter what the range of the variable x is and no matter what element of this range s denotes.

(c) This does follow. For let P stand for '311 is prime' and Q stand for '311 is odd'. Then the argument becomes

$$\begin{array}{l} P \\ \neg P \\ \hline \therefore Q \end{array}$$

But because P and $\neg P$ cannot both be true, *if* they are both true *then* Q will be true, *no matter what* P, Q *stand for*. So this conclusion does follow from the premises.

(d) This does follow. For let P stand for 'Montevideo is the capital of Uruguay' and and Q stand for 'you gotta go' Then the argument becomes

$$\begin{array}{l} P \\ \hline \therefore Q \to Q \end{array}$$

But no matter what Q stands for $Q \to Q$ is true (such an assertion is called a *tautology*) so certainly this conclusion is always true when P is true, no matter what P stands for.

2 (a) This is a formula since

$P(x_1)$, $R(x_1, x_2) \in FL$ by L1,

$(R(x_1, x_2) \to P(x_1)) \in FL$ by L2,

$\forall w_3(R(w_3, x_2) \to P(w_3)) \in FL$ by L3.

(b) This is a formula since

$P(x_1)$, $R(x_1, x_1) \in FL$ by L1,

$\forall w_1 P(w_1)$, $\exists w_1 R(w_1, w_1) \in FL$ by L3,

$(\exists w_1 R(w_1, w_1) \to \forall w_1 P(w_1)) \in FL$ by L2.

(c) This is not a formula. The idea is to state some property $\mathcal{P}$ for which we can prove by induction on the length of formulae that all formulae have $\mathcal{P}$ but $P(w_3)$ does not. (In answering an exam question it would be enough to simply state such a property without actually *proving* that it works.) There are lots of different properties we could choose here for $\mathcal{P}$, for example that if some w_i occurs in the expression then so must either $\forall$ or $\exists$.

So suppose that $\theta \in FL$ and $\mathcal{P}$ holds for all formulae of length less that $|\theta|$. As in the example on page 12 there are 7 cases:

Case 1 $\theta = R(\vec{x})$ for some relation symbol R of L. In this case no w_i is mentioned in θ so $\mathcal{P}$ holds vacuously.

Case 2 $\theta = (\phi \wedge \psi)$. In this case if some w_i occurs in θ then it must occur in one of ϕ or ψ. Without loss of generality suppose it is ϕ. Then since $|\phi| < |\theta|$ $\mathcal{P}$ must hold for ϕ. In other words one of $\exists, \forall$ must occur in ϕ and hence in θ. The cases for the other connectives $\neg, \vee, \rightarrow$ are exactly similar. [In such situations just say this rather than plodding through each case separately.]

Case 3 $\theta = \exists w_j \phi(w_j/x_i)$ where $\phi \in FL$ does not mention w_j. In this case θ does mention $\exists$ so the required property $\mathcal{P}$ holds trivially for θ (and similarly for $\theta = \forall w_j \phi(w_j/x_i)$).

So by induction on the length of formulae every formula of L must satisfy $\mathcal{P}$. But $P(w_3)$ does not satisfy $\mathcal{P}$ so it cannot be a formula of L.

(d) This is not a formula. To see this let $\mathcal{P}$ be the property of containing the same number of left parentheses '(' as right parentheses ')'. Then $\mathcal{P}$ fails for

$$((((P(x_1) \wedge P(x_2)) \wedge P(x_3)) \wedge (R(x_1, x_2) \wedge (x_2, x_3)))$$

so it is enough to show by induction on the length of formulae that $\mathcal{P}$ holds for all formulae. This is obvious of simple inspection (and in answering an exam question it would be enough to leave it at that) but if you want to go through some details there are the usual 7 cases:

If $\theta = R(\vec{x})$ then θ has $\mathcal{P}$ since θ contains one '(' and one ')'.

If $\theta = (\phi \wedge \psi)$ then by inductive hypothesis the number, l_ϕ, of '(' in ϕ is the same as the number, r_ϕ, of ')' in ϕ and similarly for ψ (since $|\phi|, |\psi| < |\theta|$). Hence

$$l_\theta = 1 + l_\phi + l_\psi = r_\phi + r_\psi + 1 = r_\theta,$$

as required. Similarly for the other connectives.

If $\theta = \forall w_j\, \phi(w_j/x_i)$ then $l_\theta = 1 + l_\phi = 1 + r_\phi$ (by IH) $= r_\theta$, as required. Similarly for $\theta = \exists s_j\, \phi(w_j/x_i)$.

(e) Not a formula. In this case take $\mathcal{P}$ to be, say, 'whenever $\forall$ appears in a formula it is followed immediately by w_j for some j'.

(f) Not a formula, but in this case the required property $\mathcal{P}$ to exclude $\exists w_1\,(R(w_1, w_1) \to \forall w_1\, P(w_1))$ from the set FL is harder to find and it seems simplest to take a different tack. So suppose that this was a formula of L. Then by the way formulae are formed it must be the case that

$$\exists w_1\,(R(w_1, w_1) \to \forall w_1\, P(w_1)) = \exists w_1\, \phi(w_1/x_i)$$

for some $\phi \in FL$ not mentioning w_1. Hence

$$\phi(w_1/x_i) = (R(w_1, w_1) \to \forall w_1\, P(w_1))$$

and since ϕ does not mention w_1 it must be the case that all the w_1 on this left hand side were x_i in ϕ, in other words

$$\phi = (R(x_i, x_i) \to \forall x_i\, P(x_i)).$$

But by the proof of (e) immediately above this right hand side is not a formula, giving the required contradiction.

3 Assume the result is true (for all $s, t \in \mathbb{N}^+$) for all formulae of length less than $|\theta|$.

If $\theta = R(x_{i_1}, \ldots, x_{i_r})$ where R is an r-ary relation symbol of L then $\theta(x_t/x_s) = R(x_{j_1}, \ldots, x_{j_r})$ where $j_k = i_k$ if $i_k \neq s$ and $j_k = t$ if $i_k = s$. Then $\theta' \in FL$ by L1.

If $\theta = (\phi \wedge \psi)$ the $\theta(x_t/x_s) = (\phi(x_t/x_s) \wedge \psi(x_t/x_s))$ and since $\phi(x_t/x_s)$, $\psi(x_t/x_s) \in FL$ by inductive hypothesis, $\theta(x_t/x_s) \in FL$ by L2. The cases for the other connectives are exactly similar. [In situations like this it is enough to just do the case for one connective, similarly for just one quantifier.]

Finally suppose that $\theta = \exists w_j\, \phi(w_j/x_i)$. If $i \neq s$ then $\phi(w_j/x_i) = \phi(x_t/x_s)(w_j/x_i)$ and $\theta(x_t/x_s) = \exists w_j\, \phi(x_t/x_s)(w_j/x_i)$ so since $\phi(x_t/x_s) \in FL$ (by Inductive Hypothesis) so $\theta(x_t/x_s) \in FL$ by L3. If $i = s$ then θ does not mention x_s so $\theta = \theta(x_t/x_s) \in FL$. If $i = t$ let k be such that x_k does not occur in ϕ and write $\phi = \phi(x_s, x_t, \vec{x})$ where $\vec{x}$ are the other free variable occurring in ϕ. Then by Inductive Hypothesis $\phi(x_k/x_t) = \phi(x_s, x_k, \vec{x}) \in FL$. Hence in turn $\{\phi(x_k/x_t)\}(x_t/x_s) = \phi(x_t, x_k, \vec{x}) \in FL$, and

$$\begin{aligned} \theta(x_t/x_s) &= \{\exists w_j\, \phi(w_j/x_t)\ \}(x_t/x_s) \\ &= \exists w_j \phi(x_t, w_j, \vec{x}) \\ &= \exists w_j \{\phi(x_t, x_k, \vec{x})(w_j/x_k)\ )\} \in FL \text{ by L3.} \end{aligned}$$

where the {,} are not part of the syntax but have been introduced here just to make clear the order of the substitutions. The case for $\forall$ is exactly similar.

4 Assume the result is true for all formulae of length less than $|\phi|$ and that ϕ has no variables, free or bound, in common with ξ.

If $\phi = R(x_{i_1}, x_{i_2}, \dots, x_{i_r})$ for some relation symbol R of L then either $R \neq S$, in which case $\phi^\star = \phi$, or $R = S$,in which case $\phi^\star = \xi(x_{i_1}, x_{i_2}, \dots, x_{i_r})$. Either way $\phi^\star \in FL$.

If $\phi = (\theta \wedge \psi)$ then $\phi^\star = (\theta^\star \wedge \psi^\star)$. Since θ, ψ must also have no variables in common with ξ, by the Inductive Hypothesis $\theta^\star, \psi^\star \in FL$ so $\phi^\star \in FL$ by L2. The cases for the other connectives are exactly similar.

If $\phi = \exists w_j\, \psi(w_j/x_i)$ then $\phi^\star = (\exists w_j\, \psi(w_j/x_i))^\star = \exists w_j\, (\psi(w_j/x_i))^\star$. We may assume that x_i does not occur in ξ, otherwise by question 3 above we can replace x_i in ψ by some x_k which does not occur in ϕ or ξ to get $\psi(x_k/x_i) \in FL$ and useinstead that $\phi = \exists w_j\, (\psi(x_k/x_i))(w_j/x_k)$. By the Inductive Hypothesis then $\psi^\star \in FL$, since $|\psi| < |\phi|$. Also $(\psi(w_j/x_i))^\star = \psi^\star(w_j/x_i)$ since because ϕ and ξ have no variables in common replacing S by ξ in ϕ to get $\phi^\star$ does not introduce any new occurrences of x_i, and hence this operation commutes with that of replacing x_i by w_j. Furthermore since w_j occurs in ϕ by assumption it does not occur in ξ, so w_j does not occur in $\psi^\star$ and $\phi^\star = \exists w_j\, (\psi(w_j/x_i))^\star = \exists w_j\, \psi^\star(w_j/x_i) \in FL$ by L3. The case for $\forall$ is exactly similar and the desired result follows by induction on the length of formulae.

5 (i) Suppose on the contrary that $(\exists w_1\, R(w_1, x_1) \to R(x_1, w_1))$ was a formula. Then the only case from Theorem 1 that can apply is (5), which means that both $\exists w_1\, R(w_1, x_1)$ and $R(x_1, w_1)$ must be formulae. But this latter does not fall under any case (not even case (1) because it contains a bound variable) so it cannot be a formula. Hence $(\exists w_1\, R(w_1, x_1) \to R(x_1, w_1))$ cannot be a formula.

(ii) Again suppose on the contrary that

$$\exists w_1\, (R(w_1, x_1) \wedge \forall w_1\, R(x_1, x_1))$$

was a formula. Then the only case in Theorem 1 that applies is (7) and we must have that $\exists w_1\, (R(w_1, x_1) \wedge \forall w_1\, R(x_1, x_1))$ is $\exists w_j\, \eta(w_j/x_i)$ for some w_j and $\eta \in FL$ with w_j not occurring in η. Clearly j must equal 1 and $\eta(w_j/x_i)$ must be $(R(w_1, x_1) \wedge \forall w_1\, R(x_1, x_1))$. Since w_j (i.e. w_1) does not occur in η it must be that η is

$$(R(x_i, x_1) \wedge \forall x_i\, R(x_1, x_i)).$$

But now by case (3) of Theorem 1 $\forall x_i\, R(x_1, x_i)$ must be a formula, which (since x_i is a free, not a bound a variable) is impossible since it does not correspond to any case in that theorem. The required conclusion follows.

6 By the Unique Readability Theorem 1, if $\exists w_j\, \phi \in FL$ then it must be the case that $\exists w_j\, \phi = \exists w_j\, \psi(w_j/x_k)$ for some k and $\psi \in FL$ in which w_j does not occur. Since as words (i.e. strings of symbols from $\exists, \forall, x_h, w_r$, (,),…etc.) these two are the same it must be that $\psi(w_j/x_k) = \phi$. Hence

$$\phi(x_i/w_j) = \{\psi(w_j/x_k)\}(x_i/w_j) = \psi(x_i/x_k)$$

and this right hand side expression is a formula by problem 3 above.

7 (a) $M \vDash R(1,2) \Leftrightarrow \langle 1,2\rangle \in R^M$ by T1 – which holds.

(b) $M \vDash (R(1,3) \to \neg R(1,1))$

$$\begin{aligned} &\Leftrightarrow \quad M \nvDash R(1,3) \text{ or } M \vDash \neg R(1,1) \quad \text{by T2} \\ &\Leftrightarrow \quad M \nvDash R(1,3) \text{ or } M \nvDash R(1,1) \quad \text{by T2} \\ &\Leftrightarrow \quad \langle 1,3\rangle \notin R^M \text{ or } \langle 1,1\rangle \notin R^M \quad \text{by T2} \end{aligned}$$

– which does not hold since both $\langle 1,3\rangle$ and $\langle 1,1\rangle$ are in R^M.

(c) $M \vDash \exists w_1 (R(w_1,2) \wedge R(w_1, w_1))$

$$\begin{aligned} \Leftrightarrow \quad & \text{for some } b \in |M| = \{1,2,3\} \\ & M \vDash R(b,2) \wedge R(b,b), \quad \text{by T3,} \\ \Leftrightarrow \quad & \text{for some } b \in |M| = \{1,2,3\}, \\ & M \vDash R(b,2) \text{ and } M \vDash R(b,b), \quad \text{by T2,} \\ \Leftrightarrow \quad & \text{for some } b \in |M| = \{1,2,3\} \\ & \langle b,2\rangle \in R^M \text{ and } \langle b,b\rangle \in R^M, \quad \text{by T1,} \end{aligned}$$

– which holds (when $b = 1$) since $\langle 1,2\rangle, \langle 1,1\rangle \in R^M$.

(d) $M \vDash \forall w_2 R(1, w_2)$

$$\begin{aligned} &\Leftrightarrow \quad \text{for all } b \in |M|,\ M \vDash R(1,b), \quad \text{by T3} \\ &\Leftrightarrow \quad M \vDash R(1,1) \text{ and } M \vDash R(1,2) \text{ and } M \vDash R(1,3) \\ &\Leftrightarrow \quad \langle 1,1\rangle \in R^M \text{ and } \langle 1,2\rangle \in R^M \text{ and } \langle 1,3\rangle \in R^M \end{aligned}$$

– which holds.

(e) $M \vDash \forall w_1 \forall w_2 ((R(w_1, w_2) \wedge R(w_2,2)) \to R(w_1,2))$

$$\begin{aligned} \Leftrightarrow \quad & \text{for all } b, c \in |M|, \text{if } M \vDash R(b,c) \text{ and } M \vDash R(c,2) \\ & \text{then } M \vDash R(b,2) \\ \Leftrightarrow \quad & \text{for all } b, c \in |M|, \text{if } \langle b,c\rangle \in R^M \text{ and } \langle c,2\rangle \in R^M \\ & \text{then } \langle b,2\rangle \in R^M. \end{aligned}$$

On the face of it we now have to check this for all $b, c \in |M| = \{1,2,3\}$. However since $\langle 1,2\rangle \in R^M$ we have right hand side of the implication for the cases for $b = 1$ (for any c). For $b = 2$, and again for $b = 3$, one of $\langle b,c\rangle$, $\langle c,2\rangle$ is not in R^M for any choice of $c \in \{1,2,3\}$, as can be easily checked. Hence the original assertion holds.

(f) $M \vDash \forall w_2 \exists w_1 \neg R(w_1, w_2)$

$\Leftrightarrow$ for each $b \in |M|$ there is a $c \in |M|$ such that $\langle c,b\rangle \notin R^M$.

This does not hold since for $b = 3$ we have $\langle 1,3\rangle$, $\langle 2,3\rangle$, $\langle 3,3\rangle, \in R^M$ so there is no $c \in |M|$ for which $\langle c, 3\rangle \notin R^M$.

(g) $M \vDash \forall w_1 (\exists w_2 \neg R(w_1, w_2) \to R(w_1, w_1))$

$\Leftrightarrow$ for all $b \in |M|$, if there is $c \in |M|$ such that $\langle b, c\rangle \notin R^M$ then $\langle b, b\rangle \in R^M$.

This fails (and so the assertion does not hold) since when $b = 2$ there is a c such that $\langle b, c\rangle \in R^M$, namely $c = 3$, but $\langle b, b\rangle$ (i.e. $\langle 2, 2\rangle$) is not in R^M.

(h) $M \vDash \forall w_1 \exists w_2 \forall w_3 (R(w_1, w_2) \to R(w_2, w_3))$

$\Leftrightarrow$ for all $a \in |M|$ there is a $b \in |M|$ such that for all $c \in |M|$ if $\langle a, b\rangle \in R^M$ then $\langle b, c\rangle \in R^M$.

We need to check cases. When $a = 1$ we can take $b = 1$. Then $\langle a, b\rangle \in R^M$ and for each choice of $c = 1, 2, 3$, $\langle a, c\rangle \in R^M$. When $a = 2$ we can take $b = 1$. Then

$$\langle a, b\rangle \in R^M \Rightarrow \langle b, c\rangle \in R^M$$

holds for any c since the left hand side is false. Similarly for $a = 3$ we can take $b = 2$. Hence the original assertion holds.

8 (a) $M \vDash \forall w_1\, P(w_1) \Leftrightarrow$ every $n \in \mathbb{N}^+$ is prime

– so this clearly is not true in M.

(b) $M \vDash \forall w_1 \exists w_2 (R(w_1, w_2) \wedge P(w_2)) \Leftrightarrow$ for every $n \in \mathbb{N}^+$ there is an $m \in \mathbb{N}^+$ such that $n < m$ and m is prime

– true since there are infinitely many (hence arbitrarily large) primes.

(c) $M \vDash \forall w_1 \forall w_2 ((P(w_1) \wedge S(w_1, w_2)) \to P(w_2)) \Leftrightarrow$ for all $n, m \in \mathbb{N}^+$, if n is prime and $m = n + 2$ then m is prime

– not true since 2 is a prime and $4 = 2 + 2$ but 4 is not prime.

(d) $M \vDash \forall w_1 \forall w_2 (S(w_1, w_2) \to R(w_1, w_2)) \Leftrightarrow$ for all $n, m \in \mathbb{N}^+$, if $m = n + 2$ then $n < m$ – true.

(e) $M \vDash \forall w_1 \forall w_2 (R(w_1, w_2) \to \neg R(w_2, w_1)) \Leftrightarrow$ for all $n, m \in \mathbb{N}^+$, if $n < m$ then (not $m < n$) – true.

(f) $M \vDash (\exists w_1\, R(w_1, w_1) \to \forall w_1\, P(w_1)) \Leftrightarrow$ if there is a number $n \in \mathbb{N}^+$ such that $n < n$ then every $m \in \mathbb{N}^+$ is primetrue, since 'there is a number $n \in \mathbb{N}^+$ such that $n < n$' is false.

(g) $M \vDash \forall w_1 \exists w_2 \exists w_3 (((R(w_1, w_2) \wedge S(w_2, w_3)) \wedge P(w_2)) \wedge P(w_3)) \Leftrightarrow$ for all $n \in \mathbb{N}^+$ there are $m, k \in \mathbb{N}^+$ such that $n < m$ and $k = m + 2$ and m, k are both primes.

Is this true?!!! [This example illustrates the point that even when you understand perfectly well what it means for a sentence to be true in a particular structure you may still not have any idea whether or not it actually is true in that structure.]

9 (i) $M \vDash \forall w_3 (R(w_3, 3) \to R(w_3, 9)) \Leftrightarrow$ for all $n \in \mathbb{N}^+$ if $n|3$ (i.e. n divides 3) then $n|9$. True.

(ii) $M \vDash \forall w_3 (R(w_3, 4) \to R(w_3, 6)) \Leftrightarrow$ for all $n \in \mathbb{N}^+$, if $n|4$ then $n|6$. False since $4|4$ but $4 \nmid 6$ (i.e. 4 does not divide 6)

(iii) $M \vDash \exists w_3\,((R(w_3, 12) \wedge R(w_3, 18)) \wedge \neg R(3, w_3)) \Leftrightarrow$ there is a number $n \in \mathbb{N}^+$ such that $n|12$ and $n|18$ but $3 \nmid n$. True, 2 is such a number.

$$M \vDash \forall w_1 \forall w_2 \exists w_3 ((R(w_3, w_1) \wedge R(w_3, w_2)) \wedge$$
$$\forall w_4 ((R(w_4, w_1) \wedge R(w_4, w_2)) \rightarrow R(w_4, w_3)))$$
$\Leftrightarrow$ for all $n, m \in \mathbb{N}^+$ there is a $k \in \mathbb{N}^+$ such that $k|n$ and $k|m$

and whenever $r \in \mathbb{N}^+$ is such that $r|n$ and $r|m$ then $r|k$,

– true, when we take for k the greatest common divisor of n and m.

Let $\phi_1(x_1, x_2) = (R(x_1, x_2) \wedge R(x_2, x_1))$, Then for $n, m \in |M|$,

$$n = m \Leftrightarrow n|m \text{ and } m|n \Leftrightarrow M \vDash \phi(n, m).$$

Let $\phi_2(x_1) = \forall w_1\, R(x_1, w_1)$. Then for $n \in |M|$,

$$n = 1 \Leftrightarrow n \text{ divides every } m \in \mathbb{N}^+ \Leftrightarrow M \vDash \psi(n).$$

Let $\phi_3(x_1, x_2) = \forall w_1\,((R(w_1, x_1) \wedge R(w_1, x_2)) \rightarrow \forall w_2\, R(w_1, w_2))$. Then for $n, m \in |M|$,

$$\begin{aligned} M \vDash \phi_3(n, m) &\Leftrightarrow \text{whenever } k|n \text{ and } k|m \text{ then } k = 1 \\ &\Leftrightarrow gcd\{n, m\} = 1. \end{aligned}$$

Let $\phi_4(x_1) = \forall w_1 \forall w_2\,((R(w_1, x_1) \wedge R(w_2, x_1)) \rightarrow (R(w_1, w_2) \vee R(w_2, w_1)))$. Then for $n \in |M|$,

$$\begin{aligned} M \vDash \phi_4(n) &\Leftrightarrow \text{whenever } k|n \text{ and } r|n \text{ then } k|r \text{ or } r|k \\ &\Rightarrow \text{any two prime divisors of } n \text{ are the same} \\ &\Rightarrow n \text{ is a power of a prime} \\ &\Rightarrow \text{whenever } k|n \text{ and } r|n \text{ then } k|r \text{ or } r|k. \\ &\Leftrightarrow M \vDash \phi_4(n). \end{aligned}$$

It is not possible to find a formula $\chi(x_1, x_2)$ of L such that

$$n < m \Leftrightarrow M \vDash \chi(n, m).$$

In short, to see this let σ be the permutation of $\mathbb{N}^+$ which maps a number with prime decomposition $2^{n_1}3^{n_2}5^{n_3}7^{n_4}\ldots p_r^{n_r}$, where p_r is the rth prime, to the number $2^{n_2}3^{n_1}5^{n_3}7^{n_4}\ldots p_r^{n_r}$, so in particular 2 gets mapped to 3. Then we can show by induction on the length of a formula $\theta(x_1, x_2, \ldots, x_m)$ that for $k_1, k_2, \ldots, k_m \in \mathbb{N}^+$,

$$M \vDash \theta(k_1, k_2, \ldots, k_m) \Leftrightarrow M \vDash \theta(\sigma(k_1), \sigma(k_2), \ldots, \sigma(k_m)).$$

Hence there can be no such $\chi(x_1, x_2)$ for if there was we would have to have

$$2 < 3 \Leftrightarrow M \vDash \chi(2,3) \Leftrightarrow M \vDash \chi(3,2) \Leftrightarrow 3 < 2,$$

– Contradiction!.

A suitable sentence η is $\exists w_1 \exists w_2(\neg R(w_1, w_2) \wedge \neg R(w_2, w_1))$ since $2 \nmid 3$ and $3 \nmid 2$, so $M \vDash \eta$ but for any $n, m \in \mathbb{N}$ either $n \leq m$ or $m \leq n$ so $K \nvDash \neg R(n, m) \wedge \neg R(m, n)$ and hence $K \nvDash \eta$.

10 The first part is proved by induction on $|\theta(\vec{x})|$. Assume the result for formulae of length less than $|\theta(\vec{x})|$. As usual there are various cases.

If $\theta(\vec{x}) = R(\vec{x})$ for some relation symbol R of L and L' then from the given condition

$$M \vDash \theta(\vec{a}) \Leftrightarrow \vec{a} \in R^M \Leftrightarrow \vec{a} \in R^{M'} \Leftrightarrow M' \vDash \theta(\vec{a}).$$

If $\theta(\vec{x}) = (\phi(\vec{x}) \wedge \psi(\vec{x})) \in FL \cap FL'$ then both $\phi(\vec{x}), \psi(\vec{x}) \in FL \cap FL'$ and by the Inductive Hypothesis

$$\begin{aligned} M \vDash \theta(\vec{a}) &\Leftrightarrow M \vDash \phi(\vec{a}) \text{ and } M \vDash \psi(\vec{a}) \\ &\Leftrightarrow M' \vDash \phi(\vec{a}) \text{ and } M' \vDash \psi(\vec{a}) \\ &\Leftrightarrow M' \vDash \theta(\vec{a}). \end{aligned}$$

The cases for the other connective follow similarly.

If $\theta(\vec{x}) = \exists w_j\, \psi(w_j, \vec{x})$ let x_i not be in $\vec{x}$. Then $|\psi(x_i, \vec{x})| < |\theta(\vec{x})|$ and by the Inductive Hypothesis

$$\begin{aligned} M \vDash \theta(\vec{a}) &\Leftrightarrow \exists b \in |M|,\, M \vDash \psi(b, \vec{a}) \\ &\Leftrightarrow \exists b \in |M'|,\, M \vDash \psi(b, \vec{a}), \text{ since } |M| = |M'|, \\ &\Leftrightarrow \exists b \in |M'|,\, M' \vDash \psi(b, \vec{a}) \\ &\Leftrightarrow M' \vDash \theta(\vec{a}). \end{aligned}$$

The case for $\forall$ follows similarly.

For the second part we shall show the contrapositive. So suppose that there is a structure M for L and assignment $x_i \mapsto a_i \in |M|$ for which each formula in Γ is true but $\theta(\vec{x})$ is not true. Define a structure M' for L' by setting $|M'| = |M|$, $R^{M'} = R^M$ for R a relation symbol common to L and L' and, say, $R^{M'} = \emptyset$ for R a relation symbol of L' which is not a relation symbol of L. Then for the interpretation of L' given by M' and the assignment $x_i \mapsto a_i \in |M| = |M'|$, by the first part, every formula in Γ is true but $\theta(\vec{x})$ is not true. Since this argument is obviously symmetric in L, L' the required conclusion now follows.

11 (i) Assume that $\Gamma, \theta \vDash \phi$. Let M be a structure for L such that $M \vDash \Gamma$. Then either $M \nvDash \theta$, in which case $M \vDash \theta \to \phi$, or $M \vDash \theta$, in which case from the assumption $\Gamma, \theta \vDash \phi$, $M \vDash \phi$, so again $M \vDash \theta \to \phi$. Hence since M was an arbitrary model of Γ, $\Gamma \vDash (\theta \to \phi)$.

In the other direction assume that $\Gamma \vDash (\theta \to \phi)$ and let $M \vDash \Gamma, \theta$. Then since $M \vDash \Gamma$, $M \vDash \theta \to \phi$ and since also $M \vDash \theta$ it must be the case that $M \vDash \phi$ (since by T2, $M \vDash \theta \to \phi$ if and only if $M \nvDash \theta$ or $M \vDash \phi$). Again since M was an arbitrary model of Γ, this shows that $\Gamma, \theta \vDash \phi$.

(ii) Assume that $\Gamma \vDash \phi$ and $\Delta \vDash \theta$ and let $M \vDash \Gamma, \Delta$. Then $M \vDash \Gamma$ so $M \vDash \phi$ (from $\Gamma \vDash \phi$) and similarly $M \vDash \Delta$ so $M \vDash \theta$. Hence, from T2, $M \vDash \theta \wedge \phi$. Since M was an arbitrary model of Γ, Δ this gives $\Gamma, \Delta \vDash \theta \wedge \phi$, as required.

(iii) Assume that $\Gamma \vDash \theta$ and $\Delta \vDash (\theta \to \phi)$. Let $M \vDash \Gamma, \Delta$. Then since $M \vDash \Gamma$, from $\Gamma \vDash \theta$, $M \vDash \theta$. Similarly since $M \vDash \Delta$, $M \vDash (\theta \to \phi)$, in other words either $M \nvDash \theta$ or $M \vDash \phi$. Since we already have that $M \vDash \theta$ it must be the case that $M \vDash \phi$. Hence since M was an arbitrary model of Γ, Δ we can conclude that $\Gamma, \Delta \vDash \phi$,as required.

12 For each of (i), (ii), (iii) we need to find a structure for L in which that sentence does not hold but the other two do.

(i), (ii) $\nRightarrow$ (iii):

Let M be the structure for L with $|M| = \mathbb{N}$ and $R^M = \{\langle n, m\rangle \in \mathbb{N} \times \mathbb{N} | n \geq m\}$. Then (i) holds (in M) since $\geq$ is transitive, (ii) holds since for $n, m \in \mathbb{N}$ either $n \geq m$ or $m \geq n$. However (iii) does not hold since if it did there would have to be a largest natural number – which there ain't!

(i), (iii) $\nRightarrow$ (ii):

Let M be the structure for L with $|M| = \{0,1\}$, $R^M = \{\langle 1,0\rangle, \langle 1,1\rangle\}$. Then by checking cases (i) holds in M and (iii) holds in M since $M \vDash \forall w_2\, R(1, w_2)$. However (ii) does not hold in M since neither $R(0, 0)$ nor $R(0, 0)$ hold $(!)$.

(ii), (iii) $\nRightarrow$ (i):

Let M be the structure for L with $|M| = \{0,1,2\}$,

$$R^M = \{\langle 0,0\rangle, \langle 0,1\rangle, \langle 1,1\rangle, \langle 1,2\rangle, \langle 2,2\rangle, \langle 2,0\rangle, \langle 0,2\rangle\}.$$

Then (ii) holds since for any $i, j \in \{0,1,2\}$ either $\langle i, j\rangle \in R^M$ or $\langle j, i\rangle \in R^M$, and (iii) holds since $M \vDash \forall w_2\, R(O, w_2)$. However (i) fails because $\langle 2,0\rangle, \langle 0,1\rangle \in R^M$ but $\langle 2,1\rangle \notin R^M$.

13 The proof is by induction on $|\phi|$ where ϕ has no variables in common with ξ . Assume that the result holds for formulae of length less than $|\phi|$.

If $\phi = R(x_{i_1}, x_{i_2}, \ldots, x_{i_r})$, where R is an r-ary relation symbol of L, then either $R \neq S$, $\phi^\star = \phi$ and

$$\begin{aligned}
M^\star \vDash \phi(a_{i_1}, a_{i_2}, \ldots, a_{i_r}) &\Leftrightarrow M^\star \vDash R(a_{i_1}, a_{i_2}, \ldots, a_{i_r}) \\
&\Leftrightarrow \langle a_{i_1}, a_{i_2}, \ldots, a_{i_r}\rangle \in R^{M^\star} \\
&\Leftrightarrow \langle a_{i_1}, a_{i_2}, \ldots, a_{i_r}\rangle \in R^M, \text{ since } R^{M^\star} = R^M, \\
&\Leftrightarrow M \vDash R(a_{i_1}, a_{i_2}, \ldots, a_{i_r}) \\
&\Leftrightarrow M \vDash \phi^\star(a_{i_1}, a_{i_2}, \ldots, a_{i_r})
\end{aligned}$$

or $R = S$ and

$$\begin{aligned}
M^{\star} \vDash \phi(a_{i_1}, a_{i_2}, \ldots, a_{i_r}) &\Leftrightarrow M^{\star} \vDash R(a_{i_1}, a_{i_2}, \ldots, a_{i_r}) \\
&\Leftrightarrow \langle a_{i_1}, a_{i_2}, \ldots, a_{i_r} \rangle \in R^{M^{\star}} \\
&\Leftrightarrow M \vDash \xi(a_{i_1}, a_{i_2}, \ldots, a_{i_r}), \text{ by definition of } S^{M^{\star}}, \\
&\Leftrightarrow M \vDash \phi^{\star}(a_{i_1}, a_{i_2}, \ldots, a_{i_r}).
\end{aligned}$$

If $\phi = (\theta \wedge \psi)$ then $\phi^{\star} = (\theta^{\star} \wedge \psi^{\star})$. Since θ, ψ must also have no variables in common with ξ, by the Inductive Hypothesis

$$\begin{aligned}
M^{\star} \vDash \phi(\vec{a}) &\Leftrightarrow M^{\star} \vDash \theta(\vec{a}) \text{ and } M^{\star} \vDash \psi(\vec{a}) \\
&\Leftrightarrow M \vDash \theta^{\star}(\vec{a}) \text{ and } M \vDash \psi^{\star}(\vec{a}) \\
&\Leftrightarrow M \vDash \phi^{\star}(\vec{a}).
\end{aligned}$$

The cases for the other connectives are exactly similar.

If $\phi(\vec{x}) = \exists w_j\, \psi(w_j/x_i, \vec{x})$ then, as in question 3, we may assume that x_i does not occur in ξ. So $\phi^\star = \exists w_j\, \psi^\star(w_j/x_i)$ with ψ and ξ having no variables in common. Hence by the Inductive Hypothesis and the fact that $|M^\star| = |M|$,

$$\begin{aligned} M^\star \vDash \phi(\vec{a}) &\Leftrightarrow \text{ for some } b \in |M^\star|,\, M^\star \vDash \psi(b,\vec{a}) \\ &\Leftrightarrow \text{ for some } b \in |M|,\, M \vDash \psi^\star(b,\vec{a}), \\ &\Leftrightarrow M \vDash \phi^\star(\vec{a}). \end{aligned}$$

The case for $\forall$ is exactly similar and the desired result follows by induction on the length of formulae.

If $\phi(\vec{x})$ is a tautology then for every structure N for L and $\vec{a} \in |N|$, $N \vDash \phi(\vec{a})$. Hence for every structure M for L and $\vec{a} \in |M|$, $M^\star \vDash \phi(\vec{a})$, so by the first part $M \vDash \phi^\star(\vec{a})$. It follows then that $\phi^\star(\vec{x})$ is a tautology.

14 Throughout let M be an arbitrary structure for the language. So to show that $\theta_1 \equiv \theta_2$ for $\theta_1, \theta_2 \in SL$ we simply need to show that $M \vDash \theta_1 \Leftrightarrow M \vDash \theta_2$

(a)
$$\begin{aligned} M \vDash \theta \vee \phi &\Leftrightarrow M \vDash \theta \text{ or } M \vDash \phi, \text{ by T2} \\ &\Leftrightarrow M \vDash \phi \text{ or } M \vDash \theta, \\ &\Leftrightarrow M \vDash \phi \vee \theta. \end{aligned}$$

(b)

$$\begin{aligned} M \vDash \forall w_1\, \psi(w_1) &\Leftrightarrow \text{ for all } b \in |M|,\, M \vDash \psi(b) \text{ by T3} \\ &\Leftrightarrow M \vDash \forall w_2\, \psi(w_2). \end{aligned}$$

(c)

$$\begin{aligned} M \vDash (\forall w_1 \psi(w_1) \wedge \theta) &\Leftrightarrow M \vDash \forall w_1 \psi(w_1) \text{ and } M \vDash \theta \text{ by T2} \\ &\Leftrightarrow \text{ for all } b \in |M|,\, M \vDash \psi(b) \text{ and} \\ &\qquad M \vDash \theta \text{ by T3} \\ &\Leftrightarrow \text{ for all } b \in |M|,\, M \vDash \psi(b) \wedge \theta \\ &\Leftrightarrow M \vDash \forall w_1\, (\psi(w_1) \wedge \theta). \end{aligned}$$

(d)

$$\begin{aligned} M \vDash (\exists w_1\, \psi(w_1) \to \theta) &\Leftrightarrow M \nvDash \exists w_1\, \psi(w_1) \text{ or } M \vDash \theta \\ &\Leftrightarrow \text{ for all } b \in |M|,\, M \nvDash \psi(b) \text{ or } M \vDash \theta \\ &\Leftrightarrow \text{ for all } b \in |M|,\, M \vDash (\psi(b) \to \theta) \\ &\Leftrightarrow M \vDash \forall w_1 (\psi(w_1) \to \theta). \end{aligned}$$

15 (a) This fails for some θ, ϕ since let, say, L have the single unary relation symbol P and let M be the structure for L with $|M|= \{0\}$, $P^M = \{0\}$. Let $\theta = \exists w_1 \neg P(w_1)$ and $\phi = \exists w_1 P(w_1)$. Then $M \vDash \neg\theta, \phi$ so $M \vDash \theta \to \neg\emptyset$ but $M \nvDash \neg(\theta \to \phi)$ (since $M \vDash \theta \to \phi$).

(b) This holds. For given a structure M and an interpretation of the free variables in M,

$$\begin{aligned} M \vDash \neg\exists w_1\, \theta(w_1) \quad &\Leftrightarrow \quad \text{it is not the case that} \\ & \quad \exists b \in |M|,\, M \vDash \theta(b) \\ &\Leftrightarrow \quad \text{for all } b \in |M|,\, M \nvDash \theta(b) \\ &\Leftrightarrow \quad \text{for all } b \in |M|,\, M \vDash \neg\theta(b) \\ &\Leftrightarrow \quad M \vDash \forall w_1 \neg\theta(w_1). \end{aligned}$$

(c) This holds since for M etc. as in (b),

$$\begin{aligned} M \vDash \forall w_1\, (\theta(w_1) \wedge \phi(w_1)) &\Leftrightarrow \forall b \in |M|,\, M \vDash \theta(b) \wedge \phi(b) \\ &\Leftrightarrow \quad \forall b \in |M|, M \vDash \theta(b) \text{ and } M \vDash \phi(b) \\ &\Leftrightarrow \quad \forall b \in |M|, M \vDash \theta(b) \text{ and } \forall b \in |M|, M \vDash \phi(b) \\ &\Leftrightarrow \quad M \vDash \forall w_1\, \theta(w_1) \text{ and } M \vDash \forall w_1 \phi(w_1)) \end{aligned}$$

(d) This fails in general. Since let L have a single unary relation symbol P and let M be the structure for L with $|M|= \{0,1\}$ and $P^M = \{0\}$. Then $M \vDash P(0)$ and $M \vDash \neg P(1)$ so $M \vDash \exists w_1\, P(w_1)$ and $M \vDash \exists w_1 \neg P(w_1)$ so $M \vDash (\exists w_1\, P(w_1) \wedge \exists w_1 \neg P(w_1))$. However, clearly, M cannot be a model of $\exists w_1\, (P(w_1) \wedge \neg P(w_1))$.

(e) This does not hold in general. To see this let M be as in

(d). Then $M \nvDash \forall w_1\, P(w_1)$ so $M \vDash (\forall w_1\, P(w_1) \to \forall w_1 \neg P(w_1))$.

However $M \nvDash P(0) \to \neg P(0)$ so $M \nvDash \forall w_1\, (P(w_1) \to \neg P(w_1))$.

(f) This holds. Since given a structure M and an interpretation of the free variables suppose that $M \vDash \exists w_1\, (\theta(w_1) \to \phi(w_1))$. Then for some $b \in |M|$, $M \vDash \theta(b) \to \phi(b)$. $\therefore M \nvDash \theta(b)$ or $M \vDash \phi(b)$. Hence $M \nvDash \forall w_1 \theta(w_1)$ or $M \vDash \exists w_1\, \phi(w_1)$, so $M \vDash \forall w_1\, \theta(w_1) \to \exists w_1\, \phi(w_1)$.

Conversely suppose that $M \vDash \forall w_1\, \theta(w_1) \to \exists w_1\, \phi(w_1)$, so either $M \nvDash \forall w_1 \theta(w_1)$ or $M \vDash \exists w_1\, \phi(w_1)$. In the former case there must be some $b \in |M|$ such that $M \nvDash \theta(b)$, in which case $M \vDash (\theta(b) \to \phi(b))$ and hence $M \vDash \exists w_1\, (\theta(w_1) \to \phi(w_1))$. In the latter case $M \vDash \phi(b)$ for some $b \in |M|$ so again $M \vDash (\theta(b) \to \phi(b))$ and hence $M \vDash \exists w_1\, (\theta(w_1) \to \phi(w_1))$. Either way then we draw this same conclusion, as required.

16 The proof is by induction on $n \in \mathbb{N}^+$. For $n = 1$,

$$
\begin{aligned}
M \vDash \bigwedge_{i=1}^{n} \theta_i(\vec{a}) \quad &\Leftrightarrow \quad M \vDash \bigwedge_{i=1}^{1} \theta_i(\vec{a}) \\
&\Leftrightarrow \quad M \vDash \theta_1(\vec{a}) \ \text{ by defn.} \\
&\Leftrightarrow \quad M \vDash \theta_i(\vec{a}) \ \text{ for all } \ 1 \le i \le 1.
\end{aligned}
$$

Now assume the result for n. Then

$$
\begin{aligned}
M \vDash \bigwedge_{i=1}^{n+1} \theta_i(\vec{a}) \quad &\Leftrightarrow \quad M \vDash \bigwedge_{i=1}^{n} \theta_i(\vec{a}) \, \text{and} \, M \vDash \theta_{n+1}(\vec{a}), \\
&\qquad \text{by T2 and denition of } \bigwedge_{i=1}^{n+1}, \\
&\Leftrightarrow \quad M \vDash \theta_i(\vec{a}) \ \text{ for } 1 \le i \le n \, \text{and} \\
&\qquad M \vDash \theta_{n+1}(\vec{a}), \ \text{ by IH} \\
&\Leftrightarrow \quad M \vDash \theta_i(\vec{a}) \ \text{ for } 1 \le i \le n+1.
\end{aligned}
$$

Similarly for disjunction, for $n = 1$,

$$\begin{aligned} M \vDash \bigvee_{i=1}^{n} \theta_i(\vec{a}) \quad &\Leftrightarrow \quad M \vDash \bigvee_{i=1}^{1} \theta_i(\vec{a}) \\ &\Leftrightarrow \quad M \vDash \theta_1(\vec{a}) \ \text{ by defn.} \\ &\Leftrightarrow \quad M \vDash \theta_i(\vec{a}) \ \text{ for some } 1 \le i \le 1 \end{aligned}$$

and assuming the result for n,

$$\begin{aligned} & M \vDash \textstyle\bigvee_{i=1}^{n+1} \theta_i(\vec{a}) \\ \Leftrightarrow \quad & M \vDash \bigvee_{i=1}^{n} \theta_i(\vec{a}) \text{ or } M \vDash \theta_{n+1}(\vec{a}), \ \text{ by T2 and dfn of } \bigvee_{i=1}^{n+1}, \\ \Leftrightarrow \quad & M \vDash \theta_i(\vec{a}) \ \text{ for some } 1 \le i \le n \text{ or } M \vDash \theta_{n+1}(\vec{a}), \ \text{ by IH,} \\ \Leftrightarrow \quad & M \vDash \theta_i(\vec{a}) \ \text{ for some } 1 \le i \le n+1. \end{aligned}$$

17 For these we use the list of 'useful logical equivalences' (ule) in the notes and Lemma 2

(a) $$\neg \exists w_1 \forall w_2\, R(w_1, w_2) \equiv \forall w_1 \neg \forall w_2\, R(w_1, w_2)$$

$$\equiv \forall w_1 \exists w_2\, \neg R(w_1, w_2) \ \text{ by Lemma 2 and a ule.}$$

(b) $$\forall w_1\, R(w_1, x_1) \equiv \forall w_2\, R(w_2, x_1)$$

Hence

$$(\forall w_1\, R(w_1, x_1) \wedge \exists w_1\, R(x_2, w_1)) \equiv (\forall w_2\, R(w_2, x_1) \wedge \exists w_1\, R(x_2, w_1)) \tag{59}$$

by Lemma 2 (and the fact that $\exists w_1\, R(x_2, w_1)) \equiv \exists w_1\, R(x_2, w_1)))$.

By Lemma 2,

$$(\forall w_2\, R(w_2, x_1) \wedge \exists w_1\, R(x_2, w_1)) \equiv \forall w_2\, (R(w_2, x_1) \wedge \exists w_1\, R(x_2, w_1)). \tag{60}$$

Again by Lemma 2,

$$(R(x_3, x_1) \wedge \exists w_1\, R(x_2, w_1)) \equiv \exists w_1\, (R(x_3, x_1) \wedge R(x_2, w_1))$$

so by this Lemma again,

$$\forall w_2\, (R(w_2, x_1) \wedge \exists w_1\, R(x_2, w_1)) \equiv \forall w_2 \exists w_1\, (R(w_2, x_1) \wedge R(x_2, w_1)). \tag{61}$$

Since $\equiv$ is an equivalence relation, putting together (59), (60), (61) gives

$$(\forall w_1\, R(w_1, x_1) \wedge \exists w_1\, R(x_2, w_1)) \equiv \forall w_2 \exists w_1\, (R(w_2, x_1) \wedge R(x_2, w_1)),$$

– a suitable logically equivalent formula in Prenex Normal Form. [Clearly this equivalent is not unique, this is but one of many correct possible answers here.]

(c) By the ule's $(\forall w_1\, R(w_1, x_1) \to \exists w_2\, R(x_2, w_2))$

$$\equiv \exists w_1\, (R(w_1, x_1) \to \exists w_2\, R(x_2, w_2)), \tag{62}$$

$$(R(x_3, x_1) \to \exists w_2\, R(x_2, w_2))$$

$$\equiv \exists w_2\, (R(x_3, x_1) \to R(x_2, w_2)),$$

so by Lemma 2,

$$\exists w_1\, (R(w_1, x_1) \to \exists w_2\, R(x_2, w_2))$$

$$\equiv \exists w_1 \exists w_2\, (R(w_1, x_1) \to R(x_2, w_2)). \tag{63}$$

Putting together (62), (63) gives

$$(\forall w_1\, R(w_1, x_1) \to \exists w_2\, R(x_2, w_2))$$

$$\equiv \exists w_1 \exists w_2\, (R(w_1, x_1) \to R(x_2, w_2)),$$

an equivalent in the required Prenex Normal Form.

18 Fill-in of justifications:

1. $\forall w_1\, P(w_1) \mid \forall w_1\, P(w_1)$ REF
2. $\forall w_1\, P(w_1) \mid P(x_1)$ $\forall$O 1,
3. $P(x_1) \mid P(x_1)$ REF
4. $P(x_1) \mid (P(x_1) \wedge P(x_1))$ AND 3, 3,
5. $\forall w_1\, P(w_1) \mid (P(x_1) \wedge P(x_1))$ AND 2, 2,
6. $\forall w_1\, P(w_1) \mid \forall w_1\, (P(w_1) \wedge P(w_1))$ $\forall$I, 5.

If we were to append to this proof the sequents

7. $\exists w_1\, P(w_1) \mid (P(x_1) \wedge P(x_1))$

8. $\exists w_1\, P(w_1) \mid \exists w_1\, (P(w_1) \wedge P(w_1))$

it would not be a correct proof because the only way we could get the left hand side of 7 is by using $\exists$O with line 4 and that would be incorrect since x_1 also occurs in the formula on the right hand side. Nevertheless we could reach the same final conclusion by appending the following lines to the initial proof:

7. $P(x_1) \mid \exists w_1\, (P(w_1) \wedge P(w_1))$, $\exists$I , 4

8. $\exists w_1\, P(w_1) \mid \exists w_1\, (P(w_1) \wedge P(w_1))$, $\exists$O , 7.

19 (a) 1. $\theta \mid \theta$, REF,

2. $\mid (\theta \to \theta)$, IMR, 1

(b) 1. $\theta, \phi \mid \theta$, REF,

2. $\theta \mid (\phi \to \theta)$, IMR, 1

3. $\mid (\theta \to (\phi \to \theta))$, IMR, 2

(c) 1. $\theta \to \phi, \theta \wedge \neg\phi \mid \theta \wedge \neg\phi$, REF

2. $\theta \to \phi, \theta \wedge \neg\phi \mid \theta \to \phi$, REF,

3. $\theta \to \phi, \theta \wedge \neg\phi \mid \theta$, AO, 1

4. $\theta \to \phi, \theta \wedge \neg\phi \mid \phi$, MP, 2, 3

5. $\theta \to \phi, \theta \wedge \neg\phi \mid \neg\phi$, AO, 1

6. $\theta \wedge \neg\phi \mid \neg(\theta \to \phi)$, NIN, 4,5.

(d) 1. $\theta, \neg\theta, \neg\phi \mid \theta$, REF,

2. $\theta, \neg\theta, \neg\phi \mid \neg\theta$, REF,

3. $\theta, \neg\theta \mid \neg\neg\phi$, NIN, 1,2

4. $\theta, \neg\theta \mid \phi$, NNO, 3

5. $\neg\theta \mid (\theta \to \phi)$, IMR, 4

6. $\mid \neg\theta \to (\theta \to \phi)$, IMR, 5

(e) 1. $\neg(\theta \wedge \phi)$, $\neg(\neg\theta \vee \neg\phi)$, $\neg\theta \mid \neg\theta$, REF

2. $\neg(\theta \wedge \phi), \neg(\neg\theta \vee \neg\phi), \neg\theta \mid (\neg\theta \vee \neg\phi)$, ORR, 1

3. $\neg(\theta \wedge \phi), \neg(\neg\theta \vee \neg\phi), \neg\theta \mid \neg(\neg\theta \vee \neg\phi)$, REF,

4. $\neg(\theta \wedge \phi), \neg(\neg\theta \vee \neg\phi) \mid \neg\neg\theta$, NIN, 2,3

5. $\neg(\theta \wedge \phi), \neg(\neg\theta \vee \neg\phi) \mid \theta$, NNO, 4

6. $\neg(\theta \wedge \phi), \neg(\neg\theta \vee \neg\phi), \neg\phi \mid \neg\phi$, REF,

7. $\neg(\theta \wedge \phi), \neg(\neg\theta \vee \neg\phi), \neg\phi \mid (\neg\theta \vee \neg\phi)$, ORR, 6

8. $\neg(\theta \wedge \phi), \neg(\neg\theta \vee \neg\phi), \neg\phi \mid \neg(\neg\theta \vee \neg\phi)$, REF,

9. $\neg(\theta \wedge \phi), \neg(\neg\theta \vee \neg\phi) \mid \neg\neg\phi$, NIN, 7,8

10. $\neg(\theta \wedge \phi), \neg(\neg\theta \vee \neg\phi) \mid \phi$, NNO, 9

11. $\neg(\theta \wedge \phi), \neg(\neg\theta \vee \neg\phi) \mid (\theta \wedge \phi)$, AND, 5,10

12. $\neg(\theta \wedge \phi), \neg(\neg\theta \vee \neg\phi) \mid \neg(\theta \wedge \phi)$, REF,

13. $\neg(\theta \wedge \phi) \mid \neg\neg(\neg\theta \vee \neg\phi)$, NIN, 11, 12

14. $\neg(\theta \wedge \phi) \mid (\neg\theta \vee \neg\phi)$, NNO, 13

(f) 1. $\forall w_1 \theta(w_1) \mid \forall w_1 \theta(w_1)$, REF,

2. $\forall w_1\, \theta(w_1) \mid \theta(x_i)$, $\forall$O, 1

3. $\forall w_1\, \theta(w_1) \mid \forall w_2\, \theta(w_2)$, $\forall$I, 2

[Here x_i is chosen so that it does not already occur in θ. This is always possible since there are infinitely many free variables but only finitely many occur in θ.]

(g) 1. $\theta(x_1) \mid \theta(x_1)$, REF,

2. $\theta(x_1) \mid \exists w_2\, \theta(w_2)$, $\exists$I, 1

3. $\exists w_1\, \theta(w_1) \mid \exists w_2\, \theta(w_2)$, $\exists$O, 2

[On line 2 notice that x_1 does not appear in $\exists w_2\, \theta(w_2)$ since in forming this formula we replaced all occurrences of x_1 in $\theta(x_1)$ by w_2.]

(h) 1. $\neg\theta(x_1), \forall w_1\, \theta(w_1) \mid \neg\theta(x_1)$, REF,

2. $\neg\theta(x_1), \forall w_1\, \theta(w_1) \mid \forall w_1\, \theta(w_1)$, REF,

3. $\neg\theta(x_1), \forall w_1\, \theta(w_1) \mid \theta(x_1)$, $\forall$O, 2

4. $\neg\theta(x_1) \mid \neg\forall w_1 \theta(w_1)$, NIN, 1, 3

5. $\exists w_1 \neg\theta(w_1) \mid \neg\forall w_1\, \theta(w_1)$, $\exists$I, 4.

[On line 4 notice that x_1 does not appear in $\exists w_1\, \theta(w_1)$ since in forming this formula we replaced all occurrences of x_1 in $\theta(x_1)$ by w_1.]

(i) 1. $\forall w_1\, \theta(w_1), \forall w_1 \neg\theta(w_1), \theta(x_1) \mid \theta(x_1)$, REF,

2. $\forall w_1 \theta(w_1), \forall w_1\ \neg\theta(w_1), \emptyset(x_1)\ \mid \forall w_1\ \neg\theta(w_1)$, REF,

3. $\forall w_1\, \theta(w_1), \forall w_1 \neg\theta(w_1), \theta(x_1) \mid \neg\theta(x_1)$, $\forall$O, 2

4. $\forall w_1\ \neg\theta(w_1), \theta(x_1) \mid \neg\forall w_1\, \theta(w_1)$, NIN, 1, 3

5. $\neg\forall w_1\, \theta(w_1), \forall w_1\ \neg\theta(w_1), \theta(x_1) \mid \theta(x_1)$, REF,

6. $\neg\forall w_1\, \theta(w_1), \forall w_1 \neg\theta(w_1), \theta(x_1) \mid \forall w_1 \neg\theta(w_1)$, REF,

7. $\neg\forall w_1\, \theta(w_1), \forall w_1 \neg\theta(w_1), \theta(x_1) \mid \neg\theta(x_1)$, $\forall$O, 6

8. $\forall w_1\ \neg\theta(w_1), \theta(x_1) \mid \neg\neg\forall w_1\, \theta(w_1)$,, NIN, 5, 7

9. $\forall w_1\ \neg\theta(w_1),\ \exists w_1\, \theta(w_1) \mid \neg\forall w_1\, \theta(w_1)$, $\exists$O, 4

10. $\forall w_1\ \neg\theta(w_1), \exists w_1\, \theta(w_1) \mid \neg\neg\forall w_1\, \theta(w_1)$,, $\exists$O, 8

11. $\forall w_1 \neg\theta(w_1) \mid \neg\exists w_1\, \theta(w_1)$, NIN, 9, 10.

[On line 9 notice that x_1 does not appear in $\forall w_1\ \ \theta(w_1)$, $\forall w_1 \theta(w_1)$

since in forming these formulae we replaced all occurrences of x_1 in $\theta(x_1)$ by w_1.]

(j) 1. $\theta(x_1) \mid \theta(x_1)$, REF,

2. $\theta(x_1) \mid \exists w_1 \theta(w_1)$, ∃I, 1

3. $\theta(x_1) \mid \exists w_1\, \theta(w_1) \vee \exists w_1\, \phi(w_1)$, ORR, 2

4. $\phi(x_1) \mid \phi(x_1)$, REF,

5. $\phi(x_1) \mid \exists w_1\, \phi(w_1)$, ∃I, 4

6. $\phi(x_1) \mid \exists w_1\, \theta(w_1) \vee \exists w_1\, \phi(w_1)$, ORR, 5

7. $(\theta(x_1) \vee \phi(x_1)) \mid \exists w_1\, \theta(w_1) \vee \exists w_1\, \phi(w_1)$, DIS, 3,6

8. $\exists w_1(\theta(w_1) \vee \phi(w_1)) \mid \exists w_1\, \theta(w_1) \vee \exists w_1\, \phi(w_1)$, ∃O, 7.

(k) 1. $\forall w_1(\theta(w_1) \to \phi(w_1)),\ \theta(x_1) \mid \theta(x_1),$ REF,

2. $\forall w_1(\theta(w_1) \to \phi(w_1)),\ \theta(x_1) \mid \forall w_1(\theta(w_1) \to \phi(w_1)),$ REF,

3. $\forall w_1(\theta(w_1) \to \phi(w_1)),\ \theta(x_1) \mid (\theta(x_1) \to \phi(x_1)),$ $\forall$O, 2

4. $\forall w_1(\theta(w_1) \to \phi(w_1)),\ \theta(x_1) \mid \phi(x_1),$ MP, 1,3

5. $\forall w_1(\theta(w_1) \to \phi(w_1)),\ \theta(x_1) \mid \exists w_1\, \phi(w_1),$ $\exists$I, 4

6. $\forall w_1(\theta(w_1) \to \phi(w_1)),\ \exists w_i\, \theta(w_1) \mid \exists w_1\, \phi(w_1),$ $\exists$O, 5.

(l)1. $\neg(\forall w_1, \theta(w_1) \vee \exists w_1\, \phi(w_1))$, $\forall w_1(\theta(w_1) \vee \phi(w_1))$ $\mid \forall w_1(\theta(w_1) \vee \phi(w_1))$ REF

2. $\phi(x_1)$, $\neg(\forall w_1, \theta(w_1) \vee \exists w_1\, \phi(w_1))$, $\forall w_1(\theta(w_1) \vee \phi(w_1)) \mid \phi(x_1)$ REF

3. $\phi(x_1)$, $\neg(\forall w_1, \theta(w_1) \vee \exists w_1\, \phi(w_1))$, $\forall w_1(\theta(w_1) \vee \phi(w_1))$ $\mid \exists w_1\, \phi(w_1)$ $\exists$I, 2

4. $\phi(x_1)$, $\neg(\forall w_1, \theta(w_1) \vee \exists w_1\, \phi(w_1))$, $\forall w_1(\theta(w_1) \vee \phi(w_1))$ $\mid \forall w_1\, \theta(w_1) \vee \exists w_1\, \phi(w_1)$ ORR 3

5. $\phi(x_1)$, $\neg(\forall w_1, \theta(w_1) \vee \exists w_1\, \phi(w_1))$, $\forall w_1(\theta(w_1) \vee \phi(w_1))$ $\mid \neg(\forall w_1, \theta(w_1) \vee \exists w_1\, \phi(w_1))$ REF

6. $\neg(\forall w_1, \theta(w_1) \vee \exists w_1\, \phi(w_1))$, $\forall w_1(\theta(w_1) \vee \phi(w_1))$ $\mid \neg\phi(x_1)$ NIN, 4, 5

7. $\neg(\forall w_1, \theta(w_1) \vee \exists w_1\, \phi(w_1))$, $\forall w_1(\theta(w_1) \vee \phi(w_1))$ $\mid \theta(x_1) \vee \phi(x_1)$ $\forall$O, 1

8. $\theta(x_1)$, $\neg\theta(x_1)$, $\neg(\forall w_1, \theta(w_1) \vee \exists w_1\, \phi(w_1))$, $\forall w_1(\theta(w_1) \vee \phi(w_1))$ $\mid \phi(x_1)$ REF

9. $\phi(x_1)$, $\neg\phi(x_1)$, $\neg(\forall w_1, \theta(w_1) \vee \exists w_1\, \phi(w_1))$, $\forall w_1(\theta(w_1) \vee \phi(w_1))$ $\mid \phi(x_1)$ REF

10. $\neg\theta(x_1)$, $\neg(\forall w_1, \theta(w_1) \vee \exists w_1\, \phi(w_1))$, $\forall w_1(\theta(w_1) \vee \phi(w_1))$ $\mid \neg\phi(x_1)$ MON, 6

11. $\phi(x_1)$, $\neg(\forall w_1, \theta(w_1) \vee \exists w_1\, \phi(w_1))$, $\forall w_1(\theta(w_1) \vee \phi(w_1))$ $\mid \neg\neg\theta(x_1)$ NIN, 9, 10

12. $\phi(x_1)$, $\neg(\forall w_1, \theta(w_1) \vee \exists w_1\, \phi(w_1))$, $\forall w_1(\theta(w_1) \vee \phi(w_1)) \mid \theta(x_1)$ NNO, 11

13. $\theta(x_1) \vee \phi(x_1)$, $\neg(\forall w_1, \theta(w_1) \vee \exists w_1\, \phi(w_1))$, $\forall w_1(\theta(w_1) \vee \phi(w_1))$ $\mid \theta(x_1)$ DIS, 8, 12

14. $\neg(\forall w_1, \theta(w_1) \vee \exists w_1\, \phi(w_1))$, $\forall w_1(\theta(w_1) \vee \phi(w_1))$ $\mid (\theta(x_1) \vee \phi(x_1)) \to \theta(x_1)$ IMR, 13

15. $\neg(\forall w_1, \theta(w_1) \vee \exists w_1\, \phi(w_1))$, $\forall w_1(\theta(w_1) \vee \phi(w_1))$ $\mid \forall w_1(\theta(w_1) \vee \phi(w_1))$ REF

16. $\neg(\forall w_1, \theta(w_1) \vee \exists w_1\, \phi(w_1))$, $\forall w_1(\theta(w_1) \vee \phi(w_1)) \mid \theta(x_1) \vee \phi(x_1)$ $\forall$O, 15

17. $\neg(\forall w_1, \theta(w_1) \vee \exists w_1\, \phi(w_1))$, $\forall w_1(\theta(w_1) \vee \phi(w_1)) \mid \theta(x_1)$ MP, 14, 16

18. $\neg(\forall w_1, \theta(w_1) \vee \exists w_1\, \phi(w_1))$, $\forall w_1(\theta(w_1) \vee \phi(w_1)) \mid \forall w_1 \theta(w_1)$ $\forall$I, 17

19. $\neg(\forall w_1, \theta(w_1) \vee \exists w_1\, \phi(w_1))$, $\forall w_1(\theta(w_1) \vee \phi(w_1)) \mid \forall w_1\, \theta(w_1) \vee \exists w_1\, \phi(w_1)$ ORR, 18

20. $\neg(\forall w_1,\, \theta(w_1) \vee \exists w_1\, \phi(w_1)),\ \forall w_1(\theta(w_1) \vee \phi(w_1))\ \mid \neg(\forall w_1\, \theta(w_1) \vee \exists w_1\, \phi(w_1))$ REF

21. $\forall w_1(\theta(w_1) \vee \phi(w_1)) \mid \neg\neg(\forall w_1\, \theta(w_1) \vee \exists w_1\, \phi(w_1))$ NIN, 19, 20

22. $\forall w_1(\theta(w_1) \vee \phi(w_1)) \mid \forall w_1\, \theta(w_1) \vee \exists w_1\, \phi(w_1)$ NNO, 21.

[We can assume that the free variable x_1 chosen does not appear anywhere in the other formulae on line 1.]

20 We have that for every $i \in \mathbb{N}^+$,

$$\phi(x_i) \vdash \theta(\vec{x}) \qquad \star$$

Since $\theta(\vec{x})$ only mentions finitely many free variables we can pick i such that x_i does not occur in $\vec{x}$. But then by Lemma 5 we can apply ∃O to $\star$ to get $\exists w_1\, \phi(w_1) \vdash \theta(\vec{x})$.

21 AND:

In this case the 'instance of the rule' is

$$\frac{\Gamma \mid \theta,\ \Delta \mid \phi}{\Gamma \cup \Delta \mid \theta \wedge \phi}$$

and we have that $\Gamma \vdash \theta$ and $\Delta \vdash \phi$, say that

$$I_1 \mid \theta_1,\, I_2 \mid \theta_2, \ldots, \Gamma_k \mid \theta_k$$

$$\Delta_1 \mid \phi_1,\, \Delta_2 \mid \phi_2, \ldots, \Delta_h \mid \phi_h$$

are proofs of these respectively, so $\Gamma_k \subseteq \Gamma$, $\theta_k = \theta$, $\Delta_h \subseteq \Delta$ and $\phi_h = \phi$. In this case

$$\Gamma_1 \mid \theta_1, \Gamma_2 \mid \theta_2, \ldots, \Gamma_k \mid \theta_k, \Delta_1 \mid \phi_1, \Delta_2 \mid \phi_2, \ldots, \Delta_h \mid \phi_h, \Gamma_k \cup \Delta_h \mid (\theta_k \wedge \phi_h)$$

is the required proof of $\Gamma \cup \Delta \vdash (\theta \wedge \phi)$ since $\Gamma_k \cup \Delta_h$ is a finite subset of $\Gamma \cup \Delta$, $(\theta_k \wedge \phi_h) = (\theta \wedge \phi)$ and the last step in this proof is justified by AND from the earlier $\Gamma_k | \theta_k$ and $\Delta_h | \phi_h$.

<u>∀I</u>

In this case the 'instance of the rule' is

$$\frac{\Gamma \mid \theta}{\Gamma \mid \forall w_j\, \theta(w_j / x_i)}$$

where x_i does not occur in any formula in Γ and we are given that $\Gamma \vdash \theta$, say

$$\Gamma_1 \mid \theta_1, \Gamma_2 \mid \theta_2, \ldots, \Gamma_k \mid \theta_k$$

is a proof of this, so $\Gamma_k \subseteq \Gamma$ and $\theta_k = \theta$. In this case

$$\Gamma_1 \mid \theta_1, \Gamma_2 \mid \theta_2, \ldots, \Gamma_k \mid \theta_k, \Gamma_k \mid \forall w_j\, \theta_k(w_j/x_i)$$

is a proof of $\Gamma \vdash \forall w_j\, \theta(w_j/x_i)$ since $\Gamma_k \subseteq \Gamma$, $\forall w_j\, \theta_k(w_j/x_i) = \forall w_j\, \theta(w_j/x_i)$, the last sequent in this proof being justified by $\forall$I from the earlier $\Gamma_k \mid \theta_k$ since x_i cannot occur in any formula in Γ_k as $\Gamma_k \subseteq \Gamma$.

DIS

In this case the 'instance of the rule' is

$$\frac{\Gamma, \theta \mid \psi, \qquad \Delta, \phi \mid \psi}{\Gamma \cup \Delta, \theta \vee \phi \mid \psi}$$

and we are given that $\Gamma, \theta \vdash \psi$, and $\Delta, \phi \vdash \psi$, say

$$\Gamma_1 \mid \theta_1, \Gamma_2 \mid \theta_2, \ldots, \Gamma_k \mid \theta_k$$
$$\Delta_1 \mid \phi_1, \Delta_2 \mid \phi_2, \ldots, \Delta_h \mid \phi_h$$

are proofs of these, so $\Gamma_k \subseteq \Gamma \cup \{\theta\}$, $\Delta_h \subseteq \Delta \cup \{\phi\}$ and $\psi = \theta_k = \phi_h$. Notice then that

$$\Gamma_k - \{\theta\} \subseteq \Gamma, \quad \Delta_h - \{\phi\} \subseteq \Delta. \tag{64}$$

In this case a suitable proof of $\Gamma \cup \Delta \cup \{\theta \vee \phi\} \vdash \psi$ is

$$\Gamma_1 \mid \theta_1, \ldots, \Gamma_k \mid \theta_k, \Delta_1 \mid \phi_1, \ldots, \Delta_h \mid \phi_h, (\Gamma_k - \{\theta\}), \theta \mid \psi,$$

$$(\Delta_h - \{\phi\}), \phi \mid \psi, (\Gamma - \{\theta\}) \cup (\Delta_h - \{\phi\}), (\theta \vee \phi) \mid \psi$$

these last three sequents following from earlier ones by MON (notice that $(\Gamma_k - \{\theta\}) \cup \{\theta\} \supseteq \Gamma_k$ etc.), MON again and DIS, since from (64),

$$\Gamma \cup \Delta \supseteq (\Gamma_k - \{\theta\}) \cup (\Delta_h - \{\phi\}).$$

22 Throughout let M be an arbitary structure for the overlying language and $\vec{a} \in |M|$.

(a) ORR: In this case the instance of the rule looks like

$$\frac{\Gamma(\vec{x}) \mid \theta(\vec{x})}{\Gamma(\vec{x}) \mid \theta(\vec{x}) \vee \phi(\vec{x})}$$

Assume that $\Gamma(\vec{x}) \vDash \theta(\vec{x})$ and suppose that $M \vDash \Gamma(\vec{a})$. Then $M \vDash \theta(\vec{a})$ so $M \vDash \theta(\vec{a}) \vee \phi(\vec{a})$. Hence since $M, \vec{a}$ are arbitrary, $\Gamma(\vec{x}) \vDash \theta(\vec{x}) \vee \phi(\vec{x})$.

(b) $\forall$O In this case the instance of the rule looks like

$$\frac{\Gamma(\vec{x}) \mid \forall w_j\, \theta(w_j, \vec{x})}{\Gamma(\vec{x}) \mid \theta(x_i, \vec{x})}$$

and we are assuming that $\Gamma(\vec{x}) \vDash \forall w_j\, \theta(w_j, \vec{x})$. Suppose that $M \vDash \Gamma(\vec{a})$. Then $M \vDash \forall w_j\, \theta(w_j, \vec{a})$ so for all $b \in |M|$, $M \vDash \theta(b, \vec{a})$. In particular then for any interpretation[46] of x_i $\theta(x_i, \vec{a}$ will be true in M. Hence $\Gamma(\vec{x}) \vDash \theta(x_i, \vec{x})$.

(c) $\exists$O In this case the instance of the rule looks like

$$\frac{\Gamma(\vec{x}),\ \phi(x_i,\vec{x}) \mid \theta(\vec{x})}{\Gamma(\vec{x}), \exists w_j\ \phi(w_j,\vec{x}) \mid \theta}$$

where x_i does not occur in $\vec{x}$ (so not on $\theta(\vec{x})$ nor any formula in $\Gamma(\vec{x})$) and (as in the usual implicit convention) w_j does not occur in $\phi(x_i,\vec{x})$. We are assuming that

$$\Gamma(\vec{x}),\ \phi(x_i,\vec{x}) \vDash \theta(\vec{x}). \tag{65}$$

Suppose that $M \vDash \Gamma(\vec{a}),\ \exists w_j\ \phi(w_j,\vec{a})$, say $b \in |M|$ is such that $M \vDash \phi(b,\vec{a})$. Then since also $M \vDash \Gamma(\vec{a})$ from (65), $\Gamma(\vec{x}),\ \phi(x_i,\vec{x})$ is true in M when $\vec{x}$ is interpreted as $\vec{a}$ and x_i is interpreted as b. [It is important to notice here that because x_i does not appear in $\vec{x}$ this is a valid interpretation. *If x_i had appeared in $\vec{x}$ then the 'interpretation' could be invalid since we might be interpreting x_i as b in one place and as the a_i for the interpretation $\vec{a}$ of $\vec{x}$ in another place.*] Hence from (65), $M \vDash \theta(\vec{a})$, confirming that $\Gamma(\vec{x}),\ \exists w_j\ \phi(w_j,\vec{x}) \vDash \theta(\vec{x})$.

23 (c) If $(\theta \wedge \phi) \in \Omega$ then $\Omega \vdash \theta \wedge \phi$ by REF (and Lemma 5(i)). Hence $\Omega \vdash \theta, \phi$ by AO (and Lemma 5(ii)). Hence from (a) of Lemma 13, $\theta, \phi \in \Omega$. Conversely suppose $\theta, \phi \in \Omega$. Then $\Omega \vdash \theta, \phi$ by REF so $\Omega \vdash \theta \wedge \phi$ by AND. By Lemma 13(a) then $(\theta \wedge \phi) \in \Omega$.

(d) Suppose that $(\theta \vee \phi) \in \Omega$. If $\theta \notin \Omega$ and $\phi \notin \Omega$ then by Lemma 13(b), $\neg\theta, \neg\phi \in \Omega$ so by (a) of this Lemma,

$$\Omega \vdash \theta, \neg\phi. \tag{66}$$

Then by MON, $\Omega, \theta \vdash \neg\theta$. Also by REF, $\Omega, \theta \vdash \theta$ so by AND,

$$\Omega, \theta \vdash \theta \wedge \neg\theta. \tag{67}$$

Also from (66) $\Omega, \neg\theta, \phi \vdash \neg\phi$ by MON and by REF $\Omega, \neg\theta, \phi \vdash \phi$ so by NIN $\Omega, \phi \vdash \neg\neg\theta$ and by NNO, $\Omega, \phi \vdash \theta$. Using (66) and MON we also have $\Omega, \phi \vdash \neg\theta$ so by AND

$$\Omega, \phi \vdash \theta \wedge \neg\theta. \tag{68}$$

Using DIS with (67) and (68) now gives $\Omega, (\theta \vee \phi) \vdash \theta \wedge \neg\theta$, i.e. $\Omega \vdash \theta \wedge \neg\theta$ since $(\theta \vee \phi) \in \Omega$. But this means that Ω is inconsistent, contradiction! So it must be that if $(\theta \vee \phi) \in \Omega$ then either $\theta \in \Omega$ or $\phi \in \Omega$.

In the other direction suppose without loss of generality that $\theta \in \Omega$. Then by (a), $\Omega \vdash \theta$, so $\Omega \vdash (\theta \vee \phi)$ by ORR and $(\theta \vee \phi) \in \Omega$ by (a), as required.

24 Suppose that Γ is satisfied in the structure K (equivalently K is a model of Γ since Γ is a set of sentences). Let M be the structure for L with $|M| = |K| \times \mathbb{N}$ and for R an r-ary relation symbol of L let

$$R^M = \{\langle b_1, b_2, \ldots, b_r\rangle \mid \langle \sigma(b_1), \sigma(b_2), \ldots, \sigma(b_m)\rangle \in R^K\},$$

where $\sigma : |M| \to |K|$ by $\sigma(\langle b, n\rangle) = b$.

Claim that for $\theta(\vec{x}) \in FL$ and $\vec{c} \in |M|$,

$$M \vDash \theta(\vec{c}) \Leftrightarrow K \vDash \theta(\sigma(\vec{c})). \tag{69}$$

Clearly this will be enough because $|M|$ is infinite and (69) ensures that $M \vDash \Gamma$ too. The proof of (69) is by induction on the length of θ. If $\theta(\vec{x}) = R(\vec{x})$ for R a relation symbol of L the result is true by definition of R^M. Assume the result for formulae shorter than θ. If $\theta(\vec{x}) = \neg\phi(\vec{x})$ then the result holds for ϕ so

$$\begin{aligned} M \vDash \theta(\vec{c}) \quad &\Leftrightarrow \quad M \nvDash \phi(\vec{c}) \\ &\Leftrightarrow \quad K \nvDash \phi(\sigma(\vec{c})) \\ &\Leftrightarrow \quad K \vDash \theta(\sigma(\vec{c})), \end{aligned}$$

as required. The cases for the other connectives are similar. Now suppose that $\theta(\vec{x}) = \exists w_j \, \phi(w_j, \vec{x})$. Then again the result holds for $\phi(x_i, \vec{x})$ and

$$\begin{aligned} M \vDash \theta(\vec{c}) \quad &\Leftrightarrow \quad \text{for some } \langle d, n \rangle \in |M|, \, M \vDash \phi(\langle d, n \rangle, \vec{c}) \\ &\Leftrightarrow \quad \text{for some } d \in |K|, \, K \vDash \phi(d, \sigma(\vec{c})), \\ &\qquad \text{by IH since } \sigma(\langle d, n \rangle) = d, \\ &\Leftrightarrow \quad K \vDash \exists w_j \, \phi(w_j, \sigma(\vec{c})) \\ &\Leftrightarrow \quad K \vDash \theta(\sigma(\vec{c})), \end{aligned}$$

as required. The case for $\forall$ is completely similar.

In contrast it is not necessarily true that Γ is satisfied some finite model. For let L as above have a single binary relation symbol R and let Γconsist of

$$\text{(i) } \forall w_1 \exists w_2 \; R(w_1, w_2), \text{ (ii) } \forall w_1 \forall w_2 \, (R(w_1, w_2) \to \neg R(w_2, w_1)),$$
$$\text{(iii) } \forall w_1 \forall w_2 \forall w_3 ((R(w_1, w_2) \wedge R(w_2, w_3)) \to R(w_1, w_3)).$$

Then if $M \vDash \Gamma$, $|M|$ must be infinite. For let $a_0 \in |M|$. Then by (i) there is some $a_1 \in |M|$ such that $M \vDash R(a_0, a_1)$ and we cannot have $a_0 = a_1$ otherwise by (ii) we would also have that $M \vDash \quad R(a_0, a_1)$. In turn there must by (i) be an $a_2 \in |M|$ such that $M \vDash R(a_1, a_2)$. By (iii) then also $M \vDash R(a_0, a_2)$ and by the same reasoning as before we cannot have that $a_1 = a_2$ or $a_0 = a_2$. Continuing in this way then we see that we can construct and infinite sequence $a_0, a_1, a_2, a_3, \ldots$ of distinct elements of $|M|$ so $|M$ must be infinite and so Γ cannot be satisfied in any finite structure.

25 Suppose on the contrary that for all m,

$$\neg\theta_0, \neg\theta_1, \ldots, \neg\theta_{m-1} \nvDash \theta_m. \tag{70}$$

Consider the set of sentences

$$\Gamma = \{\neg\theta_n \mid n \in \mathbb{N}\}.$$

Let Δ be a finite subset of Γ, say,

$$\Delta = \{\neg\theta_{j_1}, \neg\theta_{j_2}, \ldots, \neg\theta_{j_s}\}$$

with $j_1 < j_2 \ldots << j_s$. Then by our assumption Δ must be satisfiable. For if not then any model of

$$\{\neg\theta_{j_1}, \neg\theta_{j_2}, \ldots, \neg\theta_{j_{s-1}}\}$$

would also have to be a model of θ_{j_s} (otherwise it would be a model of $\neg\theta_{j_s}$ and hence of Δ). In other words

$$\neg\theta_{j_1}, \neg\theta_{j_2}, \ldots, \neg\theta_{j_{s-1}} \vDash \theta_{j_s}$$

contradicting (70).

Having shown that any finite subset of Γ must be satisfiable (under assumption (70) of course) we conclude by the Compactness Theorem that Γ must be satisfiable, say M is a model of Γ. But then $M \vDash \neg\theta_n$ for all $n \in \mathbb{N}$, contradicting the fact that every structure for L satisfies some θ_n. We conclude then that the assumption (70) must be false and hence that for some m

$$\neg\theta_0, \neg\theta_1, \ldots, \neg\theta_{m-1} \vDash \theta_m.$$

26 Let $\Gamma, \Delta \subseteq SL$ be such that for any structure M for L,

$$M \vDash \Gamma \Leftrightarrow M \nvDash \Delta \qquad \star.$$

Assume on the contrary that there do not exist finite $\Gamma' \subseteq \Gamma$ and finite $\Delta' \subseteq \Delta$ such that for any structure M for L,

$$M \vDash \Gamma' \Leftrightarrow M \nvDash \Delta'.$$

Let $\Omega \subseteq \Gamma \cup \Delta$ be finite and $\Gamma' = \Omega \cap \Gamma$, $\Delta' = \Omega \cap \Delta$, so Γ' is a finite subset of Γ, Δ' is a finite subset of Δ and $\Gamma' \cap \Delta' = \Omega$. By the assumption there is a structure M such that

$$M \vDash \Gamma' \text{ and } M \vDash \Delta' \qquad \dagger$$

or

$$M \nvDash \Gamma' \text{ and } M \nvDash \Delta' \qquad \ddagger$$

If $\ddagger$ then clearly $M \nvDash \Gamma$ and $M \nvDash \Delta$ which contradicts the given fact $\star$ that

$$M \vDash \Gamma \Leftrightarrow M \nvDash \Delta.$$

Hence it must be that $\dagger$ holds. Since Ω was an arbitrary finite subset of $\Gamma \cup \Delta$ the Compactness Theorem now gives that $\Gamma \cup \Delta$ is satisfiable.So there is a structure N for L such that $N \vDash \Gamma$ and $N \vDash \Delta$. But that too contradicts $\star$. It follows then that such Γ', Δ' must exist.

27 Suppose on the contrary that a sentence ψ such that

$$M \vDash \psi \Leftrightarrow \Gamma \text{ is satisfiable in } M. \qquad (71)$$

did exist.

We shall show that $\Gamma \cup \{\neg\psi\}$ is satisfiable. Let $\Delta \subseteq \Gamma \cup \{\neg\psi\}$ be finite, say $m \in \mathbb{N}^+$ is such that

$$\Delta \subseteq \{R_n(x_1) \mid 1 \le n \le m\} \cup \{\neg\psi\}.$$

Let M_m be the structure for L with $|M_m| = \{0\}$, $R_n^{M_m} = \{0\}$ for £ $m\}$ and $R_n^{M_m} = \phi$ for $n > m$. Then $M \vDash R_n(0)$ for $n \le m$ so $\{R_n(x_1) \mid 1 \le n \le m\}$ is satisfied in M_m by $x_1 \mapsto 0$.

Also $M_m \vDash \neg\psi$ from (71) since, for example, $R_{m+1}(x_1)$ is not satisfied in M_m. Hence Δ is satisfied in M_m.

By Compactness then $\Gamma \cup \{\neg\psi\}$ is satisfiable, say in the structure M. But then trivially Γ is satisfiable in M so $M \vDash \psi$ by (71), contradicting the fact that $\neg\psi$ is satisfiable i.e. true, since $\neg\psi$ is a sentence, in M. We conclude that no such sentence ψ can exist.

28 Suppose on the contrary there was such a sentence θ. Then we claim that Γ, the set of formulae

$$\{\neg\exists w_1,\ldots,w_n((R(x_1,w_1)\wedge R(w_n,x_2))\wedge\bigwedge_{i=1}^{n-1}R(w_i,w_{i+1}))\mid n\in\mathbb{N}^+\}\cup\{\theta,\neg R(x_1,x_2)\},$$

is satisfiable. By the Compactness Theorem it is enough to show that every finite subset of Γ is. So let Δ be a finite subset of Γ and let $k\in\mathbb{N}$ be an upper bound on the subscripts of bound variables w_i appearing in Δ. Then Δ is a subset of the set Γ_k of formulae

$$\{\neg\exists w_1,\ldots,w_n((R(x_1,w_1)\wedge R(w_n,x_2))\wedge\bigwedge_{i=1}^{n-1}R(w_i,w_{i+1}))\mid 0<n\leq k\}\cup\{\theta,\neg R(x_1,x_2)\},$$

and it is enough to show that Γ_k is satisfiable. But it clearly is, by $x_1\mapsto a_1$, $x_2\mapsto a_{k+2}$, in the structure M_k for L with $|M_k|=\{a_1,a_2,\ldots,a_k,a_{k+1},a_{k+2}\}$,

$$R^{M_k}=\{\langle a_i,a_j\rangle \mid |\, i-j\,|\leq 1,\ 1\leq i,\ j\leq k+2\}$$

– notice that $M_k\vDash\theta$ by assumption on θ since M_k *is* connected.

Having established that Γ is satisfiable suppose it is satisfied by c_1,c_2 in the structure M for L. Then since $\Gamma\vDash\theta$, by assumption on θM is connected. So either $M\vDash R(c_1,c_2)$ or for some $n\geq 1$ and $b_1,b_2,\ldots,b_n\in|M|$,

$$M\vDash R(c_1,b_1),R(b_1,b_2),R(b_2,b_3),\ldots,R(b_{n-1},b_n),R(b_n,c_2)$$

so

$$M\vDash\exists w_1,w_2,\ldots,w_n((R(c_1,w_1)\wedge R(w_n,c_2))\wedge\bigwedge_{i=1}^{n-1}R(w_i,w_{i+1})).$$

But either way this contradicts the assumption that c_1,c_2 satisfies Γ, contradiction. We conclude that such a sentence θ cannot exist.

29 (i) $f(g(f(x_1,x_1)),c)$ is a term of L since $x_1\in TL$ by Te1, $c\in TL$ by Te2. $\therefore f(x_1,x_1)\in TL$ by Te3, and $g(f(x_1,x_1))\in TL$ by Te3 again. Finally $f(g(f(x_1,x_1)),c)\in TL$ by Te3.

(ii) $gg(c)$ is not a term of L. To prove this we show by induction on the length of a term t that the number f_t of function symbols occurring in t equals the number r_t of occurrences of the right round bracket ')' in t. For clearly this is true if $t = x_i$ or $t = c$ (there are none of either) and if we assume $t = f(t_1, t_2, \ldots, t_m)$ where f is an m-ary function symbol in L and $t_1, t_2, \ldots, t_m$ are terms of (necessarily of length less than t), then

$$\begin{aligned} f_t &= 1 + f_{t_1} + f_{t_2} + \ldots f_{t_m} \\ &= 1 + r_{t_1} + r_{t_2} + \ldots r_{t_m} \quad \text{by IH} \\ &= r_t, \end{aligned}$$

as required.

However since this property is not satisfied by $gg(c)$ it cannot be a term of L.

(iii) $f(f(x_1, w_1), g(x_1))$ is not a term of L. We prove by induction on the length of a term t that no w_j occurs in t. Again this is true if $t = x_i$ or $t = w_j$, and if $t = f(t_1, t_2, \ldots, t_m)$ and we assume the Inductive Hypothesis for the shorter terms $t_1, t_2, \ldots, t_m$ then it holds for t. Hence this property holds for all terms. But it does not hold for $f(f(x_1, w_1), g(x_1))$ so this cannot be a term of L.

(iv) $f(f(g(f(c, f(f(g(f(x_1, f(g(x_2), g(g(x_3))))))), c)), x_2)$ is not a term of L by the same proof as in (ii).

30 (i) $t^M(2, -5) = (f(g(2), -5))^M = f^M(g^M(2), -5) = g^M(2) - (-5) = (2)^2 - (-5) = 9$.

(ii) $t^M(2, -5) = (f(f(g(c), 2), -5))^M = f^M(f^M(g^M(c^M), 2), -5) = (g^M(c^M) - 2) - (-5) =$
$((4)^2 - 2) + 5 = 19.$

(iii) $t^M(2, -5) = (g(f(f(2, c), g(-5))))^M =$
$g^M(f^M(f^M(2, c^M), g^M(-5))) = (((2 - 4) - (-5)^2)^2 = 729$.

31 (a)

1. $\forall w_1\, R(w_1) \mid \forall w_1\, R(w_1)$, REF
2. $\forall w_1\, R(w_1) \mid R(f(x_1))$, $\forall$O, 1
3. $\forall w_1\, R(w_1) \mid \forall w_1\, R(f(w_1))$, $\forall$I, 2

(b)

1. $R(f(x_1)) \mid R(f(x_1))$, REF
2. $R(f(x_1)) \mid \exists w_1\, R(w_1)$, $\exists$I, 1
3. $\exists w_1\, R(f(w_1)) \mid \exists w_1\, R(w_1)$, $\exists$O, 2

32 We first prove by induction on the length of the term $t(\vec{x})$ that if M and K have the same universe and interpret all the constant and function symbols in $t(\vec{x})$ the same then $t^M(\vec{a}) = t^K(\vec{a})$ for $\vec{a} \in |M| = |K|$. [Clearly this is vacuously true if constant or function symbols occur in $t(\vec{x})$ on which M and K do not agree so we can limit attention to those terms which do satisfy this (and similarly for the case of formulae which comes next).] If $t(\vec{x}) = x_i$ then

$$t^M(\vec{a}) = a_i = t^K(\vec{a}), \qquad \text{as required.}$$

If $t(\vec{x}) =$ constant c then

$$t^M(\vec{a}) = c^M = c^K = t^K(\vec{a}), \qquad \text{as required,}$$

since M and K agree on c. Finally suppose that

$$t(\vec{x}) = f(t_1(\vec{x}), \ldots, t_m(\vec{x})).$$

Then since the $|t_i| < |t|$ and M, K must also agree on all the constant and function symbols occurring in these t_i, by Inductive Hypothesis $t_i^M(\vec{a}) = t_i^K(\vec{a})$ for $i = 1, 2, \ldots, m$ and hence

$$t^M(\vec{a}) = f^M(t_1^M(\vec{a}), \ldots, t_m^M(\vec{a})) = f^K(t_1^K(\vec{a}), \ldots, t_m^K(\vec{a})) = t^K(\vec{a}),$$

as required.

For the formula $\phi(\vec{x})$ we analogously prove it by induction on the length of $\phi(\vec{x})$. In case $\phi(\vec{x}) = R(t_1(\vec{x}),\dots,t_m(\vec{x}))$ we have

$$\begin{aligned} M \vDash (\vec{a}) &\Leftrightarrow \langle t_1^M(\vec{a}),\dots,t_m^M(\vec{a})\rangle \in R^M \\ &\Leftrightarrow \langle t_1^K(\vec{a}),t_m^K(\vec{a})\rangle \in R^K \\ &\Leftrightarrow K \vDash \phi(\vec{a}) \end{aligned}$$

since $R^M = R^K$ and $t_i^M(\vec{a}) = t_i^K(\vec{a})$ for $i = 1,2,\dots,m$ by the early result for terms. [Notice that M and K must agree on the function and constant symbols in the t_i since obviously these must occur too in ϕ.]

The remaining cases for ϕ a negation, conjunction etc. follow immediately from T2-3.

(i) Let M be a structure for L and let $b \in |M|$. Let K be the structure for L which is the same as M except [47] that $c^K = b$. Then since $\vDash \theta(c)$, $K \vDash \theta(c^K)$ so by Lemma 18, $K \vDash \theta(b)$. Since c no longer appears in $\theta(x_1)$, by the first part of this question then $M \vDash \theta(b)$. Hence since $b \in |M|$ was arbitrary $M \vDash \forall w_j\, \theta(w_j)$ and hence $\vDash \forall w_j\, \theta(w_j)$ since M was an arbitrary structure for L.

(ii) Let $I_1(c) \mid \phi_1(c)$, $I_2(c) \mid \phi_2(c),\dots,\Gamma_k(c) \mid \phi_k(c)$ be a proof of $\vdash \theta(c)$ where we have explicitly exhibited the occurrences of the constant symbol c. So $\Gamma_k(c) = \phi$ and $\phi_k(c) = \theta(c)$. Assume that the free variable x_s does not occur in any formula in this proof –there must be such an s since there are only finitely many formulae (hence free variables) mentioned in the proof. We claim that is also a proof.

$$\Gamma_1(x_s) \mid \phi_1(x_s), \Gamma_2(x_s) \mid \phi_2(x_s),\dots,\Gamma_k(x_s) \mid \phi_k(x_s) \tag{72}$$

To see this we consider the justification for $\Gamma_i(c) \mid \phi_i(c)$ being in the original proof and show that the same justification applies here in (72).

If the justification is REF then $\phi(c) \in \Gamma_i(c)$, so $\phi_i(x_s) \in \Gamma_i(x_s)$.

If the justification is that $\Gamma_i(c) \mid \phi_i(c)$ follows by $\forall$O from $\Gamma_r(c) \mid \phi_r(c)$ (with $r < i$) then $\phi_i(c) = \forall w_j\, \phi_r(c, w_j/x_h)$ for some x_h not mentioned in $\Gamma_r(c)$. If $h \neq s$ then $\phi_i(x_s) = \forall w_j\, \phi_r(x_s)(w_j/x_h)$ and x_h still cannot occur in any any formula in $\Gamma_r(x_s)$ so again this step in (72) can be justified by $\forall$O. If $h = s$ then in fact x_h never did occur in $\phi_r(c)$. In this case pick g so that $g \neq s$ and x_g appears in no formula in the original proof. Then

$$\phi_i(c) = \forall w_j\, \phi_r(c, w_j/x_h) = \forall w_j\, \phi_r(c, w_j/x_g)$$

and we are essentially back in the case where $h \neq s$(!!)

If the justification is that $\Gamma_i(c) \mid \phi_i(c)$ follows by $\exists$I from $\Gamma_r(c) \mid \phi_r(c)$ (with $r < i$) then $\phi_i(c) = \exists w_j\, \phi_r{}'(c)$ where $\phi_r{}'(c)$ is the result of replacing some occurrences of a term $t(c)$ in $\phi_r(c)$ by w_j. In that case $\phi_i(x_s) = \exists w_j\, \phi_r{}'(x_s)$ where $\phi_r{}'(x_s)$ is the result of replacing these corresponding occurrences of, now, $t(x_s)$ in $\phi_r(x_s)$ by w_j so this step again has the same justification in (72) as it had in the original proof. The remaining cases go through similarly.

Since (72) is a proof we now have that $\vdash \phi_k(x_s)$, equivalently $\vdash \theta(x_s)$ and hence by $\forall$I, $\vdash \forall w_j\, \theta(w_j)$, as required.

33 Suppose that $\{\theta(c_1, c_2)\}$ is inconsistent, so $\theta(c_1, c_2) \vdash \phi, \neg\phi$ for some ϕ and hence by NIN, $\vdash \neg\theta(c_1, c_2)$. By the previous question then $\vdash \theta(c_1, c_1)$. Hence by MON $\theta(c_1, c_1) \vdash \neg\theta(c_1, c_1)$ and by REF $\theta(c_1, c_1) \vdash \theta(c_1, c_1)$ so $\{\theta(c_1, c_1)\}$ is inconsistent.

The converse is not true. For consider the language with just a unary relation symbol P and constants c_1, c_2 and let $\theta(c_1, c_2) = P(c_1) \wedge \neg P(c_2)$. Then $\theta(c_1, c_2)$ is certainly satisfiable, for example in the structure M for L with $|M| = \{0,1\}$, $c_1^M = 0$, $c_2^M = 1$ and $P^M = \{1\}$, so $\{\theta\}$ must be consistent by the precursor to the Completeness Theorem. However $\{\theta(c_1, c_1)\} = \{P(c_1) \wedge \neg P(c_1)\}$ is certainly not consistent since by REF

$$\{P(c_1) \wedge \neg P(c_1)\} \vdash P(c_1) \wedge \neg P(c_1)\}$$

34 Let $\Delta \subset \Omega$ be finite, say m is such that if the constant symbol c_n appears in a sentence in Δ then $n \leq m$. Then

$$\Delta \subseteq \Gamma = \{\theta \in SL \mid \mathcal{R} \vDash \theta\} \cup \{R_<(c_0, \varepsilon)\} \cup \{R_<(f_\times(c_n, \varepsilon), c_1) \mid n \leq m\}.$$

Let $\mathcal{K}$ be the structure for $L(\varepsilon)$ with $|\mathcal{K}| = \mathbb{R}$ which agrees with $\mathcal{R}$ on $f_+, f_\times, R_<, c_0, c_1, c_2, \ldots$ and interprets $\varepsilon^{\mathcal{K}} = (m+1)^{-1}$.

Then for $\theta \in SL, \mathcal{K} \vDash \theta$ whenever $\mathcal{R} \vDash \theta$ (by problem 28 above). Also $0 < \varepsilon^{\mathcal{K}}$ and for $n \leq m$, $f_\times^{\mathcal{K}}(c_n^{\mathcal{K}}, \varepsilon^{\mathcal{K}}) = n \times (m+1)^{-1} < 1 = c_1^{\mathcal{K}}$ so

$$\mathcal{K} \vDash R_<(c_0, \varepsilon) \text{ and } \mathcal{K} \vDash R_<(f_\times(c_n, \varepsilon), c_1) \text{ for } n \leq m.$$

Hence Γ, and so Δ, is satisfiable and by the Compactness Theorem Ω is satisfiable, equivalently has a model since $\Omega \subseteq SL$.

35 (i) $\forall w_1(x_1 = w_1)$ is not a formula of L. To confirm this we can, for example, prove by induction on its length that for a formula θ of L the number of left round brackets '(' occurring in θ equals the number of occurrences of the binary connectives $\wedge, \vee, \rightarrow$ in θ plus the number of relation symbols different from $=$ occurring in θ plus the number of occurrences of function symbols in θ. For the base cases $t_1 = t_2$ and $R(t_1, t_2, \dots, t_m)$ with $t_1, t_2, \dots, t_m \in TL$ we use the result proved in (ii) of the previous question. The cases when θ is one of $\neg\phi$, $(\phi \vee \psi)$, $(\phi \wedge \psi)$, $(\phi \rightarrow \psi)$, $\forall w_j\, \phi(w_j/x_i)$, $\exists w_j \phi(w_j/x_i)$ are now easy to check. So all formulae of L have this property, and hence $\forall w_1(x_1 = w_1)$ cannot be a formula of L since it does not possess this property.

(ii) $\forall w_1(x_1 = w_1 \vee x_1 = w_1)$ is a formula of L, since $x_1 = x_2$ is a formula of L by L1 and so $(x_1 = x_2 \vee x_1 = x_2)$ is by L2 and $\forall w_1(x_1 = w_1 \vee x_1 = w_1)$ is by L3.

(iii) $\exists w_3 f(w_3, x_1)$ is not a formula of L. An easy way to see this is to show by induction on $|\theta|$ for $\theta \in FL$ that the equality symbol '=' or one of the other relation symbols must occur in θ (which obviously fails for $\exists w_3\, f(w_3, x_1)$). Clearly this is true for θ of the form $R(t_1, \dots, t_r)$ or of the form $t_1 = t_2$. Assuming the result for all formulae shorter than θ, if $\theta = (\phi \vee \psi)$ with $\phi, \psi \in FL$ then it is true for ϕ, since $|\phi| < |\theta|$, and hence true for θ. The cases for the other connectives are similar. Finally if $\theta = \forall w_j\, \phi(w_j/x_i)$ (or $\exists w_j\, \phi(w_j/x_i)$) then by the Inductive Hypothesis ϕ mentions some relation symbol (possibly $=$) of L and this is still there when we go to $\forall w_j\, \phi(w_j/x_i)$, i.e. θ.

(iv) $\forall w_1(R(x_1, w_1) \to w_1 = x_2)$ is a formula of L since $x_3 = x_2, R(x_1, x_3) \in FL$ by Ll, $(R(x_1, x_3) \to x_3 = x_2) \in FL$ by L2 and $\forall w_1(R(x_1, w_1) \to w_1 = x_2)) \in FL$ by L3.

(1) $M \vDash \forall w_1\, f(w_1, w_1) = c \Leftrightarrow \forall n \in \mathbb{N}^+ n + n = 2,$

which is obviously false.

(2) $M \vDash \exists w_1\, c = g(w_1) \Leftrightarrow \exists n \in \mathbb{N}^+ 2 = n^2,$

which again is clearly false.

(3) $M|= \forall w_1 \forall w_2\ (R(w_1,w_2)\ \to R(w_1,g(w_2)))$

$\Leftrightarrow \forall n, m \in \mathbb{N}^+$ if $n|m$ then $n|m^2$,

–which is true.

(4) $M|= \exists w_1\ \forall w_2 \forall w_3\ (R(w_2,f(w_1,w_3))\ \to R(w_2,w_3))$

$\Leftrightarrow \exists n \in \mathbb{N}^+$ such that $\forall m, k \in \mathbb{N}^+$ if $m \mid (n+k)$ then $m|k$,

– which is false since for any $n \in \mathbb{N}^+$, $2n \mid n + n$ but $2n|n$.

Choices for $\theta_1(x_1)$, $\theta_2(x_1)$, $\theta_3(x_1)$, $\theta_4(x_1, x_2, x_3)$, $\theta_5(x_1)$,

$\theta_6(x_1, x_2, x_3) \in FL$ with the required properties are:

$\theta_1(x_1) : x_1 = g(c),$

$\theta_2(x_1) : \exists w_1(f(w_1, c) = x_1 \wedge f(w_1, x_1) = g(c)),$

$\theta_3(x_1) : \exists w_1 \exists w_2 x_1 = f(g(w_1), g(w_2)),$

$\theta_4(x_1, x_2, x_3)\ : ((R(x_1, x_2) \wedge R(x_1, x_3))$
$\wedge \forall w_1((R(w_1, x_2) \wedge R(w_1, x_3)) \to R(w_1, x_1))),$

$\theta_5(x_1) : (\neg x_1 = g(x_1) \wedge \forall w_1(R(w_1, x_1)$
$\to (w_1 = x_1 \vee w_1 = g(w_1)))),$

$\theta_6(x_1, x_2, x_3) :\ g(f(x_2, x_3)) = f(f(g(x_2), g(x_3)), f(x_1, x_1)).$

For the last part there are many possible ϕ. One such is

$$\exists w_1 \exists w_2 (\neg w_1 = w_2) \wedge (\neg R(w_1, w_2) \wedge \neg R(w_2, w_1)))$$

which holds in M (such w_1, w_2 here are 2, 3 for example, but fails in K since for any two rational numbers p, q either $p = q$ or $p < q$ or $q < p$.

36 $\theta_1 : \exists w_1 \exists w_2 \exists w_3 \forall w_4 (w_4 = w_1 \vee (w_4 = w_2 \vee w_4 = w_3))$

$\theta_2 : \exists w_1 \exists w_2 \exists w_3 (\neg w_1 = w_2 \wedge (\neg w_1 = w_3 \wedge \neg w_2 = w_3))$

$\theta_3 : (\theta_1 \wedge \theta_2)$

For the second part let M be the (normal) structure for L with $|M| = \mathbb{N}$ and for $n \in \mathbb{N}$, $f^M(n) = n + 1$. Then

$$f^M(n) = f^N(m) \Rightarrow n + 1 = m + 1 \Rightarrow n = m$$

so $M \models \forall w_1 \forall w_2 \; (f(w_1) = f(w_2) \rightarrow w_1 = w_2)$. Also for all $n \in \mathbb{N}$, $f^N(n) = n + 1 \neq 0$ so $M \vDash \exists w_1 \forall w_2 \quad f(w_2) = w_1$ (and hence is a model of the conjunction of these two sentences).

However suppose that K was a normal model of

$$\forall w_1 \forall w_2 (f(w_1) = f(w_2) \rightarrow w_1 = w_2) \wedge \exists w_1 \forall w_2 \neg f(w_2) = w_1 \tag{73}$$

and $|K|$ was finite, say $|K| = \{a_1, a_2, \ldots, a_m\}$. Then from the first conjunct of (73) the function f^K maps $\{a_1, a_2, \ldots, a_m\}$ *one-to-one* into $\{a_1, a_2, \ldots, a_m\}$ whilst from the second conjunct f^K does not map $\{a_1, a_2, \ldots, a_m\}$ *onto* $\{a_1, a_2, \ldots, a_m\}$. But this contradicts the pigeon-hole principle!

37 $\theta_1(x_1, x_2) : (\neg R(x_1, x_2) \wedge \neg R(x_2, x_1))$

$\theta_2(x_1) : \forall w_1 (\neg w_1 = x_1 \rightarrow R(w_1, x_1))$

$\theta_3 : \forall w_1 \exists w_2 (\neg w_2 = w_1 \wedge \neg R(w_1, w_2))$

$\theta_4 : \exists w_1 \exists w_2 ((\neg w_1 = w_2 \wedge \neg R(w_1, w_2)) \wedge$

$\qquad \forall w_3 ((w_3 = w_1 \quad w_3 = w_2) \quad R(w_1, w_3)))$

38 Let M be a normal structure for the (default) language L with equality and suppose that

$$M \vDash \forall w_1 (\theta(w_1) \rightarrow w_1 = c) \tag{74}$$

$$M \vDash \neg\theta(c). \tag{75}$$

Let $a \in |M|$ and suppose that $M \vDash \theta(a)$. From (74),

$$\forall b \in |M|, M \vDash (\theta(b) \rightarrow b = c)$$

so $M \vDash a = c$ and

$$M \vDash a = c^M \tag{76}$$

by Lemma 17. From Eq7 we know that

$$M \vDash a = c^M \to (\theta(a) \leftrightarrow \theta(c^M))$$

Hence with (76) and our assumption we have $M \vDash \theta(c^M)$ and hence $M \vDash \theta(c)$ my Lemma 17, contradiction! We conclude that $M \vDash \neg\theta(a)$ and hence since a was an arbitrary element of $|M|$, $M \vDash \forall w_1 \neg\theta(w_1)$. Since M was an arbitrary model of $\forall w_1(\theta(w_1) \to w_1 = c)$, $\neg\theta(c)$ we conclude that $\forall w_1(\theta(w_1) \to w_1 = c)$, $\neg\theta(c) \vDash \forall w_1 \neg\theta(w_1)$.

39 (a)

1. $\mid \forall w_1 \forall w_2 (w_1 = w_2 \to f(w_1) = f(w_2))$, Eq5,
2. $\mid \forall w_2 (s = w_2 \to f(s) = f(w_2)))$, $\forall$O, 1
3. $\mid (s = t \to f(s) = f(t))$, $\forall$O, 2
4. $s = t \mid s = t$, REF,
5. $s = t \mid f(s) = f(t)$, MP, 3, 4.

(b) 1. $\mid \forall w_1, w_2, w_3, w_4((w_1 = w_3 \wedge w_2 = w_4) \rightarrow (R(w_1, w_2) \leftrightarrow R(w_3, w_4)),$ Eq4

2. $\mid \forall w_2, w_3, w_4((x_1 = w_3 \wedge w_2 = w_4) \rightarrow (R(x_1, w_2) \leftrightarrow R(w_3, w_4)),$ ∀O, 1

3. $\mid \forall w_3, w_4((x_1 = w_3 \wedge c = w_4) \rightarrow (R(x_1, c) \leftrightarrow R(w_3, w_4)),$ ∀O, 2

4. $\mid \forall w_4((x_1 = c \wedge c = w_4) \rightarrow (R(x_1, c) \leftrightarrow R(c, w_4)),$ ∀O, 3

5. $\mid ((x_1 = c \wedge c = c) \rightarrow (R(x_1, c) \leftrightarrow R(c, c)),$ ∀O, 5

6. $\mid \forall w_1, w_1 = w_1,$ Eq1

7. $\mid c = c,$ ∀O, 6

8. $x_1 = c \mid x_1 = c,$ REF

9. $x_1 = c \mid (x_1 = c \wedge c = c),$ AND, 7, 8

10. $x_1 = c \mid R(x_1, c) \leftrightarrow R(c, c),$ MP, 5, 9

11. $x_1 = c \mid R(x_1, c) \rightarrow R(c, c),$ AO, 10.

(c) 1. $\mid \forall w_1, w_2(w_1 = w_2 \rightarrow (\theta(w_1) \leftrightarrow \theta(w_2))),$ Eq4

2. $\mid \forall w_2(x_1 = w_2 \rightarrow (\theta(x_1) \leftrightarrow \theta(w_2))),$ ∀O, 1

3. $\mid (x_1 = c \rightarrow (\theta(x_1) \leftrightarrow \theta(c))),$ ∀O, 2

4. $x_1 = c \mid x_1 = c,$ REF

5. $x_1 = c \mid \theta(x_1) \leftrightarrow \theta(c),$ MP, 3,4

6. $x_1 = c \mid \theta(x_1) \rightarrow \theta(c),$ AO, 5

7. $\neg\theta(c), \theta(x_1) \mid \theta(x_1),$ REF

8. $x_1 = c, \neg\theta(c), \theta(x_1) \mid \theta(c),$ MP, 6,7

9. $x_1 = c, \neg\theta(c), \theta(x_1) \mid \neg\theta(c),$ REF

10. $\neg\theta(c), \theta(x_1) \mid \neg x_1 = c,$ NIN, 8,9

11. $\neg\theta(c), \theta(x_1) \mid (\theta(x_1) \wedge \neg x_1 = c),$ AND, 7, 10

12. $\neg\theta(c), \theta(x_1) \mid \exists w_1(\theta(w_1) \wedge \neg w_1 = c),$ ∃I, 11

13. $\neg\theta(c), \exists w_1\theta(x_1) \mid \exists w_1(\theta(w_1) \wedge \neg w_1 = c),$ ∃O, 12

(d) 1. $\forall w_1(\theta(w_1) \to w_1 = c), \neg\theta(c) \mid \forall w_1(\theta(w_1) \to w_1 = c)$, REF

2. $\forall w_1(\theta(w_1) \to w_1 = c), \neg\theta(c) \mid (\theta(x_1) \to x_1 = c)$, $\forall$O, 1

3. $\theta(x_1) \mid \theta(x_1)$, REF

4. $\theta(x_1), \forall w_1(\theta(w_1) \to w_1 = c), \neg\theta(c) \mid x_1 = c$, MP, 2,3

5. $\mid \forall w_1, w_2(w_1 = w_2 \to (\theta(w_1) \leftrightarrow \theta(w_2)))$, Eq7

6. $\mid \forall w_2(x_1 = w_2 \to (\theta(x_1) \leftrightarrow \theta(w_2)))$, $\forall$O, 5

7. $\mid (x_1 = c \to (\theta(x_1) \leftrightarrow \theta(c)))$, $\forall$O, 6

8. $\theta(x_1), \forall w_1(\theta(w_1) \to w_1 = c), \neg\theta(c) \mid \theta(x_1) \leftrightarrow \theta(c)$, MP, 4, 7

9. $\theta(x_1), \forall w_1(\theta(w_1) \to w_1 = c), \neg\theta(c) \mid \theta(x_1) \to \theta(c)$, AO, 8

10. $\theta(x_1), \forall w_1(\theta(w_1) \to w_1 = c), \neg\theta(c) \mid \theta(c)$, MP, 3, 9

11. $\theta(x_1), \forall w_1(\theta(w_1) \to w_1 = c), \neg\theta(c) \mid \neg\theta(c)$, REF

12. $\forall w_1(\theta(w_1) \to w_1 = c), \neg\theta(c) \mid \neg\theta(x_1)$, NIN, 10, 11

13. $\forall w_1(\theta(w_1) \to w_1 = c), \neg\theta(c) \mid \neg\forall w_1\neg\theta(w_1)$, $\forall$I, 12

40 (a) Let M be the normal structure for L with $|M| = \{0,1\}$, $f^M(0) = f^M(1) = 0$, $c^M = 0$ and

$R^M = \{\langle 0,0\rangle, \langle 1,1\rangle\}$.

Then $M \vDash f(0) = f(1)$ since $f^M(0) = f^M(1)$ but $M \nvDash 0 = 1$

since M is normal (and of course $M \vDash EqL$) so

$$EqL, f(x_1) = f(x_2) \nvDash x_1 = x_2$$

and by the Completeness Theorem

$$EqL, f(x_1) = f(x_2) \nvDash x_1 = x_2.$$

(b) Let M be as in (a). Then $M \vDash (\neg 1 = c \wedge R(1,1))$ since $1 \neq c^M$ and $\langle 1,1\rangle \in R^M$, so $M \vDash \exists w_1(\neg w_1 = c \wedge R(w_1, w_1))$. However $M \vDash R(c, c)$ since $c^M = 0$ and $\langle 0,0\rangle \in R^M$ so $M \nvDash \neg R(c, c)$. Thus

$$EqL, \exists w_1(\neg w_1 = c \wedge R(w_1, w_1)) \nvDash R(c, c)$$

and the result follows by the Completeness Theorem.

(c) Let M be a structure for L with $|M| = \{0, 1, 2\}$ and $=^M$ the set of pairs

$$\{\langle 0,0\rangle, \langle 1,1\rangle, \langle 2,2\rangle, \langle 0,1\rangle, \langle 1,2\rangle, \langle 2,0\rangle\}.$$

Then it is easy to check that $M \vDash$ Eql, Eq3. However $M \nvDash$ Eq2 since $M \vDash 0 = 1$ (i.e. $\langle 0,1\rangle \in =^M$) but $M \nvDash 1 = 0$ (i.e. $\langle 1, 0\rangle \notin =^M$). The result follows by the Completeness Theorem.

41 When there is just one copy of ±,

$$\pm(\underline{1}, \pm(\underline{1}, \pm(\ldots, \pm(\underline{1}, \pm(\underline{1}, \pm(\underline{1}, \underline{1})))\ldots))^N = \pm(\underline{1}, \underline{1})^N = 1 + 1 = 2.$$

Now suppose by induction that for n copies of ±,

$$\pm(\underline{1}, \pm(\underline{1}, \pm(\ldots, \pm(\underline{1}, \pm(\underline{1}, \pm(\underline{1}, \underline{1})))\ldots))^N = n + 1.$$

Then for $n + 1$ copies

$$\begin{aligned}&\pm(\underline{1}, \pm(\underline{1}, \pm(\ldots, \pm(\underline{1}, \pm(\underline{1}, \pm(\underline{1}, \underline{1})))\ldots))^N \\ &\quad = -^N(\underline{1}^N, [-(\ldots - (\underline{1} - (\underline{1} - (\underline{1} - \underline{1})))\ldots)^N]),\end{aligned}$$

where the expression in square brackets has n ±'s, and by inductive hypothesis this is

$$\pm^N(\underline{1}^N, n+1) = 1 + (n+1) = n+2.$$

Hence by induction for n copies of ±,

$$\pm(\underline{1}, +(\underline{1}, \pm(\ldots, \pm(\underline{1}, \pm(\underline{1}, \pm(\underline{1}, \underline{1}))) \ldots))^N = n+1.$$

Denote the left hand side term here as $\underline{n+1}$ so now we have that for *all* $n \in \mathbb{N}$, $\underline{n}^N = n$. Now let

$$\Gamma(x_1) = TA \cup \{\neg \underline{n} = x_1 | n \in \mathbb{N}\},$$

Appealing to the Compactness Theorem we show that $\Gamma(x_1)$ is satisfiable (in a normal structure) by showing that every finite subset $\Delta(x_1)$ of $\Gamma(x_1)$ is so satisfiable.

For let $\Delta(x_1)$ be such a subset. Then there must be some $k \in \mathbb{N}$ such that

$$\Delta(x_1) \subset \Gamma_k(x_1) = TA \cup \{\neg \underline{n} = x_1 | n \leq k\}$$

and it is enough to show that $\Gamma_k(x_1)$ is satisfiable. But clearly it is, in N by $x_1 \mapsto k+1$.

Given that $\Gamma(x_1)$ is satisfiable let K be a structure for LA and $b \in |K|$ such that $K \vDash \Gamma(b)$. Since $K \vDash TA$ K is a model of true arithmetic. Indeed if $\phi(x_1, x_2, \ldots, x_m) \in FLA$ and $k_1, k_2, \ldots, k_m \in \mathbb{N}$ then

$$\begin{aligned}
N \vDash \phi(k_1, k_2, \ldots, k_m) &\Leftrightarrow N \vDash \phi(\underline{k_1}^N, \underline{k_2}^N, \ldots, \underline{k_m}^N) \\
&\Leftrightarrow N \vDash (\underline{k_1}, \underline{k_2}, \ldots, \underline{k_m}) \text{ by Lemma 17,} \\
&\Leftrightarrow \phi(\underline{k_1}, \underline{k_2}, \ldots, \underline{k_m}) \in TA \\
&\Leftrightarrow K \vDash \phi(\underline{k_1}, \underline{k_2}, \ldots, \underline{k_m}) \\
&\Leftrightarrow K \vDash \phi(\underline{k_1}^K, \underline{k_2}^K, \ldots, \underline{k_m}^K).
\end{aligned}$$

In particular then for $n, m, k \in \mathbb{N}$,

$$\begin{aligned}
n + m = k &\iff N \models \underline{+}(\underline{n}, \underline{m}) = \underline{k} \\
&\iff K \models \underline{+}(\underline{n}, \underline{m}) = \underline{k} \\
&\iff \underline{+}^K(\underline{n}^K, \underline{m}^K) = \underline{k}^K \text{ etc.}
\end{aligned}$$

and we now see that the $\underline{0}^K, \underline{1}^K, \underline{2}^K, \underline{3}^K, \underline{4}^K$, , … look and act (with respect to the plus $\pm^K$ and product $\cdot^K$ of K) just like $0,1,2,3,4,\dots$ act with respect to the standard plus and product of N. However for $n \in \mathbb{N}$, $K \vDash \neg \underline{n} = b$ so the element b of $|K|$ is not equal to any of these $\underline{0}^K, \underline{1}^K, \underline{2}^K, \underline{3}^K, \underline{4}^K, , , \dots$. Clearly any element n of $\mathbb{N}$ is equal to one of $0,1,2,3,\dots$ (!!) so it follows that K and N cannot be 'isomorphic'.

With a little more work we can show that in the sense of K this b must be larger than all the $\underline{n}^K$, i.e. all the standard n. We refer to b as a *non-standard natural number* and K as a *non-standard model of true arithmetic*. Finally notice that the construction in the proof of the Completeness Theorem would actually produce a K here that was countable.

42 Sketch proof: Let the language L have a unary relation symbol P_n for each $n \in \mathbb{N}$ and constants $c_{\overline{a}}$ for each $\overline{a} = a_0a_1a_2\dots a_k \in H$. Set $|a_0a_1a_2\dots a_k|$ to be the length of $a_0a_1a_2\dots a_k$, i.e. $k + 1$.

Let $\Gamma \subseteq FL$ consist of:

i) $\bigvee_{\substack{\overline{a}\in H \\ |\overline{a}|=k+1}} \bigwedge_{n=0}^{k}(P_n(x_1) \leftrightarrow P_n(c_{\overline{a}})), \quad k \in \mathbb{N},$

ii) $P_n(c_{\overline{a}})$, whenever $\overline{a} = a_0a_1\dots a_k \in H$ and $n \leq k$; $a_n = 1$,

iii) $\neg P_n(c_{\overline{a}})$, whenever $\overline{a} = a_0a_1\dots a_k \in H$ and $n \leq k$; $a_n = 0$.

Then every finite subset Δ of Γ is satisfiable – indeed if m is maximal such that some $c_{\overline{a}}$ occurs in a formula in Δ then Δ is satisfied in the structure K given by $|K| = H$, $P_n^K = \{b_0b_1b_2\dots b_r \in H \mid n \quad r, b_n = 1\}$, $c\dfrac{K}{a} = \overline{a}$ when $x_1 \mapsto \overline{e}$ for any (it doesn't matter which) $\overline{e} = e_0e_1e_2\dots e_m \in H$.

Now by the Compactness Theorem let M be a structure for L in which Γ is satisfied by some $d \in |M|$ and set

$$d_n = \begin{cases} 1 & \text{if } M \vDash P_n(d), \\ 0 & \text{otherwise.} \end{cases}$$

Then because d satisfies the formulae in (i) in M, for each $k \in \mathbb{N}$ there is an $\overline{a} = a_0a_1\dots a_k \in H$ such that for all $n = 0,1,\dots,k$

$$M \vDash P_n(d) \leftrightarrow P_n(c_{\overline{a}})$$

and with the fact that (ii), (iii) hold in M this forces

$$d_0d_1d_2\dots d_k = a_0a_1a_2\dots a_k \in H.$$

Hence $d_0d_1d_2\dots$ is the infinite sequence we are looking for.

Appendix

For the sake of simplicity we will start by considering a finite language L with just the relation symbols $R_1, R_2, \ldots, R_m$. The idea is to code each formula $\theta \in FL$ with a unique number $\sharp\theta \in \mathbb{N}^+$ We begin by giving a code to each of the symbols which can appear in a formula as follows:

Symbol X	R_j	x_n	w_m	$\neg$	$\wedge$	V	$\to$	$\exists$	$\forall$	(	)
Code $\# X$	2^j	3^n	5^m	7	11	13	17	19	23	29	31

So for example R_3 gets code $2^3 = 8$ and w_2 gets code $5^2 = 25$. Clearly given the code for a symbol we can recover that symbol. Some numbers, such as 6 for example, are not the codes for any symbol but that's of no concern.

We now code a finite sequence of symbols $X_1 X_2 X_3 \ldots X_k$ by the number

$$\sharp(X_1 X_2 X_3 \ldots X_k) = 2^{\sharp X_1} 3^{\sharp X_2} 5^{\sharp X_3} \ldots p_k^{\sharp X_k}$$

where p_k is the k'th prime, starting at $p_1 = 2$.

Notice that because the decomposition of a number into a product of powers of primes is unique we can recover $\sharp X_1, \sharp X_2, \sharp X_3, \ldots, \sharp X_k$ from $\sharp(X_1 X_2 X_3 \ldots X_k)$ and in turn then recover the original $X_1 X_2 X_3 \ldots X_k$. In other words the map

$$X_1 X_2 X_3 \ldots X_k \mapsto \sharp(X_1 X_2 X_3 \ldots X_k)$$

is injective.

Since the formulae of L are themselves such words (though of course not all such words are formulae) we can now produce a list $\theta_1, \theta_2, \theta_3, \ldots$ in which every formula of L appears at least once by, say, picking one particular formula ϕ of L and setting

$$\theta_n = \begin{cases} \psi & \text{if } \sharp\psi = n \text{ and } \psi \in FL \\ \phi & \text{otherwise.} \end{cases}$$

In this case our list will in fact contain infinitely many copies of ϕ. For the proof of the Completeness Theorem this is not problem but in any case we can refine our list to avoid repeats by simply deleting every occurrence of ϕ after the first and then telescoping down to fill the gaps.

From this proof it should be clear that the same trick will work if we start off with any countably infinite language, that is one where we can list, possibly with repeats, the symbols of the language as S_n for $n \in \mathbb{N}^+$ (Indeed with enough Set Theory at our disposal we can produce lists, albeit uncountable, even for uncountable languages, and with some minor adjustments still prove the Completeness Theorem.)

Endnotes

1. In this course 0 is taken to be a natural number, so $0 \in \mathbb{N}$.
2. Commas are treated as invisible, they're there simply for our convenience.
3. In practice we often omit the word 'symbol' in this context.
4. In this subject some practitioners use the word 'language' in a different sense.
5. To simplify this account we will not include 0-ary relation symbols in our language, though if we did they would just act like the propositional variables of Propositional Logic. 0-ary functions are just the same thing as constants so there is no need to allow their inclusion.
6. The whole point here is showing that it could not so arise in *more than one way.*
7. They need not all actually appear in ϕ
8. We leave it open here exactly what the t_j are because we will use this notation in a number of different contexts.
9. I've introduced this convention (not all presentations have it) in order avoid the messy issue of interpreting formulae such as $\forall w_1 \, \exists w_1 \, Q(w_1, w_1)$. Similar the use of w_i for bound variables and x_i for free variables (again most accounts don't do this) avoids the even more messy problem of determining whether a variable is or is not bounded by a quantifier.
10. Commonly outside of this course M is also often used instead of $|M|$. This could cause confusion because M is being used for two different things, the structure and the universe of the structure. In practice however one quickly sees which of the two is meant.
11. Had we allowed 0-ary relation symbols in our language then $|M|$ would have to specify for each of them a truth value, true or false. In this way M would look like an extension of the valuations of Propositional Logic and in turn the resulting development would show Predicate Logic to be an extension of the Propositional version.
12. Notice that we adopted the shorthand convention of omitting the outermost parentheses from $P(7) \wedge \neg Q(4,7)$. However we need to make sure we include it when we subsequently introduce the existential quantier.
13. So the 'two barred turnstile' $\models$ gets used in two different ways, for 'logical consequence' and for 'truth in an interpretation'.
14. A possibly worrying feature of 'logical consequence' as a candidate for capturing our intuitive notion of 'follows' is that it appears to depend on the overlying language L, since it talks about 'structures for L', whilst this seems irrelevant as far as our intuitive notion is concerned. Fortunately there is no such dependency, logical consequence is independent of the overlying language, the proof of which is left as Exercise 10 on page 133.
15. Since the left hand side here is supposed to be a set we should enclose it in braces $\{,\}$.
However we drop these if it cannot cause any confusion. Similarly if the left hand side is empty we may omit it altogether rather than writing $\emptyset \models \ldots$
16. Again we should really write this second left hand side as $\Gamma \cup \{\theta(\vec{x})\}$.
17. This is why interpretations are split up into structures and assignments to free variables.
18. As usual this last left hand side is an abbreviation for $\Gamma \cup \Delta \cup \{\theta \vee \phi\}$, etc..
19. Take it as read in such cases that x_i does not also appear in $\vec{x}$
20. Although we give this here for relational languages is holds mutatis mutandis when we add functions, constants and equality.
21. Unary relation symbols are sometimes referred to as *predicate* symbols.

22. Because you cannot find a formula in the empty set which is false in that interpretation, can you?!
23. You may at this point feel that they are not obviously exhaustive.
24. They could be infinite but it would make the notation trickier and we don't need that strengthening in any case.
25. Notice that x_i does not occur on the right hand side either because we chose $\theta \in SL$.
26. To have such an enumeration it is enough that L is countable (exercise!). Essentially the same proof of the Completeness Theorem that we shall give here goes through for general languages L provided L can be well-ordered (as it can be assuming AC), the only real difference then is that we define the Δ_α by transfinite induction rather than standard induction on ω_0.
27. If you ever think you've proved such a result suspect you've made a mistake!
28. Notice that this formula is actually a sentence, i.e. mentions no free variables, so we do not need to specify any assignment to the free variables.
29. As with relational languages the notion of logical consequence is independent of the overlying language. This can be proved just as for Exercise 10 on page 133 except that we must first proved that if M, M' are structures for L, L' respectively, $|M| = |M'|$ and the interpretations of constant and function symbols common to both languages is the same then for $t(\vec{x}) \in TL \cap TL'$ and $\vec{a} \in |M| = |M'|$, $t^M(\vec{a}) = t^{M'}(\vec{a})$.
30. Commonly shortened to $M \vDash \Gamma$.
31. The formulae involved are all sentences so we don't need to bother about assignment to the free variables.
32. To avoid lots of subscripts here we have chosen the free variables to be $x_1, x_2, \ldots, x_n$ though it should be clear that replacing them by distinct $x_{i_1}, x_{i_2}, \ldots, x_{i_n}$ would make no difference.
33. Recall the footnote on page 83.

34. Again for the proof we give we need to assume that we can make a list $\theta_1, \theta_2, \theta_3, \ldots$ containing all the formulae of L though with a little set theory we can dispense with this without introducing any new difficulties.
35. As before it is important to appreciate here that on the left hand side we are evaluating f^M applied to the elements $s_1, s_2, \ldots, s_r$ of $|M|$ whilst on the right hand side we are thinking of the $s_1, s_2, \ldots, s_r$ as simply terms of L. A similar splitting of roles happens frequently in what follows.
36. For example we may know that there are only Lebesgue measure zero real numbers of a certain sort, and hence uncountably many which are not of that sort, still that does not necessarily help us exhibit even one such number.
37. Functions with this role is usually referred to as a *Skolem Functions* after their originator the Norwegian Logician Thoralf Skolem.
38. In forming formulae we usually write $t_1 = t_2$ rather than $= (t_1, t_2)$ which we would use if we were thinking of $=$ *as just another relation symbol* R.
39. Here $w_1 \cdot w_2$ etc. should be taken as shorthand for the formally correct but less immediately comprehensible $\cdot(w_1, w_2)$.
40. Where $\forall w_1, w_{2n} \ldots$ is short for $\forall w_1 \forall w_2 \ldots$ etc. and $\bigwedge_{i=1}^{r}$ has been explained in the Exercises.
41. Here we are using the Completeness Theorem already proved. We are assuming nothing about $=$, it is just an arbitrary binary relation symbol at this point.
42. For $\Gamma, \Delta \subseteq FL, \Gamma \vdash \Delta$ stands for $\Gamma \vdash \phi$ for each $\phi \in \Delta$, as you would have expected by analogy with $\Gamma \vDash \Delta$. Also when referring to sets of axioms we tend to use + instead of $\cup$, so e.g. Eq6 +$\mathrm{Eq7}$ is another notation for $\mathrm{Eq6} + \mathrm{Eq7}$, alternatively Eq6, Eq7.
43. Level 3 students will not be asked to produce proofs involving equality.
44. Recall that for $\sim$ an equivalence relation,

    $a \sim b \Leftrightarrow a \in [b] \Leftrightarrow [a] = [b] \Leftrightarrow [a] \cap [b]\ {}^{1}\emptyset$.
45. So as far as statements we can formulate in L are concerned M looks just like the $\mathcal{R}$, the reals with the usual natural numbers, $+, \times$ and $<$. However in M the element ε^M looks like a positive *infinitesimal.* Structures like M have been studied quite extensively in the past 50 years because they offer an alternative approach to Analysis (called Non-standard Analysis) which uses infinitesimals in place of limits.
46. Of course if x_i is in $\vec{x}$ then the interpretation of x_i is already given – but that doesn't change anything.
47. Of course $K = M$ if by chance $c^M = b$!