Applied Mathematics by Example: Exercises

Jeremy Pickles

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Contents

Editor's Note

This is the accompanying volume to *Applied Mathematics by Example – Book 1: Theory*, and comprises a set of problems (together with solutions) covering each topic in the aforementioned title. These may be attempted to consolidate understanding, provide practice and develop familiarity with the subject of applied mathematics. More challenging questions are indicated by the presence of a *.

Jeremy left a number of the problems unsolved, and subsequently John F. Macqueen provided solutions to many of the vector problems. I have also provided a few solutions to other unsolved problems as well as contributing a handful of questions and answers to the chapters on motion in a circle and gravitation, both of which were underrepresented.

James Bedford, 2010

A note on symbols

A number of mathematical symbols are used in this text, which will be familiar to many readers. For the benefit of the younger or more inexperienced reader, however, here are a few words of explanation regarding some of the symbols used.

Basic symbols: $\approx$ means *approximately equal to*, while $\Rightarrow$ stands for *implies* (often used between steps of working where equations are being simplified for example) and $\therefore$ stands for *therefore*. Multiplication is denoted in the usual ways – $a \times b$, $a \cdot b$ or ab – as is division: $a \div b$, a/b or $\frac{a}{b}$. The universal constant relating the diameter of a circle to its circumfrence is given the usual symbol π. It has the numerical value $3.14159\ldots$.

Vectors: For most of the text, vectors are treated informally. However, in *Sections 1.12* and *2.12*, the notation of writing vectors in boldface is generally employed: **a**. The notation $\overrightarrow{\mathrm{AB}}$, meaning the vector taking one from A to B, is also used.

Angles: The greek letters θ and α are often used to label angles. Occasionally they are denoted by ABC, meaning the angle formed by going from point A to point B and then to point C. Angles are treated almost exclusively in degrees, *e.g.* $90°$. For conversion to radians (a particularly 'natural' way to measure angles, but only occasionally used in the text), one may simply remember that a full circle, *i.e.* $360°$, is equivalent to 2π radians. Thus $90° \rightarrow 90 \times 2\pi/360 = \pi/2$ radians. Similarly π radians $\rightarrow \pi \times 360/2\pi = 180°$.

1 Questions

1.1 Kinematics

1. A ball is dropped from the top of the Leaning Tower of Pisa, 56 m high. How long does it take to hit the ground? At what speed is it then travelling?

2. A stone falls from rest. Calculate the distances through which it has fallen at times of 1, 2, 3, 4 seconds after being released. Show (as did Galileo) that the distances travelled during the 1^{st}, 2^{nd}, 3^{rd}, 4^{th} second after being released are in the ratios $1:3:5:7$. Find a formula for the distance fallen in the n^{th} second.

3. An astronaut on an unknown planet throws a stone vertically upwards with a speed of 20 m/s. After 10 seconds it returns to the ground. What is the acceleration due to gravity on the planet?

4. In a police speed trap a straight length of road AB, where AB = 186 m, is kept under surveillance. A car passes point A travelling at 26.5 m/s and accelerating at a constant rate of $1.5\,\text{m/s}^2$.

   (a) What is the speed of the car when it passes B?

   (b) What is the time taken to cover this distance?

   If the speed limit is 70 mph, determine whether the speed limit has been broken

   (c) based on the average speed of the car over the whole distance, or

   (d) based on its speed at the moment it passes C, the mid-point of AB. (Take 1 km = 0.622 miles).

5. In a record attempt, a vehicle starts out over a measured 5 kilometre run travelling at a speed of 300 m/s. It is still, however, accelerating at $2\,\text{m/s}^2$. Substitute these data into the formula $s = ut + \frac{1}{2}at^2$ to find a quadratic equation for the time t taken to cover the measured distance. Determine t in seconds to 2 places of decimals.

6. **(a)** A stone is thrown vertically *downwards* from the top of a cliff with speed 15 m/s. It subsequently hits the sea at a speed of 35 m/s. Find the height of the cliff.

   (b) A stone is thrown *upwards*, almost vertically, from the top of another cliff with speed 15 m/s. It subsequently hits the sea at a speed of 35 m/s. Find the height of the cliff.

7. Two cars are approaching each other, one travelling at 20 m/s and the other at 25 m/s, along a straight single track road. They are 100 m apart when the drivers see each other and apply the brakes. Supposing that each car is capable of decelerating at $5\,\text{m/s}^2$ (and that the road has high walls on either side), determine whether a collision is inevitable.

8. Headmaster H has French Windows 2 m high in his study. An apple is dropped from a room above and passes by his window. It is observed that the apple hits the ground exactly 0.1 seconds after coming into view at the top of the window. Deduce the height from which the apple was dropped.

9. A parachutist jumps from an aeroplane and falls freely for 3 seconds before pulling the rip-cord. His parachute then opens and his speed is reduced instantaneously to 5 m/s. He then continues to fall with his speed constant at this value. Sketch his speed-time graph. How far has he fallen in total in the first 10 seconds?

10. According to the Highway Code, a car travelling at 50 km/hr requires a total distance of 24.4 metres to come to a halt in an emergency stop. This comprises 9.7 metres "thinking distance" and 14.7 m "braking distance".

    (a) Convert 50 km/hr to metres per second.

    (b) Show that the thinking distance value is consistent with the car travelling at constant speed for the driver's reaction time of 0.7 seconds, before the brakes are applied.

    (c) Calculate the time taken to stop once the brakes are applied.

    (d) Calculate the deceleration which occurs while the car is braking.

    (e) Sketch the velocity-time graph for the motion, supposing that $t = 0$ is the time at which the driver sees the hazard.

    (f) Check, using the same assumptions for thinking distance and deceleration, that the total stopping distance when travelling at 110 km/hr is 92.6 metres.

11. A train leaves station A and accelerates at a uniform rate until reaching maximum speed. It then immediately decelerates at a uniform rate before coming to a halt at station B. The distance between A and B is 3 km and the time taken for the journey is 5 minutes. What was the maximum speed attained?

12. * Sprinter A, running in a 100 metre race, accelerates at $6\,\text{m/s}^2$ for the first 2 seconds, maintains a constant speed of 12 m/s for the next 1.5 seconds, and decelerates at $0.5\,\text{m/s}^2$ for the remainder of the race. Draw the velocity-time graph for his motion. What distance does he cover in the first 2 seconds? In the first 3.5 seconds? Use the $s = ut + \frac{1}{2}at^2$ formula to find a quadratic equation for the time t needed to cover the remainder of the race. Solve for t and hence find A's total time for the 100 m.

Sprinter B, with less power but greater stamina, accelerates at $5.5\,\mathrm{m/s^2}$ for the first 2 seconds, maintains constant speed for the next 4 seconds, and then decelerates at $0.25\,\mathrm{m/s^2}$ for the remainder of the distance.

How far behind is B after 2 seconds? After 3.5 seconds? After what time does B begin to catch up, *i.e.* at what time does his speed first exceed that of A? How far behind is he at this point? Who wins? (Note: an accurate calculation is required, as the winning margin is less than 0.01 seconds.) What is the approximate margin of victory, expressed as a distance?

1.2 Projectiles

1. A tennis court is 23.8 m long and the net in the middle is 0.91 m high. A player standing on the centre point of the baseline hits a service at speed V m/s from a point 2.25 m above ground level. The ball is aimed straight down the centre of the court and leaves his racket travelling horizontally.

   (a) If $V = 25$ m/s, how long does the ball take to cover the horizontal distance of 11.9 m to the net?

   (b) How far will the ball have fallen below its original horizontal line of motion by this time?

   (c) By what margin will it clear the net? (Speed still 25 m/s.)

   (d) Show that to clear the net the speed of the ball must be at least 23 m/s.

   (e) To land in the service court, the ball must clear the net but hit the ground within 6.4 m on the other side. Find the maximum speed with which the ball can be hit if it is to land "in".

2. The pilot of an aeroplane travelling horizontally at 200 m/s at an altitude of 250 m releases a free-fall bomb when the target on the ground appears straight ahead at an angle of 10° below the horizontal.

   (a) What is the horizontal distance to the target at the moment of release?

   (b) How long does the bomb take to reach the ground?

   (c) By what margin – in the absence of air resistance – would the bomb miss?

3. A broken tile slides down the slope of a pitched roof of angle 30°. It leaves the roof at a height of 7 m above ground travelling at 4.2 m/s. How far from the wall does the tile land?

4. In a cricket match, Mr E strikes the ball in the direction of the square leg boundary with a velocity of 21 m/s at an angle of arcsin(3/5) to the horizontal. Let x and y be respectively the distances from E in the horizontal and vertical directions after time t.

   (a) What are the (x, y) co-ordinates of the ball after 1 second? After 2 seconds?

   (b) Find the two times t at which $y = 0$. What is the time of flight?

   (c) What is the horizontal distance travelled from E before the ball hits the ground?

5. In 2005, the athletics world record for the hammer throw was nearly 87 m. Estimate the speed with which the hammer must be thrown to achieve this distance.

6. A fielder in a cricket match returns the ball (full toss) from the boundary to the wicket-keeper 50 metres away. If he can throw the ball with speed 28 m/s, what is the minimum time of flight?

7. A stone is thrown with speed 30 m/s and two seconds later just clears a wall of height 5 metres. Calculate its speed and direction of motion at this instant.

8. A projectile travels a horizontal distance of 120 metres and reaches a maximum height of 40 metres. What was its initial speed and angle of projection?

9. Mr F is attempting to land a penalty goal in a rugby match. He kicks the ball with speed $V = 15$ m/s and the ball moves away at an angle of $\theta = 45°$ to the horizontal. If x and y are respectively the distances in the horizontal and vertical directions after time t:

   (a) Find the equation of the trajectory of the ball assuming the only force acting on the ball during flight is gravity.

   (b) Calculate y when $x = 15$ and hence show that if Mr F is 15 m away from the posts his kick will clear the crossbar (height 3 m) with 2.2 m to spare.

   (c) Suggest how you might allow for the size of the ball in this calculation.

   (d) If the angle $\theta = 45°$ remains constant, what is the minimum initial speed V needed to clear the crossbar?

10. Robin Hood wishes to shoot an arrow through the Sheriff of Nottingham's window, to land on his dining table. The window is 150 m away and 15 m above ground level, and to hit the table the arrow must enter the window at an angle of 30° below the horizontal (during the descending part of its trajectory). With what speed and at what angle should Robin shoot his arrow?

11. In an Olympic shot put competition, a thrower releases the shot from a point 2.5 m above ground level at a speed of 14 m/s. Calculate the distance achieved if the shot is projected at an angle of

    (a) 45° and

    (b) 40° to the horizontal.

    (c)* What is the furthest distance attainable with the optimum angle of projection?

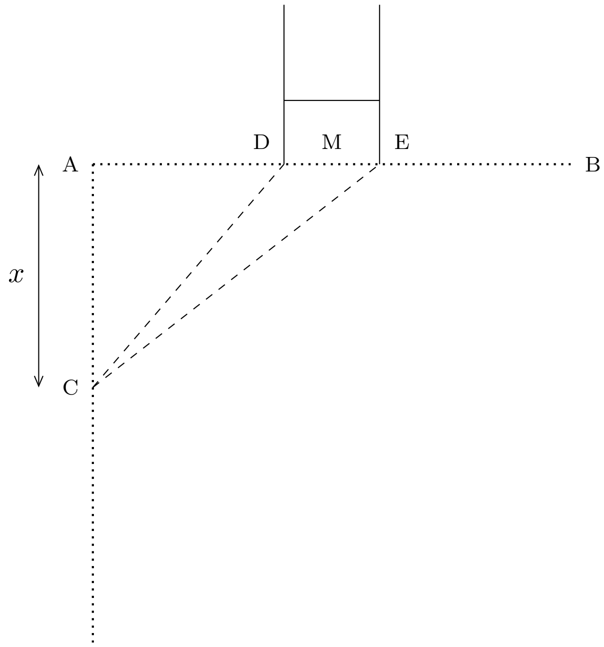
    Hint: for **(c)** *you can just experiment with different angles θ, or more elegantly use the relation* $\sec^2\theta = 1 + \tan^2\theta$ *in the trajectory equation and find the greatest distance which still allows real solutions to the quadratic equation in* $\tan\theta$.

12. * Romeo, standing in the street at $(0, 0)$, throws a parcel to Juliet, on her balcony at $(2, 4.8)$, where distances are measured in metres. If he throws the parcel with speed $V = 7\sqrt{2}$ m/s at an angle θ, show that $\tan\theta$ must satisfy the equation

    $$\tan^2\theta - 10\tan\theta + 25 = 0 .$$

    Find θ. How does the fact that there is only one solution show that the chosen V is the minimum possible to reach the balcony?

13. * AB = 40 m is the try-line of a rugby pitch. M is its midpoint and the two posts D and E are symmetrically placed about M, with DE = 5.6 m. A try is scored in the corner at A and the conversion may be attempted at any point C such that AC is perpendicular to AB. Suppose x is the distance AC. What value of x makes the target angle DCE for the conversion biggest? What other factors might influence the chances of success?



Hint: This is not necessarily a projectile question! Use any method you like to get the best answer you can.

1.3 Forces

1. A bust of Napoleon Bonaparte, weight 250 N, rests on a plinth, weight 450 N, which rests on the floor. Draw separate diagrams showing the magnitude and direction of the forces on

   (a) the bust and

   (b) the plinth.

2. A crocodile C, weight 1200 N, floats half submerged in the river. What is the buoyancy force exerted by the water?

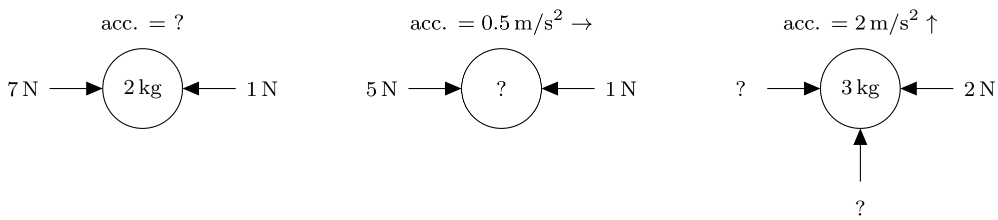
3. Mr D, mass 90 kg, puts his foot down on the accelerator. His car has a mass of 1,110 kg and the tractive (driving) force supplied by the engine is 2,400 newtons.

   (a) Calculate the acceleration of the car. (Allow for the mass of Mr D who is inside the car).

   (b) 100 m further down the road he reaches a speed of 100 km/hr. What was his initial speed?

   (c) What is the forward force (horizontal) exerted on Mr D by his seat while the car maintains this acceleration?

4. Fill in the missing forces, masses and accelerations.



5. A rocket of mass 12,000 kg takes off vertically from its launch pad attaining a height of 200 m in 5 seconds. Assuming constant acceleration, what is the thrust force generated by the engine?

6. Mr E, of mass 80 kg, goes up in a lift while standing on a set of bathroom scales. The scales register an apparent mass of 88 kg. What is the acceleration of the lift?

7. Spy D, mass 90 kg, spy E, mass 80 kg, and spy F, mass m kg, are on board a hot air balloon, whose mass, including the basket, is 110 kg. The balloon is floating in equilibrium when after a struggle D and E eject F from the basket. The balloon begins to accelerate upwards at a rate of $2.1\,\text{m/s}^2$. Deduce m.

8. Mr F, mass 55 kg, is descending by parachute. The mass of the parachute is 5 kg and the upward drag force of the air on the parachute is 570 N. What is his downward acceleration? To what extent can he reduce this acceleration by kicking off his boots which weigh 0.75 kg each?

9. A car, mass 1400 kg, is pulling a trailer, mass 200 kg, and accelerating at $0.6\,\mathrm{m/s^2}$.

   (a) Determine the tractive force of the engine.

   (b) A load of x kg is placed in the trailer, which reduces the acceleration to $0.48\,\mathrm{m/s^2}$. Assuming the tractive force remains the same, determine x.

   (c) Suppose now the load is removed but because the trailer has an underinflated tyre there is a drag force on the trailer of 160 N. How fast does the car accelerate now (assuming the same tractive force as before)?

10. Mass A (3 kg) and mass B (2 kg) are joined by a light inextensible string which passes over a smooth pulley fixed at the edge of a smooth horizontal table. Initially, A is held at rest on the table while B hangs freely over the side.

    (a) Calculate the acceleration which the system will have when mass A is released.

    (b) Find the tension in the string.

    (c) When mass B is replaced by mass C (M kg), the acceleration is observed to be $4.9\,\mathrm{m/s^2}$. Calculate M.

11. * A story famously recited by Gerard Hoffnung concerns a builder Mr B, mass 65 kg, who attempts to lower a barrel of bricks from the roof of a house. Initially, the barrel, of mass 5 kg, contains 70 kg of bricks, and is held in place 12.6 metres above ground, just below a pulley, by a rope which passes over the pulley and whose other end is fixed at ground level.

(a) At $t = 0$, Mr B, standing on the ground, unties the rope. The barrel is heavier than he is, and so starts to descend, raising Mr B into the air. Unwisely, Mr B holds on until he reaches the top, where his fingers jam in the pulley. Calculate the time at which this happens.

(b) As Mr B reaches the top, the barrel reaches the ground, where the bricks spill out. He is now heavier than the barrel, and starts to descend. What is the further time taken before he lands on the bricks?

(c) Losing his presence of mind, Mr B now lets go of the rope. Calculate the time taken before the barrel lands on his head.

(d) State some of the mathematical modelling assumptions you have made in your calculation.

1.4 Resistance forces

1. Mr C, mass 70 kg, does a parachute jump. His parachute has mass 5 kg and when deployed is of a circular cross-section with radius 4 m. Assuming a drag coefficient of 0.8 and taking the density of air as 1.2 kg/m^3, calculate the steady speed at which he descends.

2. A table tennis ball is released from the bottom of a swimming pool, and rises to the surface under the influence of a buoyancy force which, according to Archimedes' principle in hydrostatics, is equal to the weight of an equivalent volume of displaced water. Calculate the terminal velocity attained. (The ball has radius 2 cm and mass 2.7 grams, the density of the water is 1000 kg/m^3, and the drag coefficient C_d may be assumed to be equal to 0.5).

3. A skater of mass 50 kg is sliding at 9.9 m/s over smooth ice. If the coefficient of friction μ is 0.05, what is the frictional force opposing the motion? How long will it be before his speed is reduced to 5 m/s?

4. Skaters B and C, with masses 80 kg and 60 kg respectively, are sliding on ice with coefficient of friction $\mu = 0.05$, both starting with speed 9.9 m/s. Who is the first to slow down to 5 m/s? Justify your answer.

5. Particle P of mass 10 kg is at rest on a polished surface. When a horizontal force of 19.8 N is applied, P accelerates at 1 m/sec^2. Calculate the coefficient of friction between P and the surface.

6. Particle Q rests on rough ground, and the coefficient of friction between Q and the ground is 0.8. A horizontal force $F = 20$ newtons, applied to Q, gives an acceleration of 2.16 m/sec^2. Calculate the mass of Q.

7. Particle S of mass m rests on rough horizontal ground with coefficient of friction μ. A force of 4.9 N is just sufficient to set S in motion, and a force of 6.9 N will give it an acceleration of 2 m/s^2. Find m and μ.

8. A car of mass 1000 kg travelling at 100 km/hr requires 60 m to come to a halt in an emergency stop once the brakes are applied. Supposing that the car decelerates at a constant rate because of the friction of its tyres on the road, deduce the coefficient of friction.

9. A tug of war takes place between team A, consisting of eight men each of weight 800 N, and team B consisting of eight men each of weight 900 N. The forces which oppose the relative motion between the mens' boots and the ground is represented by a coefficient of friction $\mu = 0.8$. What happens when

   (a) the tension in the tug-of-war rope is 5000 N and

   (b) when the tension is 5500 N?

10. Mass A (2 kg) and mass B (3 kg) are joined by a light inextensible string which passes over a smooth pulley fixed at the edge of a smooth horizontal table. Initially, A is held at rest on the table while B hangs freely over the side of the table. The coefficient of friction between the table and mass A is $\mu = 0.5$.

    (a) By applying Newton's second law to A and to B, show that the magnitude of the acceleration which occurs when the system is released is $3.92\,\mathrm{m/s^2}$.

    (b) Find the tension in the string.

    (c) On another table which gives a lower frictional force on mass A, the acceleration is observed to be $4.9\,\mathrm{m/s^2}$. Calculate the corresponding value for μ.

11. Mass A (2 kg) and mass B (3 kg) are joined by a light inextensible string 1.5 metres long which passes over a smooth pulley fixed at the edge of a horizontal table 1 metre high. Initially, A is held at rest on the table, 1 metre from the edge, while B hangs freely over the side of the table, 0.5 metres from the ground. The coefficient of friction between the table and mass A is $\mu = 0.5$.

    (a) Calculate the acceleration which the system will have when mass A is released.

    (b) Find the time taken for B to hit the ground.

    (c) Calculate the speed which has been acquired by mass A when B hits the ground.

    (d) After B hits the ground, the string becomes slack, and the tension is zero. Mass A therefore starts to decelerate because of the friction with the table. Determine whether A will shoot over the edge of table, or come to rest before reaching the edge.

12. The coefficient of friction between the wheels of Mr A's car and a snow-covered road is $\mu = 0.15$. What is the maximum acceleration achievable in these circumstances?

13. * A block B of weight 1000 N, resting on level ground, is subjected to a force P inclined at an angle α to the horizontal. If the coefficient of friction between the ground and the block is $\mu = 0.5$, find a formula for the minimum P necessary to make the block slide (*i.e.* express P in terms of α). Investigate how this minimum P varies as the angle α is changed, and find (either by drawing a careful graph of P against α, or by using calculus and/or trigonometry) the best choice of α if it is desired to slide the block with the minimum force.

14. * A military handbook suggests the formula

$$V = 4.7\frac{\sqrt{m}}{d} ,$$

    for the terminal velocity, in metres per second, of a parachutist of mass m kg using a parachute of diameter d metres. The density of the air is assumed to be that of a standard atmosphere (15° C at sea level), viz. $1.225\,\mathrm{kg/m^3}$. Deduce the value which has been assumed for the drag coefficient of the parachute.

15. * A mediæval war bow would shoot an arrow of mass 60 grams with a speed of about $55\,\mathrm{m/s}$.

(a) Show that the maximum range achievable would in the absence of air resistance be about 310 metres.

(b) A formula which allows for the effects of air resistance on an arrow of mass m and initial speed v is

$$\frac{v^2}{g} \times \left(1 + \frac{cv^2}{mg}\right)^{-0.74},$$

where c is a constant equal to $10^{-4}\,\mathrm{N\,s^2m^{-2}}$. Use the formula to calculate a revised estimate of the maximum range.

1.5 Resolving forces

1. A railway truck of mass 2400 kg is free to move along a straight level track. Calculate the acceleration of the truck if a force of 3600 N is applied:

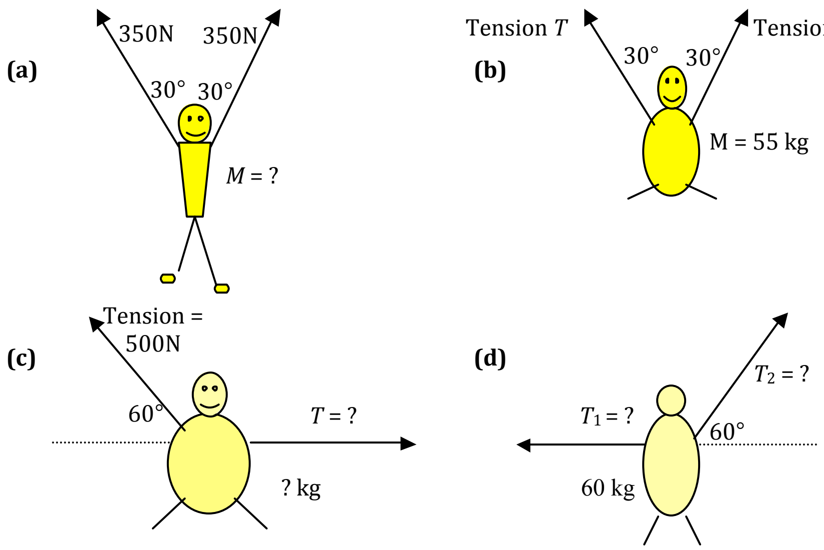
   (a) Directly behind the truck in the direction of the track, parallel with the track.

   (b) At an angle of 60° to the line of the track.

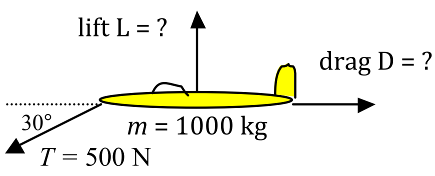
   (c) Calculate the accelerations **(a)** and **(b)** also in the case where the motion of the truck along the rails is resisted by a force of 600 N, acting parallel to the track in the direction opposite to its motion.

   (d) As in **(b)**, a force of 3600 N is applied to the truck at an angle of 60° to the line of the track. What is the magnitude of the reaction force (from the rails) which keeps the truck on the rails instead of being pushed off sideways?

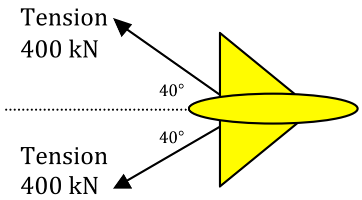
2. These gymnasts are suspended in equilibrium by ropes as shown. Calculate the unknown masses and/or tensions.



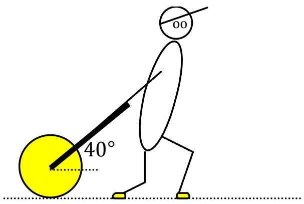
3. A glider is towed at constant speed, as shown. Calculate the drag force D and the lift force L.



4. An aeroplane, mass 15 tonnes, lands on an aircraft carrier with an arrestor hook system. What is the deceleration at the instant depicted?



5. Mr D begins to roll the cricket pitch. The roller has a mass of 100 kg and the handle is inclined at an angle of 40° to the horizontal. Mr D pulls so that the tension in the handle is 200 N and manages to give the roller an acceleration of $0.1\,\text{m/s}^2$.

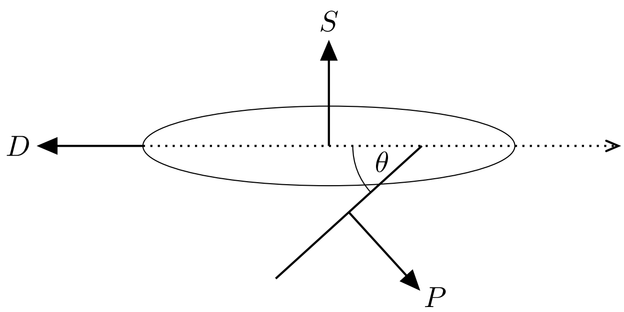


Calculate

(a) the horizontal resistance force opposing the motion of the roller and

(b) the normal (*i.e.* vertically upwards) reaction of the ground on the roller.

6. A sailing dinghy of mass 300 kg, moving through the water at a constant speed, is acted on by three forces in the horizontal plane: **(i)** a force P from the sail which acts perpendicularly to the sail **(ii)** a drag force D opposing the forward motion of the boat and **(iii)** a transverse force S from the keel which prevents the boat from drifting sideways. $P = 1000\,\text{N}$ and the angle between the sail and the line of motion of the boat is $\theta = 40°$.



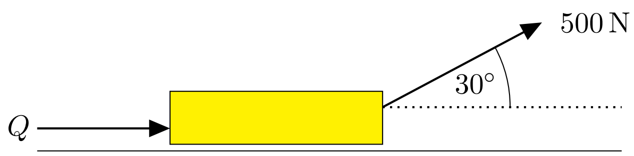
By treating the dinghy as a particle in equilibrium,

(a) calculate D and S.

(b) In a sudden gust of wind, P increases to 1150 N while the drag force D remains the same. What is the acceleration of the boat along its line of motion?

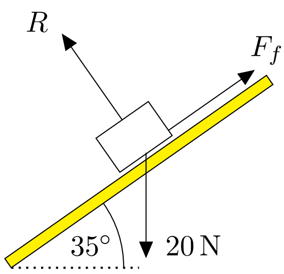
(c) Apart from P, D and S, what other forces are acting on the dinghy?

7. A block B of weight 1000 N, resting on a smooth horizontal surface, is pulled by a force $P = 500$ N inclined at an angle of 30° to the horizontal. What is the normal (perpendicular) reaction on the block from the ground? What is the horizontal frictional force from the ground which is needed to prevent the block from sliding?



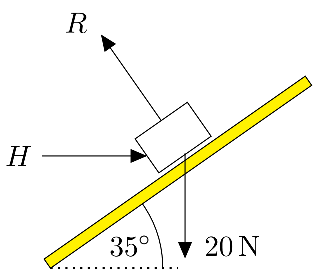
The coefficient of friction between the block and the ground is μ. Show that if $\mu = 0.5$ the block will slide along the ground but if $\mu = 0.6$ it will stay put. For the case $\mu = 0.6$, an additional horizontal force Q is applied behind the block. What is the minimum force Q necessary to make the block slide?

8. A brick of weight $W = 20\,\mathrm{N}$ is resting in equilibrium on a rough plank which is inclined at an angle 35° to the horizontal (see diagram).



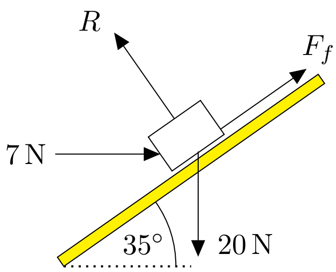
The normal reaction on the brick from the plank is R and the frictional force is F_f. By resolving forces parallel to the plank, calculate F_f. By resolving forces normal to the plank, calculate R.

9. The same brick is supported in equilibrium on a smooth surface (no friction), also inclined at an angle 35° to the horizontal, by a horizontal applied force H.

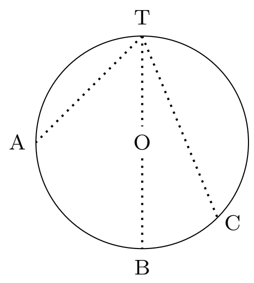


By resolving forces vertically, calculate R. By resolving forces horizontally, calculate H.

10. A new equilibrium is established with the brick on a rough 35° slope with a horizontal applied force H of 7 newtons. By resolving in suitable directions, calculate the normal reaction R and the frictional force F_f.



11. * TABC is a circle in a vertical plane with radius 2.45 m. Its centre is at O.



Calculate:

(a) The time taken to fall from T to B.

(b) The time taken to slide on a slope from T to A (level with O), if the slope is perfectly smooth.

(c) Verify that these times are both equal, and show that the time taken to slide on a smooth slope to any point C on the circumference of the circle is also the same (a theorem proved by Galileo).

12. * Two mountaineers, both of mass 80 kg, are roped together climbing directly up a glacier inclined at 20° to the horizontal. The rope joining them is 15 m long. A is leading and B is 15 m behind when he (B) suddenly falls into a deep crevasse, causing A to fall and slide downhill. Supposing that friction both between A and the ground and between the rope and the lip of the crevasse are negligible,

(a) calculate the initial acceleration of A and B.

After 1 second, A deploys his ice-axe as a brake.

(b) How far has he travelled down the slope by this time?

(c) How fast is he travelling?

Once deployed, the ice-axe exerts a constant braking force of 1400 N.

(d) Is this sufficient to stop A following B into the crevasse?

13. * An artificial ski slope drops through a total vertical distance of 30 m between start S and finish F, while the horizontal separation of S and F is 60 m. The slope is designed so that for the first part of the hill, SM, the inclination of the slope is 35° to the horizontal. See *Figure 1.1.*

A skier steps on to the (frictionless) slope at S, and, starting from an initial speed of zero, begins to accelerate down the slope. Calculate the time taken to reach M. Assuming the final speed along SM to be the initial speed along MF, calculate also the time taken to traverse the slope MF. What is the total time to cover the full distance SMF?

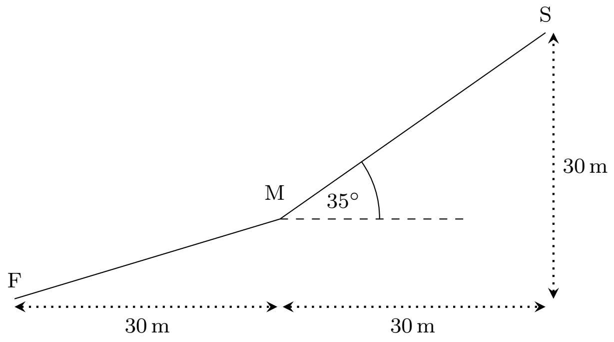


Figure 1.1: *An artificial ski-slope consisting of two parts.*

By changing the profile of the slope, for example by varying the height of M, can you reduce the time needed to get from S to F?

The perfect profile for the quickest descent from S to F is a curve known as the brachistochrone. Newton was sent this problem as a challenge in 1697, and received it on returning home one afternoon from his work at the Royal Mint. "There needs to be found out the curved line SMF in which a heavy body shall, under the force of its own weight, most swiftly descend from a given point S to any other given point F." He found the exact equation of the curve before going to bed at 4 am.

1.6 Rigid bodies

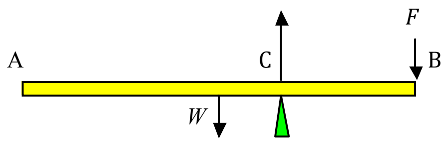
1. Edgar (weight 500 N) and Ferdinand (weight 700 N) are at opposite ends of a see-saw of length 3 m whose fulcrum is at its centre.

   (a) Where should George (weight 600 N) sit if the see-saw is to balance?

   (b) Where should the fulcrum be if Edgar and Ferdinand are to balance without the assistance of George?

   (c) What is the vertical force up from the fulcrum in cases **(a)** and **(b)**?

2. In this question, AB is a uniform rod of length 4 metres which is balanced in equilibrium on a fulcrum at C. The weight of the rod, which acts at its centre, is W newtons, and it is held in equilibrium by a downward force F newtons applied at B.



   (a) Calculate F if $W = 150$ newtons and BC $= 1$ metre.

   (b) Calculate W if $F = 50$ newtons and BC $= 1.5$ metres.

   (c) Calculate BC if $W = 150$ newtons and $F = 250$ newtons.

3. Andrea and Ben are sitting on a plank of weight 200 N which acts as a see-saw. The plank is 2.5 m long and the fulcrum is 1.0 m from Ben.

   (a) If the see-saw is in balance, and Andrea weighs 400 N, estimate the weight of Ben.

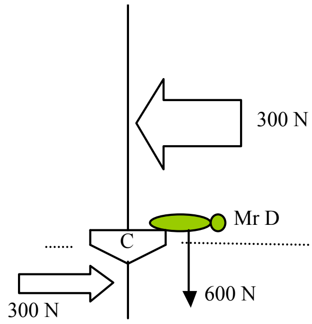
   (b) What is the force up from the fulcrum on the plank in this situation?

   (c) Where would the fulcrum have to be if the see-saw is to balance when Charlie, weight 250 N, sits on Andrea's lap?

   (d) In these calculations, what assumptions have you made about Andrea, Ben, and Charlie?

   (e) What assumptions have you made about the plank?

4. Mr D, who weighs 600 N, sails in a boat with a sail 6 m high. The wind exerts a horizontal force on the sail of 300 N which we presume acts at a point half way up. An opposite force acting on the centreboard at a point halfway down its 2 m depth prevents the boat from drifting sideways through the water.



To stop the boat toppling over Mr D leans over the edge of the boat so that his weight acts downwards at a distance x metres from the centre-line of the boat. Find x.

5. The Ruritanian army's cruise missile, 5 m long, sits on its 6 m long trailer as shown. The front and rear axles of the trailer are 0.5 metres from the ends.

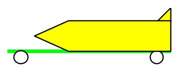


Figure 1.2: *The Ruritanian army's cruise missile on its trailer.*

(a) Show that, if the centre of gravity of the missile is assumed to be mid-way along its length, and the weight of the trailer can be ignored, the load carried by the rear wheels is 50% more than the load carried by the front wheels.

(b) Spy S receives information that the load carried by the rear wheels is actually three times greater than the load carried at the front. What is his revised estimate of the position of the centre of gravity of the missile?

6. The diagram shows a bottle opener on which the user exerts an upward force at B.

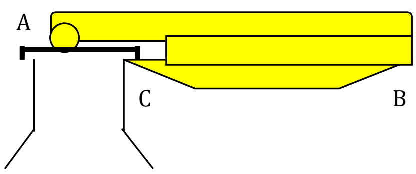


Figure 1.3: *A bottle opener being used to open a bottle.*

At A the opener presses down on the bottle top (which exerts an equal upwards force on the opener) and at C the opener exerts an upward force on the rim of the bottle top (which exerts a downward force on the opener).

(a) Draw a simplified diagram of the opener as a rod showing the forces on it at A, B, and C (the rod is your mathematical model).

(b) Using reasonable estimates of the dimensions, what are the forces at A and C if the force at B is 10 newtons?

7. Pirate P (weight 700 N) is being obliged to walk the plank. The plank is 4 m long and weighs 200 N, and is placed so that a length of 2.5 m projects over the side of the ship while pirate R (weight 1000 N) sits on the inboard end.

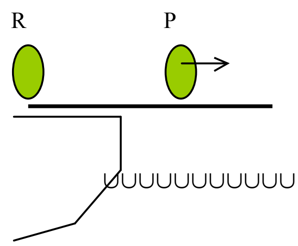
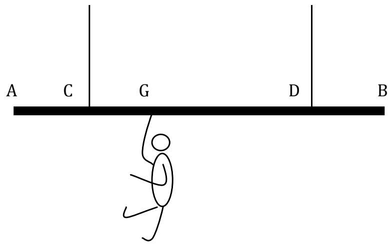


Figure 1.4: *Pirate P walking the plank.*

(a) How far out along the plank will P be able to walk before it becomes unbalanced?

(b) How heavy would R have to be to give P a chance of reaching the far end?

8. A gymnast of weight 600 newtons hangs from a point G on a uniform bar AB of length 5 metres. The bar weighs 200 newtons and is supported by two ropes attached at C and D which are 1 metre distant from A and B respectively.



   (a) Calculate the tension in the ropes at C and at D if CG = 1 metre.

   (b) Where would the gymnast have to be to make the tension in rope C equal to 600 newtons?

   (c) What are the two tensions when the gymnast hangs from H, midway between A and C?

9. * A bookshelf 2 m long is supported by brackets at its ends A and B. The shelf is of negligible weight but the space between A and C, where AC = 1 m, is filled with books of total weight 60 N.

   (a) Assuming the weight is evenly distributed between A and C, what are the loads on the brackets at A and B?

   (b) When the shelf is filled (with the same type of books) as far as X, where AX = x metres, the load at A is 48 N. Calculate x.

10. A door of size 2 m × 0.8 m and weight 100 N hangs from two hinges which are 0.2 m each from the top and bottom. Show that the force exerted by the door on the top hinge has a horizontal component of 25 N directed out from the door post. What is the horizontal force on the lower hinge?

11. Mr B, mass 80 kg, climbs to the top of a ladder of mass 20 kg. The ladder rests against a smooth wall at an angle θ to the vertical and the coefficient of friction between the ladder and the ground is $\mu = 0.5$.

   (a) By taking moments about the bottom of the ladder, show that the reaction force from the wall on the ladder, R newtons, is determined by the formula $R = 90g \tan\theta$.

   (b) Show that Mr B can safely go up to the top of the ladder provided $\theta \leq 29°$.

   (c) How does the answer to **(b)** change if Mr C, mass 60 kg, stands on the bottom rung of the ladder?

12. * A square cat flap of side 20 cm and weight 20 N, and hinged at the top, is held partly open at an angle of 60° to the downward vertical by a cat who exerts a force, perpendicular to its surface, at its centre.

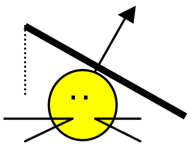


Figure 1.5: *A cat going through a cat flap.*

(a) What is the magnitude of the force exerted by the cat?

(b) What are the horizontal and vertical components of the force on the hinge?

13. A light step ladder has legs 1.5 m long, meeting at the top and both inclined at an angle of 20° to the vertical. The legs are tied together by a horizontal string attached to each leg at a distance 0.25 m from the lower end. The step ladder stands on smooth ground and Mr B, weight 784 N, stands on the top of the step ladder. What is the tension in the string?

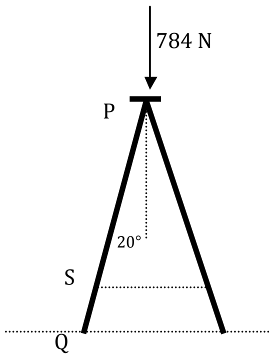
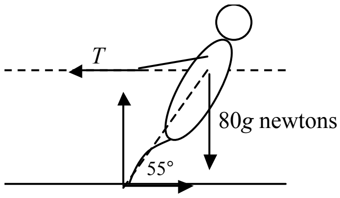


Figure 1.6: *Mr B stands at* P *on a step-ladder whose legs are tied by a horizontal string at* S.

14. Mr B, mass 80 kg, is participating in a tug-of-war. He exerts a horizontal force T on the rope and experiences an equal and opposite reaction force as shown. There are also vertical and horizontal reaction forces where his feet meet the ground. If he is leaning backwards at an angle of 55° to the horizontal, calculate T.



1.7 Centres of gravity

1. A ping pong bat, total mass 150 g, is made up from a handle 11 cm long joined to a "bat" section of mass 90 g which can be regarded as a uniform circular lamina of diameter 14 cm. Where is its centre of gravity?

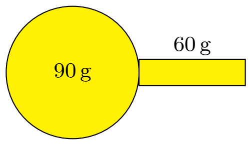
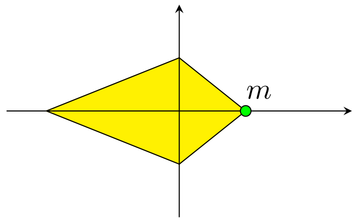


Figure 1.7: *A ping pong bat.*

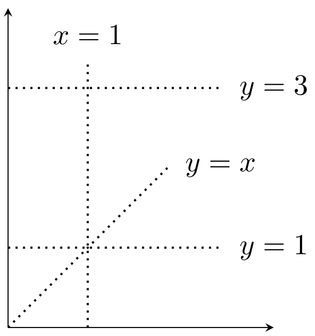
2. A kite made of material of density 1 unit per unit area is made in the shape of a flat four sided figure with corners at $(-3, 0)$, $(0, 1)$, $(0, -1)$, $(1, 0)$.



   (a) Where is its centre of gravity?

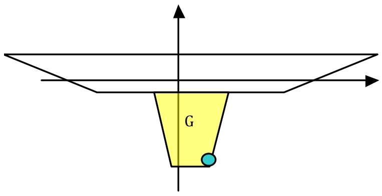
   (b) A small mass m is to be fixed to the nose at $(1, 0)$ so that the centre of gravity of the combined (kite + mass) is at $(0, 0)$. What value of m is required to achieve this?

3. Mr D is designing a centreboard for his dinghy, which is to be cut out of a uniform sheet of metal. The proposed shape as shown is enclosed between the lines $x = 0$, $x = 1$, $x = y$, $y = 3$.



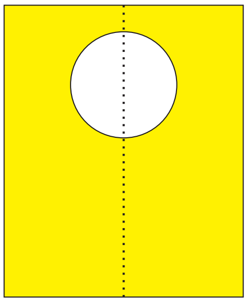
   What is the total area of the centreboard? Where is the centre of gravity of the rectangular part of the centreboard (between $x = 0$, $x = 1$, $y = 1$, $y = 3$)? Where is the centre of gravity of the triangular part (enclosed between $x = 0$, $x = y$, $y = 1$)? Where is the centre of gravity of the complete centreboard?

4. After extensive research Mr D arrives at a design for the centreboard as shown.



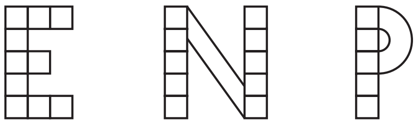
It weighs 300 N and its centre of gravity G, relative to co-ordinates centred on the middle of the boat, is at $x = 0.5$, $y = -1$. As a final adjustment he decides to add a lump of lead, weight 100 N, at the bottom, $x = 1.3$, $y = -1.8$. What is the new position of the centre of gravity?

5. A circular hole is made in a uniform square plate of side 10 cm.



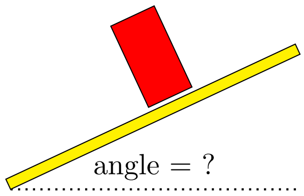
The radius of the circle is 2 cm and its centre is 3 cm in from the mid-point of one of the edges. Where is the centre of gravity of the plate?

6. Find the centres of gravity of uniform laminæ in the shape of the letters E, N and P, relative to co-ordinates with origin at the bottom left-hand corner of each letter.



The "E" measures 5 units high by 3 wide, the "N" is 5 by 3.5, and the "P" (whose curved boundaries have radii 0.5 and 1.5) is 5 by 2.5. (The centre of gravity of a semicircular lamina of radius r lies on the axis of symmetry at a distance $4r/3\pi$ from the straight edge.)

7. A brick, 9 inches long and 4.5 inches wide, is standing upright on a plank. The plank is tilted at a gradually increasing angle until the brick topples over. What is the angle between the plank and the horizontal when this occurs?



8. ABC is a triangular framework, made from three uniform rods AB, BC, CA, each of unit mass per unit length. AB is of length 40 cm, BC is 50 cm, CA is 30 cm.

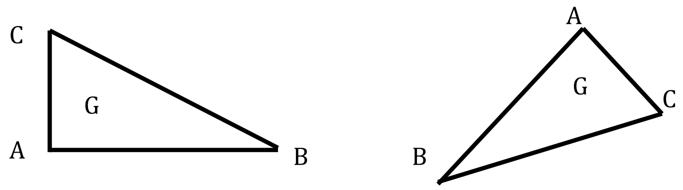


Figure 1.8: *The triangular framework (left). Suspended from* A *(right).*

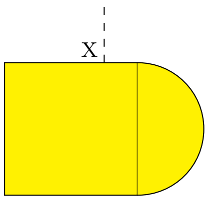
(a) Find the co-ordinates of the centre of gravity G, relative to an origin at A.

(b) What is the angle BAG?

(c) The framework is suspended from corner A, with G vertically below A. What is the angle between AB and the downward vertical?

(d) If instead the framework is suspended from C, what is the angle between AB and the horizontal?

9. A uniform lamina is made by joining a semicircle to a square of side 10 cm. as shown. From what point X on the perimeter of the lamina should it be suspended if its straight sides are to run horizontally and vertically as shown?



(The centre of gravity of a semicircular lamina of radius r lies on the axis of symmetry at a distance $4r/3\pi$ from the straight edge.)

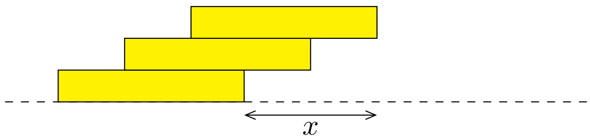
10. A rectangular crate $0.4\,\mathrm{m} \times 0.4\,\mathrm{m} \times 0.6\,\mathrm{m}$ and mass $8\,\mathrm{kg}$ is placed upright (*i.e.*, with one of its square ends down) on a sloping ramp. The ramp is now tilted at an angle α so that the crate just topples over. Find α.

    The angle α is now increased to $45°$ but a weight of mass $M\,\mathrm{kg}$ is fastened to the exterior of the crate in order to stop it from toppling. Draw a diagram to show the most effective place to attach the weight. Assuming that the centre of gravity of the crate is at its geometric centre, what is the smallest possible value of M? Explain any approximations you have made.

11. Where is the centre of gravity of **(a)** a hollow cube where the sides are made from thin sheets of of uniform density and **(b)** the same cube with one face missing?

12. * If the cube in *Question 11* **(a)** above is suspended from a corner, at what angle are its sides inclined to the vertical?

13. * Three dominoes are to balanced on top of one another so that the end of the top domino protrudes as far as possible beyond the end of the bottom domino, without toppling over.

    (a) What is the greatest distance x it can reach? Justify your answer. What also is the greatest distance x that can be achieved using **(b)** 4 dominoes, **(c)** 5 dominoes, **(d)** an infinite number of dominoes?

1.8 Momentum/Impulse/Collisions

1. Snooker ball A, of mass 0.15 kg, travelling at speed 0.5 m/s, hits an identical stationary ball B head-on. After the collision B moves off at a speed 0.45 m/s. Use the law of conservation of momentum to calculate the magnitude and direction of the velocity of A after the impact.

2. Find the missing masses or velocities

(a) Before	1 kg	2 kg	After	1 kg	2 kg
	$\longrightarrow$	$\longleftarrow$		$\longleftarrow$	
	3 m/s	2 m/s		1 m/s	?
(b) Before	2 kg	?	After	2 kg	?
	$\longrightarrow$	$\longleftarrow$		$\longleftarrow$	$\longrightarrow$
	3 m/s	1 m/s		1 m/s	1 m/s
(c) Before	M	$2M$	After	M	$2M$
	$\longrightarrow$			$\longleftarrow$	$\longrightarrow$
	$3U$	?		U	U
(d) Before	M	$2M$	After	M	$2M$
	$\longrightarrow$	$\longrightarrow$		$\longrightarrow$	
	$3U$	$2U$		$2U$	?
(e) Before	M	?	After	M	?
	$\longrightarrow$	$\longrightarrow$		$\longleftarrow$	$\longrightarrow$
	$4U$	U		$2U$	$3U$
(f) Before	M	?	After	$M+?$	
	$\longrightarrow$	$\longrightarrow$		$\longrightarrow$	
	$4U$	U		$2U$	

3. An empty truck of mass M kg rolls along a track with speed 3 m/s and hits a stationary truck of the same type which is loaded with 2000 kg of coal. After the impact, the two trucks couple together and move off with the same speed 1 m/s. Find M.

4. A football of mass 0.45 kg falls vertically to hit the floor at a speed of 5 m/s, and rebounds with a speed of 3 m/s. Calculate the impulse exerted on the ball by the floor.

5. A snooker ball, mass 0.15 kg, travelling at speed 0.3 m/s hits the cushion head-on and receives an impulse of 0.075 N s. Calculate the speed at which it bounces back.

6. An airliner of mass 250 tonnes touches down on the runway at a speed of 270 km/hr. The engines apply reverse thrust for 10 seconds after which the speed is halved. Calculate the magnitude of the reverse thrust.

7. A V-2 rocket of the 1939 – 1945 war had a mass of 4000 kg with a further 8000 kg of fuel. Fuel was burnt at a rate of 135 kg per second and the combustion products were ejected backwards at a speed of 2000 m/s relative to the rocket. Calculate the propulsive force exerted on the rocket.

8. Particles A, B and C are of masses 100 g, 200 g and 400 g respectively and are initially all at rest in a straight line ABC on a smooth table with AB = 0.2 m and BC = 0.2 m. A is now set moving with speed 0.3 m/s towards B. After A collides with B, B moves off towards C with speed 0.2 m/s. What is the speed of A? After B hits C, C moves off with speed 0.1 m/s. What is the speed of B? Check that the total momentum of A, B and C after both collisions have happened is still the same as their total momentum before the collisions. Will there be any more collisions?

9. * In *Question 8*, show also that the distance between A and B immediately after the second collision is 0.3 m.

10. A spherical egg of diameter 5 cm and mass 60 g is dropped from a height of 2 m onto a concrete floor.

    (a) Calculate the speed at impact.

    (b) What is the change in momentum of the egg?

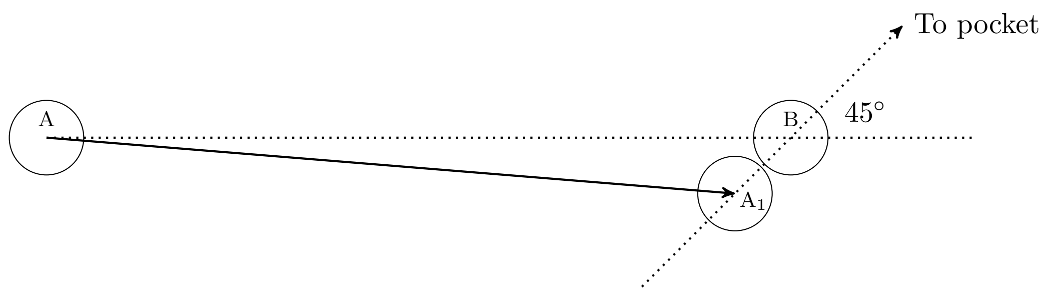
    (c) Show that the time difference between the instant when the bottom of the egg first touches the floor and the instant when the top of the egg follows it down to floor level is about 8 milliseconds.

    (d) Estimate the average force applied to the egg while the impact lasts.

11. * Ten identical railway trucks are lined up on a railway track with 10 m spacing between consecutive trucks. The first truck is set moving towards the second with a speed of 5 m/s. After impact, the trucks couple together. What is their combined speed? The two trucks now impact on the third, after which all three move off together and hit the fourth, and so on. Eventually the ten trucks, coupled together, move off together down the track. What is their final speed? What is the time between the first collision and the last one?

    Hint: We already know the speed of the original moving truck. Work out the speed of the two coupled trucks after the first collision, and the three coupled trucks after the second collision, and look to see if there is a pattern in the numbers.

12. * Snooker player P is attempting to pot ball B. If he is to be successful, ball B must move off at an angle of 45° after being struck by the cue ball A. This in turn requires that at the moment of impact the centres of the two balls must be in alignment with the pocket, so that cue ball A must travel in the direction AA_1 (see diagram). If the initial separation of A and B is 2 metres, and the balls have radius 2.6 cm, use the geometry of the triangle ABA_1 to calculate the angle $\theta = BAA_1$.



If, in fact, B will be successfully potted provided that it moves off at an angle of $45 \pm 1°$, determine the permissible margin of error in the angle θ.

Coefficients of restitution

13. Snooker ball S of mass 150 g, travelling at speed 1 m/s, collides head-on with an identical ball T which is at rest. After the collision T moves off in the direction in which S was previously travelling with speed 0.98 m/s. Use the law of conservation of momentum to calculate the speed of S after the collision. What is the coefficient of restitution in this collision?

14. Snooker ball A travelling at speed 3 m/s collides head on with an identical ball B travelling at 2m/s in the opposite direction. After the collision the speed of A is 1.8m/s and its direction of motion is reversed. Determine **(a)** the velocity of B after the collision, **(b)** the separation speed and **(c)** the coefficient of restitution.

15. A railway truck T1 of mass 4000 kg travelling at speed 2 m/s collides with a stationary truck T2 of mass 6000 kg. The coefficient of restitution is $e = 0.75$. What is the impulse of T1 on T2?

16. A railway truck of mass M, travelling at speed U, collides with a stationary truck of mass $2M$. What are the final speeds of the two trucks if **(a)** the coefficient of restitution $e = 1$, and **(b)** if $e = 0$. **(c)** Determine the range of possible values of e consistent with the observation that the first truck continues to move in its original direction after the collision.

17. Snooker ball A collides head-on with a similar stationary ball B. The coefficient of restitution for the collision is $e = 0.95$. After the collision B moves away towards the cushion 0.5 metres away, returning along the same path. The coefficient of restitution for the impact with the cushion is $e = 0.5$. How far will A have travelled before B hits it again?

18. A billiard ball B, of mass 0.15 kg, travelling at speed 0.1 m/s, strikes the cushion at an angle of 30°. The coefficient of restitution between the ball and the cushion is $e = 0.8$. Calculate **(a)** the angle at which it rebounds, **(b)** its speed after the impact and **(c)** the impulse it exerts on the cushion.

19. A cricket ball travelling at 25 m/s hits the pitch at an angle of 17° to the horizontal. **(a)** If $e = 0.7$, at what speed and in what direction is it travelling immediately after it has bounced? **(b)** Will it clear the stumps if these are 0.6 m high and a horizontal distance of 3 m from the point where the ball bounces? (Treat the ball as a projectile moving under gravity).

20. * A billiard ball travelling at speed U strikes the cushion at an angle θ. The point of impact is close to the corner of the table so that the ball also undergoes a second impact with a second cushion which is perpendicular to the first one. **(a)** If the coefficient of restitution is $e = 1$, give a geometrical argument to prove that the final direction of motion of the ball is reversed (exactly) by the impacts. **(b)** Prove that this conclusion remains true even if e is not equal to 1. **(c)** In the case $e \neq 1$, what is the final speed of the ball?

21. * Ten identical trucks are spaced equally 10 m apart along a railway line. The first truck is given a velocity 5 m/s towards the second truck, colliding with it so that the second truck moves off and hits the third *etc.* If the coefficient of restitution in each collision is $e = 0.5$, determine:

(a) The speed of truck No.2 after it has been set in motion by truck No.1.

(b) The speed of truck No.3 after it has been set in motion by truck No.2.

(c) From the pattern of the answers, deduce the final speed of truck No.10 after it has been set in motion by truck No.9.

(d) What is the total time between the first collision and the last one?

1.9 Energy/Work/Power

1. In the Tunguska event of June 1908 a comet or meteorite on a collision course with the Earth's orbit exploded over Siberia. About 15 megatons of energy was released (1 megaton = chemical energy stored in 106 tons of TNT $= 4 \times 10^{15}$ J). The impact velocity was estimated to be about 25 km/sec. Assuming that the energy of the explosion was derived from the kinetic energy of the comet/meteorite, estimate its mass. Assuming further that it was of a spherical shape and made of ice, density 920 kg/m^3, estimate its size.

2. A snowball is balanced close to the top of the dome of St Paul's cathedral, 112 m above ground level. Dislodged from this position, it starts to slide down the smooth ($\mu = 0$) icy surface of the dome, gathering speed as the slope of the curved surface gets steeper. Eventually it takes off and falls to the ground below. What is its speed at impact?

3. A batsman strikes a cricket ball with a speed of 20 m/s and it just clears the boundary 40 m away. **(a)** Use the principle of conservation of energy to calculate its speed on impact with the ground. The greatest height above the ground reached by the ball is 10 m. **(b)** What is its speed at this point?

4. In a simple model of the pole vault, assume that the vaulter runs down the runway to gain KE, uses the bending of the pole to transform the KE into elastic energy, and finally allows this elastic energy to be released again as gravitational potential energy to gain the height to clear the bar. If this transfer worked perfectly, what height would he clear if his sprinting speed was 10 m/s?

5. William Thompson, later Lord Kelvin, after whom the absolute scale of temperature is named, was for 50 years Professor of Physics at Glasgow University. On occasion he would demonstrate the law of conservation of momentum to his students by suspending a wood block, pendulum style, from the ceiling at one end of the lecture hall and firing a gun at it from the other end. If the mass of the block was 3 kg, the length of the pendulum 1 metre, and the pendulum was deflected through an angle of 10°, what was the speed of the block immediately after the impact of the bullet? Deduce the speed of the bullet if its mass was 3 grams.

   (The authorities eventually banned the demonstration on grounds of safety.)

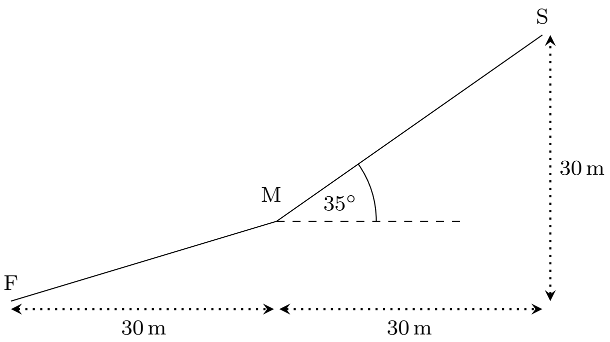
6. A car of mass 800 kg, travelling at speed 15 m/s, skids on a road for a distance of 8 m. When the driver regains control, the speed has been reduced to 10 m/s.

   (a) How much work has been done against the frictional force between the tyres and the road?

   (b) Calculate the magnitude of the frictional force.

   (c) Deduce the coefficient of friction between the tyres and the road.

7. An artificial ski slope drops through a total vertical distance of 30 m between start S and finish F, while the horizontal separation of S and F is 60 m. The slope is designed so that for the first part of the hill, SM, the inclination of the slope is 35° to the horizontal.



A skier steps on to the slope at S and slides down to the finish F. If the coefficient of friction between the skis and the slope is $\mu = 0.1$, and other resistance forces may be neglected, calculate the final speed. Repeat the calculation for **(a)** a slope with a straight line profile SF and **(b)** any other profile of your own choosing, and verify that the final speed at F remains the same in each case.

8. A particle of mass m is projected with speed V up a rough slope which is inclined at an angle θ to the horizontal. The coefficient of friction between the particle and the slope is μ. Show that the maximum distance x travelled up the slope by the particle before it starts to slide down again is given by the formula

$$x = \frac{V^2}{2g(\sin\theta + \mu\cos\theta)} .$$

If $\theta = \arcsin(3/5)$ and $\mu = 0.5$, show that the particle will have lost 80% of its initial kinetic energy by the time it returns to the bottom of the slope. What happens if $\mu > 0.75$?

9. * A particle P is released from rest at the rim of a hemispherical bowl of radius r and slides down inside it. The coefficient of friction between P and the bowl is $\mu = \frac{1}{2}$. Show that P first comes to rest at a height $r/5$ above the bottom of the bowl.

10. Mr C, mass 80 kg, runs up a flight of 30 steps in 10 seconds. Each step has a vertical rise of 17 cm. What is the power generated by Mr C?

11. An escalator 40 metres long inclined at an angle of 30° to the horizontal has a capacity of 100 passengers per minute. Assuming the average passenger has a mass of 70 kg, what is the power required to drive the escalator?

12. A car of mass 800 kg is travelling on a level road at a speed of 20 m/s against a resistance force of 250 newtons. Driver J puts his foot on the accelerator causing the car to gain speed at an initial rate of 1.5 m/s^2. Assuming no resistance forces, calculate

    (a) the tractive force of the engine at this instant and

    (b) the power output of the engine.

13. A car of mass 800 kg has an engine with a maximum power output of 50 kW. What is the fastest speed the car can attain:

    (a) On the flat, working against a constant resistance force of 1000 N.

    (b) With no resistance, but up a slope of 10°.

    (c) Up a slope of 10°, and with a resistance of 500 N.

    (d) Going down a slope of 2° against a resistance of 1500 N.

14. An Olympic sprinter of mass 80 kg is capable for a short period of generating a power output of 8 kW. When running on the flat at speed v, the resistance to motion is $60v$ newtons. Calculate the sprinter's maximum speed **(a)** on level ground and **(b)** up a slope of 5°.

15. * When Mr E cycles along a horizontal road at a speed v m/s the resistance to the motion is $(16 + \frac{1}{4}v^2)$ newtons.

(a) Find the rate at which Mr E must work to maintain a steady speed of 6 m/s.

(b) The maximum rate at which Mr E can work is 410 W. What is the maximum speed he can attain on the flat? (*Hint: it is a whole number of metres per sec.*)

(c) Mr E encounters a hill which slopes upwards at an angle 5.74° (*i.e.* arcsin(0.1)) to the horizontal. Taking $g = 9.8\,\text{m/s}^2$, and supposing the combined mass of Mr E and his cycle to be 70 kg, calculate the total rate of work (against gravity and the resistance force) required to maintain a steady speed of **(i)** 4 m/s and **(ii)** 5 m/s. Deduce that the maximum speed Mr E can achieve up the hill is between 4 and 5 m/s.

(d) Mr E reaches level ground at the top of the hill travelling at 4 m/s and celebrates reaching a level road again by exerting his full power of 410 W. What is his initial acceleration?

(e) Find the maximum speed at which Mr E could freewheel down the hill.

1.10 Motion in a circle

1. Estimate, using your own judgement of reasonable values for angular speed *etc.*, the maximum tension in the wire when an Olympic athlete throws the hammer. ($M = 7.26\,\text{kg}$, wire $= 1.21\,\text{m}$ long.)

2. A car of mass 800 kg travelling at 20 m/s negotiates a bend which has a radius of curvature of 250 m. Because it is raining and the road is wet, it is not possible for the tyres to get any transverse grip on the road. Fortunately, the road is banked at an angle α so that the car is able to negotiate the corner.

   (a) By resolving vertically find an equation which relates N, the normal reaction of the road on the car, to α and the mass of the car.

   (b) By resolving horizontally find the inwards force responsible for producing the centripetal acceleration.

   (c) Deduce what the value of α must be if the car is to stay on the road.

3. * Calculate the apparent reduction in the acceleration due to gravity g at the equator which is produced by the rotation of the Earth.

4. Boy B whirls a conker in a vertical circle on a string of length 1 m. Assuming that energy is conserved during each revolution, what is the minimum speed that the conker should have at the bottom of the circle to ensure that the string remains taut when it is at the top?

5. * An aircraft flies in a parabolic path $y = 5000 - 10^{-4}x^2$, where y is height in metres and x is horizontal distance measured along the line of flight. What are the co-ordinates of the point H where the aircraft reaches its greatest height? Show that a circle, centred at $(0, 0)$ and passing through H has, at H, the same values of $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ as the trajectory of the aircraft.

   The pilot weighs 588 N. By using the circle as an approximation to the trajectory which is valid when the aircraft is near H, find the reaction force between the pilot and his seat at H, if the speed of the aircraft is 100 m/s. At what speed would the pilot experience "zero g" at H?

6. * Monkey M, mass m, holds one end A of a light inextensible string, length L, whose other end is fixed at O. O is at the top of a tall tree OG and M is, initially, at the top of another tall tree of equal height, the two trees being a distance L apart. M now jumps from his tree and, holding the string, falls in a circular arc with centre O. If the angle between OA and the vertical is θ, express in terms of θ:

   (a) the speed of the monkey, and

   (b) the tension in the string.

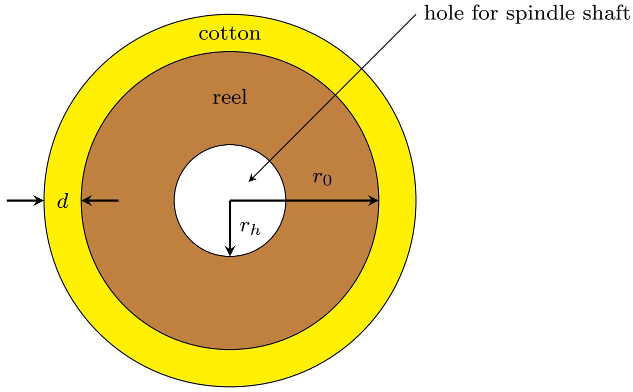
   The string breaks when the tension in it exceeds $\frac{12}{5}mg$. Show that at this instant $\theta = \arcsin(3/5)$. Show also that M hits the trunk of tree OG after a further time $\frac{3}{8}\sqrt{\frac{5L}{2g}}$. Find the distance below O that the impact occurs.

7. A grandfather clock (pendulum driven) is taken to the Moon, where the acceleration due to gravity is $g_{\mathrm{M}} = 1.6\,\mathrm{m/s}^2$. It is set running with the hands indicating 12 o'clock. What time does it indicate one (true) hour later?

8. An executive in a tower block stares out of a window 2 m wide. Immediately outside the centre of the window there appears a window cleaner suspended by long ropes from the top of the building. A gust of wind now sets the window cleaner swinging from side to side. Over a period of several swings, the executive observes that **(i)** the window cleaner is in view for exactly one third of the time and **(ii)** that each passage of the window cleaner across the field of view from one side of the window to the other takes 1.5 seconds.

   Assuming that the swinging motion of the window cleaner can be modelled by that of a pendulum, determine **(a)** the period of the motion, **(b)** its amplitude and **(c)** the maximum speed. **(d)** Assuming the formula $T = 2\pi\sqrt{\frac{L}{g}}$ for the period T of a pendulum of length L, estimate the vertical distance to the top of the building.

9. For the design of a sewing machine it is useful to know the speed at which cotton thread will come off a reel if the reel is rotated at a constant frequency f (*i.e.* a constant number of revolutions per second). The cotton reel comprises a solid cylinder of radius r_0 cm, out of which has been drilled a hole of radius $r_h < r_0$ for the spindle shaft. The cotton is then layered around the reel to a depth of d cm as shown in the diagram below:



Using values of $r_0 = 2\,\mathrm{cm}$, $d = 0.25\,\mathrm{cm}$ and $f = 5\,\mathrm{s}^{-1}$ calculate:

(a) the speed at which the thread comes off a new reel (*i.e.* when there is a depth d of cotton still on the reel), and

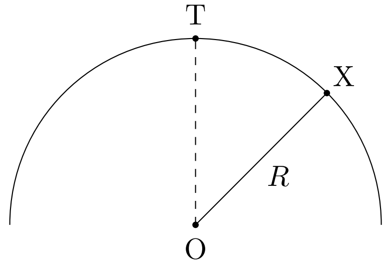
(b) the speed at which the cotton comes off the reel when the thread is almost all 'used up'.

(c) How will the speed of delivery of the cotton from the reel to the sewing machine operator vary over the lifetime of a reel if it is rotated at constant frequency?

(d) Consider now the reel itself spinning without any cotton on it. What is the speed of a point at radius $r_h < r < r_0$ if it is spun at a frequency f?

(e) * A solid object is rotated about an axis passing through the object at a constant frequency. By referring to your answer to part **(d)** or otherwise, sketch a graph of how the speed of the parts of the object vary with their perpendicular distance from the axis of rotation.

10. A toboggan rider X slides down a perfectly smooth hemispherical hill, starting almost from complete rest from a position T right on top. The centre of the hill is O and its radius is R.



(a) Find an equation which, so long as the toboggan remains in contact with the ground, relates θ, the angle TOX, to the vertical height h through which the tobogganist has fallen.

(b) Using the principle of conservation of energy, deduce an equation relating his speed v to θ, R and the acceleration due to gravity g.

(c) Given that he is moving at speed v, and assuming he is still in contact with the ground and is therefore moving in a circle with centre O, what is his acceleration towards O?

(d) By resolving along the radius OX, show that the normal reaction from the ground on X is

$$N = mg(3\cos\theta - 2) ,$$

where m is the combined mass of the tobogganist and toboggan.

(e) At what point in the motion will X take off?

(f) Show that at the instant of take-off his speed is $\sqrt{2gR/3}$.

11. A mass M hangs on an inelastic string of length L vertically below a fixed point O. As the result of a sudden impact, it acquires a speed V and begins to move in a circle (in a vertical plane) about O. In other words, the set up is similar to that of a person initially sitting at rest on a swing who is then given a sudden push.

    Suppose that the mass reaches the top of the circle without the string going slack (when it is directly above O) and that at this point its speed is v.

    (a) Use the principle of conservation of energy to write down an equation relating v to V, L and the acceleration due to gravity g.

    (b) What is the acceleration of the mass towards O when it is at the top of the circle?

    (c) What are the two forces acting on the mass at this point? **(d)** Write down Newton's second law as it applies in this case and so derive an equation relating the sum of the two forces to the acceleration of the mass.

    If it is possible to perform complete orbits, the string must not go slack, so the Tension $T \geq 0$.

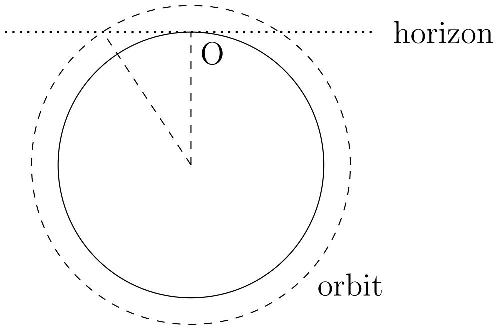
    (e) Deduce the minimum value of v if this motion is possible. **(f)** Deduce also that the minimum value of the initial speed V is $\sqrt{5gL}$.

    (g) What is the tension in the string when the mass once again reaches its original position?

1.11 Gravitation

1. An artificial satellite orbits the Earth at a height of 250 km.

   (a) Taking the radius of the Earth as 6370 km, find its orbital period.



   (b) If the satellite passes directly over an observer at O, estimate, by considering the geometry of the orbit, the time for which the satellite remains above the horizon.

2. Taking the acceleration due to gravity on the Earth and on the Moon as $9.8\,\mathrm{m/s^2}$ and $1.6\,\mathrm{m/s^2}$ respectively, their radii as 6370 km and 1730 km, and the mean density of the Earth as $5500\,\mathrm{kg/m^3}$, calculate the mean density of the Moon.

3. A satellite in a geostationary orbit has an orbital period of 24 hours and remains directly above the same place on the Earth's surface as the Earth revolves. Calculate the radius of its orbit, assumed circular.

4. Jupiter's moon Ganymede is one of the four discovered by Galileo in 1610. Its orbital radius is 1,070,000 km and its period is 7.15 Earth days. Find the mass of Jupiter ($G = 6.673 \times 10^{-11}\mathrm{N\,m^2/kg^2}$).

5. * The planet Pluto was discovered by observing it as a faint point of light moving slowly against the background of the fixed stars.

   (a) Given that Pluto takes about 250 years to complete one (circular) orbit, through what angle would Pluto move in the course of a week? (Assume that the angular movement as seen by an astronomer on Earth is approximately the same as the angular movement that would be seen from the viewpoint of the Sun).

   (b) Why is this a good approximation?

   (c) Using Kepler's third law, calculate the radius of Pluto's orbit around the Sun, assuming it to be a circle (radius of Earth's orbit is 150 million km).

   Sometime later it was discovered that Pluto has a moon (called Charon). As seen from the Earth, the distance across the diameter of its orbit around Pluto corresponds to an angle of 0.0003 degrees. (The astronomer can of course only directly observe angles and changes in angles, not masses or even distances.) The time for Charon to complete one revolution in its orbit around Pluto is $6\frac{1}{2}$ days. Estimate:

(d) the radius of Charon's orbit around Pluto, and

(e) the mass of Pluto.

6. Taking the masses of the Earth and the Moon as 5.98×10^{24} kg and 7.35×10^{22} kg respectively, and the distance between their centres as 384,400 km, calculate the position of the centre of gravity of the combined Earth-Moon system. Given also the period of revolution of the Moon as 27.32 days, calculate the centripetal acceleration of the Moon, relative to the combined centre of gravity.

7. * The simplest atom in Nature is the Hydrogen atom consisting of one proton and one neutron. In a simple physical model of the Hydrogen atom, the electron ($m_e = 9.11 \times 10^{-31}$ kg) is assumed to orbit the proton ($m_p = 1.67 \times 10^{-27}$ kg) at a radius of $r_{\mathrm{B}} = 5.29 \times 10^{-11}$ m (the so-called Bohr radius) in a manner akin to the orbit of a planet around the Sun.

   (a) What is the magnitude of the gravitational force on the electron due to the proton? In which direction does this force point?

   In addition to interacting gravitationally, the proton and electron are both electrically charged and therefore experience an electrostatic force of

$$F = \frac{Q_p Q_e}{4\pi\varepsilon_0 r^2} ,$$

   where $Q_p = +1.6 \times 10^{-19}$ coulombs is the electric charge carried by the proton, $Q_e = -1.6 \times 10^{-19}$ coulombs is the electric charge carried by the electron and r is their separation. $\varepsilon_0 = 8.85 \times 10^{-12}$ farads/metre is a constant known as the 'permittivity of a vacuum'.

   (b) What is the magnitude of the electrostatic force on the electron due to the proton? In which direction does this force point?

   (c) Comment on the relative magnitudes of the gravitational and electrostatic forces that you find.

   Hint: For part **(b)** *the units given are such that the force will come out in newtons.*

8. If the eccentricity of Mars' orbit is $\varepsilon = 0.093$ and its mean distance from the Sun (*i.e.* the semi-major axis of its orbit) is 1.524 astronomical units, find its nearest and farthest distances from the Sun. Take 1 a.u. = 149.6 million km.

   Pluto is observed to have a closest approach to the Sun of 4,437 million km and a farthest point of 7,302 million km from the Sun. What is the eccentricity of its orbit?

9. An estimate for the mass of the visible matter in the galaxy NGC 3198 is $M \approx 3.6 \times 10^6 M_\odot$, where $M_\odot = 1.989 \times 10^{30}$ kg is the mass of the Sun. By considering all of this mass to be concentrated at the centre of the galaxy, calculate the expected speed of a star orbiting in a circle around the centre at a distance of 25,000 parsecs. The parsec is a commonly used astronomical measure of distance; 1 parsec $= 3.086 \times 10^{16}$ m. Compare the answer you obtain with the observed value of $v \approx 150$ km/s.

10. An astronomer observes that a moon of the planet Utopia has a circular orbit of radius 200,000 km. Its orbital period is observed to be 314.16 hours.

    (a) Calculate the speed of the moon in its orbit.

    (b) Assuming Newton's law of Gravitation in the form $F = GMm/r^2$ for the gravitational force F between two masses M and m, separated by a distance r, determine the mass of Utopia. (Take $G = 6.673 \times 10^{-11}\,\mathrm{N\,m^2/kg^2}$.)

11. * A body moves at constant speed in a straight line from point A to point B in a certain time interval.

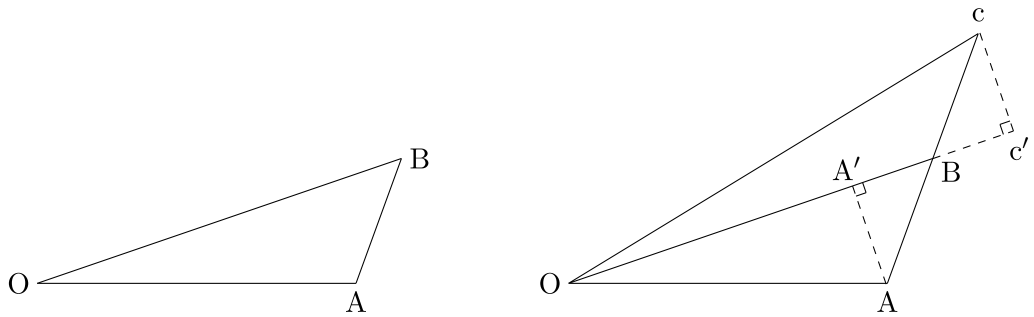


Figure 1.9: *A body moving at constant speed in a straight line from* A *to* B *(left) and then from* B *to* c *(right).*

With respect to an origin at O it therefore sweeps out a triangle OAB as shown in *Figure 1.9.* After a second identical time interval it reaches point c thus sweeping out another triangle OBc.

(a) What can you say about the relationship between the distances AB and Bc?

(b) By considering AA′ and cc′ as altitudes of triangles OAB and OBc respectively, or otherwise, show that the areas of OAB and OBc are equal.

If, instead, the body receives an impulse at B directed towards O along BO, then it moves to point C in the second time interval (again identical to the first time interval) as shown in *Figure 1.10.*

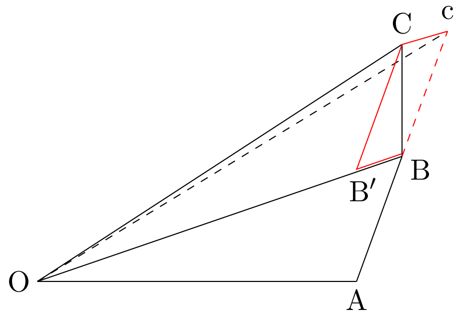


Figure 1.10: *Instead, after receiving an impulse at* B *directed towards* O, *the body continues to* C.

(c) Explain how the combination of the impulse and the original motion produce the new motion from B to C. What geometrical figure is BcCB′?

(d) By constructing a relevant altitude for the triangle OBC, show that its area is equal to the area of triangle OBc. What does this imply about the areas of OBC and OAB?

(e) How is this relevant to Kepler's laws?

1.12 Vectors

1. ABC is a triangle whose corners have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

   (a) In terms of vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, what are the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{AC}$, $\overrightarrow{CA}$?

   (b) What is the vector sum $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$?

   (c) Find the position vectors for F, the midpoint of AB, and D, the midpoint of BC.

   (d) Calculate the length FD and show that $FD = \frac{1}{2}AC$.

   (e) How can you see that the vector $\overrightarrow{FD}$ is parallel to $\overrightarrow{AC}$?

2. ABC is a triangle in which A has position vector $\mathbf{a} = 2\mathbf{i} + \mathbf{j}$, B has position vector $\mathbf{b} = 6\mathbf{i} + 13\mathbf{j}$, and C has position vector $\mathbf{c} = 10\mathbf{i} + 7\mathbf{j}$.

   (a) In terms of the unit vectors $\mathbf{i}$ and $\mathbf{j}$, what are the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{AC}$, $\overrightarrow{CA}$?

   (b) What is the vector sum $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$?

   (c) Find the position vectors for F, the midpoint of AB, and D, the midpoint of BC.

   (d) Calculate the length FD and show that $FD = \frac{1}{2}AC$.

   (e) How can you see that the vector $\overrightarrow{FD}$ is parallel to $\overrightarrow{AC}$?

3. In the triangle ABC of *Question 1*, the centroid G of the triangle is the point with position vector $\mathbf{g} = \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$.

   (a) Express $\mathbf{g}$ in terms of the unit vectors $\mathbf{i}$ and $\mathbf{j}$.

   (b) Calculate the vectors $\overrightarrow{AG}$ and $\overrightarrow{AD}$ and show that these both lie in the same direction.

   (c) Explain why **(b)** shows that G must lie on the line AD.

   (d) Using similar reasoning show that G also lies on the lines BE and CF, where E is the mid-point of AC.

   (e) Explain how **(d)** and **(e)** together show that the three lines AD, BE and CF all meet at the same point. What point is this?

   (f) Calculate the ratios of the lengths AG:AD, BG:BE, CG:CF.

4. ABCD is a parallelogram with DC parallel to AB and AD parallel to BC. The position vectors of its corners are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ respectively.

   (a) Calculate $\mathbf{d}$ in terms of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

   (b) Find the position vector M of the mid-point of the diagonal AC.

   (c) Show that M also lies on the other diagonal BD and divides it into two equal parts BM and MD.

5. ABCD is a parallelogram with DC parallel to AB and AD parallel to BC. A has position vector $\mathbf{a} = \mathbf{i} + 2\mathbf{j}$, B has position vector $\mathbf{b} = 3\mathbf{i} + 4\mathbf{j}$, C has position vector $\mathbf{c} = 2\mathbf{i} + 10\mathbf{j}$.

   (a) If point D has position vector $\mathbf{d}$, calculate $\mathbf{d}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

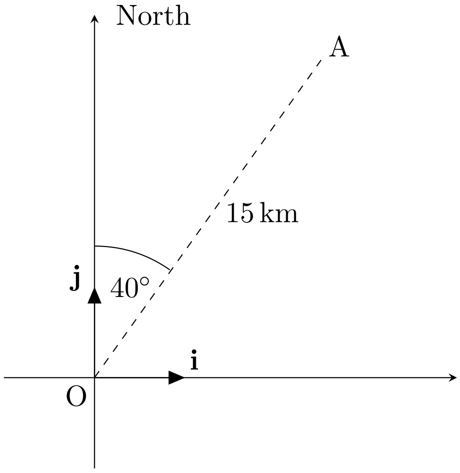
   (b) Find the length of the diagonal AC.

   (c) Find the position vector M of the mid-point of the diagonal AC.

   (d) Show that M also lies on the other diagonal BD and divides it into two equal parts BM and MD.

6. Show that the triangle ABC with corners at $\mathbf{a} = -9\mathbf{i} + 11\mathbf{j}$, $\mathbf{b} = -\mathbf{j}$, $\mathbf{c} = 6.4\mathbf{i} + 3.8\mathbf{j}$ has a right angle. At which vertex is it? Calculate also the other angles of the triangle.

7. Seen from a lighthouse at the origin O, a yacht A lies on a bearing of 40° at a distance of 15 km. Calculate its position vector relative to the lighthouse.



8. The position vectors of ships A and B, relative to a lighthouse at O, are $\mathbf{r}_{\mathrm{A}} = 20\mathbf{i} + 10\mathbf{j}$ and $\mathbf{r}_{\mathrm{B}} = -20\mathbf{i} + 30\mathbf{j}$. (Distances are in km, $\mathbf{i}$ and $\mathbf{j}$ are unit vectors east and north).

   (a) Calculate the distances OA, OB, and AB.

   (b) Calculate the bearing of A, seen from O, the bearing of O, seen from A, and the bearing of B from A.

9. Particle P1 starts at time $t = 0$ from a point with position vector $2\mathbf{i}+3\mathbf{j}$, and moves at a constant speed to arrive at the point $12\mathbf{i} + 27\mathbf{j}$ at time $t = 2$.

   (a) Express its velocity as a vector.

   (b) Calculate its speed.

   (c) What is the angle between the direction of motion and the unit vector $\mathbf{i}$?

10. Particle P2 starts at a point with position vector $2\mathbf{i} + 3\mathbf{j}$ and moves at a constant speed to arrive at the point $12\mathbf{i} - 7\mathbf{j}$ in a time $t = 4$ seconds.

    (a) Express its velocity as a vector.

    (b) Calculate its speed.

    (c) What was the position vector of P2 at time $t = 3$?

    (d) If P2 continues with the same velocity, it will pass through two of the following two points: $22\mathbf{i} - 17\mathbf{j}$, $27\mathbf{i} - 23\mathbf{j}$, $47\mathbf{i} - 42\mathbf{j}$. Which is the odd one out? Justify your answer.

11. Particle P3 has velocity $2\mathbf{i} + 3\mathbf{j}$ m/s at time $t = 0$ and velocity $12\mathbf{i} - 7\mathbf{j}$ m/s at time $t = 5$ seconds. Calculate the vector which represents its acceleration. What is the magnitude of the acceleration vector?

12. Mr B is adrift on the ocean, with his position at $t = 0$ (today) being expressed in vector form as $-12\mathbf{i} + 2\mathbf{j}$. ($\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the easterly and northerly directions respectively and distances are measured in kilometres). He is being carried along by an ocean current with velocity $4\mathbf{i}$, measured in km/day. Draw a sketch to represent this information. What will Mr B's position vector be tomorrow (*i.e.* at $t = 1$)?

    Mr B desires to reach an island with position vector $10\mathbf{j}$. Explain why his nearest approach to the island is after 3 days. By what distance does he miss the island? The wind is coming from the south, and by hoisting a sail Mr B can change his velocity to $4\mathbf{i}+k\mathbf{j}$, where he can choose k by varying the size of the sail. If he hoists the sail tomorrow, at $t = 1$, what is the value of k which will allow him to reach the island? What will be the total distance travelled?

13. Crow C flies at a speed of 12 m/s from tree A with position vector $20\mathbf{i} + 10\mathbf{j}$ to tree B with position vector $308\mathbf{i} + 94\mathbf{j}$. (Distances are given in metres). Determine

    (a) the distance AB,

    (b) the time of flight,

    (c) the velocity vector for C and

    (d) the direction of motion, expressed as a bearing (unit vector $\mathbf{j}$ is due north).

14. A cricket ball is thrown with speed 28 m/s at an angle of 30° to the horizontal. If $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the horizontal and vertical directions respectively, express the initial velocity of the ball in vector form.

15. Mr D takes part in an orienteering competition. He starts at the origin and his intention is to run (in a straight line at constant speed) to the point X, at position $\mathbf{r}_X = 0.9\mathbf{i} + 4\mathbf{j}$, in a time of half an hour (distances are expressed here in km).

    (a) What should his velocity vector be? What is his speed?

    (b) Because of a navigational error Mr D starts off with velocity $\mathbf{v} = 8\mathbf{i} + 1.8\mathbf{j}$ and after half an hour finds he is at Y, which is the wrong place. What is the position vector of Y? How far is it from X?

    (c) Mr D now sets off from Y to X (in the correct direction), with the same speed as before. When he finally reaches X, how much time will have been wasted?

16. Mr B is out for a walk with his dog F. Starting from the entrance to the park at E, with position vector $\mathbf{r}_{\mathrm{E}} = 2\mathbf{j}$, Mr B walks with speed 1 metre per second towards the point X with position vector $\mathbf{r}_{\mathrm{X}} = 20\mathbf{i} + 17\mathbf{j}$.

    (a) Calculate the distance EX.

    (b) Deduce the time taken for Mr B to reach X.

    On entering the park F immediately darts off at constant speed to investigate a bush at Y, with position vector $\mathbf{r}_{\mathrm{Y}} = -16\mathbf{i}+65\mathbf{j}$. He inspects the bush for 5 seconds before dashing back on a straight line course (at the same constant speed) to intercept Mr B at X. Calculate

    (c) the total distance travelled by F and

    (d) his speed.

17. * The position vectors $\mathbf{r}_{\mathrm{A}} = 20\mathbf{i} + 10\mathbf{j}$ and $\mathbf{r}_{\mathrm{B}} = -20\mathbf{i} + 30\mathbf{j}$ represent the positions of ships A and B at time $t = 0$ (hours). Ship A has velocity $10\mathbf{j}$ km/hr and ship B has velocity $10\mathbf{i} + 5\mathbf{j}$ km/hr.

    (a) Show that, at time t, $\mathbf{r}_{\mathrm{A}} = 20\mathbf{i} + (10 + 10t)\mathbf{j}$, and find a similar formula for $\mathbf{r}_{\mathrm{B}}$.

    (b) What are the distances between the two ships at $t = 1$ and $t = 2$?

    (c) Calculate the bearing of B, seen from A, at times $t = 1$ and $t = 2$. What is the evidence that the ships are on a collision course?

    (d) Verify that the ships will collide at $t = 4$.

18. * At $t = 0$, ships A and B have position vectors $\mathbf{r}_{\mathrm{A}} = 20\mathbf{i}+10\mathbf{j}$ and $\mathbf{r}_{\mathrm{B}} = -20\mathbf{i}+30\mathbf{j}$. Ship A has velocity $10\mathbf{j}$ km/hr and ship B has velocity $8\mathbf{i} + 4\mathbf{j}$ km/hr. (This is the same data as *Question 17*, except for the velocity of ship B.)

    (a) Calculate formulæ for the position vectors $\mathbf{r}_{\mathrm{A}}$ and $\mathbf{r}_{\mathrm{B}}$ at time t.

    (b) Check that the bearing of B, seen from A, changes between $t = 1$ and $t = 2$, confirming that the ships will not collide.

    (c) Calculate the distance of B from A at time $t = 2$ and find a general formula which gives the distance at time t.

    (d) Find out the time at which this distance is smallest and verify that the distance of closest approach of the two ships is 8 km.

    Hint: if the distance AB *is d, the mathematics may work out most easily if you consider how d^2 varies with time.*

19. Particle P has velocity $10\mathbf{i} + k\mathbf{j}$ m/s. If the speed of P is 26 m/s, what is k?

20. Particle Q has velocity $5\mathbf{i} + k(\mathbf{i} + 2\mathbf{j})$ m/s. What value of k is needed to make the velocity of Q **(a)** parallel to the vector $\mathbf{i} + \mathbf{j}$, **(b)** parallel to the vector $\mathbf{i}$ and **(c)** parallel to the vector $\mathbf{j}$.

21. * A motor boat moves on a curved path so that its velocity vector at time t is $\mathbf{v} = (1+t^2)\mathbf{i} + (2t+4)\mathbf{j}$. What is the speed at $t = 0$? At what time is the velocity parallel to the vector $\mathbf{i}+\mathbf{j}$? At what time is the velocity parallel to the vector $\mathbf{i}+3\mathbf{j}$? Sketch the path the boat follows.

22. The corners of a rugby field are at O, with position vector $0\mathbf{i}+0\mathbf{j}$, A with position vector $40\mathbf{i}+0\mathbf{j} = 40\mathbf{i}$, B $(40\mathbf{i}+80\mathbf{j})$, and C $(80\mathbf{j})$. Player P receives the ball on the touchline at the point $40\mathbf{i}+40\mathbf{j}$ and heads for B on the try line BC. If the dimensions given are in metres, and he can run at $7\,\text{m/s}$, write down his velocity in vector form.

    Simultaneously player Q, from the opposing team, starts from the point $19\mathbf{i}+52\mathbf{j}$ and makes for the corner flag in order to cut P off. Draw a diagram showing the positions of P and Q on the field and the directions in which they will start to run.

    If Q can run at $6\,\text{m/s}$, explain why his velocity can be expressed as the vector $(18/5)\mathbf{i}+(24/5)\mathbf{j}\,\text{m/s}$. Another defender, S, sets off from the point $15\mathbf{i}+55\mathbf{j}$ one second after P and Q, with speed $7.5\,\text{m/s}$. What is his velocity vector? What are the position vectors of P, Q and S, 5 seconds after P and Q have started running? Will either Q or S succeed in intercepting P?

23. A particle P has velocity $3\mathbf{i}+4\mathbf{j}\,\text{m/s}$, where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors along the x- and y-axes respectively. What is the speed of P? What is the angle between its direction of motion and **(a)** the x-axis and **(b)** the y-axis?

    P receives an impulse in a collision after which its velocity becomes $5\mathbf{i}+4\mathbf{j}\,\text{m/s}$. What is **(i)** the change in velocity of P, **(ii)** the increase in speed and **(iii)** the angle through which its direction of motion has been deflected?

24. Three forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ are represented by vectors $10\mathbf{i}+20\mathbf{j}$, $20\mathbf{i}-30\mathbf{j}$ and $-6\mathbf{i}+20\mathbf{j}$ newtons respectively. Draw a sketch showing their magnitudes and directions. Determine their resultant, expressed as a vector, and find its magnitude. An additional force $\mathbf{F}_4 = x\mathbf{j}$ is now applied so that the resultant of $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ and $\mathbf{F}_4$ has magnitude $30\,\text{N}$. What is x?

25. Two forces $\mathbf{F}_1 = 2\mathbf{i}+3\mathbf{j}\,\text{N}$ and $\mathbf{F}_2 = -10\mathbf{i}+17\mathbf{j}\,\text{N}$ act on a particle P of mass $0.5\,\text{kg}$. Express the acceleration of P in vector form.

26. Particle Q is in equilibrium under the action of forces $\mathbf{F}_1 = 2p\mathbf{i}+4q\mathbf{j}$, $\mathbf{F}_2 = p\mathbf{i}-10\mathbf{j}$ and $\mathbf{F}_3 = 3\mathbf{i}+q\mathbf{j}$. Find p and q. What are the magnitudes of the three forces?

27. Billiard ball A has mass $0.15\,\text{kg}$ and velocity vector $0.2\mathbf{i}\,\text{m/s}$. Express its momentum in vector form.

    Ball A strikes a glancing blow on an identical stationary ball B, and after the impact, ball B is observed to have velocity vector $0.1\mathbf{i}+0.1\mathbf{j}$. Use the law of conservation of momentum, in vector form, to find the new velocity vector of ball A. What are the speeds of A and B after the collision? What is the angle between their directions of motion?

28. * In this question, $\mathbf{i}$ and $\mathbf{j}$ represent unit vectors in the easterly and northerly directions respectively, while $\mathbf{k}$ represents a unit vector in the direction vertically upwards. Distances are measured in metres. At time $t = 0$, a radar station at the origin (ground level) detects a hostile aircraft 5 km away to the west, at an altitude of 200 metres. The aircraft is travelling on a level course with velocity $\mathbf{v}_{\mathrm{A}} = 200\mathbf{i} + 150\mathbf{j}$. 10 seconds later a missile is fired to intercept the aircraft. The missile has velocity $\mathbf{v}_{\mathrm{M}} = 250\mathbf{j} + 12.5\mathbf{k}$ and interception is achieved if the missile (considered as a particle) gets closer than 15 m to the aircraft (considered as a particle). Does it succeed?

Solutions

2 Solutions

2.1 Kinematics

1. Using $s = ut + \frac{1}{2}at^2$, $s = 56$, $u = 0$, $a = 9.8$, time = 3.4 seconds. Using $v^2 = u^2 + 2as$, speed = 33 m/s.

2. Distances travelled after 1, 2, 3, 4 seconds are 4.9, 19.6, 44.1, 78.4 metres. Distances travelled during 1^{st}, 2^{nd}, 3^{rd}, 4^{th} seconds are 4.9, 14.7. 24.5, 34.3 metres. In n^{th} second, distance travelled = $4.9(2n + 1)$ metres.

3. Using $s = ut + \frac{1}{2}at^2$, with $s = 0$, $u = 20$, $t = 10$, $a = ?$, gives $a = -4\,\text{m/s}^2$. The acceleration due to gravity is $4\,\text{m/s}^2$.

4. **(a)** Using $v^2 = u^2 + 2as$, $s = 186$, $u = 26.5$, $a = 1.5$, v = speed at B = 35.5 m/s.

   (b) Using $s = \frac{1}{2}(u + v)t$, now knowing $v = 35.5$, $t = 6$ seconds.

   (c) Average speed over AB = 186/6 = 31 m/s = 69.4 mph. Speed limit not broken.

   (d) From $v^2 = u^2 + 2as$, with $s = 93$, speed at C = 31.32 m/s = 70.1 mph. Speed limit broken.

5. Quadratic equation is $5000 = 300t + t^2$. Time t for measured distance = 15.83 seconds.

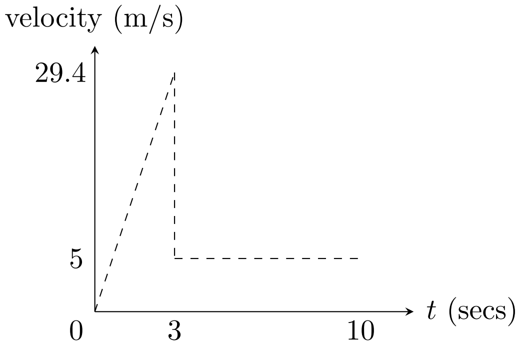
6. **(a)** Using $v^2 = u^2 + 2as$, with down = +, $v = 35$, $u = 15$, $a = 9.8$, $s = ?$, gives $s = 51$ m.

   (b) If the stone is thrown at 15 m/s upwards, instead of downwards, then the same method applies but with $u = -15$, and since $(-15)^2 = (+15)^2$, the solution is the same, $s = 51$ m. Alternatively, it is valid to split the flight of the stone into two parts, upwards and downwards, and combine the results, but this is more laborious than is really necessary. Let the equations do the work!

7. Using $v^2 = u^2 + 2as$ for each car, we find their stopping distances are 40 m and 62.5 m respectively. Since these add to more than the available total distance 100 m, we know that there will be a collision. The question does not ask for the exact position where the collision occurs so no further calculation is necessary.

8. Use $s = vt - \frac{1}{2}at^2$ with $s = 2$, $t = 0.1$, $a = 9.8$, gives v = speed of impact on ground = 20.49 m/s. Then $v^2 = u^2 + 2as$ with $u = 0$ gives height fallen = s = 21.4 metres.

9. Distance in first 3 seconds $= 1/2 \times 3 \times 9.8 = 14.7$ metres. Distance covered between $t = 3$ and $t = 10$ is $7 \times 5 = 35$ metres. Total distance for first 10 seconds $= 14.7 + 35 = 49.7$ metres.



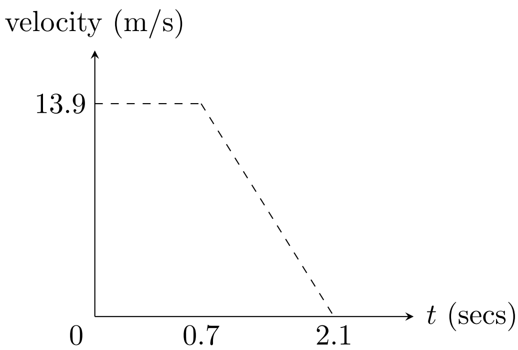
10. **(a)** $50\,\text{kph} = 50 \times 1000/3600 = 13.89\,\text{m/s} = 13.9\,\text{m/s}$ to 3 s.f.

(b) Speed unchanged at $13.9\,\text{m/s}$ for 0.7 seconds, thinking distance $= 13.9 \times 0.7 = 9.72$ metres.

(c) $s = \frac{1}{2}(u + v)t$ with $s = 14.7$, $u = 13.89$, $v = 0$, $t = ?$, gives $t = 2.117$ seconds.

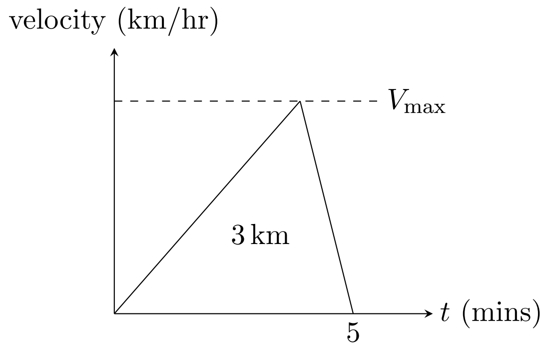
(d) $a = (v - u)/t = (0 - 13.89)/2.117 = -6.561\,\text{m/s}^2$, deceleration $= 6.56\,\text{m/s}^2$ to 3 s.f.

(e)

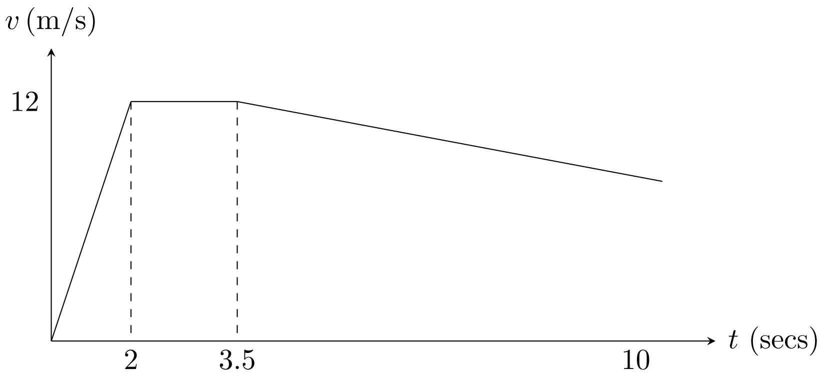


(f) $110\,\text{kph} = 30.56\,\text{m/s}$, thinking distance $= 30.56 \times 0.7 = 21.39\,\text{m}$, braking distance $= (30.56)^2/(2 \times 6.56) = 71.17\,\text{m}$, total stopping distance $= 92.6\,\text{m}$.

11. Distance covered $= 3\,\text{km} = \text{area} = 1/2 \times \text{base} \times \text{height} = 1/2 \times (5/60) \times V_{\text{max}}$, $V_{\text{max}} = 72\,\text{km/hr}$. We do not know where the apex of the triangle is, but we can still use the area formula $1/2 \times \text{base} \times \text{height}$.



12. Distance A covers in first 2 seconds $= 1/2 \times 6 \times 2^2 = 12\,\text{m}$. Distance covered in first 3.5 seconds $= 12 + 1.5 \times 12 = 30\,\text{m}$. Quadratic equation $70 = 12t - (0.5 \times 0.5t^2)$ has positive solution $t = 6.795$ giving a total time of $2 + 1.5 + 6.795 = 10.295\,\text{s}$ to cover 100 m.



In 2 seconds B covers 11 m, in 3.5 seconds 27.5 m. B starts to catch up when his speed equals that of A, *i.e.* after 5.5 seconds when speeds of both are 11 m/s. At this instant A has covered 53 m and B 49.5 m, so B is 3.5 m behind. Time for B to cover 45 m distance remaining after $t = 6$ is solution of quadratic equation $45 = 11t - (0.5 \times 0.25t^2)$, $t = 4.301\,\text{s}$. Total time for B $= 6 + 4.301 = 10.301\,\text{s}$, so A wins. The winning margin is $0.006\,\text{s}$ in time or approximately $0.006\,\text{s} \times 10\,\text{m/s} = 6\,\text{cm}$ in distance.

2.2 Projectiles

1. **(a)** Time to reach net $= 11.9/25 = 0.476\,\text{s}$.

   (b) Distance fallen $= 1/2 \times 9.8 \times 0.476^2 = 1.11\,\text{m}$.

   (c) Height of ball above ground as it passes net $= 2.25 - 1.11 = 1.14\,\text{m}$. Ball clears net by $1.14 - 0.91 = 0.23\,\text{m}$.

   (d) The ball can fall through $2.25 - 0.91 = 1.34\,\text{m}$ and still clear net. If the speed of the ball is V then the time to reach the net is $11.9/V$, distance fallen $= 1/2 \times 9.8 \times (11.9/V)^2$. If the ball gets over, $1.34 > 1/2 \times 9.8 \times (11.9/V)^2$. Solving, $V > 22.8\,\text{m/s}$.

   (e) To land in the service court, the ball must have time to fall through a vertical height $2.25\,\text{m}$ (or more) while travelling a horizontal distance $11.9 + 6.4 = 18.3\,\text{m}$. If speed is V, $2.25 < 1/2 \times 9.8 \times (18.3/V)^2$, $V < 27.0\,\text{m/s}$.

2. **(a)** The bomb is released when the aeroplane is $250/\tan(10°) = 1418\,\text{m}$ short of the target.

   (b) The bomb takes a time $\sqrt{(2 \times 250/g)} = 50/7$ seconds to reach the ground,

   (c) during which time it travels horizontally $200 \times 50/7 = 1429\,\text{m}$ and so overshoots by about $11\,\text{m}$.

3. Taking down $= +$, the time for tile to reach ground is t where $7 = 4.2\sin(30°)t + (0.5 \times 9.8)t^2$, simplifies to $7 = 2.1t + 4.9t^2$, factor 0.7 cancels to leave quadratic $7t^2 + 3t - 10 = 0$, factorises to $(7t + 10)(t - 1) = 0$, with $t = 1$ second the relevant solution. Horizontal distance covered in this time is $4.2\cos(30°) \times 1 = 3.6\,\text{m}$.

4. Horizontal and vertical components of initial velocity are 16.8 m/s and 12.6 m/s.

   (a) At $t = 1$, co-ordinates are $(16.8, 7.7)$, at $t = 2$, $(33.6, 5.6)$.

   (b) If height y is zero after time t, $0 = 12.6t - 4.9t^2$, $t = 0$ or $18/7 = 2.57\,\text{s}$. Time of flight $= 2.57\,\text{s}$.

   (c) Horizontal distance travelled $= 16.8 \times 18/7 = 43.2\,\text{m}$.

5. If the hammer is thrown at 45° inclination to horizontal, which gives the maximum range for a projectile, range is $V^2/g = 87\,\text{m}$, $V \approx 29\,\text{m/s}$.

6. To reach the wicket-keeper, range $= V^2 \sin 2\theta/g = 28^2 \sin 2\theta/9.8 = 50$. Therefore $\sin 2\theta = 5/8$, smaller solution for $\theta = 19.3°$, time of flight $= 50/(28\cos\theta) = 1.9$ seconds.

7. Let angle of projection be θ. Considering vertical motion at $t = 2$, $5 = 30 \sin\theta \times 2 - 1/2 \times 9.8 \times 2^2$, $\sin\theta = 0.41$. At $t = 2$, vertical velocity $= 30\sin\theta - 9.8 \times 2 = -7.3\,\text{m/s}$, horizontal velocity $=$ constant $= 30\cos\theta = 27.36\,\text{m/s}$. Speed at $t = 2$ is $\sqrt{(7.3^2 + 27.36^2)} = 28.3\,\text{m/s}$, inclined at angle $\arctan(7.3/27.36) = 14.9°$ below the horizontal.

8. Let initial speed be V, angle of projection be θ. Maximum height $= V^2 \sin^2\theta/2g = 40$, range $= 2V^2 \sin\theta\cos\theta/g = 120$. Divide the first equation by the second to find $\tan\theta = 4/3$, $\sin\theta = 4/5$, $V = 35\,\text{m/s}$, $\theta = 53.1°$.

9. **(a)** Equation of the trajectory is $y = x\tan(45°) - 1/2 \times 9.8 \times x^2/(\cos^2(45°) \times 15^2)$, which simplifies to $y = x - 98x^2/2250$.

   (b) When $x = 15$, $y = 5.2\,\text{m}$, so ball will clear crossbar (height 3 m) with 2.2 m to spare.

   (c) To be on the safe side, we could require an extra margin $\approx$ size of ball, say 30 cm.

   (d) For clearance, $y > 3$ at $x = 15 \Rightarrow 3 < 15 - 9.8 \times 15^2/V^2$, $V > 13.6\,\text{m/s}$.

10. One way to visualize the solution here is to consider the reversed problem in which an arrow is projected along the same trajectory as Robin Hood's arrow but in the opposite direction. The reverse arrow has a known angle of projection, 30°, and its velocity V_R must be chosen so that it passes through the point, 15 m lower down and 150 m distant, from which Robin Hood is shooting. The trajectory equation then gives

$$-15 \;=\; 150\tan(30°) - \frac{1}{2} \times 9.8 \times \frac{150^2}{\cos^2(30°) \times V_\text{R}^2}\;,$$

with the solution $V_\text{R} = 38.04\,\text{m/s}$. The time of flight will be $t = 150/(V_\text{R}\cos(30°)) = 4.553$ seconds and the vertical and horizontal velocity components for the reverse arrow as it hits the ground are $V_\text{R}\sin(30°) - 9.8t = -25.61\,\text{m/s}$ and $V_\text{R}\cos(30°) =$

32.94 m/s. Taken in the forwards direction, these are the velocity components for Robin's own arrow (with a sign difference for the vertical component); which therefore must have speed $\sqrt{(25.61^2 + 32.94^2)} = 41.7\,\mathrm{m/s}$ and angle of projection $\arctan(25.61/32.94) = 37.9°$.

11. **(a)** If the distance achieved is x, and the point of projection is taken as the origin, the shot lands at $(x, -2.5)$. The trajectory equation gives

$$-2.5 \;=\; x\tan(45°) - \frac{1}{2} \times 9.8 \times \frac{x^2}{\cos^2(45°) \times 14^2}\;,$$

which simplifies – the factors of 7 in 9.8 and 14^2 cancel nicely – to the quadratic $x^2 - 20x - 50 = 0$ with solution $x = 22.25\,\mathrm{m}$.

(b) The same method with the angle 45° replaced by 40° gives the quadratic $0.852x^2 - 16.78x - 50 = 0$ with solution $x = 22.32\,\mathrm{m}$.

(c) For a general angle θ, the trajectory equation is

$$-2.5 \;=\; x\tan\theta - \frac{1}{2} \times 9.8 \times \frac{x^2}{\cos^2\theta \times 14^2}\;.$$

Simplifying, and putting $1/\cos^2\theta \;=\; \sec^2\theta \;=\; 1 + \tan^2\theta$

$$x^2(1 + \tan^2\theta) - 40x\tan\theta - 100 \;=\; 0\;,$$

which can also be considered as a quadratic equation for $\tan\theta$

$$x^2(\tan\theta)^2 - 40x\tan\theta + (x^2 - 100) \;=\; 0\;.$$

At maximum range, the two solutions for θ coincide. The given quadratic has equal roots for $\tan\theta$, and therefore equal solutions for θ, when "$b^2 = 4ac$", or $1600x^2 = 4x^2(x^2 - 100)$, $x = \sqrt{500} = 22.36\,\mathrm{m}$. $\theta = 41.8° \approx 42°$.

12. With $V = 7\sqrt{2}$, equation of trajectory is $y = x\tan\theta - 1/2 \times 9.8 \times x^2/(\cos^2\theta \times (7\sqrt{2})^2)$. Expressing $1/\cos^2\theta = \sec^2\theta = 1 + \tan^2\theta$, substituting $x = 2$, $y = 4.8$, the equation becomes $4.8 = 2\tan\theta - 1/2 \times 9.8 \times 2^2 \times (1 + \tan^2\theta)/98$ and simplifies to $\tan^2\theta - 10\tan\theta + 25 = 0$. This is a quadratic in $\tan\theta$ with a double root $\tan\theta = 5$, so that the minimum angle of projection for a successful throw is the same as the maximum angle, $\theta = \arctan(5) = 78.7°$. This is the only feasible angle.

13. There are many ways to tackle this problem. Using geometry or calculus will produce elegant and exact solutions. For the average student however the main difficulty is getting started, and the important point to realise is that, as in many practical problems, there is no single "approved" method. There is nothing wrong in trying something very simple.

Clearly, taking $x = 0$, *i.e.* attempting the conversion from the try line, gives a target angle of zero. And attempting the conversion from the far end of the pitch gives a very small angle (quite apart from the practical difficulty of getting the range). The optimum must be somewhere in-between, and there is nothing to stop us trying a few values.

From the geometry of the pitch, AD $= 20 - 2.8 = 17.2\,\mathrm{m}$, AE $= 20 + 2.8 = 22.8\,\mathrm{m}$. The target angle DCE is the difference between the angle ACE $= \arctan(\mathrm{AE}/x)$ and the angle ADE $= \arctan(\mathrm{AD}/x)$. If say we try $x = 10$, then ACE $= 66.3°$, ADE $= 59.8°$, DCE $= 6.5°$.

Similarly with $x = 20$, angle DCE $= 8.0°$, and with $x = 30$, angle DCE $= 7.4°$. We can plot these values on a graph (along with the known point at $x = 0$) which will suggest more refined trial values for x, and a solution good enough for practical purposes.

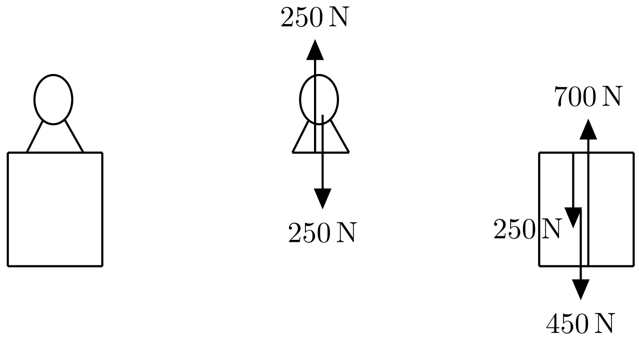
Some students, though, will be commendably dissatisfied until they find an exact answer, more precise than practically necessary, but the complete solution of the mathematical problem originally posed. One more systematic approach is to make use of the compound angle formula, $\tan(\theta_1 - \theta_2) = (\tan\theta_1 - \tan\theta_2)/(1 + \tan\theta_1 \tan\theta_2)$. Taking θ_1 = angle ACE, θ_2 = angle ACD, $(\theta_1 - \theta_2)$ = target angle DCE, gives

$$\tan(\mathrm{DCE}) = \frac{(\mathrm{AE}/x - \mathrm{AD}/x)}{1 + (\mathrm{AE}/x)(\mathrm{AD}/x)} = \frac{(\mathrm{AE} - \mathrm{AD})}{(x + \mathrm{AE} \cdot \mathrm{AD}/x)}\ .$$

The target angle DCE is largest when tan(DCE) is largest and tan(DCE) is largest when the denominator $(x + \mathrm{AE} \cdot \mathrm{AD}/x)$ is smallest. This occurs when $x = \sqrt{(\mathrm{AE} \cdot \mathrm{AD})}$ $= 19.8\,\mathrm{m}$, and angle DCE $= 8.0°$. The trial value $x = 20$ was evidently very close.

2.3 Forces

1. From left to right: The bust on the plinth, the forces on the bust and the forces on the plinth.



2. Since the crocodile is in equilibrium, the upward buoyancy force must be equal to its weight, 1200 N. The fact that the croc is half submerged is irrelevant.

3. **(a)** Applying $F = ma$ to the car plus Mr D, $F = 2400$ N, $m = 1110 + 90 = 1200$ kg, $a = F/m = 2\,\text{m/s}^2$.

   (b) Speed after 100 m is 100 km/hr = $100 \times 1000/3600 = 27.78$ m/s, and using constant acceleration formula $v^2 = u^2 + 2as$ with $v = 27.78$, $u = ?$, $a = 2$, $s = 100$, initial speed $u = 19.3$ m/s $= 69.4$ km/hr.

   (c) Applying $F = ma$ to Mr D, his mass is 90 kg, and his acceleration is the same as the acceleration of the car, $a = 2\,\text{m/s}^2$, so forward force exerted is $90 \times 2 = 180$ newtons.

4. **(a)** acc. $= 3\,\text{m/s}^2$,

   (b) $m = 8$ kg,

   (c) Force $\rightarrow = 2$ N, Force $\uparrow = 6$ N.

5. $200 = s = \frac{1}{2}at^2 = \frac{1}{2}a5^2 \therefore a = 16\,\text{m/s}^2$, $F - 12{,}000g = 12{,}000a \therefore F = 310$ kN.

6. Downward force on scales = upward force on Mr E = $88g$ newtons. Newton II applied to Mr E gives $88g - 80g = 80a$, upward acceleration $a = g/10 = 0.98\,\text{m/s}^2$.

7. When the balloon was in equilibrium, buoyancy force = total weight = $(110 + 90 + 80 + m)g = (280 + m)g$ newtons. With the same buoyancy, even after F is ejected, Newton II applied to the balloon gives $(280 + m)g - 280g = 280 \times \text{acc.} = 280 \times 2.1$, so $m = 60$.

8. Downward force on Mr F plus parachute $= (55 + 5) \times 9.8 = 588$ N, upward force = 570 N. Downward acceleration of Mr F is determined by Newton's 2nd law where $F = 588 - 570 = ma = (55 + 5)a$, so $a = 0.3\,\text{m/s}^2$. After Mr F kicks off boots

downward force is $(55 + 5 - (2 \times 0.75)) \times 9.8 = 573.3\,\mathrm{N}$. Newton II gives $F = 573.3 - 570 = ma = (55 + 5 - 1.5)a$, $a = 0.056\,\mathrm{m/s^2}$. Notice that the absence of the boots alters both the force F and the mass m.

9. **(a)** Mass of car plus trailer $m = 1400 + 200 = 1600\,\mathrm{kg}$, $a = 0.6\,\mathrm{m/s^2}$, gives the tractive force as $F = ma = 960\,\mathrm{N}$.

   (b) Let the unknown load be x kg. Then $F = 960\,\mathrm{N} = (1600 + x) \times 0.48 = ma$, so $x = 400$.

   (c) Now $F = 960 - 160 = ma = 1600a$, $a = 0.5\,\mathrm{m/s^2}$.

10. **(a)** Let acceleration of system be a and tension in string be T. Newton II applied to A gives $T = 3a$, and applied to B gives $2g - T = 2a$. Solving, $a = 2g/5 = 3.92\,\mathrm{m/s^2}$. We should avoid making the instinctive but incorrect assumption that the lighter mass B will be unable to shift the heavier mass A, which is based on experience when friction is present.

    (b) $T = 3a = 11.76\,\mathrm{N}$.

    (c) Equations of motion when C replaces B are $T = 3a$, $Mg - T = Ma$, giving $M = 3a/(g - a)$. With a given as $4.9\,\mathrm{m/s^2} = \frac{1}{2}g$, $M = 3$.

11. **(a)** The equations of motion for Mr B's ascent are

$$\text{Mr B: } T - 65g = 65a$$

$$\text{Barrel plus bricks: } (5 + 70)g - T = (5 + 70)a\ .$$

    Solving, the upward acceleration of Mr B $= a = g/14 = 0.7\,\mathrm{m/s^2}$, while the time to reach top is given by $s = ut + \frac{1}{2}at^2$. With $u = 0$, $s = 12.6$ so time $t = 6$ seconds.

    (b) Equations of motion for Mr B's descent are

$$\text{Mr B: } 65g - T = 65a$$

$$\text{Barrel only: } T - 5g = 5a\ .$$

    Downward acceleration of Mr B is $6g/7 = 8.4\,\mathrm{m/s^2}$, time to reach bottom given by $s = ut + \frac{1}{2}at^2$. With $u = 0$, $s = 12.6$ and time $t = \sqrt{3} = 1.73$ seconds.

    (c) Further time taken for the barrel to fall freely from the pulley down to Mr B at ground level is given by $s = ut + \frac{1}{2}at^2$, with $u = 0$, $s = 12.6$, $a = g = 9.8$, so time $t = \sqrt{18/7} = 1.60$ seconds.

    (d) One of the main approximations in this calculation is the treatment of Mr B as a particle of zero size. We have assumed that his upward journey to the pulley is 12.6 m but more realistically he will start from a standing position with his centre of gravity nearly a metre above ground level. When his fingers reach the pulley his centre of gravity will also be some distance below pulley level. Both of these considerations will reduce the effective value of s used in the calculation.

B's downward journey will be similarly reduced at the start, and also at its finish if he lands on a pile of spilt bricks.

Another factor to consider is the possibility of Mr B colliding with the barrel at the halfway point on both his upward and downward journeys.

Air resistance is often quoted as something which could be allowed for in more refined models but in this instance it is unlikely to be as significant as the factors mentioned above.

2.4 Resistance forces

1. The force balance equation, drag force $= D = \frac{1}{2}C_d A\rho v_t^2 = mg =$ weight, here takes the form $\frac{1}{2}\times 0.8\times\pi(4)^2\times 1.2\times v_t^2 = (70+5)\times 9.8$. Solving for the terminal velocity v_t, $v_t = 5.5\,\mathrm{m/s}$.

2. Buoyancy force $= 4\pi/3\times(0.02)^3\times 1000\times 9.8 = 0.328\,\mathrm{N}$, weight $= 0.026\,\mathrm{N}$, resultant upward force $= 0.302\,\mathrm{N}$, force balance equation is drag force $= D = \frac{1}{2}C_d A\rho v_t^2 =$ buoyancy force $= 0.302\,\mathrm{N}$, so $\frac{1}{2}\times 0.5\times\pi(0.02)^2\times 1000\times v_t^2 = 0.302$, $v_t = 0.98\,\mathrm{m/s}$.

3. The normal reaction R is $50\times 9.8 = 490\,\mathrm{N}$ and the frictional force is $F_f = \mu R = 0.05\times 490 = 24.5\,\mathrm{N}$. The deceleration of the skater, supposing he makes no effort to maintain his speed, is $F_f/m = 24.5/50 = 0.49\,\mathrm{m/s^2}$. The time taken for the speed to drop from $9.9\,\mathrm{m/s}$ to $5\,\mathrm{m/s}$ is 10 seconds.

4. Normal reaction is $N = mg$. In the case of limiting friction, $F_f = \mu N = \mu mg = -ma$ (Newton 2). Therefore $a = -\mu g$ which implies that the deceleration is independent of the mass. Both skaters stop at the same time!

5. Applying $F = ma$, $19.8 - \mu R = 19.8 - 98\mu = 10\times 1$. Solve to find $\mu = 0.1$.

6. Let m kg be the mass of Q. Then $F - \mu mg = ma$ gives $20 - (0.8\times 9.8m) = 2.16m$. Solve, giving $m = 2$.

7. Two equations: (i) $4.9 = \mu mg = \mu m\times 9.8$. (ii) $6.9 - \mu mg = ma \Rightarrow 6.9 - 9.8\mu m = 2m$. Thus $6.9 - 4.9 = 2m \Rightarrow m = 1$ kg. $\mu = 4.9/9.8 = 1/2$.

8. $100\,\mathrm{km/hr} = 27.78\,\mathrm{m/s} = u$, $v = 0$, $s = 60$, so $v^2 = u^2 + 2as$ gives $a = -6.43\,\mathrm{m/s^2}$. frictional force $F_f =$ mass$\times$deceleration $= 6430\,\mathrm{N}$, $F_f = \mu mg = 9800\mu$, so $\mu = 0.66$.

9. **(a)** The limiting frictional force for team A is 5120 N, and for team B is 5760 N. When the tension in the rope is 5000 N, the teams remain in equilibrium with the frictional force of both A and B equal to 5000 N.

   (b) The higher tension of 5500 N is higher than the limiting friction for team A so that team B will win.

10. **(a)** Let the tension in the string be T and the acceleration of the system be a. Newton II for A gives $T - 2\mu g = T - 0.5\times 2g = 2a$, and for B gives $3g - T = 3a$. Eliminate T to find $a = 2g/5 = 3.92\,\mathrm{m/s^2}$.

    (b) Substitute $a = 3.92\,\mathrm{m/s^2}$ to find $T = 17.64\,\mathrm{N}$.

    (c) On the second table, the equations of motion for A and B are $T - 2\mu g = 2a$ and $3g - T = 3a$ respectively. Adding, and substituting the given value $a = 4.9\,\mathrm{m/s^2}$, $3g - 2\mu g = 5a = 2.5g$, so $\mu = 0.25$.

11. **(a)** From the solution to the previous question, $a = 3.92\,\mathrm{m/s^2}$.
    (b) $s = 0t + \frac{1}{2}at^2$ gives $t = 0.505$ seconds.
    (c) $v = 0 + at = 3.92 \times 0.505 = 1.98\,\mathrm{m/s}$.
    (d) Once B hits the floor, the only force on A is the friction with the table and its deceleration is $F_f/m = 0.5mg/m = 4.9\,\mathrm{m/s^2}$. Using $v^2 = u^2 + 2as$, the additional distance which A must travel before coming to rest is $(0^2 - 1.98^2)/(2 \times (-4.9)) = 0.40\,\mathrm{m}$. Adding the 0.5 m distance travelled before B hits the floor gives a total of 0.9 m, so A will come to rest before it reaches the edge of the table.

12. In *Applied Mathematics by Example, Book 1: Theory, Section 4.5*, we saw that friction is actually the driving force of a car. As $F_f \leq \mu N$, the maximum acceleration occurs when $F_f = \mu N$: $\mu N = \mu \frac{M}{4} g = \frac{M}{4} a \Rightarrow a = \mu g = 0.15 \times 9.8 = 1.47\,\mathrm{m/s^2}$.

13. In equilibrium, just before sliding occurs, $P\cos\alpha = F_f = \mu R$ (with $\mu = 0.5$) and $R = 1000 + P\sin\alpha$ is the balance of horizontal and vertical forces. Taken together these imply that $P\cos\alpha = 500 + \frac{1}{2}P\sin\alpha \Rightarrow P(\cos\alpha - \frac{1}{2}\sin\alpha) = 500$. Differentiating this implicitly with respect to α gives:

$$\frac{\mathrm{d}P}{\mathrm{d}\alpha}(\cos\alpha - \frac{1}{2}\sin\alpha) + P(-\sin\alpha - \frac{1}{2}\cos\alpha) = 0 ,$$

and recalling that the condition for a minimum is $\frac{\mathrm{d}P}{\mathrm{d}\alpha} = 0$ then implies that $\sin\alpha = -\frac{1}{2}\cos\alpha \Rightarrow \tan\alpha = -1/2 \Rightarrow \alpha = \arctan(-1/2) = 26.6°$.

14. The force balance equation for the parachute is drag force $= D = \frac{1}{2}C_d A\rho v_t^2 = mg =$ weight. Making the terminal velocity the subject gives $v_t = \sqrt{(2mg/C_d A\rho)}$, and for a circular parachute of area $A = \pi d^2/4$ this is equivalent to

$$\begin{aligned} v_t &= \sqrt{\frac{8mg}{C_d \pi d^2 \rho}} \\ &= \sqrt{\frac{8g}{C_d \pi \rho}} \times \frac{\sqrt{m}}{d} \ , \end{aligned}$$

where the first factor must be numerically equivalent to the factor 4.7 in the formula from the handbook. Equating these factors and solving for C_d,

$$C_d = \frac{8g}{4.7 \times 2\pi\rho} = 0.92 \ .$$

15. **(a)** The maximum range achievable in the absence of air resistance is given by the formula $v^2/g = 55^2/9.8 = 308.7 \approx 310$ metres.

(b) The air resistance formula shows that the ideal range is reduced by a factor

$$\left(1 + \frac{cv^2}{mg}\right)^{-0.74} = \left(1 + \frac{10^{-4} \times 55^2}{0.06 \times 9.8}\right)^{-0.74} = 0.736 \ ,$$

so the estimate of maximum range, taking into account air resistance, is $0.736 \times 308.7 = 227.0 \approx 230$ metres.

2.5 Resolving forces

1. **(a)** Resolving along the line of the track, $3600 = F = ma = 2400a$, giving $a = 1.5\,\mathrm{m/s^2}$.

   (b) $3600\cos(60°) = F = ma = 2400a$, so $a = 0.75\,\mathrm{m/s^2}$.

   (c) Case **(a)**: $3600 - 600 = F = ma = 2400a$, $a = 1.25\,\mathrm{m/s^2}$. Case **(b)**: $3600\cos(60°) - 600 = F = ma = 2400a$, $a = 0.5\,\mathrm{m/s^2}$.

   (d) Considering the component of force perpendicular to the track, the force from the rails is $3600\sin(60°) = 3120\,\mathrm{N}$ (to 3 s.f.)

2. **(a)** $61.9\,\mathrm{kg}$

   (b) $311\,\mathrm{N}$

   (c) $M = 44.2\,\mathrm{kg}$, $T = 250\,\mathrm{N}$

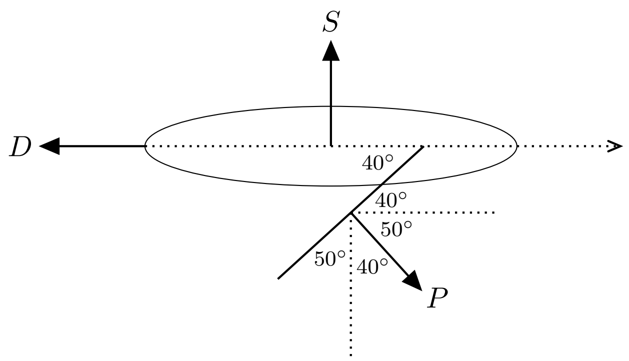
   (d) $T_1 = 339\,\mathrm{N}$, $T_2 = 679\,\mathrm{N}$

3. $L = 10{,}050\,\mathrm{N}$, $D = 433\,\mathrm{N}$.

4. $F = -2 \times 400 \times \cos(40°) = -612.8\,\mathrm{kN}$. Deceleration $= -a = -F/m = -621.8 \times 1000/(15 \times 1000) = 40.9\,\mathrm{m/s^2}$.

5. Resistance force $= 200\cos(40°) - (100 \times 0.1) = 143\,\mathrm{N}$. Normal reaction $= (100 \times 9.8) - 200\sin(40°) = 851\,\mathrm{N}$.

6. The angle between the sail and the line of motion is 40° and most of the angles in the diagram are either 40° or 50°. The crafty approach is to draw your sketch with the 40° angle somewhat less than the true 40° and the 50° angles somewhat larger than the true 50°. Then the 40°s and 50°s will be clearly differentiated. The angle between the force $P = 1000\,\mathrm{N}$ from the sail and the line of motion of the boat, which is easy to get wrong, is then clearly seen to be 50°.



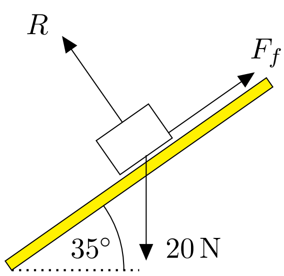
   (a) Resolving along the line of motion, $1000\cos(50°) = D = 643\,\mathrm{N}$, and perpendicular to the line of motion $1000\cos(40°) = S = 766\,\mathrm{N}$.

   (b) $1150\cos(50°) - 643 = F = ma = 300a$, $a = 0.32\,\mathrm{m/s^2}$.

(c) Other forces acting on the boat are its weight acting downwards and the buoyancy force acting upwards

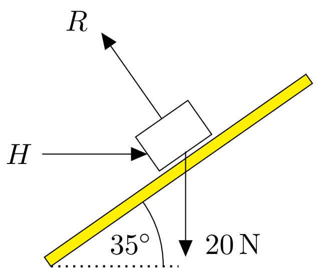
7. If normal reaction is R, resolving vertically gives $R+P\sin(30°) = R+500\sin(30°) =$ weight $= 1000\,\mathrm{N}$, so $R = 750\,\mathrm{N}$. The horizontal component of P is $500\cos(30°) = 433\,\mathrm{N}$, and if $\mu = 0.5$, $433\,\mathrm{N} >$ limiting friction $F_f = \mu R = 0.5 \times 750 = 375\,\mathrm{N}$, and the block will move. If $\mu = 0.6$, $433\,\mathrm{N} <$ limiting friction $F_f = \mu R = 0.6 \times 750 = 450\,\mathrm{N}$, and the block will not move. The additional force required for movement is $17\,\mathrm{N}$. As an alternative solution here, $433/750 = \mu = 0.58 \Rightarrow \mu = 0.5$ too small, while $\mu = 0.6$ too big.

8. The easiest method is to resolve the 20 newton weight into a force of $20\sin(35°)$ newtons down the slope and a force $20\cos(35°)$ newtons perpendicularly into the slope. Then, balancing forces up and down the slope, $20\sin(35°) = F_f$, and balancing forces in the direction perpendicular to the slope, $20\cos(35°) = R$. Hence $F_f = 11.5\,\mathrm{N}$, $R = 16.4\,\mathrm{N}$.



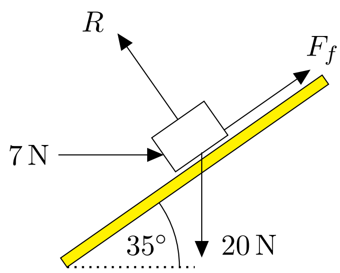
Alternatively, it is valid, though more long-winded, to balance forces in the horizontal and vertical directions. For the horizontal balance, $R\cos(55°) = F_f\sin(55°)$, and for the vertical balance, $R\cos(35°) + F_f\cos(55°) = 20$. These are simultaneous equations for the two unknowns R and F_f leading to the same values as derived above.

9. The question invites us to resolve forces vertically and horizontally, *i.e.* to check the balance of forces in these directions. It is usually a good idea to follow such advice. We get for the vertical balance $R\cos(35°) = 20$, so $R = 20/\cos(35°) = 24.4\,\text{N}$, and then for the horizontal balance $R\cos(55°) = H$, so knowing R, $H = 14.0\,\text{N}$.



Alternatively, it is equally valid, having found R, to resolve parallel to the slope, *i.e.* check the balance of forces up and down the slope. This tells us $H\cos(35°) = 20\cos(55°)$ $(= 20\sin(35°))$, which gives the same H as before, but without having to calculate R as an intermediate step.

10. Here the two forces which we do not know are R and F_f. We are free to consider the balance of forces in any direction we choose. Probably, it will be best to consider the directions **(a)** parallel to the slope, because R will not be involved and we will get an equation involving only F_f, and **(b)** perpendicular to the slope because F_f will not be involved and we will get an equation involving only R.



Trying this

(a) $7\cos(35°) + F_f = 20\cos(55°) = 20\sin(35°)$, so $F_f = 5.7\,\text{N}$.

(b) $R = 20\cos(35°) + 7\cos(55°)$, $R = 20.4\,\text{N}$.

11. **(a)** If the diameter TB $= 2 \times$ radius $2.45 = 4.9\,\text{m}$, the time taken to fall from T to B is t where $s = 4.9 = \frac{1}{2}gt^2 = 4.9t^2$, $t = 1$.

(b) The distance TA is $\sqrt{(2.452 + 2.452)} = 2.45\sqrt{2}$, and the acceleration down the slope TA is $g\sin(45°) = g/\sqrt{2}$. The required time is t with $s = 2.45\sqrt{2} = \frac{1}{2}(9.8/\sqrt{2})t^2$, and again $t = 1$.

(c) In the general case, suppose the angle TOC is 2θ. Then in the isosceles triangle TOC the angles at OTC and OCT are both equal to $90° - \theta$ and angle of inclination of the slope TC to the horizontal is $(90° - \text{angle OTC}) = (90° - (90° - \theta)) = \theta$. The acceleration down the slope TC is $g \sin\theta$, and, the distance TC is $2 \times \text{radius} \times \cos(90° - \theta) = 4.9 \sin\theta$. The travel time along TC is therefore t with $s = 4.9 \sin\theta = \frac{1}{2} \times 9.8 \sin\theta\, t^2$, giving $t = 1$ once more.

12. After B falls into the crevasse, suppose tension in rope is T and acceleration of climbers is a.

(a) Newton II gives for A, $T + 80g\sin(20°) = 80a$, while for B, $80g - T = 80a$. So, acceleration $a = g(1 + \sin(20°))/2 = 6.58\,\text{m/s}^2$.

(b) After 1 second, distance covered $= ut + \frac{1}{2}at^2 = 3.29\,\text{m}$.

(c) Speed acquired $v = u + at = 6.58\,\text{m/s}$.

(d) Once A has deployed his ice-axe, there are new values of a and T. Newton II gives for A, $T + 80g\sin(20°) - 1400 = 80a$, while for B, $80g - T = 80a$. Acceleration $a = -2.17\,\text{m/s}^2$. With initial speed $u = 6.58\,\text{m/s}$, and deceleration $-2.17\,\text{m/s}^2$, A will come to rest ($v = 0$) after covering a distance s, where $s = (v^2 - u^2)/2a = 9.94\,\text{m}$. Total distance travelled by A is $3.29 + 9.94 = 13.23\,\text{m}$, so he comes to rest $1.77\,\text{m}$ short of the crevasse.

13. From the geometry, SM $= 36.62\,\text{m}$, MF $= 31.32\,\text{m}$. The time taken to cover SM, inclined at an angle $35°$ to the horizontal, is t where $36.62 = \frac{1}{2} \times 9.8\sin(35°) \times t^2$, giving $t = 3.610$, and the speed at M is $20.28\,\text{m/s}$. Similarly the time taken to cover MF inclined at an angle $16.7°$ to the horizontal is t where $31.32 = 20.28t + (\frac{1}{2} \times 9.8\sin(16.7°) \times t^2)$, giving $t = 1.406\,\text{s}$. The total time to traverse SF is therefore $3.610 + 1.406 = 5.02$ seconds.

There are many possible candidate designs for the ski slope. One simple alternative choice is to take M level with the finish point so that the skier drops the full 30 m in the first 30 m of horizontal displacement and then finishes along the flat from M to F. The time required for SM now comes out as 3.499 s with a further 1.237 s for MF, a total of 4.74 s. Evidently, it pays to gain speed early in the descent, even at the expense of a geometrically longer path.

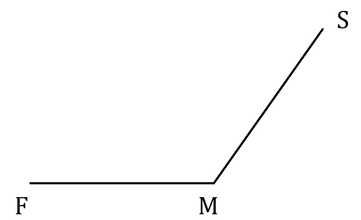


Figure 2.1: *A candidate slope where the skier drops the full* 30 m *in the first* 30 m *of horizontal displacement and then finishes along the flat from* M *to* F.

Newton's solution (known also to Bernoulli, or he wouldn't have issued the challenge) is a segment of a curve known as the cycloid. For our ski slope, it can be expressed in "parametric" form where x and y are referred to an origin at the finish point F.

$$\begin{aligned} x &= 60 - 15.515(t - \sin t) \\ y &= 30 - 15.515(1 - \cos t) \ , \end{aligned}$$

where t is measured in radians. (For a parametric curve, you pick a range of values of the parameter t, and plot the corresponding (x, y) points. In this case the parameter t is equivalent to the time in seconds since leaving S). The time to reach F is 3.5084 seconds as can be seen by substituting this value in the expressions for x and y.

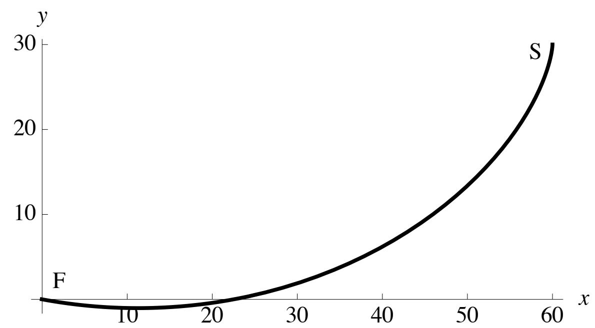


Figure 2.2: *Newton's solution to the problem: the cycloid connecting* S *and* F.

An interesting feature of the curve is that the maximum vertical fall along the path is more than 30 metres, so that the last part of the path actually rises gently to finish at F.

2.6 Rigid bodies

1. **(a)** Let George sit a distance x metres from the fulcrum on the same side as Edgar. Taking moments about the fulcrum, $500 \times 1.5 + 600x = 700 \times 1.5$, $x = 0.5$.

   (b) If Edgar sits distance y metres from the fulcrum, $500y = 700(3 - y)$, $y = 1.75$.

   (c) Vertical force for case **(a)** is $500\,\mathrm{N} + 600\,\mathrm{N} + 700\,\mathrm{N} = 1800\,\mathrm{N}$, for case **(b)** is $700\,\mathrm{N} + 500\,\mathrm{N} = 1200\,\mathrm{N}$.

2. **(a)** The weight of the rod acts at its centre. If BC = 1 metre, WC = 1 metre at a distance 0.5 metres from B. Then taking moments about C, $W \times 1 = F \times 1$, $F = W = 150\,\mathrm{N}$.

   (b) BC = 1.5, WC = 0.5, taking moments about C again, $W \times 0.5 = F \times 1.5$, $W = F \times 3 = 150\,\mathrm{N}$.

   (c) Let BC = x metres, then $W \times (2 - x) = F \times x$, so $150 \times (2 - x) = 250 \times x$, and $x = 0.75$.

3. **(a)** The weight of the plank, if uniform, acts at its centre, 0.25 m from the fulcrum. Taking moments about the fulcrum, $400 \times 1.5 + 200 \times 0.25 = 1 \times$ Ben, weight of Ben is 650 N.

   (b) Upward force $= 400 + 200 + 650 = 1250\,\mathrm{N}$.

   (c) The combined weight of Andrea and Charlie is 650 N, the same as the weight of Ben, so the fulcrum must be in the middle of the plank.

   (d) Andrea, Ben, and Charlie have been treated as particles of zero size, so that the weight of Ben acts, for example, exactly at the end of the plank.

   (e) The plank has been assumed to be both uniform and rigid.

4. Taking moments about C, weight of Mr D $\times\, x =$ (wind force $\times\, 0.5 \times 6$) + (force on centreboard $\times 0.5 \times 2$) $= (300 \times 3) + (300 \times 1) = 1200$, $x = 2\,\mathrm{m}$. Notice we have assumed that all other forces acting on the boat act along lines passing through the centre C so that they make no contribution to the moments equation.

5. **(a)** Let the weight of the missile be W. If the force up from the ground on the front wheels is F, and the force up on the rear wheels is R, then $F + R = W$. If the centre of gravity of the missile is at its mid-point, it is 3 m to the rear of the front axle and 2 m in front of the rear axle. Take moments about (say) the front axle. Then $3W = 5R$. So $R = 3W/5$, F must be $2W/5$ and R is 50% larger.

   (b) Suppose the centre of gravity of the missile is x metres behind the front axle, and therefore $(5 - x)$ metres in front of the rear axle. The force balance $F + R = W$ together with the new information $R = 3F$ now give $R = 3W/4$. The moments equation is $xW = 5R$, so $x/5 = 3/4$, $x = 3.75$.

6. Your mathematical model should look like this:

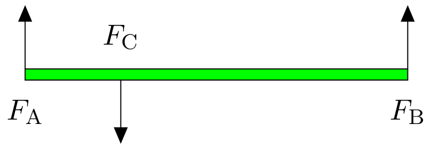


Figure 2.3: *The bottle opener force diagram.*

If say AB = 12.5 cm, AC = 2.5 cm, and forces are F_A, F_B, F_C respectively, taking moments about A gives $2.5F_C = 12.5F_B$, and if $F_B = 10$ newtons, $F_C = 50$ newtons. The force balance $F_A + F_B = F_C$ then gives $F_A = 40$ newtons.

7. Suppose that pirate P manages to get a distance x metres beyond the side of the ship before the plank becomes unbalanced. From the geometry, we note that pirate R sits 1.5 m inboard from the side while the centre of gravity of the plank, if assumed uniform, is 0.5 m outside.

   (a) Taking moments about the edge, $1000 \times 1.5 = (200 \times 0.5) + (700 \times x)$, $x = 2$. P therefore gets to within 0.5 metres of the far end of the plank.

   (b) Here we set $x = 2.5$ in the moments equation and calculate the weight of R: $W_R \times 1.5 = (200 \times 0.5) + (700 \times 2.5) = x$, $W_R = 1850/1.5 = 1233$ N.

8. For part **(a)**, we need two equations, to find the two unknown tensions T_{C} and T_{D}. An easy one to start with is the force balance $T_{\mathrm{C}} + T_{\mathrm{D}} = 600 + 200 = 800$. For the other equation, we can take moments about D:

$$600\,(\text{gymnast}) \times 2 + 200\,(\text{bar, weight at mid-point}) \times 1.5 \;=\; T_{\mathrm{C}} \times 3\ ,$$

giving $T_{\mathrm{C}} = 500\,\mathrm{N}$, and therefore $T_{\mathrm{D}} = 300\,\mathrm{N}$.

For part **(b)**, we can refer back to the same moments equation as in part **(a)**, but now we know $T_{\mathrm{C}} = 600$ newtons, but dont know the position of the gymnast, so we write "x" instead of "2" to represent the distance GD.

$$600\,(\text{gymnast}) \times x + 200\,(\text{bar, weight at mid-point}) \times 1.5 \;=\; 600 \times 3\ ,$$

giving $x = 2.5$ metres.

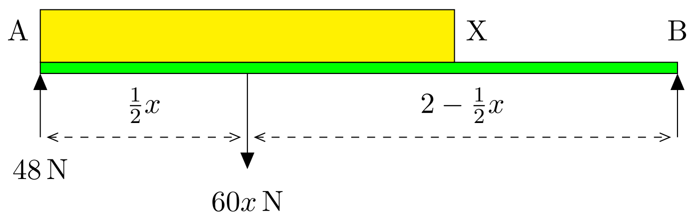
For part **(c)**, we can use the moments equation again with $\mathrm{GD} = 3.5\,\mathrm{m}$.

$$600\,(\text{gymnast}) \times 3.5 + 200\,(\text{bar, weight at mid-point}) \times 1.5 \;=\; T_{\mathrm{C}} \times 3\ ,$$

giving $T_{\mathrm{C}} = 800\,\mathrm{N}$, $T_{\mathrm{D}} = 0$. Alternatively, we may just "see" that G, weight 600 N and positioned 0.5 m to the left of C, balances perfectly with the weight of the bar, 200 N, acting 1.5 m to the right of C, so that the rope at D is superfluous; $T_{\mathrm{D}} = 0$, and so $T_{\mathrm{C}} = 800\,\mathrm{N}$.

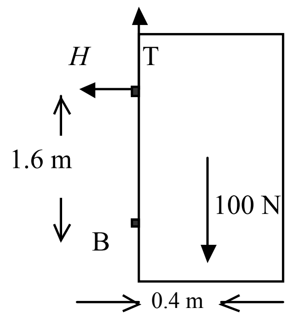
9. **(a)** The centre of gravity of the books is 0.5 m from A, so taking moments about A, $0.5 \times 60 = 2 \times F_{\mathrm{B}}$, $F_{\mathrm{B}} = 15\,\mathrm{N}$. The force balance $F_{\mathrm{A}} + F_{\mathrm{B}} = 60$ now gives $F_{\mathrm{A}} = 45\,\mathrm{N}$.

   (b) Note that x metres of books should weigh $60x$ newtons. So the diagram of forces on the shelf will look like this:



   Taking moments about B, $(2-\frac{1}{2}x)\times 60x = 2\times 48$, giving a quadratic equation $120x - 30x^2 = 96$. This simplifies to $5x^2 - 20x + 16 = 0$, with a physical solution (x must be less than 2) of $x = \frac{2}{5}(5 - \sqrt{5}) \approx 1.11$.

10. From the information given, the distance between the bottom hinge B and the top hinge T is 1.6 metres.



If the required horizontal force (exerted *by* the door *on* the hinge, which is equal and opposite to that exerted *by* the hinge *on* the door) is H, taking moments about B gives $H \times 1.6 = 100$ (weight of door) $\times\, 0.4$, $H = 25\,\mathrm{N}$. And considering the balance of forces horizontally, the horizontal force at B must be 25 N exerted outwards *by* the door.

11. **(a)** Let the length of the ladder be L, while R is the reaction force from the wall on the ladder. Taking moments about the bottom of the ladder, $80gL\sin\theta + 20g \cdot \frac{1}{2}L\sin\theta = RL\cos\theta$, so that $R = 90g\tan\theta$.

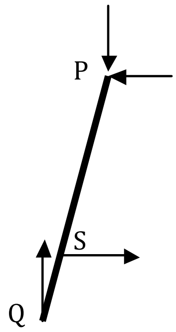
    (b) Resolving horizontally, $R = F_f$, where F_f is the frictional force from the ground. Resolving vertically, $N = 80g + 20g = 100g$, where N is the normal reaction from the ground. From the law of friction, $F_f \le \mu N$, $\mu = 0.5$, $90g\tan\theta \le 0.5 \times 100g$, $\tan\theta \le 5/9$, $\theta \le 29°$.

    (c) If Mr C, mass 60 kg, stands on the bottom rung of the ladder, R is unchanged, but N increases to $160g$. Limiting condition on θ becomes $90g\tan\theta \le 0.5 \times 160g$, $\tan\theta \le 8/9$, $\theta \le 42°$.

12. **(a)** Let the force exerted by the cat be F newtons. Taking moments about the hinge, $F \times 0.1 = 20$ (weight of cat flap) $\times\, 0.1\cos(30°)$, $F = 17.3\,\mathrm{N}$.

    (b) Let the horizontal and vertical components of the force on the hinge be H and V respectively. Resolving horizontally, $H = F\cos(60°) = 8.66\,\mathrm{N}$. Resolving vertically, $F\cos(30°) + V = 20$, $V = 5\,\mathrm{N}$.

13. The easiest way to solve this is to consider the forces on just one side of the step ladder, say PQ. These are a vertical force downwards from Mr B of 392 newtons, acting at P (we suppose the burden of Mr B's weight to be shared equally between the two sides of the ladder), the reaction force from the other side of the step ladder, also acting at P, the reaction upwards from the floor at Q, and the tension in the string at S.



Resolving vertically, reaction at Q, $R_{\mathrm{Q}} = 392\,\mathrm{N}$. Taking take moments about P, $392 \times 1.5\sin(20°) = T\,(\text{tension}) \times 1.25\cos(20°)$, $T = 171\,\mathrm{N}$.

14. Let L be the length from Mr B's feet to the line of the rope through his body (as shown by the dotted line in the diagram in the question). Taking moments about his feet:

$$\begin{aligned} 80gL\cos(55°) &= TL\sin(55°) \\ \Rightarrow T &= 80g\cot(55°) \\ &= 550\,\mathrm{N}\,. \end{aligned}$$

2.7 Centres of gravity

1. Measure distances from $x = 0$ at the tip of the bat to $x = 25$ at the base of the handle. The centre of gravity of the bat section is at its centre and of the handle, if we suppose it to be uniform, half way along its length. Then if the combined C of G is at $\bar{x}$,

$$(90 + 60)\bar{x} = 90 \times 7 + (14 + 5.5) \times 60 \ ,$$

giving $\bar{x} = 12$.

2. **(a)** Divide the kite along the y-axis into two triangles. The larger triangle has an area of 3 units, and, we shall suppose, a mass of 3 units, and its centre of gravity, one third of the way up from its "base" on the y-axis, is at $x = -1$. The smaller triangle has area 1 unit, mass 1 unit, and its C of G is at $x = 1/3$. **(b)** If the added mass at $x = 1$ is m, and the kite balances about a centre of gravity at the origin,

$$0 = (3 \times -1) + (1 \times 1/3) + (m \times 1) \, ,$$

and $m = 22/3$. Measuring x from 0:

$$\begin{aligned} (3+1)\bar{x} &= 3 \cdot (-1) + 1 \cdot 1/3 \\ \Rightarrow \bar{x} &= \frac{1}{4}\left(\frac{1}{3} - \frac{9}{3}\right) \\ &= \frac{1}{4} \cdot \frac{-8}{3} \\ &= -\frac{2}{3} \, . \end{aligned}$$

3. Rectangle has area 2, C of G is at $(1/2, 2)$. Triangle has area $1/2$, C of G at $(1/3, 2/3)$. Total area $= 3/2$, C of G has x co-ordinate $= [(2\times 1/2)+(1/2\times 1/3)] \div 3/2, = 7/15$, y co-ordinate, $[(2 \times 2) + (1/2 \times 2/3)] \div 3/2 = 26/15$.

4. Final C of G has x co-ordinate $[(300 \times 0.5) + (100 \times 1.3)]/400 = 0.7$, y co-ordinate $[(300 \times -1) + (100 \times -1.8)]/400 = -1.2$.

5. The difficulty with the shape considered here is that it cannot be built up in the usual way by combining standard shapes. But there are two ways of proceeding. Option 1 is to argue that the actual plate (that is the square, less the missing circle), plus the circle, together make a complete square. Taking co-ordinates with an origin at the centre, the calculation can be laid out in the following table:

	Actual plate	Circle	Complete square
m	$100 - 4\pi$	4π	100
y	$\bar{y}$	2	0
my	$(100 - 4\pi)\bar{y}$	8π	0

giving $(100 - 4\pi)\bar{y} + 8\pi = 0$, $\bar{y} = -0.29$.

Option 2 is to argue that a complete square, plus an "anti-matter" circle of negative mass, combine to make the actual plate. The table for the corresponding calculation is:

	Complete square	Circular hole	Actual plate
m	100	-4π	$100 - 4\pi$
y	0	2	$\bar{y}$
my	0	-8π	$(100 - 4\pi)\bar{y}$

giving $-8\pi = (100 - 4\pi)\bar{y}$, and of course the same solution $\bar{y} = -0.29$. Notice that the centre of gravity is a little below the centre of the square, which is to be expected since the missing mass comes from the region with positive y.

Of course, because of the symmetry, $\bar{x} = 0$.

6. For "E", y co-ordinate of C of G is, by symmetry, 2.5. Taking the "E" as made of a vertical stem, plus three horizontal branches, x co-ordinate is $[(5 \times 0.5) + (2 \times 2) + (1 \times 1.5) + (2 \times 2)]/10 = 1.2$.

   For "N", C of G is by symmetry at centre point, $(1.75, 2.5)$.

   For "P", consider the letter as made of a vertical stem, area 5 units, C of G at $(0.5, 2.5)$, a positive semi-circle, area $1/2 \times \pi \times (1.5)^2$, C of G at $(1 + \frac{2}{\pi}, 3.5)$, and a negative semi-circle, area $1/2 \times \pi \times (0.5)^2$, C of G at $(1 + \frac{2}{3\pi}, 3.5)$.

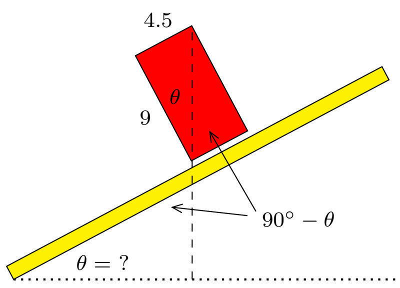
   Combined C of G is at

$$\begin{aligned} x &= \frac{(5 \times 0.5) + [(1/2 \times \pi \times (1.5)^2) \cdot (1 + 2/\pi)] - [(1/2 \times \pi \times (0.5)^2) \cdot (1 + 2/3\pi)]}{5 + (1/2 \times \pi \times (1.5)^2) - (1/2 \times \pi \times (0.5)^2)} \\ &= \frac{41/6 + \pi}{5 + \pi} \\ &= 1.23\ , \end{aligned}$$

   and

$$\begin{aligned} y &= \frac{(5 \times 2.5) + [(1/2 \times \pi \times (1.5)^2) \times 3.5] - [(1/2 \times \pi \times (0.5)^2) \times 3.5]}{5 + (1/2 \times \pi \times (1.5)^2) - (1/2 \times \pi \times (0.5)^2)} \\ &= \frac{12.5 + 3.5\pi}{5 + \pi} \\ &= 2.89\ . \end{aligned}$$

7. In the borderline case, a vertical line through the centre of the brick passes through its bottom corner.



   Finding the inclination of the plank in this case is an exercise in chasing angles round the diagram. It helps, as previously mentioned, if your sketch makes the distinction between θ and $90°-\theta$ clear. Here we see the critical angle θ is $\arctan(4.5/9) = 26.6°$.

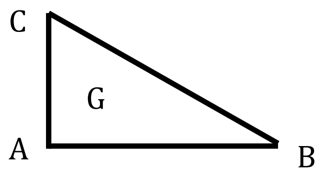
8. **(a)** Each of the component rods AB, BC, CA has its centre of gravity at its centre. Setting up a table for the centre of gravity calculation, with an origin at A, we have

	AB	BC	CA	Whole framework
M	40	50	30	120
x	20	20	0	$\bar{x}$
Mx	800	1000	0	$120\bar{x}$
y	0	15	15	$\bar{y}$
My	0	750	450	$120\bar{y}$

This gives us $120\bar{x} = 1800$, $\bar{x} = 15$ and $120\bar{y} = 1200$, $\bar{y} = 10$.

(b) The angle BAG is $\arctan(\bar{y}/\bar{x}) = \arctan(2/3) = 33.7°$.

(c) When the framework is suspended from corner A, G is vertically below A. The angle between AB and the downward vertical is just the angle between AB and AG, which is 33.7°.



(d) If instead the framework is suspended from C, we calculate the angle ACG = $\arctan(\bar{x}/(\text{CA} - \bar{y})) = \arctan(1.5/2) = 36.9°$. This will be the angle by which CA departs from the vertical and AB, perpendicular to CA, will depart from the horizontal by the same angle.

9. For the lamina to hang in equilibrium we must have X $= \bar{x}$. Therefore suppose the point X is a distance $\bar{x}$ from the 'free' end of the square. The square has area 100 sq. cm, equivalent we shall say to a mass of 100 units, and the hemispherical portion of the lamina has area $\frac{1}{2}\pi \times 5^2 = 12.5\pi$ and its centre of gravity is at a distance ($\frac{4}{3\pi} \times$ radius of 5) from its diameter. Then

$$(100 + 12.5\pi)\bar{x} \;=\; (100 \times 5) + 12.5\pi \left(10 + \frac{20}{3\pi}\right) ,$$

giving $\bar{x} = 7.01\,\mathrm{cm}$.

10. **(a)** $\tan\alpha = 0.4/0.6$, $\alpha = 33.7°$.

(b) The added mass M is best positioned at corner D so that the moment of its weight about corner C balances the moment of the weight of the crate acting through G. The perpendicular distance from C to the line of action of the weight Mg is CD $\cos(45°)$, and the perpendicular distance of the line of action of the weight $8g$ is GX $\cos(45°)$ − CX $\cos(45°)$, where X is the mid-point of CD. With CD $= 0.4\,\mathrm{m}$, GX $= 0.3\,\mathrm{m}$, CX $= 0.2\,\mathrm{m}$ and

$$8g(0.3\cos(45°) - 0.2\cos(45°)) \;=\; Mg(0.4\cos(45°)) ,$$

giving $M = 2$. The calculation assumes that the mass M is a particle of zero size, or a rod of zero thickness running perpendicular to the plane of the diagram.

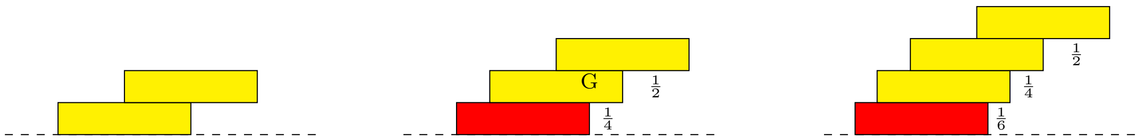
An alternative method of approach is to separate the weights of the crate and the added mass into components $8g\sin(45°) = 8g/\sqrt{2}$ and $Mg\sin(45°) = Mg/\sqrt{2}$ down the slope and $8g\cos(45°) = 8g/\sqrt{2}$ and $Mg\sin(45°) = 8g/\sqrt{2}$ into the slope. Taking moments about C (recall that moment = force × *perpendicular* distance) then gives $8g/\sqrt{2} \times \mathrm{CE} = (8g/\sqrt{2} \times \mathrm{CX}) + (Mg/\sqrt{2} \times \mathrm{CD})$, and again $M = 2$.

11. **(a)** By symmetry, the centre of gravity of the hollow cube will be at its centre.

(b) If the cube has side a, and the missing face is say at the top, then the cube consists of four sides each of mass say m and C of G at a height $a/2$, and a base of mass m with its C of G at ground level. The combined C of G will stay directly above the centre of the base and its height will be $\bar{y}$ where $5m\bar{y} = (4m \times a/2) + (m \times 0)$, $\bar{y} = 2a/5$.

12. Let A be the corner of the cube from which it is suspended. Let B be one of the three corners nearest to A. Let C be the corner of the cube diametrically opposite to A. Then AC $= \sqrt{3}$AB, BC $= \sqrt{2}$AB. The centre of gravity of the cube G lies on AC and when the cube is suspended, AC will be vertical. The angle between AB and AC is $\arccos(\mathrm{AB}/\mathrm{AC}) = \arccos(1/\sqrt{3}) = 54.7°$. The same argument applies to each of the three sides of the cube which meet at A, and also to any of the other sides, each of which is parallel to one or other of the first three.

13. **(a)** In non-standard problems like this it often pays to consider the very simplest case. What would be the distance x that we could achieve with just two dominoes? Clearly, the upper domino would protrude beyond the lower one by exactly one half of a domino length, so that its centre of gravity remained supported. Now, let us take these two dominoes as a single entity, and place them on top of a third domino, so that their combined centre of gravity G still remains supported. This gives us an extra extension of one quarter of a domino, so the required distance $x = \frac{3}{4}$.



(b) Where now is the centre of gravity of these three dominoes taken together? The top two dominoes, mass $2m$, have their C of G at a distance $\frac{3}{4}$ measured from the right hand end of the top domino. The bottom domino, mass m, has its C of G at a distance $1\frac{1}{4}$. The combined C of G is located at a distance x where $3mx = (2m \times \frac{3}{4}) + (m \times 1\frac{1}{4})$, giving $x = \frac{11}{12}$. If these three dominoes in their same relative positions are placed on top of a fourth, the span attained is therefore $x = \frac{11}{12}$, a further increment of $\frac{1}{6}$ compared with **(b)**.

(d) Now it is a question of repeating the same process. The C of G of these four dominoes, taken together, will be at distance x where $4mx = (3m \times \frac{11}{12}) + (m \times \frac{15}{12})$, $x = \frac{11}{24}$. The span attained is $x = \frac{11}{24}$, an increment of $\frac{1}{8}$ compared with **(c)**.

(e) With an infinite number of dominoes, we expect a span of $x = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \ldots$, a series which pure mathematics tells us sums to infinity.

2.8 Momentum/Impulse/Collisions

1. Using $m_1u_1+m_2u_2 = m_1v_1+m_2v_2$, $(0.15\times0.5)+(0.15\times0) = (0.15\times v_1)+(0.15\times0.45)$, so v_1 = velocity of A after impact = 0.05 m/s in the original direction of motion.

2. **(a)** $0\,\mathrm{m/s}$ **(b)** $4\,\mathrm{kg}$ **(c)** $U \leftarrow$ **(d)** $2\frac{1}{2}U \rightarrow$ **(e)** $3M$ **(f)** $2M$.

3. $(M \times 3) + (M + 2000) \times 0 = (M + M + 2000) \times 1$. Solve to find $M = 2000\,\mathrm{kg}$.

4. Impulse $I = mv - mu$, and if the upwards direction is taken as positive, $v = 3\,\mathrm{m/s}$, $u = -5\,\mathrm{m/s}$, $I = (0.45 \times 3) - (0.45 \times -5) = 3.6\,\mathrm{N\,s}$.

5. If v is speed of rebound, $0.075\,\mathrm{N\,s} = I = mv - mu = 0.15(v - (-0.3))$, $v = 0.2\,\mathrm{m/s}$.

6. For the aeroplane, $u = 270\,\mathrm{km/hr} = 75\,\mathrm{m/s}$, $v = 37.5\,\mathrm{m/s}$, $I = Ft = mv - mu = 250\times1000\times(37.5-75) = -9{,}375{,}000\,\mathrm{N\,s}$, $F = I/t = -9{,}375{,}000/10 = -937{,}500\,\mathrm{N}$, reverse thrust $\approx 940\,\mathrm{kN}$.

7. Consider a time interval of one second during which 135 kg of fuel would have been burnt. The backwards momentum imparted to the combustion products would be mass $\times$ change in velocity $= 135 \times 2000 = 270{,}000\,\mathrm{N\,s}$ and this must balance the forwards momentum imparted to the rocket. The impulse $I = Ft$ exerted on the rocket in the 1 second time interval is therefore 27,000 N s and the propulsive force is 270 kN.

8. $m_\mathrm{A}u_\mathrm{A} + m_\mathrm{B}u_\mathrm{B} = m_\mathrm{A}v_\mathrm{A} + m_\mathrm{B}v_\mathrm{B}$ for the first collision gives $(100 \times 0.3) + (200 \times 0) = (100 \times v_\mathrm{A}) + (200 \times 0.2)$, $v_\mathrm{A} = -0.1$, so the speed of A is 0.1 m/s and its direction of motion has been reversed.

   $m_\mathrm{B}u_\mathrm{B}+m_\mathrm{C}u_\mathrm{C} = m_\mathrm{B}v_\mathrm{B}+m_\mathrm{C}v_\mathrm{C}$ for the second collision gives $(200\times0.2)+(400\times0) = (200 \times v_\mathrm{B}) + (400 \times 0.1)$, $v_\mathrm{B} = 0$, so B is brought to rest.

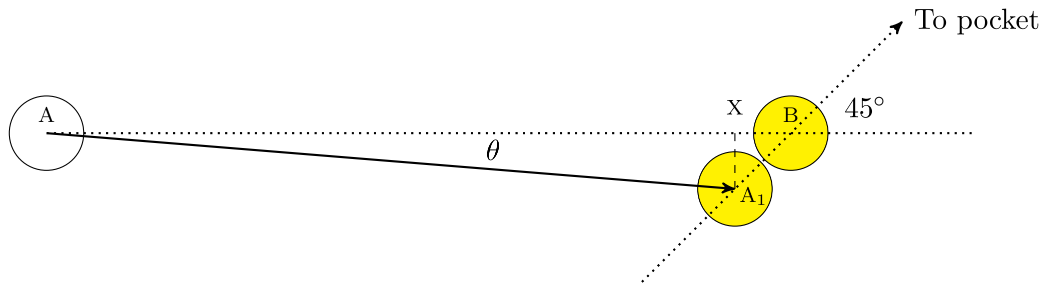
   Total momentum at start $= (100\times0.3)+(200\times0)+(400\times0) = 30$ units $= 0.03\,\mathrm{N\,s}$ (particle masses are measured in grams). Total momentum after all collisions $= (100 \times -0.1) + (200 \times 0) + (400 \times 0.1) = 30$ units $= 0.03\,\mathrm{N\,s}$. There will be no more collisions since A and C are moving in opposite directions and B is at rest between them.

9. After A hits B, B has to travel 0.2 m towards C at a speed of 0.2 m/s, taking a time of 1 second. During this time interval A, with speed 0.1 m/s in the opposite direction, will travel 0.1 m. The separation of A and B will therefore be $0.1 + 0.2 = 0.3\,\mathrm{m}$.

10. **(a)** $v^2 = 0^2 + 2as$, $s = 2$, $v = 6.26\,\mathrm{m/s}$. **(b)** $mv - mu = (0.06 \times 0) - (0.06 \times 6.26) = -0.376\,\mathrm{N\,s}$, change in momentum $\approx 0.38\,\mathrm{N\,s}$ in magnitude. **(c)** If top of egg continues its downward motion after the bottom hits the floor, additional time required $\approx$ distance/speed $= 0.05/6.26 = 0.00799\,\mathrm{s} \approx 8\,\mathrm{ms}$. **(d)** $Ft = I$ = change in momentum, $F \times 0.00799 = 0.376$, $F = 47\,\mathrm{N}$.

11. After the first collision, the initial momentum of truck 1 is shared with truck 2 and they move together at speed $5/2 = 2.5\,\mathrm{m/s}$. After the next collision, trucks 1, 2 and 3 move together with speed $5/3\,\mathrm{m/s}$, after which trucks 1, 2, 3 and 4 move with speed $5/4\,\mathrm{m/s}$, and so on. This implies that the final speed, $V_{\mathrm{final}} = 5/10\,\mathrm{m/s}$. After the first collision between trucks 1 and 2, the time taken to cover the 10 m distance to truck 3 is $10/(5/2) = 4\,\mathrm{s}$. Trucks 1, 2 & 3 then take $10/(5/3) = 6$ seconds to reach truck 4. Trucks 1, 2, 3 & 4 take $10/(5/4) = 8\,\mathrm{s}$ to reach truck 5, and so on until trucks 1–9 take $10/(5/9) = 18\,\mathrm{s}$ to reach truck 10. The total time between the first collision and the last is the sum of the arithmetical progression $4+6+8+10+12+14+16+18 = \frac{1}{2} \times 8 \times (4+18) = 88$ seconds.

12. This problem has more to do with geometry than momentum. It helps to draw $\mathrm{A_1X}$ perpendicular to AB in the diagram.



The required angle $\theta = \mathrm{BAA_1}$ has $\tan\theta = \mathrm{A_1X/AX} = \mathrm{A_1B}\sin(45°)/(\mathrm{AB} - \mathrm{A_1B}\cos(45°)) = (2 \times 0.026 \sin(45°)/(2 - 2 \times 0.026\cos(45°)) = 0.01873$, $\theta = 1.073°$. And if instead ball B is directed at an angle 44° or 46°, instead of 45°, the solutions for θ are 1.054° and 1.091°. So if the margin of error in the direction of B is 1°, the margin error in the direction of A is about 0.02°.

Coefficients of restitution

13. Using $m_\mathrm{S}u_\mathrm{S} + m_\mathrm{T}u_\mathrm{T} = m_\mathrm{S}v_\mathrm{S} + m_\mathrm{T}v_\mathrm{T}$, $(0.15 \times 1) + (0.15 \times 0) = (0.15 \times v_\mathrm{S}) + (0.15 \times 0.98)$, so v_S = velocity of S after impact $= 0.02\,\mathrm{m/s}$ in the original direction of motion. Coefficient of restitution $e = (v_\mathrm{T} - v_\mathrm{S})/(u_\mathrm{S} - u_\mathrm{T}) = (0.98 - 0.02)/(1.00 - 0.00) = 0.96$.

14. **(a)** Using $m_\mathrm{A}u_\mathrm{A} + m_\mathrm{B}u_\mathrm{B} = m_\mathrm{A}v_\mathrm{A} + m_\mathrm{B}v_\mathrm{B}$, $(0.15\times3)+(0.15\times-2) = (0.15\times-1.8)+(0.15\times v_\mathrm{B})$, so v_B = velocity of B after impact $= 2.8\,\mathrm{m/s}$ in the original direction of motion of A. **(b)** Separation speed $= 2.8 - (-1.8) = 4.6\,\mathrm{m/s}$. **(c)** Approach speed $= 2 - (-3) = 5\,\mathrm{m/s}$, coefficient of restitution $= 4.6/5 = 0.92$.

15. From momentum conservation, $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$, $(4000\times2)+(6000\times0) = 4000v_1 + 6000v_2$, which cancels to $2v_1 + 3v_2 = 4$, and from the restitution equation $v_2 - v_1 = 2e = 1.5$. Solving simultaneously gives $v_1 = -0.1\,\mathrm{m/s}$, $v_2 = 1.4\,\mathrm{m/s}$. Impulse on T2 = gain in momentum of T2 $= 6000 \times 1.4 = 8400\,\mathrm{N\,s}$.

16. **(a)** From momentum conservation, $(m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2)$, $MU = Mv_1 + 2Mv_2$, which cancels to $v_1 + 2v_2 = U$, and from the restitution equation, if $e = 1$, $v_2 - v_1 = U$. Solving simultaneously gives $v_1 = -U/3$, $v_2 = 2U/3$. **(b)** If $e = 0$ then $v_2 - v_1 = 0$ and $v_1 = v_2 = U/3$. **(c)** The case of general e requires the simultaneous solution of $v_1 + 2v_2 = U$, and $v_2 - v_1 = eU$, giving $v_1 = (1-2e)U/3$, $v_2 = (1+e)U/3$. The first truck continues in its original direction after the collision, that is to say v_1 is positive, if $v_1 = (1-2e)U/3 > 0$, *i.e.* $e < \frac{1}{2}$.

17. This question does not specify the initial speed of ball A. We do not need to know this, since it does not affect the location of the second collision, only the time taken before it happens. For convenience in our working we can call it U, but expect that it will cancel out during the course of the working. The first step is to apply the usual momentum conservation and restitution equations to the initial collision of A and B, as in *Questions 15* and *16.* The speeds of A and B after the first collision turn out to be $0.025U$ and $0.975U$ respectively, both in the original direction of A. B will now travel 0.5 m towards the cushion before rebounding with speed $0.5 \times 0.975U$ to meet A again. Suppose A travels a further distance x in an additional time t before the second collision with B. Applying time = distance/speed to the motion of A gives

$$t = \frac{x}{0.025U} \ .$$

In the same time interval B accomplishes two journeys, to the cushion and back to meet A, so

$$t = \frac{0.5}{0.975U} + \frac{(0.5 - x)}{(0.5 \times 0.975U)} \ .$$

Equate the two formulæ for t, and multiply through by a factor $0.975U$:

$$\frac{0.975x}{0.025} = 0.5 + \frac{(0.5 - x)}{0.5} \ .$$

Notice how U has disappeared as we expected. Simplifying further, $39x = 1.5 - 2x$, $x = 0.037$, so ball A travels 3.7 cm. All this assumes that balls A and B are ideal particles; a more detailed calculation would have to make allowance for their size (5.2 cm diameter).

18. Before impact, the components of the velocity of B are $0.1\cos(30°) = 0.0866\,\mathrm{m/s}$ parallel to the cushion and $0.1\sin(30°) = 0.05\,\mathrm{m/s}$ perpendicular to the cushion. After impact, the velocity component perpendicular to the cushion is $e \times 0.05 = 0.04\,\mathrm{m/s}$. **(a)** The angle at which the ball rebounds is $\arctan(0.04/0.0866) = 24.8°$. **(b)** Speed after impact $= \sqrt{(0.042 + 0.08662)} = \sqrt{0.0091} = 0.095\,\mathrm{m/s}$. **(c)** The impulse on the cushion is the same in magnitude as the change in momentum of the ball, in the direction perpendicular to the cushion, $mv - mu = (0.15 \times 0.04) - (0.15 \times -0.05) = 0.0135\,\mathrm{N\,s}$.

19. **(a)** Using the method of *Question 18*, velocity components after bouncing are $23.9\,\mathrm{m/s}$ parallel to ground and $5.12\,\mathrm{m/s}$ perpendicular to ground, speed $= 24.5\,\mathrm{m/s}$ at an angle $12.1°$ to the horizontal. **(b)** Time required to travel $3\,\mathrm{m}$ horizontally to the stumps is $t = 3/23.9 = 0.125\,\mathrm{s}$, height after time t is $s = ut - \frac{1}{2}gt^2 = (5.12 \times 0.125) - \frac{1}{2} \times 9.8 \times (0.125)^2 = 0.565\,\mathrm{m}$, so the ball just clips the top of the stumps, height $0.6\,\mathrm{m}$.

20. **(a)** If $e = 1$, the ball is reflected perfectly at each impact with the cushion. Tracing the angles round the diagram demonstrates that the exit velocity is in the reverse direction to the incoming velocity.

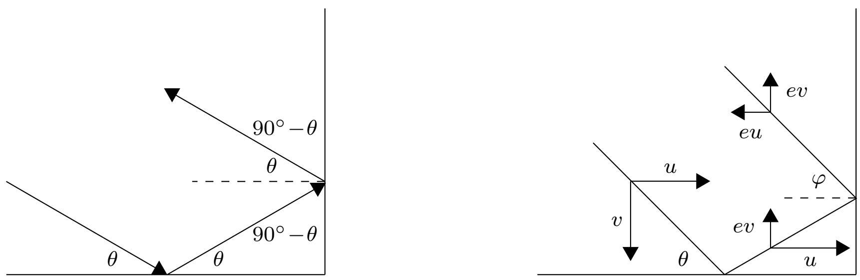


Figure 2.4: *Left:* $e = 1$ *case. Right:* $e \neq 1$ *case.*

(b) When the coefficient of restitution is not unity, velocity components perpendicular to the cushion are reduced by a factor e at each impact. Here, $\tan\varphi = (ev)/(eu) = v/u = \tan\theta$, so again the final direction is exactly reversed even though the intermediate angles are changed.

(c) $V_{\text{final}} = \sqrt{e^2u^2 + e^2v^2} = e\sqrt{u^2 + v^2} = eV_{\text{initial}}$.

21. Consider first the collision of truck 1 with truck 2. With an initial speed of 5 m/s for truck 1, the momentum equation is $5m = mv_1 + mv_2$, which cancels to $v_1 + v_2 = 5$, and from the restitution equation $v_2 - v_1 = 5e = 2.5$. Solving, $v_1 = 1.25$, $v_2 = 3.75$ m/s. The outgoing speed of truck 2 is reduced by a factor $\frac{3}{4}$ compared with the incoming speed of truck 1. Similarly, when truck 2 hits truck 3, the outgoing speed of truck 3 will be reduced by a factor $\frac{3}{4}$ compared with the incoming speed of truck 2, or by a factor $(\frac{3}{4})^2$ compared with the original 5 m/s imparted to truck 1. The same relationship applies all the way down to truck 9 which moves off towards truck 10 with speed $(\frac{3}{4})^8 \times 5$ m/s. Truck 10 eventually moves off with speed $(\frac{3}{4})^9 \times 5 = 0.375$ m/s.

    With a 10 m spacing between trucks, the time required for truck 2 to reach truck 3, after the first collision, is $10/(3/4 \times 5) = 2 \times \frac{4}{3}$ seconds. The additional time required for truck 3 to reach truck 4 is $10/((3/4)^2 \times 5) = 2 \times (\frac{4}{3})^2$, and similarly for subsequent trucks, truck 9 requiring time $10/((3/4)^8 \times 5) = 2 \times (\frac{4}{3})^8$ seconds to reach truck 10. The total time between the first collision and the last is therefore

$$2 \times \left[\left(\frac{4}{3}\right) + \left(\frac{4}{3}\right)^2 + \left(\frac{4}{3}\right)^3 + \ldots + \left(\frac{4}{3}\right)^8\right] .$$

    This is a geometrical progression which, says pure mathematics, sums to

$$2 \times \frac{4}{3} \times \frac{\left[\left(\frac{4}{3}\right)^8 - 1\right]}{\left(\frac{4}{3} - 1\right)} = 71.9 \text{ seconds} .$$

2.9 Energy/Work/Power

1. Suppose the mass of the meteorite was m kg. Then $\frac{1}{2}mv^2 = \frac{1}{2}m(25{,}000)^2 = 15 \times 4 \times 10^{15}$, $m = 1.92 \times 10^8$. Mass = volume × density and if r is radius, $\frac{4\pi}{3}r^3 \times 920 = 1.92 \times 10^8$, $r = 36.8\,\text{m} \approx 40$ metres to 1 s.f.

2. In the absence of friction, as given, and also of air resistance, energy is conserved throughout the complicated path of the snowball. Kinetic energy gained $\frac{1}{2}mv^2 =$ potential energy lost mgh. m cancels, $v = \sqrt{2gh} = \sqrt{(2 \times 9.8 \times 112)} = 46.9 \approx 47\,\text{m/s}$.

3. **(a)** Conservation of energy $\Rightarrow \frac{1}{2}m(20)^2 + 0 = \frac{1}{2}mv^2 + 0$, so $v = 20\,\text{m/s}$. **(b)** Also by the conservation of energy, initial KE + initial PE = KE at greatest height + PE at greatest height, $\frac{1}{2}m(20)^2 + 0 = \frac{1}{2}mv^2 + (m \times 9.8 \times 10)$, $v = 14.3\,\text{m/s}$. The figure for distance to the boundary is a red herring, not needed for the calculation.

4. If vaulter has mass m, KE $= \frac{1}{2}mv^2 = \frac{1}{2}m(10)^2 =$ elastic energy = potential energy $= mgh = m \times 9.8 \times h$, $h = 5.1$ metres. A height greater than this perhaps ought to be achieved since at take-off the centre of gravity of the vaulter is above ground level, but below the bar as he clears it.

5. Potential energy gained by block (+ bullet) $= 3.003 \times 9.8 \times (1 - \cos(10°))$ = kinetic energy acquired from impact of bullet $= \frac{1}{2}mv^2 = \frac{1}{2} \times 3.003 \times v^2$, speed v acquired by block $= 0.546\,\text{m/s}$, and by conservation of momentum for impact of bullet, $0.003 \times V + 0 = 3.003 \times 0.546$, speed of bullet $V \approx 550\,\text{m/s}$.

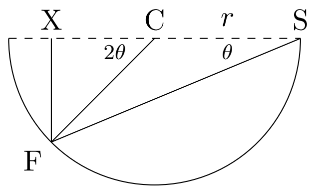
6. **(a)** Work done by frictional force $= F_f \times d =$ loss of KE of car $= \frac{1}{2}mv^2 - \frac{1}{2}mu^2 = \frac{1}{2} \times 800 \times (15^2 - 10^2) = 50{,}000\,\text{J} = 50\,\text{kJ}$. **(b)** Distance $d = 8\,\text{m}$, $\Rightarrow F_f = 50{,}000/8 = 6250\,\text{N}$. **(c)** $F_f = \mu mg = \mu \times 800 \times 9.8$, $\mu = 6250/(800 \times 9.8) = 0.80$.

7. From the geometry, SM = 36.62 m, MF = 31.32 m, SM is inclined at an angle 35° to the horizontal and MF at 16.7°. If the skier has mass m kg the normal reaction force along SM will be $mg\cos(35°)$ and the work done vs. friction will be $\mu mg\cos(35°) \times \text{SM} = 0.1 \times 9.8m \times 0.8192 \times 36.62 = 29.40m$ joules. Similarly the work done vs. friction along MF will be $\mu mg\cos(16.7°) \times \text{SM} = 0.1 \times 9.8m \times 0.9578 \times 31.32 = 29.40m$ joules. Using the work-energy principle, KE at F $= \frac{1}{2}mv^2 =$ PE at S − work done vs. friction along SM − and work done vs. friction along MF $= (mg \times 30) - 29.40m - 29.40m = 235.2m$ J. Solving for velocity v, $v = 21.7\,\text{m/s}$.

   A similar calculation applies for any other slope profile. Consider any part of the slope which falls uniformly through a height h in a horizontal distance d. The angle of this part of the slope is $\theta = \arctan(h/d)$ and its length is $d/\cos\theta$. The normal reaction force is $mg\cos\theta$ and the work done against friction is $\mu mg\cos\theta \times d/\cos\theta = \mu mgd$ independently of h or θ. If we consider a general profile SF to be made up of segments with possibly different slopes, covering horizontal distances d_1, d_2, d_3, *etc.*, the work done vs. friction will be $\mu mgd_1 + \mu mgd_2 + \mu mgd_3 + \ldots = \mu mgD$, where D is the total horizontal separation between S and F, and so is the same for any profile.

8. If the particle travels a distance x up the slope, the energy balance is: initial KE = PE gained + work done vs. friction, giving $\frac{1}{2}mV^2 = mgx\sin\theta + \mu mgx\cos\theta$. Solving for x gives the equation required. With $\theta = \arcsin(3/5)$ and $\mu = 0.5$, $x = V^2/2g$. Work done vs. friction going up and then down the slope $= \mu mg\cos\theta \times 2x = 0.5 \times mg \times 4/5 \times 2 \times V^2/2g = \frac{2}{5}mV^2 = 80\%$ of initial $\frac{1}{2}mV^2$. If $\mu > 0.75$, the frictional force is greater than the component of weight down the slope so the particle stays put having reached its maximum height.

9. The solution to this problem depends on the idea explored in *Question 7*, viz. that on sliding down a rough slope from S to F, falling through a height h and traversing a horizontal distance d, a particle of mass m loses potential energy mgh and does work against friction equal to μmgd, where μ is the coefficient of friction. This is true for a slope of any profile, curved or linear.



Here, if the particle starts at S and comes to rest at F, the KE at F is zero and the PE lost exactly equals the work done vs. friction, $mgh = \mu mgd$. If $\mu = \frac{1}{2}$, $d = 2h$. In the diagram, $\tan\theta = h/d = \frac{1}{2}$, $\tan 2\theta = 4/3$, $\text{FX} = r\sin 2\theta = 4r/5$, where r is the radius of the bowl. So F is at a height $r/5$ above the bottom of the bowl.

10. Power generated by Mr C $= mgh/T = (80 \times 9.8 \times 30 \times 0.17)/10 = 400$ watts.

11. Vertical height gained $= 40\sin(30°) = 20$ metres, potential energy gained per passenger $= 70 \times 9.8 \times 20$ joules $= 13{,}720$ J, power = energy/time $= 100 \times 13{,}720/60 = 22.9$ kW.

12. **(a)** If force from engine is F, $F - 250 = 800 \times 1.5 = 1200$, $F = 1450$ newtons. **(b)** Power = force × velocity $= 1450 \times 20 = 29{,}000$ watts $= 29$ kW.

13. **(a)** $P = Fv$, $50{,}000 = 1000v$, $v = 50$ m/s. **(b)** $P = Fv = 800g\sin(10°)v$, $v = 36.7$ m/s. **(c)** $P = Fv = (800g\sin(10°) + 500)v$, $v = 26.9$ m/s. **(d)** $P = Fv = (-800g\sin(2°) + 1500)v$, $v = 40.8$ m/s.

14. **(a)** At maximum speed, $8000 = P = Fv = 60v^2$, $v = \sqrt{8000/60} = 11.5$ m/s. **(b)** $8000 = P = Fv = (80g\sin(5°) + 60v)v$, giving a quadratic equation $60v^2 + 68.3v - 8000 = 0$, $v = 11.0$ m/s.

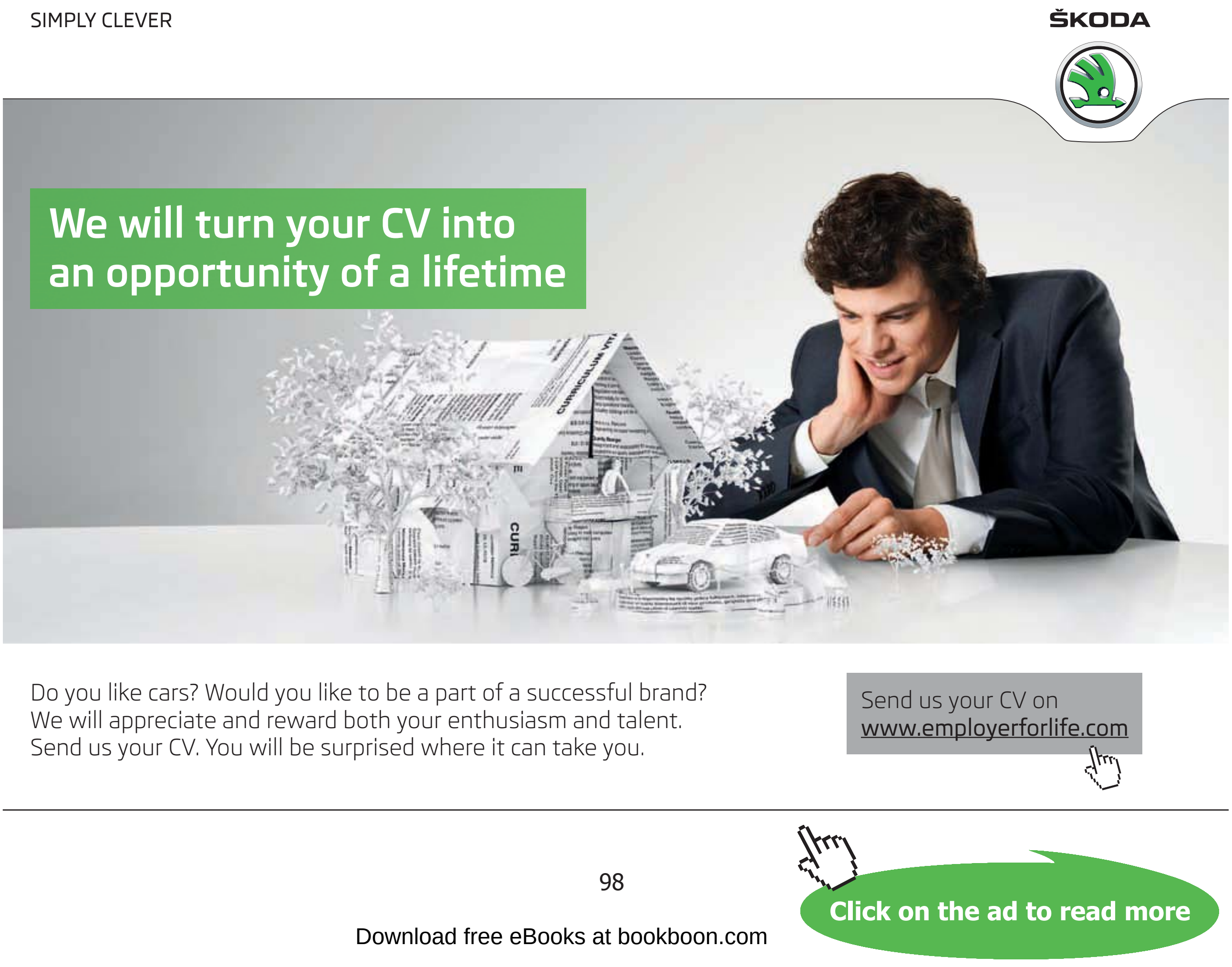
15. **(a)** $P = Fv = v(16 + \frac{1}{4}v^2)$, so putting $v = 6\,\text{m/s}$, $P = 150$ watts.

(b) If maximum speed is V m/s, $410 = V(16+\frac{1}{4}V^2)$, $1640 = 640+1000 = 64V+V^3$, solution is $V = 10$.

(c) For speed v, rate of working against gravity $= 70 \times 9.8 \times v \times \sin(5.74°) = 70 \times 9.8 \times v \times 0.1 = 68.6v$ watts. Rate of working against resistance $= v(16 + \frac{1}{4}v^2)$. For $v = 4$, total rate of working required $= 354$ watts, for $v = 5$ rate of working $= 454$ watts. If maximum power available $= 410$ watts, speed attainable is intermediate between these values.

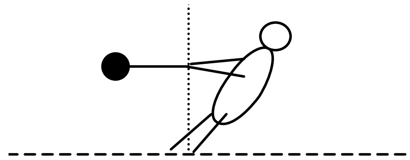
(d) On level ground with $v = 4$, resistance force $= (16+\frac{1}{4}v^2) = 20\,\text{N}$, force exerted by Mr E $= P/v = 102.5\,\text{N}$, $F = 102.5 - 20 = ma = 70a$, $a = 1.2\,\text{m/s}^2$.

(e) If Mr E freewheels, the component of his weight acting down the slope equals the resistance force at speed U, $70 \times 9.8 \times 0.1 = (16 + \frac{1}{4}U^2)$, $U = 15\,\text{m/s}$.



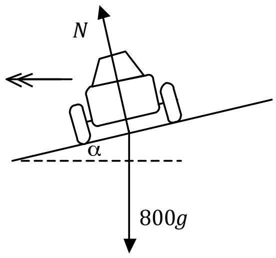
2.10 Motion in a circle

1. The thrower spins in the throwing circle in a leaning position, so that his body and the hammer rotate about a common axis.



A reasonable estimate of the distance of the hammer head from the axis might be one metre or a little more. The speed of the hammer can be estimated from the distance achieved. A world record 87 m throw corresponds to a speed V with $V^2/g = 87$, $V = 29\,\mathrm{m/s}$. The tension in the wire can be estimated as $MV^2/r = 7.26 \times 292/1 \approx 6000\,\mathrm{N}$.

2. **(a)** Resolving vertically, $N\cos\alpha = 800g$.



(b) Resolving horizontally, $N\sin\alpha = mv^2/r = 800 \times 202/250 = 1280\,\mathrm{N}$.

(c) Dividing equation **(b)** by equation **(a)**, $\tan\alpha = 1280/800g$, $\alpha = 9.3°$.

3. On the equator, any body which remains at a fixed position in relation to the ground is moving in a circle of radius 6370 km over a period of 24 hours. The acceleration towards the centre of the circle is

$$\left(\frac{2\pi \times 6370 \times 1000}{24 \times 3600}\right)^2 \div (6370 \times 1000) \;=\; 0.034\,\mathrm{m/s}^2\ .$$

To fall and reach the ground, an acceleration over and above the value is required, so the effective value of g is reduced by this same amount.

4. Call mass of conker m, length of string l, tension in string at top of circle T, speed of conker at top of circle v. By Newton II at top of circle, $mg + T = mv^2/l$, $T \geq 0$ if string remains taut $\Rightarrow v^2 \geq lg$. If speed of conker at bottom of circle is u, conservation of energy requires $\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + 2mgl \Rightarrow u^2 \geq 5lg$, so $u \geq 7\mathrm{m/s}$ when $l = 1$ metre.

5. The highest point on the aircrafts trajectory is $(0, 5000)$. The equation of this circle is $x^2 + y^2 = 5000^2$. Differentiating once $2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, differentiating twice $2 + 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$, giving values at $x = 0$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{1}{y} = -\frac{1}{5000} = -2 \times 10^{-4}$. These are the same as for the parabolic trajectory.

   If the circle is taken as an approximation to the path of the aircraft, the radius of curvature is 5000 metres. If the reaction force upwards from the pilot's seat on the pilot (mass 60 kg) is R, Newton's second law applied to the pilot gives $R - 588 = -mv^2/r = -60 \times 100^2/5000 = -120$, $R = 468\,\mathrm{N}$. At "zero g", the reaction force R will be zero, requiring a speed $\sqrt{5000g} = 220\,\mathrm{m/s}$.

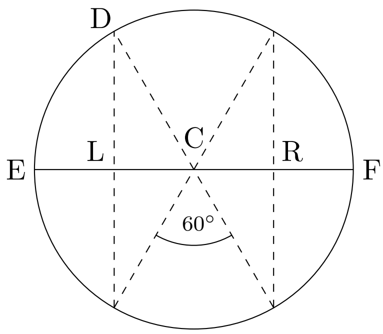
6. **(a)** Equating PE lost to KE gained, $mgL\cos\theta = \frac{1}{2}mv^2$, speed of monkey $= \sqrt{2gL\cos\theta}$.

   (b) Newton II applied to monkey, resolving along the inward radius, $T - mg\cos\theta = mv^2/L$, substituting for v gives tension T as $3mg\cos\theta$.

   The string breaks when the tension in it exceeds $12mg/5$, so $\cos\theta = 4/5$, $\theta = \arcsin(3/5)$. When $\theta = \arcsin(3/5)$, horizontal component of monkey's velocity is $v\cos\theta = \sqrt{(2gL\cos\theta)} \times \cos\theta = (4/5)\sqrt{(8gL/5)}$. Horizontal distance from monkey to left-hand tree is $L\sin\theta = 3L/5$, time taken to reach tree is $t = (3L/5) \div [(4/5)\sqrt{(8gL/5)}] = (3/8)\sqrt{(5L/2g)}$. Distance below O after this time interval t is $L\cos\theta + vt\sin\theta + \frac{1}{2}gt^2$ where $\theta = \arcsin(3/5)$. Substituting for θ and for t gives distance $= (1825/1280)L = (365/256)L$.

7. The period of oscillation for a simple pendulum is $T = 2\pi\sqrt{\frac{L}{g}}$ which varies inversely as $\sqrt{g}$. On the Moon, g is smaller, T is larger and it takes longer to complete each oscillation. The clock therefore runs slow. A true period of 1 hour is shown as only 1 hour $\times \sqrt{\frac{1.6}{9.8}} = 24$ minutes, so the hands show a time of 12:24.

8. **(a)** Let E and F be the extreme points of the window cleaner's oscillation, C the centre point, L and R the points at which he disappears from view. The two 1.5 sec. intervals for which he is in view constitute one third of the period of each complete oscillation, and correspond to two 60° arcs of the auxiliary circle. The full circle of 360° corresponds to a complete oscillation of period 9 seconds.



(b) The width of the window LR is 2 m, so CL = CR = 1 m. From the geometry, CL/CD = cos(60°) = $\frac{1}{2}$, CD = radius of auxiliary circle = 2CL = 2 m, amplitude of swing = CE = CF = radius = 2 m.

(c) Maximum speed = speed of motion round auxiliary circle = $(2\pi \times 2)/9 =$ 1.4 m/s.

(d) Given $T = 2\pi\sqrt{L/g} = 9$, $L =$ estimated the vertical distance to the top of the building = 20 m.

9. If the time for one complete revolution (the period of rotation) is $T\,(=1/f)$, then a point at a distance r from the axis of rotation will have a speed of $v = 2\pi r/T = 2\pi r f = r\omega$, where $\omega = 2\pi f$.

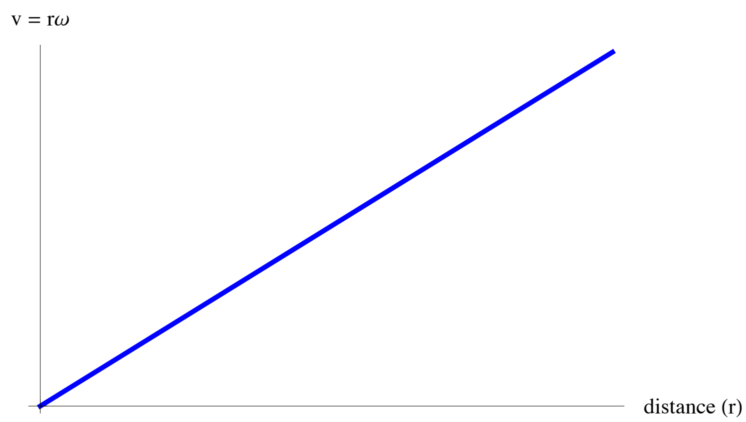
   (a) $v = (r_0 + d)\omega = (2 + 0.25) \times 2\pi \times 5 = 70.7\,\mathrm{cm/s}$.

   (b) $v = r_0\omega = 2 \times 2\pi \times 5 = 62.8\,\mathrm{cm/s}$.

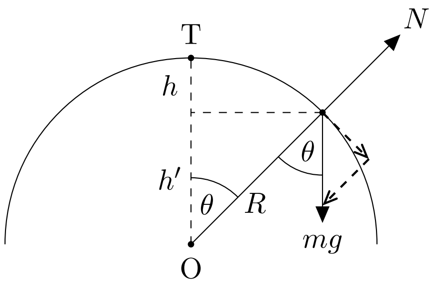
   (c) The speed of delivery will clearly decrease over the lifetime of the reel (a reduction of approximately 10% in this case) if it is revolved at constant frequency.

   (d) $v = 2\pi f r = r\omega$ as above.

   (e) Following **(d)**, the velocity varies linearly with distance from rotation axis:



10. **(a)** $h' = R\cos\theta$, so $h = R - h' = R(1-\cos\theta)$.



(b) Energy of X at T = energy of X at angle θ, so $mgR = \frac{1}{2}mv^2 + mgh'$ and $v^2 = 2gR(1-\cos\theta)$.

(c) Motion in a circle so use "v^2/r" formula: $a = v^2/R = 2g(1-\cos\theta)$.

(d) Normal reaction, N, acts outwards perpendicularly to the tangent to the slope at X. It reduces the component of the weight acting towards O, which is $mg\cos\theta$. Therefore

$$\begin{aligned} mg\cos\theta - N &= ma \\ &= m\frac{v^2}{R} \\ &= 2mg(1-\cos\theta)\ , \end{aligned}$$

so

$$\begin{aligned} N &= mg\cos\theta - 2mg(1-\cos\theta) \\ &= mg(3\cos\theta - 2)\ . \end{aligned}$$

(e) Just at the point of take-off the normal reaction will go to zero (the hill cannot exert a force on X if he is no-longer in contact with it). Therefore X will take off when $N = mg(3\cos\theta - 2) = 0$, so $\cos\theta = 2/3$ and $\theta = \arccos(2/3) \approx 48.2°$.

(f) At the instant of take-off, $v^2 = 2gR(1-\cos\theta) = 2gR(1-2/3)$, so $v = \sqrt{2gR/3}$.

11. The motion is as shown in the diagram:

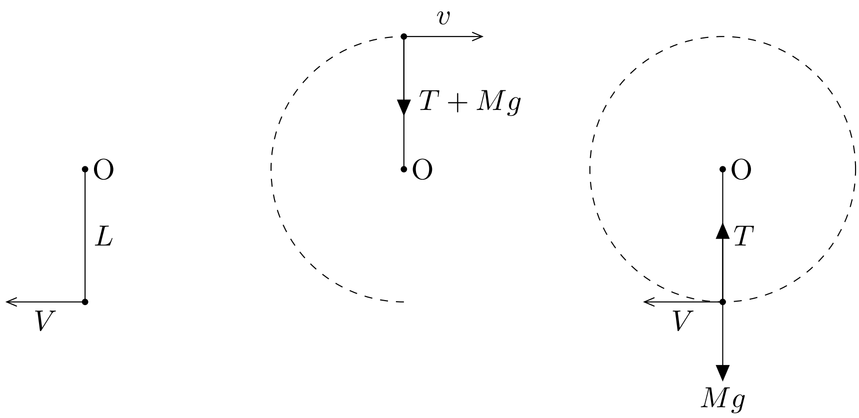


Figure 2.5: *A complete orbit in a vertical circle, beginning vertically below* O *(left), passing through the top of the circle (middle) and returning to the starting point (right).*

(a) Energy at start = KE = $\frac{1}{2}MV^2$. Energy at end of considered motion = KE + PE = $\frac{1}{2}Mv^2 + 2MgL$. Conservation of energy therefore says that $\frac{1}{2}MV^2 = \frac{1}{2}Mv^2 + 2MgL$, so $v^2 = V^2 - 4gL$.

(b) accn. = "v^2/r" $= v^2/L = V^2/L - 4g$.

(c) The two forces are the tension in the string, T, and the weight of the mass $W = Mg$, both acting downwards towards O. **(d)** "$F = ma$", with "F" being $T + Mg$ and "ma" being $Ma = Mv^2/L = MV^2/L - 4Mg$ gives $T + Mg = MV^2/L - 4Mg \Rightarrow T = M(V^2/L - 5g)$.

(e) For complete orbits need $T \geq 0 \Rightarrow M(V^2/L - 5g) \geq 0$. In the limiting case, $M(V_{\min}^2/L - 5g) = 0$ which can be rearranged into $V_{\min}^2 - 4gL = gL$ and since $v_{\min}^2 = V_{\min}^2 - 4gL$, $v_{\min} = \sqrt{gL}$. **(f)** Clearly $V_{\min}$ results from $V_{\min}^2/L = 5g$ and is $V_{\min} = \sqrt{5gL}$.

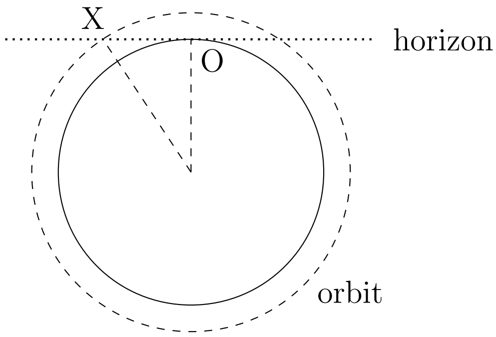
(g) Having executed a full revolution and come back to its original position (having speed v', say, on its return), the energy of M has gone from $\frac{1}{2}MV^2 \to \frac{1}{2}Mv^2 + 2MgL \to \frac{1}{2}Mv'^2$. Conservation of energy between the three points requires that $\frac{1}{2}MV^2 = \frac{1}{2}Mv^2 + 2MgL = \frac{1}{2}Mv'^2$, so $\frac{1}{2}MV^2 = \frac{1}{2}Mv'^2$ and $v' = V$. Newton's second law in the vertical direction is now $T - Mg = Mv'^2/L = MV^2/L$. Therefore $T = M(V^2/L + g)$. In the limiting case where $V = V_{\min}$, this would be $T_{\min} = 6Mg$.

2.11 Gravitation

1. **(a) (i)** At the surface of the Earth, radius 6370 km, the acceleration due to gravity is $9.8\,\mathrm{m/s^2}$. For the satellite, at altitude 250 km, the centripetal acceleration is, by Kepler's third law, $9.8 \times [6370/(6370+250)]^2 = 9.07\,\mathrm{m/s^2}$. If satellite speed is v, orbital period T, "v^2/r" $= (2\pi r/T)^2/r = 4\pi^2 r/T^2 = 4\pi^2(6.37\times10^6 + 2.5\times10^5)/T^2 = 9.07$, $T = 5367$ seconds ≈ 89 minutes.

   The rationale for this is as follows: $T^2 = 4\pi^2 R/g \Rightarrow g = 4\pi^2 R/T^2 = 4\pi^2 R/CR^3 \equiv D/R^2$ for some constants C and D. Thus the gravitational attraction at the position of the satellite g_S can be worked out from $g_\mathrm{S}/g_\mathrm{E} = R_\mathrm{E}^2/R_\mathrm{S}^2 \Rightarrow g_\mathrm{S} = 9.8\times(6370/[6370+250])^2$. The period of the satellite then follows straightforwardly from $T_\mathrm{S}^2 = 4\pi^2 R_\mathrm{S}/g_\mathrm{S} = 89$ minutes.

   (ii) As an alternative way of tackling the problem (perhaps relying more on 'brute force'), Kepler's third law says that $T^2 \propto R^3 \Rightarrow T^2 = CR^3$ for some constant C. Now: $T = 2\pi R/v$ and $Mg = Mv^2/R$ so that $T^2 = 4\pi^2R^2/v^2 = 4\pi^2R^2/gR = 4\pi^2R/g$. Therefore $T_\mathrm{E}^2 = 4\pi^2R_\mathrm{E}/9.8 = CR_\mathrm{E}^3 \Rightarrow C = 4\pi^2/(9.8R_\mathrm{E}^2)$. This means that $T_\mathrm{S}^2 = 4\pi^2/(9.8R_\mathrm{E}^2)R_\mathrm{S}^3 \Rightarrow T_\mathrm{S} = \surd(4\pi^2\times(6370+250)^3\times(1000)^3/[9.8\times(6370\times10^3)^2]) = 5367$ seconds $\simeq 89$ minutes.



   (b) From the "intersecting chords" theorem in the diagram, $\mathrm{OX}^2 = 250\times(2\times(6370+250)) \Rightarrow \mathrm{OX} = 1802$ km (or via Pythagoras' theorem $\mathrm{OX}^2 = (6370+250)^2 - 6370^2 = 3247500$). Angle subtended by OX at the centre of the Earth $= \arcsin(1802/(6370+250)) = 15.8°$. The satellite is above the horizon for an observer at O for a fraction $2\times15.8°/360° = 0.088$ of its orbit, or for a time $0.088\times89 = 7.8 \approx 8$ minutes. Note that it is also possible to work out the angle without having to know the length of OX: angle $= \arccos(6370/[6370+250])$ by elementary trigonometry.

2. Gravitational acceleration on the surface of a sphere of mass M, radius R, is GM/R^2, or for a sphere of uniform density ρ,

$$\frac{4\pi G\rho R^3}{3R^2} = \frac{4\pi}{3}G\rho R \propto \rho R \, .$$

Density of Moon is therefore (density of Earth)×(radius of Earth/radius of Moon) $\times(g_{\mathrm{E}}/g_{\mathrm{M}}) = 5500 \times (6370/1730) \times (1.6/9.8) = 3300\,\mathrm{kg/m^3}$.

3. A geostationary orbit has a period of 24 hours = 24 × 3600 seconds. If its radius is R metres, speed is V m/s, and the radius of the Earth is taken as 6,370,000 metres, centripetal acceleration $= V^2/R = (2\pi R/\text{period})^2/R = 4\pi^2 R/(24 \times 3600)^2 = 9.8 \times (6.37 \times 10^6)^2/R^2$. Solving for R gives $R = 4.2 \times 107$, orbital radius = 42,000 km.

4. For Ganymede, $T = 7.15 \times 24 \times 3600$, $R = 1.07 \times 10^9$, centripetal acceleration $= V^2/R = (2\pi R/T)^2/R = 4\pi^2 R/T^2 = 0.11\,\mathrm{m/s^2}$, which equals gravitational acceleration GM/R^2, $M = 0.11 \times R^2/G = 1.9 \times 10^{27}$ kg, 320 times the mass of the Earth.

5. **(a,b)** Angle through which Pluto is displaced in the course of a week = 360° ÷ 250 ÷ 52 = 0.0277°, assuming the observer on Earth is close to the centre of Pluto's orbit, which is the case since the Earth's orbit is much smaller.

   (c) Relating Pluto's orbit to the Earth's orbit via Kepler's third law, radius (Pluto's orbit) $= (250/1)^{2/3} \times$ radius (Earth's orbit) $= 39.7 \times 150 = 6 \times 10^9$ km.

   (d) In radian measure, angle subtended by Charon's orbit $= 0.0003 \times 2\pi/360 = 5.2 \times 10^{-6}$, diameter across orbit $= (6000 \times 10^6) \times (5.2 \times 10^{-6}) = 31{,}200$ km, radius = 15,600 ≈ 16,000 km.

   (e) Orbital period for Charon = 6.5 × 24 × 3600 seconds, centripetal acceleration $= V^2/R = (2\pi R/T)^2/R = 4\pi^2 R/T^2 = 4\pi^2 \times (15.6 \times 10^6)/(6.5 \times 24 \times 3600)^2 = 0.00195\,\mathrm{m/s^2}$, which equals the gravitational acceleration GM/R^2, $M = 0.00195 \times R^2/G = 7 \times 10^{21}$ kg.

   (In fact, Charon's mass is a significant fraction of Pluto's mass, so that taking Pluto as the centre of Charon's orbit is only a rough approximation. A better value for Pluto's mass is 1.3×10^{22} kg).

6. The distance of the combined Earth-Moon C of G from the Earth's centre is $(7.35 \times 10^{22} \times 38.44 \times 10^4)/(5.98 \times 10^{24} + 7.35 \times 10^{22}) = 4667$ km. This is inside the interior of the Earth and a distance 379,733 km from the Moon. The orbital speed of the Moon around this centre is $(2\pi \times 379733 \times 1000)/(27.32 \times 24 \times 3600) = 1011$ m/s. The centripetal acceleration is "v^2/r" $= 1011^2/(379733 \times 1000) = 0.00269\,\mathrm{m/s^2}$.

7. **(a)** The gravitational force is given by

$$F_{\mathrm{G}} = -\frac{Gm_p m_e}{r_{\mathrm{B}}^2} \ ,$$

where the minus sign indicates that the force is attractive; that is, the gravitational force on the electron due to the proton points towards the proton. Its magnitude is simply $Gm_p m_e/r_{\mathrm{B}}^2 = 6.673 \times 10^{-11} \times 1.67 \times 10^{-27} \times 9.11 \times 10^{-31}/(5.29\times10^{-11})^2 = (6.67\times1.67\times9.11/5.29^2)\times10^{-47} = 3.6\times10^{-47}$ newtons.

(b) The electrostatic force on the other hand is

$$F_{\mathrm{Q}} = -\frac{(1.6 \times 10^{-19})^2}{4\pi\varepsilon_0 r_{\mathrm{B}}^2} \ ,$$

and since the overall sign is negative (as the proton and electron have charges of equal magnitude but opposite sign), the force is again attractive. That is, the electrostatic force on the electron due to the proton also points towards the proton. Its magnitude is given by $(1.6 \times 10^{-19})^2/(4\pi \times 8.85 \times 10^{-12} \times (5.29 \times 10^{-11})^2) = (1.6^2/(4\pi \times 8.85 \times 5.29^2)) \times 10^{-4} = 8.2 \times 10^{-8}$ newtons.

(c) Clearly $3.6 \times 10^{-47} \ll 8.2 \times 10^{-8}$, so the electrostatic force between the proton and electron in Hydrogen is very much greater (by a factor of about 10^{39}) than the gravitational force between them. The understanding of this vast difference in strengths between these two forces is one of the outstanding problems of physics and is sometimes referred to as the 'hierarchy problem'.

8. Closest approach = $(1-\varepsilon)\times$ (mean distance); farthest point = $(1+\varepsilon)\times$(mean distance). So, for Mars: nearest point is $(1-0.093)\times 1.524\times 149.6 = 206.8$ million km, farthest point is $(1+0.093)\times 1.524\times 149.6 = 249.2$ million km, a difference of about 42 million km.

   For Pluto, call the nearest distance d_1, the farthest distance d_2 and the mean distance (semi-major axis) a. Then $d_1 = (1-\varepsilon)a$ and $d_2 = (1+\varepsilon)a$. Eliminating a gives

$$\begin{aligned} \frac{d_1}{d_2} &= \frac{1-\varepsilon}{1+\varepsilon} \\ \Rightarrow (1+\varepsilon)d_1 &= (1-\varepsilon)d_2 \\ \Rightarrow \varepsilon &= \frac{d_2-d_1}{d_2+d_1}\ , \end{aligned}$$

   and substituting for d_1 and d_2 we arrive at $\varepsilon = (7302-4437)/(7302+4437) \approx 0.244$.

9. The gravitational force on the star (mass m) from the galaxy (mass $M = 3.6\times 10^{10}M_\odot$) is GMm/r^2, where $r = 25\,\text{kpc}$ is the distance from the centre of the galaxy to the star. The force is directed towards the centre of the galaxy. Using Newton's second law, "$F = ma$", with $a = v^2/r$ as appropriate for circular motion (also directed towards the centre of the galaxy) we then have

$$F = \frac{GMm}{r^2} = ma = m\frac{v^2}{r}\ .$$

   Rearranging for v, this gives:

$$\begin{aligned} v &= \sqrt{\frac{GM}{r}} \\ &= \left(\frac{6.67\times 10^{-11}\cdot 3.6\times 10^{10}\cdot 1.989\times 10^{30}}{25\times 10^3\cdot 3.086\times 10^{16}}\right)^{\frac{1}{2}} \\ &= \left(\frac{6.67\times 3.6\times 1.989}{25\times 3.086}\times 10^{10}\right)^{\frac{1}{2}} \\ &\approx 80\,\text{km/s}\ . \end{aligned}$$

   The observed figure is $v_{\text{obs}} \approx 150\,\text{km/s}$, so clearly $v_{\text{theoretical}} < v_{\text{obs}}$; see *Figure 2.6* below. This, in fact, is an indication of one of the central problems in cosmology. The observed speed of rotation towards the edge of galaxies is observed to be much greater than that expected theoretically. As can be seen from the formula $v = \sqrt{GM/r}$, the speed of rotation that we calculate would be greater if M were greater; that is, if there were more matter in the galaxy than we currently observe. This is one of the reasons for suspecting the existence of 'dark matter'; so-called because it would not emit much (if any) light, otherwise we would already have observed it.

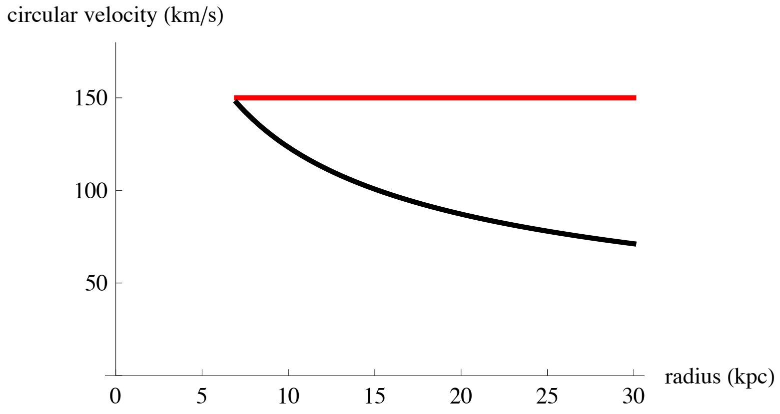
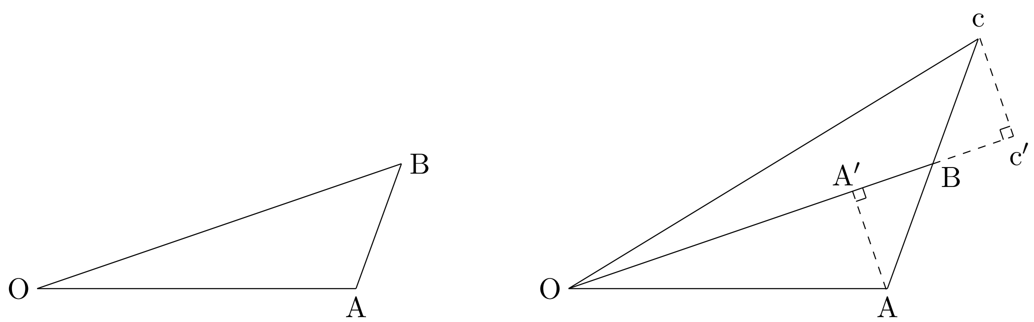


Figure 2.6: *A sketch of the theoretical (black) and observed (red) galactic rotation curves for the* NGC 3198 *galaxy. The (circular) velocity in kilometres per second is plotted against the distance from the centre of the galaxy in kiloparsecs for distances* $\gtrsim 10\,\text{kpc}$.

10. **(a)** In one orbital period, T, the moon travels a distance of $2\pi r$, where r is the radius of its orbit around Utopia. Therefore $v = 2\pi r/T = 2\pi \times (2\times 10^8)/(314.16 \times 60 \times 60) \approx 1,111\,\text{m/s}$. **(b)** If M is the mass of Utopia and m the mass of its moon, then $F = GMm/r^2$ is the force providing a centripetal acceleration of "v^2/r". Thus $GMm/r^2 = mv^2/r \Rightarrow M = rv^2/G = 2\times 10^8 \times (1,111)^2/(6.673\times 10^{-11}) \approx 3.7\times 10^{24}\,\text{kg}$.

11. **(a)** Since the body is travelling at constant speed and for equal time intervals from A to B and then from B to c, the lengths AB and Bc are equal.



(b) Considering triangles AA′B and Bc′c, they have one equal side (AB=Bc), the *angles* AA′B and Bc′c are equal (both being 90°) and the angles ABA′ and cBc′ are also both equal (since opposite angles formed by the crossing of two straight lines are equal "opposite angles are equal"). Since they share an equal side and two equal angles, AA′B and Bc′c are congruent triangles and therefore the two altitudes are equal, AA′ = cc′.

The area of a triangle is $\frac{1}{2}\times$ base $\times$ height, so considering AA′ as the height of triangle OAB and cc′ as the height of triangle OBc, their areas are equal since $AA' = cc'$ (their heights are equal) and they share a common base, OB.

(c) The impulse vector directed along BO and the original velocity vector directed along Bc combine by vector addition and generate the parallelogram $BcCB'$, whose diagonal gives the resultant motion.

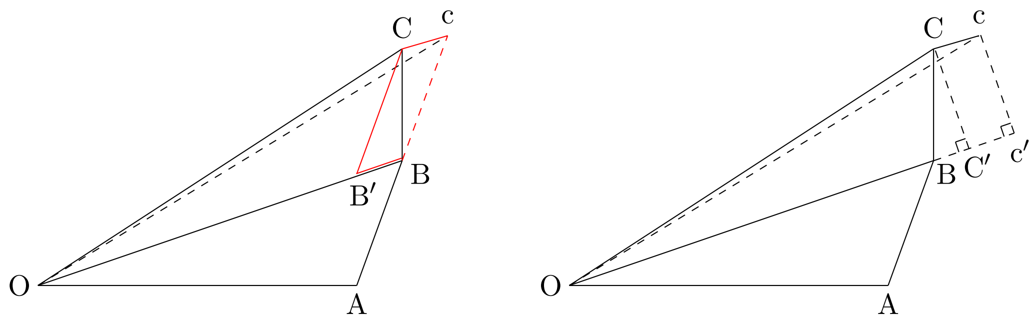


Figure 2.7: $BcCB'$ *is a parallelogram, while* $Ccc'C'$ *is a rectangle.*

(d) $BcCB'$ is a parallelogram and hence Cc is parallel to $B'B$, and hence parallel to OB and hence to OB's continuation to c' (see *Figure 2.7*). Cc and $C'c'$ are therefore parallel, and CC' and cc' are perpendiculars connecting two parallel lines and are therefore equal in length. Hence triangles OBc and OBC have the same height. They share the same base OB and their areas are therefore equal by the formula area $= \frac{1}{2}\times$ base $\times$ height. Furthermore, since the area of OBc is the same as the area of OAB, the area of OBC is also equal to the area of OAB.

(e) This is a proof of Kepler's second law. In fact it is the same as Newton's proof presented in his Principia. If O is the position of the Sun and the 'moving body' is a planet, then the fact that the area of OAB is equal to the area of OBC is the fact that equal areas are swept out in equal times. The gravitational force of the Sun is modelled as an impulse on the body in question acting discretely at B (and thereafter at C, D, …). As the time interval is taken to be smaller and smaller the approximation of the force acting discretely approaches the real setting of a continuous force with greater and greater accuracy. For the purposes of the area law, this discrete approximation is perfectly valid, however. Notice that the only (other) assumption that had to be made was that the force acts along the line connecting the body to the Sun.

2.12 Vectors

1. ABC has corner position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

   (a) In terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, $\overrightarrow{\mathrm{AB}} = \mathbf{b} - \mathbf{a}$; similarly for $\overrightarrow{\mathrm{BC}}$, $\overrightarrow{\mathrm{AC}}$, $\overrightarrow{\mathrm{CA}}$.

   (b) Vector sum $\overrightarrow{\mathrm{AB}} + \overrightarrow{\mathrm{BC}} + \overrightarrow{\mathrm{CA}} = (\mathbf{b} - \mathbf{a}) + (\mathbf{c} - \mathbf{b}) + (\mathbf{a} - \mathbf{c}) = \mathbf{0}$ (zero vector).

   (c) F, midpoint of AB so $\overrightarrow{\mathrm{AF}} = \overrightarrow{\mathrm{FB}}$; hence $(\mathbf{f} - \mathbf{a}) = (\mathbf{b} - \mathbf{f})$; so $\mathbf{f} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$; similarly, D, midpoint of BC so $\mathbf{d} = \frac{1}{2}(\mathbf{b} + \mathbf{c})$.

   (d) FD = length($\overrightarrow{\mathrm{FD}}$) = length($\mathbf{d} - \mathbf{f}$) = length($\frac{1}{2}(\mathbf{b}+\mathbf{c}) - \frac{1}{2}(\mathbf{a}+\mathbf{b})$) = length($\frac{1}{2}(\mathbf{c} - \mathbf{a})$) $= \frac{1}{2}$AC.

   (e) From **(d)**, $\overrightarrow{\mathrm{FD}} = \mathbf{d} - \mathbf{f} = \frac{1}{2}(\mathbf{c} - \mathbf{a}) = \frac{1}{2}\overrightarrow{\mathrm{AC}}$; so $\overrightarrow{\mathrm{FD}}$ parallel to $\overrightarrow{\mathrm{AC}}$.

2. ABC is triangle with corners $\mathbf{a} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{b} = 6\mathbf{i} + 13\mathbf{j}$, $\mathbf{c} = 10\mathbf{i} + 7\mathbf{j}$ so use solution to *Question 1* thus:

   (a) $\overrightarrow{\mathrm{AB}} = \mathbf{b} - \mathbf{a} = (6\mathbf{i} + 13\mathbf{j}) - (2\mathbf{i} + \mathbf{j}) = (4\mathbf{i} + 12\mathbf{j})$; similarly $\overrightarrow{\mathrm{BC}} = (4\mathbf{i} - 6\mathbf{j})$, $\overrightarrow{\mathrm{AC}} = (8\mathbf{i} + 6\mathbf{j})$ and $\overrightarrow{\mathrm{CA}} = (-8\mathbf{i} - 6\mathbf{j}) = -\overrightarrow{\mathrm{AC}}$.

   (b) Vector sum $\overrightarrow{\mathrm{AB}} + \overrightarrow{\mathrm{BC}} + \overrightarrow{\mathrm{CA}} = 0\mathbf{i} + 0\mathbf{j} = \mathbf{0}$.

   (c) Position vector of F (midpoint of AB) $= \frac{1}{2}(\mathbf{a} + \mathbf{b}) = \frac{1}{2}(8\mathbf{i} + 14\mathbf{j}) = 4\mathbf{i} + 7\mathbf{j}$; similarly position vector of D (midpoint of BC) $= 8\mathbf{i} + 10\mathbf{j}$.

   (d) Length of $\overrightarrow{\mathrm{FD}}$ = length($\mathbf{d} - \mathbf{f}$) = length($4\mathbf{i} + 3\mathbf{j}$) $= \sqrt{(4^2 + 3^2)} = 5$; length of $\overrightarrow{\mathrm{AC}}$ = length($\mathbf{c} - \mathbf{a}$) = length($8\mathbf{i} + 6\mathbf{j}$) $= \sqrt{(8^2 + 6^2)} = 10$; so FD $= \frac{1}{2}$AC.

   (e) Vector $\overrightarrow{\mathrm{FD}} = 4\mathbf{i} + 3\mathbf{j}$; $\overrightarrow{\mathrm{AC}} = 8\mathbf{i} + 6\mathbf{j} = 2\,\overrightarrow{\mathrm{FD}}$ so vectors are parallel.

3. In triangle ABC of *Question 2*, centroid G has position vector $\mathbf{g} = \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$.

   (a) $\mathbf{g} = \frac{1}{3}((2\mathbf{i} + \mathbf{j}) + (6\mathbf{i} + 13\mathbf{j}) + (10\mathbf{i} + 7\mathbf{j})) = \frac{1}{3}(18\mathbf{i} + 21\mathbf{j}) = 6\mathbf{i} + 7\mathbf{j}$.

   (b) $\overrightarrow{\mathrm{AG}} = \mathbf{g} - \mathbf{a} = 4\mathbf{i} + 6\mathbf{j}$; $\overrightarrow{\mathrm{AD}} = \mathbf{d} - \mathbf{a} = 6\mathbf{i} + 9\mathbf{j}$; $3\,\overrightarrow{\mathrm{AG}} = 2\,\overrightarrow{\mathrm{AD}}$ so parallel.

   (c) AG and AD are parallel and A is common point – so A,D,G collinear.

   (d) Similar reasoning shows G also lies on lines BE and CF.

   (e) From **(c)** and **(d)**, AD, BE and CF all meet at G.

   (f) From **(b)** 3AG = 2AD so AG : AD = 2 : 3; same ratio for BG : BE, CG : CF.

4. In parallelogram ABCD, DC is parallel to AB, AD parallel to BC and vertex position vectors are **a**, **b**, **c**, and **d** respectively.

   (a) DC = AB; DC parallel to AB; thus $\overrightarrow{DC} = \overrightarrow{AB}$ so $\mathbf{c}-\mathbf{d} = \mathbf{b}-\mathbf{a}$ so $\mathbf{d} = \mathbf{a}-\mathbf{b}+\mathbf{c}$.

   (b) Position vector M (mid-point AC) $= \mathbf{m} = \frac{1}{2}(\mathbf{a}+\mathbf{c})$.

   (c) Position vector M′ (mid-point BD) $= \mathbf{m}' = \frac{1}{2}(\mathbf{b}+\mathbf{d}) = \frac{1}{2}(\mathbf{b}+\mathbf{a}-\mathbf{b}+\mathbf{c}) = \mathbf{m} =$ position vector M (mid-point AC); so midpoints M and M′ coincide; so M lies on BD and BM = MD as required.

5. Parallelogram ABCD has DC parallel to AB and AD parallel to BC; position vectors $\mathbf{a} = \mathbf{i}+2\mathbf{j}$, $\mathbf{b} = 3\mathbf{i}+4\mathbf{j}$ and $\mathbf{c} = 2\mathbf{i}+10\mathbf{j}$. Use *Question 4* solution thus

   (a) Position vector of D $= \mathbf{d} = \mathbf{a}-\mathbf{b}+\mathbf{c} = (\mathbf{i}+2\mathbf{j})-(3\mathbf{i}+4\mathbf{j})+(2\mathbf{i}+10\mathbf{j}) = 0\mathbf{i}+8\mathbf{j} = 8\mathbf{j}$.

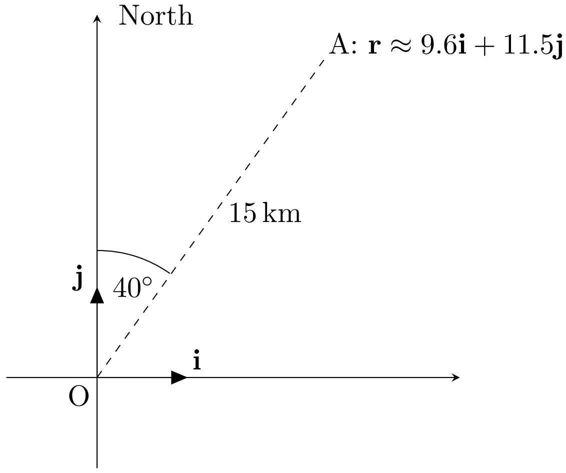
   (b) Length of AC = length$(\mathbf{c}-\mathbf{a})$ = length$(\mathbf{i}+8\mathbf{j}) = \sqrt{(1^2+8^2)} = \sqrt{65}$.

   (c) Position vector M (mid-point AC) $= \frac{1}{2}(\mathbf{a}+\mathbf{c}) = \frac{1}{2}(3\mathbf{i}+12\mathbf{j})$.

   (d) Position vector M (mid-point BD) $= \frac{1}{2}(\mathbf{b}+\mathbf{d}) = \frac{1}{2}(3\mathbf{i}+12\mathbf{j}) =$ position vector M; so M is mid point of BD (*i.e.* M is on BD and BM = MD as required).

6. Triangle ABC vertices at $\mathbf{a} = -9\mathbf{i}+11\mathbf{j}$, $\mathbf{b} = -\mathbf{j}$, $\mathbf{c} = 6.4\mathbf{i}+3.8\mathbf{j}$. Hence AB = length$(\overrightarrow{AB})$ = length$(\mathbf{b}-\mathbf{a})$ = length$(9\mathbf{i}-12\mathbf{j}) = 15$; similarly BC = 8, AC = 17. Hence AC is hypotenuse and vertex B is right angle. Trigonometry gives sin(CAB) = 8/17, CAB = 28° and so ABC = 62°.

7. Seen from a lighthouse at the origin O, a yacht A lies on a bearing of 40° at a distance of 15 km. Its position vector relative to the lighthouse is $\mathbf{r}$, where $\mathbf{r} = 15\sin(40°)\mathbf{i} + 15\cos(40°)\mathbf{j}$.



8. The position vectors of ships A and B, relative to a lighthouse at O, are $\mathbf{r}_\mathrm{A} = 20\mathbf{i} + 10\mathbf{j}$ and $\mathbf{r}_\mathrm{B} = -20\mathbf{i} + 30\mathbf{j}$. (Distances are in km, $\mathbf{i}$ and $\mathbf{j}$ are unit vectors east and north).

   (a) OA = length$(20\mathbf{i} + 10\mathbf{j}) = \sqrt{500} \approx 22.4$; similarly OB $= \sqrt{1300} \approx 36.1$; AB = length$(\mathbf{r}_\mathrm{B} - \mathbf{r}_\mathrm{A})$ = length$(-40\mathbf{i} + 20\mathbf{j}) = \sqrt{2000} \approx 14.1$.

   (b) Bearing of A from O = arctan(20/10) = 063°; bearing of O from A = 180° + 63° = 243°; bearing of B from A = 270° + arctan(20/40) = 297°.

9. P1 starts at time $t = 0$ from $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$ and arrives at $\mathbf{b} = 12\mathbf{i} + 27\mathbf{j}$ when $t = 2$.

   (a) Velocity vector $\mathbf{v}$ = (position change)/(time taken) $= (\mathbf{b} - \mathbf{a})/2 = 5\mathbf{i} + 12\mathbf{j}$.

   (b) Speed = length$(\mathbf{v}) = \sqrt{(5^2 + 12^2)} = 13$.

   (c) Angle between direction of motion ($\mathbf{v}$) and $\mathbf{i}$ is arctan(12/5) = 67°.

10. P2 starts at $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$ moves at constant speed to $\mathbf{b} = 12\mathbf{i} - 7\mathbf{j}$ in $t = 4$ seconds.

   (a) Velocity vector $= (\mathbf{b} - \mathbf{a})/4 = (10\mathbf{i} - 10\mathbf{j})/4 = (2.5\mathbf{i} - 2.5\mathbf{j})$ m/s.

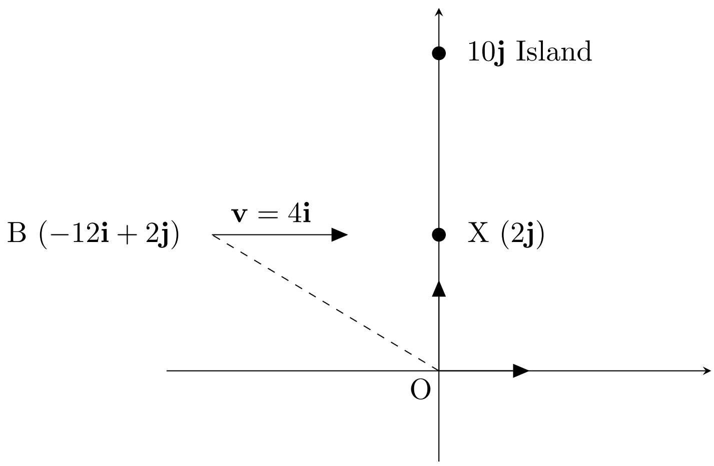
   (b) Speed $= \sqrt{(2.5^2 + 2.5^2)} = 5/\sqrt{2} = 3.54$ m/s.

   (c) At time $t = 3$, position $\mathbf{r} = \mathbf{a} + t\mathbf{v} = (2\mathbf{i} + 3\mathbf{j}) + 3(2.5\mathbf{i} - 2.5\mathbf{j}) = 9.5\mathbf{i} - 4.5\mathbf{j}$.

   (d) Let $\mathbf{x} = 22\mathbf{i} - 17\mathbf{j}$, $\mathbf{y} = 27\mathbf{i} - 23\mathbf{j}$, $\mathbf{z} = 47\mathbf{i} - 42\mathbf{j}$; then $(\mathbf{x} - \mathbf{a}) = 20(\mathbf{i} - \mathbf{j})$ parallel to $\mathbf{v}$, $(\mathbf{y} - \mathbf{a}) = (25\mathbf{i} - 26\mathbf{j})$ not parallel to $\mathbf{v}$, $(\mathbf{z} - \mathbf{a}) = 25(\mathbf{i} - \mathbf{j})$ parallel to $\mathbf{v}$; so particle does not pass through $\mathbf{y}$.

11. P3 velocity $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}\,\mathrm{m/s}$ at time $t = 0$ and velocity velocity $\mathbf{v} = 12\mathbf{i} - 7\mathbf{j}\,\mathrm{m/s}$ at time $t = 5$ seconds; acceleration vector = (velocity change)/time = $(\mathbf{v} - \mathbf{u})/t = (10\mathbf{i} - 10\mathbf{j})/5 = (2\mathbf{i} - 2\mathbf{j})\,\mathrm{m\,s^{-2}}$; magnitude of acceleration vector $= \sqrt{(2^2 + 2^2)} = \sqrt{8} = 2.83\,\mathrm{m\,s^{-2}}$.

12. Sketch:



Mr B at $t = 0$ is $\mathbf{a} = -12\mathbf{i} + 2\mathbf{j}$ with velocity $\mathbf{v} = 4\mathbf{i}$. Island is at $10\mathbf{j}$. Mr B's position vector at time t is $\mathbf{r} = \mathbf{a} + \mathbf{v}t$; when $t = 1$, $\mathbf{r} = -12\mathbf{i} + 2\mathbf{j} + 4\mathbf{i} = -8\mathbf{i} + 2\mathbf{j}$. B nearest when due south of island – *i.e.* at X ($= 2\mathbf{j}$) when $-12\mathbf{i} + 2\mathbf{j} + 4t\mathbf{i} = 2\mathbf{j}$ so $t = 3$. Thus B misses island by 8 km (= 10 − 2) km. After hoisting sail, position on day t is $\mathbf{r} = -8\mathbf{i} + 2\mathbf{j} + (t - 1)(4\mathbf{i} + k\mathbf{j})$; to reach island, $\mathbf{r} = 10\mathbf{j}$ so equating coefficients of $\mathbf{i}$ and $\mathbf{j}$ gives $-8 + 4(t - 1) = 0$ and $2 + (t - 1)k = 10$ so that $t = 3$, $k = 4$. Speed on day 1 is 4; speed after that is $\sqrt{(4^2 + 4^2)}$, so distance is $(1 \times 4) + (2 \times 4\sqrt{2}) = 4(1 + 2\sqrt{2}) \approx 15.3\,\mathrm{km}$.

13. C flies at 12 m/s from A $(20\mathbf{i} + 10\mathbf{j})$ to B $(308\mathbf{i} + 94\mathbf{j})$.

   (a) AB = length$(\overrightarrow{\mathrm{AB}})$ = length$(288\mathbf{i} + 84\mathbf{j}) = 12\sqrt{(24^2 + 7^2)} = 300\,\mathrm{m}$.

   (b) Time of flight = distance/speed = 300/12 = 25 s.

   (c) Velocity vector $\mathbf{v} = \overrightarrow{\mathrm{AB}}/\text{time} = (288\mathbf{i} + 84\mathbf{j})/25 = \frac{12}{25}(24\mathbf{i} + 7\mathbf{j})$.

   (d) Direction of motion (bearing) = arctan(24/7) = 074°.

14. Speed 28 m/s at angle of 30° to horizontal. So horizontal and vertical components of velocity vector $\mathbf{v}$ are $28\cos(30°)$ and $28\sin(30°)$; thus $\mathbf{v} = 14\sqrt{3}\mathbf{i} + 14\mathbf{j}$.

15. D starts at origin to run to X at $\mathbf{r}_X = 0.9\mathbf{i} + 4\mathbf{j}$ km in 0.5 h.

    (a) His velocity should be $(0.9\mathbf{i} + 4\mathbf{j})/(0.5) = 1.8\mathbf{i} + 8\mathbf{j}$; speed $= \sqrt{(1.8^2 + 8^2)} = 8.2\,\mathrm{km\,h^{-1}}$.

    (b) Position Y $= \frac{1}{2}(8\mathbf{i} + 1.8\mathbf{j}) = 4\mathbf{i} + 0.9\mathbf{j}$; XY $=$ length$(-3.1\mathbf{i} + 3.1\mathbf{j}) = 3.1\sqrt{2} = 4.38$ km.

    (c) Time wasted is distance(XY)/speed $= 4.38/8.2 = 0.535\,\mathrm{h} = 32$ minutes.

16. B walks from E ($\mathbf{r}_E = 2\mathbf{j}$) at 1 m/s towards X ($\mathbf{r}_X = 20\mathbf{i} + 17\mathbf{j}$).

    (a) Distance EX $=$ length$(\mathbf{r}_X - \mathbf{r}_E) = \sqrt{(20^2 + (17-2)^2)} = 25$ m.

    (b) Time taken for B to reach X $=$ disance/speed $= 25\,\mathrm{m}/1\,\mathrm{m\,s^{-1}} = 25$ s.

    F darts off to Y ($\mathbf{r}_Y = -16\mathbf{i} + 65\mathbf{j}$), inspects for 5 seconds, dashes to X.

    (c) Distance travelled by F $=$ EY $+$ YX $=$ length$(\mathbf{r}_Y - \mathbf{r}_E) +$ length$(\mathbf{r}_X - \mathbf{r}_Y) = \sqrt{((-16)^2 + (65-2)^2)} + \sqrt{((20+16)^2 + (17-65)^2)} = \sqrt{4225} + \sqrt{3600} = 65 + 60 = 125$ m.

    (d) F darts for 20 s ($= 25 - 5$); therefore speed of F $= 125\,\mathrm{m}/20\,\mathrm{s} = 6.25\,\mathrm{m\,s^{-1}}$.

17. At $t = 0$, A and B are at $\mathbf{a} = 20\mathbf{i} + 10\mathbf{j}$ and $\mathbf{b} = -20\mathbf{i} + 30\mathbf{j}$ with velocities $\mathbf{v}_A = 10\mathbf{j}$ and $\mathbf{v}_B = 10\mathbf{i} + 5\mathbf{j}$ (km/hr).

    (a) At time t, $\mathbf{r}_A = \mathbf{a} + t\mathbf{v}_A = 20\mathbf{i} + (10 + 10t)\mathbf{j}$; $\mathbf{r}_B = \mathbf{b} + t\mathbf{v}_B = (-20 + 10t)\mathbf{i} + (30 + 5t)\mathbf{j}$.

    (b) At $t = 1$, AB = length$(\mathbf{r}_B - \mathbf{r}_A)$ = length$((-10\mathbf{i} + 35\mathbf{j}) - (20\mathbf{i} + 20\mathbf{j}))$ = length$(-30\mathbf{i} + 15\mathbf{j}) = 15\sqrt{5}$ km; similarly, when $t = 2$, AB = length$(\mathbf{r}_B - \mathbf{r}_A)$ = length$((0\mathbf{i} + 40\mathbf{j}) - (20\mathbf{i} + 30\mathbf{j})) = 10\sqrt{5}$ km.

    (c) Bearing of B from A (at $t = 1$) is $360° - \arctan(30/15) = 360° - \arctan(2) \approx 297°$. Bearing of B from A (at $t = 2$) is $360° - \arctan(20/10) = 360° - \arctan(2)$. Bearing unchanged so A and B on collision course.

    (d) Using **(a)** and $t = 4$, $\mathbf{r}_A = 20\mathbf{i} + 50\mathbf{j}$; $\mathbf{r}_B = 20\mathbf{i} + 50\mathbf{j}$; same position, so collision.

18. At $t = 0$, $\mathbf{r}_A = \mathbf{a} = 20\mathbf{i} + 10\mathbf{j}$, $\mathbf{v}_A = 10\mathbf{j}$ and $\mathbf{r}_B = \mathbf{b} = -20\mathbf{i} + 30\mathbf{j}$, $\mathbf{v}_B = 8\mathbf{i} + 4\mathbf{j}$ km/hr. Ship A has velocity $10\mathbf{j}$ km/hr and ship B has velocity $8\mathbf{i} + 4\mathbf{j}$ km/hr.

    (a) at t, $\mathbf{r}_A = \mathbf{a} + t\mathbf{v}_A = 20\mathbf{i} + (10 + 10t)\mathbf{j}$; similarly $\mathbf{r}_B = \mathbf{b} + t\mathbf{v}_B = (-20 + 8t)\mathbf{i} + (30 + 4t)\mathbf{j}$.

    (b) At time t, $\mathbf{r}_B - \mathbf{r}_A = (-40 + 8t)\mathbf{i} + (20 - 6t)\mathbf{j}$; so bearing of B from A = $\arctan(-32/14)$ at $t = 1$ and $\arctan(-24/8)$ at $t = 2$, difference confirms that will not collide.

    (c) At $t = 2$, AB = length$(\mathbf{r}_B - \mathbf{r}_A)$ = length$(-24\mathbf{i} + 8\mathbf{j}) = \sqrt{640} = 8\sqrt{10}$ km; in general, AB = d = length$(\mathbf{r}_B - \mathbf{r}_A) = \sqrt{(-40 + 8t)^2 + (20 - 6t)^2)} = \sqrt{((10t - 44)^2 + 8^2)}$ (completing the square).

    (d) Distance is smallest when d^2 is minimum; from **(c)** this occurs when $t = 4.4$ h and then $d^2 = 64$; thus closes approach is $d = 8$ km, see *Figure 2.8.*

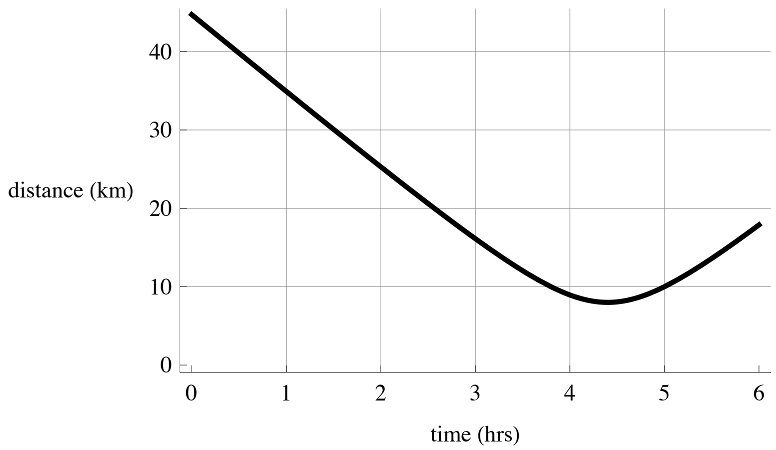
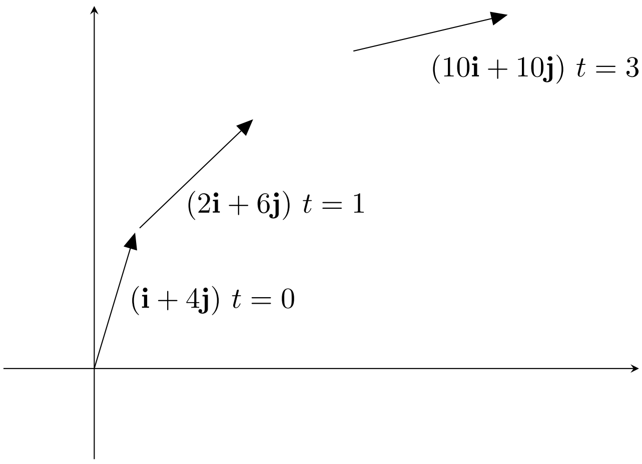


Figure 2.8: *Distance of approach of ships* A *and* B *in kilometres as a function of the time elapsed in hours.*

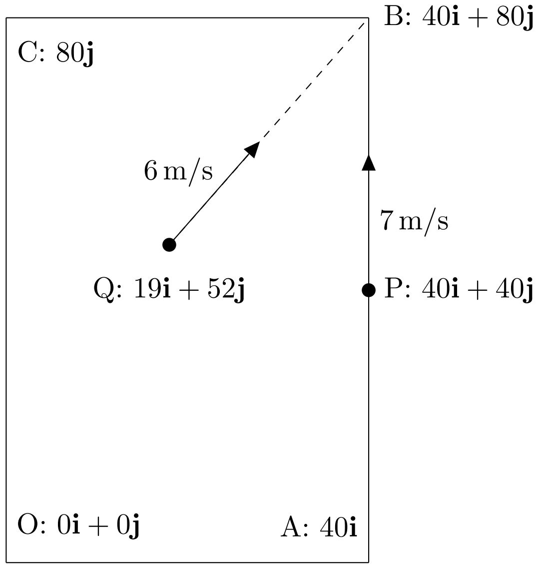
19. Velocity $= 10\mathbf{i} + k\mathbf{j}$ so speed $= \sqrt{(10^2 + k^2)} = 26$; hence $k^2 = 26^2 - 10^2 = 576$ and thus $k = 24$.

20. $\mathbf{v} = 5\mathbf{i} + k(\mathbf{i} + 2\mathbf{j})$ **(a)** $\mathbf{v}$ parallel to $\mathbf{i} + \mathbf{j}$ requires $5 + k = 2k$ so $k = 5$ (giving $\mathbf{v} = 10(\mathbf{i} + \mathbf{j})$); **(b)** $\mathbf{v}$ parallel to $\mathbf{i}$ requires $2k = 0$ so $k = 0$ (giving $v = 5\mathbf{i}$); **(c)** $\mathbf{v}$ parallel to $\mathbf{j}$ requires $5 + k = 0$ so $k = -5$ (giving $\mathbf{v} = -10\mathbf{j}$).

21. Velocity at time t is $\mathbf{v} = (1 + t^2)\mathbf{i} + (2t + 4)\mathbf{j}$. At $t = 0$, $\mathbf{v} = \mathbf{i} + 4\mathbf{j}$, so speed $= \sqrt{17}$; $\mathbf{v}$ parallel to $\mathbf{i} + \mathbf{j}$ when $(1 + t^2) = (2t + 4)$ *i.e.* $(t - 3)(t + 1) = 0$ so that $t = 3$ ($t = -1$ not valid). Similarly, $\mathbf{v}$ parallel to $\mathbf{i} + 3\mathbf{j}$ when $3(1 + t^2) = (2t + 4)$ *i.e.* $(3t + 1)(t - 1) = 0$ so that $t = 1$ ($t = -1/3$ not valid). Sketch as shown below.



22. Corners at O ($0\mathbf{i}+0\mathbf{j}$), A ($\mathbf{a} = 40\mathbf{i}+0\mathbf{j} = 40\mathbf{i}$), B ($\mathbf{b} = 40\mathbf{i}+80\mathbf{j}$), and C ($\mathbf{c} = 80\mathbf{j}$). P starts at $\mathbf{p} = 40\mathbf{i}+40\mathbf{j}$ and heads for B at $7\,\mathrm{m/s}$; velocity $\mathbf{v}_\mathrm{P}$ parallel to $\overrightarrow{\mathrm{PB}} = 40\mathbf{j}$ and speed 7 implies $\mathbf{v}_\mathrm{P} = 7\mathbf{j}$. Similarly, Q starts at $19\mathbf{i}+52\mathbf{j}$ for B at $6\,\mathrm{m/s}$. So $\mathbf{v}_\mathrm{Q} = 6\,\overrightarrow{\mathrm{QB}}/\mathrm{length}(\overrightarrow{\mathrm{QB}}) = 6(21\mathbf{i}+28\mathbf{j})/35 = \frac{6}{5}(3\mathbf{i}+4\mathbf{j})\,\mathrm{m/s}$. The diagram below shows positions of P and Q on the field and the directions in which they will start to run.

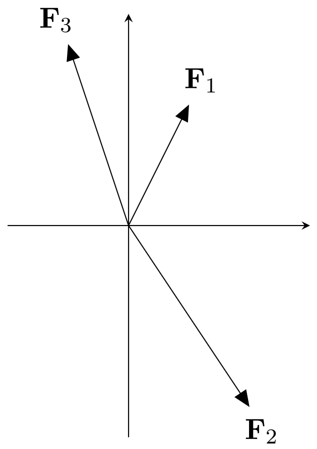


S leaves $\mathbf{s} = 15\mathbf{i}+55\mathbf{j}$, 1 s after P and Q, with speed $7.5\,\mathrm{m\,s^{-1}}$. $\mathbf{v}_\mathrm{S} = 7.5(\overrightarrow{\mathrm{SB}})\div \mathrm{length}(\overrightarrow{\mathrm{SB}}) = 7.5(25\mathbf{i}+25\mathbf{j})/(25\sqrt{2}) = \frac{15}{4}\sqrt{2}(\mathbf{i}+\mathbf{j})$. At $t = 5$, P is at $\mathbf{p}+5\mathbf{v}_\mathrm{P} = 40\mathbf{i}+40\mathbf{j}+5(7\mathbf{j}) = 40\mathbf{i}+75\mathbf{j}$; similarly, Q is at $37\mathbf{i}+76\mathbf{j}$; for S, $t = 4\,\mathrm{s}\,(= 5-1)$ so S is at $15(1+\sqrt{2})\mathbf{i}+(55+15\sqrt{2})\mathbf{j}$.

P reaches B at $t = \mathrm{BP}/(\text{speed P}) = \mathrm{length}(\mathbf{b}-\mathbf{p})/7 = \mathrm{length}(40\mathbf{j})/7 = 40/7 = 5.7143\,\mathrm{s}$; Q reaches B at $t = \mathrm{BQ}/(\text{speed Q}) = \mathrm{length}(\mathbf{b}-\mathbf{q})/6 = \mathrm{length}(21\mathbf{i}+28\mathbf{j})/6 = 35/6 = 5.833\,\mathrm{s}$; S reaches B at $t = 1+\mathrm{BS}/(\text{speed S}) = 1+\mathrm{length}(\mathbf{b}-\mathbf{s})/7.5 = 1+\mathrm{length}(25\mathbf{i}+25\mathbf{j})/7.5 = 1+50\sqrt{2}/15 = 5.7140\,\mathrm{s}$. Thus, S reaches B just before P (*i.e.* intercepts P) and Q arrives after P (*i.e.* misses P).

23. P-velocity: $\mathbf{u}_\mathrm{P} = 3\mathbf{i}+4\mathbf{j}$ so P-speed $= \sqrt{(4^2+3^2)} = 5\,\mathrm{m\,s^{-1}}$. Angle between $\mathbf{u}_\mathrm{P}$ and **(a)** x-axis is $\arctan(4/3) = 53°$ and **(b)** y-axis is $90° - \arctan(4/3) = 37°$. After impulse P-velocity is $\mathbf{v}_\mathrm{P} = 5\mathbf{i}+4\mathbf{j}\,\mathrm{m\,s^{-1}}$. **(i)** velocity change $= \mathbf{v}_\mathrm{P} - \mathbf{u}_\mathrm{P} = 2\mathbf{i}$. **(ii)** initial speed $= 5$, final speed $= \sqrt{(5^2+4^2)} = \sqrt{41}$, so speed increase $= \sqrt{41}-5 = 1.40\,\mathrm{m/s}$. **(iii)** the angle deflected $= \arctan(4/3) - \arctan(4/5) = 14.5°$.

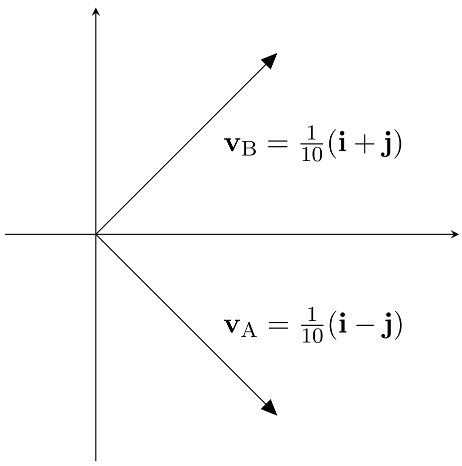
24. $\mathbf{F}_1 = 10\mathbf{i}+20\mathbf{j}$, $\mathbf{F}_2 = 20\mathbf{i}-30\mathbf{j}$, $\mathbf{F}_3 = -6\mathbf{i}+20\mathbf{j}$. Resultant $\mathbf{F} = \mathbf{F}_1+\mathbf{F}_2+\mathbf{F}_3 = (10\mathbf{i}+20\mathbf{j}) + (20\mathbf{i} - 30\mathbf{j}) + (-6\mathbf{i} + 20\mathbf{j}) = 24\mathbf{i} + 10\mathbf{j}$; magnitude of $\mathbf{F}$ is $F = \sqrt{(24^2 + 10^2)} = 26\,\mathrm{N}$. Magnitude of $(\mathbf{F} + \mathbf{F}_4)$ is $30\,\mathrm{N}$; so magnitude$(24\mathbf{i} + (10 + x)\mathbf{j}) = 30$; hence $24^2 + (10 + x)^2 = 30^2$ and so $(10 + x)^2 = 18^2$ and so $x = 8$ or $x = -28$. Sketch as below.



25. $\mathbf{F}_1 = 2\mathbf{i} + 3\mathbf{j}\,\mathrm{N}$ and $\mathbf{F}_2 = -10\mathbf{i} + 17\mathbf{j}\,\mathrm{N}$ give P ($m = 0.5\,\mathrm{kg}$) acceleration $\mathbf{a}$. From Newton's second law, ($\mathbf{F} = m\mathbf{a}$), $0.5\mathbf{a} = (2\mathbf{i} + 3\mathbf{j}) + (-10\mathbf{i} + 17\mathbf{j}) = -8\mathbf{i} + 20\mathbf{j}$; thus $\mathbf{a} = -16\mathbf{i} + 40\mathbf{j}$.

26. Q in equilibrium under $\mathbf{F}_1 = 2p\mathbf{i} + 4q\mathbf{j}$, $\mathbf{F}_2 = p\mathbf{i} - 10\mathbf{j}$, $\mathbf{F}_3 = 3\mathbf{i} + q\mathbf{j}$. Hence resultant $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = 0$; thus $(2p + p + 3)\mathbf{i} + (4q - 10 + q)\mathbf{j} = \mathbf{0}$ – *i.e.* $3p + 3 = 0$ and $5q - 10 = 0$ – giving $p = -1$, $q = 2$; so $F_1 = \text{magnitude}(\mathbf{F}_1) = \text{mag}(-2\mathbf{i} + 8\mathbf{j}) = \sqrt{((-2)^2 + 8^2)} = \sqrt{68}\,\text{N}$; similarly, $F_2 = \sqrt{((-1)^2 + (-10)^2)} = \sqrt{101}\,\text{N}$ and $F_3 = \sqrt{(3^2 + 2^2)} = \sqrt{13}\,\text{N}$.

27. Ball A mass $m\,(= 0.15\,\text{kg})$ and velocity $\mathbf{v}\,(= 0.2\mathbf{i}\,\text{m}\,\text{s}^{-1})$ has momentum $\mathbf{P} = m\mathbf{v} = 0.03\mathbf{i}\,\text{N}\,\text{s}$. Ball A strikes ball B; after impact, B has velocity $(\mathbf{i} + \mathbf{j})/10$ and A has velocity $\mathbf{u}$. Momentum conservation means $(m/5)\mathbf{i} + m\mathbf{0} = m\mathbf{u} + (m/10)(\mathbf{i} + \mathbf{j})$. Hence $\mathbf{u} = (\mathbf{i} - \mathbf{j})/10$. A-speed $= \text{mag}((\mathbf{i} - \mathbf{j})/10) = \sqrt{2}/10$; B-speed $= \text{mag}((\mathbf{i} + \mathbf{j})/10) = \sqrt{2}/10$. As shown in sketch (below), velocity of A is 45° clockwise from x-axis; velocity of B is 45° anti-clockwise from x-axis, so 90° is total angle between A and B directions of motion.



28. At $t = 0$, aircraft A is at $\mathbf{a} = -5000\mathbf{i} + 200\mathbf{k}$ with velocity $\mathbf{v}_\mathrm{A} = 200\mathbf{i} + 150\mathbf{j}$. At $t = 10$, missile M leaves origin with $\mathbf{v}_\mathrm{M} = 250\mathbf{j} + 12.5\mathbf{k}$. Position of A at time t is $\mathbf{r}_\mathrm{A} = \mathbf{a} + t\mathbf{v}_\mathrm{A}$ and position of M is $\mathbf{r}_\mathrm{M} = (t-10)\mathbf{v}_\mathrm{M}$. So position of M relative to A at time t is

$$\begin{aligned}\mathbf{r}_\mathrm{M} - \mathbf{r}_\mathrm{A} &= (t-10)\mathbf{v}_\mathrm{M} - (\mathbf{a} + t\mathbf{v}_\mathrm{A}) \\ &= (t-10)(250\mathbf{j} + 12.5\mathbf{k}) + 5000\mathbf{i} - 200\mathbf{k} - t(200\mathbf{i} + 150\mathbf{j}) \\ &= (5000 - 200t)\mathbf{i} + (-2500 + 100t)\mathbf{j} + (-325 + 12.5t)\mathbf{k}\ .\end{aligned}$$

If missile-aircraft separation (distance AM) is s, and $S = s^2$, then

$$S = s^2 = (5000 - 200t)^2 + (2500 - 100t)^2 + (325 - 12.5t)^2\ .$$

At minimum separation s, $\frac{\mathrm{d}S}{\mathrm{d}t} = 0$, and so differentiating S above w.r.t. t yields

$$\begin{aligned}(-200)(5000 - 200t) + (-100)(2500 - 100t) + (-12.5)(325 - 12.5t) &= 0 \\ 16(400 - 16t) + 8(200 - 8t) + (26 - t) &= 0 \\ 8026 - 321t &= 0\ .\end{aligned}$$

Thus closest approach occurs when $t = 8026/321 = 25.003 \approx 25$ seconds. At this time, the separation is s where $S = s^2 = (-0.62)^2 + (-0.31)^2 + (12.46)^2 = 155.8$. Hence closest approach $s = \sqrt{155.8} = 12.5\,\mathrm{m}$; this is a success ($s < 15\,\mathrm{m}$).

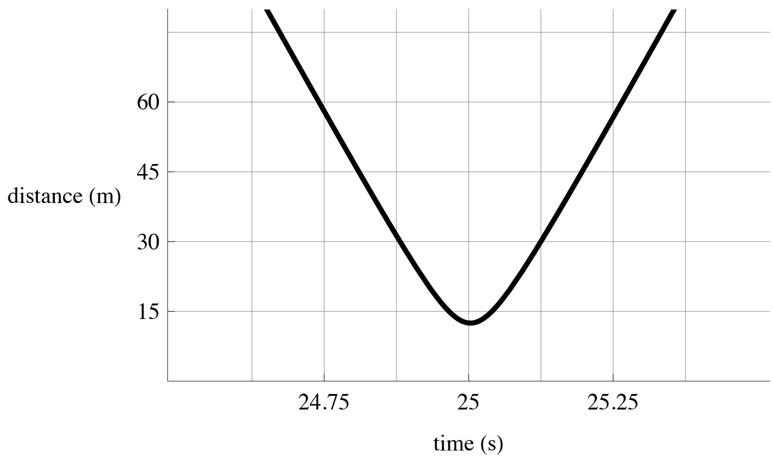


Figure 2.9: *Missile-aircraft separation in metres as a function of time elapsed in seconds.*