An introduction to partial differential equations

R.S. Johnson

R.S. Johnson

An introduction to partial differential equations

An introduction to partial differential equations

ISBN 978-87-7681-969-9

Contents

Preface to these three texts

The three texts in this one cover, entitled 'First-order partial differential equations' (Part I), 'Partial differential equations: classification and canonical forms' (Part II) and 'Partial differential equations: method of separation of variables and similarity & travelling-wave solutions' (Part III), are three of the 'Notebook' series available as additional and background reading to students at Newcastle University (UK). These three together present an introduction to all the ideas that are usually met in a fairly comprehensive study of partial differential equations, as encountered by applied mathematicians at university level. The material in some of Part I, and also some of Part II, is likely to be that encountered by all students; the rest of the material expands on this, going both further and deeper. The aim, therefore, has been to present the standard ideas on a broader canvas (but as relevant to the methods employed in applied mathematics), and to show how the subject can be developed. All the familiar topics are here, but the text is intended, primarily, to broaden and expand the experience of those who already have some knowledge and interest in the subject.

Each text is designed to be equivalent to a traditional text, or part of a text, which covers the relevant material, but in a way that moves beyond an elementary discussion. The development is based on careful derivations and descriptions, supported by many worked examples and a few set exercises (with answers provided). The necessary background is described in the preface to each Part, and there is a comprehensive index, covering the three parts, at the end.

Part I

First-order partial differential equations

List of examples

Here is a list of the various examples that are discussed in this text.

Preface

This text is intended to provide an introduction to the standard methods that are used for the solution of first-order partial differential equations. Some of these ideas are likely to be introduced, probably in a course on mathematical methods during the second year of a degree programme with, perhaps, more detail in a third year. The material has been written to provide a general – but broad – introduction to the relevant ideas, and not as a text closely linked to a specific module or course of study. Indeed, the intention is to present the material so that it can be used as an adjunct to a number of different courses – or simply to help the reader gain a deeper understanding of these important techniques. The aim is to go beyond the methods and details that are presented in a conventional study, but all the standard ideas are discussed here (and can be accessed through the comprehensive index).

It is assumed that the reader has a basic knowledge of, and practical experience in, the methods for solving elementary ordinary differential equations, typically studied in the first year of a mathematics (or physics or engineering) programme. However, the development of the relevant and important geometrical ideas is not assumed; these are carefully described here. This brief text does not attempt to include any detailed, 'physical' applications of these equations; this is properly left to a specific module that might be offered in a conventional applied mathematics or engineering or physics programme. However, a few important examples of these equations will be included, which relate to specific areas of (applied) mathematical interest.

The approach that we adopt is to present some general ideas, which might involve a notation, or a definition, or a method of solution, but most particularly detailed applications of the ideas explained through a number of carefully worked examples – we present 13. A small number of exercises, with answers, are also offered, although it must be emphasised that this notebook is not designed to be a comprehensive text in the conventional sense.

1 Introduction

The study of partial differential equations (PDEs), both first and second order, has a long and illustrious history. In the very early days, second order equations received the greater attention (essentially because they appeared more naturally and directly in problems with a physical basis). It was in the 1770s that Lagrange turned to the problem of solving first order PDEs. [J.-L. Lagrange, 1736-1813, an Italian-born, French mathematician, made contributions to the theory of functions, to the solution of various types of equations, to number theory and to the calculus of variations. However, his most significant work was on analytical mechanics.]

All the essential results (in two independent variables) were developed by Lagrange, although Clairaut (in 1739) and d'Alembert (in 1744) had considered some simpler, special PDEs. The bulk of what we describe here is based on Lagrange's ideas, although important interpretations and generalisations were added by Monge, in particular. The extension to higher dimensions was completed by Cauchy (in 1819), and his 'method of characteristics' is often the terminology used to describe Lagrange's approach.

1.1 Types of equation

We shall concern ourselves with single equations in one unknown – so we exclude, for example, two coupled equations in two unknowns – and, almost exclusively, in two independent variables. Thus we shall seek solutions, $u(x,y)$, of first order PDEs, expressed most generally as

$$f(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y})=0,$$

although we shall mention, briefly, the corresponding problem for $u(x_1,x_2,....,x_n)$, where appropriate. It will be convenient, however, to start with a simpler, general equation:

$$a(x,y,u)\frac{\partial u}{\partial x}+b(x,y,u)\frac{\partial u}{\partial y}=c(x,y,u),$$

the *quasi-linear* equation i.e. one that is linear in the two first partial derivatives. This case includes the *linear* first order PDE:

$$a(x,y)\frac{\partial u}{\partial x}+b(x,y)\frac{\partial u}{\partial y}=c(x,y)u,$$

and the *semi-linear* equation

$$a(x,y)\frac{\partial u}{\partial x}+b(x,y)\frac{\partial u}{\partial y}=c(x,y,u),$$

each for suitable, given functions *a*, *b* and *c*.

The quasi-linear structure – linear in u_x and u_y, using the shorthand notation – leads to a fairly straightforward method of solution. This is generalised and extended in order to solve the most general PDE of this type, which involves u_x and/or u_y no longer of degree 1. Thus we shall discuss the solution of equations that may look like

$$2y\frac{\partial u}{\partial x}+u\frac{\partial u}{\partial y}=2yu^2$$

or

$$x\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=2xyu^2.$$

As we shall see, the description and construction of the solution requires some relevant geometrical ideas.

Exercises 1

Classify these equations as *linear*, *semi-linear*, *quasi-linear* or *general*:

(a) $(y-u)u_x+xu_y=xy+u$; (b) $uu_x^2-xu_y=\dfrac{2}{x}u^3$; (c) $x^2u_x+(y-x)u_y=y\sin u$;

(d) $(\sin y)u_x-\mathrm{e}^x u_y=\mathrm{e}^y u$; (e) $u_x+\sin(u_y)=u$.

****

****

2 The quasi-linear equation

We develop first, with care, the solution of the most general equation written in the form

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u),$$

where we have used subscripts to denote the partial derivatives. The coefficients a, b and c need not be analytic functions of their arguments for the most general considerations. However here, in order to present the conventional and complete theory, we shall assume that each of a, b and c is in the class of C^1 functions i.e. those that possess continuous first partial derivatives in all three arguments (at least, in some domain D). Further, we will normally aim to seek solutions subject to a given boundary condition, namely, that $u(x,y)$ is prescribed on a known curve in the (x, y)-plane (which will need to be in D for a solution to exist); this is usually called the *Cauchy problem*.

2.1 Of surfaces and tangents

Let us suppose that we have a solution $u = u(x,y)$ which is represented by the surface $z = u(x,y)$ in Cartesian 3-space. Now it is a familiar (and readily derived) property that the surface $u(x,y) - z = 0$ has a normal vector $(u_x, u_y, -1)$ at every point on the surface. Further, we introduce the vector (a,b,c), where a, b and c are the given coefficients of the quasi-linear equation i.e.

$$au_x + bu_y - c = 0 \text{ becomes } (a,b,c)\cdot(u_x, u_y, -1) = 0.$$

Thus the vectors (a,b,c) and $(u_x, u_y, -1)$ are orthogonal, and because $(u_x, u_y, -1)$ is normal to the surface $u(x,y) - z = 0$, so $z = u(x,y)$ is in the tangent plane defined by the vector (a,b,c). In other words, the surface which is a solution, $z = u(x,y)$, is the surface which has (a,b,c) as its tangent everywhere: the tangent plane.

The solution of the quasi-linear equation can therefore be expressed by the description of the tangent plane in terms of the slope of this surface:

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{c}{a} \text{ and } \frac{\mathrm{d}z}{\mathrm{d}y} = \frac{c}{b}.$$

However, the required solution (surface) is $z = u(x,y)$, and so we may write

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{c(x,y,u)}{a(x,y,u)} \text{ and } \frac{\mathrm{d}u}{\mathrm{d}y} = \frac{c(x,y,u)}{b(x,y,u)},$$

or, equivalently, one of this pair together with

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b(x,y,u)}{a(x,y,u)}.$$

This last equation defines a family of curves (but dependent on u) that sit in the solution surface, and are usually called *characteristics* (after Cauchy); the set of equations is usually called the *characteristic equations* of the PDE. The result of the integration of these two (coupled) ordinary differential equations (ODEs) is a two-parameter family – the two arbitrary constants of integration – and then a general solution can be obtained by invoking a general functional relation between the two constants. This is sufficient information, at this stage, to enable us to discuss an example.

Example 1

Find a general solution of $2yu_x + uu_y = 2yu^2$.

First we have the ODE $\dfrac{\mathrm{d}u}{\mathrm{d}y} = \dfrac{2yu^2}{u} = 2yu$ (for $u \neq 0$), which gives

$$\int \frac{\mathrm{d}u}{u} = 2\int y\,\mathrm{d}y \text{ and so } \ln|u| = y^2 + \text{constant} \text{ i.e. } u = Ae^{y^2},$$

where A is an arbitrary constant. Now consider $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{u}{2y} = \dfrac{Ae^{y^2}}{2y}$, then

$$2\int ye^{-y^2}\mathrm{d}y = A\int \mathrm{d}x \text{ or } -e^{-y^2} = Ax - B \text{ i.e. } Ax + e^{-y^2} = B,$$

where B is the second arbitrary constant. The general solution is expressed by writing $A = F(B)$, where

$$B = Ax + e^{-y^2} = (1 + xu)e^{-y^2},$$

which gives

$$u(x, y) = e^{y^2} F[(1 + xu)e^{-y^2}],$$

an implicit relation for $u(x, y)$, where $F(.)$ is an arbitrary function. Note that the zero solution is recovered by setting $F = 0$.

Comment: That we have obtained an implicit, rather than explicit, representation of the solution is to be expected: the original PDE is nonlinear. In the light of this complication, it is a useful exercise to confirm, by direct substitution, that we do indeed have a solution of the equation for arbitrary F; this requires a little care.

Example 2

Show that $u(x, y) = e^{y^2} F[(1 + xu)e^{-y^2}]$ is a solution of $2yu_x + uu_y = 2yu^2$, for arbitrary F.

First, we see that $u_x = \mathrm{e}^{y^2}(xu_x + u)\mathrm{e}^{-y^2}F'(\xi)$ (where $\xi = (1+xu)\mathrm{e}^{-y^2}$), which gives

$$u_x = \frac{uF'}{1-xF'} \quad (\text{for } 1 - xF' \neq 0);$$

similarly, we obtain $u_y = 2y\mathrm{e}^{y^2}F + \mathrm{e}^{y^2}[xu_y\,\mathrm{e}^{-y^2} - 2y(1+xu)\mathrm{e}^{-y^2}]F'$, which gives

$$u_y = \frac{2y\mathrm{e}^{y^2}F - 2y(1+xu)F'}{1-xF'}.$$

Now we form

$$\begin{aligned} 2yu_x + uu_y &= \frac{2yuF' + 2yu\mathrm{e}^{y^2}F - 2yu(1+xu)F'}{1-xF'} \\ &= \frac{2yu\mathrm{e}^{y^2}F - 2xyu^2F'}{1-xF'} \\ &= \frac{2yu\mathrm{e}^{y^2}F - 2xyu\mathrm{e}^{y^2}FF'}{1-xF'} \\ &= \frac{2yu\mathrm{e}^{y^2}F(1-xF')}{1-xF'} = 2yu\mathrm{e}^{y^2}F = 2yu^2, \end{aligned}$$

which recovers the original equation: we do indeed have a general solution (for arbitrary, differentiable *F*, and where $1 - xF' \neq 0$).

In this example, we observe that the characteristic lines are

$$(1 + xu)e^{-y^2} = \text{constant},$$

along which ue^{-y^2} is a constant. We now examine more carefully the connection between the characteristic lines and the curve on which specific initial data is prescribed.

2.2 The Cauchy (or initial value) problem

The most convenient way to proceed is to introduce a parametric representation of the various curves in the problem – both those required for the general solution and that associated with the initial data. First, let the initial data (i.e. *u* given on a prescribed curve) be expressed as

$$u = u(t) \text{ on } x = x(t),\ \ y = y(t),$$

where *t* is a parameter that maps out the curve, and the *u* on it. Of course, this is equivalent to stating that $u(x, y)$ is given on a curve, $g(x, y) = 0$, say. Now we turn our attention to the characteristic lines, and the general solution defined on them.

The pair of ODEs that describe the solution can be recast in the symmetric form

$$\frac{\mathrm{d}x}{a(x,y,u)} = \frac{\mathrm{d}y}{b(x,y,u)} = \frac{\mathrm{d}u}{c(x,y,u)},$$

and then any two pairings produce an appropriate pair of ODEs (as in §2.1). However, we may extend this variant of the equations by introducing the parameter, *s*, defined by the construction

$$\frac{\mathrm{d}x}{a} = \frac{\mathrm{d}y}{b} = \frac{\mathrm{d}u}{c} = \mathrm{d}s;$$

in other words, we may write

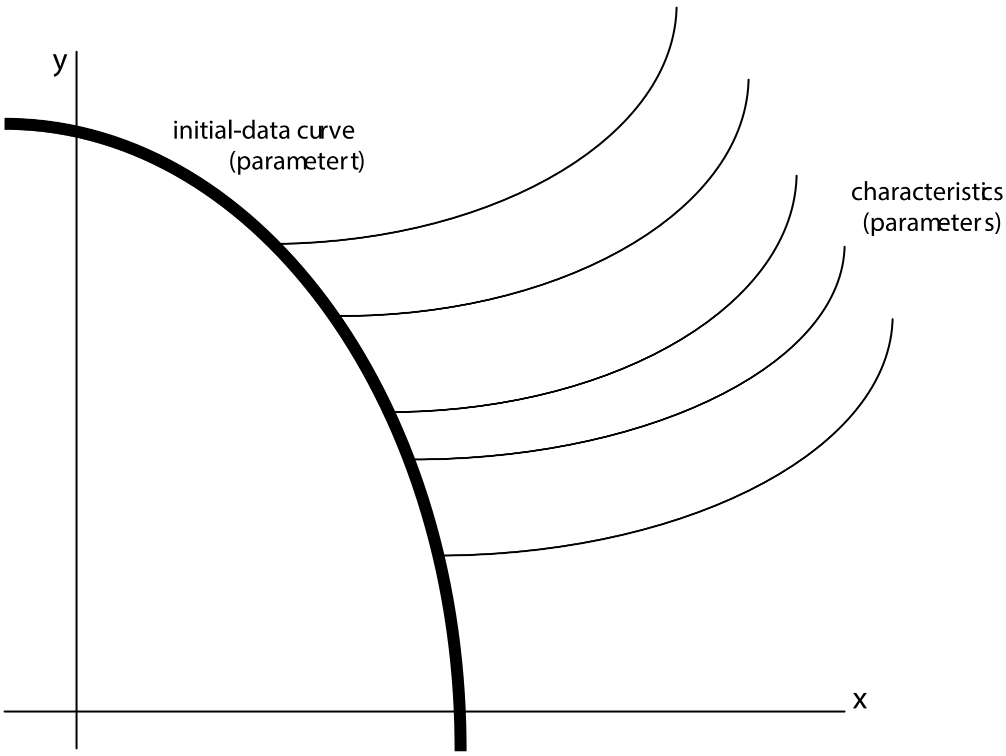
$$\frac{\mathrm{d}x}{\mathrm{d}s} = a(x,y,u),\ \frac{\mathrm{d}y}{\mathrm{d}s} = b(x,y,u),\ \frac{\mathrm{d}u}{\mathrm{d}s} = c(x,y,u).$$

These three (coupled) ODEs are the parametric equivalent to the pair of equations derived in §2.1. A solution of this set will then be expressed as

$$u = u(s) \text{ on } x = x(s),\ y = y(s),$$

which mirrors the description of the initial data.

Now a solution that satisfies the PDE and the initial data will necessarily depend on the two parameters: *t* chooses a point on the initial-data curve, and then *s* moves the solution away from this curve along a characteristic; see the sketch below.



Thus a complete solution, which satisfies given initial data, will depend on the two parameters, *s* and *t*:

$$u = u(s,t) \text{ on } x = x(s,t),\ y = y(s,t).$$

The only possible complication that might arise is when we attempt to construct $u(x, y)$ (which, presumably, is what we are seeking).

To accomplish this construction, in principle, we must solve $x = x(s,t)$ and

$y = y(s,t)$ to find $s = s(x, y)$ and $t = t(x, y)$; these are then used to obtain $u(x, y)$. The solution for *s* and *t* exists provided that the Jacobian of the transformation is non-zero i.e.

$$J \equiv \frac{\partial(x, y)}{\partial(s,t)} = \begin{vmatrix} x_s & y_s \\ x_t & y_t \end{vmatrix} = x_s y_t - x_t y_s \neq 0.$$

If $J = 0$ anywhere on the initial-data curve, then the solution fails (for this particular data): the solution does not exist. So, provided $J \neq 0$, a (unique) solution satisfying the PDE and the initial data exists; but what if we do find that $J = 0$?

If $J = 0$ then

$$x_s y_t = x_t y_s \text{ or } \frac{x_t}{y_t} = \frac{x_s}{y_s} = \frac{a}{b};$$

let us construct $$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}x}u(x,y) = u_x x_t + u_y y_t$$

$$= u_x \frac{a y_t}{b} + u_y y_t = \frac{1}{b}(a u_x + b u_y) y_t$$

and so $$\frac{u_t}{c} = \frac{a u_x + b u_y}{c} \frac{y_t}{b} = \frac{a u_x + b u_y}{a u_x + b u_y} \frac{y_t}{b} = \frac{y_t}{b}.$$

Similarly, we can obtain

$$\frac{u_t}{c} = \frac{a u_x + b u_y}{c} \frac{x_t}{a} = \frac{a u_x + b u_y}{a u_x + b u_y} \frac{x_t}{a} = \frac{x_t}{a};$$

that is, if $J = 0$, we have

$$\frac{u_t}{c} = \frac{x_t}{a} = \frac{y_t}{b}$$

which is equivalent to the original equations i.e. $\frac{\mathrm{d}u}{c} = \frac{\mathrm{d}x}{a} = \frac{\mathrm{d}y}{b}$. Thus the initial data, in this case, must itself be a solution of the original PDE; in other words, the curve on which u is prescribed is one of the characteristics, and the u on it must satisfy the PDE. The solution is therefore completely determined on this particular curve (characteristic line), but is otherwise non-unique on all the other characteristics. These various points are included in the next example.

Example 3

Find the general solution of $xu_x + 2xuu_y = u$ and then seek those solutions (if they exist) subject to: (a) $u = 2x$ on $y = 2x^2 + 1$; (b) $u = 2x^2$ on $y = 3x^3$; (c) $u = x^2$ on $y = x^3 - 1$.

The equation is equivalent to the system

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}y}{2xu} = \frac{\mathrm{d}u}{u} \text{ e.g. } \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{u}{x} \text{ and } \frac{\mathrm{d}y}{\mathrm{d}x} = 2u.$$

Thus $\int \frac{\mathrm{d}u}{u} = \int \frac{\mathrm{d}x}{x}$ and so $u = Ax$ (A an arbitrary constant),

and then $\frac{\mathrm{d}y}{\mathrm{d}x} = 2u = 2Ax$ which gives $y = Ax^2 + B$ (B the second arbitrary constant).

The general solution is described by $A = F(B)$, with $B = y - Ax^2 = y - xu$, i.e.

$$u(x,y) = xF(B) = xF(y - xu).$$

(a) This requires $2x = F(2x^2 + 1 - 2x^2) = xF(1)$, so that $F(1) = 2$: the solution is $u = 2x$ on $y = 2x^2 + 1$, and non-unique elsewhere.

(b) Now we have $2x^2 = xF(3x^3 - 2x^3) = xF(x^3)$, and so $F(z) = 2z^{1/3}$; the solution is then

$$u(x, y) = 2x(y - xu)^{1/3},$$

which can be recast as a cubic, and analysed further (if appropriate):

$$u^3 + 8x^4 u = 8yx^3.$$

(c) Finally, we require $x^2 = xF(x^3 - 1 - x^3)$ or $F(-1) = x$, and this is clearly impossible (for x is an independent variable): no solution exists that satisfies this particular initial data.

Comment: The characteristic lines here are $y - xu = \text{constant}$ which, as we see, depend on u. In (a), with $u = 2x$, the characteristic lines become $y - 2x^2 = \text{constant}$, and $u = 2x$ with $y = 2x^2 + \text{constant}$ is a solution of the original equation. In (b), the characteristic lines evaluated for the initial data are $y - 2x^3 = \text{constant}$, but the data is on $y - 3x^3 = 0$. In the last exercise, (c), the characteristic lines for this initial data become $y - x^3 = \text{constant}$, and the data is given on $y - x^3 = 1$, but with $u = x^2$ this is *not* a solution of the original PDE.

2.3 The semi-linear and linear equations

The semi-linear and linear equations are described by

$$a(x,y)u_x + b(x,y)u_y = \begin{cases} c(x,y,u), \text{ semi-linear} \\ c(x,y)u, \text{ linear} \end{cases}$$

for which the underlying system of ODEs is, correspondingly,

$$\frac{\mathrm{d}x}{a(x,y)} = \frac{\mathrm{d}y}{b(x,y)} = \begin{cases} \mathrm{d}u/c(c,y,u) \\ \mathrm{d}u/uc(x,y). \end{cases}$$

The first ODE, in both cases, is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b(x,y)}{a(x,y)},$$

which describes the characteristic lines, now defined independently of the solution $u(x,y)$. The second equation of the pair is, for example,

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{c(x,y,u)}{a(x,y)} \text{ (semi-linear) or } \frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{c(x,y)}{a(x,y)} \ (u \neq 0) \text{ (linear)},$$

and hence we may expect the construction of the solutions in these two cases to be particularly straightforward.

Example 4

Find the general solution of $u_x + 2xu_y = u^2$.

Here we have the set $\frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y}{2x} = \frac{\mathrm{d}u}{u^2}$, which we choose to write as the pair

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x \text{ and } \frac{\mathrm{d}u}{\mathrm{d}x} = u^2;$$

these are integrated directly to give $y = x^2 + A$ and $-\frac{1}{u} = x - B$ or $u = \frac{1}{B-x}$, respectively, where *A* and *B* are the arbitrary constants of integration. The general solution is then recovered from $B = F(A)$ i.e

$$u(x,y) = \frac{1}{F(y-x^2)-x},$$

for the arbitrary function *F*(.).

In this example, the characteristic lines are $y - x^2 = \text{constant}$; for reference, which we shall recall below, we observe the way in which the arbitrary function appears in this solution. Now we consider an example of a linear equation.

Example 5

Find the general solution of $yu_x - xu_y = 2xyu$.

The system of ODEs is $\dfrac{\mathrm{d}x}{y} = \dfrac{\mathrm{d}y}{-x} = \dfrac{\mathrm{d}u}{2xyu}$ e.g. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{x}{y}$ and $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 2xu$ ($y \neq 0$); these give, respectively,

$$y^2 + x^2 = A \text{ and } \ln|u| = x^2 + \text{constant} \text{ or } u = B\mathrm{e}^{x^2},$$

where *A* and *B* are the arbitrary constants of integration. The general solution is then given by the choice $B \quad F(A)$:

$$u(x, y) = \mathrm{e}^{x^2} F(x^2 + y^2).$$

Comment: This linear PDE possesses a solution in which the arbitrary function in the general solution appears in the form $u \propto F$ i.e. *u* is *linear* in *F*; this is a necessary property of the linear PDE. This is to be compared with the way in which the arbitrary function appears in the solution constructed in Example 4 – this equation was *nonlinear*.

2.4 The quasi-linear equation in *n* independent variables

The development described in §2.1 is readily extended to *n* independent variables, x_i ($i = 1, 2, \ldots n$). The quasi-linear equation in *n* independent variables is written as

$$\sum_{i=1}^{n} a_i(x_1, x_2, \ldots, x_n, u) \frac{\partial u}{\partial x_i} = c(x_1, x_2, \ldots, x_n, u),$$

and then the slopes of the tangents on the *n*-dimensional solution surface, a surface sitting in $n+1$ dimensions, generate the system of ODEs:

$$\frac{\mathrm{d}x_1}{a_1} = \frac{\mathrm{d}x_2}{a_2} = \ldots\ldots = \frac{\mathrm{d}x_n}{a_n} = \frac{\mathrm{d}u}{c}.$$

It is clear that the formulation we gave for the 2-D surface in 3-space goes over to higher dimensions in the obvious (and rather neat) way detailed above. The general solution is then described by a general functional relation between the *n* arbitrary constants of integration; we show this in the next example.

Example 6

Find the general solution of $u_x + \mathrm{e}^x u_y + \mathrm{e}^z u_z - (2x + \mathrm{e}^x)\mathrm{e}^u = 0$.

The system of ODEs is

$$\frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y}{\mathrm{e}^x} = \frac{\mathrm{d}z}{\mathrm{e}^z} = \frac{\mathrm{d}u}{(2x+\mathrm{e}^x)u};$$

e.g. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x$, $\dfrac{\mathrm{d}z}{\mathrm{d}x} = \mathrm{e}^z$ and $\dfrac{\mathrm{d}u}{\mathrm{d}x} = (2x+\mathrm{e}^x)\mathrm{e}^u$.

The solutions are, respectively,

$$y - \mathrm{e}^x = A,\ x + \mathrm{e}^{-z} = B,\ \mathrm{e}^{-u} + x^2 + \mathrm{e}^x = C;$$

the general solution is then obtained by writing $C = F(A,B)$:

$$\mathrm{e}^{-u} = F(y-\mathrm{e}^x, x+\mathrm{e}^{-z}) - x^2 - \mathrm{e}^x,$$

from which u follows directly by taking logarithms; the function $F(.,.)$ is arbitrary.

We have explored the simplest type of first order PDEs: those linear in the first partial derivatives. It should come as no surprise that, if the PDE is not linear in both u_x and u_y, then complications are to be expected. In the next chapter we investigate, in detail, the nature and construction of solutions of the most general, first order PDE in two independent variables.

Exercises 2

1. Find the general solution of $(1-xu)u_x + y(2x^2+u)u_y = 2x(1-xu)$, and then that solution which satisfies $u = \mathrm{e}^y$ on $x = 0$.

2. Find the general solution of $\mathrm{e}^{2y}u_x + xu_y = xu^2$, and then that solution which satisfies $u = \exp(x^2)$ on $y = 0$.

3. Find the general solution of $u_x - 2xuu_y = 0$, and then the solutions (if they exist) that satisfy (a) $u = x^{-1}$ on $y = 2x$; (b) $u = x$ on $y = x^3$.

****

****

3 The general equation

This equation is written in the form

$$f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0,$$

which is conventionally expressed in the notation

$$f(x, y, u, p, q) = 0 \quad \left(p \equiv \frac{\partial u}{\partial x}, q \equiv \frac{\partial u}{\partial y}\right).$$

Here, we will show how to extend the methods described in Chapter 2 to equations that are nonlinear in p and/or q; that is, we consider equations for which $|f_p| + |f_q|$ is a function of p and/or q. (The equation linear in p and q produces an expression for $|f_p| + |f_q|$ which, of course, depends on only (x, y, u).)

3.1 Geometry again

We start, as before, with the specification of the solution surface in the form

$$z = u(x, y) \quad \text{or} \quad u(x, y) - z = 0;$$

in this latter form, the normal to this surface is $(u_x, u_y, -1)$ i.e. $(p, q, -1)$. As we have seen, the quasi-linear equation (expressed as $ap + bq - c = 0$) has a tangent plane which is completely determined by the vector (a, b, c); this is no longer the case for the general equation. Now, at any point on a characteristic line, *p* and *q* are related by the requirement to satisfy the original PDE:

$$f(x, y, u, p, q) = 0.$$

To be specific, consider the point (x_0, y_0, u_0) in the solution surface and the associated values $p = p_0$ and $q = q_0$; these five quantities are necessarily related by

$$f(x_0, y_0, u_0, p_0, q_0) = 0,$$

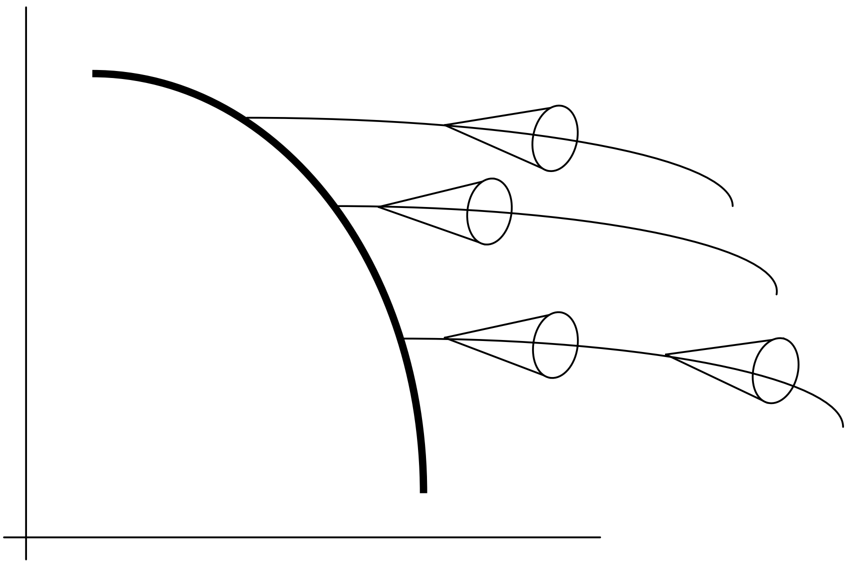
which provides a functional relation between p_0 and q_0.

Sufficiently close to the point $x = x_0$, $y = y_0$, $z = u_0$, the tangent plane in the solution surface takes the form

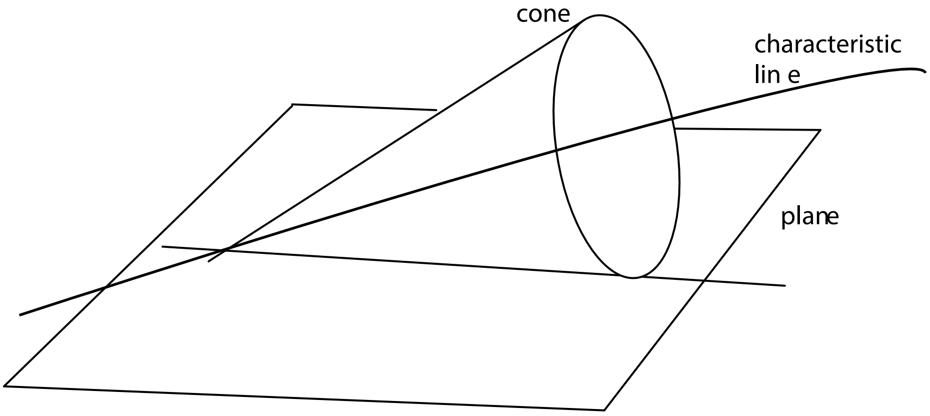
$$z - u_0 = (x - x_0)p_0 + (y - y_0)q_0,$$

where we assume that p_0 and q_0 are not both identically zero. (If they are both identically zero, we have a trivial, special case that we do not need to pursue.) Now as p_0 and q_0 vary, but always satisfying $f(x_0, y_0, u_0, p_0, q_0) = 0$, the tangent plane at (x_0, y_0, u_0), being a one-parameter family, maps out a cone with its vertex at (x_0, y_0, u_0): this is called the *Monge cone*. [G. Monge, 1746-1818, French mathematician and scientist (predominantly chemistry), made contributions to the calculus of variations, the theory of partial differential equations and combinatorics, but is remembered, mainly, for his work on infinitesimal, descriptive and analytical geometry. He was, of his time, one of the most wide-ranging scientists, working on e.g. chemistry (nitrous acid, iron, steel, water with Lavoisier), diffraction, electrical discharge in gases, capillarity, optics.]

Some characteristic lines, and associated Monge cones, are depicted in the figure below.



Then a specific, possible tangent (solution) plane (which will touch a Monge cone along a generator), in the neighbourhood of a particular Monge cone, is shown below.



3.2 The method of solution

We are now in a position to describe the construction of the system of ODEs that corresponds to the solution of the general, first order PDE

$$f(x, y, u, p, q) = 0;$$

this method embodies the characteristic structure that we met in Chapter 2, together with the requirements to admit the Monge cones. Because we now start with a point on the characteristic lines, but add the requirement to sit on a plane through the generator of the cone, the solution will be defined on a surface in the neighbourhood of the characteristic line. This surface, which contains a characteristic line, is called a *characteristic strip*. As we shall see, this process involves a subtle and convoluted development that requires careful interpretation, coupled with precise application of the differential calculus. The most natural way to proceed is to describe the solution of $f = 0$, for p and q, given x, y and u in terms of a parameter (as we used in the discussion of the Monge cone): we set $p = p(\tau)$, $q = q(\tau)$, so that

$$f(x, y, u, p(\tau), q(\tau)) = 0.$$

The τ -derivative of this equation yields

$$\frac{\partial f}{\partial p}\frac{\mathrm{d}p}{\mathrm{d}\tau} + \frac{\partial f}{\partial q}\frac{\mathrm{d}q}{\mathrm{d}\tau} = 0,$$

which is more conveniently expressed, upon the suppression of the parameter, as

$$f_p + f_q \frac{\mathrm{d}q}{\mathrm{d}p} = 0 \tag{A}$$

on a particular cone where the PDE provides $q = q(p)$ (since x, y and u are fixed).

In addition, following our earlier discussion, consider a solution which is described by the solution surface represented by a parameter *s*: $x = x(s)$, $y = y(s)$, $z = z(s)$. A tangent vector in this surface is then given by $(x'(s), y'(s), z'(s))$; this surface, containing a solution of the PDE, is $z = u(x, y)$ i.e. $u(x, y) - z = 0$, which has the normal $(u_x, u_y, -1)$ or, equivalently, $(p, q, -1)$. Thus we have

$$(x', y', z') \cdot (p, q, -1) = 0$$

which gives

$$px' + qy' - z' = 0 \ \text{ or } \ z' = px' + qy'; \tag{B}$$

this is an alternative representation of the Monge cone, at a point on a characteristic line, mapped out for some $q = q(p)$. Again, on a particular cone, which uses $q = q(p)$ for fixed (x', y', z'), we take the *p* derivative of equation (B):

$$x' + y' \frac{\mathrm{d}q}{\mathrm{d}p} = 0.$$

Thus, on a Monge cone, we have

$$\frac{\mathrm{d}q}{\mathrm{d}p} = -\frac{x'}{y'} = -\frac{\mathrm{d}x}{\mathrm{d}y};$$

and when this is combined with equation (A), we obtain

$$\frac{\mathrm{d}q}{\mathrm{d}p} = -\frac{\mathrm{d}x}{\mathrm{d}y} = -\frac{f_p}{f_q} \tag{C}$$

We return to (B) and now interpret this equation in a different way: first

$$z' = px' + qy' \text{ becomes } \frac{\mathrm{d}z}{\mathrm{d}x} = p + q\frac{\mathrm{d}y}{\mathrm{d}x}$$

which holds on a Monge cone with tangent planes on which $z = u$ (so on a solution/tangent plane we have $\delta z = \delta u$) i.e.

$$\frac{\mathrm{d}u}{\mathrm{d}x} = p + q\frac{\mathrm{d}y}{\mathrm{d}x} \text{ on a cone.} \tag{D}$$

Thus, with (C), we have

$$\frac{\mathrm{d}u}{\mathrm{d}x} = p + q\frac{f_q}{f_p} \text{ or } f_p\frac{\mathrm{d}u}{\mathrm{d}x} = pf_p + qf_q\,. \tag{E}$$

The calculation thus far has been explicitly associated with a Monge cone, on which solutions $z = u(x, y)$ must sit. But solutions clearly must also satisfy the original PDE:

$$f(x, y, u, p, q) = 0\,;$$

we take $\partial/\partial x$ of this equation:

$$f_x + f_u\frac{\partial u}{\partial x} + f_p\frac{\partial p}{\partial x} + f_q\frac{\partial q}{\partial x} = 0\,,$$

and then with $\dfrac{\partial q}{\partial x} = \dfrac{\partial^2 u}{\partial x\partial y} = \dfrac{\partial p}{\partial y}$ (where we assume that u_{xy} exists for our solutions), we obtain

$$f_x + pf_u = -(p_x f_p + p_y f_q)\,. \tag{F}$$

Also, again introducing s (as for (B)), we have

$$p(s) = \frac{\partial u}{\partial x}(x(s), y(s))$$

and so

$$\frac{\mathrm{d}p}{\mathrm{d}s} = u_{xx}\frac{\mathrm{d}x}{\mathrm{d}s} + u_{xy}\frac{\mathrm{d}y}{\mathrm{d}s} = p_x x' + p_y y'\,. \tag{G}$$

Now we form $p'/(f_x + pf_u)$, by combining (F) and (G):

$$\frac{p'}{f_x + pf_u} = -\frac{p_x x' + p_y y'}{p_x f_p + p_y f_q}\,,$$

and then use (C) to eliminate f_q:

$$\frac{p'}{f_x + pf_u} = -\frac{p_x x' + p_y y'}{p_x f_p + p_y \dfrac{\mathrm{d}y}{\mathrm{d}x} f_p}$$

$$= -\frac{1}{f_p}\left(\frac{p_x x' + p_y y'}{p_x f_p + p_y \dfrac{y'}{x'}}\right)$$

$$= -\frac{x'}{f_p}\left(\frac{p_x x' + p_y y'}{p_x x' + p_y y'}\right) = -\frac{x'}{f_p}. \tag{H}$$

A corresponding result, obtained by taking $\partial/\partial y$, gives

$$\begin{aligned} f_y + qf_u &= -(p_y f_p + q_y f_q) \\ &= -(q_x f_p + q_y f_q); \end{aligned}$$

cf. equation (F). The development that produces (H) can be followed through to give

$$\frac{q'}{f_y + qf_u} = -\frac{y'}{f_q}. \tag{J}$$

When we elect to write (E) in the form

$$\frac{u'}{pf_p + qf_q} = \frac{x'}{f_p}$$

and then combine with (C), (H) and (J), we obtain the set

$$\frac{x'}{f_p} = \frac{y'}{f_q} = \frac{u'}{pf_p + qf_q} = -\frac{p'}{f_x + pf_u} = -\frac{q'}{f_y + qf_u},$$

or, upon the suppression of the dependence on the parameter *s*:

$$\frac{\mathrm{d}x}{f_p} = \frac{\mathrm{d}y}{f_q} = \frac{\mathrm{d}u}{pf_p + qf_q} = -\frac{\mathrm{d}p}{f_x + pf_u} = -\frac{\mathrm{d}q}{f_y + qf_u}. \tag{K}$$

This is the final, required set of (four coupled) ODEs that represent the solution of the PDE. (These ODEs are sometimes referred to as *Charpit's equations*, although, strictly, this title implies only when they appear in a different context within the solution framework for the PDE; see §3.4.2. They are, more properly, the *characteristic equations* for the general PDE, first derived by Lagrange.)

Comment: The case of the quasi-linear equation corresponds to

$$f = a(x,y,u)u_x + b(x,y,u)u_y - c(x,y,u) = ap + bq - c,$$

and then the general system above becomes

$$\frac{\mathrm{d}x}{a} = \frac{\mathrm{d}y}{b} = \frac{\mathrm{d}u}{ap+bq} = -\frac{\mathrm{d}p}{a_x p + b_x q - c_x + p(a_u p + b_u q - c_u)}$$

$$= -\frac{\mathrm{d}q}{a_y p + b_y q - c_y + q(a_u p + b_u q - c_u)}.$$

The first three terms, with $ap + bq = c$, become

$$\frac{\mathrm{d}x}{a} = \frac{\mathrm{d}y}{b} = \frac{\mathrm{d}u}{c}$$

– the system derived in §2.1, but written in the form given in §2.2. In the next term – the fourth – we use

$$\frac{\partial}{\partial x}(ap + bq - c) = (a_x + a_u u_x)p + ap_x + (b_x + b_u u_x)q + bq_x - (c_x + c_u u_x) = 0,$$

to reduce that term to $\dfrac{\mathrm{d}p}{ap_x + bq_x} = \dfrac{\mathrm{d}p}{ap_x + bp_y}$;

similarly, the last term can be written as $\dfrac{\mathrm{d}q}{aq_x + bq_y}$.

Now we have, for example,

$$\frac{\mathrm{d}x}{a} = \frac{\mathrm{d}y}{b} = \frac{\mathrm{d}p}{ap_x + bp_y},$$

but on a solution curve we have

$$\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}p(x,y(x)) = p_x + p_y\frac{\mathrm{d}y}{\mathrm{d}x} = p_x + p_y\frac{b}{a};$$

thus

$$\frac{\mathrm{d}x}{a} = \frac{\mathrm{d}p}{ap + bp_y} \text{ becomes } ap_x + bp_y = a\frac{\mathrm{d}p}{\mathrm{d}x} = a\left(p_x + p_y\frac{b}{a}\right),$$

which is an identity. A corresponding calculation applies to the term $\mathrm{d}q\big/(aq_x + bq_y)$; thus the last two expressions in the system of ODEs are redundant: we are left with $\dfrac{\mathrm{d}x}{a} = \dfrac{\mathrm{d}y}{b} = \dfrac{\mathrm{d}u}{c}$, as expected.

We are now in a position to be able formulate and, in principle, solve the system to find a solution of the PDE; for our later development, it is convenient to revert to the use of the parameter, s, and so rewrite the system (K) as

$$\frac{\mathrm{d}x}{\mathrm{d}s} = f_p, \ \frac{\mathrm{d}y}{\mathrm{d}s} = f_q, \ \frac{\mathrm{d}u}{\mathrm{d}s} = pf_p + qf_q, \ \frac{\mathrm{d}p}{\mathrm{d}s} = -(f_x + pf_u), \ \frac{\mathrm{d}q}{\mathrm{d}s} = -(f_y + qf_u). \qquad \text{(L)}$$

Example 7

Solve the system of ODEs that describe the solution of $u_x u_y = u$ i.e. $pq = u$.

Here we have $f(x, y, u, p, q) \equiv pq - u = 0$, so that

$$f_x = f_y = 0, \ \ f_u = -1, \ f_p = q, \ f_q = p;$$

the system (L) of ODEs is therefore

$$\frac{\mathrm{d}x}{\mathrm{d}s} = q, \ \frac{\mathrm{d}y}{\mathrm{d}s} = p, \ \frac{\mathrm{d}u}{\mathrm{d}s} = 2pq, \ \frac{\mathrm{d}p}{\mathrm{d}s} = p, \ \frac{\mathrm{d}q}{\mathrm{d}s} = q.$$

The fourth and fifth equations give, respectively,

$$p = A\mathrm{e}^{s} \text{ and } q = B\mathrm{e}^{s} \quad (A,\, B \text{ arbitrary constants});$$

the third equation then becomes $\dfrac{\mathrm{d}u}{\mathrm{d}s} = 2AB\mathrm{e}^{2s}$, and so $u = AB\mathrm{e}^{2s} + C$ (where C is another arbitrary constant). The first two equations can now be integrated directly:

$$x = B\mathrm{e}^{s} + D \text{ and } y = A\mathrm{e}^{s} + E,$$

where D and E are the last two arbitrary constants.

However, we are not yet in a position to produce a solution – general or otherwise – without some further considerations. We do note, in this case, that we may write

$$u = A\mathrm{e}^{s} B\mathrm{e}^{s} + C = (y - E)(x - D) + C$$

it is then a simple exercise to confirm that this function, $u(x, y)$, is indeed a solution of the original PDE, for arbitrary D and E, but with $C = 0$.

Comment: The solution described by equations (L) has been developed on the basis, in the main, of a geometrical argument (which may appear rather obscure in places). However, now that we have these equations, we can readily confirm that, together, they are equivalent to the original PDE. To demonstrate this, consider a solution expressed in terms of a parameter s, and then construct

$$Q \equiv \frac{\mathrm{d}}{\mathrm{d}s} f(x(s), y(s), u(s), p(s), q(s))$$
$$= f_x x' + f_y y' + f_u u' + f_p p' + f_q q'.$$

Write $\quad u'(s) = \dfrac{\mathrm{d}}{\mathrm{d}s} u(x(s), y(s)) = u_x x' + u_y y' = px' + qy'$

(which corresponds to the first and second equations of the set, used in the third), then we obtain

$$Q = (f_x + pf_u)x' + (f_y + qf_u)y' + f_p p' + f_q q'$$

$$= (f_x + pf_u)x' + (f_y + qf_u)y' - (f_x + pf_u)f_p - (f_y + qf_u)f_q$$

when we use the fourth and fifth equations of the set. Finally, we write

$$Q = (x' - f_p)(f_x + pf_u) + (y' - f_q)(f_y + qf_u)y'$$

$$= 0$$

by virtue of the first and second equations. The result of using all five equations is therefore

$$Q = \frac{\mathrm{d}f}{\mathrm{d}s} = 0 \text{ and so } f = \text{constant},$$

and the PDE selects this constant to be zero: the set (L) satisfies the PDE.

It is now clear that we can use the construction described above to *derive* the set of underlying ODEs directly. This requires pairing off terms, and introducing factors, so that $\mathrm{d}Q/\mathrm{d}s = 0$ when appropriate choices are made. The only remaining issue in such an approach is to confirm that the choice of ODEs constitutes a consistent set for all the unknowns.

3.3 The general PDE with Cauchy data

The task now is to seek a solution of

$$f(x, y, u, p, q) = 0$$

subject to

$$u = u(t) \text{ on } x = x(t),\ y = y(t).$$

However, if this initial data is to be consistent with the PDE, we require suitable choices to be made for p and q, i.e.

$$f(x(t), y(t), u(t), p(t), q(t)) = 0; \tag{M}$$

further, with $u(t) = u(x(t), y(t))$, we must have

$$\frac{\mathrm{d}u}{\mathrm{d}t} = u_x \frac{\mathrm{d}x}{\mathrm{d}t} + u_y \frac{\mathrm{d}y}{\mathrm{d}t} = px' + qy'. \tag{N}$$

Equations (M) and (N) are used to determine $p(t)$ and $q(t)$, consistent with the given initial data and the PDE. This information, in conjunction with the initial data itself, provides all the necessary information for the integration of the characteristic equations from, say, $s = 0$.

Example 8

Find the solution of $u_x u_y = u$, subject to $u = t^2$ on $x = t$, $y = 1 + t$.

The initial data requires

$$pq = t^2 \text{ and } \frac{\mathrm{d}u}{\mathrm{d}t} = 2t = p + q,$$

and so $p(t) = q(t) = t$. From the solution developed in Example 7, we see that

$$p = te^s \text{ and } q = te^s,$$

both satisfying the initial data on $s = 0$. Then we have $u = t^2 e^{2s}$ with $x = te^s$ and $y = te^s + 1$; thus, eliminating t and s – which cannot be done uniquely – we obtain $u = x^2$ or $u = (y-1)^2$ or $u = x(y-1)$, but only the third option satisfies the original PDE; hence $u(x, y) = x(y-1)$.

A second example, based on a classical equation, is provided by $u_x^2 + u_y^2 = 1$. This equation – the *eikonal* equation – arises in geometrical optics (and we have normalised the speed of light here).

Example 9

Find the solution of $u_x^2 + u_y^2 = 1$, subject to $u = \lambda t$ on $x = y = t$, where λ is a constant.

The equation is $f \equiv p^2 + q^2 - 1 = 0$, and so we have

$$f_x = f_y = f_u = 0, f_p = 2p, f_q = 2q.$$

The initial data must satisfy

$$p^2+q^2=1 \text{ and } \frac{\mathrm{d}u}{\mathrm{d}t}=\lambda=p\frac{\mathrm{d}x}{\mathrm{d}t}+q\frac{\mathrm{d}y}{\mathrm{d}t}=p+q\,;$$

in view of the first equation of this pair, it is convenient to introduce $p=\sin\alpha$, $q=\cos\alpha$, and then we must have $\sin\alpha+\cos\alpha=\lambda$. Thus solutions exist only if λ is such as to allow the determination of α (a real parameter). The solution is described by the set of ODEs:

$$\frac{\mathrm{d}x}{\mathrm{d}s}=2p,\ \frac{\mathrm{d}y}{\mathrm{d}s}=2q,\ \frac{\mathrm{d}u}{\mathrm{d}s}=2p^2+2q^2=2,\ \frac{\mathrm{d}p}{\mathrm{d}s}=\frac{\mathrm{d}q}{\mathrm{d}s}=0\,.$$

Directly and immediately we see that

$$p=\sin\alpha,\ q=\cos\alpha,\ u=2s+\lambda t,\ x=2s\sin\alpha+t,\ y=2s\cos\alpha+t\,,$$

all satisfying the initial data on $s=0$. The last pair of equations yields

$$s=\frac{x-y}{2(\sin\alpha-\cos\alpha)} \text{ and } t=\frac{x\cos\alpha-y\sin\alpha}{\cos\alpha-\sin\alpha}$$

(and we shall assume that $\alpha\neq\pi/4,5\pi/4$, so avoiding the zero in the denominator); the solution is therefore

$$\begin{aligned}
u=2s+\lambda t&=\frac{y-x+\lambda(x\cos\alpha-y\sin\alpha)}{\cos\alpha-\sin\alpha}\\
&=\frac{x(\lambda\cos\alpha-1)+y(1-\lambda\sin\alpha)}{\cos\alpha-\sin\alpha}\\
&=\frac{x(\sin\alpha\cos\alpha-\sin^2\alpha)+y(\cos^2\alpha-\sin\alpha\cos\alpha)}{\cos\alpha-\sin\alpha}\\
&=x\sin\alpha+y\cos\alpha\,.
\end{aligned}$$

It is clear that we have developed an accessible method for solving the Cauchy problem for general, first order PDEs. Of course, any satisfactory and complete outcome of such an integration process requires that we are able to solve the various ODEs that arise – but that is a purely technical detail. There is, however, a more fundamental issue that we must now address: in what sense, if any, does this approach produce the general solution? Even if this is the case – and we must hope and expect that it does – we do not have a precise definition of the general solution. Furthermore, are there any other solutions *not* accessible from the general solution? (This possibility, e.g. singular solutions, is a familiar one in the theory of nonlinear ODEs.) Thus we now consider the solution of the general PDE from a different perspective.

3.4 The complete integral and the singular solution

Our starting point is to present a formal definition of the various solutions that, in general, are possessed by the general, first order PDE:

$$f(x, y, u, p, q) = 0.$$

Consider the function

$$\phi(x, y, u, a, b) = 0,$$

which is treated as defining a function $u(x, y, a, b)$ (implicitly), dependent on two independent variables, (x, y), and on two parameters, (a, b). We take, separately, the *x* and *y* partial derivatives, to give

$$\phi_x + \phi_u u_x = 0 \text{ and } \phi_y + \phi_u u_y = 0.$$

These two equations, together with $\phi = 0$ itself, can be used to eliminate *a* and *b*, to produce a first order PDE

$$f(x, y, u, p, q) = 0 \;\; (p \equiv u_x, q \equiv u_y);$$

if this is our PDE, then $\phi(x, y, u, a, b) = 0$ is a solution. (The elimination of *a* and *b* is possible provided that the appropriate Jacobian, $\phi_{xa}\phi_{yb} - \phi_{ya}\phi_{xb}$, is not zero, which we assume in this case.) Thus we have a two-parameter family of solutions, on the basis of which we can be precise about the nature of all solutions of the PDE.

1. A solution that involves both independent parameters, (a, b), is called the *complete integral*. (The construction of a complete integral will be described in detail below.)
2. A solution that is described by $b = b(a)$, for arbitrary functions *b*, is called a *general solution*. This can be a particular case of the general solution, by choosing a specific $b(a)$, which can be further particularised by choosing a value of *a*, or by constructing the envelope generated by varying *a* (see below).
3. If an envelope of the solutions in (1) exist, then this is a *singular solution*. (This solution is obtained by eliminating *a* and *b* between

$$\phi_a = 0, \phi_b = 0 \text{ and } \phi(x, y, u, a, b) = 0;$$

In case (2), the procedure is, of course, adopted for the one parameter *a*, but this is no longer a singular solution.)

The fundamental question now is: how do we construct the complete integral? Once we have developed these ideas, we shall be in a position to discuss all these possible solutions.

3.4.1 Compatible equations

The procedure that leads to the construction of a complete integral requires, first, the notion of *compatible equations*. We are given the PDE that we wish to solve:

$$f(x, y, u, p, q) = 0.$$

Let there be another equation, involving a parameter *a*, which has a solution that is *also* a solution of $f = 0$; we shall write this equation as

$$g(x, y, u, p, q, a) = 0.$$

The equations $f=0$ and $g=0$ are said to be *compatible*.

These two equations can be solved for $p(x,y,u)$ and $q(x,y,u)$ (provided that $J\equiv\partial(f,g)/\partial(p,q)\neq 0$); then we have a p and q which are consistent with a solution $u=u(x,y)$. A convenient way to express this is to consider a solution described by a parameter s:

$$u(s)=u(x(s),y(s)),$$

and so $\dfrac{\mathrm{d}u}{\mathrm{d}s}=u_x\dfrac{\mathrm{d}x}{\mathrm{d}s}+u_y\dfrac{\mathrm{d}y}{\mathrm{d}s}=px'+qy'$.

This can be expressed in the equivalent form of an exact differential as

$$\mathrm{d}u=p\mathrm{d}x+q\mathrm{d}y;$$

the integral of this equation generates a second arbitrary constant, b, and hence a solution $\phi(x,y,u,a,b)=0$. Additionally, in order that $p\mathrm{d}x+q\mathrm{d}y-\mathrm{d}u=0$ be completely integrable – as it must be – there must exist suitable conditions on p and q; we now find these.

Let the integral of

$$p\mathrm{d}x+q\mathrm{d}y-\mathrm{d}u=0$$

be $\psi(x,y,u)=0$; the differential form of this is

$$\psi_x \mathrm{d}x+\psi_y \mathrm{d}y+\psi_u \mathrm{d}u=0,$$

and these two versions must be identical. This will be the case if

$$\psi_x=\alpha p,\ \psi_y=\alpha q,\ \psi_u=-\alpha,$$

where $\alpha(x,y,u)$ ($\neq 0$) is an arbitrary function. Thus we obtain the identities

$$\psi_{xy}=(\alpha p)_y=(\alpha q)_x;\ \psi_{xu}=(\alpha p)_u=-\alpha_x;\ \psi_{yu}=(\alpha q)_u=-\alpha_y,$$

which give the set

$$\alpha_y p-\alpha_x q=\alpha(q_x-p_y);\ \alpha_u p+\alpha_x=-\alpha p_u;\ \alpha_u q+\alpha_y=-\alpha q_u.$$

Eliminating α_x and α_y yields

$$p(-\alpha q_u-q\alpha_u)-q(-\alpha p_u-p\alpha_u)=\alpha(q_x-p_y)$$

$$\text{i.e. } qp_u-pq_u+p_y-q_x=0,$$

which is a necessary condition that ensures integrability. This equation will provide the basis for constructing a compatible equation $g(x,y,u,p,q,a)=0$ and then, finally, a complete integral.

3.4.2 Equation for g

The given equation $f(x,y,u,p(x,y,u),q(x,y,u))=0$, when we take $\partial/\partial x$ and, separately, $\partial/\partial u$, gives the pair of equations

$$f_x+f_p p_x+f_q q_x=0 \text{ and } f_u+f_p p_u+f_q q_u=0.$$

Now we form f_x+pf_u:

$$f_x+pf_u=-(f_p p_x+f_q q_x+pf_p p_u+pf_q q_u)$$

$$=-(p_x+pp_u)f_p-(q_x+pq_u)f_q.$$

The corresponding calculation for $g(x,y,u,p(x,y,u),q(x,y,u))=0$ gives

$$g_x+pg_u=-(p_x+pp_u)g_p-(q_x+pq_u)g_q,$$

and then eliminating (p_x+pp_u), we obtain an expression for (q_x+pq_u):

$$q_x+pq_u=\frac{1}{J}\Big[(f_x+pf_u)g_p-(g_x+pg_u)f_p\Big],$$

where $J \equiv \dfrac{\partial(f,g)}{\partial(p,q)} \equiv f_p g_q - f_q g_p \neq 0$ (as previously assumed).

The calculation is repeated by taking $\partial/\partial y$ and then $\partial/\partial u$, which produces

$$p_y + qp_u = \frac{1}{J}\left[(g_y + qg_u)f_q - (f_y + qf_u)g_q\right].$$

These two identities are now used in the integrability condition

$$p_y + qp_u = q_x + pq_u$$

to give the equation

$$(g_y + qg_u)f_q - (f_y + qf_u)g_q = (f_x + pf_u)g_p - (g_x + pg_u)f_p.$$

This is the equation for g, which we may write in the form

$$f_p g_x + f_q g_y + (pf_p + qf_q)g_u - (f_x + pf_u)g_p - (f_y + qf_u)g_q = 0;$$

this is a linear equation for $g(x, y, u, p, q, a) = 0$. The methods of §2.4 provide the solution expressed by the system of ODEs:

$$\mathrm{d}x \quad \mathrm{d}y \quad \mathrm{d}u \quad \mathrm{d}p \quad \mathrm{d}q$$

which are precisely the equations for the solution of $f(x,y,u,p,q)=0$, (L)! These – now – familiar equations, derived in this way as the equation for $g(x,y,u,p,q,a)=0$, are usually called *Charpit's equations*. [Not much is known about Paul Charpit de Villecourt, a Frenchman; he is often described as a 'young mathematician'; he died in 1784, probably 'young' – his date of birth is not available. He submitted a paper to the French Royal Academy of Sciences in 1784 – note! – but it has never been published; in it he described his method for solving PDEs. His work was known to a very small group of mathematicians in France, his methods first being presented by Lacroix in 1814. His work is, ultimately, a recasting and alternative interpretation of Lagrange's.]

At first sight it might appear that we are no further forward; we have merely, by a devious route, produced the same solution method as for our original PDE ($f(x,y,u,p,q)=0$). However, there is an important difference here: *any* integral of this system is sufficient to define a compatible equation, $g(x,y,u,p,q,a)=0$. Once we have this equation, we may solve it with $f(x,y,u,p,q)=0$ for *p* and *q*; finally we integrate

$$\mathrm{d}u = p\mathrm{d}x + q\mathrm{d}y$$

to find a complete integral of our original equation. (Note that, although any integral that generates a *g* will suffice, it must necessarily contain *p* and/or *q*.)

Example 10

Find a complete integral of $u_x u_y = u$.

We have $f(x,y,u,p,q) \equiv pq-u=0$, so that

$$f_x = f_y = 0,\ f_u = -1,\ f_p = q,\ f_q = p$$

$$\text{i.e. } \frac{\mathrm{d}x}{q} = \frac{\mathrm{d}y}{p} = \frac{\mathrm{d}u}{2pq} = \frac{\mathrm{d}p}{p} = \frac{\mathrm{d}q}{q}.$$

The last equality has the integral $p/q=a$, where *a* is an arbitrary constant; this is therefore a compatible equation. This equation, with $pq=u$, yields the solution

$$p = \pm\sqrt{au},\ q = \pm\sqrt{\frac{u}{a}} \text{ (signs ordered).}$$

Then

$$\mathrm{d}u = p\mathrm{d}x + q\mathrm{d}y \text{ becomes } \mathrm{d}u = \pm(\sqrt{au}\,\mathrm{d}x + \sqrt{u/a}\,\mathrm{d}y)$$

which can be written

$$\frac{du}{\sqrt{u}} = \pm(\sqrt{a}\,\mathrm{d}x + \frac{1}{\sqrt{a}}\mathrm{d}y) \text{ or } 2\sqrt{u} = \pm(\sqrt{a}\,x + \frac{1}{\sqrt{a}}y) \pm 2b,$$

where *b* is the second arbitrary constant of integration (and the additional ± 2 is merely a convenience). Thus we have a complete integral

$$u(x,y)=\left(b+\tfrac{1}{2}(\sqrt{a}\,x+\frac{1}{\sqrt{a}}y)\right)^2,$$

where $a\ (>0)$ and *b* are the two parameters in the solution.

Comment: This solution should be compared with that obtained in Example 7. A general solution is described by arbitrarily assigning $b=b(a)$. Both the *a* and *b* partial derivatives, each set equal to zero, requires that $b+\tfrac{1}{2}(\sqrt{a}\,x+\frac{1}{\sqrt{a}}y)=0$ i.e. $u=0$ is a singular solution. That this is a singular solution is confirmed by the observation that there is no choice of the two parameters, *a* and *b*, that recovers the zero solution.

Let us explore these various properties of solutions by considering another example.

Example 11

Find a complete integral, a general solution and a singular solution of $u_x^2+u_y^2=1+2u$.

We have $f\equiv p^2+q^2-2u-1=0$, so that

$$f_x=f_y=0,\ f_u=-2,\ f_p=2p,\ f_q=2q;$$

the system of ODEs is

$$\frac{\mathrm{d}x}{2p}=\frac{\mathrm{d}y}{2q}=\frac{\mathrm{d}u}{2(p^2+q^2)}=-\frac{\mathrm{d}p}{-2p}=-\frac{\mathrm{d}q}{-2q}.$$

The last pair here give, directly, $p/q=a$ and then, with the original PDE, we obtain

$$(1+a^2)q^2=1+2u \text{ and so } q=\pm\sqrt{\frac{1+2u}{1+a^2}},\ p=\pm a\sqrt{\frac{1+2u}{1+a^2}} \text{ (signs ordered).}$$

Thus $\mathrm{d}u=p\mathrm{d}x+q\mathrm{d}y$ becomes $\mathrm{d}u=\pm\sqrt{\frac{1+2u}{1+a^2}}\,(a\mathrm{d}x+\mathrm{d}y)$, which gives

$$\frac{\mathrm{d}u}{\sqrt{1+2u}}=\pm\frac{a\mathrm{d}x+\mathrm{d}y}{\sqrt{1+a^2}} \text{ and so } \sqrt{1+2u}=\pm\frac{ax+y}{\sqrt{1+a^2}}\pm b,$$

where *b* is the second arbitrary constant (and the additional $\pm$ is a convenience). We have a complete integral

$$u=\frac{1}{2}\left(\frac{ax+y}{\sqrt{1+a^2}}+b\right)^2-\frac{1}{2}.$$

The general solution is represented by setting $b=b(a)$. The singular solution (requiring the $\partial/\partial a$ and $\partial/\partial b$ derivatives each set to zero) has the property

$\frac{ax+y}{\sqrt{1+a^2}}+b=0$, so that the singular solution is $u=-\frac{1}{2}$.

The only significant issue that we have yet to address is how we apply Cauchy data to a solution described in terms of a complete integral. This will also enable us to explain more fully how the designation $b=b(a)$ is used.

3.4.3 The complete integral with Cauchy data

Suppose that we have a complete integral, $\phi(x,y,u,a,b)=0$, of the general equation $f(x,y,u,p,q)=0$. Further, let the Cauchy data be represented by

$$u=u(t) \text{ on } x=x(t),\ y=y(t);$$

a solution of $f=0$, satisfying this data, will be one of three possibilities (provided that a solution does indeed exist). These are

1. A particular and specific choice of the values for *a* and *b*.
2. The singular solution (which, of course, is completely determined).
3. A choice of $b(a)$, followed by the construction of the envelope solution generated as a varies. (A choice of $b(a)$, followed by a specific value chosen for *a* is equivalent to case (1).)

Cases (1) and (2) are exceptional and are easily checked in particular exercises; the case that requires careful analysis is (3). This is the situation that arises most often.

Once we have excluded the possibility that specific values of *a* and *b* can admit the given initial data, we have excluded particular versions of the complete integral. However, another possible solution will be one in which the data curve *touches* each of the surfaces associated with the complete integral or, at least, a subset of them. The requirement that the Cauchy data be consistent with the complete integral gives

$$\phi(x(t),y(t),u(t),a,b)=0,$$

and that it touches each of these implies that any solution for *t* must be a repeated (double) root. This is most easily expressed by imposing the additional condition

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(x(t),y(t),u(t),a,b)=0.$$

The elimination of t between $\phi = 0$ and $\mathrm{d}\phi/\mathrm{d}t = 0$ yields a relation between a and b: this prescribes $b = b(a)$. However, we have already excluded the relevance of specific values of a and b, thus we must generate another solution consistent will all the preceding requirements. This is the envelope generated by the variation of a, given the $b(a)$ that is consistent with the initial data that generates double roots for t. (The situation that corresponds to a single root is when the data curve *intersects* a surface that contains a complete integral; this is the case when specific values of a and b allow the initial data to be satisfied.) Geometrically, this means that the initial-data curve everywhere touches all – or a subset of – the surfaces that correspond to the family of complete integrals: the envelope of this complete integral.

We shall explore these various aspects in the next two examples.

Example 12

Find the solution of $xu_x u_y + yu_y^2 = 1$ which satisfies $u = t$ on $x = 2t$, $y = 0$.

The equation is $f \equiv xpq + yq^2 - 1 = 0$, and so

$$f_x = pq\,,\ f_y = q^2\,,\ f_u = 0\,,\ f_p = xq\,,\ f_q = xp + 2yq\,;$$

the system of ODEs is then

$$\frac{\mathrm{d}x}{xq}=\frac{\mathrm{d}y}{xp+2yq}=\frac{\mathrm{d}u}{xpq+q(xp+2yq)}=-\frac{\mathrm{d}p}{pq}=-\frac{\mathrm{d}q}{q^2}.$$

The last pair of terms integrate to give $p/q=a$ – a compatible equation – and then the original PDE requires

$$(ax+y)q^2=1 \text{ and so } q=\pm\frac{1}{\sqrt{ax+y}},\ p=\pm\frac{a}{\sqrt{ax+y}} \text{ (signs ordered).}$$

Now $\mathrm{d}u=p\mathrm{d}x+q\mathrm{d}y$ becomes $\mathrm{d}u=\pm\frac{a\mathrm{d}x+\mathrm{d}y}{\sqrt{ax+y}}$ i.e. $u=\pm 2\sqrt{ax+y}+b$;

A complete integral can be written as

$$(u-b)^2=4(ax+y),$$

where a and b are the two parameters.

It is immediately clear that there is no choice of a and b that permits the initial data to be satisfied by the complete integral, so we require the appropriate envelope solution.

The initial data requires $(t-b)^2=8at$, and the double root for t that $t-b=4a$ (being the t-derivative of the preceding equation); the elimination of t yields $b=-2a$. Thus the relevant complete integral, with this $b(a)$, becomes

$$(u+2a)^2=4(ax+y),$$

and then the envelope solution requires $4(u+2a)=4x$ i.e. $a=(x-u)/2$. The solution is therefore given by

$$x^2=4\left(\frac{x-u}{2}\right)x+4y \text{ or } u(x,y)=\frac{x^2+4y}{2x}.$$

Example 13

Find a complete integral of $y(u_x^2-u_y^2)+uu_y=0$, and then the solutions that satisfy

(a) $u=3t$ on $x=2t$, $y=t$; (b) $u=2t$ on $x=t^2$, $y=0$.

The PDE is $f\equiv y(p^2-q^2)+uq=0$, so that

$$f_x=0,\ f_y=p^2-q^2,\ f_u=q,\ f_p=2yp,\ f_q=-2yq+u,$$

and the system of ODES is therefore

$$\frac{\mathrm{d}x}{2yp}=\frac{\mathrm{d}y}{-2yq+u}=\frac{\mathrm{d}u}{2yp^2+q(u-2yq)}=-\frac{\mathrm{d}p}{pq}=-\frac{\mathrm{d}q}{p^2-q^2+q^2}.$$

The last pair of terms are $\dfrac{\mathrm{d}p}{q}=\dfrac{\mathrm{d}q}{p}$, which integrate to give $p^2-q^2=a$,

which is a compatible equation. Thus the PDE becomes $ay+uq=0$, so

$$q=-\frac{ay}{u} \text{ and } p=\pm\sqrt{a+\frac{a^2y^2}{u^2}}.$$

Now $\mathrm{d}u=p\mathrm{d}x+q\mathrm{d}y$ becomes

$$\mathrm{d}u=\pm\sqrt{a+\frac{a^2y^2}{u^2}}\,\mathrm{d}x-\frac{ay}{u}\mathrm{d}y \text{ or } \frac{\mathrm{d}u+\frac{ay}{u}\mathrm{d}y}{\sqrt{a+\frac{a^2y^2}{u^2}}}=\pm\mathrm{d}x,$$

which is rewritten as $\dfrac{u\mathrm{d}u+ay\mathrm{d}y}{\sqrt{u^2+ay^2}}=\pm\sqrt{a}\,\mathrm{d}x$.

This can be integrated directly, to give

$$\sqrt{u^2+ay^2}=\pm\sqrt{a}\,x\pm b \text{ or } u^2=(b+x\sqrt{a})^2-ay^2;$$

this is a complete integral, with parameters a and b.

(a) The initial data, applied directly to this complete integral, yields

$$(3t)^2=(b+2t\sqrt{a})^2-at^2,$$

which is satisfied with $b=0$ and $a=3$; we have a solution expressed as

$$u^2=3(x^2-y^2).$$

(b) In this case we need a suitable envelope solution; first use the initial data in the complete integral:

$$4t^2=(b+t^2\sqrt{a})^2;$$

the t-derivative (for a double root) of this is $8t=4t(b+t^2\sqrt{a})\sqrt{a}$. Thus (selecting $t\neq 0$)

$$t^2=\frac{1}{\sqrt{a}}\left(\frac{2}{\sqrt{a}}-b\right), \text{ then } \frac{4}{\sqrt{a}}\left(\frac{2}{\sqrt{a}}-b\right)=\left(\frac{2}{\sqrt{a}}\right)^2 \text{ and so } b=\frac{1}{\sqrt{a}}.$$

The complete integral, with this $b(a)$, becomes $u^2=(1/\sqrt{a}+x\sqrt{a})^2-ay^2$; the a- derivative (for an envelope) requires

$$0=2\left(\frac{1}{\sqrt{a}}+x\sqrt{a}\right)\left(-\frac{1}{2}\frac{1}{{}^{3/2}}+\frac{1}{2}\frac{1}{\sqrt{a}}\right)-y$$

so $\quad y^2 = x^2 - \dfrac{1}{a^2}$ or $a = \dfrac{1}{\sqrt{x^2 - y^2}}$,

where the positive root is selected, ensuring that $\sqrt{a}$ is real.

The solution, satisfying the given Cauchy data, is conveniently expressed as

$$u^2 = \left(1 + \frac{x}{\sqrt{x^2 - y^2}}\right)^2 \sqrt{x^2 - y^2} - \frac{y^2}{\sqrt{x^2 - y^2}}.$$

This concludes the presentation of the standard techniques for solving the general, first order PDE. Some exercises follow, with answers provided at the end of the text, which should help the reader to master these ideas.

Exercises 3

1. Find a complete integral of $uu_x u_y = u_x + u_y$.
2. Find a complete integral of $u_x^2 + u_y^2 = xu$.
3. Find a complete integral of $x(u_x^2 + u_y^2) = uu_x$, and then the solutions satisfying(a) $u = 5t$ on $x = 2t$, $y = t$; (b) $u = 2t$ on $x = 0$, $y = t^2$.
4. Find a complete integral of $4uu_x - u_y^3 = 0$, and then that solution which satisfies $u = 4t$ on $x = 0$, $y = t$.

****

****

Answers

Exercise 1

(a) quasi-linear; (b) general; (c) semi-linear; (d) linear; (e) general.

Exercises 2

1. $u = x^2 + F(y(1-ux))$; $u = x^2 + \exp[y(1-ux)]$.

2. $u = \dfrac{1}{F(x^2 - e^{2y}) - y}$; $u = \dfrac{1}{\exp(e^{2y} - x^2 - 1) - y}$.

3. $u = F(y - ux^2)$; (a) $u = \dfrac{y}{2x^2} \pm \dfrac{1}{x}\sqrt{\dfrac{y^2}{4x^2} - 1}$ (note: can be written explicitly); (b) does not exist.

Exercises 3

1. $u = \sqrt{2x + a} + \sqrt{2y + b}$.

2. $u = \dfrac{1}{4}\left[\dfrac{2}{3}(x-a)^{3/2} \pm \sqrt{a}\, y + b\right]^2$.

3. $u^2 = a^2x^2 + (ay+b)^2$; (a) $u = \sqrt{5}\sqrt{x^2 + y^2}$; (b) $(u^2 - 2y)^2 = 4(x^2 + y^2)$.

4. $u = \left[\sqrt{a}\,(ax + y) + b\right]^2$; (a) $u = \dfrac{1}{6x}(\sqrt{y^2 + 12x} - y)\left[\dfrac{1}{6}(\sqrt{y^2 + 12x} - y) + y + \dfrac{6x}{\sqrt{y^2 + 12x} - y}\right]^2$,

and the boundary condition is satisfied as $\to$.

Part II

Partial differential equations: classification and canonical forms

List of Equations

This is a list of the types of equation, and specific examples, whose solutions are discussed.

Preface

This text is intended to provide an introduction to the techniques that lie behind the classification of second-order partial differential equations which, in turn, leads directly to the construction of the canonical form. This topic is likely to be included, albeit only in outline, in any mathematics degree programme that involves a discussion of applied mathematical methods. The material here has been written to provide a general – but broad – introduction to the relevant ideas, and not as a text closely linked to a specific course of study. We start with a brief overview of the relevant ideas associated with first-order equations (and there is far more detail in Part I); this is then developed and applied to second-order equations. The main intention is to present the material so that it can be used as an adjunct to a number of different courses or modules – or simply to help the reader gain a deeper understanding of these important mathematical ideas. The aim is to go beyond the methods and techniques that are usually presented, but all the standard ideas are discussed here (and can be accessed through the comprehensive index).

It is assumed that the reader has a basic knowledge of, and practical experience in, the methods for solving elementary ordinary differential equations. This brief text does not attempt to include any detailed applications of these equations; this is properly left to a specific module that might be offered in a conventional applied mathematics or engineering or physics programme. However, some important examples of these equations will be included, which relate to specific areas of (applied) mathematical interest.

The approach adopted here is to present some general ideas, which might involve a notation, or a definition, or a classification, but most particularly detailed applications of the ideas explained through a number of carefully worked examples – we present 20. A small number of exercises, with answers, are also offered, although it must be emphasised that this notebook is not designed to be a comprehensive text in the conventional sense.

1 Introduction

The study of partial differential equations, and the construction of their solutions, is at the heart of almost every problem in applied mathematics (and many problems in physics and engineering). In this text, we will describe the general character of an important class of second order equations – a class that includes the vast majority of equations that are of interest and practical relevance. This class will be carefully defined and described, as will the very significant *classification* of the equations. This classification has far-reaching consequences for both the method of solution and the nature of the solution. In addition, these ideas lead to a standard representation of the equation – the *canonical form* – which often, in turn, leads to the construction of the general solution.

We shall start by briefly examining a class of simple first-order partial differential equations in order to introduce the concept of a *characteristic variable*. This is then developed further, thereby providing the basis for both the classification and the canonical form of second-order equations. These results are interpreted in the context of the three classical equations of mathematical physics, but they are also applied to more general – and important – equations. Additionally, some ideas and techniques that relate to methods of solution and interpretation (specifically *simple waves* and *Riemann invariants*) will be presented.

1.1 Types of equation

We first describe the types of equation – both first and second order – that we shall analyse in this text. We consider functions, $u(x,y)$, defined by suitable partial differential equations; in the case of first-order equations, these are represented by the relation

$$f\left(u, x, y, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0.$$

We should note, at this stage, that we shall limit our discussion to functions of two variables, although some of the ideas go over to higher dimensions; detailed methods of solution (e.g. separation of variables) in two and higher dimensions are to be found in the text 'Partial differential equations: method of separation of variables and similarity and travelling-wave solutions' in *The Notebook Series*.

We start with linear, homogeneous equations that contain only derivative terms:

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = 0,$$

and then extend the ideas to *quasi-linear* equations:

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u)$$

i.e. linear in the two derivatives, but otherwise nonlinear and inhomogeneous. The second-order equations that we discuss are linear in the highest derivatives, in the form

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + 2b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} = d\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right);$$

this is the *semi-linear* equation. (The equation in which a, b and c also depend on u and its two first partial derivatives is the *quasi-linear* equation, which will not be discussed here, although many of the principles that we develop work equally well in this case.) Examples of the three types mentioned above are

$$x\frac{\partial u}{\partial x}+(x+y)\frac{\partial u}{\partial y}=0\,;$$

$$x\frac{\partial u}{\partial x}+u\frac{\partial u}{\partial y}=u\,;$$

$$y\frac{\partial^2 u}{\partial x^2}+(x+y)\frac{\partial^2 u}{\partial x\partial y}+x^2\frac{\partial^2 u}{\partial y^2}=xu+\frac{\partial u}{\partial x}+y\left(\frac{\partial u}{\partial y}\right)^2,$$

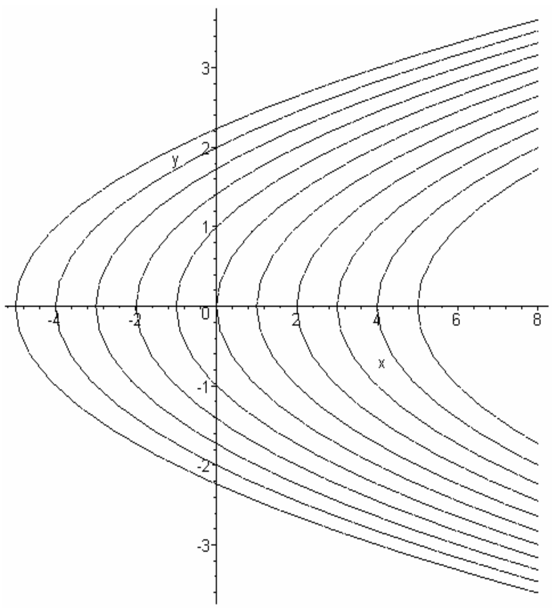
respectively.

2 First-order equations

The fundamental idea that we exploit is best introduced *via* a couple of elementary functions. First, let us suppose that we are given any function, $f(x)$, then it is obvious (and apparently of no significance) that $f(x)$ is constant whenever x is constant; this is usually described by stating that 'f is constant on lines $x = \text{constant}$'. Expressed like this, we are simply re-interpreting the description in terms of the conventional rectangular Cartesian coordinate system: $y = f(x)$, and then $y = f(x) = \text{constant}$ when $x = \text{constant}$. To take this further, let us now suppose that we have a function of two variables, $f(x, y)$, but one that depends on a specific combination of x and y e.g.

$$f(x, y) = \sin\left(x - y^2\right).$$

This function changes as x and y vary (independently), but it has the property that $f = \text{constant}$ on lines $x - y^2 = \text{constant}$; these lines are shown below.



The function takes (in general different) constant values on each line. It is quite apparent that this interpretation of the function provides more information than simply to record that it is some function of the two variables. But we may take this still further.

The conventional Cartesian coordinate axes, and the lines parallel to them, are defined by lines $x = \text{constant}$ and $y = \text{constant}$, which is usually regarded as sufficient and appropriate for describing functions $f(x, y)$. However, functions such as our example above, $\sin\left(x - y^2\right)$, are better described by lines $x - y^2 = \text{constant}$ (perhaps together with either $x = \text{constant}$ or $y = \text{constant}$). These should then be the appropriate lines to use (in place of $x = \text{constant}$, $y = \text{constant}$); these 'coordinate' lines are those reproduced in the figure above. The essence of our approach to solving partial differential equations is to find these special coordinate lines, usually called *characteristic lines*.

2.1 The linear equation

Here we consider the equation

$$a(x,y)\frac{\partial u}{\partial x}+b(x,y)\frac{\partial u}{\partial y}=0,$$

where the coefficients a and b will be assumed continuous throughout the domain (D) where the solution, $u(x,y)$, is to exist. To proceed, we seek a solution that depends on x (say) and $\xi=\xi(x,y)$; the solution is defined in 2-space so, in general, we must transform into some corresponding 2-space (that is, using two independent variables). The aim, now, is to determine the function $\xi(x,y)$ so that the equation for u becomes sufficiently simple, allowing it to be integrated; indeed, we hope that this results in a solution that depends – essentially – on only one variable (namely, ξ). We note, in passing, that if we choose $\xi=y$, then we simply recover the original problem (which does confirm that a transformation exists).

Let us write, for clarity,

$$u(x,y)\equiv U\left[x,\xi(x,y)\right]$$

then

$$\frac{\partial u}{\partial x}=\frac{\partial U}{\partial x}+\frac{\partial \xi}{\partial x}\frac{\partial U}{\partial \xi}\ \text{ and }\ \frac{\partial u}{\partial y}=\frac{\partial \xi}{\partial y}\frac{\partial U}{\partial \xi};$$

thus the equation for *u* (now *U*) becomes

$$a(x,y)\left(U_x+\xi_x U_\xi\right)+b(x,y)\,\xi_y U_\xi=0,$$

where we have used subscripts to denote the partial derivatives. Now we choose $\xi(x,y)$ such that $a\xi_x+b\xi_y=0$ (and we note that is no more than the original partial differential equation!) which leaves the equation for *U* as simply $aU_x=0$,

and so provided $a\neq 0$ throughout D, then

$$U_x=0 \text{ or } U=F(\xi) \text{ i.e. } u(x,y)=F\left[\xi(x,y)\right],$$

where *F* is an arbitrary function; this constitutes the general solution and confirms that we may, indeed, seek a solution that depends on one (specially chosen) variable. (It should be clear that arbitrary constants in the solution of ordinary differential equations go over to arbitrary functions in the solution of partial differential equations.) The function $\xi(x,y)$ is determined completely when we consider lines $\xi=$ constant for then

$$\xi_x+\frac{\mathrm{d}y}{\mathrm{d}x}\xi_y=0 \text{ or } \frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{\xi_x}{\xi_y}=\frac{b(x,y)}{a(x,y)}.$$

Note that, since we now know that we may describe the solution as $u=$constant on certain lines, we may equally write directly that $u=$constant on lines $y'=b/a$, without the need to introduce ξ at all. Nevertheless, as we shall see, the introduction of characteristic lines is fundamental to any generalisation of this technique. So the integration of the *ordinary* differential equation $y'=b(x,y)/a(x,y)$ yields the characteristic lines $\xi(x,y)=\text{constant}$ (this being the arbitrary constant of integration), and then $u=F(\xi)$ is the required general solution (which is equivalent, of course, to $u=\text{constant}$ on lines $\xi(x,y)=\text{constant}$).

Example 1

Find the general solution of the partial differential equation $yu_x+x^2u_y=0$.

The characteristic lines are given by the solution of the ordinary differential equation

$$y'=\frac{x^2}{y}\ (y\neq 0) \text{ and so } \tfrac{1}{2}y^2=\tfrac{1}{3}x^3+\text{constant},$$

or $\xi(x,y)=3y^2-2x^3=\text{constant}$, which describes the characteristic lines. Thus the general solution is

$$u(x,y)=F(3y^2-2x^3),$$

where *F* is an arbitrary function.

Comment: We can check this solution directly (at least, if F is a differentiable function), for then $u_x = -6x^2 F'(\xi)$ and $u_y = 6yF'(\xi)$, so that

$$yu_x + x^2 u_y = -6yx^2 F' + 6yx^2 F' = 0,$$

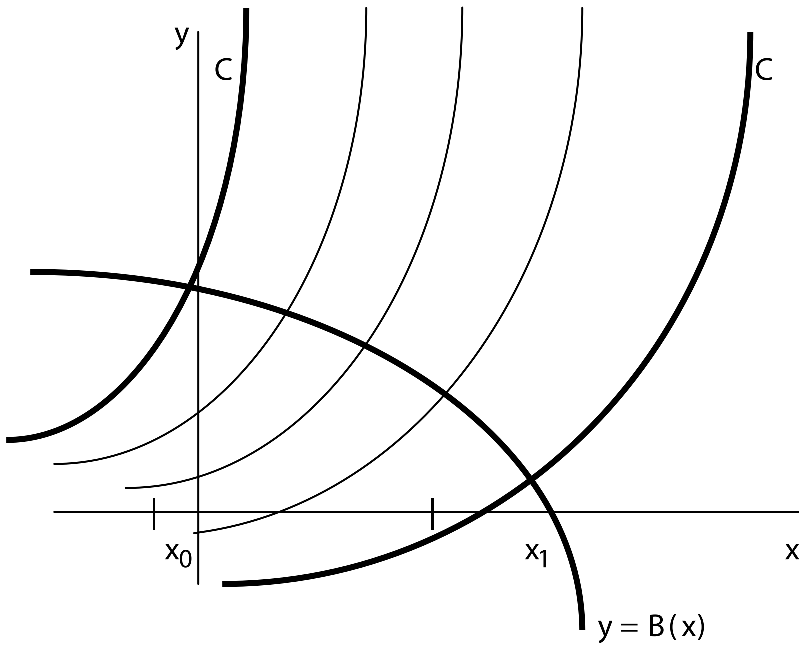
and observe that this does not require the condition $y \neq 0$.

The next issue that we must address is how the arbitrary function, F, is determined in order to produce – we hope – a unique solution of a particular problem.

2.2 The Cauchy problem

Let us suppose that there is a line, $y = B(x)$, $x_0 \leq x \leq x_1$, along which $u = V(x)$ is given (which might be expressed in terms of y or x and y); this describes the boundary condition (given on the boundary curve, $y = B(x)$) that should determine the unique solution of a first-order partial differential equation. This constitutes the *Cauchy problem*. [A.-L. Cauchy, 1789-1857, the leading French mathematician of his day.] However, two cases can arise and they are significantly different. First, if the line $y = B(x)$ is one of the characteristic lines, then $V(x)$ must be constant on this line; otherwise the solution does not exist: we would have an inconsistency. If it does exist, the solution is undetermined elsewhere; we simply know that $u = \text{constant}$ on lines $\xi(x, y) = \text{constant}$.

On the other hand, if $y = B(x)$ is not a characteristic line then, wherever it intersects a characteristic line, u takes the constant value (along that characteristic line) that is determined by $V(x)$ evaluated at the point of intersection. The solution is then defined along $y = B(x)$, $x_0 \leq x \leq x_1$, and wherever the corresponding characteristic lines develop in the (x, y)-plane; this is depicted in the figure below.



The solution exists in the region bounded by the two characteristic lines, labelled C; data is given on the line $y = B(x)$, between $x = x_0, x_1$.

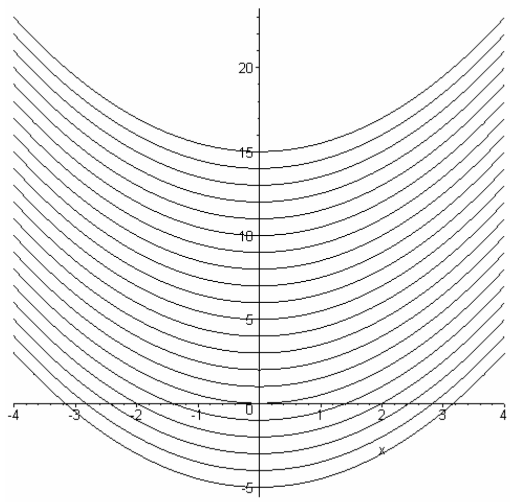
Example 2

Find the solution of the partial differential equation $u_x + xu_y = 0$ which satisfies the boundary condition $u = \sin y$, $-\infty < y < \infty$, on $x = 0$. Where is this solution defined?

The characteristic lines are given by the solution of $\frac{\mathrm{d}y}{\mathrm{d}x} = x$ i.e. $y - \frac{1}{2}x^2 = \text{constant}$, and so the general solution is $u(x, y) = F\left(y - \frac{1}{2}x^2\right)$. Now on $x = 0$ we are given $u(0, y) = F(y) = \sin y$, so the required solution is

$$u(x, y) = \sin\left(y - \frac{1}{2}x^2\right).$$

The lines $y - \frac{1}{2}x^2 = \text{constant}$ span the whole plane, and the data is on all of the *y*-axis, so the solution exists throughout the (*x*, *y*)-plane; this is clear from the figure below.



All the characteristics intersect the *y*-axis, and so the whole plane is covered.

2.3 The quasi-linear equation

The first-order quasi-linear equation is linear in the first partial derivatives only, so its most general form is

$$a(x, y, u)\frac{\partial u}{\partial x} + b(x, y, u)\frac{\partial u}{\partial y} = c(x, y, u).$$

Although this equation appears to be much more complicated than the previous one, we may still apply the technique developed in §2.1; we introduce $\xi = \xi(x, y)$ and then seek a solution $u(x, y) = U[x, \xi(x, y)]$, so that we obtain

$$a\left(U_x + \xi_x U_\xi\right) + b\xi_y U_\xi = c.$$

On lines

$$a\xi_x + b\xi_y = 0 \text{ i.e. lines given by } \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b}{a},$$

we have

$$aU_x = c;$$

thus we now have a *pair* of simultaneous ordinary differential equations:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b(x,y,U)}{a(x,y,U)} \text{ and } \frac{\mathrm{d}U}{\mathrm{d}x} = \frac{c(x,y,U)}{a(x,y,U)},$$

because the y-dependence in the second equation is prescribed by the first.

Example 3

Find the general solution of the partial differential equation $xu_x + uu_y = u$.

In this case, we have the pair of equations

$$\frac{dy}{dx} = \frac{u}{x}, \quad \frac{du}{dx} = \frac{u}{x} \quad (x \neq 0)$$

so we obtain $\ln|u| = \ln|x| + \text{constant}$ or $u = Ax$

and then $\dfrac{dy}{dx} = A$ or $y = Ax + B$,

where A and B are arbitrary constants. Thus we may express the solution as

$$\frac{u}{x} = A \text{ is constant on lines } y - u = B = \text{constant},$$

which can be written as the general solution

$$u = xF(y-u),$$

where F is an arbitrary function.

Comments: This solution is, in general, *implicit* for $u(x,y)$; only for special initial data will it be possible to write the solution *explicitly*. For example, given that $u = \frac{1}{2}y$ on $x = 1$, then $\frac{1}{2}y = F\left(\frac{1}{2}y\right)$ i.e. $F(z) = z$; thus

$$u = x(y-u) \text{ or } u = \frac{xy}{1+x}.$$

Further, we can check that we have obtained the general solution by computing u_x and u_y (at least, for differentiable F), although some care is required:

$$u_x = F - xu_x F' \;\&\; u_y = x(1-u_y)F' \text{ and so } u_x = \frac{F}{1+xF'} \;\&\; u_y = \frac{xF'}{1+xF'};$$

it then follows directly that $xu_x + uu_y = u$.

We now use the idea of characteristic lines to explore the corresponding problems for second-order partial differential equations, which is the main concern of this Notebook. However, we shall first explore the special – and simple – case afforded by the wave equation. When we have seen how the technique for first-order equations is readily employed (Chapter 3) for this equation, we will generalise the approach.

Exercises 2

1. Find the general solutions of these partial differential equations:

a) $3u_x + 2xu_y = x$; (b) $u_x + 2u_y = u$; (c) $u_t + uu_x = 0$.

2. Find the solutions of these partial differential equations that satisfy the given boundary conditions:

a) $u_x + u_y = u^2$ with $u = x$ on $y = -x$;

b) $u_t + uu_x = 0$ with $u = \tanh x$ on $y = 0$, and then decide if u_x is defined for $\forall t \in [0, \infty)$ and $\forall x \in (-\infty, \infty)$.

****

****

3 The wave equation

We motivate the more general analysis of an important class of second-order equations (in two independent variables) by considering the equation:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0,$$

the classical wave equation. This equation arises in many elementary studies of wave propagation; it describes the amplitude, $u(x,t)$, of a wave as it propagates in one dimension. (Because we prefer a natural choice of notation for distance (x) and time (t), these have been used here, rather than the more conventional x and y, although we shall revert to these in our analysis of the general equation.) The equation contains a positive constant, c, which will play a significant rôle in the interpretation of the resulting solution.

3.1 Connection with first-order equations

To proceed with our wave equation, we make an important observation:

$$\frac{\partial^2}{\partial x^2} - c^2 \frac{\partial^2}{\partial t^2} \equiv \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right),$$

demonstrating that this second-order differential operator can be factorised. Thus the wave equation can be written as

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}\right) = 0,$$

or, equivalently, $\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right) = 0.$

Clearly a solution of the former equation can be obtained from

$$\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = 0,$$

and from $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$

when we use the latter. Thus in the first case we have

$u = \text{constant}$ on lines $\frac{\mathrm{d}x}{\mathrm{d}t} = c$

i.e. $u = \text{constant}$ on $x - ct = \text{constant}$

so $u(x,t) = F(x - ct)$;

correspondingly, the second gives

$u = \text{constant}$ on lines $x + ct = \text{constant}$ so $u(x,t) = G(x + ct)$.

All this suggests that we seek a solution of the wave equation in the form $u(x,t) = U(\xi,\eta)$, where $\xi = x - ct$ and $\eta = x + ct$; thus we obtain

$$u_x = U_\xi + U_\eta \text{ and } u_t = -cU_\xi + cU_\eta,$$

and then

$$u_{tt} - c^2 u_{xx} = c^2\left(U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}\right) - c^2\left(U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}\right) = -4c^2 U_{\xi\eta}.$$

Thus our wave equation

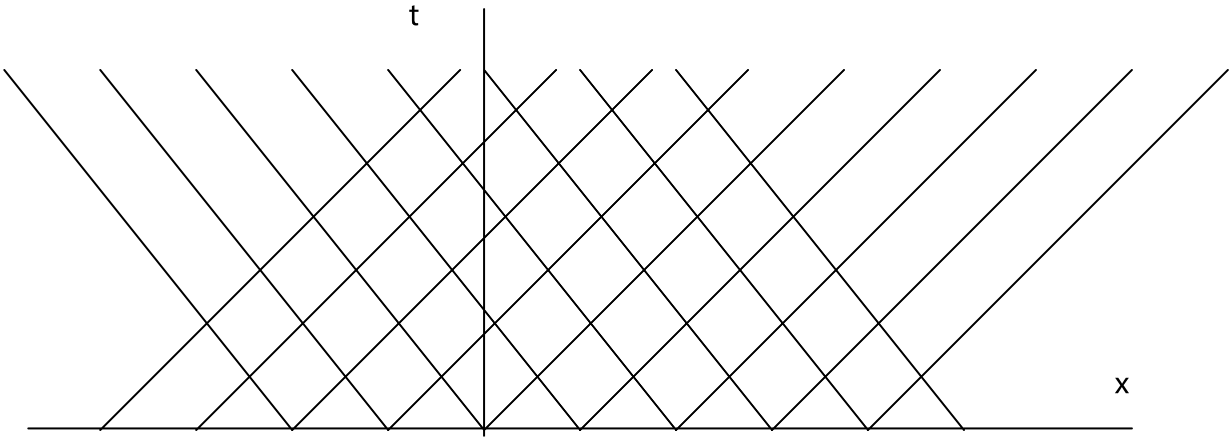
$$u_{tt} - c^2 u_{xx} = 0 \text{ becomes } U_{\xi\eta} = 0,$$

and this latter version of the wave equation has the general solution

$$U(\xi,\eta) = F(\xi) + G(\eta) \text{ or } u(x,t) = F(x - ct) + G(x + ct),$$

where *F* and *G* are arbitrary functions; this solution is usually referred to as *d'Alembert's solution* of the wave equation. [Jean le Rond d'Alembert (1717-1783) was a foundling, discovered on the steps of a church in Paris. He had a good upbringing, financed by a courtier (who was, it is assumed, his father). He trained first as a lawyer, then spent a year on medical studies, before eventually settling on mathematics as a career. He was the first to analyse, in 1746, the motion of a vibrating string, introducing both his general solution of the wave equation and the decomposition into harmonic components; he also gave a convincing derivation of the wave equation for this phenomenon. It is fair to record that he probably initiated the study of partial differential equations. He also made important contributions to dynamics – d'Alembert's principle – to complex analysis and to celestial mechanics.]

We now have two sets of characteristic lines, and each member of each set (in this case) is a straight line, one set with slope $1/c$ and the other with slope $-1/c$ (from $t = \pm x/c + \text{constant}$); see the figure below



3.2 Initial data

Suppose that we have Cauchy data on $t = 0$ (the most common situation for the classical wave equation: *initial data*) of the form

$$u(x,0) = p(x)\,,\; u_t(x,0) = q(x) \text{ both for } -\infty < x < \infty\,,$$

although we should note that a special case, of some importance, can be included here: $p(x)$ and $q(x)$ both zero outside some finite domain. (This is usually described as 'data on compact support'.) The general solution, expressed in terms of *F* and *G*, then requires

$$F(x) + G(x) = p(x) \text{ and } -cF'(x) + cG'(x) = q(x)\,,$$

and this second condition can be integrated to give

$$G(x) - F(x) = \frac{1}{c}\int^{x} q(x')\, dx'$$

which then leads to

$$F(x) = \frac{1}{2}p(x) - \frac{1}{2c}\int^{x} q(x')\, dx'\,;\; G(x) = \frac{1}{2}p(x) + \frac{1}{2c}\int^{x} q(x')\, dx'.$$

Thus, in terms of the given data, the solution becomes

$$u(x,t) = \frac{1}{2}\left\{p(x-ct) + p(x+ct) + \frac{1}{c}\int_{x-ct}^{x+ct} q(x')\, dx'\right\},$$

which takes a more readily recognisable form if we assume that there exists a $Q(x)$ such that $q(x) = Q'(x)$, for then

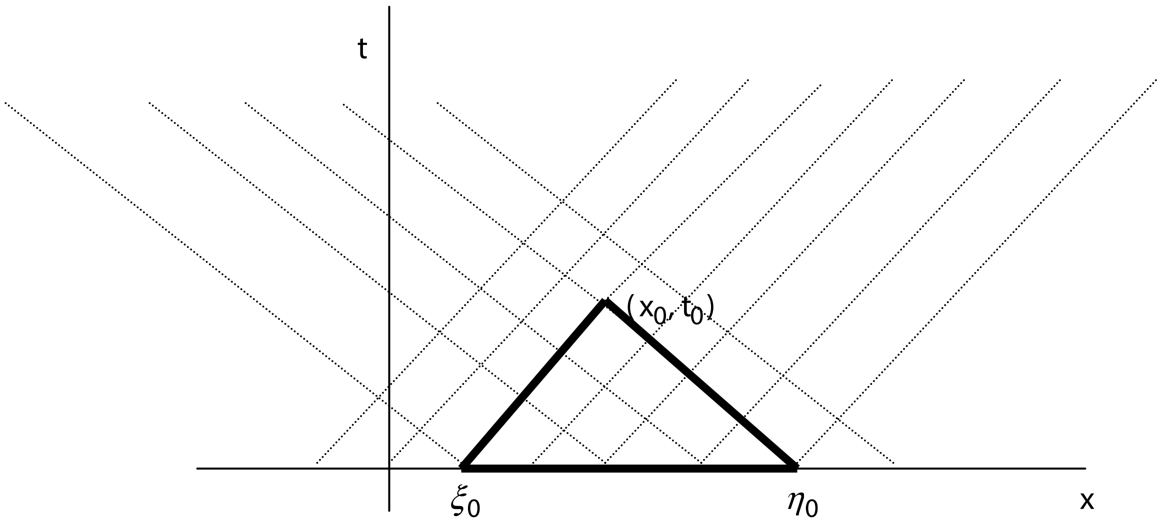
$$u(x,t) = \frac{1}{2}\left\{p(x-ct) - \frac{1}{c}Q(x-ct)\right\} + \frac{1}{2}\left\{p(x+ct) + \frac{1}{c}Q(x+ct)\right\}.$$

This solution, as we have already noted, comprises two components, one constant along lines $x - ct = \text{constant}$ and the other along lines $x + ct = \text{constant}$; these lines, and hence the solution, cover the half plane $t \geq 0$, $-\infty < x < \infty$. Observe that, with the identification of *t* for time and *x* for distance, then the profile (shape) represented by $F(x)$ (at $t = 0$) is a wave component that moves with unchanging form along lines $x - ct = \text{constant}$, which implies that this component moves at a speed *c* (to the right because $c > 0$); correspondingly, *G* moves at the speed *c* to the left. These two waves do not interact. Further, if the data is on compact support then, after a finite time, the two waves travelling in opposite directions will no longer overlap.

Now consider a point, (x_0, t_0), in the domain where the solution is defined; one of each of the pair of characteristic lines cross at this point. These two lines are

$$x - ct = x_0 - ct_0 \ (= \xi_0) \ \text{ and } \ x + ct = x_0 + ct_0 \ (= \eta_0)$$

so they meet the datum line, $t = 0$, at $x = \xi_0$ and $x = \eta_0$, respectively; see the figure below.



The triangular region bounded by these two characteristic lines and the segment of the datum line is called the *domain of dependence*: it is the region of the plane in which the solution, *u*, depends on the data given on the line segment from $x = \xi_0$ to $x = \eta_0$. A particularly illuminating problem that develops this idea is afforded by the following example, especially when we take a limit that results in a concentrated disturbance i.e. a disturbance at a point.

Example 4

Find the solution of the wave equation with $u(x,0) = p(x) \equiv 0$ and $u_t(x,0) = q(x) = \frac{1}{2a} e^{-|x|/a}$ where $a > 0$ is a constant, and then examine the form taken by this solution as $a \to 0$. (Note that

$$\int_{-\infty}^{\infty} q(x)\, dx = \frac{1}{2a}\left\{\int_{-\infty}^{0} e^{x/a} dx + \int_{0}^{\infty} e^{-x/a} dx\right\} = \frac{1}{2}\left[e^{x/a}\right]_{-\infty}^{0} + \frac{1}{2}\left[-e^{-x/a}\right]_{0}^{\infty} = 1,$$

so that the area is 1 for all *a* (> 0); as $a \to 0$ the amplitude (height) increases and the 'width' decreases: this is therefore a model for the *delta function*.)

The solution is, directly,

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} q(x')\,dx'$$

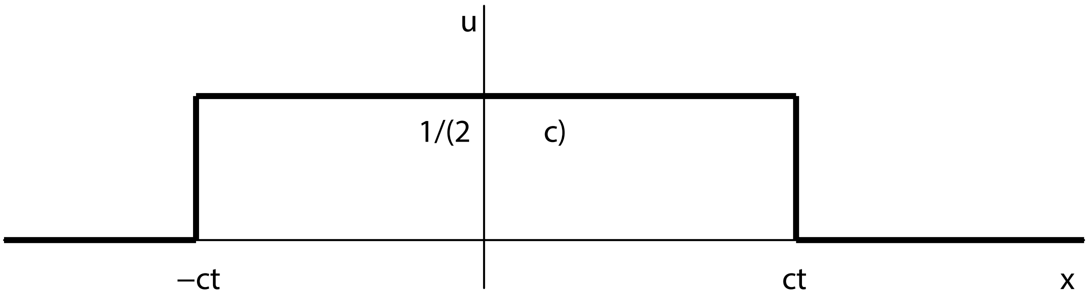
$$= \begin{cases} \dfrac{1}{4ac}\displaystyle\int_{x-ct}^{x+ct} e^{-x'/a}\,dx', & x > ct \\ \dfrac{1}{4ac}\displaystyle\int_{x-ct}^{0} e^{x'/a}\,dx' + \dfrac{1}{4ac}\displaystyle\int_{0}^{x+ct} e^{-x'/a}\,dx', & -ct \le x \le ct \\ \dfrac{1}{4ac}\displaystyle\int_{x-ct}^{x+ct} e^{x'/a}\,dx', & x < -ct \end{cases}$$

$$= \begin{cases} \dfrac{1}{4c}\left(e^{-(x-ct)/a} - e^{-(x+ct)/a}\right), & x > ct \\ \dfrac{1}{4c}\left(2 - e^{(x-ct)/a} - e^{-(x+ct)/a}\right), & -ct \le x \le ct \\ \dfrac{1}{4c}\left(e^{(x+ct)/a} - e^{(x-ct)/a}\right), & x < -ct \end{cases}$$

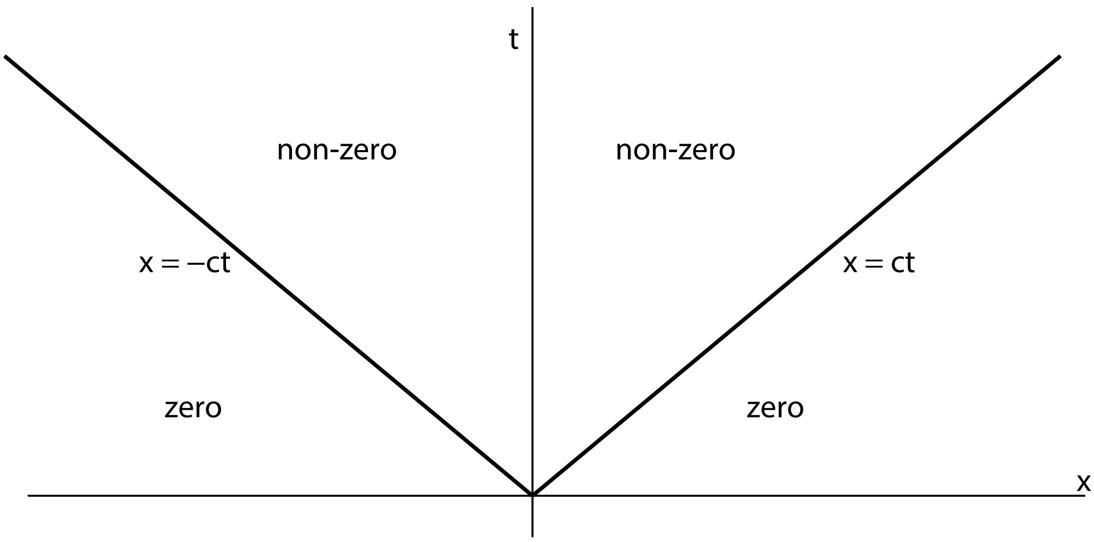
and as $a \to 0^+$, this gives (for $t > 0$)

$$u(x,t) \to \begin{cases} 0, & x > ct \\ 1/4c & \text{on} \quad x = ct \\ 1/2c, & -ct < x < ct \\ 1/4c & \text{on} \quad x = -ct \\ 0, & x < -ct. \end{cases}$$

Thus, for any $t > 0$, the solution in this limit can be represented diagrammatically as



and at $t = 0$ we have the delta function centred at $x = 0$. The solution is then non-zero in the sector $-ct \le x \le ct$:



Comment: This sector, where the solution is non-zero, but which emanates from a point on the axis $t = 0$, is called the *domain of influence* i.e. every point in this region is influenced (affected) by the disturbance at the point.

To complete this initial discussion, we consider one more illuminating example that involves a simple application of the wave equation.

Example 5

Find the solution of the classical wave equation which satisfies $u = a\sin(\omega t)$ and $u_x = 0$ on $x = \lambda ct$ (where $0 < \lambda < 1$, a and ω are constants). (This corresponds to a disturbance – an oscillation in time – which is generated at a point moving at a speed of λc in the positive x-direction.)

The general (d'Alembert) solution is

$$u(x,t) = F(x-ct) + G(x+ct)$$

and we require

$$a\sin(\omega t) = F(\lambda ct - ct) + G(\lambda ct + ct) \text{ and } 0 = F'(\lambda ct - ct) + G'(\lambda ct + ct).$$

This latter condition can be integrated once (in t) to give

$$\frac{1}{c(\lambda-1)}F[(\lambda-1)ct] + \frac{1}{c(\lambda+1)}G[(\lambda+1)ct] = \frac{A}{c},$$

where A is an arbitrary constant. Upon solving for F and G, these yield

$$F[(\lambda-1)ct] = -\tfrac{1}{2}a(\lambda-1)\sin(\omega t) + \tfrac{1}{2}\left(\lambda^2-1\right)A;$$

$$G[(\lambda+1)ct] = \tfrac{1}{2}a(\lambda+1)\sin(\omega t) - \tfrac{1}{2}\left(\lambda^2-1\right)A, ;$$

$$F(\alpha) = -\tfrac{1}{2}a(\lambda-1)\sin\left[\frac{\alpha\omega}{(\lambda-1)c}\right] + \tfrac{1}{2}\left(\lambda^2-1\right)A;$$

or $$G(\beta) = \tfrac{1}{2}a(\lambda+1)\sin\left[\frac{\beta\omega}{(\lambda+1)c}\right] - \tfrac{1}{2}\left(\lambda^2-1\right)A.$$

Thus the required solution is

$$u(x,t) = \tfrac{1}{2}a\left\{(1+\lambda)\sin\left[\frac{\omega}{c(1+\lambda)}(x+ct)\right] - (1-\lambda)\sin\left[\frac{\omega}{c(1-\lambda)}(x-ct)\right]\right\}.$$

Comment: This solution describes the familiar *Doppler effect (shift)*: the wave moving in the direction of motion (positive x) has a higher frequency ($\omega/(1-\lambda)$) than the wave moving in the opposite direction ($\omega/(1+\lambda)$). This phenomenon was first proposed, on purely physical grounds, by an Austrian physicist C.J. Doppler in 1842; his idea was tested in Utrecht in 1845 by moving an open railway carriage, at speed, containing a group of trumpeters who played the same note as other trumpeters who stood at the side of the track!

Exercises 3

1. Find the solution of the classical wave equation, $u_{tt} - c^2 u_{xx} = 0$, which satisfies

$$u(x,0) = 0,\ u_t(x,0) = \frac{1}{1+x^2}\ (-\infty < x < \infty).$$

2. Find the solution of the *signalling problem* for the classical wave equation (given in Q.1), for which the boundary and initial conditions are

$$u(x,0) = 0 \text{ and } u_t(x,0) = 0 \text{ both in } x > 0;\ u(0,t) = f(t) \text{ for } t \geq 0.$$

(You may find it convenient to introduce the *Heaviside step function*:

$$\mathrm{H}(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

in your representation of the solution.)

****

****

4 The general semi-linear partial differential equation in two independent variables

The general equation that we consider here is

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} = d(x,y,u,u_x,u_y)$$

where a, b, c and d are given; we have written partial derivatives using the subscript notation. (The appearance of the '2' here is both a small convenience and a standard convention in these equations.) The basic procedure follows that which was so successful for the wave equation, namely, to find a suitable transformation of variables. This will necessitate the consideration of three cases, which leads to the essential classification of these equations and then to the standard (*canonical*) versions of the equation (which might then be susceptible to simple solution-methods).

4.1 Transformation of variables

Although we eventually require the solution $u(x,y)$, we represent this in the form

$$u(x,y) \equiv U\left[\xi(x,y),\eta(x,y)\right]$$

for suitable choices of the new coordinates

$$\xi(x,y) = \text{constant}\,,\ \eta(x,y) = \text{constant}\,,$$

which replace the conventional Cartesian set: $x = \text{constant}$, $y = \text{constant}$. Thus we have, for example,

$$u_x = \xi_x U_\xi + \eta_x U_\eta;\ u_y = \xi_y U_\xi + \eta_y U_\eta,$$

and then U_ξ and U_η exist provided that the *Jacobian* $J = \xi_x\eta_y - \xi_y\eta_x \neq 0$ (and note that the choice $\xi = x$, $\eta = y$ – which is no transformation at all, of course – generates $J = 1$, so some ξ, η certainly do exist). [K.G.J. Jacobi, 1804-1851, German mathematician, who did much to further the theory of elliptic functions.] However, we also require second partial derivatives; for example, expressed as differential operators, we have

$$\frac{\partial^2}{\partial x^2} \equiv \frac{\partial}{\partial x}\left(\xi_x\frac{\partial}{\partial\xi} + \eta_x\frac{\partial}{\partial\eta}\right) \equiv \left(\xi_x\frac{\partial}{\partial\xi} + \eta_x\frac{\partial}{\partial\eta}\right)\left(\xi_x\frac{\partial}{\partial\xi} + \eta_x\frac{\partial}{\partial\eta}\right),$$

and we may choose to use either the first version, or the second, or a mixture of the two. In particular, we elect to use the former when we differentiate ξ_x and η_x, but the latter when we operate on $\partial/\partial\xi$ and $\partial/\partial\eta$; the result is

$$\frac{\partial^2}{\partial x^2} \equiv \xi_{xx}\frac{\partial}{\partial\xi} + \eta_{xx}\frac{\partial}{\partial\eta} + \xi_x^2\frac{\partial^2}{\partial x^2} + 2\xi_x\eta_x\frac{\partial^2}{\partial\xi\partial\eta} + \eta_x^2\frac{\partial^2}{\partial\eta^2};$$

there are corresponding results for $\partial^2/\partial x\partial y$ and $\partial^2/\partial y^2$. Our original equation now takes the form

$$AU_{\xi\xi} + 2BU_{\xi\eta} + CU_{\eta\eta} = D(\xi, \eta, U, U_\xi, U_\eta),$$

where the coefficients on the left-hand side are given by

$$A = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2; \ B = a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y;$$
$$C = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2,$$

and D is a combination of d evaluated according to the transformation and the first-derivative terms that arise from the transformation used on the left-hand side.

The first observation that we make concerns the coefficients A, B and C; in particular, we form $B^2 - AC$ (which, as we shall see shortly, naturally arises – or a version of it – in what we do later). This gives

$$\begin{aligned}
B^2 - AC = {} & \left[a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y\right]^2 \\
& -\left(a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2\right)\left(a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2\right)
\end{aligned}$$

$$\begin{aligned}
&= a^2\left(\xi_x^2\eta_x^2 - \xi_x^2\eta_x^2\right) + b^2\left[\left(\xi_x\eta_y + \xi_y\eta_x\right)^2 - 4\xi_x\xi_y\eta_x\eta_y\right] \\
&\quad + c^2\left(\xi_y^2\eta_y^2 - \xi_y^2\eta_y^2\right) + ac\left(2\xi_x\xi_y\eta_x\eta_y - \xi_x^2\eta_y^2 - \xi_y^2\eta_x^2\right) \\
&\quad + ab\left[2\xi_x\eta_x\left(\xi_x\eta_y + \xi_y\eta_x\right) - 2\xi_x^2\eta_x\eta_y - 2\eta_x^2\xi_x\xi_y\right] \\
&\quad + bc\left[2\xi_y\eta_y\left(\xi_x\eta_y + \xi_y\eta_x\right) - 2\xi_y^2\eta_x\eta_y - 2\eta_y^2\xi_x\xi_y\right] \\
&= b^2\left(\xi_x\eta_y - \xi_y\eta_x\right)^2 - ac\left(\xi_x\eta_y - \xi_y\eta_x\right)^2 = \left(b^2 - ac\right)J^2
\end{aligned}$$

where $J = \xi_x\eta_y - \xi_y\eta_x$ is the Jacobian introduced above. For the transformation from (x, y) to (ξ, η) to exist, we must have $J \neq 0$, and then the sign of $B^2 - AC$ is identical to the sign of $b^2 - ac$ (which uses the coefficients given in the original equation). Thus, no matter what (valid) transformation we choose to use, the sign of $B^2 - AC$ is controlled by that of $b^2 - ac$, and this suggests that this property of $b^2 - ac$ is fundamental to the construction of a solution; the intimate connection with the method of solution will be demonstrated in the next section.

4.2 Characteristic lines and the classification

Let us address the question of how to choose the new coordinates, ξ and η; lines $\xi(x,y) = \text{constant}$ imply that on them

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\xi_x}{\xi_y}$$

(and correspondingly $\mathrm{d}y/\mathrm{d}x = -\eta_x/\eta_y$ on lines $\eta(x,y) = \text{constant}$). With these identifications, we may write

$$A = \xi_y^2\left[a\left(\frac{dy}{dx}\right)^2 - 2b\left(\frac{dy}{dx}\right) + c\right] \text{ and } C = \eta_y^2\left[a\left(\frac{dy}{dx}\right)^2 - 2b\left(\frac{dy}{dx}\right) + c\right],$$

and then both A and C are zero if we elect to use as the definition of the characteristic lines

$$a\left(\frac{dy}{dx}\right)^2 - 2b\left(\frac{dy}{dx}\right) + c = 0 \text{ i.e. } \frac{dy}{dx} = \frac{1}{a}\left(b \pm \sqrt{b^2 - ac}\right).$$

Thus if $b^2 > ac$ we have two real families of curves (defined by the solution of the ordinary differential equation) and we may identify one family as $\xi = \text{constant}$ and the other as $\eta = \text{constant}$: we have determined a choice of ξ and η that simplifies the original equation – it now becomes simply

$$2BU_{\xi\eta} = D.$$

Further, it is clear that we have three cases: $b^2 > ac$, $b^2 = ac$ and $b^2 < ac$, and we should note that $b^2 - ac$ will, in general, vary over the (x, y)-plane, so there should be no expectation that it will remain single-signed. These three cases provide the classification.

(a) $b^2 - ac > 0$ **(hyperbolic)**

This is the most straightforward case, as we have just seen. The characteristic lines, $\xi(x,y) = \text{constant}$ and $\eta(x,y) = \text{constant}$, are defined by the two (real) solutions of the first-order equation

$$\frac{dy}{dx} = \frac{1}{a}\left(b \pm \sqrt{b^2 - ac}\right);$$

this is referred to as the *hyperbolic* case, and the partial differential equation is then said to be of hyperbolic type (a terminology that will be explained below).

Example 6

Find the characteristic lines of the wave equation $u_{xx} - k^2 u_{yy} = 0$ ($k > 0$, constant).

Here we have $a = 1$, $b = 0$ and $c = -k^2$, so that $b^2 - ac = k^2 > 0$ (and so the equation is hyperbolic everywhere); thus the characteristic lines are given by

$$\frac{dy}{dx} = \pm\sqrt{k^2} = \pm k \quad \text{i.e.} \quad y \mp kx = \text{constant}.$$

Comment: This recovers, with a change of notation, the results obtained in Chapter 3.

(b) $b^2 - ac = 0$ **(parabolic)**

We now have only one solution of the ordinary differential equation, because we have repeated roots; we call this the *parabolic* case. To proceed, we choose one characteristic, ξ say, which is defined by the solution of $y' = b/a$; the other is defined in any suitable way, provided that it is independent of the family $\xi(x,y) = \text{constant}$ i.e. it results in $J \neq 0$. Typically, the choice $\eta = x$ is made, although other choices may be convenient for particular equations.

Example 7

Find the characteristic lines for the heat conduction (diffusion) equation $u_y = ku_{xx}$ ($k > 0$, constant).

First write the equation as $ku_{xx} = u_y$, then we identify $a = k$, $b = c = 0$ which gives $b^2 - ac = 0$, so parabolic everywhere. Thus $y' = 0$; we may use $\xi = y = \text{constant}$ with $\eta = x = \text{constant}$, which is no transformation at all. Thus the original equation is already, as one of parabolic type, written in its simplest form.

(c) $b^2 - ac < 0$ **(elliptic)**

This case presents us – or so it would appear – with a much more difficult situation: the equation defining the characteristic lines is no longer real, so we might hazard that no transformation exists in this case. It is clear that, because we have the identity $B^2 - AC = (b^2 - ac)J^2 < 0$, then A and C must have the same sign and cannot be zero; thus we choose to define the transformation to produce $A = C$ and $B = 0$ i.e.

$$A - C = a\left(\xi_x^2 - \eta_x^2\right) + 2b\left(\xi_x\xi_y - \eta_x\eta_y\right) + c\left(\xi_y^2 - \eta_y^2\right) = 0$$

and

$$B = a\xi_x\eta_x + b\left(\xi_x\eta_y + \xi_y\eta_x\right) + c\xi_y\eta_y = 0.$$

Let us define the complex quantity $\chi = \xi + \mathrm{i}\eta$, then we have

$$\chi_x = \xi_x + \mathrm{i}\eta_x \text{ and } \chi_y = \xi_y + \mathrm{i}\eta_y,$$

and so

$$\begin{aligned} &a\chi_x^2 + 2b\chi_x\chi_y + c\chi_y^2 \\ &= a\left(\xi_x^2 - \eta_x^2\right) + 2b\left(\xi_x\xi_y - \eta_x\eta_y\right) + c\left(\xi_y^2 - \eta_y^2\right) \\ &\qquad + 2\mathrm{i}\left[a\xi_x\eta_x + b\left(\xi_x\eta_y + \xi_y\eta_x\right) + c\xi_y\eta_y\right] \\ &= 0; \end{aligned}$$

thus lines $\chi(x, y) = \text{constant}$ are exactly as before: solutions of

$$\frac{dy}{dx} = \frac{1}{a}\left(b \pm \sqrt{b^2 - ac}\right) = \frac{1}{a}\left(b \pm \mathrm{i}\sqrt{ac - b^2}\right).$$

However, the solution of this differential equation is necessarily complex-valued (called the *elliptic* case), so we write this as

$$\chi(x, y) = \xi(x, y) + \mathrm{i}\eta(x, y) = \alpha + \mathrm{i}\beta = \text{constant}$$

where $\alpha + \mathrm{i}\beta$ is a complex constant. Thus the choice of the new coordinates is given by $\xi(x, y) = \alpha = \text{constant}$ and $\eta(x, y) = \beta = \text{constant}$ (both real!) i.e. we follow the procedure used in the hyperbolic case, but here we apply the principle to the real and imaginary parts separately. So there is a transformation, even though the characteristic lines, defined by the ordinary differential equation, are certainly not real.

Example 8

Find the characteristic lines for Laplace's equation: $u_{xx} + u_{yy} = 0$.

Here we have $a = c = 1$ and $b = 0$, so $b^2 - ac = -1 < 0$ i.e. elliptic everywhere, and then $y' = \pm \mathrm{i}$ or $y \mp \mathrm{i}x = \text{constant}$. Thus we may choose the transformation $\xi = y$ and $\eta = x$ (or *vice versa*); as in the previous example, this is no transformation – the Laplace equation is already in its simplest form.

The simple results obtained in the last two examples lead naturally to the notion of the canonical form.

4.3 Canonical form

The general equation, following a general transformation, is

$$AU_{\xi\xi} + 2BU_{\xi\eta} + CU_{\eta\eta} = D$$

and then the three cases give

a) hyperbolic ($A = C = 0$): $2BU_{\xi\eta} = D$;
b) parabolic (e.g. $A = 0$, then $B^2 - AC = 0 \Rightarrow B = 0$): $CU_{\eta\eta} = D$;
c) elliptic ($A = C$, $B = 0$): $A(U_{\xi\xi} + U_{\eta\eta}) = D$.

These constitute the *canonical* forms (and so we confirm that both $ku_{yy} = u_x$ and $u_{xx} + u_{yy} = 0$, Examples 7 and 8, are already in this form). Here, we use the word 'canonical' in the sense of 'standard' or 'accepted'. The terminology (hyperbolic, parabolic, elliptic) as applied to the classification of partial differential equations, was introduced in 1889 by Paul du Bois-Reymond (1831-1889, French mathematician) because he interpreted the underlying differential equation

$$a\left(\frac{dy}{dx}\right)^2 - 2b\left(\frac{dy}{dx}\right) + c = 0$$

as being associated with the algebraic form

$$ay^2 - 2bxy + cx^2 = \text{terms linear in } x \text{ and } y.$$

Then $a = c = 0$ gives e.g. $xy = \text{constant}$, the rectangular hyperbola; $a = b = 0$ gives e.g. $x^2 = y$, a parabola; $a \propto c$, $b = 0$ gives e.g. $x^2 + k^2y^2 = \text{constant}$, an ellipse (and a circle if $a = c$).

The construction of the canonical form, via the appropriate characteristic variables, will be explored in three further examples (and then we will briefly examine a few specific and relevant applications of these ideas).

Example 9

Show that the equation $y^2u_{xx} - 4x^2u_{yy} = 0$ is of hyperbolic type (for $x \neq 0$, $y \neq 0$), find the characteristic variables and hence write the equation in canonical form.

We have '$b^2 - ac$'$= 4x^2y^2 > 0$ for $x \neq 0$, $y \neq 0$, so everywhere else the equation is of hyperbolic type. The characteristic lines, where the equation is hyperbolic, are given by the solution of the equation $y' = \pm\sqrt{4x^2y^2}\big/y^2 = \pm 2x/y$ (and note that $y = 0$ must be avoided here, anyway) so that $y^2 \mp 2x^2 = \text{constant}$; we set $\xi = y^2 - 2x^2$ and $\eta = y^2 + 2x^2$, to give

$$\frac{\partial}{\partial x} \equiv 4x\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right) \text{ and } \frac{\partial}{\partial y} \equiv 2y\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)$$

Then we obtain

$$\frac{\partial^2}{\partial x^2} \equiv 4\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right) + 16x^2\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right)\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right)$$

and

$$\frac{\partial^2}{\partial y^2} \equiv 2\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) + 4y^2\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right);$$

the original equation becomes, with $u(x,y) \equiv U(\xi,\eta)$,

$$4y^2\left(U_\eta - U_\xi\right) + 16x^2y^2\left(U_{\eta\eta} - 2U_{\eta\xi} + U_{\xi\xi}\right) \\ -8x^2\left(U_\xi + U_\eta\right) - 16x^2y^2\left(U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}\right) = 0 .$$

Thus

$$64x^2y^2U_{\xi\eta} = 4\left(y^2 - 2x^2\right)U_\eta - 4\left(y^2 + 2x^2\right)U_\xi$$

where we now write $y^2 = \frac{1}{2}(\xi + \eta)$ and $x^2 = \frac{1}{4}(\eta - \xi)$, giving

$$2\left(\eta^2 - \xi^2\right)U_{\xi\eta} = \xi U_\eta - \eta U_\xi ,$$

which is the canonical form of the equation (because the only second-order derivative is $U_{\xi\eta}$).

Example 10

Show that the equation $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 0$ is of parabolic type, choose appropriate characteristic variables, write the equation in canonical form and hence find the general solution.

Here we have $'b^2 - ac' = (xy)^2 - x^2y^2 = 0$, so the equation is parabolic (everywhere). One characteristic line is given by the solution of $y' = xy/x^2 = y/x$ (so, technically, we must avoid $x = 0$) i.e. $y/x = \text{constant}$; thus we introduce $\xi = x/y$ (slightly more convenient than y/x) and choose $\eta = x$, to give

$$\frac{\partial}{\partial x} \equiv \frac{1}{y}\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \text{ and } \frac{\partial}{\partial y} \equiv -\frac{x}{y^2}\frac{\partial}{\partial \xi} .$$

Then we obtain

$$\frac{\partial^2}{\partial x^2} \equiv \left(\frac{1}{y}\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(\frac{1}{y}\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right); \quad \frac{\partial^2}{\partial x \partial y} \equiv -\frac{1}{y^2}\frac{\partial}{\partial \xi} - \frac{x}{y^2}\left(\frac{1}{y}\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\frac{\partial}{\partial \xi};$$

$$\frac{\partial^2}{\partial y^2} \equiv \frac{2x}{y^3}\frac{\partial}{\partial \xi} + \frac{x^2}{y^4}\frac{\partial^2}{\partial \xi^2} ,$$

and so the equation becomes, with $u(x,y) \equiv U(\xi,\eta)$,

$$x^2\left(\frac{1}{y^2}U_{\xi\xi}+\frac{2}{y}U_{\xi\eta}+U_{\eta\eta}\right)+2xy\left(-\frac{1}{y^2}U_{\xi}-\frac{x}{y^3}U_{\xi\xi}-\frac{x}{y^2}U_{\xi\eta}\right)$$
$$+y^2\left(\frac{2x}{y^3}U_{\xi}+\frac{x^2}{y^4}U_{\xi\xi}\right)=0$$

This simplifies to .

$$U_{\eta\eta}=0 \text{ and so } U=F(\xi)+\eta G(\xi),$$

where F and G are arbitrary functions; thus the general solution is

$$u(x,y)=F(x/y)+xG(x/y).$$

Example 11

Show that the equation

$$y^2u_{xx}+2xyu_{xy}+\left(x^2+4x^4\right)u_{yy}=\frac{2y^2}{x}u_x+\frac{1}{y}\left(y^2+x^2+4x^4\right)u_y$$

(for $x\neq 0$, $y\neq 0$) is of elliptic type, find suitable characteristic variables and hence write the equation in canonical form.

We have $'b^2-ac'=(xy)^2-y^2\left(x^2+4x^4\right)=-4x^4y^2<0$ for $x\neq 0$, $y\neq 0$, so elliptic and the characteristic lines are given by the solution of the equation

$$y'=\frac{xy\pm\sqrt{-4x^4y^2}}{y^2}=\frac{x}{y}\pm 2\mathrm{i}\frac{x^2}{y}$$

.

Thus $\frac{1}{2}y^2=\frac{1}{2}x^2\pm\frac{2}{3}\mathrm{i}x^3+\text{constant}$ or $y^2-x^2\mp\frac{4}{3}\mathrm{i}x^3=\text{constant}$;

We choose $\xi=y^2-x^2$ and $\eta=x^3$, although we could use just $\eta=x$; the current choice will produce the simplest version of the canonical form – indeed, we could even include the factor $4/3$ (and we comment on this later). Thus

$$\frac{\partial}{\partial x}\equiv -2x\frac{\partial}{\partial\xi}+3x^2\frac{\partial}{\partial\eta}\ \text{ and }\ \frac{\partial}{\partial y}\equiv 2y\frac{\partial}{\partial\xi},$$

and then

$$\frac{\partial^2}{\partial x^2}\equiv -2\frac{\partial}{\partial\xi}+6x\frac{\partial}{\partial\eta}-2x\left(-2x\frac{\partial}{\partial\xi}+3x^2\frac{\partial}{\partial\eta}\right)\frac{\partial}{\partial\xi}+3x^2\left(-2x\frac{\partial}{\partial\xi}+3x^2\frac{\partial}{\partial\eta}\right)\frac{\partial}{\partial\eta}$$

with $\dfrac{\partial^2}{\partial x\partial y}\equiv 2y\left(-2x\dfrac{\partial}{\partial\xi}+3x^2\dfrac{\partial}{\partial\eta}\right)\dfrac{\partial}{\partial\xi}$ and $\dfrac{\partial^2}{\partial y^2}\equiv 2\dfrac{\partial}{\partial\xi}+4y^2\dfrac{\partial^2}{\partial\xi^2}$.

Thus the original equation, with $u(x,y)\equiv U(\xi,\eta)$, becomes

$$\begin{aligned}&y^2\left(-2U_\xi+6xU_\eta+4x^2U_{\xi\xi}-12x^3U_{\xi\eta}+9x^4U_{\eta\eta}\right)\\&\quad+2xy\left(-4xyU_{\xi\xi}+6x^2yU_{\xi\eta}\right)+\left(x^2+4x^4\right)\left(2U_\xi+4y^2U_{\xi\xi}\right)\\&\qquad=\frac{2y^2}{x}\left(-2xU_\xi+3x^2U_\eta\right)+\frac{1}{y}\left(y^2+x^2+4x^4\right)\left(2yU_\xi\right)\end{aligned}$$

which simplifies to give

$$x^4 y^2\left(9U_{\eta\eta} + 16U_{\xi\xi}\right) = 0 \text{ or } 9U_{\eta\eta} + 16U_{\xi\xi} = 0.$$

This equation is essentially the classical Laplace equation (and therefore the required canonical form); it can be written in precisely the conventional form if, for example, we replace η by $(3/4)\eta$ i.e. the new $\eta = (4/3)x^3$, which is exactly the transformation suggested by the solution of the ordinary differential equation.

4.4 Initial and boundary conditions

Any differential equation will normally be provided with additional constraints on the solution: the given boundary and/or initial data (as appropriate). Indeed, any physical problem or practical application will almost always have such auxiliary conditions. However, what forms these should take in order to produce a well-posed problem for partial differential equations is not a trivial investigation. (By 'well-posed' we mean the conditions necessary for a unique solution to exist.) We have already touched on this aspect for first order equations (§2.5) and for the wave equation (Chapter 3); we will now discuss these ideas a little further (although it is beyond the scope of this text to produce any formal proofs of the various assertions of uniqueness and existence).

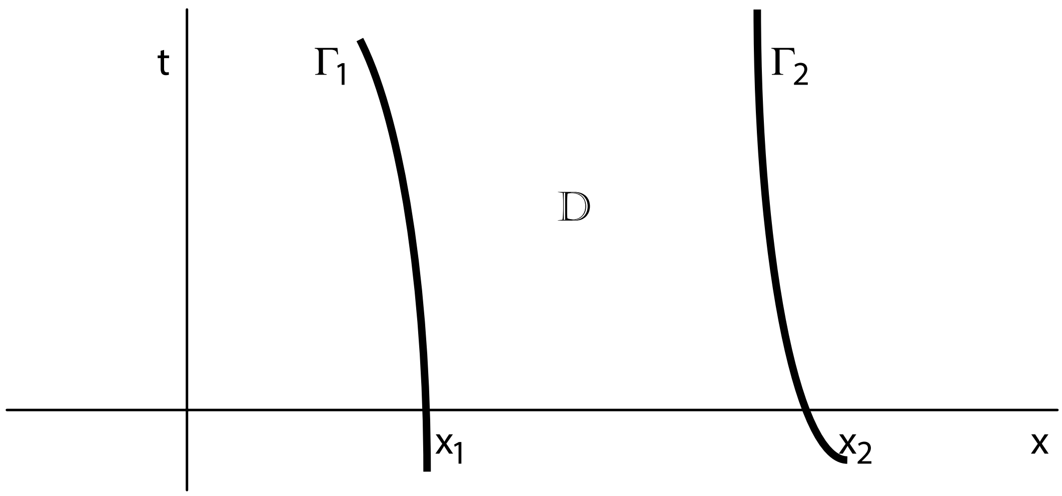
(a) Hyperbolic equations

The standard type of data – Cauchy data – is to be given both *u* and $\partial u/\partial n$ on some curve, Γ, which intersects the characteristic lines i.e. at no point is Γ parallel to a characteristic line (so a characteristic line and Γ do not have a common tangent at any point). Here, $\partial u/\partial n$ is the normal derivative on Γ (and this situation is exactly what we encountered for the wave equation: $u(x,0)$ and $\dfrac{\partial u}{\partial t}(x,0)$ were prescribed). Further, it is quite usual to seek solutions that move away (along characteristic lines) from the curve Γ on one side only (again, as we did in Chapter 3, where we had data on $t = 0$ and we required a solution in $t > 0$).

A degenerate version of this problem, closely related to the conventional Cauchy problem, is when *u* alone is prescribed on Γ, but some additional information (usually *u* again) is given on one of the characteristic lines (C_0, say). This is possible provided that all the data is consistent with the equations and characteristics, because we have the two sets of information needed to determine a solution: one set on the other characteristic lines parallel to C_0 and intersecting Γ, and one set on the characteristic lines of the other family, provided that they intersect C_0. This is called a *Goursat* problem. [E.J.-B. Goursat, 1858-1936, French mathematician who made some significant contributions to analysis.]

(b) Parabolic equations

It will be helpful, in this brief overview, to consider the canonical form of the parabolic equation, written with x (distance) replacing η and t (time) replacing ξ (see §4.3); the simplest such equation is $u_{xx} = u_t$. The characteristic lines, as we have seen, appear as a repeated pair defined by $\mathrm{d}t/\mathrm{d}x = 0$ (see Example 7); interpreting this in the form $\mathrm{d}x/\mathrm{d}t$, we see that propagation on the characteristic lines $t = \text{constant}$ is at infinite speed, implying that the whole domain is affected instantaneously (although often to an exponentially small degree well away from the initial disturbance). Then we may have data on $t = 0$ (initial data) and, if the solution is defined in the domain $t \geq 0$, $-\infty < x < \infty$, no further information is required (although a boundedness condition may need to be invoked e.g. the solution decays as $|x| \to \infty$). However, more often than not, the region is bounded, usually by one or two lines $x = \text{constant}$, although any pair of curves in (x, t)-space will suffice to describe the region where the solution is to exist; see the figure below.



The solution is in D, bounded by the curves Γ_1, Γ_2 and $x_1 < x < x_2$ (on $t = 0$).

Note, however, that no part of the curves, Γ_1 and Γ_2, must be parallel to the characteristic lines i.e. no point of these curves must have a slope parallel to the x-axis.

The data given on the curves, Γ_1 and Γ_2, will be either u (the *Dirichlet* problem) or $\partial u/\partial n$ (the *Neumann* problem) or a mixture of the two, each on different sections of Γ_1 and Γ_2 (the *mixed* problem). (A linear combination of u and $\partial u/\partial n$ is also allowed; this is usually called the *Robin* problem.) [P.G.L. Dirichlet, 1805-1859, German mathematician who made important contributions to both analysis and number theory; C.G. Neumann, 1832-1925, German mathematician; G. Robin was a French mathematician.]

(c) Elliptic equations

This class of problems is the easiest to describe in terms of boundary conditions. First, initial data has no meaning here, for the two variables – see §4.3 – appear symmetrically and there are no real characteristics, so there is no exceptional variable such as 'time'. Indeed, elliptic equations in two variables arise exclusively in two spatial dimensions. Then we simply need to prescribe u (Dirichlet) or $\partial u / \partial n$ (Neumann), or a mix of these two, or a linear combination of them (Robin) on the boundary of a region, D, in order to define a unique solution throughout D. (Note that, by the very nature of Neumann data, the solution in this case will be known only up to an arbitrary constant.)

Exercises 4

Write these partial differential equations in canonical form.

a) $u_{yy} - x u_{xx} = 0$ for $x > 0$;

b) $4 y u_{xx} + 2(y-1) u_{xy} - u_{yy} = 0$;

c) $x y u_{xx} + 4x^2 y u_{xy} + 4x^3 y u_{yy} = u$;

d) $u_{xx} + 2u_{xy} + 2\left(1 - 2x + 2x^2\right) u_{yy} = 0$.

****

****

5 Three examples from fluid mechanics

The study of fluid mechanics has a long and involved history; it has produced, arguably, the greatest range of applied mathematical problems of any branch of study – and it certainly encompasses all the types of partial differential equation that we are discussing here. Although other fields of study, e.g. gravitation, elasticity, quantum mechanics, to name but three, could be cited, fluid mechanics rather readily provides all the examples that we could ever wish for. Thus the Euler equation of inviscid fluid mechanics gives rise to both elliptic and hyperbolic problems, and the Navier-Stokes equation is essentially parabolic. We shall describe three simple examples, together with some relevant background, that arise as suitable models for inviscid fluid flows; so that we can describe a fair amount of detail, these are problems that are hyperbolic, or become hyperbolic under certain conditions.

5.1 The Tricomi equation

For steady, compressible flow of a gas, we have

$$\nabla\cdot(\rho\mathbf{u})=0 \text{ and } \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}=-\frac{1}{\rho}\nabla p \ \left(\text{where } \frac{\mathrm{D}}{\mathrm{D}t}\equiv\frac{\partial}{\partial t}+\mathbf{u}\cdot\nabla \ \ (=\mathbf{u}\cdot\nabla \text{ here})\right)$$

in the absence of body forces; the usual notation has been used, and we shall assume that the gas is modelled by $p=k\rho^{\gamma}$ (where k and $1<\gamma<2$ are constants). The two displayed equations are, respectively, the equation of mass conservation and Euler's equation. Let us consider flow in the (x, y)-plane which is a perturbation from the uniform state: $\mathbf{u}\equiv(U,0)$, $p=p_0$ and $\rho=\rho_0$, where U, p_0 and ρ_0 are constants. We write the solution as the uniform state plus a perturbation in the form

$$\mathbf{u}\equiv(U+u,v),\ \rho=\rho_0+r,\ p=p_0+P,$$

and treat u, v, P and r (and their derivatives) as small. The three scalar equations, upon the elimination of P (*via* $p=k\rho^{\gamma}$), become approximately

$$\rho_0\left(u_x+v_y\right)+U\rho_x=0;\ Uu_x+\frac{\gamma P_0}{\rho_0^2}\rho_x=0;\ Uv_x+\frac{\gamma P_0}{\rho_0^2}\rho_y=0$$

where $\gamma P_0/\rho_0$ is usually written as a_0^2, a_0 being the sound speed associated with the uniform state. Let us further assume that the flow is irrotational, so that $(u,v)=\nabla\phi$ and then $\rho=-\left(\rho_0 U/a_0^2\right)\phi_x$ (where we take all perturbations to be zero if one is). Thus we obtain

$$\rho_0\left(\phi_{xx}+\phi_{yy}\right)-\left(\rho_0 U^2/a_0^2\right)\phi_{xx}=0 \text{ or } \left(1-M^2\right)\phi_{xx}+\phi_{yy}=0,$$

where $M=U/a_0$ is the *Mach number* of the flow. We see immediately that, for $M<1$ (subsonic), the equation for ϕ is elliptic; on the other hand, for $M>1$ (supersonic), the equation is hyperbolic. This is our first observation, and it is very significant. [E. Mach, 1838-1916, Austrian philosopher and theoretical physicist; in 1887 he published some photographs of projectiles, showing the shock waves.]

On the other hand, if M is close to unity (usually called *transonic flow*), the simple-minded approximation that we have used above is not valid; in this special case the appropriate approximation results in the nonlinear equation

$$\phi_x \phi_{xx} + \phi_{yy} = 0.$$

Now this equation is not of the form

$$a(x,y)\,u_{xx} + b(x,y)\,u_{yy} = 0,$$

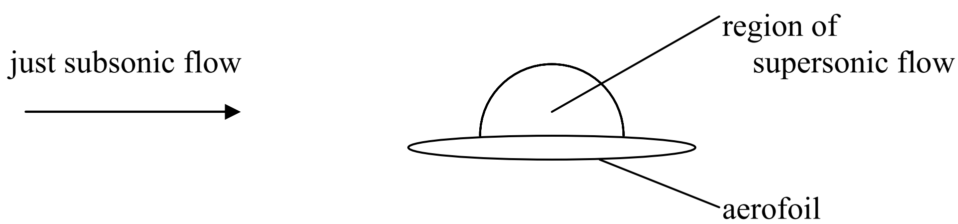
so our development that was expounded in Chapter 4 is not applicable. But this equation can be transformed into an alternative version that *is* of the form that we have previously discussed; this is possible by virtue of the *hodograph transformation*. This important idea and procedure is described in §5.4, which provides an appendix to this chapter. In the case under discussion here, we form the equation for

$$x = X(u,v) \text{ and } y = Y(u,v) \text{ (where } u = \phi_x,\ v = \phi_y)$$

to replace that for $\phi(x,y)$; this produces the equation

$$Y_{uu} + uY_{vv} = 0,$$

as described in §5.4. This is the *Tricomi* equation (F. Tricomi, an Italian mathematician, who published his analywsis of this equation in 1922) for which $'b^2 - ac' = -u$, so the equation is hyperbolic for $u < 0$ and elliptic for $u > 0$. These two cases correspond to supersonic and subsonic flow, respectively: the fundamental nature of the problem changes across the sonic line ($u = 0$) – on one side there are real characteristics and on the other they are imaginary (i.e. the elliptic case). Tricomi introduced this problem as a means for analysing this change of type; however, in the context of compressible flow, the situation is rather more complicated e.g. just subsonic flow past an aerofoil produces a region – whose determination is part of the problem – where the flow is supersonic, as depicted in the figure below.



(A Tricomi-type problem was encountered in Exercises 4.)

5.2 General compressible flow

We suppose that a flow of a gas (described by $p = k\rho^{\gamma}$, as in §5.1) satisfies Bernoulli's equation, in the form

$$a^2 + \tfrac{1}{2}(\gamma - 1)\left(u^2 + v^2\right) = \text{constant}$$

where $a(\rho) = \sqrt{\mathrm{d}p/\mathrm{d}\rho} = \sqrt{\gamma\, p/\rho} = \sqrt{k\gamma\rho^{\gamma-1}}$, the local sound speed, and $\mathbf{u} \equiv (u, v)$. Further, we assume that the flow is irrotational (so $u = \phi_x$, $v = \phi_y$, again as in §5.1) and that the usual mass conservation applies:

$$(\rho u)_x + (\rho v)_y = 0,$$

all written in two-dimensional Cartesian coordinates. Thus we may obtain

$$2aa_x + (\gamma - 1)(uu_x + vv_x) = 0,$$

and correspondingly for the y-derivative, where e.g.

$$a_x = \tfrac{1}{2}(\gamma-1)\sqrt{k\gamma}\rho^{(\gamma-1)/2}\rho_x = \tfrac{1}{2}(\gamma-1)\frac{a}{\rho}\rho_x$$

which gives $\qquad \rho_x = -\dfrac{\rho}{a^2}(uu_x + vv_x).$

Hence we may write

$$(\rho u)_x + (\rho v)_y = -\frac{\rho}{a^2}(uu_x + vv_x)u + \rho u_x - \frac{\rho}{a^2}(uu_y + vv_y)v + \rho v_y$$

and so we have the equation

$$\left(a^2-u^2\right)u_x - uv\left(u_y+v_x\right)+\left(a^2-v^2\right)v_y=0.$$

Note that this derivation has not required us to make any approximations about, for example, a uniform state (as we did in §5.1): within this model, this is exact (and so, not surprisingly, it is highly nonlinear).

Now we write $u=\phi_x$, $v=\phi_y$ in the derivative terms (only) and so obtain

$$\left(a^2-u^2\right)\phi_{xx} - 2uv\phi_{xy}+\left(a^2-v^2\right)\phi_{yy}=0;$$

it is convenient to define $\Phi = x\phi_x + y\phi_y - \phi = xu + yv - \phi$ (usually called a *Legendre transformation*) and then to apply a hodograph transformation: $\Phi = \Phi(u,v)$ (see §5.4). This eventually produces

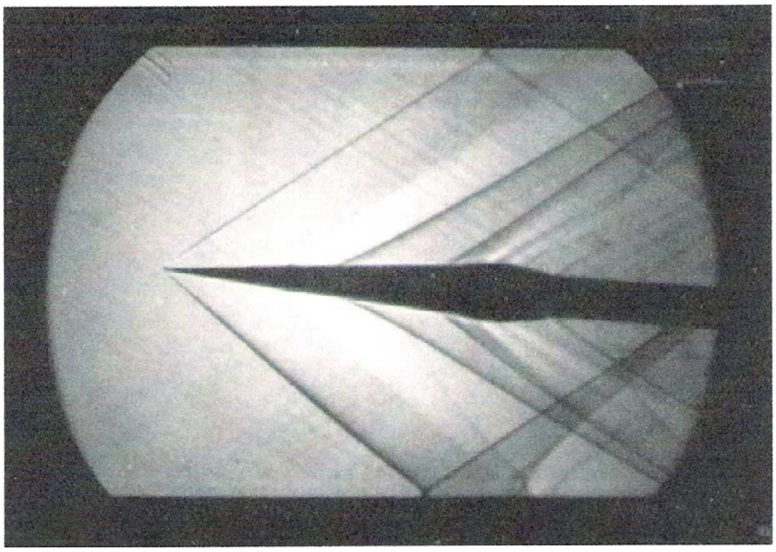
$$\left(a^2-u^2\right)\Phi_{vv} + 2uv\Phi_{uv}+\left(a^2-v^2\right)\Phi_{uu}=0,$$

for which

$$'b^2 - ac' = (uv)^2 - \left(a^2-u^2\right)\left(a^2-v^2\right) = a^2\left(u^2+v^2-a^2\right),$$

so the equation is hyperbolic if $u^2+v^2-a^2>0$ i.e. $\left(u^2+v^2\right)/a^2 = M^2>1$ (supersonic) and elliptic if $u^2+v^2-a^2<0$ i.e. $M^2<1$ (subsonic). [A.-M. Legendre, 1752-1833, French mathematician who worked on number theory, celestial mechanics and elliptic functions.]

This example, because it avoids any 'small disturbance' approximation, explains completely the hyperbolic/elliptic nature of the inviscid, compressible flow problem as it appears in classical fluid mechanics. One quite notable result of this analysis, in the hyperbolic case, is that weak disturbances to the flow field, it turns out, propagate along the (real) characteristics (usually called *Mach lines* in this context); larger disturbances produce *shock waves*. These weak disturbances give rise to minute changes in density which, nevertheless, can be measured and photographed: the characteristic lines of the hyperbolic problem can be observed! This phenomenon is shown in the photograph reproduced below.



This shows a flat wedge in a supersonic flow, Mach number $1 \cdot 8$; it has an included angle of $10 \cdot 2^o$ and its centre line is inclined at an angle of $3 \cdot 3^o$. The faint lines, mainly emanating from the upper and lower walls of the wind tunnel are the characteristic lines for this problem/flow; they are visible because of the small blemishes on the surface of the walls. Across these lines is a very small change in the density of the air which, in turn, changes the refractive index of the air; this change can be captured photographically, by a technique known as *Schlieren* photography. The darker lines are the paths of the shock waves.

5.3 The shallow-water equations

The assumption of shallow water or, equivalently, long waves, in the classical water-wave problem implies that the pressure (p) in the flow relative to the hydrostatic pressure distribution is proportional to the amplitude of the surface wave. Thus, written in non-dimensional variables, we set $p = \eta$, where $\eta(x,t)$ is the wave height measured relative to the undisturbed surface of the water; thus we obtain

$$\frac{\mathrm{D}u}{\mathrm{D}t} = -p_x \text{ which becomes } u_t + uu_x + wu_z = -\eta_x$$

(all non-dimensional) with the equation of mass conservation for an incompressible fluid:

$$u_x + w_z = 0.$$

The boundary conditions are

$$w = \frac{\mathrm{D}\eta}{\mathrm{D}t} = \eta_t + u\eta_x \text{ on } z = 1 + \eta \text{ and } w = 0 \text{ on } z = 0,$$

where the free surface is represented by $z = 1 + \eta$ and the impermeable bed is $z = 0$. The solution that is relevant here is recovered by writing

$$w = -z u_x(x,t) = z\left(\frac{\eta_t + u\eta_x}{1+\eta}\right),$$

and so we are left with the pair of equations (which are independent of *z*):

$$u_t + uu_x + h_x = 0\,;\; h_t + uh_x + hu_x = h_t + (uh)_x = 0,$$

the *shallow-water equations*, where $h = 1 + \eta(x,t)$ (the local depth). These equations – which are coupled and nonlinear – can be analysed in a number of different ways (and the methods described in the next chapter are particularly relevant here), but for the present, let us content ourselves with the following observations.

Although it is possible to eliminate either *u* or *h* between this pair of equations (most easily done by first setting $u = \phi_x$), this is not very instructive (and is mildly tiresome); indeed, it turns out to be more useful to retain the pair at this stage, but to introduce $c = \sqrt{h(x,t)}$, and certainly $h > 0$ in any realistic problem. (It transpires that this quantity, *c*, is directly associated with the speed of propagation of the surface waves.) With the use of *c*, our pair of equations becomes

$$u_t + uu_x + 2cc_x = 0\,;\; c_t + uc_x + \tfrac{1}{2}cu_x = 0,$$

and then we apply a hodograph transformation (§5.4) in the form

$$t = T(u,c),\; x = X(u,c).$$

This gives, for example,

$$u_t = -X_c / J \text{ and } c_t = X_u / J \, ,$$

where $J = X_u T_c - X_c T_u$ ($\neq 0$) is the Jacobian; there are corresponding results for the x-derivatives. All this leads to

$$X_c - uT_c + 2cT_u = 0 \, ; \; X_u - uT_u + \tfrac{1}{2}cT_c = 0 \, ,$$

which are *linear* equations in X and T; to proceed, we eliminate X – the most convenient (by forming X_{uc}), to produce

$$4T_{uu} - T_{cc} = \frac{3}{c}T_c \, ,$$

which has the (real) characteristics $u \pm 2c$ – we have a hyperbolic problem. This was to be expected, since we had hoped to have formulated a problem that self-evidently described wave propagation.

5.4 Appendix: The hodograph transformation

The essential idea that underpins the hodograph transformation is surprisingly simple: it is the interchange of dependent and independent variables. This is almost a standard procedure for ordinary differential equations; consider, for example, the equation

$$\left[xa(y) + b(y)\right]\frac{\mathrm{d}y}{\mathrm{d}x} = 1 \, ,$$

which is, in general, nonlinear, non-separable and not of homogeneous type. However, if we treat $x = x(y)$, then the equation becomes

$$\frac{\mathrm{d}x}{\mathrm{d}y} - a(y)x = b(y) \, ,$$

which is linear in x, so it can be solved completely using standard (and elementary) methods.

We consider the second-order partial differential equation, of the form

$$a(\phi_x, \phi_y)\phi_{xx} + 2b(\phi_x, \phi_y)\phi_{xy} + c(\phi_x, \phi_y)\phi_{yy} = 0 \, ;$$

first we introduce the *Legendre transformation*

$$\Phi = x\phi_x + y\phi_y - \phi \, ,$$

and treat $\Phi = \Phi(\alpha, \beta)$ where $\alpha = \phi_x$ and $\beta = \phi_y$. Thus we have $\Phi = x\alpha + y\beta - \phi$, and so

$$\Phi_\alpha = x + \alpha x_\alpha + \beta y_\alpha - \left(x_\alpha \phi_x + y_\alpha \phi_y\right)$$

$$= x + \alpha x_\alpha + \beta y_\alpha - \left(\alpha x_\alpha + \beta y_\alpha\right) = x$$

and, correspondingly, $\Phi_\beta = y$. Then we find

$$\frac{\partial}{\partial x}\left(\Phi_\beta\right) = \frac{\partial}{\partial x}(y) = 0 \quad \text{so} \quad \alpha_x \Phi_{\alpha\beta} + \beta_x \Phi_{\beta\beta} = 0$$

where $\alpha_x = \phi_{xx}$ and $\beta_x = \phi_{xy}$, which produces the expression

$$\phi_{xx}\Phi_{\alpha\beta} + \phi_{xy}\Phi_{\beta\beta} = 0.$$

Similarly $\frac{\partial}{\partial y}\left(\Phi_\alpha\right) = \frac{\partial}{\partial y}(x) = 0$, and then we have

$$\phi_{xy}\Phi_{\alpha\alpha} + \phi_{yy}\Phi_{\alpha\beta} = 0;$$

thus, for $\Phi_{\alpha\beta} \neq 0$, we obtain the identities

$$\phi_{xx} = -\frac{\Phi_{\beta\beta}}{\Phi_{\alpha\beta}}\phi_{xy} \quad \text{and} \quad \phi_{yy} = -\frac{\Phi_{\alpha\alpha}}{\Phi_{\alpha\beta}}\phi_{xy}.$$

We now transform

$$a\phi_{xx} + 2b\phi_{xy} + c\phi_{yy} = -a\frac{\Phi_{\beta\beta}}{\Phi_{\alpha\beta}}\phi_{xy} + 2b\phi_{xy} - c\frac{\Phi_{\alpha\alpha}}{\Phi_{\alpha\beta}}\phi_{xy}$$

and hence, for $\phi_{xy} \neq 0$, our equation

$$a\phi_{xx} + 2b\phi_{xy} + c\phi_{yy} = 0$$

becomes

$$a(\alpha,\beta)\Phi_{\beta\beta} - 2b(\alpha,\beta)\Phi_{\alpha\beta} + c(\alpha,\beta)\Phi_{\alpha\alpha} = 0;$$

this is now in standard form, and so amenable to the methods described earlier. We see that the equation uses $\alpha = \phi_x$ and $\beta = \phi_y$ as the independent variables; this is the essence of the hodograph transformation (although, as we have seen, the general partial differential equation also requires a Legendre transformation).

In the early applications of this technique, α and β were velocity components e.g. $\alpha = u = \phi_x$ and $\beta = v = \phi_y$. To represent a curve in terms of the velocity components, rather than the conventional position vector, is called a *hodograph*. (This terminology comes from the Greek word όδος – 'hodos' – which means 'way' or 'road' and is used to indicate that the curve is mapped out other than in the usual fashion *via* the position vector.)

1. As a first example, let us consider the general compressible-flow equation (§5.2):

$$\left(a^2 - \phi_x^2\right)\phi_{xx} - 2\phi_x\phi_y\phi_{xy} + \left(a^2 - \phi_y^2\right)\phi_{yy} = 0,$$

which is of the general type described above. Hence we obtain directly

$$\left(a^2 - u^2\right)\Phi_{vv} + 2uv\Phi_{uv} + \left(a^2 - v^2\right)\Phi_{uu} = 0,$$

where $\Phi = xu + yv - \phi$ and $(u, v) = \left(\phi_x, \phi_y\right)$.

2. The Tricomi equation (§5.1)

$$\phi_x \phi_{xx} + \phi_{yy} = 0$$

is initially treated in the same manner; the general result, applied in this case, gives immediately

$$u\Phi_{vv} + \Phi_{uu} = 0.$$

But with $\Phi_v = y$, we can take advantage of the particular form of this equation, and elect to introduce

$$y = \Phi_v(u,v) = Y(u,v);$$

thus, when we write

$$u\Phi_{vvv} + \Phi_{uuv} = 0 \text{ we obtain } Y_{uu} + uY_{vv} = 0,$$

which simply uses the transformation of the coordinates alone (and note that $x = X(u,v)$ is not needed here).

3. Finally, we consider the shallow-water equations, §5.3, in the form

$$u_t + uu_x + 2cc_x = 0;\ c_t + uc_x + \tfrac{1}{2}cu_x = 0,$$

and because we have not developed a single equation (for *u* or *c*), we have no natural choice for a Legendre transformation. Thus we exploit the interpretation employed for the Tricomi equation by introducing, directly, only a change of variable:

$$x = X(u,c),\ t = T(u,c),$$

which is equivalent to working with a single equation in one variable – either *u* or *c* – as we did for the Tricomi equation. Thus we obtain

$$\frac{\partial}{\partial x}(X) = \frac{\partial}{\partial x}(x) = 1 \text{ so } X_u u_x + X_c c_x = 1$$

and $\quad \dfrac{\partial}{\partial t}(X) = \dfrac{\partial}{\partial t}(x) = 0 \text{ so } X_u u_t + X_c c_t = 0;$

correspondingly, we have

$$\frac{\partial}{\partial t}(T) = \frac{\partial}{\partial t}(t) = 1 \text{ so } T_u u_t + T_c c_t = 1$$

and $$\frac{\partial}{\partial x}(T) = \frac{\partial}{\partial x}(t) = 0 \text{ so } T_u u_x + T_c c_x = 0.$$

The first and fourth of these equations give

$$u_x = \frac{T_c}{J} \text{ and } c_x = -\frac{T_u}{J},$$

where $J = X_u T_c - X_c T_u \neq 0$ is the Jacobian. Similarly, the second and third equations yield

$$u_t = -\frac{X_c}{J} \text{ and } c_t = \frac{X_u}{J}.$$

Thus our two shallow-water equations can be written as

$$X_c - uT_c + 2cT_x = 0;\ X_u - uT_u + \tfrac{1}{2}cT_c = 0,$$

and then we may form $$X_{uc} = \frac{\partial}{\partial u}\left(uT_c - 2cT_u\right) = \frac{\partial}{\partial c}\left(uT_u - \tfrac{1}{2}cT_c\right)$$

which gives $$4T_{uu} - T_{cc} = \frac{3}{c}T_c$$

because the cross-derivative term (T_{uc}) cancels identically.

Exercise 5

Equations that model a slender, axisymmetric jet in an inviscid fluid are

$$u_t + uu_x = R^{-2}R_x \text{ and } \left(R^2\right)_t + \left(uR^2\right)_x = 0,$$

where $R(x,t)$ is its radius and $u(x,t)$ is the speed of the flow in the axial (x) direction. Use the hodograph transformation $x = X(u,R)$, $t = T(u,R)$, to show that

$$X_R - uT_R - R^{-2}T_u = 0 \text{ and } \tfrac{1}{2}RT_R - uT_u + X_u = 0.$$

Find the equation satisfied by $T(u,R)$, and classify it.

****

****

6 Riemann invariants and simple waves

We conclude by presenting a brief commentary on two important aspects of the solution of coupled, hyperbolic equations, even though the properties that we describe are not always available to us. Let us restrict ourselves to two unknowns in two independent variables e.g. our equations for shallow-water waves (§5.3):

$$u_t + uu_x + 2cc_x = 0\,;\; c_t + uc_x + \frac{1}{2}cu_x = 0.$$

If there exists some function of $u(x,t)$ and $c(x,t)$ which is constant along characteristic lines, then this is called a *Riemann invariant*. (The invariant is, of course, the function of u and c that is constant.) Indeed, in a problem such as the one we have quoted, we might expect *a pair* of functions that are constant, one associated with each characteristic. We will explore the details, for the shallow-water equations, shortly. [G.F.B. Riemann,1826-1866, German mathematician who probably had no equal in laying the foundations for 20th century mathematics and theoretical physics.]

A very significant special case, that makes direct use of the Riemann invariants, is the following. A flow field has at least two regions, in one of which the solution can be described by a constant state; the Riemann invariant associated with (i.e. on characteristic lines emanating from) this region are therefore all the *same* constant. In this situation, the problem posed on the other set of characteristic lines can be solved completely (and quite straightforwardly). The waves that represent the solution of such problems are called *simple waves*; we will give an example below.

6.1 Shallow-water equations: Riemann invariants

It is convenient, first, to write our two equations in the form $u_t + uu_x + 2cc_x = 0$; $2c_t + 2uc_x + cu_x = 0$

which are added to give

$$(u+2c)_t + u(u+2c)_x + c(u+2c)_x = 0,$$

and subtracted to give

$$(u-2c)_t + u(u-2c)_x - c(u-2c)_x = 0.$$

These two equations are therefore

$$\left\{\frac{\partial}{\partial t} + (u+c)\frac{\partial}{\partial x}\right\}(u+2c) = 0 \text{ and } \left\{\frac{\partial}{\partial t} + (u-c)\frac{\partial}{\partial x}\right\}(u-2c) = 0$$

where we have the characteristic lines defined by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u + c \ \text{ and } \ \frac{\mathrm{d}x}{\mathrm{d}t} = u - c,$$

respectively; also note the appearance of $u \pm 2c$; cf. §5.3. Thus we have that

$$u + 2c \text{ is constant on lines } C^+ : \frac{\mathrm{d}x}{\mathrm{d}t} = u + c$$

and $\quad u - 2c$ is constant on lines $C^- : \dfrac{\mathrm{d}x}{\mathrm{d}t} = u - c$;

the Riemann invariants for this problem are therefore $(u+2c)$ and $(u-2c)$. It is useful to express the solution in the alternative form

$u + 2c = F(\alpha)$, where α is constant on $\dfrac{\mathrm{d}x}{\mathrm{d}t} = u + c$ (i.e. on C^+)

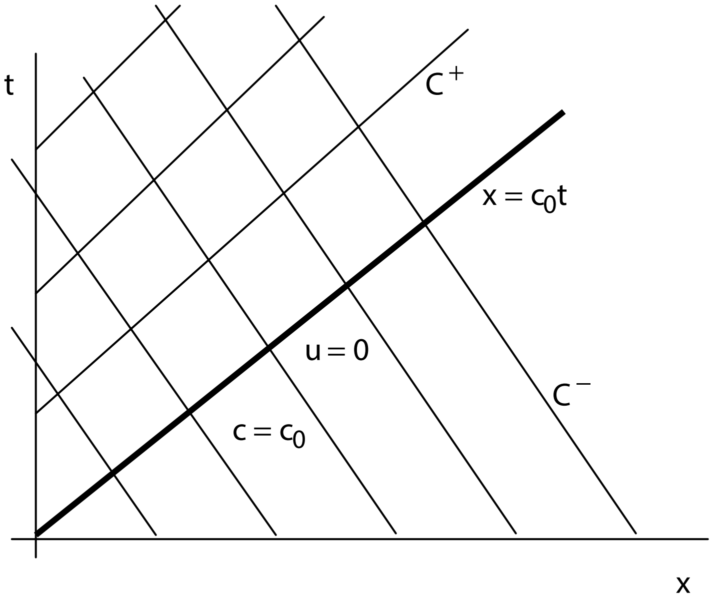
and $\quad u - 2c = G(\beta)$, where β is constant on $\dfrac{\mathrm{d}x}{\mathrm{d}t} = u - c$ (i.e. on C^-),

where *F* and *G* are arbitrary functions. We may note that the characteristic lines here describe propagation that is at a speed *c* upstream/downstream ($-c/+c$) relative to the speed of the flow (u).

6.2 Shallow-water equations: simple waves

Let us consider the problem of a wave that is moving to the right into a region ($x > 0$) of stationary water (so $u = 0$) of constant depth h_0 (so $c = c_0 = \sqrt{h_0}$ here). Then all the C^- characteristics emanate from this region of undisturbed flow

(see figure below)



and so on the characteristic lines C^- we have

$$u - 2c = G(\beta) = -2c_0.$$

Thus $u - 2c = -2c_0$ is constant everywhere (because this is the value taken on *all* C^- characteristics, no matter where they sit in the (x, y)-plane); also $u + 2c$ is constant on each C^+ characteristic. These two conditions imply that both u and c are constant on C^+ lines, so

$\dfrac{\mathrm{d}x}{\mathrm{d}t} = u + c$ can be integrated to give $x - (u + c)t = \alpha$

and then $\quad u + 2c = F(\alpha) = F\left[x - (u + c)t\right].$

Let us prescribe the surface (wave) profile at $t = 0$ – the initial data – by writing

$$h(x,0) = H(x),$$

which is to be consistent with $h = h_0$ in $x > 0$. Thus at $t = 0$ we have

$$F(x) = (u + 2c)\big|_{t=0} = (4c - 2c_0)\big|_{t=0} = 4\sqrt{H(x)} - 2\sqrt{h_0},$$

and then for $t > 0$, this gives

$$u + 2c = F[x - (u + c)t]$$
$$= 4\sqrt{H[x - (u + c)t]} - 2\sqrt{h_0}.$$

Thus $\quad 4c - 2c_0 = 4\sqrt{h} - 2\sqrt{h_0} = 4\sqrt{H\left[x - \left(u + \sqrt{h}\right)t\right]} - 2\sqrt{h_0}$

which requires that

$$h(x, t) = H\left[x - \left(u + \sqrt{h(x, t)}\right)t\right]$$

with

$$u(x, t) = 4\sqrt{H\left[x - \left(u + \sqrt{h}\right)t\right]} - 2\sqrt{h_0} - 2\sqrt{h(x, t)}$$
$$= 4\sqrt{h} - 2\sqrt{h_0} - 2\sqrt{h}$$
$$= 2\left[\sqrt{h(x, t)} - \sqrt{h_0}\right].$$

Finally, when we use this expression for *u*, we obtain the complete (implicit) solution for $h(x, t)$:

$$h(x, t) = H\left[x - \left(3\sqrt{h(x, t)} - 2\sqrt{h_0}\right)t\right].$$

Thus we have produced a description of the solution, in this example of a simple wave, but this exact solution – of the original nonlinear, coupled equations – necessarily is implicit, in general. Indeed, if the initial wave profile is one of elevation, then the solution (for *h*, for example) will 'break' in a finite time i.e. the characteristic lines will cross, requiring some appropriate shock (discontinuity) condition to be invoked.

Answers

Exercises 2

1. (a) $u(x,y)=F(3y-x^2)+\frac{1}{6}x^2$; (b) $u(x,y)=\mathrm{e}^x F(y-2x)$; (c) $u(x,t)=F(x-ut)$.

2. (a) $u(x,y)=\dfrac{2(x-y)}{y^2-x^2+4}$;

(b) $u(x,t)=\tanh(x-ut)$ and $u_x=\dfrac{\operatorname{sech}^2(x-ut)}{1+t\operatorname{sech}^2(x-ut)}$, which exists for $\forall t\geq 0$.

Exercises 3

1. $u(x,t)=\dfrac{1}{2c}\left[\arctan(x+ct)-\arctan(x-ct)\right]$.

2. $u(x,t)=f(t-x/c)H(t-x/c)$.

Exercises 4

(a) $2u_{\xi\eta}=\left(\dfrac{1}{\eta-\xi}\right)(u_\xi-u_\eta)$, $\xi=y-2\sqrt{x}$, $\eta=y+2\sqrt{x}$, hyperbolic;

(b) $2(1+\xi+\eta)u_{\xi\eta}=-u_\eta$, $\xi=2y-x$, $\eta=y^2+x$, hyperbolic;

(c) $\eta(\xi+\eta^2)u_{\eta\eta}=u+2\eta(\xi+\eta^2)u_\xi$, $\xi=y-x^2$, $\eta=x$, parabolic;

(d) $u_{\xi\xi}+u_{\eta\eta}=-\left(\dfrac{2}{1+4\eta}\right)u_\eta$, $\xi=y-x$, $\eta=x^2-x$, elliptic.

Exercise 5

$2T_{uu}+R^3T_{RR}+3R^2T_R=0$ so $'b^2-ac'=-2R^3<0$ i.e. elliptic.

Part III

Partial differential equations: method of separation of variables and similarity & travelling-wave solutions

List of Equations

This is a list of the types of equation, and specific examples, whose solutions are discussed.

Method of separation of variables:

Travelling-wave solutions:

Similarity solutions:

Preface

This text is intended to provide an introduction to the standard, elementary methods that are used for the solution of partial differential equations (mainly of the second order). Some of these methods and ideas are likely to be mentioned in a degree programme that includes mathematical methods, probably first met in the second year. The material has been written to provide a general – but broad – introduction to the relevant ideas, and not as a text closely linked to a specific course of study. Indeed, the intention is to present the material so that it can be used as an adjunct to a number of different courses or modules – or simply to help the reader gain a deeper understanding of these important techniques. The aim is to go beyond the methods that are presented in a conventional study, but all the standard ideas are discussed here (and can be accessed through the comprehensive index).

It is assumed that the reader has a basic knowledge of, and practical experience in, the methods for solving elementary ordinary differential equations. In addition, it is assumed that the reader is also familiar with eigenvalue problems; these are not developed here, although the ideas will be mentioned. This brief text does not attempt to include any detailed applications of these equations; this is properly left to a specific module that might be offered in a conventional applied mathematics or engineering or physics programme. However, some important examples of these equations will be included, which relate to specific areas of (applied) mathematical interest. The use of transform methods is not included.

The approach that we adopt is to present some general ideas, which might involve a notation, or a definition, or a method of solution, but most particularly detailed applications of the ideas explained through a number of carefully worked examples – we present 32. A small number of exercises, with answers, are also offered, although it must be emphasised that this notebook is not designed to be a comprehensive text in the conventional sense.

1 Introduction

The formulation of virtually every problem in modern – and classical – applied mathematics results in partial differential equations (of various types). The ultimate aim is then, usually, to obtain explicit solutions relevant to the investigation being undertaken. Of course, quite often, no solutions can be found (even assuming that appropriate solutions exist) because the equation, or the set of equations, are too difficult to solve. Although this may be the eventual outcome, the starting point for any such considerations is the collection of standard techniques that enable simple equations to be solved. In this volume in *The Notebook Series*, we shall give an overview of these methods. It is not our intention to produce, for example, all the details of Sturm-Liouville theory that are required for the method of separation of variables; this can be found in the volume in this series entitled 'An introduction to Sturm-Liouville theory'. Rather, we shall outline how the basic method unfolds when applied to various fairly common partial differential equations of applied mathematics and mathematical physics, and particularly when expressed in various coordinate systems.

The emphasis here will be on how partial differential equations are reduced to one or more ordinary differential equations, by a suitable transformation or assumption about the form of the solution. Of course, quite often, other – more general – solutions may exist, obtained by applying less restrictive assumptions; some of these ideas can be found in the volume 'Partial differential equations: classification and canonical forms'. In this volume, we will develop the techniques that produce ordinary differential equations by invoking the method of separation of variables, or by seeking solutions of travelling-wave or similarity form. As we shall find, one great advantage over more general methods (e.g. as applied to general, semi-linear equations in two independent variables) is that solutions can often be found for more complicated variants of such equations. Thus we may be able to cope with highly nonlinear equations, equations of higher order or equations in more than two independent variables. Indeed, we shall present examples taken from all these categories, many of them being important equations in their own right.

1.1 The Laplacian and coordinate systems

The simplest class of second-order partial differential equations that we commonly meet in classical applied mathematics and theoretical physics is based on the Laplacian ($\nabla\cdot\nabla\equiv\nabla^2$) of the unknown function. So we have

$$\nabla^2 u = 0\,,\ u_t = k\nabla^2 u \ \text{and}\ u_{tt} - c^2\nabla^2 u = 0\,,$$

which are Laplace's equation, the heat conduction (diffusion) equation and the wave equation, respectively; here t is time and k and c are positive constants, and we have denoted partial derivatives by subscripts. The physical systems that might be modelled by any one of these equations could be written in any suitable coordinate system; typically, these will be either rectangular, Cartesian coordinates (x, y, z), or cylindrical coordinates (r, θ, z), or spherical coordinates (r, θ, ϕ), although other choices are possible, perhaps even general curvi-linear coordinates. The simplest, and most familiar, is rectangular, Cartesian coordinates, giving rise to the operator

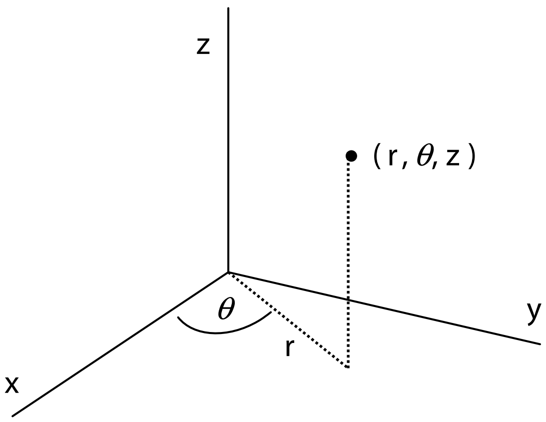
$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\,.$$

Then, with this choice of coordinates and a suitable reduction in the number of spatial dimensions, examples of our three equations above are

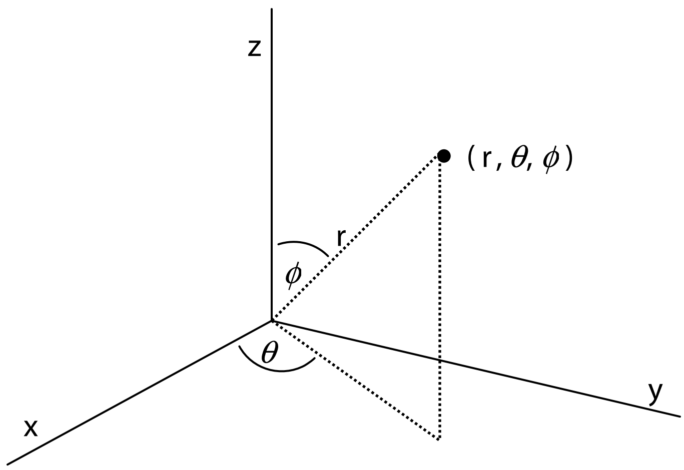
$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0\,,\quad \frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2}\,,\quad \frac{\partial^2 u}{\partial t^2}-c^2\frac{\partial^2 u}{\partial x^2}=0\,,$$

which are the simplest equations that are classified as *elliptic*, *parabolic* and *hyperbolic*, respectively. (The background and relevant details of this classification can be found in any introductory text on partial differential equations, or in the volume in this series: 'Partial differential equations: classification and canonical forms'.)

The other two, standard, coordinate systems are defined by



for cylindrical coordinates (where $x = r\cos\theta$, $y = r\sin\theta$), and



for spherical coordinates ($x = r\sin\phi\cos\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\phi$). Note that we have elected to use the symbols r and θ in each, although r has a different meaning in each; there should be no confusion, because either one or the other will be used exclusively in a particular application. The corresponding Laplace operators are then

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

in cylindrical coordinates (which is equivalently plane polars plus z), and

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{1}{r^2}\cot\phi\frac{\partial}{\partial \phi} + \frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial \theta^2}$$

in spherical coordinates. (We quote these results here; a derivation can be found in any good text on the vector calculus which includes a description of general curvi-linear coordinates.)

1.2 Overview of the methods

The reduction of partial differential equations to ordinary differential equations can be accomplished in one of three ways, although sometimes we are simply generating the same solution in a different way; it is these three approaches that we shall discuss in this volume. The first, and most familiar, is *separation of variables*; suppose that $u = u(x,t)$ is defined by a partial differential equation, then we seek a solution $u(x,t) = X(x)T(t)$ – the variables have been *separated*. Now this may appear to be a very special construct – it is! – yet quite general problems, based on suitable linear equations, can be solved when we invoke, in addition, the method of Fourier series. As we shall see, the method can be extended to any number of variables, so we may write e.g. $u(x,y,z,t) = X(x)Y(y)Z(z)T(t)$.

The second and third methods are essentially two different variants of just one: instead of separating the variables, *all* the variables are combined together into one functional group. However, there are two very different ways of doing this. In one, we restrict ourselves to a linear combination of the variables e.g. ($kx + \ell t$); this is called the *travelling-wave* form, because it mirrors precisely the type of solution obtained in wave problems e.g. with solutions such as $F(x - ct)$, written in the usual notation, which *is* a travelling wave (of unchanging form). (This terminology is quite often used even if time is not involved, although any resulting solutions are usually of no great importance.) In the other combination, all the independent variables are multiplied together, allowing for general powers e.g. xt^n; this is called the *similarity* (or *self-similar*) form. (Note that $F(x^m t^p)$ can be replaced by $G(xt^n)$, for suitable *G* and *n*, so there is no need to raise both *x* and *t* to general powers, in the case of two independent variables.) The solution expressed as $F(xt^n)$ has the property that $F = \text{constant}$ on lines $xt^n = \text{constant}$; this description of a solution i.e. constant on a given family of curves, is usually called *self-similarity*. A self-similar object is one which essentially looks the same on different scales; a classic example is a *fractal*. In our case, $F(xt^n)$ looks the same for any $F(\text{constant})$, and this applies for any combination of *x* and *t* ('different scales') for which $xt^n = \text{constant}$.

We should comment that, although the method of separation of variables enables very general solutions to be constructed (satisfying rather general types of initial and boundary conditions), the other two solutions are not so accommodating. The travelling-wave solution is a special solution of the equation which would normally be generated only by special initial (and possibly boundary) conditions, although such solutions might appear eventually (e.g. as time increases) in certain cases. This is typically the situation that is encountered in the appearance of solitary-wave solutions from general initial data in a 'soliton' problem. The situation that obtains for similarity solutions is even more extreme. Now the initial and boundary conditions must be very special indeed, and quite often cannot be easily predicted before the solution is known. However, it does sometimes happen that such solutions, suitably interpreted, can be used to derive more general solutions of the same equation, but satisfying more general initial and/or boundary conditions.

Finally we mention that another important method for finding solutions of partial differential equations, also usually by generating ordinary differential equations, is to employ transforms. This is too large a topic – there are many different transforms – for incorporation in this text; it is hoped to devote a separate volume to a description of transform methods.

2 The method of separation of variables

The history of this method is hidden in the fog that inevitably surrounds the past; it was used by a number of mathematicians, but usually applied only to very specific problems, from about the middle of the 18th century. Certainly L'Hospital, d'Alembert, Daniel Bernoulli and Euler employed the technique, and a case can be made that L'Hospital was the first. However, it was not until J.B.J. Fourier (1768-1830) that a complete and systematic development was presented (in about 1807, for the problem of heat conduction); thereafter it became a standard procedure in the armoury of applied mathematicians, at least for certain types of partial differential equation.

2.1 Introducing the method

We will describe the fundamental principles that underpin the method of separation of variables by considering the classical wave equation

$$u_{tt} - c^2 u_{xx} = 0 \ (c > 0\text{, constant}),$$

which is of hyperbolic type. The separable solution is written as $u(x,t) = X(x)T(t)$, for suitable functions X and T; the wave equation then becomes

$$XT'' - c^2 X''T = 0,$$

where the primes denote derivatives with respect to the corresponding arguments of the functions. It is convenient to divide throughout by XT:

$$\frac{T''}{T} - c^2 \frac{X''}{X} = 0 \text{ or } \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T}$$

and, although this manoeuvre requires $XT \neq 0$, we can dispense with this restriction when we have seen how the method proceeds. (We may note that this structure of the equation has produced another version of the separable property.) Thus we have

$$\frac{X''}{X} \text{ (a function of only } x) = \frac{1}{c^2}\frac{T''}{T} \text{ (a function of only } t) = \lambda\text{, say,}$$

but x and t are, by definition, independent variables i.e. they are assigned arbitrarily on the appropriate domains. It is clear, therefore, that any one of the choices

λ is a function of only x; λ is a function of only t; λ is a function of x and t, leads to an inconsistency. The only possible choice is $\lambda = \text{constant}$ (usually called the *separation constant*), and so we obtain

$$\frac{X''}{X} = \lambda \text{ and } \frac{T''}{T} = c^2\lambda .$$

The problem for $X(x)$ now becomes, with suitable boundary conditions, an eigenvalue (*Sturm-Liouville*) problem, which will have appropriate solutions only if

$$\lambda = -\omega^2 < 0$$

i.e. $$X'' + \omega^2 X = 0 \text{ and then } T'' + c^2\omega^2 T = 0 ,$$

for specific values of λ, the *eigenvalues*. The general solution for both $X(x)$ and $T(t)$ involve trigonometric functions. Finally we observe, from the original separation of variables, namely

$$XT'' - c^2 X''T = 0 \text{, that we must have e.g. } X'' \propto X ,$$

and so $T'' \propto T$: the equation is separable without the need to require $XT \neq 0$. A more general solution can now be obtained, in the familiar way, by summing over all the eigenvalues of the underlying Sturm-Liouville problem (permitted because the partial differential equation is linear).

This method goes over directly to the other two elementary, standard equations of applied mathematics/theoretical physics.

Example 1

Apply the method of separation of variables to the heat conduction (diffusion) equation $u_t = ku_{xx}$ ($k > 0$, constant).

We set $u(x,t) = X(x)T(t)$, which gives $XT' = kX''T$ and we require for separability and consistency that $X'' \propto X$; let $X'' = \lambda X$. The Sturm-Liouville problem, with suitable boundary conditions, then requires that $\lambda = -\omega^2 < 0$ i.e. $X'' + \omega^2 X = 0$, which (again) has trigonometric solutions. This leaves $T' = -k\omega^2 T$, so that

$$T(t) = A\exp\left(-k\omega^2 t\right),$$

where *A* is an arbitrary constant: the *x*-dependence is oscillatory, but the *t*-dependence is a decaying exponential (because $\omega \neq 0$ and is real, and $k > 0$).

Example 2

Apply the method of separation of variables to the Laplace equation $u_{xx} + u_{yy} = 0$.

We write $u(x,y) = X(x)Y(y)$, and so obtain $X''Y + XY'' = 0$, and then with $X'' \propto X$ (or, indeed, $Y'' \propto Y$), we have

$$X'' + \lambda X = 0 \text{ and } Y'' - \lambda Y = 0.$$

Depending on the boundary conditions, either $X(x)$ or $Y(y)$ will be described by a Sturm-Liouville problem. If this is $X(x)$, then $\lambda = \omega^2 > 0$ and $X(x)$ is a trigonometric function, but then $Y(y)$ is a hyperbolic (exponential) function; on the other hand, if $Y(y)$ is the Sturm-Liouville problem, the rôles of $X(x)$ and $Y(y)$ are reversed. However, there will always be one of this pair of functions that is trigonometric and the other hyperbolic.

Comment: In summary, we see that we have

$u_{tt} - c^2 u_{xx} = 0$ (hyperbolic type); both $X(x)$ and $T(t)$ are trigonometric;

$ku_{xx} = u_t$ (parabolic type); $X(x)$ is trigonometric and $T(t)$ is exponential;

$u_{xx} + u_{yy} = 0$ (elliptic type); $X(x)$ is trigonometric/hyperbolic, $Y(y)$ is hyperbolic/trigonometric.

The three examples that we have discussed above have been second order, in two independent variables, and with constant coefficients. It should be plain that linear, constant coefficient equations are all susceptible to the method of separation of variables. However, if the equation is nonlinear, it is not clear if we can proceed (but, exceptionally, this is possible, as we shall demonstrate later); if the equation has variable coefficients then, in general, we cannot separate the variables. Consider the equation

$$u_{tt} - [c(x,t)]^2 u_{xx} = 0,$$

which is a wave equation with variable speed, then separability clearly fails i.e. $XT'' = \left[c(x,t)^2\right]X''T$ is not separable unless the $c(x,t)$ is itself separable, but then we will certainly generate a more difficult pair of ordinary differential equations to solve. One example, of this general type, that is readily solved, is the heat conduction equation with a variable conductivity:

$$u_t = k(t)u_{xx},$$

for then we obtain $XT' = k(t)X''T$, and so $X'' = -\lambda X$ yields

$$T' = -\lambda k(t)T \text{ i.e. } \ln|T| = -\lambda \int^{t} k(t')\,\mathrm{d}t'$$

$$\text{or } T(t) = A\exp\left[-\lambda \int_0^t k(t')\,\mathrm{d}t'\right].$$

In this case, all that has happened is that the variable t has been replaced by $\int_0^t k(t')\,\mathrm{d}t'$ (provided that this integral exist; otherwise we simply work with $\int^t k(t')\,\mathrm{d}t'$). It is instructive to note that we may perform this transformation (replacing t by the integral) in the original equation, no matter what method of solution we might adopt. In some of what follows, variable coefficients do arise, but by virtue of the forms of the Laplacian that appear in the equations.

2.2 Two independent variables: other coordinate systems

We introduce this extension of the method by first considering the heat conduction (diffusion) equation with cylindrical symmetry:

$$u_t = k\left(u_{rr} + \tfrac{1}{r}u_r\right) \ (k > 0, \text{ constant}).$$

The separable solution takes the form $u(r,t) = R(r)T(t)$, which gives

$$RT' = k\left(R'' + \tfrac{1}{r}R'\right)T$$

and so with $T' = -\lambda T$, we obtain

$$-\lambda R = k\left(R'' + \tfrac{1}{r}R'\right).$$

Almost all physically realistic solutions will require that u decays exponentially in time (t), so we shall impose $\lambda > 0$. We have

$$R'' + \frac{1}{r}R' + \frac{\lambda}{k}R = 0$$

in which we may write $R(r) = S\left(r\sqrt{\lambda/k}\right)$, to give

$$\frac{\lambda}{k}S'' + \frac{1}{r}\sqrt{\frac{\lambda}{k}}S' + \frac{\lambda}{k}S = 0$$

$$\text{or } \rho S'' + S' + \rho S = 0 \text{ (where } \rho = r\sqrt{\lambda/k}\text{),}$$

which is a zero-order *Bessel equation* (with solutions J_0 and Y_0; see e.g. the volume 'The series solution of second order, ordinary differential equations and special functions' in *The Notebook Series*). The solution that is bounded as $\rho \to 0$ is J_0, so one solution of the partial differential equation is

$$u(r,t) = A\mathrm{J}_0\left(r\sqrt{\lambda/k}\right)\exp(-\lambda t),$$

where A is an arbitrary constant. (Typically, the eigenvalues are associated with the zeros of J_0 or J_0'.)

The same technique can be applied to similar equations, as we now demonstrate.

Example 3

Apply the method of separation of variables to $u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0$ (which is Laplace's equation written in plane polar coordinates).

We set $u(r,\theta)=R(r)\Theta(\theta)$, to give

$$\left(R''+\frac{1}{r}R'\right)\Theta+\frac{1}{r^2}R\Theta''=0,$$

and we write $\Theta''=-\lambda\Theta$; if the problem is described by $0\le\theta\le 2\pi$, then we must have solutions that are periodic (period at least 2π) if they are to be continuous for $\theta\in[0,2\pi]$. On the other hand, if $\theta\in[0,\theta_0]$, $0<\theta_0<2\pi$, then no such condition can be imposed; nevertheless, we normally expect $\lambda>0$ so that the solutions are trigonometric in θ (rather than exponential or linear). Let us write $\lambda=\omega^2>0$, then we obtain

$$\Theta(\theta)=A\sin(\omega\theta)+B\cos(\omega\theta),$$

where A and B are arbitrary constants.

The equation for $R(r)$ becomes

$$r^2R''+rR'-\omega^2R=0$$

which is of *Euler* type; the solution, for $\omega>0$, is of the form $R(r)=r^{\alpha}$, so

$$\alpha(\alpha-1)+\alpha-\omega^2=\alpha^2-\omega^2=0 \text{ i.e. } \alpha=\pm\omega,$$

which leads to the general solution

$$R(r)=Cr^{\omega}+Dr^{-\omega},$$

where C and D are arbitrary constants. Thus we have a solution for u:

$$\theta)=\left(Cr^{\omega}+Dr^{-\omega}\right)\left(A\sin(\omega\theta)+B\cos(\omega\theta)\right).$$

Comment: We can simplify the dependence on the arbitrary constants; for example, if all the constants in the above solution are non-zero, then (for new arbitrary constants) we may write

$$u(r,\theta)=A\left(r^{\omega}+Br^{-\omega}\right)\left(\sin(\omega\theta)+C\cos(\omega\theta)\right).$$

Example 4

Apply the method of separation of variables to $u_{rr}+\frac{2}{r}u_r+\frac{1}{r^2}u_{\phi\phi}+\frac{1}{r^2}(\cot\phi)u_{\phi}=0$, which is Laplace's equation with symmetry about the 'polar' axis in spherical coordinates i.e. u does not depend on θ.

We seek a solution in the form $u(r,\phi) = R(r)\Phi(\phi)$, to obtain

$$\left(R'' + \frac{2}{r}R'\right)\Phi + \frac{1}{r^2}\left(R\Phi'' + R\Phi'\cot\phi\right) = 0,$$

in which we write $R'' + \frac{2}{r}R' = \lambda\frac{1}{r^2}R$, leaving the equation for Φ:

$$\Phi'' + \Phi'\cot\phi + \lambda\Phi = 0.$$

Now this latter equation is more usually expressed in terms of $\Phi(\phi) = \Psi(\cos\phi)$, so that we have

$$\Phi' = -(\sin\phi)\Psi'(x) \text{ where } x = \cos\phi,$$

and then $\Phi'' = -(\cos\phi)\Psi' + (\sin\phi)^2\Psi'' = -x\Psi' + \left(1 - x^2\right)\Psi''$;

thus the equation for Φ becomes

$$\left(1 - x^2\right)\Psi'' - 2x\Psi' + \lambda\Psi = 0.$$

This is *Legendre's equation* (see e.g. the volume 'The series solution of second order, ordinary differential equations and special functions') and, for $-1 \le x \le 1$ i.e. $0 \le \phi \le 2\pi$, bounded solutions exist only for $\lambda = n(n+1)$ (with $n = 0, 1, 2, \dots$); the corresponding solutions are the *Legendre polynomials*, $P_n(x) = P_n(\cos\phi)$.

With this choice of λ, the equation for $R(r)$ is

$$r^2R'' + 2rR' - n(n+1)R = 0,$$

which is again an Euler equation. Thus we seek a solution $R(r) = r^\alpha$, which gives

$$\alpha(\alpha - 1) + 2\alpha - n(n+1) = \alpha^2 + \alpha - n(n+1) = 0 \text{ i.e. } \alpha = n \text{ or } -(n+1),$$

which is valid for all the allowed *n* (so no repeated roots); the general solution is therefore

$$R(r) = Ar^n + \frac{B}{r^{n+1}}.$$

A solution of the partial differential equation is thus

$$u(r,\phi) = \left(Ar^n + Br^{-(n+1)}\right)P_n(\cos\phi).$$

Comment: If $r = 0$ is in the domain of the solution, then we shall presumably require $B = 0$; correspondingly, if $r \to \infty$, then we shall need to select $A = 0$ (unless $n = 0$) for a solution bounded at infinity.

2.3 Linear equations in more than two independent variables

The method of separation of variables can be extended to any number of variables, at least if the equation is linear and with constant coefficients; to see this, let us consider, as an introductory example, the heat conduction (diffusion) equation in two spatial variables:

$$u_t = k\left(u_{xx} + u_{yy}\right) \;\; (k > 0\text{, constant}).$$

First, we write $u(x, y, t) = T(t)\phi(x, y)$, to give

$$T'\phi = kT\left(\phi_{xx} + \phi_{yy}\right),$$

and the separation of the t- and the (x, y)-dependences requires that $T' = -\lambda T$ (where λ is our familiar separation constant). This leaves

$$k\left(\phi_{xx} + \phi_{yy}\right) + \lambda\phi = 0,$$

which is an equation of elliptic type, solved in any appropriate way. But one way is clearly by separation of variables, obtained by writing $\phi(x,y) = X(x)Y(y)$:

$$k\left(X''Y + XY''\right) + \lambda XY = 0,$$

and then we may set, for example, $X'' = \mu X$ (where μ is a second separation constant) which gives

$$kY'' + (\lambda + k\mu)Y = 0.$$

It is now clear that we may seek a solution, *a priori*, in the form

$$u(x,y,t) = X(x)Y(y)T(t);$$

further, it follows that we may extend this procedure to any number of independent variables (if the equation admits such a solution). Let us return to a consideration of the details of the current equation.

We expect that $\lambda > 0$, so that the solution remains bounded as $t \to \infty$, and then we have *two* eigenvalue (Sturm-Liouville) problems:

first $X'' + \omega^2 X = 0$ (where we have written $\mu = -\omega^2$)

and then $Y'' + \left(\dfrac{\lambda}{k} - \omega^2\right)Y = 0$,,

which becomes a second conventional eigenvalue problem if $(\lambda/k) - \omega^2 > 0$. For example, the boundary conditions

$$X(0) = X(a) = 0 \text{ and } Y(0) = Y(b) = 0$$

require the choices $\omega = n\pi/a$ ($n = 1, 2, \ldots$) and $(\lambda/k) - \omega^2 = (m\pi/b)^2$ ($m = 1, 2, \ldots$) so that we have

$$\lambda = k\pi^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right) > 0.$$

In this case we have a general solution, obtained by summing over all *m* and *n*, that can be written as

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn} \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)\exp\left[-k\pi^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right)t\right],$$

where the A_{mn} is a set of arbitrary constants; note that this solution comprises a *double Fourier series* (in *x* and *y*).

The corresponding wave equation provides us with a closely similar example, as we show below.

Example 5

Apply the method of separation of variables to $u_{tt} - c^2\left(u_{xx} + u_{yy}\right) = 0$ ($c > 0$, constant).

We set $u(x,y,t) = X(x)Y(y)T(t)$, which gives

$$XYT'' - c^2\left(X''Y + XY''\right)T = 0,$$

and then choose to write $T'' = -\omega^2 c^2 T$, to leave

$$\omega^2 XY + X''Y + XY'' = 0.$$

Now we set $X'' = -\Omega^2 X$, which produces the equation for *Y*:

$$Y'' + \left(\omega^2 + \Omega^2\right)Y = 0.$$

Thus we can have a solution that is oscillatory (i.e. trigonometric) in *t*, in *x* and in *y*.

Comment: This solution would be relevant, for example, to the vibrations of a rectangular membrane.

Now we turn to the consideration of three independent variables as they appear in cylindrical coordinates.

Example 6

Apply the method of separation of variables to the Laplace equation

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} = 0.$$

We seek a solution of the form $u(r,\theta,z) = R(r)\Theta(\theta)Z(z)$, then

$$R''\Theta Z + \frac{1}{r}R'\Theta Z + \frac{1}{r^2}R\Theta''Z + R\Theta Z'' = 0;$$

we choose to write $\Theta'' = -\omega^2\Theta$ and $Z'' = \lambda Z$, which leaves

$$R'' + \frac{1}{r}R' - \frac{1}{r^2}\omega^2 R + \lambda R = 0.$$

This equation, written in the more usual form, becomes

$$r^2 R'' + rR' + \left(\lambda r^2 - \omega^2\right)R = 0,,$$

which is a Bessel equation.

Comment: Although we are likely to require $\omega^2 > 0$ (so that $\Theta(\theta)$ is a trigonometric function), we might have $\lambda < 0$ (so $Z(z)$ is also trigonometric) or $\lambda > 0$, in which case $Z(z)$ will be an exponential function. These choices then control the particular type (and order) of the solution of the Bessel equation (which is described in the volume 'The series solution of second order, ordinary differential equations and special functions').

Example 7

Apply the method of separation of variables to the heat conduction (diffusion) equation

$$u_t = k\left(u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}u_{\phi\phi} + \frac{1}{r^2}(\cot\phi)u_\phi + \frac{1}{r^2\sin^2\phi}u_{\theta\theta}\right) \text{ } (k > 0\text{, constant}).$$

We seek a solution in the form $u(r,\theta,\phi,t) = R(r)\Theta(\theta)\Phi(\phi)T(t)$, which gives

$$R\Theta\Phi T' = k\left(R''\Theta\Phi T + \frac{2}{r}R'\Theta\Phi T + \frac{1}{r^2}R\Theta\Phi'' T + \frac{\cot\phi}{r^2}R\Theta\Phi' T + \frac{1}{r^2\sin^2\phi}R\Theta''\Phi T\right);$$

further, we set $T' = -\lambda T$ ($\lambda > 0$) and $\Theta'' = -\omega^2\Theta$, so we obtain

$$-\lambda R\Phi = k\left(R''\Phi + \frac{2}{r}R'\Phi + \frac{1}{r^2}R\Phi'' + \frac{\cot\phi}{r^2}R\Phi' - \frac{\omega^2}{r^2\sin^2\phi}R\Phi\right).$$

At this stage, we elect to write

$$\Phi'' + (\cot\phi)\Phi' - \frac{\omega^2}{\sin^2\phi}\Phi = -\mu\Phi,$$

which leads to the equation for *R*:

$$-\lambda R = k\left(R'' + \frac{2}{r}R' - \frac{\mu}{r^2}R\right).$$

The equation for $\Phi(\phi) = \Psi(\cos\phi)$ is (cf. Example 4)

$$\left(1 - x^2\right)\Psi'' - 2x\Psi' + \left(\mu - \frac{\omega^2}{1 - x^2}\right)\Psi = 0 \text{ } (x = \cos\phi),$$

which is a *generalised Legendre equation*; the equation for $R(r)$ is

$$r^2R'' + 2rR' + \left(\frac{\lambda}{k}r^2 - \mu\right)R = 0,$$

which can be recast as a Bessel equation (by writing $R = r^{-1/2}S(r)$).

Comment: Note that the equation for $R(r)$ is not a Bessel equation, because of the factor $2r$; to be a Bessel equation this must be simply *r*. The proposed transformation ensures that $S(r)$ does satisfy a Bessel equation.

2.4 Nonlinear equations

We conclude our presentation of various applications of the method of separation of variables by showing that, occasionally, nonlinear equations can have separable solutions. One particularly simple type of problem, in two independent variables (*x* and *t*, say), occurs when we have

$$L_t(u) = M_x(u),$$

where L_t is a suitable (even nonlinear) differential operator in *t*, and M_x is a corresponding operator in *x*.

Example 8

Apply the method of separation of variables to $\left(u^p\right)_t = u_{xx}$, $p > 0$ (constant), which is a nonlinear diffusion equation (of a type that can arise in the study of plasmas).

We seek a solution $u(x,t) = X(x)T(t)$, which gives and then we set $T' = \lambda T^{2-p}$ i.e.

$$\frac{T^{p-1}}{p-1} = \lambda t + A \ (p \neq 1),$$

where A is an arbitrary constant. (Note that the case $p = 1$ recovers the linear equation.) The equation for $X(x)$ becomes

$$X'' = \lambda X^p$$

which can be integrated once, by first multiplying by X':

$$X''X' = \lambda X^p X' \text{ so } X'^2 = \frac{2\lambda}{p+1}\left(X^{p+1} + B\right)$$

where B is a second arbitrary constant. This allows us to represent the solution for X in an implicit form:

$$\int^X \frac{dX'}{\sqrt{(X')^{p+1} + B}} = t\sqrt{\frac{2\lambda}{p+1}},$$

which requires $\lambda > 0$ for a real solution.

Comment: This example demonstrates that the technique can be used, but we note that the resulting equations are no longer likely to be trivial (or even standard).

A rather more sophisticated type of example arises when we must first transform the equation before seeking a separable solution. An interesting (but rather involved) such example is presented in the next exercise.

Example 9

Transform the sine-Gordon equation, $u_{xx} - u_{tt} = \sin u$, according to $u(x,t) = 4\arctan\left[v(x,t)\right]$, and then seek a separable solution for $v(x,t)$ (which is most conveniently expressed as $v(x,t) = X(x)/T(t)$).

We are given $u(x,t) = 4\arctan\left[v(x,t)\right]$, so we obtain

$$u_x = \frac{4v_x}{1+v^2} \text{ and } u_{xx} = \frac{4v_{xx}}{1+v^2} - \frac{8vv_x}{\left(1+v^2\right)^2},$$

and correspondingly for the t-derivatives. Thus the sine-Gordon equation can be written

$$\begin{aligned}\frac{4(v_{xx}-v_{tt})}{1+v^2}-\frac{8v\left(v_x^2-v_t^2\right)}{\left(1+v^2\right)^2}&=\sin u=2\sin(u/2)\cos(u/2)\\&=4\sin(u/4)\cos(u/4)\left[\cos^2(u/4)-\sin^2(u/4)\right]\\&=4\cos^4(u/4)\tan(u/4)\left[1-\tan^2(u/4)\right]\\&=\frac{4\tan(u/4)\left[1-\tan^2(u/4)\right]}{\left[\sec^2(u/4)\right]^2}\\&=\frac{4\tan(u/4)\left[1-\tan^2(u/4)\right]}{\left[1+\tan^2(u/4)\right]^2}=\frac{4v\left(1-v^2\right)}{\left(1+v^2\right)^2}.\end{aligned}$$

Thus we find the equation for $v(x,t)$ to be

$$(v_{xx}-v_{tt})\left(1+v^2\right)-2v\left(v_x^2-v_t^2\right)=v\left(1-v^2\right)=v\left(1+v^2\right)-2v^3,,$$

or

$$\left(1+v^2\right)(v_{xx}-v_{tt}-v)-2v\left(v_x^2-v_t^2-v^2\right)=0,$$

which does not look promising! However, we do note that, although the original contained both linear $(u_{xx}-u_{tt})$ and highly nonlinear $(\sin u)$ terms, the equation for v has a more uniform structure. We set $v(x,t)=X(x)/T(t)$, to give

$$\left[1+\left(\frac{X}{T}\right)^2\right]\left(\frac{X''}{T}+\frac{XT''}{T^2}-\frac{2XT'^2}{T^3}-\frac{X}{T}\right)-\frac{2X}{T}\left[\left(\frac{X'}{T}\right)^2-\left(\frac{XT'}{T^2}\right)^2-\left(\frac{X}{T}\right)^2\right]=0.$$

This is more conveniently written as

$$\frac{X}{T^3}\left(X^2+T^2\right)\left(\frac{X''}{X}+\frac{T''}{T}\right)-\frac{2X}{T^3}\left(X'^2+T'^2\right)+\frac{X}{T^3}\left(X^2-T^2\right)=0$$

and then collecting appropriate terms, this becomes

$$\left(X^2+T^2\right)\frac{X''}{X}-2X'^2+X^2+\left(X^2+T^2\right)\frac{T''}{T}-2T'^2-T^2=0,$$

which still does not look very hopeful. Although more general solutions are possible, we will content ourselves with a simple choice: $X' = \lambda X$; let us investigate what this produces.

First, we observe that $X'' = \lambda X' = \lambda^2 X$; now, because of the near-symmetry between *X* and *T*, we also write $T' = \mu T$ (and then $T'' = \mu^2 T$) to give

$$\left(1-\lambda^2+\mu^2\right)X^2+\left(\lambda^2-\mu^2-1\right)T^2=0.$$

Thus we do indeed have a separable solution, provided that $\mu^2 = \lambda^2 - 1$; this gives

$$X' = \lambda X \text{ and } T' = \pm\sqrt{\lambda^2-1}\,T \text{ (for } |\lambda|>1\text{)},$$

or

$$X = Ae^{\lambda x} \text{ and } T = Be^{\pm t\sqrt{\lambda^2-1}},$$

where *A* and *B* are arbitrary constants. A solution of the sine-Gordon equation is therefore

$$u(x,t) = 4\arctan\left[C\exp\left(\lambda x \pm t\sqrt{\lambda^2-1}\right)\right]$$

where $C = A/B$ and λ (with $|\lambda|>1$) are two arbitrary constants (so we do not have a conventional eigenvalue in this problem).

Comment: The solution that we have constructed is a wave, propagating with a speed $\sqrt{\lambda^2-1}\big/|\lambda|$, either to the left or to the right. This is a solitary-wave (so-called 'kink') solution of the sine-Gordon equation, one of the important 'soliton'equations.

Exercises 2

1. Apply the method of separation of variables to the wave equation

$$u_{tt}-c^2\left(u_{rr}+\tfrac{1}{r}u_r\right)=0..$$

2. Seek a solution of $u_{tt}-c^2\left(u_{rr}+\tfrac{2}{r}u_r\right)=0$ in the form $u(r,t)=r^{-1}v(r,t)$, and comment on the resulting equation for *v*.

3. Apply the method of separation of variables to $u_t + uu_x = 0$.

****

****

3 Travelling-wave solutions

Suppose that we have a function of two variables, $u(x,t)$, where it will be convenient to use the familiar interpretation: x is distance and t is time. This will enable us to give a description that is consistent with the notion of a *travelling wave*. Further, let us suppose that the function takes the form

$$u(x,t) = F(x - Vt),$$

for some constant V and some function F, either or both of which may be arbitrary (but it is not unusual to find that they are prescribed). We now have a representation that can be expressed as

$$u = \text{constant} \;\text{ on lines }\; x - Vt = \text{constant} \;\text{ i.e. on lines }\; \frac{\mathrm{d}x}{\mathrm{d}t} = V\,;$$

thus all points on the profile (shape), $u = F(x)$, move at the same speed (V). This is therefore a wave that propagates at speed V (to the right if $V > 0$), with unchanging shape. (We may note that this is precisely the form of each of the two components that constitute d'Alembert's solution of the wave equation:

$$u(x,t) = F(x - ct) + G(x + ct), \text{ the general solution of } u_{tt} - c^2 u_{xx} = 0.)$$

A solution that can be written as

$$u(x,t) = F(x - Vt)$$

is called a *travelling-wave* solution.

3.1 The classical, elementary partial differential equations

We have just mentioned the connection with the classical wave equation; indeed, we may use this approach to recover d'Alembert's solution. We seek a solution $u(x,t) = F(x - Vt)$ of

$$u_{tt} - c^2 u_{xx} = 0 \text{ (where } c > 0 \text{ is constant);}$$

thus we obtain

$$V^2 F'' - c^2 F'' = 0,$$

where the prime denotes the derivative with respect to $(x - Vt)$. Thus for arbitrary $F''(x - Vt)$ ($\neq 0$), we require

$$V^2 = c^2 \text{ so that } V = \pm c;$$

then, because the wave equation is linear, we may use this result to construct the general solution of the wave equation

$$u(x,t) = F(x - ct) + G(x + ct)$$

where F and G are arbitrary functions (although this argument is only valid for functions whose second derivatives exist and are non-zero).

Example 10

Seek a travelling-wave solution of the heat conduction (diffusion) equation $u_t = ku_{xx}$ (for $k > 0$, constant).

We write $u(x,t) = F(x - Vt)$, where V is an unknown constant, and then we obtain

$$-VF' = kF'' \text{ so that } kF' = -VF + AV,$$

where A is an arbitrary constant. The equation for F has the general solution

$$F = A + B\mathrm{e}^{-(V/k)(x - Vt)},$$

where B is a second arbitrary constant; the travelling-wave solution, in this case, is necessarily an exponential function.

Comment: A more usual (and useful) form of this solution is obtained when we set $-V/k = \mathrm{i}\ell$ (where ℓ is real), which gives

$$F = A + B\exp\left[\mathrm{i}\ell(x + \mathrm{i}k\ell t)\right] = A + B\exp\left[\mathrm{i}\ell x - k\ell^2 t\right]$$

which represents a non-propagating harmonic wave ($\exp(\mathrm{i}\ell x)$) that decays in time. We note that, because the equation is linear, we may elect to use this complex form, or take the real or the imaginary part.

A problem, based on the equation of heat conduction, that more closely corresponds to a travelling wave is described in the next example.

Example 11

Find a travelling-wave solution of the problem: $u_t = ku_{xx}$ in $x < ct$, with $u = 0$ and $u_x = -c$ both on $x = ct$, where *c* is a constant. (This is a single phase, one dimensional *Stefan* problem, which might model a moving front, with e.g. water on one side, and ice on the other side of the front $x = ct$, and *u* the temperature (taken to be zero in the ice phase).)

We seek a solution of the form $u(x,t) = F(x - Vt)$, then we obtain

$$-VF' = kF'' \text{ with the general solution (see above) } F = A + B\mathrm{e}^{-(V/k)(x - Vt)}.$$

This solution satisfies the two conditions on $x = ct$ if $V = c$, $A + B = 0$ and $B(V/k) = c$, which produces

$$u(x,t) = k\left(\mathrm{e}^{-(c/k)(x-ct)} - 1\right).$$

Comment: If $c > 0$, then in the region behind the front, we have $u \to \infty$ as $(x - ct) \to -\infty$ (so the water temperature is above zero – usually regarded as the stable configuration). On the other hand, if $c < 0$, then $u \to -k$: the water is colder than the ice, which suggests that we have a problem! (It turns out that the existence of appropriate solutions is in doubt.)

Example 12

Seek a 'travelling-wave' solution of Laplace's equation: $u_{xx} + u_{yy} = 0$.

We write $u(x,y) = F(ax + by)$, to give

$$a^2 F'' + b^2 F'' = 0,$$

and so for a and b real (and at least one non-zero), we require $F'' = 0$ i.e.

$$u(x,y) = F = A + B(ax+by),$$

where each of A, B, a and b are arbitrary constants. This is not a very illuminating or important result!

Comment: If we allowed ourselves the liberty of writing $b = \pm \mathrm{i}a$ (a real), then we have a solution for arbitrary functions, F. Thus we may then write down the complex-valued version of the general solution (cf. d'Alembert's solution of the wave equation mentioned earlier in this paragraph):

$$u(x,y) = F(x+\mathrm{i}y) + G(x-\mathrm{i}y).$$

The further choice $G \equiv 0$ then leads to the fundamental notion of a function of a complex variable, the Cauchy-Riemann relations, and so on.

3.2 Equations in higher dimensions

The obvious exemplar under this heading is the wave equation written in rectangular, Cartesian coordinates:

$$u_{tt} - c^2\left(u_{xx} + u_{yy} + u_{zz}\right) = 0.$$

The travelling-wave solution then takes the form $u(x,y,z,t) = F(kx+ly+mz-Vt)$,

where k, l, m and V are constants; this gives

$$\left[V^2 - c^2\left(k^2 + l^2 + m^2\right)\right]F'' = 0,$$

and so for $F'' \neq 0$ (but otherwise arbitrary) we require

$$V = \pm c\sqrt{k^2 + l^2 + m^2}.$$

This solution describes a wave, of arbitrary shape, that propagates with speed V (either forwards or backwards); the wave profile is such that

$$u = F = \text{constant} \text{ on lines } kx + ly + mz - Vt = \text{constant},$$

which are parallel planes in (x, y, z)-space, at any time, t. Such a solution is often referred to as a *plane wave*. Indeed, if we recast the solution as

$$u(\mathbf{x},t) = F(\mathbf{k}\cdot\mathbf{x} - Vt),$$

where $\mathbf{k} \equiv (k,l,m)$ is the *wave-number vector*, then lines $\mathbf{k}\cdot\mathbf{x} - Vt = \text{constant}$ can be described by

$$\mathbf{k}\cdot\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = V = \pm c\sqrt{k^2+l^2+m^2} \quad \text{i.e. } \hat{\mathbf{k}}\cdot\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \pm c,$$

where $\hat{\mathbf{k}}$ is the unit vector in the $\mathbf{k}$ -direction. Thus the component of the velocity of the wave, in the direction of the wave-number vector, is *c* (either forwards or backwards), and elementary considerations confirm that the wave propagates in the $\mathbf{k}$ -direction, with $\mathbf{k}$ normal to the wave fronts (defined by the lines $\mathbf{k}\cdot\mathbf{x} - Vt = \text{constant}$).

Example 13

Find the travelling-wave solution of the heat conduction (diffusion) equation in two spatial dimensions: $u_t = k(u_{xx}+u_{yy})$ ($k > 0$, constant).

We seek a solution $u(x,y,t) = F(px+qy-Vt)$, where *p* and *q* are constants, to give

$$-VF' = k(p^2+q^2)F'',$$

which therefore repeats Example 10, but with *k* replaced by $k(p^2+q^2)$.

3.3 Nonlinear equations

A class of equations that often pose significant difficulties if we are aiming to find general solutions are nonlinear equations, but many of them are susceptible to more direct methods. Here, we will introduce this aspect of travelling-wave solutions by considering a classical example of this type:

$$u_t + uu_x = ku_{xx} \ \ (k > 0\text{, constant}),$$

the *Burgers equation*. This is an important equation, particularly relevant to models of sound waves in a viscous gas and to the structure of shock waves, although it was first introduced as a model for one-dimensional turbulence (by J.M. Burgers in 1948).

We seek a solution $u(x,t) = F(x - Vt)$, and then we obtain

$$-VF' + FF' = kF'' \ \text{ or } \ -VF + \tfrac{1}{2}F^2 = kF' + A,$$

where A is an arbitrary constant. The solution usually of interest is the one that satisfies conditions such as

$$u = F \to u_0 \text{ as } x \to \infty;\ u = F \to u_1 \text{ as } x \to -\infty,$$

which are typically the initial conditions for the unsteady problem i.e. $u = u_0$ in $x > 0$ and $u = u_1$ in $x < 0$, at $t = 0$. These conditions require that the two roots of

$$\tfrac{1}{2}F^2 - VF - A = 0 \text{ are } F = u_0, u_1,$$

so we have $F^2 - 2VF - 2A = (F - u_0)(F - u_1)$

i.e. $V = \frac{1}{2}(u_0 + u_1)$ and $A = -\frac{1}{2}u_0 u_1$,

and this expression for V is a significant result: the wave propagates at a speed that is the average of the conditions ahead and behind. Now we have the equation

$$2kF' = (F - u_0)(F - u_1)$$

and so we obtain $\displaystyle \frac{1}{(u_0 - u_1)}\int\left(\frac{1}{F - u_0} - \frac{1}{F - u_1}\right)\mathrm{d}F = \int \frac{\mathrm{d}\xi}{2k}$,

where $\xi = x - Vt$; thus

$$\ln\left|\frac{F - u_0}{F - u_1}\right| = \frac{u_0 - u_1}{2k}\xi + B$$

$$\text{or} \quad \frac{F - u_0}{F - u_1} = C\exp\left[\left(\frac{u_0 - u_1}{2k}\right)\xi\right],$$

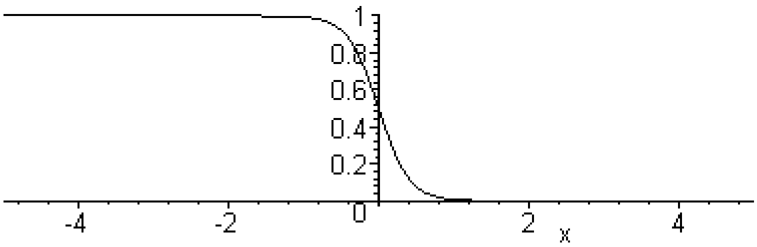
where the arbitrary constant C is most naturally associated with an arbitrary origin shift (x_0). Thus we may write

$$u(x,t) = F(x - VT) = \frac{u_0 + u_1 \exp\left[\left(\frac{u_0 - u_1}{2k}\right)(x - Vt - x_0)\right]}{1 + \exp\left[\left(\frac{u_0 - u_1}{2k}\right)(x - Vt - x_0)\right]}$$

which is more conveniently expressed as

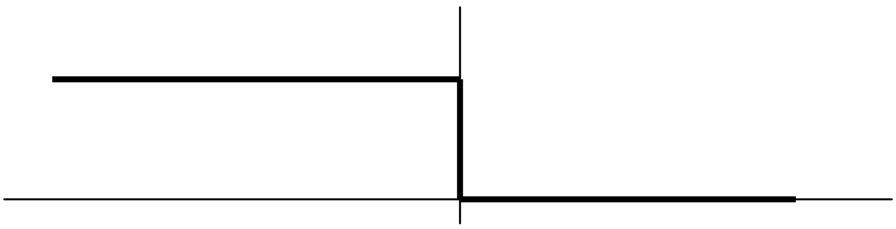
$$u(x,t) = \tfrac{1}{2}(u_0 + u_1) - \tfrac{1}{2}(u_0 - u_1)\tanh\left[\left(\frac{u_0 - u_1}{4k}\right)(x - Vt - x_0)\right]$$

where $V = \tfrac{1}{2}(u_0 + u_1)$; an example is reproduced below. (This is often referred to as the *Taylor shock profile*.)



The tanh profile $y = \frac{1}{2}[1 - \tanh(x/0\cdot 4)]$.

It is instructive to observe that, as $k \to 0^+$ (equivalently decreasing the viscosity in the fluids context), so the wave front steepens; in the limit we shall obtain a jump (discontinuous) solution, depicted below.



We record the significant fact that the Burgers equation can be solved completely, by using a transformation (the *Hopf-Cole transform*) that maps the equation into the classical, linear heat conduction equation. It is then possible to find the solution, for example, to the general initial-value problem (but this goes well beyond the material of this Notebook).

Example 14

Find a travelling-wave solution, satisfying $u \to 0$ as $|x| \to \infty$, of the Korteweg-de Vries (KdV) equation: $u_t + 6uu_x + u_{xxx} = 0$.

We seek a solution $u(x,t) = F(x - Vt)$, then we obtain

$$-VF' + 6FF' + F''' = 0 \text{ or } -VF - 3F^2 + F'' = A;$$

we assume that $u \to 0$ (at infinity) sufficiently smoothly so that $u_x \to 0$ and $u_{xx} \to 0$ there, and so we must have the arbitrary constant $A = 0$. Now we form

$$-VFF' + 3F^2F' + F''F' = 0 \text{ which gives } -\tfrac{1}{2}VF^2 + F^3 + \tfrac{1}{2}(F')^2 = B,$$

and by the same argument as above, the second arbitrary constant (B) must also be zero. This final equation for F can be integrated directly, although this is mildly tiresome; it is simpler to confirm that a suitable sech^2 function is the relevant solution. Let us write

$$F(\xi) = a\operatorname{sech}^2(b\xi) \quad (\xi = x - VT - x_0)$$

where a, b and x_0 are constants; we then have

$$F' = -2ab\operatorname{sech}^2(b\xi)\tanh(b\xi),$$

and so our equation gives

$$-\tfrac{1}{2}Va^2\mathrm{sech}^4(b\xi)+a^3\mathrm{sech}^6(b\xi)+\tfrac{1}{2}4a^2b^2\mathrm{sech}^4(b\xi)\left[1-\mathrm{sech}^2(b\xi)\right]=0.$$

We therefore have a solution when we make the choices

$$V=4b^2 \text{ and } a=2b^2,$$

for arbitrary b (and also arbitrary x_0, which simply represents an origin shift); thus the solution is

$$u(x,t)=2b^2\mathrm{sech}^2\left[b\left(x-4b^2t-x_0\right)\right].$$

Comment: This is the famous *solitary-wave* solution of the Korteweg-de Vries equation; it plays an important rôle in water waves and most particularly, in 'soliton' theory.

We conclude with one more example, of some practical significance.

Example 15

Find a travelling-wave solution, $u=F(x-Vt)$, of the equation $u_t=\left(u^n u_x\right)_x$ $(n>0,$ ($n>0$, constant), which is zero for $x\geq Vt$ (i.e. ahead of the wave front).

We write $u(x,t)=F(x-Vt)$, to give

$$-VF'=\left(F^nF'\right)' \text{ and so } -VF=F^nF'+A,$$

and for $F=0$ (and F' finite) on $\xi=x-Vt=0$, we require the arbitrary constant $A=0$. Thus we have

$$F^{n-1}F'=-V \text{ and so } \frac{1}{n}F^n=-V\xi+B,$$

and the same condition just used now requires $B=0$; we have the solution

$$u(x,t)=\begin{cases}\left[nV(Vt-x)\right]^{1/n}, & x\leq Vt\\ 0, & x>Vt.\end{cases}$$

Comment: This solution, although continuous, does not have continuous first or higher derivatives if $n\geq 1$, so we would need to reconsider the applicability of the equation in this case. The equation describes the (one dimensional) flow of a perfect gas through a porous medium, where u is the speed of the gas; the equation, with the given condition ahead, is regarded as a good model for $0<n<1$.

Exercises 3

1. Find a travelling-wave solution of $u_t = ku_{xx} + \lambda u$, where $k > 0$ and λ are constants. What condition ensures that oscillatory solutions exist?

2. Find the equation satisfied by the travelling-wave solution, $u(x,t) = F(x - Vt)$, of the nonlinear Schrödinger equation: $\mathrm{i}u_t + \alpha u_{xx} + \beta |u|^2 u = 0$ (where α and β are real constants). Hence show that there is a solution $F = a\exp[\mathrm{i}k(x - Vt)]$, for a and k real, and determine V.

****

****

4 Similarity solutions

We mentioned at the start of this Notebook that similarity solutions are very special, although without doubt interesting and important. The basic idea is introduced by considering the classical example of this type – the heat conduction equation – and then we shall make some general observations about the fundamental property that underpins the method. Also, as we have done in previous chapters, we will consider both simple, linear, classical partial differential equations and some nonlinear examples. Finally, we will outline how these special solutions can be extended and then used to obtain more general solutions.

4.1 Introducing the method

Let us consider the heat conduction (diffusion) equation in one dimension:

$$u_t = ku_{xx} \quad (k > 0\text{, constant}),$$

and seek a solution $u(x,t) = F\left(xt^n\right)$ where n is a constant to be determined; we must also expect that this form of solution is valid only for certain functions F. We therefore have

$$u_t = nxt^{n-1}F'(\eta) \text{ and } u_{xx} = t^{2n}F''(\eta),$$

where $\eta = xt^n$ is the *similarity variable*; the equation becomes

$$nxt^{n-1}F'(\eta) = kt^{2n}F''(\eta).$$

Now for this to be a consistent and meaningful equation, it must be an ordinary differential equation for $F(\eta)$. First, let us introduce η in place of, for example, x i.e. set $x = \eta/t^n$. (We could equally replace all the t-dependence by writing $t = (\eta/x)^{1/n}$,, but our choice is the simpler route.) Thus we obtain the equation in the form

$$n\eta t^{-1}F'(\eta) = kt^{2n}F''(\eta)$$

and this can be made consistent only if $2n = -1$, so $n = -1/2$ (and we should note that the equation for F poses serious difficulties on $t = 0$ – which was not evident in the original partial differential equation – so we must use $t > 0$). With this prescription for n, we obtain

$$F'' = -\frac{1}{2}\frac{\eta}{k}F' \text{ and so } F' = A\exp\left[-\frac{1}{4}\frac{\eta^2}{k}\right],$$

where A is an arbitrary constant; finally we have

$$F(\eta) = A\int^{\eta}\exp\left[-\frac{\eta'^2}{4k}\right]d\eta' \text{ i.e. } u(x,t) = A\int^{x/\sqrt{t}}\exp\left[-\frac{\eta^2}{4k}\right]d\eta.$$

However, it is slightly neater to change the integration variable by writing $\eta = 2\sqrt{k}\,y$ to give

$$u(x,t) = B\int^{x/2\sqrt{kt}} \exp\left(-y^2\right) dy\,,$$

where B is a redefined version of A, and a second arbitrary constant is implied by the indefinite integral.

To see how special all this is, let us suppose that we wished to use this method to find the solution for the temperature, $u(x,t)$, in an infinite rod ($x \ge 0$), described by

$$u_t = ku_{xx} \text{ with } u(0,t) = u_0,\ u \to u_1 \text{ as } x \to \infty\text{, for all } t > 0\text{, and } u(x,0) = u_2\,,$$

where u_0, u_1 and u_2 are constants. First, we note that all the given temperatures are constant, whereas the most general problem with this configuration would be

$$u_0 = u_0(t)\,,\ u_1 = u_1(t) \text{ and } u_2 = u_2(x)\,,$$

so we have already simplified the problem. Further, in terms of the similarity variable ($\eta = x/\sqrt{t}$), we see that

$$\eta = 0 \text{ on } x = 0 \text{ for } t > 0$$

and

$$\eta \to \infty \text{ as } x \to \infty \text{ for any finite } t > 0,$$

but that η is undefined on $t = 0$. We can proceed, therefore, only by replacing the given initial-boundary-value problem by the following:

$$u(0,t) = u_0 \text{ for } t > 0;$$

$$u \to u_1 \text{ as } x \to \infty \text{ for any finite } t > 0;$$

$$u \to u_2 \text{ as } t \to 0^+ \text{ for } x > 0.$$

With this interpretation, the first condition is unchanged i.e. $\eta = 0$ on $x = 0$ ($t > 0$), but the second and third conditions are now equivalent:

$$\eta \to \infty \quad \begin{cases} \text{as} \quad x \to \infty & \text{for } t > 0 \\ \text{as} \quad t \to 0^+ & \text{for } x > 0. \end{cases}$$

This in turn requires that, for a similarity solution to exist, we must have $u_1 = u_2$, so very special indeed; with all these choices, we may formulate the problem for $u = F(\eta)$:

$$u(x,t) \equiv F(\eta) = B \int^{\eta/2\sqrt{k}} \exp\left(-y^2\right) dy,$$

with $F(0) = u_0$ and $F \to u_1$ as $\eta \to \infty$. The solution is therefore

$$u(x,t) = u_0 + \left(u_1 - u_0\right) \frac{\int_0^{x/2\sqrt{kt}} \exp\left(-y^2\right) dy}{\int_0^{\infty} \exp\left(-y^2\right) dy}$$

which, with $\int_0^{\infty} \exp\left(-y^2\right) dy = \frac{1}{2}\sqrt{\pi}$, is more neatly written as

$$u(x,t) = u_0 + \frac{2}{\sqrt{\pi}}\left(u_1 - u_0\right) \int_0^{x/2\sqrt{kt}} \exp\left(-y^2\right) dy.$$

Before we investigate an important underlying property of partial differential equations that, when satisfied, implies the existence of a similarity form, we consider, as examples, our other two elementary, classical equations.

Example 16

Seek a similarity solution, in the form $u(x,t) = F\left(x\,t^n\right)$, of the wave equation $u_{tt} - c^2 u_{xx} = 0$ (where $c > 0$ is a constant).

With $u(x,t) = F\left(x\,t^n\right)$ (although we can note that a function of $t\,x^n$ would work equally well – indeed, the symmetry in *x* and *t* might suggest something more), we obtain

$$u_{xx} = t^{2n} F'' \text{ and } u_{tt} = n(n-1)xt^{n-2}F' + n^2x^2t^{2n-2}F'',$$

which gives

$$n(n-1)xt^{n-2}F' + n^2x^2t^{2n-2}F'' - c^2t^{2n}F'' = 0.$$

We introduce $x = \eta/t^n$, and so

$$n(n-1)\eta\,t^{-2}F' + n^2\eta^2t^{-2}F'' - c^2t^{2n}F'' = 0,$$

which is consistent only if $n=-1$ (and then $t \neq 0$); thus we have

$$\left(\eta^2 - c^2\right)F'' + 2\eta F' = 0 \text{ and so } \left(\eta^2 - c^2\right)F' = A,$$

where A is an arbitrary constant. Hence

$$F(\eta) = \frac{A}{2c}\int\left(\frac{1}{\eta - c} - \frac{1}{\eta + c}\right)d\eta = \frac{A}{2c}\ln\left|\frac{\eta - c}{\eta + c}\right| + B,,$$

where B is a second arbitrary constant. This solution, with $\eta = x/t$, can be expressed as

$$u(x,t) = C\ln\left|\frac{x-ct}{x+ct}\right| + B = C\left[\ln|x-ct| - \ln|x+ct|\right] + B,$$

where C has replaced $A/2c$; the essential symmetry in x and t is now evident. Indeed, this solution is no more than a special example of d'Alembert's general solution:

$$u = F(x-ct) + G(x+ct).$$

Example 17

Seek a similarity solution, in the form $u(x,y) = F(xy^n)$, of Laplace's equation $u_{xx} + u_{yy} = 0$.

This is virtually identical to Example 16; we therefore obtain

$$y^{2n}F'' + n(n-1)\eta y^{-2}F' + n^2\eta^2 y^{-2}F'' = 0$$

and then, as before, we require $n = -1$:

$$F'' + 2\eta F' + \eta^2 F = 0 \text{ and so } \left(1+\eta^2\right)F' = A$$

which gives $F(\eta) = A\arctan(\eta) + B$,

where *A* and *B* are arbitrary constants. Thus we have the solution

$$u(x,y) = A\arctan\left(\frac{x}{y}\right) + B .$$

Comment: It is clear, by virtue of the symmetry in *x* and *y*, that another solution is

$$u(x,y) = C\arctan\left(\frac{y}{x}\right) + D$$

where *C* and *D* are arbitrary constants. That these two solutions are the same solution is clear:

$$\tan\left(\frac{u-B}{A}\right) = \frac{x}{y} \text{ and so } \cot\left(\frac{u-B}{A}\right) = \frac{y}{x},$$

$$\cot\left(-\frac{u-D}{C} + \frac{\pi}{2}\right) = \tan\left(\frac{u-D}{C}\right) \text{ i.e. } u = C\arctan\left(\frac{y}{x}\right) + D .$$

In other words, a choice of the arbitrary constants in the first ($A = -C$, $B/A = -\left(D/C + \pi/2\right)$) recovers the second.

4.2 Continuous (Lie) groups

In order to motivate these ideas, let us return to the heat conduction equation:

$$u_t = ku_{xx},$$

and transform the variables according to

$$u = \alpha U \,,\; t = \beta T \,,\; x = \gamma X \,,$$

where α, β and γ are real, non-zero constants. We now regard $U = U(X, T)$, so that the equation for U becomes

$$\frac{\alpha}{\beta} U_T = k \frac{\alpha}{\gamma^2} U_{XX} \,,$$

and then for the choice $\beta = \gamma^2$, and any α $(\neq 0)$, we recover the same equation but written in the new notation:

$$U_T = k U_{XX} \,.$$

We say that the equation is *invariant* under the transform

$$u = \alpha U \,,\; x = \gamma X \,,\; t = \gamma^2 T \;\text{ usually expressed as }\; u \to \alpha u \,,\; x \to \gamma x \,,\; t \to \gamma^2 t \,,$$

where '$\to$' means 'replace the old by the new', avoiding the need to change variables (and this reinforces the property of leaving the equation invariant). Note that this holds for arbitrary, real, non-zero α and γ; further, because the equation in u is linear, the use of α here is redundant, so it is sufficient to state that the equation is invariant under the transformation $x \to \gamma x$, $t \to \gamma^2 t$. In nonlinear equations, however, α will normally play a critical rôle, as we shall now see.

We consider the Korteweg-de Vries equation

$$u_t + uu_x + u_{xxx} = 0,$$

and transform exactly as above, using α, β and γ:

$$\frac{\alpha}{\beta}u_t + \frac{\alpha^2}{\gamma}uu_x + \frac{\alpha}{\gamma^3}u_{xxx} = 0.$$

The equation is therefore invariant under the transformation

$$\alpha = \gamma^{-2} \text{ and } \beta = \gamma^3.$$

What we have described are two examples of a *continuous* or *Lie group*, which we will now define more precisely. [Marius Sophus Lie, 1842-1899, Norwegian mathematician who, with Felix Klein, developed the notion of groups in geometry. He also discovered contact transformations, and applied the ideas to integration and the classification of partial differential equations.]

Let the transformations that we have just introduced be represented by T_γ (for real $\gamma \neq 0$); further, we consider the application of successive transformations: T_γ, T_δ, T_ε, etc., and examine the consequences.

1. We apply T_γ followed by T_δ; the result is the same as applying $T_{\gamma\delta}$, so we immediately have the *multiplication* law: $T_\gamma T_\delta = T_{\gamma\delta}$.
2. Now we apply T_δ followed by T_γ; this is also the same as applying $T_{\gamma\delta}$ ($= T_{\delta\gamma}$), so the multiplicative law is *commutative*: $T_\gamma T_\delta = T_\delta T_\gamma$.
3. We can now apply a third transformation, T_ε, to produce the equivalent of applying $T_{\gamma\delta\varepsilon}$.
4. From (3) we see that the multiplication law is *associative*:
$$\mathrm{T}_\varepsilon(\mathrm{T}_\delta\mathrm{T}_\gamma) = \mathrm{T}_\varepsilon\mathrm{T}_{\gamma\delta} = \mathrm{T}_{\gamma\delta\varepsilon} = \mathrm{T}_{\delta\varepsilon}\mathrm{T}_\gamma = (\mathrm{T}_\varepsilon\mathrm{T}_\delta)\mathrm{T}_\gamma.$$
5. There is an *identity* transformation,: T_1 i.e. $T_1 T_\gamma = T_\gamma T_1 = T_\gamma$.
6. There exists an inverse of T_γ, namely, $T_{1/\gamma}$, because $T_\gamma T_{1/\gamma} = T_1$ and also $T_{1/\gamma} T_\gamma = T_1$, so $T_{1/\gamma}$ is both the left and right inverse of T_γ.

The conditions and properties (1)-(6) show that the elements of T_γ, for all real $\gamma \neq 0$, form an *infinite*, *continuous* (*Lie*) *group*, where γ is usually called the *parameter of the group*.

The property that an equation is invariant under such a transformation suggests that solutions of the equation might be constructed that possess the same property i.e. the solution is also invariant under the transformation. For example, the heat conduction equation (see above) is invariant under the transformation $x \to \gamma x$, $t \to \gamma^2 t$, so we might seek a solution that is a function of x^2/t or, equivalently, $x/\sqrt{t}$ – which is exactly what we found in §4.1. Similarly, the wave equation and Laplace's equation are clearly invariant under the transformation $x \to \gamma x$, $t \to \gamma t$ and $x \to \gamma x$, $y \to \gamma y$, respectively, so solutions that are functions of x/t and x/y, respectively, are to be expected; see Examples 16 and 17.

Example 18

Find the transformation that leaves $u_t + u_{xxx} = 0$ (the linearised Korteweg-de Vries equation) invariant, and hence seek an appropriate solution (but do not solve the resulting ordinary differential equation).

Given $u_t + u_{xxx} = 0$, we transform according to

$$t \to \alpha t\,,\ x \to \beta x \text{ and then } \alpha^{-1}u_t + \beta^{-3}u_{xxx} = 0\,,$$

(and we do not need to do anything more because the equation is linear); so we must choose $\alpha = \beta^3$ i.e. use x^3/t or, more conveniently, $x/t^{1/3}$. Hence we seek a solution

$$u(x,t) = F\left(x/t^{1/3}\right)$$

which gives

$$-\frac{1}{3}xt^{-4/3}F' + \left(t^{-1/3}\right)^3 F''' = 0 \text{ or } F''' - \tfrac{1}{3}\eta F' = 0 \ (\ \eta = x/t^{1/3}\ ).$$

Comment: this ordinary differential equation can b e solved; we set $F'(\eta) = G\left(\eta/3^{1/3}\right)$, which gives $G'' - \zeta G = 0$ where $\zeta = \eta/3^{1/3}$; this is *Airy's equation* with a solution $G = \mathrm{A_i}(\zeta)$ (which is related to Bessel functions of order $1/3$). Thus we have a solution for *u*:

$$u(x,t) = C\int^{x/(3t)^{1/3}} \mathrm{A_i}(\zeta)\,d\zeta$$

where *C* is an arbitrary constant.

4.3 Similarity solutions of other equations

In order to demonstrate how this approach can be extended, let us consider, for example, the higher-dimensional heat conduction (diffusion) equation

$$u_t = k\left(u_{xx} + u_{yy}\right);$$

see §2.3 and Example 13. First we transform according to

$$t \to \alpha t\,,\ x \to \beta x\,,\ y \to \gamma y$$

(and we note that the equation is linear), and so we obtain

$$\alpha^{-1}u_t = k\left(\beta^{-2}u_{xx} + \gamma^{-2}u_{yy}\right),$$

which leads to the choice $\alpha = \beta^2 = \gamma^2$. Let us therefore seek a solution using a *pair* of similarity variables:

$$u(x, y, t) = F\left(x/\sqrt{t}, y/\sqrt{t}\right),$$

which gives $-\frac{1}{2}xt^{-3/2}F_\eta - \frac{1}{2}yt^{-3/2}F_\xi = k\left(t^{-1}F_{\eta\eta} + t^{-1}F_{\xi\xi}\right)$,

where $\eta = x/\sqrt{t}$, $\xi = y/\sqrt{t}$. This equation can be simplified (for $t > 0$), so that

$$-\tfrac{1}{2}\left(\eta F_\eta + \xi F_\xi\right) = k\left(F_{\eta\eta} + F_{\xi\xi}\right),$$

which we may solve in any suitable way – by separation of variables, for example! We set $F(\eta,\xi) = E(\eta)X(\xi)$:

$$-\tfrac{1}{2}\eta E'X - \tfrac{1}{2}\xi EX' = k\left(E''X + EX''\right) \text{ or } -\frac{1}{2}\eta\frac{E'}{E} - \frac{1}{2}\xi\frac{X'}{X} = k\left(\frac{E''}{E} + \frac{X''}{X}\right),$$

so we introduce $kE'' + \frac{1}{2}\eta E' = \lambda E$, where λ is the separation constant, leaving $kX'' + \frac{1}{2}\xi X' = -\lambda X$. One simple solution is available if we choose $\lambda = 0$, for then

$$E(\eta) = A\int^{\eta}\exp\left(-\eta'^2/4k\right)d\eta'\,;\;\; X(\xi) = B\int^{\xi}\exp\left(-\xi'^2/4k\right)d\xi',$$

and so we obtain

$$u(x, y, t) = C\left(\int^{x/2\sqrt{kt}} \exp\left(-\eta^2\right) d\eta\right)\left(\int^{y/2\sqrt{kt}} \exp\left(-\xi^2\right) d\xi\right),$$

where A, B and then C are arbitrary constants.

Finally, we consider three equations that, even though we have discussed them previously, warrant further investigation under this new heading.

Example 19

Seek a similarity solution of the Burgers equation $u_t + uu_x = ku_{xx}$ (see §3.3), $k > 0$ constant, that decays as $|x| \to \infty$.

First we transform $u \to \alpha u$, $t \to \beta t$, $x \to \gamma x$, to give

$$\beta^{-1}u_t + \alpha\gamma^{-1}uu_x = k\gamma^{-2}u_{xx} \text{ and so } \alpha = \gamma^{-1},\ \beta = \gamma^2.$$

Thus we seek a solution $u(x, t) = \frac{1}{\sqrt{t}} F\left(x/\sqrt{t}\right)$, where we have interpreted the choice of α $(= \gamma^{-1} = \beta^{-1/2})$ in terms of t rather than x, because the equation involves only one derivative in t; this yields

$$-\tfrac{1}{2}t^{-3/2}F - \tfrac{1}{2}xt^{-2}F' + t^{-3/2}FF' = kt^{-3/2}F'',$$

which, for $t > 0$, becomes

$$2FF' - \left(F + \eta F'\right) = kF'' \quad (\eta = x/\sqrt{t}).$$

This equation can be integrated once, to give

$$F^2 - \eta F = kF' + A,$$

where A is an arbitrary constant. However, given that $F \to 0$ (and we shall assume that F' does likewise) as $|\eta| \to \infty$, we must have $A = 0$. Thus we now have

$$F^2 - \eta F = kF',$$

which is a simple Riccati equation; to solve this, we set $F = \mu\phi'/\phi$, where μ is a constant to be chosen, then

$$\mu\left(\frac{\phi'}{\phi}\right)^2 - \eta\frac{\phi'}{\phi} = k\left(\frac{\phi''}{\phi} - \frac{\phi'^2}{\phi^2}\right)$$

and we select $\mu = -k$. We are left with

$$k\phi'' = -\eta\phi' \text{ i.e. } \phi(\eta) = A\int^{\eta} \exp\left(-\eta'^2 \big/ 4k\right) d\eta' ;$$

see §4.1 (again, A is an arbitrary constant). Thus we have a solution

$$u(x, t) = -\frac{k}{\sqrt{t}} \frac{\exp\left(-x^2 \big/ 4kt\right)}{B + \int_0^{x/2\sqrt{kt}} \exp\left(-y^2\right) dy},$$

where B is an arbitrary constant (obtained by electing to define the lower limit of the integral and then dividing throughout by A).

Our next example also refers back to one of our important earlier equations, discussed in Example 14.

Example 20

Find the ordinary differential equation associated with the similarity solution of the KdV equation $u_t + 6uu_x + u_{xxx} = 0$.

We transform as in the previous example, to produce

$$\beta^{-1}u_t + 6\alpha\gamma^{-1}uu_x + \gamma^{-3}u_{xxx} = 0 \text{ and so } \alpha = \gamma^{-2},\ \beta = \gamma^3 .$$

Thus we seek a solution $u(x, t) = \frac{1}{(3t)^{1/3}} F\left(\frac{x}{(3t)^{1/3}}\right)$,, where the '3' is purely a convenience; we obtain

$$-\tfrac{2}{3}3^{-2/3}t^{-5/3}F - (3t)^{-2/3}x(3t)^{-4/3}F' + 6(3t)^{-5/3}FF' + (3t)^{-5/3}F''' = 0 .$$

For $t > 0$, this simplifies to give

$$F''' + (6F - \eta)F' - 2F = 0 \text{ where } \eta = x \big/ (3t)^{1/3} ,$$

which is the required equation.

Comment: This equation can be transformed into a *Painlevé equation of the second kind*, about which much is known.

Example 21

Find a similarity solution of $u_t = (uu_x)_x$ (cf. Example 15) which satisfies $\int_{-\infty}^{\infty} u(x,t)\,dx = \text{constant}$ and which is zero at a finite, non-zero value of the similarity variable.

The transformation (as used above) gives

$$\beta^{-1}u_t = \alpha\gamma^{-2}(uu_x)_x \text{ and so } \beta = \gamma^2/\alpha;$$

on the basis of this, the most general form that the similarity solution can take is

$$u(x,t) = t^{n/(2-n)} F\left(x/t^{1/(2-n)}\right),$$

for any constant n ($\neq 2$). Now let us examine the integral constraint, for any $t > 0$:

$$\int_{-\infty}^{\infty} u(x,t)\,dx = t^{n/(2-n)} \int_{-\infty}^{\infty} F\left(x/t^{1/(2-n)}\right)dx$$

$$= t^{(n+1)/(n-2)} \int_{-\infty}^{\infty} F(\eta)\,d\eta \qquad \left(\text{where } \eta = x/t^{1/(n-2)}\right),$$

which is a constant only if $(n+1)/(2-n)=0$, so $n=-1$, producing the similarity form

$$u(x,t)=t^{-1/3}F\left(x/t^{1/3}\right).$$

Thus we obtain

$$-\tfrac{1}{3}t^{-4/3}F-\tfrac{1}{3}t^{-1/3}xt^{-4/3}F'=t^{-4/3}(FF')'$$

or (for $t>0$) $\quad (FF')'+\tfrac{1}{3}(F+\eta F')=0 \quad (\eta=x/t^{1/3}),$

which integrates immediately to give

$$3FF'+\eta F=A,$$

where A is an arbitrary constant. Now for $F=0$ at finite η (assuming that F' is finite at this same point) requires $A=0$, so we obtain for every point where $F\neq 0$:

$$3F'=-\eta \text{ and so } F(\eta)=\tfrac{1}{6}\left(B-\eta^2\right)$$

which is zero at $\eta=\sqrt{B}$, where $B>0$ is another arbitrary constant. (We can see that $F'\left(\sqrt{B}\right)$ is finite.) Thus $u(x,t)=t^{-1/3}\left(B-x^2/t^{2/3}\right)/6$..

4.4 More general solutions from similarity solutions

We describe this important idea by analysing a classical example of this type: the heat conduction (diffusion) equation

$$u_t=ku_{xx} \ \ (k>0\text{, constant}).$$

This equation has a similarity solution (see §4.1) of the form

$$u(x,t)=F\left(x/\sqrt{t}\right);$$

further, the equation is linear, so the transformation $u\to\alpha u$ plays no obvious rôle (but note what we did in Example 21). Let us investigate the consequences of seeking a more general solution, for some F:

$$u(x,t)=t^nF\left(x/\sqrt{t}\right),$$

which we should expect is permitted because of the freedom to use any α in the group transformation. Thus we obtain

$$nt^{n-1}F-\tfrac{1}{2}xt^{n-3/2}F'=kt^{n-1}F'' \text{ or } nt^{n-1}F-\tfrac{1}{2}\eta t^{n-1}F'=kt^{n-1}F''$$

which, for $t > 0$ and any *n*, simplifies to

$$nF - \tfrac{1}{2}\eta F' = kF''.$$

It is convenient to write $F(\eta) = \mathrm{e}^{-m\eta^2} G(\eta)$, for some constant *m*, and then

$$nG - \tfrac{1}{2}\eta\left(G' - 2m\eta G\right) = k\left(G'' - 4m\eta G' - 2mG + 4m^2\eta^2 G\right)$$

where we choose $m = 1/4k$ and elect to work with the case $n = -1/2$ (and note that we may choose any *n*, and then investigate the solution for *G*). The equation for *G* now becomes

$$\tfrac{1}{2}\eta G' = kG'' \text{ and so } G' = A\exp\left(\eta^2/4k\right),$$

where *A* is an arbitrary constant. However, this solution implies that $F = \mathrm{e}^{-\eta^2/4k}G$ will not decay at infinity – indeed, in general, it will grow linearly in η; thus we must choose $A = 0$ and then $G = \text{constant} = B$. Thus we have the solution

$$u(x,t) = \frac{B}{\sqrt{t}}\exp\left(-x^2/4kt\right).$$

This new solution, which is an extension of the classical similarity solution, has an important property. Let us examine

$$\int_{-\infty}^{\infty} u(x,t)\,dx = \frac{b}{\sqrt{t}}\int_{-\infty}^{\infty}\exp\left(-x^2/4kt\right)dx,$$

which corresponds to the calculation that we presented in Example 21; we change the integration variable ($x = 2\sqrt{kt}\,y$) to give

$$\int_{-\infty}^{\infty} u(x,t)\,dx = 2B\sqrt{k}\int_{-\infty}^{\infty}\mathrm{e}^{-y^2}dy = 2B\sqrt{\pi k},$$

which is a constant i.e. not a function of time. How can we exploit this?

First, let us observe that the original equation, $u_t = ku_{xx}$, is unchanged if we perform a constant origin shift (in *x* or *t*); for example, on setting $x = a + X$, where *a* is a constant, the equation becomes

$$u_t = ku_{XX}.$$

Thus, given that $u = \frac{B}{\sqrt{t}}\exp\left(-x^2/4kt\right)$ or, rather, $\frac{B}{\sqrt{t}}\exp\left(-X^2/4kt\right)$ is a solution, then so is

$$u(x,t) = \frac{B}{\sqrt{t}}\exp\left[-(x-a)^2/4kt\right].$$

But then another solution is

$$u(x,t) = \frac{B}{\sqrt{t}}\int\exp\left[-(x-a)^2/4kt\right]da,$$

because the original partial differential equation is linear with constant coefficients, and so we may apply an integral operator to the equation, providing the integration is not with respect to either x or t (and the integral exists, of course). By the same token, yet another solution is

$$u(x,t) = \frac{1}{\sqrt{t}} \int B(a) \exp\left[-(x-a)^2 / 4kt\right] da \,,$$

at least for suitable $B(a)$.

Let us therefore examine the solution (which, if it is thought necessary, can be verified as such by direct substitution into the equation):

$$u(x,t) = \frac{1}{\sqrt{t}} \int_{-\infty}^{\infty} B(a) \exp\left[-\frac{(x-a)^2}{4kt}\right] da \,;$$

in particular we ask what value this takes on $t = 0$. To accomplish this, it is most natural to change the integration variable; for any $t > 0$, we write $a - x = 2\sqrt{kt}\, y$ and so obtain

$$u(x,t) = 2\sqrt{k} \int_{-\infty}^{\infty} B(x + 2\sqrt{kt}\, y)\, e^{-y^2} dy \,.$$

This is readily evaluated on $t = 0$, and yields

$$u(x,0) = 2\sqrt{k} B(x) \int_{-\infty}^{\infty} e^{-y^2} dy = 2\sqrt{k\pi} B(x) \,,$$

which demonstrates that our new solution, based on a similarity structure, provides the solution to the general initial-value problem. That is, given $u(x,0)$ (with suitable conditions at infinity), the solution is as just described, provided that the resulting integral exists (which will very often be the case because of the strong exponential decay at infinity). We see that we have evaluated the transformed integral at $t = 0$, which may seem a slightly dubious manoeuvre (because the change of variable requires $t > 0$); a more careful analysis adopted for $t \to 0^+$, using more sophisticated methods, confirms the correction of our result.

Example 22

Use the result in Example 18, and in the comment following it, to find the solution of $u_t + u_{xxx} = 0$ which satisfies $u(x,0) = f(x)$.

The conventional similarity solution can be expressed as

$$u(x,t) = F\left(x / t^{1/3}\right),$$

but because the equation is linear, we may seek a generalised version of this:

$$u(x,t) = t^n F\left(x / t^{1/3}\right).$$

Thus we get

$$nt^{n-1}F - \tfrac{1}{3}xt^{n-4/3}F' + t^{-1}t^{n}F''' = 0$$

which simplifies to $\quad nF - \tfrac{1}{3}\eta F' + F''' = 0$;

guided by the introductory example (so on this basis we would hope to find that $n = -1/3$), we try a solution $F = \mathrm{A_i}\left(\eta/3^{1/3}\right)$, where $\eta = x/t^{1/3}$ and $\mathrm{A_i}(\zeta)$ is a solution of the Airy equation: $w'' - \zeta w = 0$. Thus we obtain

$$n\mathrm{A_i} - \tfrac{1}{3}3^{-1/3}\eta\mathrm{A_i}' + \tfrac{1}{3}\mathrm{A_i}''' = 0 \text{ and so } \mathrm{A_i}''' - (\zeta\mathrm{A_i})' = (\mathrm{A_i}'' - \zeta\mathrm{A_i})' = 0,$$

if we select $n = -1/3$ (as expected). Hence we have a solution

$$u(x,t) = \frac{C}{t^{1/3}}\mathrm{A_i}\left(x/\left(3t^{1/3}\right)\right),$$

where C is an arbitrary constant.

But we may replace x by $x - a$ (an origin shift), and then write $C = C(a)$; integration over all a then yields another solution:

$$u(x,t) = \frac{1}{t^{1/3}}\int_{-\infty}^{\infty} C(a)\mathrm{A_i}\left(\frac{x-a}{\left(3t^{1/3}\right)}\right)da\,.$$

We now introduce $x - a = (3t)^{1/3} y$, to give

$$u(x, t) = 3^{1/3} \int_{-\infty}^{\infty} C\left(x - (3t)^{1/3} y\right) A_i(y)\, dy$$

and then evaluation on $t = 0$ yields

$$u(x,0) = 3^{1/3} C(x) \int_{-\infty}^{\infty} A_i(y)\, dy .$$

Let us write $\int_{-\infty}^{\infty} A_i(y)\, dy = k$ (it turns out that $k = 1$), then we require

$$3^{1/3} kC(x) = f(x)$$

and so the initial-value problem has the solution

$$u(x, t) = \frac{1}{(3t)^{1/3} k} \int_{-\infty}^{\infty} f(a) A_i\left(\frac{x - a}{(3t)^{1/3}}\right) da .$$

Exercises 4

1. Construct the continuous group transformation for the equation $u_t + uu_x = tu_{xx}$, and hence seek an appropriate similarity solution (by deriving the ordinary differential equation satisfied by this solution).

2. Repeat Q.1 for the nonlinear Schrödinger equation $iu_t + u_{xx} + |u|^2 u = 0$.

****

****

Answers

Exercises 2

1. $u(r,t)=R(r)T(t)$: $T''+\omega^2c^2T=0$, $rR''+R'+\omega^2R=0$ (a Bessel equation).

2. $u(r,t)=r^{-1}v(r,t)$: $v_{tt}-c^2v_{xx}=0$ – the classical wave equation.

3. $u(x,t)=X(x)T(t)$: $u=(C+\lambda x)/(1+\lambda t)$.

Exercises 3

1. $u(x,t)=F(x-Vt)$: $kF''+VF'+\lambda F=0$ with condition $V^2<4\lambda k^2$.

2. $u(x,t)=F(x-Vt)$: $-\mathrm{i}VF'+\alpha F''+\beta|F|^2F=0$ and then $V=\alpha k-\dfrac{\beta}{k}a^2$.

Exercises 4

1. $u(x,t)=F(x/t)$: $F''+(\eta-F)F'=0$ where $\eta=x/t$.

2. $u(x,t)=\dfrac{1}{\sqrt{t}}F\left(x/\sqrt{t}\right)$: $F''-\frac{1}{2}\mathrm{i}(\eta F)'+|F|^2F=0$ where $\eta=x/\sqrt{t}$.

****

****

Index