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A First Course in Ordinary Differential Equations

Norbert Euler

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Preface

In many cases continuous dynamical processes which occur in nature and in engineering sciences can successfully be described by mathematical models that involve differential equations or systems of differential equations. The solution of the initial-value problem or the boundary-value problem described by a differential equation with some limited information about its solutions, can provide the full details and description of the dynamical process. For example, Newton's Second Law of Motion is in the form of a second-order ordinary differential equation that relates the second time-derivative of the position of a mass and the total forces that act on the mass as it moves under these forces. Solving this second-order differential equation provides information on the velocity and the position of the mass in terms of time. In fact, most physical theories are based on some fundamental differential equation and are usually named after the scientist who first derived the equation: in quantum mechanics it is *Schrödinger's equation*, in fluid dynamics it is the *Navier-Stokes equation*, in electrodynamics it is *Maxwell's equations*, in general relativity theory it is *Einstein's field equations*, in relativistic quantum mechanics it is *Dirac's equations*, etc. The mentioned equations are all very interesting differential equations and their solutions model many important natural processes. We should however point out that the mentioned equations are mostly *partial differential equations or systems* (meaning that their dependent variables depend on several independent variables) and are moreover often **nonlinear** and, therefore, are much more advanced than the differential equations that we study in the current set of lecture notes. In order to provide an introduction to the general theory of differential equations, we need to start with the simplest type of equations, which are the **linear ordinary differential equations**. Hereafter, referred to simply as *linear differential equations.*

The lecture notes presented here are intended for engineering and science students as a first course on differential equations. It is assumed that the students have already read a course on linear algebra, that included a discussion of general vector spaces, as well as a course on integral calculus for functions that depend on one variable. However, no previous knowledge of differential equations is required to read and understand this material. Many examples have been included in these notes and the proof of most statements are done in full details. The aim of the notes is to provide the student with a thorough understanding of the methods to obtain solutions of certain classes of differential equations, rather than the qualitative understanding of solutions and their existence. With the exception of some nonlinear first-order differential equations, we concentrate on linear differential equations and the derivation of their solutions.

In Chapter 1 we provide the theoretical basis of the solution structure for linear differential equations. This is an important part of the notes, as this chapter introduces the concept of a linearly independent set of functions (or solutions) as well as the concept of linear superposition. The Wronskian is introduced here to establish the linear independence. In fact, the Wronskian plays a central role in the study of linear differential equations and it appears in many solution formulas throughout these lecture notes. The solution methods described in Chapters 2 to 4 mostly involve Ansätze for the solutions of the differential equations and, in some cases, we also need to introduce a change of the variables in order to derive the solutions. In Appendix A we introduce an alternative method to solve linear differential equations based on first-order linear operators and their integral operators. This method is free from any Ansatz and can be viewed as an alternative to the solution methods proposed in Chapters 2 to 4. Appendix B sums up the different techniques of integration, whereas Appendix C gives some references to books on differential equations. In Appendix D we give the full solutions of a selection of exercises and in Appendix E we list the answers of all the exercises.

The four chapters included in this material can be taught in 15 lectures, which corresponds to about 50% of a quarter-semester (8 weeks) course in Engineering Mathematics.

Norbert Euler *Luleå, June 2015*

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A List of mathematical symbols:

$\mathbb{R}$:	The set of all real numbers.
$\mathbb{N}$:	The set of all natural numbers.
$\mathbb{Z}$:	The set of all integer numbers.
$P_m(x)$:	A polynomial of degree m.
$\mathcal{D} \subseteq \mathbb{R}$:	$\mathcal{D}$ is a subset of real numbers, which may be the set of all real numbers.
$\mathcal{C}^n(\mathcal{D})$:	The vector space of all continuously differentiable functions of order n on $\mathcal{D}$.
$\mathcal{C}^\infty(\mathcal{D})$:	The vector space of all continuously differentiable functions of all orders on $\mathcal{D}$.
$\mathcal{C}^\infty(\mathbb{R})$:	The vector space of all continuously differentiable functions of all orders on $\mathbb{R}$.
$W[\phi_1, \phi_2, \ldots, \phi_n](x)$:	The Wronskian of the set of functions $\{\phi_1(x), \phi_2(x), \ldots, \phi_n(x\})$ for all x in some given interval.

Chapter 1

Linear differential equations and linearly independent solutions

In this chapter we define the different types of solutions that we will encounter in our studies of differential equations. We do not describe or propose in this chapter any methods to solve differential equations, as this is the main subject of the remaining chapters in these notes. However, we prove here several fundamental results regarding the solution structure of linear differential equations and we also introduce the very important Wronskian of a set of differentiable functions, which makes it easy to establish the linear independence of sets of solutions. This paves the way for several solution-methods for linear differential equation, studied in detail in chapters 2 to 4.

1.1 Solutions of differential equations

An ordinary differential equation of **order** n, where n is a natural number, is an equation of the general form

$$F\left(x, y(x), y'(x), y''(x), y^{(3)}(x), \ldots, y^{(n)}(x)\right) = 0, \tag{1.1.1}$$

where $y' = dy/dx,\ y'' = d^2y/dx^2, \ldots, y^{(n)} = d^ny/dx^n$ and F is a given function of the arguments as shown.

Definition 1.1.1. *A* **solution** *of (1.1.1) is a function $\phi(x)$ such that $y(x) = \phi(x)$ satisfies (1.1.1). Here ϕ is a function that is n times differentiable on $\mathcal{D} \subseteq \mathbb{R}$ and therefore belongs to the vector space $C^n(\mathcal{D})$. That is, the solution $\phi(x)$ is such that*

$$F\left(x, \phi(x), \phi'(x), \phi''(x), \phi^{(3)}(x), \ldots, \phi^{(n)}(x)\right) = 0.$$

The interval $\mathcal{D}$ is known as the solution domain of ϕ for (1.1.1) and the domain of all the solutions of (1.1.1) is called the **solution domain of the differential equation**.

In this course we will deal with different types of solutions, namely *general solutions*, *special solutions* and *singular solutions*. There also exist several methods to solve differential equations numerically (or to approximate solutions numerically). This subject is, however, outside the scope of this course and these notes.

Definition 1.1.2.

a) *A* **general solution** *of (1.1.1) on some domain* $\mathcal{D} \subseteq \mathbb{R}$, *is a function,* $\phi(x;\ c_1, c_2, \ldots, c_n) \in \mathcal{C}^n(\mathcal{D})$, *which satisfies the differential equation for every* $x \in \mathcal{D}$ *and which contains* n *arbitrary and independent constants* $c_1, c_2, \ldots, c_n$, *called* **constants of integration**.

b) *Those solutions of (1.1.1) on the interval* $\mathcal{D}$ *which follow from a given general solution* $\phi(x;\ c_1, c_2, \ldots, c_n)$ *by choosing fixed values for the constants of integration* $c_1, c_2, \ldots, c_n$, *are called* **special solutions** *of (1.1.1).*

c) *Those solutions of (1.1.1) that cannot be obtained by choosing fixed values for the constants of integration* $c_1, c_2, \ldots, c_n$ *in a given general solution* $\phi(x;\ c_1, c_2, \ldots, c_n)$, *are called* **singular solutions** *of (1.1.1) with respect to that general solution.*

d) *Equation (1.1.1) may admit solutions in the form* $\Psi(x,\ y(x)) = 0$, *where* y *cannot be solved explicitly in terms of* x *for a given function* Ψ. *Such solutions are called* **implicit solutions** *of (1.1.1). If the implicit solution contains* n *arbitrary constants, then this relation gives a* **general implicit solution** *of (1.1.1).*

Graphically solutions of differential equations may be depicted as curves in the XY-plane on some interval $\mathcal{D} \subseteq \mathbb{R}$ of the X-axis. For a first-order equation, a general solution, $y(x) = \phi(x; c_1)$, contains one arbitrary constant (or parameter) c_1. These solutions are then one-parameter family of curves in the XY-plane. That is, for every fixed choice of c_1 we obtain an explicit solution curve. This family of one-parameter solution curves are also known as **level curves**. For second-order differential equations a general solution, $y(x) = \phi(x; c_1, c_2)$, contains two arbitrary constants, c_1 and c_2, so that this results in a two-parameter family of curves in the XY-plane. The same holds for nth-order differential equations. Special solutions of a differential equation are then the explicit solution curves that result when choosing fixed values for the constants of integration $c_1, c_2, \ldots, c_n$ in the given general solution. A singular solution of a differential equation is a curve in the XY-plane that does not belong to the family of curves as given by a general solution of that equation. The singular solution curve may be an asymptote to the family of solution curves given by a general solution.

Example 1.1.1.

a) Consider the first-order differential equation

$$y' = y^2. \tag{1.1.2}$$

It can easily be verified that a general solution of this equation is

$$y(x) = \frac{1}{c - x}, \tag{1.1.3}$$

for all $c \in \mathbb{R}$. However, this solution does not contain the solution $y = 0$, which is clearly also a solution for (1.1.2). Thus the solution $y = 0$ is a singular solution for (1.1.2).

b) As a second example of a singular solution, we consider the equation

$$y' + y^2 = \left(2x + \frac{1}{x}\right) y - x^2, \qquad x > 0. \tag{1.1.4}$$

A general solution of (1.1.4) is

$$y(x) = \frac{x(x^2 + 2 + c)}{x^2 + c} \tag{1.1.5}$$

for all $c \in \mathbb{R}$. However, we can verify that $y(x) = x$ is also a solution of (1.1.4) and that this solution is not contained in the general solution (1.1.5). For example, there is no value for c such that $y(2) = 2$. The solution $y(x) = x$ is thus a singular solution for (1.1.4). In fact, there exists no value for c such that $y(a) = a$ for every $a > 0$, since

$$y(a) = \frac{a(a^2 + 2 + c)}{a^2 + c} = a \text{ leads to the contradiction that } 2 = 0.$$

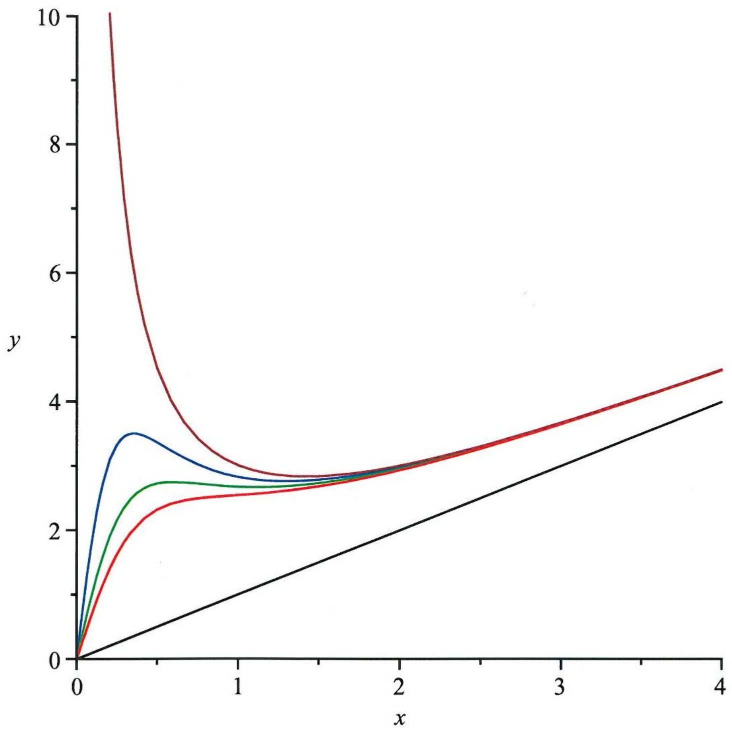

Figure 1.1: Some solution curves of (1.1.4)

Figure 1.1 depicts some solution curves of the general solution (1.1.5) for the values $c = 0.1$ (blue), $c = 0.2$ (green), $c = 0.3$ (red) and $c = 0$ (brown). The singular solution $y = x$ is indicated in black.

Equation (1.1.4) is an example of the so-called Riccati equation, which we introduce in Section 3.2, where we also show that this type of singular solutions always exist for the Riccati equation.

d) We can verify that the second-order differential equation

$$y'' + 4y = 4x\cos x \tag{1.1.6}$$

admits the solutions

$$y(x) = c_1\sin(2x) + c_2\cos(2x) + \frac{8}{9}\sin x + \frac{4}{3}x\cos x \tag{1.1.7}$$

for all $x \in \mathbb{R}$, where c_1 and c_2 are two arbitrary constants. Some solution curves for the values $\{c_1 = 1,\ c_2 = 1\}$ (blue), $\{c_1 = 2,\ c_2 = 3\}$ (green), and $\{c_1 = 3,\ c_2 = 4\}$ (red) are shown in Figure 1.2.

c) We can verify that the first-order differential equation

$$y' = \frac{x}{y} - 1 + \frac{y}{x}, \quad x \neq 0 \tag{1.1.8}$$

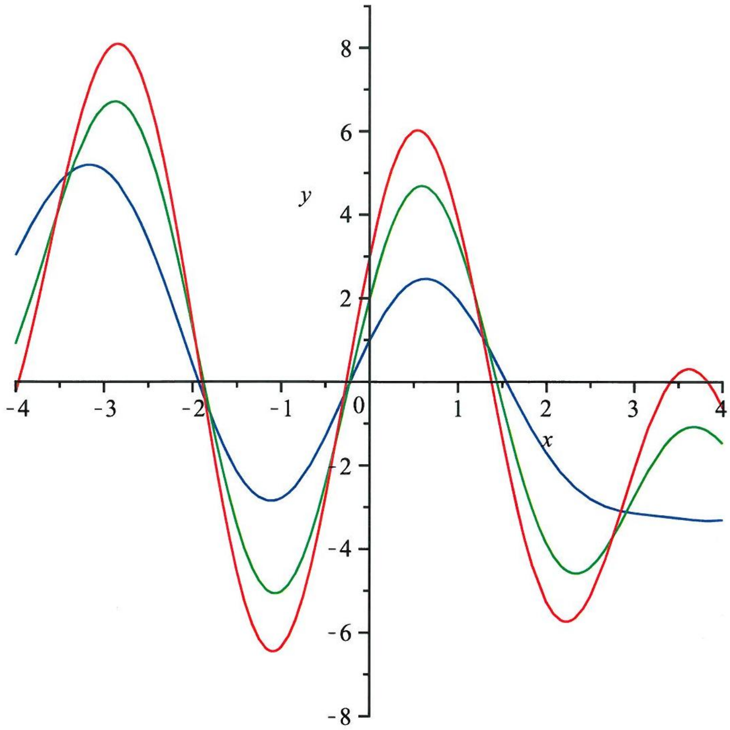

Figure 1.2: Some solution curves of (1.1.6)

admits the following general implicit solution

$$\frac{y}{x} + \ln\left|1 - \frac{y}{x}\right| = -\ln|cx|,$$

where c is an arbitrary constant. A singular solution is $y = x$.

Consider now a special form of (1.1.1), namely the so-called **linear homogeneous ordinary differential equation** of order n, which has the following general form:

$$\boxed{p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0} \tag{1.1.9}$$

Here $p_j(x)$ ($j = 0, 1, 2, \ldots, n$) are real-valued continuous functions given on some common domain $\mathcal{D} \subseteq \mathbb{R}$, $n \geq 1$ and $p_n(x) \neq 0$ for all $x \in \mathcal{D}$.

Let

$$\{\phi_1(x),\ \phi_2(x), \ldots, \phi_s(x)\} \tag{1.1.10}$$

be a set of solutions of (1.1.9) on $\mathcal{D}$. That is

$$p_n(x)\phi_j^{(n)} + p_{n-1}(x)\phi_j^{(n-1)} + \cdots + p_1(x)\phi_j' + p_0(x)\phi_j = 0, \qquad j = 1, 2, \ldots, s \tag{1.1.11}$$

and $\phi_j \in C^n(\mathcal{D})$.

Proposition 1.1.1. (Linear Superposition Principle): *Any linear combination of the set of solutions (1.1.10) for (1.1.9) on $\mathcal{D}$, i.e.*

$$c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_s\phi_s(x), \tag{1.1.12}$$

are solutions for (1.1.9) on $\mathcal{D}$ for any $c_j \in \mathbb{R}$ $(j = 1, 2, \ldots, s)$.

Proof: We assume that the set of functions (1.1.10) are solutions of (1.1.9) and show that

$$y(x) = c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_s\phi_s(x) \tag{1.1.13}$$

satisfies (1.1.9). Differentiating (1.1.13) n times, respectively, we obtain

$$\begin{aligned}
y'(x) &= c_1\phi_1' + c_2\phi_2' + \cdots + c_s\phi_s' \\
y''(x) &= c_1\phi_1'' + c_2\phi_2'' + \cdots + c_s\phi_s'' \\
&\vdots \\
y^{(n)}(x) &= c_1\phi_1^{(n)} + c_2\phi_2^{(n)} + \cdots + c_s\phi_s^{(n)}.
\end{aligned}$$

Inserting the above expressions for y, $y', \ldots, y^{(n)}$ into (1.1.9) we obtain

$$\begin{aligned}
&p_n(x)\left[c_1\phi_1^{(n)} + c_2\phi_2^{(n)} + \cdots + c_s\phi_s^{(n)}\right] \\
&\quad +p_{n-1}(x)\left[c_1\phi_1^{(n-1)} + c_2\phi_2^{(n-1)} + \cdots + c_s\phi_s^{(n-1)}\right] \\
&\quad +\cdots \\
&\quad +p_1(x)\left[c_1\phi_1' + c_2\phi_2' + \cdots + c_s\phi_s'\right] \\
&\quad +p_0(x)\left[c_1\phi_1 + c_2\phi_2 + \cdots + c_s\phi_s\right] \\
&\quad = c_1\left[p_n(x)\phi_1^{(n)} + p_{n-1}(x)\phi_1^{(n-1)} + \cdots + p_1(x)\phi_1' + p_0(x)\phi_1\right] \\
&\quad +c_2\left[p_n(x)\phi_2^{(n)} + p_{n-1}(x)\phi_2^{(n-1)} + \cdots + p_1(x)\phi_2' + p_0(x)\phi_2\right] \\
&\quad +\ldots \\
&\quad +c_s\left[p_n(x)\phi_s^{(n)} + p_{n-1}(x)\phi_s^{(n-1)} + \cdots + p_1(x)\phi_s' + p_0(x)\phi_s\right] \\
&= c_1\,0 + c_2\,0 + \cdots + c_s\,0 \quad (\text{since } \phi_1,\ \phi_2, \ldots, \phi_n \text{ are solutions for (1.1.9)}) \\
&= 0. \qquad \square
\end{aligned}$$

To find a general solution for the n-th order linear differential equation, (1.1.9), we have to find a set of n linearly independent solutions for (1.1.9). The linear combination of this set of solutions will then describe the general solution of the equation. This is stated in Proposition 1.1.5 below. To establish this, we start with the definition of a linearly independent set of functions in the vector space $\mathcal{C}(\mathcal{D})$.

Definition 1.1.3. *Consider the set S of n continuous functions on some domain $\mathcal{D} \subseteq \mathbb{R}$:*

$$\{\phi_1(x),\ \phi_2(x), \ldots, \phi_n(x)\}. \tag{1.1.14}$$

That is, $\phi_j(x)$ $(j = 1, 2, \ldots, n)$ belong to the vector space of continuous functions, $\mathcal{C}(\mathcal{D})$. The set S is a **linearly dependent** *set in the vector space $\mathcal{C}(\mathcal{D})$ if there exist constants $c_1,\ c_2, \ldots,\ c_n$, not all zero, such that*

$$c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_n\phi_n(x) = 0 \qquad \text{for all } x \in \mathcal{D}. \tag{1.1.15}$$

The set (1.1.14) is **linearly independent** *in $\mathcal{C}(\mathcal{D})$ if equation (1.1.15) can only be satisfied on $\mathcal{D}$ when all constants $c_1,\ c_2, \ldots,\ c_n$ are zero.*

Example 1.1.2.

a) Consider the set $S = \{1,\ e^x\} \in \mathcal{C}(\mathbb{R})$. Then it is clear that the equation

$$c_1\, 1 + c_2\, e^x = 0$$

can only be satisfied for all $x \in \mathbb{R}$, if $c_1 = 0$ and $c_2 = 0$. Therefore we conclude that S is a linearly independent set.

b) Consider the set $S = \{\cos^2 x,\ \sin^2 x,\ 1\} \in \mathcal{C}(\mathbb{R})$. Then the equation

$$c_1 \cos^2 x + c_2 \sin^2 x + c_3\, 1 = 0$$

is satisfied for all $x \in \mathbb{R}$ if, for example, $c_1 = c_2 = 1$ and $c_3 = -1$. Therefore we conclude that S is a linearly dependent set in $\mathcal{C}(\mathbb{R})$.

To determine whether a set of functions are linearly dependent on some interval $\mathcal{D} \subseteq \mathbb{R}$ in the vector space $\mathcal{C}^n(\mathcal{D})$, it is useful to introduce the so-called Wronskian.

Historical Note: (source: Wikipedia)
Józef Maria Hoene-Wroński (1776 –1853) was a Polish Messianist philosopher who worked in many fields of knowledge, not only as philosopher but also as mathematician. The Wronskian was introduced by Hoene-Wronski in 1812 and was named as such by Thomas Muir in 1882

Józef Maria Hoene-Wroński (1776 –1853)

Definition 1.1.4. *Consider the set*

$$S = \{\phi_1(x),\ \phi_2(x), \dots, \phi_n(x)\} \text{ in } \mathcal{C}^n(\mathcal{D}). \tag{1.1.16}$$

The determinant

$$W[\phi_1, \phi_2, \dots, \phi_n](x) := \begin{vmatrix} \phi_1 & \phi_2 & \dots & \phi_n \\ \phi_1' & \phi_2' & \dots & \phi_n' \\ \vdots & \vdots & \dots & \vdots \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \dots & \phi_n^{(n-1)} \end{vmatrix} \tag{1.1.17}$$

is defined as the **Wronskian** *of the set (1.1.16), where $W[\phi_1, \phi_2, \dots, \phi_n](x)$ is a differentiable function on $\mathcal{D}$.*

Example 1.1.3.

Consider $\phi_1(x) = x$ and $\phi_2(x) = \cos x$ for all $x \in \mathbb{R}$. Then

$$W[\phi_1, \phi_2](x) = \begin{vmatrix} x & \cos x \\ 1 & -\sin x \end{vmatrix} = -x\sin x - \cos x. \tag{1.1.18}$$

To determine whether a set of functions is linearly independent, we can use the following

Proposition 1.1.2. *Let $S = \{\phi_1(x),\ \phi_2(x), \dots, \phi_n(x)\}$ be a set of n nonzero functions in $\mathcal{C}^{(n)}(\mathcal{D})$. If the set S is linearly dependent on the interval $\mathcal{D}$, then the Wronskian $W[\phi_1, \dots, \phi_n](x) = 0$ for all $x \in \mathcal{D}$. Therefore, if $W[\phi_1, \dots, \phi_n](x_0) \neq 0$ at some point $x_0 \in \mathcal{D}$, then S is a linearly independent set on $\mathcal{D}$.*

Proof: Consider the set $S = \{\phi_1(x),\ \phi_2(x), \dots, \phi_n(x)\}$ in $\mathcal{C}^{(n)}(\mathcal{D})$ and the equation

$$\lambda_1\phi_1(x) + \lambda_2\phi_2(x) + \dots + \lambda_n\phi_n(x) = 0, \tag{1.1.19}$$

where $\lambda_j,\ j = 1, 2, \dots, n$, are unspecified constants. Differentiating relation (1.1.19) $(n-1)$-times, respectively, we obtain

$$\begin{aligned} &\lambda_1\phi_1' + \lambda_2\phi_2' + \dots + \lambda_n\phi_n' = 0 \\ &\lambda_1\phi_1'' + \lambda_2\phi_2'' + \dots + \lambda_n\phi_n'' = 0 \\ &\qquad\vdots \\ &\lambda_1\phi_1^{(n-1)} + \lambda_2\phi_2^{(n-1)} + \dots + \lambda_n\phi_n^{(n-1)} = 0. \end{aligned} \tag{1.1.20}$$

The above n equations, (1.1.19) and (1.1.20), can be written as follows:

$$\begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \cdots & \phi_n' \\ \vdots & & \vdots & \vdots \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{1.1.21}$$

We denote the $n \times n$ coefficient matrix of (1.1.21) as matrix A. Now, if the set S is linearly dependent for all $x \in \mathcal{D}$, then there exist nonzero solutions for at least two of the constants λ_j that satisfy equation (1.1.19), so that A is singular (A^{-1} does not exist) and $\det A = 0$ for all $x \in \mathcal{D}$. On the other hand, if $\det A \neq 0$ at some point $x_0 \in \mathcal{D}$, then A is not singular in that point, so that the only solution for any λ_j that satisfies equation (1.1.19) for all $x \in \mathcal{D}$ is the trivial solution, $\lambda_1 = 0$, $\lambda_2 = 0, \ldots, \lambda_n = 0$. We note that $\det A = W[\phi_1, \phi_2, \ldots, \phi_n](x)$. Therefore we conclude that, if $W[\phi_1, \phi_2, \ldots, \phi_n](x_0) \neq 0$ at some $x_0 \in \mathcal{D}$, then S is a linearly independent set on the interval $\mathcal{D}$. $\square$

Example 1.1.4.

a) In Example 1.2b we have shown that the set $S = \{\phi_1 = \cos^2 x,\ \phi_2 = \sin^2 x,\ \phi_3 = 1\}$ is linearly dependent for all $x \in \mathbb{R}$. Then

$$W[\phi_1, \phi_2, \phi_3](x) = \begin{vmatrix} \cos^2 x & \sin^2 x & 1 \\ -2\cos x \sin x & 2\sin x \cos x & 0 \\ 2\sin^2 x - 2\cos^2 x & 2\cos^2 x - 2\sin^2 x & 0 \end{vmatrix}$$

$$= -2\cos x \sin x \left(2\cos^2 x - 2\sin^2 x\right) - 2\sin x \cos x \left(2\sin^2 x - 2\cos^2 x\right)$$

$$= 0 \text{ for all } x \in \mathbb{R}$$

as stated by Proposition 1.1.2

b) Consider the two exponential functions, namely

$$\phi_1(x) = e^{\alpha_1 x}, \qquad \phi_2(x) = e^{\alpha_2 x},$$

where α_1 and α_2 are any real numbers. To show that ϕ_1 and ϕ_2 are linearly independent on $\mathbb{R}$ we evaluate the Wronskian of ϕ_1 and ϕ_2 in the point $x = 0$:

$$W[\phi_1, \phi_2](0) = \begin{vmatrix} e^{\alpha_1 x} & e^{\alpha_2 x} \\ \alpha_1 e^{\alpha_1 x} & \alpha_2 e^{\alpha_2 x} \end{vmatrix}_{x=0} = \alpha_2 - \alpha_1.$$

Since $W[\phi_1, \phi_2](x) \neq 0$ in the point $x = 0$ for $\alpha_1 \neq \alpha_2$, it follows by Proposition 1.1.2 that ϕ_1 and ϕ_2 are linearly independent on $\mathbb{R}$ for $\alpha_1 \neq \alpha_2$.

c) Consider the complex function

$$f(x) = e^{(\alpha + i\beta)x}, \qquad \alpha \in \mathbb{R},\ \beta \in \mathbb{R},\ \ i^2 := -1.$$

Since

$$e^{(\alpha+i\beta)x} = e^{\alpha x}\left(\cos \beta x + i \sin \beta x\right)$$

we have

$$\phi_1(x) := \mathrm{Re}\,[f(x)] = e^{\alpha x} \cos \beta x, \qquad \phi_2(x) := \mathrm{Im}\,[f(x)] = e^{\alpha x} \sin \beta x.$$

Calculating $W[\phi_1,\ \phi_2](0)$, we obtain

$$W[\phi_1,\ \phi_2](0) = \begin{vmatrix} e^{\alpha x}\cos(\beta x) & e^{\alpha x}\sin(\beta x) \\ \alpha e^{\alpha x}\cos(\beta x) - \beta e^{\alpha x}\sin(\beta x) & \alpha e^{\alpha x}\sin(\beta x) + \beta e^{\alpha x}\cos(\beta x) \end{vmatrix}(0)$$

$$= \begin{vmatrix} 1 & 0 \\ \alpha & \beta \end{vmatrix} = \beta$$

Hence, it follows by Proposition 1.1.2 that the set $\{\phi_1(x),\ \phi_2(x)\}$ is linearly independent on $\mathbb{R}$ for $\beta \neq 0$.

Proposition 1.1.3. *Let* $\{\phi_1(x), \phi_2(x), \ldots, \phi_n(x)\}$ *be a set of* n *nonzero solutions of*

$$p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0$$

on some interval $\mathcal{D} \subseteq \mathcal{R}$. *Then either*

$$W[\phi_1, \phi_2, \ldots, \phi_n](x) = 0$$

for every $x \in \mathcal{D}$, *or*

$$W[\phi_1, \phi_2, \ldots, \phi_n](x) \neq 0$$

for every $x \in \mathcal{D}$.

Proof: We give the proof for the case $n = 2$. The general case is proved in the Appendix to Chapter 1. For $n = 2$, equation (1.1.9) is

$$p_2(x)y'' + p_1(x)y' + p_0(x)y = 0, \tag{1.1.22}$$

where $p_2(x) \neq 0$ for every $x \in \mathcal{D}$. Let $\phi_1(x)$ and $\phi_2(x)$ be two solutions for (1.1.22) on the interval $\mathcal{D}$. Then

$$p_2(x)\phi_1'' + p_1(x)\phi_1' + p_0(x)\phi_1 = 0 \tag{1.1.23a}$$

$$p_2(x)\phi_2'' + p_1(x)\phi_2' + p_0(x)\phi_2 = 0. \tag{1.1.23b}$$

Multiplying (1.1.23a) by $-\phi_2$ and (1.1.23b) by ϕ_1 and then adding the resulting equations (1.1.23a) and (1.1.23b), we obtain

$$p_2(x)(\phi_1\phi_2'' - \phi_2\phi_1'') + p_1(x)(\phi_1\phi_2' - \phi_2\phi_1') = 0. \tag{1.1.24}$$

We recall that

$$W[\phi_1,\ \phi_2](x) = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = \phi_1\phi_2' - \phi_2\phi_1'$$

and, furthermore, we note that

$$W'[\phi_1, \phi_2](x) = \phi_1\phi_2'' - \phi_2\phi_1'' = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1'' & \phi_2'' \end{vmatrix}. \tag{1.1.25}$$

Therefore equation (1.1.24) can be written in the form

$$p_2(x)W' + p_1(x)W = 0$$

or, since $p_2(x) \neq 0$ for all $x \in \mathcal{D}$, we can write

$$W' + \frac{p_1(x)}{p_2(x)}W = 0. \tag{1.1.26}$$

Now, either $W = 0$ for all $x \in \mathcal{D}$, or $W \neq 0$ for all $x \in \mathcal{D}$, as we will now show by integrating (1.1.26): Equation (1.1.26) can be integrated:

$$\int \frac{dW}{W} = -\int \frac{p_1(x)}{p_2(x)}\, dx + \ln|c|, \quad \text{so that a general solution is}$$

$$W[\phi_1,\ \phi_2](x) = c\exp\left[-\int \frac{p_1(x)}{p_2(x)}\, dx\right], \tag{1.1.27}$$

where c is an arbitrary constant of integration. Since

$$\exp\left[-\int \frac{p_1(x)}{p_2(x)}\, dx\right] \neq 0 \quad \text{for every } x \in \mathcal{D}$$

and $c \neq 0$, it follows that $W[\phi_1,\ \phi_2](x) \neq 0$ for every $x \in \mathcal{D}$ (except for the singular solution $W = 0$). Thus the statement is established for the case $n = 2$. $\square_{n=2}$

It is important to remark that Proposition 1.1.3 is essential in the methods to construct solutions of linear differential equations, as we will see in the chapters that follow.

One may want to construct a differential equation from a given set of solution-functions. This can be done by the use of the Wronskian: Let S be the given set of linearly independent solutions for

$$p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0,$$

for all $x \in \mathcal{D}$. Then, by the linear superposition principle, any linear combination of these solutions is also a solution of this equation, i.e.,

$$y(x) = c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_n\phi_n(x), \tag{1.1.28}$$

where $c_1, c_2, \ldots .c_n$ are arbitrary constants. However, the set

$$Q = \{\phi_1(x), \phi_2(x), \ldots, \phi_n(x),\, y\},$$

is clearly linearly dependent in $\mathcal{C}^n(\mathcal{D})$. Differentiating now (1.1.28) n times, respectively, we obtain

$$\begin{aligned}
&c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n - y = 0\\
&c_1\phi_1' + c_2\phi_2' + \cdots + c_n\phi_n' - y' = 0\\
&\qquad\qquad\qquad\vdots\\
&c_1\phi_1^{(n)} + c_2\phi_2^{(n)} + \cdots + c_n\phi_n^{(n)} - y^{(n)} = 0
\end{aligned}$$

or equivalently

$$\begin{pmatrix} \phi_1 & \phi_2 & \ldots & \phi_n & y \\ \phi_1' & \phi_2' & \ldots & \phi_n' & y' \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ \phi_1^{(n)} & \phi_2^{(n)} & \ldots & \phi_n^{(n)} & y^{(n)} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^{n+1}.$$

Since the set Q is linearly dependent, the $(n+1) \times (n+1)$ matrix on the left side of the above relation must be a singular matrix for all $x \in \mathcal{D}$. Hence its determinant must be zero for all $x \in \mathcal{D}$ and this determinant is the Wronskian for Q. This leads to the following

Proposition 1.1.4. *Consider a set of n linearly independent solutions, $S = \{\phi_1(x), \phi_2(x), \ldots, \phi_n(x)\}$ in $\mathcal{C}^n(\mathcal{D})$ for*

$$p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0, \tag{1.1.29}$$

where $\mathcal{D} \subseteq \mathcal{R}$. Then this differential equation can equivalently be written in the form

$$W[\phi_1, \phi_2, \ldots, \phi_n, y](x) = 0 \tag{1.1.30}$$

for all $x \in \mathcal{D}$, where W is the Wronskian of the set of functions $\{\phi_1, \phi_2, \ldots, \phi_n, y\}$, namely

$$W[\phi_1, \phi_2, \ldots, \phi_n, y](x) := \begin{vmatrix} \phi_1 & \phi_2 & \ldots & \phi_n & y \\ \phi_1' & \phi_2' & \ldots & \phi_n' & y' \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ \phi_1^{(n)} & \phi_2^{(n)} & \ldots & \phi_n^{(n)} & y^{(n)} \end{vmatrix}. \tag{1.1.31}$$

Proof: Consider $n = 1$ with $p_1(x) \neq 0$. Then the general linear first-order homogeneous equation is

$$p_1(x)y' + p_0(x)y = 0. \tag{1.1.32}$$

Assume now that $\phi(x) \in \mathcal{C}^1(\mathcal{D})$ is a solution of (1.1.32), i.e.

$$p_1(x)\phi' + p_0(x)\phi = 0 \quad \text{or} \quad \phi' = -\left(\frac{p_0(x)}{p_1(x)}\right)\phi. \tag{1.1.33}$$

We show that $W[\phi, y](x) = 0$ is equivalent to (1.1.32). Now

$$W[\phi, y](x) = \begin{vmatrix} \phi & y \\ \phi' & y' \end{vmatrix} = \phi y' - \phi' y = 0. \tag{1.1.34}$$

Inserting ϕ' from (1.1.33) into (1.1.34), we obtain

$$\phi y' - \left[-\left(\frac{p_0}{p_1}\right)\phi\right] y = 0 \quad \text{or} \quad p_1(x)y' + p_0(x)y = 0.$$

Consider now $n = 2$. Then the general linear second-order homogeneous equation is

$$p_2(x)y'' + p_1(x)y' + p_0(x)y = 0. \tag{1.1.35}$$

Assume now that $\phi_1(x) \in \mathcal{C}^1(\mathcal{D})$ and $\phi_2(x) \in \mathcal{C}^1(\mathcal{D})$ are two linearly independent solutions of (1.1.35), i.e.

$$p_2(x)\phi_j'' + p_1(x)\phi_j' + p_0(x)\phi_j = 0, \quad j = 1, 2. \tag{1.1.36}$$

The equation $W[\phi_1, \phi_2, y](x) = 0$ gives

$$\phi_1\phi_2'y'' + \phi_2\phi_1''y' + \phi_1'\phi_2''y - \phi_2'\phi_1''y - \phi_1\phi_2''y' - \phi_2\phi_1'y'' = 0. \tag{1.1.37}$$

Substituting ϕ_1'' and ϕ_2'' from (1.1.36) into (1.1.37), we obtain

$$p_2(x)\left(\phi_1\phi_2' - \phi_2\phi_1'\right)y'' + p_1(x)\left(\phi_1\phi_2' - \phi_2\phi_1'\right)y'$$
$$+p_0(x)\left(\phi_1\phi_2' - \phi_2\phi_1'\right)y = 0. \qquad (1.1.38)$$

Since the set $\{\phi_1(x), \phi_2(x)\}$ is a linearly independent set of solutions for (1.1.35), we have that $W[\phi_1, \phi_2](x) \neq 0$ for all $x \in \mathcal{D}$. Hence (1.1.38) reduces to (1.1.36).
The same method of proof can be used for all natural numbers n (see Exercises 1.1.1). □

Example 1.1.5.

Consider the set of functions $S = \{\phi_1(x), \phi_2(x)\}$, where

$$\phi_1(x) = e^{x^2}, \quad \phi_2(x) = e^{-x^2},$$

so that

$$W[\phi_1, \phi_2](x) = -4x.$$

Applying Proposition 1.1.4 we can construct a second-order linear homogeneous differential equation with solutions S for all $x \in \mathbb{R}\backslash\{0\}$. It follows that

$$W[\phi_1, \phi_2, y](x) = -4xy'' + 4y' + 16x^3y.$$

Hence the differential equation with the given solution set S has the form

$$xy'' - y' - 4x^3y = 0.$$

Finally we have

Proposition 1.1.5. *Let $S = \{\phi_1(x),\ \phi_2(x), \dots, \phi_n(x)\} \in \mathcal{C}^{(n)}(\mathcal{D})$ be a linearly independent set of n solutions for equation*

$$p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0,$$

where $p_j(x)$ are continuous functions on the interval $\mathcal{D}$. Then the linear combination

$$y(x) = c_1\phi_1(x) + c_2\phi_2(x) + \ldots + c_n\phi_n(x),$$

is the general solution of this equation on $\mathcal{D}$, where $c_1, \dots, c_n$ are n arbitrary real constants.

In order to give a rigorous proof of Proposition 1.1.5 we need the following theorem on the existence and uniqueness of the solutions of (1.1.9) (the subject of existence and uniqueness is outside the scope of these lecture notes and we will therefore not provide the proof)

Proposition 1.1.6. *(Existence and uniqueness theorem)*

Consider the nth order homogeneous equation

$$p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0,$$

where $p_j(x)$ *are continuous and bounded on an interval* $\mathcal{D}$*. For a given* $x_0 \in \mathcal{D}$ *and given numbers* $b_1, b_2, \ldots, b_n$*, there exists a unique solution* $y(x)$ *on* $\mathcal{D}$ *such that*

$$y(x_0) = b_1, \quad y'(x_0) = b_2, \quad \ldots, \quad y^{(n-1)}(x_0) = b_n. \tag{1.1.39}$$

Note that Proposition (1.1.6) is also true for linear equations of the form

$$p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x),$$

where $p_j(x)$ and $f(x)$ are continuous and bounded on $\mathcal{D}$.

Example 1.1.6.

One may verify that

$$\phi_1(x) = x^{-2}\cos(3\ln x) \qquad \text{and} \qquad \phi_2(x) = x^{-2}\sin(3\ln x)$$

are solutions of

$$x^2y'' + 5xy' + 13y = 0 \tag{1.1.40}$$

on the interval $\mathcal{D} = \{x \in \mathbb{R} : \ x > 0\}$. Now

$$W[\phi_1, \phi_2](x)$$

$$= \begin{vmatrix} x^{-2}\cos(3\ln x) & x^{-2}\sin(3\ln x) \\ -x^{-3}\left[3\sin(3\ln x) + 2\cos(3\ln x)\right] & x^{-3}\left[3\cos(3\ln x) - 2\sin(3\ln x)\right] \end{vmatrix}$$

A convenient point in $\mathcal{D}$ to evaluate the above Wronskian is at $x = 1$. Thus

$$W[\phi_1, \phi_2](1) = \begin{vmatrix} 1 & 0 \\ -2 & 3 \end{vmatrix} = 3.$$

Since $W[\phi_1, \phi_2](1) \neq 0$, it follows by Proposition 1.1.2 that ϕ_1 and ϕ_2 are linearly independent on $\mathcal{D}$. The general solution of (1.1.40) is then given by

$$y(x) = c_1x^{-2}\cos(3\ln x) + c_2x^{-2}\sin(3\ln x)$$

for all $x \in \mathcal{D}$, where c_1 and c_2 are arbitrary constants.

1.1.1 Exercises

[Solutions of those Exercises marked with a * are given in Appendix D].

1. Determine whether the following sets of functions, $\{f_1,\ f_2,\ f_3\ \ldots\}$, are linearly dependent or linearly independent on the interval $\mathcal{D}$:

   a) $f_1(x) = e^x,\ f_2(x) = e^{2x},\ f_3(x) = e^{3x}, \quad \mathcal{D} := \mathbb{R}$

   b)* $f_1(x) = \ln(x),\ f_2(x) = \ln(x^2),\ f_3(x) = e^{3x}, \quad \mathcal{D} := (0, \infty)$

   c) $f_1(x) = \cos x,\ f_2(x) = \sin x,\ f_3(x) = x\cos x,\ f_4(x) = x\sin x, \quad \mathcal{D} := \mathbb{R}$

   d)* $f_1(x) = e^x,\ f_2(x) = e^{-x},\ f_3(x) = xe^x,\ f_4(x) = xe^{-x}, \quad \mathcal{D} := \mathbb{R}$

e) $f_1(x) = e^x \cos x,\ f_2(x) = e^x \sin x, \quad \mathcal{D} := \mathbb{R}$

f) $f_1(x) = e^{\sin x},\ f_2(x) = e^x,\ f_3(x) = \sin x, \quad \mathcal{D} := \mathbb{R}$

g) $f_1(x) = \sec^2 x,\ f_2(x) = \tan^2 x,\ f_3(x) = -5, \quad \mathcal{D} := \mathbb{R}$

h) $f_1(x) = \mathrm{cosec}^2 x,\ f_2(x) = \cot^2 x,\ f_3(x) = \pi, \quad \mathcal{D} := \mathbb{R}$

i) $f_1(x) = \cos(2x),\ f_2(x) = 2\cos^2 x,\ f_3(x) = 3\sin^2 x, \quad \mathcal{D} := \mathbb{R}$

j) $f_1(x) = \sin(2x),\ f_2(x) = 2\cos x \sin x,\ f_3(x) = 1,\ f_4(x) = e^x, \quad \mathcal{D} := \mathbb{R}$

2. * Consider the following two functions

$$f_1(x) = x^2, \qquad f_2(x) = x|x|$$

on $\mathbb{R}$. Show that the Wronskian $W[f_1,\ f_2](x)$ is identically zero for all $x \in \mathbb{R}$ and show furthermore that the set $S = \{f_1(x),\ f_2(x)\}$ is in fact linearly independent on $\mathbb{R}$. This shows that we cannot conclude linear dependence on an interval for a set of functions if the Wronskian is zero on that interval.

3. Show that

a) $e^x + e^{-y(x)} = c$ is a general solution of the first-order differential equation $y' = e^{x+y}$, where c is an arbitrary constant.

b)* $y(x) = c_1 \cos(2x) + c_2 \sin(2x)$
is the general solution of the second-order linear differential equation $y'' + 4y = 0$, where c_1 and c_2 are arbitrary constants.

c) $y(x) = c_1 e^x + c_2 \sin x + c_3 \cos x$ is the general solution of the third-order differential equation $y^{(3)} - y'' + y' - y = 0$, where c_1, c_2 and c_3 are arbitrary constants.

d) $y^2(x) + 2y(x) = x^2 + 2x + c$ is a general solution of the first-order differential equation $y' = \dfrac{x+1}{y+1}$, where c is an arbitrary constant.

e) $(2c - x)y^2 = x^3$ is a general solution of the first-order differential equation $2x^3 y' - 3x^2 y - y^3 = 0$, where c is an arbitrary constant.

f) $e^{x^2} + \ln[y(x) + \sqrt{1 + y^2}] = c$ is a general solution of the first-order differential equation $e^{-x^2} y' + 2x\sqrt{1 + y^2} = 0$, where c is an arbitrary constant.

g) $y(x) = c_1 e^x + c_2 \sin x + c_3 \cos x + e^x \left(\dfrac{1}{4}x^2 - \dfrac{1}{2}x\right)$ is the general solution of the third-order differential equation $y^{(3)} - y'' + y' - y = x\, e^x$, where c_1, c_2 and c_3 are arbitrary constants.

h) $y(x) = \sum_{j=1}^{n} c_j x^{j-1}$ is the general solution of the n-th order differential equation $y^{(n)} = 0$, where $c_1,\ c_2,\ \ldots,\ c_n$ are arbitrary constants.

4. Use the following set of functions,

$$f_1(x) = e^{-x}, \quad f_2(x) = e^{3x}, \quad f_3(x) = e^{4x}, \quad f_4(x) = e^{x}$$

to construct a general solution for the equation

$$y''' - 6y'' + 5y' + 12y = 0.$$

5. Consider the following sets of functions and construct in each case, if possible, the linear homogeneous differential equation for which a linear combination of the given set of functions gives the general solution of the differential equation and establish the solution domain of the so constructed differential equation.
Hint: Make use of Proposition 1.1.4.

a) $S = \{e^{\sqrt{x}},\ e^{-\sqrt{x}}\}, \quad x > 0$

b) $S = \{x\, e^{a/x},\ x\, e^{-a/x}\}$, where $x \neq 0$ and $a \in \mathbb{R}\backslash\{0\}$.

c)* $S = \{x\, \cos(1/x),\ x \sin(1/x)\}$, $x \neq 0$.

d) $S = \{\frac{1}{x}\, e^{x},\ \frac{1}{x}\, e^{-x}\}$, $x \neq 0$.

e) $S = \{\frac{1}{\sqrt{x}},\ \frac{1}{\sqrt{x}}\, e^{-x^2/2}\}$, $x > 0$.

f) $S = \{x,\ x^2,\ x \ln x\}$, $x > 0$.

g) $S = \{x^3,\ \cos(\ln x),\ \sin(\ln x)\}$, $x > 0$.

6. Prove (1.3.1), namely

$$W'[\phi_1,\ \phi_2, \ldots, \phi_n](x) = \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \cdots & \phi_n' \\ \vdots & \vdots & \cdots & \vdots \\ \phi_1^{(n-2)} & \phi_2^{(n-2)} & \cdots & \phi_n^{(n-2)} \\ \phi_1^{(n)} & \phi_2^{(n)} & \cdots & \phi_n^{(n)} \end{vmatrix},$$

where $\{\phi_1,\ \phi_2, \ldots, \phi_n\}$ are functions in $\mathcal{C}^n(\mathcal{D})$ and W' denotes the x-derivative of the Wronskian W.

Remark: In the theory of determinants, the following result is established: *If the elements $a_{ij}(x)$ of the determinant of an $n \times n$ matrix A are differentiable functions* *of the variable* x*, then*

$$\frac{d}{dx}(\det A) = \frac{d}{dx}\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a'_{11} & a'_{12} & \cdots & a'_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a'_{21} & a'_{22} & \cdots & a'_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \cdots$$

$$+ \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a'_{n1} & a'_{n2} & \cdots & a'_{nn} \end{vmatrix}$$

7. Prove Proposition 1.1.4 for $n = 3$ and consequently for all natural numbers n.

1.2 The solution space of linear homogeneous differential equations

We consider the linear homogeneous differential equation (1.1.9)

$$p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0, \tag{1.2.1}$$

where $p_j(x)$ $(j = 0, 1, 2, \ldots, n)$ are given continuous functions on some common domain $\mathcal{D} \subseteq \mathbb{R}$ and $p_n(x) \neq 0$ for all $x \in \mathcal{D}$. For convenience we write (1.2.1) in the following form:

$$\boxed{L\,y(x) = 0} \tag{1.2.2}$$

Here L denotes the following **linear differential operator** of order n:

$$L := p_n(x)\frac{d^n}{dx^n} + p_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + p_1(x)\frac{d}{dx} + p_0(x). \tag{1.2.3}$$

Acting L on $y(x) \in \mathcal{C}^n(\mathcal{D})$, we have

$$L\,y(x) = p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y. \tag{1.2.4}$$

Consider now the transformation T, such that

$$T: \ \mathcal{C}^n(\mathcal{D}) \to \mathcal{C}(\mathcal{D}), \tag{1.2.5}$$

where $\mathcal{D} \subseteq \mathbb{R}$. Recall that $\mathcal{C}^n(\mathcal{D})$ is the vector space of n-times differentiable functions on the interval $\mathcal{D}$ and $\mathcal{C}$ is the vector space of continuous functions on the interval $\mathcal{D}$. In particular, we define T as follows:

$$T: \ y(x) \mapsto L\,y(x). \tag{1.2.6}$$

We now prove

Proposition 1.2.1. *The transformation T, namely $T: \ y(x) \mapsto L\,y(x)$ with L defined by (1.2.3), is a linear transformation.*

Proof: Let $y_1(x)$ and $y_2(x)$ be any two functions in $\mathcal{C}^n(\mathcal{D})$. Then

$$\begin{aligned}
&T(y_1(x) + y_2(x)) = L\,(y_1(x) + y_2(x))\\
&= \left(p_n(x)\frac{d^n}{dx^n} + p_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + p_1(x)\frac{d}{dx} + p_0(x)\right)(y_1(x) + y_2(x))\\
&= p_n(x)y_1^{(n)} + p_n(x)y_2^{(n)} + p_{n-1}(x)y_1^{(n-1)} + p_{n-1}(x)y_2^{(n-1)} + \cdots + p_0(x)y_1(x) + p_0(x)y_2(x)\\
&= L\,y_1(x) + L\,y_2(x) = T(y_1(x)) + T(y_2(x)).
\end{aligned}$$

Moreover, for any real constant c we have

$$\begin{aligned}
&T(cy_1(x)) = L\ (cy_1(x))\\
&= \left(p_n(x)\frac{d^n}{dx^n} + p_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + p_1(x)\frac{d}{dx} + p_0(x)\right)(cy_1(x))\\
&= cp_n(x)y_1^{(n)} + cp_{n-1}(x)y_1^{(n-1)} + \cdots + cp_0(x)y_1(x)\\
&= cL\,y_1(x) = cT(y_1(x)).
\end{aligned}$$

We conclude that T is a linear transformation. □

We recall that the **kernel** of T consists of all those functions $y(x)$ for which

$$T: \ y(x) \mapsto 0. \tag{1.2.7}$$

That is, for the linear transformation (1.2.6), the kernel of T contains all solutions of the equation $L\,y(x) = 0$. See Figure 1.3.

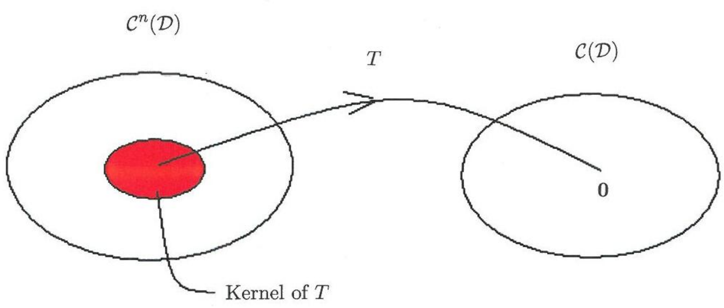

Figure 1.3: The solution space of $L\,y(x) = 0$

Proposition 1.2.2. *Let $T:\ y(x) \mapsto L\,y(x)$ with L defined by (1.2.3). Then the kernel of T is an n-dimensional subspace of $\mathcal{C}^n(\mathcal{D})$ with basis*

$$B = \{\phi_1(x),\ \phi_2(x),\ \ldots,\ \phi_n(x)\}, \tag{1.2.8}$$

where $\phi_1(x),\ \phi_2(x),\ \ldots,\ \phi_n(x)$ are linearly independent solutions of $L\,y(x) = 0$.

Proof: Let $y_1(x)$ and $y_2(x)$ be any two functions in the kernel of T. Since T is a linear transformation, it follows that

$$T(y_1(x) + y_2(x)) = T(y_1(x)) + T(y_2(x)) = 0 \qquad \text{and}$$
$$T(cy_1(x)) = c\,T(y_1(x)) = c\,0 = 0 \qquad \text{for all } c \in \mathbb{R}$$

so that the kernel of T is a subspace of $\mathcal{C}^n(\mathcal{D})$. By Proposition 1.1.5 a general solution of (1.2.2) is of the form

$$y(x) = c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_n\phi_n(x) \quad \text{for all } c_j \in \mathbb{R}\ (j = 1, 2, \ldots, n), \tag{1.2.9}$$

where every $\phi_j(x)$ is a solution of (1.2.2) and the set $\{\phi_1(x),\ \phi_2(x),\ \cdots,\ \phi_n(x)\}$ is linearly independent in $\mathcal{C}^n(\mathcal{D})$. Since (1.2.2) includes all the solutions of (1.2.2), the set

$$\{\phi_1(x),\ \phi_2(x),\ \cdots,\ \phi_n(x)\} \tag{1.2.10}$$

spans the kernel of T and the finite set (1.2.10) is thus a basis for this n-dimensional subspace of $\mathcal{C}^n(\mathcal{D})$. $\square$

This leads to

Definition 1.2.1. *The kernel of T, where $T : y(x) \mapsto L\,y(x)$ with L defined by (1.2.3), is called the* **solution space** *of the homogeneous linear differential equation $L\,y(x) = 0$.*

Remark: *If the linear homogeneous differential equation (1.2.2) contains only constant coefficients* p_j *(rather than functions* $p_j(x)$*) in the differential operator* L*, then the solutions* $\phi_j(x)$ *of the equation are all (depending on the values of the constant coefficients) of the form*

$$x^s e^{rx} \quad \text{or} \quad x^s e^{rx}\cos(qx) \quad \text{or} \quad x^s e^{rx}\sin(qx),$$

where s *is a natural number, whereas* q *and* r *are real numbers. These solutions are functions that can be differentiated indefinitely many times for all values of* $x \in \mathbb{R}$*, so that the* n*-dimensional solution space of (1.2.2) is in fact a subspace of* $\mathcal{C}^\infty(\mathbb{R})$*, rather than just* $\mathcal{C}^n(\mathcal{D})$*.*

1.2.1 Exercises

[Solutions of those Exercises marked with a * are given in Appendix D].

1. Consider the equation

$$y'' - 4y' + 13y = 0 \tag{1.2.11}$$

and the functions

$$\phi_1(x) = e^{2x}\cos(3x), \qquad \phi_2(x) = e^{2x}\sin(3x).$$

a) Show that ϕ_1 and ϕ_2 are solutions of (1.2.11).

b) Show that the set $S = \{\phi_1(x),\ \phi_2(x)\}$ is a linearly independent set in the space $\mathcal{C}^2(\mathbb{R})$ and give the general solution of (1.2.11).

c) Give the linear transformation $T:\ \mathcal{C}^2(\mathbb{R}) \to \mathcal{C}(\mathbb{R})$ for which the kernel of T defines the solution space of (1.2.11).

d) Give a basis for the solution space of (1.2.11).

e) Find that function in the solution space of (1.2.11) for which $y(0) = 4$ and $y'(0) = -1$.

f) Find that function in the solution space of (1.2.11) for which $y(0) = 1$ and $y(\pi/6) = 2$.

2. * Show that

$$\phi_1(x) = 3\,e^{-x}\cos x, \qquad \phi_2(x) = \pi\,e^{-x}\sin x$$

are functions that belong to the solution space of the differential equation

$$y'' + 2y' + 2y = 0$$

and give the general solution of this differential equation as well as a basis and the dimension of the solution space.

3. Show that

$$\phi_1(x) = e^{x}, \qquad \phi_2(x) = e^{-2x}, \qquad \phi_3(x) = 2e^{x}, \qquad \phi_4(x) = -3e^{-2x},$$

are functions that belong to the solution space of the differential equation

$$y'' + y' - 2y = 0$$

and give the general solution of this differential equation, as well as a basis and the dimension of the solution space.

4. In the exercises below, let $S = \{\phi_1(x),\ \phi_2(x),\ \ldots\}$ be a basis for the solution space of a second-order homogeneous differential equation with constant coefficients. Find the corresponding differential equation, if it exists, and give the general solution of this equation, as well as the dimension of the solution space.

   a) $\phi_1(x) = \cos x,\ \phi_2(x) = \sin x$

   b) $\phi_1(x) = e^{-x}\cos(2x),\ \phi_2(x) = e^{-x}\sin(2x)$

   c) $\phi_1(x) = 1,\ \phi_2(x) = e^{-2x}$

   d) $\phi_1(x) = e^{x},\ \phi_2(x) = \sin x,\ \phi_3(x) = \cos x$

   e) $\phi_1(x) = 1,\ \phi_2(x) = e^{-x},\ \phi_3(x) = \sin(2x),\ \phi_4(x) = \cos(2x)$

5. Show that there exists no differential equation of the form

$$y'' + ay' + by = 0$$

   for which the solution space has a basis $S = \{\phi_1(x),\ \phi_2(x)\}$, where

$$\phi_1(x) = e^{x}, \qquad \phi_2(x) = e^{x}\cos x.$$

1.3 Appendix to Chapter 1

We prove Proposition 1.1.3 for all natural numbers n:

Proposition 1.1.3 *Let* $\{\phi_1(x), \phi_2(x), \ldots, \phi_n(x)\}$ *be a set of* n *nonzero solutions of*

$$p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0$$

in some interval $\mathcal{D} \subseteq \mathcal{R}$. *Then either*

$$W[\phi_1, \phi_2, \ldots, \phi_n](x) = 0$$

for every $x \in \mathcal{D}$, *or*

$$W[\phi_1, \phi_2, \ldots, \phi_n](x) \neq 0$$

for every $x \in \mathcal{D}$.

Proof for all natural numbers n: (the proof of the case $n = 2$ is given in section 1.1)

In order to prove the statement for all n, we take a second look at the derivation of equation (1.1.26) as given in the proof in section 1.1: Consider (1.1.25), i.e.

$$W'[\phi_1, \phi_2](x) = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1'' & \phi_2'' \end{vmatrix}.$$

Replace now ϕ_1'' and ϕ_2'' from equations (1.1.23a) – (1.1.23b), in the second row by

$$\phi_1'' = -\frac{p_1}{p_2}\phi_1' - \frac{p_0}{p_2}\phi_1, \quad \phi_2'' = -\frac{p_1}{p_2}\phi_2' - \frac{p_0}{p_2}\phi_2,$$

respectively, to obtain

$$W'[\phi_1, \phi_2](x) = \begin{vmatrix} \phi_1 & \phi_2 \\ -\frac{p_1}{p_2}\phi_1' - \frac{p_0}{p_2}\phi_1 & -\frac{p_1}{p_2}\phi_2' - \frac{p_0}{p_2}\phi_2 \end{vmatrix}.$$

Multiplying the first row in the above determinant by $\frac{p_0}{p_2}$ and adding this to the second row (which does not change the value of the determinant), we obtain

$$W'[\phi_1, \phi_2](x) = \begin{vmatrix} \phi_1 & \phi_2 \\ -\frac{p_1}{p_2}\phi_1' & -\frac{p_1}{p_2}\phi_2' \end{vmatrix}.$$

Factoring out $-\frac{p_1}{p_2}$ from the second row in the above determinant, we obtain

$$W'[\phi_1, \phi_2](x) = -\frac{p_1(x)}{p_2(x)} \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = -\frac{p_1(x)}{p_2(x)} W[\phi_1,\ \phi_2](x).$$

To prove the statement for the nth-order equation (1.1.9) we use the same strategy. We need $W'[\phi_1,\ \phi_2,\ldots,\phi_n](x)$, which is of the form (the proof is left as an exercise: see Exercise 1.1.1 nr. 6)

$$W'[\phi_1,\ \phi_2,\ldots,\phi_n](x) = \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \cdots & \phi_n' \\ \vdots & \vdots & \cdots & \vdots \\ \phi_1^{(n-2)} & \phi_2^{(n-2)} & \cdots & \phi_n^{(n-2)} \\ \phi_1^{(n)} & \phi_2^{(n)} & \cdots & \phi_n^{(n)} \end{vmatrix}. \tag{1.3.1}$$

We consider n solutions, $\phi_1(x)$, $\phi_2(x)$, $\ldots$, $\phi_n(x)$ for the nth-order equation

$$p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0,$$

so that

$$\begin{aligned} \phi_1^{(n)} &= -\frac{p_{n-1}(x)}{p_n(x)}\,\phi_1^{(n-1)}(x) - \frac{p_{n-2}(x)}{p_n(x)}\,\phi_1^{(n-2)}(x) - \cdots - \frac{p_1(x)}{p_n(x)}\,\phi_1'(x) - \frac{p_0(x)}{p_n(x)}\,\phi_1 \\ &= -\sum_{k=1}^{n} \frac{p_{n-k}(x)}{p_n(x)}\,\phi_1^{(n-k)}. \end{aligned}$$

Also

$$\phi_2^{(n)} = -\sum_{k=1}^{n} \frac{p_{n-k}(x)}{p_n(x)}\,\phi_2^{(n-k)},\ \ldots,\ \phi_n^{(n)} = -\sum_{k=1}^{n} \frac{p_{n-k}(x)}{p_n(x)}\,\phi_n^{(n-k)}.$$

Substituting now the above values of $\phi_1^{(n)},\ \phi_2^{(n)},\ldots,\ \phi_n^{(n)}$ into the last row of the determinant of $W'[\phi_1,\ \phi_2,\ldots,\ \phi_n](x)$ in (1.3.1), we obtain

$$\begin{aligned} &W'[\phi_1,\ \phi_2,\ldots,\phi_n](x) \\ &= \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \cdots & \phi_n' \\ \vdots & \vdots & \cdots & \vdots \\ \phi_1^{(n-2)} & \phi_2^{(n-2)} & \cdots & \phi_n^{(n-2)} \\ -\sum_{k=1}^{n} \frac{p_{n-k}(x)}{p_n(x)}\,\phi_1^{(n-k)} & -\sum_{k=1}^{n} \frac{p_{n-k}(x)}{p_n(x)}\,\phi_2^{(n-k)} & \cdots & -\sum_{k=1}^{n} \frac{p_{n-k}(x)}{p_n(x)}\,\phi_n^{(n-k)} \end{vmatrix}. \end{aligned} \tag{1.3.2}$$

If we now multiply the first row, the second row,..., the $(n-1)$ row of the determinant (1.3), respectively, by

$$\frac{p_0(x)}{p_n(x)},\ \frac{p_1(x)}{p_n(x)},\ \ldots,\ \frac{p_{n-2}(x)}{p_n(x)},$$

and then add this result to the last row, we obtain

$$W'[\phi_1,\ \phi_2,\ldots,\phi_n](x)$$

$$= \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \cdots & \phi_n' \\ \vdots & \vdots & \cdots & \vdots \\ \phi_1^{(n-2)} & \phi_2^{(n-2)} & \cdots & \phi_n^{(n-2)} \\ -\frac{p_{n-1}(x)}{p_n(x)}\,\phi_1^{(n-1)} & -\frac{p_{n-1}(x)}{p_n(x)}\,\phi_2^{(n-1)} & \cdots & -\frac{p_{n-1}(x)}{p_n(x)}\,\phi_n^{(n-1)} \end{vmatrix}. \tag{1.3.3}$$

Factoring out $-\frac{p_{n-1}(x)}{p_n(x)}$ from the last row of the above determinant (1.3), we obtain

$$W'[\phi_1,\ \phi_2,\ \ldots,\ \phi_n](x) = -\frac{p_{n-1}(x)}{p_n(x)}\,W[\phi_1,\ \phi_2,\ \ldots,\ \phi_n](x). \tag{1.3.4}$$

The solution of (1.3.4) is either $W = 0$ for all $x \in \mathcal{D}$, or we have the general solution

$$W[\phi_1,\ \phi_2,\ \ldots,\ \phi_n](x) = c\ \exp\left[-\int \frac{p_{n-1}(x)}{p_n(x)}\,dx\right], \tag{1.3.5}$$

where c is an arbitrary nonzero constant of integration. Since

$$\exp\left[-\int \frac{p_{n-1}(x)}{p_n(x)}\,dx\right] \neq 0 \quad \text{for every } x \in \mathcal{D}$$

and $c \neq 0$, it follows that either $W \neq 0$ for all $x \in \mathcal{D}$ or $W = 0$ for all $x \in \mathcal{D}$ (which corresponds to the singular solution of (1.3.4). Thus the statement is established for arbitrary n. □

Chapter 2

First-order differential equations

2.1 Introduction: the initial-value problem

A first-order differential equation is of the following general form:

$$F(x, y(x), y'(x)) = 0. \tag{2.1.1}$$

In this section we introduce the so-called *initial-value problem* for first-order equations and then consider two types of first-order differential equations, namely the so-called *separable first-order equations* and the *linear first-order equation*. We also include some cases of first-order differential equations which can be written in the form of a separable differential equation or a first-order linear differential equation by introducing a new dependent variable.

The **initial-value problem** for first-order differential equations:

Let

$$y = \phi(x;\, c) \tag{2.1.2}$$

be a general solution of (2.1.1) on some interval $\mathcal{D} \subseteq \mathbb{R}$, where ϕ contains an arbitrary real constant c. The initial-value problem states the problem to find the solution curve of (2.1.1) which contains the point

$$y(x_0) = b, \tag{2.1.3}$$

where x_0 is a given point in the solution domain of the differential equation and b is a given real number. The relation (2.1.3) is known as the **initial data** and x_0 is the **initial value** for the solutions of the differential equation. If the point x_0 is in the solution domain of the given general solution $\phi(x;\, c)$ and the general solution satisfies the initial data, then the solution of the initial-value problem is obtained by solving the constant c from the algebraic relation

$$\phi(x_0,\, c) = b. \tag{2.1.4}$$

The solution of this initial-value problem is then given by the general solution of (2.1.1) where c is the explicit (unique) number that has been solved from the relation (2.1.4). If the given initial data cannot be satisfied by the given general solution $y = \phi(x;\, c_1)$, that is, if

$$\phi(x_0,\, c) \neq b \quad \text{for all } c \in \mathbb{R}, \tag{2.1.5}$$

then the initial data may be in the domain of a singular solution for (2.1.1), say

$$y(x) = \psi(x), \tag{2.1.6}$$

for which $\psi(x_0) = b$. In this case the solution to the initial-value problem is given by the singular solution $y(x) = \psi(x)$. If neither the general solution, nor any of the singular solutions for (2.1.1), satisfy the initial data $y(x_0) = b$, then we say that this initial data is **inconsistent** with the differential equation, which means that the differential equation does not contain the point $y(x_0) = b$ for any of its solutions.

Several examples to illustrate the initial-value problem for first-order differential equations are given in the sections that follow.

2.2 Separable first-order differential equations

A **separable first-order differential equation** is of the form

$$\boxed{\frac{dy}{dx} = g(x)h(y)} \tag{2.2.1}$$

where $g(x)$ and $h(y)$ are given continuous functions of their arguments.

The method to integrate (2.2.1) is as follows:

Divide (2.2.1) by $h(y)$, i.e. $\dfrac{1}{h(y)}\dfrac{dy}{dx} = g(x)$ and integrate then with respect to x:

$$\int \left(\frac{1}{h(y)}\frac{dy}{dx}\right) dx = \int g(x)\,dx + c, \tag{2.2.2}$$

where c is a constant of integration. We let $y(x) = p(x)$, so that $\dfrac{dy}{dx} = p'(x)$ (or $dy = p'(x)dx$, i.e. $\dfrac{dy}{dx}\,dx = dy$) and (2.2.2) takes the form

$$\boxed{\int \frac{1}{h(y)}\,dy = \int g(x)\,dx + c}. \tag{2.2.3}$$

Relation (2.2.3) is an **integral-solution formula** for (2.2.1). Since this formula contains one arbitrary constant it represents a general solution of (2.2.1).

Example 2.2.1.

Solve the initial-value problem for the equation

$$(1 + e^x)y' = e^x e^y, \tag{2.2.4}$$

with initial data $y(0) = 1$. Using the integral-solution formula (2.2.3) we have

$$\int e^y\,dy = \int \frac{e^x}{1+e^x}\,dx + c$$

and a general solution becomes

$$y(x) = \ln\left[\ln(1+e^x) + c\right] \quad \text{for all } c \in \mathbb{R}. \tag{2.2.5}$$

Use now the given initial data to solve c:

$$y(0) = \ln\left[\ln(1+e^0) + c\right] = 1 \tag{2.2.6}$$

so that $c = e - \ln(2)$ and the solution of the initial-value problem is

$$y(x) = \ln\left[\ln\left(\frac{1+e^x}{2}\right) + e\right]. \tag{2.2.7}$$

In some cases we can write a given first-order differential equation in the form of a separable first-order differential equation by introducing a new dependent variable.

Proposition 2.2.1. *The first-order equation*

$$y' = f\left(\frac{y}{x}\right), \qquad x \in \mathbb{R}\backslash\{0\} \tag{2.2.8}$$

where f is a continuous function of the argument y/x, reduces to the separable equation

$$v' = \frac{1}{x}\left(f(v) - v\right) \tag{2.2.9}$$

by the substitution

$$y(x) = x\,v(x) \tag{2.2.10}$$

Proof: Differentiating (2.2.10) with respect to x we have

$$y'(x) = v(x) + xv'(x)$$

so that (2.2.8) becomes $v(x) + xv'(x) = f(v(x))$, i.e., the equation in v is of the form (2.2.9) and is therefore separable in the variables x and $v(x)$. □

Example 2.2.2.

We find a general solution of

$$xy\,y' = x^2 - xy + y^2$$

for all $x \in \mathbb{R}\backslash\{0\}$ and then solve the initial-value problem $y(1/2) = 0$.

The equation can equivalently be written in the form

$$y' = \frac{x}{y} - 1 + \frac{y}{x}.$$

By the substitution (2.2.10), i.e. $y = xv(x)$, this equation reduces to

$$v' = \frac{1}{x}\left(\frac{1-v}{v}\right)$$

which is a separable first-order equation in the variables x and $v \neq 1$. Thus

$$\int \frac{v}{1-v}\,dv = \int \frac{1}{x}\,dx + \ln|c| \quad \text{or} \quad v + \ln|1-v| = -\ln|cx|.$$

A general solution of the given equation is now in the following implicit form:

$$\frac{y}{x} + \ln\left|1 - \frac{y}{x}\right| = -\ln|cx|,$$

where c is the constant of integration. Applying the initial data $y(1/2) = 0$, we have

$$0 + \ln(1) = -\ln\left|\frac{c}{2}\right|$$

so that $c = 2$. The solution of the initial-value problem is thus

$$\frac{y}{x} + \ln\left|1 - \frac{y}{x}\right| = -\ln|2x|.$$

For the case $v = 1$, we obtain the singular solution $y = x$.

2.2.1 Exercises

[Solutions of those Exercises marked with a * are given in Appendix D].

1. Find general solutions of the following differential equations:

   a) $y' = e^{x+y}$

   b) $x + \dfrac{1+x^2}{1+2y}y' = 0$

c) $xy^2 + x + (y - x^2 y)y' = 0$

d) $xy = (3 - x)y'$, $x \neq 3$.

e)* $y' + \dfrac{1-y^2}{1-x^2} = 0, \quad x > 1$

f) $y' + \dfrac{1-y^2}{1+x^2} = 0$

g) $y' + \dfrac{(y-1)x^2}{(x-1)^2} = 0, \quad x \neq 1$

h) $x\sqrt{a^2 - y^2} - (x^4 + 1)(1 + y)\,y' = 0, \quad a \neq 0$

i) $x^2 + y^3\,e^{x+y}\,y' = 0$

2. Solve the following initial-value problems:

a) $y' - \dfrac{1+y}{1-x^2} = 0, \quad y(0) = 1$

b) $y' - \dfrac{1+y^2}{1+x^2} = 0, \quad y(0) = 1$

c) $e^y y' = x, \quad y(0) = 0$

d) $e^y(y' + 1) = 1, \quad y(0) = 0$

e) $x \sin y = y'(1 + x^2)\cos y, \quad y(1) = \dfrac{\pi}{4}$

3. Use the substitution $y(x) = xv(x)$ to find a general solution of the following differential equations:

a) $y + xe^{(y/x)} - xy' = 0$

b)* $y^2 - x^2 + xyy' = 0$

c) $x - 2y + yy' = 0$

4. Solve the following initial-value problems using the substitution $y(x) = xv(x)$:

a) $y' = \dfrac{y^2 + xy}{x^2}, \quad y(1) = 1$

b) $y' = \dfrac{4y^2 + 3xy}{x^2}, \quad y(2) = 1$

2.3 Linear first-order differential equations

The **linear first-order differential equation** is of the form

$$\boxed{y' + g(x)y = h(x)} \tag{2.3.1}$$

where g and h are continuous functions on an interval $\mathcal{D} \subseteq \mathbb{R}$. A general solution can then be given in terms of an integral-solution formula:

Proposition 2.3.1. *A general solution of (2.3.1), i.e. equation*

$$y' + g(x)y = h(x),$$

is

$$y(x) = e^{-G(x)} \left[\int h(x)e^{G(x)}dx + c \right], \tag{2.3.2}$$

where c is an arbitrary constant and $G(x)$ is an anti-derivative of $g(x)$, i.e.

$$G(x) = \int g(x)dx. \tag{2.3.3}$$

Proof: We now prove this proposition in two ways, which provides two methods to solve the linear equation (2.3.1).

The method of integrating factors:

Multiplying

$$y' + g(x)y = h(x)$$

by the expression $e^{G(x)}$, where $G(x) = \int g(x)\,dx$, we obtain

$$y'e^{G(x)} + g(x)ye^{G(x)} = h(x)e^{G(x)} \quad \text{or equivalently}$$

$$\frac{d}{dx}\left(ye^{G(x)}\right) = h(x)e^{G(x)}.$$

The factor $e^{G(x)}$ is known as an integrating factor (see Definition 2.3.1 below). Integrating the previous relation over x, we obtain

$$ye^{G(x)} = \int h(x)e^{G(x)}\,dx + c,$$

where c is a constant of integration. Since $e^{G(x)} \neq 0$ for any $x \in \mathbb{R}$, we can divide by this term to obtain the integral formula (2.3.2). $\square_1$

The method of variation of constants:

To find a general solution for

$$y' + g(x)y = h(x)$$

we first consider the homogeneous equation

$$y' + g(x)y = 0, \tag{2.3.4}$$

which is a separable first-order equation. We can now integrate (2.3.4) for $y \neq 0$, to obtain

$$\int \frac{dy}{y} = -\int g(x)\,dx + \ln|k| \quad \text{so that}$$

$$y(x) = k\,e^{-\int g(x)\,dx}, \tag{2.3.5}$$

where k is an arbitrary nonzero constant. Consider now the variation of the constant k, namely we consider k as a function of x, i.e.

$$k = k(x).$$

We then insert

$$y(x) = k(x)\,e^{-\int g(x)\,dx} \tag{2.3.6}$$

and its derivative,

$$y'(x) = k'(x)\,e^{-\int g(x)\,dx} - k(x)g(x)\,e^{-\int g(x)\,dx},$$

in the full first-order linear equation

$$y' + g(x)y = h(x)$$

and obtain

$$k'(x)\,e^{-\int g(x)\,dx} = h(x).$$

Integrating the above expression over x, we get

$$k(x) = \int e^{\int g(x)\,dx} h(x)\,dx + c, \tag{2.3.7}$$

where c is a constant of integration. Inserting $k(x)$ given by (2.3.7) into (2.3.6), we obtain the integral formula (2.3.2), namely

$$y(x) = e^{-G(x)} \left[\int h(x) e^{G(x)} dx + c \right],$$

where $G(x) = \int g(x)\,dx$. $\square_2$

Definition 2.3.1. *The factor* $e^{G(x)}$*, where*

$$G(x) = \int g(x)dx,$$

which results in the integration of the linear equation (2.3.1), is known as the **integrating factor** *of the linear equation (2.3.1).*

Remark: *A remark regarding Integrating Factors is in order: By the general theory of ordinary differential equations it is known that for any differential equation that can be integrated in closed form at least once (i.e. a differential equation of order* n *for which the order can be reduced by integration), there exists some integrating factor that brings the equation to such an integrable form. To find an integrating factor for a given differential equation is in general a difficult problem (depending strongly on the type of differential equation). In the case of first-order linear differential equations, the integrating factor,* $\exp[\int g(x)\,dx]$*, reduces the first-order equation to a zero-order differential equation, i.e. we obtain a relation between* $y(x)$ *and* x *and an arbitrary constant* c *from the integration; hence we obtain a general solution of the first-order differential equation. It is a difficult problem to find integrating factors for nonlinear first-order differential equations, since the problem is in general under-determined (in this sense first-order differential equations are more complex than higher-order differential equations, which is due to the geometry or symmetry properties of these differential equations).*

Example 2.3.1.

Consider the linear first-order equation

$$xy' = -(2x+1)y + xe^{-2x}, \qquad x > 0.$$

We find a general solution and solve the initial-value problem $y(1) = 2$.

Dividing the given equation by x, it takes the following form:

$$y' + \left(\frac{2x+1}{x}\right) y = e^{-2x}. \tag{2.3.8}$$

The integrating factor for (2.3.8) is $e^{G(x)}$, where

$$G(x) = \int \frac{2x+1}{x}\, dx = 2x + \ln x.$$

Thus

$$e^{G(x)} = xe^{2x}$$

and upon multiplying (2.3.8) with this integrating factor we have

$$xe^{2x}y' + e^{2x}(2x+1)y = x \quad \text{or} \quad \frac{d}{dx}\left(xe^{2x}y\right) = x.$$

Integrating the last expression we obtain

$$xe^{2x}y = \frac{1}{2}x^2 + c,$$

where c is the constant of integration. Thus the general solution is

$$y(x) = x^{-1}e^{-2x}\left(\frac{x^2}{2} + c\right). \tag{2.3.9}$$

We can now solve c for the given initial data $y(1) = 2$ to obtain

$$y(1) = e^{-2}\left(\frac{1}{2} + c\right) = 2, \quad \text{so that} \quad c = 2e^2 - \frac{1}{2}.$$

The solution of the initial-value problem is therefore

$$y(x) = x^{-1}e^{-2x}\left(\frac{x^2}{2} + 2e^2 - \frac{1}{2}\right) \quad \text{for all } x > 0.$$

2.3.1 Exercises

[Solutions of those Exercises marked with a * are given in Appendix D].

1. Find general solutions of the following differential equations.

   a) $y' - y = 2x - x^2$

   b) $y' + y - \sin x = 0$

   c) $y' + x^2 y + x^2 = 0$

   d)* $y' + y + \sin x + x^3 = 0$

   e) $y' + 3y = x^2 + 1$

   f) $y' + y \cos x = e^{-\sin x}$

   g) $y' - \dfrac{xy}{x^2+1} = \left(\dfrac{x^2 - x + 1}{x^2+1}\right) e^x$

2. Solve the following initial-value problems.

   a) $y' - y = x - 1, \quad y(0) = 1$

   b) $y' + xy = x^3, \quad y(0) = -2$

   c)* $xy' + y = x \cos x, \quad y(\pi/2) = 1$

   d) $x \ln x \, y' + y = 2 \ln x, \quad y(e) = 0$

3. Assume that $y_1(x)$ and $y_2(x)$ are two solutions for

   $$y' + g(x)y = h(x)$$

   on some interval $\mathcal{D} \subseteq \mathbb{R}$, where $y_1(x) \neq y_2(x)$. Find a formula for a general solution using these two solutions, without performing any integration.

2.4 Some linearizable first-order equations

2.4.1 A rather general case

Consider the following

Proposition 2.4.1. *The first-order nonlinear equation*

$$\frac{df(y)}{dy}\frac{dy}{dx} + f(y)P(x) = Q(x) \tag{2.4.1}$$

where $f(y)$ *is any differentiable function of* y *and* P *and* Q *are continuous functions of* x *on some domain* $\mathcal{D} \subseteq \mathbb{R}$*, can be linearized in*

$$\frac{dv}{dx} + P(x)v = Q(x). \tag{2.4.2}$$

by the following substitution:

$$v(x) = f(y(x)) \tag{2.4.3}$$

Proof: The first derivative of $v(x)$, given by the substitution (2.4.3), is

$$\frac{dv}{dx} = \frac{df}{dy}\frac{dy}{dx}, \tag{2.4.4}$$

so that (2.4.1) takes the form (2.4.2) in terms of the new dependent variable $v(x)$. A general (possibly implicit) solution of (2.4.2) then leads to a general solution for (2.4.1) by the relation (2.4.3). □

Example 2.4.1.

We linearize the equation

$$\frac{dy}{dx} + 1 = 4e^{-y}\sin x. \tag{2.4.5}$$

An equivalent form of (2.4.5) is

$$e^y\frac{dy}{dx} + e^y = 4\sin x$$

so that, following Proposition 2.4.1, a suitable new dependent variable is

$$v(x) = e^y, \qquad \text{with} \qquad \frac{dv}{dx} = e^y\frac{dy}{dx}.$$

Equation (2.4.5) then takes the linear form

$$\frac{dv}{dx} + v = 4\sin x. \tag{2.4.6}$$

A general solution of (2.4.6) is

$$v(x) = 2(\sin x - \cos x) + ce^{-x}.$$

Thus a general solution for (2.4.5) is then

$$y(x) = \ln v = \ln\left(2(\sin x - \cos x) + ce^{-x}\right).$$

2.4.2 The Bernoulli equation

The **Bernoulli equation** is an important special case of (2.4.1), namely

$$\boxed{\frac{dy}{dx} = f(x)y + g(x)y^n, \qquad n \in \mathbb{R}\backslash\{0,1\}} \tag{2.4.7}$$

Here $f(x)$ and $g(x)$ are any given continuous functions on some domain $\mathcal{D}$.

Remark: *Note that $y = 0$ is always a solution of (2.4.7). Moreover, if $y(x) = \phi(x)$ is a solution of (2.4.7), then $y(x) = -\phi(x)$ is also a solution of (2.4.7) if and only if the equation admits the discrete symmetry $y \mapsto -y$, $x \mapsto x$ for all $x \in \mathcal{D}$ (see Example 2.4.2 below).*

Proposition 2.4.2. *The Bernoulli equation (2.4.7) can be linearized for all* $n \in \mathbb{R}\backslash\{0,1\}$ *in terms of a new dependent variable* $v(x)$*, by the substitution*

$$v(x) = y^{1-n}(x). \tag{2.4.8}$$

Proof: Assume that $y \neq 0$. Multiplying equation (2.4.7) by y^{-n}. Then the equation takes the form

$$y^{-n}\frac{dy}{dx} - f(x)y^{1-n} = g(x). \tag{2.4.9}$$

By comparing (2.4.9) with (2.4.1) we note that (2.4.7) is linearizable in terms of the new dependent variable $v(x)$, where

$$v(x) = y^{1-n}(x) \qquad \text{so that} \qquad \frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}. \tag{2.4.10}$$

In terms of the dependent variable $v(x)$, (2.4.7) then takes the linear form

$$\frac{dv}{dx} - (1-n)f(x)v = (1-n)g(x) \tag{2.4.11}$$

□

Historical Note: (source: Wikipedia)
The Bernoulli equation is named after Jacob Bernoulli (1654 – 1705), who described this equation in 1695. Jacob Bernoulli, born in Basel, Switzerland, was one of the prominent mathematicians in the Bernoulli family. He is known for his numerous contributions to calculus and along with his brother Johann, was one of the founders of the calculus of variations.

Jacob Bernoulli (1654 – 1705)

Example 2.4.2.

We find a general solution for the following first-order equation:

$$y' + y = xy^3. \tag{2.4.12}$$

We recognize that (2.4.12) is a Bernoulli equation of the form (2.4.7) with $n = 3$, $f(x) = -1$ and $g(x) = x$. We therefore introduce a new dependent variable $v(x)$ as $v(x) = y^{-2}(x)$. Now

$$v' = -2y^{-3}y'$$

and the equation in $v(x)$ takes the linear form

$$v' - 2v = -2x.$$

Solving this linear equation we obtain

$$v(x) = x + \frac{1}{2} + ce^{2x},$$

where c is an arbitrary constant and, since $y(x) = v^{-1/2}(x)$, a general solution of (2.4.12) is

$$y(x) = \left(x + \frac{1}{2} + ce^{2x}\right)^{-1/2}.$$

As pointed out in the above Remark, $y = 0$ is also a solution and, since (2.4.12) admits the symmetry $y \mapsto -y$, $x \mapsto x$ for all $x \in \mathbb{R}$, another nontrivial solution of (2.4.12) is

$$y(x) = -\left(x + \frac{1}{2} + ce^{2x}\right)^{-1/2}.$$

2.4.3 The Riccati equation

The **Riccati equation** is of the form

$$\boxed{\frac{dy}{dx} = f(x)y^2 + g(x)y + h(x)} \tag{2.4.13}$$

where f, g and h are any given continuous functions on some domain $\mathcal{D} \subseteq \mathbb{R}$. One of the remarkable properties of the Riccati equation is that it can be linearized in a first-order differential equation if any solution of (2.4.13) is known. In particular, we assume that $\phi(x)$ is a solution of (2.4.13) and introduce a new dependent variable $z(x)$ as follows:

$$y(x) = \phi(x) + z(x) \qquad \text{with} \qquad \frac{dy}{dx} = \frac{d\phi}{dx} + \frac{dz}{dx} \tag{2.4.14}$$

Since $\phi(x)$ is assumed to be a solution of (2.4.13), i.e.

$$\frac{d\phi}{dx} = f(x)\phi^2 + g(x)\phi + h(x), \tag{2.4.15}$$

we obtain, with (2.4.14), the following equation in the dependent variable z:

$$\frac{dz}{dx} = [2\phi f(x) + g(x)]\, z + f(x)z^2. \tag{2.4.16}$$

We recognize (2.4.16) as a special Bernoulli equation, (2.4.7), which can be linearized by introducing a new dependent variable $v(x)$ as follows:

$$z(x) = \frac{1}{v(x)} \qquad \text{with} \qquad \frac{dz}{dx} = -\frac{1}{v(x)^2}\frac{dv}{dx}. \tag{2.4.17}$$

Inserting (2.4.17) in (2.4.16) we obtain the linear equation

$$\frac{dv}{dx} + [2\phi(x)f(x) + g(x)]\, v = -f(x). \tag{2.4.18}$$

This leads to

Proposition 2.4.3. *The Riccati equation*

$$\frac{dy}{dx} = f(x)y^2 + g(x)y + h(x)$$

can be linearized in the first-order linear equation

$$\frac{dv}{dx} + [2\phi(x)f(x) + g(x)]\, v = -f(x) \tag{2.4.19}$$

by the following change of dependent variable

$$y(x) = \phi(x) + \frac{1}{v(x)}$$

where ϕ is any solution of the Riccati equation and v satisfies the linear equation (2.4.19).

Regarding singular solutions of the Riccati equation, we have the following

Proposition 2.4.4. *Consider the Riccati equation (2.4.13), i.e.*

$$\frac{dy}{dx} = f(x)y^2 + g(x)y + h(x)$$

with general solution of the form

$$y(x;\, c) = \phi(x) + \frac{1}{v(x;\, c)},$$

where $v(x;\, c)$ is a general solution of the linear equation

$$\frac{dv}{dx} + [2\phi(x)f(x) + g(x)]\, v = -f(x)$$

and $\phi(x)$ is a special solution of the Riccati equation. Then the special solution

$$y(x) = \phi(x)$$

is a singular solution with respect to the initial data $y(x_0) = \phi(x_0)$ for every x_0 in the solution domain of the Riccati equation. The solution of this initial-value problem is then given by the singular solution, $y(x) = \phi(x)$.

Historical Note: (source: Wikipedia)
The Riccati equation is named after the Italian mathematician Jacopo Francesco Riccati (1676–1754), who was born in Venice. Riccati received various academic offers, amongst those was an invitation by Peter the Great of Russia for president of the St. Petersburg

Academy of Sciences as well as some professorships, but he declined all offers in order to devote his full attention to the study of mathematical analysis.

Jacopo Francesco Riccati (1676–1754)

Example 2.4.3.

We find a general solution of the Riccati equation

$$y' + y^2 = \left(2x + \frac{1}{x}\right) y - x^2, \qquad (2.4.20)$$

where $\phi(x) = x$ is a special solution for this equation. We then solve two initial-value problems: i) we use the initial data $y(1) = 2$ and ii) the initial data $y(1) = 1$.

As stated in Proposition 2.4.3, we make a change of the dependent variable

$$y(x) = x + \frac{1}{v(x)}. \qquad (2.4.21)$$

That is $y' = 1 - v^{-2}v'$, so that (2.4.20) takes the linear form

$$v' + \frac{v}{x} = 1,$$

which admits the general solution

$$v(x) = \frac{1}{2}x + \frac{c}{x},$$

where c is a constant of integration. Inserting the obtained expression for $v(x)$ into the relation (2.4.21) we obtain a general solution for (2.4.20) in the form

$$y(x) = \frac{x(x^2 + 2c + 2)}{x^2 + 2c} \quad \text{for all } x \in \mathbb{R}\backslash\{0\}. \qquad (2.4.22)$$

i) Using the initial data $y(1) = 2$ we obtain, from the above general solution,

$$y(1) = \frac{1(1^2 + 2c + 2)}{1^2 + 2c} = 2 \quad \text{or } c = \frac{1}{2}.$$

The solution for this initial-value problem is therefore

$$y(x) = \frac{x(x^2 + 3)}{x^2 + 1}.$$

ii) It is clear that the singular solution $y(x) = x$ passes through the point of the initial data, $y(1) = 1$. Therefore, $y(x) = x$ is the solution for this initial-value problem in this case. Note that, if we use the general solution (2.4.22) for this initial data we obtain a contradiction:

$$y(1) = \frac{3 + 2c}{1 + 2c} = 1 \quad \text{or } 3 = 1$$

as stated in Proposition 2.4.4.

2.4.4 Exercises

[Solutions of those Exercises marked with a * are given in Appendix D].

1. Find a general solution of the following nonlinear differential equation by a suitable linearization:

$$\sin y \frac{dy}{dx} = \cos x \left(2\cos y - \sin^2 x\right).$$

Hint:
a) See Proposition 2.4.1.
b) For the integral $\int \cos x \, \sin^2 x \, e^{2\sin x} \, dx$ use the substitution $t = \sin x$.

2. Find general solutions of the following Bernoulli equations:

a) $y' - y = x^2 y^3$

b) $y' + \frac{2}{x} y = e^x \sqrt{y}$

c)* $xy' + y = y^2 \ln(x), \quad x > 0$

d) $2xyy' + x = y^2$

e) $y' + y + y^2 \sin x = 0$

f) $y' - \frac{y}{2x} + \frac{y^3}{2\sqrt{1-x^2}} = 0, \quad -1 < x < 1, \; x \neq 0$

g)* $y' + \frac{xy}{1-x^2} = x\sqrt{y}, \quad x > 1$

3. Solve the following initial-value problems:

a) $y' + 2xy = 2xy^2$. Consider two cases:

i) $y(0) = 2$
ii) $y(0) = 1$

b) $y' + \left(\frac{3}{x}\right) y = xy^{1/3}, \; y(1) = 3$

4. Linearize the following Riccati equations and find their general solutions:

a) $y' = -\frac{y^2}{4} - \frac{1}{x^2}$. A special solution is $\phi(x) = \frac{2}{x}$.

b) $y' = y^2 - (2x+1)\,y + x^2 + x + 1$. A special solution is $\phi(x) = x$.

5. Solve the initial-value problem of the following Riccati equations:

a) $y' = y^2 - 2xy + x^2 + 1,\ \ y(0) = \dfrac{1}{2}$. A special solution is $\phi(x) = x$.

b) $y' = y^2 - \left(\dfrac{1}{x}\right) y - \dfrac{1}{x^2},\ \ y(1) = 2$. A special solution is $\phi(x) = \dfrac{1}{x}$.

c) $y' = y^2 - \left(\dfrac{1}{x}\right) y - \dfrac{1}{x^2},\ \ y(1) = 1$. A special solution is $\phi(x) = \dfrac{1}{x}$.

6. * Consider the Riccati equation

$$(x - x^4)y' - x^2 - y + 2xy^2 = 0, \quad x \in \mathbb{R}\backslash[0,\ 1]. \tag{2.4.23}$$

Find a value of the constant k, such that $\phi(x) = kx^2$ is a special solution of this equation. Solve then the initial-value problem, where $y(2) = 1$.

7. Consider the Riccati equation

$$x^2y' - x^2y^2 + 5xy - 3 = 0.$$

Find a value of the constant k, such that $\phi(x) = x^k$ is a special solution of this equation. Solve then the initial-value problem for two initial data. i) $y(1) = 2$ and ii) $y(2) = 1/2$.

8. Consider the Riccati equation

$$x^2y' - x^2y^2 = xy + 1.$$

Find a value of the constant k, such that $\phi(x) = \dfrac{k}{x}$ is a special solution for this equation and find then a general solution of this equation.

9. Show that the general Riccati equation (2.4.13) **linearizes** to a second-order linear equation in terms of the dependent variable $w(x)$ given by the following relation:

$$y(x) = -\frac{w'(x)}{w(x)}\frac{1}{f(x)}. \tag{2.4.24}$$

10. It can be shown that any Riccati equation (2.4.13) admits the following **nonlinear superposition formula**:

$$y(x) = \frac{c[y_1(x) - y_2(x)]y_3(x) - [y_1(x) - y_3(x)]y_2(x)}{c[y_1(x) - y_2(x)] - [y_1(x) - y_3(x)]}, \tag{2.4.25}$$

where $y_1(x)$, $y_2(x)$ and $y_3(x)$ are any distinct solutions of the Riccati equation (2.4.13) and c is an arbitrary constant. Since c is an arbitrary constant in (2.4.25), this superposition formula (2.4.25) provides a general solution to the Riccati equation for three given distinct solutions.

Using this superposition formula, find a general solution of

$$y' = -y^2 + \left(2x + \frac{1}{x}\right) y - x^2,$$

which admits the following three solutions:

$$y_1(x) = x, \quad y_2(x) = \frac{x^3}{x^2 - 2}, \quad y_3(x) = \frac{x^3 + x}{x^2 - 1}.$$

Chapter 3

Second-order linear differential equations

3.1 Introduction: the initial- and boundary-value problem

A second-order differential equation is of the general form

$$F(x, y(x), y'(x), y''(x)) = 0 \tag{3.1.1}$$

and its general solution is a function $\phi \in \mathcal{C}^2(\mathcal{D})$ which contains two arbitrary constants, c_1 and c_2, and satisfies the differential equation. We write

$$y(x) = \phi(x;\, c_1,\, c_2). \tag{3.1.2}$$

The **initial value problem** requires the following initial data in a point x_0 in the solution domain of the equation:

$$y(x_0) = b_0, \qquad y'(x_0) = b_1, \tag{3.1.3}$$

where b_0 and b_1 are given real numbers. This data is then used to fix the constants c_1 and c_2 in the general solution (if the initial data is within the domain of the general solution and this data can be satisfied by the general solution) by solving the (nonlinear) algebraic system of equations

$$y(x_0) = \phi(x_0;\, c_1,\, c_2) = b_0, \quad y'(x_0) = \left.\frac{d\phi(x;\, c_1,\, c_2)}{dx}\right|_{x=x_0} = b_1. \tag{3.1.4}$$

If the differential equation (3.1.1) is linear, then the algebraic system (3.1.4) is a system of two linear algebraic equations in c_1 and c_2. This is clear since the constants c_1 and c_2 appear as weights in the linear combination of linearly independent solutions.

For the so-called **boundary-value problem** we require **boundary data** in two points, x_1 and x_2, in the equations' solution domain, namely

$$y(x_1) = b_1, \qquad y(x_2) = b_2,$$

where b_1 and b_2 are given real numbers. If this boundary data lies within the domain of the general solution and can be satisfied by the general solution, then the constants c_1 and c_2 can be fixed by solving the (nonlinear) algebraic system

$$y(x_1) = \phi(x_1;\, c_1,\, c_2) = b_1, \quad y(x_2) = \phi(x_2;\, c_1,\, c_2) = b_2. \tag{3.1.5}$$

If the differential equation is linear, then the algebraic system (3.1.5) is also a linear system in c_1 and c_2.

Example of initial-value problems and boundary-value problems are given int the sections that follow.

3.2 Second-order linear homogeneous equations with constant coefficients

Consider the equation

$$\boxed{y'' + py' + qy = 0} \tag{3.2.1}$$

where p and q are real constants. To find the general solution of (3.2.1) we make use of the Ansatz

$$\boxed{y(x) = e^{\lambda x}} \tag{3.2.2}$$

where λ is in general a complex number that needs to be determined such that (3.2.2) satisfies equation (3.2.1). Inserting the Ansatz (3.2.2) and its derivatives

$$y' = \lambda e^{\lambda x}, \quad y'' = \lambda^2 e^{\lambda x}$$

in (3.2.1), we obtain

$$\left(\lambda^2 + p\lambda + q\right) e^{\lambda x} = 0.$$

Since $e^{\lambda x} \neq 0$ for all complex λ and all real x, we remain with the condition

$$\boxed{\lambda^2 + p\lambda + q = 0} \tag{3.2.3}$$

which is called the **characteristic equation** (or auxiliary equation) of (3.2.1). The form of the solution of (3.2.1) depends on the algebraic solution of (3.2.3) and hence on the values of p and q. The cases are given in the following

Proposition 3.2.1. *Consider equation (3.2.1), i.e.*

$$y'' + py' + qy = 0,$$

where p and q are real constants. Let λ_1 and λ_2 denote the roots of the characteristic equation (3.2.3), i.e. $\lambda^2 + p\lambda + q = 0$.

a) If λ_1 and λ_2 are real and distinct roots of (3.2.3), which is the case when $\boxed{p^2 > 4q}$, then the general solution of (3.2.1) is given by

$$\boxed{y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \quad \textit{for all } x \in \mathbb{R}} \tag{3.2.4}$$

where c_1 and c_2 are arbitrary constants and

$$\lambda_1 = \frac{1}{2}\left(-p + \sqrt{p^2 - 4q}\right) \in \mathbb{R}, \qquad \lambda_2 = \frac{1}{2}\left(-p - \sqrt{p^2 - 4q}\right) \in \mathbb{R}.$$

b) If λ_1 and λ_2 are real and equal roots of (3.2.3), which is the case when $\boxed{p^2 = 4q}$, then the general solution of (3.2.1) is given by

$$\boxed{y(x) = (c_1 + c_2 x)\, e^{\lambda_1 x} \quad \textit{for all } x \in \mathbb{R}} \tag{3.2.5}$$

where c_1 and c_2 are arbitrary constants and $\lambda_1 = \lambda_2 = -\dfrac{p}{2} \in \mathbb{R}$.

c) If λ_1 and λ_2 are complex roots of (3.2.3), which is the case when $\boxed{p^2 < 4q}$, then the general solution of (3.2.1) is given by

$$\boxed{y(x) = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x) \equiv c_1\, Re\left\{e^{\lambda_1 x}\right\} + c_2\, Im\left\{e^{\lambda_1 x}\right\}} \tag{3.2.6}$$

for all $x \in \mathbb{R}$, where c_1 and c_2 are arbitrary constants and

$$\alpha = -\frac{p}{2} \in \mathbb{R}, \qquad \beta = \frac{1}{2}\sqrt{4q - p^2} \in \mathbb{R}.$$

Here the complex roots of (3.2.3) are $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$.

Proof: We consider the three different cases, which is a result of the three different types of solutions of the characteristic equation (3.2.3), i.e., $\lambda^2 + p\lambda + q = 0$.

Case a: Let $p^2 > 4q$. Then the characteristic equation (3.2.3) has two distinct real roots, namely

$$\lambda_1 = \frac{1}{2}\left(-p + \sqrt{p^2 - 4q}\right) \in \mathbb{R}, \qquad \lambda_2 = \frac{1}{2}\left(-p - \sqrt{p^2 - 4q}\right) \in \mathbb{R}.$$

Thus, by the Ansatz $y(x) = e^{\lambda x}$, the two solutions for (3.2.1) are

$$\phi_1(x) = e^{\lambda_1 x}, \quad \phi_2(x) = e^{\lambda_2 x}.$$

The Wronskian for these two solutions in the point $x = 0$ is $W[\phi_1,\ \phi_2](0) = \lambda_2 - \lambda_1 \neq 0$, so that $\{\phi_1(x),\ \phi_2(x)\}$ is a linearly independent set in the vector space $\mathcal{C}^\infty(\mathbb{R})$. Thus the general solution of (3.2.1) is a linear combination of these two solutions:

$$y(x) = c_1\, e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \quad \text{for all } x \in \mathbb{R}. \tag{3.2.7}$$

Case b: Let $p^2 = 4q$. Then the characteristic equation (3.2.3) has one real solution (or twice the same real solution), namely

$$\lambda_1 = \lambda_2 = -\frac{p}{2}.$$

This leads to only one real solution for (3.2.1)

$$\phi(x) = e^{-(p/2)x}.$$

We now find the general solution for (3.2.1) by the Ansatz

$$y(x) = w(x)e^{-(p/2)x}, \tag{3.2.8}$$

where $w(x)$ is a twice differentiable function that needs to be determined such that the Ansatz satisfies (3.2.1). Differentiating the Ansatz (3.2.8) twice, we obtain

$$y'(x) = e^{-(p/2)x}\left[w' - \left(\frac{p}{2}\right)w\right] \tag{3.2.9a}$$

$$y''(x) = e^{-(p/2)x}\left[w'' - pw' + \left(\frac{p^2}{4}\right)w\right]. \tag{3.2.9b}$$

Inserting (3.2.8), (3.2.9a) and (3.2.9b) in the differential equation (3.2.1), we obtain

$$e^{-(p/2)x}\left\{w'' - pw' + \left(\frac{p^2}{4}\right)w + p\left[w' - \left(\frac{p}{2}\right)w\right] + qw\right\} = 0.$$

Since $e^{-(p/2)x} \neq 0$ for all $x \in \mathbb{R}$ and $p^2 = 4q$, the previous expression reduces to

$$w''(x) = 0. \tag{3.2.10}$$

Integrating (3.2.10) twice over x, we obtain

$$w(x) = c_1 x + c_2, \tag{3.2.11}$$

where c_1 and c_2 are constants of integration. Thus the Ansatz (3.2.8) leads to the following solution for (3.2.1):

$$y(x) = (c_1 x + c_2)e^{-(p/2)x}. \tag{3.2.12}$$

The set $\{\psi_1(x) = x\,e^{-(p/2)x},\ \psi_2(x) = e^{-(p/2)x}\}$ is linearly independent on $\mathcal{D}$, since $W[\psi_1(x), \psi_2(x)](0) = -1$ in $\mathcal{C}^\infty(\mathbb{R})$. Therefore (3.2.12) is the general solution of (3.2.1).

Case c: Let $p^2 < 4q$. Then the characteristic equation (3.2.3) has two distinct complex solutions, namely

$$\lambda_1 = \frac{1}{2}\left(-p + \sqrt{p^2 - 4q}\right) = \frac{1}{2}\left(-p + i\sqrt{4q - p^2}\right)$$

$$\lambda_2 = \frac{1}{2}\left(-p - \sqrt{p^2 - 4q}\right) = \frac{1}{2}\left(-p - i\sqrt{4q - p^2}\right)$$

We set

$$\alpha = -\frac{p}{2}, \quad \beta = \frac{1}{2}\sqrt{4q - p^2},$$

so that the two solutions λ_1 and λ_2 take the form

$$\lambda_1 = \alpha + i\beta, \quad \lambda_2 = \alpha - i\beta. \tag{3.2.13}$$

Using λ_1 we obtain the complex solution $\phi_c(x)$ for (3.2.3), namely

$$\phi_c(x) = e^{(\alpha+i\beta)x} = e^{\alpha x}e^{i\beta x} = e^{\alpha x}\left[\cos(\beta x) + i\sin(\beta x)\right]. \tag{3.2.14}$$

We note that the solution $\phi_c(x)$ is a linear combination of two functions,

$$\psi_1(x) = e^{\alpha x}\cos(\beta x) \text{ and } \psi_2(x) = e^{\alpha x}\sin(\beta x).$$

By the linear superposition principle, it follows that $\psi_1(x)$ and $\psi_2(x)$ must also be solutions of (3.2.3). This can also be verified directly for the equation (3.2.3) by showing that $\psi_1(x)$ and $\psi_(x)$ satisfy (3.2.3). Moreover,

$$W[\psi_1,\ \psi_2](0) = \begin{vmatrix} \psi_1 & \psi_2 \\ \psi_1' & \psi_2' \end{vmatrix}(0) = \begin{vmatrix} 1 & 0 \\ \alpha & \beta \end{vmatrix} = \beta \neq 0.$$

Hence $\{\psi(x),\ \psi_2(x)\}$ is a linearly independent set in $\mathcal{C}^\infty(\mathbb{R})$ and therefore the general solution of (3.2.1) is

$$y(x) = c_1e^{\alpha x}\cos(\beta x) + c_2e^{\alpha x}\sin(\beta x) \equiv c_1\,\mathrm{Re}\left\{e^{\lambda_1 x}\right\} + c_2\,\mathrm{Im}\left\{e^{\lambda_1 x}\right\}. \qquad \square$$

Example 3.2.1.

a) We find the general solution for

$$y'' - 4y' + 13y = 0 \tag{3.2.15}$$

for all $x \in \mathbb{R}$, and then solve the initial-value problem with the initial data

$$y(0) = 4, \qquad y'(0) = -1 \tag{3.2.16}$$

as well as the boundary-value problem with boundary data

$$y(0) = 1, \qquad y(\pi/6) = 2. \tag{3.2.17}$$

With the Ansatz $y(x) = e^{\lambda x}$, the characteristic equation becomes

$$\lambda^2 - 4\lambda + 13 = 0,$$

which admits two complex roots

$$\lambda_1 = 2 + 3i, \qquad \lambda_2 = 2 - 3i.$$

By (3.2.6) the general solution of (3.2.15) is then

$$y(x) = e^{2x}\left(c_1\cos(3x) + c_2\sin(3x)\right) \tag{3.2.18}$$

To solve the stated initial-valued problem we differentiate (3.2.18) and utilize the given initial data (3.2.16): We have

$$4 = y(0) = e^{2x}\left(c_1\cos(3x) + c_2\sin(3x)\right)\Big|_{x=0} = c_1$$

$$-1 = y'(0) = e^{2x}\left[(2c_1 + 3c_2)\cos(3x) + (2c_2 - 3c_1)\sin(3x)\right]\Big|_{x=0} = 2c_1 + 3c_2$$

so that

$$c_1 = 4, \qquad c_2 = -3$$

and the solution of the initial-value problem (3.2.15) – (3.2.16) is

$$y(x) = e^{2x}\left(4\cos(3x) - 3\sin(3x)\right).$$

For the boundary data (3.2.17) we obtain

$$1 = y(0) = e^{2x}\left(c_1\cos(3x) + c_2\sin(3x)\right)\Big|_{x=0} = c_1$$

$$2 = y(\pi/6) = e^{2x}\left(c_1\cos(3x) + c_2\sin(3x)\right)\Big|_{x=\pi/6} = e^{\pi/3}c_2,$$

so that $c_1 = 1$ and $c_2 = 2e^{-\pi/3}$ and the solution of the boundary-value problem is

$$y(x) = e^{2x}\left(\cos(3x) + 2e^{-\pi/3}\sin(3x)\right).$$

b) We find the general solution for

$$y'' + 2y' + y = 0. \tag{3.2.19}$$

With the Ansatz $y(x) = e^{\lambda x}$, the characteristic equation becomes

$$\lambda^2 + 2\lambda + 1 = 0,$$

which admits the same root, $\lambda = -1$ twice. By (3.2.12) the general solution of (3.2.19) is then

$$y(x) = c_1 e^{-x} + c_2 x\, e^{-x}.$$

c) We find the general solution for

$$y'' - y' = 0. \tag{3.2.20}$$

With the Ansatz $y(x) = e^{\lambda x}$, the characteristic equation becomes

$$\lambda^2 - \lambda = 0,$$

which admits the two real roots, $\lambda_1 = 0$ and $\lambda_2 = 1$. By (3.2.7) the general solution of (3.2.20) is then

$$y(x) = c_1 + c_2\, e^x.$$

3.2.1 Exercises

1. Find the general solutions of the following equations:

   a) $y'' + 4y' + 13y = 0$

   b) $y'' - 4y' + 4y = 0$

   c) $3y'' + 5y' + 2y = 0$

   d) $y'' + 3y = 0$

   e) $y'' + 3y' = 0$

   f) $y'' - ay' + y = 0$ for i) $a^2 > 4$ and ii) $a^2 < 4$.

2. Solve the following initial-value problems:

   a) $y'' + 2y' + 3y = 0, \quad y(0) = 0,\ y'(0) = \sqrt{2}$

   b) $y'' + 6y' + 9y = 0, \quad y(0) = 1,\ y'(0) = -1$

   c) $y'' - 2y = 0, \quad y(0) = 0,\ y'(0) = 2$

   d) $y'' - 2y' = 0, \quad y(3) = 1,\ y'(3) = 1$

   e) $y'' + 9y = 0, \quad y(\pi/3) = 1,\ y'(\pi/3) = 5$

3. Solve the following boundary-value problems:

   a) $y'' - y = 0, \quad y(0) = 1,\ y(1) = 5$

   b) $y'' - 4y' = 0, \quad y(1) = 2,\ y(2) = 3$

   c) $y'' = 0, \quad y(3) = 1,\ y(-1) = 3$

4. Show that the only solution of the differential equation

   $$y'' + py' + qy = 0$$

   with the initial conditions $y(x_0) = 0$ and $y'(x_0) = 0$, is

   $$y(x) = 0$$

   for all $p,\ q \in \mathbb{R}$ and all $x_0 \in \mathbb{R}$.

3.3 Particular solutions of nonhomogeneous linear second-order differential equations

We consider the linear second-order equation

$$\boxed{y'' + g(x)y' + h(x)y = f(x)} \tag{3.3.1}$$

where $g(x)$, $h(x)$ and $f(x)$ are given continuous functions on some interval $\mathcal{D} \subseteq \mathbb{R}$. When $f(x)$ is not the zero function, then equation (3.3.1) is known as a **nonhomogeneous second-order linear differential equation** and a **homogeneous second-order linear differential equation** when $f(x)$ is the zero function.

Definition 3.3.1. *Any function $y_p(x)$ which satisfies the nonhomogeneous equation (3.3.1) on an interval $\mathcal{D}$ and which does not contain two arbitrary constants, is known as a* **particular solution** *for (3.3.1) on $\mathcal{D}$.*

The following proposition follows directly from the linear superposition principle:

Proposition 3.3.1.

a) A general solution of (3.3.1), i.e. equation

$$y'' + g(x)y' + h(x)y = f(x),$$

is of the form

$$y(x;\, c_1,\, c_2) = \phi_H(x; c_1, c_2) + y_p(x), \tag{3.3.2}$$

where ϕ_H is the general solution of the associated homogeneous equation $y'' + g(x)y' + h(x)y = 0$ and y_p is a particular solution of the nonhomogeneous equation (3.3.1).

b) A particular solution $y_p(x)$ for the nonhomogeneous equation

$$y'' + g(x)y' + h(x)y = f_1(x) + f_2(x), \tag{3.3.3}$$

where $g(x)$, $h(x)$, $f_1(x)$ and $f_2(x)$ are given continuous functions on $\mathcal{D}$, is given by the sum

$$y_p(x) = y_1(x) + y_2(x), \tag{3.3.4}$$

where $y_1(x)$ is a particular solution for $y'' + g(x)y' + h(x)y = f_1(x)$ and $y_2(x)$ is a particular solution for $y'' + g(x)y' + h(x)y = f_2(x)$.

The proof is left as an exercise (see Exercises 3.3.2).

There exist several methods for finding particular solutions of (3.3.1) and we study here two of those methods, namely the *method of variation of parameters* as well as the *method of undetermined coefficients*. The former method can be applied for any continuous functions $g(x)$, $h(x)$ and $f(x)$, whereas the latter method is useful only if the coefficient functions $g(x)$ and $h(x)$ are constants and the function $f(x)$ is of special type. The advantage for the method of undetermined coefficients is that it does not involve any integration as all steps are purely algebraic.

3.3.1 Particular solutions: the method of variation of parameters

In this section we present a general method to find particular solutions and derive an integral-solution formula for particular solutions for the linear equation

$$y'' + g(x)y' + h(x)y = f(x), \tag{3.3.5}$$

where g, h and f are continuous functions on some domain $\mathcal{D} \subseteq \mathbb{R}$.

The following method to construct particular solutions, as described in the proof of Proposition 3.3.2 below, is known as **the method of variation of parameters**.

Proposition 3.3.2. *Assume that two linearly independent solutions of the homogeneous equation*

$$y'' + g(x)y' + h(x)y = 0 \tag{3.3.6}$$

are given by $\phi_1(x)$ *and* $\phi_2(x)$ *on the interval* $\mathcal{D} \subseteq \mathbb{R}$. *Then a particular solution* $y_p(x)$ *of*

$$y'' + g(x)y' + h(x)y = f(x)$$

is

$$y_p(x) = w_1(x)\phi_1(x) + w_2(x)\phi_2(x), \tag{3.3.7}$$

where $w_1(x)$ *and* $w_2(x)$ *have the following form:*

$$w_1(x) = -\int \frac{f(x)\phi_2(x)}{W[\phi_1, \phi_2](x)}\, dx, \qquad w_2(x) = \int \frac{f(x)\phi_1(x)}{W[\phi_1, \phi_2](x)}\, dx.$$

Here $W[\phi_1, \phi_2](x)$ *is the Wronskian.*

Proof: Consider two linearly independent solutions, $\phi_1(x)$ and $\phi_2(x)$, of the homogeneous equation (3.3.6) and use the Ansatz

$$y_p(x) = w_1(x)\phi_1(x) + w_2(x)\phi_2(x), \tag{3.3.8}$$

to seek for a particular solution of the nonhomogeneous equation (3.3.5). Differentiating (3.3.8), we obtain

$$y_p' = w_1'\phi_1 + w_1\phi_1' + w_2'\phi_2 + w_2\phi_2'.$$

Let now

$$w_1'\phi_1 + w_2'\phi_2 = 0, \tag{3.3.9}$$

so that y_p' takes the form

$$y_p' = w_1\phi_1' + w_2\phi_2'. \tag{3.3.10}$$

Differentiating y_p' one more time, we obtain

$$y_p'' = w_1'\phi_1' + w_1\phi_1'' + w_2'\phi_2' + w_2\phi_2''. \tag{3.3.11}$$

Inserting (3.3.8) for y_p, (3.3.10) for y_p', and (3.3.11) for y_p'' into (3.3.5), we obtain

$$w_1\left[\phi_1'' + g(x)\phi_1' + h(x)\phi_1\right] + w_2\left[\phi_2'' + g(x)\phi_2' + h(x)\phi_2\right] + w_1'\phi_1' + w_2'\phi_2' = f(x). \tag{3.3.12}$$

Since ϕ_1 and ϕ_2 satisfies the homogeneous equation (3.3.6), the above expression reduces to

$$w_1'\phi_1' + w_2'\phi_2' = f(x). \tag{3.3.13}$$

We conclude that the Ansatz (3.3.8) is a particular solution for the nonhomogeneous equation (3.3.6) if and only if the two conditions on w_1 and w_2, namely (3.3.9) and (3.3.13) are satisfied, i.e.

$$\begin{cases} w_1'\phi_1 + w_2'\phi_2 = 0 \\ w_1'\phi_1' + w_2'\phi_2' = f(x) \end{cases}$$

This system of equations can be written in the form

$$\begin{pmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{pmatrix} \begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f(x) \end{pmatrix}. \tag{3.3.14}$$

We note that the determinant of the coefficient matrix of the above system is the Wronskian of the functions ϕ_1 and ϕ_2, namely

$$W[\phi_1, \phi_2](x) = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = \phi_1\phi_2' - \phi_2\phi_1'.$$

By Proposition 1.1.3 we know that $W[\phi_1, \phi_2](x) \neq 0$ for all $x \in \mathcal{D}$, as ϕ_1 and ϕ_2 are solutions and are linearly independent functions on $\mathcal{D}$ by assumption. Therefore the coefficient matrix of (3.3.14) is nonzero and **Cramer's rule** for the unique algebraic solution of w_1' and w_2' from system (3.3.14) applies. We obtain

$$w_1'(x) = \frac{1}{W[\phi_1, \phi_2](x)} \begin{vmatrix} 0 & \phi_2 \\ f(x) & \phi_2' \end{vmatrix} = -\frac{f(x)\phi_2(x)}{W[\phi_1, \phi_2](x)}$$

$$w_2'(x) = \frac{1}{W[\phi_1, \phi_2](x)} \begin{vmatrix} \phi_1 & 0 \\ \phi_1' & f(x) \end{vmatrix} = \frac{f(x)\phi_1(x)}{W[\phi_1, \phi_2](x)}.$$

Integrating the previous expressions over x we obtain

$$w_1(x) = -\int \frac{f(x)\phi_2(x)}{W[\phi_1, \phi_2](x)}\, dx, \qquad w_2(x) = \int \frac{f(x)\phi_1(x)}{W[\phi_1, \phi_2](x)}\, dx. \qquad \square$$

Historical Note: (source: Wikipedia)
Gabriel Cramer (1704 – 1752) was a Swiss mathematician, born in Geneva. Cramer showed promise in mathematics from an early age. At 18 he received his doctorate and at 20 he was co-chair of mathematics. Cramer's Rule for linear algebraic systems is named after Gabriel Cramer, as he published the rule for an arbitrary number of unknowns in 1750, although Colin Maclaurin also published special cases of the rule in 1748 (and possibly knew of it as early as 1729). Cramer published his best-known work in his forties in his treatise on algebraic curves (1750).

Example 3.3.1.

We find a particular solution $y_p(x)$ for

$$y'' - 2y' + y = x^{-1}e^x \quad \text{on } x \in \mathbb{R}\backslash\{0\}.$$

Gabriel Cramer (1704 – 1752)

Following Proposition 3.3.2 we use the Ansatz

$$y_p(x) = w_1(x)\phi_1(x) + w_2(x)\phi_2(x),$$

where the two linearly independent solutions for the associated homogeneous equation are

$$\phi_1(x) = e^x, \qquad \phi_2(x) = xe^x.$$

The Wronskian for ϕ_1 and ϕ_2 is

$$W[\phi_1, \phi_2](x) = \begin{vmatrix} e^x & xe \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x}.$$

Thus, by the integral formulas given in Proposition 3.3.2, we obtain

$$w_1(x) = -\int \frac{(xe^x)(x^{-1}e^x)}{e^{2x}}\,dx = -\int dx = -x$$

$$w_2(x) = \int \frac{(e^x)(x^{-1}e^x)}{e^{2x}}\,dx = \ln|x|.$$

A particular solution is thus

$$y_p(x) = -x\,e^x + x\ln|x|\,e^x.$$

The following proposition gives a method to find a general solution of a linear second-order nonhomogeneous differential equation when three particular solutions for the equation are known:

Proposition 3.3.3. *Assume that three distinct particular solutions, namely* $y_1(x)$, $y_2(x)$ *and* $y_2(x)$, *are given for the nonhomogeneous equation*

$$y'' + g(x)y' + h(x)y = f(x)$$

on the interval $\mathcal{D} \subseteq \mathbb{R}$. *Consider*

$$\phi_1(x) = y_2(x) - y_1(x) \quad \text{and} \quad \phi_2(x) = y_2(x) - y_3(x). \tag{3.3.15}$$

If $W[\phi_1, \phi_2](x_0) \neq 0$ *in any point* $x_0 \in \mathcal{D}$, *then* $\phi_1(x)$ *and* $\phi_2(x)$ *are two linearly independent solutions of the associated homogeneous equation*

$$y'' + g(x)y' + h(x)y = 0 \tag{3.3.16}$$

on $\mathcal{D}$. *A general solution for*

$$y'' + g(x)y' + h(x)y = f(x)$$

is then

$$y(x) = c_1\phi_1(x) + c_2\phi_2(x) + y_j(x)$$

for all $x \in \mathcal{D}$, *where* y_j *is any particular solution of this equation for all* $x \in \mathcal{D}$ *and* c_1 *and* c_2 *are two arbitrary constant.*

Proof: Consider $\phi_1(x) = y_2(x) - y_1(x)$, and assume that $y_1(x)$ and $y_2(x)$ are particular solutions on an interval $\mathcal{D}$ for the nonhomogeneous equation

$$y'' + g(x)y' + h(x)y = f(x).$$

Then $\phi_1' = y_2' - y_1'$ and $\phi_1'' = y_2'' - y_1''$, and we have

$$\begin{aligned}\phi_1'' + g(x)\phi_1' + h(x)\phi_1 &= y_2'' - y_1'' + g(x)[y_2' - y_1'] + h(x)[y_2 - y_1] \\ &= y_2'' + g(x)y_2' + h(x)y_2 - \left[y_1'' + g(x)y_1' + h(x)y_1\right] \\ &= f(x) - f(x) = 0,\end{aligned}$$

so that we can conclude that $\phi_1(x)$ is a solution of the homogeneous equation

$$y'' + g(x)y' + h(x)y = 0.$$

The same is true for $\phi_2(x) = y_2(x) - y_3(x)$, where $y_3(x)$ is also a particular solution of the nonhomogeneous equation. If $W[\phi_2,\ \phi_2[(x_0) \neq 0$ for some $x_0 \in \mathcal{D}$, then $\{\phi_1(x),\ \phi_2(x)\}$ is a linearly independent set. The general solution $\phi_H(x; c_1, c_2)$ of the homogeneous equation

$$y'' + g(x)y' + h(x)y = 0$$

is $\phi_H(x; c_2, c_2) = c_1\phi_1(x) + c_2\phi_2(x)$ for all $x \in \mathcal{D}$ and the general solution of the nonhomogeneous equation is $\phi_H(x; c_1, c_2)) + y_j$ for any particular solution of the nonhomogeneous equation. $\square$

Remark: *From Proposition 3.3.3 we can conclude that a second-order nonhomogeneous differential equation on the domain $\mathcal{D}$ that admits three distinct particular solutions, $\{y_1(x),\ y_2(x),\ y_3(x)\}$ for all $x \in \mathcal{D}$, can identically be constructed for all $x \in \mathcal{D}$ under the conditions sated in Proposition 3.3.3*

Example 3.3.2.

Consider the nonhomogeneous equation

$$y'' - 4y' + 4y = x\,e^x. \tag{3.3.17}$$

It is easy to verify that the equation admits the following three particular solutions:

$$y_1(x) = e^x(x+2), \quad y_2(x) = e^{2x} + e^x(x+2), \quad y_3(x) = x\,e^{2x} + e^x(x+2)$$

for all $x \in \mathbb{R}$. Let now

$$\phi_1(x) = y_2(x) - y_1(x) = e^{2x}, \quad \phi_2(x) = y_2(x) - y_3(x) = e^{2x}(x+2).$$

Calculating the Wronskian in the point $x = 0$, we obtain

$$W[\phi_1,\ \phi_2](0) = -1,$$

so that the set $\{\phi_1(x),\ \phi_2(x)\}$ is linearly independent in $\mathcal{C}^\infty(\mathbb{R})$. The general solution $\phi_H(x)$ of the homogeneous equation

$$y'' - 4y' + 4y = 0$$

is therefore

$$\phi_H(x) = a_1 e^{2x} + a_2 e^{2x}(1 - x) = (a_1 + a_2)e^{2x} - a_2 x e^{2x},$$

or we can rename the constant $a_1,\ a_2$ as follows: $c_1 = a_1 + a_2,\ c_2 = -a_2$, to obtain the general solution in the form

$$\phi_H(x) = c_1 e^{2x} + c_2 x e^{2x}.$$

To get a general solution for the given nonhomogeneous equation (3.3.17) we can add any particular solution, say y_1, so that the general solution is $y(x) = \phi_H(x) + y_1(x)$. If we use any other particular solution, $y_j(x)$, in the sum $\phi_H(x) + y_j(x)$, then the general solution will remain the same (up to a change of the constants), since the constants c_1 and c_2 are arbitrary and can always be re-defined in terms of some other arbitrary constants.

3.3.2 Exercises

[Solutions of those Exercises marked with a * are given in Appendix D].

1. Find general solutions of the following equations:

   a) $y'' + 4y' + 4y = \sqrt{x}e^{-2x}, \qquad x > 0$

   b) $y'' - 2y' + y = \sqrt{x-1}\, e^{x}, \qquad x > 1$

   c) $y'' + 2y' + y = e^{-x} \ln(x), \qquad x > 0$

   d) $y'' + 2y' + y = x^{-2}e^{-x}, \qquad x > 0$

   e) $y'' - 9y = e^{x} + 1$

   f) $y'' + 4y = \tan x$ for all those $x \in \mathbb{R}$ such that $\cos x \neq 0$.

   g) $y'' + y = \dfrac{1}{\cos x}$ for all those $x \in \mathbb{R}$ such that $\cos x \neq 0$.

   h)* $y'' + 4y = 9\cos^2 x$ for all $x \in \mathbb{R}$.

2. Solve the initial-value problem for

   $$y'' - y = (e^{x} + 1)^{-2}e^{2x},$$

   where $y(0) = \ln(2)$ and $y'(0) = \dfrac{7}{2} - \ln(2)$.

3. Solve the initial-value problem

   $$y'' - 3y' + 2y = (e^{2x} + 1)^{-1}\, e^{3x}$$

   where $y(0) = \dfrac{\pi}{4}$ and $y'(0) = \dfrac{\pi}{2}$.

4. Find a general solution for

   $$xy'' - (2x+1)y' + (x+1)y = 2x^2e^{x}\ln(x), \qquad x > 0$$

   where $\phi_1(x) = x^2e^{x}$ and $\phi_2(x) = e^{x}$ are solutions of the associated homogeneous equation.

5. Find a general solution on the interval $|x| > 1$ for

   $$(x^2+1)y'' - 2xy' + 2y = (x^2+1)^2,$$

   where $\phi_1(x) = x^2 - 1$ and $\phi_2(x) = x$ are solutions of the associated homogeneous equation.

6. Find a general solution for

$$(1-x)y'' + xy' - y = 2(x-1)^2 e^{-x},$$

where $\phi_1(x) = e^x$ and $\phi_2(x) = x$ are solutions of the associated homogeneous equation.

7. In the following exercises, the nonhomogeneous equations and three particular solutions are given. Find in each case a general solution of the given equations.

   a) $y'' + 4y = \cos x$ with particular solutions

   $$y_1(x) = \frac{1}{3}\cos x, \quad y_2(x) = 2\cos(2x) + \frac{1}{3}\cos x, \quad y_3(x) = -3\sin(2x) + \frac{1}{3}\cos x.$$

   b) $y'' - 6y' + 9y = \dfrac{9x^2 + 6x + 2}{x^3}$ with particular solutions

   $$y_1(x) = e^{3x} + \frac{1}{x}, \quad y_2(x) = 2xe^{3x} + \frac{1}{x}, \quad y_3(x) = \frac{1}{x}.$$

   c) $x^4y'' + 2x^3y' - 4y = \dfrac{1}{x}$ with particular solutions

   $$y_1(x) = e^{2/x} - \frac{1}{4x}, \quad y_2(x) = 2e^{-2/x} - \frac{1}{4x}, \quad y_3(x) = -\frac{1}{4x}.$$

8. Find nonhomogeneous second-order differential equations for which the following particular solutions are known:

   a) $y_1(x) = x\cos x + x^2\sin x, \quad y_2(x) = 3\cos x + x\cos x + x^2\sin x,$

   $y_3(x) = \sin x + x\cos x + x^2\sin x.$

   b) $y_1(x) = x + \dfrac{3}{4}x^3, \quad y_2(x) = x\ln x + \dfrac{3}{4}x^3, \quad y_3(x) = \dfrac{3}{4}x^3, \quad x > 0.$

   c) $y_1(x) = x^2\ln x + \dfrac{x^4}{2}, \quad y_2(x) = 4x^2 + x^2\ln x + \dfrac{x^4}{2},$

   $y_3(x) = x^3 + x^2\ln x + \dfrac{x^4}{2}, \quad x > 0.$

3.3.3 Particular solutions: the method of undetermined coefficients

We now consider a method to construct particular solutions for the equation

$$\boxed{y'' + py' + qy = f(x)} \tag{3.3.18}$$

where p and q are given real numbers and $f(x)$ is one of the following three special functions:

Case I: $f(x) = P_m(x)$, where P_m is an m-th degree polynomial;

Case II: $f(x) = e^{\alpha_1 x}\cos(\alpha_2 x)\,P_m(x)$ or $f(x) = e^{\alpha_1 x}\sin(\alpha_2 x)\,P_m(x)$, where $\alpha_1 \in \mathbb{R}$, $\alpha_2 \in \mathbb{R}$ and P_m is an m-th degree polynomial;

Case III: $f(x) = e^{\alpha x}P_m(x)$, where $\alpha \in \mathbb{R}$. and $P_m(x)$ is an m-th degree polynomial.

Let us now study the above three cases in detail.

Case I: $f(x) = P_m(x)$:

Consider

$$\boxed{y'' + py' + qy = P_m(x)} \tag{3.3.19}$$

where P_m is an mth-degree polynomial, i.e.,

$$P_m(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0. \tag{3.3.20}$$

Here a_j, $(j = 0, 1, \ldots, m)$ are given real coefficients and m is a given natural number. We now make Ansätze to find particular solutions for (3.3.19). We need to distinguish between three different subcases:

Case Ia: Let $\boxed{q \neq 0}$. The Ansatz for a particular solution of (3.3.19) is then

$$y_p(x) = A_m x^m + A_{m-1}x^{m-1} + \cdots + A_1 x + A_0 := Q_m(x), \tag{3.3.21}$$

where the real constants, A_j, $(j = 0, 1, \ldots, m)$, are to be determined such that the Ansatz (3.3.21) satisfies (3.3.19).

Case Ib: Let $\boxed{q = 0 \text{ and } p \neq 0}$. The Ansatz for a particular solution of (3.3.19) is then

$$y_p(x) = x\, Q_m(x), \tag{3.3.22}$$

where $Q_m(x)$ is given by (3.3.21).

Case Ic: Let $\boxed{q = 0 \text{ and } p = 0}$. The Ansatz for a particular solution of (3.3.19) is then

$$y_p(x) = x^2 Q_m(x), \tag{3.3.23}$$

where $Q_m(x)$ is given by (3.3.21).

Example 3.3.3.

We find a general solution for

$$y'' + 4y = 8x^2. \tag{3.3.24}$$

First we find the general solution ϕ_H of the associated homogeneous equation

$$y'' + 4y = 0. \tag{3.3.25}$$

Using the Ansatz $y(x) = e^{\lambda x}$, we obtain

$$\phi_H(x) = c_1 \cos(2x) + c_2 \sin(2x). \tag{3.3.26}$$

For a particular solution we need to use the Ansatz proposed in Case I a) due to the presence of the term $4y$. The Ansatz is thus

$$y_p(x) = A_2 x^2 + A_1 x + A_0, \tag{3.3.27}$$

so that

$$y_p'(x) = 2A_2 x + A_1, \quad y_p''(x) = 2A_2.$$

Inserting the above into (3.3.24), we obtain

$$2A_2 + 4\left(A_2 x^2 + A_1 x + A_0\right) = 8x^2. \tag{3.3.28}$$

Equating coefficients of x^2, x and 1 in the above relation leads to the following set of linear algebraic equations for A_2, A_1 and A_0:

$$\begin{aligned} x^2: &\quad 4A_2 = 8 \\ x^1: &\quad 4A_1 = 0 \\ 1\ : &\quad 2A_2 + 4A_0 = 0. \end{aligned}$$

Solving this algebraic system, we obtain $A_0 = -1$, $A_1 = 0$, $A_2 = 2$. Hence, a particular solution for (3.3.24) is

$$y_p(x) = 2x^2 - 1. \tag{3.3.29}$$

A general solution of (3.3.24) is therefore

$$y(x) = c_1 \cos(2x) + c_2 \sin(2x) + 2x^2 - 1. \tag{3.3.30}$$

Case II: $f(x) = e^{\alpha_1 x}\cos(\alpha_2 x)\, P_m(x)$ **or** $f(x) = e^{\alpha_1 x}\sin(\alpha_2 x)\, P_m(x)$**,** $\alpha_1, \alpha_2 \in \mathbb{R}$

We consider the following linear **complex differential equation** with dependent complex variable $y_c(x)$:

$$\boxed{y_c'' + py_c' + qy_c = e^{\alpha x}P_m(x), \qquad \alpha := \alpha_1 + i\alpha_2, \quad \alpha_1 \in \mathbb{R},\ \alpha_2 \in \mathbb{R}} \tag{3.3.31}$$

where P_m is an mth-degree polynomial, i.e.,

$$P_m(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0 \tag{3.3.32}$$

with a_j, $(j = 0, 1, \ldots, m)$ real coefficients. Here α is a complex number, such that

$$\alpha = \alpha_1 + i\alpha_2, \ \text{with } \alpha_1,\ \alpha_2 \ \text{ real.}$$

Every differentiable complex function $y_c(x)$ can be written in the form

$$y_c(x) = u(x) + iv(x), \tag{3.3.33}$$

where u and v are real differentiable functions on some domain $\mathcal{D} \subseteq \mathbb{R}$ and

$$y_c'(x) = u'(x) + iv'(x)$$

$$y_c''(x) = u''(x) + iv''(x).$$

Using the Ansatz (3.3.33) and and its derivatives for reduces (3.3.31) to the form

$$\begin{aligned} &u'' + iv'' + p(u' + iv') + q(u + iv) \\ &\qquad = e^{\alpha_1 x}\cos(\alpha_2 x)\, P_m(x) + ie^{\alpha_1 x}\sin(\alpha_2 x)\, P_m(x), \end{aligned} \tag{3.3.34}$$

where we have used the relation

$$e^{(\alpha_1+i\alpha_2)x} = e^{\alpha_1 x}\left(\cos(\alpha_2 x) + i\sin(\alpha_2 x)\right). \tag{3.3.35}$$

Comparing the real parts and the imaginary parts of (3.3.34), we obtain the following two real nonhomogeneous differential equations in u and v:

$$\boxed{u'' + pu' + qu = e^{\alpha_1 x}\cos(\alpha_2 x)\, P_m(x)} \tag{3.3.36}$$

and

$$\boxed{v'' + pv' + qv = e^{\alpha_1 x}\sin(\alpha_2 x)\, P_m(x)} \tag{3.3.37}$$

To find particular solutions for (3.3.36) and (3.3.37), the proposition that follows is useful:

Proposition 3.3.4. *A convenient Ansatz for a complex particular solution* $y_{cp}(x)$ *of*

$$y_c'' + py_c' + qy_c = e^{\alpha x} P_m(x), \qquad \alpha := \alpha_1 + i\alpha_2, \quad \alpha_1 \in \mathbb{R},\ \alpha_2 \in \mathbb{R}$$

is

$$y_{cp}(x) = e^{\alpha x} w_c(x), \tag{3.3.38}$$

where $w_c(x)$ *is any complex solution of the equation*

$$w_c'' + (2\alpha + p)w_c' + (\alpha^2 + \alpha p + q)w_c = P_m(x). \tag{3.3.39}$$

A real particular solution $u_p(x)$ *of*

$$u'' + pu' + qu = e^{\alpha_1 x} \cos(\alpha_2 x)\, P_m(x)$$

is then given by the real part of y_{cp}*, i.e.*

$$u_p(x) = Re\,[y_{cp}], \tag{3.3.40}$$

whereas a real particular solution $v_p(x)$ *of*

$$v'' + pv' + qv = e^{\alpha_1 x} \sin(\alpha_2 x)\, P_m(x)$$

is give by the imaginary part of $y_{cp}(x)$*, i.e.*

$$v_p(x) = Im\,[y_{cp}]. \tag{3.3.41}$$

Proof: Differentiating the Ansatz (3.3.38) we obtain

$$y_{cp}'(x) = \alpha e^{\alpha x} w_c(x) + e^{\alpha x} w_c'(x)$$

$$y_{cp}''(x) = \alpha^2 e^{\alpha x} w_c(x) + 2\alpha e^{\alpha x} w_c'(x) + e^{\alpha x} w_c''(x).$$

Inserting (3.3.38) and the above derivatives, y_{cp}' and y_{cp}'', in

$$y_c'' + py_c' + qy_c = e^{\alpha x} P_m(x), \qquad \alpha := \alpha_1 + i\alpha_2, \quad \alpha_1 \in \mathbb{R},\ \alpha_2 \in \mathbb{R}$$

leads to condition (3.3.39). $\square$

To find a solution $w_c(x)$ of (3.3.39), i.e. equation

$$w_c'' + (2\alpha + p)w_c' + (\alpha^2 + \alpha p + q)w_c = P_m(x),$$

we use the same Ansätze as listed in **Case I** since the nonhomogeneous part of (3.3.39) is a polynomial, albeit we now need to evaluate complex coefficients in the polynomial Ansatz $Q_m(x)$. The following three cases may appear:

Case IIa: Let $\boxed{\alpha^2 + \alpha p + q \neq 0}$. An Ansatz for a solution w_c of (3.3.39) is then

$$w_c(x) = B_m x^m + B_{m-1} x^{m-1} + \cdots + B_1 x + B_0 := S_m(x), \tag{3.3.42}$$

where B_j $(j = 0, 1, \ldots, m)$ are **complex constants** which need to be determined for the Ansatz.

Case IIb: Let $\boxed{\alpha^2 + \alpha p + q = 0 \text{ and } 2\alpha + p \neq 0}$. An Ansatz for a solution w_c of (3.3.39) is then

$$w_c(x) = x S_m(x), \tag{3.3.43}$$

where $S_m(x)$ is given by (3.3.42).

Case IIc: Let $\boxed{\alpha^2 + \alpha p + q = 0 \text{ and } 2\alpha + p = 0}$. An Ansatz for a solution, w_c, of (3.3.39) is then

$$w_c(x) = x^2 S_m(x), \tag{3.3.44}$$

where $S_m(x)$ is given by (3.3.42).

Example 3.3.4.

We find a particular solution for the differential equation

$$y'' - y = \sin x. \tag{3.3.45}$$

Since $\mathrm{Im}\left[e^{ix}\right] = \sin x$, we need to consider the complex differential equation

$$y_c'' - y_c = e^{-ix}, \tag{3.3.46}$$

where the complex function $y_c(x)$ is

$$y_c(x) = u(x) + iy(x) \quad \text{and} \quad \mathrm{Im}[y_c(x)] = y(x).$$

We now seek a particular solution $y_{cp}(x)$ for the complex equation (3.3.46) by the Ansatz

$$y_{cp}(x) = e^{ix}\, w_c(x). \tag{3.3.47}$$

A particular solution $y_p(x)$ for the real equation (3.3.45) is then

$$y_p(x) = \mathrm{Im}\left[y_{cp}(x)\right].$$

Differentiating the Ansatz (3.3.47) twice, we obtain

$$y_{cp}' = e^{ix}\left(iw_c + w_c'\right), \quad y_{cp}'' = e^{ix}\left(-w_c + 2iw_c' + w_c''\right). \tag{3.3.48}$$

Inserting the Ansatz (3.3.47) and (3.3.48) in the complex equation (3.3.46), we obtain

$$e^{ix}\left(w_c'' + 2iw_c' - 2w_c\right) = e^{ix}$$

or, since $e^{ix} \neq 0$ for all $x \in \mathbb{R}$, we have

$$w_c'' + 2iw_c' - 2w_c = 1. \tag{3.3.49}$$

An Ansatz for w_c of (3.3.49) is given by **Case II a**, namely

$$w_c(x) = B_0 \tag{3.3.50}$$

where B_0 is a constant (in general B_0 is complex, but in this case it is clearly real). Inserting the Ansatz (3.3.50) into (3.3.49) we obtain $-2B_0 = 1$ so that $w_c(x) = -\frac{1}{2}$. Thus we have obtained a complex particular solution for (3.3.46), namely

$$y_{cp}(x) = -\frac{1}{2}\,e^{ix}$$

so that a real particular solution $y_p(x)$ for the real equation (3.3.45) is

$$y_p(x) = \operatorname{Im}\left[-\frac{1}{2}\,e^{ix}\right] = \operatorname{Im}\left[-\frac{1}{2}\left(\cos x + i\sin x\right)\right] = -\frac{1}{2}\sin x.$$

For a second example of Case II, we consider a slightly more complicated equation:

Example 3.3.5.

We find a particular solution for the differential equation

$$y'' + 4y = (10x - 1)e^x \cos x. \tag{3.3.51}$$

Since

$$\text{Re}\left[(10x - 1)e^{(1+i)x}\right] = (10x - 1)e^x \cos x \tag{3.3.52}$$

we need to consider the complex equation

$$y_c'' + 4y_c = (10x - 1)e^{(1+i)x} \tag{3.3.53}$$

and construct a complex particular solution $y_{cp}(x)$ by the Anstaz

$$y_{cp} = e^{(1+i)x} w_c(x). \tag{3.3.54}$$

A real particular solution $y_p(x)$ of (3.3.51) then follows by

$$y_p(x) = \text{Re}\,[y_{cp}]. \tag{3.3.55}$$

Differentiating the Ansatz (3.3.55) we obtain

$$\begin{aligned} y_{cp}' &= (1+i)e^{(1+i)x} w_c + e^{(1+i)x} w_c' \\ y_{cp}'' &= (1+i)^2 e^{(1+i)x} w_c + 2(1+i)e^{(1+i)x} w_c' + e^{(1+i)x} w_c'' \end{aligned}$$

and the condition on w_c becomes

$$w_c'' + 2(1+i)w_c' + (4+2i)w_c = 10x - 1. \tag{3.3.56}$$

An appropriate Ansatz for (3.3.56) is provided by Case IIa, i.e.

$$w_c(x) = B_1 x + B_0, \qquad B_0 \in \mathcal{C},\ B_1 \in \mathcal{C}. \tag{3.3.57}$$

Equation (3.3.56) then takes the form

$$2(1+i)B_1 + (4+2i)(B_1 x + B_0) = 10x - 1 \tag{3.3.58}$$

and equating coefficients of x and 1 leads to

$$(4+2i)B_1 = 10, \qquad 2(1+i)B_1 + (4+2i)B_0 = -1.$$

We find

$$B_0 = -\frac{8}{5} + \frac{3}{10}i, \qquad B_1 = 2 - i.$$

Thus a complex solution for (3.3.56) is

$$w_c(x) = (2-i)x - \frac{8}{5} + \frac{3}{10}i \tag{3.3.59}$$

and a complex particular solution of (3.3.53) is

$$\begin{aligned} y_{cp}(x) &= e^{(1+i)x}\left[(2-i)x - \frac{8}{5} + \frac{3}{10}i\right] \\ &= e^x\left[\left(2x - \frac{8}{5}\right)\cos x + \left(x - \frac{3}{10}\right)\sin x\right] \\ &\quad + ie^x\left[\left(-x + \frac{3}{10}\right)\cos x + \left(2x - \frac{8}{5}\right)\sin x\right] \end{aligned} \tag{3.3.60}$$

A real particular solution $y_p(x)$ of (3.3.51) is then

$$y_p(x) = \mathrm{Re}\,[y_{cp}] = \left(2x - \frac{8}{5}\right)e^x\cos x + \left(x - \frac{3}{10}\right)e^x\sin x. \tag{3.3.61}$$

Case III $f(x) = e^{\alpha x} P_m(x), \quad \alpha \in \mathbb{R}$

We consider the equation

$$\boxed{y'' + py' + qy = e^{\alpha x} P_m(x), \qquad \alpha \in \mathbb{R}} \tag{3.3.62}$$

where P_m is an m-th-degree polynomial with real coefficients. We note that this is in fact a special case of (3.3.31), with $\alpha \in \mathbb{R}$. Here $y(x)$ is a real function and the same Ansatz, (3.3.38), for a particular solution $y_p(x)$ of (3.3.62) is valid, namely

$$y_p(x) = e^{\alpha x}\, w(x) \tag{3.3.63}$$

with the same condition on w as given by (3.3.39), namely

$$\boxed{w'' + (2\alpha + p)w' + (\alpha^2 + \alpha p + q)w = P_m(x)} \tag{3.3.64}$$

To find a solution $w(x)$ of (3.3.64) we distinguish between three cases:

Case IIIa: Let $\boxed{\alpha^2 + \alpha p + q \neq 0}$. An Ansatz for a solution $w(x)$ of (3.3.64) is then

$$w(x) = A_m x^m + A_{m-1} x^{m-1} + \cdots + A_1 x + A_0 := Q_m(x), \tag{3.3.65}$$

where A_j $(j = 0, 1, \ldots, n)$ are **real constants** which need to be determined for the Ansatz.

Case IIIb: Let $\boxed{\alpha^2 + \alpha p + q = 0 \text{ and } 2\alpha + p \neq 0}$. An Ansatz for a solution $w(x)$ of (3.3.64) is then

$$w(x) = x Q_m(x), \tag{3.3.66}$$

where $Q_m(x)$ is given by (3.3.65).

Case IIIc: Let $\boxed{\alpha^2 + \alpha p + q = 0 \text{ and } 2\alpha + p = 0}$. An Ansatz for a solution $w(x)$ of (3.3.64) is then

$$w(x) = x^2 Q_m(x), \tag{3.3.67}$$

where $Q_m(x)$ is given by (3.3.65).

Example 3.3.6.

We find a particular solution for

$$y'' - 9y = 8x^3 e^x. \tag{3.3.68}$$

The Ansatz (3.3.63) for $y_p(x)$ is

$$y_p(x) = e^x\, w(x)$$

and the derivatives are

$$y_p' = e^x\left(w + w'\right), \qquad y_p'' = e^x\left(w + 2w' + w''\right)$$

Inserting this Ansatz into the given equation, we obtain

$$e^x\left(w'' + 2w' + w - 9w\right) = 8x^3e^x.$$

Since $e^x \neq 0$ for all $\mathbb{R}$, we have

$$w'' + 2w' - 8w = 8x^3. \tag{3.3.69}$$

An Ansatz for $w(x)$ is given by Case IIIa, namely

$$w(x) = A_0 + A_1x + A_2x^2 + A_3x^3$$

with

$$w' = 3A - 3x^2 + 2A - 2x + A_1, \quad w'' = 6A_3x + 2A_2.$$

Inserting the above Ansatz into (3.3.69), we obtain

$$6A_3x + 2A_2 + 2\left(3A_3x^2 + 2A_2x + A_1\right) - 8\left(A_3x^3 + A_2x^2 + A_1x + A_0\right) = 8x^3.$$

Equating the coefficients x^3, x^2, x and x^0 we obtain

$$\begin{aligned}
x^3 &: \quad -8A_3 = 8\\
x^2 &: \quad 6A_3 - 8A_2 = 0\\
x\ &: \quad 6A_3 + 4A_2 - 8A_1 = 0\\
x^0 &: \quad 2A_2 + 2A_1 - 8A_0 = 0.
\end{aligned}$$

This algebraic system has the unique solution

$$A_0 = -\frac{15}{32}, \quad A_1 = -\frac{9}{8}, \quad A_2 = -\frac{3}{4}, \quad A_3 = -1.$$

Thus the particular solution for (3.3.68) takes the form

$$y_p(x) = -e^x\left(x^3 + \frac{3}{4}x^2 + \frac{9}{8}x + \frac{15}{32}\right).$$

3.3.4 Exercises

[Solutions of those Exercises marked with a * are given in Appendix D].

1. Find general solutions of the following differential equations:

a) $y'' + y = 2(1 - x)$.

b) $y'' - 7y = (x - 1)^2$.

c) $y'' - 4y' + 4y = xe^x$.

d) $y'' + y = xe^{2x}$.

e) $y'' - 3y' - 4y = 5e^{4x}$.

f) $y'' + 2y' + y = xe^x \cos x$.

g) $y'' + 4y = \sin 2x$.

h) $y'' - 6y' + 9y = \left(3x^7 - 5x^4\right) e^{3x}$.

i) $y'' - y = e^x - 3$.

j) $y'' + 3y' + 2y = 7e^{2x} + e^{-x}$.

k) $y'' + 4y = 9\cos^2 x$.

l) $y'' - 9y = 5\sin^2 x$.

m) $y'' + k^2 y = k$ for all real constant k.

n) $y'' + k^2 y = k\sin(kx + \alpha)$ for all real constants k and α.

o) $4y'' - 3y' = xe^{(3/4)x}$.

p) $y'' + 25y = \cos(5x)$.

q) $y'' + 6y' + 13y = e^{-3x}\cos(2x)$.

r) $y'' - 2y' - 3y = 2x + e^{-x} - 2e^{3x}$.

s) $y'' - 2y' + y = 2 + e^x \sin x$.

t)* $y'' - 9y = 40xe^{3x}\cos(2x)$.

2. Solve the following initial-value problems:

a) $y'' - 4y' + 4y = x^2, \quad y(0) = 0,\ y'(0) = 0$.

b) $y'' + y = 2(1 - x), \quad y(0) = 2,\ y'(0) = -2$.

c) $y'' - 6y' + 9y = 9x^2 - 12x + 2, \quad y(1) = 1,\ y'(1) = 3$.

d) $y'' - 5y' + 6y = (12x - 7)e^{-x} \quad y(0) = y'(0) = 0$.

e) $y'' + y = 4x \cos x \quad y(0) = 0,\; y'(0) = 1.$

f) $y'' - 6y' + 9y = 16e^{-x} + 9x - 6, \quad y(0) = 1,\; y'(0) = 1.$

g) $y'' - 2y' + 2y = 4e^{x} \cos x, \quad y(\pi) = 1,\; y'(\pi) = 1.$

h) $y'' - y' = -5e^{-x} (\sin x + \cos x), \quad y(0) = -4,\; y'(0) = 5.$

3. Find the linear nonhomogeneous second-order differential equations with constant coefficients which admit the following general solutions (c_1 and c_2 are arbitrary constants):

   a) $y(x) = c_1 e^x + c_2 e^{-x} + x^5$

   b) $y(x) = c_1 \cos x + c_2 \sin x + x e^x$

   c) $y(x) = c_1 e^{3x} \cos(2x) + c_2 e^{3x} \sin(2x) + 2x e^{-4x}$

   d) $y(x) = c_1 e^{3x} + c_2 x e^{3x} + x \sin x - 7$

   e) $y(x) = c_1 \sin(2x) + c_2 \cos(2x) + 3 \sin^2 x$

   f) $y(x) = c_1 e^{-x} \cos(3x) + c_2 e^{-x} \sin(3x) + (5x^2 + 6) e^{-x} + x^3 e^{2x}$

   g) $y(x) = c_1 + c_2 e^{-4x} + \sin x + e^{4x} + x e^{2x} + 4$

   h) $y(x) = c_1 e^x \sin(2x) + c_2 e^x \cos(2x) + 3 e^{3x} \sin^2 x - 2 e^{3x} \cos^2 x$

4. Prove Proposition 3.3.1.

3.4 The second-order Cauchy-Euler equation

The general Cauchy-Euler equation of second order is of the form

$$\boxed{x^2 y'' + p x y' + q y = f(x)} \tag{3.4.1}$$

where p and q are given real numbers and $f(x)$ is a given continuous function on some interval $\mathcal{D} \subseteq \mathbb{R}$. To solve this equation we can introduce a **new independent variable**, z, and transform the Cauchy-Euler equation into a linear second-order nonhomogeneous differential equation with **constant coefficients** in terms of this new independent variable. We have

Proposition 3.4.1.

a) For $x > 0$, the Cauchy-Euler equation

$$x^2\frac{d^2y}{dx^2} + px\frac{dy}{dx} + qy(x) = f(x)$$

can be written in the form

$$\frac{dy^2}{dz^2} + (p-1)\frac{dy}{dz} + qy(z) = f(e^z), \tag{3.4.2}$$

where the new independent variable z is given by the relation

$$\begin{cases} x = e^z \Leftrightarrow z = \ln(x) \\ y(x) = y(z). \end{cases} \tag{3.4.3}$$

b) For $x < 0$, the Cauchy-Euler equation

$$x^2\frac{d^2y}{dx^2} + px\frac{dy}{dx} + qy(x) = f(x)$$

can be written in the form

$$\frac{dy^2}{dz^2} + (1-p)\frac{dy}{dz} + qy(z) = f(-e^z) \tag{3.4.4}$$

where the new independent variable z is given by the relation

$$\begin{cases} x = -e^z \Leftrightarrow z = \ln(-x) \\ y(x) = y(z). \end{cases} \tag{3.4.5}$$

Proof: Consider $x > 0$. Find now $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ for

$$\begin{cases} x = e^z \Leftrightarrow z = \ln(x) \\ y(x) = y(z). \end{cases}$$

By the chain rule we have

$$\frac{dy(x)}{dx} = \frac{dy(z)}{dz}\,\frac{dz}{dx} = \frac{dy(z)}{dz}\,\frac{1}{x} \tag{3.4.6}$$

and

$$\begin{aligned}
\frac{d^2y(x)}{dx^2} &= \frac{d}{dx}\left(\frac{dy(z)}{dz}\ \frac{1}{x}\right) \\
&= \left(\frac{d^2y(z)}{dz^2}\ \frac{dz}{dx}\right)\frac{1}{x} + \frac{dy(z)}{dz}\ \left(-\frac{1}{x^2}\right) \\
&= \frac{d^2y(z)}{dz^2}\ \frac{1}{x^2} - \frac{dy(z)}{dz}\ \frac{1}{x^2}. \qquad (3.4.7)
\end{aligned}$$

Insert now (3.4.3), (3.4.6) and (3.4.7) in the Cauchy-Euler equation (3.4.1). This leads to

$$x^2\left(\frac{d^2y}{dz^2}\ \frac{1}{x^2} - \frac{dy}{dz}\ \frac{1}{x^2}\right) + px\left(\frac{dy}{dz}\ \frac{1}{x}\right) + qy(z) = f(e^z)$$

and, upon simplification, we obtain the constant-coefficient equation (3.4.2). The proof of (3.4.4) is similar. □

Historical Note: (source: Wikipedia)
Baron Augustin-Louis Cauchy (1789 – 1857) was a French mathematician who was an early pioneer of analysis. He started the project of formulating and proving the theorems of infinitesimal calculus in a rigorous manner. He defined continuity in terms of infinitesimals and gave several important theorems in complex analysis and initiated the study of permutation groups in abstract algebra. He wrote approximately eight hundred research articles

Augustin-Louis Cauchy (1789 – 1857)

Example 3.4.1.

We find a general solution for the following Cauchy-Euler equation:

$$x^2y'' + xy' - y = \ln(x), \quad x > 0.$$

Using Proposition 3.4.1, we obtain 3.4.2) with $p = 1$, $q = -1$ and $f(e^z) = \ln(e^z) = z$ by the change of independent variable (3.4.3). That is the constant-coefficients equation

$$\frac{d^2y(z)}{dz^2} - y(z) = z,$$

which admits the general solution

$$y(z) = c_1e^z + c_2e^{-z} - z.$$

Replacing now $z = \ln(x)$ we obtain the general solution

$$y(x) = c_1x + \frac{c_2}{x} - \ln(x),$$

of the given Cauchy-Euler equation.

Historical Note: (source: Wikipedia)
Leonhard Euler (1707–1783) was a pioneering Swiss mathematician and physicist. He made important discoveries in fields as diverse as infinitesimal calculus and graph theory. He also introduced much of the modern mathematical terminology and notation, particularly for mathematical analysis such as the notion of a mathematical function. He is also renowned for his work in mechanics, fluid dynamics, optics, and astronomy. Euler is considered to be the pre-eminent mathematician of the 18th century and one of the greatest mathematicians ever. He is also one of the most prolific mathematicians; his collected works fill 80 volumes. He spent most of his adult life in St. Petersburg, Russia, and in Berlin, Prussia. A statement attributed to Pierre-Simon Laplace expresses Euler's influence on mathematics: "Read Euler, read Euler, he is the master of us all."

Leonhard Euler (1707–1783)

3.4.1 Exercises

[Solutions of those Exercises marked with a * are given in Appendix D].

1. Find general solutions of the following Cauchy-Euler equations for $x > 0$:

   a) $x^2y'' - xy' - 3y = 0$

   b) $x^2y'' + 2xy' - 6y = 0$

   c) $x^2y'' + 3xy' + 2y = x^3$

   d) $x^2y'' - 4xy' + 6y = 12 - x^2$

   e) $x^2y'' + 7xy' + 9y = \dfrac{4}{x^3}$

f) $x^2y'' - 4xy' + 6y = x^4 - x^2$

g)* $x^2y'' - 3xy' - 5y = x^2 \ln x$

h) $x^2y'' - 2xy' + 2y = x^5 \ln x$

2. Solve the initial-value problem

$$x^2y'' + xy' + y = \cos(\ln x),$$

where $y(1) = 2$ and $y'(1) = 3$

3. Find the second-order Cauchy-Euler equation which admits the following general solution:

$$y(x) = c_1x + c_2x\ln x + \frac{3}{4}x^3.$$

4. a) Consider the differential equation

$$a(\alpha + \beta x)^2y'' + b(\alpha + \beta x)y' + cy = f(x), \tag{3.4.8}$$

where $a,\ b,\ c,\ \alpha,\ \beta$ are constants and $a \neq 0,\ \beta \neq 0.$ Show that

$$\begin{cases} \alpha + \beta x = e^z, \quad x > -\dfrac{\alpha}{\beta} \\ y(x) = y(z) \end{cases} \tag{3.4.9}$$

reduces (3.4.8) into the following equation:

$$a\beta^2\frac{d^2y}{dz^2} + (b\beta - a\beta^2)\frac{dy}{dz} + cy = f\left(\frac{e^z - \alpha}{\beta}\right). \tag{3.4.10}$$

b) Find now a general solution of

$$(2 + 3x)^2y'' - 3(2 + 3x)y' + 9y = 81x, \quad x > -\frac{2}{3}$$

3.5 On second-order linear homogeneous equations with non-constant coefficients

We consider the equation

$$y'' + g(x)y' + h(x)y = 0$$

where g and h are given continuous functions on some domain $\mathcal{D} \subseteq \mathbb{R}$.

We show that this equation can be reduced to a first-order linear differential equation for all $g(x)$ and $h(x)$ if one solution is known.

Proposition 3.5.1.
Suppose that $\phi_1(x)$ *is a nonconstant special solution of*

$$y'' + g(x)y' + h(x)y = 0, \tag{3.5.1}$$

for every x *on some interval* $\mathcal{D} \subseteq \mathbb{R}$*. Then the following statements hold:*

a) The Ansatz $y(x) = \phi_2(x)$*, where*

$$\phi_2(x) = v(x)\phi_1(x), \tag{3.5.2}$$

and v *is nonconstant function on* $\mathcal{D}$*, reduces (3.5.1) on* $\mathcal{D}$ *to a first-order homogeneous linear equation in* $w(x)$*, namely*

$$\phi_1 w' + w(2\phi_1' + g(x)\phi_1) = 0, \tag{3.5.3}$$

where $w(x) = v'(x)$*.*

b) A second solution $\phi_2(x)$ of (3.5.1) on $\mathcal{D}$ is given by

$$\phi_2(x) = \phi_1(x) \int \phi_1^{-2}(x)\, e^{\xi(x)}\, dx, \tag{3.5.4}$$

where $\xi(x) = -\int g(x)\, dx$.

c) The solutions $\phi_1(x)$ and $\phi_2(x)$ given by (3.5.4) are linearly independent on $\mathcal{D}$.

d) A general solution $y(x)$ of (3.5.1) is given by the linear combination of the given solution $\phi_1(x)$ and the derived solution (3.5.4), i.e.

$$y(x) = c_1\phi_1(x) + c_2\phi_2(x) \tag{3.5.5}$$

for all $x \in \mathcal{D}$.

Proof: Insert the Ansatz (3.5.2) for a second solution $\phi_2(x)$ into (3.5.1). This leads to

$$\phi_1 v'' + v'(2\phi_1' + g\phi_1) = 0. \tag{3.5.6}$$

With the substitution $v'(x) = w(x)$ we obtain (3.5.3). Equation (3.5.3) is a separable equation and, since $w(x) \neq 0$, we can integrate this equation as follows:

$$\int \frac{dw}{w} = -\int \left(2\frac{\phi_1'}{\phi_1} + g\right) dx,$$

so that

$$w(x) = \exp\left[-\int \left(\frac{2\phi_1'}{\phi_1} + g\right) dx\right].$$

Since

$$\int \frac{\phi_1'}{\phi_1}\, dx = \int \frac{d}{dx} \ln|\phi_1(x)|\, dx = \ln|\phi_1(x)|,$$

we obtain

$$w(x) = \phi_1^{-2} \exp\left[-\int g(x)dx\right]. \tag{3.5.7}$$

Recall that $v'(x) = w(x)$. Integrating (3.5.7) one more time over x, we obtain

$$v(x) = \int \phi_1^{-2} e^{\xi(x)} dx, \quad \text{where} \quad \xi(x) = -\int g(x)dx$$

and the second solution, (3.5.4), then follows from the (3.5.2). To prove the linear independence we evaluate the Wronskian

$$W[\phi_1, v\phi_1](x) = \begin{vmatrix} \phi_1 & v\phi_1 \\ \phi_1' & v'\phi_1 + v\phi_1' \end{vmatrix}(x) = v(x)'\phi_1^2(x).$$

Thus $W[\phi_1, v\phi_1](x) \neq 0$, since $v(x)$ is not constant and $\phi_1 \neq 0$. Hence $\phi_1(x)$ and $\phi_2(x) = v(x)\phi_1(x)$ given by (3.5.4) are linearly independent on $\mathcal{D}$. The general solution of (3.5.1) then follows from the superposition principle. $\square$

Example 3.5.1.

Consider the equation

$$xy'' - (2x+1)y' + (x+1)y = 0$$

on the interval $x > 0$. One solution, ϕ_1, is given by

$$\phi_1(x) = e^x.$$

We find a second solution, ϕ_2, by the formula (3.5.4). Dividing the given equation by x we identify

$$g(x) = -\frac{2x+1}{x},$$

so that

$$\xi(x) = 2x + \ln|x|$$

and

$$\phi_2(x) = e^x\left[\int e^{-2x}e^{2x+\ln x}dx\right] = e^x\int x dx = \frac{1}{2}x^2\, e^x.$$

3.5.1 Exercises

[Solutions of those Exercises marked with a * are given in Appendix D].

1. In the following differential equations, assume that the coefficient of y'' is positive and find then a general solutions of the given equations:

   a) $(2x - x^2)y'' + (x^2 - 2)y' + 2(1-x)y = 0,$ where one solution is $\phi_1(x) = x^2$

   b) $xy'' + 2y' + xy = 0,$ where one solution is $\phi_1(x) = \dfrac{\sin x}{x}$

   c) $x^2y'' - 2x(x+1)y' + 2(1+x)y = 0,$ where one solution is $\phi_1(x) = xe^{2x}$

   d) $(2x+1)y'' + (4x-2)y' - 8y = 0,$ where one solution is $\phi_1(x) = e^{-2x}$

   e)∗ $x^2y'' - 2xy' + (4x^2+2)y = 0,$ where one solution is $\phi_1(x) = x\cos(2x)$

   f) $xy'' - (2x+1)y' + (x+1)y = 0,$ where one solution is $\phi_1(x) = e^x$

2. Let $\phi_1(x)$ and $\phi_2(x)$ be two linearly independent solutions on an interval $\mathcal{D} \subseteq \mathbb{R}$ for the differential equation

$$y'' + g(x)y' + h(x)y = 0, \tag{3.5.8}$$

where g and h are any given continuous functions on $\mathcal{D}$.

   a) Show that the Wronskian, $W[\phi_1, \phi_2](x)$, satisfies the first-order differential equation

$$\frac{dW}{dx} + g(x)W = 0. \tag{3.5.9}$$

   b) Integrate (3.5.9) to derive the formula (3.5.4), namely

$$\phi_2(x) = \phi_1(x) \int \phi_1^{-2}(x)\, e^{\xi(x)}\, dx, \qquad \xi(x) = -\int g(x)\, dx$$

   given in Proposition 3.5.1, for the second solution $\phi_2(x)$ in terms of $\phi_1(x)$.

   Hint: $\phi_1^2 \frac{d}{dx}\left(\frac{\phi_2}{\phi_1}\right) = \phi_1\phi_2' - \phi_2\phi_1'$

3. Show that

$$\frac{d^2y}{dx^2} + g(x)\frac{dy}{dx} + h(x)y = 0,$$

can be transformed to

$$\frac{d^2y}{dz^2} + a\frac{dy}{dz} + bu = 0, \qquad a,\ b \in \mathbb{R}, \quad b \neq 0,$$

by the change of the independent variable, x, as

$$z = b^{-1/2} \int h^{1/2}(x)\, dx,$$

where $h(x)$ is a positively defined function that satisfies the following Bernoulli equation:

$$\frac{dh}{dx} + 2g(x)h = b^{-1/2} a\, h^{3/2}.$$

Integrate this Bernoulli equation to find the explicit form of $h(x)$ in terms of $g(x)$.

4. a) Show that

$$\frac{d^2y}{dx^2} + g(x)\frac{dy}{dx} + h(x)y = 0$$

   can be transformed to

$$\frac{d^2v}{dx^2} + H(x)v = 0,$$

by the following change of the dependent variable:

$$\ln|y(x)| = \ln|v(x)| - \frac{1}{2}\int g(x)\,dx,$$

where

$$H(x) = h(x) - \frac{1}{2}\frac{dg}{dx} - \frac{1}{4}g^2(x).$$

b) Using the result of a), find a general solution of the following equations:

$$i) \quad xy'' - (2x+1)y' + (x+1)y = 0, \qquad x > 0$$

$$ii) \quad y'' + \left(\frac{2}{x}\right)y' + y = 0, \qquad x > 0.$$

5. a) Consider the second-order linear equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \tag{3.5.10}$$

and the linear first-order equation

$$M(x)y' + N(x)y = c, \tag{3.5.11}$$

where c is an arbitrary constant and $a_0,\ a_1,\ a_2$ are differentiable functions. Find the condition on $a_0,\ a_1,\ a_2$ such that (3.5.10) can be integrated once to obtain (3.5.11) and express N and M explicitly in terms of $a_0,\ a_1,\ a_2$. The expression $I = M(x)y' + N(x)y$ is known as a **first integral** of the equation (3.5.10).

b) Find a first integral and a general solution of the following equation:

$$(x^2 + 2x)y'' + 4(x+1)y' + 2y = 0.$$

Chapter 4

Higher-order linear differential equations

4.1 Introduction: the initial-value problem

In this chapter we consider the nth order ($n \geq 2$) **linear homogeneous ordinary differential equation**

$$\boxed{p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0} \tag{4.1.1}$$

and the **linear nonhomogeneous ordinary differential equation**

$$\boxed{p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x)} \tag{4.1.2}$$

Here $p_j(x)$ $(j = 0, 1, 2, \ldots, n)$ and $f(x)$ are real-valued continuous functions given on some common domain $\mathcal{D} \subseteq \mathbb{R}$, $n \geq 1$ and $p_n(x) \neq 0$ for all $x \in \mathcal{D}$.

We know from Proposition 1.1.5 in Chapter 1, that the general solution of the homogeneous equation (4.1.1) is given by the linear combination of n linearly independent solutions

$$S = \{\phi_1(x),\ \phi_2(x), \ldots, \phi_n(x)\} \tag{4.1.3}$$

in $\mathcal{C}^n(\mathcal{D})$. That is, the general solution of (4.1.1) is

$$y(x) = c_1\phi_1(x) + c_2\phi_2(x) + \ldots + c_n\phi_n(x), \tag{4.1.4}$$

where $c_1, \ldots, c_n$ are n arbitrary real constants. Moreover, we know from Proposition 3.3.1 in Chapter 3, that the general solution of the second-order nonhomogeneous equation of the form (4.1.2) (with $n = 2$) is given by the general solution of its homogeneous part plus any particular solution $y_p(x)$ of the nonhomogeneous differential equations. This is also true for the n-th order nonhomogeneous equation (4.1.2), so that the general solution of (4.1.2) is of the form

$$y(x) = c_1\phi_1(x) + c_2\phi_2(x) + \ldots + c_n\phi_n(x) + y_p(x) \tag{4.1.5}$$

where all functions are in $\mathcal{C}^n(\mathcal{D})$ (see Section 4.3 below).

In our study of equations (4.1.1) and (4.1.2), we shall mainly restrict ourselves to equations with constant coefficients and concentrate on generalizing the methods described for linear second-order equations in Chapter 3 to n-th order linear equations. In a sense this generalization is straight forward, although there are of course complications. For example, for n-th order linear homogeneous differential equations the characteristic equation resulting from the Ansatz $y(x) = e^{\lambda x}$, becomes an n-th degree polynomial, and to find all roots of such a polynomial is in general not possible even though we know that those exist.

The initial-value problem for (4.1.2) requires the following **initial data** at a point x_0 in the solution domain of the equation:

$$y(x_0) = b_1,\ y'(x_0) = b_2,\ y''(x_0) = b_3,\ \ldots,\ y^{(n-1)}(x_0) = b_n, \tag{4.1.6}$$

where $b_1,\ b_2, \ldots,\ b_n$ are given real numbers. This data is then used to fix the constants of integration $c_1,\ c_2,\ \ldots,\ c_n$ in the general solution (4.1.5) by solving the linear algebraic system of equations

$$\begin{aligned}
y(x_0) &= c_1\phi_1(x_0) + c_2\phi_2(x_0) + \ldots + c_n\phi_n(x_0) + y_p(x_0) = b_1 \\
y'(x_0) &= \frac{d}{dx}\left[c_1\phi_1(x) + c_2\phi_2(x) + \ldots + c_n\phi_n(x) + y_p(x)\right]\Bigg|_{x=x_0} = b_2 \\
&\vdots \\
y^{(n-1)}(x_0) &= \frac{d^{n-1}}{dx^{n-1}}\left[c_1\phi_1(x) + c_2\phi_2(x) + \ldots + c_n\phi_n(x) + y_p(x)\right]\Bigg|_{x=x_0} = b_n.
\end{aligned}$$

This algebraic system can be written in the form

$$A\mathbf{c} = \mathbf{b} - \mathbf{y}_p, \tag{4.1.7}$$

where

$$A = \begin{pmatrix} \phi_1(x_0) & \phi_2(x_0) & \ldots & \phi_n(x_0) \\ \phi_1'(x_0) & \phi_2'(x_0) & \ldots & \phi_n'(x_0) \\ \vdots & \vdots & \ldots & \vdots \\ \phi_1^{(n-1)}(x_0) & \phi_2^{(n-1)}(x_0) & \ldots & \phi_n^{(n-1)}(x_0) \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \tag{4.1.8a}$$

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, \qquad \mathbf{y}_p = \begin{pmatrix} y_p(x_0) \\ y_p'(x_0) \\ \vdots \\ y_p^{(n-1)}(x_0) \end{pmatrix}. \tag{4.1.8b}$$

We note that $\det A = W[\phi_1, \phi_2, \ldots, \phi_n](x_0)$, where W is the Wronskian of the set S given by (4.1.3). Since S is by assumption a linearly independent set for all x in (4.1.2) solution domain $\mathcal{D}$, we have that

$$W[\phi_1, \phi_2, \ldots, \phi_n](x) \neq 0 \quad \text{for all } x \in \mathcal{D}.$$

Hence A is an invertible matrix in $\mathcal{D}$ so that the algebraic system (4.1.7) has a unique solution. We recall, Proposition 1.1.6, given in Chapter 1, for the existence and uniqueness of the solutions of linear differential equations.

In the sections that follow we give several examples of initial-value problems, although we mainly consider linear constant coefficient equations.

4.2 Linear homogeneous constant coefficients equations

In this section we discuss the problem to find the general solution for the n-th order linear homogeneous ordinary differential equation with **constant coefficients**, that is

$$\boxed{p_n y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_1 y' + p_0 y = 0} \tag{4.2.1}$$

where p_j $(j = 0, 1, \ldots, n)$ are given real constants and $p_n \neq 0$.

To find the general solution of (4.2.1) we make use of the following

Proposition 4.2.1.

a) *The Ansatz*

$$y(x) = e^{\lambda x}, \qquad \lambda \in \mathbb{C} \tag{4.2.2}$$

is a (possibly complex) solution of

$$p_n y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_1 y' + p_0 y = 0$$

for all $x \in \mathbb{R}$, for every (possibly complex) solution λ of the algebraic equation

$$P_n(\lambda) := p_n \lambda^n + p_{n-1} \lambda^{n-1} + \cdots + p_1 \lambda + p_0 = 0. \tag{4.2.3}$$

Equation (4.2.3) is known as the **characteristic equation** *of the differential equation (4.2.1) regarding the Ansatz (4.2.2). The n-th degree polynomial $P_n(\lambda)$ is known as the* **characteristic polynomial** *and the solutions of (4.2.3) are the* **roots** *of $P_n(\lambda)$.*

b) If λ is a complex solution of the characteristic equation (4.2.3), i.e,

$$\lambda = \alpha + i\beta \ (\alpha \in \mathbb{R},\ \beta \in \mathbb{R})$$

then the associated complex solution

$$\phi_c(x) = \psi_1(x) + i\psi_2(x)$$

of (4.2.1) results in two real solution for (4.2.1), namely

$$\psi_1(x) = \mathrm{Re}[\phi_c(x)] = e^{\alpha x}\cos(\beta x), \qquad \psi_2(x) = \mathrm{Im}[\phi_c(x)] = e^{\alpha x}\sin(\beta x).$$

Proof:

a) Differentiating the Ansatz

$$y(x) = e^{\lambda x}, \quad (\lambda \in \mathbb{C})$$

n times, we obtain

$$y'(x) = \lambda e^{\lambda x}, \quad y''(x) = \lambda^2 e^{\lambda x}, \ \ldots, \ y^{(n)}(x) = \lambda^n e^{\lambda x}, \tag{4.2.4}$$

so that (4.2.1) leads to

$$e^{\lambda x}\left[p_n\lambda^n + p_{n-1}\lambda^{n-1} + \cdots + p_1\lambda + p_0\right] = 0.$$

Since $e^{\lambda x} \neq 0$ for all $x \in \mathbb{R}$, we conclude that λ must be a root of the n-th degree characteristic polynomial $P_n(\lambda)$ (4.2.3) in order for $y(x) = e^{\lambda x}$ to satisfy the n-th order differential equation (4.2.1).

b) Assume that $\phi_c(x) = \psi_1(x) + i\psi_2(x)$ is a solution for (4.2.1). Then

$$p_n\phi_c^{(n)} + p_{n-1}\phi_c^{(n-1)} + \cdots + p_1\phi_c' + p_0\phi_c = 0$$

and

$$p_n\psi_1^{(n)} + p_{n-1}\psi_1^{(n-1)} + \cdots p_1\psi_1' + p_0\psi_1$$
$$+i\left[p_n\psi_2^{(n)} + p_{n-1}\psi_2^{(n-1)} + \cdots + p_1\psi_2' + p_0\psi_2\right] = 0 + i\,0.$$

Since two complex functions are equal if and only if their real- and imaginary parts are equal, we have that

$$p_n\psi_1^{(n)} + p_{n-1}\psi_1^{(n-1)} + \cdots + p_1\psi_1' + p_0\psi_1 = 0$$
$$p_n\psi_2^{(n)} + p_{n-1}\psi_2^{(n-1)} + \cdots + p_1\psi_2' + p_0\psi_2 = 0,$$

from which we conclude that $\psi_1(x)$ and $\psi_2(x)$ must be solutions of (4.2.1). Let now

$$\phi_c(x) = e^{\lambda x} = e^{(\alpha + i\beta)x} = e^{\alpha x}\left[\cos(\beta x) + i\sin(\beta x)\right],$$

where λ is a root of $P_n(\lambda)$. It follows that

$$\psi_1(x) = \mathrm{Re}[\phi_c(x)] = e^{\alpha x}\cos(\beta x), \qquad \psi_2(x) = \mathrm{Im}[\phi_c(x)] = e^{\alpha x}\sin(\beta x),$$

as stated. $\square$

Example 4.2.1.

Consider the third-order equation

$$y^{(3)} - y'' + y' - y = 0. \tag{4.2.5}$$

We first find the general solution and then solve the initial-value problem for the initial data

$$y(0) = 1, \quad y'(0) = 2, \quad y''(0) = 3.$$

Applying the Ansatz (4.2.2), we obtain the characteristic equation

$$P_3(\lambda) := \lambda^3 - \lambda^2 + \lambda - 1 = 0$$

which obviously admits the root $\lambda = 1$. Dividing $P_3(\lambda)$ by $\lambda - 1$, we obtain $\lambda^2 + 1$. Hence $P_3(\lambda)$ factorizes in the form

$$P_3(\lambda) = (\lambda - 1)(\lambda^2 + 1) = 0$$

and the three roots (one real- and two complex roots) are

$$\lambda_1 = 1, \quad \lambda_2 = i, \quad \lambda_3 = -i \equiv \bar{\lambda}_2 \text{ (bar denotes the complex conjugate).}$$

This leads to three solutions for (4.2.5), namely

$$\phi_1(x) = e^x, \quad \phi_{c1}(x) = e^{ix}, \quad \phi_{c2}(x) = e^{-ix}.$$

Since $e^{\pm ix} = \cos x \pm i \sin x$, the two complex solutions take the form

$$\phi_{c2}(x) = \cos x + i \sin x, \quad \phi_{c3}(x) = \cos x - i \sin x,$$

The real- and imaginary parts of both ϕ_{2c} and ϕ_{3c} are solutions of (4.2.5), i.e. we obtain four real solutions for (4.2.5) from ϕ_{2c} and ϕ_{3c}, but obviously only two of those are linearly independent, namely

$$\phi_2(x) = \cos x = \mathrm{Re}[\phi_{2c}] \equiv \mathrm{Re}[\phi_{3c}], \quad \phi_3(x) = \sin x = \mathrm{Im}[\phi_{2c}] \equiv \mathrm{Im}[-\phi_{3c}]$$

We now have a set of three real solutions, $S = \{\phi_1(x) = e^x,\ \phi_2(x) = \cos x,\ \phi_2(x) = \sin x\}$, for (4.2.5). It is easy to check that S is a linearly independent set in $\mathcal{C}^\infty(\mathbb{R})$, e.g. $W[\phi_1,\ \phi_2,\ \phi_3](0) = 2$. Thus the general solution of (4.2.5) is

$$y(x) = c_1 e^x + c_2 \cos x + c_3 \sin x$$

for all $x \in \mathbb{R}$ with $c_1,\ c_2,\ c_3$ arbitrary constants. We now solve the initial-value problem for the given initial data. Differentiating the general solution twice, we obtain

$$y'(x) = c_1 e^x - c_2 \sin x + c_3 \cos x$$

$$y''(x) = c_1 e^x - c_2 \cos x - c_3 \sin x.$$

For the given initial data we now have the following linear algebraic system:

$$y(0) = c_1 + c_2 = 1$$

$$y'(0) = c_1 + c_3 = 2$$

$$y''(0) = c_1 - c_2 = 3,$$

which has the unique solution

$$c_1 = 2, \quad c_2 = -1, \quad c_3 = 0.$$

Hence the solution of the initial-value problem is

$$y(x) = 2e^x - \cos x$$

for all $x \in \mathbb{R}$.

We note that the root $\lambda_3 = -i$ was not needed for the general solution, since this is the complex conjugate of the root λ_2 and it gives the same (up to sign) real solutions for its real- and imaginary parts. This is a general property:

If λ_k is a complex root of the characteristic equation, then the complex conjugate of this root is also a root of the characteristic polynomial (see Proposition 4.2.2), but this complex conjugate root does not lead to new linearly independent solutions for the differential equation.

What are the difficulties?

One should point out that the difficulty of the task to find the general solution of an n-th order homogeneous equation with constant coefficients is due to the following questions:

1. *How can we find all the roots of the n-th degree characteristic polynomial equation*

   $$P_n(\lambda) := p_n\lambda^n + p_{n-1}\lambda^{n-1} + \cdots + p_1\lambda + p_0 = 0?$$

   Although we know that n roots always exists, it is in general not possible to find them. However, there are some properties of n-degree polynomials that can be useful to find the roots (see Proposition 4.2.2 below).

2. *How can we select n linearly independent solutions from the list of all solutions obtained by the Ansatz*

   $$y(x) = e^{\lambda x},$$

   for the roots λ_j of the corresponding characteristic equation?

   This is in fact not a difficult problem and the answer is provided by Propositions 4.2.3.

3. *How should we construct a sufficient number of linearly independent solutions for the case when the characteristic polynomial $P_n(\lambda)$ admits roots with multiplicity $k > 1$, i.e. when the same root appears k times ($k \in \mathbb{N}$)?*

   We provide a statement for this construction in Proposition 4.2.4.

We now address the above questions and provide general statements that will help overcome the mentioned difficulties.

Proposition 4.2.2. Properties of n-degree polynomials with real coefficients:

A polynomial of degree n,

$$P_n(\lambda) = p_n\lambda^n + p_{n-1}\lambda^{n-1} + \cdots + p_1\lambda + p_0. \tag{4.2.6}$$

where p_j are all real numbers, has the following properties:

a) *$P_n(\lambda)$ has exactly n roots (including multiplicity, i.e., roots with the same values), $\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n$ (real and/or complex) and the resulting factorization*

$$P_n(\lambda) = p_n(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n).$$

b) *If λ_k is a complex root of $P_n(\lambda)$, then the complex conjugate of λ_k, denoted by $\bar{\lambda}_k$, is also a root of $P_n(\lambda)$.*

c) *$P_n(\lambda)$ can be factorized in terms of first-degree polynomial factors that have only real coefficients and/or second-degree polynomial factors that have only real coefficients.*

d) *The n-th degree polynomial $P_n(\lambda) = \lambda^n - a_0$ admits the following n distinct roots:*

$$\lambda_k = |a_0|^{1/n}\left[\cos\left(\frac{\theta+2\pi k}{n}\right) + i\sin\left(\frac{\theta+2\pi k}{n}\right)\right],$$

where $k = 0,\ 1,\ 2,\ \ldots,\ n-1$ and $\theta = \pi$ for $a_0 < 0$ or $\theta = 0$ for $a_0 > 0$.

e) **Viéta's Theorem**: *$P_n(\lambda)$ admits the following relationship between its roots $\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n$ and its coefficients $p_n,\ p_{n-1},\ \ldots,\ p_0$:*

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \sum_{i=1}^{n}\lambda_i = -\frac{p_{n-1}}{p_n}$$

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \cdots + \lambda_{n-1}\lambda_n = \sum_{i<j,\ \{i,\ j\}=1}^{n}\lambda_i\lambda_j = \frac{p_{n-2}}{p_n}$$

$$\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \cdots + \lambda_{n-2}\lambda_{n-1}\lambda_n = \sum_{i<j<k,\ \{i,\ j,k\}=1}^{n}\lambda_i\lambda_j\lambda_k = -\frac{p_{n-3}}{p_n}$$

$$\vdots$$

$$\lambda_1\lambda_2\ldots\lambda_n = (-1)^n\frac{p_0}{p_n}.$$

Remark: *Properties a) and e) listed in Proposition 4.2.2, also hold for $P_n(\lambda)$ with complex* *coefficients* $p_j \in \mathbb{C}$*, whereas properties b) and c) only hold when all coefficients* p_j *of* $P_n(\lambda)$ *are real. Property d) holds also for* a_0 *complex, but then* θ *is the argument of the complex number* a_0*.*

A well known application of the last relation in Viéta's Theorem, namely the statement $\lambda_1\lambda_2\ldots\lambda_n = (-1)^n \dfrac{p_0}{p_n}$, is the following.

Viéta's statement for integer roots: *Consider* $P_n(\lambda)$ *which contain only real coefficients. Assume that all roots* λ_j *of* $P_n(\lambda)$ *belong to the set of integers* $\mathbb{Z}$*. If* $q = \dfrac{p_0}{p_n} \in \mathbb{Z}$*, then* $\dfrac{q}{\lambda_j} \in \mathbb{Z}$ *for every root of* $P_n(\lambda)$*.*

This means that in the case where $q = \dfrac{p_0}{p_n} \in \mathbb{Z}$ for a given $P_n(\lambda)$, we can search for roots α_j by checking all the divisors $\alpha_j \in \mathbb{Z}$ of q, i.e. all α_j such that $\dfrac{q}{\alpha_j} \in \mathbb{Z}$.

Example 4.2.2.

Consider the 4th degree polynomial

$$P_4(\lambda) = \lambda^4 - 6\lambda^3 + 3\lambda^2 + 26\lambda - 24.$$

In this case $q = \dfrac{p_0}{p_4} = -24$. Hence q is divisible by the following set of numbers:

$$S = \{\pm 1,\ \pm 2,\ \pm 3,\ \pm 4,\ \pm 6,\ \pm 8,\ \pm 12,\ \pm 24\}.$$

Inserting each number in the above set S into $P_4(\lambda)$ we find four roots, since

$$P_4(1) = 0,\ P_4(-2) = 0,\ P_4(3) = 0,\ P_4(4) = 0.$$

Hence, the four roots of $P_4(\lambda)$ are

$$\lambda_1 = 1,\ \lambda_2 = -2,\ \lambda_3 = 3,\ \lambda_4 = 4.$$

Historical Note: (source: Wikipedia)
Francois Viéte (1540 – 1603) was a French mathematician whose work on new algebra was an important step towards modern algebra, due to its innovative use of letters as parameters in equations. Viéta's formulas are formulas that relate the coefficients of a polynomial to sums and products of its roots.

We now address the problem to select the set of linearly independent real solutions for

$$p_n y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_1 y' + p_0 y = 0 \tag{4.2.7}$$

(where p_j are real constants) provided by the Ansätze

$$y(x) = e^{\lambda x} \quad \text{and } y(x) = w(x) e^{\lambda x} \tag{4.2.8}$$

for every root of the characteristic equation λ of $P_n(\lambda)$.

The following proposition states the linear independence of several sets of functions in $\mathcal{C}^\infty(\mathbb{R})$. These include all the functions that may appear as solution of (4.2.7) from the Ansätze (4.2.8). To prove this we can use the Wronskian and show that the Wronskian is non-zero in any point $x_0 \in \mathbb{R}$, say the point $x_0 = 0$. The proof is left as an exercise.

Francois Viéte (1540 – 1603)

Proposition 4.2.3. *Let $\lambda_1,\ \lambda_2, \dots,\ \lambda_k$ be distinct real numbers ($k \in \mathbb{N}$). Then the following sets are linearly independent in the vector space $\mathcal{C}^\infty(\mathbb{R})$:*

1. $S_1 = \left\{e^{\lambda_1 x},\ e^{\lambda_2 x},\ \dots,\ e^{\lambda_k x}\right\}$

2. $S_2 = \left\{e^{\lambda_j x},\ x\, e^{\lambda_j x},\ x^2\, e^{\lambda_j x}, \dots,\ x^m\, e^{\lambda_j x}\right\}$ *for every fixed λ_j and $m \in \mathbb{N}$.*

3. $S_3 = \Big\{e^{\lambda_1 x},\ e^{\lambda_2 x},\ \dots,\ e^{\lambda_k x},\ \ x\, e^{\lambda_1 x},\ x\, e^{\lambda_2 x},\ \dots,\ x\, e^{\lambda_k x},\ \dots,$

   $x^m\, e^{\lambda_1 x},\ x^m\, e^{\lambda_2 x},\ \dots,\ x^m\, e^{\lambda_k x}\Big\}, \quad m \in \mathbb{N}.$

Clearly S_1 and S_2 are subsets of S_3.

Let α_j be distinct nonzero real numbers for $j = 1, 2, \dots$ and let β_j be distinct nonzero real numbers for $j = 1, 2, \dots$. Then the following sets are linearly independent in the vector space $\mathcal{C}^\infty(\mathbb{R})$:

4. $Q = \{\cos(\beta_j x),\ \sin(\beta_j x),\ x\, \cos(\beta_j x),\ x\, \sin(\beta_j x),\ x^2\, \cos(\beta_j x),\ x^2\, \sin(\beta_j x),\ \dots,$

   $x^m\, \cos(\beta_j x),\ x^m\, \sin(\beta_j x),\ e^{\alpha_j x} \cos(\beta_j x),\ e^{\alpha_j x} \sin(\beta_j x),\ x\, e^{\alpha_j x} \cos(\beta_j x),$

   $x\, e^{\alpha_j x} \sin(\beta_j x),\ x^2\, e^{\alpha_j x} \cos(\beta_j x),\ x^2\, e^{\alpha_j x} \sin(\beta_j x),\ \dots,\ x^n\, e^{\alpha_j x} \cos(\beta_j x),$

   $x^n\, e^{\alpha_j x} \sin(\beta_j x)\},$

   for all $j \in \mathbb{N}$, $m = 0,\ 1,\ 2,\ \dots$ and $n = 0,\ 1,\ 2,\ \dots$.

The set $\{Q,\ S_3\}$ is also linearly independent, as is any subset of this.

See Exercises 4.2.1 for some special cases of the sets S_3 and Q.

Example 4.2.3.

The following sets are linearly independent on $\mathbb{R}$:

$\{e^{2x},\ \cos x,\ \sin x,\ x\cos x,\ x\sin x\}$

$\{1,\ e^{-x},\ x^2\cos x,\ e^{-x}\sin x,\ x\cos x,\ x\,e^{x}\sin x\}$

$\{1,\ e^{2x},\ x^2\,e^{2x},\ e^{3x}\}$

In the next example we make use of Propositions 4.2.2 and 4.2.3 to find the general solution of a 6-th order homogeneous equation.

Example 4.2.4.

We find the general solution for the 6-th order equation

$$y^{(6)} + y = 0. \tag{4.2.9}$$

The characteristic equation for the Ansatz $y(x) = e^{\lambda x}$ is now

$$P_6(\lambda) = \lambda^6 + 1 = 0.$$

Following statement d) of Proposition 4.2.2, we have

$$\lambda_k = |-1|^{1/6}\left[\cos\left(\frac{\pi + 2\pi k}{6}\right) + i\sin\left(\frac{\pi + 2\pi k}{6}\right)\right]$$

$$k = 0,\ 1,\ \ldots,\ 5,$$

so that

$$k = 0: \quad \lambda_0 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + i\frac{1}{2}$$

$$k = 1: \quad \lambda_1 = \cos\left(\frac{\pi}{6} + \frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{\pi}{3}\right) = i$$

$$k = 2: \quad \lambda_2 = \cos\left(\frac{\pi}{6} + \frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2} + i\frac{1}{2}.$$

Following statement b) of Proposition 4.2.2, we know that $P_6(\lambda)$ admits the complex conjugate roots $\bar{\lambda}_0$, $\bar{\lambda}_1$, and $\bar{\lambda}_2$. These follow also from the above formula, as λ_3, λ_4 and λ_5, respectively. However, complex conjugate roots lead to the same real solutions (up to sign) as for λ_0, λ_1, and λ_2 and do therefore not make a contribution to the linearly independent set of solutions for (4.2.9). Using now the three complex roots, λ_0, λ_1 and λ_2, we have the following complex solutions for (4.2.9):

$$\phi_{1c}(x) = e^{\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)x} = e^{\frac{\sqrt{3}}{2}x}\, e^{i\frac{1}{2}x} = e^{\frac{\sqrt{3}}{2}x}\left[\cos\left(\frac{x}{2}\right) + i\sin\left(\frac{x}{2}\right)\right]$$

$$\phi_{2c}(x) = e^{ix} = \cos x + i\sin x$$

$$\phi_{3c}(x) = e^{\left(\frac{-\sqrt{3}}{2} + i\frac{1}{2}\right)x} = e^{\frac{-\sqrt{3}}{2}x}\, e^{i\frac{1}{2}x} = e^{-\frac{\sqrt{3}}{2}x}\left[\cos\left(\frac{x}{2}\right) + i\sin\left(\frac{x}{2}\right)\right].$$

The real solutions for equation (4.2.9) are then

$$\psi_1(x) = \mathrm{Re}[\phi_{1c}] = e^{\frac{\sqrt{3}}{2}x}\cos\left(\frac{x}{2}\right)$$

$$\psi_2(x) = \mathrm{Im}[\phi_{1c}] = e^{\frac{\sqrt{3}}{2}x}\sin\left(\frac{x}{2}\right)$$

$$\psi_3(x) = \mathrm{Re}[\phi_{2c}] = \cos x$$

$$\psi_4(x) = \mathrm{Im}[\phi_{2c}] = \sin x$$

$$\psi_5(x) = \mathrm{Re}[\phi_{3c}] = e^{\frac{-\sqrt{3}}{2}x}\cos\left(\frac{x}{2}\right)$$

$$\psi_6(x) = \mathrm{Im}[\phi_{3c}] = e^{\frac{-\sqrt{3}}{2}x}\sin\left(\frac{x}{2}\right).$$

By Proposition 4.2.3, the set

$$S = \{\psi_1(x),\ \psi_2(x),\ \psi_3(x),\ \psi_4(x),\ \psi_5(x),\ \psi_6(x)\}$$

is linearly independent in the vector space $\mathcal{C}^\infty(\mathbb{R})$. Thus the general solution of (4.2.9) is

$$y(x) = c_1\psi_1(x) + c_2\psi_2(x) + c_3\psi_3(x) + c_4\psi_4(x) + c_5\psi_5(x) + c_6\psi_6(x)$$

for all $x \in \mathbb{R}$ with six arbitrary constants $c_1,\ c_2, \ldots,\ c_6$.

Roots with multiplicity of degree $k > 1$:

The characteristic polynomial $P_n(\lambda)$ may admit roots, say λ_0, with **multiplicity** of degree $k \in \mathbb{N}$, where $k \leq n$. This means that $P_n(\lambda)$ admits the same root k times and that $P_n(\lambda)$ factorizes in the form

$$P_n(\lambda) = (\lambda - \lambda_0)^k Q_{n-k}(\lambda),$$

where Q_{n-k} is a polynomial of degree $n-k$. If $\lambda_0 \in \mathbb{R}$, then the Ansatz $y(x) = e^{\lambda_0 x}$ provides only one distinct real solution for the equation

$$p_n y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_1 y' + p_0 y = 0, \qquad p_j \in \mathbb{R}$$

and, if $\lambda_0 \in \mathbb{C}$ it provides two real linearly independent solutions. More solutions can then be obtained for the differential equation by the Ansatz

$$y(x) = e^{\lambda_0 x} w(x),$$

where $w(x)$ is a function in $\mathcal{C}^\infty(\mathbb{R})$ that needs to be determined for this Ansatz. This results in the following

Proposition 4.2.4.

a) Suppose that the characteristic equation $P_n(\lambda) = 0$ of the homogeneous equation

$$p_n y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_1 y' + p_0 y = 0, \qquad p_j \in \mathbb{R} \tag{4.2.10}$$

admits the real root $\lambda = \lambda_0 \in \mathbb{R}$ with multiplicity of degree $k \in \mathbb{N}$, with $k \leq n$. Then the Ansatz

$$y(x) = e^{\lambda_0 x} w(x), \qquad w(x) \in \mathcal{C}^\infty(\mathbb{R}),$$

leads to the following set of linearly independent solutions for (4.2.10):

$$\left\{ e^{\lambda_0 x},\ x e^{\lambda_0 x},\ x^2 e^{\lambda_0 x},\ \ldots,\ x^{k-1} e^{\lambda_0 x} \right\}. \tag{4.2.11}$$

b) Suppose that the characteristic equation $P_n(\lambda) = 0$ of the homogeneous equation (4.2.10) admits the complex root $\lambda_0 = \alpha + i\beta$ ($\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$) with multiplicity of degree $k \in \mathbb{N}$, with $k \leq n$. Then the Ansatz

$$y(x) = e^{\lambda_0 x} w(x), \qquad w(x) \in \mathcal{C}^\infty(\mathbb{R}),$$

leads to the following set of linearly independent solutions for (4.2.10):

$$\Big\{ e^{\alpha x} \cos(\beta x),\ e^{\alpha x} \sin(\beta x),\ x\, e^{\alpha x} \cos(\beta x),\ x\, e^{\alpha x} \sin(\beta x), \tag{4.2.12}$$

$$x^2\, e^{\alpha x} \cos(\beta x),\ x^2\, e^{\alpha x} \sin(\beta x),\ \ldots, x^{k-1}\, e^{\alpha x} \cos(\beta x),\ x^{k-1}\, e^{\alpha x} \sin(\beta x) \Big\}.$$

The proof is left as an exercise. See Exercises 4.2.1 numbers 4 and 5 for two special cases of the general third- and fourth-order equations.

Example 4.2.5.

a) We find the general solution of the third-order equation

$$y^{(3)} - 6y'' + 12y' - 8y = 0. \tag{4.2.13}$$

For the Ansatz $y(x) = e^{\lambda x}$, the characteristic equation takes the form

$$P_3(\lambda) = \lambda^3 - 6\lambda^2 + 12\lambda - 8 = (\lambda - 2)^3 = 0.$$

Hence this third degree characteristic polynomial $P_3(\lambda)$ admits only one root, namely the real root $\lambda = 2$ and this root has multiplicity of degree 3. By Proposition 4.2.4

the root $\lambda = 2$ leads to a set of three linearly independent solutions for the equation (4.2.13), namely

$$\left\{e^{2x},\ xe^{2x},\ x^2e^{2x}\right\},$$

so that the general solution of (4.2.13) is of the form

$$y(x) = c_1e^{2x} + c_2xe^{2x} + c_3x^2e^{2x}$$

for all $x \in \mathbb{R}$.

b) We find the general solution of the fifth-order equation

$$y^{(5)} - 2y^{(4)} + 2y^{(3)} - 4y'' + y' - 2y = 0. \tag{4.2.14}$$

For the Ansatz $y(x) = e^{\lambda x}$, the characteristic equation takes the form

$$P_5(\lambda) = \lambda^5 - 2\lambda^4 + 2\lambda^3 - 4\lambda^2 + \lambda - 2 = (\lambda - 2)(\lambda^2 + 1)^2 = 0.$$

Hence $P_5(\lambda)$ admits the following roots:

$$\lambda_1 = 2,\ \lambda_2 = i,\ \lambda_3 = i,\ \lambda_4 = -i,\ \lambda_5 = -i.$$

Note that the complex root i has a multiplicity of degree two and so does the complex root $-i$. These roots give the following solutions of (4.2.14):

$\lambda_1 = 2:$ $\quad \phi_1(x) = e^{2x}$

$\lambda_2 = i:$ $\quad$ complex solution: $\phi_{1c}(x) = e^{ix}$; real solutions: $\psi_1(x) = \cos x,\ \psi_2(x) = \sin x$

$\lambda_3 = i:$ $\quad$ real solutions due to 2nd-degree multiplicity: $\psi_3(x) = x\cos x,\ \psi_4(x) = x\sin x$.

The five solutions $\{\phi_1(x),\ \psi_1(x),\ \psi_2(x),\ \psi_3(x),\ \psi_4(x)\}$ form a linearly independent set in $\mathcal{C}^\infty(\mathbb{R})$, so that the general solution of (4.2.14) is

$$y(x) = c_1e^{2x} + c_2\cos x + c_3\sin x + c_4x\cos x + c_5x\sin x$$

for all $x \in \mathbb{R}$.

c) We find the general solution of the fourth-order equation

$$y^{(4)} + 4y^{(3)} + 8y'' + 8y' + 4y = 0. \tag{4.2.15}$$

For the Ansatz $y(x) = e^{\lambda x}$, the characteristic equation takes the form

$$P_4(\lambda) = \lambda^4 + 4\lambda^3 + 8\lambda^2 + 8\lambda + 4 = (\lambda^2 + 2\lambda + 2)^2$$

$$= (\lambda + 1 + i)^2(\lambda + 1 - i)^2 = 0.$$

Hence $P_4(\lambda)$ admits the following roots:

$$\lambda_1 = -1 - i,\ \lambda_2 = -1 - i,\ \lambda_3 = -1 + i,\ \lambda_4 = -1 + i.$$

The root $-1-i$ gives the complex solution

$$\phi_{1c}(x) = e^{-x}(\cos x - i\sin x) \quad \text{and hence two real solutions}$$

$$\psi_1(x) = e^{-x}\cos x, \quad \psi_2(x) = e^{-x}\sin x.$$

Since the root $-1-i$ has the multiplicity of degree two, two more real solutions are

$$\psi_3(x) = xe^{-x}\cos x, \quad \psi_4(x) = xe^{-x}\sin x.$$

The root $\lambda_3 = \lambda_4$ is the complex conjugates of the root $\lambda_1 = \lambda_2$ so that it does not contribute to more linearly independent solutions for (4.2.15). The set $S = \{\psi_1(x),\ \psi_2(x),\ \psi_3(x),\ \psi_4(x)\}$ is linearly independent, so that the general solution of (4.2.15) is

$$y(x) = c_1e^{-x}\cos x + c_2e^{-x}\sin x + c_3xe^{-x}\cos x + c_4xe^{-x}\sin x$$

for all $x \in \mathbb{R}$.

4.2.1 Exercises

1. Find the general solutions, or solve the initial-value problems in case initial data is given, for the following third-order equations:

   a) $y^{(3)} + 2y'' - 5y' - 6y = 0$

   b) $y^{(3)} + 3y'' - 4y' - 12y = 0, \quad y(0) = 1,\ y'(0) = 2,\ y''(0) = 3$

   c) $y^{(3)} - 5y'' + 3y' + 9y = 0$

   d) $y^{(3)} + y'' + 9y' + 9y = 0, \quad y(\pi) = 5,\ y'(\pi) = 1,\ y''(\pi) = 1$

   e) $y^{(3)} + 6y'' + 12y' + 8y = 0, \quad y(1) = 3,\ y'(1) = 0,\ y''(1) = 2$

   f) $y^{(3)} - 2y'' = 0$

   g) $y^{(3)} + 27y = 0$

   h) $y^{(3)} - 4y' = 0$

2. Find the general solutions, or solve the initial-value problems in case initial data is given, for the following fourth-order equations:

   a) $y^{(4)} - 13y'' + 36y = 0, \quad y(0) = 1,\ y'(0) = 2,\ y''(0) = 3,\ y^{(3)}(0) = 4$

   b) $y^{(4)} - 3y^{(3)} = 0, \quad y(1) = 4,\ y'(1) = 0,\ y''(1) = 2,\ y^{(3)}(1) = 1$

   c) $y^{(4)} + 8y^{(3)} + 24y'' + 32y' + 16y = 0$

   d) $y^{(4)} - 4y^{(3)} + 6y'' - 4y' + y = 0$

   e) $y^{(4)} - 2y^{(3)} - 3y'' + 4y' + 4y = 0$

   f) $y^{(4)} + 9y'' = 0, \quad y(0) = 2,\ y'(0) = 1,\ y''(0) = 2,\ y^{(3)}(0) = 1$

   g) $y^{(4)} - 3y^{(3)} = 0$

   h) $y^{(4)} - 4y^{(3)} + 4y'' = 0$

3. Find the general solutions, or solve the initial-value problems in case initial data is given, for the following fifth- or sixth-order equations:

   a) $y^{(5)} - 16y' = 0$

   b) $y^{(5)} - 3y^{(4)} + 2y^{(3)} = 0$

c) $y^{(5)} - 4y^{(4)} = 0$

d) $y^{(6)} - 5y^{(5)} + 4y^{(4)} = 0$

e) $y^{(6)} + 2y^{(5)} = 0$

f) $y^{(6)} + 3y^{(4)} = 0,$

$$y(0) = 1,\ y'(0) = 2,\ y''(0) = 3,\ y^{(3)}(0) = 4,\ y^{(4)}(0) = 5,\ y^{(5)}(0) = 6$$

4. Consider the general 3-rd order linear homogeneous differential equation with constant coefficients,

$$p_3 y^{(3)} + p_2 y'' + p_1 y' + p_0 y = 0$$

and assume that the characteristic polynomial $P_3(\lambda)$ of this equation for the Ansatz $y(x) = e^{\lambda x}$ admits a real root λ_0 of multiplicity with degree three. Use the Ansatz

$$y(x) = w(x) e^{\lambda_0 x}$$

to find the general solution for this differential equation.

Note: This is a special case of Proposition 4.2.4a) and your result would therefore verify Proposition 4.2.4a) for third-order equations, where the characteristic polynomial admits roots with multiplicity of degree three.

5. Consider the general 4-th order linear homogeneous differential equation with constant coefficients,

$$p_4 y^{(4)} + p_3 y^{(3)} + p_2 y'' + p_1 y' + p_0 y = 0$$

and assume that the characteristic polynomial $P_4(\lambda)$ of this equation for the Ansatz $y(x) = e^{\lambda x}$ admits a complex root λ_0 of multiplicity with degree two. Use the Ansatz

$$y(x) = w(x) e^{\lambda_0 x}$$

to find the general solution for this differential equation.

Note: This is a special case of Proposition 4.2.4b) and your result would therefore verify Proposition 4.2.4b) for fourth-order equations, where the characteristic polynomial admits complex roots with multiplicity of degree two.

6. Show that the following two sets, which are special cases of the set given in Proposition 4.2.3, are linearly independent:

a) $\left\{e^{\lambda_1 x},\ e^{\lambda_2 x},\ x\, e^{\lambda_1 x},\ x\, e^{\lambda_2 x},\ x^2\, e^{\lambda_1 x},\ x^2\, e^{\lambda_2 x}\right\},\ \lambda_1 \neq \lambda_2,\ \lambda_1 \in \mathbb{R},\ \lambda_2 \in \mathbb{R}.$

This is a special case of the set S_3 in Proposition 4.2.3, with $k = 2$ and $m = 2$.

b) $\{\cos(\beta_1 x),\ \sin(\beta_1 x),\ x\cos(\beta_1 x),\ x\sin(\beta_1 x),\ e^{\alpha_1 x}\cos(\beta_1 x),\ e^{\alpha_1 x}\sin(\beta_1 x)\},$ $\alpha_1 \in \mathbb{R}\backslash\{0\},\ \beta_1 \in \mathbb{R}\backslash\{0\}.$

This is a special case of the set Q in Proposition 4.2.3, with $j = 1$, $m = 1$ and $n = 0$.

4.3 Higher-order linear nonhomogeneous equations

We consider the linear n-th order equation

$$\boxed{p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x)} \tag{4.3.1}$$

Here $p_j(x)$ $(j = 0, 1, 2, \ldots, n)$ and $f(x)$ are real-valued continuous functions given on some common domain $\mathcal{D} \subseteq \mathbb{R}$, $n \geq 1$ and $p_n(x) \neq 0$ for all $x \in \mathcal{D}$.

Definition 4.3.1. *Any function $y_p(x)$ which satisfies the nonhomogeneous equation (4.3.1) on an interval $\mathcal{D}$, is known as a* **particular solution** *for (4.3.1) on $\mathcal{D}$.*

The statement below follows directly from the linear superposition principle:

Proposition 4.3.1.

a) The general solution of (4.3.1), i.e.

$$p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x)$$

is of the form

$$y(x) = \phi_H(x; c_1, c_2, \ldots, c_n) + y_p(x), \tag{4.3.2}$$

where ϕ_H is the general solution of the associated homogeneous equation,

$$p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0$$

and y_p is a particular solution of the nonhomogeneous equation (4.3.1).

b) A particular solution $y_p(x)$ for the nonhomogeneous equation

$$p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f_1(x) + f_2(x), \tag{4.3.3}$$

where $p_j(x)$, $f_1(x)$ and $f_2(x)$ are given continuous functions on $\mathcal{D}$, is given by the sum of a particular solution for $f_1(x)$ and a particular solution for $f_2(x)$, i.e.

$$y_p(x) = y_1(x) + y_2(x), \tag{4.3.4}$$

where $y_1(x)$ is a particular solution for

$$p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f_1(x)$$

and $y_2(x)$ is a particular solution for

$$p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f_2(x).$$

The proof is left as an exercise (see Exercises 4.3.2).

We'll now describe two methods to construct particular solutions for nonhomogeneous equations, namely the **method of undetermined coefficients** and the **method of variation of parameters**. These methods have already been described in detail for second-order equations, so here we'll just need to generalize these methods to higher-order equations.

4.3.1 Particular solutions: the method of undetermined coefficients

Consider the n-th order linear nonhomogeneous equation with constant coefficients

$$p_n y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_1 y' + p_0 y = f(x) \tag{4.3.5}$$

where $p_j \in \mathbb{R}$, $p_n \neq 0$ and $f(x)$ is a continuous function on some domain $\mathcal{D} \subseteq \mathbb{R}$. As in the case of second-order equations, we consider special forms of the function $f(x)$ and propose Ansätze for particular solutions $y_p(x)$ in each case.

Case I: $f(x) = P_m(x)$**:**

Consider

$$\boxed{p_n y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_1 y' + p_0 y = P_m(x)} \tag{4.3.6}$$

where P_m is an mth-degree polynomial, i.e.,

$$P_m(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0. \tag{4.3.7}$$

Here a_j ($j = 0, 1, \ldots, m$) are given real coefficients. We can now propose Ansätze to find particular solutions of (4.3.6) and we need to distinguish between $n+1$ different subcases:

Case I (1): Let $\boxed{p_0 \neq 0}$. The Ansatz for a particular solution of (4.3.6) is then

$$y_p(x) = A_m x^m + A_{m-1} x^{m-1} + \cdots + A_1 x + A_0 := Q_m(x), \tag{4.3.8}$$

where the real constants, $A_j,\ j = 0, 1, \ldots, m$, are to be determined for the Ansatz (4.3.8).

Case I (2): Let $\boxed{p_0 = 0 \text{ and } p_1 \neq 0}$. The Ansatz for a particular solution of (4.3.6) is then

$$y_p(x) = x Q_m(x), \tag{4.3.9}$$

where $Q_m(x)$ is given by (4.3.8).

Case I (3): Let $\boxed{p_0 = 0 \text{ and } p_1 = 0 \text{ and } p_2 \neq 0}$. The Ansatz for a particular solution of (4.3.6) is then

$$y_p(x) = x^2 Q_m(x), \tag{4.3.10}$$

where $Q_m(x)$ is given by (4.3.8).

$\vdots$

Case I (n+1): Let $\boxed{p_n \neq 0 \text{ and } p_k = 0 \text{ for } k = 0, 1, 2 \ldots, n-1}$. The Ansatz for a particular solution of (4.3.6) is then

$$y_p(x) = x^n Q_m(x), \tag{4.3.11}$$

where $Q_m(x)$ is given by (4.3.8).

Case II: $f(x) = e^{\alpha_1 x}\cos(\alpha_2 x)\,P_m(x)$ **or** $f(x) = e^{\alpha_1 x}\sin(\alpha_2 x)\,P_m(x)$**,** $\alpha_1, \alpha_2 \in \mathbb{R}$

We consider the following linear **complex differential equation** with dependent complex variable $y_c(x)$:

$$\boxed{p_n y_c^{(n)} + p_{n-1} y_c^{(n-1)} + \cdots + p_1 y_c' + p_0 y_c = e^{\alpha x} P_m(x),\ \alpha := \alpha_1 + i\alpha_2} \tag{4.3.12}$$

where $\alpha_1 \in \mathbb{R}$, $\alpha_2 \in \mathbb{R}$ and P_m is an mth-degree polynomial, i.e.,

$$P_m(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0 \tag{4.3.13}$$

with $a_j,\ j = 0, 1, \ldots, m$ real coefficients. Here α is a complex number, such that $\alpha = \alpha_1 + i\alpha_2$, with $\alpha_1,\ \alpha_2$ real. Every differentiable complex function $y_c(x)$ can be written in the form

$$y_c(x) = u(x) + iv(x), \tag{4.3.14}$$

where u and v are real differentiable functions on some domain $\mathcal{D} \subseteq \mathbb{R}$ and

$$y_c^{(k)}(x) = u^{(k)}(x) + iv^{(k)}(x), \quad k = 1, 2, \ldots, n. \tag{4.3.15}$$

Using (4.3.14) and (4.3.15), equation (4.3.12) takes the form

$$\begin{aligned} &p_n\left(u^{(n)} + iv^{(n)}\right) + p_{n-1}\left(u^{(n-1)} + iv^{(n-1)}\right) + \cdots + p_0(u + iv) \\ &\qquad = e^{\alpha_1 x}\cos(\alpha_2 x)\,P_m(x) + ie^{\alpha_1 x}\sin(\alpha_2 x)\,P_m(x), \end{aligned} \tag{4.3.16}$$

where we have used the relation

$$e^{(\alpha_1 + i\alpha_2)x} = e^{\alpha_1 x}\left(\cos(\alpha_2 x) + i\sin(\alpha_2 x)\right). \tag{4.3.17}$$

Comparing the real- and imaginary parts of (4.3.1), respectively, we obtain the following two real nonhomogeneous differential equations in u and v:

$$\boxed{p_n u^{(n)} + p_{n-1} u^{(n-1)} + \cdots + p_1 u' + p_0 u = e^{\alpha_1 x}\cos(\alpha_2 x)\,P_m(x)} \tag{4.3.18}$$

and

$$\boxed{p_n v^{(n)} + p_{n-1} v^{(n-1)} + \cdots + p_1 v' + p_0 v = e^{\alpha_1 x}\sin(\alpha_2 x)\,P_m(x)} \tag{4.3.19}$$

This leads to

Proposition 4.3.2. *A convenient Ansatz for a complex particular solution $y_{cp}(x)$ of (4.3.12), namely*

$$p_n y_c^{(n)} + p_{n-1} y_c^{(n-1)} + \cdots + p_1 y_c' + p_0 y_c = e^{\alpha x} P_m(x)$$

is

$$y_{cp}(x) = e^{\alpha x}\, w_c(x), \tag{4.3.20}$$

where $w_c(x)$ is a complex function that needs to be determined such that the Ansatz satisfies (4.3.12). The condition on $w_c(x)$ is a linear nonhomogeneous equation with nonhomogeneous part $P_m(x)$, so that a solution for $w_c(x)$ can be constructed by the Ansätze listed in **Case I**, *albeit* **with complex coefficients** *for Q_m in (4.3.8), i.e.*

$$Q_m = B_m x^m + B_{m-1} x^{m-1} + \cdots + B_1 x + B_0$$

with $B_j \in \mathbb{C}$ $(j = 0,\ 1,\ 2,\ \ldots,\ m)$.

Example 4.3.1.

We find the general solution of the third-order equation

$$y^{(3)} + y = e^{2x} \sin(3x). \tag{4.3.21}$$

Using the Ansatz $y(x) = e^{\lambda x}$ for the homogeneous equation, the characteristic equation is

$$\lambda^3 + 1 = 0$$

and the three roots are $\lambda_1 = -1$, $\lambda_2 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $\lambda_3 = \bar{\lambda}_2$. This leads to the following general solution $\phi_H(x; c_1, c_2, c_3)$ of the homogeneous equation:

$$\phi_H(x; c_1, c_2, c_3) = c_1 e^{-x} + c_2 e^{x/2} \cos(\sqrt{3}x/2) + c_3 e^{x/2} \sin(\sqrt{3}x/2).$$

To find a particular solution for (4.3.21) we consider the complex differential equation

$$y_c^{(3)} + y_c = e^{(2+3i)x}, \tag{4.3.22}$$

where

$$y_c(x) = u(x) + iy(x),$$

so that the imaginary part of the complex equation (4.3.22) is the given real equation (4.3.22). Therefore, an Ansatz for the complex particular solution $y_{cp}(x)$ of (4.3.22) is given by Proposition 4.3.2, i.e.

$$y_{cp}(x) = e^{(2+3i)x} w_c(x). \tag{4.3.23}$$

A real particular solution $y_p(x)$ for (4.3.22) is then

$$y_p(x) = \text{Im}\left[y_{cp}(x)\right].$$

Inserting Ansatz (4.3.23) into the complex equation (4.3.22) we obtain the following condition on $w_c(x)$:

$$w_c^{(3)} + 3(2+3i)w_c'' + 3(2+3i)^2 w_c' + \left((2+3i)^3 + 1\right) w_c = 1. \tag{4.3.24}$$

Following **Case I** we should make an Ansatz for $w_c(x)$ of a zero-degree polynomial with a complex coefficient, i.e.

$$w_c(x) = B_0, \tag{4.3.25}$$

where B_0 is a complex constant that needs to be determined such that (4.3.25) satisfies (4.3.24). Inserting this Ansatz into (4.3.24), we obtain

$$\left((2+3i)^3 + 1\right) B_0 = 1 \quad \text{or } B_0 = -\frac{5}{234} - i\left(\frac{1}{234}\right) = w_c(x).$$

Thus the complex particular solution for (4.3.22) is

$$y_{cp}(x) = e^{(2+3i)x}\left(-\frac{5}{234} - \frac{i}{234}\right) = -e^{2x}\left(\cos(3x) + i\sin(3x)\right)\left(-\frac{5}{234} - \frac{i}{234}\right)$$

so that the real particular solution $y_p(x)$ for (4.3.21) becomes

$$y_p(x) = \text{Im}\left[y_{cp}(x)\right] = -\frac{1}{234}e^{2x}\cos(3x) - \frac{5}{234}e^{2x}\sin(3x).$$

The general solution of (4.3.21) is thus

$$y(x) = c_1 e^{-x} + c_2 e^{x/2}\cos(\sqrt{3}x/2) + c_3 e^{x/2}\sin(\sqrt{3}x/2)$$
$$-\frac{1}{234}e^{2x}\cos(3x) - \frac{5}{234}e^{2x}\sin(3x).$$

Case III $f(x) = e^{\alpha x}P_m(x), \quad \alpha \in \mathbb{R}$

We consider the equation

$$\boxed{p_n y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_1 y' + p_0 y = e^{\alpha x}P_m(x), \qquad \alpha \in \mathbb{R},} \tag{4.3.26}$$

where P_m is an m-th-degree polynomial with real coefficients. We note that this is in fact a special case of (4.3.12), with $y_c(x)$ a real function $y(x)$ and $\alpha \in \mathbb{R}$. The same Ansatz (4.3.20) is valid, albeit for a real particular solution $y_p(x)$ of (4.3.26), namely

$$\boxed{y_p(x) = e^{\alpha x}\, w(x)} \tag{4.3.27}$$

where the condition on w is a linear nonhomogeneous equation with nonhomogeneous part $P_m(x)$. To find a solution $w(x)$ of this equation we use the same Ansätze as those listed in **Case I**.

Example 4.3.2.

We find the general solution of the third-order equation

$$y^{(3)} - y'' + y' - y = xe^x. \tag{4.3.28}$$

First we use the Ansatz $y(x) = e^{\lambda x}$ to find the general solution of the homogeneous equation

$$y^{(3)} - y'' + y' - y = 0.$$

The characteristic equation is

$$\lambda^3 - \lambda^2 + \lambda - 1 = 0 \quad \text{or } (\lambda - 1)(\lambda^2 + 1) = 0.$$

The root $\lambda_1 = 1$ gives the real solution e^x, whereas the complex root $\lambda_2 = i$ gives two real solutions $\{\cos x,\ \sin x\}$, so that the general homogeneous solution $\phi_H(x;\, c_1, c_2, c_3)$ is

$$\phi_H(x;\, c_1, c_2, c_3) = c_1 e^x + c_2 \cos x + c_3 \sin x$$

for all $x \in \mathbb{R}$ and arbitrary constants $c_1,\ c_2$ and c_3. For a particular solution of (4.3.28) we make use of the Ansatz proposed in **Case III** above, namely $y_p(x) = e^x w(x)$. This leads to the condition

$$w^{(3)} + 2w'' + 2w' = x$$

and by **Case I** we use the Ansatz $w(x) = x\,(A_1 x + A_0)$. This results in the condition $4A_1 + 4A_1 x + 2A_0 = x$, so that $4A_1 = 1$ and $4A_1 + 2A_0 = 0$. Hence $A_1 = \dfrac{1}{4}$ and $A_0 = -\dfrac{1}{2}$, so that $w(x) = \dfrac{1}{4}x^2 - \dfrac{1}{2}x$ and finally $y_p(x) = e^x \left(\dfrac{1}{4}x^2 - \dfrac{1}{2}x\right)$. The general solution of (4.3.28) is thus

$$y(x) = c_1 e^x + c_2 \cos x + c_3 \sin x + e^x \left(\frac{1}{4}x^2 - \frac{1}{2}x\right).$$

4.3.2 Exercises

[Solutions of those Exercises marked with a * are given in Appendix D].

1. Find the general solutions of the following equations:

   a) $y^{(3)} + y' = 2$

   b) $y^{(3)} + y'' = 3$

   c) $y^{(4)} - y'' = 4$

   d) $y^{(4)} + 4y^{(3)} + 4y'' = 1$

   e) $y^{(3)} - 2y'' + y' = 2x$

   f) $y^{(4)} + y'' = x^2 + x$

   g) $y^{(4)} + 2y^{(3)} + y'' = e^{4x}$

   h) $y^{(4)} + 2y^{(3)} + y'' = xe^{-x}$

   i) $y^{(3)} - y = \sin x$

   j) $y^{(3)} - 3y'' + 3y' - y = e^x \cos(2x)$

   k) $y^{(4)} - 2y'' + y = \cos x$

   l) $y^{(3)} + 4y' = 1 - \sin(2x) + e^{2x} \cos(2x)$

   m) $y^{(4)} - 16y'' = x \sin x$

   n) $y^{(5)} - y^{(4)} = xe^x - 1$

   o) $y^{(5)} + y^{(3)} = x + 2e^{-x}$

   p)* $y^{(4)} + 2y^{(3)} + y'' = (x + 1)^2$

   q)* $y^{(3)} + 3y'' + 3y' + y = (x + 1)e^{-x}$

2. Solve the following initial-value problems:

   a) $y^{(3)} - y' = -2x, \quad y(0) = 0,\ y'(0) = 1,\ y''(0) = 2.$

   b) $y^{(4)} - y = 8e^x, \quad y(0) = -1,\ y'(0) = 0,\ y''(0) = 1,\ y^{(3)}(0) = 0.$

   c) $y^{(3)} - 2y'' - y' + 2y = 2x^2 + 4x - 9, \quad y(0) = 2,\ y'(0) = -4,\ y''(0) = -1.$

d) $y^{(3)} + 2y'' + 5y' = 20e^{-x}\cos(2x), \quad y(0) = 1,\ y'(0) = 2,\ y''(0) = -3.$

e) $y^{(4)} + 5y'' + 4y = 40\cos(3x), \quad y(\pi/2) = 2,\ y'(\pi/2) = 2,\ y''(\pi/2) = 1,$

$y^{(3)}(\pi/2) = 4.$

3. Prove Proposition 4.3.1.

4. Consider Proposition 4.3.2 and give the explicit condition on $w_c(x)$ for the case $n = 3$. Then classify the different Ansätze that apply to obtain solutions for $w_c(x)$ for your obtained condition.

4.3.3 Particular solutions: the method of variation of parameters

We now describe the **method of variation of parameters** by generalizing Proposition 3.3.2 of Chapter 3, where we have studied this method for second-order equations. We should recall that the method of variation of parameters is more general than the method of undetermined coefficients, as it is applicable to equations which have continuous functions as coefficients and it does not require a special form of the functions $f(x)$ of the nonhomogeneous part of the equation, neither does it require lots of different Ansätze. However, the derivation of particular solutions with this method does require the calculation of integrals, which can be difficult and tedious at times.

In order to state the next proposition it is convenient to introduce a new notation: Consider a linearly independent set of functions $S = \{\phi_1(x),\ \phi_2(x),\ \ldots,\ \phi_n(x)\}$ in $\mathcal{C}^n(\mathcal{D})$ and a continuous function $f(x)$ on the interval $\mathcal{D}$. We now define

$$W_j[\phi_1, \ldots, \phi_{j-1}, (f), \phi_{j+1}, \ldots, \phi_n](x) \tag{4.3.29}$$

$$= \begin{vmatrix} \phi_1 & \ldots & \phi_{j-1} & 0 & \phi_{j+1} & \ldots & \phi_n \\ \phi_1' & \ldots & \phi_{j-1}' & 0 & \phi_{j+1}' & \ldots & \phi_n' \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ \phi_1^{(n-2)} & \ldots & \phi_{j-1}^{(n-2)} & 0 & \phi_{j+1}^{(n-2)} & \ldots & \phi_n^{(n-2)} \\ \phi_1^{(n-1)} & \ldots & \phi_{j-1}^{(n-1)} & f(x) & \phi_{j+1}^{(n-1)} & \ldots & \phi_n^{(n-1)} \end{vmatrix}, \qquad j = 1, 2, \ldots, n.$$

For example,

$$W_2[\phi_1, (f), \ldots, \phi_n](x) = \begin{vmatrix} \phi_1 & 0 & \ldots & \phi_n \\ \phi_1' & 0 & \ldots & \phi_n' \\ \vdots & \vdots & \vdots & \vdots \\ \phi_1^{(n-2)} & 0 & \ldots & \phi_n^{(n-2)} \\ \phi_1^{(n-1)} & f(x) & \ldots & \phi_n^{(n-1)} \end{vmatrix},$$

Proposition 4.3.3. *Consider the n-th order nonhomogeneous linear differential equation*

$$p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x), \tag{4.3.30}$$

where $p_j(x)$ $(j = 0, 1, 2, \ldots, n)$ and $f(x)$ are continuous functions given on some common domain $\mathcal{D} \subseteq \mathbb{R}$, $n \geq 1$ and $p_n(x) \neq 0$ for all $x \in \mathcal{D}$. Assume that n linearly independent solutions of the homogeneous equation

$$p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0 \tag{4.3.31}$$

are given by the set of $\mathcal{C}^n(\mathcal{D})$ functions $S = \{\phi_1(x),\ \phi_2(x),\ \ldots,\ \phi_n(x)\}$ on the interval $\mathcal{D} \subseteq \mathbb{R}$. Then a particular solution $y_p(x)$ of (4.3.30) is

$$y_p(x) = w_1(x)\phi_1(x) + w_2(x)\phi_2(x) + \cdots + w_n(x)\phi_n(x), \tag{4.3.32}$$

where $w_j(x)$ $(j = 1, 2, \ldots, n)$ have the following form:

$$w_1(x) = \int \frac{W_1[(f), \phi_2, \phi_3, \ldots, \phi_n](x)}{W[\phi_1, \phi_2, \ldots, \phi_n](x)}\, dx$$

$$w_2(x) = \int \frac{W_2[\phi_1, (f), \phi_3, \ldots, \phi_n](x)}{W[\phi_1, \phi_2, \ldots, \phi_n](x)}\, dx$$

$$\vdots$$

$$w_n(x) = \int \frac{W_n[\phi_1, \phi_2, \ldots, \phi_{n-1}, (f)](x)}{W[\phi_1, \phi_2, \ldots, \phi_n](x)}\, dx$$

Here $W[\phi_1, \phi_2, \ldots, \phi_n](x)$ is the Wronskian of the set S and the notation $W_j[\phi_1, \ldots, \phi_{j-1}, (f), \phi_{j+1}, \ldots, \phi_n](x)$ is defined by (4.3.29).

Proof: We consider first the case $n = 3$, that is the nonhomogeneous equation

$$p_3(x)y^{(3)} + p_2(x)y'' + p_1(x)y' + p_0(x)y = f(x). \tag{4.3.33}$$

Assume that three linearly independent solutions, namely

$$\{\phi_1(x),\ \phi_2(x),\ \phi_3(x)\},$$

are given for the homogeneous equation

$$p_3(x)y^{(3)} + p_2(x)y'' + p_1(x)y' + p_0(x)y = 0. \tag{4.3.34}$$

We now use the following Ansatz for a particular solution of (4.3.33):

$$y_p(x) = w_1\phi_1(x) + w_2(x)\phi_2(x) + w_3(x)\phi_3(x). \tag{4.3.35}$$

The first derivative $y_p'(x)$ is

$$y_p'(x) = w_1'\phi_1 + w_1\phi_1' + w_2'\phi_2 + w_2\phi_2' + w_3'\phi_3 + w_3\phi_3'. \tag{4.3.36}$$

Let now

$$w_1'\phi_1 + w_2'\phi_2 + w_3'\phi_3 = 0, \tag{4.3.37}$$

so that (4.3.36) reduces to

$$y_p'(x) = w_1\phi_1' + w_2\phi_2' + w_3\phi_3'. \tag{4.3.38}$$

Differentiating (4.3.38) one more time, we obtain

$$y_p''(x) = w_1'\phi_1' + w_1\phi_1'' + w_2'\phi_2' + w_2\phi_2'' + w_3'\phi_3' + w_3\phi_3''. \tag{4.3.39}$$

Let now

$$w_1'\phi_1' + w_2'\phi_2' + w_3'\phi_3' = 0, \tag{4.3.40}$$

so that (4.3.39) reduces to

$$y_p''(x) = w_1\phi_1'' + w_2\phi_2'' + w_3\phi_3''. \tag{4.3.41}$$

Differentiating (4.3.41) one more time, we obtain

$$y_p^{(3)}(x) = w_1'\phi_1'' + w_1\phi_1^{(3)} + w_2'\phi_2'' + w_2\phi_2^{(3)} + w_3'\phi_3'' + w_3\phi_3^{(3)}. \tag{4.3.42}$$

Inserting the Ansatz (4.3.35) and its derivatives, (4.3.38), (4.3.41), and (4.3.42) into the third-order equation (4.3.33), we obtain

$$\begin{aligned}
&w_1\left[p_3(x)\phi_1^{(3)} + p_2(x)\phi_1'' + p_1(x)\phi_1' + p_0(x)\phi_1\right]\\
&\quad + w_2\left[p_3(x)\phi_2^{(3)} + p_2(x)\phi_2'' + p_1(x)\phi_2' + p_0(x)\phi_2\right]\\
&\quad + w_3\left[p_3(x)\phi_3^{(3)} + p_2(x)\phi_3'' + p_1(x)\phi_3' + p_0(x)\phi_3\right]\\
&\quad + w_1'\phi_1'' + w_2'\phi_2'' + w_3'\phi_3'' = f(x).
\end{aligned}$$

Since $\phi_1(x)$, $\phi_2(x)$ and $\phi_3(x)$ are solutions of the homogeneous equation (4.3.34), the previous expression reduces to

$$w_1'\phi_1'' + w_2'\phi_2'' + w_3'\phi_3'' = f(x). \tag{4.3.43}$$

We thus remain with three conditions on $w_1(x)$, $w_2(x)$ and $w_3(x)$, namely the relations (4.3.37), (4.3.40) and (4.3.43), which can conveniently be expressed in matrix form

$$\begin{pmatrix} \phi_1 & \phi_2 & \phi_3 \\ \phi_1' & \phi_2' & \phi_3' \\ \phi_1'' & \phi_2'' & \phi_3'' \end{pmatrix} \begin{pmatrix} w_1' \\ w_2' \\ w_3' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ f(x) \end{pmatrix}. \tag{4.3.44}$$

Note that the determinant of the coefficient matrix on the left side is the Wronskian $W[\phi_1,\ \phi_2,\ \phi_3](x)$, which is nonzero by the assumption that the set $\{\phi_1(x),\ \phi_2(x),\ \phi_3(x)\}$ is linearly independent. Thus the algebraic system (4.3.44) has a unique solution for $w_1',\ w_2'$ and w_3' and this unique solution can be obtained from Cramer's rule as follows:

$$w_1'(x) = \frac{1}{W[\phi_1, \phi_2\phi_3](x)} \begin{vmatrix} 0 & \phi_2 & \phi_3 \\ 0 & \phi_2' & \phi_3' \\ f(x) & \phi_2'' & \phi_3'' \end{vmatrix} \equiv \frac{W_1[(f), \phi_2, \phi_3](x)}{W[\phi_1, \phi_2, \phi_3](x)}$$

$$w_2'(x) = \frac{1}{W[\phi_1, \phi_2, \phi_3](x)} \begin{vmatrix} \phi_1 & 0 & \phi_3 \\ \phi_1' & 0 & \phi_3' \\ \phi_1'' & f(x) & \phi_3'' \end{vmatrix} \equiv \frac{W_2[\phi_1, (f), \phi_3](x)}{W[\phi_1, \phi_2, \phi_3](x)}$$

$$w_3'(x) = \frac{1}{W[\phi_1, \phi_2, \phi_3](x)} \begin{vmatrix} \phi_1 & \phi_2 & 0 \\ \phi_1' & \phi_2' & 0 \\ \phi_1'' & \phi_3'' & f(x) \end{vmatrix} \equiv \frac{W_3[\phi_1, \phi_2, (f)](x)}{W[\phi_1, \phi_2, \phi_3](x)}.$$

Integrating the above expressions with respect to x establishes the Proposition for the case $n = 3$. In the same way we can show that the formulas hold for $n \geq 4$, but this is straightforward so we leave it as an exercise (See Exercises 4.3.4). $\square$.

Example 4.3.3.

We find the general solution for the equation

$$y^{(3)} - 3y'' + 3y' - y = 3\sqrt{x}\, e^x \tag{4.3.45}$$

For the homogeneous equation

$$y^{(3)} - 3y'' + 3y' - y = 0,$$

we use the Ansatz $y(x) = e^{\lambda x}$ to obtain the characteristic equation

$$(\lambda - 1)^3 = 0.$$

The three linearly independent solutions are then

$$\{\phi_1(x) = e^x,\ \phi_2(x) = xe^x,\ \phi_3(x) = x^2 e^x\}$$

so that the general solution ϕ_H of the homogeneous equation becomes

$$\phi_H(x; c_1, c_2, c_3) = c_1 e^x + c_2 x e^x + c_3 x^2 e^x.$$

For a particular solution $y_p(x)$ we apply the method of variation of parameters. Following Proposition 4.3.3 we use the Ansatz

$$y_p(x) = w_1(x)e^x + w_2(x)xe^x + w_3(x)x^2 e^x \tag{4.3.46}$$

where

$$w_1'(x) = \frac{W_1[(3\sqrt{x}\, e^x),\ \phi_2,\ \phi_3](x)}{W[\phi_1,\ \phi_2,\ \phi_3](x)}$$

$$w_2'(x) = \frac{W_2[\phi_1,\ (3\sqrt{x}\, e^x),\ \phi_3](x)}{W[\phi_1,\ \phi_2,\ \phi_3](x)}$$

$$w_3'(x) = \frac{W_3[\phi_1,\ \phi_2,\ (3\sqrt{x}\, e^x)](x)}{W[\phi_1,\ \phi_2,\ \phi_3](x)}.$$

Calculating the Wronskian W as well as $W_1,\ W_2$ and W_3, we obtain

$$W[\phi_1,\ \phi_2,\ \phi_3](x) = \begin{vmatrix} e^x & xe^x & x^2e \\ e^x & e^x + xe^x & 2xe^x + x^2e^x \\ e^x & 2e^x + xe^x & 2e^x + 4xe^x + x^2e^x \end{vmatrix} = 2e^{3x}$$

$$W_1[(3\sqrt{x}\,e^x),\ \phi_2,\ \phi_3](x) = \begin{vmatrix} 0 & xe^x & x^2e \\ 0 & e^x + xe^x & 2xe^x + x^2e^x \\ 3\sqrt{x}\,e^x & 2e^x + xe^x & 2e^x + 4xe^x + x^2e^x \end{vmatrix} = 3x^{5/2}\ e^{3x}$$

$$W_2[\phi_1,\ (3\sqrt{x}\,e^x),\ \phi_3](x) = \begin{vmatrix} e^x & 0 & x^2e \\ e^x & 0 & 2xe^x + x^2e^x \\ e^x & 3\sqrt{x}\,e^x & 2e^x + 4xe^x + x^2e^x \end{vmatrix} = -6x^{3/2}e^{3x}$$

$$W_3[\phi_1,\ \phi_2,\ (3\sqrt{x}\,e^x)](x) = \begin{vmatrix} e^x & xe^x & 0 \\ e^x & e^x + xe^x & 0 \\ e^x & 2e^x + xe^x & 3\sqrt{x}\,e^x \end{vmatrix} = 3\sqrt{x}e^{3x}.$$

Thus we have

$$w_1(x) = \frac{3}{2}\int x^{5/2}\,dx = \frac{3}{7}x^{7/2}$$

$$w_2(x) = -3\int x^{3/2}\,dx = -\frac{6}{5}x^{5/2}$$

$$w_3(x) = \frac{3}{2}\int x^{1/2}\,dx = x^{3/2}.$$

Inserting this back into the Ansatz (4.3.46) we obtain the following particular solution for (4.3.45)

$$y_p(x) = \frac{8}{35}x^{7/2}e^x$$

and hence the general solution of (4.3.45) is

$$y(x) = c_1e^x + c_2xe^x + c_3x^2e^x + \frac{8}{35}x^{7/2}e^x$$

for all $x > 0$.

4.3.4 Exercises

[Solutions of those Exercises marked with a * are given in Appendix D].

1. Find the general solutions of the following equations:

   a) $y^{(3)} + y'' = \dfrac{x-1}{x^2}$

   b) $y^{(3)} + 3y'' + 3y' + y = e^{-x}\ln(x), \quad x > 0$

   c)* $y^{(3)} - 6y'' + 11y' - 6y = \dfrac{e^{3x}}{e^{2x}+1}$

2. Consider the following equation:

   $$y^{(4)} - 4y^{(3)} + 6y'' - 4y' + y = \frac{e^x}{x^n}, \quad x > 0,\ n \in \mathbb{R}.$$

   Find particular solutions for all $n \in \mathbb{R}$. Note that there exist five essentially different cases!

3. Solve the following initial-value problems:

   a) $y^{(3)} - 3y'' + 2y' = 4x - 8 + \dfrac{2e^{2x}}{e^x+1}, \quad y(0) = 1,\ y'(0) = -1,\ y''(0) = 2.$

   b) $y^{(4)} - y = 8e^x, \quad y(0) = 0,\ y'(0) = 2,\ y''(0) = 4,\ y^{(3)}(0) = 6.$

   c) $y^{(5)} = \dfrac{288}{x}, \quad x > 0,\ y(1) = 7,\ y'(1) = 2,\ y''(1) = 0,\ y^{(3)}(1) = 0,\ y^{(4)}(1) = 0.$

4. Prove Proposition 4.3.3 for all $n \geq 3$.

4.4 The higher-order Cauchy-Euler equation

The n-th order linear equation

$$\boxed{p_n x^n y^{(n)} + p_{n-1} x^{n-1} y^{(n-1)} + \cdots + p_1 x y' + p_0 y = f(x)} \tag{4.4.1}$$

where $p_0,\ p_1, \ldots, p_n$ are constants, $p_n \neq 0$, and the function $f(x)$ is continuous on some interval $\mathcal{D} \subseteq \mathbb{R}$, is called the **Cauchy-Euler equation of order** n.

Similar to the second-order Cauchy-Euler equation discussed in Chapter 3, equation (4.4.1) also can be transformed into a linear nonhomogeneous equation with constant coefficients of its homogeneous part. For $x > 0$ the change of variables is

$$\begin{cases} x = e^z \Leftrightarrow z = \ln(x) \\ y(x) = y(z). \end{cases}$$

Differentiating y with respect to x we can identify a formula for the n-th derivative under this change of variables:

$$\begin{aligned}
\frac{dy(x)}{dx} &= \frac{dy}{dz}\,\frac{dz}{dx} = \frac{dy}{dz}\,\frac{1}{x} = \frac{dy}{dz}\,e^{-z} \\
&= e^{-z}\left(\frac{d}{dz}\right) y \\
\frac{d^2y(x)}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dz}\,e^{-z}\right) = \frac{d^2y}{dz^2}\,\frac{dz}{dx}\,e^{-z} + \frac{dy}{dz}\left(-e^{-z}\,\frac{dz}{dx}\right) = \frac{d^2y}{dz^2}\,e^{-2z} - \frac{dy}{dz}\,e^{-2z} \\
&= e^{-2z}\,\frac{d}{dz}\left(\frac{d}{dz} - 1\right) y \\
\frac{d^3y(x)}{dx^3} &= \frac{d}{dz}\left[e^{-2z}\left(\frac{d^2y}{dx^2} - \frac{dy}{dz}\right)\right] = e^{-3z}\left(\frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}\right) \\
&= e^{-3z}\,\frac{d}{dz}\left(\frac{d}{dz} - 1\right)\left(\frac{d}{dz} - 2\right) y \\
\frac{d^4y(x)}{dx^4} &= e^{-4z}\,\frac{d}{dz}\left(\frac{d}{dz} - 1\right)\left(\frac{d}{dz} - 2\right)\left(\frac{d}{dz} - 3\right) y \\
&\ \ \vdots \\
\frac{d^ny(x)}{dx^n} &= e^{-nz}\,\frac{d}{dz}\left(\frac{d}{dz} - 1\right)\left(\frac{d}{dz} - 2\right)\cdots\left(\frac{d}{dz} - n + 1\right) y.
\end{aligned}$$

This leads to the following

Proposition 4.4.1. *The transformation*

$$\begin{cases} x = e^z \Leftrightarrow z = \ln(x), \qquad x > 0 \\ y(x) = y(z). \end{cases} \tag{4.4.2}$$

reduces the Cauchy-Euler equation

$$p_n x^n y^{(n)} + p_{n-1} x^{n-1} y^{(n-1)} + \cdots + p_1 x y' + p_0 y = f(x), \qquad x > 0 \tag{4.4.3}$$

to an equation with constant coefficients of the form

$$b_n \frac{d^n y}{dz^n} + b_{n-1} \frac{d^{n-1} y}{dz^{n-1}} + \cdots + b_1 \frac{dy}{dz} + b_0 y(z) = f(e^z), \tag{4.4.4}$$

where $b_0,\ b_1,\ \ldots,\ b_n$ *are constants which are related to the constants* $p_0,\ p_1.\ \ldots,\ p_n$. *Moreover, the transformation*

$$\begin{cases} x = -e^z \Leftrightarrow z = \ln(-x), \qquad x < 0 \\ y(x) = y(z). \end{cases} \tag{4.4.5}$$

reduces the Cauchy-Euler equation

$$p_n x^n y^{(n)} + p_{n-1} x^{n-1} y^{(n-1)} + \cdots + p_1 x y' + p_0 y = f(x), \qquad x < 0 \tag{4.4.6}$$

to an equation with constant coefficients of the form

$$c_n \frac{d^n y}{dz^n} + c_{n-1} \frac{d^{n-1} y}{dz^{n-1}} + \cdots + c_1 \frac{dy}{dz} + c_0 y(z) = f(-e^z), \tag{4.4.7}$$

where $c_0,\ c_1,\ \ldots,\ c_n$ *are constants which are related to the constants* $p_0,\ p_1.\ \ldots,\ p_n$. *For both transformations (4.4.2) and (4.4.5) the* k*-th derivative of* y *with respect to* x *is transformed by the following formula:*

$$\frac{d^k y(x)}{dx^k} = e^{-kz} \frac{d}{dz} \left(\frac{d}{dz} - 1 \right) \left(\frac{d}{dz} - 2 \right) \cdots \left(\frac{d}{dz} - k + 1 \right) y(z). \tag{4.4.8}$$

For the special cases $n = 3$ and $n = 4$ we have included exercises to derive the explicit 3-rd- and 4-th order constant coefficient equations (see Exercises 4.4.1).

Example 4.4.1.

We find the general solution for the 3-rd order Cauchy-Euler equation

$$x^3 y^{(3)} - 6x^2 y'' + 18 x y' - 24 y = 4x^4, \qquad x > 0. \tag{4.4.9}$$

Applying Proposition 4.4.1, namely the transformation (4.4.2) and the formula (4.4.8), we have

$$y^{(3)}(x) = e^{-3z}\frac{d}{dz}\left(\frac{d}{dz}-1\right)\left(\frac{d}{dz}-2\right)y(z)$$

$$= e^{-3z}\frac{d}{dz}\left(\frac{d}{dz}-1\right)\left(\frac{dy}{dz}-2y(z)\right)$$

$$= e^{-3z}\frac{d}{dz}\left(\frac{d^2y}{dz^2}-2\frac{dy}{dz}-\frac{dy}{dz}+2y(z)\right)$$

$$= e^{-3z}\left(\frac{d^3y}{dz^3}-3\frac{d^2y}{dz^2}+2\frac{dy}{dz}\right)$$

$$y''(x) = e^{-2z}\frac{d}{dz}\left(\frac{d}{dz}-1\right)y(z) = e^{-2z}\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right)$$

$$y'(x) = e^{-z}\left(\frac{d}{dz}\right)y(z) = e^{-z}\frac{dy}{dz}.$$

Equation (4.4.9) then transforms into

$$\frac{d^3y}{dz^3}-9\frac{d^2y}{dz^2}+26\frac{dy}{dz}-24y(z) = 4e^{4z}. \tag{4.4.10}$$

For the homogeneous part of this equation, i.e.

$$\frac{d^3y}{dz^3}-9\frac{d^2y}{dz^2}+26\frac{dy}{dz}-24y(z) = 0$$

we make the Ansatz

$$y(z) = e^{\lambda z}$$

and obtain the characteristic equation

$$\lambda^3 - 9\lambda^2 + 26\lambda - 24 = 0$$

with the roots $\lambda_1 = 2,\ \lambda_2 = 3,\ \lambda_3 = 4$. Hence the general homogeneous solution $\phi_H(z;\, c_1,\, c_2,\, c_3)$ is

$$\phi_H(z;\, c_1,\, c_2,\, c_3) = c_1e^{2z} + c_2e^{3z} + c_3e^{4z}.$$

For a particular solution of (4.4.10) we make the Ansatz (see **Case II** in Section 4.3.1 of the method of undetermined coefficients)

$$y_p(z) = w(z)e^{4z}$$

which leads to

$$\frac{d^3w}{dz^3}+3\frac{d^2w}{dz^2}+2\frac{dw}{dz} = 4.$$

To find a solution for $w(z)$ that satisfies this condition we use the Ansatz (see **Case I** in Section 4.3.1 of the method of undetermined coefficients)

$$w(z) = zA_0$$

(A_0 is a constant) and obtain $A_0 = 2$, so that $w(z) = 2z$ and a particular solution for (4.4.10) is

$$y_p(z) = 2ze^{4z}.$$

The general solution of (4.4.10) is then

$$y(z) = c_1e^{2z} + c_2e^{3z} + c_3e^{4z} + 2ze^{4z}$$

and back-substituting $z = \ln x$ we obtain the general solution of (4.4.9), namely

$$y(x) = c_1x^2 + c_2x^3 + c_3x^4 + 2x^4 \ln x.$$

The Cauchy-Euler equation can be generalized to the equation

$$\boxed{p_n(\alpha+\beta x)^n y^{(n)} + p_{n-1}(\alpha+\beta x)^{n-1} y^{(n-1)} + \cdots + p_1(\alpha+\beta x)y' + p_0 y = f(x)} \tag{4.4.11}$$

where $p_j \in \mathbb{R},\ j = 0, 1, \ldots, n)$ and $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$. We call this the **generalized Cauchy-Euler equation of order** n. Similar to Proposition 4.4.1, the generalized Cauchy-Euler equation can also be reduced to a linear nonhomogeneous equation of order n with constant coefficients.

Proposition 4.4.2. *The transformation*

$$\begin{cases} \alpha+\beta x = e^z \Leftrightarrow z = \ln(\alpha+\beta x), & x > -\dfrac{\alpha}{\beta} \\ y(x) = y(z). \end{cases} \tag{4.4.12}$$

reduces the generalized Cauchy-Euler equation

$$\begin{aligned} &p_n(\alpha+\beta x)^n y^{(n)} + p_{n-1}(\alpha+\beta x)^{n-1} y^{(n-1)} + \cdots + p_1(\alpha+\beta x)y' + p_0 y = f(x) \\ &x > -\frac{\alpha}{\beta} \end{aligned} \tag{4.4.13}$$

to an equation with constant coefficients of the form

$$b_n \frac{d^n y}{dz^n} + b_{n-1}\frac{d^{n-1}y}{dz^{n-1}} + \cdots + b_1 \frac{dy}{dz} + b_0 y(z) = f\left(\frac{e^z - \alpha}{\beta}\right), \tag{4.4.14}$$

where $b_0,\ b_1,\ \ldots,\ b_n$ are constants which are related to $p_j,\ \alpha$ and β.

The transformation

$$\begin{cases} \alpha+\beta x = -e^z \Leftrightarrow z = \ln[-(\alpha+\beta x)], & x < -\dfrac{\alpha}{\beta} \\ y(x) = y(z). \end{cases} \tag{4.4.15}$$

reduces the generalized Cauchy-Euler equation

$$\begin{aligned} &p_n(\alpha+\beta x)^n y^{(n)} + p_{n-1}(\alpha+\beta x)^{n-1} y^{(n-1)} + \cdots + p_1(\alpha+\beta x)y' + p_0 y = f(x) \\ &x < -\frac{\alpha}{\beta} \end{aligned} \tag{4.4.16}$$

to an equation with constant coefficients of the form

$$c_n \frac{d^n y}{dz^n} + c_{n-1}\frac{d^{n-1}y}{dz^{n-1}} + \cdots + c_1 \frac{dy}{dz} + c_0 y(z) = f\left(-\frac{e^z + \alpha}{\beta}\right), \tag{4.4.17}$$

where $c_0,\ c_1,\ \ldots,\ c_n$ are constants which are related to $p_j,\ \alpha$ and β.

For the special cases $n = 3$ and $n = 4$ we have included exercises to derive the explicit 3-rd- and 4-th order constant coefficient equations (see Exercises 4.4.1).

4.4.1 Exercises

[Solutions of those Exercises marked with a * are given in Appendix D].

1. Show that under the transformation $x = e^z$ and $y(x) = y(z)$, $x > 0$, the third-order Cauchy-Euler equation

$$p_3x^3\frac{d^3y}{dx^3} + p_2x^2\frac{d^2y}{dx^2} + p_1x\frac{dy}{dx} + +p_0y = f(x) \tag{4.4.18}$$

$(p_j \in \mathbb{R},\ j = 0, 1, \ldots, 3)$ transforms into the third-order equation

$$p_3\frac{d^3y}{dz^3} + (p_2 - 3p_3)\frac{d^2y}{dz^2} + (p_1 - p_2 + 2p_3)\frac{dy}{dz} + p_0y(z) = f(e^z). \tag{4.4.19}$$

2. Show that under the transformation $x = e^z$ and $y(x) = y(z)$, $x > 0$, the fourth-order Cauchy-Euler equation

$$p_4x^4\frac{d^4y}{dx^4} + p_3x^3\frac{d^3y}{dx^3} + p_2x^2\frac{d^2y}{dx^2} + p_1x\frac{dy}{dx} + p_0y = f(x) \tag{4.4.20}$$

$(p_j \in \mathbb{R},\ j = 0, 1, \ldots, 4)$ transforms into the fourth-order equation

$$p_4\frac{d^4y}{dz^4} + (p_3 - 6p_4)\frac{d^3y}{dz^3} + (p_2 - 3p_3 + 11p_4)\frac{d^2y}{dz^2} + (p_1 - p_2 + 2p_3 - 6p_4)\frac{dy}{dz}$$
$$+p_0y(z) = f(e^z). \tag{4.4.21}$$

3. Find the general solutions of the following Cauchy-Euler equations for $x > 0$:

a) $x^3y^{(3)} + xy' - y = 0$

b) $x^3y^{(3)} - 3x^2y'' + 6xy' - 6y = 0$

c) $x^4y^{(4)} + 10y = 0$

d) $x^3y^{(3)} - xy' - 3y = x^2$

e) $x^3y^{(3)} - x^2y'' + 2xy' - 2y = x^3 + 3x$

f) $x^3y^{(3)} + 8x^2y'' + 12xy' = \ln x$

g) $x^3y^{(3)} - 6x^2y'' + 18xy' - 24y = 48 - 2x^2 + 4x^4$

h)* $x^4y^{(4)} + 12x^3y^{(3)} + 38x^2y'' + 32xy' + 4y = \dfrac{2}{x} + \dfrac{4}{x^2}$

i) $x^4y^{(4)} + 6x^3y^{(3)} + 9x^2y'' + 3xy' + y = 8\cos(\ln x)$

j) $x^5y^{(5)} + 10x^4y^{(4)} + 25x^3y^{(3)} + 15x^2y'' + xy' = \dfrac{288}{\ln x}$

4. Consider the 3-rd order generalized Cauchy-Euler equation

$$p_3(\alpha+\beta x)^3y^{(3)} + p_2(\alpha+\beta x)^2y'' + p_1(\alpha+\beta x)y' + p_0y = f(x) \tag{4.4.22}$$

$$x > -\frac{\alpha}{\beta}$$

and find the explicit 3-rd order linear constant coefficient nonhomogeneous differential equation that results when (4.4.22) is transformed under the transformation

$$\begin{cases} \alpha+\beta x = e^z \Leftrightarrow z = \ln(\alpha+\beta x), & x > -\dfrac{\alpha}{\beta} \\ y(x) = y(z). \end{cases}$$

5. Show that the 4-th order generalized Cauchy-Euler equation

$$p_4(\alpha+\beta x)^4y^{(4)} + p_3(\alpha+\beta x)^3y^{(3)} + p_2(\alpha+\beta x)^2y''$$
$$+p_1(\alpha+\beta x)y' + p_0y = f(x), \quad x > -\frac{\alpha}{\beta} \tag{4.4.23}$$

is transformed into

$$p_4\,\beta^4\,\frac{d^4y}{dz^4} + (p_3\beta^3 - 6p_4\beta^4)\,\frac{d^3y}{dz^3} + (p_2\beta^2 - 3p_3\beta^3 + 11p_4\beta^4)\,\frac{d^2y}{dz^2}$$
$$+(p_1\beta - p_2\beta^2 + 2p_3\beta^3 - 6p_4\beta^4)\,\frac{dy}{dz} + p_0y(z) = f\left(\frac{e^z-\alpha}{\beta}\right) \tag{4.4.24}$$

under the transformation

$$\begin{cases} \alpha+\beta x = e^z \Leftrightarrow z = \ln(\alpha+\beta x), & x > -\dfrac{\alpha}{\beta} \\ y(x) = y(z). \end{cases}$$

6. Find the general solution of

$$(4+x)^4y^{(4)} + 6(4+x)^3y^{(3)} = x, \quad x > -4$$

Appendix A

Integral operators: an alternative approach for solving linear differential equations

In this appendix we make use of linear operators to derive formulae for the general solution of linear differential equations of order n. This provides an alternative method, to those proposed in the previous chapter, to find the (general) solutions of these equations without the need of any Ansätze.. The key lies in the factorization of the linear differential operators that determine the differential equation in terms of first-order differential operators. For linear equations with nonconstant coefficients, the condition for the factorization of the operators is in the form of Riccati equations. This method also provides an alternative method to the *method of variation of parameters* and the *method of undetermined coefficients* for the calculation of particular solutions of linear nonhomogeneous equations, which we studied in Chapters 3 and 4.

A.1 The definition of $\hat{L}$

In this section we introduce linear operators and introduce an integral operator that corresponds to a general first-order linear differential operator. This integral operator is the key to the integration of the linear equations.

We remind that $\mathcal{C}(\mathcal{D})$ denotes the vector space of all continuous functions on some domain $\mathcal{D} \subseteq \mathcal{R}$ and $\mathcal{C}^n(\mathcal{D})$ the subspace of $\mathcal{C}(\mathcal{D})$ consisting of all n-continuously differentiable functions on $\mathcal{D}$.

Definition A.1.1. We define the linear transformation

$$L : \mathcal{C}^n(\mathcal{D}) \to \mathcal{C}(\mathcal{D})$$

for all $f(x) \in \mathcal{C}^n(\mathcal{D})$ on the interval $\mathcal{D} \subseteq \mathcal{R}$ as

$$L : f(x) \mapsto Lf(x), \tag{A.1.1}$$

where L is the linear differential operator of order n

$$L := p_n(x)D_x^n + p_{n-1}(x)D_x^{(n-1)} + \cdots + p_1(x)D_x + p_0(x). \tag{A.1.2}$$

Here $n \in \mathcal{N}$ and $p_j(x) \in \mathcal{C}^n(\mathcal{D})$ $(j = 0, 1, \ldots, n)$ with

$$D_x^{(k)} := \frac{d^k}{dx^k}, \qquad D_x \equiv D_x^{(1)}$$

so that

$$Lf(x) = p_n(x)f^{(n)}(x) + p_{n-1}(x)f^{(n-1)}(x) + \cdots + p_1(x)f'(x) + p_0(x)f(x).$$

Example A.1.1.

Consider the second-order linear operator

$$L = \cos x D_x^2 + e^x D_x + x^2.$$

As an example, let us act L on both e^{2x} and on $u(x)\,e^{-x}$:

$$L\,e^{2x} = 4e^{2x}\cos x + 2e^{3x} + x^2e^{2x} \quad \text{and}$$

$$L\left(u(x)\,e^{-x}\right) = \left(e^{-x}\cos x\right)u'' + \left(1 - 2e^{-x}\cos x\right)u' + \left(e^{-x}\cos x + x^2e^{-x} - 1\right)u$$
$$\equiv \tilde{L}\,u(x),$$

where $\tilde{L}$ is another linear operator given by

$$\tilde{L} := e^{-x}\cos x D_x^2 + \left(1 - 2e^{-x}\cos x\right)D_x + e^{-x}\cos x + x^2e^{-x} - 1.$$

Next we define the composite linear operator and the integral operator D_x^{-1}:

Definition A.1.2.

a) Consider two linear differential operators of the form (A.1.2), namely L_1 of order m and L_2 of order n, and consider a function $f(x) \in \mathcal{C}^{m+n}(\mathcal{D})$ with $\mathcal{D} \subseteq \mathcal{R}$. The **composite operator** $L_1 \circ L_2$ is defined by

$$L_1 \circ L_2\, f(x) := L_1\left(L_2 f(x)\right) \tag{A.1.3}$$

where $L_1 \circ L_2$ is a linear differential operator of order $m + n$.

b) The **integral operator**, D_x^{-1}, is defined by the linear mapping

$$D_x^{-1}:\ \mathcal{C}(\mathcal{D}) \to \mathcal{C}^1(\mathcal{D})$$

for all $f(x) \in \mathcal{C}(\mathcal{D})$ on the interval $\mathcal{D} \subseteq \mathcal{R}$ as

$$D_x^{-1}: f(x) \mapsto D_x^{-1} f(x) \tag{A.1.4}$$

where

$$D_x^{-1} f(x) := \int f(x)dx \tag{A.1.5}$$

Note that, in general, $L_1 \circ L_2\ f(x) \neq L_2 \circ L_1 f(x)$.

Example A.1.2.

We consider the two linear differential operators

$$L_1 = D_x + x^2, \qquad L_2 = xD_x^2 + 1.$$

Then

$$L_2 \circ L_1 \, f(x) = \left(xD_x^2 + 1\right)\left(f' + x^2 f\right)$$
$$= xf^{(3)} + x^3 f'' + (4x^2 + 1)f' + (x^2 + 2x)f \equiv L_3 \, f(x),$$

where $L_3 := xD_x^3 + x^3 D_x^2 + (4x^2 + 1)D_x + x^2 + 2x$. Furthermore

$$L_1 \circ L_2 \, f(x) = xf^{(3)} + (x^3 + 1)f'' + f' + x^2 f \equiv L_4 \, f(x)$$

where $L_4 := xD_x^3 + (x^3 + 1)D_x^2 + D_x + x^2$. Clearly

$$L_2 \circ L_1 \, f(x) \neq L_1 \circ L_2 \, f(x).$$

Let f be a differentiable function. Then, following Definition A.1.2 b), we have

$$D_x^{-1} \circ D_x f(x) = D_x^{-1} f'(x) = f(x) + c, \tag{A.1.6}$$

where c is an arbitrary constant of integration and, furthermore,

$$D_x \circ D_x^{-1} f(x) = \frac{d}{dx}\left(\int f(x)\, dx + c\right) = f(x). \tag{A.1.7}$$

In terms of the linear operator (A.1.2), the nth-order linear nonhomogeneous differential equation,

$$p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x) \tag{A.1.8}$$

takes the form

$$L\, y(x) = f(x) \tag{A.1.9}$$

where L is the linear operator (A.1.2).

In the next section we introduce a method to solve (A.1.9) by factorizing L into first-order linear operators and then act the corresponding integral operators to eliminate all derivatives. For this purpose the following definition plays a central role:

Definition A.1.3. Let a and b be continuous functions on some interval $\mathcal{D} \subseteq \mathcal{R}$, such that $a(x) \neq 0$ for all $x \in \mathcal{D}$. Consider the first-order linear operator

$$L = a(x)D_x + b(x). \tag{A.1.10}$$

The **integral operator corresponding to** L, denoted by $\hat{L}$, is defined as follows:

$$\hat{L} := e^{-\xi(x)} D_x^{-1} \circ \frac{1}{a(x)}\, e^{\xi(x)}, \tag{A.1.11}$$

where

$$\frac{d\xi(x)}{dx} = \frac{b(x)}{a(x)}. \tag{A.1.12}$$

By Definition A.1.3 follows

Proposition A.1.1. *For any continuous function, $f(x)$, we have*

$$\hat{L} f(x) = e^{-\xi(x)} \left[\int \frac{f(x)}{a(x)}\, e^{\xi(x)}\, dx + c \right] \tag{A.1.13}$$

and

$$\hat{L} \circ L\, f(x) = f(x) + e^{-\xi(x)}\, c, \tag{A.1.14}$$

where L is defined by (A.1.10), $\hat{L}$ is defined by (A.1.11), and c is an arbitrary constant of integration.

Proof: We show that relation (A.1.14) holds:

$$\begin{aligned} \hat{L} \circ L\, f(x) &= \hat{L} \left(a(x) f'(x) + b(x) f(x) \right) \\ &= e^{-\xi(x)} \left(\int \frac{a(x) f'(x) + b(x) f(x)}{a(x)}\, e^{\xi(x)}\, dx + c \right) \\ &= e^{-\xi(x)} \left(\int f'(x) e^{\xi(x)}\, dx + \int \frac{b(x)}{a(x)} f(x)\, e^{\xi(x)}\, dx + c \right). \end{aligned}$$

Note that $f(x)' e^{\xi(x)} + f(x) \left(e^{\xi(x)} \right)' = \left(f(x) e^{\xi(x)} \right)'$, so that (A.1.14) follows. $\square$

Example A.1.3.

We consider $L = xD_x + x^2$ and $f(x) = e^{-x^2/2}$. Then the corresponding integral operator is

$$\hat{L} = e^{-x^2/2} D_x^{-1} \circ \frac{e^{x^2/2}}{x},$$

so that $\hat{L}\, e^{-x^2/2} = e^{-x^2/2}\,(\ln|x| + c)$ and $\hat{L} \circ L\, e^{-x^2/2} = e^{-x^2/2}(1 + c)$.

Consider now the first-order linear differential equation in the form

$$y' + g(x)y = h(x). \tag{A.1.15}$$

In terms of a linear differential operator (A.1.2) we can write (A.1.15) in the form

$$Ly(x) = h(x), \tag{A.1.16}$$

where L is the first-order linear operator

$$L = D_x + g(x). \tag{A.1.17}$$

Following Definition A.1.3 we now apply the corresponding integral operator $\hat{L}$ on (A.1.16) to gain the general solution of (A.1.15). We demonstrate this explicitly in the next example.

Example A.1.4.

We find the general solution of (A.1.16), i.e.

$$y' + g(x)y = h(x).$$

Following Definition A.1.3 we apply $\hat{L}$, given by

$$\hat{L} = e^{-\xi(x)} D_x^{-1} \circ e^{\xi(x)} \quad \text{with} \quad \xi(x) = \int g(x)dx, \tag{A.1.18}$$

to the left-hand side and the right-hand side of (A.1.16). For the left-hand side we obtain

$$\hat{L} \circ Ly(x) = y(x) + c_1 e^{-\xi(x)}, \quad (c_1 \text{ is an arbitrary constant})$$

and for the right-hand side

$$\hat{L}h(x) = e^{-\xi(x)} \int \left[h(x) e^{\xi(x)} dx + c_2 \right] \qquad (c_2 \text{ is an arbitrary constant}).$$

Thus

$$y(x) + c_1 e^{-\xi(x)} = e^{-\xi(x)} \int \left[h(x) e^{\xi(x)} dx + c_2 \right],$$

or, equivalently

$$y(x) = e^{-\xi(x)} \left[\int h(x) e^{\xi(x)} dx + c \right],$$

where $c = c_2 - c_1$ is an arbitrary constant and $\xi(x) = \int g(x)dx$.

A.2 Higher-order linear constant-coefficient differential equations

In order to apply the method of linear operators and their corresponding integral operators to solve higher-order linear differential equations, we need to factorize the higher-order linear operators that determine the differential equations in terms of first-order operators. This is in principle always possible, but in practise there are some obstacles.

The linear operators for the constant-coefficient homogeneous equation factorizes in terms of first-order differential operators in the same manner as the characteristic equation in λ for the Ansatz $y(x) = e^{\lambda x}$ does. Take, for example, the second-order equation

$$y'' + py' + qy = f(x), \qquad p \in \mathcal{R},\ q \in \mathcal{R}, \tag{A.2.1}$$

with characteristic equation

$$\lambda^2 + p\lambda + q = 0 \tag{A.2.2}$$

and roots

$$\lambda_1 = \frac{1}{2}\left(-p + \sqrt{p^2 - 4q}\right), \qquad \lambda_2 = \frac{1}{2}\left(-p - \sqrt{p^2 - 4q}\right). \tag{A.2.3}$$

Equation (A.2.1) can then be presented in the form

$$L\,y(x) = f(x), \tag{A.2.4}$$

where

$$L = D_x^2 + pD_x + q. \tag{A.2.5}$$

It is now easy to show that the second-order operator (A.2.5) factorizes as

$$L = (D_x - \lambda_1) \circ (D_x - \lambda_2) \equiv L_1 \circ L_2, \tag{A.2.6}$$

since

$$\begin{aligned} L_1 \circ L_2\, y(x) &= (D_x - \lambda_1)(y' - \lambda_2 y) \\ &= y'' - (\lambda_1 + \lambda_2)y' + \lambda_1\lambda_2 y = 0 \end{aligned}$$

and, by (A.2.3), $\lambda_1 + \lambda_2 = -p$ and $\lambda_1\lambda_2 = q$. Thus

$$\begin{aligned} L\,u(x) &= L_1 \circ L_2\, y(x) \\ &= y'' + py' + qy = f(x). \end{aligned}$$

This directly extends to linear constant-coefficient equations of any order n:

Proposition A.2.1. *Consider a constant coefficient nth-order nonhomogeneous differential equation of the form*

$$L\,y(x) = f(x), \tag{A.2.7}$$

where

$$L = a_n D_x^n + a_{n-1} D_x^{n-1} + \cdots + a_1 D_x + a_0, \quad a_j \in \mathcal{R},\ j = 0, 1, \ldots, n. \tag{A.2.8}$$

The characteristic equation of (A.2.7), following the Ansatz $y(x) = e^{\lambda x}$, is the nth degree polynomial

$$P_n(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0 \tag{A.2.9}$$

which admits n roots, $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, ($\lambda \in \mathbb{R}$ or $\mathbb{C}$) so that (A.2.9) can be factorized as

$$(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n) = 0.$$

Then equation (A.2.7) factorizes in the form

$$L_1 \circ L_2 \circ \cdots \circ L_n \, y(x) = f(x), \tag{A.2.10}$$

where

$$L_j = D_x - \lambda_j, \qquad j = 1, 2, \ldots, n. \tag{A.2.11}$$

The general solution of (A.2.7) then follows by applying, successively, the corresponding integral operators $\hat{L}_1, \hat{L}_2, \ldots, \hat{L}_n$ to (A.2.10).

Applying now Proposition A.2.1 to second-order equations leads to the following Proposition:

Proposition A.2.2. *Consider the 2nd-order equation*

$$y'' + py' + qy = f(x), \qquad p,\ q \in \mathcal{R} \tag{A.2.12}$$

with characteristic equation

$$P_2(\lambda) = \lambda^2 + p\lambda + q = 0. \tag{A.2.13}$$

a) If the two roots, λ_1 and λ_2 of (A.2.13) are real and distinct numbers, then the general solution of (A.2.12) is

$$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + y_p(x), \tag{A.2.14}$$

where $y_p(x)$ is a particular solution of (A.2.12) given by

$$y_p(x) = \left(\frac{1}{\lambda_1 - \lambda_2}\right)\left(e^{\lambda_1 x}\int f(x)e^{-\lambda_1 x}dx - e^{\lambda_2 x}\int f(x)e^{-\lambda_2 x}dx\right) \tag{A.2.15}$$

and c_1 and c_2 are arbitrary constants.

b) If the two roots, λ_1 and λ_2 of (A.2.13) are complex numbers, then $\lambda_2 = \bar{\lambda}_1$ (λ_2 is the complex conjugate of λ_1) and the general solution of (A.2.12) is

$$y(x) = c_1\, Re\left\{e^{\lambda_1 x}\right\} + c_2\, Im\left\{e^{\lambda_1 x}\right\} + y_p(x), \tag{A.2.16}$$

where $y_p(x)$ is a particular solution of (A.2.12) given by

$$y_p(x) = \left(\frac{1}{\lambda_1 - \bar{\lambda}_1}\right)\left(e^{\lambda_1 x}\int f(x)e^{-\lambda_1 x}dx - e^{\bar{\lambda}_1 x}\int f(x)e^{-\bar{\lambda}_1 x}dx\right) \tag{A.2.17}$$

and c_1 and c_2 are arbitrary constants. Note that although λ_1 and $\bar{\lambda}_1$ are complex numbers, the solution $y(x)$ is always a real-valued function.

c) If the two roots for (A.2.13) are equal, i.e. $\lambda_1 = \lambda_2 \in \mathcal{R}$, then the general solution of (A.2.12) is

$$y(x) = c_1 e^{\lambda_1 x} + c_2\, x\, e^{\lambda_1 x} + y_p(x) \tag{A.2.18}$$

where c_1 and c_2 are arbitrary constants and a particular solution $y_p(x)$ of (A.2.12) is

$$y_p(x) = e^{\lambda_1 x}\int\left(\int f(x)e^{-\lambda_1 x}dx\right)dx. \tag{A.2.19}$$

Proof: Equation (A.2.12) can be written in the form

$$L_1 \circ L_2\, y(x) = f(x), \quad L_1 = D_x - \lambda_1, \quad L_2 = D_x - \lambda_2, \tag{A.2.20}$$

where λ_1 and λ_2 are the roots of the characteristic equation (A.2.13). The corresponding integral operators for L_1 and L_2 are

$$\hat{L}_1 = e^{\lambda_1 x} D_x^{-1} \circ e^{-\lambda_1 x}, \quad \hat{L}_2 = e^{\lambda_2 x} D_x^{-1} \circ e^{-\lambda_2 x}, \tag{A.2.21}$$

respectively. Acting $\hat{L}_1$ on (A.2.20), i.e. $\hat{L}_1 \circ L_1 \circ L_2\, y(x) = \hat{L}_1 f(x)$, we obtain

$$L_2 y(x) + e^{\lambda_1 x} k_1 = e^{\lambda_1 x} \int f(x) e^{-\lambda_1 x}\, dx + k_2 e^{\lambda_1 x}$$

$$\text{or} \quad L_2 y(x) = e^{\lambda_1 x} \int f(x) e^{-\lambda_1 x}\, dx + k_3 e^{\lambda_1 x}, \tag{A.2.22}$$

where $k_3 \equiv k_2 - k_1$ is a constant of integration. Applying now $\hat{L}_2$ on (A.2.22) leads to

$$y(x) + e^{\lambda_2 x} k_4 = e^{\lambda_2 x} \int \left(k_3 e^{(\lambda_1 - \lambda_2)x} + e^{(\lambda_1 - \lambda_2)x} F(x) \right) dx + k_5 e^{\lambda_2 x}$$

$$\text{or} \quad y(x) = e^{\lambda_2 x} \int \left(k_3 e^{(\lambda_1 - \lambda_2)x} + e^{(\lambda_1 - \lambda_2)x} F(x) \right) dx + k_6 e^{\lambda_2 x}, \tag{A.2.23}$$

where $k_6 \equiv k_5 - k_4$ is a constant of integration and

$$F(x) := \int f(x) e^{-\lambda_1 x}\, dx. \tag{A.2.24}$$

If $\lambda_1 \neq \lambda_2$, then (A.2.23) reduces (after integration by parts) to (A.2.14) for $\lambda_1 \in \mathcal{R}$ and $\lambda_2 \in \mathcal{R}$ and to (A.2.16) for complex roots λ_1 and $\lambda_2 = \bar{\lambda}_1$, or to (A.2.18) for equal roots $\lambda_1 = \lambda_2 \in \mathcal{R}$. $\square$

Example A.2.1.

We find the general solution of

$$y'' + 4y = 8x^2. \tag{A.2.25}$$

The characteristic equation and its roots are $\lambda^2 + 4 = 0$ and $\lambda_1 = 2i, \ \lambda_2 = -2i$ so (A.2.25) can be presented in factorized form

$$L_1 \circ L_2\, y(x) = 8x^2, \quad \text{where} \quad L_1 = D_x - 2i, \quad L_2 = D_x + 2i.$$

Following Proposition A.2.2, the general solution, $\phi_H(x)$, of the homogeneous part of (A.2.25) is

$$\phi_H(x; c_1, c_2) = c_1 \cos(2x) + c_2 \sin(2x). \tag{A.2.26}$$

For a particular solution $y_p(x)$ we use formula (A.2.17) and calculate the integrals:

$$y_p(x) = \frac{1}{4i} \left(e^{2ix} \int 8x^2 e^{-2ix}\, dx - e^{-2ix} \int 8x^2 e^{2ix}\, dx \right) = 2x^2 - 1.$$

The general solution of (A.2.25) is thus

$$y(x) = c_1 \cos(2x) + c_2 \sin(2x) + 2x^2 - 1,$$

where c_1 and c_2 are arbitrary constants.

For an nth-order linear constant-coefficient equations we then have

Proposition A.2.3. *Consider a constant coefficient nth-order nonhomogeneous differential equation with $n > 2$ of the form*

$$L\, y(x) = f(x), \tag{A.2.27}$$

where

$$L = a_n D_x^n + a_{n-1} D_x^{n-1} + \cdots + a_1 D_x + a_0 \tag{A.2.28}$$

$$a_j \in \mathcal{R}, \quad j = 0, 1, \ldots, n.$$

Equation (A.2.27) factorizes in the form

$$L_1 \circ L_2 \circ \cdots \circ L_n\, y(x) = f(x), \tag{A.2.29}$$

where $L_j = D_x - \lambda_j$ $(j = 1, 2, \ldots, n)$ and $\{\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n\}$ ($\lambda_j \in \mathbb{R}$ or $\mathbb{C}$) are the roots of its characteristic equation. The general solution of (A.2.27) is then

$$y(x) = \phi_H(x; c_1, c_2, \ldots, c_n) + y_p(x), \tag{A.2.30}$$

where $\phi_H(x)$ is the general solution of the homogeneous part of (A.2.27),

$$\phi_H(x;\, c_1, c_2, \ldots, c_n) = c_1 e^{\lambda_n x} \left(\int e^{(\lambda_{n-1} - \lambda_n)x} G_{12}^{\{n-2\}}(x)\, dx \right) + c_2 e^{\lambda_n x} G_{23}^{\{n-2\}}(x)$$

$$+ c_3 e^{\lambda_n x} G_{34}^{\{n-3\}}(x) + \cdots + c_{n-1} e^{\lambda_n x} G_{(n-1)n}^{\{1\}}(x) + c_n e^{\lambda_n x} \tag{A.2.31}$$

($c_1,\ c_2,\ \ldots,\ c_n$ are arbitrary constants) and $y_p(x)$ is a particular solution of (A.2.27), namely

$$y_p(x) = e^{\lambda_n x} \int \left(e^{(\lambda_{n-1} - \lambda_n)x} F_{n-1}(x)\, dx \right). \tag{A.2.32}$$

Here

$$G^{\{1\}}_{k\ell}(x) := \int e^{(\lambda_k - \lambda_\ell)x}\,dx, \qquad G^{\{j\}}_{k\ell}(x) := \int e^{(\lambda_{\ell+j-2} - \lambda_{\ell+j-1})x} G^{\{j-1\}}_{k\ell}(x)\,dx,$$

$$j = 2, 3, \ldots; \quad k = 1, 2, \ldots; \quad \ell = 2, 3, \ldots\ .$$

$$F_1(x) := \int f(x) e^{-\lambda_1 x}\,dx, \qquad F_i(x) := \int e^{(\lambda_{i-1} - \lambda_i)x} F_{i-1}(x)\,dx$$

$$i = 2, 3, \ldots\ .$$

To prove Proposition A.2.3 we apply the corresponding integral operators on (A.2.29) and identify the patterns. The details are left as an exercise.

Remark: *For complex roots,* $\{\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n\}$*, the expression for* $\phi_H(x; c_1, c_2, \ldots, c_n)$*, (A.2.31), will be a complex-valued solution for (A.2.27) for which the real- and the imaginary parts are real-valued solutions of (A.2.27). One therefore needs to combine the real- and imaginary parts of* ϕ_H *such that one remains with* n *linear independent real-valued solutions. We remark further that (A.2.32) is always a real particular solution for (A.2.27), even for the case where* $\{\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n\}$ *are complex roots of the characteristic equation.*

Example A.2.2.

We find the general solution of

$$y^{(3)} - y'' + y' - y = e^{2x} \cos x. \tag{A.2.33}$$

The characteristic equation is $\lambda^3 - \lambda^2 + \lambda - 1 = 0$, with roots $\lambda_1 = 1$, $\lambda_2 = i$, $\lambda_3 = -i$. Thus (A.2.33) can be presented in the factorized form

$$L_1 \circ L_2 \circ L_3\, y(x) = e^{2x} \cos x, \quad \text{where} \quad L_1 = D_x - 1, \quad L_2 = D_x - i, \quad L_3 = D_x + i.$$

Following Proposition A.2.3 the general solution of the homogeneous part of (A.2.33) is

$$\phi_H(x;\, c_1, c_2, c_3) = c_1 e^{\lambda_3 x} \left(\int e^{(\lambda_2 - \lambda_3)x} G_{12}^{\{1\}}\, dx \right) + c_2 e^{\lambda_3 x} G_{23}^{\{1\}} + c_3 e^{\lambda_3 x},$$

where

$$G_{12}^{\{1\}} = \int e^{(\lambda_1 - \lambda_2)x}\, dx = \int e^{(1-i)x}\, dx = \left(\frac{1}{1-i} \right) e^{(1-i)x}$$

$$G_{23}^{\{1\}} = \int e^{(\lambda_2 - \lambda_3)x}\, dx = \int e^{2ix}\, dx = \left(\frac{1}{2i} \right) e^{2ix}.$$

We find

$$\phi_H(x;\, c_1, c_2, c_3) = \left(\frac{1}{2} \right) c_1 e^{x} + \left(\frac{1}{2i} \right) c_2 e^{ix} + c_3 e^{-ix},$$

where

$$\text{Re}\,\{\phi_h\} = \frac{1}{2} c_1 e^{x} + \frac{1}{2} c_2 \sin x + c_3 \cos x$$

$$\text{Im}\,\{\phi_h\} = -\frac{1}{2} c_2 \cos x - c_3 \sin x$$

are real-valued solutions of (A.2.33). Since $\{e^x,\ \sin x,\ \cos x\}$ is a linearly independent set of functions for all $x \in \mathcal{R}$, the general solution of the homogeneous part of (A.2.33) is

$$\phi_H(x;\, a_1, a_2, a_3) = a_1 e^{x} + a_2 \sin x + a_3 \cos x,$$

where a_1, a_2 and a_3 are arbitrary constants. Following Proposition A.2.3, a particular solution for (A.2.33) is of the form

$$y_p(x) = e^{-ix} \left(\int e^{2ix} F_2(x)\, dx \right),$$

where

$$F_2(x) = \int e^{(1-i)x} F_1(x)\, dx, \qquad F_1(x) = \int e^{x} \cos x\, dx.$$

Calculating the integrals we obtain

$$F_1(x) = \frac{1}{2} e^x (\cos x + \sin x)$$

$$F_2(x) = \frac{1}{8} e^{(2-i)x} (\cos x + 2\sin x + i \sin x)$$

so that the particular solution becomes

$$y_p(x) = e^{-ix} \left[\int \frac{1}{8} e^{2x} e^{ix} (\cos x + 2\sin x + i\sin x)\ dx \right] = \frac{1}{8} e^{2x} \sin x.$$

The general solution of (A.2.33) is thus

$$y(x) = a_1 e^x + a_2 \sin x + a_3 \cos x + \frac{1}{8} e^{2x} \sin x.$$

A.3 Higher-order linear nonconstant coefficient differential equations

First we consider second-order linear nonhomogeneous equations of the form

$$y'' + g(x)y' + h(x)y = f(x), \tag{A.3.1}$$

where f, g and h are differentiable functions on some common interval $\mathcal{D} \subseteq \mathcal{R}$. Assume a factorization in terms of two linear first-order differential operators,

$$L_1 = D_x + q_1(x), \qquad L_2 = D_x + q_2(x), \tag{A.3.2}$$

such that (A.3.1) is equivalent to

$$L_1 \circ L_2\, y(x) = f(x). \tag{A.3.3}$$

Now (A.3.3) takes the form

$$y'' + (q_1 + q_2)y' + (q_1 q_2 + q_2')y = f(x) \tag{A.3.4}$$

and, comparing (A.3.4) to (A.3.1) leads to the condition

$$q_1 + q_2 = g(x), \qquad q_2' + q_1 q_2 = h(x),$$

or, equivalently,

$$q_2' = q_2^2 - g(x)q_2 + h(x), \qquad q_1(x) = g(x) - q_2(x).$$

We note that the condition on $q_2(x)$ is a **Riccati equation**. To find the general solution of (A.3.3), we apply the corresponding integral operators $\hat{L}_1$ and $\hat{L}_2$, successively. This leads to

Proposition A.3.1. *The 2nd-order linear equation*

$$y'' + g(x)y' + h(x)y = f(x) \tag{A.3.5}$$

can be written in the factorized form

$$L_1 \circ L_2\, y(x) = f(x), \tag{A.3.6}$$

where $L_1 = D_x + q_1(x)$ and $L_2 = D_x + q_2(x)$, if and only if $q_2(x)$ satisfies the Riccati equation

$$q_2' = q_2^2 - g(x)q_2 + h(x). \tag{A.3.7}$$

Then $q_1(x) = g(x) - q_2(x)$. Applying the corresponding integral operators, $\hat{L}_1$ and $\hat{L}_2$ successively on (A.3.6), leads to the general solution of (A.3.5), namely

$$y(x) = c_1 e^{-\xi_2(x)} + c_2 e^{-\xi_2(x)} \int e^{\xi_2(x)} e^{-\xi_1(x)} dx + y_p(x), \tag{A.3.8}$$

where $y_p(x)$ is a particular solution of (A.3.5) given by

$$y_p(x) = e^{-\xi_2(x)} \int e^{\xi_2(x)} e^{-\xi_1(x)} F(x) dx. \tag{A.3.9}$$

Here c_1 and c_2 are arbitrary constants and

$$F(x) := \int f(x) e^{\xi_1(x)} dx, \quad \xi_1(x) := \int q_1(x) dx, \quad \xi_2(x) := \int q_2(x) dx.$$

Example A.3.1.

We consider the **second-order Cauchy-Euler equation**

$$ax^2 y'' + bxy' + cy = f(x), \qquad x \neq 0, \tag{A.3.10}$$

where $a \neq 0$, b and c are real constants and f is a continuous function on some interval $\mathcal{D} \subseteq \mathcal{R}$. Equation (A.3.10) can equivalently be presented in the form

$$y'' + \left(\frac{b}{ax}\right) y' + \left(\frac{c}{ax^2}\right) y = \frac{f(x)}{ax^2} \tag{A.3.11}$$

Comparing (A.3.11) and (A.3.5) we identify

$$g(x) = \frac{b}{ax}, \qquad h(x) = \frac{c}{ax^2},$$

so that, by Proposition A.3.1, equation (A.3.10) can be factorized in the form (A.3.6) if q_2 satisfies the following Riccati equation:

$$q_2' = q_2^2 - \left(\frac{b}{ax}\right) q_2 + \frac{c}{ax^2}. \tag{A.3.12}$$

A solution of (A.3.12) is of the form

$$q_2(x) = \alpha x^\beta$$

with $\beta = -1$ and α satisfying the quadratic equation

$$\alpha^2 + \left(1 - \frac{b}{a}\right) \alpha + \frac{c}{a} = 0. \tag{A.3.13}$$

As an explicit example we consider $a = 1$, $b = -1$ and $c = 1$. This corresponds to the equation

$$x^2 y'' - xy' + y = x^3. \tag{A.3.14}$$

A solution for (A.3.13) is then $\alpha = -1$, so that

$$q_2(x) = -\frac{1}{x}, \quad q_1(x) = 0, \quad \xi_1(x) = 0, \quad \xi_2(x) = -\ln|x|.$$

Thus the equation

$$y'' - \left(\frac{1}{x}\right) y' + \left(\frac{1}{x^2}\right) y = x$$

factorizes in the form

$$\left(D_x\right) \circ \left(D_x - \frac{1}{x}\right) y(x) = x$$

so that, by the solution formula (A.3.8), the general solution of (A.3.14) becomes

$$y(x) = c_1 x + c_2 x \ln|x| + \frac{1}{4} x^3,$$

where c_1 and c_2 are arbitrary constants.

We give another example of a second-order nonconstant-coefficient equation:

Example A.3.2.

Consider the equation

$$y'' + 2xy' + (x^2 + 1)y = e^{-x^2/2}. \tag{A.3.15}$$

Comparing (A.3.15) and (A.3.5) we identify

$$g(x) = 2x, \qquad h(x) = x^2 + 1,$$

so that, by Proposition A.3.1, equation (A.3.15) can be factorized in the form (A.3.6), where q_2 satisfies the following Riccati equation:

$$q_2' = q_2^2 - 2xq_2 + x^2 + 1. \tag{A.3.16}$$

A special solution for (A.3.16) is

$$q_2(x) = x \quad \text{so that} \quad q_1(x) = x.$$

Hence (A.3.15) takes the factorized form

$$(D_x + x)(D_x + x)\, y(x) = e^{-x^2/2}.$$

By the solution formula (A.3.8), the general solution of (A.3.15) becomes

$$y(x) = c_1 e^{-x^2/2} + c_2 x e^{-x^2/2} + \frac{1}{2}x^2 e^{-x^2/2},$$

where c_1 and c_2 are arbitrary constants.

An extension to higher-order nonconstant-coefficient linear equations is possible, although the general condition for the factorization into first-order linear operators becomes rather complicated. To demonstrate this, we consider the third-order case:

$$y^{(3)} + k(x)y'' + g(x)y' + h(x)y = f(x), \tag{A.3.17}$$

or, equivalently

$$L\, y(x) = f(x), \tag{A.3.18}$$

where $L = D_x^3 + k(x)D_x^2 + g(x)D_x + h(x)$. Assume now a factorization in the form

$$(D_x + q_1(x)) \circ (D_x + q_2(x)) \circ (D_x + q_3(x))\, y(x) = f(x),$$

which leads to

$$y^{(3)} + (q_1 + q_2 + q_3)y'' + \left(q_2' + 2q_3' + q_1q_2 + q_2q_3 + q_1q_3\right) y' + \left(q_3'' + q_2'q_3 + q_3'q_1 + q_3'q_2 + q_1q_2q_3\right) y = f(x). \tag{A.3.19}$$

Hence, we have the following relations between the coefficients in (A.3.17) and (A.3.19):

$$k(x) = q_1 + q_2 + q_3 \tag{A.3.20a}$$

$$g(x) = q_2' + 2q_3' + q_1q_2 + q_2q_3 + q_1q_3 \tag{A.3.20b}$$

$$h(x) = q_3'' + q_2'q_3 + q_3'q_1 + q_3'q_2 + q_1q_2q_3 \tag{A.3.20c}$$

We now consider the special case $q_3 = 1$. The relations (A.3.20a) – (A.3.20c) lead to the condition

$$h(x) + k(x) - g(x) - 1 = 0.$$

Thus we can state that the equation

$$y^{(3)} + k(x)y'' + g(x)y' + (1 + g(x) - k(x))y = f(x)$$

factorizes in the form

$$(D_x + q_1(x)) \circ (D_x + q_2(x)) \circ (D_x + 1)\, y(x) = f(x)$$

if and only if q_2 satisfies the **Riccati equation**

$$q_2' = q_2^2 + (1 - k(x))q_2 + 1 + g(x) - k(x)$$

Then $q_1(x)$ is given by the relation

$$q_1(x) = k(x) - q_2(x) - 1.$$

Appendix B

Methods of integration

We sum up some of the important methods and substitutions which can be applied to integrate certain continuous functions of one variable.

B.1 The method of substitution

Indefinite Integrals:

$$\int f(g(x))\, g'(x)dx = \int f(u)du, \qquad \text{where} \quad u = g(x); \quad \frac{du}{dx} = g'(x).$$

Definite Integrals:

$$\int_a^b f(g(x))\, g'(x)dx = \int_{g(a)}^{g(b)} f(u)du, \qquad \text{where} \quad u = g(x); \quad \frac{du}{dx} = g'(x).$$

Examples for the method of substitution:

1. We calculate $\int x^2(x^3+2)^{2/3}dx$

Make the substitution $u = x^3 + 2$. Then $\frac{du}{dx} = 3x^2$, so that $du = 3x^2dx$. Thus, in terms of u, the integral becomes

$$\frac{1}{3}\int u^{2/3}\,du = \frac{1}{3}\left(\frac{3}{5}u^{5/3}\right) + c = \frac{1}{5}(x^3+2)^{5/3} + c,$$

where c is an arbitrary constant (a constant of integration).

For the definite integral

$\int_{-1}^{2} x^2(x^3+2)^{2/3}dx$ we have

$$\frac{1}{3}\int_{1}^{10} u^{2/3}\,du = \frac{1}{3}\left(\frac{3}{5}u^{5/3}\right)\Bigg|_{1}^{10} = \frac{1}{5}\left(10^{5/3}-1\right).$$

2. We calculate $\int \frac{\sin^3 x}{\cos^5 x}\,dx$

Consider $\int \frac{\sin^2 x}{\cos^5 x}\sin x dx = \int \frac{1-\cos^2 x}{\cos^5 x}\sin x dx$ and make the substitution $u = \cos x$. Then $\frac{du}{dx} = -\sin x$, so that $du = -\sin x dx$. Thus, in terms of u, the integral becomes

$$-\int \frac{1-u^2}{u^5}\,du = \int \left(-u^{-5}+u^{-3}\right)\,du = \frac{1}{4}u^{-4} - \frac{1}{2}u^{-2} + c = \frac{1}{4}\cos^{-4}x - \frac{1}{2}\cos^{-2}x + c,$$

where c is a constant of integration.

B.2 Inverse substitution

Indefinite Integrals:

$$\int f(x)dx = \int f(g(u))\,g'(u)du, \qquad \text{where} \quad x = g(u), \quad \frac{dx}{du} = g'(u).$$

Definite Integrals:

$$\int_a^b f(x)dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u))\,g'(u)du, \qquad \text{where} \quad x = g(u), \quad \frac{dx}{du} = g'(u), \quad g \text{ is 1-1}.$$

B.2.1 Common trigonometric inverse substitutions

$$\text{For} \quad \sqrt{a^2-x^2}, \quad a>0 \qquad \text{use} \quad x = a\sin\theta, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \tag{B.2.1}$$

$$\text{For} \quad \sqrt{a^2+x^2} \quad \text{or} \quad \frac{1}{a^2+x^2}, \quad a>0 \qquad \text{use} \quad x = a\tan\theta \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \tag{B.2.2}$$

$$\text{For} \quad \sqrt{x^2-a^2}, \quad a>0 \qquad \text{use} \quad x = a\sec\theta, \quad 0 \le \theta < \frac{\pi}{2} \quad \text{or} \quad \pi \le \theta < \frac{3\pi}{2} \tag{B.2.3}$$

An example for the method of inverse substitution:

We calculate $\displaystyle\int_{-1}^{0} \sqrt{2 - x^2}\, dx$

Let $x = \sqrt{2}\sin\theta$, where $-\pi/2 \leq \theta \leq \pi/2$. Then $\dfrac{dx}{d\theta} = \sqrt{2}\cos\theta$, so that $dx = \sqrt{2}\cos\theta\, d\theta$. For $x = -1$, we have $\theta = -\pi/4$ and for $x = 0$, we have $\theta = 0$. The integral then becomes

$$\int_{-\pi/4}^{0} \sqrt{2 - 2\sin^2\theta}\sqrt{2}\cos\theta\, d\theta = 2\int_{-\pi/4}^{0} \cos^2\theta\, d\theta = 2\int_{-\pi/4}^{0} \frac{1+\cos 2\theta}{2}\, d\theta = \left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_{-\pi/4}^{0}$$

$$= \frac{\pi}{4} + \frac{1}{2}$$

B.2.2 Substitutions by completing the square

In some cases, where the integrand, $f(x)$, in

$$\int f(x)\,dx$$

contains a second-degree polynomial,

$$ax^2 + bx + c, \qquad (a,\ b,\ c: \quad \text{some given real constants})$$

we can find a suitable substitution by completing the square. That is

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right].$$

A suitable substitution could then be

$$u = x + \frac{b}{2a}.$$

An example: completing the square, substitution and inverse substitution:

We calculate $\displaystyle\int \frac{1}{(x^2+2x+3)^{3/2}}\,dx$

First we complete the square for the term $x^2 + 2x + 3$, i.e.

$$x^2 + 2x + 3 = (x+1)^2 + 2.$$

We make the substitution $u = x + 1$. Then the integral becomes

$$\int \frac{1}{(u^2+2)^{3/2}}\,du.$$

Now we use the inverse substitution $u = \sqrt{2}\tan\theta$, so that

$$\frac{du}{d\theta} = \frac{\sqrt{2}}{\cos^2\theta}$$

and

$$u^2 + 2 = 2 + 2\tan^2\theta = 2 + 2\frac{\sin^2\theta}{\cos^2\theta} = \frac{2}{\cos^2\theta}.$$

Thus the integral becomes

$$\int \frac{1}{\left(\dfrac{2}{\cos^2\theta}\right)^{3/2}}\frac{\sqrt{2}}{\cos^2\theta}\,d\theta = \frac{1}{2}\int \cos\theta\,d\theta = \frac{1}{2}\sin\theta + c.$$

Recall that $\tan\theta = \dfrac{u}{\sqrt{2}}$ and $u = x+1$, so that

$$\sin\theta = \frac{u}{\sqrt{u^2+2}} = \frac{x+1}{\sqrt{(x+1)^2+2}}.$$

Hence

$$\int \frac{1}{(x^2+2x+3)^{3/2}}\,dx = \frac{1}{2}\frac{x+1}{\sqrt{(x+1)^2+2}} + c,$$

where c is a constant of integration.

B.2.3 Substitutions for n root expressions

In some cases, where the integrand, $f(x)$, in

$$\int f(x)\,dx$$

contains an n-root expressions, e.g.

$$(ax+b)^{m/n}, \qquad (a,\ b: \quad \text{real constants},\ \ m \neq n: \ \text{integers } \mathbb{Z}\backslash\{-1,0,1\},$$

a suitable substitution could be

$$ax + b = u^{n/m}.$$

An example for n root expressions of the type $(ax+b)^{m/n}$, $m \neq n; m,n \in \mathbb{Z}\backslash\{-1,0,1\}$:

We calculate $\displaystyle\int \frac{x^2}{(2x+3)^{3/5}}\,dx$

We use the substitution $u = (2x+3)^{3/5}$, i.e.

$$x = \frac{1}{2}u^{5/3} - \frac{3}{2},$$

so that

$$dx = \frac{5}{6}u^{2/3}\,du.$$

The integral then becomes

$$\int \frac{1}{u}\left(\frac{1}{2}u^{5/3} - \frac{3}{2}\right)^2 \left(\frac{5}{6}\right) u^{2/3}\,du = \frac{5}{6}\int \left(\frac{1}{4}u^3 - \frac{3}{2}u^{4/3} + \frac{9}{4}u^{-1/3}\right) du$$

$$= \frac{5}{6}\left(\frac{1}{16}u^4 - \frac{9}{14}u^{7/3} + \frac{27}{8}u^{2/3}\right) + c$$

$$= \frac{5}{6}\left(\frac{1}{16}(2x+3)^{12/5} - \frac{9}{14}(2x+3)^{7/5} + \frac{27}{8}(2x+3)^{2/5}\right) + c$$

B.2.4 The tan $(\theta/2)$ substitution

In some cases, where the integrand, $f(\theta)$, in

$$\int f(\theta)\, d\theta$$

is a rational function of $\cos\theta$ and $\sin\theta$, a suitable substitution could be

$$x = \tan\frac{\theta}{2}.$$

It follows that

$$\frac{dx}{d\theta} = \frac{1}{2}\sec^2\frac{\theta}{2}, \quad \text{or} \quad d\theta = \frac{2}{1+x^2}\,dx$$

$$\cos\theta = \frac{1-x^2}{1+x^2}$$

$$\sin\theta = \frac{2x}{1+x^2}.$$

An example for the tan $(\theta/2)$ substitution:

We calculate $\displaystyle\int \frac{1}{\cos\theta}\,d\theta$.

With the substitution $x = \tan\dfrac{\theta}{2}$, we have

$$\cos\theta = \frac{1-x^2}{1+x^2}, \quad d\theta = \frac{2}{1+x^2}\,dx,$$

so that the integral becomes

$$\int \frac{1}{\left(\frac{1-x^2}{1+x^2}\right)}\frac{2}{1+x^2}\,dx = \int \frac{2}{1-x^2}\,dx = \int \frac{2}{(1-x)(1+x)}\,dx = \int \left(\frac{1}{1-x}+\frac{1}{1+x}\right)dx$$

$$= -\ln|1-x| + \ln|1+x| + c$$

$$= \ln\left|\frac{1+x}{1-x}\right| + c$$

$$= \ln\left|\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}\right| + c.$$

B.3 Integration by parts

Consider two differentiable functions, $f(x)$ and $g(x)$, with continuous derivatives. Then

$$\frac{d}{dx}\left[f(x)\,g(x)\right] = f'(x)g(x) + f(x)g'(x).$$

Integrating the above, we obtain

$$f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx \qquad \text{or}$$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx,$$

which is the integration-by-parts formula for the indefinite integral of $f(x)g'(x)$. The definite integration-by-parts formula is

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx,$$

An example for integration by parts:

We calculate $\displaystyle\int_0^1 (1+x)^{-2}\ln(1+x)\,dx.$

Using the integration by parts formula $\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$, we let

$$f(x) = \ln(1+x), \quad g'(x) = (1+x)^{-2},$$

so that

$$f'(x) = \frac{1}{1+x}, \quad g(x) = -(1+x)^{-1}.$$

We then have

$$\begin{aligned}
&\int_0^1 (1+x)^{-2}\ln(1+x)\,dx \\
&= -(1+x)^{-1}\ln(1+x)\Big|_0^1 - \int_0^1 \left(\frac{1}{1+x}\right)\left(\frac{-1}{1+x}\right)dx \\
&= -(1+x)^{-1}\ln(1+x)\Big|_0^1 + \int_0^1 \frac{1}{(1+x)^2}\,dx \\
&= -(1+x)^{-1}\ln(1+x)\Big|_0^1 - (1+x)^{-1}\Big|_0^1 \\
&= \frac{1}{2} - \frac{1}{2}\ln(2).
\end{aligned}$$

B.4 Integration of rational functions

Integrals of rational functions

$$\int \frac{P(x)}{Q(x)}\,dx,$$

where $P(x)$ is a polynomial of degree n and $Q(x)$ is a polynomial of degree m (we write Deg $(P) = n$ and Deg $(Q) = m$), can be treated in the following manner:

Case A: Here Deg $(P) \geq$ Deg (Q):

By long division, divide Q into P. This will result in an expression of the form

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)},$$

where S and R are also polynomials, with Deg $(R) <$ Deg (Q) and Deg $(S) =$ Deg $(P)-$ Deg (Q) and R being the rest of this division. We then have

$$\int \frac{P(x)}{Q(x)}\,dx = \int S(x)\,dx + \int \frac{R(x)}{Q(x)}\,dx.$$

Case B: Here Deg $(P) <$ Deg (Q):

Note: By theorem, any polynomial with real coefficients can be factored in terms of $a_j x + b_j$ and $p_j x^2 + q_j x + s_j$ for some real constants a_j, b_j, p_j, q_j, s_j, where $p_j x^2 + q_j x + s_j$ is irreducible (cannot be factored further, i.e. $q_j^2 - 4p_j s_j < 0$).

Given this fact, we can write $\dfrac{P(x)}{Q(x)}$ as a sum of partial fractions, albeit we need to consider different cases; depending on the factorization properties of $Q(x)$:

Case I: Let $Q(x)$ be a product of k distinct linear factors. That is

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_kx + b_k).$$

Then $Q(x)$ can be written as the following sum of partial fractions:

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}.$$

The constants, A_j, must then be determined such that the above relation is satisfied. This is done by multiplying the above expression by Q and then comparing coefficients of different powers of x. This will result in a linear system of algebraic equations which determines $A_1,\ A_2,\ \ldots, A_k$.

We then have

$$\int \frac{P(x)}{Q(x)}\,dx = \int \frac{A_1}{a_1x + b_1}\,dx + \int \frac{A_2}{a_2x + b_2}\,dx + \cdots + \int \frac{A_k}{a_kx + b_k}\,dx.$$

Case II: Let $Q(x)$ be a product of k linear factors, where some are repeated $r \leq k$ times.

For example, say Q is of degree 4 which factorizes as follows:

$$Q(x) = (a_1x + b_1)^2(a_2x + b_2)(a_3x + c_3)$$

Then the sum of partial fractions would be of the form

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \frac{A_3}{a_2x + b_2} + \frac{A_4}{a_3x + b_3}.$$

Again, the constants A_j can be determined by multiplying the above expression by $Q(x)$ and comparing coefficients of different powers of x.

Case III: Assume that $Q(x)$ can be factored in terms of linear factors and irreducible second-degree factors of the form

$$p_jx^2 + q_jx + s_j, \qquad q_j^2 - 4p_js_j < 0,$$

where none of the irreducible factors appear more than once in the product.

For example, say Q is of degree 6 and has factors of the form

$$Q(x) = (a_1x + b_1)(a_2x + b_2)(p_1x^2 + q_1x + s_1)(p_2x^2 + q_2x + s_2).$$

Then the sum of partial fractions would be of the form

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \frac{B_1x + C_1}{p_1x^2 + q_1x + s_1} + \frac{B_2x + C_2}{p_2x^2 + q_2x + s_2}.$$

The constants A_j, B_j, C_j are determined by multiplying the above expression by $Q(x)$ and comparing coefficients of different powers of x.

Case IV: Assume that $Q(x)$ can be factored in terms of linear factors and irreducible second-degree factors of the form

$$p_j x^2 + q_j x + s_j, \qquad q_j^2 - 4p_j s_j < 0,$$

where some of the irreducible factors appear more than once in the product.

For example, say Q is of degree 6 and has factors of the form

$$Q(x) = (a_1 x + b_1)(a_2 x + b_2)(p_1 x^2 + q_1 x + s_1)^2.$$

That is, the irreducible second-degree factor appears two times. Then the sum of partial fractions would be of the form

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1 x + b_1} + \frac{A_2}{a_2 x + b_2} + \frac{B_1 x + C_1}{p_1 x^2 + q_1 x + s_1} + \frac{B_2 x + C_2}{(p_1 x^2 + q_1 x + s_1)^2}.$$

The constants A_j, B_j, C_j are determined by multiplying the above expression by $Q(x)$ and comparing coefficients of different powers of x.

In view of the above Case III and Case IV, one should point out that

$$\int \frac{1}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + k, \qquad (k \text{ is an arbitrary constant, } a \neq 0) \tag{B.4.1}$$

as well as the following two Statements:

Statement 1:

Let $ax^2 + bx + c$ *be an irreducible second-degree polynomial, i.e.* $b^2 - 4ac < 0$. *Then*

$$\int \frac{1}{ax^2 + bx + c}\, dx = \frac{2}{\sqrt{4ac - b^2}} \tan^{-1}\left(\frac{2ax + b}{\sqrt{4ac - b^2}}\right) + k \tag{B.4.2}$$

where k *is a constant of integration.*

Statement 2:

Let

$$I_n = \int \frac{1}{(x^2 + a^2)^n}\, dx, \tag{B.4.3}$$

where $n = 2,\ 3,\ \ldots$. *Then the following reduction formula is valid:*

$$I_n = \frac{1}{2a^2(n-1)} \left[\frac{x}{(x^2 + a^2)^{n-1}} + (2n-3)\ I_{n-1}\right]. \tag{B.4.4}$$

Examples for integrals of rational functions:

1. We calculate $\displaystyle\int \frac{x^3 + 4x^2 + 17x + 21}{(x+3)^2(2x-1)(x+2)}\, dx.$

We write the given rational function as the following sum of partial fractions:

$$\frac{x^3 + 4x^2 + 17x + 21}{(x+3)^2(2x-1)(x+2)} = \frac{A_1}{x+3} + \frac{A_2}{(x+3)^2} + \frac{A_3}{2x-1} + \frac{A_4}{x+2}.$$

We multiply the above relation by the denominator on the right-hand side, i.e. by $(x + 3)^2(2x - 1)(x + 2)$, to obtain the polynomial relation

$$x^3 + 4x^2 + 17x + 21$$

$$= A_1(x+3)(2x-1)(x+2) + A_2(2x-1)(x+2) + A_3(x+3)^2(x+2) + A_4(x+3)^2(2x-1).$$

Equating coefficients of x^3, x^2, x^1 and x^0, respectively, we obtain the following relation for the constants A_j:

$$2A_1 + A_3 + 2A_4 = 1$$
$$9A_1 + 2A_2 + 8A_3 + 11A_4 = 4$$
$$7A_1 + 3A_2 + 21A_3 + 12A_4 = 17$$
$$-6A_1 - 2A_2 + 18A_3 - 9A_4 = 21.$$

The solution of the above system of linear algebraic equations (using for example Gauss elimination) gives the following unique solution for the constants A_j:

$$A_1 = -1, \quad A_2 = -3, \quad A_3 = 1, \quad A_4 = 1.$$

Thus

$$\frac{x^3 + 4x^2 + 17x + 21}{(x+3)^2(2x-1)(x+2)} = -\frac{1}{x+3} - \frac{3}{(x+3)^2} + \frac{1}{2x-1} + \frac{1}{x+2}$$

and the integral becomes

$$\begin{aligned}
&\int \frac{x^3 + 4x^2 + 17x + 21}{(x+3)^2(2x-1)(x+2)}\, dx \\
&= \int \left(-\frac{1}{x+3} - \frac{3}{(x+3)^2} + \frac{1}{2x-1} + \frac{1}{x+2} \right) dx \\
&= -\ln|x+3| + \frac{3}{x+3} + \frac{1}{2}\ln|2x-1| + \ln|x+2| + c.
\end{aligned}$$

2. We calculate $\int \frac{2x^3 + 2x^2 + 3x - 1}{(x+1)(2x-1)(x^2+1)^2}\, dx.$

We write the given rational function as the following sum of partial fractions:

$$\frac{2x^3 + 2x^2 + 3x - 1}{(x+1)(2x-1)(x^2+1)^2} = \frac{A_1}{x+1} + \frac{A_2}{2x-1} + \frac{B_1 x + C_1}{x^2+1} + \frac{B_2 x + C_2}{(x^2+1)^2}.$$

Multiplying this relation by the denominator on the left-hand side and equating different powers of x, we obtain the following linear system of equations for the constants A_j, B_j and C_j:

$$\begin{aligned}
&2A_1 + A_2 + 2B_1 = 0 \\
&-A_1 + A_2 + B_1 + 2C_1 = 0 \\
&4A_1 + 2A_2 + B_1 + C_1 + 2B_2 = 2 \\
&-2A_1 + 2A_2 + B_1 + B_2 + C_1 + C_2 = 2 \\
&2A_1 + A_2 - B_1 - B_2 + C_1 + C_2 = 3 \\
&-A_1 + A_2 - C_1 - C_2 = -1,
\end{aligned}$$

with the unique solution

$$A_1 = \frac{1}{3}, \quad A_2 = \frac{8}{15}, \quad B_1 = -\frac{3}{5}, \quad B_2 = 0, \quad C_1 = \frac{1}{5}, \quad C_2 = 1$$

The integral now becomes

$$\int \frac{2x^3+2x^2+3x-1}{(x+1)(2x-1)(x^2+1)^2}\,dx$$
$$= \frac{1}{3}\int \frac{1}{x+1}\,dx + \frac{8}{15}\int \frac{1}{2x-1}\,dx - \frac{3}{5}\int \frac{x}{x^2+1}\,dx + \frac{1}{5}\int \frac{1}{x^2+1}\,dx + \int \frac{1}{(x^2+1)^2}\,dx$$
$$= \frac{1}{3}\ln|x+1| + \frac{4}{15}\ln|2x-1| - \frac{3}{10}\ln(x^2+1) + \frac{7}{10}\tan^{-1}x + \frac{1}{2}\frac{x}{x^2+1} + c.$$

Note that the last two terms in the above expression have been obtained by using **Statement 2**, i.e.

$$\int \frac{1}{(x^2+1)^2}\,dx = \frac{1}{2}\frac{x}{x^2+1} + \frac{1}{2}\tan^{-1}x.$$

Some important trigonometric identities:

$$\cos^2 x = \frac{1}{2}(1+\cos 2x)$$

$$\sin^2 x = \frac{1}{2}(1-\cos 2x)$$

$$\sin^2 x + \cos^2 x = 1$$

$$\sec^2 x - \tan^2 x = 1$$

$$\mathrm{cosec}^2 x - \cot^2 x = 1$$

$$\cos(x+y) = \cos x\cos y - \sin x \sin y$$

$$\sin(x+y) = \sin x\cos y + \cos x\cos y$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x\,\tan y}$$

where

$$\tan x = \frac{\sin x}{\cos x}$$

$$\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

$$\mathrm{cosec}\, x = \frac{1}{\sin x}$$

$$\sec x = \frac{1}{\cos x}$$

$$\cos(-x) = \cos x, \qquad \sin(-x) = -\sin x.$$

Appendix C

Some references on differential equations

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Appendix D

Solutions to some of the exercises

Exercise 1.1.1: nr. 1 b

We consider the set

$$S = \{f_1(x) = \ln x,\ f_2(x) = \ln x^2,\ f_3(x) = e^{3x}\}$$

on the interval $\mathcal{D} = (0,\ \infty)$. Since $\ln x^2 = 2\ln x$, the set S is clearly a linearly dependent set, as the equation

$$c_1 \ln x + 2c_2 \ln x + c_3 e^{3x} = 0$$

has nontrivial solutions for c_1 and c_2. For example the above relation is true if $c_1 = -2$, $c_2 = 1$ and $c_3 = 0$ for all $x \in \mathcal{D}$. Therefore it follows by **Proposition 1.1.2** that the Wronskian is zero for all $x \in \mathcal{D}$. We verify this:

$$W[f_1,\ f_2,\ f_3](x) := \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{vmatrix} = \begin{vmatrix} \ln x & 2\ln x & e^{3x} \\ 1/x & 2/x & 3e^{3x} \\ -1/x^2 & -2/x^2 & 9e^{3x} \end{vmatrix}$$

$$= 18\left(\frac{\ln x}{x}\right)e^{3x} - 6\left(\frac{\ln x}{x^2}\right)e^{3x} - \left(\frac{2}{x^3}\right)e^{3x}$$

$$+ \left(\frac{2}{x^3}\right)e^{3x} - 18\left(\frac{\ln x}{x}\right)e^{3x} + 6\left(\frac{\ln x}{x^2}\right)e^{3x}$$

$$\equiv 0 \quad \text{for all } x \in (0,\ \infty).$$

Exercise 1.1.1: nr. 1 d

We consider the set

$$S = \{f_1(x) = e^x,\ f_2(x) = e^{-x},\ f_3(x) = xe^x,\ f_4(x) = xe^{-x}\}$$

on the interval $\mathcal{D} = \mathbb{R}$. Let us calculate the Wronskian of the set S in the point $x = 0$:

$$W[f_1,\ f_2,\ f_3,\ f_4](0) := \begin{vmatrix} f_1(0) & f_2(0) & f_3(0) & f_4(0) \\ f_1'(0) & f_2'(0) & f_3'(0) & f_4'(0) \\ f_1''(0) & f_2''(0) & f_3''(0) & f_4''(0) \\ f_1^{(3)}(0) & f_2^{(3)}(0) & f_3^{(3)}(0) & f_4^{(3)}(0) \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & 2 & -2 \\ 1 & -1 & 3 & 3 \end{vmatrix}$$
$$= -16.$$

Since the Wronskian is nonzero in a point in the interval $\mathbb{R}$ (in this case we have chosen $x = 0$), it follows by **Proposition 1.1.2** that the set S is linearly independent.

Exercise 1.1.1: nr. 2

We consider the set

$$S = \{f_1(x) = x^2,\ ,f_2(x) = x|x|\}$$

on $\mathbb{R}$. For $x > 0$ the Wronskian for S is

$$W[f_1,\ f_2](x) = \begin{vmatrix} x^2 & x^2 \\ 2x & 2x \end{vmatrix} = 2x^3 - 2x^3 = 0$$

for all $x \in (0,\ \infty)$. For $x < 0$ the Wronskian of S is

$$W[f_1,\ f_2](x) = \begin{vmatrix} x^2 & -x^2 \\ 2x & -2x \end{vmatrix} = -2x^3 + 2x^3 = 0$$

for all $x \in (-\infty,\ 0)$. We note that for $f_2(x) = x|x|$, the derivative of $f_2(x)$ in the point $x = 0$ is

$$f_2'(0) = \lim_{h\to 0} \frac{f_2(x+h) - f_2(x)}{h}\bigg|_{x=0} = \lim_{h\to 0} \frac{(x+h)|x+h| - x|x|}{h}\bigg|_{x=0} = \lim_{h\to 0} \frac{h|h|}{h}$$
$$= \lim_{h\to 0} |h| = 0.$$

Hence the derivative of $f_1(x)$ and $f_2(x)$ exists at $x = 0$ and we can calculate the Wronskian for S at $x = 0$:

$$W[f_1,\ f_2](0) = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0.$$

From the above it follows that the Wronskian for S is zero for all $x \in \mathbb{R}$. However, **Proposition 1.1.2** does not make a conclusion form this fact. So to establish whether the set S is linearly independent or linearly dependent on $\mathbb{R}$, we need to consider the equation

$$c_1 x^2 + c_2 x|x| = 0$$

and investigate the possibilities for c_1 and c_2 to satisfy this relation. We first consider the case $x > 0$. Then we have

$$c_1 x^2 + c_2 x^2 = 0$$

which leads to the condition $c_1 + c_2 = 0$ for all $x \in [0,\ \infty)$. Consider now the case $x < 0$, for which we have

$$c_1 x^2 - c_2 x^2 = 0.$$

This leads to the condition $c_1 - c_2 = 0$ for all $x \in (-\infty,\ 0]$. For the case $x = 0$, we have

$$c_1 0 - c_2 0 = 0.$$

for which there are obviously no conditions on c_1 or on c_2. Now, for all $x \in \mathbb{R}$ both conditions, $c_1 + c_2 = 0$ and $c_1 - c_2 = 0$, have to be satisfied, which is possible only if $c_1 = 0$ and $c_2 = 0$. Hence by **Definition 1.1.3** the set S is linearly independent on $\mathbb{R}$.

Exercise 1.1.1: nr. 3 b

We show that

$$y(x) = c_1 \cos(2x) + c_2 \sin(2x), \qquad (c_1,\ c_2 \text{ arbitrary constants})$$

is the general solution for the equation

$$y'' + 4y = 0$$

for all $x \in \mathbb{R}$. We apply **Proposition 1.1.5**. Thus we need to show that the set of functions $S = \{\phi_1)(x) = \cos(2x),\ \phi_2(x) = \sin(2x)\}$ is a linearly independent set for all $x \in \mathbb{R}$ and that $\phi_1(x)$ and $\phi_2(x)$ satisfy the given differential equation. We calculate the Wronskian in $x = 0$:

$$W[\phi_1,\ \phi_2](0) = \begin{vmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{vmatrix}_{x=0} = 2.$$

By **Proposition 1.1.2** it then follows that the set S is linearly independent on $\mathbb{R}$. We now differentiate $\phi_1(x)$ and $\phi_2(x)$ twice and insert those functions and their derivatives into the differential equation to verify that $\phi_1(x)$ and $\phi_2(x)$ are indeed solutions of the equation. It then follows by **Proposition 1.1.5** that $y(x) = c_1 \cos(2x) + c_2 \sin(2x)$ is the general solution of $y'' + 4y = 0$ for all $x \in \mathbb{R}$ and that the solution domain is $\mathbb{R}$.

Exercise 1.1.1: nr. 5 c

We construct a second-order homogeneous differential equation that admits the following set of solutions:

$$S = \{\phi_1(x) = x\cos(1/x),\ \phi_2(x) = x\sin(1/x)\}.$$

We apply **Proposition 1.1.4**. To construct the equation, we first need to show that the set S is linearly independent. The differential equation with dependent variable y then follows from the relation $W[\phi_1,\ \phi_2,\ y](x) = 0$, where W denotes the Wronskian of the set $\{\phi_1(x),\ \phi_2(x),\ y(x)\}$.

To establish the linear independence of the set S, we apply **Proposition 1.1.2** and calculate the Wronskian in the point $x = 1/\pi$:

$$W[\phi_1,\ \phi_2](1/\pi) = \begin{vmatrix} -1/\pi & 0 \\ -1 & \pi \end{vmatrix} = -1.$$

By **Proposition 1.1.2** it then follows that S is linearly independent on $\mathbb{R}\backslash\{0\}$. Applying now **Proposition 1.1.4**, we calculate the following Wronskian:

$$W[\phi_1,\,\phi_2,\,y](x) = \begin{vmatrix} \phi_1(x) & \phi_2(x) & y(x) \\ \phi_1'(x) & \phi_2'(x) & y'(x) \\ \phi_1''(x) & \phi_2''(x) & y''(x) \end{vmatrix}$$

$$= \begin{vmatrix} x\cos(1/x) & x\sin(1/x) & y(x) \\ \cos(1/x) + (1/x)\sin(1/x) & \sin(1/x) - (1/x)\cos(1/x) & y'(x) \\ -(1/x^3)\cos(1/x) & -(1/x^3)\sin(1/x) & y''(x) \end{vmatrix}$$

$$= -y''\left[\cos^2(1/x) + \sin^2(1/x)\right] - \frac{1}{x^4}\left[\cos^2(1/x) + \sin^2(1/x)\right] y$$

$$= -y'' - \left(\frac{1}{x^4}\right) y,$$

where we have used the identity

$$\cos^2(1/x) + \sin^2(1/x) = 1 \quad \text{for all } x \in \mathbb{R}\backslash\{0\}.$$

By **Proposition 1.1.4**, the differential equation which admits the general solution given by the linearly independent set of functions S, is then given by the relation $W[\phi_1,\,\phi_2,\,y](x) = 0$. Hence the equation is

$$y'' + \left(\frac{1}{x^4}\right) y = 0 \quad \text{for all } \in \mathbb{R}\backslash\{0\}.$$

Exercise 1.2.1: 2

The functions

$$\phi_1(x) = e^{-x}\cos x, \qquad \phi_2(x) = e^{-x}\sin x$$

satisfy the differential equation

$$y'' + 2y' + 2y = 0$$

so that these functions belong to the solution space of this differential equation. Moreover, these functions form a linearly independent set,

$$B = \{e^{-x}\cos x,\ e^{-x}\sin x\}.$$

It can easily be verified that the set B is a linearly independent set by calculating the Wronskian in, for example, the point $x = 0$. This gives

$$W[\phi_1,\, \phi_2](0) = 3\pi,$$

so that, by **Proposition 1.1.2**, the set B is linearly independent. Furthermore, by **Proposition 1.2.2**, the set B is a basis for the solution space of the differential equation $y'' + 2y' + 2y = 0$. Since this basis consists of two functions, which are vectors in $\mathcal{C}^\infty(\mathbb{R})$, the dimension of the solution space is two. The solution space is therefore a 2-dimensional subspace of $\mathcal{C}^\infty(\mathbb{R})$ given by the kernel of T, where T is the linear transformation

$$T\,:\,y(x) \mapsto Ly(x), \qquad L = \frac{d^2}{dx^2} + 2\frac{d}{dx} + 2.$$

Exercise 2.2.1: 1 e

The task is to find a general solution of the following separable first-order differential equation:

$$y' + \frac{1-y^2}{1-x^2} = 0$$

for all $x > 1$. First consider the case

$$y(x) = \pm 1$$

which are obviously two solutions of the given differential equation for all $x > 1$. Consider now the case $y(x) \neq \pm 1$: We can separate the $x-$ and y-variables and write the differential equation in the form

$$\frac{dy}{1-y^2} = \frac{dx}{x^2-1}.$$

To find a general solution, we now have to integrate this relation:

$$\int \frac{dy}{1-y^2} = \int \frac{dx}{x^2-1} + c_1,$$

where c_1 is a constant of integration. Note that $\dfrac{1}{1-y^2}$ can be written as the following sum of partial fractions:

$$\frac{1}{(1-y)(1+y)} = \frac{A}{1-y} + \frac{B}{1+y}$$

or

$$1 = A(1+y) + B(1-y),$$

so that $A = 1/2$ and $B = 1/2$. That is

$$\frac{1}{(1-y)(1+y)} = \frac{1}{2}\left(\frac{A}{1-y}\right) + \frac{1}{2}\left(\frac{B}{1+y}\right)$$

In the same way we can write

$$\frac{1}{x^2-1} = \frac{1}{2}\left(\frac{1}{x-1}\right) - \frac{1}{2}\left(\frac{1}{x+1}\right).$$

Thus the above integrals take the form

$$\int \frac{dy}{1-y} + \int \frac{dy}{1+y} = \int \frac{dx}{x-1} - \frac{dx}{x+1} + c_1,$$

so that

$$-\ln|1-y| + \ln|1+y| = \ln|x-1| - \ln|x+1| + c_1$$

where c_1 is an arbitrary constant that plays the role of the constant of integration. This can be simplified to

$$\ln\left|\frac{1+y}{1-y}\right| = \ln\left(\left|\frac{x-1}{x+1}\right|\right) + c_1$$

or

$$\left|\frac{1+y}{1-y}\right| = e^{c_1}\left|\frac{x-1}{x+1}\right|.$$

Since $x > 1$, we have

$$\left|\frac{x-1}{x+1}\right| = \frac{x-1}{x+1} > 0$$

so that

$$\left|\frac{1+y}{1-y}\right| = e^{c_1}\left(\frac{x-1}{x+1}\right).$$

For $\dfrac{1+y}{1-y} > 0$ we solve $y(x)$ from the relation

$$\frac{1+y}{1-y} = e^{c_1}\left(\frac{x-1}{x+1}\right),$$

to obtain a general solution of the given differential equation in the form

$$y(x) = \frac{e^{c_1}(x-1) - x - 1}{e^{c_1}(x-1) + x + 1}.$$

For $\dfrac{1+y}{1-y} < 0$ we solve $y(x)$ from the relation

$$-\left(\frac{1+y}{1-y}\right) = e^{c_1}\left(\frac{x-1}{x+1}\right)$$

to obtain a general solution of the given differential equation in the form

$$y(x) = \frac{e^{c_1}(x-1) + x + 1}{e^{c_1}(x-1) - x - 1}.$$

Note that the two solutions $y(x) = \pm 1$ are singular solutions for this equation.

Exercise 2.2.1: 3 b

We find a general solution for

$$y^2 - x^2 + xyy' = 0$$

by making the substitution

$$y(x) = xv(x),$$

where $v(x)$ is a new dependent variable. The x-derivative is

$$y' = v + xv'.$$

Substituting this into the differential equation we obtain a differential equation in the new dependent variable $v(x)$, namely the following first-order equation:

$$xvv' + 2v^2 - 1 = 0,$$

which is a separable equation. Separating the variables, leads to the following integrals:

$$\int \frac{v}{1-2v^2}dv = \int \frac{1}{x}dx + c_1,$$

where $x \neq 0$. Evaluating these integrals, we obtain

$$-\frac{1}{4}\ln\left|1-2v^2\right| = \ln|x| + c_1,$$

or, by multiplying with -4 and introducing a new arbitrary constant $c_2 = -4c_1$, we obtain

$$\ln\left|1-2v^2\right| = \ln\left(\frac{1}{x^4}\right) + c_2,$$

Upon inverting $\ln\left|1-2v^2\right|$, we obtain

$$v^2 = \frac{1}{2}\left(1 - x^{-4}e^{c_2}\right).$$

We now write the answer in terms of the original dependent variable $y(x)$ by replacing $v(x) = y(x)/x$, so that a general solution takes the form

$$y^2(x) = \frac{1}{2x^2}\left(x^4 - e^{c_2}\right) \quad \text{for all } x \in \mathbb{R}.$$

Exercise 2.3.1: 1 d

We find the general solution of the linear first-order equation

$$y' + y + \sin x + x^3 = 0 \quad \text{for all } x \in \mathbb{R}.$$

The integrating factor is

$$e^{\int 1\,dx} = e^x.$$

We multiply the equation by this integrating factor:

$$e^x y' + e^x y = -e^x \sin x - e^x x^3,$$

which has the equivalent form

$$\frac{d}{dx}\left(e^x y\right) = -e^x \sin x - e^x x^3.$$

Integrating both sides of this equation with respect to x, we obtain

$$\begin{aligned} e^x y &= -\int e^x \sin x \, dx - \int e^x x^3 \, dx + c_1 \\ &= -\left(-\frac{1}{2}e^x \cos x + \frac{1}{2}e^x \sin x\right) - e^x x^3 - 3e^x x^2 + 6e^x x - 6e^x + c_1. \end{aligned}$$

Thus the general solution is

$$y = \frac{1}{2}\cos x - \frac{1}{2}\sin x - x^3 + 3x^2 - 6x + 6 + c_1 e^{-x} \quad \text{for all } x \in \mathbb{R},$$

where c_1 is an arbitrary constant.

Exercise 2.3.1: 2 c

We solve the initial-value problem

$$xy' + y = x \cos x, \qquad y(\pi/2) = 1.$$

Let $x \neq 0$. Then we divide the equation by x, so it takes the form

$$y' + \frac{1}{x}\, y = \cos x.$$

The integrating factor is

$$e^{\int (1/x)\, dx} = e^{\ln |x|} = |x|.$$

Consider $x > 0$: Multiplying the equation by the integrating factor x we obtain

$$\begin{aligned} &xy' + y = x \cos x, \quad \text{or} \\ &\frac{d}{dx}\left(xy\right) = x \cos x. \end{aligned}$$

Integrating with respect to x, we obtain

$$\begin{aligned} xy &= \int x \cos x \, dx + c_1 \\ &= x \sin x + \cos x + c_1. \end{aligned}$$

Thus the general solution is

$$y = \sin x + \frac{1}{x}\cos x + \frac{c_1}{x}$$

For $x > 0$ we have the integrating factor $-x$, so that the equation takes the form

$$-xy' - y = -x \cos x,$$

which is, after multiplying by -1, the same as in the case $x > 0$. Thus the general solution that we have obtained above holds for $x > 0$ and $x < 0$. Using now the initial condition $y(\pi/2) = 1$, we determine the constant that picks out the curve which contains this point $(\pi/2,\ 1)$ from the family of one-parameter curves given by the above general solution with the arbitrary parameter c_1. We have

$$y(\pi/2) = sin(\pi/2) + \frac{2}{\pi} \cos(\pi/2) + \frac{2c_1}{\pi} = 1,$$

so that $c_1 = 0$. Thus the solution of the initial-value problem is

$$y = \sin x + \frac{1}{x} \cos x \quad \text{for all } x \in \mathbb{R}\backslash\{0\}.$$

Exercise 2.4.4: 2 c

We find a general solution of the Bernoulli equation

$$xy' + y = y^2 \ln x, \qquad x > 0.$$

By **Proposition 2.4.2** we make use of the substitution

$$v(x) = y^{-1}(x)$$

to linearize the given equation. The derivative becomes

$$v' = -\frac{y'}{y^2}.$$

To do the substitution, it is easier to first divide the given equation by xy^2, so the equation takes the form

$$\frac{y'}{y^2} + \frac{1}{xy} = \frac{1}{x} \ln x.$$

Making now the substitution, we obtain the following linear equation:

$$v' - \frac{v}{x} = -\frac{1}{x} \ln x.$$

We now solve this linear equation: An integrating factor for the linear equation is

$$e^{\int -(1/x)dx} = e^{-\ln x} = \frac{1}{x}.$$

Multiplying the equation with this integrating factor, we obtain

$$\frac{d}{dx}\left(\frac{v}{x}\right) = -\frac{1}{x^2} \ln x$$

and, upon integrating

$$\frac{v}{x} = -\int x^{-2} \ln x \, dx + c_1,$$

we have (doing integration by parts)

$$\frac{v}{x} = \frac{1}{x} \ln x + \frac{1}{x} + c_1.$$

Substituting back, $v(x) = y^{-1}(x)$, a general solution for all $x > 0$ takes the form

$$y(x) = (\ln x + 1 + c_1 x)^{-1}.$$

Exercise 2.4.4: 2 g

We find a general solution for the Bernoulli equation

$$y' + \frac{xy}{1 - x^2} = x\sqrt{y}, \qquad x > 1.$$

By **Proposition 2.4.2** we make use of the substitution $v(x) = y^{1/2}(x)$ to linearize the given equation. The substitution and its derivative can also be written as

$$v^2(x) = y(x), \qquad y' = 2vv'$$

Then the linear equation takes the form

$$v' + \frac{xv}{2(1-x^2)} = \frac{x}{2}$$

We now solve this linear equation: An integrating factor for the linear equation is

$$e^{(1/2)\int x/(1-x^2)\,dx} = e^{-(1/4)\ln(1-x^2)} = \frac{1}{(1-x^2)^{1/4}}.$$

Multiplying the linear equation with this integrating factor, we obtain

$$\frac{d}{dx}\left[(1-x^2)^{-1/4}\,v\right] = \frac{1}{2}(1-x^2)^{-1/4}$$

and, upon integrating

$$(1-x^2)^{-1/4}\,v = \int \frac{1}{2}(1-x^2)^{-1/4}\,dx + c_1$$

we have $\dfrac{v}{x} = -\dfrac{1}{3}(1-x^2)^{3/4} + c_1$. Substituting back, $v(x) = y^{1/2}(x)$, a general solution for all $x > 1$ takes the form

$$y(x) = \left(-\frac{1}{3}(1-x^2) + c_1(1-x^2)^{1/4}\right)^2.$$

Exercise 2.4.4: 6

We solve the initial-value problem for the following Riccati equation:

$$(x-x^4)y' - x^2 - y + 2xy^2 = 0, \quad x \in R\backslash[0,\ 1],$$

where $y(2) = 1$. To find a special solution $\phi(x)$, we make use of the Ansatz

$$\phi(x) = kx^2,$$

where k is an unknown real number. Inserting this Ansatz into the given equation, we obtain

$$(x-x^4)2kx - x^2 - kx^2 + 2x\left(kx^2\right)^2 = 0$$

or, after some simplifications,

$$(k-1)\left(x^2 + 2kx^5\right) = 0.$$

Thus, the Ansatz $\phi(x) = kx^2$ is a solution of the given equation for $k = 1$, i.e.

$$\phi(x) = x^2$$

satisfies the given Riccati equation. Following **Proposition 2.4.3** we are now able to linearize the given Riccati equation in terms of a new dependent variable $v(x)$ by the substitution

$$y(x) = x^2 + \frac{1}{v(x)}.$$

The derivative of this substitution is

$$y' = 2x - \frac{v'}{v^2}.$$

Substituting this into the given Riccati equation, we obtain

$$(x-x^4)\left(2x - \frac{v'}{v^2}\right) - x^2 - \left(x^2 + \frac{1}{v}\right) + 2x\left(x^2 + \frac{1}{v}\right)^2 = 0$$

or, after some simplifications,

$$v' + \frac{4x^3-1}{x^4-x}\, v + \frac{2x}{x^4-x} = 0,$$

which is a linear first-order differential equation in the dependent variable $v(x)$. An integrating factor for this linear equation is

$$e^{\int (4x^3-1)/(x^4-x)\, dx} = e^{\ln(x^4-x)} = x^4 - x.$$

Note that $x^4 - x > 0$ in the interval $\mathbb{R}\backslash[0,\ 1]$. Multiplying the linear equation with this integrating factor, we have

$$(x^4-x)v' + (4x^3-1)v = -2x \quad \text{or} \quad \frac{d}{dx}\left[(x^4-x)v\right] = -2x.$$

Integrating the previous relation with respect to x, we obtain

$$(x^4 - x)v = -x^2 + c_1,$$

so that the general solution of the linear equation becomes

$$v(x) = \frac{c_1 - x^2}{x(x^3 - 1)} \quad \text{for all } x \in \mathbb{R}\backslash[0,\ 1].$$

A general solution of the given Riccati equation is therefore

$$y(x) = x^2 + \frac{x(x^3 - 1)}{c_1 - x^2}$$

With the initial condition $y(2) = 1$, we obtain

$$y(2) = 4 + \frac{2(2^3 - 2)}{c_1 - 4} = 1$$

so that $c_1 = -2/3$. The solution of this initial-value problem is therefore

$$y(x) = x^2 + \frac{x(x^3 - 1)}{-(2/3) - x^2} = \frac{x(2x + 3)}{3x^2 + 2}$$

for all $x \in R\backslash[0,\ 1]$.

<u>**Exercise 3.3.2: 1 h**</u>

We find the general solution of the equation

$$y'' + 4y = 9\cos^2 x$$

for all $x \in \mathbb{R}$. To find a particular solution we will make use the **method of variation of parameters** as given in **Proposition 3.3.2**. First we calculate the general solution of the associated homogeneous equation

$$y'' + 4y = 0$$

by the Ansatz $y(x) = e^{\lambda x}$, $\lambda \in \mathbb{C}$. This leads to the condition $\lambda^2 + 4 = 0$, which has the two solutions $\lambda_1 = 2i$, $\lambda_2 = -2i$. The general homogeneous solution is therefore [see **Proposition 3.2.1 c)**]

$$\phi_H(x; c_1, c_2) = c_1 \operatorname{Re}\left\{e^{2ix}\right\} + c_2 \operatorname{Im}\left\{e^{2ix}\right\} = c_1 \cos(2x) + c_2 \sin(2x) \quad \text{for all } x \in \mathbb{R}.$$

To find a particular solution $y_p(x)$ by the method of variation of parameters, we make the Ansatz

$$y_p(x) = w_1(x)\phi_1(x) + w_2(x)\phi_2(x),$$

where $\phi_1(x)$ and $\phi_2(x)$ are the two linearly independent solutions of the above homogeneous equation, i.e.

$$\phi_1(x) = \cos(2x), \qquad \phi_2(x) = \sin(2x),$$

whereas $w_1(x)$ and $w_2(x)$ are given in **Proposition 3.3.2** as

$$w_1(x) = -\int \frac{9\cos^2 x\, \phi_2(x)}{W[\phi_1,\, \phi_2](x)}\, dx, \quad w_2(x) = \int \frac{9\cos^2 x\, \phi_1(x)}{W[\phi_1,\, \phi_2](x)}\, dx.$$

Here $W[\phi_1,\, \phi_2](x)$ is the Wronskian

$$W[\phi_1,\, \phi_2](x) = \begin{vmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{vmatrix} = 2\cos^2(2x) + 2\sin^2(2x) = 2.$$

Thus we have

$$\begin{aligned} w_1(x) &= -\frac{9}{2}\int \sin(2x)\cos^2 x\, dx = -\frac{9}{2}\int \sin(2x)\left(\frac{1+\cos(2x)}{2}\right) dx \\ &= -\frac{9}{4}\left(\int \sin(2x)\, dx + \int \sin(2x)\cos(2x)\, dx\right) \\ &= -\frac{9}{4}\left(-\frac{1}{2}\cos(2x)\right) - \frac{9}{4}\int \frac{1}{2}\sin(4x)\, dx \\ &= \frac{9}{8}\cos(2x) + \frac{9}{32}\cos(4x) \end{aligned}$$

Also, in a similar way, we obtain

$$w_2(x) = \frac{9}{8}\sin(2x) + \frac{9}{32}\sin(4x) + \frac{9}{8}x.$$

A particular solution is therefore

$$\begin{aligned} y_p(x) &= \left(\frac{9}{8}\cos(2x) + \frac{9}{32}\cos(4x)\right)\cos(2x) \\ &\quad + \left(\frac{9}{8}\sin(2x) + \frac{9}{32}\sin(4x) + \frac{9}{8}x\right)\sin(2x) \\ &= \frac{9}{32}\cos(2x) + \frac{9}{8}x\sin(2x) + \frac{9}{8} \end{aligned}$$

so that, by **Proposition 3.3.1 a)**, the general solution of the given nonhomogeneous equation is

$$\begin{aligned} y(x) &= \phi_H(x;\, c_1, c_2) + y_p(x) \\ &= c_1\cos(2x) + c_2\sin(2x) + \frac{9}{32}\cos(2x) + \frac{9}{8}x\sin(2x) + \frac{9}{8} \\ &= \tilde{c}_1\cos(2x) + c_2\sin(2x) + \frac{9}{8}x\sin(2x) + \frac{9}{8} \end{aligned}$$

for all $x \in \mathbb{R}$, where $\tilde{c}_1$ and $\tilde{c}_2$ are two arbitrary constants (Note: $\tilde{c}_1 = c_1 + 9/32$).

Exercise 3.3.4: 1 t

We find the general solution of the equation

$$y'' - 9y = 40xe^{3x}\cos(2x)$$

for all $x \in \mathbb{R}$. The general homogeneous solution $\phi_H(x)$ of the corresponding homogeneous equation

$$y'' - 9y = 0$$

is obtained by the Ansatz $y(x) = e^{\lambda x}$, so that $\lambda^2 - 9 = 0$ and

$$\phi_H(x) = c_1 e^{3x} + c_2 e^{-3x}.$$

To find a particular solution of the given nonhomogeneous equation we make use of the **method of undetermined coefficients** and consider the equation in the complex form, i.e.

$$y_c'' - 9y_c = 40xe^{(3+2i)x},$$

where $y_c(x)$ is a complex function and

$$y(x) = \mathrm{Re}\,\{y_c(x)\}.$$

An Ansatz for a complex particular solution $y_{cp}(x)$ of this complex equation is (see **Proposition 3.3.4**)

$$y_{cp}(x) = e^{(3+2i)x} w_c(x),$$

where $w_c(x)$ is an unknown complex function. Inserting this Ansatz into the above complex equation, we obtain

$$w_c'' + 2(3+2i)w_c' + (-4+12i)w_c = 40x.$$

To find a solution for this equation we use the Ansatz

$$w_c(x) = B_1 x + B_0, \qquad B_j \in \mathbb{C}$$

which leads to

$$(3+2i)B_1 + (-2+6i)B_1 x + (-2+6i)B_0 = 20x.$$

Equating coefficients of x and 1, we obtain

$$(-2+6i)B_1 = 20, \qquad (3+2i)B_1 + (-2+6i)B_0 = 0,$$

so that

$$B_1 = -1-3i, \qquad B_0 = \frac{9}{5} - \frac{1}{10}\,i.$$

Thus a solution for $w_c(x)$ is

$$w_c(x) = -(1+3i)x + \frac{9}{5} - \frac{1}{10}\,i$$

and a complex particular solution is

$$y_c(x) = \left(-(1+3i)x + \frac{9}{5} - \frac{1}{10}\,i\right) e^{(3+2i)x}.$$

A particular solution $y_p(x)$ of the given real differential equation is then the real part of this complex particular solution $y_{cp}(x)$. We obtain

$$y_p(x) = \left[\left(\frac{9}{5} - x\right)\cos(2x) + \left(3x + \frac{1}{10}\right)\sin(2x)\right] e^{3x}$$

so that the general solution takes the form

$$y(x) = c_1 e^{3x} + c_2 e^{-3x} + \left[\left(\frac{9}{5} - x\right)\cos(2x) + \left(3x + \frac{1}{10}\right)\sin(2x)\right] e^{3x}$$

for all $x \in \mathbb{R}$.

Exercise 3.4.1: 1 g

We find the general solution of the following second-order Cauchy-Euler equation

$$x^2 y'' - 3xy' - 5y = x^2 \ln x \quad \text{for all } x > 0.$$

Using **Proposition 3.4.1** we introduce a new independent variable z as

$$x = e^z, \qquad y(x) = y(z).$$

The given equation then becomes

$$\frac{d^2y}{dz^2} - 4\frac{dy}{dz} - 5y(z) = z\,e^{2z}.$$

We now need to find the general solution $y(z)$ of this equation. We first find the general solution $\phi_H(z;\, c_1, c_2)$ of the associated homogeneous equation

$$\frac{d^2y}{dz^2} - 4\frac{dy}{dz} - 5y(z) = 0.$$

With the Ansatz

$$y(z) = e^{\lambda z}, \qquad \lambda \in \mathbb{C}$$

we obtain

$$P_2(\lambda) = \lambda^2 - 4\lambda - 5 = (\lambda+1)(\lambda-5) = 0$$

so $\lambda_1 = -1$ and $\lambda_2 = 5$. Thus

$$\phi_H(z;\, c_1, c_2) = c_1\, e^{-z} + c_2\, e^{5z}.$$

To find a particular solution $y_p(z)$ of

$$\frac{d^2y}{dz^2} - 4\frac{dy}{dz} - 5y(z) = z\,e^{2z}.$$

we make use of the **method of undetermined coefficients** as described in paragraph **3.3.3 Case III** and use the Ansatz

$$y_p(z) = e^{2z} w(z).$$

The derivatives are

$$\frac{dy}{dz} = \left(\frac{dw}{dz} + 2w\right) e^{2z}$$

$$\frac{d^2y}{dz^2} = \left(\frac{d^2w}{dz^2} + 4\frac{dw}{dz} + 4w\right) e^{2z}$$

and the equation then becomes

$$\frac{d^2w}{dz^2} - 9w = z.$$

To find a solution for $w(z)$ we use the Ansatz

$$w(z) = A_1 z + A_0$$

as suggested by the **method of undetermined coefficients** discussed in **Case Ia** in paragraph **3.3.3**. We have

$$-9A_1 z - 9A_0 = z.$$

Equating coefficients of z and 1, we get $A_1 = -1/9$ and $A_0 = 0$. Thus a solution for $w(z)$ is

$$w(z) = -\frac{1}{9}\,z$$

and a particular solution is therefore

$$y(z) = -\frac{1}{9}\,ze^{2z}$$

so that he general solution of the differential equation, with z as its independent variable, becomes

$$y(z) = c_1\,e^{-z} + c_2\,e^{5z} - \frac{1}{9}\,ze^{2z}.$$

Substituting now $z = \ln(x)$, we obtain the general solution in terms of x, namely

$$y(x) = c_1\,x^{-1} + c_2 x^5 - \frac{1}{9}\,x^2\ln x$$

for all $x > 0$.

Exercise 3.5.1: 1 e

We find the general solution of the second-order homogeneous equation

$$x^2y'' - 2xy' + (4x^2 + 2)y = 0$$

for all $x \in \mathbb{R}\backslash\{0\}$, where one solution is given as

$$\phi_1(x) = x\cos(2x).$$

Using **Proposition 3.5.1** we make the Ansatz $y(x) = \phi_2(x)$, with

$$\phi_2(x) = v(x)\phi_1(x) = v(x)x\cos(2x)$$

for a second linearly independent solution of the given equation. This leads to the following condition on $v(x)$:

$$\cos(2x)\,v'' - 4\sin(2x)\,v' = 0.$$

We let

$$v'(x) = z(x)$$

so that the equation becomes a separable first-order equation, namely

$$z'(x) = 4z(x)\frac{\sin(2x)}{\cos(2x)}$$

with general solution

$$|z(x)| = e^{c_1}\cos^{-2}(2x).$$

Integrating this expression again, we find

$$v(x) = \pm\frac{1}{2}\frac{\sin(2x)}{\cos(2x)}$$

so that a second solution of the given equation takes the form

$$\phi_2(x) = \pm\frac{1}{2}x\sin(2x).$$

Since $\phi_1(x)$ and $\phi_2(x)$ are linearly independent solutions, we have obtained the general solution of the given differential equation, namely

$$y(x) = c_1 x\cos(2x) + c_2 x\sin(2x)$$

for all $x \in \mathbb{R}\backslash\{0\}$.

Exercise 4.2.1: 2 f

We solve the following initial-value problem:

$$y^{(4)} + 9y'' = 0, \quad y(0) = 2,\ y'(0) = 1,\ y''(0) = 2,\ y^{(3)}(0) = 1.$$

We first calculate the general solution by the use of the Ansatz

$$y(x) = e^{\lambda x}, \qquad \lambda \in \mathbb{C}.$$

This leads to the following condition on λ:

$$P_4(\lambda) = \lambda^4 + 9\lambda^2 = \lambda^2(\lambda^2 + 9) = 0,$$

which has the four roots

$$\lambda_1 = \lambda_2 = 0, \quad \lambda_3 = 3i, \quad \lambda_4 = -3i.$$

Now $\lambda_1 = 0$ and $\lambda_2 = 0$ gives two real solutions, namely

$$\phi_1(x) = 1, \qquad \phi_2(x) = x \quad \text{for all } x \in \mathbb{R},$$

whereas the complex root $\lambda_3 = 3i$ gives two real solutions [see **Proposition 4.2.1 b)**], namely

$$\phi_3(x) = \text{Re}\left\{e^{3ix}\right\} = \cos(3x), \quad \phi_4(x) = \text{Im}\left\{e^{3ix}\right\} = \sin(3x) \quad \text{for all } x \in \mathbb{R}.$$

The general solution of the given equation for all $x \in \mathbb{R}$ is thus

$$y(x) = c_1 + c_2 x + c_3\cos(3x) + c_4\sin(3x).$$

The derivatives of the general solution are as follows:

$$y'(x) = c_2 - 3c_3\sin(3x) + 3c_4\cos(3x)$$

$$y''(x) = -9c_3\cos(3x) - 9c_4\sin(3x)$$

$$y^{(3)}(x) = 27c_3\sin(3x) - 27c_4\cos(3x)$$

and, for the initial conditions in the point $x = 0$, we obtain

$$c_1 + c_3 = 2, \qquad c_2 + 3c_4 = 1$$

$$-9c_3 = 2, \qquad -27c_4 = 1$$

with the solution

$$c_1 = \frac{20}{9}, \quad c_2 = \frac{10}{9}, \quad c_3 = -\frac{2}{9}, \quad c_4 = -\frac{1}{27}.$$

The solution of the given initial-value problem is thus

$$y(x) = \frac{20}{9} + \frac{10}{9}\,x - \frac{2}{9}\cos(3x) - \frac{1}{27}\sin(3x) \qquad \text{for all } x \in \mathbb{R}.$$

Exercise 4.3.2: 1 p

We find the general solution of the following fourth-order linear nonhomogeneous equation:

$$y^{(4)} + 2y^{(3)} + y'' = (x+1)^2,$$

for all $x \in \mathbb{R}$. To find the general solution $\phi_H(x\, c_1, c_2, c_3, c_4)$ of the corresponding homogeneous equation

$$y^{(4)} + 2y^{(3)} + y'' = 0,$$

we use the Ansatz $y(x) = e^{\lambda x}$, which leads to the condition

$$P_4(\lambda) = \lambda^4 + 2\lambda^3 + \lambda^2 = 0 \quad \text{or} \quad \lambda^2(\lambda+1)^2 = 0.$$

The four roots of characteristic polynomial $P_4(\lambda)$ are

$$\lambda_1 = \lambda_2 = 0, \quad \lambda_3 = \lambda_4 = -1$$

so that

$$\phi_H(x\, c_1, c_2, c_3, c_4) = c_1 + c_2 x + c_3 e^{-x} + c_4 x e^{-x}.$$

To find a particular solution $y_p(x)$, we make use of the **method of undetermined coefficients** and use the Ansatz

$$y_p(x) = x^2\left(A_2 x^2 + A_1 x + A_0\right) = A_2 x^4 + A_1 x^3 + A_0 x^2.$$

as given by **Case I (3)** in paragraph **4.3.1.**, due to the fact that the degree of the polynomial on the right side of the given equation is two and the coefficients of y and y' are zero. Inserting this Ansatz into the given nonhomogeneous differential equation, we obtain

$$24A_2 + 2(24A_2 x + 6A_1) + 12Ax^2 + 6A_1 x + 2A_0 = x^2 + 2x + 1.$$

By equating the coefficients x^2, x and 1, we obtain

$$\begin{aligned} &12A_2 = 1 \\ &48A_2 + 6A_1 = 2 \\ &24A_2 + 12A_1 + 2A_0 = 1 \end{aligned}$$

with the unique solution $A_2 = 1/12$, $A_1 = -1/3$ and $A_0 = 3/2$. Thus a particular solution is

$$y_p(x) = \frac{1}{12}\, x^4 - \frac{1}{3}\, x^3 + \frac{3}{2}\, x^2$$

and by **Proposition 4.3.1** the general solution of the given equation takes the form

$$y(x) = c_1 + c_2 x + c_3 e^{-x} + c_4 x e^{-x} + \frac{1}{12}\, x^4 - \frac{1}{3}\, x^3 + \frac{3}{2}\, x^2$$

for all $x \in \mathbb{R}$.

Exercise 4.3.2: 1 q

We find the general solution of the third-order equation

$$y^{(3)} + 3y'' + 3y' + y = (x+1)e^{-x}$$

To find the general solution $\phi_H(x;\, c_1, c_2, c_3)$ of the associated homogeneous equation

$$y^{(3)} + 3y'' + 3y' + y = 0$$

we use the Ansatz $y(x) = e^{\lambda x}$, which leads to the condition

$$P_3(\lambda) = \lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0 \quad \text{or} \quad (\lambda+1)^3 = 0.$$

The four roots of the characteristic polynomial $P_3(\lambda)$ are

$$\lambda_1 = \lambda_2 = \lambda_3 = -1$$

so that

$$\phi_H(x\, c_1, c_2, c_3, c_4) = \left(c_1 + c_2 x + c_3 x^2\right) e^{-x}.$$

To find a particular solution $y_p(x)$, we make use of the **method of undetermined coefficients** and use the Ansatz

$$y_p(x) = e^{-x} w(x),$$

where $w(x)$ is an unknown function. This Ansatz is given by **Case III** in paragraph **4.3.1**. For the derivatives, we obtain

$$y_p' = \left(w' - w\right) e^{-x}$$

$$y_p'' = \left(w'' - 2w' + w\right) e^{-x}$$

$$y_p^{(3)} = \left(w^{(3)} - 3w'' + 3w' - w\right) e^{-x}.$$

Inserting this Ansatz into the given nonhomogeneous differential equation, we obtain

$$w^{(3)} = x + 1.$$

To find a solution $w(x)$ of this third-order nonhomogeneous equation we can make use of the **method on undetermined coefficients** discussed in **Case I** in paragraph **4.3.1**. Since the coefficients of w'', w' and w are zero, the Ansatz is

$$w(x) = x^3(A_1 x + A_0),$$

with

$$w' = 4A_1 x^3 + 3A_0 x^2$$

$$w'' = 12A_1 x^2 + 6A_0 x$$

$$w^{(3)} = 24A_1 x + 6A_0,$$

so that

$$24A_1x + 6A_0 = x + 1.$$

Equating coefficients of x and 1, we obtain

$$24A_1 = 1, \qquad 6A_0 = 1$$

so that $A_1 = 1/24$ and $A_0 = 1/6$, and a solution for w takes the form

$$w(x) = \frac{1}{24}x^4 + \frac{1}{6}x^3.$$

Hence a particular solution $y_p(x)$ for the given equation is

$$y_p(x) = \left(\frac{1}{24}x^4 + \frac{1}{6}x^3\right)e^{-x} \quad \text{for all } x \in \mathbb{R}.$$

The general solution is then

$$\begin{aligned} y(x) &= \phi_H(x; c_1, c_2, c_3) + y_p(x) \\ &= \left(c_1 + c_2x + c_3x^2 + \frac{1}{24}x^4 + \frac{1}{6}x^3\right)e^{-x} \quad \text{for all } x \in \mathbb{R}. \end{aligned}$$

Exercise 4.3.4: 1 c

We find the general solution of the following third-order nonhomogeneous differential equation

$$y^{(3)} - 6y'' + 11y' - 6y = \frac{e^{3x}}{e^{2x}+1}$$

for all $x \in \mathbb{R}$. First we calculate the general solution $\phi_H(x;\, c_1, c_2, c_3)$ of the associated homogeneous equation

$$y^{(3)} - 6y'' + 11y' - 6y = 0.$$

Using the Ansatz $y(x) = e^{\lambda x},\ \lambda \in \mathbb{C}$, we obtain the condition

$$P_3(\lambda) = \lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda-1)(\lambda-2)(\lambda-3) = 0.$$

The three roots of the polynomial $P_3(\lambda)$ are

$$\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 3$$

which gives three linearly independent solutions of the homogeneous equation, namely

$$\phi_1(x) = e^x, \quad \phi_2(x) = e^{2x}, \quad \phi_3(x) = e^{3x}.$$

The general solution of the homogeneous equation becomes

$$\phi_H(x;\, c_1, c_2, c_3) = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} \quad \text{for all } x \in \mathbb{R}.$$

To find a particular solution for the given nonhomogeneous equation, we make us of the **method of variation of parameters** as described in **Proposition 4.3.3**. A particular solution $y_p(x)$ is given by

$$y_p(x) = w_1(x)\phi_1(x) + w_2(x)\phi_2(x) + w_3(x)\phi_3(x),$$

where $w_j(x)$ $(j = 1, 2, 3)$ have the following form:

$$w_1(x) = \int \frac{W_1[(f), \phi_2, \phi_3](x)}{W[\phi_1, \phi_2, \phi_3](x)}\, dx$$

$$w_2(x) = \int \frac{W_2[\phi_1, (f), \phi_3](x)}{W[\phi_1, \phi_2, \phi_3](x)}\, dx$$

$$w_3(x) = \int \frac{W_3[\phi_1, \phi_2, (f)](x)}{W[\phi_1, \phi_2, \phi_3](x)}\, dx.$$

Here $W[\phi_1, \phi_2, \phi_3](x)$ is the Wronskian of the set of solutions $\{\phi_1(x),\, \phi_2(x),\, \phi_3(x)\}$ and

$$f = \frac{e^{3x}}{e^{2x}+1}.$$

The notation $W_j[\phi_1, \ldots, \phi_{j-1}, (f), \phi_{j+1}, \ldots, \phi_n](x)$ is defined in (4.3.29) (see paragraph **4.3.3.**). We obtain the following

$$W[\phi_1, \phi_2, \phi_3](x) = 2e^{6x}, \quad W_1[(f), \phi_2, \phi_3](x) = \frac{e^{8x}}{e^{2x}+1}$$

$$W_2[\phi_1, (f), \phi_3](x) = -\frac{2e^{7x}}{e^{2x}+1}, \quad W_3[\phi_1, \phi_2, (f)](x) = \frac{e^{6x}}{e^{2x}+1}.$$

This leads to

$$w_1(x) = \frac{1}{2}\int \frac{e^{2x}}{e^{2x}+1}\,dx = \frac{1}{4}\ln\left(e^{2x}+1\right)$$

$$w_2(x) = -\int \frac{e^{x}}{e^{2x}+1}\,dx = -\arctan\left(e^{x}\right)$$

$$w_3(x) = \frac{1}{2}\int \frac{1}{e^{2x}+1}\,dx = \frac{1}{2}\,x - \frac{1}{4}\ln\left(e^{2x}+1\right).$$

A particular solution is thus

$$y_p(x) = \frac{1}{4}\ln\left(e^{2x}+1\right)\,e^{x} - \arctan\left(e^{x}\right)\,e^{2x} + \left(\frac{1}{2}\,x - \frac{1}{4}\ln\left(e^{2x}+1\right)\right)\,e^{3x}$$

for all $x \in \mathbb{R}$. The general solution then takes the form

$$\begin{aligned} y(x) &= \phi_H(x;\, c_1, c_2, c_3) + y_p(x) \\ &= c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x} + \frac{1}{4}\ln\left(e^{2x}+1\right)\,e^{x} - \arctan\left(e^{x}\right)\,e^{2x} \\ &\quad + \left(\frac{1}{2}\,x - \frac{1}{4}\ln\left(e^{2x}+1\right)\right)\,e^{3x} \quad \text{for all } x \in \mathbb{R}. \end{aligned}$$

Exercise 4.4.1: 3 h

We find the general solution of the following fourth-order Cauchy-Euler equation:

$$x^4 y^{(4)} + 12x^3 y^{(3)} + 38x^2 y'' + 32xy' + 4y = \frac{2}{x} + \frac{4}{x^2}$$

for all $x > 0$. Using **Proposition 4.4.1** we introduce a new independent variable z as

$$x = e^{z}, \qquad y(x) = y(z).$$

This leads to the equation

$$\frac{d^4y}{dz^4} + 6\frac{d^3y}{dz^3} + 13\frac{d^2y}{dz^2} + 12\frac{dy}{dz} + 4y(z) = 2e^{-z} + 4e^{-2z}.$$

Solving this equation we obtain the general solution

$$y(z) = (c_1 + c_2 z)e^{-2z} + (c_3 + c_4 z)e^{-z} + z^2 e^{-z} + 2z^2 e^{-2z}$$

so that the general solution of the given equation becomes

$$y(x) = (c_1 + c_2 \ln x)x^{-2} + (c_3 + c_4 \ln x)x^{-1} + x^{-1}(\ln x)^2 + 2x^{-2}(\ln x)^2$$

for all $x \in \mathbb{R}$.

Appendix E

Answers to the exercises

Exercise 1.1.1

1. a) Linearly independent.

   b) Linearly dependent.

   c) Linearly independent.

   d) Linearly independent.

   e) Linearly independent.

   f) Linearly independent.

   g) Linearly dependent.

   h) Linearly dependent.

   i) Linearly dependent.

   j) Linearly dependent.

5. a) $4xy'' + 2y' - y = 0, \quad x > 0.$

   b) $x^4y'' - a^2y = 0, \quad a \neq 0, \quad x \in \mathbb{R}.$

   c) $x^4y'' + y, \quad x \in \mathbb{R}\backslash\{0\}.$

   d) $xy'' + 2y' - xy = 0, \quad x \in \mathbb{R}\backslash\{0\}.$

   e) $4x^2y'' + 4x^3y' + (2x^2 - 3)y = 0, \quad x > 0.$

   f) $x^3y^{(3)} - x^2y'' + 2xy' - 2y = 0, \quad x > 0.$

   g) $x^3y^{(3)} - xy' - 3y = 0, \quad x > 0.$

Exercise 1.2.1

1. b) $y(x) = c_1 e^{2x}\cos(3x) + c_2 e^{2x}\sin(3x)$.

   c) $T:\ y(x) \mapsto L\,y(x)$ where $L = \dfrac{d^2}{dx^2} - 4\dfrac{d}{dx} + 13$.

   d) $B = \{e^{2x}\cos(3x),\ e^{2x}\sin(3x)\}$.

   e) $y(x) = 4e^{2x}\cos(3x) - 3e^{2x}\sin(3x)$.

   f) $y(x) = e^{2x}\cos(3x) + 2e^{-\pi/3}e^{2x}\sin(3x)$.

2. General solution: $y(x) = c_1 e^{-x}\cos x + c_2 e^{-x}\sin x$.

   Basis of the solution space: $B = \{e^{-x}\cos x,\ e^{-x}\sin x\}$.

   Dimension of the solution space: Two.

3. General solution: $y(x) = c_1 e^{x} + c_2 e^{-2x}$.

   Basis of the solution space: $B = \{e^{x},\ e^{-2x}\}$.

   Dimension of the solution space: Two.

4. a) Differential equation: $y'' + y = 0$.

   General solution: $y(x) = c_1 \cos x + c_2 \sin x$.

   Basis of the solution space: $B = \{\cos x,\ \sin x\}$.

   Dimension of the solution space: Two.

   b) Differential equation: $y'' + 2y' + 5y = 0$.

   General solution: $y(x) = c_1 e^{-x} \cos(2x) + c_2 e^{-x} \sin(2x)$.

   Basis of the solution space: $B = \{e^{-x} \cos(2x),\ e^{-x} \sin(2x)\}$.

   Dimension of the solution space: Two.

   c) Differential equation: $y'' + 2y' = 0$.

   General solution: $y(x) = c_1 + c_2 e^{-2x}$.

   Basis of the solution space: $B = \{1,\ e^{-2x}\}$.

   Dimension of the solution space: Two.

   d) Differential equation: $y^{(3)} - y'' + y' - y = 0$.

   General solution: $y(x) = c_1 e^{x} + c_2 \sin x + c_3 \cos x$.

   Basis of the solution space: $B = \{e^{x},\ \sin x,\ \cos x\}$.

   Dimension of the solution space: Three.

   e) Differential equation: $y^{(4)} + y^{(3)} + 4y'' + 4y' = 0$.

   General solution: $y(x) = c_1 + c_2 e^{-x} + c_3 \cos(2x) + c_4 \sin(2x)$.

   Basis of the solution space: $B = \{1,\ e^{-x},\ \cos(2x),\ \sin(2x)\}$.

   Dimension of the solution space: Four.

Exercise 2.2.1

1. a) $y(x) = \ln\left(\dfrac{1}{c - e^{x}}\right)$.

   b) $y(x) = \dfrac{1}{2}\left(\dfrac{c - x^2}{1 + x^2}\right)$.

   c) $y^2(x) = cx^2 - c - 1$.

   d) $y(x) = \dfrac{ce^{-x}}{(x-3)^3}$.

   e) $y(x) = \dfrac{e^{c}(x-1) - x - 1}{e^{c}(x-1) + x + 1}$ or $y(x) = \dfrac{e^{c}(x-1) + x + 1}{e^{c}(x-1) - x - 1}$.
   Singular solutions: $y = \pm 1$.

f) $\left|\dfrac{y-1}{y+1}\right| = |c|e^{2\arctan(x)}$. Singular solution: $y=-1$.

g) $y(x) = 1 + \dfrac{c}{(x-1)^2}\, e^{(x^2-x-1)/(1-x)}$.

h) $\dfrac{1}{2}\arctan(x^2) + \sqrt{a^2-y^2} - \arctan\left(\dfrac{y}{\sqrt{a^2-y^2}}\right) = c$ or $y = \pm a$.

i) $(x^2+2x+2)\,e^{-x} + (6-6y+3y^2-y^3)\,e^{y} = c$.

2. a) $y(x) = -1 + \dfrac{2(1+x)}{\sqrt{1-x^2}}$.

b) $y(x) = \tan\left(\arctan(x) + \dfrac{\pi}{4}\right)$.

c) $y(x) = \ln\left(\dfrac{x^2}{2}+1\right)$.

d) $y(x) = 0$.

e) $y(x) = \arcsin\left[\dfrac{1}{2}(1+x^2)^{1/2}\right]$.

3. a) $e^{y/x} = \dfrac{1}{\ln|1/(cx)|}$, $c \neq 0$.

b) $y^2 = \dfrac{1}{2x^2}\left(x^4 - e^c\right)$.

c) $\dfrac{x}{y-x} - \ln\left|\dfrac{y-x}{x}\right| = \ln|cx|$.

4. a) $y(x) = \dfrac{x}{1-\ln x}$.

b) $y(x) = \dfrac{1}{2}\left(\dfrac{x^3}{8-x^2}\right)$.

Exercise 2.3.1

1. a) $y(x) = x^2 + ce^x$.

b) $y(x) = -\dfrac{1}{2}\cos x + \dfrac{1}{2}\sin x + ce^{-x}$.

c) $y(x) = -1 + ce^{-x^3/3}$.

d) $y(x) = \dfrac{1}{2}\cos x - \dfrac{1}{2}\sin x - x^3 + 3x^2 - 6x + 6 + ce^{-x}$.

e) $y(x) = \dfrac{11}{27} - \dfrac{2}{9}x + \dfrac{1}{3}x^2 + ce^{-3x}$.

f) $y(x) = xe^{-\sin x} + ce^{-\sin x}$.

g) $y(x) = e^x + c\sqrt{1+x^2}$.

2. a) $y(x) = e^x - x$.

   b) $y(x) = x^2 - 2$.

   c) $y(x) = \dfrac{1}{x}(\cos x + x\sin x)$.

   d) $y(x) = \ln x - \dfrac{1}{\ln x}$, $x \neq 1$.

3. $y(x) = y_1(x) + c\,[y_2(x) - y_1(x)]$

Exercise 2.4.4

1. $\cos y = \dfrac{1}{4}\left(2\sin^2 x - 2\sin x + 1\right) + c\, e^{-2\sin x}$

2. a) $y^2(x) = 4\left(-2 - 4x^2 + 4x + c\, e^{-2x}\right)^{-1}$.

   b) $y(x) = \dfrac{1}{x^2}\left(\dfrac{1}{2}(x-1)e^x + c\right)^2$.

   c) $y(x) = (1 + \ln(x) + cx)^{-1}$.

   d) $y^2(x) = -x\ln|x| + cx$.

   e) $y(x) = -2(\cos x + \sin x - c\, e^x)^{-1}$.

   f) $y^2(x) = \dfrac{x(1-x)^{1/2}(1+x)^{1/2}}{[-1 + x^2 + c(1-x)^{1/2}(1+x)^{1/2}]^2}$.

   g) $y(x) = \left[-\dfrac{1}{3}(1-x^2) + c(1-x^2)^{1/4}\right]^2$.

3. a) i $y(x) = 2\left(2 - e^{x^2}\right)^{-1}$.

   a) ii $y(x) = 1$.

   b) $y(x) = \dfrac{\sqrt{6}}{36}\dfrac{(x^4 + 6(3)^{2/3} - 1)^{3/2}}{x^3}$.

4. a) Linear equation: $4xv' - 4v - x = 0$.

   Solution of linear equation: $v(x) = x\left(\dfrac{1}{4}\ln|x| + c\right)$.

   Solution of Riccati equation: $y(x) = \dfrac{2(\ln|x| + c + 2)}{x(\ln|x| + c)}$.

   b) Linear equation: $v' - v + 1 = 0$.

   Solution of linear equation: $v(x) = 1 + c\, e^x$.

   Solution of Riccati equation: $y(x) = \dfrac{x + cx\, e^x + 1}{1 + c\, e^x}$.

5. a) $y(x) = \dfrac{-2x - 1 + x^2}{x - 2}$.

   b) $y(x) = -\dfrac{1}{x}\left[\tanh(\ln|x|) - \operatorname{arctanh}(2)\right]$.

   c) $y(x) = \dfrac{1}{x}$.

6. $k = 1$. Then $y(x) = \dfrac{x(2x+3)}{3x^2 + 2}$

7. $k=-1$

   i) $y(x)=\dfrac{3+x^2}{x(x^2+1)}$.

   ii) $y(x)=\dfrac{1}{x}$.

8. $k=-1$. Then $y(x)=-\dfrac{\ln|x|-c+1}{x(\ln|x|-c)}$

9. $w''-\left(\dfrac{f'(x)}{f(x)}+1\right)w'+f(x)h(x)w=0$.

10. $y(x)=\dfrac{x(cx^2+c-x^)2}{cx^2-c-x^2+2}$.

Exercise 3.2.1

1. a) $y(x)=c_1\,e^{-2x}\sin(3x)+c_2\,e^{-2x}\cos(3x)$.

   b) $y(x)=c_1\,e^{2x}c_2x\,e^{2x}$.

   c) $y(x)=c_1\,e^{-x}+c_2\,e^{-2x/3}$.

   d) $y(x)=c_1\sin(\sqrt{3}x+c_2\cos(\sqrt{3}x)$.

   e) $y(x)=c_1+c_2\,e^{-4x}$.

   f) i. $y(x)=c_1\,e^{ax/2+\sqrt{a^2-4}\,x/2}+c_2\,e^{ax/2-\sqrt{a^2-4}\,x/2}$.

   f) ii. $y(x)=c_1\,e^{ax/2}\cos\left(\sqrt{4-a^2}\,x/2\right)+c_2\,e^{ax/2}\sin\left(\sqrt{4-a^2}\,x/2\right)$.

2. a) $y(x)=e^{-x}\sin(\sqrt{2}\,x)$.

   b) $y(x)=\ e^{-3x}+2x\,e^{-3x}$.

   c) $y(x)=\dfrac{\sqrt{2}}{2}\,e^{\sqrt{2}\,x}-\dfrac{\sqrt{2}}{2}\,e^{-\sqrt{2}\,x}$.

   d) $y(x)=\dfrac{1}{2}+\dfrac{1}{2}\,e^{2x-6}$.

   e) $y(x)=-\dfrac{5}{3}\sin(3x)-\cos(3x)$.

3. a) $y(x)=-\dfrac{5\,e^{x}}{e^{-1}-e}+\dfrac{5\,e^{-x}}{e^{-1}-e}$.

   b) $y(x)=-\dfrac{2\,e^{8}-3\,e^{4}}{e^{8}-e^{4}}+\dfrac{e^{4x}}{e^{8}-e^{4}}$.

   c) $y(x)=-\dfrac{1}{2}\,x+\dfrac{5}{2}$.

Exercise 3.3.2

1. a) $y(x) = c_1 e^{-2x} + c_2 x\, e^{-2x} + \dfrac{4}{15} x^{5/2}\, e^{-2x}.$

   b) $y(x) = c_1 e^{x} + c_2 x\, e^{x} + \dfrac{4}{15} (x-1)^{5/2}\, e^{x}.$

   c) $y(x) = c_1 e^{-x} + c_2 x\, e^{-x} + \dfrac{1}{4} x^2 (2\ln(x) - 3)\, e^{-x}.$

   d) $y(x) = c_1 e^{-x} + c_2 x\, e^{-x} - (2\ln(x) + 1)\, e^{-x}.$

   e) $y(x) = c_1\, e^{3x} + c_2\, e^{-3x} - \dfrac{1}{8} e^{x} - \dfrac{1}{9}.$

   f) $y(x) = c_1 \cos(2x) + c_2 \sin(2x) + \dfrac{1}{2}(\cos x \sin x - x) \cos(2x)$
   $\quad -\dfrac{1}{2}(\cos^2 x - \ln(|\cos x|) \sin(2x).$

   g) $y(x) = c_1 \cos x + c_2 \sin x + x \sin x + \cos x \ln|\cos x|.$

   h) $y(x) = c_1 \cos(2x) + c_2 \sin(2x) + \left(\dfrac{9}{8}\cos(2x) + \dfrac{9}{32}\cos(4x)\right)\cos(2x)$
   $\quad + \left(-\dfrac{9}{8}\sin(2x) - \dfrac{9}{32}\sin(4x) + \dfrac{9}{16}x\right)\sin(2x)$ for all $x \in \mathbb{R}$.

2. $y(x) = 2\,e^x - \dfrac{3}{2}\,e^{-x} - \dfrac{1}{2}\left[2\,e^x - 2\ln(e^x+1) - 1\right]\,e^{-x}.$

3. $y(x) = e^{2x}\arctan(e^x) - \dfrac{1}{2}\,e^x\ln(1+e^{2x}) + \dfrac{1}{2}\,e^x\ln(2).$

4. $y(x) = c_1\,e^x + c_2\,x^2e^x + \dfrac{2}{9}\,e^x x^3(3\ln x - 4).$

5. $y(x) = c_1x + c_2(x^2-1) + \dfrac{1}{6}x^4 + \dfrac{1}{2}.$

6. $y(x) = c_1\,e^x + c_2x - \dfrac{1}{2}(2x-1)\,e^{-x}.$

7. a) $y(x) = c_1\cos(2x) + c_2\sin(2x) + \dfrac{1}{3}\cos x.$

   b) $y(x) = c_1\,e^{3x} + c_2x\,e^{3x} + \dfrac{1}{x}.$

   c) $y(x) = c_1\,e^{-2/x} + c_2e^{2/x} - \dfrac{1}{4x}.$

8. a) $y'' + y = 4x\cos x.$

   b) $x^2y'' - xy' + y = 3x^3.$

   c) $x^2y'' - 4xy' + 6y = x^2(x^2-1).$

Exercise 3.3.4

1. a) $y(x) = c_1\sin x + c_2\cos x + 2 - 2x.$

   b) $y(x) = c_1\,e^{\sqrt{7}\,x} + c_2\,e^{-\sqrt{7}\,x} - \dfrac{9}{49} + \dfrac{2}{7}x - \dfrac{1}{7}x^2.$

   c) $y(x) = c_1\,e^{2x} + c_2x\,e^{2x} + (2+x)\,e^x.$

   d) $y(x) = c_1\cos x + c_2\sin x + \dfrac{1}{25}\,(5x-4)\,e^{2x}.$

   e) $y(x) = c_1\,e^{-x} + c_2\,e^{4x} + x\,e^{4x}.$

   f) $y(x) = c_1\,e^{-x} + c_2x\,e^{-x} + \dfrac{1}{125}\,e^x\,(-4\cos x + 15x\cos x - 22\sin x + 20x\sin x).$

   g) $y(x) = c_1\cos(2x) + c_2\sin(2x) - \dfrac{1}{4}x\cos(2x).$

   h) $y(x) = c_1\,e^{3x} + c_2x\,e^{3x} + \dfrac{x^6}{24}(x^3-4)e^{3x}.$

   i) $y(x) = c_1\,e^x + c_2\,e^{-x} + \dfrac{1}{2}xe^x + 3.$

   j) $y(x) = c_1\,e^{-x} + c_2\,e^{-2x} + xe^{-x} + \dfrac{7}{12}e^{2x}.$

k) $y(x) = c_1 \cos(2x) + c_2 \sin(2x) + \dfrac{9}{8} x \sin(2x) + \dfrac{9}{8}$.

l) $y(x) = c_1\, e^{-3x} + c_2\, e^{3x} + \dfrac{5}{26} \cos(2x) - \dfrac{5}{18}$.

m) For $k \neq 0$: $y(x) = c_1 \cos(kx) + c_2 \sin(kx) + \dfrac{1}{k}$.

For $k = 0$: $y(x) = c_1 + c_2 x$.

n) For $k \neq 0$: $y(x) = c_1 \cos(kx) + c_2 \sin(kx) + \dfrac{1}{4k} [\sin(kx + \alpha) - 2kx \cos(kx + \alpha)]$.

For $k = 0$: $y(x) = c_1 + c_2 x$.

o) $y(x) = c_1 + c_2\, e^{3x/4} + e^{3x/4} \left(-24x + 9x^2\right)$.

p) $y(x) = c_1 \cos(5x) + c_2 \sin(5x) + \dfrac{1}{10} x \sin(5x)$.

q) $y(x) = c_1\, e^{-3x} \cos(2x) + c_2\, e^{-3x} \sin(2x) + \dfrac{1}{4} e^{-3x} x \sin(2x)$.

r) $y(x) = c_1\, e^{-x} + c_2\, e^{3x} - \dfrac{1}{144} \left[9 + 36x - 64e^{x} + 96xe^{x} - 18e^{4x} + 72xe^{4x}\right] e^{-x}$.

s) $y(x) = c_1\, e^{x} + c_2 x\, e^{x} - e^{x} \sin x + 2$.

t) $y(x) = c_1 e^{3x} + c_2 e^{-3x} + \left[\left(\dfrac{9}{5} - x\right) \cos(2x) + \left(3x + \dfrac{1}{10}\right) \sin(2x)\right] e^{3x}$.

2. a) $y(x) = -\frac{3}{8}\,e^{2x} + \frac{1}{4}x\,e^{2x} + \frac{3}{8} + \frac{1}{2}x + \frac{1}{4}x^2.$

   b) $y(x) = 2 - 2x.$

   c) $y(x) = -e^{3x-3} + x\,e^{3x-3} + x^2.$

   d) $y(x) = e^{2x} - e^{3x} + x\,e^{-x}.$

   e) $y(x) = x\cos x + x^2 \sin x.$

   f) $y(x) = x\,e^{3x} + e^{-x} + x.$

   g) $y(x) = -e^{x-\pi}\,(1 + 2\pi e^{\pi}) \sin x + 2e^x x \sin x.$

   h) $y(x) = -2e^{-x}\cos x + e^{-x}\sin x + 2\,e^x - 4.$

3. a) $y'' - y = 20x^3 - x^5.$

   b) $y'' + y = 2\,e^x + 2x\,e^x.$

   c) $y'' - 6y' + 13y = (-28 + 106x)\,e^{-4x}.$

   d) $y'' - 6y' + 9y = (8x - 6)\sin x + (-6x + 2)\cos x - 63.$

   e) $y'' + 4y = 6.$

   f) $y'' + 2y' + 10y = (45x^2 + 64)\,e^{-x} + 6x(3x^2 + 3x + 1)\,e^{2x}.$

   g) $y'' + 4y' = (8 + 12x)\,e^{2x} + 32\,e^{4x} - \sin x + 4\cos x.$

   h) $y'' - 2y' + 5y = 2\,e^{3x}\big(20\sin x\cos x - 10\cos^2 x + 7\big).$

Exercise 3.4.1

1. a) $y(x) = c_1\frac{1}{x} + c_2x^3.$

   b) $y(x) = c_1x^2 + c_2\frac{1}{x^3}.$

   c) $y(x) = c_1\frac{1}{x}\sin(\ln x) + c_2\frac{1}{x}\cos(\ln x) + \frac{1}{17}x^3.$

   d) $y(x) = c_1x^2 + c_2x^3 + x^2\ln x + x^2 + 2.$

   e) $y(x) = c_1\frac{1}{x^3} + c_2\frac{1}{x^3}\ln x + \frac{2}{x^3}\ln^2 x.$

   f) $y(x) = c_1x^2 + c_2x^3 + \frac{1}{2}\big(x^2 + 2\ln x + 2\big).$

   g) $y(x) = c_1x^5 + c_2\frac{1}{x} - \frac{1}{9}x^2\ln x.$

h) $y(x) = c_1 x + c_2 x^2 - \frac{1}{144} x^5 (12 \ln x - 7)$.

2. $y(x) = 3\sin(\ln x) + 2\cos(\ln x) + \frac{1}{2}\sin(\ln x)\ \ln x$.

3. $x^2 y'' - xy' + y = 3x^3$.

4. b) $y(x) = c_1\left(x + \frac{2}{3}\right) + c_2\left(x + \frac{2}{3}\right)\ln\left(x + \frac{2}{3}\right) + \frac{1}{2}(6+9x)\ln^2(2+3x)$
$-\frac{1}{2}(6+9x)\ln^2(3) - 6$.

Exercise 3.5.1

1. a) $y(x) = c_1 x^2 + c_2\, e^x$.

b) $y(x) = c_1 \frac{1}{x}\sin x + c_2 \frac{1}{x}\cos x$.

c) $y(x) = c_1 x + c_2 x e^{2x}$.

d) $y(x) = c_1(1 + 4x^2) + c_2\, e^{-2x}$.

e) $y(x) = c_1 x \sin(2x) + c_2 x \cos(2x)$.

f) $y(x) = c_1\, e^x + c_2 x^2\, e^x$.

3. $\frac{1}{\sqrt{h}} = -\frac{a}{2\sqrt{b}}\, e^{G(x)} \int e^{-G(x)}\, dx + c$, where $G(x) = \int g(x)\, dx$.

4. b i) $y(x) = c_1\, e^x + c_2 x^2\, e^x$.

b ii) $y(x) = c_1 \frac{1}{x}\sin x + c_2 \frac{1}{x}\cos x$.

5. a) $M = a_0(x),\ N = a_1(x) - a_0'(x)$ if and only if $a_0'' - a_1' + a_2 = 0$.

b) $I = (x^2 + 2x)y' + (2x+2)y =$ constant. Then $y(x) = \frac{c_1 x + c_2}{x(x+2)}$.

Exercise 4.2.1

1. a) $y(x) = c_1\, e^{-3x} + c_2\, e^{-x} + c_3\, e^{2x}$.

b) $y(x) = \frac{1}{4}\, e^{-2x} - \frac{1}{5}\, e^{-3x} + \frac{19}{20}\, e^{2x}$.

c) $y(x) = c_1\, e^{-x} + c_2\, e^{3x} + c_3 x\, e^{3x}$.

d) $y(x) = \frac{23}{5}\, e^{\pi - x} - \frac{28}{15}\sin(3x) - \frac{2}{5}\cos(3x)$.

e) $y(x) = 4\, e^{2-2x} - 8x\, e^{2-2x} + 7x^2\, e^{2-2x}$.

f) $y(x) = c_1 + c_2 x + c_3\, e^{2x}$.

g) $y(x) = c_1\, e^{-3x} + c_2\, e^{3x/2} \sin\left(\frac{3\sqrt{3}}{2}x\right) + c_3\, e^{3x/2} \cos\left(\frac{3\sqrt{3}}{2}x\right)$.

h) $y(x) = c_1 + c_2\, e^{-2x} + c_3\, e^{2x}$.

2. a) $y(x) = \frac{1}{30}\, e^{-3x} + \frac{13}{10}\, e^{2x} - \frac{1}{10}\, e^{-2x} - \frac{7}{30}\, e^{3x}$.

b) $y(x) = \frac{265}{54} - \frac{16}{9}x + \frac{5}{6}x^2 + \frac{1}{27}\, e^{3x-3}$.

c) $y(x) = c_1\, e^{-2x} + c_2 x\, e^{-2x} + c_3 x^2\, e^{-2x} + c_4 x^3\, e^{-2x}$.

d) $y(x) = c_1\, e^{x} + c_2 x\, e^{x} + c_3 x^2\, e^{x} + c_4 x^3\, e^{x}$.

e) $y(x) = c_1\, e^{2x} + c_2 x\, e^{2x} + c_3\, e^{-x} + c_4 x\, e^{-x}$.

f) $y(x) = \frac{20}{9} + \frac{10}{9}x - \frac{1}{27}\sin(3x) - \frac{2}{9}\cos(3x)$.

g) $y(x) = c_1 + c_2 x + c_3 x^2 + c_4\, e^{3x}$.

h) $y(x) = c_1 + c_2 x + c_3\, e^{2x} + c_4 x\, e^{2x}$.

3. a) $y(x) = c_1 + c_2\,e^{-2x} + c_3\,e^{2x} + c_4\sin(2x) + c_5\cos(2x)$.

   b) $y(x) = c_1 + c_2x + c_3x^2 + c_4\,e^{x} + c_5\,e^{2x}$.

   c) $y(x) = c_1 + c_2x + c_3x^2 + c_4x^3 + c_5\,e^{4x}$.

   d) $y(x) = c_1 + c_2x + c_3x^2 + c_4x^3 + c_5\,e^{x} + c_6\,e^{4x}$.

   e) $y(x) = c_1 + c_2x + c_3x^2 + c_4x^3 + c_5x^4 + c_6\,e^{-2x}$.

   f) $y(x) = \frac{4}{9} + \frac{4}{3}x + \frac{7}{3}x^2 + x^3 + \frac{2\sqrt{3}}{9}\sin(\sqrt{3}x) + \frac{5}{9}\cos(\sqrt{3}x)$.

Exercise 4.3.2

1. a) $y(x) = c_1 + c_2\sin x + c_3\cos x + 2x$.

   b) $y(x) = c_1 + c_2x + c_3\,e^{-x} + \frac{3}{2}x^2$.

   c) $y(x) = c_1 + c_2x + c_3\,e^{x} + c_4\,e^{-x} - 2x^2$.

   d) $y(x) = c_1 + c_2x + c_3\,e^{-2x} + c_4x\,e^{-2x} + \frac{1}{8}x^2$.

   e) $y(x) = c_1 + c_2\,e^{x} + c_3x\,e^{x} + x^2 + 4x$.

   f) $y(x) = c_1 + c_2x + c_3\cos x + c_4\sin x + \frac{1}{6}x^3 + \frac{1}{12}x^4 - x^2$.

   g) $y(x) = c_1 + c_2x + c_3e^{-x} + c_4xe^{-x} + \frac{1}{400}e^{4x}$.

   h) $y(x) = c_1 + c_2x + c_3e^{-x} + c_4xe^{-x} + \frac{1}{6}\left(24 + 18x + 6x^2 + x^3\right)e^{-x}$.

   i) $y(x) = c_1\,e^{x} + c_2\,e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right) + c_3\,e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right) - \frac{1}{2}\sin x$
   $+\frac{1}{2}\cos x$.

   j) $y(x) = c_1\,e^{x} + c_2x\,e^{x} + c_3x^2\,e^{x} - \frac{1}{8}\,e^{x}\sin(2x)$.

   k) $y(x) = c_1\,e^{x} + c_2\,e^{-x} + c_3x\,e^{x} + c_4x\,e^{-x} + \frac{1}{4}\cos x$.

   l) $y(x) = c_1 + c_2\cos(2x) + c_3\sin(2x) - \frac{1}{80}\,e^{2x}\cos(2x) + \frac{3}{80}\,e^{2x}\sin(2x)$
   $+\frac{1}{8}x\sin(2x) + \frac{1}{4}x$.

   m) $y(x) = c_1 + c_2x + c_3\,e^{-4x} + c_4\,e^{4x} + \frac{36}{289}\cos x + \frac{1}{17}x\sin x$.

   n) $y(x) = c_1 + c_2x + c_3x^2 + c_4\,e^{x} - 4x\,e^{x} + \frac{1}{2}x^2\,e^{x} + \frac{1}{24}x^4$.

o) $y(x) = c_1 + c_2 x + c_3 x^2 + c_4 \sin x + c_5 \cos x + \dfrac{1}{24} x^4 - e^{-x}.$

p) $y(x) = c_1 + c_2 x + c_3 e^{-x} + c_4 x e^{-x} + \dfrac{1}{12}\, x^4 - \dfrac{1}{3}\, x^3 + \dfrac{3}{2}\, x^2.$

q) $y(x) = \left(c_1 + c_2 x + c_3 x^2\right) e^{-x} + \left(\dfrac{1}{24}\, x^4 + \dfrac{1}{6}\, x^3\right) e^{-x}.$

2. a) $y(x) = \dfrac{1}{2}\, e^{x} - \dfrac{1}{2}\, e^{-x} + x^2.$

b) $y(x) = -3\, e^{x} + 2x\, e^{x} + \cos x + 2 \sin x - e^{-x}.$

c) $y(x) = -1 + 3x + x^2 + e^{x} + 4\, e^{-x} - 2\, e^{2x}.$

d) $y(x) = \dfrac{3}{2}\, e^{-x} \sin(2x) - e^{-x} \cos(2x) - 2x\, e^{-x} \cos(2x) - x\, e^{-x} \sin(2x) + 2.$

e) $y(x) = \cos(3x) - 9 \cos x + 3 \sin x + \cos(2x) + 5 \sin(2x).$

Exercise 4.3.4

1. a) $y(x) = c_1 + c_2 x + c_3\, e^{-x} - x + x \ln x.$

b) $y(x) = c_1\, e^{-x} + c_2 x\, e^{-x} + c_3 x^2\, e^{-x} + \dfrac{1}{6} x^3\, e^{-x} \ln x - \dfrac{11}{36} x^3\, e^{-x}.$

c) $y(x) = c_1\, e^{x} + c_2\, e^{2x} + c_3\, e^{3x} + \dfrac{1}{4}\, e^{x} \ln(e^{2x} + 1) + \dfrac{1}{2} x\, e^{3x} - \dfrac{1}{4}\, e^{3x} \ln(e^{2x} + 1)$

$-e^{2x} \arctan\left(e^{x}\right).$

2. $y_p(x) = \dfrac{x^{4-n}\, e^{x}}{(n-1)(n-2)(n-3)(n-4)}$ for all $n \in \mathbb{R} \backslash \{1,\ 2,\ 3,\ 4\}.$

For $n = 1 : y_p(x) = -\dfrac{11}{36} x^3\, e^{x} + \dfrac{1}{6} x^3\, e^{x} \ln x.$

For $n = 2 : y_p(x) = \dfrac{1}{4} x^2\, e^{x} - \dfrac{1}{2} x^2\, e^{x} \ln x.$

For $n = 3 : y_p(x) = \dfrac{1}{4} x\, e^{x} + \dfrac{1}{2} x\, e^{x} \ln x.$

For $n = 4 : y_p(x) = -\dfrac{11}{36}\, e^{x} - \dfrac{1}{6} x\, e^{x} \ln x.$

3. a) $y(x) = (e^{2x} + e^{x} + 1) \ln 2 - (e^{2x} + e^{x} + 1) \ln(e^{x} + 1) + x^2 + e^{x} + x\, e^{2x} - x.$

b) $y(x) = 2x\, e^{x}.$

c) $y(x) = 12 x^4 \ln x - 25 x^4 + 48 x^3 - 36 x^2 + 18 x + 2.$

Exercise 4.4.1

3. a) $y(x) = c_1 x + c_2 x \ln x + c_3 x \ln^2 x.$

   b) $y(x) = c_1 x + c_2 x^2 + c_3 x^3.$

   c) $y(x) = c_1 \sin(\ln x) + c_2 \cos(\ln x) + c_3 x^3 \sin(\ln x) + c_4 x^3 \cos(\ln x).$

   d) $y(x) = c_1 x^3 + c_2 \cos(\ln x) + c_3 \sin(\ln x) - \dfrac{1}{5} x^2.$

   e) $y(x) = c_1 x + c_2 x^2 + c_3 x \ln x - \dfrac{1}{4} x(-x^2 + 12 \ln x + 6 \ln^2 x + 12).$

   f) $y(x) = c_1 + c_2 \dfrac{1}{x^2} + c_3 \dfrac{1}{x^3} + \dfrac{1}{12} \ln^2 x - \dfrac{5}{36} \ln x.$

   g) $y(x) = c_1 x^2 + c_2 x^3 + c_3 x^4 - 2 - x^2 \ln x + 2x^4 \ln x.$

   h) $y(x) = c_1 \dfrac{1}{x} + c_2 \dfrac{1}{x^2} + c_3 \dfrac{\ln x}{x^2} + c_4 \dfrac{\ln x}{x} + \dfrac{\ln^2 x}{x} + \dfrac{2 \ln^2 x}{x^2}.$

   i) $y(x) = c_1 \cos(\ln x) + c_2 \sin(\ln x) + c_3 \ln x \cos(\ln x) + c_4 \ln x \sin(\ln x)$
   $- \ln^2 x \cos(\ln x).$

   j) $y(x) = c_1 \ln x + c_2 \ln^2 x + c_3 \ln^3 x + c_4 \ln^4 x + c_5 + 12 \ln^4 x \ln(\ln x).$

6. $y(x) = c_1 + c_2 x + c_3 x^2 + c_4 \dfrac{1}{(4+x)^3} - \dfrac{5}{3} \ln(4+x) - \dfrac{x}{4} \ln(4+x) + 1 + \dfrac{x}{4}$
   $-\dfrac{16}{45(4+x)^3}.$

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