The Laplace Transformation I – General Theory

Complex Functions Theory a-4

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ISBN 978-87-7681-718-3

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Introduction

We have in *Ventus: Complex Functions Theory a-1, a-2, a-3* given the most basic of the theory of analytic functions:

a-1 The book *Elementary Analytic Functions* is defining the battlefield. It introduces the analytic functions using the *Cauchy-Riemann equations*. Furthermore, the powerful results of the *Cauchy Integral Theorem* and the *Cauchy Integral Formula* are proved, and the most elementary analytic functions are defined and discussed as our building stones. The important applications of Cauchy's two results mentioned above are postponed to **a-2**.

a-2 The book *Power Series* is dealing with the correspondence between an analytic function and its complex power series. We make a digression into the theory of *Harmonic Functions*, before we continue with the *Laurent series* and the *Residue Calculus*. A handful of simple rules for computing the residues is given before we turn to the powerful applications of the residue calculus in computing certain types of *trigonometric integrals*, *improper integrals* and the sum of some not so simple series.

a-3 The book *Stability, Riemann surfaces, and Conformal maps* starts with pointing out the connection between analytic functions and *Geometry*. We prove some classical criteria for stability in *Cybernetics*. Then we discuss the inverse of an analytic function and the consequence of extending this to the so-called *multi-valued functions*. Finally, we give a short review of the *conformal maps* and their importance for solving a *Dirichlet problem*.

In the following volumes we describe some applications of this basic theory. We start in this book with the general theory of the *Laplace Transformation Operator*, and continue in *Ventus, Complex Functions Theory a-5* with applications of this general theory.

The author is well aware of that the topics above only cover the most elementary parts of *Complex Functions Theory*. The aim with this series has been hopefully to give the reader some knowledge of the mathematical technique used in the most common technical applications.

Leif Mejlbro
December 5, 2010

1 The Lebesgue Integral

1.1 Null sets and null functions

The theory of the Laplace transformation presented here relies heavily on *residue calculus*, cf. *Ventus, Complex Functions Theory a-2* and the *Lebesgue integral*. For that reason we start this treatise with a very short (perhaps too short?) introduction of the most necessary topics from *Measure Theory* and the theory of the *Lebesgue integral*.

We start with the definition of a *null set*, i.e. a set with no length (1 dimension), no area (2 dimension) or no volume (3 dimensions). Even if Definition 1.1.1 below seems to be *obvious* most of the problems of understanding *Measure Theory* and the *Lebesgue integral* can be traced back to this definition.

Definition 1.1.1 *Let $N \subset \mathbb{R}$ be a subset of the real numbers. We call N a* null set*, if one to every $\varepsilon > 0$ can find a sequence of (not necessarily disjoint) intervals I_n, each of length $\ell(I_n)$, such that*

$$N \subseteq \bigcup_{n=1}^{+\infty} I_n \qquad \textit{and} \qquad \sum_{n=1}^{+\infty} \ell(I_n) \leq \varepsilon.$$

Definition 1.1.1 is easily extended to the n-dimensional space $\mathbb{R}^n$ by defining a *closed interval* by

$$I := [a_1, b_1] \times \cdots \times [a_n, b_n], \qquad \text{where } a_j < b_j \text{ for all } j = 1, \dots, n.$$

If $n = 2$, then $I = [a_1, b_1] \times [a_2, b_2]$ is a rectangle, and $m(I) := (b_1 - a_1) \cdot (b_2 - a_2)$ is the *area* of this rectangle. In case of $n \geq 3$ we talk of n-dimensional *volumes* instead.

We first prove the following simple theorem.

Theorem 1.1.1 *Every finite or countable set is a null set.*

PROOF. Every subset of a null set is clearly again a null set, because we can apply the same ε-coverings of Definition 1.1.1 in both cases. It therefore suffices to prove the claim in the countable case. Assume that $N = \{x_n \mid n \in \mathbb{N}\}$, $x_n \in \mathbb{R}$, is countable. Choose any $\varepsilon > 0$ and define the following sequence of closed intervals

$$I_n := \left[x_n - \varepsilon \cdot 2^{-n-1}, x_n + \varepsilon \cdot 2^{-n-1}\right], \qquad \text{for all } n \in \mathbb{N}.$$

Then $x_n \in I_n$ and $\ell(I_n) = \varepsilon \cdot 2^{-n}$, so

$$N \subseteq \bigcup_{n=1}^{+\infty} I_n \qquad \text{and} \qquad \sum_{n=1}^{+\infty} \ell(I_n) = \sum_{n=1}^{+\infty} \varepsilon \cdot 2^{-n} = \varepsilon.$$

Since ε was chosen arbitrarily, it follows from Definition 1.1.1 that N is a null set. □

Example 1.1.1 The set of rational numbers $\mathbb{Q}$ are dense in $\mathbb{R}$, because given any real numbers $r \in \mathbb{R}$ and $\varepsilon > 0$ we can always find $q \in \mathbb{Q}$, such that $|r - q| < \varepsilon$. This is of course very convenient for many applications, because we in most cases can replace a real number r by a neighbouring rational number $q \in \mathbb{Q}$ only making an error $< \varepsilon$ in the following computations.

However, $\mathbb{Q}$ is countable, hence a null set by Theorem 1.1.1, while $\mathbb{R}$ clearly is *not* a null set, so points from a large set in the sense of measure can be approximated by points from a small set in the sense of measure, in the present case even of measure 0.

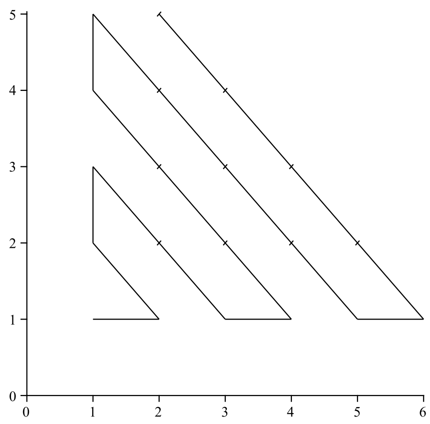

Figure 1: Proof of $\mathbb{N} \times \mathbb{N}$ being countable.

That $\mathbb{Q}$ is countable is seen in the following way. Since countability relies on the rational numbers $\mathbb{N}$, the set $\mathbb{N}$ is of course countable. Then $\mathbb{N} \times \mathbb{N} := \{(m,n) \mid m \in \mathbb{N},\, n \in \mathbb{N}\}$ is also countable.

The points of $\mathbb{N} \times \mathbb{N}$ are illustrated on Figure 1, where we have laid a broken line mostly following the diagonals, so it goes through every point of $\mathbb{N} \times \mathbb{N}$. Starting at $(1,1) \sim 1$ and $(2,1) \sim 2$ and $(1,2) \sim 3$ following this broken line we see that we at the same time have numbered all points of $\mathbb{N} \times \mathbb{N}$, so this set must be countable.

An easy modification of the proof above shows that $\mathbb{Z} \times \mathbb{N}$ is also countable. The reader is urged as an exercise to describe the extension and modification of Figure 1, such that the broken line goes through all points of $\mathbb{Z} \times \mathbb{N}$.

To any given $(m,n) \in \mathbb{Z} \times \mathbb{N}$ there corresponds a unique rational number $q := \dfrac{m}{n} \in \mathbb{Q}$, and to every $q = \dfrac{m}{n} \in \mathbb{Q}$ there corresponds infinitely many pairs $(p \cdot m, p \cdot n) \in \mathbb{Z} \times \mathbb{N}$ for $p \in \mathbb{N}$. Therefore, $\mathbb{Q}$ contains at most as many points as $\mathbb{Z} \times \mathbb{N}$, so $\mathbb{Q}$ is *at most* countable. On the other hand, $\mathbb{Q} \supset \mathbb{N}$, so $\mathbb{Q}$ is also *at least* countable. We therefore conclude that $\mathbb{Q}$ is countable, and $\mathbb{Q}$ is a null set. $\Diamond$

Example 1.1.2 Life would be easier if one could conclude that is a set is uncountable, then it is not a null set. *Unfortunately, this is not the case!!!* The simplest example is probably the (classical) set of points

$$N := \left\{ x \in [0,1] \;\middle|\; x = \sum_{n=1}^{+\infty} a_n \cdot 3^{-n},\, a_n \in \{0,2\} \right\}.$$

The set N is constructed by dividing the interval $[0,1]$ into three subintervals

$$\left[0, \frac{1}{3}\right], \quad \left]\frac{1}{3}, \frac{2}{3}\right[, \quad \left[\frac{2}{3}, 1\right],$$

and then remove the middle one. Then repeat this construction on the smaller intervals, etc.. At each step the length of the remaining set is multiplied by $\dfrac{2}{3}$, so N is at step n contained in a union of intervals of a total length $\left\{\dfrac{2}{3}\right\}^n \to 0$ for $n \to +\infty$, so N is a null set.

On the other hand, we define a bijective map $\varphi : N \to M$ by

$$\varphi\left(\sum_{n=1}^{+\infty} a_n \cdot 3^{-n}\right) := \sum_{n=1}^{+\infty} \frac{a_n}{2} \cdot 2^{-n} = \sum_{n=1}^{+\infty} b : n \cdot 2^n, \qquad \text{where } b_n := \frac{a_n}{2} \in \{0,1\}.$$

Clearly, every point $y \in [0,1]$ can be written in the form

$$y = \sum_{n=1}^{+\infty} b_n \cdot 2^{-n}, \qquad b_n \in \{0,1\},$$

so we conclude that $M = [0,1]$. Since $\varphi : N \to [0,1]$ is surjective, N and $[0,1]$ must have the same number of elements, or more precisely, N has *at least* as many elements as $[0,1]$, but since $N \subset [0,1]$ it also must have *at most* as many elements as $[0,1]$. The interval $[0,1]$ is not a null set, because its length is 1, so it follows from Theorem 1.1.1 that $[0,1]$ is not countable. Hence, N is a non-countable null set. $\Diamond$

Examples 1.1.1 and 1.1.2 above show that null sets are more difficult to understand than one would believe from the simple Definition 1.1.1. The reason is that there is a latent aspect of Geometry in this definition, which has never been clearly described, although some recent attempts have been done in the *Theory of Fractals*. So after this warning the reader is recommended always to stick to the previous Definition 1.1.1 and in the simple cases apply Theorem 1.1.1, and not speculate too much of the Geometry of possible null sets.

The next definition is building on Definition 1.1.1.

Definition 1.1.2 *A function f defined on $\mathbb{R}$ is called a* null function*, if the set $\{x \in \mathbb{R} \mid f(x) \neq 0\}$ is a null set, i.e. if the function is zero outside a null set.*
When f is a null function, we define *its integral as 0, i.e.*

$$\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x = 0, \qquad \text{if } f \text{ is a null function.}$$

That this is a fortunate definition is illustrated by the following example.

Example 1.1.3 Given a subset $A \subseteq \mathbb{R}$, we define its *indicator function* $\chi_A : \mathbb{R} \to \{0,1\}$ by

$$\chi_A(x) = \begin{cases} 1 & \text{for } x \in A, \\ 0 & \text{for } x \notin A. \end{cases}$$

The indicator function is in some textbooks also called the *characteristic function* of the set A, and denoted by 1_A.

It follows from the above that A is a null set, if and only if χ_A is a null function.

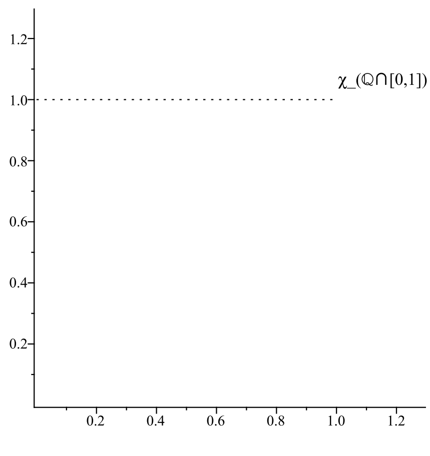

Figure 2: The indicator function of $\mathbb{Q} \cap [0,1]$ is a null function, which is not Riemann integrable.

If we choose $A = \mathbb{Q} \cap [0,1]$, then A is countable, hence a null set, so by the *definition* above,

$$\int_{-\infty}^{+\infty} \chi_{\mathbb{Q}\,\cap\,[0,1]}(x)\,\mathrm{d}x = 0.$$

On the other hand, $\chi_{\mathbb{Q}\,\cap\,[0,1]}$ cannot be Riemann integrable. In fact, any approximation from below is ≤ 0, while any approximation from above is ≥ 1. This illustrates that Definition 1.1.2 cannot be derived from the well-known Riemann integral.

Clearly, if the integral of f exists and is finite, and $g = f + h$, where h is a null function, then we put

$$\int_{-\infty}^{+\infty} g(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} \{f(x)+h(x)\}\,\mathrm{d}x = \int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x + \int_{-\infty}^{+\infty} h(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x,$$

so we can always change a function on a null set without changing the value of its integral. This observation will often be convenient in the following.

More generally, we *define* the following *pointwise* product

$$0 \cdot (+\infty) = 0 \cdot (-\infty) = 0,$$

so a *"pointwise"* 0 will always dominate the values $\pm\infty$. (The reader should be aware of that these rules cannot be applied when we are dealing with limits in general.) Then we shall also allow the (real) functions to have infinite values $\pm\infty$. Combining the definitions above we get for a null set A that

$$\int_{-\infty}^{+\infty} (+\infty) \cdot \chi_A(x)\, \mathrm{d}x = (+\infty) \cdot \int_{-\infty}^{+\infty} \chi_A(x)\, \mathrm{d}x = (+\infty) \cdot 0 = 0,$$

so $(+\infty) \cdot \chi_A(x)$ is again a null function.

Furthermore, if g is any real function, and f is a null function, then the pointwise product $f(x)g(x)$ is again a null function. $\Diamond$

If f is a null function, we write $f = 0$ *a.e.*, where "a.e." is an abbreviation of "almost everywhere". If $f - g = 0$ *a.e.*, we also write $f = g$ *a.e.*. If they both are integrable, their integrals are equal.

Remark 1.1.1 The rules of calculation of $\pm\infty$ defined above are not universal. There exist other forms of infinity, which do not obey these rules. One example is given by the so-called *"Dirac's δ function"*, which is 0 at any $x \neq 0$, and a "higher form of infinity" at 0, because one customary puts

$$\text{“}\int_{-\infty}^{+\infty} \delta(x)\, \mathrm{d}x = 1\text{”},$$

which is not possible, if $\delta(x)$ was an ordinary function, because $\{0\}$ is a null set. This enigma is solved by identifying δ as a *point measure* at 0, and *not* as a function. We shall return to δ in *Ventus, Complex Functions Theory a-5.* $\Diamond$

1.2 The Lebesgue integral

As mentioned earlier we shall base our theory of the *Laplace transformation* on the *Lebesgue integral*, and not on the more familiar *Riemann integral.* We shall only give the most necessary definitions and results without any proper proof, because this book is not meant to be a course in *Measure Theory* and *Lebesgue integral.*

We shall only consider the so-called *measurable functions.* This is actually no problem, because every function stemming from practical applications in *Physics* or *Engineering Sciences* are necessarily measurable, and non-measurable functions only exist in an ideal mathematical world. They do exist, but it is not possible to construct just one of them without using some doubtful methods. Therefore, we shall in the following tacitly assume that every function under consideration is indeed measurable, even if we do not make their definition precise.

Concerning the *Lebesgue integral* a workable definition for our purposes, though most mathematicians would not like this version, is the following:

Definition 1.2.1 *Assume that* $f \geq 0$ *is a non-negative function. If there exists a* Riemann integrable *function* g*, such that* $f = g$ *a.e., then we call* f Lebesgue integrable *(or just integrable), and we define*

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{+\infty} g(x)\,dx.$$

Let f be a real function. We define

$$f^{+}(x) = \max\{f(x), 0\} = (f \vee 0)(x), \qquad f^{-}(x) = \min\{f(x), 0\} = (f \wedge 0)(x),$$

where $\vee$, resp. $\wedge$, is a shorthand for max, resp. min.

Definition 1.2.2 *A real function* f *is (Lebesgue) integrable, if both* $f^{+} \geq 0$ *and* $-f^{-} \geq 0$ *are integrable in the sense of Definition 1.2.1.*
If this is the case, we define

(1)
$$\int_{-\infty}^{+\infty} f(x)\,dx := \int_{-\infty}^{+\infty} f^{+}(x)\,dx - \int_{-\infty}^{+\infty} (-f^{-}(x))\,dx = \int_{-\infty}^{+\infty} (f(x) \vee 0)\,dx - \int_{-\infty}^{+\infty} \{-(f(x) \wedge 0)\}\,dx.$$

It follows from Definition 1.2.2 that

$$\int_{-\infty}^{+\infty} (f \wedge 0)(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} \min\{f(x), 0\}\,\mathrm{d}x = -\int_{-\infty}^{+\infty} \{-(f \wedge 0)(x)\}\mathrm{d}x,$$

so (1) can be written in the usual way,

$$\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x := \int_{-\infty}^{+\infty} f^{+}(x)\,\mathrm{d}x + \int_{-\infty}^{+\infty} f^{-}(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} (f \vee 0)(x)\,\mathrm{d}x + \int_{-\infty}^{+\infty} (f \wedge 0)(x)\,\mathrm{d}x.$$

We define in a similar way the Lebesgue integral of complex functions by requiring that both $\Re f$ and $\Im f$ are Lebesgue integrable.

Consider any (measurable) function f and any number $p \in [1, +\infty[$. If $|f|^p$ is integrable, we define the *p-norm* of f by

$$\|f\|_p := \left\{\int_{-\infty}^{+\infty} |f(x)|^p\,\mathrm{d}x\right\}^{1/p} \qquad (< +\infty).$$

The most important cases are $p = 1$ and $p = 2$, and $p = (+)\infty$, the latter being defined in the following way,

$$\|f\|_\infty := \text{ess.sup}\,\{|f(x)| \mid x \in \mathbb{R}\},$$

where "ess.sup" means the "essential supremum". It is defined as the uniquely determined number $\|f\|_\infty$, for which

$$\{x \in \mathbb{R} \mid f(x) > a\} \quad \begin{cases} \text{is a null set, if } a > \|f\|_\infty, \\ \text{is not a null set, if } a < \|f\|_\infty. \end{cases}$$

If $f(x)$ is continuous and bounded, then of course

$$\|f\|_\infty = \sup\{|f(x)| \mid x \in \mathbb{R}\}.$$

The set of all functions f on $\mathbb{R}$, for which $\|f\|_p < +\infty$, is denoted by $L^p(\mathbb{R})$.

Of course the above is easily generalized to $\mathbb{R}^n$ in the most obvious way.

Finally we mention without proofs in the present section the following main theorems from the theory of the Lebesgue integral.

Theorem 1.2.1 Theorem of majoring convergence. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions from $L^1(\mathbb{R})$, and assume that the pointwise sequence $(f_n(x))_{n\in\mathbb{N}}$ is convergent for almost every (fixed) $x \in \mathbb{R}$ with the limit function $f(x)$, defined almost everywhere.*
Assume that there is a function $g \in L^1(\mathbb{R})$, such that

$$|f_n(x)| \le g(x) \qquad \text{for all } n \in \mathbb{N} \text{ and almost every } x \in \mathbb{R}.$$

Then the limit function $f(x) = \lim_{n\to+\infty} f_n(x)$, defined a.e. is Lebesgue integrable, and we have

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{+\infty} \lim_{n\to\infty} f_n(x)\,dx = \lim_{n\to+\infty} \int_{-\infty}^{+\infty} f_n(x)\,dx,$$

so in this case the limit process and the integration process can be interchanged.

This theorem is one of the advantages of the Lebesgue integral. It does not hold for the Riemann integral, unless we replace the pointwise convergence is replaces by the much stronger uniform convergence.

Theorem 1.2.2 Fubini's theorem. *Assume that $f(\mathbf{x}) = f(x_1, x_2)$ for $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ is a Lebesgue integrable function.*
For almost every fixed $x_1 = a \in \mathbb{R}$ the function $f(a, x_2)$ is Lebesgue integrable in x_2. Its integral $\int_{-\infty}^{+\infty} f(x_1, x_2)\,dx_2$ (writing x_1 again instead of a) is an (almost everywhere defined) Lebesgue integrable function, and we have

$$\int\!\!\int_{\mathbb{R}^2} f(\mathbf{x})\,d\mathbf{x} = \int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} f(x_1, x_2)\; dx_2 \right\} dx_1.$$

Fubini's theorem states that if $f \in L^1(\mathbb{R}^2)$, then the order of integration can be interchanged,

$$\int\!\!\int_{\mathbb{R}^2} f(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} f(x_1, x_2)\;\mathrm{d}x_2 \right\} \mathrm{d}x_1 = \int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} f(x_1, x_2)\;\mathrm{d}x_1 \right\} \mathrm{d}x_2.$$

this is a very important result, because formally the three integrations

$$\int\!\!\int_{\mathbb{R}^2} f(\mathbf{x})\,\mathrm{d}\mathbf{x}, \qquad \int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} f(x_1, x_2)\;\mathrm{d}x_2 \right\} \mathrm{d}x_1, \qquad \int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} f(x_1, x_2)\;\mathrm{d}x_1 \right\} \mathrm{d}x_2,$$

are constructed in three different ways, and yet they give the same result for all $f \in L^1(\mathbb{R}^2)$.

2 The Laplace transformation

2.1 Definition of the Laplace transformation using complex functions theory

We shall in the following only consider functions defined *a.e.* on the nonnegative real axis $[0,+\infty[$ and shall not always specify that $f(x)=0$ for $x<0$.

Most readers have first met the Laplace transformation more or less given by the following definition.

Definition 2.1.1 *The class of functions $\mathcal{E}$ consists of all piecewise continuous functions $f:[0,+\infty[\,\to\mathbb{C}$, for which there are constants $A>0$ and $B\in\mathbb{R}$, such that*

(2) $|f(t)|\le A\,e^{Bt}$ *for every* $t\in[0,+\infty[$.

Using the quantors $\forall$ (= "for all") and $\exists$ (="there exists") we define

(3) $\varrho(f)=\inf\left\{B\in\mathbb{R}\mid\exists A>0\,\forall t\ge 0:|f(t)|\le A\,e^{Bt}\right\}.$

In many simple cases concerning the Laplace transformation it suffices just to consider functions from $\mathcal{E}$. However, we shall here introduce a larger class of functions $\mathcal{F}\supset\mathcal{E}$, on which the Laplace transformation is equally well defined and with the same properties. We shall here use our knowledge of *Measure Theory* from Chapter 1.

Definition 2.1.2 *The class of functions $\mathcal{F}$ consists of all (measurable) functions*

$$f : [0, +\infty[\to \mathbb{C}^\star = \mathbb{C} \cup \{\infty\},$$

for which there is a constant $\sigma \in \mathbb{R}$, such that

$$(4)\quad \int_0^{+\infty} |f(t)|\, e^{-\sigma t}\, \mathrm{d}t < +\infty.$$

We put

$$(5)\quad \sigma(f) = \inf \left\{ \sigma \in \mathbb{R} \mid \int_0^{+\infty} |f(t)|\, e^{-\sigma t}\, \mathrm{d}t < +\infty \right\}.$$

By introducing the Lebesgue integral in (4) we see that (5) is less complicated than (3). Furthermore, if (4) holds for some $\sigma_0 \in \mathbb{R}$, then it clearly holds for all $\sigma \geq \sigma_0$, because $e^{-\sigma t} \leq e^{-\sigma_0 t}$. We even get that if (4) holds for some $\sigma \in \mathbb{R}$, then

$$\int_0^{+\infty} f(t)\, e^{-zt}\, \mathrm{d}t, \qquad \text{for } z := \sigma + i\tau$$

is convergent for every $\tau \in \mathbb{R}$.

Let $f \in \mathcal{E}$, i.e. f satisfies condition (2). For given $\varepsilon > 0$ we choose $\sigma = B + \varepsilon$, from which we get

$$\int_0^{+\infty} |f(t)|\, e^{-(B+\varepsilon)t}\, \mathrm{d}t \leq A \int_0^{+\infty} e^{-\varepsilon t}\, \mathrm{d}t < +\infty,$$

and (4) holds for every $\sigma = B + \varepsilon$, $\varepsilon > 0$, from which we conclude that $\mathcal{E} \subset \mathcal{F}$ and $\sigma(f) \leq \varrho(f)$.

Example 2.1.1 The function defined by

$$f(x) = \begin{cases} \dfrac{1}{\sqrt{x}} & \text{for } x > 0, \\[2ex] 0 & \text{for } x \leq 0, \end{cases}$$

belongs to $\mathcal{F}$. In fact, choose any $\sigma > 0$, then by a change of variable for $\varepsilon > 0$,

$$\int_\varepsilon^{+\infty} |f(t)|\, e^{-\sigma t}\, \mathrm{d}t = \int_\varepsilon^{+\infty} \frac{1}{\sqrt{t}}\, e^{-\sigma t}\, \mathrm{d}t = 2 \int_{\sqrt{\varepsilon}}^{+\infty} e^{-\sigma\, u^2}\, \mathrm{d}u \to 2 \int_0^{+\infty} e^{-\sigma\, u^2}\, \mathrm{d}u < +\infty \qquad \text{for } \varepsilon \to 0+,$$

and (4) is proved.

On the other hand, since $f(t) \to +\infty$ for $t \to 0+$, there is absolutely no hope for (2) to be fulfilled, so $f \in \mathcal{F} \setminus \mathcal{E}$. $\Diamond$

Whenever $f \in \mathcal{F} \setminus \mathcal{E}$, we put $\varrho(f) = +\infty$.

It follows immediately from Definition 2.1.2 that $\alpha f + \beta g \in \mathcal{F}$ for all f, $g \in \mathcal{F}$ and all α, $\beta \in \mathbb{C}$, so $\mathcal{F}$ is a vector space over $\mathbb{C}$.

We notice that if $f \in \mathcal{F}$ and $g \in \mathcal{E}$, then it follows from a trivial estimate that the pointwise product $f \cdot g \in \mathcal{F}$. This result is not true in general if both f, $g \in \mathcal{F}$.

Definition 2.1.3 *Let $f \in \mathcal{F}$. We define the* Laplace transform $\mathcal{L}\{f\}$ *of f as the complex function given by*

$$(6)\quad \mathcal{L}\{f\}(z) := \int_0^{+\infty} e^{-zt} f(t)\, dt,$$

where z belongs to the set of complex numbers, for which the integral on the right hand side of (6) is convergent.

Remark 2.1.1 It follows from Definition 2.1.2 that the set of points z for which the right hand side of (6) is convergent, is not the empty set. This shows that the Laplace transform $\mathcal{L}\{f\}$ exists for every $f \in \mathcal{F}$. $\Diamond$

Remark 2.1.2 In some very rare and simple applications of the Laplace transformation it suffices only to consider a real variable. However, in most cases the use of a complex variable is almost inevitable. $\Diamond$

Remark 2.1.3 It is customary in the literature to denote the variable by s, so one writes $\mathcal{L}\{f\}(s)$. However, in order to keep in line with *Complex Functions Theory* we have decided in *Ventus, Complex Functions Theory a-4 and a-5* to write z instead, in order to emphasize that the Laplace transform $\mathcal{L}\{f\}(z)$ is even an analytic function, so we can apply all the previous results from *Ventus, Complex Functions Theory a-1, a-2, a-3.* More precisely, we prove in the following that $\mathcal{L}\{f\}(z)$ defined by (6) is an analytic function in at least an half plane of the form $\Re z > \Re z_0$. It will therefore often be possible to extend $\mathcal{L}\{f\}(z)$ analytically to larger sets than just to this half plane. We shall call any such analytic extension a Laplace transform $\mathcal{L}\{f\}(z)$ of f. $\Diamond$

We shall in the following examples derive the simplest Laplace transforms $\mathcal{L}\{f\}(z)$.

Example 2.1.2 The constant function $f(t) = 1$ for $t \geq 0$ belongs to the class $\mathcal{E} \subset \mathcal{F}$, and we have $\varrho(1) = \sigma(1) = 0$, and

$$\mathcal{L}\{1\}(z) = \int_0^{+\infty} e^{-zt} \cdot 1\, \mathrm{d}t = \left[-\frac{1}{z}\, e^{-zt}\right]_{t=0}^{+\infty} = \frac{1}{z} \qquad \Re z > 0.$$

Obviously, $\mathcal{L}\{1\}(z) = \dfrac{1}{z}$ is defined in $\mathbb{C} \setminus \{0\}$, so we have a unique analytic extension from the half plane $\Re z > 0$ to the deleted plane $\mathbb{C} \setminus \{0\}$, cf. Remark 2.1.3 above. $\Diamond$

Example 2.1.3 Let $f(t) = t^n$, $t \geq 0$ and $n \in \mathbb{N}$. Since the exponential dominates the power function, we see that even $t^n \in \mathcal{E}$ with $\varrho\left(t^n\right) = \sigma\left(t^n\right) = 0$. For $\Re z > 0$ we get by partial integration,

$$\mathcal{L}\left\{t^n\right\}(z) = \int_0^{+\infty} t^n\, e^{-zt}\, \mathrm{d}t = \left[-\frac{1}{z}\, t^n\, e^{-zt}\right]_{t=0}^{+\infty} + \frac{n}{z} \int_0^{+\infty} t^{n-1}\, e^{-zt}\, \mathrm{d}t = \frac{n}{z}\, \mathcal{L}\left\{t^{n-1}\right\}(z),$$

so we get by recursion,

$$\mathcal{L}\left\{t^n\right\}(z) = \frac{n}{z} \cdot \frac{n-1}{z} \cdots \frac{1}{z} \cdot \mathcal{L}\{1\}(z) = \frac{n!}{z^n} \cdot \frac{1}{z} = \frac{n!}{z^{n+1}} \qquad \text{for } \Re\, z > 0.$$

Here we have for the first time used the common convention to write $\mathcal{L}\{f(t)\}(z)$ instead of the formally more "correct" $\mathcal{L}\{f\}$. $\Diamond$

Example 2.1.4 Let $a \in \mathbb{C}$ be a constant, and consider the function $f(t) = e^{at}$ for $t \geq 0$. Then $e^{at} \in \mathcal{E}$ and $\varrho\left(e^{at}\right) = \sigma\left(e^{at}\right) = \Re\, a$.

We compute for $\Re\, z > \Re\, a$,

$$\int_0^{+\infty} e^{-zt}\, f(t)\, \mathrm{d}t = \int_0^{+\infty} e^{-(z-a)t}\, \mathrm{d}t = \left] -\frac{1}{z-a}\, e^{-(z-a)t} \right]_0^{+\infty} = \frac{1}{z-a},$$

thus

$$\mathcal{L}\left\{e^{at}\right\}(x) = \frac{1}{z-a} \qquad \text{for } \Re\, z > \Re\, a.$$

Then it follows for $\Re\, z > |\Re\, a|$ that

$$\begin{aligned}
&(7) \quad \mathcal{L}\{\cosh(at)\}(z) &=& \frac{1}{2}\,\mathcal{L}\left\{e^{at}\right\}(z) + \frac{1}{2}\,\mathcal{L}\left\{e^{-at}\right\}(z) = \frac{1}{2}\,\frac{1}{z-a} + \frac{1}{2}\,\frac{1}{z+a},\\
&(8)\\
&(9) &=& \frac{z}{z^2-a^2} \qquad \text{for } \Re\, z > |\Re\, a|,
\end{aligned}$$

and similarly,

$$\begin{aligned}
&(10) \quad \mathcal{L}\{\sinh(at)\}(z) &=& \frac{1}{2}\,\mathcal{L}\left\{e^{at}\right\}(z) - \frac{1}{2}\,\mathcal{L}\left\{e^{-at}\right\}(z) = \frac{1}{2}\,\frac{1}{z-a} - \frac{1}{2}\,\frac{1}{z+a},\\
&(11)\\
&(12) &=& \frac{a}{z^2-a^2} \qquad \text{for } \Re\, z > |\Re\, a|.
\end{aligned}$$

Then put $a = ib$, so

$$\Re\, a = \Re(ib) = -\Im\, b,$$

and

$$\cosh(at) = \cosh(ibt) = \cos(bt), \qquad \text{and} \qquad \sinh(at) = \sinh(ibt) = i\,\sin(bt),$$

cf. also *Ventus, Complex Functions Theory a-1*, Chapter 4. Therefore, by this simple substitution it follows from the above that

$$\mathcal{L}\{\cos(bt)\}(z) = \frac{z}{z^2+b^2} \qquad \text{for } \Re\, z > |\Im\, b|,$$

and

$$\mathcal{L}\{\sin(bt)\}(z) = \frac{b}{z^2+b^2} \qquad \textit{for } \Re\, z > |\Im\, b|. \qquad \Diamond$$

The three examples above cover most of the *elementary* applications of the Laplace transformation. We shall in this first book on the Laplace transformation mainly consider examples which can be derived from these three examples, leaving more advanced applications to *Ventus, Complex Functions Theory a-5, Laplace Transformation II.* We collect these simple cases in Table 1. A larger table is given in Section 4, page 101.vs
We shall now return to the general theory, where the next important topic is to prove that the Laplace transform $\mathcal{L}\{f\}(z)$ is indeed an analytic function in the half plane $\Re z > \sigma(f)$. Once we have proved this result, we can rely on the theory of analytic functions as derived already in *Ventus, Complex Functions Theory, a-1 – a-3.*

In the proof we shall need the following lemma.

Lemma 2.1.1 *Assume that* $\mathcal{L}\{f\}(z_0)$ *defined by (6) exists at the point* z_0. *Choose any* $\Theta \in \left[0, \frac{\pi}{2}\right[$, *and denote by* $S_\Theta(z_0)$ *the angular sector*

$$S_\Theta(z_0) := \{z \in \mathbb{C} \mid Arg(z - z_0) \in [-\Theta, \Theta]\}.$$

Then the integral $\int_0^{+\infty} e^{-zt} f(t)\,\mathrm{d}t$ *is uniformly convergent in* z *for* $z \in S_\Theta(z_0)$.

Notice that the vertex $z_0 \notin S_\Theta(z_0)$.

	$f(t)$	$\mathcal{L}\{f\}(z)$	$\sigma(f)$
1	1	$\dfrac{1}{z}$	0
2	t^n	$\dfrac{n!}{z^{n+1}}$	0
3	e^{at}	$\dfrac{1}{z-a}$	$\Re(a)$
4	$\sin(at)$	$\dfrac{a}{z^2+a^2}$	$\lvert\Im(a)\rvert$
5	$\cos(at)$	$\dfrac{z}{z^2+a^2}$	$\lvert\Im(a)\rvert$
6	$\sinh(at)$	$\dfrac{a}{z^2-a^2}$	$\lvert\Re(a)\rvert$
7	$\cosh(at)$	$\dfrac{z}{z^2-a^2}$	$\lvert\Re(a)\rvert$

Table 1: The simplest examples of Laplace transforms.

PROOF. We introduce an auxiliary function $h(x)$ by

$$(13)\quad h(x) := \int_0^x e^{-z_0 t}\, f(t)\, \mathrm{d}t - \int_0^{+\infty} e^{-z_0 t}\, f(t)\, \mathrm{d}t = -\int_x^{+\infty} e^{-z_0 t}\, f(t)\, \mathrm{d}t,$$

where x only occurs in the limits of integration. Then clearly, $h(x) \to 0$ for $x \to +\infty$.

We shall prove that for any given $\varepsilon > 0$ there is a t_0, such that

$$\left| \int_{t_1}^{t_2} e^{-zt}\, f(t)\, \mathrm{d}t \right| < \varepsilon \qquad \text{for every } t_1,\, t_2 \geq t_0 \text{ and every } z \in S_\Theta(z_0),$$

because then we have proved that $\int_0^x e^{-zt}\, f(t)\, \mathrm{d}t$ is uniformly convergent in z on $S_\Theta(z_0)$ for $x \to +\infty$.

Let $z \in S_\Theta(z_0)$ be fixed, and let $h(x)$ be given by (13) above. Then by a simple rearrangement and a partial integration,

$$(14)\quad \begin{aligned} \int_{t_1}^{t_2} e^{-zt}\, f(t)\, \mathrm{d}t &= \int_{t_1}^{t_2} e^{-(z-z_0)t} \left\{ e^{-z_0 t}\, f(t) \right\} \mathrm{d}t \\ &= e^{(z-z_0)t_2} \cdot h(t_2) - e^{(z-z_0)t}\, h(t_1) + (z - z_0) \int_{t_1}^{t_2} e^{-(z-z_0)t}\, h(t)\, \mathrm{d}t. \end{aligned}$$

When $\varepsilon > 0$ is given, we choose t_0, such that

$$|h(t)| < \frac{\varepsilon}{3} \cdot \cos\Theta \qquad \text{for all } t \geq t_0.$$

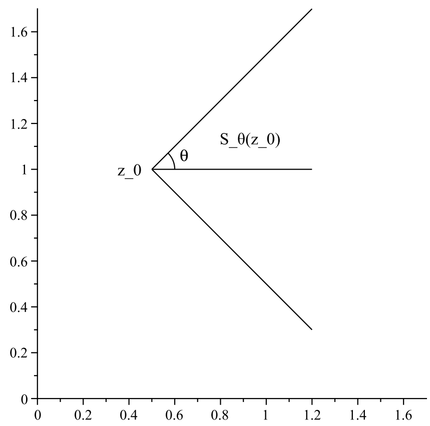

Figure 3: The angular sector $S_\Theta(z_0)$.

Since $\Re z > \Re z_0$ for all $z \in S_\Theta(z_0)$, it follows that

$$\left| e^{-(z-z_0)t_2} \right| = e^{-\{\Re z - \Re z_0\}t_2} \leq 1,$$

hence for all $t_2 > t_0$ and all $z \in S_\Theta(z_0)$,

$$\left| e^{-(z-z_0)t_2} \cdot h(t_2) \right| \leq |h(t_2)| < \frac{\varepsilon}{3} \cos\Theta \leq \frac{\varepsilon}{3}.$$

Analogously we get for $t_1 > t_0$ that

$$\left| e^{-(z-z_0)t_1} \cdot h(t_1) \right| < \frac{\varepsilon}{3},$$

and we have estimated the first two terms on the right hand side of (14).

Concerning the estimate of the third term on the right hand side of (14) we put for short

$$x = \Re z \qquad \text{and} \qquad x_0 = \Re z_0,$$

and then get the estimate

$$\left| (z - z_0) \int_{t_1}^{t_2} e^{-(z-z_0)t} h(t)\, \mathrm{d}t \right| \leq |z - z_0| \cdot \frac{\varepsilon}{3} \cdot \cos\Theta \cdot \left| \int_{t_1}^{t_2} e^{-(x-x_0)t}\, \mathrm{d}t \right|.$$

Since $z \in S_\Theta(z_0)$, we have $x > x_0$, and it follows by an elementary geometric consideration, cf. Figure 4, that we have the estimate

$$\begin{aligned} |z - z_0| \cdot \frac{\varepsilon}{3} \cdot \cos\Theta \cdot \left| \int_{t_1}^{t_2} e^{-(x-x_0)t}\, \mathrm{d}t \right| &= \frac{\varepsilon}{3} \cdot \cos\Theta \cdot \frac{|z - z_0|}{x - x_0} \cdot \left| e^{-(x-x_0)t_1} - e^{-(x-x_0)t_2} \right| \\ &\leq \frac{\varepsilon}{3} \cdot \cos\Theta \cdot \frac{1}{\cos\Theta} \cdot 1 = \frac{\varepsilon}{3}, \end{aligned}$$

and the lemma follows by inserting these estimates into (14). $\square$.

We shall later on need the following classical theorem from *Complex Functions Theory*.

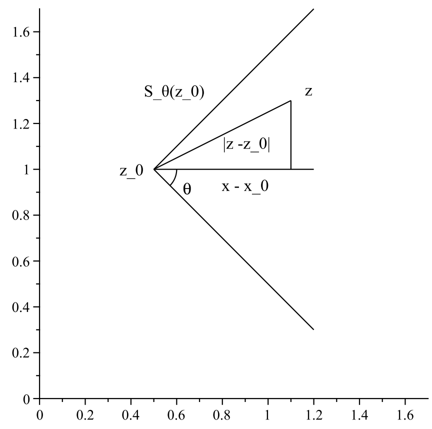

Figure 4: The geometry of the sector $S_\Theta(z_0)$ in the proof of Lemma 2.1.1.

Theorem 2.1.1 (Karl Weierstraß, appr. 1860). *Assume that $\{f_n\}$ is a sequence of analytic functions in an open domain Ω, where this sequence converges* uniformly *towards a function f in every closed disc contained in Ω.*
Then the limit function f is analytic in Ω.
Furthermore, $f_n' \to f'$ pointwise in Ω, and uniformly in every closed disc contained in Ω.

We shall not prove Theorem 2.1.1, because its proof relies on *Morera's theorem*, which again requires a lot of preparations, and the aim of this book would be in danger of being lost, if we also gave a proof of Theorem 2.1.1.

Theorem 2.1.1 is used in the proof of the following important theorem.

Theorem 2.1.2 *If $f \in \mathcal{F}$, then there is a uniquely determined number $\sigma = \sigma(f) \in [-\infty, +\infty]$ (including $\pm\infty$ as possibilities), such that the integral*

$$(15)\quad \int_0^{+\infty} e^{-zt} f(t)\, dt \quad \textit{is} \quad \begin{cases} \textit{convergent for } \Re z > \sigma, \\ \\ \textit{divergent for } \Re z < \sigma. \end{cases}$$

The number σ is called the abscissa of convergence *of $\mathcal{L}\{f\}(z)$, which is defined by (15) for $\Re z > \sigma$.*

The function $\mathcal{L}\{f\}(z)$ is analytic in the open half plane

$$(16)\quad \Omega := \{z \in \mathbb{C} \mid \Re z > \sigma\},$$

and we have for $z \in \Omega$,

$$(17)\quad \frac{d}{dz}\, \mathcal{L}\{f\}(z) = -\int_0^{+\infty} t\, e^{-zt}\, f(t)\, dt,$$

i.e. when $z \in \Omega$ it is legal to differentiate (15) under the integral sign.
If $f \in \mathcal{E}$, then $\sigma \leq \varrho$.

The set Ω given above by (16) is called the *half plane of convergence* of $\mathcal{L}\{f\}$. It is clearly in the extreme case when $\sigma = -\infty$ not a half plane, though even better, namely all of $\mathbb{C}$.

Usually is is not possible to decide, whether the integral (15) is convergent or divergent on the vertical line $\Re z = \sigma$.

Notice that it is possible to give examples of functions $f \in \mathcal{E}$, for which $\sigma < \varrho$.

PROOF. Define

$$(18)\quad \sigma := \inf\left\{ x \in \mathbb{R} \;\middle|\; \int_0^{+\infty} e^{-xt} f(t)\,\mathrm{d}t \text{ is convergent} \right\}.$$

We first use Lemma 2.1.1 to conclude that if $\int_0^{+\infty} e^{-z_0 t} f(t)\,\mathrm{d}t$ is convergent for some $z_0 \in \mathbb{C}$, then $\int_0^{+\infty} e^{-zt} f(t)\,\mathrm{d}t$ is convergent for every z, for which $\Re z > \Re z_0$. In fact, given z_0 and z, such that $\Re z > \Re z_0$, we can always find a $\Theta \in \left[0, \frac{\pi}{2}\right[$, such that $z \in S_\Theta(z_0)$.

Let $\Re z > \sigma$. By the definition (18), there always exists an $x_0 < \Re z$, such that $\int_0^{+\infty} e^{-x_0 t} f(t)\,\mathrm{d}t$ is convergent, so it follows from Lemma 2.1.1 that $\int_0^{+\infty} e^{-zt} f(t)\,\mathrm{d}t$ is also convergent.

Then assume that $\Re z < \sigma$ and choose $x \in]\Re z, \sigma[$. If the integral $\int_0^{+\infty} e^{-zt} f(t)\,\mathrm{d}t$ were convergent, then it would follow from Lemma 2.1.1 that $\int_0^{+\infty} e^{-xt} f(t)\,\mathrm{d}t$ is convergent. Hence, $\sigma \le x$ by the definition (18) of σ. This is contradicting the choice of $x < \sigma$. We therefore conclude that $\int_0^{+\infty} e^{-zt} f(t)\,\mathrm{d}t$ is divergent for every z, for which $\Re z < \sigma$, and the first part of Theorem 2.1.2 is proved.

Clearly, σ defined by (18), is identical with $\sigma(f)$ given in Definition 2.1.2.

Consider Ω given by (16). The functions

$$g_n(z) := \int_0^n e^{-zt} f(t)\,\mathrm{d}t, \qquad n \in \mathbb{N},$$

are analytic in Ω with the sequence of derivatives

$$g_n'(z) = \lim_{\zeta \to z} \frac{g_n(\zeta) - g_n(z)}{\zeta - z} = -\int_0^n t\, e^{-zt} f(t)\,\mathrm{d}t,$$

where we have used that the interval of integration is bounded for every $n \in \mathbb{N}$. Clearly, $g_n(z) \to \mathcal{L}\{f\}(z)$ pointwise in Ω for $n \to +\infty$. According to Theorem 2.1.1 we shall only prove that $g_n \to \mathcal{L}\{f\}$ uniformly for $n \to +\infty$ on every closed disc contained in the open half plane Ω.

Now, every closed disc in the open set Ω must lie in some angular sector $S_\Theta\left(z_0\right)$ for some $z_0 \in \Omega$ and some $\Theta \in \left[0, \frac{\pi}{2}\right[$, so Lemma 2.1.1 assures that the convergence is uniform on every closed disc. Then by Theorem 2.1.1, $\mathcal{L}\{f\}$ is analytic in Ω, and

$$\frac{d}{dz}\,\mathcal{L}\{f\}(z) = -\int_0^{+\infty} t\, e^{-zt} f(t)\,\mathrm{d}t \qquad \text{for } \Re z > \sigma.$$

In particular, this integral description of the derivative of $\mathcal{L}\{f\}$ is convergent for $\Re z > \sigma$. By iteration, the same is true for derivatives of any order,

$$(19)\quad \frac{d^n}{dz^n}\,\mathcal{L}\{f\}(z) = (-1)^n \int_0^{+\infty} t^n e^{-zt} f(t)\,\mathrm{d}t, \qquad \text{for } \Re z > \sigma.$$

We shall still prove that if $f \in \mathcal{E}$, then $\sigma \le \varrho(f)$. It suffices to prove that if $|f(t)| \le A\,e^{Bt}$ for all $t \in [0, +\infty[$, then $\sigma \le B$. According to the above this must be the case, if $\int_0^{+\infty} e^{-zt} f(t)\,\mathrm{d}t$ is convergent, when $\Re z > B$. This follows from the following simple estimates,

$$\left|\int_0^{+\infty} e^{-zt} f(t)\,\mathrm{d}t\right| \le \int_0^{+\infty} \left|e^{-zt} f(t)\right|\,\mathrm{d}t = \int_0^{+\infty} \left|e^{-(z-B)} \cdot e^{-Bt}\, f(t)\right|\,\mathrm{d}t$$

$$\le \int_0^{+\infty} e^{-(\Re z - B)t} \cdot A\,\mathrm{d}t = \frac{A}{\Re z - B}. \qquad \square$$

We have proved that

$$(20)\quad \mathcal{L}\{f\}(z) = \int_0^{+\infty} e^{-zt} f(t)\,\mathrm{d}t \qquad \text{for } \Re z > \sigma(f),$$

is an analytic function. Therefore, it is tempting to use the following procedure:

1) First use the integral on the right hand side of (20) to define the uniquely defined analytic function $\mathcal{L}\{f\}(z)$ on the left hand side of (20) in the set $\Omega = \{z \in \Omega \mid \Re z > \sigma\}$.

2) Then extend the analytic function $\mathcal{L}\{f\}(z)$ to a larger set which contains the half plane of convergence Ω.

It will therefore be convenient to extend the definition of the Laplace transform $\mathcal{L}\{f\}(z)$ of a function $f \in \mathcal{F}$.

Definition 2.1.4 *Given $f \in \Omega$ and its half plane of convergence $\Omega = \{z \in \Omega \mid \Re z > \sigma\}$. Let $F(z)$ be an analytic function in an open set $\tilde{\Omega} \subseteq \Omega$, such that*

$$F(z) = \int_0^{+\infty} e^{-zt} f(t)\, dt \qquad \textit{for every } z \in \Omega.$$

Then $F(z)$ is also called a Laplace transform *of f, and we write $F(z) = \mathcal{L}\{f\}(z)$, even if $z \in \tilde{\Omega} \setminus \Omega$.*

The essence of Definition 2.1.4 is that $\tilde{\Omega}$ contains the half plane of convergence Ω. Obviously, $\tilde{\Omega}$ is usually not unique. If e.g. $f(t) = \dfrac{1}{\sqrt{t}}$ for $t > 0$, then

$$\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\}(z) = \int_0^{+\infty} \frac{1}{\sqrt{t}}\, e^{-tz}\, \mathrm{d}t = \sqrt{\frac{\pi}{z}} \qquad \text{for } \Re z > 0.$$

The analytic function $\sqrt{\dfrac{\pi}{z}}$ can of course be extended analytically to larger sets which also include subsets of the left half plane, but the extension is not unique, because the branch cut of the square root is not unique.

Theorem 2.1.3 *If $f \in \mathcal{F}$, then*

$$\mathcal{L}\{f\}(z) \to 0 \qquad \textit{for } \Re z \to +\infty.$$

Proof. When $f \in \mathcal{F}$, there is a $\sigma_0 \in \mathbb{R}$, such that

$$\int_0^{+\infty} |f(t)|\, e^{-\sigma_0 t}\, \mathrm{d}t < +\infty.$$

Since the integral is convergent, we can to every given $\varepsilon > 0$ find a constant $a > 0$, such that

$$\int_0^a |f(t)| e^{-\sigma_0 t}\, \mathrm{d}t < \frac{\varepsilon}{2}.$$

Then choose $\sigma_1 > \sigma_0$, such that

$$\int_a^{+\infty} |f(t)| e^{-\sigma_1 t}\, \mathrm{d}t = \int_a^{+\infty} |f(t)| e^{-\sigma_0 t} e^{-(\sigma_1-\sigma_0)t}\, \mathrm{d}t \le e^{-(\sigma_1-\sigma_0)a} \int_a^{+\infty} |f(t)|\, e^{-\sigma_0 t}\, \mathrm{d}t < \frac{\varepsilon}{2}.$$

It follows for $\Re z \ge \sigma_1$ that

$$\begin{aligned} \left| \int_0^{+\infty} f(t)\, e^{-zt}\, \mathrm{d}t \right| &\le \int_0^{+\infty} |f(t)|\, e^{-\sigma_1 t}\, \mathrm{d}t \\ &\le \int_0^a |f(t)|\, e^{-\sigma_0 t}\, \mathrm{d}t + \int_a^{+\infty} |f(t)|\, e^{-\sigma_1 t}\, \mathrm{d}t < \varepsilon. \end{aligned}$$

This is true for every $\varepsilon > 0$, and the claim follows. $\Box$

Example 2.1.5 The simplest example of an analytic function, which does not satisfy the condition of Theorem 2.1.1, is the constant function $F(z) = 1$. Hence, it cannot be the Laplace transform of any function $f \in \mathcal{F}$. In the technical sciences one claims that the impulse "function" (or Dirac's "function") δ has $F(z) = 1$ as its Laplace transform. This is not quite true according to the usual *Theory of Distributions*, but the theory can be modified. Anyway, since $F(z) =$ "$\mathcal{L}\{\delta\}(z)$" $= 1$ does not converge towards 0 for $\Re z \to +\infty$, Theorem 2.1.3 shows that δ cannot be a function from $\mathcal{F}$. $\Diamond$

Example 2.1.6 The proof of Theorem 2.1.3 showed that to every $\varepsilon > 0$ there is a $\sigma_1 \in \mathbb{R}$, such that

$$(21)\quad |\mathcal{L}\{f\}(z)| \le \int_0^{+\infty} |f(t)| e^{-\sigma_1 t}\,\mathrm{d}t < \varepsilon \qquad \text{for } \Re z > \sigma_1,$$

i.e. the estimate (21) holds in a right half plane. As a consequence, the function $\exp\left(-z^2\right)$ cannot be the Laplace transform of any function $f \in \mathcal{F}$. The estimate (21) is of course true for real $z = x \ge \sigma_1$. However, if we put $z = \lambda(1+i)$ for $\lambda > 0$, then $z^2 = 2i\lambda^2$, and we get

$$\left|\exp\left(z^2\right)\right| = \left|\exp\left(-2i\lambda^2\right)\right| = 1 \qquad \text{for all } \lambda > 0,$$

so $\left|\exp\left(z^2\right)\right|$ does not tend towards 0 for $\lambda \to +\infty$, or $\Re z \to +\infty$. $\Diamond$

2.2 Some important properties of Laplace transforms

We shall here list some common and important properties of Laplace transforms.

Let $\mathcal{A}$ denote the set of all analytic functions with their domains containing some right half planes. Then the *Laplace transformation* is an operator $\mathcal{L} : \mathcal{F} \to \mathcal{A}$.

Theorem 2.2.1 . The linearity property.*The operator $\mathcal{L} : \mathcal{F} \to \mathcal{A}$ is linear in the following sense: If λ, $\mu \in \mathbb{C}$ are constants, and f, $g \in \mathcal{F}$, then*

$$\mathcal{L}\{\lambda f + \mu g\}(z) = \lambda\mathcal{L}\{f\}(z) + \mu\mathcal{L}\{g\}(z) \qquad \textit{for } \Re z > max\{\sigma(f), \sigma(g)\}.$$

Sketch of proof. Let $\Re z > \max\{\sigma(f), \sigma(g)\}$. Prove that the integral (6) corresponding to $\mathcal{L}\{\lambda f + \mu g\}(z)$ is convergent, and then use that the integral is linear, where one uses that the two integrals on the right hand side obviously are convergent for $\Re z > \max\{\sigma(f), \sigma(g)\}$. The simple missing details are left to the reader. $\Box$

Example 2.2.1 We get for $\Re z > 0$ by using the linearity property above and Table 1 that

$$\mathcal{L}\left\{t^2 - 2\cos t - 2e^{-t}\right\}(z) = \mathcal{L}\left\{t^2\right\}(z) - 2\mathcal{L}\{\cos t\}(z) - 2\mathcal{L}\left\{e^{-t}\right\}(z)$$

$$= \frac{2!}{z^3} - 2\cdot\frac{z}{z^2+1} - 2\cdot\frac{1}{z+1} = 2\left\{\frac{1}{z^3} - \frac{z}{z^2+1} - \frac{1}{z+1}\right\}$$

The right hand side is clearly analytic in $\mathbb{C} \setminus \{0, -1, i, -i\}$. $\Diamond$

Theorem 2.2.2 First translation or shifting property. *Let* $f \in \mathcal{F}$ *and* $a \in \mathbb{C}$. *Then*

$$\mathcal{L}\left\{e^{at} f(t)\right\}(z) = \mathcal{L}\{f\}(z-a) \qquad \textit{for } \Re\, z > \sigma(f) + \Re\, a.$$

Proof. Assume that $\Re(z-a) = \Re\, z - \Re\, a > \sigma(f)$. Then by Definition 2.1.3,

$$\mathcal{L}\left\{e^{at} f(t)\right\}(z) = \int_0^{+\infty} e^{-zt} \cdot e^{at} f(t)\, \mathrm{d}t = \int_0^{+\infty} e^{-(z-a)t} f(t)\, \mathrm{d}t = \mathcal{L}\{f\}(z-a). \qquad \square$$

Example 2.2.2 We proved in Example 2.1.4 that

$$\mathcal{L}\{\cos t\}(z) = \frac{z}{z^2+1} \qquad \text{for } \Re\, z > 0.$$

Then by Theorem 2.2.2 above,

$$\mathcal{L}\left\{e^{t} \cos t\right\}(z) = \frac{z-1}{(z-1)^2+1} = \frac{z-1}{z^2-2z+2} \qquad \text{for } \Re\, z > 1.$$

By Definition 2.1.4, the analytic function $\dfrac{z}{z^2+1}$ can be considered as the Laplace transform of $\cos t$ in the set $\Omega = \mathbb{C} \setminus \{i, -i\}$, so

$$\mathcal{L}\left\{e^t \cos t\right\}(z) = \frac{z-1}{z^2-2z+2} \qquad \text{for } z \in \mathbb{C} \setminus \{1+i, 1-i\}. \qquad \Diamond$$

Theorem 2.2.3 Second translation or shifting property. *Let* $f \in \mathcal{F}$ *and* $a > 0$*, and define* $f_a : [0, +\infty[\to \mathbb{C}$ *by*

$$f_a(t) = \begin{cases} f(t-a) & \text{for } t \geq a, \\ 0 & \text{for } t < a. \end{cases}$$

Then $f_a \in \mathcal{F}$*, and*

$$\mathcal{L}\left\{f_a\right\}(z) = e^{-az}\mathcal{L}\{f\}(z) \qquad \text{for } \Re z > \sigma(f).$$

Proof. This follows from the simple computation,

$$\mathcal{L}\left\{f_a\right\}(z) = \int_a^{+\infty} e^{-zt} f(t-a)\,\mathrm{d}t = \int_0^{+\infty} e^{-z(\tau+a)} f(\tau)\,\mathrm{d}\tau = e^{-az}\mathcal{L}\{f\}(z),$$

which is valid for $\Re z > \sigma(f)$. $\square$

Example 2.2.3 It follows from Example 2.1.2 that $\mathcal{L}\{1\}(z) = \dfrac{1}{z}$ for $\Re z > 0$. Then by the second shifting property,

$$\mathcal{L}\left\{\chi_{[a,+\infty[}\right\}(z) = \frac{e^{-az}}{z} \qquad \text{for } \Re z > 0,$$

where we define

$$\chi_{[a,+\infty[}(t) = \begin{cases} 1 & \text{for } t \geq a, \\ 0 & \text{for } t < a. \end{cases}$$

Obviously, $\mathcal{L}\left\{\chi_{[a,+\infty[}\right\}(z) = \dfrac{e^{-az}}{z}$ is analytic for $z \in \mathbb{C} \setminus \{0\}$. $\Diamond$

Theorem 2.2.4 Change of scale property. *Let* $f \in \mathcal{F}$ *and* $k > 0$*. Then*

$$\mathcal{L}\{f(k \cdot t)\}(z) = \frac{1}{k}\,\mathcal{L}\{f\}\left(\frac{z}{k}\right) \qquad \text{for } \Re z > k \cdot \sigma(f).$$

Proof. This follows again from a simple computation, where it is important that $k > 0$,

$$\mathcal{L}\{f(k \cdot t)\}(z) = \int_0^{+\infty} e^{-zt} f(k \cdot t)\,\mathrm{d}t = \frac{1}{k}\int_0^{+\infty} \exp\left(-\frac{z}{k} \cdot \tau\right) f(\tau)\,\mathrm{d}\tau = \frac{1}{k}\,\mathcal{L}\{f\}\left(\frac{z}{k}\right)$$

for $\Re\left(\dfrac{z}{k}\right) = \dfrac{1}{k} \cdot \Re z > \sigma(f)$, i.e. for $\Re z > k \cdot \sigma(f)$. Here we have used the change of variables $\tau = k \cdot t$. $\square$

Example 2.2.4 It follows from Example 2.2.4 that

$$\mathcal{L}\left\{e^t \cos t\right\}(z) = \frac{z-1}{z^2-2z+2} \qquad \text{for } \Re z > 1.$$

Choose $k = 2 > 0$ to get

$$\mathcal{L}\left\{e^{2t} \cos 2t\right\}(z) = \frac{1}{2} \cdot \frac{\frac{z}{2}-1}{\left(\frac{z}{2}\right)^2 - 2\cdot\frac{z}{2}+2} = \frac{z-2}{z^2-4z+8} \qquad \text{for } \Re z > 2. \qquad \Diamond$$

The importance of the Laplace transformation in the technical applications lies in the fact that it transforms problems of differential equations (actually also more general problems) into algebraic problems, which are easier to solve. After the derived algebraic problem has been solved, we imply the *inverse Laplace transformation* to obtain the solution of the original problem. We shall first extend the concept of differentiability in order to meet our more general demands.

Assume that f is a *continuous* function, which is piecewise of class C^1. Then the set of points, where f is *not* of class C^1, a null set, so it is quite natural to let f' denote the function, which is given by

$$f'(x) := \begin{cases} \text{the usual derivative, if } f \text{ is } C^1 \text{ at } x, \\ \\ \text{any value, if } f \text{ is not } C^1 \text{ at } x. \end{cases}$$

The philosophy is of course that we can freely change the value of $f'(x)$ on a null set, and we use this principle on the null set where the usual derivative does not exist.

Notice that $f'(x)$ then can be considered as a piecewise continuous function.

Theorem 2.2.5 Laplace transformation of derivatives. *Assume that $f \in \mathcal{E}$ is continuous and piecewise C^1. Then $f' \in \mathcal{F}$, and*

$$\mathcal{L}\{f'\}(z) = z \cdot \mathcal{L}\{f\}(z) - f(0).$$

Proof. The proof below requires actually that $f \in \mathcal{E}$ and that we apply $\varrho(f)$ as the bound of the half plane of convergence instead of $\sigma(f)$.

Let $\{t_n\}$ denote the null set of points, in which f is not of class C^1, and put $t_0 = 0$. We shall assume that $\{t_n\}$ is infinite, because the proof is modified trivially, if $\{t_n\}$ is finite.

We get by partial integration,

$$\begin{aligned}
\mathcal{L}\{f'\}(z) &= \sum_{n=0}^{+\infty} \int_{t_n}^{t_{n+1}} e^{-zt} f'(t)\,\mathrm{d}t = \sum_{n=0}^{+\infty} \left\{ \left[e^{-zt} f(t)\right]_{t_n}^{t_{n+1}} + \int_{t_n}^{t_{n+1}} z\, e^{-zt} f(t)\,\mathrm{d}t \right\} \\
&= \lim_{n\to+\infty} \left[e^{-zt} f(t)\right]_0^{t_{n+1}} + z \int_0^{+\infty} e^{-zt} f(t)\,\mathrm{d}t \\
&= \lim_{n\to+\infty} e^{-zt_{n+1}} \cdot f\left(t_{n+1}\right) - f(0) + z\,\mathcal{L}\{f\}(z).
\end{aligned}$$

Here we can only say for sure that $\lim_{n\to+\infty} e^{-zt_{n+1}} \cdot f(t_{n+1})$ exists, if $f \in \mathcal{E}$ and $\Re z > \varrho(f)$, so we add these assumptions. Then it follows from the definition of $\varrho(f)$ that we can find real constants $A > 0$ and B, where furthermore $\varrho(f) < B < \Re z$, such that

$$\left|e^{-Bt} f(t)\right| \le A \qquad \text{for all } t > 0.$$

Then

$$\left|e^{-zt} f(t)\right| = \left|e^{-(z-B)t}\right| \cdot \left|e^{-Bt} f(t)\right| \le e^{-(\Re z - B)t} A \to 0 \qquad \text{for } t \to +\infty,$$

from which follows that even

$$\lim_{n\to+\infty} e^{-zt_{n+1}} \cdot f(t_{n+1}) = 0.$$

and the theorem is proved. □

Even if the proof of Theorem 2.2.5 *required* that $\Re z > \varrho(f)$, it is obvious that the analytic function $\mathcal{L}\{f'\}(z) = z \cdot \mathcal{L}\{f\}(z) - f(0)$ may be extended analytically to a larger open set Ω. Now, the analytic extension to a given open set Ω is always unique, if it exists. This implies that we *could* define a more general form of differentiability of functions in $\mathcal{F}$ by its Laplace transform, i.e.

(22) $\mathcal{L}\{f'\}(z) := z \cdot \mathcal{L}\{f\}(z) - f(0), \qquad \text{for } f \in \mathcal{F}.$

There is a lot of truth in (22), though it is not the full story. We shall here point out some pitfalls in the adoption of (22) as a definition of a more general derivative.

1) Due to the term $f(0)$ we must require that $f \in \mathcal{F}$ is continuous at $t = 0$, i.e.

$$\lim_{t\to 0+} f(t) = f(0).$$

It is of course a strange condition to require continuity at one specific point and not anywhere else. This is, however, caused by the fact that the *Laplace transformation* in the *Theory of Distributions* has a hidden discontinuity at $t = 0$, so we must compensate by requiring that $f(t)$ is continuous at $t = 0$.

2) The new function f' defined by the *inverse Laplace transformation* of (22) does not have to belong to $\mathcal{F}$. It even does not have to be an ordinary function

Example 2.2.5 To illustrate some of the problems above we notice the trivial factorization $1 = z \cdot \dfrac{1}{z}$ for $z \neq 0$. Now,

$$\mathcal{L}\{1\}(z) = \mathcal{L}\left\{\chi_{[0,+\infty[}\right\}(z) = \frac{1}{z},$$

so we guess that

$$1 = \mathcal{L}\left\{\chi'_{[0,+\infty[}\right\}(z) = \mathcal{L}\{\delta\}(z).$$

This can actually be justified, *but not by using the argument above!* The hidden obstacle is that the discontinuity of the function $\chi_{[0,+\infty[}$ at $t = 0$ coincides with the discontinuity of the Laplace transformation itself. ♢

Example 2.2.6 We found in Example 2.1.4 that

$$\mathcal{L}\{\sin t\}(z) = \frac{1}{z^2+1} \qquad \text{and} \qquad \mathcal{L}\{\cos t\}(z) = \frac{z}{z^2+1}.$$

These results are in agreement with Theorem 2.2.5, because

$$\mathcal{L}\{\cos t\}(z) = \mathcal{L}\left\{\frac{d}{dt}\sin t\right\}(z) = z \cdot \mathcal{L}\{\sin t\}(z) - \sin 0 = \frac{z}{z^2+1},$$

and

$$\mathcal{L}\{\sin t\}(z) = \mathcal{L}\left\{-\frac{d}{dt}\cos t\right\}(z) = -z \cdot \mathcal{L}\{\cos t\}(z) + \cos 0 = -z \cdot \frac{z}{z^2+1} + 1 = \frac{1}{z^2+1}. \qquad \diamond$$

Obviously, if $f \in \mathcal{E}$ is of class C^1 and piecewise C^2, then it follows by iteration of Theorem 2.2.5 that

$$(23)\quad \mathcal{L}\{f''\}(z) = z \cdot \mathcal{L}\{f'\}(z) - f'(0) = z^2 \mathcal{L}\{f\}(z) - z \cdot f(0) - f'(0).$$

It follows by induction that if $f \in C^{n-1}$ is piecewise C^n, then

Theorem 2.2.6 Laplace transformation of integrals. *Assume that $f \in \mathcal{F}$ is piecewise continuous and define*

$$g(t) := \int_0^t f(\tau)\, d\tau.$$

Then even $g \in \mathcal{E}$, and

$$\mathcal{L}\{g\}(z) = \mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\}(z) = \frac{1}{z}\,\mathcal{L}\{f\}(z) \qquad \text{for } \Re z > \max\{0, \sigma(f)\}.$$

The simple proof is left to the reader. Here we only prove that if $f \in \mathcal{F}$, then $g \in \mathcal{E}$. In fact, it follows from the assumption $f \in \mathcal{F}$ that

$$\int_0^{+\infty} |f(t)| e^{-\sigma t}\, \mathrm{d}t = C < +\infty \qquad \text{for some } \sigma \in \mathbb{R}.$$

This implies that $\int_0^t f(\tau)\, \mathrm{d}\tau$ is continuous in $t \in [0, +\infty[$, and that

$$|g(t)| = \left|\int_0^t f(\tau)\, \mathrm{d}\tau\right| \leq \int_0^t |f(\tau)|\, \mathrm{d}\tau \leq \int_0^t |f(\tau)| e^{-\sigma\tau} e^{\sigma t} \leq C \cdot e^{\sigma t},$$

and we have shown that $g(t) = \int_0^t f(\tau)\, \mathrm{d}\tau \in \mathcal{E}$.

Example 2.2.7 At this stage we only know very few Laplace transforms, so we can only apply Theorem 2.2.6 on very simple examples, where we also have other possibilities of solution. We have proved in Example 2.1.3 that $t^n \in \mathcal{E}$ with $\varrho\left(t^n\right) = \sigma\left(t^n\right) = 0$ and

$$\mathcal{L}\left\{t^n\right\}(z) = \frac{n!}{z^{n+1}} \qquad \text{for } \Re z > 0.$$

It follows from Theorem 2.2.6 and $g(t) = \int_0^t \tau^n\, \mathrm{d}\tau = \dfrac{t^{n+1}}{n+1}$ that

$$\mathcal{L}\left\{\frac{t^{n+1}}{n+1}\right\}(z) = \frac{n!}{z^{n+2}}, \qquad \text{for } \Re z > 0,$$

so when we multiply by $n+1$ we get

$$\mathcal{L}\left\{t^{n+1}\right\}(z) = \frac{(n+1)!}{z^{(n+1)+1}}, \qquad \text{for } \Re z > 0,$$

proving that Example 2.1.4 is consistent. $\Diamond$

Example 2.2.8 Similarly, if we consider $f(t) = \sin t \in \mathcal{E}$, where $\varrho(f) = \sigma(f) = 0$, and $f(t)$ is continuous, then

$$g(t) = \int_0^t \sin\tau\, \mathrm{d}\tau = [-\cos\tau]_0^t = 1 - \cos t,$$

so

$$\mathcal{L}\{1-\cos t\}(z) = \mathcal{L}\left\{\int_0^t \sin\tau \,\mathrm{d}\tau\right\}(z) = \frac{1}{z}\cdot\mathcal{L}\{\sin t\}(z) = \frac{1}{z\,(z^2+1)}.$$

For comparison we also have

$$\mathcal{L}\{1-\cos t\}(z) = \mathcal{L}\{1\}(z) - \mathcal{L}\{\cos t\}(z) = \frac{1}{z} - \frac{z}{z^2+1} = \frac{z^2+1-z^2}{z\,(z^2+1)} = \frac{1}{z\,(z^2+1)}. \qquad \Diamond$$

Theorem 2.2.7 Multiplication by t^n. *Let $f \in \mathcal{F}$. Then for every $n \in \mathbb{N}$,*

$$\mathcal{L}\left\{t^n f\right\}(z) = (-1)^n \frac{d^n}{dz^n}\mathcal{L}\{f\}(z) \qquad \textit{for } \Re z > \sigma(f).$$

Proof. This follows immediately from (19). $\square$.

Example 2.2.9 It was shown in Example 2.1.2 that $\mathcal{L}\{1\}(z) = \dfrac{1}{z}$, and furthermore in Example 14 that $\mathcal{L}\left\{t^n\right\}(z) = \dfrac{n!}{z^{n+1}}$. This is consistent with the above, because

$$\mathcal{L}\left\{t^n\right\}(z) = \mathcal{L}\left\{t^n\cdot 1\right\}(z) = (-1)^n \frac{d^n}{dz^n}\left\{\frac{1}{z}\right\} = \frac{n!}{z^{n+1}} \qquad \text{for } \Re z > 0. \qquad \Diamond$$

Theorem 2.2.8 Division by t. *Assume that $f \in \mathcal{F}$ and that $\lim_{t\to 0+}\dfrac{f(t)}{t}$ exists and is finite. Then also $\dfrac{f(t)}{t} \in \mathcal{F}$, and if $x > \sigma(f)$ is* real, *then*

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\}(x) = \int_x^{+\infty}\mathcal{L}\{f\}(\xi)\,d\xi.$$

More generally, if z is complex and $\Re z > \sigma(f)$, then

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_{\Gamma_z}\mathcal{L}\{f\}(\zeta)\,d\zeta,$$

where Γ_z is any continuous and piecewise differential curve in the half plane $\Re z > \sigma(f)$ from the initial point z to the real $+\infty$ as its end point, e.g. along the curve consisting of the line segment from z to $\Re z$, joined with the (real) interval $[\Re z, +\infty[$.

Proof. We shall first prove that $\dfrac{f(t)}{t} \in \mathcal{F}$. It follows from the existence and finiteness of $\lim_{t\to 0+}\dfrac{f(t)}{t}$, where $f \in \mathcal{F}$ that $\int_0^1 \left|\dfrac{f(t)}{t}\right| \mathrm{d}t < +\infty$. If $\sigma > \sigma(f)$, then

$$\int_0^{+\infty}\left|\frac{f(t)}{t}\right| e^{-\sigma t}\,\mathrm{d}t \le \int_0^1 \left|\frac{f(t)}{t}\right| e^{-\sigma t}\,\mathrm{d}t + \int_1^{+\infty}\left|\frac{f(t)}{t}\right| e^{-\sigma t}\,\mathrm{d}t < +\infty,$$

proving that $\dfrac{f(t)}{t} \in \mathcal{F}$.

Put $g(t) = \dfrac{f(t)}{t}$. Then $f(t) = t \cdot g(t)$, and it follows from Theorem 2.2.7 that

$$\mathcal{L}\{f\}(z) = \mathcal{L}\{t \cdot g(t)\}(z) = -\frac{d}{dz}\,\mathcal{L}\{g\}(z),$$

thus $\mathcal{L}\{f\}(z)$ is an integral of $-\mathcal{L}\{f\}(z)$. It follows from

$$\mathcal{L}\{f\}(z) \to 0 \quad \text{and} \quad \mathcal{L}\{g\}(z) = \mathcal{L}\left\{\frac{f(t)}{t}\right\}(z) \to 0 \qquad \text{for } \Re z \to +\infty,$$

that

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\}(z) = \int_{\Gamma_z} \mathcal{L}\{f\}(z)\,\mathrm{d}z. \qquad \square$$

Example 2.2.10 We know that

$$\mathcal{L}\{\sin t\}(z) = \frac{1}{z^2+1} \qquad \text{and} \qquad \lim_{t\to 0+} \frac{\sin t}{t} = 1 < +\infty.$$

Hence, it follows for real positive $x > 0$ from Theorem 2.2.8 that

$$\begin{aligned}\mathcal{L}\left\{\frac{\sin t}{t}\right\}(x) &= \int_0^{+\infty} \frac{\sin t}{t}\, e^{-xt}\,\mathrm{d}t = \int_x^{+\infty} \frac{\mathrm{d}\xi}{1+\xi^2} = [\operatorname{Arctan}\,\xi]_x^{+\infty} \\ &= \frac{\pi}{2} - \operatorname{Arctan}\,x = \operatorname{Arccot}\,x, \qquad \text{for } x > 0.\end{aligned}$$

If $x = 1$, then in particular,

$$\int_0^{+\infty} \frac{\sin t}{t}\, e^{-t}\,\mathrm{d}t = \operatorname{Arccot}\,1 = \frac{\pi}{4},$$

which is a result which is not possible to derive by only using plain elementary calculus.

Notice also that we by an *unjustified* limit process $x \to 0+$ under the integral sign obtain the result, which by other methods can be proved to be correct,

$$(24) \quad \int_0^{+\infty} \frac{\sin t}{t}\,\mathrm{d}t = \lim_{x\to 0+} \int_0^{+\infty} \frac{\sin t}{t}\, e^{-xt}\, e^{-xt}\,\mathrm{d}t = \lim_{x\to 0+} \operatorname{Arccot}\,x = \frac{\pi}{2}.$$

We emphasize that (24) is not a strict proof, because $\dfrac{\sin t}{t} \notin L^1$, so we do not know if it is legal to interchange the integration and the limit process to get the *left* hand side of (24).

Since $\dfrac{\sin t}{t}$ is bounded and continuous, we see that $\dfrac{\sin t}{t} \in \mathcal{E}$. Define $g(t) := \int_0^t \dfrac{\sin \tau}{\tau}\,\mathrm{d}\tau$. Then we conclude for $x > 0$ from Theorem 2.2.6 that

$$\mathcal{L}\left\{\int_0^t \frac{\sin\tau}{\tau}\,\mathrm{d}\tau\right\}(x) = \frac{1}{x}\,\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \frac{1}{x}\,\operatorname{Arccot}\,x.$$

The function $\operatorname{Si}(t) := \int_0^t \dfrac{\sin\tau}{\tau}\,d\tau$ is a new transcendental function, not known from elementary calculus. However, its Laplace transform $\dfrac{1}{x}\operatorname{Arccot} x$ is fairly simple. Its natural extension $\dfrac{1}{z}\operatorname{Arccot} z$ has a simple pole at $z = 0$ and branch points at $\pm i$. Putting the branch cuts along the imaginary axis we see that $\dfrac{1}{z}\operatorname{Arccot} z$ is well-defined as an analytic function in the right half plane $\Re z > 0$. This example shows that we need some other special functions which are not known from elementary calculus. The most important of these more advanced transcendental functions will be introduced in *Ventus: Complex Functions Theory a-5, Laplace Transformation II.* $\Diamond$

Theorem 2.2.9 Periodic functions. *Assume that $f \in \mathcal{F}$ is a periodic function of period $T > 0$ for $t \geq 0$, thus $f(t+T) = f(t)$ for every $t \geq 0$. Then*

$$\mathcal{L}\{f\}(z) = \frac{1}{1-e^{-zT}} \int_0^T e^{-zt} f(t)\, dt \qquad \text{for } \Re z > 0.$$

Proof. Notice that $e^{-znT} = \left(e^{-zT}\right)^n$ and $\left|e^{-zT}\right| = e^{-\Re z \cdot T} < 1$ for $\Re z > 0$. Then we have the following simple computation

$$\begin{aligned}\mathcal{L}\{f\}(z) &= \int_0^{+\infty} e^{-zt} f(t)\,\mathrm{d}t = \sum_{n=0}^{+\infty} \int_{nT}^{(n+1)T} e^{-zt} f(t)\,\mathrm{d}t = \sum_{n=0}^{+\infty} \int_0^T e^{-z(t+nT)} f(t+nT)\,\mathrm{d}t \\ &= \sum_{n=0}^{+\infty} e^{-znT} \int_0^T e^{-zt} f(t)\,\mathrm{d}t = \frac{1}{1-e^{-zT}} \int_0^T e^{-zt} f(t)\,\mathrm{d}t \qquad \text{for } \Re z > 0. \qquad \square\end{aligned}$$

The *proof* of Theorem 2.2.9 relies heavily on the assumption that $\Re z > 0$. However, by the usual analytic extension we conclude that the result of Theorem 2.2.9 holds if $e^{-zT} \neq 1$, thus for

$$z \in \mathbb{C} \setminus \left\{ i \cdot \frac{2p\pi}{T} \;\middle|\; p \in \mathbb{Z} \right\},$$

because it is almost trivial that the integral $\int_0^T e^{-zt} f(t)\,\mathrm{d}t$ over the bounded interval $[0,T]$ is defined for every $f \in \mathcal{F}$ and every $z \in \mathbb{C}$.

Example 2.2.11 In this example we show that even if we obtain the structure

$$F(z) = \frac{1}{1-e^{-zT}}\, \varphi(z) = \mathcal{L}\{f\}(z)$$

of the Laplace transform, this does *not* necessarily imply that the function $f(t)$ is periodic.

For $t \in \mathbb{R}$, let $[t] \in \mathbb{Z}$ denote the largest integer $\leq t$,

$$[t] := \max\{p \in \mathbb{Z} p \leq t\}, \qquad t \in \mathbb{R},$$

and put $f(t) = 1 + [t]$, where the constant 1 has only been added in order to simplify the following computation. Then we get

$$\begin{aligned}\mathcal{L}\{f\}(z) &= \int_0^1 e^{-zt}\,\mathrm{d}t + 2\int_1^2 e^{-zt}\,\mathrm{d}t + \cdots + n\int_{n-1}^n e^{-zt}\,\mathrm{d}t + \cdots \\ &= \sum_{n=1}^{+\infty} \int_{n-1}^n n\, e^{-zt}\,\mathrm{d}t = \sum_{n=1}^{+\infty} n \left[-\frac{1}{z}\, e^{-zt}\right]_{n-1}^n = \frac{1}{z} \sum_{n=1}^{+\infty} \left\{e^{-(n-1)z} - e^{-nz}\right\} \\ &= \frac{e^z - 1}{z} \sum_{n=1}^{+\infty} n\, e^{-nz} = \frac{e^z-1}{z} \sum_{n=1}^{+\infty} n \left(e^{-z}\right)^n.\end{aligned}$$

The series is clearly convergent for $\Re z > 0$.

Write for short $w = e^{-z}$ for $\Re z > 0$. Then $|w| < 1$, and $\dfrac{1}{1-w} = \sum_{n=0}^{+\infty} w^n$, hence by a differentiation,

$$\frac{1}{(1-w)^2} = \sum_{n=1}^{+\infty} n\, w^{n-1}, \qquad \text{thus} \qquad \sum_{n=1}^{+\infty} \frac{w}{(1-w)^2} \qquad \text{for } |w| < 1.$$

Returning to $w = e^{-z}$ for $\Re z > 0$ we get by insertion,

$$\mathcal{L}\{f\}(z) = \frac{e^z - 1}{z} \cdot \frac{w}{(1-w)^2} = \frac{e^z - 1}{z} \cdot \frac{e^{-z}}{(1-e^{-z})^2} = \frac{1}{1-e^{-z}} \cdot \frac{1}{z}.$$

Obviously, $f \in \mathcal{E}$. The question is now:

Does there exist a periodic function $g \in \mathcal{E}$, such that $\int_0^T e^{-zt} g(t)\, dt = \dfrac{1}{z}$?

The answer is "no", because if $g(t)$ were periodic, then $\mathcal{L}\{f\}(z) = \mathcal{L}\{g\}(z)$, thus $\mathcal{L}\{f-g\}(z) = 0$. We shall later prove the *uniqueness theorem* which states that 0 (almost everywhere) is the only function which has the Laplace transform 0. Hence, $f(t) = g(t)$ a.e. and it is obvious that the step function $f(t)$ is *not* periodic.

Another point is that even if $f(t)$ *is* periodic, its Laplace transform in its simplest form does not have the structure of Theorem 2.2.9. We showed in Example 2.1.4 that

$$\mathcal{L}\{\sin t\}(z) = \frac{1}{z^2+1} \qquad \text{for } \Re z > 0.$$

The function $\sin t$ has the period 2π, so in order to obtain the same structure as in Theorem 2.2.9 we should write

$$\mathcal{L}\{\sin t\}(z) = \frac{1}{1-e^{-2\pi z}} \cdot \frac{1-e^{-2\pi z}}{z^2+1} = \frac{1}{1-e^{-2\pi z}} \int_0^{2\pi} e^{-zt} \sin t \, \mathrm{d}t,$$

which is a little farfetched, although we by identification get the unexpected result

$$\int_0^{2\pi} e^{-zt} \sin t \, \mathrm{d}t = \frac{1-e^{-2\pi z}}{z^2+1} \qquad \text{for } \Re z > 0. \qquad \Diamond$$

Theorem 2.2.10 Initial value theorem. *Let $f \in \mathcal{F}$, and assume that $\lim_{t\to 0+} f(t) = f(0)$ exists and is finite. Then the real limit $\lim_{x\to+\infty} x\mathcal{L}\{f\}(x)$ exists, and*

$$\lim_{t\to 0+} f(t) = \lim_{x\to+\infty} x\mathcal{L}\{f\}(x).$$

If $f \in C^1$ and $f,\ f' \in \mathcal{E}$, then even

$$\lim_{t\to 0+} f(t) = \lim_{\Re z\to+\infty} z\mathcal{L}\{f\}(z).$$

Proof. Using that $f \in \mathcal{F}$, choose $x_0 > 0$, such that

$$M := \int_0^{+\infty} |f(t)| e^{-x_0 t} \, \mathrm{d}t < +\infty.$$

Let $x \geq x > 0$ and change variable $\xi = xt$ to get

$$x\mathcal{L}\{f\}(x) = x \int_0^{+\infty} f(t) e^{-xt}\,\mathrm{d}t = \int_0^{+\infty} f\left(\frac{\xi}{x}\right) e^{-\xi}\,\mathrm{d}\xi,$$

where the improper integrals clearly are convergent.

Now, $f(t) \to f(0)$ for $t \to 0+$, so to every $\varepsilon > 0$ we can find a $\delta > 0$, such that

$$|f(t) - f(0)| < \frac{\varepsilon}{3} \qquad \text{for } t \in [0, \delta].$$

Corresponding to this $\delta > 0$, choose an $x_1 \geq x_0$, such that for all $x \geq x_1$,

$$x \cdot e^{-\delta x} \cdot e^{\delta x_0} \cdot M < \frac{\varepsilon}{3} \qquad \text{and} \qquad |f(0)| \cdot e^{-\delta z} < \frac{\varepsilon}{3}.$$

Then we get the following estimate for all $x \geq x_1$,

$$\begin{aligned}
|x\mathcal{L}\{f\}(x) - f(0)| &= \left| \int_0^{+\infty} \left\{ f\left(\frac{\xi}{x}\right) - f(0) \right\} e^{-\xi}\,\mathrm{d}\xi \right| \\
&\leq \left| \int_0^{x\delta} \left\{ f\left(\frac{\xi}{x}\right) - f(0) \right\} e^{-\xi}\,\mathrm{d}\xi \right| + \int_{x\delta}^{+\infty} \left| f\left(\frac{\xi}{x}\right) \right| e^{-\xi}\,\mathrm{d}\xi + |f(0)| \int_{x\delta}^{+\infty} e^{-\xi}\,\mathrm{d}\xi \\
&\leq \frac{\varepsilon}{3} \int_0^{x\delta} e^{-\xi}\,\mathrm{d}\xi + x \int_\delta^{+\infty} |f(t)| e^{-xt}\,\mathrm{d}t + |f(0)| \cdot e^{-\delta x} \\
&\leq \frac{\varepsilon}{3} + x \cdot e^{-\delta x} \cdot e^{\delta x_0} \int_\delta^{+\infty} |f(t)| e^{-x_0 t}\,\mathrm{d}t + \frac{\varepsilon}{3} \\
&\leq \frac{\varepsilon}{3} + x \cdot e^{-\delta x} e^{\delta x_0} \cdot M + \frac{\varepsilon}{3} = \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

We conclude that $|x \cdot \mathcal{L}\{f\}(x) - f(0)| < \varepsilon$, whenever $x \geq x_1 = x_1(\varepsilon, x_0)$, from which follows that

$$\lim_{x \to +\infty} x \cdot \mathcal{L}\{f\}(x) = f(0).$$

Then assume that $f \in C^1$ and that f and $f' \in \mathcal{E}$. It follows from Theorem 2.2.5 that

$$\mathcal{L}\{f'\}(z) = z \cdot \mathcal{L}\{f\}(z) - f(0).$$

Using that $f' \in \mathcal{E}$ it follows from Theorem 2.1.3 that $\mathcal{L}\{f'\}(z) \to 0$ for $\Re z \to +\infty$, and the theorem is proved. $\square$

Example 2.2.12 Choose $f(t) = \cos t$. Then $f(0) = 1$, and

$$\lim_{\Re z \to +\infty} z\mathcal{L}\{f\}(z) = \lim_{\Re z \to +\infty} z \cdot \frac{z}{z^2+1} = \lim_{\Re z \to +\infty} \left\{ 1 - \frac{1}{z^2+1} \right\} = 1 = \cos 0.$$

Choose $f(t) = e^t$. Then $f(0) = 1$ and $\mathcal{L}\{e^t\}(z) = \dfrac{1}{z-1}$ for $\Re z > 1$, hence

$$\lim_{\Re z \to +\infty} z \cdot \frac{1}{z-1} = \lim_{\Re z \to +\infty} \left\{ 1 + \frac{1}{z-1} \right\} = 1. \qquad \Diamond$$

Theorem 2.2.11 Final value theorem. *Let $f \in \mathcal{F}$, where $\sigma(f) \le 0$. If $\lim_{t \to +\infty} f(t) = c \in \mathbb{C}$ exists, then the real limit $\lim_{x \to 0+} x\mathcal{L}\{f\}(x)$ also exists, and*

$$\lim_{t \to +\infty} f(t) = \lim_{x \to 0+} x\mathcal{L}\{f\}(x).$$

If $f \in \mathcal{E} \cap C^1$ and $|f'(t)| \le A$ for all $t \ge 0$, then

$$\lim_{t \to +\infty} f(t) = \lim_{\substack{z \to 0 \\ \Re z > 0}} z\mathcal{L}\{f\}(z).$$

Proof. Since $\lim_{t \to +\infty} f(t) = c \in \mathbb{C}$ we can to every $\varepsilon > 0$ find a $T = T(\varepsilon) > 0$, such that

$$|f(t) - c| < \frac{\varepsilon}{3} \qquad \text{for every } t \ge T.$$

The improper integral $\int_0^{+\infty} |f(t)| e^{-xt}\,\mathrm{d}t$ is convergent for every $x > 0$, hence

$$M_T := \int_0^T |f(t)|\mathrm{d}t \le e^{xT} \int_0^T |f(t)| e^{-xt}\mathrm{d}t < +\infty, \qquad \text{where } x > 0 \text{ is small.}$$

Choose $x_0 > 0$, such that for all $x \in \,]0, x_0]$,

$$x \cdot M_T < \frac{\varepsilon}{3} \qquad \text{and} \qquad |c| \cdot \left(1 - e^{-xT}\right) < \frac{\varepsilon}{3}.$$

Then we get the following estimate for $0 < x \le x_0$,

$$\begin{aligned}
|x\mathcal{L}\{f\}(x) - c| &= \left| x \int_0^{+\infty} \{f(t) - c\} e^{-xt}\,\mathrm{d}t \right| \\
&\le x \int_0^T |f(t) - c| e^{-xt}\,\mathrm{d}t + x \int_T^{+\infty} |f(t) - c| e^{-xt}\,\mathrm{d}t \\
&< x \int_0^T |f(t)| e^{-xt}\,\mathrm{d}t + |c| \cdot x \int_0^T e^{-xt}\,\mathrm{d}t + \frac{\varepsilon}{3} \int_T^{+\infty} x\, e^{-xt}\,\mathrm{d}t \\
&< x \int_0^T |f(t)|\,\mathrm{d}t + |c| \cdot \left(1 - e^{-xT}\right) + \frac{\varepsilon}{3} \\
&< x \cdot M_T + \frac{\varepsilon}{3} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

This is true for every $\varepsilon > 0$, so we conclude that

$$\lim_{t \to +\infty} f(t) = c = \lim_{x \to 0+} x\mathcal{L}\{f\}(x).$$

Then assume that also $f \in \mathcal{E} \cap C^1$ and $|f'| \le A$. It follows from Theorem 2.2.5 that

$$\mathcal{L}\{f'\}(z) = z\mathcal{L}\{f\}(z) - f(0),$$

because the additional assumptions above imply that $f(0)$ exists.

Finally, by interchanging limit and integration,

$$\begin{aligned}
\lim_{\substack{z \to 0 \\ \Re z > 0}} z\mathcal{L}\{f\}(z) &= f(0) + \lim_{\substack{z \to 0 \\ \Re z > 0}} \mathcal{L}\{f'\}(z) \\
&= f(0) + \lim_{\substack{z \to 0 \\ \Re z > 0}} \int_0^{+\infty} e^{-zt} f'(t)\,\mathrm{d}t = f(0) + \int_0^{+\infty} f'(t)\,\mathrm{d}t \\
&= f(0) + \lim_{T \to +\infty} \int_0^T f'(t)\,\mathrm{d}t = \lim_{T \to +\infty} f(T) = c,
\end{aligned}$$

and the theorem is proved. □

Example 2.2.13 In case of $f(t) = e^{-t}$ we get

$$\lim_{t \to +\infty} e^{-t} = 0 \qquad \text{and} \qquad \lim_{\substack{z \to 0 \\ \Re z > 0}} z\mathcal{L}\left\{e^{-t}\right\}(z) = \lim_{\substack{z \to 0 \\ \Re z > 0}} \frac{z}{z+1} = 0,$$

so the conclusion of Theorem 2.2.11 holds in this case.

Note, however, that e.g. $\cos t$ does not have a limit for $t \to \infty$, while of course

$$\lim_{\substack{z \to 0 \\ \Re z > 0}} z\mathcal{L}\{\cos t\}(z) = \lim_{\substack{z \to 0 \\ \Re z > 0}} z \cdot \frac{z}{z^2+1} = 0.$$

This shows that $\lim_{x \to 0+} x\mathcal{L}\{f\}(x)$ may exist, while the limit of $f(t)$ for $t \to +\infty$ does not. $\Diamond$

We shall later return to the *Initial value theorem* and the *Final value theorem*, when we have become able to handle more complicated examples.

2.3 The complex inversion formula I

In practical applications one typically uses the theorems of Section 2.2 in order to compute the *Laplace transform* $F \in \mathcal{A}$ of the unknown function $f(t)$. It is therefore important also to establish some methods by which one can reconstruct $f(t)$, $t > 0$, from the analytic function $F(z)$.

In this section we prove the simplest versions of the *Laplace inversion formula.* They are very easy to apply, because they rely on the well known *residuum calculus*, cf. *Ventus: Complex Functions Theory a-2.* Furthermore, in many technical sciences, like e.g. in *Cybernetics* and *Elementary Circuit Theory* the functions $F \in \mathcal{A}$ will be of a type where the methods of this section suffice.

We shall later in Section 3.5 give a more general inversion formula where we only require that $F \in \mathcal{A}$ some very reasonable growth conditions in a right half plane. We shall in this general case even allow that $F(z)$ is a branch of a many valued function.

We mention without proof

Theorem 2.3.1 Weierstraß's approximation theorem. *Let $f : [a, b] \to \mathbb{R}$ be a continuous real function on the closed and bounded (i.e. compact) interval $[a, b]$, $a < b$.*
There exists a sequence of polynomials (P_n) which converges uniformly *towards f on $[a, b]$.*
This means that to every $\varepsilon > 0$ we can find $n_0 \in \mathbb{N}$, such that for every $n \geq n_0$ and every $x \in [a, b]$ we have the estimate

$$|f(x) - P_n(x)| < \varepsilon, \qquad \left(\textit{or more compactly,} \qquad \max_{x \in [a,b]} |f(x) - P_n(x)| < \varepsilon\right).$$

It is possible to give an elementary proof using only calculus in one variable, but it is very long. There is also a much shorter proof, but it requires knowledge of *Chebyshev's inequality.*

We use Theorem 2.3.1 to prove the following lemma.

Lemma 2.3.1 *Assume that $f : [a, b] \to \mathbb{C}$ is continuous.*
If $\int_a^b t^n f(t)\, dt = 0$ for every $n \in \mathbb{N}_0$, then $f \equiv 0$ on $[a, b]$,

Proof. We can always consider the real and the imaginary parts separately, so it suffices to prove the lemma for $f : [a, b] \to \mathbb{R}$ real and continuous.

Choose by Theorem 2.3.1 a sequence (P_n) of polynomials converging *uniformly* towards f. Since every $P_n(t)$ is a polynomial, it follows from the assumption that also

$$\int_a^b P_n(t) f(t)\, \mathrm{d}t = 0 \qquad \text{for every } n \in \mathbb{N}_0,$$

For given $\varepsilon > 0$ choose $n_0 \in \mathbb{N}$, such that

$$\max_{x \in [a,b]} |f(x) - P_n(x)| \cdot \int_a^b |f(t)|\, \mathrm{d}t < \varepsilon \qquad \text{for all } n \geq n_0.$$

Then for $n \geq n_0$,

$$\begin{aligned}\int_a^b \{f(t)\}^2\,\mathrm{d}t &= \left|\int_a^b \{f(t)\}^2\,\mathrm{d}t\right| = \left|\int_a^b P_n(t)f(t)\,\mathrm{d}t + \int_a^b \{f(t)-P_n(t)\}\,f(t)\,\mathrm{d}t\right| \\ &\leq 0 + \int_a^b |f(t)-P_n(t)|\cdot|f(t)|\,\mathrm{d}t \leq \max_{x\in[a,b]} |f(x)-P_n(x)|\cdot\int_a^b |f(t)|\,\mathrm{d}t < \varepsilon.\end{aligned}$$

This holds for every $\varepsilon > 0$, hence $\int_a^b \{f(t)\}^2\,\mathrm{d}t = 0$. Since the integrand $\{f(t)\}^2 \geq 0$ is continuous, this is only possible if $f(t) = 0$ for all $t \in [a,b]$. □

Theorem 2.3.2 *Assume that $f \in \mathcal{E}$ is* continuous*. If there is a τ, such that*

$$\mathcal{L}\{f\}(z) = 0 \qquad \textit{for } \Re\, z > \tau,$$

then $f(t) = 0$ for every $t \in [0,+\infty[$.

Proof. It follows from the assumption that

$$\mathcal{L}\{f\}(z) = \int_0^{+\infty} e^{-zt} f(t)\,\mathrm{d}t = 0 \qquad \text{for } \Re\, z > \tau.$$

Choose a fixed $x_0 > \tau + 1$. Then $x_0 - 1 > \tau$ and

$$\mathcal{L}\{f\}\,(x_0-1) = \int_0^{+\infty} e^{-x_0 t + t} f(t)\,\mathrm{d}t = 0.$$

In particular, $e^{-x_0 t+t} f(t) \to 0$ for $t \to +\infty$, because $f \in \mathcal{E}$. If we put $s = e^{-t}$, $s \in]0,1]$, and

$$h(s) = e^{-x_0 t + t} f(t), \qquad t = \ln \frac{1}{s},$$

we see that $h(s)$ can be extended continuously to $[0,1]$ by adding the value

$$h(0) = \lim_{t\to+\infty} e^{-x_0 t+t} f(t) = 0.$$

Now $x_0 + n > \tau$ for every $n \in \mathbb{N}_0$, so for every $n \in \mathbb{N}_0$ we get by the assumption,

$$\int_0^1 s^n h(s)\,\mathrm{d}s = -\int_{+\infty}^0 e^{-nt}\, e^{x_0 t + t}\, f(t)\,\mathrm{d}t = \int_0^{+\infty} e^{-(x_0+n)t} f(t)\,\mathrm{d}t = \mathcal{L}\{f\}\,(x_0+n) = 0.$$

Finally, Lemma 2.3.1 implies that $h(s) = e^{-x_0 t+t} f(t) = 0$, hence also $f(t) = 0$. □

Remark 2.3.1 The assumption of Theorem 2.3.2 can be relaxed to the following: *There is a $z_0 \in \mathbb{C}$ and a $\lambda > 0$, such that*

$$\mathcal{L}\{f\}\,(z_0 + \lambda n) = 0 \qquad \text{for every } n \in \mathbb{N}_0.$$

The proof is only a modification of the proof of Theorem 2.3.2. From this follows that if $F \in \mathcal{A}$ is not identically 0, then $F(z)\sin z$ can never be the Laplace transform of any continuous function. The proof is very simple: By choosing $z_n = n\pi$ for $n \geq n_0$ we trivially get $F(n\pi)\sin n\pi = 0$, so the only possible continuous function is $f(t) \equiv 0$. However, $F(z) \neq 0$ by assumption, so $F(z)\sin z \neq 0 = \mathcal{L}\{0\}(z)$. ◊

We shall next get rid of the assumption that $f \in \mathcal{E}$ is continuous.

Theorem 2.3.3 *Let $f \in \mathcal{E}$, and assume that there is a τ, such that $\mathcal{L}\{f\}(z) = 0$ for $\Re z > \tau$. Then $f(t) = 0$ for almost every $t \in [0, +\infty[$.*

Proof. It follows from $f \in \mathcal{E}$ that $g(t) := \int_0^t f(\tau)\,\mathrm{d}\tau \in \mathcal{E}$ is continuous. Then we conclude from Theorem 2.2.6 that

$$\mathcal{L}\{g\}(z) = \mathcal{L}\left\{\int_0^t f(\tau)\,\mathrm{d}\tau\right\}(z) = \frac{1}{z}\,\mathcal{L}\{f\}(z) = 0 \qquad \text{for } z > \max\{0, \tau\}.$$

Finally, by Theorem 2.3.2,

$$g(t) = \int_0^t f(\tau)\,\mathrm{d}\tau \equiv 0,$$

which is only possible if $f(\tau) = 0$ almost everywhere. □

Remark 2.3.2 If we knew more about of the Lebesgue integral than just what is given in Chapter 1, we could prove that it suffices in Theorem 2.3.3 to assume that $f \in \mathcal{F}$. It cannot be done in the present book, so we just mention that such a result exists. ◊

Corollary 2.3.1 *If f and $g \in \mathcal{E}$, and*

$$\mathcal{L}\{f\}(z) = \mathcal{L}\{g\}(z) \qquad \text{for } \Re z > \tau,$$

then $f = g$ almost everywhere in $[0, +\infty[$.

Proof. The assumption implies by linearity that $\mathcal{L}\{f - g\}(z) = 0$ for $\Re z > \tau$, where $f - g \in \mathcal{E}$, so the corollary follows immediately from Theorem 2.3.3. □

Remark 2.3.3 Combining Corollary 2.3.1 and Remark 2.3.2 it follows that the conclusion of Corollary 2.3.1 also holds, if f, $g \in \mathcal{F}$. As mentioned above we cannot here give the correct proof, so this information is just for reference. ♢

We shall use Corollary 2.3.1 in the investigation of the following problem:

> *Given an analytic function $F \in \mathcal{A}$ defined in an open domain Ω which contains a right half plane $\Re z > \tau$. Does there exist a function $f \in \mathcal{F}$, such that $\mathcal{L}\{f\}(z) = F(z)$?*

According to Remark 2.3.3 there is – apart from addition of a *null function* – at most one such function $f \in \mathcal{F}$.

If there exists a function $f \in \mathcal{F}$, such that $\mathcal{L}\{f\}(z) = F(z)$, then we call f the *inverse Laplace transform* of the analytic function $F \in \mathcal{A}$, and we write

$$f(t) := \mathcal{L}^{-1}\{F\}(t), \qquad t \geq 0.$$

In practice it will cause no problem that $f(t)$ is "only" determined *almost everywhere*, because we are always allowed to change $f(t)$ on a null set with no consequences for the concrete physical or technical situation we are trying to model.

Our first result of finding $\mathcal{L}^{-1}\{F\}(t)$ is the following theorem.

Theorem 2.3.4 Residuum inversion formula. *Let $F(z)$ be analytic in $\mathbb{C} \setminus \{z_1, \ldots, z_n\}$, i.e. in the whole complex plane except for a finite number of singularities. Assume that there are positive constants M, R, $\alpha > 0$, such that we have the estimate*

$$|F(z)| \leq \frac{M}{|z|^\alpha} \qquad \text{for } |z| \geq R.$$

Then $F(z)$ has a (uniquely determined) inverse Laplace transform. It is given by the residuum formula

$$(25) \quad f(t) := \mathcal{L}^{-1}\{F\}(t) = \sum_{j=1}^{n} res\left(e^{zt} F(z); z_j\right), \qquad \text{for } t > 0.$$

Remark 2.3.4 Theorem 2.3.4 is in particular useful in *Cybernetics* and in *Circuit Theory*,, because in these sciences one shall typically find the inverse Laplace transform of a fraction between two polynomials, where the denominator has higher degree than the numerator. Furthermore, formula (25) is easy to use, once one knows the simple rules of finding a residuum, cf. also *Ventus: Complex Functions Theory a-2*. This is advantageous, when the use of tables becomes too complicated. On the other hand, tables are of course useful, when the assumptions of Theorem 2.3.4 are not met, so we cannot rely on formula (25). We mention that residuum formulæ usually give a wrong result when not all of the assumptions are fulfilled. It is therefore of paramount importance *always to check the assumptions of a residuum formula*, because otherwise the error may be very dramatic indeed, without any element at all of an "approximation" of the right result. ◊

PROOF. Fix z, such that $\Re z = \beta > \sigma := \max\{\Re z_j \mid j = 1, \dots, n\}$. Then choose γ, such that $\sigma < \gamma < \beta$. Finally, consider any $r > R$ and define the closed path of integration $C_{r,\gamma}$, which is given on Figure 5, i.e. $C_{r,\gamma}$ is composed of a circular arc of centre 0 and radius r, and a segment of the vertical line $\Re \zeta = \gamma$.

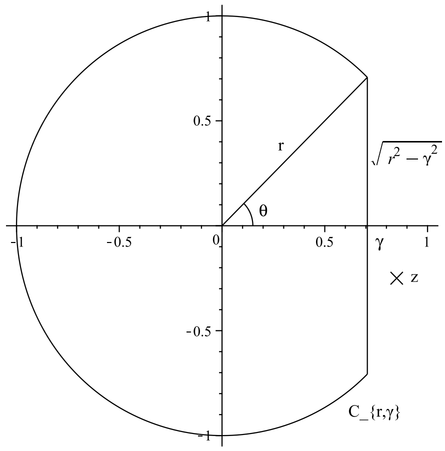

Figure 5: The closed path of integration $C_{r,\gamma}$, where $\sigma \geq 0$. The case $\sigma < 0$ is similar.

Due to the estimate $|F(z)| \leq M \cdot |z|^{-\alpha} < +\infty$ for $|z| \geq r > R$, all singularities of $F(z)$ lie inside $C_{r,\gamma}$. Using (25) as a *definition* of $f(t)$, it follows from the *Residuum Theorem*, cf. *Ventus, Complex Functions Theory a-2*, that

$$(26)\quad \oint_{C_{r,\gamma}} e^{\zeta t} F(\zeta)\, \mathrm{d}\zeta = 2\pi i \sum_{j=1}^{n} \operatorname{res}\left(e^{\zeta t} F(\zeta); z_j\right) = 2\pi i \cdot f(t).$$

On the other hand, it follows for every $\varrho > 0$, when $\Re z > \gamma > \sigma$ that the finite integral $\int_0^{\varrho} e^{-zt} f(t)\, \mathrm{d}t$

is computed in the following way

$$\begin{aligned}\int_0^{\varrho} e^{-zt} f(t)\,\mathrm{d}t &= \frac{1}{2\pi i}\int_0^{\varrho} e^{-zt}\left\{\oint_{C_{r,\gamma}} e^{\zeta t}F(\zeta)\,\mathrm{d}\zeta\right\}\mathrm{d}t = \frac{1}{2\pi i}\oint_{C_{r,\gamma}}\left\{\int_0^{\varrho} e^{(\zeta-z)t}F(\zeta)\,\mathrm{d}t\right\}\mathrm{d}\zeta \\ (27) \qquad &= \frac{1}{2\pi i}\oint_{C_{r,\gamma}} \frac{e^{(\zeta-z)\varrho}-1}{\zeta-z}\,F(\zeta)\,\mathrm{d}\zeta = -\frac{1}{2\pi i}\oint_{C_{r,\gamma}}\frac{F(\zeta)}{\zeta-z}\,\mathrm{d}z + \frac{1}{2\pi i}\oint_{C_{r,\gamma}}\frac{e^{(\zeta-z)\varrho}}{\zeta-z}\,F(\zeta)\,\mathrm{d}\zeta,\end{aligned}$$

where the change of order of integration in the first line of (27) is legal because the integrand is continuous, and the two integration paths $C_{r,\gamma}$ and $[0,\varrho]$ are both closed and bounded sets (i.e. compact).

Concerning the last line of (27) we notice that there exists a constant M_1, which only depends on R, α, $\beta = \Re z$ and γ, such that

$$\left|\frac{F(\zeta)}{\zeta-z}\right| \le M \qquad \text{for } \zeta \in C_{r,\gamma} \text{ and } t \in [0,\varrho].$$

Since $\Re\zeta \le \gamma < \beta = \Re z$ for every $\zeta \in C_{r,\gamma}$, we get the estimate

$$\left|\frac{1}{2\pi i}\oint_{C_{r,\gamma}} e^{(\zeta-z)\varrho}\cdot\frac{F(\zeta)}{\zeta-z}\,\mathrm{d}\zeta\right| \le \frac{1}{2\pi}\,e^{-(\beta-\gamma)\varrho}\cdot M_1\cdot 2\pi i \to 0 \qquad \text{for } \varrho \to +\infty.$$

It therefore follows from (27) that

$$\begin{aligned}\int_0^{+\infty} e^{-zt}f(t)\,\mathrm{d}t &= \lim_{\varrho\to+\infty}\int_0^{\varrho} e^{-zt}f(t)\,\mathrm{d}t \\ &= \lim_{\varrho\to+\infty}\left\{-\frac{1}{2\pi i}\oint_{C_{r,\gamma}}\frac{F(\zeta)}{\zeta-z}\,\mathrm{d}\zeta + \frac{1}{2\pi i}\oint_{C_{r,\gamma}} e^{(\zeta-z)\varrho}\,\frac{F(\zeta)}{\zeta-z}\,\mathrm{d}\zeta\right\} \\ &= -\frac{1}{2\pi i}\oint_{C_{r,\gamma}}\frac{F(\zeta)}{\zeta-z}\,\mathrm{d}\zeta + 0 = \oint_{-C_{r,\gamma}}\frac{F(\zeta)}{\zeta-z}\,\mathrm{d}\zeta \\ &= \operatorname{res}\left(\frac{F(\zeta)}{\zeta-z};z\right) + \operatorname{res}\left(\frac{F(\zeta)}{\zeta-z};\infty\right) = F(z)+0 = F(z),\end{aligned}$$

where we have used, cf. *Ventus, Complex Functions Theory a-2*, that $\zeta = z$ (a simple pole) is the only singularity of $\dfrac{F(\zeta)}{\zeta-z}$ outside $C_{r,\gamma}$, and that

$$\operatorname{res}\left(\frac{F(\zeta)}{\zeta-z};\infty\right) = -\lim_{\zeta\to\infty}\zeta\cdot\frac{F(\zeta)}{\zeta-z} = 0,$$

because

$$\left|\zeta\cdot\frac{F(\zeta)}{\zeta-z}\right| \le \frac{|\zeta|}{|\zeta|-|z|}\cdot\frac{M}{|\zeta|^{\alpha}} \to 0 \qquad \text{for } \zeta\to\infty.$$

This holds for every fixed $z \in \mathbb{C}$, such that $\Re z > \sigma$, so we have proved that

$$\mathcal{L}\{f\}(z) = \int_0^{+\infty} e^{-zt}f(t)\,\mathrm{d}t = F(z) \qquad \text{for } \Re z > \sigma.$$

Finally, we get $\mathcal{L}\{f\}(z) = F(z)$ for $z \in \mathbb{C} \setminus \{z_1, \dots, z_n\}$ by the unique analytic continuation. Then of course the integral representation is no longer convergent, when $\Re z < \sigma$. □

Example 2.3.1 We shall give some warning examples of what happens if the assumptions of Theorem 2.3.4 are not all fulfilled, and we in spite of this apply formula (25).

1) First consider $F(z) \equiv 1$, in which case we do not have an estimate of the form

$$|F(z)| \leq \frac{M}{|z|^\alpha} \qquad \text{for } |z| > R,$$

so (25) is not valid. In fact, using (25) we get

$$f(t) = \sum_{j=1}^{n} \operatorname{res}\left(e^{zt}F(z); z_j\right) \equiv 0,$$

because $F(z)$ does not have any singularity at all. It is obvious that

$$\mathcal{L}\{f\}(z) = \mathcal{L}\{0\}(z) = 0 \neq 1 = F(z).$$

2) One may add that $F(z) \equiv 1 \notin \mathcal{A}$, because it does not satisfy the necessary condition of $F(z) \to 0$ for $\Re z \to \infty$ of being a Laplace transform of a function from $\mathcal{F}$, cf. Theorem 2.1.3, so one may believe that this is the reason why (25) does not hold. This is, however, not the case, which we shall prove now by choosing $f(t) = \chi_{[0,1]}(t)$, where clearly $f \in \mathcal{F}$. We get immediately by a small computation,

$$\mathcal{L}\{f\}(z) = \int_0^1 e^{-zt}\,\mathrm{d}t = \frac{1-e^{-z}}{z} = F(z) \qquad \text{for } z \in \mathbb{C},$$

because the singularity at $z = 0$ is removable, $F(0) = 1$. Hence, $\sigma = -\infty$, and $F(z)$ is analytic in all of $\mathbb{C}$, and an application of (25) gives

$$f_1(t) = \mathcal{L}^{-1}\{F\}(t) = \sum_{j=1}^{n} \operatorname{res}\left(e^{zt}F(z); z_j\right) \equiv 0 \neq f(t) = \chi_{[0,1]}(t).$$

We therefore conclude that at least one of the assumptions of Theorem 2.3.4 is not fulfilled.

In this case the villain is the exponential, because

$$\frac{1-e^{-z}}{z} = \frac{1-e^{-x}}{x} = \frac{e^{|x|}-1}{|x|} \to +\infty \qquad \text{for } z = x \to -\infty,$$

when $z = x$ is chosen real and negative. It is again the growth condition

$$|F(z)| \leq \frac{M}{|z|^{\alpha}} \qquad \text{for } |z| > R,$$

which is not fulfilled. $\Diamond$

It was mentioned above that Theorem 2.3.4 is in particular useful when $F(z) = \dfrac{P(z)}{Q(z)}$ is a *rational function*, where $P(z)$ and $Q(z)$ are polynomials, and $\deg Q > \deg P$. If z_0 is a zero of multiplicity n of Q, then it follows from one of the residuum theorems in *Ventus, Complex Functions Theory a-2* that

$$\operatorname{res}\left(e^{zt}\cdot\frac{P(z)}{Q(z)}; z_0\right) = \frac{1}{(n-1)!}\lim_{z\to z_0}\frac{\mathrm{d}^{n-1}}{dz^{n-1}}\left\{(z-z_0)^n\cdot\frac{P(z)}{Q(z)}\cdot e^{tz}\right\},$$

from which we conclude that there exist n complex constants $a_0, a_1, \dots, a_{n-1}$, such that

$$\operatorname{res}\left(e^{zt}\cdot\frac{P(z)}{Q(z)}; z_0\right) = \left\{a_0 + a_1 t + \cdots + a_{n-1}t^{n-1}\right\}e^{z_0 t}.$$

Since z_0 may also be complex, we conclude that the inverse Laplace transform of a rational function $F(z)$, which satisfies $F(z) \to 0$ for $z \to \infty$, is a linear sum of terms of the form

$$t^k e^{at}, \qquad t^k e^{at}\cos bt \qquad \text{and} \qquad t^k e^{at}\sin bt,$$

corresponding to the complete solution of some linear homogeneous ordinary differential equation of constant coefficients. Therefore, if the Laplace transform $F(z)$ of some unknown function proves to be a rational function satisfying $F(z) \to 0$ for $z \to \infty$, then it is theoretically possible to reformulate the original problem to a linear ordinary differential equation of constant coefficients, where one already knows the complete solution from *Elementary Calculus*, cf. e.g. *Ventus, Calculus 1.* For such problems we may therefore apply the simple old solution method known from *Elementary Calculus* without involving the theory of Laplace transformation, so also excluding the residuum calculus in the form of Theorem 2.3.4. The two methods certainly give the same result. The former is easy to understand, while the latter requires some more advanced knowledge of mathematics. It can, however, be proved that both methods rely on the problem of finding the roots of the same so-called *characteristic polynomial* of the problem.

From the discussion above the reader should therefore conclude that in the simple case of a rational function $F(z)$, where $F(z) \to 0$ for $z \to \infty$ one might *alternatively* just as well solve a linear ordinary differential equation by elementary calculus instead of using the Laplace transformation, where the elementary calculus sometimes may even be easier to use. However, when engineers nevertheless prefer to use the Laplace transformation instead of elementary calculus, the reason is that one intrinsically build the given *boundary conditions* $f(0)$, $f'(0)$, etc. into $F(z)$ by repeating the rule

$$\mathcal{L}\{f'\}(z) = z \cdot \mathcal{L}\{f\}(z) - f(0).$$

A trivial, though important consequence of Theorem 2.3.4 is the following

Corollary 2.3.2 Heaviside's expansion theorem. *Let $P(z)$ and $Q(z)$ be polynomials, where* $\deg Q > \deg P$, *and assume that the denominator $Q(z)$ has only simple roots, $z_1, \dots, z_n$. Then the inverse Laplace transform of the rational function $F(z) = \dfrac{P(z)}{Q(z)}$ is given by*

$$f(t) = \sum_{j=1}^{n} \frac{P(z_j)}{Q'(z_j)} \cdot e^{z_j t} \qquad \text{for } t > 0,$$

and $\sigma(f) = \max\{\Re z_j \mid j = 1, \dots n\}$.

Example 2.3.2 We shall here show that Theorem 2.3.4 can be applied to other analytic functions than just the rational functions, so Theorem 2.3.4 is therefore more general than one would expect from the discussion above of the rational functions.

We choose

$$F(z) = \frac{1}{z}\exp\left(\frac{1}{z}\right) \qquad \text{for } z \in \mathbb{C} \setminus \{0\}.$$

It is obvious that $z = 0$ is the only singularity and that it is an *essential singularity.* Concerning the necessary estimate we get for $|z| \geq R \geq 1$,

$$|F(z)| = \frac{1}{|z|} \cdot \left|\exp\left(\frac{1}{z}\right)\right| \leq \frac{1}{R} \cdot \exp\left(\frac{1}{R}\right) \leq \frac{e}{R},$$

so the assumption of Theorem 2.3.4 is fulfilled for $M = e$ and $\alpha = 1$ and $R \geq 1$.

Using (25) we get the inverse Laplace transform of $F(z)$,

$$f(t) = \operatorname{res}\left(e^{zi}F(z); 0\right) = \operatorname{res}\left(\frac{1}{z}\, e^{zt}\exp\left(\frac{1}{z}\right); 0\right) \qquad \text{for } t > 0.$$

The residuum is equal to the coefficient $a_{-1}(t)$ of the Laurent series in z (and fixed t) of the function $\dfrac{1}{z}\exp(zt)\exp\left(\dfrac{1}{z}\right)$. Using that the summations in the Cauchy multiplication below can be interchanged, we get the Laurent series

$$\frac{1}{z}\,e^{zt}\exp\frac{1}{z}=\frac{1}{z}\sum_{n=0}^{+\infty}\frac{1}{n!}\,t^n z^n\cdot\sum_{k=0}^{+\infty}\frac{1}{k!}\,z^k=\sum_{n=0}^{+\infty}\sum_{k=0}^{+\infty}\frac{t^n}{n!k!}\,z^{n-k-1},\qquad z\in\mathbb{C}\setminus\{0\},$$

and it follows that we get the coefficient $a_{-1}(t)$ by collecting all terms for which $k=n$, hence

$$f(t)=\operatorname{res}\left(\frac{1}{z}\,e^{zt}\exp\left(\frac{1}{z}\right);0\right)=a_{-1}(t)=\sum_{nm=0}^{+\infty}\frac{1}{\{n!\}^2}\,t^n\qquad\text{for } t>0.$$

This series can be expressed by means of the *modified Bessel function of order 0* $I_0(t)$ as the function $f(t)=I_0(2\sqrt{t})$ for $t>0$. Here $I_0(t)$ is explicitly defined by

$$I_0(t):=\sum_{n=0}^{+\infty}\frac{1}{(n!)^2}\left\{\frac{t}{2}\right\}^{2n}\qquad\text{for } t\in\mathbb{R}\text{ or } t\in\mathbb{C}.$$

Thus we have proved that

$$\mathcal{L}\left\{I_0(2\sqrt{2})\right\}(z)=\frac{1}{z}\,\exp\left(\frac{1}{z}\right)\text{ for } z\in\mathbb{C}\setminus\{0\}.\qquad\Diamond$$

Example 2.3.3 If $F(z)=\dfrac{1}{z+a}$, then it follows from Corollary 2.3.2 that $\sigma(f)=-\Re\,a$ and

$$f(t)=\operatorname{res}\left(\frac{e^{zt}}{z+a};-a\right)=e^{-at}\qquad\text{for } t>0.$$

This result is in line with Example 2.1.4, because

$$\mathcal{L}\left\{e^{-at}\right\}(z)=\frac{1}{z+a}\qquad\text{for }\Re\,z>-\Re\,a.\qquad\Diamond$$

Example 2.3.4 Let $F(z)=\dfrac{1}{(z+a)^n}$. Then the growth condition at ∞ of Theorem 2.3.4 is trivially fulfilled and $\sigma(f)=-\Re\,a$, so formula (25) can be applied,

$$f(t)=\operatorname{res}\left(\frac{e^{zt}}{(z+a)^n};-a\right)=\frac{1}{(n-1)!}\lim_{z\to-a}\frac{d^{n-1}}{dz^{n-1}}\,e^{zt}=\frac{t^{n-1}}{(n-1)!}\,e^{-at}\qquad\text{for } t\geq 0.$$

If we put $a=0$ and replace n by $n+1$, we get

$$F(z)=\frac{1}{z^{n+1}}\qquad\text{and}\qquad f(t)=\frac{t^n}{n!}\qquad\text{and}\qquad\sigma(f)=0.$$

This shows that

$$\mathcal{L}\left\{t^n\right\}(z)=\frac{n!}{z^{n+1}}\qquad\text{for } n\in\mathbb{N}_0\text{ and }\Re\,z>0,$$

which is the same result as in Example 2.1.3. $\Diamond$

Example 2.3.5 Let $F(z) = \dfrac{z}{z^2+a^2}$, $a \in \mathbb{C} \setminus \{0\}$ a constant. Then $\sigma(f) = |\Im a|$, and

$$f(t) = \operatorname{res}\left(\frac{ze^{zt}}{z^2+a^2}; ia\right) + \operatorname{res}\left(\frac{ze^{zt}}{z^2+a^2}; -ia\right) = \frac{iae^{iat}}{2ia} + \frac{-iae^{-iat}}{-2ia} = \frac{1}{2}\left\{e^{iat}+e^{-iat}\right\} = \cos at,$$

cf. Example 2.1.4.

If in particular $a = ib$ is purely imaginary, i.e. $F(z) = \dfrac{z}{z^2-b^2}$, $b \in \mathbb{R} \setminus \{0\}$, then $\sigma(f) = |b|$ and

$$f(t) = \cos(ibt) = \cosh(bt). \qquad \Diamond$$

Example 2.3.6 If $F(z) = \dfrac{a}{z^2+a^2}$ for $a \in \mathbb{C} \setminus \{0\}$ constant (cf. Example 2.1.4), then $\sigma(f) = |\Im\, a|$, and

$$f(t) = \operatorname{res}\left(\frac{ae^{zt}}{z^2+z^2}; ia\right) + \operatorname{res}\left(\frac{ae^{zt}}{z^2+z^2}; ia\right) = \frac{ae^{iat}}{2ia} + \frac{ae^{-iat}}{-2ia} = \frac{1}{2i}\left\{e^{iat}-e^{-iat}\right\} = \sin at. \qquad \Diamond$$

Example 2.3.7 Let $F(z) = \dfrac{z}{\left(z^2+1\right)^2}$. Then $\sigma(f) = 0$, and

$$\begin{aligned}
f(t) &= \operatorname{res}\left(\frac{ze^{zt}}{\left(z^2+1\right)^2};i\right) + \operatorname{res}\left(\frac{ze^{zt}}{\left(z^2+1\right)^2};-i\right) = \lim_{z\to i}\frac{d}{dz}\left\{\frac{ze^{zt}}{(z+i)^2}\right\} + \lim_{z\to -i}\frac{d}{dz}\left\{\frac{ze^{zt}}{(z-i)^2}\right\} \\
&= \lim_{z\to i}\left\{\frac{e^{zt}+tze^{zt}}{(z+i)^2} - 2\,\frac{ze^{zt}}{(z+i)^3}\right\} + \lim_{z\to -i}\left\{\frac{e^{zt}+tze^{zt}}{(z-i)^2} - 2\,\frac{ze^{zt}}{(z-i)^3}\right\} \\
&= \left\{\frac{(1+it)e^{it}}{(2i)^2} - \frac{2ie^{it}}{(2i)^3}\right\} + \left\{\frac{(1-it)e^{-it}}{(-2i)^2} - \frac{(-2i)e^{-it}}{(-2i)^3}\right\} \\
&= \frac{ite^{it}}{-4} - \frac{ite^{-it}}{-4} = \frac{1}{2}\,t\left\{\frac{e^{it}-e^{-it}}{2i}\right\} = \frac{1}{2}\,t\,\sin t.
\end{aligned}$$

It is easy to check this result, because $\mathcal{L}\{\sin t\}(z) = \dfrac{1}{z^2+1}$, so

$$\mathcal{L}\left\{\frac{1}{2}\,t\,\sin t\right\} = \frac{1}{2}\cdot(-1)\cdot\frac{d}{dz}\left\{\frac{1}{z^2+1}\right\} = -\frac{1}{2}\cdot(-1)\cdot\frac{2z}{\left(z^2+1\right)^2} = \frac{z}{\left(z^2+1\right)^2}. \qquad \Diamond$$

2.4 Convolutions

Convolutions were originally introduced in *Number Theory*, but it was soon proved that it was also useful in *Mathematical Analysis*, because the discrete and the continuous formula were of the same structure, and the continuous formula also occurred naturally in solution formulæ. For that reason one kept the vocabulary from *Number Theory*, although it is difficult to see in the continuous case without this historical background why we call this new operation a convolution.

Definition 2.4.1 *Let f and g be two functions, both 0 for $t<0$ negative. The* convolution $f \star g$ *of the two functions is defined by the integral*

$$(f\star g)(t) = \int_{-\infty}^{+\infty} f(t-\tau)g(\tau)\,d\tau, \qquad t\in\mathbb{R},$$

where we also allow infinite values of the integral.

In general, the requirement that the functions are 0 for $t<0$ negative, is not at all necessary. It is only added here, because we are dealing with the Laplace transformation, so unless specified otherwise we only consider in the following functions which are 0 on the negative half axis.

Given the assumptions of Definition 2.4.1, it follows that the integrand

$$f(t-\tau)g(\tau) = 0 \qquad \text{for } \tau\notin[0,1] \qquad \text{for } t>0 \text{ fixed},$$

and of cause for every $\tau\in\mathbb{R}$, if $t\le 0$, because then either $f(t-\tau)=0$ or $g(\tau)=0$. This implies that Definition 2.4.1 can be reduced to

$$(f\star g)(t) = \begin{cases} \int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau & \text{for } t>0, \\ 0 & \text{for } t\le 0. \end{cases}$$

Notice in particular that if both f and g are 0 for $t < 0$, then the convolution $f \star g$ is also 0 for $t < 0$. We have an even better result.

Theorem 2.4.1 *If f, $g \in \mathcal{F}$ are both 0 for $t < 0$, then also $f \star g \in \mathcal{F}$, and the convolution $f \star g$ is 0 for $t < 0$.*

Proof. From f, $g \in \mathcal{F}$ follow the existence of $\sigma > 0$ such that

$$\int_0^{+\infty} |f(t)|e^{-\sigma t}|\,\mathrm{d}t < +\infty \qquad \text{and} \qquad \int_0^{+\infty} |g(t)|e^{-\sigma t}|\,\mathrm{d}t < +\infty,$$

where we of course can use the same $\sigma > 0$ in both integrals.

Using this $\sigma > 0$ we shall prove that also

$$\int_0^{+\infty} |(f \star g)(t)|e^{-\sigma t}|\,\mathrm{d}t < +\infty.$$

This is done in the following way, where we use *Fubini's theorem* from Chapter 1, when we below interchange the order of integration. Notice that $0 \le \tau \le t < +\infty$, when this interchange takes place:

$$\begin{aligned}
\int_0^{+\infty} |(f \star g)(t)|e^{-\sigma t}|\,\mathrm{d}t &= \int_0^{+\infty} \left| \int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau \right| e^{-\sigma t}\,\mathrm{d}t \\
&\le \int_0^{+\infty} \left\{ \int_\tau^{+\infty} |f(t-\tau)| \cdot e^{-\sigma(t-\tau)}\,\mathrm{d}t \right\} \cdot |g(\tau)| \cdot e^{-\sigma\tau}\,\mathrm{d}\tau \\
&= \int_0^{+\infty} |f(u)|e^{-\sigma u}\,\mathrm{d}u \cdot \int_0^{+\infty} |g(\tau)|\,e^{-\sigma\tau}\,\mathrm{d}\tau < +\infty,
\end{aligned}$$

where we also have used the substitution $u = t - \tau$. $\square$

Remark 2.4.1 It is easy to prove (cf. e.g. *Ventus, Complex Functions Theory c-11*) that if f, $g \in \mathcal{E}$, then also $f \star g \in \mathcal{E}$. Notice also that $f \star g \in \mathcal{F}$ implies that $(f \star g)(t)$ is finite almost everywhere. $\Diamond$

The importance of the convolution in connection with the Laplace transformation is described by the following theorem, which also shows that $\star$ behaves like some sort of multiplication between functions in $\mathcal{F}$.

Theorem 2.4.2 Laplace transformation of a convolution.

1) *Convolution is commutative on $\mathcal{F}$. Thus, if f, $g \in \mathcal{F}$, then*

$$(f \star g)(t) = (g \star f)(t) \qquad \textit{almost everywhere.}$$

2) *If f, $g \in \mathcal{F}$, then*

$$\mathcal{L}\{f \star g\}(z) = \mathcal{L}\{f\}(z) \cdot \mathcal{L}\{g\}(z) \qquad \textit{for } \Re z > \max\{\sigma(f), \sigma(g)\}.$$

Thus, the Laplace transformation of a convolution of functions from $\mathcal{F}$ is the pointwise product of their Laplace transforms $\mathcal{L}\{f\}$ and $\mathcal{L}\{g\}$. An analytic extension shows that this rule holds, whenever the product $\mathcal{L}\{f\}(z) \cdot \mathcal{L}\{g\}(z)$ is defined.

Therefore, Theorem 2.4.2 implies that a complicated operation like convolution (which often occurs in solution formulæ) by the Laplace transformation is transformed into a simple pointwise multiplication, where one knows the rules of computation from elementary calculus.

Proof. It follows from the proof of Theorem 2.4.1 that it is allowed to interchange the order of integration, provided that $\Re z > \max\{\sigma(f), \sigma(g)\}$. Then just copy the computation of the proof of Theorem 2.4.1 with trivial modifications to get

$$\begin{aligned}\mathcal{L}\{f \star g\}(z) &= \int_0^{\infty} (f \star g)(t) e^{-zt}\,\mathrm{d}t = \int_0^{+\infty} \left\{ \int_0^t f(t-\tau) g(\tau) \right\} \cdot e^{-zt} \mathrm{d}t \\ &= \int_0^{+\infty} \left\{ \int_{\tau}^{+\infty} f(t-\tau) e^{-z(t-\tau)} \mathrm{d}t \right\} \cdot g(\tau) e^{-z\tau}\,\mathrm{d}\tau \\ &= \int_0^{+\infty} f(u) e^{-zu}\,\mathrm{d}u \cdot \int_0^{+\infty} g(\tau) e^{-z\tau}\,\mathrm{d}\tau = \mathcal{L}\{f\}(z) \cdot \mathcal{L}\{g\}(z).\end{aligned}$$

When f and g are interchanged in the computation above we also get

$$\mathcal{L}\{g \star f\}(z) = \mathcal{L}\{g\}(z) \cdot \mathcal{L}\{f\}(z) = \mathcal{L}\{f\}(z) \cdot \mathcal{L}\{g\}(z) = \mathcal{L}\{f \star g\}(z),$$

because pointwise multiplication is commutative. Using that the Laplace transformation on $\mathcal{F}$ is injective (apart from values of the functions on a null set, a result still to be proved), we finally conclude that

$$(g \star f)(t) = (f \star g)(t) \qquad \text{almost everywhere,}$$

so the convolution is commutative. □

Example 2.4.1 It was proved in Example 2.3.7 that the inverse Laplace transform of $F(z) = \dfrac{z}{\left(z^2+1\right)^2}$ was given by $\dfrac{1}{2}\, t \sin t$. From Example 2.3.6 follows that the inverse Laplace transform of $\dfrac{1}{z^2+1}$ is $\sin t$, and from Example 2.3.5 follows that the inverse Laplace transform of $\dfrac{z}{z^2+1}$ is $\cos t$. Hence,

$$\mathcal{L}\{\sin \star \cos\}(z) = \mathcal{L}\{\sin\}(z) \cdot \mathcal{L}\{\cos\}(z) = \frac{z}{\left(z^2+1\right)^2} = \mathcal{L}\left\{\frac{1}{2}\, t \sin t\right\}(z).$$

We therefore conclude that

$$(\sin \star \cos)(t) = \frac{1}{2}\, t \sin t \qquad \text{for } t > 0,$$

where we have written $\sin t$ and $\cos t$ instead of the more correct $\chi_{\mathbb{R}_+}(t) \cdot \sin t$ and $\chi_{\mathbb{R}_+}(t) \cdot \cos t$. In this simple case it is not hard to prove that this result is indeed correct. In fact,

$$
\begin{aligned}
(\sin\star\cos)(t) \;&:=\; \int_0^t \sin(t-\tau)\cos\tau\,\mathrm{d}\tau = \int_0^t \{\sin t\cdot\cos\tau - \cos t\cdot\sin\tau\}\cos\tau\,\mathrm{d}\tau \\
&= \sin t\cdot\int_0^t \cos^2\tau\,\mathrm{d}\tau - \cos t\int_0^t \sin\tau\cdot\cos\tau\,\mathrm{d}\tau = \sin t\int_0^t \frac{1+\cos 2\tau}{2}\,\mathrm{d}\tau - \cos\cdot\left[\sin^2\tau\right]_0^t \\
&= \frac{1}{2}\,t\,\sin t + \sin t\cdot\left[\frac{\sin 2\tau}{4}\right]_0^t - \frac{1}{2}\,\cos t\cdot\sin^2 t \\
&= \frac{1}{2}\,t\,\sin t + \frac{1}{4}\,\sin t\cdot\sin 2t - \frac{1}{4}\,\sin t\cdot\sin 2t = \frac{1}{2}\,t\,\sin t. \qquad \Diamond
\end{aligned}
$$

Example 2.4.2 The *Bessel function* $J_0(t)$ *of order 0* is defined by the following series which is convergent everywhere in $\mathbb{C}$,

(28) $$J_0(t) := \sum_{n=0}^{+\infty} (-1)^n \frac{t^{2n}}{2^{2n}(n!)^2} = \sum_{n=0}^{+\infty} (-1)^n \frac{1}{n!\,n!} \left\{\frac{t}{2}\right\}^{2n}, \qquad t \in \mathbb{C}.$$

It follows by termwise differentiation that

(29) $$J_0'(t) = \sum_{n=0}^{+\infty} (-1)^n \frac{2n}{2^{2n}(n!)^2}\, t^{2n-1} \qquad \text{and} \qquad J_0''(t) = \sum_{n=0}^{+\infty} (-1)^n \frac{2n(2n-1)}{2^{2n}(n!)^2}\, t^{2n-2}.$$

1) We shall prove that $J_0(t)$ is a *power series solution* of the *Bessel differential equation of order 0*,

(30) $$t \cdot J_0''(t) + J_0'(t) + t \cdot J_0(t) = 0.$$

This follows from the computation

$$\begin{aligned}
&t \cdot J_0''(t) + J_0'(t) + t \cdot J_0(t) \\
&= \sum_{n=1}^{+\infty} (-1)^n \frac{2n(2n-1)}{2^{2n}(n!)^2}\, t^{2n-1} + \sum_{n=1}^{+\infty} (-1)^n \frac{2n}{2^{2n}(n!)^2}\, t^{2n-1} + \sum_{n=0}^{+\infty} (-1)^n \frac{1}{2^{2n}(n!)^2}\, t^{2n+1} \\
&= \sum_{n=1}^{+\infty} (-1)^n \frac{2n \cdot 2n}{2^{2n}(n!)^2}\, t^{2n-1} + \sum_{n=0}^{+\infty} (-1)^n \frac{1}{2^{2n}(n!)^2}\, t^{2n+1} \\
&= \sum_{n=1}^{+\infty} (-1)^n \frac{1}{2^{2(n-1)}((n-1)!)^2}\, t^{2(n-1)+1} + \sum_{n=0}^{+\infty} (-1)^n \frac{1}{2^{2n}(n!)^2}\, t^{2n+1} \\
&= \sum_{n=0}^{+\infty} (-1)^{n+1} \frac{1}{2^{2n}(n!)^2}\, t^{2n+1} + \sum_{n=0}^{+\infty} (-1)^n \frac{1}{2^{2n}(n!)^2}\, t^{2n+1} = 0,
\end{aligned}$$

and (30) is proved.

It follows from (28) and (29) that $J_0(0) = 1$ and $J_0'(0) = 0$.

2) We shall then prove that $|J_0(t)| \le 1$ for every $t \in \mathbb{R}$. This is done by proving below the integral representation

(31) $$J_0(t) = \frac{1}{\pi} \int_0^\pi \cos(t \cdot \sin\Theta)\, d\Theta \qquad \text{for } t \in \mathbb{R},$$

because (31) would imply the following trivial estimate

$$|J_0(t)| \le \frac{1}{\pi} \int_0^\pi |\cos(t \cdot \sin\Theta)|\, d\Theta \le \frac{1}{\pi} \int_0^\pi 1\, d\Theta = 1 \qquad \text{for all } t \in \mathbb{R}.$$

Define

$$\varphi(t) := \frac{1}{\pi} \int_0^\pi \cos(t \cdot \sin\Theta)\, d\Theta \qquad \text{for } t \in \mathbb{R}.$$

We shall prove that $\varphi(t)$ fulfils the Bessel equation (30) and the initial conditions $\varphi(0) = 1$ and $\varphi'(0) = 0$, because then it follows from the *Existence and Uniqueness Theorem of Linear Ordinary Differential Equations of Second Order* that $\varphi(t) = J_0(t)$.

First we get by differentiating under the integral sign that

$$\varphi'(t) = -\frac{1}{\pi}\int_0^\pi \sin(t\cdot\sin\Theta)\sin\Theta\,\mathrm{d}\Theta,$$

$$\varphi''(t) = -\frac{1}{\pi}\int_0^\pi \cos(t\cdot\sin\Theta)\sin^2\Theta\,\mathrm{d}\Theta.$$

In particular,

$$\varphi(0) = \frac{1}{\pi}\int_0^\pi \cos 0\,\mathrm{d}\Theta = 1 \qquad \text{and} \qquad \varphi'(0) = -\frac{1}{\pi}\int_0^\pi \sin 0\,\mathrm{d}\Theta = 0,$$

and $\varphi(t)$ fulfils the two boundary conditions.

Then by insertion into the Bessel equation (30),

$$\begin{aligned}
&t\,\varphi''(t) + \varphi'(t) + t\,\varphi(t)\\
&\quad = -\frac{t}{\pi}\int_0^\pi \cos(t\cdot\sin\Theta)\sin^2\Theta\,\mathrm{d}\Theta - \frac{1}{\pi}\int_0^\pi \sin(t\cdot\sin\Theta)\,\mathrm{d}\Theta + \frac{t}{\pi}\int_0^\pi \cos(t\,\sin\Theta)\,\mathrm{d}\Theta\\
&\quad = \frac{t}{\pi}\int_0^\pi \cos(t\cdot\Theta)\cos^2\Theta\,\mathrm{d}\Theta + \frac{1}{\pi}\int_{\Theta=0}^\pi \sin(t\cdot\sin\Theta)\,\mathrm{d}\cos\Theta\\
&\quad = \frac{1}{\pi}\int_{\Theta=0}^\pi (\cos\,\mathrm{d}\{\sin(t\cdot\sin\Theta)\} + \sin(t\cdot\sin\Theta)\,\mathrm{d}\cos\Theta)\\
&\quad = \frac{1}{\pi}\int_{\Theta=0}^\pi \mathrm{d}\{\cos\Theta\cdot\sin(t\cdot\sin\Theta)\} = \frac{1}{\pi}\left[\cos\Theta\cdot\sin(t\cdot\sin\Theta)\right]_{\Theta=0}^\pi = 0.
\end{aligned}$$

We have proved that $\varphi(t)$ is a solution of the Bessel equation of order 0 satisfying the same boundary conditions as $J_0(t)$. Hence, $\varphi(t) = J_0(t)$ by the *Existence and Uniqueness Theorem of Ordinary Differential Equations*, and we have proved (31), from which follows, as already shown above, that $|J_0(t)| \le 1$ for $t \in \mathbb{R}$.

3) Finally, the Laplace transform $\mathcal{L}\{J_0\}(z)$ of $J_0(t)$ is found by applying the Laplace transformation on the Bessel equation (30) and then use the various rules of computation derived in Section 2.2, i.e. "multiplication by t", "differentiation", linearity and formula (23), which we repeat here,

$$\mathcal{L}\{f''\}(z) = z^2\mathcal{L}\{f\}(z) - z\cdot f(0) - f'(0).$$

The computation goes as follows,

$$\begin{aligned}
0 &= \mathcal{L}\left\{t\,J_0''(t)+J_0'(t)+t\,J_0(t)\right\}=\mathcal{L}\left\{t\,J_0''(t)\right\}+\mathcal{L}\left\{J_0'(t)\right\}+\mathcal{L}\left\{J_0(t)\right\}\\
&= -\frac{d}{dz}\,\mathcal{L}\left\{J_0''\right\}+\mathcal{L}\left\{J_0'\right\}-\frac{d}{dz}\,\mathcal{L}\left\{J_0\right\}\\
&= -\frac{d}{dz}\left\{z^2\mathcal{L}\left\{J_0\right\}(z)-z\cdot J_0(0)-J_0'(0)\right\}+\left\{z\cdot\mathcal{L}\left\{J_0\right\}(z)-J_0(0)\right\}-\frac{d}{dz}\,\mathcal{L}\left\{J_0\right\}(z)\\
&= -\frac{d}{dz}\left\{z^2\mathcal{L}\left\{J_0\right\}(z)+\mathcal{L}\left\{J_0\right\}(z)\right\}+1+z\mathcal{L}\left\{J_0\right\}(z)-1\\
&= -2z\cdot\mathcal{L}\left\{J_0\right\}(z)-\left(z^2+1\right)\frac{d}{dz}\,\mathcal{L}\left\{J_0\right\}(z)+z\cdot\mathcal{L}\left\{J_0\right\}(z)\\
&= -\left(z^2+1\right)\frac{d}{dz}\,\mathcal{L}\left\{J_0\right\}(z)-z\cdot\mathcal{L}\left\{J_0\right\}(z).
\end{aligned}$$

Choose $\sqrt{\cdot}$ as the branch of the square root, which has its branch cut lying along the negative real half axis, and for which $\sqrt{1}=+1$, cf. Figure 6, and *Ventus, Complex Functions Theory a-3.*

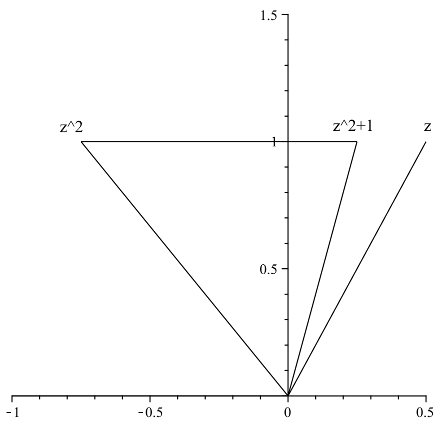

Figure 6: Domain of the chosen branch of the square root.

If $\Re\, z>0$, then $\sqrt{z^2+1}$ is uniquely determined, and $\sqrt{z^2+1}\neq 0$ in this right half plane. Thus, by dividing the equation above by $-\sqrt{z^2+1}$ we get for $\Re\, z>0$,

$$0=\sqrt{z^2+1}\,\mathrm{d}z\,\mathcal{L}\left\{J_0\right\}+\frac{1}{\sqrt{z^2+1}}\,\mathcal{L}\left\{J_0\right\}=\frac{d}{dz}\left\{\sqrt{z^2+1}\cdot\mathcal{L}\left\{J_0\right\}(z)\right\},$$

hence by integration, $\sqrt{z^2+1}\cdot\mathcal{L}\left\{J_0\right\}(z)=c$, i.e.

$$\mathcal{L}\left\{J_0\right\}(z)=\frac{c}{\sqrt{z^2+1}},\qquad \Re\, z>0.$$

The constant c is found by an application of the *initial value theorem*. Choosing $z = x > 0$ real and positive, we get

$$1 = \lim_{t \to 0+} J_0(t) = \lim_{x \to +\infty} x\,\mathcal{L}\left\{J_0\right\}(x) = \lim_{x \to +\infty} \frac{cx}{\sqrt{x^2+1}} = c,$$

thus $c = 1$, and we conclude that

$$\mathcal{L}\left\{J_0\right\}(z) = \frac{1}{\sqrt{z^2+1}}, \qquad \Re\, z > 0.$$

The argument above is a typical application of the *initial value theorem*. In general, the Laplace transformation transforms some linear differential equation in t into another linear differential equation in z, which hopefully is simpler, so it can easily be solved. The complete solution of the transformed equation will of course contain arbitrary constants, which usually are determined, either by the *initial value theorem* or the *final value theorem*, depending on the conditions on the original function $f(t)$.

4) As an application of the result above we see that it then follows immediately from Theorem 2.4.2 that

$$\mathcal{L}\left\{J_0 \star J_0\right\}(z) = \frac{1}{\sqrt{z^2+1}} \cdot \frac{1}{\sqrt{z^2+1}} = \frac{1}{z^2+1} = \mathcal{L}\{\sin\}(z), \qquad \Re z > 0.$$

Then it follows from the injectivity of the Laplace transformation that

$$(32) \quad \left(J_0 \star J_0\right)(t) = \sin t \qquad \text{for almost every } t \geq 0,$$

and since both $J_0(t)$ and $\sin t$ are continuous, (32) must hold for every $t \geq 0$. In other words,

$$\int_0^t J_0(t-\tau) J_0(\tau)\, \mathrm{d}\tau = \sin t \qquad \text{for } t \geq 0. \qquad \Diamond$$

2.5 Linear ordinary differential equations

In many cases linear ordinary differential equations are easily solved by using the Laplace transformation. The reader should, however, be aware of that the more primitive methods from *Elementary Calculus* must not be forgotten, because they sometimes give the solution in an even easier way.

We shall here describe the solution method by means of the Laplace transformation.

We shall without any justification *assume* that there exist solutions from $\mathcal{F}$, and we shall aim at finding a solution formula, in which the Laplace transformation enters. Once we have found a *candidate* of the solution, we check it by insertion into the differential equation and the boundary conditions, because this check will justify our computations.

From time to time we may obtain "solutions" which are not differentiable, so they do not fulfil the differential equation in the usual sense. We shall consider these as *generalized solutions*. A further investigation of these will lead to the *Theory of Distributions*, from which we shall only consider *Dirac's δ "function"* in a few cases.

The method is based on the result

$$(33) \quad \mathcal{L}\{f'\}(z) = z\,\mathcal{L}\{f\}(z) - f(0), \qquad \text{for } f \in \mathcal{F},$$

proved in Section 2.2. We mention in this connection that if $f \in \mathcal{F}$ is only *continuous* for $t \to 0+$, then $f'(t)$ does not exist, but ... The right hand side of (33) makes sense! Therefore, (33) can in such cases be considered as the *definition* of $\mathcal{L}\{f'\}(z)$, where f' here means *differentiation in a generalized sense*, fixed by equation (33). In scientific applications this extension of the differentiability is very convenient, as long as one does not speculate too much about the difference between the usual differentiation and differentiation in this generalized sense.

Example 2.5.1 We shall solve the differential equation with boundary value conditions,

$$(34) \quad \begin{cases} \varphi''(t) + 4\varphi'(t) + 3\varphi(t) = 0 & \text{for } t \geq 0 \\[2mm] \varphi(0) = 0 \qquad \text{and} \qquad \varphi'(0) = 1. \end{cases}$$

1) *The sophisticated method.* Rewrite the equation in the following way:

$$\varphi''(t)+4\varphi'(t)+3\varphi(t)=\{\varphi''(t)+3\varphi'(t)\}+\{\varphi'(t)+3\varphi(t)\}$$

$$=e^{-t}\,\frac{d}{dt}\left\{e^{t}(\varphi'(t)+3\varphi(t))\right\}=e^{-t}\,\frac{d}{dt}\left\{e^{-2t}\,\frac{d}{dt}\left(e^{3t}\varphi(t)\right)\right\},$$

so by reduction and two succeeding integrations,

$$\frac{d}{dt}\left\{e^{-2t}\,\frac{d}{dt}\left(e^{3t}\varphi(t)\right)\right\}=0,$$

$$e^{-2t}\,\frac{d}{dt}\left(e^{3t}\varphi(t)\right)=2C_1,\qquad\text{thus}\qquad\frac{d}{dt}\left(e^{3t}\varphi(t)\right)=2C_1e^{2t},$$

$$e^{3t}\varphi(t)=C_2+C_1e^{2t},\qquad\text{thus}\qquad\varphi(t)=C_1e^{-t}+C_2e^{-3t},$$

where C_1 and C_2 are arbitrary constants. Then it follows from the initial conditions that

$$\varphi(0)=C_1+C_2=0\qquad\text{and}\qquad\varphi'(t)=-C_1-3C_2=1,$$

so $C_2=-\dfrac{1}{2}$ and $C_1=\dfrac{1}{2}$, and the solution of (34) becomes

$$\varphi(t)=\frac{1}{2}\left\{e^{-t}-e^{-3t}\right\}=e^{-2t}\sinh t.$$

2) *The classical method.* The differential equation of (34) is linear and homogeneous of constant coefficients. It therefore has a *characteristic polynomial*, which is obtained by replacing the derivative $\varphi^{(j)}(t)$ by λ^j, where $\lambda\in\mathbb{C}$. We get

$$P(\lambda)=\lambda^2+4\lambda+3=(\lambda+1)(\lambda+3).$$

The characteristic polynomial $P(\lambda)$ has the roots $\lambda=-1$ and $\lambda=-3$, so the complete solution of the differential equation is given by

$$\varphi(t)=C_1e^{-t}+C_2e^{3t},\qquad C_1,\,C_2\text{ arbitrary constants.}$$

Then it follows from the initial conditions that

$$\varphi(0)=C_1+C_2=0\qquad\text{and}\qquad\varphi'(t)=-C_1-3C_2=1,$$

so $C_2=-\dfrac{1}{2}$ and $C_1=\dfrac{1}{2}$, and the solution of (34) becomes

$$\varphi(t)=\frac{1}{2}\left\{e^{-t}-e^{-3t}\right\}=e^{-2t}\sinh t.$$

3) *The method of Laplace transformation.* Since $\varphi(0)=0$ and $\varphi'(0)=1$, we get

$$\mathcal{L}\{\varphi'\}(z)=z\,\mathcal{L}\{\varphi\}(z)-\varphi(0)=z\,\mathcal{L}\{\varphi\}(z),$$

and

$$\mathcal{L}\{\varphi''\}(z)=z\,\mathcal{L}\{\varphi'\}(z)-\varphi'(0)=z^2\mathcal{L}\{\varphi\}(z)-1.$$

Thus an application of the Laplace transformation on (34) gives

$$\left(z^2\mathcal{L}\{\varphi\}(z)-1\right)+4z\mathcal{L}\{\varphi\}(z)+3\mathcal{L}\{\varphi\}=0,$$

or, by a rearrangement of this equation,

$$\left(z^2+4z+3\right)\mathcal{L}\{\varphi\}(z)=1.$$

We notice that the factor z^2+4z+3 is precisely the *characteristic polynomial* found by the second method, so the characteristic polynomial seems inevitable.

Then we get for $z \neq -1$ and $z \neq -3$,

$$\mathcal{L}\{\varphi\}(z)=\frac{1}{z^2+4z+3}=\frac{1}{(z+1)(z+3)}.$$

The poles $z=-1$ and $z=-3$ are simple, so we get by Corollary 2.3.2 that

$$\varphi(t)=\operatorname{res}\left(\frac{e^{zt}}{(z+1)(z+3)};-1\right)+\operatorname{res}\left(\frac{e^{zt}}{(z+1)(z+3)};-3\right)=\frac{e^{-1}}{2}-\frac{e^{-3t}}{2}=e^{-2t}\sinh t.$$

It is up to the reader to decide which method is the easiest one to apply. $\Diamond$

Example 2.5.2 We define the *Heaviside function* $H(t)$ by

$$H(t)=\begin{cases}1 & \text{for } t \geq 0,\\ \star & \text{for } t=0,\\ 0 & \text{for } t \leq 0,\end{cases}$$

where $\star$ can be any real number. Since $\{0\}$ is a null set, the choice of the value of $\star$ will not influence the rest of the example.

We shall solve the equation

$$\varphi'(t)-\varphi(t)=H(t-1) \qquad \text{for } t \geq 0 \text{ and } \varphi(0)=0.$$

Notice that $H(t-1)$ has a discontinuity at $t=1$, so one should be careful using elementary calculus methods, although the method as shown below is possible.

1) One solution method is to multiply by $e^{-t}>0$, because then

$$e^{-t}H(t-1)=e^{-t}\varphi'(t)-e^{-t}\varphi(t)=\frac{d}{dt}\left\{e^{-t}\varphi(t)\right\},$$

or

$$\frac{d}{dt}\left\{e^{-t}\varphi(t)\right\}=\begin{cases}e^{-t} & \text{for } t \geq 1,\\ 0 & \text{for } t<1.\end{cases}$$

We get by integration,

$$e^{-t}\varphi(t) = \begin{cases} c_1 - e^{-t} & \text{for } t \geq 1, \\ \\ c_2 & \text{for } t < 1. \end{cases}$$

It follows from $\varphi(0) = 0$ that $c_2 = 0$. Then by continuity,

$$e^{-1}\varphi(1) = 0 = c_1 - e^{-1}, \qquad \text{hence } c_1 = e^{-1},$$

and

$$e^{-t}\varphi(t) = \begin{cases} e^{-1} - e^{-t} & \text{for } t \geq 1, \\ \\ 0 & \text{for } t < 1, \end{cases}$$

from which

$$\varphi(t) = \begin{cases} e^{t-1} - 1 & \text{for } t \geq 1, \\ \\ 0 & \text{for } t < 1. \end{cases}$$

2) We shall now see what happens, if we instead apply the Laplace transformation, where we *define* f' by the equation

$$\mathcal{L}\{\varphi'\}(z) := z\mathcal{L}\{\varphi\}(z) - \varphi(0) = z\mathcal{L}\{\varphi\}(z).$$

We get

$$\begin{aligned} z\mathcal{L}\{\varphi\}(z) - \mathcal{L}\{\varphi\}(z) &= (z-1)\mathcal{L}\{\varphi\}(z) = \mathcal{L}\{H(t-1)\varphi(t)\}(z) \\ &= \int_1^{+\infty} e^{-tz}\,\mathrm{d}z = \frac{e^{-z}}{z} \qquad \text{for } \Re z > 0, \end{aligned}$$

so

$$\mathcal{L}\{\varphi\}(z) = \frac{e^{-z}}{z(z-1)}.$$

The inverse Laplace transform of $\dfrac{1}{z(z-1)} = \dfrac{1}{z-1} - \dfrac{1}{z}$ is $e^{-t} - 1$, and the factor e^{-z} implies according to Theorem 2.2.3, *Second translation or shifting property*, that the variable t is here replaced by $t-1$, so we get the solution

$$\varphi(t) = \begin{cases} e^{t-1} - 1 & \text{for } t \geq 1, \\ 0 & \text{for } t < 1. \end{cases} \qquad \Diamond$$

Example 2.5.3 We shall by the Laplace transformation find a solution formula for the linear inhomogeneous initial value problem

$$\begin{cases} \varphi''(t) + 2\varphi'(t)0 + 2\varphi(t) = f(t), \\ \varphi(0) = \varphi'(0) = 0, \end{cases}$$

for general $f \in \mathcal{F}$.

By the Laplace transformation,

$$z^2\mathcal{L}\{\varphi\}(z) + 2z\mathcal{L}\{\varphi\}(z) + 2\mathcal{L}\{\varphi\}(z) = \mathcal{L}\{f\}(z),$$

so

$$\mathcal{L}\{\varphi\}(z) = \frac{\mathcal{L}\{f\}(z)}{z^2+2z+2}.$$

The inverse Laplace transform of

$$\frac{1}{z^2+2z+2} = \frac{1}{(z+1)^2+1}$$

is $g(t) = e^{-t}\sin t$, hence

$$\mathcal{L}\{\varphi\}(z) = \mathcal{L}\{g\}(z) \cdot \mathcal{L}\{f\}(z),$$

so we get from Theorem 2.4.2 concerning convolutions that

$$\begin{aligned}
\varphi(t) &= (g \star g)(t) = \int_0^t g(t-\tau) f(\tau)\,\mathrm{d}\tau = \int_0^t e^{-t+\tau}\sin(t-\tau) f(\tau)\,\mathrm{d}\tau \\
&= e^{-t}\sin t \int_0^t e^{\tau}\cos\tau \cdot f(\tau)\,\mathrm{d}\tau - e^{-t}\cos t \int_0^t e^{\tau}\sin\tau \cdot f(\tau)\,\mathrm{d}\tau.
\end{aligned} \tag{35}$$

We see that the structure of the solution is that of a convolution. It can be proved that this is always the case for linear ordinary inhomogeneous differential equations.

Notice that (35) is the usual solution formula in this specific example, when we use the *Wroński method*. $\Diamond$

If the linear ordinary differential equation has variable constants, one can only hope for a successful application of the Laplace transformation, when the coefficients are at most polynomials of degree at most 2. This was actually the case in Example 2.4.2, where we found the Laplace transform $\mathcal{L}\{J_0\}(z)$ by transforming the Bessel function (30), i.e.

$$t\,J_0''(t) + J_0'(t) + t\,J_0(t) = 0,$$

and then derive the differential equation

$$\left(z^2+1\right)\frac{d}{dz}\,\mathcal{L}\{J_0\}(z) + z\,\mathcal{L}\{J_0\}(z) = 0.$$

The order of the differential equation of $\mathcal{L}\{J_0\}(z)$ is the same as the highest degree of the polynomial coefficients of the original equation.

At the same time it becomes obvious that if the coefficients of the governing equation are *not* polynomials or constants, then there is almost no hope of solving it by means of the Laplace transformation.

3 Other transformations and the general inversion formula

3.1 The two-sided Laplace transformation

We proved in Chapter 2 some computational rules for the Laplace transformation. Among these the Laplace transformation of a derivative, i.e.

(36) $\mathcal{L}\{f'\}(z) = z\,\mathcal{L}\{f\}(z) - f(0) \qquad \text{for } f \in \mathcal{E},$

where we assume that f is piecewise of class C^1, and $f(0) = \lim_{t\to 0+} f(t)$ exists, has a strange mathematical structure. In particular, two facts are indeed odd.

1) If $f \in \mathcal{F}$, then $z\,\mathcal{L}\{f\}(z)$ is well-defined, while the classical definition of the derivative,

   (37) $f'(t_0) = \displaystyle\lim_{t\to t_0} \frac{f(t) - f(t_0)}{t - t_0}$

   is not sufficient general in all cases to specify f' as a well-defined element. It is, however, possible to give a convenient definition of f' as a so-called *generalized function* such that (36) makes sense, while (37) may not be defined. It is unfortunately not possible here to go through the necessary arguments from *Distribution Theory*, so we shall only use heuristic considerations, when we are dealing with e.g. the δ "function", which is the simplest such element which is not an ordinary function.

2) The second problem of (36) is of course the constant $-f(0)$, where f is not transformed, and 0 is a t-value. Thus, (36) contains both the function f itself and its Laplace transform $\mathcal{L}\{f\}$ as well as the variables z and t, so in this sense formula (36) is rather messy. The reason is of course the contribution from the lower bound $t = 0$, when $\int_0^{+\infty} e^{-zt} f'(t)\,\mathrm{d}t$ is integrated partially.

The easiest way to avoid the second problem pointed out above is to extend the integration to all of $\mathbb{R}$, because then the lower bound becomes $-\infty$. We shall in this section consider this possibility, so we introduce

Definition 3.1.1 *Let $f(t)$, $t \in \mathbb{R}$, be a (measurable) function. Assume that there is a non-empty open domain Ω in CC, such that*

$$\int_{-\infty}^{+\infty} e^{-zt} f(t)\,dt \qquad \textit{is convergent for all } z \in \Omega.$$

Then we say that $f(t)$ has a two-sided Laplace transform $\mathcal{L}_2\{f\}(z)$ *in Ω, and it is defined by*

$$\mathcal{L}_2\{f\}(z) := \int_{-\infty}^{+\infty} e^{-zt} f(t)\,dt \qquad \textit{for } z \in \Omega.$$

If f has a two-sided Laplace transform, then

(38) $\mathcal{L}_2\{f\}(z) = \displaystyle\int_{-\infty}^{+\infty} e^{-zt} f(t)\,\mathrm{d}t = \int_{-\infty}^{0} e^{-zt} f(t)\,\mathrm{d}t + \int_{0}^{+\infty} e^{-zt} f(t)\,\mathrm{d}t,$

where the two improper integrals are both convergent. The latter integral on the right hand side of (38) is the usual Laplace transform of $f(t)H(t)$, where

$$H(t) := \begin{cases} 1 & \text{for } t \geq 0, \\ 0 & \text{for } t < 0, \end{cases}$$

denotes the *Heaviside function*. Hence, the domain of convergence of the latter integral is the usual half plane of convergence $\Re z > a$, possibly all of $\mathbb{C}$, and the integral representation is an analytic function for $\Re z > a$.

The former integral is treated similarly, only t is negative in the domain of integration, hence the half plane of convergence must be given by $\Re z < b$, possibly all of $\mathbb{C}$.

If z fulfils $a < \Re z < b$, then both improper integrals on the right hand side of (38) are convergent. Hence, if $a < b$, then $f(t)$ has a two-sided Laplace transform given by

$$(39)\quad \mathcal{L}_2\{f\}(z) = \int_{-\infty}^{+\infty} e^{-zt} f(t)\,\mathrm{d}t \qquad \text{for } \Re z \in \,]a, b[,$$

and the domain of convergence is usually a vertical parallel strip in $\mathbb{C}$ with $a = -\infty$ or $b = +\infty$ as trivial exceptions.

When z belongs to the strip of convergence, then $\mathcal{L}_2\{f\}(z)$ is an analytic function of z so we may again have the possibility of an analytic extension of $\mathcal{L}_2\{f\}(z)$ to larger subsets of $\mathbb{C}$.

Remark 3.1.1 In some sense the extension above of the laplace transformation to the two-sided Laplace transformation resembles the extension of Taylor series to Laurent series. We know e.g. that an analytic function have different Laurent expansions in different annuli. Concerning the two-sided Laplace transformation, the same analytic function $F(z)$ may be the two-sided Laplace transform of two different functions when restricted to two different strips. Therefore, if we want an inversion formula of the two-sided Laplace transformation, we must always specify the strip of convergence. $\Diamond$

Example 3.1.1 Let $a > 0$, and put $f(t) = e^{-a|t|}$. We shall find the two.sided Laplace transform of f. Assuming the convergence of the improper integrals we get

$$\mathcal{L}_2\{f\}(z) = \int_{-\infty}^{0} e^{-zt}e^{at}\,\mathrm{d}t + \int_{0}^{+\infty} e^{-zt}e^{-at}\,\mathrm{d}t = \left[\frac{1}{a-z}\,e^{(a-z)t}\right]_{-\infty}^{0} + \left[-\frac{1}{a+z}\,e^{-(a+z)t}\right]_{0}^{+\infty}.$$

It follows that the conditions of convergence is $\Re\,(a-z) > 0$ and $\Re\,(-(a+z)) < 0$, hence $\Re\,z \in \,]-a,a[$, and when this is fulfilled, then

$$\mathcal{L}_2\{f\}(z) = \frac{1}{a-z} + \frac{1}{a+z} = \frac{2a}{a^2-z^2} \qquad \text{for } |\Re\,z| < a.$$

Choose instead $g(t) = -2\sinh(at)\cdot H(t)$. Then

$$\mathcal{L}_2\{g\}(z) = -2\mathcal{L}\{\sinh(at)\}(z) = \frac{2a}{a^2-z^2} \qquad \text{for } \Re\,z > a,$$

because $g(t) = 0$ for $t < 0$, and where we have used Example 2.1.4.

Finally, if $h(t) = 2\sinh(at)\cdot H(-t)$, then

$$\mathcal{L}_2\{h\}(z) = 2\int_{-\infty}^{0} \sinh(at)e^{-zt}\,\mathrm{d}t = \frac{2a}{a^2-z^2} \qquad \text{for } \Re\,z < -a.$$

The three totally different functions $f(t)$, $g(t)$ and $h(t)$ have formally the same two-sided Laplace transform $\dfrac{2a}{a^2-z^2}$, where the only difference is the specification of their disjoint strips of convergence. The danger here is that if each of the three two-sided Laplace transforms is extended analytically to its largest possible domain $\mathbb{C}\setminus\{-a,a\}$, then they are identical. Fortunately, it can be shown that if one is given the analytic function

$$F(z) = \frac{2a}{a^2-z^2} \qquad \text{for } z \in \Omega = \mathbb{C}\setminus\{-a,a\},$$

and then restrict it to one of the three possible vertical strips

$$\Re\,z < -a, \qquad -a < \Re\,z < a, \qquad \Re\,z > a,$$

contained in Ω, then we get the right function

$$h(t), \qquad f(t), \qquad g(t) \qquad \text{resp.}$$

as inverse two-sided Laplace transform.

There is, however, no need here to derive a theory of the inverse two-sided Laplace transformation $\mathcal{L}_2^{-1}$, because it can be derived from the general inverse Laplace transformation, presented in Section 3.5. $\Diamond$

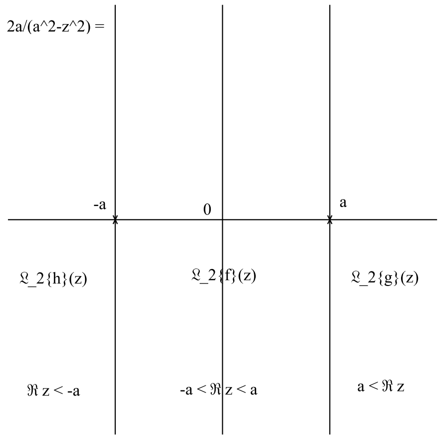

Figure 7: Three possible functions having formally the same two-sided Laplace transform

$$F(z) = \frac{2a}{a^2 - z^2}.$$

Remark 3.1.2 We computed in Example 3.1.1 directly the two-sided Laplace transform of a given function. In general, it is more standard to use the change of variable $\tau = -t$ in one of the integrals, because then

$$\begin{aligned}\mathcal{L}_2\{f\}(z) &= \int_{-\infty}^{0} e^{-zt} f(t)\,\mathrm{d}t + \int_{0}^{+\infty} e^{-zt} f(t)\,\mathrm{d}t = \int_{0}^{+\infty} e^{z\tau} f(-\tau)\,\mathrm{d}\tau + \int_{0}^{+\infty} e^{-zt} f(t)\,\mathrm{d}t \\ &= \int_{0}^{+\infty} e^{-zt} f(t)\,\mathrm{d}t + \int_{0}^{+\infty} e^{-(-z)t} f(-t)\,\mathrm{d}t = \mathcal{L}\{f(t)\}(z) + \mathcal{L}\{f(-t)\}(-z),\end{aligned}$$

where $\mathcal{L}\{f\}$ denotes the usual one-sided Laplace transform computed as an integral over $[0,+\infty[$. This formula is often used, when one explicitly computes $\mathcal{L}_2\{f\}(z)$. $\diamondsuit$

3.2 The Fourier transformation

If $a = b$ in (39), page 64 in Section 3.1, then the strip of convergence is obviously empty. Nevertheless, the improper integral

$$\int_{-\infty}^{+\infty} e^{-zt} f(t)\,\mathrm{d}t = \int_{-\infty}^{+\infty} e^{-at} e^{-i\xi t} f(t)\,\mathrm{d}t = \int_{-\infty}^{+\infty} e^{-i\xi t}\left\{e^{-at} f(t)\right\}\,\mathrm{d}t$$

may still be convergent for $z = a + i\xi$, $\xi \in \mathbb{R}$. This integral cannot in general be extended analytically in the variable ξ. Now, the real part a of z is fixed, so by writing $f(t)$ instead of $e^{-at} f(t)$ it follows that we may assume that $a = 0$. This heuristic analysis shows that it may also be of interest to consider another transformation of the form

$$\mathcal{F}\{f\}(\xi) := \int_{-\infty}^{+\infty} e^{-i\xi t} f(t)\,\mathrm{d}t, \qquad \text{for } \xi \in \mathbb{R}.$$

This is the definition of the *Fourier transformation*, which we in general shall consider in this section.

Definition 3.2.1 *Let $f(x)$ be a (measurable) function. If the improper integral $\int_{-\infty}^{+\infty} e^{-ix\xi} f(x)\,dx$ is convergent for almost every $\xi \in \mathbb{R}$, then f has a* Fourier transform, *denoted by either $\mathcal{F}\{f\}(\xi)$ or by $\hat{f}(\xi)$, and it is given by*

$$\text{(40)}\quad \hat{f}(i) = \mathcal{F}\{f\}(\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi} f(x)\,dx \qquad \text{for almost every } \xi \in \mathbb{R}.$$

Remark 3.2.1 We have in (40) used the most common definition of the Fourier transformation. The reader should, however, be aware of that other definitions may also be met, like e.g.

$$\int_{-\infty}^{+\infty} e^{ix\xi} f(x)\,\mathrm{d}x, \qquad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ix\xi} f(x)\,\mathrm{d}x, \qquad \int_{-\infty}^{+\infty} e^{-2\pi tx\xi} f(x)\,\mathrm{d}x.$$

No matter which definitions is used, the *qualitative* results will always be the same, but we may have different constants involved. In case of using another definition the reader should start with identifying the necessary constants. $\Diamond$

Most of the rules of computations of the Laplace transformation have an analogue for the Fourier transformation. It is, however, obvious that the Fourier transform of a periodic function is only defined in the trivial case, when f is a null function.

Theorem 3.2.1 Linearity. *Assume that f and g both have a Fourier transform. Let λ, $\mu \in \mathbb{C}$ be complex constants. Then $\lambda f + \mu g$ have a Fourier transform, which is given by*

$$\mathcal{F}\{\lambda f + \mu g\}(\xi) = \lambda \mathcal{F}\{f\}(\xi) + \mu \mathcal{F}\{g\}(\xi) \qquad \text{for almost every } \xi \in \mathbb{R}.$$

Proof. Let $\mathcal{F}\{f\}(\xi)$ be defined for $\xi \in \mathbb{R} \setminus N_1$, and let $\mathcal{F}\{f\}(\xi)$ be defined for $\xi \in \mathbb{R} \setminus N_2$, where N_1 and N_2 are null sets. Then $N_1 \cup N_2$ is a null set, and when $\xi \in \mathbb{R} \setminus \{N_1 \cup N_2\}$, then

$$\begin{aligned} \int_{-\infty}^{+\infty} \{\lambda f(x) + \mu g(x)\} e^{-ix\xi}]\,\mathrm{d}x &= \lambda \int_{-\infty}^{+\infty} e^{-ix\xi} f(x)\,\mathrm{d}x + \mu \int_{-\infty}^{+\infty} e^{-ix\xi} g(x)\,\mathrm{d}x \\ &= \lambda \mathcal{F}\{f\}(\xi) + \mu \mathcal{F}\{g\}(\xi). \end{aligned}$$

Hence, the integral on the left hand side is convergent in (at least) $\mathbb{R} \setminus (N_1 \cup N_2)$, where $N_1 \cup N_2$ is a null set, so it is equal to $\mathcal{F}\{\lambda f + \mu g\}(\xi)$, and the theorem is proved. $\Box$

Theorem 3.2.2 First Shifting property. *Assume that f has a Fourier transform. Then*

$$\mathcal{F}\left\{e^{iax} f(x)\right\}(\xi) = \mathcal{F}\{f\}(\xi - a) = \hat{f}(\xi - a) \qquad \text{for all } a \in \mathbb{R}.$$

Proof. In order to make the proofs easier we shall no longer bother with the null sets. It follows by computing from the right to the left that

$$\hat{f}(\xi - a) := \int_{-\infty}^{+\infty} e^{-i(\xi - a)x} f(x)\,\mathrm{d}x = \int_{.\infty}^{+\infty} e^{-ix\xi} \left\{e^{iax} f(x)\right\}\,\mathrm{d}x = \mathcal{F}\left\{e^{iax} f(x)\right\}(\xi). \qquad \Box$$

Theorem 3.2.3 Change of scale property. *If $k \in \mathbb{R} \setminus \{0\}$, then*

$$\mathcal{F}\left\{f\left(\frac{x}{k}\right)\right\}(\xi) = |k|\,\hat{f}(k\xi).$$

PROOF. We shall use the change of variable $x = kt$.

1) If $k > 0$, then

$$\mathcal{F}\left\{f\left(\frac{x}{k}\right)\right\}(\xi) = \int_{-\infty}^{+\infty} f\left(\frac{x}{k}\right) e^{-ix\xi}\,\mathrm{d}x = k\int_{-\infty}^{+\infty} f(t)e^{-it\cdot k\xi}\,\mathrm{d}t = k\,\hat{f}(k\xi) = |k|\,\hat{f}(k\xi).$$

2) If $k < 0$, then

$$\mathcal{F}\left\{f\left(\frac{x}{k}\right)\right\}(\xi) = \int_{-\infty}^{+\infty} f\left(\frac{x}{k}\right) e^{-ix\xi}\,\mathrm{d}x = k\int_{+\infty}^{-\infty} f(t)e^{-it\cdot k\xi}\,\mathrm{d}t = -k\,\hat{f}(k\xi) = |k|\,\hat{f}(k\xi). \qquad \square$$

Theorem 3.2.4 Second shifting property. *For all* $a \in \mathbb{R}$,

$$\mathcal{F}\{f(x+a)\}(\xi) = e^{ia\xi}\,\hat{f}(xi).$$

Proof. This follows from the change of variable $t = x + a$ and the computation

$$\begin{aligned}\mathcal{F}\{f(x+a)\}(\xi) &= \int_{-\infty}^{+\infty} e^{-ix\xi} f(x+a)\,\mathrm{d}x = \int_{-\infty}^{+\infty} e^{-i(t-a)\xi} f(t)\,\mathrm{d}t \\ &= e^{ia\xi}\int_{-\infty}^{+\infty} e^{-it\xi} f(t)\,\mathrm{d}t = e^{ia\xi}\,\hat{f}(\xi). \qquad \square\end{aligned}$$

If both f and f' have a Fourier transform, then it can be proved (the proof is not given here) that $f(x) \to 0$ for $x \to +\infty$ as well as for $x \to -\infty$. Using this result it is easy to prove

Theorem 3.2.5 Fourier transformation of derivatives. *If* $f \in C^1$ *and* $f' \in C^0$ *both have a Fourier transform, then*

$$\mathcal{F}\{f'\}(\xi) = i\xi\,\hat{f}(\xi).$$

Proof. since $f(x) \to 0$ for $x \to \pm\infty$, this follows by a partial integration,

$$\begin{aligned}\mathcal{F}\{f'\}(\xi) &= \int_{-\infty}^{+\infty} f'(x)e^{-ix\xi}\,\mathrm{d}x = \left[f(x)e^{ix\xi}\right]_{x=-\infty}^{+\infty} - \int_{-\infty}^{+\infty}(-i\xi)f(x)e^{-ix\xi}\,\mathrm{d}x \\ &= 0 + i\xi\int_{-\infty}^{+\infty} e^{-ix\xi} f(x)\,\mathrm{d}x = i\xi\,\hat{f}(\xi). \qquad \square\end{aligned}$$

Theorem 3.2.6 Fourier transformation of integrals. *If* $f \in C^0$ *and* $\int_{-\infty}^{x} f(t)\,dt$ *both have a Fourier transform, then*

$$\mathcal{F}\left\{\int_{-\infty}^{x} f(t)\,dt\right\}(\xi) = \frac{1}{i\xi}\,\hat{f}(\xi).$$

Sketch of proof. Replace $f(x)$ by $\int_{-\infty}^{x} f(t)\,\mathrm{d}t$, and $f'(x)$ by $f(x)$ in Theorem 3.2.5. The obvious exceptional point $\xi = 0$ is a null set. Furthermore, it is possible though hard to prove that $\int_{-\infty}^{x} f(t)\,\mathrm{d}t \to 0$ for $x \to +\infty$, while this claim is trivial for $x \to -\infty$. Then the theorem follows from Theorem 3.2.5. $\square$

Theorem 3.2.7 Multiplication by x^n. *Assume for a given* $n \in \mathbb{N}$ *that both* $f(x)$ *and* $x^n f(x)$ *have a Fourier transform. Then the n-th derivative* $\hat{f}^{(n)}(\xi)$ *exists and is given by*

$$\frac{d^n}{d\xi^n}\,\hat{f}(\xi) = \frac{1}{i^n}\,\mathcal{F}\left\{x^n f(x)\right\}(\xi).$$

Sketch of proof. Notice that

$$|x|^j \le 1 + |x|^n \qquad \text{for all } x \in \mathbb{R} \text{ and all } j = 0,\, 1,\, \dots\, ,\, n.$$

Since both $\int_{-\infty}^{+\infty} e^{-ix\xi} f(x)\,\mathrm{d}x$ and $\int_{-\infty}^{+\infty} e^{-ix\xi} x^n f(x)\,\mathrm{d}x$ are convergent (except possibly in a null set), one can use the estimate above to prove that

$$\int_{-\infty}^{+\infty} e^{-ix\xi} x^j f(x)\,\mathrm{d}x \qquad \text{is convergent for every } j = 0,\, 1,\, \dots\, ,\, n.$$

The proof is tricky, so it is not given here.

Then differentiate under the integral sign to get

$$\frac{d}{d\xi}\,\hat{f}(\xi) = \frac{d}{d\xi} \int_{-\infty}^{+\infty} e^{-ix\xi} f(x)\,\mathrm{d}x = \frac{1}{i} \int_{-\infty}^{+\infty} x e^{-ix\xi} f(x)\,\mathrm{d}x = \frac{1}{i}\mathcal{F}\{x f(x)\}(\xi),$$

from which the claim follows by induction. □

Remark 3.2.2 We emphasize once more that the proof above is not strictly correct, though the main ingredients are mentioned. Also, when $j = n$ is considered, the derivative $\hat{f}^{(n)}(\xi)$ may only exist in a generalized sense, because the ordinary derivative of $\hat{f}^{(n-1)}(\xi)$ probably does not exist. ♢

Theorem 3.2.8 Division by x. *Assume that* $\hat{f}(\xi) \to 0$ *for* $\xi \to -\infty$ *and that* $\lim_{x\to 0} \dfrac{f(x)}{x}$ *exists and is finite. Then*

$$\mathcal{F}\left\{\frac{f(x)}{x}\right\}(\xi) = \frac{1}{i} \int_{-\infty}^{\xi} \hat{f}(\tau)\,d\tau.$$

Sketch of proof. Using multiplication by x we get

$$\frac{d}{d\xi}\,\mathcal{F}\left\{\frac{f(x)}{x}\right\}(\xi) = \frac{1}{i}\,\mathcal{F}\{f(x)\}(\xi) = \frac{1}{i}\,\hat{f}(\xi),$$

and the result follows by an integration. □

Remark 3.2.3 The proof above relies on Theorem 3.2.7. Since the proof of Theorem 3.2.7 was not correct, so is the proof above of Theorem 3.2.8 not correct.

On the other hand, the assumption that $\lim_{x\to 0} \dfrac{f(x)}{x}$ exists and is finite, can be relaxed to the assumption that $\dfrac{f(x)}{x}$ is bounded in a deleted neighbourhood of 0. ♢

In case of the Fourier transform, the *convolution* is defined by integrating over all of $\mathbb{R}$, thus

$$(f \star g)(x) := \int_{-\infty}^{+\infty} f(x-t) g(t)\,\mathrm{d}t,$$

whenever the right hand side is defined almost everywhere. Then

Theorem 3.2.9 Fourier transformation of a convolution. *Assume that $f \star g$ exists and that f, g and $f \star g$ all have a Fourier transform. Then*

$$\mathcal{F}\{f \star g\}(\xi) = \mathcal{F}\{f\}(\xi) \cdot \mathcal{F}\{g\}(\xi).$$

The proof of Theorem 3.2.9 is far more difficult than any other of the proofs above, so we shall not produce it here.

We give some Fourier transforms in Table 4 in Section 4, page 102. Although we have not defined *Dirac's δ "function"*, we shall include some of the results concerning this element, because it is so frequently used in the technical sciences. Concerning the notation, $\delta^{(n)}$ means the n-th derivative of δ, and $\delta_{(h)}$ means δ translated to $x = h$, i.e. in the usual (incorrect, though understandable) engineering notation, "$\delta_{(h)}(x) = \delta(x-h)$".

3.3 The Fourier transformation on $L^1(\mathbb{R})$.

We start by proving a simple lemma.

Lemma 3.3.1

$$\int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}\,x^2\right) \mathrm{d}x = \sqrt{2\pi}.$$

Proof. Consider the improper plane integral

$$\iint_{\mathbb{R}^2} \exp\left(-\frac{1}{2}\left(x^2+y^2\right)\right) \mathrm{d}x\mathrm{d}y = \iint_{\mathbb{R}^2} \exp\left(-\frac{1}{2}\,x^2\right) \exp\left(-\frac{1}{2}\,y^2\right) \mathrm{d}x\mathrm{d}y = \left\{\int_{\mathbb{R}} \exp\left(-\frac{1}{2}\,t^2\right) \mathrm{d}t\right\}^2,$$

which clearly is convergent. Using instead polar coordinates, we also get

$$\iint_{\mathbb{R}^2} \exp\left(-\frac{1}{2}\left(x^2+y^2\right)\right) \mathrm{d}x\mathrm{d}y = \int_0^{2\pi} \left\{\int_0^{+\infty} \exp\left(-\frac{1}{2}\,r^2\right) \cdot r\,\mathrm{d}r\right\} \mathrm{d}\Theta = 2\pi,$$

thus

$$\left\{\int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}\,t^2\right) \mathrm{d}t\right\}^2 = 2\pi,$$

and the lemma follows. □

Let $f(x)$, $x \in \mathbb{R}$, be any (measurable) function of real or complex values. We say that f belongs to the class of functions $L^1 = L^1(\mathbb{R})$, if

$$(41) \quad \int_{-\infty}^{+\infty} |f(x)|\,\mathrm{d}x < +\infty.$$

If (41) holds, then we define a norm $\|\cdot\|_1$ on $L^1(\mathbb{R})$ by

$$(42) \quad \|f\|_1 := \int_{-\infty}^{+\infty} |f(x)|\,\mathrm{d}x \qquad \text{for } f \in L^1(\mathbb{R}).$$

All the rules of computation given in Section 3.2 assumed vaguely that the functions involved all allowed Fourier transforms. We shall in the following show that every $f \in L^1(= L^1(\mathbb{R}))$ indeed has a Fourier transform $\mathcal{F}\{f\}(\xi)$. In fact, whenever $\xi \in \mathbb{R}$, then

$$(43)\quad \left|\hat{f}(\xi)\right| = \left|\int_{-\infty}^{+\infty} e^{-ix\xi} f(x)\,\mathrm{d}x\right| \le \int_{-\infty}^{+\infty} |f(x)|\,\mathrm{d}x = \|f\|_1,$$

hence the improper integral (40) is convergent for *every* $\xi \in \mathbb{R}$, and $\hat{f}(\xi)$ is bounded.

We shall prove that $\hat{f}(\xi)$ is also continuous. In fact, for every $\varepsilon > 0$ we can find $A > 0$, such that

$$\int_{-\infty}^{-A} |f(x)|\,\mathrm{d}x + \int_{A}^{+\infty} |f(x)|\,\mathrm{d}x < \frac{\varepsilon}{3},$$

and $\delta > 0$, such that for all $|x| < A$ and $|\xi - \eta| < \delta$,

$$\left|e^{-ix\xi} - e^{-ix\eta}\right| = \left|e^{ix\eta}\left\{e^{-i(\xi-\eta)} - 1\right\}\right| = \left|e^{-ix(\xi-\eta)} - 1\right| < \frac{\varepsilon}{3} \cdot \frac{1}{\|f\|_1}$$

for $\|f\|_1 > 0$. Notice that if $\|f\|_1 = 0$, then $\hat{f}(\xi) \equiv 0$, and there is nothing to prove, so we assume that $\|f\|_1 > 0$ in the following. Then for $|\xi - \eta| < \delta$,

$$\begin{aligned}\left|\hat{f}(\xi) - \hat{f}(\eta)\right| &= \left|\int_{-\infty}^{+\infty} e^{-ix\xi} f(x)\,\mathrm{d}x - \int_{-\infty}^{+\infty} e^{-ix\eta} f(x)\,\mathrm{d}x\right| \\ &\le 2\int_{-\infty}^{-A} |f(x)|\,\mathrm{d}x + 2\int_{A}^{+\infty} |f(x)|\,\mathrm{d}x + \int_{-A}^{A} \left|e^{ix\xi} - e^{-ix\xi}\right| \cdot |f(x)|\,\mathrm{d}x \\ &\le \frac{2}{3}\,\varepsilon + \frac{\varepsilon}{3} \cdot \frac{1}{\|f\|_1} \int_{-A}^{A} |f(x)|\,\mathrm{d}x \le \frac{2}{3}\,\varepsilon + \frac{1}{3}\,\varepsilon = \varepsilon,\end{aligned}$$

and it follows that $\hat{f}(\xi)$ is continuous. Hence we have proved above,

Lemma 3.3.2 *If $f \in L^1$, then $\hat{f}(\xi)$ is a bounded and continuous function in $\xi \in \mathbb{R}$.*

We shall now prove that if $f \in L^1$, then we also have $\hat{f}(\xi) \to 0$ for $\xi \to \pm\infty$. We start by proving the following simple lemma.

Lemma 3.3.3 *Assume that $f(x)$ is piecewise constant in the interval $[-a, a]$ with a finite number of discontinuities. The*

$$\lim_{\xi \to \pm\infty} \int_{-a}^{a} f(x) e^{-ix\xi}\, d\xi = 0.$$

Proof. First assume that $f(x) \equiv 1$ on $[-a, a]$. Then we get for $\xi \neq 0$,

$$\int_{-a}^{a} f(x) e^{-ix\xi}\,\mathrm{d}x = \int_{-a}^{a} e^{-ix\xi}\,\mathrm{d}x = \left[-\frac{1}{i\xi}\, e^{-ix\xi}\right]_{-a}^{a} = \frac{2\sin(a\xi)}{\xi},$$

and since $|2\sin(a\xi)| \le 2$ for all $\xi \in \mathbb{R}$, the claim follows.

Then consider the function

$$f(x) = \begin{cases} \alpha & \text{for } x \in [-a, b[, \\ \beta & \text{for } x \in]b, a]. \end{cases}$$

We notice that since $\{b\}$ is a null set, there is no need to specify $f(b)$.

In the same way as above we get

$$\int_{-a}^{a} f(x) e^{-ix\xi}\,\mathrm{d}x = \left[-\frac{\alpha}{i\xi}\, e^{-ix\xi}\right]_{-a}^{b} + \left[-\frac{\beta}{i\xi}\, e^{-ix\xi}\right]_{b}^{a} = \frac{1}{\xi} \cdot C_{a,b,\alpha,\beta},$$

where we have the estimate $|C_{a,b,\alpha,\beta}| \le 4\,\max\{|\alpha|, |\beta|\}$.

It now follows by induction that if the piecewise constant function $f(x)$ has n different values, then

$$\left|\int_{-a}^{a} f(x) e^{-ix\xi}\,\mathrm{d}x\right| \le \frac{2n}{|\xi|} \max_{x \in [-a,a]} |f(x)|,$$

and the lemma is proved. $\Diamond$

We mention without proof the following simple approximation theorem.

Theorem 3.3.1 *Given $f \in L^1([-a,a])$. To every $\varepsilon > 0$ there exists a piecewise constant function g with only finite many different values, such that*

$$\|f-g\|_1 = \int_{-a}^{a} |f(t)-g(t)|\,dt < \varepsilon.$$

This approximation theorem is used in the proof of the following

Theorem 3.3.2 *If $f \in L^1(\mathbb{R})$, then its Fourier transform $\hat{f}(\xi)$ is continuous and bounded,*

$$\left|\hat{f}(\xi)\right| \le \|f\|_1 \qquad \text{for all } \xi \in \mathbb{R},$$

and

$$\lim_{\xi \to -\infty} \hat{f}(\xi) = 0, \qquad \lim_{\xi \to +\infty} \hat{f}(\xi) = 0.$$

Proof. The first two claims were proved in Lemma 3.3.2, so we shall only prove that

$$\hat{f}(\xi) \to 0 \qquad \text{for } \xi \to -\infty \text{ or } \xi \to +\infty.$$

From $f \in L^1(\mathbb{R})$ follows that to every given $\varepsilon > 0$ we can find $a > 0$, such that

$$\int_{-\infty}^{-a} |f(x)|\,\mathrm{d}x + \int_{a}^{+\infty} |f(x)|\,\mathrm{d}x < \frac{\varepsilon}{3}.$$

Then apply Theorem 3.3.1 using the truncated function

$$f(x) \cdot \chi_{[-a,a]}(x) \in L^1([-a,a]),$$

from which follows that there exists a piecewise constant function $g(x)$ of finitely many values on $[-a,a]$, such that

$$\int_{-a}^{a} |f(x)-g(x)|\,\mathrm{d}x < \frac{\varepsilon}{3}.$$

Finally, apply Lemma 3.3.3 on $g(x)$. Thus, there exists a $\Xi > 0$, such that

$$\left|\int_{-a}^{a} g(x)e^{-ix\xi}\,\mathrm{d}x\right| < \frac{\varepsilon}{3} \qquad \text{for } \xi \in \mathbb{R} \text{ and } |\xi| > \Xi.$$

Then for ξ real and $|\xi| > \Xi$,

$$\begin{aligned}
\left|\hat{f}(\xi)\right| &= \left|\int_{-\infty}^{+\infty} e^{-ix\xi} f(x)\,\mathrm{d}x\right| \\
&\le \left|\int_{-\infty}^{-a} e^{-ix\xi} f(x)\,\mathrm{d}x\right| + \left|\int_{a}^{+\infty} e^{-ix\xi} f(x)\,\mathrm{d}x\right| + \left|\int_{-a}^{a} e^{-ix\xi} f(x)\,\mathrm{d}x\right| \\
&\le \int_{-\infty}^{-a} |f(x)|\,\mathrm{d}x + \int_{a}^{+\infty} |f(x)|\,\mathrm{d}x + \left|\int_{-a}^{a} e^{-ix\xi}\{f(x)-g(x)\}\,\mathrm{d}x\right| + \left|\int_{-a}^{a} e^{-ix\xi} g(x)\,\mathrm{d}x\right| \\
&< \frac{\varepsilon}{3} + \int_{-a}^{a} |f(x)-g(x)|\,\mathrm{d}x + \frac{\varepsilon}{3} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\end{aligned}$$

and since $\varepsilon > 0$ was arbitrary, the claim is proved. $\square$

Lemma 3.3.4 *Assume that* f, $g \in L^1$. *Then* $\hat{f}(\xi) \cdot g(\xi) \in L^1$, *and*

$$\mathcal{F}\left\{\hat{f} \cdot g\right\}(-x) := \int_{-\infty}^{+\infty} \hat{f}(\xi) g(\xi)\, e^{ix\xi}\, \mathrm{d}\xi = \int_{-\infty}^{+\infty} \hat{g}(t) f(x+t)\, \mathrm{d}t.$$

Proof. From $f \in L^1$ follows that $\left|\hat{f}(\xi)\right| \leq \|f\|_1$ for every $\xi \in \mathbb{R}$, so we get the estimate

$$\int_{-\infty}^{+\infty} \left|\hat{f}(\xi) g(\xi)\right| \mathrm{d}\xi \leq \|f\|_1 \cdot \int_{-\infty}^{+\infty} |g(\xi)|\, \mathrm{d}\xi = \|f\|_1 \cdot \|g\|_1 < +\infty,$$

and we have proved that $\hat{f} \cdot g \in L^1$, so it has also a Fourier transform. In the computation of this we use *Fubini's theorem* from Chapter 1 to interchange the order of integration,

$$\begin{aligned}\int_{-\infty}^{+\infty} \hat{f}(\xi)g(\xi)e^{ix\xi}\,\mathrm{d}\xi &= \int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty} f(t)e^{-it\xi}\,\mathrm{d}t\right\}g(\xi)e^{ix\xi}\,\mathrm{d}\xi \\ &= \int_{-\infty}^{+\infty} f(t)\left\{\int_{-\infty}^{+\infty} g(\xi)e^{-i(t-x)\xi}\,\mathrm{d}\xi\right\}\mathrm{d}t \\ &= \int_{-\infty}^{+\infty} f(t)\,\hat{g}(t-x)\,\mathrm{d}t = \int_{-\infty}^{+\infty} \hat{g}(t)f(x+t)\,\mathrm{d}t,\end{aligned}$$

and the lemma is proved. □

Lemma 3.3.5 *If* $g(x) = \exp\left(-\frac{1}{2}\,x^2\right)$*, then* $\hat{g}(\xi) = \sqrt{2\pi}\cdot g(\xi)$*, thus*

$$\int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}\,x^2 - ix\xi\right)\mathrm{d}x = \sqrt{2\pi}\cdot\exp\left(-\frac{1}{2}\,\xi^2\right).$$

Proof. Obviously, $g \in L^1$, so $\hat{g}$ exists and is bounded. Then by a direct computation,

$$\begin{aligned}\hat{g}(\xi) &= \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}\,x^2 - ix\xi\right)\mathrm{d}x = \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}(x+i\xi)^2\right)\cdot\exp\left(-\frac{1}{2}\,\xi^2\right)\mathrm{d}x \\ &= \exp\left(-\frac{1}{2}\,\xi^2\right)\int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}(x+i\xi)^2\right)\mathrm{d}x,\end{aligned}$$

so the lemma follows if we can prove that

$$(44)\quad \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}(x+i\xi)^2\right)\mathrm{d}x = \sqrt{2\pi} \qquad \text{for every } \xi \in \mathbb{R}.$$

We proved in Lemma 3.3.1 that (44) holds for $\xi = 0$. When $\xi \neq 0$, we choose the path of integration C_R in the complex plane as shown in Figure 8, where we assume that $\xi > 0$. The proof for $\xi < 0$ is similar, and it *alternatively* also follows from the change og variable $t = -x$ in the integral.

Using *Cauchy's integral theorem*, cf. *Ventus, Complex Functions Theory a-1*, that

$$\oint_{C_R} \exp\left(-\frac{1}{2}\,z^2\right)\mathrm{d}z = 0,$$

or, written in all details,

(45)

$$\int_{-R}^{R} \exp\left(-\frac{1}{2}\,x^2\right)\mathrm{d}x - \int_{-R}^{R} \exp\left(-\frac{1}{2}(x+i\xi)^2\right)\mathrm{d}x + \int_{\Gamma_1} \exp\left(-\frac{1}{2}\,z^2\right)\mathrm{d}z + \int_{\Gamma_2} \exp\left(-\frac{1}{2}\,z^2\right)\mathrm{d}z = 0.$$

In order to prove that

$$\int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}(x+i\xi)^2\right)\mathrm{d}x = \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}\,x^2\right)\mathrm{d}x = \sqrt{2\pi},$$

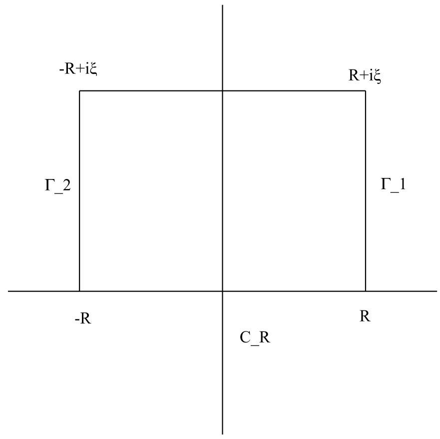

Figure 8: Path of integration C_R in the proof of Lemma 3.3.5.

we shall only prove that the latter two integrals of (45) tend to 0 for $R \to +\infty$. It suffices to estimate the integral along Γ_1, given by its parametric description $z = R + it$, $t \in [0, \xi]$, because the estimate of the integral along Γ_2 is similar. Using the parametric description above we get

$$\begin{aligned}\left|\int_{\Gamma_1} \exp\left(-\frac{1}{2}\,z^2\right) \mathrm{d}z\right| &= \left|\int_0^{\xi} \exp\left(-\frac{1}{2}(R+it)^2\right) i\,\mathrm{d}t\right| = \left|\int_0^{\xi} \exp\left(-\frac{1}{2}\left\{R^2 - t^2 + 2iRt\right\}\right) i\,\mathrm{d}t\right| \\ &\le \int_0^{\xi} \exp\left(-\frac{1}{2}\left\{R^2 - t^2\right\}\right) \mathrm{d}t.\end{aligned}$$

Here $\xi > 0$ is fixed, so when $R \ge 2\xi$, i.e. $t \le \xi \le \frac{1}{2}\,R$, we get the trivial estimate

$$\begin{aligned}\left|\int_{\Gamma_1} \exp\left(-\frac{1}{2}\,z^2\right) \mathrm{d}z\right| &\le \int_0^{\xi} \exp\left(-\frac{1}{2}\,R^2 + \frac{1}{2}\,t^2\right) \mathrm{d}t \le \int_0^{\xi} \exp\left(-\frac{1}{2}\,R^2 + \frac{1}{8}\,R^2\right) \mathrm{d}t \\ &= \xi \cdot \exp\left(-\frac{3}{8}\,R^2\right) \to 0 \qquad \text{for } R \to +\infty,\end{aligned}$$

and the claim is proved.

Notice in particular that

$$(46)\quad \int_{-\infty}^{+\infty} \hat{g}(\xi)\,\mathrm{d}\xi = \sqrt{2\pi} \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}\,\xi^2\right) \mathrm{d}\xi = 2\pi. \qquad \square$$

After the preparations above we are finally able to prove the main result of this section.

Theorem 3.3.3 Fourier's inversion formula. *If $f \in L^1$, then*

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ix\xi}\,\hat{f}(\xi)\,d\xi \qquad \textit{for almost every } x \in \mathbb{R}.$$

Remark 3.3.1 The inversion formula is also written

$$f(x) = \mathcal{F}^{-1}\left\{\hat{f}\right\}(x) = \frac{1}{2\pi}\,\mathcal{F}\{\mathcal{F}\{f\}\}(-x).$$

The inversion formula implies that if f and g have the same Fourier transform, then they are equal almost everywhere. $\Diamond$

Proof. Since we have not developed the full *Lebesgue Theory of Integration*, we can only give a correct proof of the inversion formula under the additional assumption that f is continuous. The general proof will, however, follow the same pattern as the proof when f is continuous.

The next obstacle is that it is not possible here to give a straightforward proof of the inversion formula. The reason is that if we compute the right hand side of the formula, then

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ix\xi}\,\hat{f}(\xi)\,\mathrm{d}\xi = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ix\xi}\left\{\int_{-\infty}^{+\infty} f(t)\,e^{-it\xi}\,\mathrm{d}t\right\}\mathrm{d}\xi = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty} f(t)\,e^{i(x-t)\xi}\,\mathrm{d}t\right\}\mathrm{d}\xi,$$

where it is obvious that we *cannot* interchange the order of integration, because the indefinite integral $\int e^{i(x-t)\xi}\,\mathrm{d}\xi$ does not have limits for $\xi \to \pm\infty$.

Instead we use the old trick of choosing an auxiliary function $g \in L^1$, which we later shall choose as the explicit function $g(x)$ in Lemma 3.3.5. For the time being g is just any function from $L^1(\mathbb{R})$.

Choose any $g \in L_1$, and let $\varepsilon > 0$ be a parameter. Defining $g_\varepsilon(\xi) := g(\varepsilon\xi)$, it follows from the *Theorem of change of scale* that

$$\hat{g}_\varepsilon(x) = \frac{1}{\varepsilon}\,\hat{g}\left(\frac{x}{\varepsilon}\right) \qquad \text{and} \qquad g_\varepsilon \in L^1.$$

Then it follows from Lemma 3.3.4 that

$$\begin{aligned}
\int_{-\infty}^{+\infty} g_\varepsilon(\xi)\,\hat{f}(\xi)\,e^{ix\xi}\,\mathrm{d}\xi &= \int_{-\infty}^{+\infty} g(\varepsilon\xi)\,\hat{f}(\xi)\,e^{ix\xi}\,\mathrm{d}\xi = \frac{1}{\varepsilon}\int_{-\infty}^{+\infty}\hat{g}\left(\frac{t}{\varepsilon}\right)f(x+t)\,\mathrm{d}t \\
&= \int_{-\infty}^{+\infty}\hat{g}(y)f(x+\varepsilon y)\,\mathrm{d}y.
\end{aligned}$$

For fixed x, both functions $g_\varepsilon(\xi)\cdot\hat{f}(\xi)$ and $\hat{g}(y)f(x+\varepsilon y)$ belong to L^1, so both integrands are absolutely integrable for every $\varepsilon > 0$.

At this step we choose $g(\xi) = \exp\left(-\frac{1}{2}\,\xi^2\right)$ with $g(0) = 1$, and then let $\varepsilon \to 0+$ under the integral signs. When we also apply (46) we get

$$\begin{aligned}
\int_{-\infty}^{+\infty}\hat{f}(\xi)\,e^{ix\xi}\,\mathrm{d}\xi &= \int_{-\infty}^{+\infty} g(0)\,\hat{f}(\xi)\,e^{ix\xi}\,\mathrm{d}\xi = \int_{-\infty}^{+\infty}\hat{g}(y)f(x+0)\,\mathrm{d}y \\
&= f(x)\int_{-\infty}^{+\infty}\hat{g}(y)\,\mathrm{d}y = 2\pi\,f(x),
\end{aligned}$$

and we have proved the inversion formula for continuous $f \in L^1$.

In general, we use the following approximation theorem, which is not proved here.

Theorem 3.3.4 *To every* $f \in L^1(\mathbb{R})$ *and every* $\varepsilon > 0$ *there exists a* continuous *function* $g \in L^1(\mathbb{R})$*, such that*

$$\|f - g\|_1 = \int_{-\infty}^{+\infty} |f(t) - g(t)| \, dt < \varepsilon.$$

One may even choose g *continuous and 0 outside a bounded (compact) interval.*

We continue by proving the following residuum inversion formula of the Fourier transform.

Theorem 3.3.5 Fourier's residuum inversion formula I. *Assume that $F(\zeta)$ is analytic in an open domain Ω, which contains the closed upper half plane $\Im\zeta \geq 0$, except for a finite number of singularities $\{\zeta_1, \dots, \zeta_n\}$, where all $\Im\zeta_j > 0$.*
Furthermore, assume that there exist positive constants R, α, $c > 0$, such that we have the estimate

$$|F(\zeta)| \leq \frac{c}{|\zeta|^\alpha} \qquad \text{for } \Im\zeta \geq 0 \text{ and } |\zeta| \geq R.$$

Then the improper integral $\int_{-\infty}^{+\infty} F(\xi)\, e^{i\xi x}\, d\xi$ is convergent for every $x > 0$, and its value is given by the residuum formula

$$(47) \quad \int_{-\infty}^{+\infty} F(\xi)\, e^{ix\xi}\, d\xi = 2\pi i \sum_{j=1}^{n} res\left(F(\zeta)\, e^{ix\zeta}; \zeta_j\right).$$

Proof. We shall prove that the limit

$$\lim_{R_1\to+\infty} \lim_{R_2\to+\infty} \int_{-R_1}^{R_2} f(\xi)\, e^{ix\xi}\, \mathrm{d}\xi$$

exists and is finite. Choose the path of integration C_{R_1,R_2} as shown in Figure 9, where R_1, $R_2 > R$. Then all singularities $\zeta_1, \dots, \zeta_n$ in the upper half plane lie inside C_{R_1,R_2}, because they all lie in the set $\{\zeta \in \mathbb{C} \mid \Im\zeta > 0, |\zeta| < R\}$.

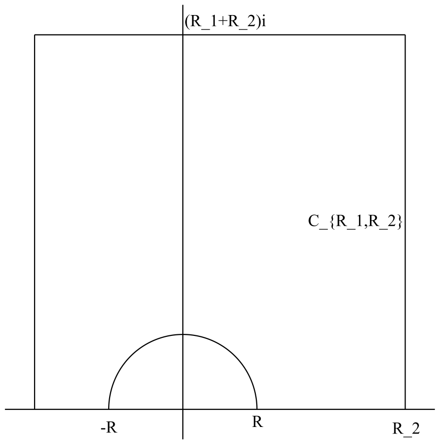

Figure 9: Path of integration C_{R_1,R_2} in the proof of Theorem 3.3.5.

Then it follows from *Cauchy's residuum theorem* that

$$(48) \quad \oint_{C_{R_1,R_2}} F(\zeta)\, e^{ix\zeta}\, \mathrm{d}\zeta = 2\pi i \sum_{j=1}^{n} \mathrm{res}\left(F(\zeta) e^{ix\zeta}; \zeta_j\right),$$

where the right hand side is a constant which is independent of R_1 and R_2, as long as they fulfil R_1, $R_2 > R$.

On the other hand, the left hand side of (48) can also be written

$$\oint_{C_{R_1,R_2}} F(\zeta)\, e^{ix\zeta}\, \mathrm{d}\zeta = \int_{-R_1}^{R_2} F(\xi)\, e^{ix\xi}\, \mathrm{d}\xi + \int_0^{R_1+R_2} F\left(R_2+it\right) e^{ix(R_2+it)} i\, \mathrm{d}t$$

(49)

$$- \int_{R_1}^{R_2} F\left(\xi + i\left(R_;1+R_2\right)\right) e^{ix(\xi+i\{R_1+R_2\})} - \int_0^{R_1+R_2} F\left(-R_1+it\right)\, e^{ix(R_1+it)}\, \mathrm{d}t,$$

When $x>0$ is kept fixed, and R_1, $R_2>R$, we get the estimates

$$\left|\int_0^{R_1+R_2} F\left(R_2+it\right) e^{ix(R_2+it)} i\, \mathrm{d}t\right| \le \int_0^{R_1+R_2} \left|F\left(R_2+it\right)\right|\, e^{-xt}\, \mathrm{d}t$$

$$\le \frac{c}{R_2^\alpha}\int_0^{R_1+R_2} e^{-xt}\, \mathrm{d}t \le \frac{c}{xR_2^\alpha} \to 0 \qquad \text{for } R_2 \to +\infty.$$

Similarly,

$$\left|\int_{-R_1}^{R_2} F\left(\xi + i\left\{R_1+R_2\right\}\right) e^{ix(\xi+i\{R_1+R_2\})}\, \mathrm{d}\xi\right| \le \int_{-R_1}^{R_2} \left|F\left(\xi+i\left\{R_1+R_2\right\}\right)\right| e^{-x(R_1+R_2)}\, \mathrm{d}\xi$$

$$\le e^{-x(R_1+R_2)} \int_{-R_1}^{R_2} \frac{c}{\left(R_1+R_2\right)^\alpha}\, \mathrm{d}\xi = c\cdot\left(R_1+R_2\right)^{1-\alpha}\cdot e^{-x(R_1+R_2)}.$$

Since the exponential dominates the power function, this expression tends towards 0 for either $R_1 \to +\infty$ or $R_2 \to +\infty$.

Also

$$\left|\int_0^{R_1+R_2} F\left(-R_1+it\right)\, e^{ix(-R_1+it)} i\, \mathrm{d}t\right| \le \int_0^{R_1+R_2} \left|F\left(-R_1+it\right)\right| e^{-xt}\, \mathrm{d}t$$

$$\le \frac{c}{R_1^\alpha}\int_0^{R_1+R_2} e^{-xt}\, \mathrm{d}t \le \frac{c}{xR_1^\alpha} \to 0 \qquad \text{for } R_1 \to +\infty.$$

Finally, when (49) is inserted into (48), we get by the two mutually independent limits that the improper integral $\int_{-\infty}^{+\infty} F(\xi)\, e^{ix\xi}\, \mathrm{d}\xi$ makes sense and that its value is indeed given by (47). □

Corollary 3.3.1 Fourier's residuum inversion formula II. *Assume that $F(\zeta)$ is analytic in $\mathbb{C}\setminus\{\zeta_1,\dots,\zeta_n\}$, where none of the singularities ζ_j lies on the real axis. Furthermore, assume that there are constants R, α, $c>0$, such that*

$$|F(\zeta)| \le \frac{c}{|\zeta|^\alpha} \qquad \textit{for } |\zeta| \ge R.$$

Then

$$(50) \qquad f(x) := \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\xi)\, e^{ix\xi}\, d\xi = \begin{cases} i\sum_{\Im\,\zeta_j>0} res\left(F(\zeta)e^{ix\zeta};\zeta_j\right) & \textit{for } x>0, \\ -i\sum_{\Im\,\zeta_j<0} res\left(F(\zeta)e^{ix\zeta};\zeta_j\right), & \textit{for } x<0. \end{cases}$$

Finally, if $f\in L^1$, then $\hat{f}(\xi) = F(\xi)$.

PROOF. If $x > 0$, then the first result of (50) was proved in Theorem 3.3.5. If $x < 0$, we modify the proof of Theorem 3.3.5 in the following way. The curve C_{R_1,R_2} is reflected in the ξ-axis, and the orientation of the curve is reversed, such that the segment on the ξ-axis is traversed from R_2 to $-R_1$, explaining the change of sign in (50). The final claim of Corollary 3.3.1 follows from the first*Fourier's inversion formula*. □

Example 3.3.1 Choose $F(\zeta) = \dfrac{1}{1+\zeta^2}$. Then $F(\zeta)$ is analytic in $\mathbb{C}\setminus\{-i, i\}$, and we have the estimate

$$|F(\zeta)| = \frac{1}{|1+\zeta^2|} \le \frac{2}{|\zeta|^2} \qquad \text{for } |\zeta| \ge 2.$$

Furthermore, $F(\xi)$ lies in $L^1(\mathbb{R})$ for $\xi \in \mathbb{R}$.

It follows from Corollary 3.3.1 that

$$f(x) = \begin{cases} +i \operatorname{res}\left(\dfrac{e^{ix\zeta}}{1+\zeta^2}; i\right) = i\,\dfrac{e^{-x}}{2i} = \dfrac{1}{2}\,e^{-x} & \text{for } x > 0, \\[2ex] -i \operatorname{res}\left(\dfrac{e^{ix\zeta}}{1+\zeta^2}; -i\right) = -i\,\dfrac{e^{+x}}{-2i} = \dfrac{1}{2}\,e^{+x} & \text{for } x < 0, \end{cases}$$

hence

$$f(x) = \frac{1}{2}\,e^{-|x|} \qquad \text{for } x \in \mathbb{R}.$$

Obviously, $f \in L^1$, so $\hat{f}(\xi) = F(\xi)$, and we have in particular proved that

$$(51) \quad \frac{1}{2}\int_{-\infty}^{+\infty} e^{-|x|-i\xi x}\,\mathrm{d}x = \frac{1}{1+\xi^2}, \qquad \xi \in \mathbb{R},$$

and

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{e^{ix\xi}}{1+\xi^2}\,\mathrm{d}\xi = e^{-|x|}, \qquad x \in \mathbb{R}.$$

From $f \in L^1$ follows that the integral in (51) is convergent. It is possible to compute it by elementary calculus. In fact,

$$\begin{aligned} \frac{1}{2}\int_{-\infty}^{+\infty} e^{-|x|-i\xi x}\,\mathrm{d}x &= \frac{1}{2}\int_{-\infty}^{+\infty} e^{-|x|}\cos(\xi x)\,\mathrm{d}x - \frac{i}{2}\int_{-\infty}^{+\infty} e^{-|x|}\sin(\xi x)\,\mathrm{d}x \\ &= \frac{1}{2}\int_{-\infty}^{+\infty} e^{-|x|}\cos(\xi x)\,\mathrm{d}x + i\cdot 0 = \int_0^{+\infty} e^{-x}\cos(\xi x)\,\mathrm{d}x \\ &= \Re\int_0^{+\infty} e^{-x}e^{i\xi x}\,\mathrm{d}x = \Re\int_0^{+\infty} e^{(-1+i\xi)x}\,\mathrm{d}x = \Re\left[\frac{e^{(-1+i\xi)x}}{-1+i\xi}\right]_{x=0}^{+\infty} \\ &= \Re\left\{-\frac{1}{-1+i\xi}\right\} = \frac{1}{1+\xi^2}. \qquad \Diamond \end{aligned}$$

Example 3.3.2 Let $F(\zeta) = -\dfrac{i}{\zeta - i}$. Then $F(\zeta)$ is analytic in $\mathbb{C} \setminus \{i\}$ and we can choose $\alpha = 1$. Thus, to every $R > 1$ there is a $c_R > 0$, such that

$$|F(\zeta)| \le \frac{c_R}{|\zeta|} \qquad \text{for } |\zeta| \ge R.$$

Since $F(\zeta)$ does not have singularities in the lower half plane, $f(x)$ given by (50) is equal to 0 for $x < 0$.

We get for $x > 0$,

$$f(x) = i \cdot \operatorname{res}\left(-i\,\frac{e^{ix\zeta}}{\zeta - i}; i\right) = i(-i)e^{ixi} = e^{-x},$$

so

$$f(x) = \begin{cases} e^{-x} & \text{for } x > 0, \\ 0 & \text{for } x < 0. \end{cases}$$

Since

$$F(\xi) = -\frac{i}{\xi - i} = \frac{1}{\xi^2+1} - \frac{i\xi}{\xi^2+1},$$

it follows that

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{ix\xi}}{\xi^2+1}\,\mathrm{d}\xi - \frac{i}{2\pi}\int_{-\infty}^{+\infty} \frac{\xi e^{ix\xi}}{\xi^2+1}\,\mathrm{d}\xi = \begin{cases} e^{-x} & \text{for } x > 0 \\ 0 & \text{for } x < 0. \end{cases}$$

We note that the latter integral is *divergent* for $x = 0$, so we get in particular that $F \notin L^1(\mathbb{R})$. A check shows that

$$\hat{f}(\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi} f(x)\,\mathrm{d}x = \int_0^{+\infty} e^{-(1+i\xi)x}\,\mathrm{d}x = \left[-\frac{1}{1+i\xi}\,e^{-(1+i\xi)x}\right]_0^{+\infty} = \frac{1}{1+i\xi} = F(\xi).$$

This is an example which shows that if $f \in L^1$ is *not* continuous, then $F(\xi) = \hat{f}(\xi)$ can *never* belong to L^1. The general proof goes as follows. *If* $\hat{f} \in L^1$, then it would follow from the inversion formula and Theorem 3.3.2 that $f(x) = \dfrac{1}{2\pi}\,\mathcal{F}\left\{\hat{f}\right\}(-x)$ is continuous, which it is not. $\Diamond$

Example 3.3.3 Let $F(\zeta) = \dfrac{1}{(\zeta - i)^2}$. Then $F(\zeta)$ is analytic in $\mathbb{C} \setminus \{i\}$. We get the estimate

$$|F(\zeta)| = \frac{1}{|\zeta - i|^2} \le \frac{2}{|\zeta|^2} \qquad \text{for } |\zeta| > 2,$$

so $F(\xi) \in L^1(\mathbb{R})$. It then follows from Theorem 3.3.2 that $f(x) = \dfrac{1}{2\pi}\mathcal{F}\{F(\xi)\}(-x)$ is continuous.

Since $F(\zeta)$ does not have singularities in the lower half plane, the function $f(x)$ given by (50) must be 0 for $x < 0$, and by the continuity, also for $x = 0$. When $x > 0$, then

$$f(x) = i \cdot \operatorname{res}\left(\frac{e^{ix\xi}}{(\zeta - i)^2}; i\right) = i \cdot \lim_{\zeta \to i} \frac{d}{d\zeta}\left\{e^{ix\zeta}\right\} = i \cdot \lim_{\zeta \to i}\left\{i\,x\,e^{ix\zeta}\right\} = -x\,e^{-x},$$

hence

$$f(x) = \begin{cases} -x\,e^{-x} & \text{for } x > 0, \\ 0 & \text{for } x > 0. \end{cases}$$

It is easy to check that if we add $f(0) = 0$, then $f(x)$ becomes continuous, and also $f \in L^1$. $\Diamond$

Example 3.3.4 We shall here consider the more complicated example of the analytic function

$$F(\zeta) = \frac{1}{(\zeta - i)^2}\exp\left(\frac{1}{\zeta - i}\right) \qquad \text{for } \zeta \in \mathbb{C} \setminus \{i\},$$

where the only singularity $\zeta = i$ is essential.

Since

$$\exp\left(\frac{1}{\zeta-i}\right) \to e^0 = 1 \qquad \text{for } \zeta \to +\infty,$$

we get the estimate

$$|F(\zeta)| = \frac{1}{|\zeta-i|^2}\left|\exp\left(\frac{1}{\zeta-i}\right)\right| \le \frac{C_R}{|\zeta|^2} \qquad \text{for } |\zeta| > R > 1.$$

Hence, $F(\xi) \in L^1(\mathbb{R})$, and $f(x)$ given by (50) is continuous in $\mathbb{R}$. Since $F(\zeta)$ does not have singularities in the lower half plane, we get $f(x) = 0$ for $x < 0$, and also $f(0) = 0$ by the continuity.

When $x > 0$, it follows from (50) that

$$\begin{aligned} f(x) &= i \cdot \operatorname{res}\left(\frac{e^{ix\zeta}}{(\zeta-i)^2}\exp\left(\frac{1}{\zeta-i}\right);i\right) = i \cdot \operatorname{res}\left(e^{-x}\cdot\frac{e^{ix(\zeta-i)}}{(\zeta-i)^2}\cdot\exp\left(\frac{1}{\zeta-i}\right);i\right) \\ &= i \cdot e^{-x}\cdot \operatorname{res}\left(\frac{e^{ix\zeta}}{\zeta^2}\cdot\exp\left(\frac{1}{\zeta}\right);0\right) = i\cdot e^{-x}\cdot a_{-1}(x), \end{aligned}$$

where $a_{-1}(x)$ is the coefficient of $\dfrac{1}{\zeta}$ in the Laurent series expansion of $\dfrac{1}{\zeta^2}\cdot e^{ix\zeta}\cdot\exp\left(\dfrac{1}{\zeta}\right)$.

This Laurent series expansion is computed in the following way,

$$\frac{1}{\zeta^2}\cdot e^{ix\zeta}\cdot\exp\left(\frac{1}{\zeta}\right) = \frac{1}{\zeta^2}\sum_{n=0}^{+\infty}\frac{1}{n!}\,(ix)^n\,\zeta^n\sum_{p=0}+\infty\frac{1}{p!}\cdot\frac{1}{\zeta^p} = \sum_{n=0}^{+\infty}\sum_{p=0}^{+\infty}\frac{(ix)^n}{n!\,p!}\,\zeta^{n-p-2},$$

from which we get $a_{-1}(x)$ by collecting all terms for which $n = p+1 \ge 1$, hence

$$a_{-1}(x) = \sum_{p=0}^{+\infty}\frac{i^{p+1}x^{p+1}}{(p+1)!\,p!} \qquad \text{for } x > 0.$$

We conclude that

$$(52)\quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\xi)\,e^{ix\xi}\,\mathrm{d}\xi = \begin{cases} \sum_{n=0}^{+\infty}\dfrac{i^{n+2}x^{n+1}}{n!(n+1)!}\,e^{-x} & \text{for } x > 0, \\ 0 & \text{for } x \le 0. \end{cases}$$

Since also $F(\xi) \in L^1(\mathbb{R})$, we get by replacing ξ by x, and x by $-\xi$ in (50) that

$$\hat{F}(\xi) = \int_{-\infty}^{+\infty} F(x)\,e^{-ix\xi}\,\mathrm{d}x = 2\pi\,f(-\xi) = \begin{cases} 0 & \text{for } \xi > 0, \\ 2\pi i\sum_{n=0}^{+\infty}\dfrac{(-i\xi)^{n+1}}{n!(n+1)!}\cdot e^{\xi} & \text{for } \xi < 0. \end{cases}$$

This example shows that the Fourier transform of an L^1 function does not have to be an analytic function, since $\hat{F}(\xi)$ obviously is not analytic at $\xi = 0$.

It is possible to prove that the series in (52) can be expressed by some Bessel function. We shall briefly return to the family of Bessel functions in *Ventus, Complex Functions Theory a-5, The Laplace Transform II*, where we also introduce other functions which somehow are connected with the Laplace transformation. $\Diamond$

3.4 The Mellin transformation

For completeness we shall also consider the so-called *Mellin transformation*, which is defined as an operator on the set of functions f by the following

$$(53)\quad \mathcal{M}\{f\}(a) := \int_0^{+\infty} f(x)\, x^a\, \frac{\mathrm{d}x}{x},$$

which is a function of the variable a, provided that the improper integral is convergent.

If the integral of (53) is *absolutely convergent*, we change variable, $x = e^{-t}$, to get

$$(54)\quad \mathcal{M}\{f\}(a) := \int_0^{+\infty} f(x)\, x^a\, \frac{\mathrm{d}x}{x} = \int_{-\infty}^{+\infty} f\left(e^{-t}\right) e^{-at}\, \mathrm{d}t,$$

where the right hand side is the *two-sided Laplace transform* of the function $g(t) := f\left(e^{-t}\right)$ at the point a. This shows the connection between the *Mellin transformation* and the *two-sided Laplace transformation*.

In order not to get into trouble of defining the right branch of x^a for a complex we shall always assume that the exponent $a \in \mathbb{R}$ is real. We shall furthermore assume that $f(z)$ is analytic in $\mathbb{C}$ except for a finite number of singularities, none of these lying on the positive real axis $\mathbb{R}$. Usually one also assumes that all singularities are poles. However, the proof below will show that we only require that a possible singularity at $z = 0$ is at most a pole. Any other singularity may be a pole or an essential singularity.

Then we turn to the factor x^a of the integrand. If $a \in \mathbb{Z}$, then z^a is defined in the usual way for $z \neq 0$ (and also for $z = 0$, if $a \in \mathbb{N}_0$). Assume that $a \in \mathbb{R} \setminus \mathbb{Z}$, and fix the following branch of the logarithm,

$$\mathrm{Log}_0\, z := \ln|z| + i\, \mathrm{Arg}_0\, z, \qquad \text{where } \mathrm{Arg}_0\, z \in]0, 2\pi[,$$

for $z \in \mathbb{C} \setminus (\mathbb{R}_+ \cup \{0\})$. The branch cut is along the positive real axis, and the argument is the angle of the vector $\overrightarrow{z} \neq \overrightarrow{0}$, measured from $\mathbb{R}_+$ in the positive sense of the plane. We define

$$(55)\quad z^a := \exp\left(a \cdot \mathrm{Log}_0\, z\right) \qquad \text{for } z \in \mathbb{C} \setminus (\mathbb{R}_+ \cup \{0\}).$$

It should be obvious, why we cannot use the principal logarithm.

Since z^a is composed of analytic functions, it is also analytic in the open domain $\Omega := \mathbb{C} \setminus (\mathbb{R}_+ \cup \{0\})$, and it is easy to prove from (55) that

$$\frac{d}{dz}\, z^a = a\, z^{a-1} \qquad \text{for } z \in \Omega,$$

which is what we also would hope for.

Put $z = r\, e^{i\Theta}$, where $r > 0$ and $0 < \Theta < 2\pi$. Then we get from (55) that

$$(56)\quad |z^a| = |\exp(a\{\ln r + i\Theta\})| = r^a,$$

because $a \in \mathbb{R}$ was chosen real.

Remark 3.4.1 It is possible to allow $a \in \mathbb{C}$, but then (56) should be modified. The details are left to the interested reader. $\Diamond$

Theorem 3.4.1 *Assume that f is analytic in $\Omega = \mathbb{C} \setminus \{z_1, \dots, z_n\}$, where none of the isolated singularities $z_1, \dots, z_n$ lies in $\mathbb{R}_+$. We assume that there exist constants α, $\beta \in \mathbb{R}$, where $\alpha < \beta$, and constants C, R_0, $r_0 > 0$, such that*

(57) $|z^{\alpha} f(z)| \leq C \qquad$ *for* $|z| \leq r_0$ *and* $z \in \Omega$,

and

(58) $\left|z^{\beta} f(z)\right| \leq C \qquad$ *for* $|z| \geq R_0$ *and* $z \in \Omega$.

If $\alpha < a < \beta$, then the improper integral $\int_0^{+\infty} f(x)\, x^a \, \frac{\mathrm{d}x}{x}$ is convergent, and its value is for $a \notin \mathbb{Z}$ given by

(59) $$\int_0^{+\infty} f(x)\, x^a \, \frac{\mathrm{d}x}{x} = -\frac{\pi\, e^{-\pi i a}}{\sin \pi a} \sum_{z_j \neq 0} res\left(f(z) z^{a-1}; z_j\right) \qquad \text{for } \alpha < a < \beta \text{ and } a \notin \mathbb{Z}.$$

Remark 3.4.2 In the literature one often requires that α, $\beta > 0$. The proof below shows that this requirement is not necessary. $\Diamond$

Proof. As usual it is only a matter of choosing the right path of integration $C_{R,r,\varphi}$ in $\Omega' := \Omega \setminus (\mathbb{R}_+ \cup \{0\})$. We choose this path as the one given in Figure 10, where we also assume that $R \geq R_0$ and $r \leq r_0$ and $0 < \varphi < \frac{\pi}{2}$ are chosen, such that all singularities $\neq 0$ of $f(z)\, z^{a-1}$ lie inside $C_{R,r,\varphi}$.

It follows from *Cauchy's residuum formula* that

$$(60)\quad 2\pi i \sum_{z_j \neq 0} \operatorname{res}\left(f(z)z^{a-1}; z_j\right) = \oint_{C_{R,r,\varphi}} f(z)z^{a-1}\,\mathrm{d}z = \int_{\Gamma_R} + \int_{-\gamma_{-\varphi}} + \int_{-\Gamma_r} + \int_{\gamma_\varphi} f(z)z^{a-1}\,\mathrm{d}z,$$

where Γ is a symbol of circular arcs oriented in the positive sense of the plane, and γ is a symbol of line segments on lines through 0 oriented away from 0.

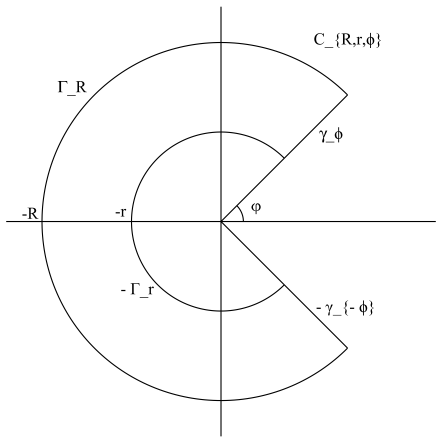

Figure 10: Path of integration $C_{R,r,\varphi}$ in the proof of Theorem 3.4.1.

We first notice that

$$\begin{aligned}
\int_{\gamma_\varphi} f(z)z^{a-1}\,\mathrm{d}z &= \int_r^R f\left(t\,e^{i\varphi}\right)\,e^{(a-1)(\ln t+i\varphi)}\,e^{i\varphi}\,\mathrm{d}t = \int_r^R f\left(t\,e^{i\varphi}\right)\,e^{(a-1)i\varphi}\,e^{i\varphi}\cdot t^{a-1}\,\mathrm{d}t \\
&\to \int_r^R f(x)x^{a-1}\,\mathrm{d}x \qquad \text{for } \varphi \to 0+,
\end{aligned}$$

because the integrand is continuous in the closed and bounded (i.e. compact) interval $[r, R]$, so we are allowed to interchange the limit and the integral.

Since the line segment $\gamma_{-\varphi}$ has the parametric description

$$z(t) = t\,e^{i(2\pi-\varphi)}, \qquad \text{for } t \in [r, R],$$

we get $\operatorname{Log}_0 z(t) = \ln t + i(2\pi - \varphi)$, so

$$\begin{aligned}
\int_{-\gamma_{-\varphi}} f(z)z^{a-1}\,\mathrm{d}z &= -\int_{\gamma_{-\varphi}} f(z)z^{a-1}\,\mathrm{d}z = +\int_r^R f\left(t\,e^{-i\varphi}\right)t^{a-1}\,e^{(a-1)(2\pi i-i\varphi)}\,e^{i(2\pi-\varphi)}\,\mathrm{d}t \\
&= \int_r^R f\left(t\,e^{-i\varphi}\right)\cdot t^{a-1}\cdot e^{ai(2\pi-\varphi)}\,\mathrm{d}t \to \int_r^R f(x)x^{a-1}\,e^{2\pi i a}\,\mathrm{d}x \qquad \text{for } \varphi \to 0+\,.
\end{aligned}$$

Hence, we get for fixed r and R,

$$\lim_{\varphi\to 0+}\left\{\int_{\gamma_\varphi}+\int_{-\gamma_{-\varphi}} f(z)z^{a-1}\,\mathrm{d}z\right\} = \int_r^R f(x)\,x^{a-1}\left\{1-e^{2\pi i a}\right\}\,\mathrm{d}x$$

$$= -2i\,e^{i\pi a}\cdot\frac{e^{i\pi a}-e^{-i\pi a}}{2i}\int_r^R f(x)\,x^{a-1}\,\mathrm{d}x = -2i\,e^{i\pi a}\,\sin\pi a\int_r^R f(x)\,x^{a-1}\,\mathrm{d}x.$$

Then we consider the estimates of the integrals along Γ_R and Γ_r. Using (56–58) we get

$$\begin{aligned}\left|\int_{\Gamma_R} f(z)z^{a-1}\,\mathrm{d}z\right| &= \left|\int_{\Gamma_R} z^\beta\, f(z)z^{a-\beta-1}\,\mathrm{d}z\right| \le 2\pi R\cdot C\cdot R^{a-\beta-1}\\ &= 2\pi C\cdot R^{-(\beta-a)}\to 0 \qquad \text{for } R\to+\infty,\end{aligned}$$

and

$$\begin{aligned}\left|\int_{\Gamma_r} f(z)z^{a-1}\,\mathrm{d}z\right| &= \left|\int_{\gamma_r} z^\alpha\, f(z)z^{a-\alpha-1}\,\mathrm{d}z\right| \le 2\pi r\cdot C\cdot r^{a-\alpha-1}\\ &= 2\pi C\cdot r^{a-\alpha}\to 0 \qquad \text{for } r\to 0+,\end{aligned}$$

where both estimates are independent of the angle $0<\varphi<\dfrac{\pi}{2}$.

When we carry out the three limits $\varphi\to 0+$ and $r\to 0+$ and $R\to+\infty$, combined with (60), it follows that the improper integral $\int_0^{+\infty} f(x)\,x^{a-1}\,\mathrm{d}x$ exists and that

$$2\pi i\sum_{z_j\neq 0}\operatorname{res}\left(f(z)z^{a-1};z_j\right) = -2i\,e^{i\pi a}\,\sin\pi a\int_0^{+\infty} f(x)\,x^{a-1}\,\mathrm{d}x,$$

so (59) follows from a rearrangement, when $a\notin\mathbb{Z}$.

If $a\in\mathbb{Z}$, and $\alpha<a<\beta$, we first compute $\int_0^{+\infty} f(x)\,x^b\,\dfrac{\mathrm{d}x}{x}$ for $\alpha<b<\beta$ and $b\notin\mathbb{Z}$, and then let $b\to a$, or, *alternatively* we may use the more direct methods developed in *Ventus, Complex Functions Theory a-2.* □

Example 3.4.1 Let $f(z)=\dfrac{1}{z^2+1}$ for $z\in\mathbb{C}\setminus\{-i,i\}$. Then none of the poles $\pm i$ lies in $\mathbb{R}_+$, and we have the estimates

$$\left|z^0 f(z)\right|\le 2 \text{ for } |z|\le r_0=\frac{1}{\sqrt{2}} \quad\text{and}\quad \left|z^2 f(z)\right|\le 2 \text{ for } |z|\ge R_0=\sqrt{2}.$$

Let $a \in]\alpha,\beta[=]0,2[$, and $a \neq 1 \in \mathbb{Z}$. Then it follows from Theorem 3.4.1 that

$$\begin{aligned}
\int_0^{+\infty} \frac{x^a}{x^2+1}\,\frac{\mathrm{d}x}{x} &= -\frac{\pi\, e^{-i\pi a}}{\sin \pi a}\left\{\operatorname{res}\left(\frac{z^{a-1}}{z^2+1};i\right)+\operatorname{res}\left(\frac{z^{a-1}}{z^2+1};-i\right)\right\} \\
&= -\frac{\pi\, e^{-i\pi a}}{\sin \pi a}\left\{\frac{\exp\left((a-1)i\,\frac{\pi}{2}\right)}{2i}+\frac{\exp\left((a-1)i\,\frac{3\pi}{2}\right)}{-2i}\right\} \\
&= -\frac{\pi\, e^{-i\pi a}}{\sin \pi a}\cdot\frac{1}{2}\left\{\exp\left(-ia\,\frac{\pi}{2}\right)+\exp\left(ia\,\frac{\pi}{2}\right)\right\} \\
&= \frac{\pi\cos\left(a\,\frac{\pi}{2}\right)}{\sin \pi a} = \frac{\pi}{2\sin\left(a\,\frac{\pi}{2}\right)}.
\end{aligned}$$

A check shows for $a=1$ that

$$\int_0^{+\infty} \frac{x^1}{x^2+1}\,\frac{\mathrm{d}x}{x} = \int_0^{+\infty}\frac{\mathrm{d}x}{x^2+1} = [\operatorname{Arctan} x]_0^{+\infty} = \frac{\pi}{2},$$

which is in agreement with the formula above, when we let $a \to 1$. Thus we have proved that

$$\int_0^{+\infty} \frac{x^{a-1}}{x^2+1}\,\mathrm{d}x = \frac{\pi}{2\sin\left(a\,\frac{\pi}{2}\right)} \qquad \text{for all } 0<a<2.$$

We notice that if $a \in \mathbb{Q}\cap]0,2[$, then this integral can *in principle*, though usually with some huge difficulties be computed by elementary calculus by using the change of variable $t = \sqrt[q]{x}$ for $x \geq 0$, where $a = \dfrac{p}{q}$ and p, $q \in \mathbb{N}$, followed by a decomposition. In practice, this method is both difficult and tedious, and even for $q=2$ we may encounter some mathematical problems. It is obvious that if $a \notin \mathbb{Q}$, e.g. if $a=\sqrt{2}$, then there is no hope of using elementary calculus when we shall find the *exact value* (and not an approximate) of the integral

$$\int_0^{+\infty} \frac{x^{\sqrt{x}-1}}{x^2+1}\,\mathrm{d}x = \frac{\pi}{2\sin\left(\frac{\pi}{\sqrt{2}}\right)}. \qquad \Diamond$$

3.5 The complex inversion formula II

We proved in Section 2.3 a residuum formula of the inverse Laplace transform of an analytic function $F(z)$, defined in a domain of the form $\mathbb{C}\setminus\{z_1,\dots,z_n\}$, and which fulfilled an estimate of the form

$$|F(z)| \leq \frac{c}{|z|^\alpha} \qquad \text{for } |z| \geq R,$$

where c, α, $R>0$ are fixed constants.

We shall in this section prove a more general inversion formula, where we only require that $F(z)$ is defined in some domain Ω, which contains a right half plane. The proof is unexpectedly using *Fourier's inversion formula.*

Theorem 3.5.1 *We make the following assumptions:*

1) *Assume that $F(z)$ is analytic in an open domain Ω which contains a right half plane $\Re z > \sigma_0$.*

2) *Assume that there exists a $\sigma_1 \geq \sigma_0$, such that for every $\delta > 0$ one can find $A > 0$, such that for every $x > \sigma_1$ and every $y \in \mathbb{R}$, for which $|y| \geq A$, we have*

$$|F(x+iy)| < \delta.$$

3) *Finally, we assume that there is an $x_0 > \sigma_0$, such that the integral*

$$(61) \quad \frac{1}{2\pi i} \int_{x_0 - i\infty}^{x_0 + i\infty} F(z)\, e^{zt}\, \mathrm{d}z, \qquad \text{for } t > 0,$$

represents a function from the class of functions $\mathcal{F}$ in the variable t. In particular, the improper integral (61) is convergent for almost every $t > 0$.

Then the function

$$(62) \quad f(t) := \frac{1}{2\pi i} \int_{x - i\infty}^{x + i\infty} F(z)\, e^{zt}\, \mathrm{d}z, \qquad \text{for } t > 0,$$

is convergent for almost every $t > 0$ and all $x > \sigma_1$.
Furthermore, except for a null set, the function $f(t)$ is independent of the choice of $x > \sigma_1$, and $f \in \mathcal{F}$, and

$$\mathcal{L}\{f\}(z) = F(z).$$

Thus, the inverse Laplace transform of $F(z)$ is, apart from a null set, given by (62).

PROOF. Let x_0 be given as above, and choose any $x > \sigma_1$. Finally, choose $t > 0$, such that the improper line integral along the vertical line through x_0 is convergent.

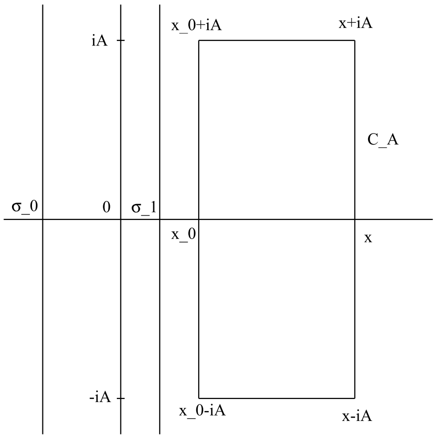

Figure 11: Path of integration C_A in the proof of Theorem 3.5.1.

Given $\delta > 0$, choose $A > 0$ as in the condition above and consider the simple, closed curve C_A as given in Figure 11. It follows from Cauchy's integral theorem that

$$\frac{1}{2\pi i}\oint_{C_A} F(z)\, e^{zt}\, \mathrm{d}z = 0,$$

hence by a rearrangement.

$$\frac{1}{2\pi i}\int_{x-iA}^{x+iA} F(z)\, e^{zt}\, \mathrm{d}t - \frac{1}{2\pi i}\int_{x_0-iA}^{x_0+iA} F(z)\, e^{zt}\, \mathrm{d}z$$
$$= \frac{1}{2\pi i}\int_{x_0-iA}^{x-iA} F(z)\, e^{zt}\, \mathrm{d}t - \frac{1}{2\pi i}\int_{x_0+iA}^{x+iA} F(z)\, e^{zt}\, \mathrm{d}z.$$

If we can prove that the right hand side of this equation tends towards zero for $A \to +\infty$, when $x > \sigma_1$ is kept fixed, then it follows that (62) is independent of the choice of $x > \sigma_1$. This follows easily from the estimate

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{x_0+iA}^{x+iA} F(z)\, e^{zt}\, \mathrm{d}t\right| &= \frac{1}{2\pi i}\left|\int_{x_0}^{x} F(\xi+iA)\, e^{\xi t}\, e^{iAt}\, \mathrm{d}z\right| \\ &\le \frac{1}{2\pi}\cdot \exp\left(\max\left\{x_0 t, xt\right\}\right)\cdot |x-x_0|\cdot\delta.\end{aligned}$$

Since x_0, x and t are fixed, and $\delta > 0$ arbitrary, and the choice of A only depends on $\delta > 0$, and since we also have

$$\left|\frac{1}{2\pi i}\int_{x_0-iA}^{x-iA} F(z)\, e^{zt}\, \mathrm{d}z\right| \le \frac{1}{2\pi}\, \exp\left(\max\left\{x_0 t, xt\right\}\right)\cdot |x-x_0|\cdot\delta,$$

the claim follows by letting $\delta \to 0+$, and we have proved that (62) is independent of $x > \sigma_1$. In particular, $f \in \mathcal{F}$, and there exists a $\sigma_2 \in \mathbb{R}$, such that

$$(63)\quad \int_0^{+\infty} e^{-\sigma t}\,|f(t)|\,\mathrm{d}t < +\infty \qquad \text{for every } \sigma > \sigma_2.$$

We define for every fixed $x > \max\{\sigma_1, \sigma_2\}$ an auxiliary function φ_x by

$$\varphi_x(t) = \begin{cases} e^{-xt}\,f(t) & \text{for } t > 0, \\[2mm] \quad 0 & \text{for } t \le 0. \end{cases}$$

It follows from 63) that $\varphi_x \in L^1(\mathbb{R})$.

If $z = x + iy$, where $x > \max\{\sigma_1, \sigma_2\}$, then

$$\mathcal{L}\{f\}(z) = \int_0^{\infty} e^{-zt}\,f(t)\,\mathrm{d}t = \int_0^{+\infty} e^{-(x+iy)t}\,f(t)\,\mathrm{d}t = \int_0^{+\infty} e^{-iyt}\,\varphi_x(t)\,\mathrm{d}t = \hat{\varphi}_x(y),$$

which shows that the Laplace transform of f at the point $z = x + iy$, where $x > \max\{\sigma_1, \sigma_2\}$, is equal to the Fourier transform of φ_x at the point $y \in \mathbb{R}$.

Since $\varphi_x \in L^1$ for $x > \sigma_2$, Fourier's inversion formula holds for $\hat{\varphi}_x$, hence

$$\varphi_x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{\varphi}_x(y)\,e^{ity}\,\mathrm{d}y = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathcal{L}\{f\}(x+iy)\,e^{ity}\,\mathrm{d}y.$$

Multiply this equation by e^{xt}. Then we get for $t > 0$ that

$$f(t) = e^{xt}\varphi_x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathcal{L}\{f\}(x+iy)\,e^{(x+iy)t}\,\mathrm{d}y = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} \mathcal{L}\{f\}(z)\,e^{zt}\,\mathrm{d}z,$$

where the line integral is taken along the straight vertical line which goes through $x > \max\{\sigma_1, \sigma_2\}$ on the x-axis.

On the other hand, $f(t)$ is given by (62), and since both improper integrals are convergent for $x > \max\{\sigma_1, \sigma_2\}$, it follows that

$$\int_{x-i\infty}^{x+i\infty} \{F(z) - \mathcal{L}\{f\}(z)\}\,e^{zt}\,\mathrm{d}z = 0 \qquad \text{for almost every } t > 0,$$

and we conclude that

$$(64)\quad F(z) = \mathcal{L}\{f\}(z) \qquad \text{for almost every } z, \text{ for which } \Re\, z > \max\{\sigma_1, \sigma_2\}.$$

Finally, both $F(z)$ and $\mathcal{L}\{f\}(z)$ are analytic functions, in particular continuous, so (64) holds for every z, for which $\Re\, z > \max\{\sigma_1, \sigma_2\}$, and the theorem is proved. □

Remark 3.5.1 Theorem 3.5.1 is more general than Theorem 43, because it also covers cases, where we consider branches of many-valued functions where the branch cuts lie in a left half plane. The integral (62) is called a *Bromwich integral.* ◊

Example 3.5.1 Consider the function

$$F(z) = \frac{1}{z\sqrt{z^2+1}}, \qquad \text{for } \Re z > 0,$$

where we have chosen the branch of the square root which has its branch cut lying along the real negative axis, and where we put $\sqrt{1} = +1$. Then $F(x+it) \in L^1(\mathbb{R})$ as a function in $t \in \mathbb{R}$ for every fixed $x > 0$, and

$$|F(z)| \le \frac{2}{|z|^2} \qquad \text{for } \Re z > 0 \text{ and } |z| \ge 2.$$

It follows that the assumptions of Theorem 3.5.1 are trivially fulfilled. Hence

$$f(t) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} \frac{e^{zt}}{z\sqrt{z^2+1}}\,\mathrm{d}z.$$

We proved in Example 2.4.2 that

$$\mathcal{L}\left\{J_0\right\}(z) = \frac{1}{\sqrt{z^2+1}} \qquad \text{for } \Re z > 0,$$

where J_0 is the Bessel function of order 0. Therefore,

$$\int_0^t J_0(\tau)\,\mathrm{d}\tau = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} \frac{e^{zt}}{z\sqrt{z^2+1}}\,\mathrm{d}z, \qquad t > 0 \text{ and } x > 0,$$

where the integral on the left hand side is computed by termwise integration of the series expansion of $J_0(\tau)$, thus

$$\int_0^t J_0(\tau)\,\mathrm{d}\tau = \int_0^t \sum_{k=0}^{+\infty} \frac{(-1)^k}{(k!)^2}\left(\frac{\tau}{2}\right)^{2k}\mathrm{d}\tau = \sum_{k=0}^{+\infty} \frac{2(-1)^k}{(k!)^2(2k+1)}\left(\frac{t}{2}\right)^{2k+1} = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} \frac{e^{zt}}{z\sqrt{z^2+1}}\,\mathrm{d}z.$$

Obviously, the branch cut along $\mathbb{R}_-$ causes that Theorem 2.3.4 cannot be applied. At the same time we also notice that the elementary functions known from Calculus are far from sufficient, when we shall find the inverse Laplace transform of $F(z)$. $\Diamond$

3.6 Laplace transformation of series

If the real function $f(t)$ has a power series expansion which is convergent in all of $\mathbb{R}$, then it is tempting to compute its Laplace transform by termwise integration. The precise result is the following.

Theorem 3.6.1 *Assume that we have the power series expansion*

$$f(t) = \sum_{n=0}^{+\infty} a_n\, t^n, \qquad \textit{for } t > 0,$$

and that

$$g(w) := \sum_{n=0}^{+\infty} n!\, a_n\, w^n \qquad \textit{is convergent for } |w| < R.$$

Then the Laplace transform of $f(t)$ is given by

$$(65)\quad \mathcal{L}\{f\}(z) = \frac{1}{z}\,g\left(\frac{1}{z}\right) = \sum_{n=0}^{+\infty} \frac{n!\,a_n}{z^{n+1}} \qquad \textit{for } |z| > \frac{1}{R}.$$

Proof. Choose a $z \in \mathbb{C}$, such that $\Re z > \dfrac{1}{R}$. The series of $f(t)$ is absolutely and uniformly convergent for $t \in [0, r]$. Hence, for every $r > 0$,

$$\int_0^r e^{-zt}\,f(t)\,\mathrm{d}t = \sum_{n=0}^{+\infty} a_n \int_0^r e^{-zt}\,t^n\,\mathrm{d}t.$$

Then by taking the limit $r \to +\infty$,

$$\mathcal{L}\{f\}(z) = \lim_{r\to+\infty} \int_0^r e^{-zt}\,f(t)\,\mathrm{d}t = \sum_{n=0}^{+\infty} a_n \int_0^{+\infty} e^{-zt}\,t^n\,\mathrm{d}t = \sum_{n=0}^{+\infty} a_n\,n! \cdot \frac{1}{z^{n+1}}.$$

To every $0 < \varepsilon < R$ there exists an n_0, such that

$$|n!\, a_n (R-\varepsilon)^n| \le 1 \qquad \text{for every } n \ge n_0,$$

thus

$$|a_n| \le \frac{1}{n!} \cdot \frac{1}{(R-\varepsilon)^n} \qquad \text{for } n \ge n_0.$$

Then for $t > 0$ we get the estimate

$$|f(t)| = \left| \sum_{n=0}^{+\infty} a_n\, t^n \right| \le \sum_{n=0}^{+\infty} \frac{1}{n!} \, \frac{t^n}{(R-\varepsilon)^n} + A_{n_0}(t) = \exp\left(\frac{t}{R-\varepsilon}\right) + A_{n_0}(t),$$

where

$$A_{n_0}(t) = \sum_{n=0}^{n_0} |a_n|\; t^n$$

is a finite sum. It follows that $f \in \mathcal{E}$, so the Laplace transform exists, and the theorem follows by an analytic extension. □

Theorem 3.6.2 *If $F(z)$ has the Laurent series expansion*

$$F(z) = \sum_{n=0}^{+\infty} b_n \cdot \frac{1}{z^{n+1}} \qquad \textit{for } |z| > \frac{1}{R},$$

then the inverse Laplace transform of $F(z)$ is given by

$$f(t) = \sum_{n=0}^{+\infty} \frac{1}{n!}\, b_n\, t^n, \qquad \textit{for } t > 0.$$

Proof. If we write $a_n = \dfrac{1}{n!}\, b_n$, then Theorem 3.6.2 is an immediate consequence of Theorem 3.6.1. □

The factor $n!$ of the transition formulæ $b_n = n!\, a_n$, or $a_n = \dfrac{1}{n!}\, b_n$, indicates once more that if we shall get an optimum benefit of the theory of the Laplace transformation, then we need some new functions, which are not included in the usual courses of elementary Calculus. These "new" transcendental functions are for historical reasons called *Special Functions*. A better word, however, would be *Useful Functions*, because they almost certainly always pop up in more advanced applications of the Laplace transformation in the technical sciences. We see here that these special functions are necessary, if we want to apply the theory of Laplace transformation on other cases than just when $F(z)$ is a quotient between two polynomials. The introduction of the most necessary special functions is given in Chapter 1 of *Ventus, Complex Functions Theory a-5, The Laplace Transformation II.*

Example 3.6.1 We shall here give an example of what we may encounter in general. We choose arbitrarily $F(z) = \sin \dfrac{1}{z}$, thus by a Laurent series expansion,

$$F(z) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{1}{z^{2n+1}} = \sum_{k=0}^{+\infty} b_k \cdot \frac{1}{z^{k+1}} \qquad \text{for } z \in \mathbb{C} \setminus \{0\}.$$

Hence,

$$b_{2n} = \frac{(-1)^n}{(2n+1)!} \quad \text{and} \quad b_{2n+1} = 0 \qquad \text{for all } n \in \mathbb{N}_0.$$

An application of Theorem 3.6.2 shows that the inverse Laplace transform $f(t)$ of $F(z)$ is given by

$$f(t) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!(2n+1)!}\, t^{2n} \qquad \text{for } t > 0,$$

which cannot be described explicitly by any function known from Elementary Calculus. ♢

Example 3.6.2 Let $\sqrt{w}$ denote the branch of the square root which has its branch cut lying along the real negative axis, and for which $\sqrt{r} > 0$ for $r > 0$. Then

$$F(z) = \frac{1}{z\sqrt{1+\dfrac{1}{z^2}}} \qquad \text{for } |z| > 1,$$

is an analytic function with no branch cut at all in the domain $|z| > 1$. Its Laurent series expansion is found by an application of the *generalized binomial series*,

$$F(z) = \frac{1}{z\sqrt{1+\dfrac{1}{z^2}}} = \frac{1}{z}\sum_{n=0}^{+\infty} \binom{-\frac{1}{2}}{n} \frac{1}{z^{2n}} = \frac{1}{z}\sum_{k=0}^{+\infty} b_k\, \frac{1}{z^k} \qquad \text{for } |z| > 1,$$

where $b_{2n+1} = 0$ and

$$\begin{aligned} b_{2n} &= \binom{-\frac{1}{2}}{n} = \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(\frac{1}{2}-n\right)}{n!} = \frac{(-1)^n}{2^n}\cdot\frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!} \\ &= \frac{(-1)^n}{2^n}\cdot\frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots(2n-1)(2n)}{n!\cdot 2\cdot 4\cdot 6\cdots 2n} = \frac{(-1)^n}{2^{2n}}\cdot\frac{(2n)!}{n!\,n!} = \frac{(-1)^n}{2^{2n}}\binom{2n}{n}. \end{aligned}$$

It follows from Theorem 3.6.2 that

$$a_{2n} = \frac{(-1)^n}{2^{2n}}\cdot\frac{1}{(n!)^2} \qquad \text{and} \qquad a_{2n+1} = 0,$$

hence

$$f(t) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(n!)^2}\left(\frac{t}{2}\right)^{2n} = J_0(t) \qquad \text{for } t > 0,$$

where we recognize the series expansion of $J_0(t)$, the *Bessel function of order 0*.

The right half plane $\Re\, z > 1$ is contained in the set of all z for which $|z| > 1$, hence

$$\mathcal{L}\left\{J_0\right\}(z) = \frac{1}{z\sqrt{1+\dfrac{1}{z^2}}} \qquad \text{for } \Re\, z > 1,$$

where the square root has it branch cut along $\mathbb{R}_-$, and where $\sqrt{1} = +1$.

It follows by an analytic extension that this also holds for $\Re z > 0$ and that $\sqrt{z^2} = z$ for $\Re z > 0$ for the chosen branch of the square root. (Notice that this is not true in the set $|z| > 1$.) We therefore conclude that

$$\mathcal{L}\{J_0\}(z) = \frac{1}{z\sqrt{1+\dfrac{1}{z^2}}} = \frac{1}{\sqrt{z^2}\sqrt{1+\dfrac{1}{z^2}}} = \frac{1}{\sqrt{z^2\left(1+\dfrac{1}{z^2}\right)}} = \frac{1}{\sqrt{z^2+1}} \qquad \text{for } \Re z > 0,$$

so by quite a different method we fortunately obtained the same result as in Example 3.6.1.

The present example also shows that we in general only obtain *uniqueness* of the Laplace transform in some right half plane, because the analytic extension depends on where the possible branch cuts are lying. Her we see that both

$$\frac{1}{z\sqrt{1+\dfrac{1}{z^2}}} \quad \text{for } |z| > 1, \qquad \text{and} \qquad \frac{1}{\sqrt{z^2+1}} \quad \text{for } \Re z > 0,$$

are indeed Laplace transforms of $J_0(t)$, and they agree on the right half plane $\Re z > 1$. However, no matter how we extend $\dfrac{1}{\sqrt{z^2+1}}$ analytically to a part of the closed left half plane $\Re z \leq 0$, this extension will not match the Laplace transform of $J_0(t)$ for $|z| > 1$. $\Diamond$

3.7 A catalogue of methods of finding the Laplace transform and the inverse Laplace transform

In this final section we shall sketch a catalogue of possible methods when we want to compute either the Laplace transform or the inverse Laplace transform of a given function. We also list some methods, where the corresponding examples are postponed to *Ventus, Complex Functions Theory a-5, The Laplace Transformation II.*

3.7.1 Methods of finding Laplace transforms

1) *Direct application of the definition*

$$\mathcal{L}\{f\}(z) := \int_0^{+\infty} e^{-zt}\, f(t)\, \mathrm{d}t.$$

The simplest examples of Laplace transforms were computed by using the definition alone. It is possible in some more complicated cases, though in far from all of them, to use this primitive approach which only requires some knowledge of practical integration.

2) *The method of series expansion.*
If

$$f(t) = \sum_{n=0}^{+\infty} a_n\, t^n \qquad \text{is convergent for all } t > 0,$$

then formally,

$$\mathcal{L}\{f\} \sum_{n=0}^{+\infty} n!\, a_n \cdot \frac{1}{z^{n+1}}.$$

If this Laurent series expansion is convergent for $|z| > R$, where $R < +\infty$ is finite, then it is indeed the Laplace transform of $f(t)$, defined in the set, where $|z| > R$. This is the result of Theorem 3.6.1 in Section 3.6.
The method requires at least that $f(t)$ is the restriction of an analytic function $f(z)$ defined in all of $\mathbb{C}$. Another requirement is of course that the indicated Laurent series of $\mathcal{L}\{f\}(z)$ is indeed convergent for $|z| > R$ for some finite R.

3) *The method of Laplace transforming a differential equation.*
This method was applied in Example 2.4.2, where we found the Laplace transform of the Bessel function $J_0(t)$ of order 0. We apply the Laplace transformation on the Bessel equation of order 0, and then use the rules of computation to get a (hopefully) simpler differential equation in the complex variable z. If it really is simpler, then we solve it by elementary methods. Finally, the constants are found either by the initial value theorem or the final value theorem.

4) *Differentiation with respect to a parameter.*
An example of this method is postponed to *Ventus, Complex Functions Theory a-5, The Laplace Transformation II*, where we shall find the Laplace transform of the function $f(t) = \ln t$, $t > 0$.

5) *Application of tables.*
This method requires that one owns a sufficiently large table to cover all the functions needed. So far we have only systematically found the Laplace transforms of Table 1 on page 17. A larger table is given by Table 2 on page 102.

3.7.2 Computation of inverse Laplace transforms

1) *Decomposition.*
This method is used, when

$$F(z) = \frac{P(z)}{Q(z)}\, e^{-az},$$

where $P(z)$ and $Q(z)$ are polynomials, for which deg $Q >$ deg P. One decomposes the fraction $\dfrac{P(z)}{Q(z)}$. The exponential e^{-az} is handled by using the *Second Translation Property.*

2) *Method of series expansion.*
If

$$\mathcal{L}\{f\}(z) = \sum_{n=0}^{+\infty} b_n \cdot \frac{1}{z^{n+1}} \qquad \text{for } |z| > R,$$

then

$$f(t) = \sum_{n=0}^{+\infty} \frac{1}{n!}\, b_n\, t^n \qquad \text{for } t > 0.$$

This method was used in Example 3.6.1 and in Example 3.6.2. It requires that the given function $\mathcal{L}\{f\}(z)$ can be written as a convergent Laurent series for $|z| > R$, and that $a_n = 0$ for every $n \in \mathbb{N}_0$.

3) *The method of applying a differential equation.*
First establish a differential equation of the given function $\mathcal{L}\{f\}(z) = F(z)$. Then use the rules of computation, given in Section 2.2, to obtain a differential equation of $f(t)$, which hopefully is simple enough to be solved.

4) *Differentiation with respect to a parameter.*
An example of this method is postponed to *Ventus, Complex Functions Theory a-5, The Laplace transformation II.*

5) *Applications of tables.*
The comments are the same as above. We shall of course use either Table 1 on page 17, or Table 2 on page 102.

6) *The complex inversion formula.*
This is given in two versions. The simpler one in Section 2.3 is a residuum formula, so one shall always be extremely careful of checking all the assumptions, because residuum formulæ usually give very wrong "results", when these are not satisfied.
The general formula of Section 3.5 is a line integral (a *Bromwich integral*) along a vertical line. This method is in particular used when $F(z)$ is a branch of a many-valued function. We shall postpone the more difficult applications of this formula to *Ventus, Complex Functions Theory a-5, The Laplace Transformation II.*

4 Tables

	$f(t), \quad t \geq 0$	$\mathcal{L}\{f\}(z) = F(z)$
1	1	$\dfrac{1}{z}$
2	e^{at}	$\dfrac{1}{z-a}$
3	$\dfrac{1}{T}\exp\left(-\dfrac{t}{T}\right)$	$\dfrac{1}{1+Tz}$
4	$\dfrac{1}{(n-1)!}\,t^{n-1}, \quad n \in \mathbb{N}$	$\dfrac{1}{z^n}, \quad n \in \mathbb{N}$
5	$\dfrac{1}{(n-1)!}\,t^{n-1}\,e^{at}, \quad n \in \mathbb{N}$	$\dfrac{1}{(z-a)^n}, \quad n \in \mathbb{N}$
6	$\dfrac{e^{at}-e^{bt}}{a-b}$	$\dfrac{1}{(z-a)(z-b)}, \quad a \neq b$
7	$\sin \omega t$	$\dfrac{\omega}{z^2+\omega^2}$
8	$\sinh \omega t$	$\dfrac{\omega}{z^2-\omega^2}$
9	$\cos \omega t$	$\dfrac{z}{z^2+\omega^2}$
10	$\cosh \omega t$	$\dfrac{z}{z^2-\omega^2}$
11	$\dfrac{e^{bt}-e^{at}}{t}$	$\mathrm{Log}\left(\dfrac{z-a}{z-b}\right)$
12	$(-1)^{n-1}$ for $t \in\,](n-1)a, na[,\ n \in \mathbb{N}$	$\dfrac{1}{z}\,\tanh\dfrac{az}{2}$

Table 2: Some Laplace transforms,

$$\mathcal{L}\{f\}(\xi) = F(z) = \int_0^{+\infty} e^{-zt}\,f(t)\,\mathrm{d}t.$$

	$f(x)$	$\mathcal{F}\{f\}(\xi) = F(\xi)$
1	$\chi_{[-T,T]}(x), \quad T > 0$	$2\,\dfrac{\sin T\xi}{\xi}$
2	$\left(1 - \dfrac{\lvert x\rvert}{T}\right)\chi_{[-T,T]}, \quad T > 0$	$\dfrac{4}{T\xi^2}\,\sin^2\left(\dfrac{T\xi}{2}\right)$
3	$\dfrac{a}{x^2+a^2}, \quad \Re\, a > 0$	$\pi\, e^{-a\lvert\xi\rvert}$
4	$\dfrac{\sin Tx}{x}, \quad T > 0$	$\pi\,\chi_{[-T,T]}(\xi)$
5	$\cos(\omega x)\cdot\chi_{[-T,T]}(x), \quad T > 0$	$\dfrac{\sin(T(\xi-\omega))}{\xi-\omega} + \dfrac{\sin(T(\xi+\omega))}{\xi+\omega}$
6	$\sin(\omega x)\cdot\chi_{[-T,T]}(x), \quad T > 0$	$\dfrac{1}{i}\left\{\dfrac{\sin(T(\xi-\omega))}{\xi-\omega} - \dfrac{\sin(T(\xi+\omega))}{\xi+\omega}\right\}$
7	$e^{-a\lvert x\rvert}, \quad \Re a > 0$	$\dfrac{2a}{\xi^2+a^2}$
8	$e^{-ax}H(x), \quad \Re\, a > 0$	$\dfrac{1}{a+i\xi}$
9	$e^{ax}H(-x), \quad \Re\, a > 0$	$\dfrac{1}{a-i\xi}$
10	$\exp\left(-ax^2\right), \quad a > 0$	$\sqrt{\dfrac{\pi}{2}}\cdot\exp\left(-\dfrac{\xi^2}{4a}\right)$
11	1	$2\pi\,\delta$
12	$x^2, \quad n \in \mathbb{N}_0$	$2\pi\, i^n\,\delta^{(n)}$
13	$e^{ihx}, \quad h \in \mathbb{R}$	$2\pi\,\delta_{(h)}$
14	$\cos(hx), \quad h \in \mathbb{R}$	$\pi\left\{\delta_{(h)} + \delta_{(-h)}\right\}$
15	$\sin(hx), \quad h \in \mathbb{R}$	$-i\pi\left\{\delta_{(h)} - \delta_{(-h)}\right\}$
16	δ	1
17	$\delta_{(h)}, \quad h \in \mathbb{R}$	$e^{-ih\xi}$
18	$\delta^{(n)}, \quad n \in \mathbb{N}_0$	$(i\xi)^n$

Table 3: Some Fourier transforms,

$$\mathcal{F}\{f\}(\xi) = F(\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi}\, f(x)\,\mathrm{d}x.$$

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