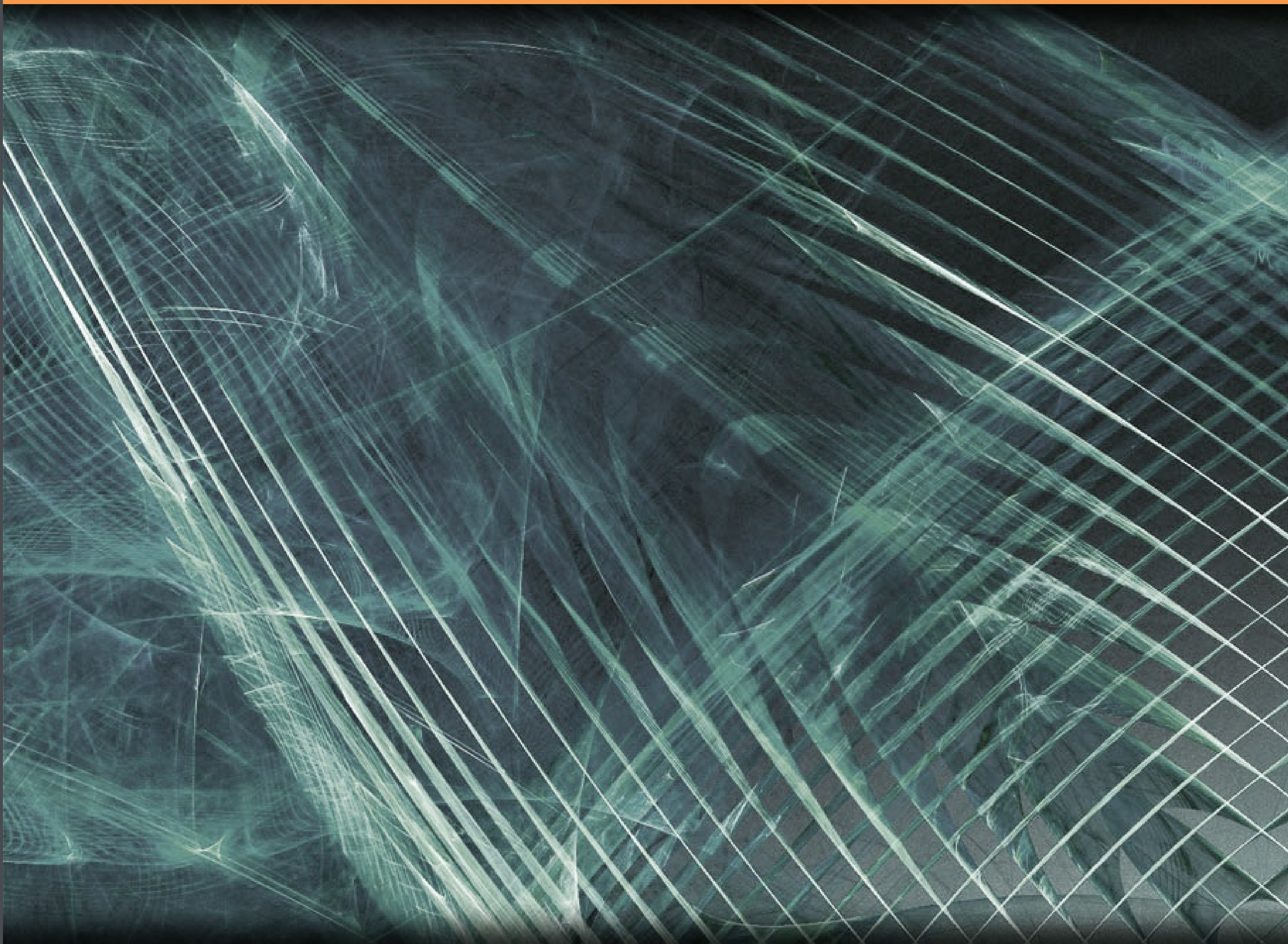
bookboon.com

Complex Functions Theory c-12

Leif Mejlbro



Download free books at

bookboon.com

Leif Mejlbro

Tha Laplace Transformation II

Complex functions theory c-12

Download free eBooks at bookboon.com

The Laplace Transformation II c-12
© 2011 Leif Mejlbro & Ventus Publishing ApS
ISBN 978-87-7681-763-3

Download free eBooks at bookboon.com

Contents

Download free eBooks at bookboon.com

Introduction

In this volume we give some examples of the elementary part of the theory of the *Laplace transformation* as described in *Ventus, Complex Functions Theory a-5, The Laplace Transformation II.* The chapters and the sections will follow the same structure as in the above mentioned book on the theory.

The examples have been collected about 30 years ago from some long forgotten book on applications. It was then pointed out by the author, and repeated here that one should not uncritically apply the Laplace transformation in all cases. Sometimes the simpler methods known from plain Calculus may be easier to apply.

Leif Mejlbro
March 31, 2011

Download free eBooks at bookboon.com

1 Special Functions

1.1 The Gamma Function

Example 1.1.1 *Compute* $\Gamma\left(-n-\frac{1}{2}\right)$ *for every* $n \in \mathbb{N}_0$.

We shall take for granted that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, and also the functional equation of the Gamma function,

$$\Gamma(z+1) = z\Gamma(z),$$

from which

$$\Gamma(z) = \frac{1}{z}\,\Gamma(z+1) \qquad \text{for } z \neq 0.$$

We get by a simple iteration,

$$\begin{aligned}\Gamma\left(-n-\frac{1}{2}\right) &= \frac{-1}{n+\frac{1}{2}}\cdot\Gamma\left(-(n-1)-\frac{1}{2}\right) = \frac{(-1)^2}{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}\,\Gamma\left(-n-\frac{1}{2}+2\right) = \cdots \\ &= \frac{(-1)^{n+1}}{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)\cdots\frac{1}{2}}\,\Gamma\left(\frac{1}{2}\right) = \frac{(-1)^{n+1}\,2^{n+1}\,\sqrt{\pi}}{(2n+1)(2n-1)\cdots 3\cdot 1} \\ &= (-1)^{n+1}\,\frac{2^{2n+1}\,n!\,\sqrt{\pi}}{(2n+1)!}. \qquad \Diamond\end{aligned}$$

Click on the ad to read more

Download free eBooks at bookboon.com

Example 1.1.2 *Compute* $\mathcal{L}\left\{\sqrt{t}+\frac{1}{\sqrt{t}}\right\}(z)$.

We get by a straightforward computation for $\Re z > 0$, that

$$\mathcal{L}\left\{\sqrt{t}+\frac{1}{\sqrt{t}}\right\}(z)=\frac{\Gamma\left(\frac{3}{2}\right)}{z^{\frac{3}{2}}}+\frac{\Gamma\left(\frac{1}{2}\right)}{z^{\frac{1}{2}}}=\frac{\frac{1}{2}\sqrt{\pi}}{z\sqrt{z}}+\frac{\sqrt{\pi}}{\sqrt{z}}=\sqrt{\frac{\pi}{z}}\cdot\left(1+\frac{1}{2z}\right). \qquad \Diamond$$

Example 1.1.3 *Compute* $\mathcal{L}\left\{\left(1+\sqrt{t}\right)^4\right\}(z)$.

We first compute

$$\left(1+\sqrt{t}\right)^4=1+4\,t^{\frac{1}{2}}+6t+4\,t^{\frac{3}{2}}+t^2.$$

From this result we then get for $\Re z > 0$,

$$\begin{aligned}\mathcal{L}\left\{\left(1+\sqrt{t}\right)^4\right\}(z) &= \frac{1}{z}+4\,\frac{\Gamma\left(\frac{3}{2}\right)}{z^{\frac{3}{2}}}+6\,\frac{\Gamma(2)}{z^2}+4\,\frac{\Gamma\left(\frac{5}{2}\right)}{z^{\frac{5}{2}}}+\frac{\Gamma(3)}{z^3}\\ &= \frac{1}{z}+\frac{4\cdot\frac{1}{2}\sqrt{\pi}}{z\sqrt{z}}+\frac{6}{z^2}+\frac{4\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{z^2\sqrt{z}}+\frac{2}{z^3}\\ &= \frac{1}{z}+\frac{2\sqrt{\pi}}{z\sqrt{z}}+\frac{6}{z^2}+\frac{3\sqrt{\pi}}{z^2\sqrt{z}}+\frac{2}{z^3},\end{aligned}$$

where $\sqrt{\cdot}$ as usual denotes the branch of the square root which is positive on $\mathbb{R}_+$, and which has its branch cut lying along $\mathbb{R}_-$. $\Diamond$

Example 1.1.4 *Compute* $\mathcal{L}\left\{t^{\frac{7}{2}}\,e^{3t}\right\}(z)$.

It follows by a straightforward computation, using one of the rules of the Laplace transformation, that

$$\begin{aligned}\mathcal{L}\left\{t^{\frac{7}{2}}\,e^{3t}\right\}(z) &= \mathcal{L}\left\{t^{\frac{7}{2}}\right\}(z-3)=\frac{\Gamma\left(\frac{9}{2}\right)}{(z-3)^{\frac{9}{2}}}=\frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{(z-3)^4\sqrt{z-3}}\\ &=\\ &= \frac{105}{16}\sqrt{\pi}\cdot\frac{1}{(z-3)^4\sqrt{z-3}} \qquad \text{for } \Re z > 3. \qquad \Diamond\end{aligned}$$

Download free eBooks at bookboon.com

Example 1.1.5 *Find all real constants a, b, α, β and λ, for which*

$$\mathcal{L}\left\{a\,t^{-\alpha}+b\,t^{-\beta}\right\}(z)=\lambda\cdot\left\{a\,z^{-\alpha}+b\,z^{-\beta}\right\}.$$

If $a=-b$ and $\alpha=\beta$, then the relation is trivial for all λ, because both the left hand side and the right hand side are 0.

We assume that this is not the case. Then we must have $0<\alpha,\ \beta<1$, and it follows that

$$\mathcal{L}\left\{a\,t^{-\alpha}+b\,t^{-\beta}\right\}(z)=a\,\frac{\Gamma(1-\alpha)}{z^{1-\alpha}}+b\,\frac{\Gamma(1-\beta)}{z^{1-\beta}}=\lambda\cdot a\cdot\frac{1}{z^{\alpha}}+\lambda\cdot b\cdot\frac{1}{z^{\beta}},$$

if one of the following two possibilities is fulfilled.

1)

$$a\,\frac{\Gamma(1-\alpha)}{z^{1-\alpha}}=\lambda\cdot a\cdot\frac{1}{z^{\alpha}}\qquad\text{and}\qquad b\cdot\frac{\Gamma(1-\beta)}{z^{1-\beta}}=\lambda\cdot b\cdot\frac{1}{z^{\beta}}.$$

We have three possibilities.

a) If $a\neq 0$ and $b\neq 0$, then $1-\alpha=\alpha$ and $1-\beta=\beta$, so $\alpha=\beta=\frac{1}{2}$, which implies that $\lambda=\Gamma(1-\alpha)=\Gamma\left(1-\frac{1}{2}\right)=\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$, and $a\neq 0$ and $b\neq 0$ arbitrary.

b) If $a=0$ and $b\neq 0$, then α is arbitrary, while we still have $\beta=\frac{1}{2}$ and $\lambda=\sqrt{\pi}$, and $b\neq 0$ arbitrary.

c) If $a\neq 0$ and $b=0$, then β is arbitrary, while we still have $\alpha=\frac{1}{2}$ and $\lambda=\sqrt{\pi}$, and $a\neq 0$ is arbitrary.

2)

$$a\cdot\frac{\Gamma(1-\alpha)}{z^{1-\alpha}}=\lambda\cdot b\cdot\frac{1}{z^{\beta}}\qquad\text{and}\qquad b\cdot\frac{\Gamma(1-\beta)}{z^{1-\beta}}=\lambda\cdot a\cdot\frac{z^{\alpha}}{,}$$

hence $\alpha+\beta=1$ (or, equivalently, $\beta=1-\alpha$), and

$$\lambda=\frac{a}{b}\,\Gamma(1-\alpha)=\frac{b}{a}\,\Gamma(1-\beta),$$

so

$$\frac{a}{b}=\pm\sqrt{\frac{\Gamma(1-\beta)}{\Gamma(1-\alpha)}}=\pm\sqrt{\frac{\Gamma(\alpha)}{\Gamma(\beta)}}=\pm\sqrt{\frac{\Gamma(\alpha)}{\Gamma(1-\alpha)}\cdot\frac{\Gamma(\alpha)}{\Gamma(\alpha)}}=\pm\Gamma(\alpha)\sqrt{\frac{\sin\alpha\pi}{\pi}}.$$

Thus,

$$\lambda=\frac{a}{b}\cdot\Gamma(1-\alpha)=\pm\Gamma(\alpha)\Gamma(1-\alpha)\sqrt{\frac{\sin\alpha\pi}{\pi}}=\pm\sqrt{\frac{\pi}{\sin\alpha\pi}}.$$

Summing up we get in this case

$$\alpha\in\,]0,1[,\qquad\text{and}\qquad\beta=1-\alpha\in\,]0,1[,$$

$$a=\pm\Gamma(\alpha)\cdot\sqrt{\frac{\sin\alpha\pi}{\pi}}\cdot b\qquad\text{and}\qquad\lambda=\pm\sqrt{\frac{\pi}{\sin\alpha\pi}}.\qquad\Diamond.$$

Download free eBooks at bookboon.com

Example 1.1.6 *1) Compute the Laplace transform of* $\dfrac{1}{\sqrt[3]{t}}\sin t$.

2) Explain why the improper integral $\int_0^{+\infty} x\,\sin\left(x^3\right)\,\mathrm{d}x$ *is convergent.*

3) Apply the result above to compute the integral $\int_0^{+\infty} x\,\sin\left(x^3\right)\,\mathrm{d}x$.

1) Assume that $\Re z > 0$. Then it follows by a straightforward computation that

$$
\begin{aligned}
\mathcal{L}\left\{\frac{1}{\sqrt[3]{t}}\sin t\right\}(z) &= \int_+^{+\infty} \frac{1}{\sqrt[3]{t}}\cdot\frac{1}{2}\left\{e^{it}-e^{-it}\right\}e^{-zt}\,\mathrm{d}t \\
&= \frac{1}{2i}\int_0^{+\infty}\frac{1}{\sqrt[3]{t}}\,e^{-(z-i)t}\,\mathrm{d}t - \frac{1}{2i}\int_0^{+\infty}\frac{1}{\sqrt[3]{t}}\,e^{-(z+i)t}\,\mathrm{d}t \\
&= \frac{1}{2i}\mathcal{L}\left\{t^{-\frac{1}{3}}\right\}(z-i) - \frac{1}{2i}\mathcal{L}\left\{t^{-\frac{1}{3}}\right\}(z+i) \\
&= \frac{1}{2i}\,\frac{\Gamma\left(\frac{2}{3}\right)}{(z-i)^{\frac{2}{3}}} - \frac{1}{2i}\,\frac{\Gamma\left(\frac{2}{3}\right)}{(z+i)^{\frac{2}{3}}} = \frac{\Gamma\left(\frac{2}{3}\right)}{2i}\cdot\frac{(z+i)^{\frac{2}{3}}-(z-i)^{\frac{2}{3}}}{\left(z^2+1\right)^{\frac{2}{3}}}.
\end{aligned}
$$

2) Next, turn to the improper integral

$$\int_0^{+\infty} x\cdot\sin\left(x^3\right)\,\mathrm{d}x.$$

We apply the change of variable $t = x^3$, thus $x = t^{\frac{1}{3}}$ and $\mathrm{d}x = \dfrac{1}{3}\,t^{-\frac{2}{3}}\,\mathrm{d}t$, to get

$$
\begin{aligned}
\int_0^{+\infty}\left|x\,\sin\left(x^3\right)\right|\,\mathrm{d}x &= \frac{1}{3}\int_0^{+\infty} t^{-\frac{1}{3}}\,|\sin t|\,\mathrm{d}t = \frac{1}{3}\sum_{n=0}^{+\infty}\int_{n\pi}^{(n+1)\pi} t^{-\frac{1}{3}}\,|\sin t|\,\mathrm{d}t \\
&= \frac{1}{3}\sum_{n=0}^{+\infty}\left|\int_{n\pi}^{(n+1)\pi} t^{-\frac{1}{3}}\,\sin t\,\mathrm{d}t\right|.
\end{aligned}
$$

We get for $n \in \mathbb{N}$,

$$
\begin{aligned}
\int_{n\pi}^{(n+1)\pi} t^{-\frac{1}{3}}\,\sin t\,\mathrm{d}t &= \left[-t^{-\frac{1}{3}}\,\cos t\right]_{n\pi}^{(n+1)\pi} + \int_{n\pi}^{(n+1)\pi}\left\{-\frac{1}{3}\right\}t^{-\frac{4}{3}}\,\cos t\,\mathrm{d}t \\
&= \frac{(-1)^n}{\sqrt[3]{(n+1)\pi}} - \frac{(-1)^n}{\sqrt[3]{n\pi}} - \frac{1}{3}\int_{n\pi}^{(n+1)\pi} t^{-\frac{4}{3}}\,\cos t\,\mathrm{d}t,
\end{aligned}
$$

hence,

$$\int_{n\pi}^{(n+1)\pi} t^{-\frac{1}{3}}\,|\sin t|\,\mathrm{d}t \le \frac{1}{\sqrt[3]{\pi}}\left\{\frac{1}{\sqrt[3]{n}} - \frac{1}{\sqrt[3]{n+1}}\right\} + \frac{1}{3}\int_{n\pi}^{(n+1)\pi} t^{-\frac{4}{3}}\,\mathrm{d}t$$

$$= \frac{1}{\sqrt[3]{\pi}}\left\{\frac{1}{\sqrt[3]{n}} - \frac{1}{\sqrt[3]{n+1}}\right\} + \frac{1}{3}\left[\frac{1}{-\frac{1}{3}}\,t^{-\frac{1}{3}}\right]_{n\pi}^{(n+1)\pi} = \frac{2}{\sqrt[3]{\pi}}\left\{\frac{1}{\sqrt[3]{n}} - \frac{1}{\sqrt[3]{n+1}}\right\}.$$

Download free eBooks at bookboon.com

We therefore conclude that

$$\begin{aligned}\int_0^{+\infty} \left|x \sin\left(x^3\right)\right| \mathrm{d}x &= \frac{1}{3}\int_0^{\pi} t^{-\frac{1}{3}}\,|\sin t|\,\mathrm{d}t + \frac{1}{3}\sum_{n=1}^{+\infty}\int_{n\pi}^{(n+1)\pi} t^{-\frac{1}{3}}\,|\sin t|\,\mathrm{d}t \\ &\le \frac{1}{3}\int_0^{\pi} t^{-\frac{1}{3}}\,\mathrm{d}t + \frac{1}{3}\cdot\frac{2}{\sqrt[3]{\pi}}\sum_{n=1}^{+\infty}\left\{\frac{1}{\sqrt[3]{n}} - \frac{1}{\sqrt[3]{n+1}}\right\} \\ &= \frac{1}{3}\left[\frac{1}{\frac{2}{3}}\,t^{\frac{2}{3}}\right]_0^{\pi} + \frac{2}{3}\cdot\frac{1}{\sqrt[3]{\pi}}\cdot 1 = \frac{1}{2}\sqrt[3]{\pi^2} + \frac{2}{3}\,\frac{1}{\sqrt[3]{\pi}},\end{aligned}$$

where we have used that the terms of the telescoping series tend towards 0 for $n \to +\infty$. This implies that $x \cdot \sin\left(x^3\right) \in L^1$, hence also that $\dfrac{1}{\sqrt[3]{t}}\,\sin t \in L^1$.

Discover the truth at www.deloitte.ca/careers
Deloitte.
© Deloitte & Touche LLP and affiliated entities.

Download free eBooks at bookboon.com
Click on the ad to read more

3) Since $f_n(t) := \frac{1}{\sqrt[3]{t}} \cdot \sin t \cdot \exp\left(-\frac{1}{n}\,t\right) \in L^1$ converges pointwise towards $f(t) := \frac{1}{\sqrt[3]{t}}\,\sin t$, and since $|f(t)|$ is an integrable majoring function, we conclude from the theorem of majoring convergence that

$$\begin{aligned}
\int_0^{+\infty} x\,\sin\left(x^3\right)\,\mathrm{d}x &= \int_0^{+\infty} \frac{1}{\sqrt[3]{t}}\,\sin t\,\mathrm{d}t = \lim_{n\to+\infty}\int_0^{+\infty} \frac{1}{\sqrt[3]{t}}\,\sin t\dot{c}ot\exp\left(-\frac{1}{n}\,t\right)\,\mathrm{d}t \\
&= \lim_{n\to+\infty}\mathcal{L}\left\{\frac{1}{\sqrt[3]{t}}\,\sin t\right\}\left(\frac{1}{n}\right) = \lim_{x\to 0+}\mathcal{L}\left\{\frac{1}{\sqrt[3]{t}}\,\sin t\right\}(x) \\
&= \frac{\Gamma\left(\frac{2}{3}\right)}{2i}\lim_{x\to 0+}\frac{(x+i)^{\frac{2}{3}}-(x-i)^{\frac{2}{3}}}{\left(x^2+1\right)^{\frac{2}{3}}} = \frac{\Gamma\left(\frac{2}{3}\right)}{2i}\cdot\frac{i^{\frac{2}{3}}-(-i)^{\frac{2}{3}}}{1} \\
&= \frac{\Gamma\left(\frac{2}{3}\right)}{2i}\left\{\exp\left(\frac{2}{3}\cdot i\,\frac{\pi}{2}\right)-\exp\left(\frac{2}{3}\cdot\left(-i\,\frac{\pi}{2}\right)\right)\right\} \\
&= \Gamma\left(\frac{2}{3}\right)\cdot\frac{1}{2i}\left\{\exp\left(i\,\frac{\pi}{3}\right)-\exp\left(-i\,\frac{\pi}{3}\right)\right\} \\
&= \Gamma\left(\frac{2}{3}\right)\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}\,\Gamma\left(\frac{2}{3}\right). \qquad\Diamond
\end{aligned}$$

Example 1.1.7 *Compute the inverse Laplace transforms of*

1) $\dfrac{1}{\sqrt{2z+3}}$,

2) $\dfrac{e^{4-3z}}{(z+4)^{\frac{5}{2}}}$.

1) It follows from the rearrangement

$$\frac{1}{\sqrt{2z+3}} = \frac{1}{\sqrt{2}}\cdot\frac{1}{\left\{z+\frac{3}{2}\right\}^{\frac{1}{2}}} = \frac{1}{\sqrt{2\pi}}\cdot\frac{\Gamma\left(\frac{1}{2}\right)}{\left(z+\frac{3}{2}\right)^{-\frac{1}{2}+1}},$$

that

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{2z+3}}\right\}(t) = \frac{1}{\sqrt{2\pi}}\,\exp\left(-\frac{3}{2}\,t\right)\cdot\frac{1}{\sqrt{t}}.$$

2) Analogously,

$$\frac{e^{4-3z}}{(z+4)^{\frac{5}{2}}} = e^4\cdot e^{-3z}\cdot\frac{1}{\Gamma\left(\frac{5}{2}\right)}\cdot\frac{\Gamma\left(\frac{5}{2}\right)}{(z+4)^{\frac{3}{2}+1}} = \frac{e^4}{\frac{3}{2}\cdot\frac{1}{2}\,\sqrt{\pi}}\cdot e^{-3z}\,\mathcal{L}\left\{t\sqrt{t}\,e^{-4t}\right\}(z),$$

Download free eBooks at bookboon.com

hence,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{e^{4-3z}}{(z+4)^{\frac{5}{2}}}\right\}(t) &= \frac{4e^4}{3\sqrt{\pi}}\,(t-3)^{\frac{3}{2}}\,e^{-t(t-3)}\,H(t-3)\\ &= \frac{4\,e^{16}}{3\sqrt{\pi}}\,e^{-4t}\cdot(t-3)\sqrt{t-3}\cdot H(t-3). \qquad \Diamond\end{aligned}$$

Example 1.1.8 *Compute the inverse Laplace transform of*

$$\left(\frac{\sqrt{z}-1}{z}\right)^2.$$

We get by a small computation,

$$\begin{aligned}\left(\frac{\sqrt{z}-1}{z}\right)^2 &= \frac{z+1-2\sqrt{z}}{z^2} = \frac{1}{z}+\frac{1}{z^2}-\frac{2}{z^{\frac{3}{2}}} = \mathcal{L}\{1\}(z)+\mathcal{L}\{t\}(z)-\frac{2}{\Gamma\left(\frac{3}{2}\right)}\cdot\frac{\Gamma\left(\frac{3}{2}\right)}{z^{\frac{3}{2}}}\\ &= \mathcal{L}\left\{1+t-\frac{2}{\frac{1}{2}\sqrt{\pi}}\sqrt{t}\right\}(z),\end{aligned}$$

hence,

$$\mathcal{L}^{-1}\left\{\left(\frac{\sqrt{z}-1}{z}\right)^2\right\}(t) = 1+t-\frac{4}{\sqrt{\pi}}\sqrt{t}. \qquad \Diamond$$

Example 1.1.9 *Compute the inverse Laplace transform of*

$$\frac{z}{(z+1)^{\frac{5}{2}}}.$$

We get by a small manipulation of the expression,

$$\begin{aligned}F(z) &:= \frac{z}{(z+1)^{\frac{5}{2}}} = \frac{z+1-1}{(z+1)^{\frac{5}{2}}} = \frac{1}{(z+1)^{\frac{3}{2}}}-\frac{1}{(z+1)^{\frac{5}{2}}}\\ &= \frac{1}{\Gamma\left(\frac{3}{2}\right)}\cdot\frac{\Gamma\left(\frac{3}{2}\right)}{(z+1)^{\frac{3}{2}}}-\frac{1}{\Gamma\left(\frac{5}{2}\right)}\cdot\frac{\Gamma\left(\frac{5}{2}\right)}{(z+1)^{\frac{5}{2}}}\\ &= \frac{1}{\frac{1}{2}\sqrt{\pi}}\,\mathcal{L}\left\{t^{\frac{1}{2}}\right\}(z+1)-\frac{1}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}\,\mathcal{L}\left\{t^{\frac{3}{2}}\right\}(z+1)\\ &= \frac{2}{\sqrt{\pi}}\,\mathcal{L}\left\{e^{-t}\sqrt{t}\right\}(z)-\frac{4}{3\sqrt{\pi}}\,\mathcal{L}\left\{e^{-t}\,t\sqrt{t}\right\}(z),\end{aligned}$$

hence

$$\mathcal{L}^{-1}\{F\}(t) = \frac{2}{\sqrt{\pi}}\,e^{-t}\sqrt{t}-\frac{4}{3\sqrt{\pi}}\,e^{-t}\,t\sqrt{t} = \frac{2}{3\sqrt{\pi}}\,e^{-t}\sqrt{t}\,(3-2t). \qquad \Diamond$$

Download free eBooks at bookboon.com

Example 1.1.10 *Compute the inverse Laplace transform of*

$$\frac{1}{\sqrt[3]{8z-27}}.$$

It follows from

$$F(z) := \frac{1}{\sqrt[3]{8z-27}} = \frac{1}{2}\cdot\frac{1}{\sqrt[3]{x-\left(\frac{3}{2}\right)^3}}, \qquad \text{and} \qquad \mathcal{L}\left\{t^{-\frac{2}{3}}\right\}(z) = \frac{\Gamma\left(\frac{1}{3}\right)}{\sqrt[3]{z}},$$

that

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt[3]{8z-27}}\right\}(t) = \frac{1}{2\,\Gamma\left(\frac{1}{3}\right)}\cdot\frac{\exp\left(\frac{27}{8}\,t\right)}{\sqrt[3]{t^2}}. \qquad \Diamond$$

Example 1.1.11 *Solve the equation*

$$\int_0^t f'(u)f(t-u)\,\mathrm{d}u = 24\,t^3, \qquad t\in\mathbb{R}_+.$$

where we assume that $f\in\mathcal{F}$ and $f'\in\mathcal{F}$, and $f(0)=0$.

First write the equation as a convolution equation

$$(f'\star f)(t) = 24\,t^3.$$

Since we have assumed that f and $f'\in\mathcal{F}$, we may apply the Laplace transformation on this equation, so

$$\mathcal{L}\left\{24\,t^3\right\}(z) = \frac{24\cdot 3!}{z^4} = \mathcal{L}\left\{f'\right\}(z)\cdot\mathcal{L}\{f\}(z) = z\cdot(\mathcal{L}\{f\}(z))^2, \qquad \text{for } \Re z>0,$$

hence, by solving after $\mathcal{L}\{f\}(z)$,

$$\mathcal{L}\{f\}(z) = \pm\frac{12}{z^{\frac{5}{2}}} = \pm\frac{12}{\Gamma\left(\frac{5}{2}\right)}\cdot\frac{\Gamma\left(\frac{5}{2}\right)}{z^{\frac{5}{2}}} = \frac{\pm 12}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}\cdot\mathcal{L}\left\{t^{\frac{3}{2}}\right\}(z),$$

from which we conclude that the two solutions are given by

$$f(t) = \pm\frac{12\cdot 4}{3\sqrt{\pi}}\,t^{\frac{3}{2}} = \pm\frac{16}{\sqrt{\pi}}\,t\sqrt{t}. \qquad \Diamond$$

Download free eBooks at bookboon.com

Example 1.1.12 *Solve the equation*

$$\int_0^t \frac{f(u)}{\sqrt{t-u}}\,\mathrm{d}u = 1+t+t^2, \qquad t \in \mathbb{R}_+,$$

where we assume that $f \in \mathcal{F}$.

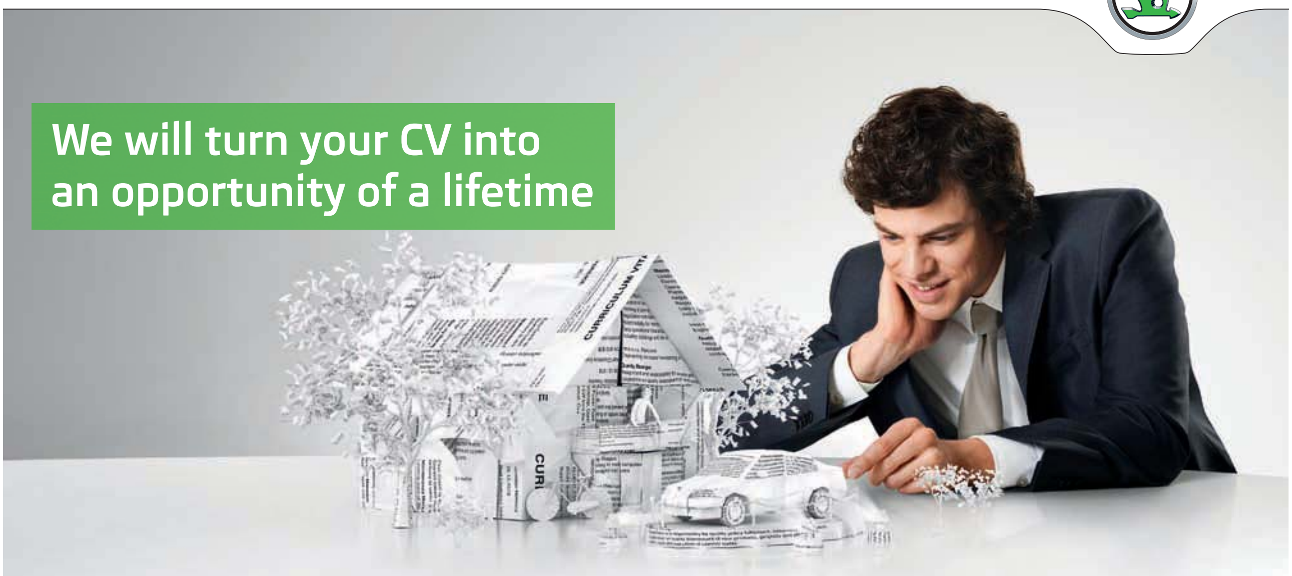
We first notice that since $g(t) = 1+t+t^2$ is not equal to 0 for $t = 0$, we cannot apply the formula, which will be derived in Example 1.2.1.

The equation can be written as the convolution equation

$$\left(f \star \frac{1}{\sqrt{t}}\right)(t) = 1+t+t^2.$$

This is mapped by the Laplace transformation into

$$\mathcal{L}\{f\}(z) \cdot \mathcal{L}\left\{t^{-\frac{1}{2}}\right\}(z) = \frac{\Gamma\left(\frac{1}{2}\right)}{z^{\frac{1}{2}}}\,\mathcal{L}\{f\}(z) = \frac{1}{z} + \frac{1}{z^2} + \frac{2}{z^3},$$

SIMPLY CLEVER
ŠKODA


Do you like cars? Would you like to be a part of a successful brand?
We will appreciate and reward both your enthusiasm and talent.
Send us your CV. You will be surprised where it can take you.
Send us your CV on
www.employerforlife.com
Click on the ad to read more

hence, by solving it with respect to $\mathcal{L}\{f\}(z)$,

$$\begin{aligned}\mathcal{L}\{f\}(z) &= \frac{1}{\sqrt{\pi}}\cdot\frac{1}{z^{\frac{1}{2}}}+\frac{1}{\sqrt{\pi}}\cdot\frac{1}{z^{\frac{3}{2}}}+\frac{2}{\sqrt{\pi}}\cdot\frac{1}{z^{\frac{5}{2}}}\\ &= \frac{1}{\pi}\cdot\frac{\Gamma\left(\frac{1}{2}\right)}{z^{\frac{1}{2}}}+\frac{1}{\sqrt{\pi}\cdot\Gamma\left(\frac{3}{2}\right)}\cdot\frac{\Gamma\left(\frac{3}{2}\right)}{z^{\frac{3}{2}}}+\frac{2}{\sqrt{\pi}\cdot\Gamma\left(\frac{5}{2}\right)}\cdot\frac{\Gamma\left(\frac{5}{2}\right)}{z^{\frac{5}{2}}}\\ &= \frac{1}{\pi}\,\mathcal{L}\left\{t^{-\frac{1}{2}}\right\}(z)+\frac{2}{\pi}\,\mathcal{L}\left\{t^{\frac{1}{2}}\right\}(z)+\frac{2}{\pi\cdot\frac{3}{2}\cdot\frac{1}{2}}\,\mathcal{L}\left\{t^{\frac{3}{2}}\right\}(z)\\ &= \frac{1}{\pi}\,\mathcal{L}\left\{\frac{1}{\sqrt{t}}+2\sqrt{t}+\frac{8}{3}\,t\sqrt{t}\right\},\end{aligned}$$

We conclude that

$$f(t)=\frac{1}{\pi}\left\{\frac{1}{\sqrt{t}}+2\sqrt{t}+\frac{8}{3}\,t\sqrt{t}\right\}=\frac{1}{\pi\sqrt{t}}\left\{1+2t+\frac{8}{3}\,t^2\right\},\qquad t\in\mathbb{R}_+.$$

Finally, it is obvious that the solution satisfies the condition that $f\in\mathcal{F}$. $\Diamond$

Example 1.1.13 *Find the solution $f\in\mathcal{F}$ of the equation*

$$\int_0^t\frac{f(u)}{\sqrt{t-u}}\,\mathrm{d}u=\sqrt{t},\qquad \textit{for } t\in\mathbb{R}_+.$$

The given equation can also be written as a convolution equation

$$\left(f\star\frac{1}{\sqrt{t}}\right)(t)=\sqrt{t},\qquad \text{for } t\in\mathbb{R}_+.$$

Given that $f\in\mathcal{F}$ and $\dfrac{1}{\sqrt{t}}\in\mathcal{F}$ and $\sqrt{t}\in\mathcal{F}$, we get by a Laplace transformation for $\Re z>\max\{0,\sigma(f)\}$ that

$$\begin{aligned}\mathcal{L}\left\{f\star\frac{1}{\sqrt{t}}\right\}(z) &= \mathcal{L}\{f\}(z)\cdot\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\}(z)=\frac{\Gamma\left(\frac{1}{2}\right)}{z^{\frac{1}{2}}}\,\mathcal{L}\{f\}(z)\\ &= \sqrt{\frac{\pi}{z}}\cdot\mathcal{L}\{f\}(z)=\mathcal{L}\left\{t^{\frac{1}{2}}\right\}(z)=\frac{\Gamma\left(\frac{3}{2}\right)}{z^{\frac{3}{2}}}=\frac{1}{2z}\sqrt{\frac{\pi}{z}},\end{aligned}$$

so a necessary condition for the solution f is that it satisfies the equation

$$\mathcal{L}\{f\}(z)=\frac{1}{2z}.$$

By the inverse Laplace transformation, the only possible solution is the constant function $f(t)=\dfrac{1}{2}$.

CHECK. It is obvious that $f(t)=\dfrac{1}{2}\in\mathcal{F}$, and $\sigma(f)=0$. Finally, we get by insertion that

$$\int_0^t\frac{f(u)}{\sqrt{t-u}}\,\mathrm{d}u=\frac{1}{2}\int_0^t\frac{\mathrm{d}u}{\sqrt{t-u}}=\left[-\sqrt{t-u}\right]_0^t=\sqrt{t},$$

so $f(t)=\dfrac{1}{2}$ is indeed a solution. $\Diamond$

Download free eBooks at bookboon.com

Example 1.1.14 *Find the solution $f \in \mathcal{F}$ if the equation*

$$\int_0^t \frac{f(u)}{(t-u)^{\frac{1}{3}}}\,\mathrm{d}u = t(1+t), \qquad \textit{for } t \in \mathbb{R}_+.$$

We shall solve the convolution equation

$$\int_0^t \frac{f(u)}{(t-u)^{\frac{1}{3}}}\,du = (t) = t(1+t).$$

Put for convenience $F(z) := \mathcal{L}\{f\}(z)$. Then by taking the Laplace transformation and using the rule of convolution,

$$\mathcal{L}\left\{f \star \frac{1}{t^{\frac{1}{3}}}\right\}(z) = F(z)\cdot\frac{\Gamma\left(1-\frac{1}{3}\right)}{z^{\frac{2}{3}}} = \mathcal{L}\left\{t+t^2\right\}(z) = \frac{1}{z}+\frac{1}{z^2},$$

from which we get

$$\begin{aligned} F(z) &= \frac{1}{\Gamma\left(\frac{2}{3}\right)}\cdot\frac{1}{z^{\frac{1}{3}}} + \frac{1}{\Gamma\left(\frac{2}{3}\right)}\cdot\frac{1}{z^{\frac{4}{3}}} \\ &= \frac{1}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}\cdot\frac{\Gamma\left(\frac{1}{3}\right)}{z^{\frac{1}{3}}} + \frac{1}{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)}\cdot\frac{\Gamma\left(\frac{4}{3}\right)}{z^{\frac{4}{3}}} \\ &= \frac{\sin\frac{\pi}{3}}{\pi}\,\mathcal{L}\left\{t^{-\frac{2}{3}}\right\}(z) + \frac{3\sin\frac{\pi}{3}}{\pi}\,\mathcal{L}\left\{t^{\frac{1}{3}}\right\}(z). \end{aligned}$$

Finally, by the inverse Laplace transformation,

$$f(t) = \frac{\sqrt{3}}{2\pi}\cdot\frac{\sqrt[3]{t}}{t} + \frac{3\sqrt{3}}{2\pi}\,\sqrt[3]{t} = \frac{\sqrt{3}}{2\pi}\cdot\frac{\sqrt[3]{t}}{t}\,(1+t). \qquad \Diamond$$

Example 1.1.15 *Given $n \in \mathbb{N}\setminus\{1\}$. Let $s \in \mathbb{R}_+$. Prove that*

$$\mathcal{L}\left\{\frac{t^{n-1}}{1-e^{-t}}\right\}(s) = \Gamma(n)\sum_{n=0}^{+\infty}\frac{1}{(s+p)^n}.$$

We derive the classical Riemann's zeta function *from the above by the definition*

$$\zeta(n) := \sum_{p=1}^{+\infty}\frac{1}{p^n} = \frac{1}{\Gamma(n)}\,\mathcal{L}\left\{\frac{t^{n-1}}{1-e^{-t}}\right\}(1) = \frac{1}{\Gamma(n)}\int_0^{+\infty}\frac{t^{n-1}}{e^t-1}\,\mathrm{d}t.$$

We get for $s > 0$,

$$\begin{aligned} \mathcal{L}\left\{\frac{t^{n-1}}{1-e^{-t}}\right\}(s) &= \int_0^{+\infty} t^{n-1}\cdot\frac{e^{-st}}{1-e^{-t}}\,\mathrm{d}t = \lim_{\varepsilon\to 0+}\int_\varepsilon^{+\infty} t^{n-1}\sum_{p=0}^{+\infty}e^{-pt}\cdot e^{-st}\,\mathrm{d}t \\ &= \lim_{\varepsilon\to 0+}\sum_{p=0}^{+\infty}\int_\varepsilon^{+\infty} y^{n-1}\,e^{-(p+s)t}\,\mathrm{d}t = \sum_{p=0}^{+\infty}\Gamma(n)\cdot\frac{1}{(p+s)^n} \\ &= \Gamma(n)\sum_{n=0}^{+\infty}\frac{1}{(z+p)^n}. \end{aligned}$$

Download free eBooks at bookboon.com

In particular we get for $s = 1$ and $n \in \mathbb{N} \setminus \{1\}$,

$$\zeta(n) := \sum_{p=1}^{+\infty} \frac{1}{p^n} = \frac{1}{\Gamma(n)} \mathcal{L}\left\{\frac{t^{n-1}}{1-e^{-t}}\right\}(1) = \frac{1}{\Gamma(n)} \int_0^{+\infty} \frac{t^{n-1}}{e^t - 1}\, \mathrm{d}t.$$

We know from e.g. the theory of Fourier series that

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Therefore, we also get

$$\frac{\pi^2}{6} = \sum_{n=1}^{+\infty} \frac{1}{n^2} = \zeta(2) = \int_0^{+\infty} \frac{t}{e^t - 1}\, \mathrm{d}t.$$

Example 1.1.16 *Prove that*

$$\mathcal{L}\left\{\int_0^{+\infty} \frac{t^u\, f(u)}{\Gamma(u+1)}\, \mathrm{d}u\right\}(s) = \frac{\mathcal{L}\{f\}(\ln s)}{s}, \qquad \textit{for } s \in \mathbb{R}_+.$$

First apply the definition of the Laplace transformation with respect to t, and then interchange the order of integration to get

$$\begin{aligned}\mathcal{L}\left\{\int_0^{+\infty} \frac{t^u\, f(u)}{\Gamma(u+1)}\, du\right\}(s) &= \int_0^{+\infty} \frac{\mathcal{L}\{t^u\}(s) \cdot f(u)}{\Gamma(u+1)}\, \mathrm{d}u \\ &= \int_0^{+\infty} \frac{1}{\Gamma(u+1)} \cdot \frac{\Gamma(u+1)}{s^{u+1}} \cdot f(u)\, \mathrm{d}u = \frac{1}{s} \int_0^{+\infty} f(u)\, e^{-u \cdot \ln s}\, \mathrm{d}u \\ &= \frac{1}{s}\, \mathcal{L}\{f\}(\ln s),\end{aligned}$$

and the claim is proved. $\Diamond$

Download free eBooks at bookboon.com

1.2 The Beta function

Example 1.2.1 *Given a constant $\alpha \in]0,1[$, and assume that $g \in \mathcal{F} \cap C^1$ and $g(0) = 0$. Prove that the solution $f \in \mathcal{F}$ of the convolution equation*

$$\int_0^t \frac{f(u)}{(t-u)^\alpha}\,\mathrm{d}u = g(t), \qquad \text{for } t \in \mathbb{R}_+,$$

is given by the solution formula

$$(1)\quad f(t) = \frac{\sin \alpha\pi}{\pi} \int_0^t g'(u)(t-u)^{\alpha-1}\,\mathrm{d}u.$$

We first check that (1) is indeed a solution. We get by insertion and an application of Fubini's theorem,

$$\begin{aligned}
f \star \frac{1}{t^\alpha} &= \int_0^t \frac{f(u)}{(t-u)^\alpha}\,\mathrm{d}u = \frac{\sin \alpha\pi}{\pi} \int_0^t \frac{1}{(t-u)^\alpha} \int_0^t \frac{1}{(t-u)^\alpha} \int : 0 g'(x)(u-x)^{\alpha-1}\,\mathrm{d}x\,\mathrm{d}u \\
&= \frac{\sin \alpha\pi}{\pi} \int_0^t g'(x) \left\{ \int_x^t \frac{1}{(t-u)^\alpha} \cdot (u-x)^{\alpha-1}\,\mathrm{d}u \right\} \mathrm{d}x \\
&= \frac{\sin \alpha\pi}{\pi} \int_0^t g'(x) \left\{ \int_0^{t-x} \frac{1}{(t-x-u)^\alpha} \cdot u^{\alpha-1}\,\mathrm{d}u \right\} \mathrm{d}x \\
&= \frac{\sin \alpha\pi}{\pi} \int_0^t g'(x) \cdot \left(\frac{1}{u^\alpha} \star u^{\alpha-1} \right)(t-x)\,\mathrm{d}x \\
&= \left(g' \star \left(\frac{\sin \alpha\pi}{\pi} \cdot \frac{1}{x^\alpha} \star x^{\alpha-1} \right) \right)(t).
\end{aligned}$$

Then we separately compute the inner convolution, where we use the change of variable $t = xu$ for $x > 0$. This gives,

$$\begin{aligned}
\frac{\sin \alpha\pi}{\pi} \cdot \frac{1}{x^\alpha} \star x^{\alpha-1} &= \frac{\sin \alpha\pi}{\pi} \int_0^x (x-t)^{-\alpha}\, t^{\alpha-1}\,\mathrm{d}t = \frac{\sin \alpha\pi}{\pi} \int_0^1 x^{-\alpha}\,(1-u)^{-\alpha} \cdot x^{\alpha-1}\,x\,\mathrm{d}u \\
&= \frac{\sin \alpha\pi}{\pi} \int_0^1 (1-u)^{(1-\alpha)-1}\,u^{\alpha-1}\,\mathrm{d}u = \frac{\sin \alpha\pi}{\pi}\, B(1-\alpha,\alpha) \\
&= \frac{\sin \alpha\pi}{\pi}\, \frac{\Gamma(1-\alpha)\Gamma(\alpha)}{\Gamma(1)} = \frac{\sin \alpha\pi}{\pi}\, \frac{\pi}{\sin \alpha\pi} = 1,
\end{aligned}$$

hence,

$$f \star \frac{1}{t^\alpha} = (g' \star H)(t) = \int_0^t g'(u) H(t-u)\,\mathrm{d}u = \int_0^t g'(u)\,\mathrm{d}u = [g(u)]_0^t = g(t) - g(0) = 0,$$

and the claim is proved.

Notice that the result is independent of whether $g \in \mathcal{F}$ or not. The important thing for this part of the proof is that $g \in C^1$ and that $g(0) = 0$.

Download free eBooks at bookboon.com

An *alternative proof* in which we apply that $g \in \mathcal{F} \cap C^1$, is the following. We shall prove that the convolution equation

(2) $f \star t^{-\alpha} = g$

has the solution

(3) $f = \dfrac{\sin \alpha\pi}{\pi}\, g' \star t^{\alpha-1}$.

When we apply the Laplace transformation on (2), then

$$\mathcal{L}\{g\}(z) = \mathcal{L}\{f\}(z) \cdot \mathcal{L}\left\{t^{-\alpha}\right\}(z) = \frac{\Gamma(1-\alpha)}{z^{1-\alpha}} \cdot \mathcal{L}\{f\}(z) = \frac{1}{\Gamma(\alpha)} \cdot \frac{\pi}{\sin \alpha\pi}\, \mathcal{L}\{f\}(z) \cdot \frac{1}{z^{1-\alpha}},$$

thus

$$\begin{aligned} \mathcal{L}\{f\}(z) &= \frac{\sin \alpha\pi}{\pi} \cdot \frac{\Gamma(\alpha)}{z^{\alpha}} \cdot z\, \mathcal{L}\{g\}(z) = \frac{\sin \alpha\pi}{\pi} \cdot \mathcal{L}\left\{t^{\alpha-1}\right\}(z) \cdot \mathcal{L}\left\{g'\right\}(z) \\ &= \mathcal{L}\left\{\frac{\sin \alpha\pi}{\pi} \cdot g' \star t^{\alpha-1}\right\}(z), \end{aligned}$$

and (3) follows.

Click on the ad to read more

Download free eBooks at bookboon.com

Remark 1.2.1 As a check we can apply the solution formula on Example 1.1.13,

$$\begin{aligned} f(t) &= \frac{\sin\frac{\pi}{2}}{\pi}\int_0^t \frac{1}{2}\cdot\frac{1}{\sqrt{u}}\cdot\frac{1}{\sqrt{t-u}}\,\mathrm{d}u = \frac{1}{2\pi}\int_0^t u^{\frac{1}{2}-1}\cdot(t-u)^{\frac{1}{2}-1}\,\mathrm{d}u \\ &= \frac{1}{2\pi}\,B\left(\frac{1}{2},\frac{1}{2}\right) = \frac{1}{2\pi}\cdot\frac{\pi}{\sin\frac{\pi}{2}} = \frac{1}{2}. \end{aligned}$$

Similarly, we get in Example 1.1.14,

$$\begin{aligned} f(t) &= \frac{\sin\frac{\pi}{3}}{\pi}\int_0^t (1+2u)\cdot(t-u)^{\frac{1}{3}-1}\,\mathrm{d}u = \frac{\sin\frac{\pi}{3}}{\pi}\int_0^t (t+2t-2(t-u))\cdot(t-u)^{-\frac{2}{3}}\,\mathrm{d}u \\ &= \frac{\sqrt{3}}{2\pi}\left\{(1+2t)\int_0^t (t-u)^{-\frac{2}{3}}\,\mathrm{d}u - 2\int_0^t (t-u)^{\frac{1}{3}}\,\mathrm{d}u\right\} \\ &= \frac{\sqrt{3}}{2\pi}\left\{(1+2t)\left[-3(t-u)^{\frac{1}{3}}\right]_0^t - 2\left[-\frac{3}{4}\,(t-u)^{\frac{4}{3}}\right]_0^t\right\} \\ &= \frac{\sqrt{3}}{2\pi}\left\{(1+2t)3\sqrt[3]{t} - \frac{3}{2}\,t\sqrt[3]{t}\right\} = \frac{3\sqrt{3}}{2\pi}\sqrt[3]{t} - \frac{9\sqrt{3}}{4\pi}\,t\sqrt[3]{t}. \qquad \diamond \end{aligned}$$

Example 1.2.2 *Compute the integrals,*

1) $\int_0^1 x^{\frac{3}{2}}(1-x)^2\,\mathrm{d}x$,

2) $\int_0^4 x^3(4-x)^{-\frac{1}{2}}\,\mathrm{d}x$,

3) $\int_0^2 x^4\sqrt{4-x^2}\,\mathrm{d}x$.

The idea is of course to use that

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,\mathrm{d}x = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

1) We get by a straightforward computation,

$$\begin{aligned} \int_0^1 x^{\frac{3}{2}}(1-x)^2\,\mathrm{d}x &= \int_0^1 x^{\frac{5}{2}-1}(1-x)^{3-1}\,\mathrm{d}x = \frac{\Gamma\left(\frac{5}{2}\right)\Gamma(3)}{\Gamma\left(\frac{11}{2}\right)} \\ &= \frac{\Gamma\left(\frac{5}{2}\right)\cdot 2!}{\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\Gamma\left(\frac{5}{2}\right)} = \frac{2^4}{9\cdot 7\cdot 5} = \frac{16}{315}. \end{aligned}$$

Download free eBooks at bookboon.com

2) In this case we apply the change of variable,

$$u=\frac{x}{4}, \qquad x=4u, \qquad dx=4du.$$

Then

$$\begin{aligned}\int_0^4 x^3(4-x)^{-\frac{1}{2}}\,\mathrm{d}x &= \int_0^1 4^3\,u^3\cdot 4^{-\frac{1}{2}}(1-u)^{-\frac{1}{2}}\cdot 4\,\mathrm{d}u = 2\cdot 4^3\int_0^1 u^{4-1}(1-u)^{\frac{1}{2}-1}\,\mathrm{d}u\\ &= 128\cdot\frac{\Gamma(4)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{9}{2}\right)} = \frac{128\cdot 3!\,\Gamma\left(\frac{1}{2}\right)}{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)} = \frac{256\cdot 2^4}{5\cdot 7} = \frac{4096}{35}.\end{aligned}$$

3) Here we apply the change of variable

$$u=\frac{1}{4}\,x^2, \qquad x=2\sqrt{u}, \qquad dx=\frac{1}{\sqrt{u}}\,du.$$

Then

$$\begin{aligned}\int_0^2 x^4\sqrt{4-x^2}\,\mathrm{d}x &= \int_0^1 2^4\cdot u^2\cdot 4^{\frac{1}{2}}\,\sqrt{1-u}\cdot\frac{1}{\sqrt{u}}\,\mathrm{d}u = 32\int_0^1 u^{\frac{5}{2}-1}(1-u)^{\frac{3}{2}-1}\,\mathrm{d}u\\ &= 32\,B\left(\frac{5}{2},\frac{3}{2}\right) = 32\,\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(4)} = 32\cdot\frac{\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)}{3!}\\ &= \frac{32\cdot 3}{6\cdot 8}\,\left(\sqrt{\pi}\right)^2 = 2\pi. \qquad \Diamond\end{aligned}$$

Example 1.2.3 *Compute* $B\left(\frac{3}{2},4\right)$.

We get straightforward,

$$B\left(\frac{3}{2},4\right) = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(4)}{\Gamma\left(\frac{11}{2}\right)} = \frac{\Gamma\left(\frac{3}{2}\right)\cdot 6}{\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\Gamma\left(\frac{3}{2}\right)} = \frac{6\cdot 16}{9\cdot 7\cdot 5\cdot 3} = \frac{32}{315}. \qquad \Diamond$$

Example 1.2.4 *Compute*

1) $\int_0^{\frac{\pi}{2}} \cos^6\Theta\,\mathrm{d}\Theta$*,*

2) $\int_0^{\frac{\pi}{2}} \sin^4\Theta\,\cos^4\Theta\,\mathrm{d}\Theta$*,*

3) $\int_0^{\pi} \sin^4\Theta\,\cos^4\Theta\,\mathrm{d}\Theta$*.*

We shall use that in general

$$(4)\quad \int_0^{\frac{\pi}{2}} \sin^{2m-1}\Theta\,\cos^{2n-1}\Theta\,\mathrm{d}\Theta = \frac{1}{2}\,B(m,n) = \frac{1}{2}\,\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \qquad \text{for } m,\,n\in\mathbb{R}_+.$$

Download free eBooks at bookboon.com

1) When we apply (4), we get

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos^6 \Theta \, d\Theta &= \int_0^{\frac{\pi}{2}} \sin^{2\cdot\frac{1}{2}-1}\Theta \cdot \cos^{2\cdot\frac{7}{2}-1}\Theta \, d\Theta = \frac{1}{2}\, B\left(\frac{1}{2},\frac{7}{2}\right) \\ &= \frac{1}{2}\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{7}{2}\right)}{\Gamma(4)} = \frac{1}{2}\cdot\frac{\sqrt{\pi}\cdot\frac{15}{8}\sqrt{\pi}}{3!} = \frac{15\pi}{16\cdot 6} = \frac{5\pi}{32}.\end{aligned}$$

2) It follows again, applying (4), that

$$\begin{aligned}\int 0^{\frac{\pi}{2}} \sin^2\Theta \cos^4\Theta \, d\Theta &= \int_0^{\frac{\pi}{2}} \sin^{2\cdot\frac{3}{2}-1}\Theta \cos^{2\cdot\frac{5}{2}-1}\Theta \, d\Theta \, d\Theta = \frac{1}{2}\, B\left(\frac{3}{2},\frac{5}{2}\right) \\ &= \frac{1}{2}\,\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(4)} = \frac{\frac{1}{2}\sqrt{\pi}\cdot\frac{3}{4}\sqrt{\pi}}{2\cdot 3!} = \frac{3\pi}{16\cdot 6} = \frac{\pi}{32}.\end{aligned}$$

3) In this case we start with a small rearrangement, before we apply (4),

$$\begin{aligned}\int_0^{\pi} \sin^4\Theta \cos^4\Theta \, d\Theta &= \frac{1}{2^4}\int_0^{4} \sin^4 2\Theta \, d\Theta = \frac{1}{32}\int_0^{2\pi} \sin^4\Theta \, d\Theta \\ &= \frac{4}{32}\int_0^{\frac{\pi}{2}} \sin^4\Theta \, d\Theta = \frac{1}{8}\int_0^{\frac{\pi}{2}} \cos^{2\cdot\frac{1}{2}-1}\Theta \sin^{2\cdot\frac{5}{2}-1}\Theta \, d\Theta \\ &= \frac{1}{8}\cdot\frac{1}{2}\, B\left(\frac{1}{2},\frac{5}{2}\right) = \frac{1}{16}\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(3)} = \frac{1}{32}\cdot\sqrt{\pi}\cdot\frac{3}{4}\sqrt{\pi} = \frac{3\pi}{128}. \qquad \Diamond\end{aligned}$$

Example 1.2.5 *Compute* $\int_0^{\frac{\pi}{2}} \cos^n\Theta \, d\Theta$ *for all* $n \in \mathbb{N}$.

We have in general,

$$\int_0^{\frac{\pi}{2}} \cos^n\Theta \, d\Theta = \int_0^{\frac{\pi}{2}} \cos^{2\cdot\frac{n+1}{2}-1}\Theta \cdot \sin^{2\cdot\frac{1}{2}-1}\Theta \, d\Theta = \frac{1}{2}\, B\left(\frac{n+1}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{n}{2}+1\right)}.$$

1) If $n = 2m$ is *even*, then

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos^{2m}\Theta \, d\Theta &= \frac{1}{2}\,\frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(m+1)} = \frac{1}{2}\,\frac{\left(m-\frac{1}{2}\right)\cdots\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{m!} \\ &= \frac{1}{2}\cdot\frac{(2m-1)\cdot(2-1)}{2^m\cdot m!}\cdot\pi = \frac{\pi}{2^{2m+1}}\cdot\frac{(2m)!}{m!m!} = \frac{\pi}{2^{2m+1}}\begin{pmatrix} 2m \\ m \end{pmatrix}.\end{aligned}$$

Download free eBooks at bookboon.com

2) If $n = 2m+1$ is *odd*, then

$$\begin{aligned}\int_0^{\pi} 2\cos^{2m+1}\Theta\, d\Theta &= \frac{1}{2}\,\frac{\Gamma(m+1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{3}{2}\right)} = \frac{1}{2}\cdot\frac{m!\,\sqrt{\pi}}{\left(m+\frac{1}{2}\right)\cdots\frac{1}{2}\,\sqrt{\pi}} \\ &= \frac{2^{m+1}}{2}\cdot\frac{m!}{(2m+1)(2m-1)\cdots 1} = 2^{2m}\cdot\frac{m!m!}{(2m+1)\cdot(2m)!} \\ &= \frac{2^{2m}}{2m+1}\cdot\frac{1}{\binom{2m}{m}}. \qquad \Diamond\end{aligned}$$

Example 1.2.6 *Apply the formula*

$$\int_0^{+\infty} \frac{x^{p-1}}{x+1}\, dx = \frac{\pi}{\sin p\pi}$$

to compute the integral $\int_0^{+\infty} \frac{y^2}{1+y^4}\, dy$.

If we apply the change of variable $x = y^4$, i.e. $y = x^{\frac{1}{4}}$, then we get

$$\int_0^{+\infty} \frac{y^2}{1+y^4}\, dy = \int_0^{+\infty} \frac{x^{\frac{1}{2}}}{1+x}\cdot\frac{1}{4}\cdot x^{\frac{1}{4}-1}\, dx = \int_0^{+\infty} \frac{x^{\frac{3}{4}-1}}{1+x}\, dx = \frac{\pi}{\sin\frac{3\pi}{4}} = \pi\sqrt{2}. \qquad \Diamond$$

Click on the ad to read more

Download free eBooks at bookboon.com

Example 1.2.7 *Prove without using the definition of the Beta function that*

$$\int_0^{\frac{\pi}{2}} \sqrt{\cot\Theta}\,\mathrm{d}\Theta = \frac{\pi}{\sqrt{2}}.$$

We shall use the substitution

$$u = \sqrt{\tan\Theta}, \qquad \text{thus} \qquad \Theta = \text{ Arctan}\left(u^2\right), \qquad \text{and} \qquad d\Theta = \frac{2u}{1+u^4}\,\mathrm{d}u,$$

which clearly should be followed by another substitution,

$$x = u^4, \qquad \text{thus} \qquad u = x^{\frac{1}{4}} \qquad \text{and} \qquad du = \frac{1}{4}\,x^{\frac{1}{4}-1}\,\mathrm{d}x.$$

Then,

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sqrt{\cos\Theta}\,\mathrm{d}\Theta &= \int_0^{+\infty}]\frac{1}{u}\cdot\frac{2u}{1+u^4}\,\mathrm{d}u = \int_0^{+\infty}\frac{2}{1+u^4}\,\mathrm{d}u = \frac{2}{4}\int_0^{+\infty}\frac{x^{\frac{1}{4}-1}}{1+x}\,\mathrm{d}x \\ &= \frac{1}{2}\cdot\frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{\sqrt{2}}. \qquad \Diamond\end{aligned}$$

Example 1.2.8 *Compute the integrals*

1) $\int_2^4 \dfrac{\mathrm{d}x}{\sqrt{(x-2)(4-x)}}$,

2) $\int_1^5 \sqrt[4]{(5-x)(x-1)}\,\mathrm{d}x$.

1) We shall use the change of variable,

$$t = \frac{1}{2}\,(x-2), \qquad \text{thus} \qquad x = 2t+2 \qquad \text{and} \qquad dx = 2dt.$$

Then

$$\begin{aligned}\int_2^4 \frac{\mathrm{d}x}{\sqrt{(x-2)(4-x)}} &= \int_0^1 \frac{2\,\mathrm{d}t}{\sqrt{2t\cdot 2(1-t)}} = \int_0^1 t^{\frac{1}{2}-1}\,(1-t)^{\frac{1}{2}-1}\,\mathrm{d}t = B\left(\frac{1}{2},\frac{1}{2}\right) \\ &= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} = \pi.\end{aligned}$$

2) In this case we use the change of variable

$$t = \frac{1}{4}\,(x-1), \qquad \text{thus} \qquad x = 4t+1 \qquad \text{and} \qquad dx = 4\,dt.$$

Then

$$\int_1^5 \sqrt[4]{(5-x)(x-1)}\,\mathrm{d}x = \int_0^1 \sqrt[4]{4(1-t)\cdot 4t}\cdot 4\,\mathrm{d}t = 8\int_0^1 t^{\frac{5}{4}-1}\,(1-t)^{\frac{5}{4}-1}\,\mathrm{d}t$$

$$= 8\,B\left(\frac{5}{4},\frac{5}{4}\right) = 8\,\frac{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{5}{2}\right)} = 8\cdot\frac{\frac{1}{4}\,\Gamma\left(\frac{1}{4}\right)\cdot\frac{1}{4}\,\Gamma\left(\frac{1}{4}\right)}{\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)} = \frac{2}{3\sqrt{\pi}}\left\{\Gamma\left(\frac{1}{4}\right)\right\}^2. \qquad \Diamond$$

Download free eBooks at bookboon.com

1.3 The sine and cosine and exponential integrals

Example 1.3.1 *Compute the Laplace transforms of*

1) $e^{2t}\,\mathrm{Si}(t)$,

2) $t\text{-}\mathrm{si}(t)$.

We shall use that

$$\mathcal{L}\{\mathrm{Si}\}(z)=\frac{1}{z}\,\mathrm{Arccot}\,z \qquad \text{for } \Re z>0.$$

1) It follows from a rule of computation that

$$\mathcal{L}\left\{\mathrm{Si}(t)e^{2t}\right\}(z)=\mathcal{L}\{\mathrm{Si}\}(z-2)=\frac{1}{z-2}\,\mathrm{Arccot}(z-2), \qquad \text{for } Re\,z>2.$$

2) It follows from the rule of multiplication by t that

$$\mathcal{L}\{t\,\mathrm{Si}(t)\}(z)=-\frac{d}{dz}\left\{\frac{1}{z}\,\mathrm{Arccot}\,z\right\}=\frac{1}{z^2}\,\mathrm{Arccot}\,z+\frac{1}{z}\cdot\frac{1}{1+z^2}. \qquad \Diamond$$

Example 1.3.2 *Prove that* $\dfrac{\sin t}{t}\notin L^1(\mathbb{R}_+)$, *i.e. that* $\int_0^{+\infty}\left|\dfrac{\sin t}{t}\right|\,dt=+\infty$.

Clearly, $|\sin t|\geq\dfrac{1}{\sqrt{2}}$ for all $t\in\left[p\pi+\dfrac{\pi}{4},p\pi+\dfrac{3\pi}{4}\right]$. We therefore have the simple estimates

$$\begin{aligned}\int_0^{+\infty}\left|\frac{\sin t}{t}\right|\mathrm{d}t &\geq \sum_{p=0}^{+\infty}\int_{p\pi+\frac{\pi}{4}}^{p\pi+\frac{3\pi}{4}}\left|\frac{\sin t}{t}\right|\mathrm{d}t\geq\frac{1}{\sqrt{2}}\sum_{p=0}^{+\infty}\int_{p\pi+\frac{\pi}{4}}^{p\pi+\frac{3\pi}{4}}\frac{\mathrm{d}t}{t}\\ &\geq \frac{1}{\sqrt{2}}\sum_{p=0}^{+\infty}\frac{1}{p\pi+\frac{3\pi}{4}}\cdot\frac{\pi}{2}\geq\frac{\pi}{2\sqrt{2}}\sum_{p=0}^{+\infty}\frac{1}{(p+1)\pi}=\frac{1}{2\sqrt{2}}\sum_{n=1}^{+\infty}\frac{1}{n}=+\infty. \qquad \Diamond\end{aligned}$$

Example 1.3.3 *Apply the trivial formula*

$$\int_a^b\frac{\sin\lambda t}{t}\,\mathrm{d}t=\int_0^b\frac{\sin\lambda t}{t}\,\mathrm{d}t-\int_0^a\frac{\sin\lambda t}{t}\,\mathrm{d}t,$$

to prove that

$$\lim_{\lambda\to+\infty}\int_a^b\frac{\sin\lambda t}{t}\,\mathrm{d}t=0.$$

Using the hint and that

$$\int_0^{\infty}\frac{\sin t}{t}\,\mathrm{d}t=\lim_{n\to+\infty}\int_0^n\frac{\sin t}{t}\,\mathrm{d}t=\frac{\pi}{2},$$

Download free eBooks at bookboon.com

this is easy,

$$\begin{aligned}\lim_{\lambda\to+\infty}\int_a^b \frac{\sin\lambda t}{t}\,\mathrm{d}t &= \lim_{\lambda\to+\infty}\int_0^b \frac{\sin\lambda t}{t}\,dt - \lim_{\lambda\to+\infty}\int_0^a \frac{\sin\lambda t}{t}\,dt\\ &= \lim_{\lambda\to+\infty}\int_0^{\lambda\,b} \frac{\sin u}{u}\,du - \lim_{\lambda\to+\infty}\int_0^{\lambda\,a} \frac{\sin u}{u}\,du = \frac{\pi}{2}-\frac{\pi}{2} = 0. \qquad \Diamond\end{aligned}$$

Example 1.3.4 *Compute the Laplace transform of* $t^2\,\mathrm{Ci}(t)$.

Given that

$$\mathcal{L}\{\mathrm{Ci}\}(z) = \frac{\mathrm{Log}\left(z^2+1\right)}{2z},$$

it follows from the rule of multiplication by t^2 that

$$\begin{aligned}\mathcal{L}\left\{t^2\mathrm{Ci}(t)\right\}(z) &= \frac{d^2}{dz^2}\left\{\frac{\mathrm{Log}\left(z^2+1\right)}{2z}\right\} = \frac{d}{dz}\left\{\frac{2z}{z^2+1}\cdot\frac{1}{2z} - \frac{\mathrm{Log}\left(z^2+1\right)}{2z^2}\right\}\\ &= -\frac{2z}{\left(z^2+1\right)^2} - \frac{2z}{z^2+1}\cdot\frac{1}{2z^2} + \frac{\mathrm{Log}\left(z^2+1\right)}{z^3}\\ &= -\frac{3z^2+1}{z\left(z^2+1\right)^2} + \frac{\mathrm{Log}\left(z^2+1\right)}{z^3}. \qquad \Diamond\end{aligned}$$

Example 1.3.5 *Compute the Laplace transforms of*

1) $e^{-3t}\,\mathrm{Ei}(t)$,

2) $t\,\mathrm{Ei}(t)$.

We shall use that

$$\mathcal{L}\{\mathrm{Ei}\}(z) = \frac{\mathrm{Log}(1+z)}{z}, \qquad \text{for } \Re z > 0.$$

1) By using a rule of computation,

$$\mathcal{L}\left\{e^{-3t}\,\mathrm{Ei}(t)\right\}(z) = \frac{\mathrm{Log}(z+4)}{z+3}\text{for } \Re z > -3.$$

2) Using the rule of multiplication by t,

$$\mathcal{L}\{t\,\mathrm{Ei}(t)\}(z) = -\frac{d}{dz}\left\{\frac{\mathrm{Log}(1+z)}{z}\right\} = \frac{\mathrm{Log}(1+z)}{z^2} + \frac{1}{z(1+z)}, \qquad \text{for } \Re z > 0. \qquad \Diamond$$

Download free eBooks at bookboon.com

Example 1.3.6 *Find the error in the following "proof" of*

$$F(z)=\mathcal{L}\{\mathrm{Ci}\}(z)=\frac{\mathrm{Log}\left(z^2+1\right)}{2z}.$$

"It follows from the definition

$$\mathrm{Ci}(t)=\int_t^{+\infty}\frac{\cos u}{u}\,\mathrm{d}u,$$

that $t\cdot(\mathrm{Ci})'(t)=-\cos t$, *thus*

$$-\frac{d}{dz}\{zF(z)-\mathrm{Ci}(0)\}=-\frac{d}{dz}\{z\,F(z)\}=-\frac{z}{z^2+1},$$

hence,

$$\frac{d}{dz}\{z\,F(z)\}=\frac{z}{z^2+1},$$

and therefore,

$$z\,F(z)=\frac{1}{2}\,Log\left(z^2+1\right)+C.$$

Then it follows from the Finite Value Theorem *that*

$$\lim_{s\to 0+} s\,F(s)=\lim_{t\to+\infty}\mathrm{Ci}(t)=0,$$

so $C=0$, *and we conclude that*

$$\mathcal{L}\{\mathrm{Ci}\}(z)=\frac{\mathrm{Log}\left(z^2+1\right)}{2z}."$$

It follows from the sketch above that Ci(0) occurs early in the proof. However, since the improper integral $\int_0^{+\infty}\frac{\cos u}{u}\,\mathrm{d}u$ is divergent, which follows from the estimate

$$\int_0^{+\infty}\left|\frac{\cos u}{u}\right|\,\mathrm{d}u\ge\cos\frac{\pi}{4}\int_0^{\frac{\pi}{4}}\frac{\mathrm{d}u}{u}=+\infty,$$

the constant Ci(0) is not defined. The sneaky thing is that this (non-existing) constant is disappearing by a later differentiation. ◊

Download free eBooks at bookboon.com

Example 1.3.7 *Prove that*

$$\int_0^{+\infty} t\,e^{-t}\,\mathrm{Ei}(t)\,\mathrm{d}t = \ln 2 - \frac{1}{2}.$$

It follows by inspection supplied with the rules of computation [we notice that $1 > 0 = \sigma(\mathrm{Ei})$] that

$$\begin{aligned}\int_0^{+\infty} t\,e^{-t}\,\mathrm{Ei}(t)\,\mathrm{d}t &= \mathcal{L}\{t\,\mathrm{Ei}(t)\}(t) = \lim_{z\to 1}\left\{-\frac{d}{dz}\,\mathcal{L}\{\mathrm{Ei}\}(z)\right\} = -\lim_{z\to 0}\frac{d}{dz}\left\{\frac{\mathrm{Log}(1+z)}{z}\right\}\\ &= \lim_{z\to 1}\left\{\frac{\mathrm{Log}(1+z)}{z^2}\right\} - \frac{1}{z(1+z)} = \ln 2 - \frac{1}{2}. \qquad \diamond\end{aligned}$$

Click on the ad to read more

Download free eBooks at bookboon.com

1.4 The error function

Example 1.4.1 *Compute the Laplace transforms of*

1) $e^{3t}\text{erf}\left(\sqrt{t}\right)$,

2) $t \cdot \text{erf}\left(2\sqrt{t}\right)$.

It follows from *Ventus, Complex Functions Theory a-6, The Laplace Transformation II* that

$$\mathcal{L}\left\{\text{erf}\left(\sqrt{t}\right)\right\}(z) = \frac{1}{z\sqrt{z+1}}, \qquad \Re z > 0.$$

1) It follows from one of the rules of computation for the Laplace transformation that

$$\mathcal{L}\left\{e^{3t}\text{erf}\left(\sqrt{t}\right)\right\}(z) = \mathcal{L}\left\{\text{erf}\left(\sqrt{t}\right)\right\}(z-3) = \frac{1}{(z-3)\sqrt{z-2}}.$$

2) We apply the rule of multiplication by t and the rule of similarity. Then for $\Re z > 0$,

$$\mathcal{L}\left\{t \cdot \text{erf}\left(2\sqrt{t}\right)\right\}(z) = -\frac{d}{dz}\mathcal{L}\left\{\text{erf}\left(\sqrt{4t}\right)\right\}(z) = -\frac{d}{dz}\left\{\frac{1}{4}\mathcal{L}\left\{\text{erf}\left(\sqrt{t}\right)\right\}\left(\frac{z}{4}\right)\right\}$$

$$= -\frac{1}{4}\frac{d}{dz}\left\{\frac{1}{\frac{z}{4}\sqrt{\frac{z}{4}+1}}\right\} = -\frac{d}{dz}\left\{\frac{2}{z\sqrt{z+4}}\right\} = -\left\{-\frac{2}{z^2\sqrt{z+4}} - \frac{1}{2}\cdot\frac{2}{z(z+4)^{\frac{3}{2}}}\right\}$$

$$= \frac{1}{z^2(z+4)^{\frac{3}{2}}}\left\{2(z+4)+z\right\} = \frac{3z+8}{z^2(z+4)^{\frac{3}{2}}}. \qquad \Diamond$$

Example 1.4.2 *Compute the Laplace transform of* $\text{erfc}\left(\sqrt{t}\right)$.

Using that $\text{erfc}\left(\sqrt{t}\right) = 1 - \text{erf}\left(\sqrt{t}\right)$ and that the Fourier transform of $\text{erf}\left(\sqrt{t}\right)$ was found in *Ventus, Complex Functions Theory a-6, The Laplace Transformation II* it follows that

$$\mathcal{L}\left\{\text{erfc}\left(\sqrt{t}\right)\right\}(z) = \frac{1}{z} - \frac{1}{z\sqrt{z+1}} = \frac{\sqrt{z+1}-1}{z\sqrt{z+1}}. \qquad \Diamond$$

Example 1.4.3 *Compute the Laplace transform of* $\int_0^t \text{erf}\left(\sqrt{u}\right)\, du$.

We use the rule of integration and that the Fourier transform of $\text{erf}\left(\sqrt{t}\right)$ was found in *Ventus, Complex Functions Theory a-6, The Laplace Transformation II* to get

$$\mathcal{L}\left\{\int_0^t \text{erf}\left(\sqrt{u}\right)\, \text{d}u\right\}(z) = \frac{1}{z^2\sqrt{z+1}}, \qquad \Re z > 0. \qquad \Diamond$$

Download free eBooks at bookboon.com

Example 1.4.4 *Prove that* $\int_0^{+\infty} e^{-t}\,\mathrm{erf}\left(\sqrt{t}\right)\,dt = \dfrac{\sqrt{2}}{2}$.

HINT. *Consider* $\mathcal{L}\left\{\mathrm{erf}\left(\sqrt{t}\right)\right\}(z)$.

We have straightforward,

$$\int_0^{+\infty} e^{-t}\,\mathrm{erf}\left(\sqrt{t}\right)\,\mathrm{d}t = \mathcal{L}\left\{\mathrm{erf}\left(\sqrt{t}\right)\right\}(1) = \left[\frac{1}{z\sqrt{z+1}}\right]_{z=1} = \frac{\sqrt{2}}{2}. \qquad \Diamond$$

Example 1.4.5 *Compute the inverse Laplace transform of* $\dfrac{1}{\sqrt{z}(z-1)}$.

It follows from the theorem of convolution applied in the opposite direction that

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{z}(z-1)}\right\}(t) &= \frac{1}{\sqrt{\pi}}\left(\frac{1}{\sqrt{t}}\star e^t\right)(t) = \frac{1}{\sqrt{\pi}}\int_0^t \frac{e^{t-u}}{\sqrt{u}}\,\mathrm{d}u = \frac{1}{\sqrt{\pi}}\,e^t\int_0^t \frac{e^{-u}}{\sqrt{u}}\,\mathrm{d}u \\ &= \frac{2}{\sqrt{\pi}}\,e^t\int_0^{\sqrt{t}} e^{-v^2}\,\mathrm{d}v = e^t\,\mathrm{erf}\left(\sqrt{t}\right). \qquad \Diamond\end{aligned}$$

Example 1.4.6 *Compute the inverse Laplace transform of* $\dfrac{\sqrt{z}}{z-1}$.

We first compute

$$\begin{aligned}\frac{\sqrt{z}}{z-1} &= z\cdot\frac{1}{\sqrt{z}\cdot(z-1)} = z\cdot\frac{1}{(z-1)\sqrt{(z-1)+1}} = z\,\mathcal{L}\left\{e^t\,\mathrm{erf}\left(\sqrt{t}\right)\right\}(z) \\ &= z\,\mathcal{L}\left\{e^t\,\mathrm{erf}\left(\sqrt{t}\right)\right\}(z) - e^0\,\mathrm{erf}\left(\sqrt{0}\right) = \mathcal{L}\left\{\frac{d}{dt}\left(e^t\,\mathrm{erf}\left(\sqrt{t}\right)\right)\right\}(z).\end{aligned}$$

We therefore conclude by the uniqueness that

$$\mathcal{L}^{-1}\left\{\frac{\sqrt{z}}{z-1}\right\}(t) = \frac{d}{dt}\left\{e^t\,\mathrm{erf}\left(\sqrt{t}\right)\right\}.$$

According to a result of an example in *Ventus, Complex Functions Theory a-6, The Laplace Transformation II* we have

$$\mathrm{erf}\left(\sqrt{t}\right) = \frac{1}{\sqrt{\pi}}\int_0^t \frac{e^{-u}}{\sqrt{u}}\,\mathrm{d}u.$$

Hence finally,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{\sqrt{z}}{z-1}\right\}(t) &= \frac{d}{dt}\left\{e^t\,\mathrm{erf}\left(\sqrt{t}\right)\right\} = e^t\cdot\mathrm{erf}\left(\sqrt{t}\right) + \frac{e^t}{\sqrt{\pi}}\cdot\frac{e^{-t}}{\sqrt{t}} \\ &= e^t\cdot\mathrm{erf}\left(\sqrt{t}\right) + \frac{1}{\sqrt{\pi t}}. \qquad \Diamond\end{aligned}$$

Download free eBooks at bookboon.com

Example 1.4.7 *Compute the inverse Laplace transform of* $\dfrac{1}{1+\sqrt{z}}$.

Assume that $\Re z > 1$. Then

$$\begin{aligned}
\frac{1}{1+\sqrt{z}} &= \frac{1}{z-1}\cdot\frac{z-1}{\sqrt{z}+1} = \frac{\sqrt{z}-1}{z-1} = \frac{\sqrt{z}}{z-1} - \frac{1}{z-1} = z\cdot\frac{1}{(z-1)\sqrt{z}} - \frac{1}{z-1}\\
&= z\,\mathcal{L}\left\{\operatorname{erf}\left(\sqrt{t}\right)\right\}(z-1) - \mathcal{L}\left\{e^t\right\}(z) = z\,\mathcal{L}\left\{e^t\operatorname{erf}\left(\sqrt{t}\right)\right\}(z) - \mathcal{L}\left\{e^t\right\}(z)\\
&= \mathcal{L}\left\{\frac{d}{dt}\left\{e^t\operatorname{erf}\left(\sqrt{t}\right)\right\} - e^t\right\}(z),
\end{aligned}$$

where we have used that $\lim_{\to 0+} e^t\operatorname{erf}\left(\sqrt{t}\right) = 0$.

Then we use the formula

$$\operatorname{erf}\left(\sqrt{t}\right) = \frac{1}{\sqrt{\pi}} = \frac{1}{\sqrt{\pi}}\int_0^t \frac{e^{-u}}{\sqrt{u}}\,\mathrm{d}u,$$

which was also applied in Example 1.4.6, to get

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{1+\sqrt{z}}\right\}(t) &= \frac{d}{dt}\left\{e^t\operatorname{erf}\left(\sqrt{t}\right)\right\} - e^t = e^t\operatorname{erf}\left(\sqrt{t}\right) + \frac{1}{\sqrt{\pi t}} - e^t\\
&= \frac{1}{\sqrt{\pi t}} - e^t\operatorname{erfc}\left(\sqrt{t}\right). \qquad \Diamond
\end{aligned}$$

Example 1.4.8 *For $a > 0$ fixed we define $f_a(t) := \dfrac{1}{|t-a|}$. Compute the Laplace transform $\mathcal{L}\{f_a\}(z)$.*

We use the rule of similarity and an example from *Ventus, Complex Functions Theory a-6, The Laplace Transformation II* to get

$$\begin{aligned}
\mathcal{L}\{f_a\}(z) &= \mathcal{L}\left\{\frac{1}{\sqrt{|t-a|}}\right\}(z) = \frac{1}{\sqrt{a}}\,\mathcal{L}\left\{\frac{1}{\sqrt{\left|\frac{t}{a}-1\right|}}\right\}(z) = \frac{a}{\sqrt{a}}\,\mathcal{L}\left\{\frac{1}{\sqrt{|t-1|}}\right\}(a\,z)\\
&= \sqrt{a\pi}\cdot\frac{e^{-az}}{\sqrt{az}}\left\{1 - i\cdot\operatorname{erf}\left(i\sqrt{az}\right)\right\} = \sqrt{\frac{\pi}{z}}\cdot e^{-az}\left\{1 - i\cdot\operatorname{erf}\left(i\sqrt{az}\right)\right\}. \qquad \Diamond
\end{aligned}$$

Download free eBooks at bookboon.com

1.5 The Bessel functions

Example 1.5.1 *Compute* $\int_0^{+\infty} J_0\left(x^2\right) \mathrm{d}x$.

Hint. *Define the auxiliary function* $f(t) := \int_0^{+\infty} J_0\left(tx^2\right) \mathrm{d}x$*, and then compute* $\mathcal{L}\{f\}(s)$ *for* $s \in \mathbb{R}_+$ *by interchanging the order of integration.*

We define as in the hint,

$$f(t) := \int_0^{+\infty} J_0\left(tx^2\right) \mathrm{d}x,$$

and then apply the Laplace transformation on f for $z = s \in \mathbb{R}_+$ real and positive. Then by interchanging the order of integration,

$$\begin{aligned}
\mathcal{L}\{f\}(s) &= \int_0^{+\infty} e^{-st}\left\{\int_0^{+\infty} J_0\left(tx^2\right) \mathrm{d}x\right\} \mathrm{d}t = \int_0^{+\infty}\left\{\int_0^{+\infty} e^{-st} J_0\left(t\,x^2\right) \mathrm{d}t\right\} \mathrm{d}x \\
&= \int_0^{+\infty} \mathcal{L}\left\{J_0\left(t \cdot x^2\right)\right\}(s) \mathrm{d}x = \int_0^{+\infty} \frac{1}{x^2} \mathcal{L}\left\{J_0\right\}\left(\frac{s}{x^2}\right) \mathrm{d}x \\
&= \int_0^{+\infty} \frac{1}{x^2} \cdot \frac{1}{\sqrt{1+\left\{\frac{s}{x^2}\right\}^2}} \mathrm{d}x = \int_0^{+\infty} \frac{\mathrm{d}x}{\sqrt{x^4+s^2}} = \frac{1}{\sqrt{s}} \int_0^{+\infty} \frac{1}{\sqrt{\left\{\frac{x}{\sqrt{s}}\right\}^4+1}} \frac{\mathrm{d}x}{\sqrt{s}} \\
&= \frac{1}{\sqrt{s}} \int_0^{+\infty} \frac{\mathrm{d}x}{\sqrt{1+x^4}} = \frac{1}{\sqrt{\pi}} \mathcal{L}\left\{\sqrt{t}\right\}(s) \cdot \int_0^{+\infty} \frac{\mathrm{d}x}{\sqrt{1+x^4}},
\end{aligned}$$

Click on the ad to read more

Download free eBooks at bookboon.com

from which we conclude that

$$f(t) = \sqrt{\frac{t}{\pi}} \int_0^{+\infty} \frac{\mathrm{d}x}{\sqrt{1+x^4}},$$

hence by choosing $t = 1$,

$$f(1) = \int_0^{+\infty} J_0\left(x^2\right) \mathrm{d}x = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \frac{\mathrm{d}x}{\sqrt{1+x^4}}.$$

Finally, we get by the substitution $x = \sqrt{\tan\Theta}$,

$$\begin{aligned}
\int_0^{+\infty} J_0\left(x^2\right) \mathrm{d}x &= \frac{1}{\sqrt{\pi}} \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1+\tan^2\Theta}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{\tan\Theta}} \cdot \frac{\mathrm{d}\Theta}{\cos^2\Theta} \\
&= \frac{1}{2\sqrt{\pi}} \int_0^{\frac{\pi}{2}} \frac{\cos\Theta}{\cos^2\Theta} \sqrt{\frac{\cos\Theta}{\sin\Theta}}\, \mathrm{d}\Theta = \frac{1}{2\sqrt{\pi}} \int_0^{\frac{\pi}{2}} \sin^{2\cdot\frac{1}{4}-1}\Theta \cdot \cos^{2\cdot\frac{1}{4}-1}\Theta \,\mathrm{d}\Theta \\
&= \frac{1}{2\sqrt{\pi}} \cdot \frac{1}{2} B\left(\frac{1}{4}, \frac{1}{4}\right) = \frac{1}{4\pi} \cdot \frac{\left\{\Gamma\left(\frac{1}{4}\right)\right\}^2}{\Gamma\left(\frac{1}{2}\right)} = \frac{\left\{\Gamma\left(\frac{1}{4}\right)\right\}^2}{4\pi}. \qquad \Diamond
\end{aligned}$$

Example 1.5.2 *Compute by using the series method the inverse Laplace transform of*

$$\frac{1}{z} J_0\left(\frac{2}{\sqrt{z}}\right).$$

When we apply the series expansion of the Bessel function J_0 we get

$$\frac{1}{z} J_0\left(\frac{2}{\sqrt{z}}\right) = \frac{1}{z} \sum_{n=0}^{+\infty} \frac{(-1)^n}{\{n!\}^2} \left(\frac{2}{\sqrt{z}}\right)^{2n} = \frac{1}{z} \sum_{n=0}^{+\infty} \frac{(-4)^n}{\{n!\}^2} \cdot \frac{1}{z^n},$$

where the series is convergent for $z \in \mathbb{C} \setminus \{0\}$.

According to a theorem in *Ventus, Complex Functions Theory a-6, The Laplace Transformation II* we have in general

$$\mathcal{L}^{-1}\left\{\sum_{n=0}^{+\infty} b_n \cdot \frac{1}{z^{n+1}}\right\}(t) = \sum_{n=0}^{+\infty} \frac{1}{n!}\, b^n\, t^n,$$

provided that the series $\sum_{n=0}^{+\infty} b_n\, z^{-(n+1)}$ is convergent for $|z| > \frac{1}{R}$. The latter condition is trivial, and we get by identification that

$$b_n = \frac{(-4)^n}{\{n!\}^2}.$$

Hence,

$$f(t) = \mathcal{L}^{-1}\left\{\sum_{n=0}^{+\infty} \frac{(-4)^n}{\{n!\}^2} \cdot \frac{1}{z^{n+1}}\right\}(t) = \sum_{n=0}^{+\infty} \frac{(-4)^n}{\{n!\}^3}\, t^n, \qquad t \in \mathbb{R}_+. \qquad \Diamond$$

Download free eBooks at bookboon.com

Example 1.5.3 *Prove that*

$$\text{1)} \quad \int_0^{+\infty} J_n(t)\,\mathrm{d}t = 1, \qquad \text{2)} \quad \int_0^{+\infty} t\,J_n(t)\,\mathrm{d}t = n,$$

where we assume (without proof) that the improper integrals are convergent.

1) If $s > 0$ is real, then

$$\mathcal{L}\{J_n\}(s) = \frac{\left(\sqrt{s^2+1}-s\right)^n}{\sqrt{s^2+1}} \to \frac{(1-0)^n}{\sqrt{0^2+1}} = 1 \qquad \text{for } s \to 0.$$

Therefore, *if* the improper integral exists, then

$$\int_0^{+\infty} J_n(t)\,\mathrm{d}t = \lim_{s\to 0+} \int_0^{+\infty} J_n(t)\,e^{-st}\,\mathrm{d}t = \lim_{s\to 0+} \mathcal{L}\{J_n\}(s) = 1,$$

according to the computation above.

2) *Analogously* we get here that *if* the improper integral exists, then its value is given by

$$\begin{aligned}\int_0^{+\infty} t\,J_n(t)\,\mathrm{d}t &= \lim_{s\to 0+} \int_0^{+\infty} t\,J_n(t)\,e^{-st}\,\mathrm{d}t = \lim_{s\to 0+} \mathcal{L}\{t\,J_n\}(s) = \lim_{s\to 0+}\left\{-\frac{d}{ds}\,\mathcal{L}\{J_n\}(s)\right\} \\ &= \lim_{s\to 0+}\left\{-\frac{n\left(\sqrt{s^2+1}-s\right)^{n-1}}{\sqrt{s^2+1}}\left(\frac{s}{\sqrt{s^2+1}}-1\right)+\frac{\left(\sqrt{s^2+1}-s\right)^n}{\left(\sqrt{s^2+1}\right)^3}\cdot s\right\} \\ &= n. \qquad \diamond\end{aligned}$$

Example 1.5.4 *Prove that*

$$\int_0^{+\infty} u\,\exp\left(-u^2\right)J_0(a\,u)\,\mathrm{d}u = \frac{1}{2}\,\exp\left(-\frac{a^2}{4}\right).$$

We put

$$\varphi(a) := \int_0^{+\infty} u\,\exp\left(-u^2\right)J_0(a\,u)\,du \qquad \text{and} \qquad \psi(u) := \frac{1}{2}\,\exp\left(-\frac{a^2}{4}\right).$$

The trivial estimate $|J_0(a\,u)| \le 1$ implies that $\varphi(a)$ is well-defined, and that $\varphi \in C^\infty(\mathbb{R})$, and we are allowed to differentiate under the sign of integration. It follows from

$$\psi'(a) = \frac{1}{2}\left\{-\frac{2a}{4}\right\}\exp\left(-\frac{a^2}{4}\right) = -\frac{a}{2}\,\psi(a), \qquad \psi(0) = \frac{1}{2},$$

that we shall only prove that $\varphi(a)$ satisfies

$$\varphi'(a) = -\frac{a}{2}\,\varphi(a) \quad \text{for } a > 0, \qquad \text{and} \qquad \varphi(0) = \frac{1}{2}.$$

Download free eBooks at bookboon.com

It follows from the computation

$$\varphi(0)=\int_0^{+\infty} u\,\exp\left(-u^2\right) J_0(0)\,\mathrm{d}u=\int_0^{+\infty} u\,\exp\left(-u^2\right)\,\mathrm{d}u=\left[-\frac{1}{2}\,\exp\left(-u^2\right)\right]_0^{+\infty}=\frac{1}{2}=\psi(0),$$

that the initial condition is fulfilled.

It follows from *Ventus, Complex Functions Theory a-6, The Laplace Transformation II*, that

$$J_0'(t)=-J_1(t) \qquad \text{and} \qquad \frac{d}{dt}\{t\,J_1(t)\}=t\,J_0(t),$$

so when the expression of $\varphi(a)$ is differentiated with respect to $a>0$, then

$$\begin{aligned}\varphi'(a) &= \int_0^{+\infty} u\,e^{-u^2}\,\frac{\partial}{\partial a}\,J_0(a\,u)\,\mathrm{d}u=\int_0^{+\infty} u\,e^{-u^2}\,u\,J_0'(a\,u)\,\mathrm{d}u=-\int_0^{+\infty} u\,e^{-u^2}\cdot u\,J_1(a\,u)\,\mathrm{d}u\\ &= \left[\frac{1}{2}\,e^{-u^2}\,u\,J_1(a\,u)\right]_0^{+\infty}-\frac{1}{2}\int_0^{+\infty} e^{-u^2}\,\frac{\partial}{\partial u}\,\{u\,J_1(a\,u)\}\,\mathrm{d}u\\ &= -\frac{1}{2}\int_0^{+\infty} e^{-u^2}\,\frac{\partial}{\partial(au)}\,\{(au)J_1(au)\}\,\mathrm{d}u\\ &= -\frac{1}{2}\int_0^{+\infty} e^{-u^2}\,(au)J_0(au)\,\mathrm{d}u=-\frac{a}{2}\int_0^{+\infty} u\,e^{-u^2}\,J_0(au)\,\mathrm{d}u=-\frac{a}{2}\,\varphi(a),\end{aligned}$$

and the claim is proved. ◊

Example 1.5.5 *Compute*

$$\mathcal{L}\left\{e^{-at}\,J_0(b\,t)\right\}(z), \qquad \textit{where } a,\,b\in\mathbb{R}_+.$$

We just apply the rules of computation for the Laplace transformation to get

$$\begin{aligned}\mathcal{L}\left\{e^{-at}\,J_0(b\,t)\right\}(z) &= \mathcal{L}\left\{J_0(b\,t)\right\}(z+a)=\frac{1}{b}\,\mathcal{L}\left\{J_0(t)\right\}\left(\frac{z+a}{b}\right)\\ &= \frac{1}{b}\cdot\frac{1}{\sqrt{1+\left\{\frac{z+a}{b}\right\}^2}}=\frac{1}{\sqrt{(z+a)^2+b^2}}, \qquad \Re\,z>-a. \qquad \Diamond\end{aligned}$$

Example 1.5.6 *Compute* $\mathcal{L}\left\{t\,J_0(2t)\right\}(z)$.

We just apply the rules of computation for the Laplace transformation to get

$$\begin{aligned}\mathcal{L}\left\{t\,J_0(2t)\right\}(z) &= -\frac{d}{dz}\,\mathcal{L}\left\{J_0(2t)\right\}(z)=-\frac{1}{2}\,\frac{d}{dz}\left(\mathcal{L}\left\{J_0\right\}\left(\frac{z}{2}\right)\right)\\ &= -\frac{1}{2}\,\frac{d}{dz}\left\{\frac{1}{\sqrt{1+\left\{\frac{z}{2}\right\}^2}}\right\}=-\frac{1}{2}\,\frac{\left(-\frac{1}{2}\right)}{\left(\sqrt{1+\left(\frac{z}{2}\right)^2}\right)^3}\cdot\frac{z}{2}=\frac{z}{\left(\sqrt{z^2+4}\right)^3}. \qquad \Diamond\end{aligned}$$

Download free eBooks at bookboon.com

Example 1.5.7 *We define the* modified Bessel function *of order 0 by* $I_0(t) := J_0(it)$, *which makes sense, because* $J_0(t)$ *has a convergent series expansion which can be extended to all of* $\mathbb{C}$.
Compute

$$\mathcal{L}\left\{I_0(a\,t)\right\}(z) \qquad \textit{for } a \in \mathbb{R}_+.$$

It follows from the rule of change of scale that

$$\mathcal{L}\left\{I_0(a\,t)\right\}(z) = \frac{1}{a}\,\mathcal{L}\left\{I_0\right\}\left(\frac{z}{a}\right).$$

It therefore suffices to compute $\mathcal{L}\left\{I_0\right\}(z)$.

Clearly,

$$I_0'(t) = i\,J_0'(i\,t), \qquad \text{thus} \qquad J_0'(i\,t) = -i\,I_0(t),$$

and

$$I_0''(t) = -J_0''(i\,t), \qquad \text{thus} \qquad J_0''(i\,t) = -I_0''(t).$$

By insertion into the Bessel equation of order 0 we obtain the following differential equation for I_0,

$$0 = it\,J_0''(it) + J_0'(i\,t) + it\,J_0(i\,t) = -it\,J_0''(t) - i\,I_0'(t) + it\,I_0(t),$$

so the differential equation of I_0 becomes

$$-t\,I_0''(t) - I_0'(t) + t\,I_0(t) = 0, \qquad \text{and } I_0(0) = 1.$$

Click on the ad to read more

Download free eBooks at bookboon.com

We apply the Laplace transformation on this differential equation to get

$$\begin{aligned} 0 &= \frac{d}{dz}\left(z^2\,\mathcal{L}\{I_0\}(z) - z\,I_0(0) - I_0'(0)\right) - \left(z\,\mathcal{L}\{I_0\}(z) - I_0(0)\right) - \frac{d}{dz}\,\mathcal{L}\{I_0\}(z) \\ &= z^2\,\frac{d}{dx}\,\mathcal{L}\{I_0\}(z) + 2z\,\mathcal{L}\{I_0\}(z) - 1 - z\,\mathcal{L}\{I_0\}(z) + 1 - \frac{d}{dz}\,\mathcal{L}\{I_0\}(z) \\ &= \left(z^2-1\right)\frac{d}{dz}\,\mathcal{L}\{I_0\}(z) + z\,\mathcal{L}\{I_0\}(z), \end{aligned}$$

the solution of which for some arbitrary constant $c \in \mathbb{C}$ is

$$\mathcal{L}\{I_0\}(z) = \frac{c}{\sqrt{z^2-1}}, \qquad \text{for } \Re\, z > 1.$$

Finally, we conclude from

$$\lim_{z\to+\infty} z\cdot\frac{c}{\sqrt{z^2-1}} = c = I_0(0) = 1,$$

that

$$\mathcal{L}\{I_0\}(z) = \frac{1}{\sqrt{z^2-1}}, \qquad \text{for } \Re\, z > 1. \qquad \Diamond$$

Example 1.5.8 *Compute the Laplace transform of* $\frac{d^2}{dt^2}\left\{e^{2t}\,J_0(2t)\right\}$.

This is the usual exercise of applications of the rules of computation. If we put $f(t) := e^{2t}\,J_0(2t)$, then $f(0) = 1$ and $f'(0) = 2$. Then for $\Re\, z > 2$,

$$\begin{aligned} \mathcal{L}\left\{\frac{d^2}{dt^2}\left(e^{2t}\,J_0(2t)\right)\right\}(z) &= z^2\,\mathcal{L}\left\{e^{2t}\,J_0(2t)\right\}(z) - z\,f(0) - f'(0) \\ &= z^2\,\mathcal{L}\{J_0(2t)\}(z-2) - z - 2 = z^2\cdot\frac{1}{2}\,\mathcal{L}\{J_0(t)\}\left(\frac{z-2}{2}\right) - z - 2 \\ &= z^2\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{1+\left(\frac{z-2}{2}\right)^2}} - z - 2 = \frac{z^2}{\sqrt{(z-2)^2+4}} - z - 2. \qquad \Diamond \end{aligned}$$

Example 1.5.9 *Compute* $\mathcal{L}\{t\,J_1(t)\}(z)$.

This is straightforward for $\Re\, z > 0$,

$$\begin{aligned} \mathcal{L}\{t\,J_1(t)\}(z) &= -\frac{d}{dz}\,\mathcal{L}\{J_1\}(z) = -\frac{d}{dz}\left\{1 - \frac{z}{\sqrt{z^2+1}}\right\} = \frac{d}{dz}\left\{\frac{z}{\sqrt{z^2+1}}\right\} \\ &= \frac{1}{\sqrt{z^2+1}} - \frac{1}{2}\,\frac{z\cdot 2z}{(z^2+1)^{\frac{3}{2}}} = \frac{1}{\sqrt{z^2+1}} - \frac{z^2}{(z^2+1)^{\frac{3}{2}}} = \frac{1}{(z^2+1)^{\frac{3}{2}}}. \end{aligned}$$

Download free eBooks at bookboon.com

We notice that also

$$\mathcal{L}\left\{\sin\star J_0\right\}(z)=\frac{1}{\left(z^2+1\right)^{\frac{3}{2}}},$$

so we have proved that

$$\left(\sin\star J_0\right)(t)=t\,J_1(t). \qquad \Diamond$$

Example 1.5.10 *Compute the integral* $\int_0^{+\infty} t\,e^{-3t}\,J_0(4t)\,\mathrm{d}t$.

Assume that $\Re z>0$. Then

$$\begin{aligned}
int_0^{+\infty} t\,e^{-3t}\,J_0(4t)\,\mathrm{d}t &= \mathcal{L}\left\{t\,J_0(4t)\right\}(z)=-\frac{d}{dz}\,\mathcal{L}\left\{J_0(4t)\right\}(z)\\
&= \frac{d}{dz}\left(\frac{1}{4}\,\mathcal{L}\left\{J_0(t)\right\}\left(\frac{z}{4}\right)\right)=-\frac{d}{dz}\left\{\frac{1}{4}\,\frac{1}{\sqrt{1+\frac{z^2}{16}}}\right\}=-\frac{d}{dz}\left(\frac{1}{\sqrt{z^2+16}}\right)\\
&= -\left(-\frac{1}{2}\right)\cdot\frac{2z}{\left(z^2+16\right)^{\frac{3}{2}}}=\frac{z}{\left(z^2+16\right)^{\frac{3}{2}}}.
\end{aligned}$$

When we choose $z=3$, we get

$$\int_0^{+\infty} t\,e^{-3t}\,J_0(4t)\,\mathrm{d}t=\frac{3}{(9+16)^{\frac{3}{2}}}=\frac{3}{5^3}=\frac{3}{125}. \qquad \Diamond$$

Example 1.5.11 *Prove that* $\int_0^{+\infty} t^2\,J_0(t)\,\mathrm{d}t=-1$.

Consider $\Re z>0$. Then

$$\begin{aligned}
\int_0^{+\infty} t^2\,e^{-zt}\,J_0(t)\,\mathrm{d}t &= \mathcal{L}\left\{t^2\,J_0(t)\right\}(z)=(.1)^2\,\frac{d^2}{dz^2}\left(z^2+1\right)^{-\frac{1}{2}}\\
&= \frac{d}{dz}\left\{-\frac{1}{2}\left(z^2+1\right)^{-\frac{3}{2}}\cdot 2z\right\}=\frac{d}{dz}\left\{-z\left(z^2+1\right)^{-\frac{3}{2}}\right\}\\
&= -\left(z^2+1\right)^{-\frac{3}{2}}+3z^2\left(z^2+1\right)^{-\frac{5}{2}}\to -1 \qquad \text{for } z\to 0+\,.
\end{aligned}$$

Strictly speaking, we should start with a proof of that the improper integral is convergent, and that the limit process gives the right value. $\Diamond$

Download free eBooks at bookboon.com

Example 1.5.12 *Prove the following formulæ,*

1) $\int_0^{+\infty} J_0\left(2\sqrt{tu}\right)\cos u\,\mathrm{d}u = \sin t$,

2) $\int_0^{+\infty} J_0\left(2\sqrt{tu}\right)\sin u\,\mathrm{d}u = \cos t$,

3) $\int_0^{+\infty} J_0\left(2\sqrt{tu}\right) J_0(u)\,\mathrm{d}u = J_0(t)$.

We assume without proof that the improper integrals are all convergent, and that we may interchange the order of integration, when we apply the Laplace transformation.

It follows from a result in *Ventus, Complex Functions Theory a-6, The Laplace Transformation II*, that

$$\mathcal{L}\left\{J_0\left(2\sqrt{t}\right)\right\}(z) = \frac{1}{z}\cdot\exp\left(-\frac{1}{z}\right), \qquad \Re z > 0.$$

Then it follows from the rule of scaling for $\lambda > 0$ a constant,

$$\mathcal{L}\left\{J_0\left(2\sqrt{\lambda t}\right)\right\}(z) = \frac{1}{\lambda}\,\mathcal{L}\left\{J_0\left(2\sqrt{t}\right)\right\}\left(\frac{z}{\lambda}\right) = \frac{1}{\lambda}\cdot\frac{\lambda}{z}\cdot\exp\left(-\frac{\lambda}{z}\right) = \frac{1}{z}\cdot\exp\left(-\frac{\lambda}{z}\right).$$

1) We get by a Laplace transformation with respect to t that

$$\mathcal{L}\left\{\int_0^{+\infty} J_0\left(2\sqrt{tu}\right)\cos u\,\mathrm{d}u\right\}(z) = \int_0^{+\infty}\mathcal{L}\left\{J_0\left(2\sqrt{tu}\right)\right\}(z)\cos u\,\mathrm{d}u$$

$$= \int_0^{+\infty}\frac{1}{z}\,\exp\left(-\frac{u}{z}\right)\cos u\,\mathrm{d}u = \frac{1}{z}\,\mathcal{L}\{\cos u\}\left(\frac{1}{z}\right) = \frac{1}{z}\cdot\frac{\frac{1}{z}}{1+\frac{1}{z^2}} = \frac{1}{z^2+1} = \mathcal{L}\{\sin t\}(z).$$

Hence, we get by the inverse Laplace transformation,

$$\int_0^{+\infty} J_0\left(2\sqrt{tu}\right)\cos u\; du = \sin t.$$

Alternatively, we get either by formal computations, or by an analytic extension that

$$\int_0^{+\infty} J_0\left(2\sqrt{tu}\right)\cos u\,\mathrm{d}u = \frac{1}{2}\int_0^{+\infty} J_0\left(2\sqrt{tu}\right)\left\{e^{iu}+e^{-iu}\right\}\mathrm{d}u$$

$$= \frac{1}{2}\left(\mathcal{L}\left\{J_0\left(2\sqrt{tu}\right)\right\}(-i) + \mathcal{L}\left\{J_0\left(2\sqrt{tu}\right)\right\}(i)\right)$$

$$= \frac{1}{2}\left(\frac{1}{t}\,\mathcal{L}\left\{J_0\left(2\sqrt{u}\right)\right\}\left(-\frac{i}{t}\right) + \frac{1}{t}\,\mathcal{L}\left\{J_0\left(2\sqrt{u}\right)\right\}\left(\frac{i}{t}\right)\right)$$

$$= \frac{1}{2t}\left\{\left(-\frac{t}{i}\right)\exp\left(\frac{t}{i}\right) + \frac{t}{i}\,\exp\left(-\frac{t}{i}\right)\right\}$$

$$= \frac{1}{2i}\left\{e^{it}-e^{-it}\right\} = \sin t.$$

Download free eBooks at bookboon.com

2) Using the same method as above we get in this case,

$$\mathcal{L}\left\{\int_0^{+\infty} J_0\left(2\sqrt{tu}\right)\sin u\,\mathrm{d}u\right\}(z) = \frac{1}{z}\,\mathcal{L}\{\sin u\}\left(\frac{1}{z}\right) = \frac{1}{z}\cdot\frac{1}{1+\frac{1}{z^2}} = \frac{z}{z^1+1} = \mathcal{L}\{\cos t\}(z).$$

Then by the inverse Laplace transformation,

$$\int_0^{+\infty} J_0\left(2\sqrt{tu}\right)\sin u\,\mathrm{d}u = \cos t.$$

Alternatively, and analogously we here get

$$\int_0^{+\infty} J_0\left(2\sqrt{tu}\right)\sin u\,\mathrm{d}u = \frac{1}{2i}\int_0^{+\infty} J_0\left(2\sqrt{tu}\right)\left\{e^{iu}-e^{-iu}\right\}\mathrm{d}u$$

$$= \frac{1}{2it}\left\{\left(-\frac{t}{i}\right)\exp\left(\frac{t}{i}\right) - \frac{t}{i}\exp\left(-\frac{t}{i}\right)\right\} = \frac{1}{2}\left\{e^{it}+e^{-it}\right\} = \cos t.$$

3) We get by a Laplace transformation with respect to t that

$$\mathcal{L}\left\{\int_0^{+\infty} J_0\left(2\sqrt{tu}\right)J_0(u)\,\mathrm{d}u\right\}(z) = \frac{1}{z}\,\mathcal{L}\left\{J_0\right\}\left(\frac{1}{z}\right) = \frac{1}{z}\cdot\frac{1}{\sqrt{1+\frac{1}{z^2}}} = \frac{1}{\sqrt{1+z^2}} = \mathcal{L}\left\{J_0(t)\right\}(z).$$

Hence, by the inverse Laplace transformation,

$$\int_0^{+\infty} J_0\left(2\sqrt{tu}\right)J_0(u)\,\mathrm{d}u = J_0(t). \qquad \Diamond$$

Click on the ad to read more

Download free eBooks at bookboon.com

Example 1.5.13 *Compute* $\int_0^t J_0(u)\,J_1(t-u)\,\mathrm{d}u$.

We apply the rule of convolution for $\Re z > 0$,

$$\mathcal{L}\left\{\int_0^t J_0(u)J_1(t-u)\,\mathrm{d}u\right\}(z) = \mathcal{L}\left\{J_0 \star J_1\right\}(z) = \mathcal{L}\left\{J_0\right\}(z)\cdot\mathcal{L}\left\{J_1\right\}(z)$$

$$= \frac{1}{\sqrt{z^2+1}}\cdot\frac{\sqrt{z^2+1}-z}{\sqrt{z^2+1}} = \frac{1}{\sqrt{z^2+1}} - \frac{z}{z^2+1}.$$

Hence by the inverse Laplace transformation,

$$\int_0^t J_0(u)\,J_1(t-u)\,\mathrm{d}u = J_0(t) - \cos t. \qquad \diamond$$

Example 1.5.14 *Compute* $\int_0^t J_0(u)\,J_2(t-u)\,\mathrm{d}u$.

It follows by the rule of convolution that

$$\begin{aligned} F(z) &= \mathcal{L}\left\{\int_0^t J_0(u)\,J_2(t-u)\,\mathrm{d}u\right\}(z) = \mathcal{L}\left\{J_0\right\}(z)\cdot\mathcal{L}\left\{J_2\right\}(z) \\ &= \frac{1}{\sqrt{z^2+1}}\cdot\frac{\left(\sqrt{z^2+1}-z\right)^2}{\sqrt{z^2+1}} = \frac{z^2+1+z^2-2z\sqrt{z^2+1}}{z^2+1} \\ &= \frac{2\left(z^2+1-z\sqrt{z^2+1}\right)-1}{z^2+1} = 2\,\frac{\sqrt{z^2+1}-z}{\sqrt{z^2+1}} - \frac{1}{z^2+1}. \end{aligned}$$

Then by the inverse Laplace transformation,

$$\int_0^t J_0(u)\,J_2(t-u)\,\mathrm{d}u = 2\,J_1(t) - \sin t. \qquad \diamond$$

Example 1.5.15 *Compute* $\int_0^t J_0(u)\,\sin(t-u)\,\mathrm{d}u$.

Again we use the rule of convolution to get

$$\mathcal{L}\left\{\int_0^t J_0(u)\sin(t-u)\,\mathrm{d}u\right\}(z) = \mathcal{L}\left\{J_0\right\}(z)\cdot\mathcal{L}\{\sin t\}(z)$$

$$= \frac{1}{(z^2+1)^{\frac{3}{2}}} = \frac{d}{dz}\left\{\frac{z}{\sqrt{z^2+1}}\right\} = -\frac{d}{dz}\left\{1-\frac{z}{\sqrt{z^2+1}}\right\}$$

$$= -\frac{d}{dz}\left\{\frac{\sqrt{z^2+1}-z}{\sqrt{z^2+1}}\right\} = -\frac{d}{dz}\,\mathcal{L}\left\{J_1\right\}(z) = \mathcal{L}\left\{t\,J_1(t)\right\}(z).$$

Then we get by the inverse Laplace transformation,

$$\int_0^t J_0(u)\,\sin(t-u)\,\mathrm{d}u = \left(J_0 \star \sin\right)(t) = t\,J_1(t). \qquad \diamond$$

Download free eBooks at bookboon.com

Example 1.5.16 *Prove that* $J_m \star J_n(t) = J_0 \star J_{m+n}(t)$ *for all* $m,\ n \in \mathbb{N}_0$.

We shall use that

$$j_n(z) := \mathcal{L}\left\{J_n\right\}(z) = \frac{\left(\sqrt{z^2+1}-z\right)^n}{\sqrt{z^2+1}}, \qquad \text{for } \Re\, z > 0.$$

Then by the rule of convolution,

$$\mathcal{L}\left\{J_m \star J_n\right\}(z) = j_m(z)\cdot j_n(z) = \frac{\left(\sqrt{z^2+1}-z\right)^{m+n}}{\left(\sqrt{z^2+1}\right)^2} = j_0(z)\cdot j_{m+n}(z) = \mathcal{L}\left\{J_0 \star J_{m+n}\right\}(z),$$

and the claim follows, when we apply the inverse Laplace transformation. ◊

Example 1.5.17 *Compute the Laplace transform of* $\dfrac{1-J_0(t)}{t}$.
Apply the result to prove that

$$\int_0^{+\infty} \frac{1-J_0(t)}{t\, e^t}\, \mathrm{d}t = \ln\left(\frac{\sqrt{2}+1}{2}\right).$$

Apply the rule of division by t to get

$$\begin{aligned}
\mathcal{L}\left\{\frac{1-J_0(t)}{t}\right\}(z) &= \int_{\Gamma_z} \mathcal{L}\left\{1-J_0(t)\right\}(z)\,\mathrm{d}z = \int_{\Gamma_z}\left\{\frac{1}{z} - \frac{1}{\sqrt{z^2+1}}\right\}\mathrm{d}z \\
&= \text{Arsinh}\, z - \text{Log}\, z + c = \text{Log}\left(z+\sqrt{z^2+1}\right) - \text{Log}\, z + c \\
&= \text{Log}\left(1+\sqrt{1+\frac{1}{z^2}}\right) + c.
\end{aligned}$$

It follows from

$$\mathcal{L}\left\{\frac{1-J_0(t)}{t}\right\}(z) \to 0 \qquad \text{for } \Re\, z \to +\infty,$$

that

$$c = -\text{Log}\left(1+\sqrt{1+0}\right) = -\ln 2,$$

so we finally get

$$\mathcal{L}\left\{\frac{1-J_0(t)}{t}\right\}(z) = \text{Log}\left(1+\sqrt{1+\frac{1}{z^2}}\right) - \ln 2 = \text{Log}\left(\frac{1+\sqrt{1+\frac{1}{z^2}}}{2}\right) = \text{Log}\left(\frac{z+\sqrt{1+z^2}}{2z}\right).$$

Finally, if we put $z=1$, then

$$\mathcal{L}\left\{\frac{1-J_0(t)}{t}\right\}(1) = \int_0^{+\infty}\frac{1-J_0(t)}{t\,e^t}\,\mathrm{d}t = \ln\left(\frac{1+\sqrt{1+1}}{2\cdot 1}\right) = \ln\left(\frac{1+\sqrt{2}}{2}\right). \qquad \Diamond$$

Download free eBooks at bookboon.com

Example 1.5.18 *Compute the Laplace transform of* $t\,e^{-t}\,J_0\left(t\sqrt{2}\right)$.

Just use the well-known rules of computations to get

$$\mathcal{L}\left\{t\,e^{-t}\,J_0\left(t\sqrt{2}\right)\right\} = -\frac{d}{dz}\,\mathcal{L}\left\{e^{-t}\,J_0\left(t\sqrt{2}\right)\right\}(z) = -\frac{d}{dz}\,\mathcal{L}\left\{J_0\left(t\sqrt{2}\right)\right\}(z+2)$$

$$= -\frac{1}{\sqrt{2}}\,\frac{d}{dz}\,\mathcal{L}\{J_0\}\left(\frac{z+2}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}\,\frac{d}{dz}\left\{\frac{1}{\sqrt{1+\left(\frac{z+2}{\sqrt{2}}\right)^2}}\right\}$$

$$= -\frac{d}{dz}\left\{\frac{1}{\sqrt{2+(z+2)^2}}\right\} = \frac{z+2}{(z^2+4z+6)^{\frac{3}{2}}}. \qquad \Diamond$$

Example 1.5.19 *Apply a series expansion to prove that*

$$\mathcal{L}\left\{J_0\left(2\sqrt{t}\right)\right\}(z) = \frac{1}{z}\,\exp\!\left(-\frac{1}{z}\right).$$

Using a termwise Laplace transformation we get

$$\mathcal{L}\left\{J_0\left(2\sqrt{t}\right)\right\}(z) = \mathcal{L}\left\{\sum_{n=0}^{+\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{2\sqrt{t}}{2}\right)^{2n}\right\}(z) = \mathcal{L}\left\{\sum_{n=0}^{+\infty}\frac{(-1)^n}{(n!)^2}\,t^n\right\}(z)$$

$$= \sum_{n=0}^{+\infty}\frac{(-1)^n}{(n!)^2}\,\mathcal{L}\{t^n\}(z) = \sum_{n=0}^{+\infty}\frac{(-1)^n}{(n!)^2}\,\frac{n!}{z^{n+1}} = \frac{1}{z}\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}\,\frac{1}{z^n} = \frac{1}{z}\,\exp\!\left(-\frac{1}{z}\right).$$

We finally notice that the series is absolutely convergent, so the termwise Laplace transformation is legal. $\Diamond$

Example 1.5.20 *Compute the Laplace transform of* $J_1\left(2\sqrt{t}\right)$.

By termwise Laplace transformation,

$$\mathcal{L}\left\{J_1\left(2\sqrt{t}\right)\right\}(z) = \mathcal{L}\left\{\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!(n+1)!}\left(\frac{2\sqrt{t}}{2}\right)^{2n+1}\right\}(z)$$

$$= \mathcal{L}\left\{\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!(n+1)!}\,t^{n+\frac{1}{2}}\right\}(z) = \sum_{n=0}^{+\infty}\frac{(-1)^n}{n!(n+1)!}\,\frac{\Gamma\left(n+\frac{3}{2}\right)}{z^{n+\frac{3}{2}}}.$$

We compute separately,

$$\frac{\Gamma\left(n+\frac{3}{2}\right)}{n!(n+1)!} = \frac{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)\cdots\frac{1}{2}\sqrt{\pi}}{n!(n+1)!} = \frac{\sqrt{\pi}}{2^{n+1}}\cdot\frac{(2n+1)(2n-1)\cdots 1}{n!(n+1)!}$$

$$= \frac{\sqrt{\pi}}{2^{2n+1}}\cdot\frac{(2n+1)!}{(n!)^2(n+1)!} = \frac{\sqrt{\pi}}{2^{2n+1}}\cdot\frac{1}{n!}\binom{2n+1}{n}.$$

Download free eBooks at bookboon.com

It follows from this computation that the series is absolutely convergent, and that

$$\mathcal{L}\left\{J_1\left(2\sqrt{t}\right)\right\}(z) = \frac{\sqrt{\pi}}{2z\sqrt{z}} \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} \binom{2n+1}{n} \frac{1}{(4z)^n}, \qquad \Re z > 0. \qquad \Diamond$$

Example 1.5.21 *Define the Laguerre polynomials* $L_n(t)$ *by*

$$L_n(t) = \frac{e^t}{n!} \frac{d^n}{dt^n} \left\{t^n e^{-t}\right\}, \qquad n \in \mathbb{N}_0.$$

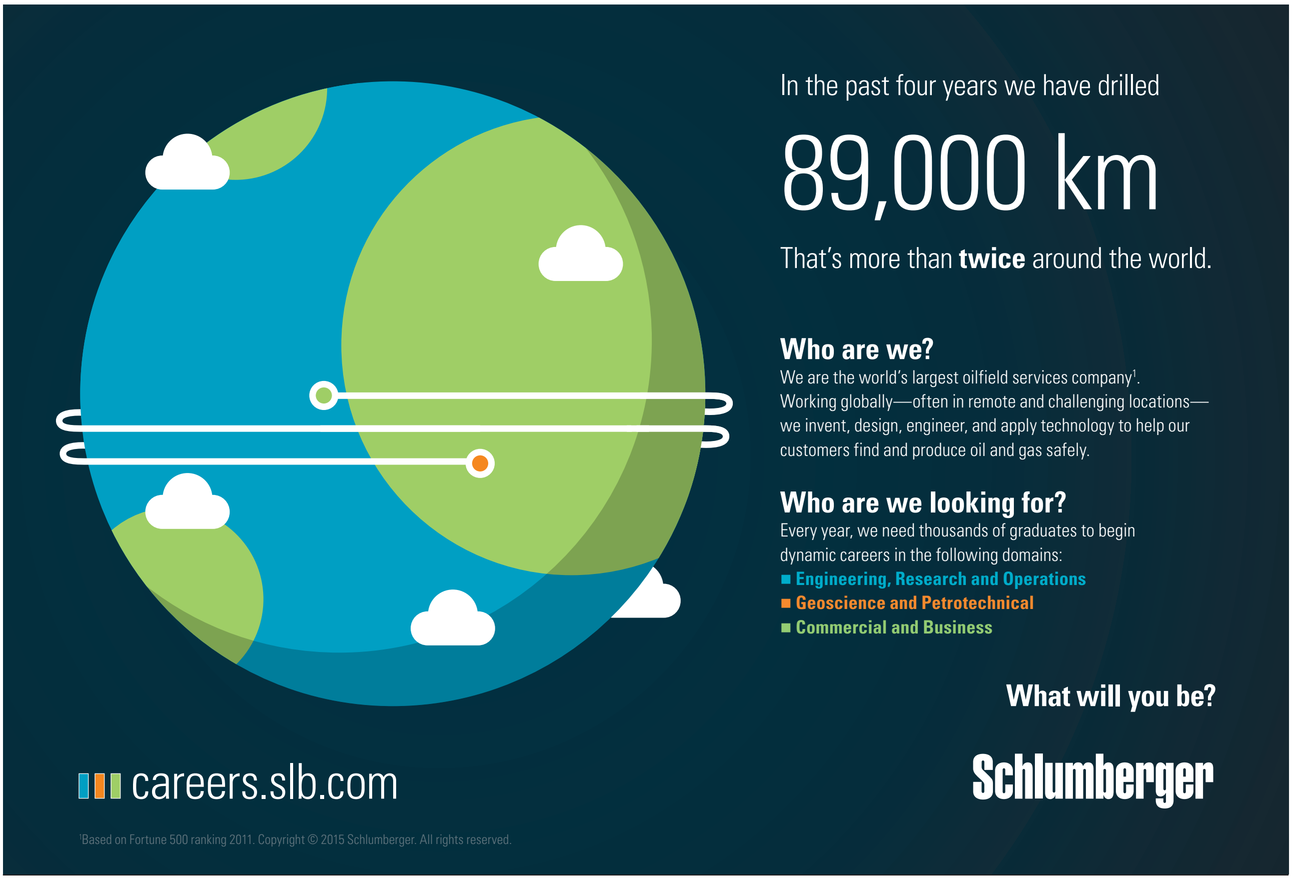
Compute $L_0(t)$, $L_1(t)$, $\dots$, $L_4(t)$.
Then compute the Laplace transform of $L_n(t)$ *for* $n \in \mathbb{N}_0$.

By straightforward computations,

$$L_0(t) = \frac{e^t}{0!} t^0 e^{-t} = 1,$$

$$L_1(t) = \frac{e^t}{1} \frac{d}{dt} \left\{t\, e^{-t}\right\} = e^t \left\{-t\, e^{-t} + e^{-t}\right\} = -t + 1,$$

$$\begin{aligned} L_2(t) &= \frac{e^t}{2!} \frac{d^2}{dt^2} \left\{t^2 e^{-t}\right\} = \frac{e^t}{2} \frac{d}{dt} \left\{-t^2 e^{-t} + 2t\, e^{-t}\right\} \\ &= \frac{e^t}{2} \left\{t^2 e^{-t} - 4t\, e^{-t} + 2e^{-t}\right\} = \frac{1}{2} t^2 - 2t + 1, \end{aligned}$$



Click on the ad to read more

Download free eBooks at bookboon.com

$$\begin{aligned} L_3(t) &= \frac{e^t}{3!}\,\frac{d^3}{dt^3}\left\{t^3\,e^{-t}\right\} = \frac{e^t}{6}\,\frac{d^2}{dt^2}\left\{-t^3\,e^{-t}+3t^2\,e^{-t}\right\} = \frac{e^t}{6}\,\frac{d}{dt}\left\{t^3\,e^{-t}-6t^2\,e^{-t}+6t\,e^{-t}\right\} \\ &= \frac{e^t}{6}\left\{-t^3\,e^{-t}+9t^2\,e^{-t}-18t\,e^{-t}+6\,e^{-t}\right\} = -\frac{t^3}{6}+\frac{3}{2}\,t^2-3t+1, \end{aligned}$$

$$\begin{aligned} L_4(t) &= \frac{e^t}{4!}\,\frac{d^4}{dt^4}\left\{t^4\,e^{-t}\right\} = \frac{e^t}{24}\,\frac{d^3}{dt^3}\left\{-t^4\,e^{-t}+4t^3\,e^{-t}\right\} \\ &= \frac{e^t}{24}\,\frac{d^2}{dt^2}\left\{t^4\,e^{-t}-8\,t^3\,e^{-t}+12t^2\,e^{-t}\right\} = \frac{e^t}{24}\,\frac{d}{dt}\left\{-t^4\,e^{-t}+12t^3\,e^{-t}-36t^2\,e^{-t}+24t\,e^{-t}\right\} \\ &= \frac{e^t}{24}\left\{t^4\,e^{-t}-16t^3\,e^{-t}+72t^2\,e^{-t}-96t\,e^{-t}+24e^{-t}\right\} = \frac{1}{24}\,t^4-\frac{2}{3}\,t^3+3t^2-4t+1. \end{aligned}$$

Finally, we use the rules of computation to find the Laplace transforms in general,

$$\mathcal{L}\left\{L_n(t)\right\}(z) = \mathcal{L}\left\{\frac{e^t}{n!}\,\frac{d^n}{dt^n}\left(t^n\,e^{-t}\right)\right\}(z) = \frac{1}{n!}\,\mathcal{L}\left\{\frac{d^n}{dt^n}\left(t^n\,e^{-t}\right)\right\}(z-1)$$

$$= \frac{1}{n!}\,(z-1)^n\,\mathcal{L}\left\{t^n\,e^{-t}\right\}(z-1) - \sum_{j=0}^{n-1}\left[\frac{d^j}{dt^j}\left(t^n\,e^{-t}\right)\right]_{t=0}\cdot(z-1)^{n-1-j}$$

$$= \frac{1}{n!}\,(z-1)^n\,\mathcal{L}\left\{t^n\right\}(z-1+1) - 0 = \frac{1}{n!}\,(z-1)^n\,\mathcal{L}\left\{t^n\right\}(z)$$

$$= \frac{1}{n!}\,(z-1)^n\cdot\frac{n!}{z^{n+1}} = \frac{(z-1)^n}{z^{n+1}} = \frac{1}{z}\left(1-\frac{1}{z}\right)^n.$$

The computations above are valid for $\Re\, z > 0$, or, by an analytic extension, for $z \in \mathbb{C}\setminus\{0\}$. $\Diamond$

Example 1.5.22 *Let $L_n(t)$ denote the Laguerre polynomials introduced in Example 1.5.21. Prove that*

$$\sum_{n=0}^{+\infty}\frac{1}{n!}\,L_n(t) = e\cdot J_0\left(2\sqrt{t}\right).$$

It follows by some combinatorics that

$$\begin{aligned} L_n(t) &= \frac{e^t}{n!}\,\frac{d^n}{dt^n}\left\{t^n\,e^{-t}\right\} = \frac{e^t}{n!}\sum_{n=0}^{n}\binom{n}{k}\frac{d^{n-k}}{dt^{n-k}}\,t^n\cdot\frac{d^k}{dt^k}\,e^{-t} \\ &= \frac{e^t}{n!}\sum_{k=0}^{n}\binom{n}{k}\frac{n!}{k!}\,t^k\cdot(-1)^k\,e^{-t} = \sum_{k=0}^{n}\frac{n!(-1)^k}{(k!)^2(n-k)!}\,t^k, \end{aligned}$$

Download free eBooks at bookboon.com

hence, by insertion,

$$\begin{aligned}\sum_{n=0}^{+\infty} \frac{1}{n!} L_n(t) &= \sum_{n=0}^{+\infty}\sum_{k=0}^{n} \frac{(-1)^k}{(k!)^2(n-k)!}\, t^k = \sum_{k=0}^{+\infty}\left\{\sum_{n=k}^{+\infty} \frac{1}{(n-k)!}\right\} \frac{(-1)^k}{(k!)^2}\left(\frac{2\sqrt{t}}{2}\right)^{2k} \\ &= \sum_{k=0}^{+\infty}\left\{\sum_{n=0}^{+\infty}\frac{1}{n!}\right\}\frac{(-1)^k}{(k!)^2}\left(\frac{2\sqrt{t}}{2}\right)^{2k} = e\sum_{k=0}^{+\infty}\frac{(-1)^k}{(k!)^2}\left(\frac{2\sqrt{t}}{2}\right)^{2k} \\ &= e\cdot J_0\left(2\sqrt{t}\right). \qquad \Diamond\end{aligned}$$

Example 1.5.23 *Let $\sqrt{\cdot}$ denote the branch of the square root, which is positive on $\mathbb{R}_+$ and which has its branch cut lying along $\mathbb{R}_-$.*

1) Compute the inverse Laplace transforms of $\dfrac{1}{\sqrt{z+i}}$ and $\dfrac{1}{\sqrt{z-i}}$.

2) Apply the results above and the rule of convolution to prove that

$$J_0(t) = \frac{1}{\pi}\int_{-1}^{1} \frac{e^{its}}{\sqrt{1-s^2}}\,\mathrm{d}s, \qquad \textit{for } t\in\mathbb{R}_+.$$

1) We shall use the well-known result

$$\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\}(z) = \sqrt{\frac{\pi}{z}}, \qquad \text{thus } \frac{1}{\sqrt{z}} = \mathcal{L}\left\{\frac{1}{\sqrt{\pi}}\cdot\frac{1}{\sqrt{t}}\right\}(z).$$

It follows from one of the rules of computation for every $a\in\mathbb{C}$ that

$$\mathcal{L}\left\{\frac{1}{\sqrt{\pi}}\cdot\frac{e^{at}}{\sqrt{t}}\right\}(z) = \frac{1}{\sqrt{z-a}}.$$

Choosing $a=\pi i$ we finally get

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{z+i}}\right\}(t) = \frac{1}{\sqrt{\pi}}\cdot\frac{e^{-it}}{\sqrt{t}} \qquad \text{and} \qquad \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{z-i}}\right\}(t) = \frac{1}{\sqrt{\pi}}\cdot\frac{e^{it}}{\sqrt{t}}.$$

2) We get by the rule of convolution,

$$\mathcal{L}\{J_0\}(z) = \frac{1}{\sqrt{z^2+1}} = \frac{1}{\sqrt{z-i}}\cdot\frac{1}{\sqrt{z+i}} = \mathcal{L}\left\{\frac{1}{\sqrt{\pi}}\cdot\frac{e^{-it}}{\sqrt{t}}\star\frac{1}{\sqrt{\pi}}\cdot\frac{e^{t}}{\sqrt{t}}\right\}(z),$$

hence

$$J_0(t) = \frac{1}{\pi}\int_0^t \frac{e^{-iu}}{\sqrt{u}}\cdot\frac{e^{i(t-u)}}{\sqrt{t-u}}\,\mathrm{d}u = \frac{1}{\pi}\int_0^t\frac{e^{i(t-2u)}}{\sqrt{u(t-u)}}\,\mathrm{d}u = \frac{1}{\pi}\int_0^t \frac{\exp\left(it\left(1-2\,\frac{u}{t}\right)\right)}{t\sqrt{\frac{u}{t}\left(1-\frac{u}{t}\right)}}\,\mathrm{d}u.$$

Download free eBooks at bookboon.com

Then by the change of variable $s = 1 - 2\,\frac{u}{t}$, thus $\frac{u}{t} = \frac{1}{2}(1-s)$ and $\mathrm{d}u = -\frac{t}{2}\,\mathrm{d}s$,

$$\begin{aligned} J_0(t) &= \frac{1}{\pi}\int_0^t \frac{\exp\left(it\left(1-2\,\frac{u}{t}\right)\right)}{t\sqrt{\frac{u}{t}\left(1-\frac{u}{t}\right)}}\,\mathrm{d}u = \frac{1}{\pi}\int_{-1}^1 \frac{e^{ist}}{t\sqrt{\frac{1}{2}(1-s)\cdot\left(1-\frac{1}{2}\{1-s\}\right)}}\left(-\frac{t}{2}\right)\mathrm{d}s \\ &= \frac{1}{\pi}\int_{-1}^1 \frac{e^{ist}}{\sqrt{1-s^2}}\,\mathrm{d}s. \qquad \Diamond \end{aligned}$$

Example 1.5.24 *Compute the inverse Laplace transform of*

$$\frac{e^{-2z}}{\sqrt{z^2+9}}.$$

We get by a small rearrangement,

$$\frac{e^{-2z}}{\sqrt{z^2+9}} = e^{-2z}\cdot\frac{1}{3}\cdot\frac{1}{\sqrt{1+\left(\frac{z}{3}\right)^2}} = e^{-2z}\,\mathcal{L}\left\{J_0(3t)\right\}(z).$$

Then by using the rule of delay,

$$f(t) = \mathcal{L}^{-1}\left\{\frac{e^{-2z}}{\sqrt{z^2+9}}\right\}(t) = \begin{cases} J_0(3(t-2)) & \text{for } t \geq 2, \\ 0 & \text{for } t < 2. \end{cases} \qquad \Diamond$$

Example 1.5.25 *Compute the inverse Laplace transform of*

$$\frac{1}{\sqrt{z^2-4z+20}}.$$

It follows from the well-known trick

$$\begin{aligned} \frac{1}{\sqrt{z^2-4z+20}} &= \frac{1}{\sqrt{(z-2)^2+4^2}} = \frac{1}{4}\cdot\frac{1}{\sqrt{1+\left(\frac{z-2}{4}\right)^2}} = \frac{1}{4}\,\mathcal{L}\left\{J_0(t)\right\}\left(\frac{z-2}{4}\right) \\ &= \mathcal{L}\left\{J_0(4t)\right\}(z-2) = \mathcal{L}\left\{e^{2t}\,J_0(4t)\right\}(z), \end{aligned}$$

hence

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{z^2-4z+20}}\right\}(t) = e^{2t}\,J_0(4t). \qquad \Diamond$$

Download free eBooks at bookboon.com

Example 1.5.26 *Compute the inverse Laplace transform of*

$$\left(z^2+2z+5\right)^{-\frac{3}{2}}.$$

It follows from

$$\mathcal{L}\left\{J_1(t)\right\}(z)=\frac{\sqrt{z^2+1}-z}{\sqrt{z^2+1}}=1-\frac{z}{\sqrt{z^2+1}},$$

that

$$\mathcal{L}\left\{t\,J_t(t)\right\}(z)=-\frac{d}{dz}\left\{1-\frac{z}{\sqrt{z^2+1}}\right\}=\frac{1}{\left(z^2+1\right)^{\frac{3}{2}}}.$$

Then we get

$$\begin{aligned}\left(z^2+2z+5\right)^{-\frac{3}{2}} &= \frac{1}{\left((z+1)^2+4\right)^{\frac{3}{2}}}=\frac{1}{4^{\frac{3}{2}}\left(1+\left\{\frac{z+1}{2}\right\}^2\right)^{\frac{3}{2}}}=\frac{1}{4}\cdot\frac{1}{2}\,\mathcal{L}\left\{t\,J_1(t)\right\}\left(\frac{z+1}{2}\right)\\ &= \frac{1}{4}\,\mathcal{L}\left\{2t\,J_1(2t)\right\}(z+1)=\frac{1}{4}\,\mathcal{L}\left\{2\,e^{-t}\,t\,J_1(2t)\right\}(z),\end{aligned}$$

and we conclude that

$$\mathcal{L}^{-1}\left\{\left(z^2+2z+5\right)^{-\frac{3}{2}}\right\}(t)=\frac{1}{2}\,e^{-t}\,t\,J_1(2t). \qquad \Diamond$$

Click on the ad to read more

Download free eBooks at bookboon.com

2 Applications

2.1 Linear ordinary differential equations

Example 2.1.1 *Find the general solution of the differential equation*

$$t\, f''(t) + 2\, f'(t) + t\, f(t) = 0.$$

We *assume that $f(t)$ and $f'(t)$ are defined by continuity for $t = 0$.* Then put $F(z) := \mathcal{L}\{f\}(z)$, and we get by the Laplace transformation of the given differential equation,

$$-\frac{d}{dz}\left\{z^2\, F(z) - z\, f(0) - f'(0)\right\} + 2\left\{z\, F(z) - f(0)\right\} - \frac{d}{dz}\, F(z) = 0,$$

hence

$$-2z\, F(z) - z^2\,\frac{dF}{dz} + f(0) + 2z\, F(z) - 2\, f(0) - \frac{dF}{dz} = 0,$$

from which

$$\frac{dF}{dz} = -\frac{f(0)}{z^2+1}.$$

On the other hand, it follows from the rule of multiplication by t, that

$$\mathcal{L}\{t\, f(t)\}(z) = -\frac{dF}{dz} = \frac{f(0)}{z^2+1} = f(0)\cdot\mathcal{L}\{\sin t\}(z),$$

from which we get by the inverse Laplace transformation,

$$f(t) = f(0)\cdot\frac{\sin t}{t}.$$

The differential equation is *singular* at the point $t = 0$, because the coefficient t of the term of highest order of differentiation, $f''(t)$, is (trivially) 0 at $t = 0$. Therefore, we cannot conclude that the differential equation has two linearly independent solutions at $t = 0$.

One may of course, using plain ordinary Calculus, compute another linearly independent solution for $t \neq 0$ by the well-known formula,

$$\begin{aligned}\varphi(t) &= \frac{\sin t}{t}\int\left(\frac{t}{\sin t}\right)^2 \exp\left(-\int\frac{2}{t}\,\mathrm{d}t\right)\mathrm{d}t = \frac{\sin t}{t}\int\frac{1}{\sin^2 t}\,\mathrm{d}t \\ &= \frac{\sin t}{t}\cdot(-\cot t) = -\frac{\cos t}{t}.\end{aligned}$$

It is obvious that $\varphi(t)$ is not defined at $t = 0$.

Remark 2.1.1 Note that the equation can also be solved directly by using the following clever rearrangement,

$$\begin{aligned}0 &= t\, f''(t) + 2\, f'(t) + t\, f(t) = \{t\, f''(t) + 1\cdot f'(t)\} + f'(t) + t\, f(t) \\ &= \frac{d}{dt}\{t\cdot f'(t)\} + f'(t) + t\, f(t) = \frac{d}{dt}\{t\cdot f'(t) + 1\cdot f'(t)\} + t\, f(t) \\ &= \frac{d^2}{dt^2}\{t\, f(t)\} + t\, f(t),\end{aligned}$$

Download free eBooks at bookboon.com

from which we immediately get $t\,f(t) = a\,\cos t + b\,\sin t$, hence

$$f(t) = a\,\frac{\cos t}{t} + b\,\frac{\sin t}{t} \qquad \text{for } t \neq 0,$$

where a and b are arbitrary constants. $\Diamond$

Example 2.1.2 *Find the bounded solution of the differential equation*

$$t^2\,f''(t) + t\,f'(t) + \left(t^2 - 1\right) f(t) = 0, \qquad f(1) = 2.$$

We immediately recognize the equation as a *Bessel equation* of first order, so its bounded solutions are given by

$$f(t) = c\,J_1(t), \qquad c \text{ arbitrary constant.}$$

It follows from $f(1) = 2$ that $s = 2/J_1(1)$, so the bounded solution is given by

$$f(t) = \frac{2}{J_1(1)}\,J_1(t). \qquad \Diamond$$

Example 2.1.3 *Solve the linear differential equation*

$$t\,f''(t) + f'(t) + 4t\,f(t) = 0, \qquad f(0) = 3, \quad f'(0) = 0.$$

When we apply the Laplace transformation on the differential equation to get

$$\begin{aligned}
0 &= \mathcal{L}\left\{t\,f''(t)\right\}(z) + \mathcal{L}\left\{f'\right\}(z) + 4\,\mathcal{L}\{t\,f(t)\}(z) \\
&= -\frac{d}{dz}\,\mathcal{L}\left\{f''\right\}(z) + z\,\mathcal{L}\{f\}(z) - f(0) - 4\,\frac{d}{dz}\,\mathcal{L}\{f\}(z) \\
&= -\frac{d}{dz}\left\{z^2\,\mathcal{L}\{f\}(z) - z\,f(0) - f'(0)\right\} - 4\,\frac{d}{dz}\,\mathcal{L}\{f\}(z) + z\,\mathcal{L}\{f\}(z) - f(0) \\
&= -z^2\,\frac{d}{dz}\,\mathcal{L}\{f\}(z) - 2z\,\mathcal{L}\{f\}(z) + f(0) - f(0) - 4\,\frac{d}{dz}\,\mathcal{L}\{f\}(z) + z\,\mathcal{L}\{f\}(z) \\
&= -\left(z^2 + 4\right)\frac{d}{dz}\,\mathcal{L}\{f\}(z) - z\,\mathcal{L}\{f\}(z).
\end{aligned}$$

We divide this equation by $-\sqrt{z^2+4}$ to get

$$0 = \sqrt{z^2+4}\,\frac{d}{dz}\,\mathcal{L}\{f\}(z) + \frac{z}{\sqrt{z^2+4}}\,\mathcal{L}\{f\}(z) = \frac{d}{dz}\left(\sqrt{z^2+4}\cdot\mathcal{L}\{f\}(z)\right),$$

hence, by an integration,

$$\sqrt{z^2+4}\cdot\mathcal{L}\{f\}(z) = c,$$

Download free eBooks at bookboon.com

from which

$$\mathcal{L}\{f\}(z)=\frac{c}{\sqrt{z^2+4}}=c\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{1+\left(\frac{z}{2}\right)^2}}=c\cdot\mathcal{L}\left\{J_0(2t)\right\}(z),$$

and we conclude by the inverse Laplace transformation,

$$f(t)=c\cdot J_0(2t).$$

Finally, it follows from $f(0)=3=c\cdot J_0(0)=c$ that $c=3$, hence the solution is given by

$$f(t)=3\,J_0(2t).\qquad\Diamond$$

Example 2.1.4 *Solve the convolution equation*

$$\int_0^t f(u)f(t-u)\,\mathrm{d}u=16\,\sin 4t,\qquad t\in\mathbb{R}_+.$$

When we apply the rule of convolution on the Laplace transform of the equation above, we get

$$\begin{aligned}\mathcal{L}\left\{\int_0^t f(u)f(t-u)\,\mathrm{d}u\right\}(z)&=(\mathcal{L}\{f\}(z))^2=\mathcal{L}\{16\sin 4t\}(z)\\&=16\cdot\frac{1}{4}\,\mathcal{L}\{\sin t\}\left(\frac{z}{4}\right)=\frac{4}{1+\left(\frac{z}{4}\right)^2},\end{aligned}$$

hence

$$\mathcal{L}\{f\}(z)=\pm 2\,\frac{1}{\sqrt{1+\left(\frac{z}{4}\right)^2}}=\pm 8\cdot\frac{1}{4}\,\frac{1}{\sqrt{1+\left(\frac{z}{4}\right)^2}}=\pm 8\,\mathcal{L}\left\{J_0(4t)\right\}(z),$$

and we conclude by an inverse Laplace transformation and a square root that we have two solutions,

$$f(t)=\pm 8\,J_0(4t).\qquad\Diamond$$

Example 2.1.5 *Solve the equation*

$$f(t)=t+\int_0^t f(u)\,J_1(t-u)\,\mathrm{d}u,\qquad t\in\mathbb{R}_+.$$

We apply the Laplace transformation on the equation above to get

$$\mathcal{L}\{f\}(z)=\frac{1}{z^2}+\mathcal{L}\{f\}(z)\cdot\frac{\sqrt{z^2+1}-1}{\sqrt{z^2+1}},$$

Download free eBooks at bookboon.com

from which we get

$$\begin{aligned}\mathcal{L}\{f\}(z) &= \frac{\sqrt{z^2+1}}{z^2} = \frac{z^2+1}{z^2\sqrt{z^2+1}} = \frac{1}{z^2\sqrt{z^2+1}} + \frac{1}{\sqrt{z^2+1}} \\ &= \frac{1}{z^2}\,\mathcal{L}\left\{J_0\right\}(z) + \mathcal{L}\left\{J_0\right\}(z) = \mathcal{L}\left\{t \star J_0\right\}(z) + \mathcal{L}\left\{J_0\right\}(z).\end{aligned}$$

This gives by the inverse Laplace transformation,

$$\begin{aligned} f(t) &= J_0(t) + \left(J_0 \star t\right)(t) = J_0(t) + \int_0^t (t-u) J_0(u)\, \mathrm{d}u \\ &= J_0(t) + t\int_0^t J_0(u)\,\mathrm{d}u - \int_0^t u^1\, J_0(u)\,\mathrm{d}u \\ &= J_0(t) + t\int_0^t J_0(u)\,\mathrm{d}u - \int_0^t \frac{d}{du}\left(u <, J_1(u)\right)\,\mathrm{d}u \\ &= J_0(t) + y\int_0^t J_0(u)\,\mathrm{d}u - t\,J_1(t). \qquad \Diamond \end{aligned}$$

Click on the ad to read more

Download free eBooks at bookboon.com

Example 2.1.6 *A particle P of mass $2\,g$ is moving along the X-axis. The particle is attracted by a force directed towards 0, and it is numerically of the size $8|x|$. The particle is at time $t=0$ lying at the point $x=0$. Find the position of P at every time $t \in \mathbb{R}_+$ in each of the following two cases:*

1) The particle P is not subjected to any other force.

2) The particle P is subjected to a damping, which numerically is 8 times the speed.

1) Based on the conditions above the problem is described by

$$2\,\frac{d^2x}{dt^2} = -8x, \qquad x(0)=0,$$

thus by a rearrangement,

$$\frac{d^2x}{dt^2} + 4x = 0, \qquad x(0)=0, \qquad x'(0) \text{ unspecified.}$$

There is no need to apply the Laplace transformation, because one immediately realizes from elementary Calculus that the differential equation has the complete solution

$$x(t) = c_1\,\sin 2t + c_2\,\cos 2t, \qquad c_1,\, c_2 \text{ arbitrary constants.}$$

Since $x(0)=0$ and $x'(0)$ is unspecified, the searched solution becomes

$$x(t) = c_1\,\sin 2t = \frac{x'(0)}{2}\,\sin 2t.$$

2) In the second case with damping the differential equation with its initial conditions becomes

$$2\,\frac{d^2x}{dt^2} = -8x - 8\,\frac{dx}{dt}, \qquad x(0)=0, \qquad x'(0) \text{ unspecified,}$$

thus by a rearrangement,

$$\frac{d^2x}{dt^2} + 4\,\frac{dx}{dt} + 4x = 0, \qquad x(0)=0, \qquad x'(0) \text{ unspecified.}$$

The Laplace transformation is not needed in this case either, because the characteristic polynomial becomes $\lambda^2+4\lambda+4$ has the root of second order $\lambda=-2$, so the complete solution of the differential equation is according to ordinary Calculus given by

$$x(t) = c_1\,t\,e^{-2t} + c_2\,e^{-2t}.$$

It follows from $x(0)=0$ that

$$x(t) = c_1\,t\,e^{-2t},$$

so from $4x'(t) = c_1(-2t+1)e^{-2t}$ follows that $c_1 = x'(0)$, and the solution is

$$x(t) = x'(0)\,t\,e^{-2t}.$$

Download free eBooks at bookboon.com

It should here be added that *if* we had applied the Laplace transformation in the two cases, then we would have obtained

1) $\left(z^2+4\right)\mathcal{L}\{f\}(z) = x'(0)\,z,$

and

2) $\left(z^2+4z+4\right)\mathcal{L}\{f\}(z) = \cdots .$

In both cases we get the characteristic polynomial as a factor of the Laplace transform $\mathcal{L}\{f\}(z)$, while the initial conditions are put on the right hand side of the transformed equation as coefficients of a polynomial of smaller degree. The results of course become the same in both methods, but an application of the Laplace transformation in such simple cases may seen a little elaborated, when the problem can be solved straightaway by plain Calculus. This example should therefore be seen as a warning that one should not forget what one has learned earlier. Such "primitive methods" could indeed in some cases be more easy to apply. $\Diamond$

Example 2.1.7 *A particle of mass m is moving along the X-axis, subjected to a force $F(t)$, which is given by*

$$F(t) = \begin{cases} \frac{2}{T}\,F_0 \cdot t & \text{for } t \in \left[0, \frac{T}{2}\right], \\ \frac{2}{T}\,F_0 \cdot (T-t) & \text{for } t \in \left[\frac{T}{2}, T\right], \\ 0 & \text{otherwise}, \end{cases} \qquad \text{where } F_0 \text{ is a constant.}$$

Assuming that the particle starts from rest at $t = 0$ at the point $x = 0$ one shall find the position and the velocity of the particle at any $t \in \mathbb{R}_+$.

The problem is described by the following initial problem,

$$m\,\frac{d^2x}{dt^2} = F(t), \qquad x(0) = 0, \qquad x'(0).$$

It is immediately judged that it like in Example 2.1.6 will be too much to apply the Laplace transformation on this problem, because we for e.g. $t \in \left[0, \frac{T}{2}\right]$ immediately get by an integration,

$$m\,\frac{dx}{dt}(t) = m\,x'(0) + \int_0^t \frac{2}{T}\,F_0\,\tau\,\mathrm{d}\tau = \frac{2}{T}\,F_0 \cdot \frac{t^2}{2},$$

thus by a rearrangement,

$$\frac{dx}{dt} = \frac{F_0}{mT}\,t^2.$$

This equation invites to another simple integration, which gives

$$x(t) = \frac{F_0}{3mT}\,t^3 \qquad \text{for } t \in \left[0, \frac{T}{2}\right].$$

The values of the solution above at the endpoint $t = \dfrac{T}{2}$ are

$$x\left(\frac{T}{2}\right) = \frac{F_0 T^2}{24m} \qquad \text{and} \qquad x'\left(\frac{T}{2}\right) = \frac{F_0 T}{4m}.$$

Download free eBooks at bookboon.com

Using these as the new initial values we get in the same way for $t \in \left[\frac{T}{2}, T\right]$,

$$\begin{aligned} m\,\frac{dx}{dt} &= m \cdot x'\left(\frac{T}{2}\right) + \int_{\frac{T}{2}}^{T} \frac{2}{T}\, F_0(T-\tau)\,\mathrm{d}\tau = \frac{F_0 T}{4} - \left[\frac{1}{T}\, F_0(T-\tau)^2\right]_{\frac{T}{2}}^{t} \\ &= \frac{F_0 T}{4} - \frac{1}{T}\, F_0/T-t)^2 + \frac{1}{T}\, F_0 \cdot \frac{T^2}{4} = \frac{F_0 T}{2} - \frac{F_0}{T}\left(T^2 - 2Tt + t^2\right) \\ &= 2F_0 t - \frac{F_0}{T}\, t^2 - \frac{F_0 T}{2}, \end{aligned}$$

thus after a rearrangement,

$$\frac{dx}{dt} = -\frac{F_0}{mT}\, t^2 + \frac{2F_0}{m}\, t - \frac{F_0 T}{2m}, \qquad \text{for } t \in \left[\frac{T}{2}, T\right].$$

Then by another integration,

$$\begin{aligned} x(t) &= x\left(\frac{T}{2}\right) + \left[-\frac{F_0}{3mT}\, t^3 + \frac{F_0}{m}\, t^2 - \frac{F_0 T}{2}\, t\right]_{\frac{T}{2}}^{T} \\ &= \frac{F_0 T^2}{24m} + \frac{F_0 T^2}{24m} - \frac{F_0 T^2}{4m} + \frac{F_0 T^2}{4m} - \frac{F_0}{3mT}\, t^3 + \frac{F_0}{m}\, t^2 - \frac{F_0 T}{2m}\, t \\ &= -\frac{F_0}{3mT}\, t^3 + \frac{F_0}{m}\, t^2 - \frac{F_0 T}{2m}\, t + \frac{F_0 T^2}{12m}. \end{aligned}$$

We get for $t = T$ that

$$x'(T) = -\frac{F_0}{mT}\, T^2 + \frac{2F_0}{m}\, T - \frac{F_0 T}{2m} = \frac{F_0 T}{2m}.$$

Since $F(t) = 0$ for $t > T$, we get

$$x'(t) = \frac{F_0 T}{2m} \qquad \text{for } t \geq T.$$

Then analogously,

$$x(T) = -\frac{F_0 T^2}{3m} + \frac{F_0 T^2}{m} - \frac{F_0 T^2}{2m} + \frac{F_0 T^2}{12m} = \frac{F_0 T^2}{4m},$$

so

$$x(t) = \frac{F_0 T^2}{4m} + \frac{F_0 T}{2m}\,(t-T) = -\frac{F_0 T^2}{4m} + \frac{F_0 T}{2m}\, t \qquad \text{for } t \geq T.$$

Summing up,

$$x(t) = \begin{cases} \frac{F_0}{3mT}\, t^3, & \text{for } t \in \left[0, \frac{T}{2}\right], \\ -\frac{F_0}{3mT}\, t^3 + \frac{F_0}{m}\, t^2 - \frac{F_0 T}{2m}\, t + \frac{F_0 T^2}{12m}, & \text{for } t \in \left[\frac{T}{2}, T\right], \\ -\frac{F_0 T^2}{4m} + \frac{F_0 T}{2m}\, t, & \text{for } t \in [T, +\infty[, \end{cases}$$

Download free eBooks at bookboon.com

and

$$x'(t) = \begin{cases} \dfrac{F_0}{mT}\,t^2, & \text{for } t \in \left[0, \frac{T}{2}\right], \\ -\frac{F_0}{mT}\,t^2 + 2\,\frac{F_0}{m} - \frac{F_0 T}{2m}, & \text{for } t \in \left[\frac{T}{2}, T\right], \\ \frac{F_0 T}{2m}, & \text{for } t \in [T, +\infty[. \qquad \Diamond \end{cases}$$

Click on the ad to read more

Download free eBooks at bookboon.com

Example 2.1.8 *A coil of 2 henry, a resistance of 16 ohm and a capacitor of 0.02 farad are connected in a circuit with an electric force of E volt. At time $t = 0$ the capacity of the capacitor is zero, and the current in the circuit is zero. Find the charging and the current at any later time in the following cases,*

1) $E = 300$ *volt;*

2) $E = 100 \cdot \sin 3t$ *volt.*

According to the given information we shall solve the following integro differential equation

$$2\,\frac{dI}{dt} + 16I + \frac{1}{0.02}\int_0^t I\,\mathrm{d}t = E(t), \qquad I(0) = 0 \quad \text{and} \quad Q(0) = 0,$$

or, equivalently,

$$\frac{dI}{dt} + 8I(t) + 25\int_0^t I(\tau)\,\mathrm{d}\tau = \frac{1}{2}\,E(t), \qquad I(0) = 0.$$

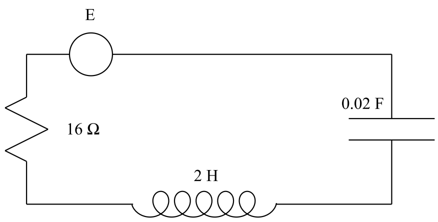


Figure 1: The circuit of Example 2.1.8.

Using that $I(0) = 0$ we get by the Laplace transformation,

$$z\,\mathcal{L}\{I\}(z) + 8\,\mathcal{L}\{I\}(z) + \frac{25}{z}\,\mathcal{L}\{I\}(z) = \frac{1}{2}\,\mathcal{L}\{E\}(z),$$

or, by a rearrangement, isolating $\mathcal{L}\{I\}(z)$,

$$(5)\quad \mathcal{L}\{I\}(z) = \frac{z}{2}\cdot\frac{\mathcal{L}\{E\}(z)}{z^2+8z+25} = \frac{z}{2}\cdot\frac{\mathcal{L}\{E\}(z)}{(z+4)^2+3^2}.$$

Once $I(t)$ has been found we find $Q(t)$ by the formula $Q(t) = \int_0^t I(\tau)\,\mathrm{d}\tau$.

Download free eBooks at bookboon.com

1) If $E = 300$, then $\mathcal{L}\{E\}(z) = \dfrac{300}{z}$, and we get from (5),

$$\mathcal{L}\{I\}(z) = \frac{150}{(z+4)^2+3^2} = 50\,\mathcal{L}\left\{e^{-4t}\sin 3t\right\}(z),$$

hence by the inverse Laplace transformation,

$$I(t) = 50\,e^{-4t}\sin 3t.$$

Finally,

$$\begin{aligned} Q(t) &= \int_0^t I(\tau)\,\mathrm{d}\tau = 50\,\Im\int_0^t e^{(-4+3i)\tau}\,\mathrm{d}\tau = 50\,\Im\left[\frac{1}{-4+3i}\,e^{(-4+3i)\tau}\right]_0^t \\ &= 50\,\Im\left\{\frac{-4-3i}{25}\,e^{-4t}(\cos 3t + i\,\sin 3t) - \frac{-4-3i}{25}\right\} \\ &= 2\,\Im\left\{e^{-4t}(-4-3i)(\cos 3t + i\,\sin 3t) + 4 + 3i\right\} \\ &= 2\cdot 3 + 2\,e^{-4t}(-4\,\sin 3t - 3\,\cos 3t) = 6 - 8\,e^{-4t}\sin 3t - 6\,e^{-4t}\cos 3t. \end{aligned}$$

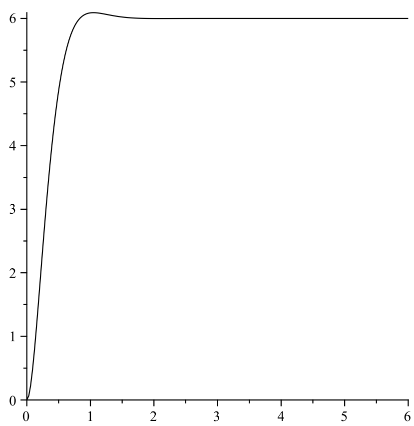


Figure 2: Graph of the charge $Q(t)$ of Example 2.1.8, 1).

2) If $E = 100\sin 3t$, then $\mathcal{L}\{E\}(z) = \dfrac{300}{z^2+3^2}$. so we get from (5) that

$$\mathcal{L}\{I\}(z) = \frac{150z}{(z^2+9)\,(z^2+8z+25)} = \frac{az+b}{z^2+9} + \frac{cz+d}{z^2+8z+25}$$

for some constants a, b, c and d. This structure shows by the inverse Laplace transformation that

$$I(t) = a\,\cos 3t + \frac{b}{3}\,\sin 3t + c\,e^{-4t}\,\cos 3t + \frac{d}{3}\,e^{-4t}\,\sin 3t.$$

Download free eBooks at bookboon.com

Then notice that it follows from $I(0)=0$ that $c=-a$, so we shall only find the three constants of

$$I(t)=a\left(1-e^{-4t}\right)\cos 3t+\frac{1}{3}\left(b+d\,e^{-4t}\right)\sin 3t.$$

If the expressions above of $\mathcal{L}\{I\}(z)$ are multiplied by the common denominator, then we get

$$\begin{aligned}150z &= (az+b)\left(z^2+8z+25\right)+(-az+d)\left(z^2+9\right)\\ &= (8a+b+d)z^2+(25a+8b-9a)z+(25b+9d)\\ &= (8a+b+d)z^2+(16a+8b)z+(25b+9d).\end{aligned}$$

When we identify the coefficients of the two polynomials, we get the following system of equations,

$$\left\{\begin{array}{rcrcrcr} 0 &=& 8a &+& b &+& d\\ 150 &=& 16a &+& 8b && \\ 0 &=& && 25b &+& 9d \end{array}\right. .$$

The determinant of the system is

$$D=\begin{vmatrix} 8 & 1 & 1\\ 16 & 8 & 0\\ 0 & 25 & 9\end{vmatrix}=8\begin{vmatrix} 1 & 0 & 1\\ 2 & 8 & 0\\ 0 & 16 & 9\end{vmatrix}=8(72+32)=8\cdot 104,$$

so by Cramer's formula,

$$a=\frac{1}{8\cdot 104}\begin{vmatrix} 0 & 1 & 1\\ 150 & 8 & 0\\ 0 & 25 & 9\end{vmatrix}=-\frac{150}{8\cdot 104}\begin{vmatrix} 1 & 1\\ 25 & 9\end{vmatrix}=+\frac{150}{8\cdot 104}\cdot 16=\frac{75}{26},$$

and

$$b=\frac{1}{8\cdot 104}\begin{vmatrix} 8 & 0 & 1\\ 16 & 150 & 0\\ 0 & 0 & 9\end{vmatrix}=\frac{150}{8\cdot 104}\begin{vmatrix} 8 & 1\\ 0 & 9\end{vmatrix}=\frac{150\cdot 72}{8\cdot 104}=\frac{75\cdot 9}{52}=\frac{675}{52}=\frac{225}{52}\cdot 3,$$

and

$$d=\frac{1}{8\cdot 104}\begin{vmatrix} 8 & 1 & 0\\ 16 & 8 & 150\\ 0 & 25 & 0\end{vmatrix}=-\frac{150}{8\cdot 104}\begin{vmatrix} 8 & 1\\ 0 & 25\end{vmatrix}=-\frac{75\cdot 25}{52}=-\frac{625}{52}\cdot 3,$$

hence

$$\begin{aligned}I(t) &= a\left(1-e^{-4t}\right)\cos 3t+\frac{1}{3}\left(b+d\,e^{-4t}\right)\sin 3t\\ &= \frac{75}{26}\cos 3t+\frac{225}{52}\sin 3t-e^{-4t}\left\{\frac{75}{26}\cos 3t+\frac{625}{52}\sin 3t\right\}.\end{aligned}$$

Download free eBooks at bookboon.com

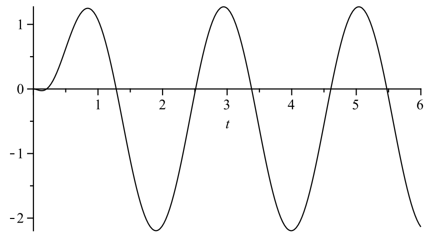


Figure 3: Graph of the charge $Q(t)$ of Example 2.1.8, 2).

Then

$$\begin{aligned}
Q(t) &= \int_0^t I(\tau)\,\mathrm{d}\tau = \frac{25}{26}\,\sin 3t + \frac{75}{52}\,(1-\cos 3t) \\
&\quad -\frac{75}{26}\,\Re\int_0^t e^{(-4+3i)\tau}\,\mathrm{d}\tau - \frac{625}{52}\,\Im\int_0^t e^{(-4+3i)\tau}\,\mathrm{d}\tau \\
&= \frac{75}{52} + \frac{25}{26}\,\sin 3t - \frac{75}{52}\,\cos 3t \\
&\quad +\frac{3}{26}\,\Re\left[(4+3i)e^{(-4+3i)\tau}\right]_0^t + \frac{25}{52}\,\Im\left[(4+3i)e^{(-4+3i)\tau}\right]_0^t \\
&= \frac{75}{52} - \frac{75}{26}\,\cos 3t + \frac{25}{26}\,\sin 3t \\
&\quad +\frac{3}{26}\,\Re\left\{(4+3i)e^{(-4+3i)t}-4-3i\right\} + \frac{25}{52}\,\Im\left\{(4+3i)e^{(-4+3i)t}-4-3i\right\} \\
&= \frac{75}{52} - \frac{75}{52}\,\cos 3t + \frac{25}{26}\,\sin 3t \\
&\quad +\frac{3}{26}\,e^{-4t}(4\cos 3t-3\sin 3t) - \frac{6}{13} + \frac{25}{52}\,e^{-4t}(4\sin 3t+3\cos 3t) - \frac{75}{52} \\
&= -\frac{6}{13} - \frac{75}{52}\,\cos 3t + \frac{25}{26}\,\sin 3t + \frac{1}{52}\,e^{-4t}\{(24+75)\cos 3t+(100-18)\sin 3t\} \\
&= -\frac{6}{13} - \frac{75}{52}\,\cos 3t + \frac{25}{26}\,\sin 3t + \frac{99}{52}\,e^{-4t}\,\cos 3t + \frac{41}{26}\,e^{-4t}\,\sin 3t. \qquad \Diamond
\end{aligned}$$

Download free eBooks at bookboon.com

Example 2.1.9 *A resistance of R ohm and a condenser of C farad are connected with a generator of E volt. The capacity of the condenser is 0 at time $t = 0$. Find the charge and the current as functions of $t > 0$, when*

1) $E = E_0$ *(a constant);*

2) $E = E_0\, e^{-\alpha t}$*, where $\alpha > 0$ is also a constant.*

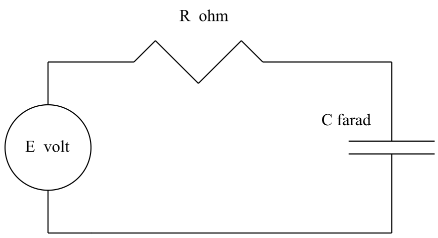


Figure 4: The circuit of Example 2.1.9.

Click on the ad to read more

Download free eBooks at bookboon.com

In this case the corresponding equation becomes

$$R\,I+\frac{1}{C}\int_0^t I\,\mathrm{d}t=E(t).$$

Assuming that $I\in\mathcal{F}$ we get by the Laplace transformation,

$$R\,\mathcal{L}\{I\}(z)+\frac{1}{C}\cdot\frac{1}{z}\,\mathcal{L}\{I\}(z)=\mathcal{L}\{E\}(z),$$

thus

$$\mathcal{L}\{I\}(z)=\frac{\mathcal{L}\{E\}(z)}{R+\frac{1}{Cz}}=\frac{\frac{1}{R}\cdot z}{z+\frac{1}{CR}}\cdot\mathcal{L}\{E\}(z).$$

1) If $E=E_0$ is a constant, then $\mathcal{L}\{E\}(z)=\dfrac{E_0}{z}$, hence

$$\mathcal{L}\{I\}(z)=\frac{E_0}{R}\cdot\frac{1}{z+\frac{1}{CR}}=\frac{E_0}{R}\,\mathcal{L}\left\{\exp\left(-\frac{t}{CR}\right)\right\}(z),$$

and it follows by the inverse Laplace transformation that

$$I(t)=\frac{E_0}{R}\,\exp\left(-\frac{t}{CR}\right),$$

and

$$Q(t)=\int_0^t I(\tau)\,\mathrm{d}\tau=C\{E(t)-R\,I(t)\}=E_0C\left\{1-\exp\left(-\frac{t}{CR}\right)\right\}.$$

2) If $E(t)=E_0\,e^{-\alpha t}$, then $\mathcal{L}\{E\}(z)=\dfrac{E_0}{z+\alpha}$, hence

$$\mathcal{L}\{I\}(z)=\frac{E_0}{R}\cdot\frac{z}{\left(z+\frac{1}{CR}\right)\cdot(z+\alpha)}.$$

We see that we have two cases, either $\alpha\neq\dfrac{1}{CR}$, or $\alpha=\dfrac{1}{CR}$.

a) If $\alpha\neq\dfrac{1}{CR}$, then it follows by a plain decomposition,

$$\begin{aligned}\mathcal{L}\{I\}(z)&=\frac{E_0}{R}\left\{\frac{-\frac{1}{CR}}{\alpha-\frac{1}{CR}}\cdot\frac{1}{z+\frac{1}{CR}}+\frac{-\alpha}{-\alpha 0\frac{1}{CR}}\cdot\frac{1}{z+\alpha}\right\}\\&=\frac{E_0}{R}\left\{\frac{1}{1-CR\alpha}\cdot\frac{1}{z+\frac{1}{CR}}-\frac{CR\alpha}{1-CR\alpha}\cdot\frac{1}{z+\alpha}\right\},\end{aligned}$$

hence, by the inverse Laplace transformation,

$$I(t)=\frac{E_0}{R}\cdot\frac{1}{1-CR\alpha}\,\exp\left(-\frac{t}{CR}\right)-\frac{E_0C\alpha}{1-CR\alpha}\,\exp(-\alpha t),$$

Download free eBooks at bookboon.com

from which we get by an integration,

$$\begin{aligned} Q(t) &= \int_0^t I(\tau)\,\mathrm{d}\tau = \frac{E_0}{R}\cdot\frac{CR}{1-CR\alpha}\left\{1-\exp\left(-\frac{t}{CR}\right)\right\} - \frac{E_0C}{1-CR\alpha}\{1-\exp(-\alpha t)\} \\ &= \frac{E_0C}{1-CR\alpha}\left\{1-\exp\left(-\frac{t}{CR}\right)-1+\exp(-\alpha t)\right\} \\ &= \frac{E_0C}{1-CR\alpha}\left\{\exp(-\alpha t)-\exp\left(-\frac{t}{CR}\right)\right\}. \end{aligned}$$

b) If instead $\alpha = \dfrac{1}{CR}$, then

$$\begin{aligned} \mathcal{L}\{I\}(z) &= \frac{E_0}{R}\cdot\frac{z+\alpha-\alpha}{(z+\alpha)^2} = \frac{E_0}{R}\cdot\frac{1}{z+\alpha} - \frac{\alpha E_0}{R}\cdot\frac{1}{(z+\alpha)^2} \\ &= \frac{E_0}{R}\cdot\mathcal{L}\left\{e^{-\alpha t}\right\}(z) - \frac{\alpha E_0}{R}\,\mathcal{L}\left\{t\,e^{-\alpha t}\right\}(z), \end{aligned}$$

and we get by the inverse Laplace transformation

$$I(t) = \frac{E_0}{R}\,(1-\alpha t)e^{-\alpha t} = \frac{E_0}{R}\left(1-\frac{t}{CR}\right)\exp\left(-\frac{t}{CR}\right),$$

and then by an integration,

$$\begin{aligned} Q(t) &= \int_0^t I(\tau)\,\mathrm{d}\tau = \frac{E_0}{R\alpha}\int_0^{\alpha t}(1-\tau)e^{-\tau}\,\mathrm{d}\tau = \frac{E_0}{R\alpha}\left\{\left[-(1-\tau)e^{-\tau}\right]_0^{\alpha t} - \int_0^{\alpha t} e^{-\tau}\,\mathrm{d}\tau\right\} \\ &= \frac{E_0}{R\alpha}\left\{-(1-\alpha t)e^{-\alpha t}+1+\left[e^{-\tau}\right]_0^{\alpha t}\right\} = \frac{E_0}{R\alpha}\left\{-e^{-\alpha t}+\alpha t e^{-\alpha t}+1-1+e^{-\alpha t}\right\} \\ &= \frac{E_0}{R\alpha}\,\alpha t\,e^{-\alpha t} = \frac{E_0}{R}\,t\,e^{-\alpha t} = E_0C\cdot\frac{t}{CR}\,\exp\left(-\frac{t}{CR}\right). \qquad \Diamond \end{aligned}$$

Example 2.1.10 *A beam is suspended as indicated on Figure 5 on page 62 with its endpoints at $x = 0$ and $x = \ell$. The beam carries a load, given by W_0 (a constant) per unit length. Find the bending at every point, i.e. solve the boundary value problem*

$$\frac{d^4f}{dx^4} = \frac{W_0}{EI}, \quad x \in \,]0,\ell[, \qquad f(0) = f''(0) = 0, \qquad f(\ell) = f''(\ell) = 0.$$

In this case there is absolutely no need to apply the Laplace transformation, and one would even get into troubles with it, because we have not specified $f'(0)$ and $f^{(3)}(0)$, which are needed.

We get by four successive integrations that the general solution is given by

$$f(x) = \frac{1}{4!}\,\frac{W_0}{EI}\,x^4 + a_3\,x^3 + a_2\,x^2 + a_1\,x + a_0,$$

where

$$f''(x) = \frac{1}{2}\,\frac{W_0}{EI}\,x^2 + 6a_3\,x + 2a_2.$$

Download free eBooks at bookboon.com

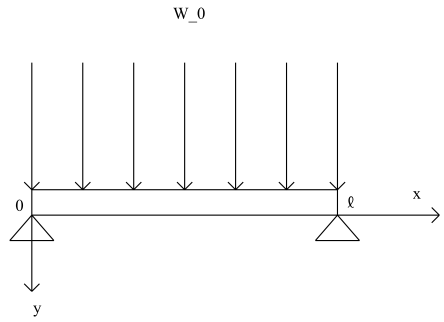


Figure 5: The beam of Example 2.1.10.

Thus, $f(0) = a_0 = 0$ and $f''(0) = 2a_2 = 0$, so the solution must have the structure

$$f(x) = \frac{1}{24} \frac{W_0}{EI} x^4 + a_3 x^3 + a_1 x,$$

where

$$f''(x) = \frac{1}{2} \frac{W_0}{EI} x^2 + 6a_3 x.$$

We then derive that

$$f''(\ell) = 0 = \frac{1}{2} \frac{W_0}{EI} \ell^2 + 6a_3 \ell, \qquad \text{thus } a_3 = -\frac{1}{12} \frac{W_0}{EI} \ell,$$

and

$$f(\ell) = 0 = \frac{1}{24} \frac{W_0}{EI} \ell^4 - \frac{1}{12} \frac{W_0}{EI} \cdot \ell \cdot \ell^3 + a_1 \ell,$$

so

$$a_1 = \frac{1}{24} \frac{W_0}{EI} \left(2\ell^3 - \ell^3\right) = \frac{1}{24} \frac{W_0}{EI} \ell^3.$$

Finally, by insertion,

$$\begin{aligned} f(x) &= \frac{1}{24} \frac{W_0}{EI} x^4 - \frac{1}{12} \frac{W_0}{EI} \ell \cdot x^3 + \frac{1}{24} \frac{W_0}{EI} \ell^3 x = \frac{1}{24} \frac{W_0}{EI} \left(x^4 - 2\ell x^3 + \ell^3 x\right) \\ &= \frac{1}{24} \frac{W_0}{EI} x(\ell - x)\left(\ell^2 - \ell x - x^2\right). \qquad \Diamond \end{aligned}$$

Download free eBooks at bookboon.com

Example 2.1.11 *A beam is fixed at $x = 0$, while the other endpoint $x = \ell$ is free. The beam carries a load which per unit length is given by*

$$W(x) = \begin{cases} W_0, & x \in \left]0, \frac{\ell}{2}\right[, \\ 0, & x \in \left]\frac{\ell}{2}, \ell\right[. \end{cases}$$

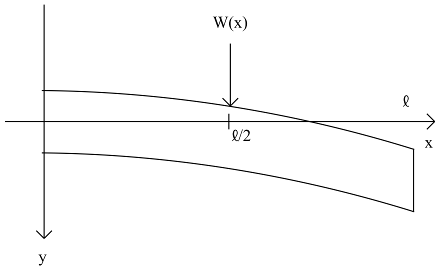


Figure 6: The beam of Example 2.1.11.

Find the bending of the beam, i.e. solve the following boundary value problem

$$\frac{d^4 f}{dx^4} = \frac{W(x)}{EI}, \quad x \in]0, \ell[, \qquad f(0) = f'(0) = 0, \quad f''(\ell) = f^{(3)}(\ell) = 0.$$

This is a *boundary value problem*, so it is not easy to solve it using the Laplace transformation. Instead we integrate

$$\frac{d^4 f}{dx^4} = \begin{cases} \dfrac{W_0}{EI}, & \text{for } x \in \left[0, \frac{\ell}{2}\right[, \\ 0 & \text{for } x \in \left]\frac{\ell}{2}, \ell\right], \end{cases}$$

to get

$$\frac{d^3 f}{dx^3} = \begin{cases} \frac{W_0}{EI}\left(x - \frac{\ell}{2}\right) + a, & \text{for } x \in \left[0, \frac{\ell}{2}\right], \\ a, & \text{for } x \in \left[\frac{\ell}{2}, \ell\right]. \end{cases}$$

It follows from the boundary condition $f^{(3)}(\ell) = 0$ that $a = 0$. Hence, we get by another integration,

$$\frac{d^2 f}{dx^2} = \begin{cases} \frac{W_0}{2EI}\left(x - \frac{\ell}{2}\right)^2 + b, & \text{for } x \in \left[0, \frac{\ell}{2}\right], \\ b, & \text{for } x \in \left[\frac{\ell}{2}, \ell\right]. \end{cases}$$

Download free eBooks at bookboon.com

Then it follows from $f^{(2)}(\ell) = 0$ that $b = 0$, so by another integration,

$$\frac{df}{dx} = \begin{cases} \frac{W_0}{6EI}\left(x - \frac{\ell}{2}\right)^3 + c, & \text{for } x \in \left[0, \frac{\ell}{2}\right], \\ c, & \text{for } x \in \left[\frac{\ell}{2}, \ell\right], \end{cases}$$

where $f'(0) = 0$ implies that $c = \frac{W_0}{48EI}\,\ell^3$. Finally,

$$f(x) = \begin{cases} \frac{W_0}{24EI}\left(x - \frac{\ell}{2}\right)^4 + \frac{W_0}{48EI}\,\ell^3 x + d, & \text{for } x \in \left[0, \frac{\ell}{2}\right], \\ \frac{W_0}{48EI}\,\ell^3 x + d, & \text{for } x \in \left[\frac{\ell}{2}, \ell\right], \end{cases}$$

where finally $f(0) = 0$ implies that $d = -\frac{W_0}{384EI}\,\ell^4$, and the solution is given by

$$f(x) = \begin{cases} \frac{W_0}{384EI}\left\{16\left(x - \frac{\ell}{2}\right)^4 + 8\ell^3 x - \ell^4\right\}, & \text{for } x \in \left[0, \frac{\ell}{2}\right], \\ \frac{W_0}{384EI}\left\{8\ell^3 x - \ell^4\right\}, & \text{for } x \in \left[\frac{\ell}{2}, \ell\right]. \end{cases} \qquad \Diamond$$

Click on the ad to read more

Download free eBooks at bookboon.com

Example 2.1.12 *Solve the boundary value problem*

$$t\,f''(t)+2\,f'(t)+t\,f(t)=0,\qquad \lim_{t\to 0+} f(t)=1,\quad f(\pi)=0.$$

We must be careful here, because we have a singular point at $t=0$, where the coefficient of the highest order term $f''(t)$ is zero. It will later follow that the problem is even ill-posed, because the solution, based on the condition $\lim_{t\to 0+} f(t)=1$ alone will automatically satisfy $f(\pi)=0$. Similarly, even if it follows from the differential equation itself by letting $t\to 0$ that $f'(0)=0$ is easily derived, and yet it is not used in the derivation of the solution. The point is, of course as mentioned above, that we have a singular point at $t=0$, and that even if the linear equation for $t\neq 0$ is spanned by two linearly independent solutions, at most one of these is also a function in the class $\mathcal{F}$ of functions, which can be Laplace transformed.

When the differential equation is Laplace transformed, we get

$$\begin{aligned}
0 &= -\frac{d}{dz}\,\mathcal{L}\left\{f''\right\}(z)+2\,\mathcal{L}\left\{f'\right\}(z)-\frac{d}{dz}\,\mathcal{L}\{f\}(z)\\
&= -\frac{d}{dz}\left(z^2\mathcal{L}\{f\}(z)-z\,f(0)-f'(0)\right)+2(z\,\mathcal{L}\{f\}(z)-f(0))-\frac{d}{dz}\,\mathcal{L}\{f\}(z)\\
&= \left(z^2+1\right)\,\frac{d}{dz}\,\mathcal{L}\{f\}(z)-2z\,\mathcal{L}\{f\}(z)+1+2z\,\mathcal{L}\{f\}(z)-2\\
&= \left(z^2+1\right)\cdot\left(-\frac{d}{dz}\,\mathcal{L}\{f\}(z)\right)-1=\left(z^2+1\right)\mathcal{L}\{t\,f(t)\}(z)-1,
\end{aligned}$$

from which

$$\mathcal{L}\{t\,f(t)\}=\frac{1}{z^2+1}=\mathcal{L}\{\sin t\}(z).$$

Using the inverse Laplace transformation we therefore conclude that $t\,f(t)=\sin t$, so

$$f(t)=\frac{\sin t}{t},\qquad t\in\mathbb{R}_+,$$

with $f(0)=\lim_{t\to 0+} f(t)=1$ by continuity. $\Diamond$

Download free eBooks at bookboon.com

2.2 Linear systems of ordinary differential equations

Example 2.2.1 *Solve the system of differential equations,*

$$\begin{cases} \dfrac{dx}{dt} = 2\,x(t) - 3\,y(t), \\ \\ y(t) - 2\,x(t), \end{cases} \qquad \begin{cases} x(0) = 8, \\ \\ y(0) = 3. \end{cases}$$

We shall assume that $x(t)$, $y(t) \in \mathcal{F}$. Then we write for short

$$X = X(z) := \mathcal{L}\{x(t)\}(z) \qquad \text{and} \qquad Y = Y(z) = \mathcal{L}\{y(t)\}(z).$$

Using the Laplace transformation we get

$$\begin{cases} zX - 8 = 2X - 3Y, \\ \\ zY - 3 = -2X + Y, \end{cases}$$

hence by a rearrangement,

$$\begin{cases} (z-2)X + 3Y &= 8, \\ \\ 2X + (z-1)T &= 3. \end{cases}$$

The corresponding determinant is

$$\Delta = \begin{vmatrix} z-2 & 3 \\ 2 & z-1 \end{vmatrix} = z^2 - 3z + 2 - 6 = z^2 - 3z - 4 = (z+1)(z-4).$$

Thus, for $\Re\, z > 4$, by Cramer's formula,

$$X(z) = \frac{1}{(z+1)(z-4)} \begin{vmatrix} 8 & 3 \\ 3 & z-1 \end{vmatrix} = \frac{8z-17}{(z+1)(z-4)} = \frac{\frac{-25}{-5}}{z+1} + \frac{\frac{32-17}{5}}{z-4} = \frac{5}{z+1} + \frac{3}{z-4},$$

and

$$Y(z) = \frac{2}{(z+1)(z-4)} \begin{vmatrix} z-2 & 8 \\ 2 & 3 \end{vmatrix} = \frac{3z-22}{(z+1)(z-4)} = \frac{\frac{-25}{-5}}{z+1} + \frac{\frac{12-22}{5}}{z-4} = \frac{5}{z+1} - \frac{2}{z-4}.$$

Finally, by the inverse Laplace transformation,

$$\begin{cases} x(t) = 5e^{-t} + 3e^{4t}, \\ \\ 5e^{-t} - 2e^{4t}. \end{cases} \qquad \Diamond$$

Example 2.2.2 *Solve the following system of linear ordinary differential equations,*

$$\begin{cases} \dfrac{d^2x}{dt^2} + \dfrac{dy}{dt} + 3x(t) \qquad = \quad 15\,e^{-t}, \\ \\ \dfrac{d^2y}{dt^2} - 4\,\dfrac{dx}{dt} + 3\,y(t) = 15\,\sin 2t, \end{cases} \qquad \begin{cases} x(0) = 35, & x'(0) = -48, \\ \\ y(0) = 27, & y'(0) = -55. \end{cases}$$

Download free eBooks at bookboon.com

We get by the Laplace transformation,

$$\begin{cases} \left(z^2 X - 35z + 48\right) + (z\,Y - 27) + 3X &= \dfrac{15}{z+1}, \\[2mm] \left(z^2 Y - 27z + 55\right) - 4(z\,X - 35) + 3Y &= \dfrac{15 \cdot 2}{z^2+4}, \end{cases}$$

hence, by a rearrangement,

$$\begin{cases} \left(z^2+3\right) X + zY &= \dfrac{15}{z+1} + 35z - 21, \\[2mm] -4zX + \left(z^2+3\right) Y &= \dfrac{30}{z^2+4} + 27z - 195. \end{cases}$$

The corresponding determinant is

$$\Delta = \left(z^2+3\right)^2 + 4z^2 = z^4 + 10z^2 + 9 = \left(z^2+1\right)\left(z^2+9\right).$$

Then, using Cramer's formula

$$\begin{aligned} X(z) &= \frac{1}{\left(z^2+1\right)\left(z^2+9\right)} \begin{vmatrix} \frac{15}{z+1} + 35z - 21 & z \\ \frac{30}{z^2+4} + 27z - 195 & z^2+3 \end{vmatrix} \\ &= \frac{1}{8}\left\{\frac{1}{z^2+1} - \frac{1}{z^2+9}\right\}\left\{\frac{15\left(z^2-1+4\right)}{z+1} + 35z^3 + 105z - 21z^2 - 63 - \frac{30z}{z^2+4} - 27z^2 + 195z\right\} \\ &= \frac{1}{8}\left\{\frac{1}{z^2+1} - \frac{1}{z^2+9}\right\}\left\{15z - 15 + \frac{60}{z+1} - \frac{30z}{z^2+4} + 35z^3 - 48z^2 + 300z - 63\right\} \\ &= \frac{60}{8}\,\frac{1}{(z+1)\left(z^2+1\right)} - \frac{60}{8}\,\frac{1}{(z+1)\left(z^2+9\right)} - \frac{30z}{8}\,\frac{1}{\left(z^2+1\right)\left(z^2+9\right)} + \frac{30z}{8}\,\frac{1}{\left(z^2+4\right)\left(z^2+9\right)} \\ &\quad + \frac{35z}{8}\left\{\frac{z^2+1-1}{z^2+1} - \frac{z^2+9-9}{z^2+9}\right\} - \frac{48}{8}\left\{\frac{z^2+1-1}{z^2+1} - \frac{z^2+9-9}{z^2+9}\right\} + \frac{315}{8}\cdot\frac{z}{z^2+1} \\ &\quad - \frac{315}{8}\cdot\frac{z}{z^2+9} - \frac{78}{8}\cdot\frac{1}{z^2+1} + \frac{78}{8}\cdot\frac{1}{z^2+9} \\ &= \frac{60}{8}\cdot\frac{1}{2}\cdot\frac{1}{z+1} + \frac{60}{8}\cdot\frac{1}{2}\cdot\frac{2-z^2-1}{(z+1)\left(z^2+1\right)} - \frac{60}{8}\cdot\frac{1}{10}\cdot\frac{1}{z+1} + \frac{60}{8}\cdot\frac{1}{10}\cdot\frac{z^2+9-10}{(z+1)\left(z^2+9\right)} \\ &\quad - \frac{30}{8}\cdot\frac{1}{3}\cdot\frac{z}{z^2+1} + \frac{30}{8}\cdot\frac{1}{3}\cdot\frac{z}{z^2+4} + \frac{30}{8}\cdot\frac{1}{5}\cdot\frac{z}{z^2+4} - \frac{30}{8}\cdot 15\cdot\frac{z}{z^2+9} - \frac{35}{8}\cdot\frac{z}{z^2+1} \\ &\quad + \frac{315}{8}\cdot\frac{z}{z^2+9} + \frac{6}{z^2+1} - \frac{54}{z^2+9} + \frac{315}{8}\cdot\frac{z}{z^2+1} - \frac{315}{8}\cdot\frac{z}{z^2+9} - \frac{78}{8}\cdot\frac{1}{z^2+1} + \frac{78}{8}\cdot\frac{1}{z^2+9}, \end{aligned}$$

Download free eBooks at bookboon.com

thus

$$\begin{aligned} X(z) &= \left\{\frac{30}{8}-\frac{6}{8}\right\}\frac{1}{z+1}+\frac{30}{8}\cdot\frac{z}{z^2+1}+\frac{30}{8}\cdot\frac{1}{z^2+1}+\frac{6}{8}\cdot\frac{z}{z^2+9}-\frac{1}{z^2+9} \\ &\quad +\left\{-\frac{10}{8}-\frac{35}{8}+\frac{315}{8}\right\}\frac{z}{z^2+1}+\left\{6-\frac{78}{8}\right\}\frac{1}{z^2+1}+\left\{\frac{10}{8}+\frac{6}{8}\right\}\frac{z}{z^2+4} \\ &\quad +\left\{-\frac{6}{8}\right\}\frac{z}{z^2+9}+\left\{-54+\frac{78}{8}\right\}\frac{1}{z^2+9} \\ &= 3\,\frac{1}{z+1}+30\frac{z}{z^2+1}-45\,\frac{1}{z^2+9}+2\,\frac{z}{z^2+4}, \end{aligned}$$

and we get by the inverse Laplace transformation,

$$x(t) = 3\,e^{-t} + 30\,\cos t - 15\,\sin 3t + 2\,\cos 2t.$$

Once we have found $x(t)$, we compute

$$\begin{aligned} \frac{dy}{dt} &= 15\,e^{-t} - \frac{d^2x}{dt^2} - 3\,x(t) \\ &= 15\,e^{-t} - \{3\,e^{-t} - 30\,\cos t + 135\,\sin 3t - 8\,\cos 2t\}\,\{9\,e^{-t} + 90\,\cos t - 45\,\sin 3t + 6\,\cos 2t\} \\ &= 3\,e^{-t} - 60\,\cos t - 90\,\sin 3t + 2\,\cos 2t, \end{aligned}$$

Need help with your dissertation?

Get in-depth feedback & advice from experts in your topic area. Find out what you can do to improve the quality of your dissertation!

Get Help Now

Go to www.helpmyassignment.co.uk for more info

Helpmyassignment

Download free eBooks at bookboon.com

Click on the ad to read more

hence by an integration

$$y(t) = -3\,e^{-t} - 60\,\sin t + 30\,\cos 3t + \sin 2t + c,$$

where

$$y(0) = -3 + 30 + c = 27, \qquad \text{hence } c = 0.$$

Summing up we get

$$\begin{cases} x(t) = 3\,e^{-t} + 30\,\cos t - 15\,\sin 3t + 2\,\cos 2t, \\ y(t) = -3\,e^{-t} - 60\,\cos t - 30\,\cos 3t + \sin 2t \end{cases} \qquad \Diamond$$

Example 2.2.3 *Solve the system of ordinary differential equations*

$$\begin{cases} y_1' + y_2 = 0, \\ y_2' + y_1 = 0, \end{cases} \qquad \begin{cases} y_1(0) = 1, \\ y_2(0) = 0. \end{cases}$$

First method It follows by inspection that

$$\frac{d}{dt}(y_1 + y_2) + (y_1 + y_2) = 0, \qquad \text{thus } y_1 + y_2 = c_1\,e^{-t},$$

and

$$\frac{d}{dt}(y_1 - y_2) - (y_1 - y_2) = 0, \qquad \text{thus } y_1 - y_2 = c_2\,e^{t}.$$

It follows from the initial conditions that $c_1 = c_2 = 1$, so

$$y_1(t) = \cosh t \qquad \text{and} \qquad y_2(t) = -\sinh t.$$

Second method It follows by the Laplace transformation that

$$\begin{cases} zY_1(z) - 1 + Y_2(z) = 0, \\ zY_2(z) - 0 + Y_1(z) = 0, \end{cases}$$

hence

$$\begin{cases} z \cdot Y_1(z) + 1 \cdot Y_2(z) = 1, \\ 1 \cdot Y_1(z) + z \cdot Y_2(z) = 0. \end{cases}$$

The determinant is $z^2 - 1$, so it follows from Cramer's formula,

$$Y_1(z) = \frac{1}{z^2-1}\begin{vmatrix} 1 & 1 \\ 0 & z \end{vmatrix} = \frac{z}{z^2-1} = \mathcal{L}\{\cosh t\}(z),$$

and

$$Y_2(z) = \frac{1}{z^2-1}\begin{vmatrix} z & 1 \\ 1 & 0 \end{vmatrix} = -\frac{1}{z^2-1} = -\mathcal{L}\{\sinh t\}(z),$$

from which we conclude that

$$y_1(t) = \cosh t \qquad \text{and} \qquad y_2(t) = -\sinh t. \qquad \Diamond$$

Download free eBooks at bookboon.com

Example 2.2.4 *Solve the system of ordinary differential equations*

$$\begin{cases} y_1' + y_2' + y_1 = 0, \\ \qquad y_2' + y_1 = 3, \end{cases} \qquad \begin{cases} y_1(0) = 0, \\ y_2(0) = 0. \end{cases}$$

First method It is obvious that y_2' can be eliminated by subtraction, so application of the Laplace transformation is totally unnecessary. We get by this subtraction that

$$y_1' = -3,$$

thus $y_1 = -3t$, using that $y_1(0) = 0$, whence

$$y_2' = 3 - y_1 = 3 + 3t,$$

from which by an integration,

$$y_2 = \frac{3}{2}\,t^2 + 3t.$$

Summing up,

$$y_1 = -3t \qquad \text{and} \qquad y_2 = \frac{3}{2}\,t^2 + 3t.$$

Second method If we instead apply the Laplace transformation, then we get

$$\begin{cases} zY_1 + zY_2 + Y_1 = 0, \\ zY_2 + Y_1 = \dfrac{3}{2}, \end{cases}$$

hence by a rearrangement,

$$\begin{cases} (z+1)Y_1 + zY_2 = 0, \\ 1 \cdot Y_1 + zY_2 = \dfrac{3}{2}. \end{cases}$$

The determinant of this system is

$$\Delta = (z+1)z - z = z^2,$$

and we get by Cramer's formula,

$$Y_1 = \frac{1}{z^2} \begin{vmatrix} 0 & z \\ \frac{3}{2} & z \end{vmatrix} = -\frac{3}{z^2} = -\mathcal{L}\{3r\}(z),$$

and

$$Y_2 = \frac{1}{z^2} \begin{vmatrix} z+1 & 0 \\ 1 & \frac{3}{z} \end{vmatrix} = \frac{3}{z^2} + \frac{3}{z^3} = \mathcal{L}\left\{\frac{3}{2}\,t^2 + 3t\right\}(z).$$

Finally, by the inverse Laplace transformation,

$$y_1 = -3t \qquad \text{and} \qquad y_2 = \frac{3}{2}\,t^2 + 3t. \qquad \Diamond$$

Download free eBooks at bookboon.com

Example 2.2.5 *Find the function $x(t)$ where $x(t)$ is given by the following system of three linear ordinary differential equations,*

$$\begin{cases} x' + y' = y + z, \\ y' + z' = z + x, \\ z' + x' = x + y, \end{cases} \qquad \begin{cases} x(0) = 2, \\ y(0) = -3, \\ z(0) = 1. \end{cases}$$

We shall use the Laplace transformation to get

$$\begin{cases} zX(z) - 2 + zY(z) + 3 = Y(z) + Z(z), \\ zY(z) + 3 + zZ(z) - 1 = Z(z) + X(z), \\ zZ(z) - 1 + zX(z) - 2 = X(z) + Y(z), \end{cases}$$

hence by some rearrangements,

$$\begin{cases} zX(z) + (z-1)Y(z) - Z(z) = -1, \\ -X(z) + zY(z) + (z-1)Z(z) = -2, \\ (z-1)X(z) - Y(z) + zZ(z) = 3. \end{cases}$$

The corresponding determinant is

$$\Delta = \begin{vmatrix} z & z-1 & -1 \\ -1 & z & z-1 \\ z-1 & -1 & z \end{vmatrix} = z^3 + (z-1)^3 - 1 - 3z(z-1) = 2\left(z^3 - 1\right).$$

Hence, for $\Re z > 1$,

$$\begin{aligned} X(z) &= \frac{1}{2\left(z^3-1\right)} \begin{vmatrix} -1 & z-1 & -1 \\ -2 & z & z-1 \\ 3 & -1 & z \end{vmatrix} \\ &= \frac{1}{2\left(z^3-1\right)} \left\{-z^2 + 3(z-1)^2 - 2 - \{-3z - 2z(z-1) + (z+1)\}\right\} \\ &= \frac{1}{2\left(z^3-1\right)} \left\{-z^2 + 3z^2 - 6z + 3 - 2 + 3z + 2z^2 - 2z + z - 1\right\} \\ &= \frac{4z^2 - 4z}{2\left(z^3-1\right)} = 2 \cdot \frac{z(z-1)}{(z-1)\left(z^2+z+1\right)} = \frac{2z}{\left(z+\frac{1}{2}\right)^2 + \frac{3}{4}} = \frac{2\left(z+\frac{1}{2}\right) - 1}{\left(z+\frac{1}{2}\right)^2 + \left\{\frac{\sqrt{3}}{2}\right\}^2} \\ &= 2 \cdot \frac{z+\frac{1}{2}}{\left(z+\frac{1}{2}\right)^2 + \left\{\frac{\sqrt{3}}{2}\right\}^2} - \frac{2}{\sqrt{3}} \cdot \frac{\frac{\sqrt{3}}{2}}{\left(z+\frac{1}{2}\right)^2 + \left\{\frac{\sqrt{3}}{2}\right\}^2}. \end{aligned}$$

Download free eBooks at bookboon.com

Then finally, by the inverse Laplace transformation,

$$
\begin{aligned}
x(t) &= 2\exp\left(-\frac{1}{2}\,t\right)\cos\left(\frac{\sqrt{3}}{2}\,t\right)-\frac{2}{\sqrt{3}}\exp\left(-\frac{1}{2}\,t\right)\sin\left(\frac{\sqrt{3}}{2}\,t\right)\\
&= \frac{4}{\sqrt{3}}\cdot\exp\left(-\frac{1}{2}\,t\right)\left\{\cos\left(\frac{\sqrt{3}}{2}\,t\right)\cdot\frac{\sqrt{3}}{2}-\sin\left(\frac{\sqrt{3}}{2}\,t\right)\cdot\frac{1}{2}\right\}\\
&= \frac{4}{\sqrt{3}}\exp\left(-\frac{1}{2}\,t\right)\cos\left(\frac{\sqrt{3}}{2}\,t+\frac{\pi}{6}\right).
\end{aligned}
$$

We notice that we shall not explicitly find $y(t)$ and $z(t)$. $\Diamond$

Click on the ad to read more

Download free eBooks at bookboon.com

Example 2.2.6 *Solve the system of ordinary differential equations*

$$\begin{cases} \dfrac{dx}{dt} + \dfrac{dy}{dt} = t, \\[2mm] \dfrac{d^2x}{dt^2} - y(t) = e^{-t}, \end{cases} \qquad \begin{cases} x(0)=3, \quad x'(0)=-2 \\ y(0)=0. \end{cases}$$

It is absolutely no reason here to solve the problem via the Laplace transformation, because it is much easier to start by integrating the first equation to get

$$x(t)+y(t)=\frac{t^2}{2}+c=\frac{t^2}{2}+3,$$

from which

$$-y(t)=x(t)-\frac{t^2}{2}-3.$$

When this is put into the second equation of the problem, we get

$$\frac{d^2x}{dt^2}+x(t)=e^{-t}+\frac{t^2}{2}+3,$$

where we guess a particular integral of the form

$$x(t)=\frac{1}{2}\,e^{-t}+\frac{1}{2}\,t^2+a.$$

Then by insertion,

$$\frac{d^2x}{dt^2}+x(t)=e^{-t}+\frac{t^2}{2}+1+a=e^{-t}+\frac{t^2}{2}+3,$$

so it is indeed a particular solution of the inhomogeneous equation, when we choose $a=2$. Then we get

$$\begin{cases} x(t) = \dfrac{1}{2}\,e^{-t}+\dfrac{1}{2}\,t^2+2+c_1\cos t+c_2\sin t, \\[2mm] x'(t) = -\dfrac{1}{2}\,e^{-t}+t-c_1\sin t+c_2\cos t, \end{cases}$$

thus

$$x(0)=3=\frac{1}{2}+2+c_1, \qquad \text{hence } c_1=\frac{1}{2},$$

and

$$x'(0)=-2=-\frac{1}{2}+c_2, \qquad \text{hence } c_2=-\frac{3}{2}.$$

We conclude that

$$x(t)=\frac{1}{2}\,e^{-t}+\frac{1}{2}\,t^2+2+\frac{1}{2}\,\cos t-\frac{3}{2}\,\sin t,$$

and

$$y(t)=\frac{t^2}{2}+3-x(t)=-\frac{1}{2}\,e^{-t}+1-\frac{1}{2}\,\cos t+\frac{3}{2}\,\sin t. \qquad \Diamond$$

Download free eBooks at bookboon.com

Example 2.2.7 *Solve the system of linear ordinary differential equations*

$$\begin{cases} \dfrac{dx}{dt} - \dfrac{dy}{dt} - 2x(t) + 2y(t) = \sin t, \\[2mm] \dfrac{d^2x}{dt^2} + 2\,\dfrac{dy}{dt} + x(t) = 0, \end{cases} \qquad \begin{cases} x(0) = x'(0) = 0, \\[2mm] y(0) = 0. \end{cases}$$

It is seen by inspection that it is not necessary to apply the Laplace transformation in this example either, because the first equation can be rewritten in the form

$$\frac{d}{dt}\,(x-y) - 2(x-y) = \sin t.$$

The complete solution of the corresponding homogeneous equation is $c\,e^{2t}$, and we guess a particular solution of the structure

$$x - y = a\,\cos t + b\,\sin t.$$

We get by insertion,

$$\frac{d}{dt}(x-y) - 2(x-y) = -a\,\sin t + b\,\cos t - 2a\,\cos t - 2b\,\sin t = (-a-2b)\sin t + (b-2a)\cos t,$$

which is equal to $\sin t$ for $b = 2a$ and $a = -\frac{1}{5}$, so $b = -\frac{2}{5}$. Since $x(0) - y(0) = 0$, the solution is

$$x(t) - y(t) = -\frac{1}{5}\,\cos t - \frac{2}{5}\,\sin t + c\,e^{2t} = \frac{1}{5}\left(-\cos t - 2\sin t + e^{2t}\right).$$

Thus,

$$2(x-y) = -\frac{2}{5}\,\cos t - \frac{4}{5}\,\sin t + \frac{2}{5}\,e^{2t},$$

and so by a differentiation,

$$2\,\frac{dx}{dt} - 2\,\frac{dy}{dt} = \frac{2}{5}\,\sin t - \frac{4}{5}\,\cos t + \frac{4}{5}\,e^{2t}.$$

When this expression is added to the second equation of the system we get

$$\frac{d^2x}{dt^2} + 2\,\frac{dx}{dt} + x(t) = \frac{2}{5}\,\sin t - \frac{4}{5}\,\cos t + \frac{4}{5}\,e^{2t}.$$

The corresponding homogeneous equation has the complete solution

$$c_1\,e^{-t} + c_2\,t\,e^{-t}.$$

A particular solution of this equation must have the structure

$$x(t) = a\,\sin t + b\,\cos t + k\,e^{2t},$$

thus

$$x'(t) = -b\,\sin t + a\,\cos t + 2k\,e^{2t},$$

Download free eBooks at bookboon.com

and

$$x''(t) = -a\,\sin t - b\,\cos t + 4k\,e^{2t},$$

and we get by insertion

$$\frac{d^2x}{dt^2} + 2\,\frac{dx}{dt} + x(t) = -2b\,\sin t + 2a\,\cos t + 9k\,e^{2t}.$$

This expression is equal to $\frac{2}{5}\,\sin t - \frac{4}{5}\,\cos t + \frac{4}{5}\,e^{2t}$ for $a = -\frac{2}{5}$ and $b = -\frac{1}{5}$ and $k = \frac{4}{45}$. Hence, the complete solution of the differential equation in $x(t)$ alone is given by

$$x(t) = -\frac{2}{5}\,\sin t - \frac{1}{5}\,\cos t + \frac{4}{45}\,e^{2t} + c_1\,e^{-t} + c_2\,t\,e^{-t}.$$

By a differentiation,

$$x'(t) = \frac{1}{5}\,\sin t - \frac{2}{5}\,\cos t + \frac{8}{45}\,e^{2t} + (c_2 - c_1)\,e^{-t} - c_2\,t\,e^{-t}.$$

Then we use the initial conditions to get

$$x(0) = 0 = -\frac{1}{5} + \frac{4}{45} + c_1, \qquad \text{thus} \qquad c_1 = \frac{1}{5} - \frac{4}{45} = \frac{5}{45} = \frac{1}{9},$$

and

$$x'(0) = 0 = -\frac{2}{5} + \frac{8}{45} + c_2 - c_1, \qquad \text{thus} \qquad c_2 = \frac{1}{9} + \frac{2}{5} - \frac{8}{45} = \frac{5+18-8}{45} = \frac{1}{3}.$$

Summing up, we have proved that

$$x(t) = -\frac{2}{5}\,\sin t - \frac{1}{5}\,\cos t + \frac{4}{45}\,e^{2t} + \frac{1}{9}\,e^{-t} + \frac{1}{3}\,t\,e^{-t},$$

from which we derive that

$$y(t) = x(t) + \frac{2}{5}\,\sin t + \frac{1}{5}\,\cos t - \frac{1}{5}\,e^{2t} = -\frac{1}{9}\,e^{2t} + \frac{1}{9}\,e^{-t} + \frac{1}{3}\,t\,e^{-t}.$$

For comparison we *alternatively* also solve the problem by using the *Laplace transformation.* Then we get

$$\begin{cases} zX - zY - 2X + 2Y &= \dfrac{1}{1+z^2}, \\[2mm] z^2\,X - 2zY + X &= 0, \end{cases}$$

thus

$$\begin{cases} (z-2)X + (-z+2)Y &= \dfrac{1}{1+z^2}, \\[2mm] \left(z^2+1\right)X + 2zY &= 0. \end{cases}$$

Download free eBooks at bookboon.com

The determinant of this system is

$$\Delta = (z-2)\left\{2z-(-1)\left(z^2+1\right)\right\} = (z-2)(z+1)^2.$$

Then use Cramer's formula to get

$$X = \frac{1}{(z-2)(z+1)^2}\begin{vmatrix} \frac{1}{1+z^2} & -(z-2) \\ 0 & 2z \end{vmatrix} = \frac{2z}{(z-2)(z+1)^2\left(z^2+1\right)},$$

and

$$Y = \frac{1}{(z-2)(z+1)^2}\begin{vmatrix} z-2 & \frac{1}{z^2+1} \\ z^2+1 & 0 \end{vmatrix} = -\frac{1}{(z-2)(z+1)^2}.$$

Click on the ad to read more

Download free eBooks at bookboon.com

By a very tedious decomposition,

$$\begin{aligned}
X &= \frac{4}{3^2 5}\,\frac{1}{z-2}+\frac{-2}{(-3)2}\,\frac{1}{(z+1)^2}+\frac{2z}{(z-2)(z+1)^2\,(z^2+1)}-1\frac{4}{45}\,\frac{1}{z-2}-\frac{1}{3}\,\frac{1}{(z+1)^2}\\
&= \frac{4}{45}\,\frac{1}{z-2}+\frac{1}{3}\,\frac{1}{(z+1)^2}\;+\frac{1}{3}\,\frac{6z-(z-2)\left(z^2+1\right)}{(z-2)(z+1)^2\,(z^2+1)}-\frac{4}{45}\,\frac{1}{z-2}\\
&= \frac{4}{45}\,\frac{1}{z-2}+\frac{1}{3}\,\frac{1}{(z+1)^2}-\frac{1}{3}\,\frac{z^3-2z^2-5z-2}{(z-2)(z+1)^2\,(z^2+1)}-\frac{4}{45}\,\frac{1}{z-2}\\
&= \frac{4}{45}\,\frac{1}{z-2}+\frac{1}{3}\,\frac{1}{(z+1)^2}-\frac{1}{3}\,\frac{z^2-3z-2}{(z-2)(z+1)\,(z^2+1)}-\frac{4}{45}\,\frac{1}{z-2}\\
&= \frac{4}{45}\,\frac{1}{z-2}+\frac{1}{3}\,\frac{1}{(z+1)^2}-\frac{1}{3}\,\frac{2}{(-3)2}\,\frac{1}{z+1}-\frac{1}{9}\,\frac{3z^2-9z-6+(z-2)\left(z^2+1\right)}{(z-2)(z+1)\,(z^2+1)}-\frac{4}{45}\,\frac{1}{z-2}\\
&= \frac{4}{45}\,\frac{1}{z-2}+\frac{1}{3}\,\frac{1}{(z+1)^2}+\frac{1}{9}\,\frac{1}{z+1}-\frac{1}{9}\,\frac{z^3+z^2-8z-8}{(z-2)(z+1)\,(z^2+1)}-\frac{4}{45}\,\frac{1}{z-2}\\
&= \frac{4}{45}\,\frac{1}{z-2}+\frac{1}{3}\,\frac{1}{(z+1)^2}+\frac{1}{9}\,\frac{1}{z+1}-\frac{1}{9}\,\frac{z^2-8}{(z-2)\,(z^2+11)}-\frac{4}{45}\,\frac{1}{z-2}\\
&= \frac{4}{45}\,\frac{1}{z-2}+\frac{1}{3}\,\frac{1}{(z+1)^2}+\frac{1}{9}\,\frac{1}{z+1}-\frac{1}{45}\,\frac{5z^2-40+4z^2+4}{(z-2)\,(z^2+1)}\\
&= \frac{4}{45}\,\frac{1}{z-2}+\frac{1}{3}\,\frac{1}{(z+1)^2}+\frac{1}{9}\,\frac{1}{z+1}-\frac{1}{45}\,\frac{9\left(z^2-4\right)}{(z-2)\,(z^2+1)}\\
&= \frac{4}{45}\,\frac{1}{z-2}+\frac{1}{3}\,\frac{1}{(z+1)^2}+\frac{1}{9}\,\frac{1}{z+1}-\frac{1}{5}\,\frac{z}{z^2+1}-\frac{2}{5}\,\frac{1}{z^2+1}.
\end{aligned}$$

Analogously,

$$\begin{aligned}
Y &= -\frac{1}{(z-2)(z+1)^2}=-\frac{1}{9}\,\frac{1}{z-2}-\frac{1}{9}\,\frac{9-(z+1)^2}{(z-2)(z+1)^2}\\
&= -\frac{1}{9}\,\frac{1}{z-2}-\frac{1}{9}\,\frac{(3-z-1)(3+z+1)}{(z-2)(z+1)^2}=-\frac{1}{9}\,\frac{1}{z-2}+\frac{1}{9}\,\frac{(z+1)+3}{(z+1)^2}\\
&= -\frac{1}{9}\,\frac{1}{z-2}+\frac{1}{9}\,\frac{1}{z+1}+\frac{1}{3}\,\frac{1}{(z+1)^2}.
\end{aligned}$$

We finally apply the inverse Laplace transformation to get

$$\begin{cases}
x(t)=\dfrac{4}{45}\,e^{2t}+\dfrac{1}{3}\,t\,e^{-t}+\dfrac{1}{9}\,e^{-t}-\dfrac{1}{5}\,\cos t-\dfrac{2}{5}\,\sin t,\\[2ex]
y(t)=-\dfrac{1}{9}\,e^{2t}+\dfrac{1}{3}\,t\,e^{-t}+\dfrac{1}{9}\,e^{-t}.
\end{cases}\qquad\Diamond$$

Download free eBooks at bookboon.com

Example 2.2.8 *Solve the system of linear ordinary differential equations*

$$\begin{cases} \dfrac{dx}{dt} + 2\,\dfrac{d^2y}{dt^2} & = & e^{-t}, \\ \\ \dfrac{dx}{dt} + x(t) - y(t) & = & 1, \end{cases} \qquad \begin{cases} x(0) = 0, \\ \\ y(0) = y'(0) = 0. \end{cases}$$

We apply the Laplace transformation,

$$\begin{cases} zX & + & 2z^2Y & = & \dfrac{1}{z+1} \\ \\ (z+1)X & - & Y & = & \dfrac{1}{z}. \end{cases}$$

The determinant of the system is

$$\Delta = -z - 2z^2(z+1) = -z\left(1 + 2z^2 + 2z\right).$$

Then by Cramer's formula,

$$Y = -\frac{1}{z\left(2z^2+2z+1\right)} \begin{vmatrix} z & \frac{1}{z+1} \\ \\ z+1 & \frac{1}{z} \end{vmatrix} = 0,$$

from which we conclude that $y \equiv 0$, hence $\dfrac{dx}{dt} = e^{-t}$ by the first equation, from which

$$x(t) = 1 - e^{-t} \qquad \text{and} \qquad y(t) = 0.$$

Alternatively, we can also find X by Cramer's formula,

$$\begin{aligned} X &= -\frac{1}{z\left(2z^2+2z+1\right)} \begin{vmatrix} \frac{1}{z+1} & 2z^2 \\ \\ \frac{1}{z} & -1 \end{vmatrix} = -\frac{1}{z\left(2z^2+2z+1\right)} \left\{-\frac{1}{z+1} - 2z\right\} \\ &= \frac{2z^2+2z+1}{z\left(2z^2+2z+1\right)(z+1)} = \frac{1}{z(z+1)} = \frac{1}{z} - \frac{1}{z+1}, \end{aligned}$$

from which $x(t) = 1 - e^{-t}$. Finally, (sketch) there is no need to apply the Laplace transformation, because a straightforward integration of the first equation gives

$$x(t) + 2\,\frac{dy}{dt} = 1 - e^{-t}, \qquad \text{thus} \qquad x(t) = 1 - e^{-t} - 2\,\frac{dy}{dt},$$

so by eliminating $x(t)$ in the second equation,

$$e^{-t} - 2\,\frac{d^2y}{dt^2} + 1 - e^{-t} - 2\frac{dy}{dt} - y(t) = 1,$$

which is reduced to

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} = 0, \qquad y(0) = y'(0) = 0,$$

from which we get $y(t) = 0$, and we proceed as above. $\Diamond$

Download free eBooks at bookboon.com

Example 2.2.9 *Solve the system of linear ordinary differential equations of variable coefficients*

$$\begin{cases} t\,x(t)+y(t)+t\,\dfrac{dy}{dt} &= (t-1)e^{-t}, \\ \dfrac{dx}{dt}-y(t) &= e^{-t}, \end{cases} \qquad x(0)=1.$$

Hint. *First find $y(0)$.*

When we put $t=0$ into the first equation, we get $y(0)=-1$. Then we see that it is absolutely no need to use the Laplace transformation, because it follows from the second equation that

$$(6)\quad y(t)=\frac{dx}{dt}-e^{-t},$$

thus

$$\frac{dy}{dt}=\frac{d^2x}{dt^2}+e^{-t},$$

and hence by insertion into the first equation,

$$t\,x(t)+\frac{dx}{dt}-e^{-t}+t\,\frac{d^2x}{dt^2}+t\,e^{-t}=(t-1)e^{-t},$$

which is reduced to the *Bessel differential equation*

$$t\,\frac{d^2x}{dt^2}+\frac{dx}{dt}+t\,x(t)=0,$$

the bounded solutions of which are given by $c_0\,J_0(t)$. Since both $J_0(0)=1$ and $x(0)=1$, we get that $x(t)=J_0(t)$. Hence, by (6),

$$y(t)=J_0'(t)+e^{-t}=-J_1(t)+e^{-t},$$

and the solutions become

$$x(t)=J_0(t) \qquad \text{and} \qquad y(t)=-J_1(t)+e^{-t} \qquad \Diamond$$

Download free eBooks at bookboon.com

Example 2.2.10 *Solve the system of ordinary differential equations*

$$\begin{cases} -3\,\dfrac{d^2x}{dt^2} + 3\,\dfrac{d^2y}{dt^2} = t\,e^{-t} - 3\,\cos t, \\[2mm] t\,\dfrac{d^2x}{dt^2} - \dfrac{dy}{dt} = \sin t, \end{cases}$$

given that

$$x(0) = -1, \qquad x'(0) = 2, \qquad y(0) = 4, \qquad \frac{d^2y}{dt^2}(0) = 0.$$

Notice that $y'(0)$ is unknown. If, however, $\dfrac{d^2x}{dt^2}(0)$ exists and is finite, then it follows from the second equation that $y'(0) = 0$. Under this assumption it follows from the first equation for $t = 0$ that

$$-3\,\frac{d^2x}{dt^2}(0) + 3 \cdot 0 = 0 - 3,$$

so we get additional,

$$\frac{d^2x}{dt^2}(0) = 1 \qquad \text{and} \qquad y'(0) = 0.$$

Click on the ad to read more

Download free eBooks at bookboon.com

We put as usual $\mathcal{L}\{x\}(z) = X$ and $\mathcal{L}\{y\}(z) = Y$. Then by the Laplace transformation,

$$\begin{cases} -3\left(z^2 X + z - 2\right) + 3\left(z^2 Y - 4z - 0\right) & = & \dfrac{1}{(z+1)^2} - \dfrac{3z}{z^2+1}, \\ -\dfrac{d}{dz}\left(z^2 X + z - 2\right) - z\,Y + 4 & = & \dfrac{1}{z^2+1}, \end{cases}$$

thus

$$\begin{cases} -3z^2 X + 3z^2 Y & = & 15z - 6 + \dfrac{1}{(z+1)^2} - \dfrac{3z}{z^2+1}, \\ -\dfrac{d}{dz}\left(z^2 X\right) - z\,Y & = & -3 + \dfrac{1}{z^2+1}. \end{cases}$$

When the second equation is multiplied by $3z$, it follows by an addition that

$$-3z\,\frac{d}{dz}\left(z^2 X\right) - 3\cdot 1\left(z^2 X\right) = 6z - 6 + \frac{1}{(z+1)^2},$$

which can also be written

$$\frac{d}{dz}\left\{-3z^3 X\right\} = 6z - 6 + \frac{1}{(z+1)^2}.$$

Then by an integration,

$$-3z^3 X = 3z^2 - 6z - \frac{1}{z+1} + C,$$

hence

$$\begin{aligned} X &= -\frac{1}{z} + \frac{2}{z^2} + \frac{1}{3}\,\frac{1}{z^3(z+1)} - \frac{C}{3}\cdot\frac{1}{z^3} = -\frac{1}{z} + \frac{2}{z^2} - \frac{C}{3}\cdot\frac{1}{z^3} - \frac{1}{3}\cdot\frac{1}{z+1} + \frac{1}{3}\cdot\frac{1+z^3}{z^3(z+1)} \\ &= -\frac{1}{z} + \frac{2}{z^2} - \frac{C}{3}\cdot\frac{1}{z^3} - \frac{1}{3}\cdot\frac{1}{z+1} + \frac{1}{3}\cdot\frac{z^2-z+1}{z^3} = -\frac{2}{3}\cdot\frac{1}{z} + \frac{5}{3}\cdot\frac{1}{z^2} + \frac{1}{3}(1-C)\frac{1}{z^3} - \frac{1}{3}\cdot\frac{1}{z+1}. \end{aligned}$$

By the inverse Laplace transformation,

$$x(t) = -\frac{2}{3} + \frac{5}{3}\,t + \frac{1}{6}(1-C)t^2 - \frac{1}{3}\,e^{-t},$$

thus

$$\frac{d^2x}{dt^2} = \frac{1}{3}(1-C) - \frac{1}{3}\,e^{-t}.$$

It follows from

$$\frac{d^2x}{dt^2}(0) = 1 = \frac{1}{3}(1-C) - \frac{1}{3} = -\frac{C}{3},$$

that $C = -3$, so

$$x(t) = -\frac{2}{3} + \frac{5}{3}\,t + \frac{2}{3}\,t^2 - \frac{1}{3}\,e^{-t},$$

Download free eBooks at bookboon.com

(notice that $x(0) = -1$ and $x'(0) = 2$) and

$$\frac{d^2x}{dt^2} = \frac{4}{3} - \frac{1}{3}\, e^{-t}.$$

By insertion into the second equation we get

$$\frac{dy}{dt} = t\,\frac{d^2x}{dt^2} - \sin t = \frac{4}{3}\, t\, e^{-t} - \sin t,$$

hence by an integration,

$$y = \frac{2}{3}\, t^2 + \frac{1}{3}\, t\, e^{-t} + \frac{1}{3}\, e^{-t} + \cos t + k.$$

We get for $t = 0$,

$$y(0) = 4 = \frac{1}{3} + 1 + k,$$

thus $c = \frac{8}{3}$. The solution is

$$\begin{cases} x(t) = -\dfrac{2}{3} + \dfrac{5}{3}\, t + \dfrac{2}{3}\, t^2 - \dfrac{1}{3}\, e^{-t}, \\[2ex] y(t) = \dfrac{8}{3} + \dfrac{2}{3}\, t^2 + \dfrac{1}{3}(t+1)e^{-t}e^{-t} + \cos t. \end{cases} \qquad \Diamond$$

Example 2.2.11 *A particle moves in the XY-plane such that its position (x, y) at time t is governed by the system of differential equations*

$$x''(t) + k_1^2\, y(t) = 0, \qquad y''(t) + k_2^2\, x(t) = 0.$$

Assume that the particle is at rest at (a, b) at time $t = 0$, when it is set free. Find the position of the particle at any later time.

We use $x(0) = a$ and $y(b) = b$ and $x'(0) = y'(0) = 0$, when we apply the Laplace transformation,

$$\begin{cases} z^2\, X - z \cdot a + k_1^2\, Y &= 0, \\ k_2^2\, X + z^2\, Y - zb &= 0, \end{cases}$$

thus

$$\begin{cases} z^2\, X + k_1^2\, Y &= a\, z, \\ k_2^2\, X + z^2\, Y &= b\, z. \end{cases}$$

The determinant of the system is

$$\Delta = z^4 - k_1^2\, k_2^2 = \left(z^2 - k_1\, k_2\right)\left(z^2 + k_a\, k_2\right).$$

Download free eBooks at bookboon.com

Then by Cramer's formula,

$$
\begin{aligned}
X &= \frac{1}{z^4 - k_1^2 k_2^2} \begin{vmatrix} az & k_1^2 \\ bz & z^2 \end{vmatrix} = \frac{z\left(az^2 - k_1^2 b\right)}{z^4 - \left(k_1 k_2\right)^2} = z \cdot \frac{az^2 - k_1^2}{\left(z^2 - k_1 k_2\right)\left(z^2 + k_1 k_2\right)} \\
&= \frac{ak_1 k_2 - k_1^2 b}{2k_1 k_2} \cdot \frac{z}{z^2 - k_1 k_2} + \frac{-ak_1 k_2 - k_1^2 b}{-2k_1 k_2} \cdot \frac{1}{z^2 + k_1 k_2} \\
&= \frac{ak_2 - bk_1}{2k_2} \cdot \frac{z}{z^2 - k_1 k_2} + \frac{ak_2 + bk_1}{2k_2} \cdot \frac{z}{z^2 + k_1 k_2},
\end{aligned}
$$

hence by the inverse Laplace transformation,

$$
x(t) = \frac{ak_2 - bk_1}{2k_2} \cdot \cosh\left(\sqrt{k_1 k_2}\, t\right) + \frac{ak_2 + bk_1}{2k_2} \cos\left(\sqrt{k_1 k_2}\, t\right),
$$

and analogously, or simply by interchanging letters and indices,

$$
y(t) = -\frac{ak_2 - bk_1}{2k_1} \cosh\left(\sqrt{k_1 k_1}\, t\right) + \frac{ak_2 + bk_1}{2k_1} \cos\left(\sqrt{k_1 k_2}\, t\right).
$$

It is easy to check these solution functions by insertion. $\Diamond$

Click on the ad to read more

Download free eBooks at bookboon.com

Example 2.2.12 *Compute the currents I, I_1 and I_2 in the circuit on the figure, when $E(t) = 110H(t)$ Volt, and the initial currents are*

$$I(0) = I_1(0) = I_2(0) = 0.$$

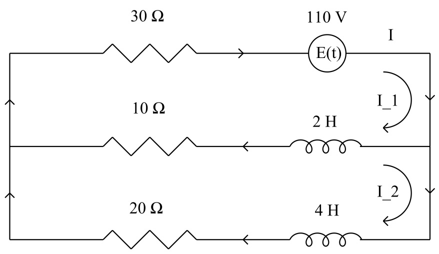


Figure 7: The circuit of Example 2.2.12.

The inductances of 2 henry and 4 henry are chosen for convenience, so the solutions do not become too complicated. It will in practice be difficult to realize these very large inductances.

The circuit I is broken down into two simple circuits I_1 and I_2, where $I = I_1 + I_2$. The governing differential equations are

$$\begin{cases} 30\,I_1 + 10\,(I_1 - I_2) + 2\,\dfrac{d}{dt}\,(I_1 - I_2) & = \quad 110\,H(t), \\ 20\,I_2 + 4\,\dfrac{d}{dt}\,I_2 + 10\,(I_2 - I_1) + 2\,\dfrac{d}{dt}\,(I_2 - I_1) & = \quad 0, \end{cases}$$

which is reduced to

$$\begin{cases} 2\,\dfrac{dI_1}{dt} + 40\,I_1 - 2\,\dfrac{dI_2}{dt} - 10\,I_2 & = \quad 110\,H(t), \\ -2\,\dfrac{dI_1}{dt} - 10\,I_1 + 6\,\dfrac{dI_2}{dt} + 30\,I_2 & = \quad 0. \end{cases}$$

Then by the Laplace transformation,

$$\begin{cases} (2z+40)\,\mathcal{L}\left\{I_1\right\}(z) - (2z+10)\,\mathcal{L}\left\{I_2\right\}(z) & = \quad \dfrac{110}{z}, \\ -(2z+10)\,\mathcal{L}\left\{I_1\right\}(z) + (6z+30)\,\mathcal{L}\left\{I_2\right\}(z) = 0. \end{cases}$$

Download free eBooks at bookboon.com

The determinant of this system is

$$\begin{aligned}\Delta &= B(2z+40)(6z+30)+(2z+10)^2=(z+5)(12z+240+4z+20)\\ &= 16(z+5)\left(z+\frac{65}{4}\right).\end{aligned}$$

Then, using Cramer's formula,

$$\begin{aligned}\mathcal{L}\{I_1\}(z) &= \frac{1}{16(z+5)\left(z+\frac{65}{4}\right)}\begin{vmatrix}\frac{110}{z} & -(2z+10)\\ 0 & 6z+30\end{vmatrix}=\frac{110\cdot 6(z+5)}{16(z+5)\left(z+\frac{65}{4}\right)z}\\ &= \frac{33}{13}\cdot\frac{1}{z}-\frac{33}{13}\cdot\frac{1}{z+\frac{65}{4}},\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}\{I_2\}(z) &= \frac{1}{16\left(z+\frac{65}{4}\right)(z+5)}\begin{vmatrix}2z+40 & \frac{110}{z}\\ -(2z+10) & 0\end{vmatrix}=\frac{110\cdot 2(z+5)}{16z\left(z+\frac{65}{4}\right)(z+5)}\\ &= \frac{11}{13}\cdot\frac{1}{z}-\frac{11}{13}\cdot\frac{1}{z+\frac{65}{4}}.\end{aligned}$$

Finally, by the inverse Laplace transformation,

$$I_1(t)=\frac{33}{13}\left\{1-\exp\left(-\frac{65}{4}\,t\right)\right\},$$

$$I_2(t)=\frac{11}{13}\left\{1-\exp\left(-\frac{65}{4}\,t\right)\right\},$$

$$I(t)=\frac{44}{13}\left\{1-\exp\left(-\frac{65}{4}\,t\right)\right\}. \qquad \Diamond$$

Example 2.2.13 *Consider the circuit of Figure 8, where $E(t)=500\sin 10t$ volt, and $R_1=R_2=10$ ohm, and $L=1$ henry, and $C=0.01$ farad. At time $t=0$ the load of the condenser is 0, and the currents I_1 and I_2 are both 0. Compute the load of the condenser for $t>0$.*

We consider the two simple circuits I_1 and I_2, indicated on Figure 8. The corresponding currents are denoted by i_1 and i_2, so analyzing the figure we get the equations,

$$\begin{cases} R_1\,i_1(t)+L\,\dfrac{di_1(t)}{dt}+\dfrac{1}{C}\int_0^t(i_1-i_2)\,\mathrm{d}t-v_C(0) = e(t),\\[2mm] \dfrac{1}{C}\int_0^t(i_2-i_1)\,\mathrm{d}t-v_C(0)+R_2\,i_2(t) = 0,\end{cases}$$

Download free eBooks at bookboon.com

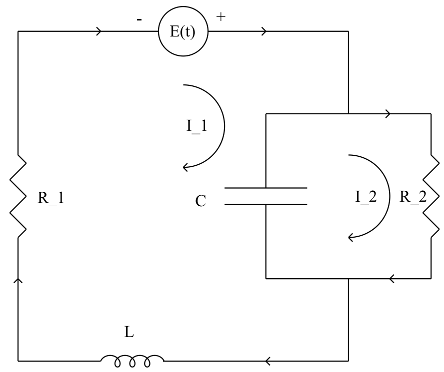


Figure 8: The circuit of Example 2.2.13.

hence by the Laplace transformation,

$$\begin{cases} 10\, I_1(z) + z\, I_1(z) + 100 \cdot \dfrac{1}{z}\, \{I_1(z) - I_2(z)\} + \dfrac{v_C(0)}{z} & = \quad E(z) = \dfrac{5000}{z^2+100}, \\ 100 \cdot \dfrac{1}{z}\, \{I_2(z) - I_1(z)\} - \dfrac{v_C(0)}{z} + z\, I_2(z) + 10\, I_2(z) & = \quad 0. \end{cases}$$

We have $v_C(t) = 0$ for $t < 0$ and

$$v_C(0) = \frac{E_0}{2} = 250 \sin 0 = 0.$$

The system is then reduced to

$$\begin{pmatrix} 10 + z + \frac{100}{z} & -\frac{100}{z} \\ -\frac{100}{z} & 10 + z + \frac{100}{z} \end{pmatrix} \begin{pmatrix} I_1(z) \\ I_2(z) \end{pmatrix} = \begin{pmatrix} \frac{5000}{z^2+100} \\ 0 \end{pmatrix}.$$

The determinant of the system is

$$\Delta = \left(10 + z + \frac{100}{z}\right)^2 - \left(\frac{100}{z}\right)^2 = \left(10 + z + \frac{200}{z}\right)(10+z) = \frac{1}{z}\left(z^2 + 10z + 200\right)(10+z),$$

and we get by Cramer's formula,

$$I_1(z) = \frac{\frac{5000}{z^2+100}\left(z + 10 + \frac{100}{z}\right) + 0}{\frac{1}{z}\left(z^2+10z+200\right)(10+z)} = 5000 \cdot \frac{z^2 + 10z + 100}{\left(z^2+100\right)\left(z^2+10z+200\right)(10+z)},$$

and

$$I_2(z) = \frac{\frac{5000}{z^2+100} \cdot \dfrac{100}{z}}{\frac{1}{z}\left(z^2+10z+200\right)(z+10)} = 5000 \cdot \frac{100}{\left(z^2+100\right)\left(z^2+10z+200\right)(z+10)}.$$

Download free eBooks at bookboon.com

It follows from $\dfrac{dq}{dt} = i_1(t) - i_2(t)$ that

$$z\,Q(z) - 0 = z\,Q(z) = I_1(z) - I_2(z) = 5000 \cdot \frac{z}{\left(z^2+100\right)\left(z^2+10z+200\right)},$$

hence

$$\begin{aligned}
Q(z) &= 5000 \cdot \frac{1}{\left(z^2+100\right)\left(z^2+10z+200\right)} \\
&= \frac{5}{8}\,\frac{z}{z^2+100} - \frac{5}{8}\,\frac{10}{z^2+100} - \frac{5}{8}\,\frac{z+12}{(z+5)^2+\left(5\sqrt{7}\right)^2} \\
&= \frac{5}{8}\,\frac{z}{z^2+10^2} - \frac{5}{8}\,\frac{10}{z^2+10^2} - \frac{5}{8}\,\frac{z+5}{(z+5)^2+\left(5\sqrt{7}\right)^2} - \frac{\sqrt{7}}{8} \cdot \frac{5\sqrt{7}}{(z+5)^2+\left(5\sqrt{7}\right)^2}.
\end{aligned}$$

Finally, we get by the inverse Laplace transformation,

$$q(t) = \frac{5}{8}\cos 10t - \frac{5}{8}\sin 10t - \frac{5}{8}\,e^{-5t}\cos\left(5\sqrt{7}\,t\right) - \frac{\sqrt{7}}{8}\,e^{-5t}\sin\left(5\sqrt{7}\,t\right). \qquad \Diamond$$

Click on the ad to read more
Download free eBooks at bookboon.com

2.3 Linear partial differential equations

Example 2.3.1 *Solve the equation*

$$\frac{ddu}{\partial t} = 2\,\frac{\partial^2 u}{\partial x^2},$$

given that

$$u(0,t) = u(5,t) = 0 \qquad \textit{and} \qquad u(x,0) = 10\,\sin 4\pi x.$$

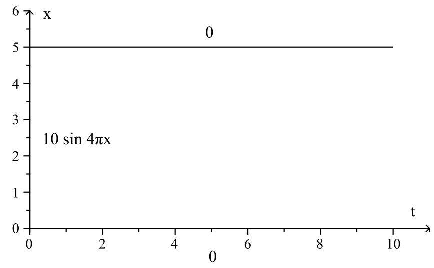


Figure 9: The boundary conditions of Example 2.3.1.

The structure of the partial differential equation is that of the *heat equation*.

We denote the partial Laplace transform of $u(x,t)$ with respect to t by $U(x,z)$. Then it follows that

$$z\,U(x,z) - u(x,0) = 2\,\frac{\partial^2 U}{\partial x^2},$$

thus

$$\frac{\partial^2 U}{\partial x^2} - \frac{z}{2}\,U(x,z) = -\frac{1}{2}\,u(x,0) = -5\,\sin 4\pi x.$$

In this equation we consider z as a parameter, so when we guess a solution of the form $c(z)\cdot\sin 4\pi x$, then we get

$$c(x)\cdot\left\{-16\pi^2 - \frac{z}{2}\right\} sin 4\pi z = -5\sin 4\pi z,$$

so a particular integral of the equation is given by

$$U_0(x,z) = \frac{5}{16\pi^2 + \frac{z}{2}}\,\sin 4\pi x = \frac{10}{32\pi^2 + z}\,\sin 4\pi x.$$

Download free eBooks at bookboon.com

If we choose the usual branch of the complex square root, then the complete solution becomes for $\Re z > 0$,

$$U(x,z) = \frac{10}{32\pi^2 + z}\,\sin 4\pi x + c_1(z)\,\exp\left(\frac{1}{\sqrt{2}}\,\sqrt{z}\,x\right) + c_2(z)\,\exp\left(-\frac{1}{\sqrt{2}}\,\sqrt{z}\,x\right).$$

Then we apply the horizontal boundary conditions.

If $x = 0$, then

$$c_1(z) + c_2(z) = 0.$$

If $x = 5$, then

$$c_1(z)\left(\exp\left(\frac{1}{\sqrt{2}}\,\sqrt{z}\,x\right)\right)^5 + c_2(z)\left(\exp\left(-\frac{1}{\sqrt{2}}\,\sqrt{z}\,x\right)\right)^5 = 0.$$

We conclude that either $c_1(z) = c_2(z) \equiv 0$, or $\exp\left(\frac{10}{\sqrt{2}}\,\sqrt{z}\right) = 1$, corresponding to $\frac{10}{\sqrt{2}}\,\sqrt{z} = 2ip\pi$. However, the latter is not possible for $z \in \mathbb{C}\setminus(\mathbb{R}_- \cup \{0\})$, because we have chosen the usual branch of the square root with its branch cut along the negative real axis. Hence, we conclude that the partial Laplace transform is uniquely given by

$$U(x,z) = \frac{10}{32\pi^2 + z}\,\sin 4\pi x,$$

corresponding to the solution

$$u(x,t) = 10\,\exp\left(-32\pi^2\right)\sin 4\pi x.$$

Check of solution! Given $u(x,t)$ above, we clearly have $u(0,t) = 0$ and $u(5,t) = 0$ and $u(x,0) = 10\,\sin 4\pi x$. Furthermore, by partial differentiations,

$$\frac{\partial u}{\partial t} = -320\,\pi^2\,\exp\left(-332\pi^2 t\right)\sin 4\pi x,$$

and

$$2\,\frac{\partial^2 u}{\partial x^2} = -16\pi^2 \cdot 2 \cdot 10 \cdot \exp\left(-32\pi^2 t\right)\sin 4\pi x = \frac{\partial u}{\partial t},$$

and the partial differential equation is also fulfilled. $\Diamond$

Download free eBooks at bookboon.com

Example 2.3.2 *Solve the linear partial differential equation*

$$\frac{\partial^2 f}{\partial t^2} = 9\,\frac{\partial^2 f}{\partial x^2},$$

given the boundary and initial conditions

$$f(0,t) = 0, \quad f(2,t) = 0, \qquad \textit{and} \qquad f(x,0) = 20\,\sin 2\pi x - 10\sin 5\pi x, \quad f_t'(x,0) = 0.$$

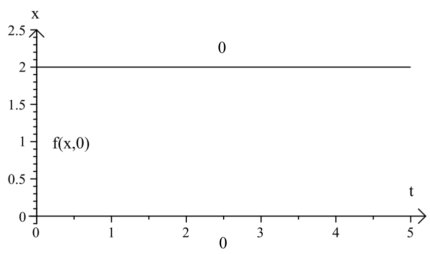


Figure 10: The boundary conditions of Example 2.3.2.

The structure of the partial differential equation is that of a *wave equation*.

Apply the partial Laplace transformation with respect to t,

$$z^2\,F(x,z) - z\,f(x,0) - f_t'(x,0) = 9\,\frac{\partial^2 F}{\partial x^2},$$

which is reduced to

$$\frac{\partial^2 F}{\partial x^2} - \left\{\frac{z}{3}\right\}^2\,F(x,z) = -20\,z\,\sin 20\pi x - 10\,z\,\sin 5\pi x.$$

We guess that some particular solution must have the structure

$$F(x,z) = a(z)\cdot\sin 2\pi x + b(z)\cdot\sin 5\pi x,$$

where we get by insertion,

$$\frac{\partial^2 F}{\partial x^2} - \frac{z^2}{9}\,F(x,z) \quad = \quad -4\pi^2\cdot a(z)\cdot\sin 2\pi x - 25\,\pi^2\cdot b(z)\cdot\sin 5\pi x$$

$$-\frac{z^2}{9}\cdot a(z)\cdot\sin 2\pi x - \frac{z^2}{9}\cdot b(z)\cdot\sin 5\pi x,$$

Download free eBooks at bookboon.com

which is equal to $-20\,z\cdot\sin 2\pi x - 10\,z\cdot\sin 5\pi x$, if and only if

$$-\left\{4\pi^2+\frac{z^2}{9}\right\}a(z) = -20\,z \qquad \text{and} \qquad -\left\{25\pi^2+\frac{z^2}{9}\right\}b(z) = -10\,z,$$

so we conclude that

$$a(z) = \frac{180z}{z^2+(6\pi)^2} \qquad \text{and} \qquad b(z) = \frac{90z}{z^2+(15\pi)^2}.$$

Hence, the complete solution is

$$F(x,z) = \frac{180z}{z^2+(6\pi)^2}\,\sin 2\pi x + \frac{90z}{z^2+(15\pi)^2}\,\sin 5\pi x + c_1(z)\exp\left(\frac{z}{3}\,x\right) + c_2(z)\exp\left(-\frac{z}{3}\,x\right).$$

If we put $x=0$, then

$$F(0,z) = 0 = c_1(z)+c_2(z).$$

If we put $x=2$, then

$$F(2,z) = 0 = c_1(z)\exp\left(\frac{2}{3}\,z\right) + c_2(z)\exp\left(-\frac{2}{3}\,z\right).$$

Click on the ad to read more

Download free eBooks at bookboon.com

Thus, $c_1(z) = c_2(z) = 0$, so

$$F(x,z) = \frac{180z}{z^2+(6\pi)^2}\,\sin 2\pi x + \frac{90z}{z^2+(15\pi)^2}\,\sin 5\pi x.$$

Finally, by the inverse Laplace transformation,

$$f(x,t) = 180\,\cos 6\pi t \cdot \sin 2\pi x + 90\,\cos 15\pi t \cdot \sin 5\pi x. \qquad \Diamond$$

Example 2.3.3 *Solve the linear partial differential equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 4u(x,t),$$

given that

$$u(0,t)=0, \quad u(\pi,t)=0, \qquad \textit{and} \qquad u(x,0) = 6\,\sin 2x - 4\sin 2x.$$

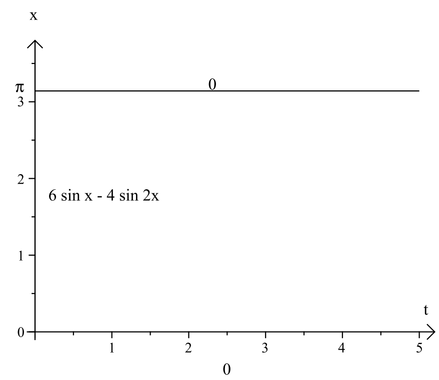


Figure 11: The boundary conditions of Example 2.3.3.

This is a variant of the *heat equation*.

We get by the partial Laplace transformation with respect to t,

$$z\,U(x,z) - 6\sin x - 4\sin 2x = \frac{\partial^2 U}{\partial x^2} - 4\,U(x,z),$$

thus by a rearrangement,

$$\frac{\partial^2 U}{\partial x^2} - (4+z)U(x,z) = -6\sin x - 4\sin 2x.$$

Download free eBooks at bookboon.com

There exists a particular integral of the structure

$$U(x,z)=a(z)\,\sin x+b(z)\,\sin 2x,$$

so by insertion,

$$\begin{aligned}\frac{\partial^2 U}{\partial x^2}-(4+z)U(x,z) &= -a(z)\,\sin x-4b(z)\,\sin 2x-(4+z)a(z)\,\sin x-(4+z)b(z)\,\sin 2x\\ &= -(5+z)a(z)\,\sin x-(8+z)b(z)\,\sin 2x=-6\sin x-4\sin 2x,\end{aligned}$$

and we conclude that

$$a(z)=\frac{6}{z+5}\qquad\text{and}\qquad b(z)=\frac{4}{z+8}.$$

The complete solution is therefore

$$U(x,z)=\frac{6}{z+5}\,\sin x+\frac{4}{z+8}\,\sin 2x+C_1(z)\exp\left(x\sqrt{z+4}\right)+C_2(z)\exp\left(-x\sqrt{z+4}\right).$$

Then by the boundary conditions,

$$U(0,z)=0=C_1(z)+C_2(z),$$

$$U(\pi,z)=0=C_1(z)\exp\left(\pi\sqrt{z+4}\right)+C_2(z)\exp\left(-\pi\sqrt{z+4}\right),$$

and we conclude that $C_1(z)=C_2(z)=0$, hence

$$U(x,z)=\frac{6}{z+5}\,\sin x+\frac{4}{z+8}\,\sin 2z.$$

Finally, by the inverse Laplace transformation,

$$u(x,t)=6\,e^{-5t}\,\sin x+4e^{-8t}\,\sin 2x.\qquad\Diamond$$

Example 2.3.4 *Find the bounded solution $f(x,t)$, $x\in\,]0,1[$, $t\in\mathbb{R}_+$, of the initial value problem*

$$\frac{\partial f}{\partial x}-\frac{\partial f}{\partial t}=1-e^{-t},\qquad f(x,0)=x.$$

We assume that the partial Laplace transform $F(x,z)$ with respect to t exists. Then

$$\frac{\partial F}{\partial x}-z\,F(x,z)+x=\mathcal{L}\left\{1-e^{-t}\right\}(z)=\frac{1}{z}-\frac{1}{z+1},$$

thus

$$\frac{\partial F}{\partial x}-z\,F(x,z)=\frac{1}{z}-\frac{1}{z+1}-x.$$

Consider z as a parameter. Then we have a linear ordinary inhomogeneous differential equation of first order in the *real* variable x, so we can apply the usual methods from real Calculus. The corresponding

Download free eBooks at bookboon.com

homogeneous equation has the complete solution $F(x,z) = C(z)e^{zx}$, and one particular integral must have the structure

$$F_0(x,z) = a(z)x + b(z).$$

We get by insertion,

$$\frac{\partial F}{\partial x} - z\,F(x,z) = a(z) - z\,a(z)\,x - z\,b(z) = -z\,a(z)\,x + \{a(z) - z\,b(z)\},$$

which is equal to $\frac{1}{z} - \frac{1}{z+1} - x$ for $a(z) = \frac{1}{z}$ and $b(z) = \frac{1}{z(z+1)} = \frac{1}{z} - \frac{1}{z+1}$, and the complete solution becomes

$$F(x,z) = \frac{1}{z} - \frac{1}{z+1} + \frac{x}{z} + C(z)e^{zx}.$$

If $\Re\, z > 0$, then the term $C(z)e^{zx}$ becomes unbounded for $x \to +\infty$, unless we choose $C(z) \equiv 0$. Therefore,

$$F(x,z) = \frac{1+x}{z} - \frac{1}{z+1} = (x+1)\mathcal{L}\{1\}(z) - \mathcal{L}\left\{e^{-t}\right\}(z),$$

and we get by the inverse Laplace transformation that

$$f(x,t) = 1 + x - e^{-t}. \qquad \Diamond$$

Example 2.3.5 *Find the bounded solution for $(x,t) \in \mathbb{R}_+ \times \mathbb{R}_+$ of the initial value problem*

$$\frac{\partial f}{\partial x} = 2\,\frac{\partial f}{\partial t} + f(x,t), \qquad f(x,0) = 6e^{-3x}.$$

Let $F(x,z)$ denote the partial Laplace transform with respect to t of $f(x,t)$. Then

$$\frac{\partial F}{\partial x} = 2zF(x,z) - 2\cdot 6\,e^{-3x} + F(x,z) = (2z+1)F(x,z) - 12e^{-3x}.$$

A particular solution is given by a well-known solution formula from real Calculus,

$$e^{(2z+1)x}\int(-12)e^{-(2z+4)x}\,\mathrm{d}x = e^{(2z+1)x} - \frac{-12}{-2(z+2)}\,e^{-(2z+4)x}$$

$$= \quad e^{-3x}\cdot\frac{6}{z+2} = 6e^{-3x}\,\mathcal{L}\left\{e^{-2t}\right\}(z),$$

and the corresponding homogeneous equation has the general solutions $c(z)e^{(2z+1)x}$. Since $e^{(2z+1)x}$ is unbounded in x, if e.g. $\Re\, z > 0$, we must have $c(z) \equiv 0$, so we conclude by the inverse Laplace transformation that

$$f(x,t) = 6\,e^{-3x-2t}, \qquad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+,$$

which is trivially bounded. $\Diamond$

Download free eBooks at bookboon.com

Example 2.3.6 *Find the bounded solution of the linear partial differential equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+,$$

for which also $u(0,t) = 1$ *and* $u(x,0) = 0$.

This is the classical heat equation.

When we apply the partial Laplace transformation with respect to t, we get

$$\frac{\partial^2 U}{\partial x^2} = z\,U(x,z) - u(x,0) = z\,U(x,z),$$

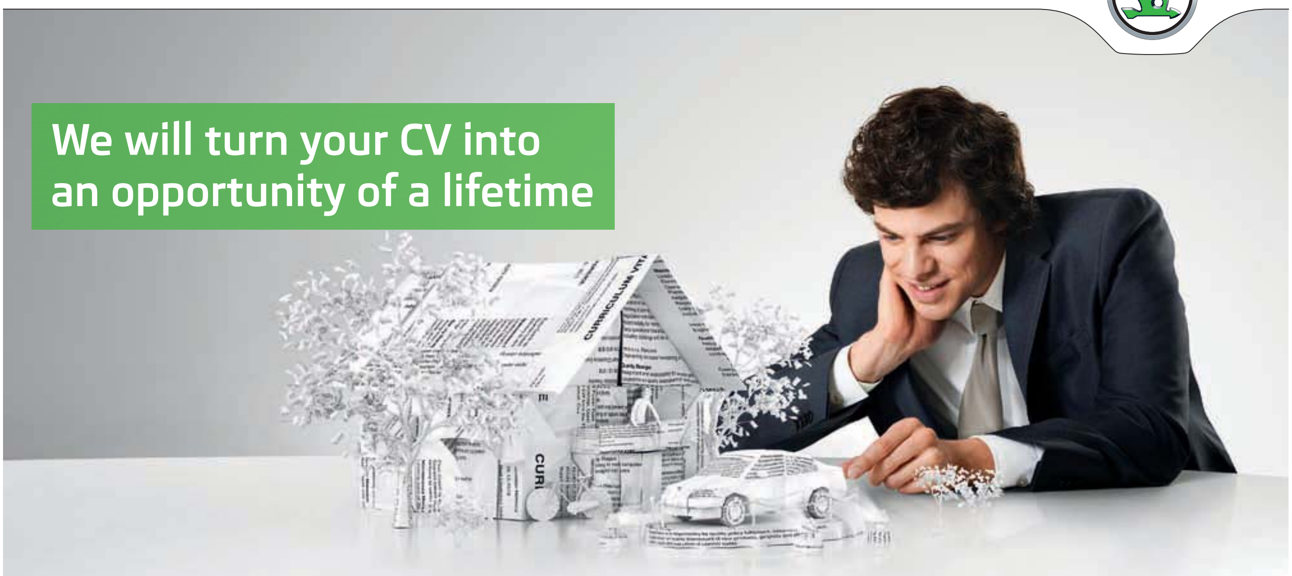
which is a simple linear homogeneous partial differential equation of parametric coefficients in x, so its complete solution is

$$U(x,z) = C_1(z)\,\exp\left(\sqrt{z}\,x\right) + C_2(z)\,\exp\left(-\sqrt{x}\,x\right).$$

If $\Re\, z > 0$, then $\Re\sqrt{z} > 0$, hence $\left|\exp\left(\sqrt{z}\,x\right)\right| \to +\infty$ for $x \to +\infty$, so we are forced to put $C_1(z) = 0$. We conclude that we shall only consider solutions of the form

$$U(x,z) = C_2(z)\exp\left(-\sqrt{z}\,x\right).$$

SIMPLY CLEVER
ŠKODA



Do you like cars? Would you like to be a part of a successful brand?
We will appreciate and reward both your enthusiasm and talent.
Send us your CV. You will be surprised where it can take you.

Send us your CV on
www.employerforlife.com

Click on the ad to read more

Download free eBooks at bookboon.com

If $x = 0$, then $u(0,t) = 1$, hence

$$U(0,z) = \frac{1}{z} = C_2(z) \cdot e^0 = C_2(z),$$

so we conclude that the bounded solution of the given initial/boundary problem has its partial Laplace transform given by

$$U(x,z) = \frac{1}{z} \exp\left(-x\sqrt{z}\right).$$

Then note that

$$\mathcal{L}\left\{\operatorname{erfc}\left(\frac{1}{2\sqrt{t}}\right)\right\}(z) = \frac{\exp\left(-\sqrt{z}\right)}{z} \qquad \text{for } \Re z > 0.$$

Thus, by the inverse Laplace transformation and a change of variable, where we put $k = x^2$,

$$\begin{aligned} u(t,x) &= \mathcal{L}_z^{-1}\left\{\frac{\exp\left(-x\sqrt{z}\right)}{z}\right\}(t) = x^2\,\mathcal{L}_z^{-1}\left\{\frac{\exp\left(-\sqrt{x^2 z}\right)}{x^2\,z}\right\}(t) = x^2 \cdot \frac{1}{x^2}\operatorname{erfc}\left(\frac{1}{2\sqrt{\frac{t}{x^2}}}\right) \\ &= \operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right) = \frac{2}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{t}}}^{+\infty} \exp\left(-u^2\right)\,du, \end{aligned}$$

which is the classical solution most frequently applied in the technical sciences. $\Diamond$

Example 2.3.7 *Solve the linear partial differential equation*

$$\frac{\partial^2 f}{\partial t^2} - 4\,\frac{\partial^2 f}{\partial x^2} + f(x,r) = 16\,x + 20\,\sin x,$$

given the boundary/initial conditions

$$f(0,t) = 0, \qquad f(\pi,t) = 16\pi, \qquad f(x,0) = 16x + 12\sin 2x - 8\sin 3x, \qquad \frac{\partial f}{\partial t}(x,0) = 0.$$

The equation is a wave equation.

When we apply the partial Laplace transformation with respect to t, then we get

$$z^2\,F(x,z) - z\,f(x,0) - \frac{\partial f}{\partial t}(x,0) - 4\,\frac{\partial^2 F}{\partial x^2} + F(x,z) = \frac{1}{z} \cdot 16x + \frac{1}{z} \cdot 20\sin x,$$

thus by a rearrangement,

$$\begin{aligned} \frac{\partial^2 F}{\partial x^2} - \frac{1}{4}\left(z^2+1\right)F(x,z) &= -\frac{4}{z}\,x - \frac{5}{z}\,\sin x - 4xz - 3z\,\sin 2x - 2z\,\sin 3x \\ &= -\left(\frac{4}{z} + 4z\right)x - \frac{5}{z}\,\sin x - 3z\,\sin 2x - 2z\,\sin 3x. \end{aligned}$$

We see that there exists a particular solution of the structure

$$F(x,z) = a(z)\,x + b(z)\,\sin x + c(z)\,\sin 2x + d(z)\,\sin 3x,$$

Download free eBooks at bookboon.com

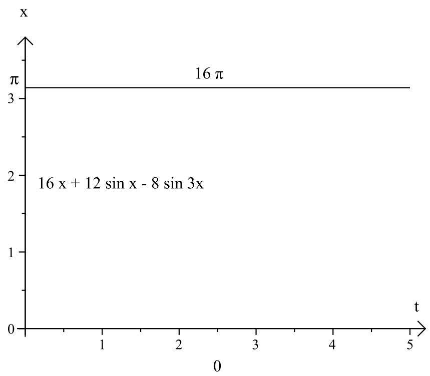


Figure 12: The initial/boundary conditions of Example 2.3.7.

where we shall find the four unknown parametric coefficients, $a(z)$, $b(z)$, $c(z)$ and $d(z)$. We get by insertion,

$$\frac{\partial^2 F}{\partial x^2} - \frac{1}{4}\left(z^2+1\right) F(x,z) = b(z)\,\sin x - 4c(z)\,\sin 2x - 9d(z)\,\sin 3x$$

$$-\frac{1}{4}\left(z^2+1\right) a(z)z - \frac{1}{4}\left(z^2+1\right) b(z)\sin x - \frac{1}{4}\left(z^2+1\right)\sin 2x - \frac{1}{4}\left(z^2+1\right)\sin 3x$$

$$= -\frac{1}{4}\left(z^2+1\right) a(z)x - \frac{1}{4}\left(z^2+5\right) b(z)\sin x - \frac{1}{4}\left(z^2+17\right) c(z)\sin 2x - \frac{1}{4}\left(z^2+37\right) d(z)\sin 3x$$

$$= -\frac{4}{z}\left(z^2+1\right) x - \frac{5}{z}\,\sin x - 3z\,\sin 2x - 2z\,\sin 3x,$$

so when we identify the coefficients we get

$$a(z) = \frac{16}{z}, \qquad b(z) = \frac{20}{z\left(z^2+5\right)}, \qquad c(z) = \frac{12z}{z^2+17}, \qquad d(z) = \frac{2z}{z^2+37}.$$

Then the complete solution of the partial Laplace transform becomes

$$F(x,z) = \frac{16}{z}\,x + \left(\frac{4}{z} - \frac{4z}{z^2+5}\right)\sin x + \frac{12z}{z^2+17}\,\sin 2x + \frac{2z}{z^2+37}\,\sin 3x$$
$$+C_1(z)\exp\left(\frac{x}{2}\sqrt{z^2+1}\right) + C_2(z)\exp\left(-\frac{x}{2}\sqrt{z^2+1}\right).$$

Then we apply the initial/boundary conditions,

$$F(0,z) = 0 = C_1(z) + C_2(z),$$

and

$$F(\pi,z) = \frac{16\pi}{z} = \frac{16\pi}{z} + C_1(z)\exp\left(\frac{\pi}{2}\sqrt{z^2+1}\right) + C_2(z)\exp\left(-\frac{\pi}{2}\sqrt{z^2+1}\right),$$

Download free eBooks at bookboon.com

from which we conclude that $C_1(z) = C_2(z) \equiv 0$. Hence the partial Laplace transform becomes

$$F(x,z) = \frac{16}{z}\,x + \left(\frac{4}{z} - \frac{4z}{z^2+5}\right)\sin x + \frac{12z}{z^2+17}\,\sin 2x + \frac{2z}{z^2+37}\,\sin 3x,$$

so by the inverse partial Laplace transformation,

$$f(x,t) = 16x + 4\sin x - 4\cos\left(\sqrt{5}\,t\right)\sin x + 12\cos\left(\sqrt{17}\,t\right)\sin 2x + 2\cos\left(\sqrt{37}\,t\right)\sin 3x. \qquad \Diamond$$

2.4 The Dirac measure δ

Example 2.4.1 *Given* $f_\varepsilon(t) := \dfrac{1}{\varepsilon}\,\chi_{[0,\varepsilon]}(t)$, *where* $\varepsilon > 0$ *is a parameter. Compute*

$$\mathcal{L}\left\{f_\varepsilon\right\}(z) \qquad \textit{and} \qquad \lim_{\varepsilon\to 0+}\mathcal{L}\left\{f_\varepsilon\right\}(z).$$

We get by a straightforward computation that

$$\begin{aligned}\mathcal{L}\left\{f_\varepsilon\right\}(z) &= \frac{1}{\varepsilon}\,\mathcal{L}\left\{\chi_{[0,\varepsilon]}\right\}(z) = \frac{1}{\varepsilon}\,\mathcal{L}\{H(t) - H(t-\varepsilon)\}(z) \\ &= \frac{1}{\varepsilon}\cdot\frac{1}{z}\left(1 - e^{-\varepsilon z}\right) = \frac{1 - e^{-\varepsilon z}}{z\varepsilon},\end{aligned}$$

where $H(t)$ denotes the Heaviside function $H(t) := \chi_{\mathbb{R}_+}$.

Then by taking the limit $\varepsilon \to 0+$,

$$\lim_{\varepsilon\to 0+}\mathcal{L}\left\{f_\varepsilon\right\}(z) = \frac{1}{z}\lim_{\varepsilon\to 0+}\frac{e^{-0\cdot z} - e^{-\varepsilon\, z}}{\varepsilon} = \frac{1}{z}\lim_{\varepsilon\to 0+}\left(+z\,e^{-\varepsilon z}\right) = 1,$$

which corresponds to $\mathcal{L}\{\delta\}(z)$, where δ denotes the Dirac measure. $\Diamond$

Example 2.4.2 *Let* $g\_\varepsilon(t) := \dfrac{1}{\varepsilon^2}\,\chi_{[0,\varepsilon]}(t) - \dfrac{1}{\varepsilon^2}\,\chi_{[2\varepsilon,3\varepsilon]}(t)$. *Compute* $\mathcal{L}\left\{g_\varepsilon\right\}(z)$, *and then the limit* $\lim_{\varepsilon\to 0+}\mathcal{L}\left\{g_\varepsilon\right\}(z)$.

By a straightforward computation,

$$\begin{aligned}\mathcal{L}\left\{g_\varepsilon\right\}(z) &= \frac{1}{\varepsilon^2}\,\mathcal{L}\{H(t) - H(t-\varepsilon) - H(t-2\varepsilon) + H(t-3\varepsilon)\}(z) \\ &= \frac{1}{\varepsilon^2}\cdot\frac{1}{z}\left(1 - e^{-\varepsilon z} - e^{-2\varepsilon z} + e^{-3\varepsilon z}\right).\end{aligned}$$

Download free eBooks at bookboon.com

Then use series expansions to proceed the computations above,

$$\begin{aligned}\mathcal{L}\left\{g_\varepsilon\right\}(z) &= \frac{1}{z}\cdot\frac{1}{\varepsilon^2}\left(1-\left\{1-\varepsilon z+\frac{1}{2}\,\varepsilon^2\,z^2+o\left(\varepsilon^2\right)\right\}-\left\{1-2\varepsilon\,z+\frac{4\varepsilon^2}{2}\,z^2+o\left(\varepsilon^2\right)\right\}\right.\\ &\qquad\left.+\left\{1-3\varepsilon\,z+\frac{9\varepsilon^2}{2}\cdot z^2+o\left(\varepsilon^2\right)\right\}\right)\\ &= \frac{1}{z}\cdot\frac{1}{\varepsilon^2}\cdot\varepsilon^2\left(-\frac{1}{2}\,z^2-\frac{4}{2}\,z^2+\frac{9}{2}\,z^2\right)+\frac{1}{z}\cdot\frac{1}{\varepsilon^2}\,o\left(\varepsilon^2\,z^2\right)\\ &= 2z+\frac{1}{z\,\varepsilon^2}\,o\left(\varepsilon^2\,z^2\right).\end{aligned}$$

We conclude by taking the limit that

$$\lim_{\varepsilon\to 0+}\mathcal{L}\left\{g_\varepsilon\right\}(z)=2z. \qquad \Diamond$$

Download free eBooks at bookboon.com

Example 2.4.3 *Find the Laplace transform of*

$$t \cdot H(t-1) + t^2\,\delta(t-1).$$

Let φ be any test function. Then

$$\int_{-\infty}^{+\infty} t^2\,\delta(t-1)\varphi(t)\,\mathrm{d}t = 1^2\varphi(1) = \int_{-\infty}^{+\infty} \delta(t-1)\varphi(t)\,\mathrm{d}t,$$

and we conclude that $t^2\,\delta(t-1) = \delta(t-1)$.

Then for $\Re z > 0$,

$$\mathcal{L}\left\{t \cdot H(t-1) + t^2\,\delta(t-1)\right\}(z) = \mathcal{L}\{t \cdot H(t-1)\}(z) + \mathcal{L}\{\delta(t-1)\}(z)$$

$$= \int_1^{+\infty} t\,e^{-zt}\,\mathrm{d}t + e^{-z} = \left[t \cdot \left(-\frac{1}{z}\right) e^{-zt}\right]_1^{+\infty} + \frac{1}{z}\int_1^{+\infty} e^{-zt}\,\mathrm{d}t + e^{-z}$$

$$= \frac{1}{z}\,e^{-z} - \left[\frac{1}{z^2}\,e^{-zt}\right]_1^{+\infty} + e^{-z} = \left(1 + \frac{1}{z} + \frac{1}{z^2}\right) e^{-z}. \qquad \Diamond$$

Example 2.4.4 *Find the Laplace transform of*

$$\cos t \cdot \ln t \cdot \delta(t-\pi).$$

We get by formal computations that

$$\begin{aligned} \mathcal{L}\{\cos t \cdot \ln t \cdot \delta(t-\pi)\}(z) &= \int_0^{+\infty} \cos t \cdot \ln t \cdot \delta(t-\pi) e^{-zt}\,\mathrm{d}t \\ &= \cos\pi \cdot \ln\pi \cdot e^{\pi z} = -\ln\pi \cdot e^{-\pi z}, \end{aligned}$$

which could also be derived directly from

$$\cos t \cdot \ln t \cdot \delta(t-\pi) = -\ln\pi \cdot \delta(t-\pi). \qquad \Diamond$$

Example 2.4.5 *Solve the differential equation*

$$f''(t) + 4\,f(t) = \delta(t-2), \qquad f(0) = 0, \qquad f'(0) = 1.$$

Assuming that $f \in \mathcal{F}$, it follows by taking the Laplace transformation that

$$z^2\,\mathcal{L}\{f\}(z) - z \cdot 0 - 1 + 4\,\mathcal{L}\{f\}(z) = e^{-2z},$$

thus

$$\mathcal{L}\{f\}(z) = \frac{1}{z^2+4} + \frac{e^{-2z}}{z^2+4} = \frac{1}{2}\,\mathcal{L}\{\sin 2t\}(z) + e^{\cdot 2z} \cdot \frac{1}{2}\,\mathcal{L}\{\sin 2t\}(z).$$

Download free eBooks at bookboon.com

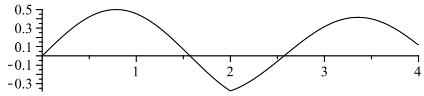


Figure 13: The graph of the solution of Example 2.4.5.

Then by the inverse Laplace transformation,

$$f(t) = \frac{1}{2}\sin 2t + \frac{1}{2}\sin(2\{t-2\})\cdot H(t-2).$$

Check of the solution. Since $f(t)$ is continuous for $x \geq 0$, and differentiable for $x \neq 2$, we get

$$f'(t) = \cos 2t + \cos(2(t-2))\cdot H(t-2) \qquad \text{for } t \neq 2.$$

We see in particular that $f(0) = 0$ and $f'(0) = 1$, so the initial conditions are fulfilled.

The derivative $f'(t)$ has a jump at $t = 2$, so the trick is to add and then subtract $H(t-2)$ to get

$$f'(t) = (\cos 2t + \{\cos(2(t-2)) - 1\}\cdot H(t-2)) + H(t-2).$$

The first term is continuous and differentiable for $t \neq 2$, so

$$f''(t) = -2\sin 2t - 2\sin(2(t-2))\cdot H(t-2) + \delta(t-2).$$

Then finally,

$$f''(t) + 4f(t) = \delta(t-2). \qquad \Diamond$$

Download free eBooks at bookboon.com

Example 2.4.6 *Solve the convolution equation*

$$\int_0^t f(u)f(t-u)\,\mathrm{d}u = t + 2\,f(t), \qquad \textit{for } t \geq 0.$$

A formal application of the Laplace transformation gives

$$(\mathcal{L}\{f\}(z))^2 = \mathcal{L}\{t\}(z) + 2\,\mathcal{L}\{f\}(z) = 2\,\mathcal{L}\{f\}(z) + \frac{1}{z^2},$$

hence by a rearrangement,

$$(\mathcal{L}\{f\}(z) - 1)^2 = 1 + \frac{1}{z^2} = \frac{z^2+1}{z^2}.$$

Choose the usual branch of the square root, which is positive on $\mathbb{R}_+$ and has its branch cut lying along $\mathbb{R}_-$. Then we get from the equation above that the Laplace transform has two solutions,

$$\mathcal{L}\{f\}(z) = 1 \pm \frac{\sqrt{z^2+1}}{z} = \begin{cases} 1 - \dfrac{\sqrt{z^2+1}}{z} = 1 - \dfrac{z^2+1}{z\sqrt{z^2+1}} = \dfrac{\sqrt{z^2+1}-z}{\sqrt{z^2+1}} - \dfrac{1}{z}\,\dfrac{1}{\sqrt{z^2+1}}, \\[2ex] 1 + \dfrac{\sqrt{z^2+1}}{z} = 2 - \left(1 - \dfrac{\sqrt{z^2+1}}{z}\right) = 2 - \dfrac{\sqrt{z^2+1}-z}{\sqrt{z^2+1}} + \dfrac{1}{z}\,\dfrac{1}{\sqrt{z^2+1}}. \end{cases}$$

We know from Section 1.5 that

$$\frac{\sqrt{z^2+1}-z}{\sqrt{z^2+1}} = \mathcal{L}\left\{J_1\right\}(z), \qquad \text{and} \qquad \frac{1}{z}\,\frac{1}{\sqrt{z^2+1}} = \mathcal{L}\{H\}(z) \cdot \mathcal{L}\left\{J_0\right\}(z) = \mathcal{L}\left\{H \star J_0\right\}(z),$$

so using the inverse Laplace transformation we obtain the two solutions

$$f(t) = \begin{cases} J_1(t) - \int_0^t J_0(u)\,\mathrm{d}u, \\[2ex] 2\delta - J_1(t) + \int_0^t J_0(u)\,\mathrm{d}u. \end{cases} \qquad \Diamond$$

Example 2.4.7 *Solve the convolution equation*

$$\int_0^t f(u)f(t-u)\,\mathrm{d}u = 2\,f(t) + \frac{1}{6}\,t^3 - 2t, \qquad \textit{for } t \geq 0.$$

Apply a formal Laplace transformation with $F(z) = \mathcal{L}\{f\}(z)$ to get

$$F(z)^2 = 2\,F(z) + \frac{1}{z^4} - \frac{2}{z^2},$$

which we write

$$F(z)^2 - 2\,F(z) + 1 = \frac{1}{z^4} - \frac{2}{z^2} + 1,$$

Download free eBooks at bookboon.com

and the equation of the Laplace transform is reduced to

$$\{F(z)-1\}^2=\left(1-\frac{1}{z^2}\right)^2.$$

We conclude that

$$F(z)=\mathcal{L}\{f\}(z)=1\pm\left(1-\frac{1}{z^2}\right)=\begin{cases}2-\dfrac{1}{z^2},\\[2mm] \dfrac{1}{z^2}.\end{cases}$$

Finally, by the inverse Laplace transformation,

$$f(t)=\begin{cases}2\,\delta-t,\\[2mm] t.\end{cases}\qquad\Diamond$$

Click on the ad to read more

Download free eBooks at bookboon.com

Example 2.4.8 *A beam has its endpoints at $x = 0$ and $x = \ell$ clamped. The beam is subjected to a vertical concentrated load P_0 at the point $x = \frac{\ell}{3}$. Find the bending of the beam, i.e. solve the boundary value problem*

$$\frac{d^4 f}{dx^4} = \frac{P_0}{EI}\,\delta\left(x - \frac{\ell}{3}\right), \qquad f(0) = f'(0) = 0, \qquad f(\ell) = f'(\ell) = 0.$$

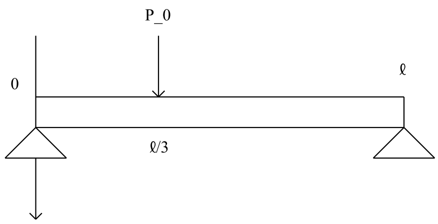


Figure 14: The beam of Example 2.4.8.

This example was found in a long forgotten book as an exercise in application of the Laplace transformation. However, a boundary value problem is considering a *finite* interval, while the Laplace transformation requires an *infinite* interval, so we cannot apply the Laplace transformation here. It is not possible to reconstruct the original exercise. It was probably included due to the occurrence of Dirac's delta function. In order not to make the reader disappointed we solve the given classical problem by simply integrating the equation successively,

$$\frac{d^3 f}{dx^3} = \frac{P_0}{EI}\,H\left(x - \frac{\ell}{3}\right) + a_1,$$

$$\frac{d^2 f}{dx^2} = \frac{P_0}{EI}\left(x - \frac{\ell}{3}\right)H\left(x - \frac{\ell}{3}\right) + a_1\,x + a_2,$$

$$\frac{df}{dx} = \frac{P_0}{EI}\cdot\frac{1}{2}\left(x - \frac{\ell}{3}\right)^2 H\left(x - \frac{\ell}{3}\right) + \frac{a_1}{2}\,x^2 + a_2\,x + a_3,$$

$$f(x) = \frac{P_0}{EI}\cdot\frac{1}{6}\left(x - \frac{\ell}{3}\right)^3 H\left(x - \frac{\ell}{3}\right) + \frac{a_1}{6}\,x^3 + \frac{a_2}{2}\,x^2 + a_3\,x + a_4,$$

where we shall use the boundary conditions to find the values of the four constants a_1, a_2, a_3 and a_4.

Download free eBooks at bookboon.com

It follows from $f(0) = 0$ that $a_4 = 0$, and from $f'(0) = 0$ that also $a_3 = 0$. The solution is therefore reduced to

$$f(x) = \frac{P_0}{EI} \cdot \frac{1}{6} \left(x - \frac{\ell}{3} \right)^3 H\left(x - \frac{\ell}{3} \right) + \frac{a_1}{6}\, x^3 + \frac{a_2}{2}\, x^2,$$

and

$$f'(x) = \frac{P_0}{EI} \cdot \frac{1}{2} \left(x - \frac{\ell}{3} \right)^2 H\left(x - \frac{\ell}{3} \right) + \frac{a_1}{2}\, x^2 + a_2\, x.$$

Then

$$f(\ell) = 0 = \frac{P_0}{EI} \cdot \frac{1}{6} \cdot \frac{8}{27}\, \ell^3 + \frac{a_1}{6}\, \ell^3 + \frac{a_2}{2}\, \ell^2,$$

thus

(7) $a_1\, \ell + 3a_2 = -\dfrac{P_0}{EI} \cdot \dfrac{8}{27}\, \ell,$

and

$$f'(\ell) = 0 = \frac{P_0}{EI} \cdot 12 \cdot \frac{4}{9}\, \ell^2 + \frac{1}{2}\, a_1\, \ell^2 + a_2\, \ell,$$

thus

(8) $a_1\, \ell + 2a_2 = -\dfrac{P_0}{EI} \cdot \dfrac{4}{9}\, \ell.$

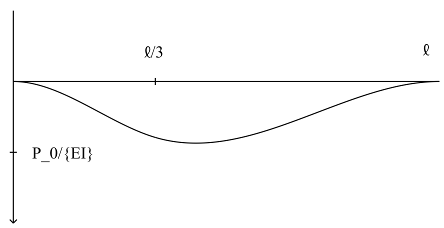


Figure 15: The graph of the solution of Example 2.4.8.

When (8) is subtracted from (7) we get

$$a_2 = \frac{P_0}{EI} \left(\frac{4}{9} - \frac{8}{27} \right) \ell = \frac{2}{27}\, \frac{P_0}{EI}\, \ell,$$

Download free eBooks at bookboon.com

from which

$$a_1 = -\frac{P_0}{EI}\left(\frac{4}{9}+\frac{8}{27}\right) = -\frac{20}{27}\,\frac{P_0}{EI},$$

and the solution is

$$f(x) = \frac{P_0}{EI}\left\{\frac{1}{6}\left(x-\frac{\ell}{3}\right)^3 H\left(x-\frac{\ell}{3}\right) - \frac{10}{81}\,x^3 + \frac{2}{27}\,\ell\, x^2\right\}. \qquad \Diamond$$

2.5 The $\mathfrak{z}$ transformation

Example 2.5.1 *Find the $\mathfrak{z}$ transform of sample period T of the function $f(t) = t^2$.*

If $|w| < 1$, then by series expansion and termwise differentiation,

$$\frac{1}{1-w} = \sum_{n=0}^{+\infty} w^n, \qquad \frac{1}{(1-w)^2} = \sum_{n=0}^{+\infty}(n+1)w^n, \qquad \frac{2}{(1-w)^3} = \sum_{n=0}^{+\infty}(n+1)(n+2)w^n,$$

from which

$$\begin{aligned}\sum_{n=0}^{+\infty} n^2\, w^n &= \sum_{n=0}^{+\infty}\{(n^2+3n+2)-3(n+1)+1\}w^n = \frac{2}{(1-w)^3} - \frac{3}{(1-w)^2} + \frac{1}{1-w} \\ &= \frac{1}{(1-w)^3}\cdot\left\{2-3(1-w)+(1-w)^2\right\} = \frac{2-3+3w+1-2w+w^2}{(1-w)^3} = \frac{w(w+1)}{(1-w)^3}.\end{aligned}$$

When $f(t) = t^2$, we get for $z = \dfrac{1}{w}$ and $|z| > 1$ that

$$\mathfrak{z}_T\left\{t^2\right\}(z) = \sum_{n=0}^{+\infty} = n^2\, T^2 \cdot \frac{1}{z^n} = T^2 \cdot \frac{\frac{1}{z}\left(\frac{1}{z}+1\right)}{\left(1-\frac{1}{z}\right)^3} = \frac{T^2\, z(z+1)}{(z-1)^3}. \qquad \Diamond$$

Example 2.5.2 *Find the $\mathfrak{z}$ transform of sample period 1 of the function $\dfrac{1}{\Gamma(1+t)}$ for $t \geq 0$.*

Just use the definition of the $\mathfrak{z}$ transformation to get

$$\begin{aligned}\mathfrak{z}_1\{f\}(z) &= \sum_{n=0}^{+\infty} f(n\cdot 1)\cdot\frac{1}{z^n} = \sum_{n=0}^{+\infty}\frac{1}{\Gamma(n+1)}\cdot\frac{1}{z^n} = \sum_{n=0}^{+\infty}\frac{1}{n!}\,\frac{1}{z^n} \\ &= \exp\left(\frac{1}{z}\right), \qquad \text{for } z \in \mathbb{C}\setminus\{0\}. \qquad \Diamond\end{aligned}$$

Download free eBooks at bookboon.com

Example 2.5.3 *Find the* $\mathfrak{z}$ *transforms of* $f(t) = \sin t$ *for* $t \geq 0$*, when*

$$(1) \quad T = \pi, \qquad (2) \quad T = \frac{\pi}{2}.$$

1) When $T = \pi$, it follows from the definition that

$$\mathfrak{z}_{\pi}\{\sin\}(z) = \sum_{n=0}^{+\infty} \sin(n\pi) \cdot \frac{1}{z^n} \equiv 0.$$

2) When $T = \frac{\pi}{2}$, then we get for $|z| > 1$ that

$$\begin{aligned}
\mathfrak{z}_{\pi/2}\{\sin\}(z) &= \sum_{n=0}^{+\infty} \sin\left(n \cdot \frac{\pi}{2}\right) \cdot \frac{1}{z^n} = \sum_{n=0}^{+\infty} \sin\left((2n+1)\frac{\pi}{2}\right) \cdot \frac{1}{z^{2n+1}} \\
&= \sum_{n=0}^{+\infty} \sin\left(\frac{\pi}{2} + n\pi\right) \cdot \frac{1}{z^{2n+1}} = \frac{1}{z} \sum_{n=0}^{+\infty} (-1)^n \left\{\frac{1}{z^2}\right\}^n \\
&= \frac{1}{z} \cdot \frac{1}{1 + \frac{1}{z^2}} = \frac{z}{z^2+1}.
\end{aligned}$$

These two examples show that if the sample period T is large compared with the oscillations of the function, then we lose a lot of information when we apply the $\mathfrak{z}$ transformation. $\Diamond$

Download free eBooks at bookboon.com

Example 2.5.4 *Find the $\mathfrak{z}$ transform of the function*

$$f(t) = \tan t, \qquad t \geq 0 \text{ and } t \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{N}_0,$$

when $T = \dfrac{\pi}{3}$.

The point is of course that $f(t)$ is not defined at $t = \dfrac{\pi}{2} + p\pi$, $p \in \mathbb{N}_0$, and that no sample time is of this form. We therefore obtain a fairly nice $\mathfrak{z}$ transform below, in spite of the fact that $f(t)$ is discontinuous at infinitely many points, so we also lose some information in this case.

We get by the definition of the zz transformation for $|z| > 1$ that

$$\begin{aligned}
\mathfrak{z}_{\pi/3}\{\tan\}(z) &= \sum_{n=0}^{+\infty} \tan\left(n \cdot \frac{\pi}{3}\right) \cdot \frac{1}{z^n} \\
&= \sum_{n=0}^{+\infty} \tan\left(3n \cdot \frac{\pi}{3}\right) \frac{1}{z^{3n}} + \sum_{n=0}^{+\infty} \tan\left((3n+1)\frac{\pi}{3}\right) \frac{1}{z^{3n+1}} + \sum_{n=0}^{+\infty} \tan\left((3n+2)\frac{\pi}{3}\right) \frac{1}{z^{3n+2}} \\
&= \sum_{n=0}^{+\infty} \tan(n\pi) \frac{1}{z^{3n}} + \sum_{n=0}^{+\infty} \tan\left(\frac{\pi}{3} + n\pi\right) \frac{1}{z^{3n+1}} + \sum_{n=0}^{+\infty} \tan\left(\frac{2\pi}{3} + n\pi\right) \frac{1}{z^{3n+2}} \\
&= 0 + \sqrt{3} \sum_{n=0}^{+\infty} \frac{1}{z^{3n+1}} - \sqrt{3} \sum_{n=0}^{+\infty} \frac{1}{z^{3n+2}} = \sqrt{3}\left\{\frac{1}{z} - \frac{1}{z^2}\right\} \sum_{n=0}^{+\infty} \left\{\frac{1}{z^3}\right\}^n \\
&= \sqrt{3} \cdot \frac{z-1}{z^2} \cdot \frac{1}{1 - \frac{1}{z^3}} = \sqrt{3} \cdot \frac{z-1}{z^2} \cdot \frac{z^3}{z^3 - 1} = \sqrt{3} \cdot \frac{z}{z^2 + z + 1}. \qquad \Diamond
\end{aligned}$$

Example 2.5.5 *Find the $\mathfrak{z}$ transform of the sequence*

$$\left(\sum_{n=1}^{n+1} \frac{1}{k} \right)_{n \in \mathbb{N}_0}.$$

We get for $|z| > 1$, using a rule of computation,

$$\begin{aligned}
\mathfrak{z}\left\{ \sum_{k=1}^{n-1} \frac{1}{k} \right\}(z) &= \mathfrak{z}\left\{ \sum_{k=0}^{n} \frac{1}{k+1} \right\}(z) = \frac{z}{z-1}\, \mathfrak{z}\left\{ \frac{1}{n+1} \right\}(z) = \frac{z}{z-1} \sum_{n=0}^{+\infty} \frac{1}{n+1} \cdot \frac{1}{z^n} \\
&= -\frac{z^2}{z-1} \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} \left\{ -\frac{1}{z} \right\}^n = -\frac{z^2}{z-1} \operatorname{Log}\left(1 - \frac{1}{z}\right) \\
&= \frac{z^2}{z-1} \operatorname{Log}\left(\frac{z}{z-1}\right). \qquad \Diamond
\end{aligned}$$

Download free eBooks at bookboon.com

Example 2.5.6 *Find the inverse* $\mathfrak{z}$ *transform of the function* $\exp\left(\frac{1}{z}\right)$, $z \neq 0$.

It follows immediately from the series expansion

$$\exp\left(\frac{1}{z}\right) = \sum_{n=0}^{+\infty} \frac{1}{n!}\,\frac{1}{z^n}, \qquad z \neq 0,$$

that $a_n = \frac{1}{n!}$, hence

$$\mathfrak{z}^{-1}\left\{\exp\left(\frac{1}{z}\right)\right\} = \left(\frac{1}{n!}\right)_{n\in\mathbb{N}_0}. \qquad \Diamond$$

Example 2.5.7 *Find the inverse* $\mathfrak{z}$ *transform of* $\cosh\left(\frac{1}{\sqrt{z}}\right)$.

It follows from the series expansion

$$\cosh\left(\frac{1}{\sqrt{z}}\right) = \sum_{n=0}^{+\infty} \frac{1}{(2n)!}\cdot\frac{1}{z^n}, \qquad \text{for } z \neq 0,$$

where $a_n = \frac{1}{(2n)!}$, so

$$\mathfrak{z}^{-1}\left\{\cosh\left(\frac{1}{\sqrt{z}}\right)\right\} = \left(\frac{1}{(2n)!}\right)_{n\in\mathbb{N}_0}. \qquad \Diamond$$

Example 2.5.8 *Find the inverse* $\mathfrak{z}$ *transform of* $\frac{z+2}{z^4-1}$.

The Laurent series of $\frac{z+2}{z^4-1}$ is for $|z| > 1$,

$$\frac{z+2}{z^4-1} = \frac{z+2}{z^4}\cdot\frac{1}{1-\frac{1}{z^4}} = \frac{z+2}{z^4}\sum_{n=0}^{+\infty}\frac{1}{z^{4n}} = \sum_{n=1}^{+\infty}\frac{z+2}{z^{4n}}.$$

We therefore get

$$\mathfrak{z}^{-1}\left\{\frac{z+2}{z^4-1}\right\} = (a_n)_{n\in\mathbb{N}_0},$$

where

$$a_n = \begin{cases} & \text{for } n = 4p-1, \qquad p \in \mathbb{N}, \\ 2 & \text{for } n = 4p, \qquad p \in \mathbb{N}, \\ 0 & \text{otherwise.} \qquad \Diamond \end{cases}$$

Download free eBooks at bookboon.com

3 Extension of the inversion formula

3.1 The inversion formula for analytic functions with branch cuts

Example 3.1.1 *Find the inverse Laplace transform of* $e^{-\sqrt{z}}$.

Assume that $\Re z > \gamma$. Then $\Re \sqrt{z} > \sqrt{\gamma}$, cf. Figure 16, hence

$$\left|e^{-\sqrt{z}}\right| = e^{-\Re\sqrt{z}} = \frac{1}{e^{\Re\sqrt{z}}} \leq \frac{C}{|\Re z|} \qquad \text{for } \Re z > k.$$

We therefore conclude by the inversion formula that

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{z\,t}\,F(z)\,\mathrm{d}z = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{zt}\,e^{-\sqrt{z}}\,\mathrm{d}z.$$

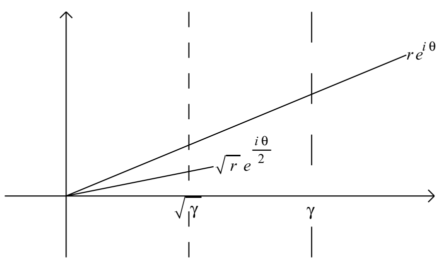


Figure 16: An analysis of the square root in Example 3.1.1.

Then choose the path of integration of Figure 17. It follows from Cauchy's integral theorem that

$$\begin{aligned}
0 &= \frac{1}{2\pi i}\oint_{C_{r,\varepsilon}} e^{zt}\,e^{-\sqrt{z}}\,\mathrm{d}z \\
&= \frac{1}{2\pi i}\int_{\gamma-iT}^{\gamma+iT} e^{zt}e^{-\sqrt{z}}\,\mathrm{d}z + \frac{1}{2\pi i}\int_{\pi}^{-\pi} \exp\left(t\,\varepsilon\,e^{i\Theta}\right)\exp\left(-\sqrt{\varepsilon}\,e^{i\,\frac{\Theta}{2}}\right) i\,\varepsilon\,e^{i\Theta}\,\mathrm{d}\Theta \\
&\quad + \frac{1}{2\pi i}\left\{\int_{\Theta_0(t)}^{\pi} + \int_{-\pi}^{-\Theta_0(r)} \exp\left(t\,r\,e^{i\Theta} - \sqrt{r}\,e^{i\,\frac{\Theta}{2}}\right) i\,r\,e^{i\Theta}\,\mathrm{d}\Theta\right\} \\
&\quad + \frac{1}{2\pi i}\int_{-r}^{-\varepsilon} e^{-i\sqrt{|x|}}\,\mathrm{d}x + \frac{1}{2\pi i}\int_{-\varepsilon}^{-r} e^{xt}\,e^{i\sqrt{|x|}}\,\mathrm{d}x,
\end{aligned}$$

Download free eBooks at bookboon.com

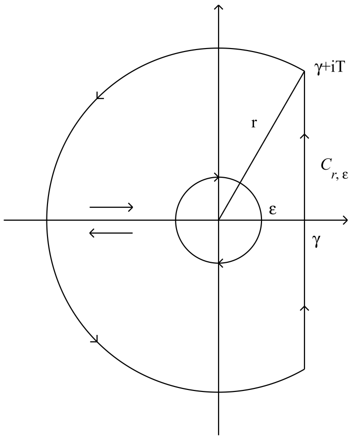


Figure 17: The path of integration in Example 3.1.1.

where $\Theta_0(r) = \text{Arccos}\,\dfrac{\gamma}{r}$. It is obvious that

$$\lim_{\varepsilon\to 0+} \frac{1}{2\pi i}\int_{\pi}^{-\pi} \exp\left(t\,\varepsilon\cdot e^{i\Theta}\right)\exp\left(-\sqrt{\varepsilon}\cdot e^{i\frac{\Theta}{2}}\right) i\,\varepsilon\, e^{i\Theta}\,\mathrm{d}\Theta = 0.$$

Furthermore,

$$\left|\frac{1}{2\pi i}\int_{\text{Arccos}\,\frac{\gamma}{r}}^{\pi} \exp\left(t\,r\,e^{i\Theta} - \sqrt{r}\,e^{i\frac{\Theta}{2}}\right) i\,r\,\mathrm{d}\Theta\right| \le \frac{1}{2\pi}\left|\int_{\text{Arccos}\,\frac{\gamma}{r}}^{\pi} \exp\left(t\,r\,\cos\Theta - \sqrt{r}\,\cos\frac{\Theta}{2}\right) r\,\mathrm{d}\Theta\right|$$

$$\le \frac{1}{2\pi}\exp\left(t\,r\left\{\frac{2}{\left(4t\sqrt{r}\right)^2} - 1\right\} - \sqrt{r}\cdot\frac{1}{4t\sqrt{r}}\right) r\,\pi \le C(t)\cdot r\,e^{-tr} \to 0 \quad \text{for } r\to+\infty \text{ and } t>0,$$

because the maximum is attained for

$$\cos\frac{\Theta_0}{2} = \frac{1}{4t\sqrt{r}},$$

if either $\Theta \in \left[\text{Arccos}\,\dfrac{\gamma}{r}, \pi\right]$, or, if r is large, for $\Theta_0 \in \left[0, \text{Arccos}\,\dfrac{\gamma}{r}\right]$.

Download free eBooks at bookboon.com

We finally get by taking the limit and use a substitution of the variable,

$$\int_{-\infty}^{0} e^{xt}\, e^{-i\sqrt{|x|}}\, \mathrm{d}x + \frac{1}{2\pi i}\int_{0}^{-\infty} e^{xt}\, e^{i\sqrt{|x|}}\, \mathrm{d}x$$

$$= \frac{1}{2\pi i}\int_{0}^{+\infty} e^{-xt}\, e^{-i\sqrt{x}}\, \mathrm{d}x - \frac{1}{2\pi i}\int_{0}^{+\infty} e^{-xt}\, e^{i\sqrt{x}}\, \mathrm{d}x = -\frac{1}{\pi}\int_{0}^{+\infty} e^{-xt}\, \sin\left(\sqrt{x}\right)\, \mathrm{d}x$$

$$= -\frac{1}{\pi}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)!}\int_{0}^{+\infty} e^{-xt}\cdot x^{n+\frac{1}{2}}\, \mathrm{d}x = -\frac{1}{\pi}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)!}\,\mathcal{L}\left\{x^{n+\frac{1}{2}}\right\}(t)$$

$$= -\frac{1}{\pi}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)!}\cdot\frac{\Gamma\left(n+\frac{3}{2}\right)}{t^{n+\frac{3}{2}}} = -\frac{1}{\pi}\cdot\frac{1}{t^{\frac{3}{2}}}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)!}\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)\cdots\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)\cdot\frac{1}{t^n}$$

$$= -\frac{1}{\sqrt{\pi}}\, e^{-\frac{3}{2}}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)!}\cdot\frac{2n{+}1)(2n{-}1)\cdots 1}{2^{n+1}}\cdot\frac{1}{t^n} = -\frac{1}{2\sqrt{\pi}}\, t^{-\frac{3}{2}}\sum_{n=0}^{+\infty}(-1)^n\cdot\frac{1}{2^n\, n!\, 2^n}\cdot\frac{1}{t^n}$$

$$= -\frac{1}{2\sqrt{\pi}}\, t^{-\frac{3}{2}}\sum_{n=0}^{+\infty}\frac{1}{n!}\left\{-\frac{1}{4t}\right\}^n = -\frac{1}{2\sqrt{\pi}}\, t^{-\frac{3}{2}}\,\exp\left(-\frac{1}{4t}\right).$$

We get by taking the limit followed by a rearrangement,

$$\mathcal{L}^{-1}\left\{e^{-\sqrt{z}}\right\}(t) = \frac{1}{2\sqrt{\pi}}\, t^{-\frac{3}{2}}\,\exp\left(-\frac{1}{4t}\right). \qquad \Diamond$$

Click on the ad to read more

Download free eBooks at bookboon.com

Example 3.1.2 *Find the inverse Laplace transform of* $\dfrac{1}{\sqrt{z}}$.

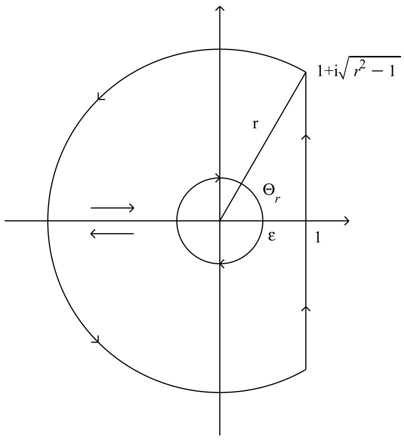


Figure 18: The path of integration in Example 3.1.2.

We choose the path of integration as given on Figure 18, where $r > 1$. Then

$$\begin{aligned}0 \;=\; & \int_{1-i\sqrt{r^2-1}}^{1+i\sqrt{r^2-1}} \frac{e^{zt}}{\sqrt{z}}\,\mathrm{d}z + \int_{\Theta_r}^{\pi} \frac{\exp\left(r\,e^{i\Theta}t\right)}{\sqrt{r}\,\exp\left(i\,\dfrac{\Theta}{2}\right)}\,r\,i\,e^{i\Theta}\,\mathrm{d}\Theta + \int_{-r}^{-\varepsilon} \frac{e^{xt}}{i\sqrt{|x|}}\,\mathrm{d}x \\ & + \int_{\pi}^{-\pi} \frac{\exp\left(\varepsilon\,e^{i\Theta}t\right)}{\sqrt{\varepsilon}\,\exp\left(i\,\frac{\Theta}{2}\,t\right)}\,\varepsilon\,i\,e^{i\Theta}\,\mathrm{d}\Theta + \int_{-\varepsilon}^{-r} \frac{e^{xt}}{-i\sqrt{|x|}}\,\mathrm{d}x + \int_{-\pi}^{\Theta_r} \frac{\exp\left(r\,e^{i\Theta}t\right)}{\sqrt{r}\,\exp\left(i\,\frac{\Theta}{2}\right)}\cdot r\,i\,e^{i\Theta}\,\mathrm{d}\Theta.\end{aligned}$$

We conclude for every fixed $t \geq 0$ that

$$\left| i\sqrt{\varepsilon}\int_{-\pi}^{\pi} \exp\left(\varepsilon\,e^{i\Theta}\,t\right)\exp\left(i\,\frac{\Theta}{2}\right)\,\mathrm{d}\Theta\right| \leq \sqrt{\varepsilon}\cdot e^{\varepsilon t}\cdot 2\pi \to 0 \qquad \text{for } \varepsilon \to 0+\,.$$

Furthermore, for $t > 0$,

$$\begin{aligned} & 2i\int_{\varepsilon}^{r} \frac{e^{-xt}}{\sqrt{x}}\,\mathrm{d}x = 4i\int_{\sqrt{\varepsilon}}^{\sqrt{r}} \exp\left(-y^2\,t\right)\,\mathrm{d}y = \frac{4i}{\sqrt{t}}\int_{\sqrt{\varepsilon t}}^{\sqrt{rt}} \exp\left(-u^2\right)\,\mathrm{d}u \\ & \qquad \to \frac{4i}{\sqrt{t}}\int_{0}^{+\infty} \exp\left(-u^2\right)\,\mathrm{d}u = \frac{4i}{\sqrt{t}}\cdot\frac{\sqrt{\pi}}{2} = \frac{2i\pi}{\sqrt{\pi t}} \qquad \text{for } \varepsilon \to 0+ \text{ and } r \to +\infty.\end{aligned}$$

Finally,

$$\left| -i\sqrt{r}\int_{\Theta_r}^{\pi} \exp\left(r\,e^{i\Theta}\,t\right)\exp\left(i\,\frac{\Theta}{2}\right)\,\mathrm{d}\Theta\right| = \sqrt{r}\left|\int_{\mathrm{Arccos}\left(\frac{1}{r}\right)}^{\pi} e^{rt\cdot\cos\Theta}\exp\left(rt\cdot\sin\Theta + \frac{\Theta}{2}\right)\,\mathrm{d}\Theta\right|,$$

where

$$e^{rt\cdot\cos\Theta} \leq \frac{1}{r} \qquad \text{for } r\cdot\cos\Theta \leq -\log r,$$

Download free eBooks at bookboon.com

thus for

$$\cos\Theta \leq -\frac{\log r}{r\cdot t}.$$

Hence,

$$\sqrt{r}\left|\int_{\operatorname{Arccos}\left(-\frac{\log r}{r\cdot t}\right)}^{\pi} e^{rt\cdot\cos\Theta}\exp\left(i\left(rt\,\sin\Theta+\frac{\Theta}{2}\right)\right)\,\mathrm{d}\Theta\right| \leq \frac{\pi}{2}\cdot\frac{1}{\sqrt{r}}\to 0 \qquad \text{for } r\to+\infty,$$

so it only remains to estimate

$$\begin{aligned}
&\sqrt{r}\left|\int_{\operatorname{Arccos}\left(\frac{1}{r}\right)}^{\operatorname{Arccos}\left(-\frac{\log r}{r\cdot t}\right)} e^{it\,\cos\Theta}\exp\left(i\left(rt\,\sin\Theta+\frac{\Theta}{2}\right)\right)\,\mathrm{d}\Theta\right| \\
&\quad\leq \sqrt{r}\cdot\exp\left(rt\,\cos\left(\operatorname{Arccos}\left(\frac{1}{r}\right)\right)\right)\left\{\operatorname{Arccos}\left(-\frac{\log r}{r\,t}\right)-\operatorname{Arccos}\left(\frac{1}{r}\right)\right\} \\
&\quad\leq \sqrt{r}\,e^{t}\int_{-\frac{\log r}{r\,t}}^{\frac{1}{r}}\frac{\mathrm{d}x}{\sqrt{1-x^2}}\leq e^{t}\sqrt{r}\left(\frac{1}{r}+\frac{\log r}{r\,t}\right)\cdot\frac{1}{\sqrt{1-\left\{\dfrac{\log r}{r\,t}\right\}^2}} \\
&\quad= e^{t}\left\{\frac{1}{\sqrt{r}}+\frac{\log r}{t\sqrt{r}}\right\}\cdot\frac{1}{\sqrt{1-\left\{\frac{\log r}{r\,t}\right\}^2}}\to 0 \qquad \text{for } r\to+\infty.
\end{aligned}$$

Summing up, we obtain by taking the limits $r\to+\infty$ and $\varepsilon\to 0+$ that

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{z}}\right\}(t) &= \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\frac{e^{zt}}{\sqrt{z}}\,\mathrm{d}z=\frac{1}{2\pi i}\lim_{r\to+\infty}\int_{1-i\sqrt{r^2-1}}^{1+\sqrt{r^2-1}}\frac{e^{zt}}{\sqrt{z}}\,\mathrm{d}z \\
&= 0+\frac{1}{\sqrt{\pi\,t}}+0=\frac{1}{\sqrt{\pi}}\cdot\frac{1}{\sqrt{t}}.
\end{aligned}$$

Notice that the main contributions to the value of the integral come from the integrations along either side of the branch cut. ♢

Example 3.1.3 *Prove that the inverse Laplace transform of* $\dfrac{1}{z\sqrt{z+1}}$ *is* $\operatorname{erf}\left\{\sqrt{t}\right\}$.

We see immediately that

$$\left|\frac{1}{z\sqrt{z+1}}\right|\leq\frac{\text{const.}}{|z|^{\frac{3}{2}}} \qquad \text{for } \Re\,z>2,$$

thus

$$\mathcal{L}^{-1}\left\{\frac{1}{z\sqrt{z+1}}\right\}(t)=\frac{1}{2\pi i}\int_{i-i\infty}^{1+i\infty}\frac{e^{zt}}{z\sqrt{z+1}}\,\mathrm{d}z.$$

Download free eBooks at bookboon.com

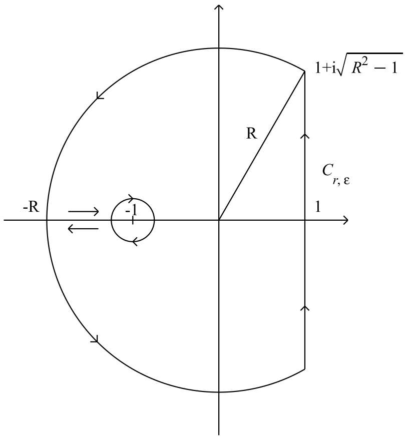


Figure 19: The path of integration in Example 3.1.3.

We choose the path of integration indicated on Figure 19, where $R > 1$ and $0 < \varepsilon < 1$, where ε is the radius of the circle surrounding -1. Then, by the residuum theorem,

$$\frac{1}{2\pi i}\oint_{C_{R,\varepsilon}} \frac{e^{zt}}{z\sqrt{z+1}}\,\mathrm{d}z = \mathrm{res}\left(\frac{e^{zt}}{z\sqrt{z+1}};0\right) = 1,$$

and therefore,

$$\begin{aligned} 1 \;=\; & \frac{1}{2\pi i}\int_{1-i\sqrt{R^2-1}}^{1+i\sqrt{R^2-1}} \frac{e^{zt}}{z\sqrt{z+1}}\,\mathrm{d}z + \frac{1}{2\pi i}\int_{\Theta_R}^{\pi} \frac{\exp\left(R\,e^{i\Theta}\,t\right)}{R\,e^{i\Theta}\sqrt{R\,e^{i\Theta}+1}}\,R\cdot i\,e^{i\Theta}\,\mathrm{d}\Theta \\ & + \frac{1}{2\pi i}\int_{-R}^{-1-\varepsilon} \frac{e^{xt}}{x\,i\sqrt{|x|-1}}\,\mathrm{d}x + \frac{1}{2\pi i}\int_{\pi}^{-\pi} \frac{\exp\left(\left(-1+\varepsilon\,e^{i\Theta}\right)t\right)}{\left(-1+\varepsilon\,e^{i\Theta}\right)\sqrt{\varepsilon}\,\exp\left(i\,\frac{\Theta}{2}\right)}\cdot\varepsilon\cdot i\,e^{i\Theta}\,\mathrm{d}\Theta \\ & + \frac{1}{2\pi i}\int_{-1-\varepsilon}^{-R} \frac{e^{xt}}{x(-i)\sqrt{|x|-1}}\,\mathrm{d}x + \frac{1}{2\pi i}\int_{-\pi}^{-\Theta_R} \frac{\exp\left(R\,e^{i\Theta}\,t\right)}{R\,e^{i\Theta}\sqrt{R\,e^{i\Theta}+1}}\cdot R\,i\,e^{i\Theta}\,\mathrm{d}\Theta. \end{aligned}$$

Here,

$$\frac{1}{2\pi i}\int_{1-i\sqrt{R^2-1}}^{1+i\sqrt{R^2-1}} \frac{e^{zt}}{z\sqrt{z+1}}\,\mathrm{d}z \to \mathcal{L}^{-1}\left\{\frac{1}{z\sqrt{z+1}}\right\}(t) \qquad \text{for } R\to+\infty,$$

and

$$\frac{1}{2\pi i}\int_{\pi}^{-\pi} \frac{\exp\left(\left(-1+\varepsilon\,e^{i\Theta}\right)t\right)}{\left(-1+\varepsilon\,e^{i\Theta}\right)\sqrt{\varepsilon}\,\exp\left(i\,\frac{\Theta}{2}\right)}\cdot\varepsilon\,i\,e^{i\Theta}\,\mathrm{d}\Theta \to 0 \qquad \text{for } \varepsilon\to 0+\,.$$

Furthermore, $\cos\Theta_R = \dfrac{1}{R}$, hence

$$\left|\frac{1}{2\pi i}\int_{\Theta_R}^{\pi} \frac{e^{R\,\cos\Theta\cdot t+i\,R\,\sin\Theta\cdot t}}{R\,e^{i\Theta}\sqrt{R\,e^{i\Theta}+1}}\cdot R\,i\,e^{i\Theta}\right| \le \frac{1}{2\pi}\int_0^{\pi}\frac{e^t}{\sqrt{R-1}}\,\mathrm{d}\Theta \to 0 \qquad \text{for } R\to+\infty,$$

Download free eBooks at bookboon.com

and analogously for the conjugated integral.

Finally,

$$\frac{1}{2\pi i}\int_{-R}^{-1-\varepsilon}\frac{e^{xt}}{xi\sqrt{|x|-1}}\,\mathrm{d}x+\frac{1}{2\pi i}\int_{-1-\varepsilon}^{-R}\frac{e^{xt}}{x(-i)\sqrt{|x|-1}}\,\mathrm{d}t=\frac{1}{\pi}(-1)\int_{R}^{1+\varepsilon}\frac{e^{-yt}}{-y\sqrt{y-1}}(-1)\,\mathrm{d}y$$
$$=\frac{1}{\pi}\int_{1+\varepsilon}^{R}\frac{e^{-yt}}{y\sqrt{y-1}}\,\mathrm{d}y\to\frac{1}{\pi}\int_{1}^{+\infty}\frac{e^{-yt}}{y\sqrt{y-1}}\,\mathrm{d}y=\frac{1}{\pi}\int_{0}^{+\infty}\frac{e^{-(y+1)t}}{(y+1)\sqrt{y}}\,\mathrm{d}y=\frac{e^{-t}}{\pi}\int_{0}^{+\infty}\frac{e^{-yt}}{(y+1)\sqrt{y}}\,\mathrm{d}y$$
$$=\frac{2}{\pi}e^{-t}\int_{0}^{+\infty}\frac{\exp\left(-x^2t\right)}{x^2+1}\,\mathrm{d}x=\frac{2}{\pi}e^{-t}e^{-t}\cdot\frac{\pi}{2}e^{t}\operatorname{erfc}\left(\sqrt{t}\right)=\operatorname{erfc}\left(\sqrt{t}\right),$$

for $\varepsilon \to 0+$ and $R \to +\infty$, where we have applied an example from *Ventus, Complex Functions Theory a-6, The Laplace Transformation.*

Summing up we get by taking these limits,

$$1=\mathcal{L}^{-1}\left\{\frac{1}{z\sqrt{z+1}}\right\}(t)+\operatorname{erfc}\left(\sqrt{t}\right),$$

hence by a rearrangement,

$$\mathcal{L}^{-1}\left\{\frac{1}{z\sqrt{z+1}}\right\}(t)=1-\operatorname{erfc}\left(\sqrt{t}\right)=\operatorname{erf}\left(\sqrt{t}\right).\qquad\Diamond$$

Click on the ad to read more

Download free eBooks at bookboon.com

Example 3.1.4 *Find the inverse Laplace transform of* $\dfrac{\sqrt{z}}{z-1}$.

If $\Re z > 1$, we get by a Laurent series expansion

$$\begin{aligned}\frac{\sqrt{z}}{z-1} &= \frac{1}{\sqrt{z}} \cdot \frac{1}{1-\frac{1}{z}} = \frac{1}{\sqrt{z}} \sum_{n=0}^{+\infty} \frac{1}{z^n} = \sum_{n=0}^{+\infty} \frac{1}{z^{n+\frac{1}{2}}} = \sum_{n=0}^{+\infty} \frac{1}{\Gamma\left(n+\frac{1}{2}\right)} \cdot \frac{\Gamma\left(n+\frac{1}{2}\right)}{z^{n+\frac{1}{2}}} \\ &= \sum_{n=0}^{+\infty} \frac{1}{\Gamma\left(\frac{2n+1}{2}\right)} \mathcal{L}\left\{t^{n-\frac{1}{2}}\right\}(z),\end{aligned}$$

hence by the inverse Laplace transformation,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{\sqrt{z}}{z-1}\right\}(t) &= \sum_{n=0}^{+\infty} \frac{1}{\Gamma\left(\frac{2n+1}{2}\right)} t^{n-\frac{1}{2}} = \frac{1}{\sqrt{t}} \sum_{n=0}^{+\infty} \frac{1}{\frac{2n+1}{2} \cdot \frac{2n-1}{2} \cdots \frac{1}{2}\sqrt{\pi}} t^n \\ &= \frac{1}{\sqrt{t}} \sum_{n=0}^{+\infty} \frac{2^{n+1}}{\sqrt{\pi}} \cdot \frac{2^n \cdot n!}{(2n+1)!} t^n = \frac{2}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{t}} \sum_{n=0}^{+\infty} \frac{n!}{(2n+1)!} (4t)^n. \qquad \Diamond\end{aligned}$$

Example 3.1.5 *Find the inverse Laplace transform of* $\operatorname{Log}\left(1+\dfrac{1}{z}\right)$ *by choosing a convenient path of integration.*

Is it possible to find the inverse Laplace transform by using more simple methods?

The estimate

$$\left|\operatorname{Log}\left(1+\frac{1}{z}\right)\right| \le \frac{C}{|z|} \qquad \text{for } |z| \ge 2,$$

shows that $\operatorname{Log}\left(1+\dfrac{1}{z}\right)$ satisfies the necessary (and also sufficient) estimate for the existence of the inverse Laplace transform.

In this case we have a branch cut along the interval $[-1, 0]$ on the real axis, so we choose the path of integration as indicated on Figure 20. Then we get

$$\begin{aligned}0 &= \frac{1}{2\pi i} \oint_{C_{r,\varepsilon}} \operatorname{Log}\left(1+\frac{1}{z}\right) e^{zt}\,\mathrm{d}z = \frac{1}{2\pi i} \int_{2-i\sqrt{r^2-4}}^{2+i\sqrt{r^2-4}} \operatorname{Log}\left(1+\frac{1}{z}\right) e^{zt}\,\mathrm{d}t \\ &\quad + \frac{1}{2\pi i} \int_{\operatorname{Arccos}\frac{2}{r}}^{2\pi-\operatorname{Arccos}\frac{2}{r}} \operatorname{Log}\left(1+\frac{1}{r\,e^{i\Theta}}\right) \cdot e^{rt(\cos\Theta+i\,\sin\Theta)} i\,r\,e^{i\Theta}\,\mathrm{d}\Theta \\ &\quad + \frac{1}{2\pi i} \int_{2\pi} 0 \operatorname{Log}\left(1+\frac{1}{-1+\varepsilon\,e^{i\Theta}}\right) \exp\left(t\left(-1+\varepsilon\,e^{i\Theta}\right)\right)\, i\,\varepsilon\,e^{i\Theta}\,\mathrm{d}\Theta \\ &\quad + \frac{1}{2\pi i} \int_{\pi}^{-\pi} \operatorname{Log}\left(1+\frac{1}{\varepsilon\,e^{i\Theta}}\right) \exp(t\,\varepsilon\,e^{i\Theta}) \cdot i\,\varepsilon\,e^{i\Theta}\,\mathrm{d}\Theta + \frac{1}{2\pi i} \int_{-1+\varepsilon}^{-\varepsilon} \left\{\ln\left|1+\frac{1}{x}\right| - i\pi\right\} e^{tx}\,\mathrm{d}x \\ &\quad - \frac{1}{2\pi} \int_{-1+\varepsilon}^{-\varepsilon} \left\{\ln\left|1+\frac{1}{x}\right| + i\pi\right\} e^{tx}\,\mathrm{d}x.\end{aligned}$$

Download free eBooks at bookboon.com

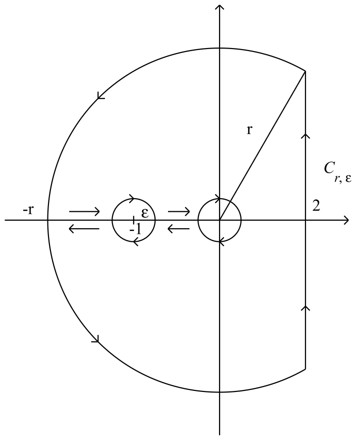


Figure 20: The path of integration in Example 3.1.5.

The first line integral on the right hand side converges towards

$$f(t) = \mathcal{L}^{-1}\left\{\text{Log}\left(1+\frac{1}{z}\right)\right\}(t)$$

for $r \to +\infty$. Concerning the second integral we have the following estimate of the integrand, when $r > 2$,

$$\left|\frac{1}{2\pi i}\,\text{Log}\left(1+\frac{1}{r\,e^{i\Theta}}\right)e^{rt(\cos\Theta + i\sin\theta)}\,i\,r\,e^{i\Theta}\right| \le \frac{1}{2\pi}\cdot\frac{C}{r}\,e^{rt\cos\Theta}\cdot r = \frac{C}{2\pi}\,e^{tr\,\cos\Theta},$$

so the integral is estimated in the following way,

$$2\,\frac{C}{2\pi}\int_{\text{Arccos}\,\frac{2}{r}}^{\text{Arccos}\left(-\frac{\ln r}{rt}\right)} e^{tr\cdot\frac{2}{r}}\,\mathrm{d}\Theta + 2\,\frac{C}{2\pi}\int_{\text{Arccos}\left(-\frac{\ln r}{rt}\right)}^{\pi}\exp\left(tr\left(-\frac{\ln r}{rt}\right)\right)\mathrm{d}\Theta$$
$$\le \frac{C}{\pi}\,e^{2t}\cdot\left\{\text{Arccos}\left(-\frac{\ln r}{rt}\right) - \text{Arccos}\left(\frac{2}{r}\right)\right\} + \frac{C}{\pi}\cdot\pi\cdot\frac{1}{r} \to 0 \qquad \text{for } r \to +\infty.$$

The next two integrals both tend towards 0 for $\varepsilon \to 0+$, because $\varepsilon\,\ln\varepsilon \to 0$ for $\varepsilon \to 0+$.

Considering the remaining two integrals we get

$$\frac{1}{2\pi i}\int_{-1+\varepsilon}^{-\varepsilon}\left\{\ln\left|1+\frac{1}{x}\right| - i\pi\right\}e^{tx}\,\mathrm{d}x - \frac{1}{2\pi i}\int_{-1+\varepsilon}^{-\varepsilon}\left\{\ln\left|1+\frac{1}{x}\right| + i\pi\right\}e^{tx}\,\mathrm{d}x$$
$$= -\int_{-1+\varepsilon}^{-\varepsilon} e^{tx}\,\mathrm{d}x = -\int_{\varepsilon}^{1-\varepsilon} e^{-tx}\,\mathrm{d}x \to -\int_0^1 e^{-tx}\,\mathrm{d}x = \left[\frac{1}{t}\,e^{-tx}\right]_0^1 = -\frac{1}{t}\left(1-e^{-t}\right).$$

We therefore get by taking the limits, followed by a rearrangement,

$$f(t) = \mathcal{L}^{-1}\left\{\text{Log}\left(1+\frac{1}{z}\right)\right\}(t) = +\frac{1}{t}\left(1-e^{-t}\right).$$

Download free eBooks at bookboon.com

An *alternative* approach is to take the Laurent series expansion,

$$\operatorname{Log}\left(1+\frac{1}{z}\right)=\sum_{n=0}^{+\infty}\frac{(-1)^n}{n+1}\,\frac{1}{z^{n+1}}=\sum_{n=0}^{+\infty}\frac{(-1)^n}{(n+1)!}\cdot\frac{n!}{z^{n+1}}=\sum_{n=0}^{+\infty}\frac{1}{(n+1)!}\,\mathcal{L}\left\{(-t)^n\right\}(z),$$

hence by the inverse Laplace transformation,

$$f(t)=\mathcal{L}^{-1}\left\{\operatorname{Log}\left(1+\frac{1}{z}\right)\right\}(t)=\sum_{n=0}^{+\infty}\frac{1}{(n+1)!}\,(-t)^n=-\frac{1}{t}\sum_{n=1}^{+\infty}\frac{1}{n!}\,(-t)^n=\frac{1}{t}\left(1-e^{-t}\right).$$

Another *alternative* is the following proof,

$$\mathcal{L}\{t\,f\}(z)=-\frac{d}{dz}\operatorname{Log}\left(1+\frac{1}{z}\right)=\frac{d}{dz}\operatorname{Log}\left(\frac{z}{1+z}\right)=\frac{1}{z}-\frac{1}{1+z}=\mathcal{L}\left\{1-e^{-t}\right\}(z),$$

from which we conclude that

$$t\,f(t)=1-e^{-t},$$

thus

$$f(t)=\frac{1}{t}\left(1-e^{-t}\right).$$

Notice that $\lim_{t\to 0+}\dfrac{1}{t}\left(1-e^{-t}\right)=1$. $\Diamond$

Example 3.1.6 *Compute the inverse Laplace transform of* $\operatorname{Log}\left(1+\dfrac{1}{z^2}\right)$ *by a Bromwich integral. Then find an alternative and simpler proof, using only elementary methods.*

First method First, the estimate

$$\left|\operatorname{Log}\left(1+\frac{1}{z^2}\right)\right|\le\frac{C}{|z|^2}\qquad\text{for }|z|\ge 2,$$

shows that the inverse Laplace transform exists and it is given by

$$f(t)=\mathcal{L}^{-1}\left\{\operatorname{Log}\left(1+\frac{1}{z^2}\right)\right\}(t)=\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\operatorname{Log}\left(1+\frac{1}{z^2}\right)e^{zt}\,dz.$$

The branch cut can in this case be chosen as the line segment on the imaginary axis from $-i$ to i, cutting through the third singularity at 0. We therefore choose the path of integration as indicated

Download free eBooks at bookboon.com

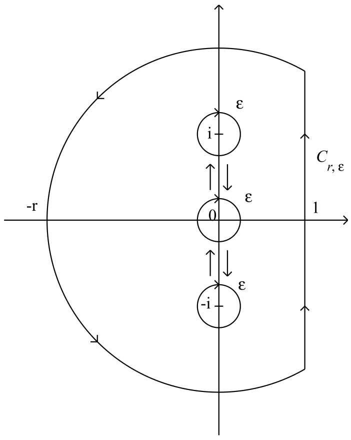


Figure 21: The path of integration in Example 3.1.6.

on Figure 21. Then

$$\begin{aligned}
0 \; = \;& \frac{1}{2\pi i}\oint_{C_{r,\varepsilon}} \operatorname{Log}\left(1+\frac{1}{z^2}\right) e^{zt}\,\mathrm{d}z = \frac{1}{2\pi i}\int_{1-i\sqrt{r^2-1}}^{1+i\sqrt{r^2-1}} \operatorname{Log}\left(1+\frac{1}{z^2}\right) e^{zt}\,\mathrm{d}z \\
&+\frac{1}{2\pi i}\int_{\text{Arccos}\,\frac{1}{r}}^{2\pi-\text{Arccos}\,\frac{1}{r}} \operatorname{Log}\left(1+\frac{1}{r^2\,e^{2i\Theta}}\right)\exp\left(tr\,e^{i\Theta}\right)\cdot i\,r\,e^{i\Theta}\,\mathrm{d}\Theta \\
&+\frac{1}{2\pi i}\int_{\frac{3\pi}{2}}^{-\frac{\pi}{2}} \operatorname{Log}\left(1+\frac{1}{\left(i+\varepsilon\,e^{i\Theta}\right)^2}\right)\exp\left(t\left(i+\varepsilon\,e^{i\Theta}\right)\right) i\,\varepsilon\,e^{i\Theta}\,\mathrm{d}\Theta \\
&+\frac{1}{2\pi i}\int_{\frac{\pi}{2}}^{-\frac{3\pi}{2}} \operatorname{Log}\left(1+\frac{1}{\left(-i+\varepsilon\,e^{i\Theta}\right)^2}\right)\exp\left(t\left(-i+\varepsilon\,e^{i\Theta}\right)\right) i\,\varepsilon\,e^{i\Theta}\,\mathrm{d}\Theta \\
&+\frac{1}{2\pi i}\int_{-\frac{\pi}{2}}^{-\frac{3\pi}{2}} \operatorname{Log}\left(1+\frac{1}{\varepsilon^2\,e^{2i\Theta}}\right)\exp\left(t\varepsilon\,e^{i\Theta}\right)\cdot i\,\varepsilon\,e^{i\Theta}\,\mathrm{d}\Theta \\
&+\frac{1}{2\pi i}\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \operatorname{Log}\left(1+\frac{1}{\varepsilon^2\,e^{2i\Theta}}\right)\exp\left(t\varepsilon\,e^{i\Theta}\right)\cdot i\,\varepsilon\,e^{i\Theta}\,\mathrm{d}\Theta \\
&+\frac{1}{2\pi i}\int_{\varepsilon}^{1-\varepsilon}\left\{\ln\left|1-\frac{1}{y^2}\right|+i\pi\right\} e^{ity}\,i\,\mathrm{d}y - \frac{1}{2\pi i}\int_{\varepsilon}^{1-\varepsilon}\left\{\ln\left|1-\frac{1}{y^2}\right|-i\pi\right\} e^{ity}\,i\,\mathrm{d}y \\
&+\frac{1}{2\pi i}\int_{-1-\varepsilon}^{-\varepsilon}\left\{\left|1-\frac{1}{y^2}\right|-i\pi\right\} e^{ity}\,i\,\mathrm{d}y - \frac{1}{2\pi i}\int_{-1-\varepsilon}^{-\varepsilon}\left\{\ln\left|1-\frac{1}{y^2}\right|+i\pi\right\} e^{ity}\,i\,\mathrm{d}y.
\end{aligned}$$

The first term tends towards $f(t)$ for $r \to +\infty$.

The next term tends towards 0 for $r \to +\infty$, because we have the estimate

$$\left|\frac{1}{2\pi i}\operatorname{Log}\left(1+\frac{1}{r^2\,e^{2i\Theta}}\right)\exp\left(tr\,e^{i\Theta}\right) i\,r\,e^{i\Theta}\right| \le \frac{1}{2\pi}\cdot\frac{C}{r^2}\,e^{t\cdot r\,\cos\Theta}\cdot r,$$

Download free eBooks at bookboon.com

because $r \cdot \Theta \leq 1$ for $\Theta \in \left[\operatorname{Arccos}\dfrac{1}{r}, 2\pi - \operatorname{Arccos}\dfrac{1}{r}\right]$.

The next four terms tend towards 0 for $\varepsilon \to 0+$, because $\varepsilon \ln \varepsilon \to 0$, when $\varepsilon \to 0+$.

The last four terms are reduced to

$$
\begin{aligned}
i \int_{\varepsilon}^{1-\varepsilon} e^{ity}\, dy - i \int_{-1-\varepsilon}^{-\varepsilon} e^{ity}\, dy &= \frac{1}{t}\left[e^{ity}\right]_{\varepsilon}^{1-\varepsilon} - \frac{1}{t}\left[e^{ity}\right]_{-1-\varepsilon}^{-\varepsilon} \\
&\to \frac{1}{t}\left\{e^{it} - 1 - 1 + e^{-it}\right\} = -\frac{2}{t}(1 - \cos t) \qquad \text{for } \varepsilon \to 0+.
\end{aligned}
$$

Finally, taking the limits followed by a rearrangement,

$$
\mathcal{L}^{-1}\left\{\operatorname{Log}\left(1 + \frac{1}{z^2}\right)\right\}(t) = \frac{2}{t}(1 - \cos t).
$$

Click on the ad to read more

Download free eBooks at bookboon.com

Second method Write for short, $f(t)=\mathcal{L}^{-1}\left\{\operatorname{Log}\left(1+\frac{1}{z^2}\right)\right\}(t)$. Then

$$\mathcal{L}\{t\,f(t)\}(z)=-\frac{d}{dz}\operatorname{Log}\left(\frac{z^2+1}{z^2}\right)=\frac{d}{dz}\operatorname{Log}\left(\frac{z^2}{z^2+1}\right)=\frac{2}{z}-\frac{2z}{z^2+1},$$

thus

$$t\,f(t)=2-2\cos t=2(1-\cos t),$$

and hence

$$f(t)=2\cdot\frac{1-\cos t}{t}.$$

Third method By a Laurent series expansion,

$$\begin{aligned}\operatorname{Log}\left(1+\frac{1}{z^2}\right) &= \sum_{n=0}^{+\infty}\frac{(-1)^n}{n+1}\cdot\frac{1}{z^{2n+1+1}}=2\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)(2n+1)!}\cdot\frac{(2n+1)!}{z^{2n+1+1}}\\ &= 2\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+2)!}\,\mathcal{L}\left\{t^{2n+1}\right\}(z),\end{aligned}$$

hence

$$\begin{aligned}\mathcal{L}^{-1}\left\{\operatorname{Log}\left(1+\frac{1}{z^2}\right)\right\}(t) &= 2\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+2)!}\,t^{2n+1}=-\frac{2}{t}\sum_{n=1}^{+\infty}\frac{(-1)^n}{(2n)!}\,t^{2n}\\ &= \frac{2}{t}\,(1-\cos t). \qquad \Diamond\end{aligned}$$

Download free eBooks at bookboon.com

3.2 The inversion formula for functions with infinitely many singularities

Example 3.2.1 *Find the inverse Laplace transform of* $\dfrac{1}{z\left(e^z+1\right)}$.

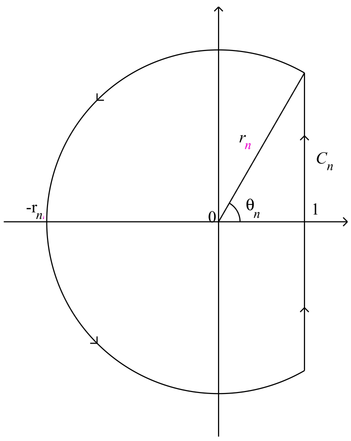


Figure 22: The path of integration in Example 3.2.1.

Clearly,

$$\left|\frac{1}{z\left(e^z+1\right)}\right| \le \frac{C}{|z|} \qquad \text{for } \Re\, z > 1, \text{ where } C = \frac{1}{e-1}.$$

Choose $r_n = 2n\pi$, $n \in \mathbb{N}$, and let C_n denote the curve on Figure 22. The singularities inside C_n are

$$z_0 = 0 \qquad \text{and} \qquad z_p = i(2p+1)\pi, \quad p = -n, \dots, n-1,$$

thus

$$\begin{aligned}\frac{1}{2\pi i}\oint_{C_n}\frac{e^{zt}}{z\left(e^z+1\right)}\,\mathrm{d}z &= \operatorname{res}\left(\frac{e^{zt}}{z\left(e^z+1\right)};0\right) + \sum_{p=-n}^{n-1}\operatorname{res}\left(\frac{e^{zt}}{z\left(e^z+1\right)};z_p\right) = \frac{1}{2} + \sum_{p=-n}^{n-1}\frac{e^{z_p t}}{z_p\, e^{z_p}} \\ &= \frac{1}{2} + \sum_{p=-n}^{n-1}\frac{e^{(2p+1)i\pi t}}{i(2p+1)\pi\cdot(-1)} = \frac{1}{2} - \frac{2}{\pi}\sum_{p=0}^{n-1}\frac{1}{2p+1}\,\sin(2p+1)\pi t.\end{aligned}$$

On the other hand,

$$\frac{1}{2\pi i}\oint_{C_n}\frac{e^{zt}}{z\left(e^z+1\right)}\,\mathrm{d}z = \frac{1}{2\pi i}\int_{1-i\sqrt{r_n^2-1}}^{1+i\sqrt{r_n^2-1}}\frac{e^{zt}}{z\left(e^z+1\right)}\,\mathrm{d}z + \frac{1}{2\pi i}\int_{\Theta_n}^{2\pi-\Theta_n}\frac{\exp\left(r_n\, e^{i\Theta}t\right) r_n\, i\, e^{i\Theta}}{r_n\, e^{i\Theta}\left(1+\exp\left(r_n e^{i\Theta}\right)\right)}\,\mathrm{d}\Theta.$$

Download free eBooks at bookboon.com

It follows from $\Theta_n = \operatorname{Arccos}\dfrac{1}{r_n}$ that

$$\left|\frac{1}{2\pi i}\int_{\Theta_n}^{2\pi-\Theta_n}\frac{\exp\left(r_n e^{i\Theta}t\right) r_n\, i\, e^{i\Theta}}{r_n e^{i\Theta}\left(1+\exp\left(r_n e^{i\Theta}\right)\right)}\,d\Theta\right| \le \frac{1}{2\pi}\cdot 2\int_{\Theta_n}^{\pi}\frac{e^{r_n t\cos\Theta}}{\left|1+\exp\left(r_n e^{i\Theta}\right)\right|}\,d\Theta$$

$$\le \frac{1}{\pi}\cdot C\int_{\operatorname{Arccos}\left(\frac{1}{r_n}\right)}^{\operatorname{Arccos}\left(-\frac{\ln r_n}{t\,r_n}\right)} e^t\,d\Theta + \frac{1}{\pi}\cdot C\int_{\frac{\pi}{2}}^{\pi} e^{-\ln r_n}\,d\Theta$$

$$= \frac{C}{\pi}\left(e^t\left\{\operatorname{Arccos}\left(-\frac{\ln r_n}{t\,r_n}\right) - \operatorname{Arccos}\left(\frac{1}{r_n}\right)\right\} + \frac{\pi}{2}\cdot\frac{1}{r_n}\right) \to 0 \qquad \text{for } n \to +\infty,$$

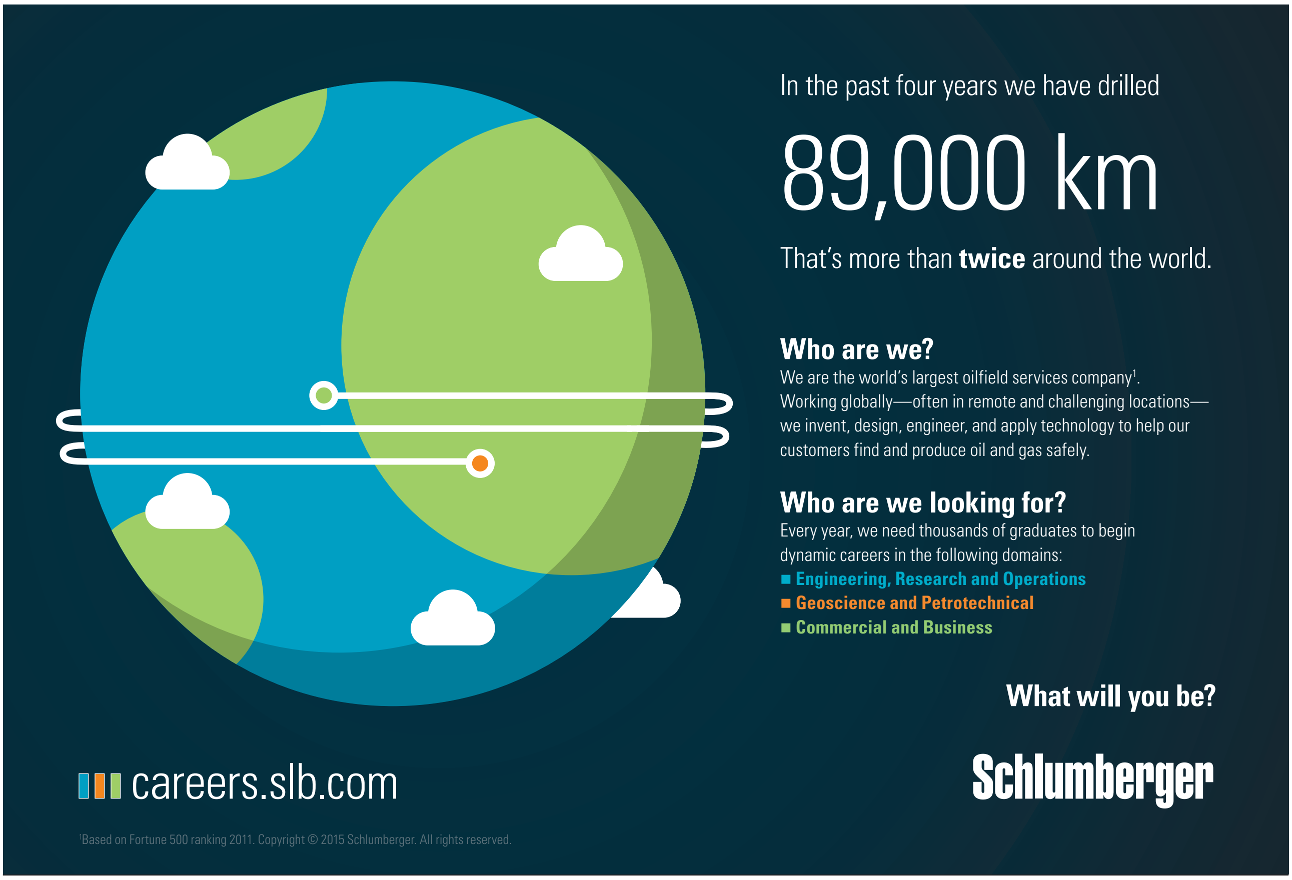
thus it follows by taking the limit $n \to +\infty$ that

$$\mathcal{L}^{-1}\left\{\frac{1}{z\left(e^z+1\right)}\right\} = \lim_{n\to+\infty}\frac{1}{2\pi i}\oint_{C_n}\frac{e^{zt}}{z\left(e^z+1\right)}\,dz$$

$$= \frac{1}{2} - \frac{2}{\pi}\sum_{p=0}^{+\infty}\frac{1}{2p+1}\sin(2p+1)\pi t.$$

Remark 3.2.1 With some knowledge of known Fourier series this expression can be reduced to

$$\mathcal{L}^{-1}\left\{\frac{1}{z\left(e^z+1\right)}\right\} = \frac{1}{2}\left\{1+(-1)^{n+1}\right\} \qquad \text{for } t \in \,]n, n+1[, \text{ where } n \in \mathbb{N}_0. \qquad \Diamond$$



Click on the ad to read more

Download free eBooks at bookboon.com

Example 3.2.2 *Prove that the inverse Laplace transform of* $\dfrac{1}{z\,\cosh z}$ *can be expressed as a Fourier series, and then find this Fourier series.*

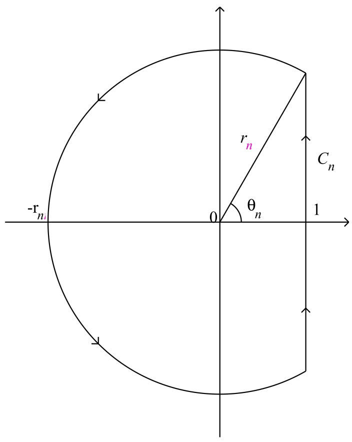


Figure 23: The path of integration in Example 3.2.2.

First find the singularities of $\dfrac{1}{z\,\cosh z}$. These are the poles

$$z=0 \qquad \text{and} \qquad z=i\left(\frac{1}{2}+n\right)\pi, \quad \text{for } n\in\mathbb{Z}.$$

We choose $r_n=n\pi$, $n\in\mathbb{N}$, and then the path of integration in Figure 23. Then

$$\frac{1}{2\pi i}\oint_{C_n}\frac{e^{zt}}{z\,\cosh z}\,\mathrm{d}z=\frac{1}{2\pi i}\int_{1-i\sqrt{r_n^2-1}}^{1+i\sqrt{r_n^2-1}}\frac{e^{zt}}{z\,\cosh z}\,\mathrm{d}z+\frac{1}{2\pi i}\int_{\mathrm{Arccos}\left(\frac{1}{n\pi}\right)}^{2\pi-\mathrm{Arccos}\left(\frac{1}{n\pi}\right)}\frac{\exp\left(r_n\,e^{i\Theta}\,t\right)\,i\,r_n\,e^{i\Theta}}{r_n\,e^{i\Theta}\cosh\left(r_n\,e^{i\Theta}\right)}\,\mathrm{d}\Theta.$$

The former term on the right hand side converges towards

$$f(t)=\mathcal{L}^{-1}\left\{\frac{1}{z\,\cosh z}\right\}(t) \qquad \text{for } n\to+\infty,$$

because

$$\left|\frac{1}{z\,\cosh z}\right|\le\frac{C}{|z|} \qquad \text{for } \Re\,z\ge 1.$$

Download free eBooks at bookboon.com

The latter term on the right hand side is estimated in the following way,

$$\begin{aligned}
&\left|\frac{1}{2\pi i}\int_{\text{Arccos}\left(\frac{1}{n\pi}\right)}^{2\pi-\text{Arccos}\left(\frac{1}{n\pi}\right)} \frac{\exp\left(n\pi\, e^{i\Theta}t\right)\, in\pi e^{i\Theta}}{n\pi e^{i\Theta}\cosh\left(n\pi e^{i\Theta}\right)}\,\mathrm{d}\Theta\right| \le \frac{C}{2\pi}\cdot 2\int_{\text{Arccos}\left(\frac{1}{n\pi}\right)}^{\pi} e^{tn\pi\cos\Theta}\,\mathrm{d}\Theta\\
&= \frac{C}{\pi}\left\{\int_{\text{Arccos}\left(\frac{1}{n\pi}\right)}^{\text{Arccos}\left(-\frac{\ln n}{tn\pi}\right)} + \int_{\text{Arccos}\left(-\frac{\ln n}{tn\pi}\right)}^{\pi} e^{tn\pi\cos\Theta}\,\mathrm{d}\Theta\right\}\\
&\le \frac{C}{\pi}\, e^{t}\left\{\text{Arccos}\left(-\frac{\ln n}{tn\pi}\right) - \text{Arccos}\left(\frac{1}{n\pi}\right)\right\} + \frac{C}{\pi}\int_{\frac{\pi}{2}}^{\pi}\exp\left(t\, n\pi\cdot\frac{(-\ln n)}{tn\pi}\right)\mathrm{d}\Theta\\
&\to 0+0=0 \qquad \text{for } n\to+\infty.
\end{aligned}$$

Hence, by taking the limit,

$$\begin{aligned}
f(t) &= \mathcal{L}^{-1}\left\{\frac{1}{z\,\cosh z}\right\}(t) = \lim_{n\to+\infty}\frac{1}{2\pi i}\oint_{C_n}\frac{e^{zt}}{z\,\cosh z}\,\mathrm{d}z\\
&= \text{res}\left(\frac{e^{zt}}{z\,\cosh z};0\right) + \lim_{n\to+\infty}\sum_{p=-n}^{n}\text{res}\left(\frac{e^{zt}}{z\,\cosh z};i\left(n+\frac{1}{2}\right)\pi\right)\\
&= 1+\sum_{n=-\infty}^{+\infty}\left[\frac{e^{zt}}{z\,\sinh z}\right]_{z=i\left(n+\frac{1}{2}\right)\pi} = 1+\sum_{n=-\infty}^{+\infty}\frac{\exp\left(i\left(n+\frac{1}{2}\right)\pi t\right)}{i\left(n+\frac{1}{2}\right)\pi\cdot\sinh\left(i\left(n+\frac{1}{2}\right)\pi\right)}\\
&= 1+\frac{4}{\pi}\sum_{n=-\infty}^{+\infty}\frac{\exp\left(i\left(n+\frac{1}{2}\right)\pi t\right)}{2i(2n+1)\cdot i\,\sin\left(\left(n+\frac{1}{2}\right)\pi\right)}\\
&= 1-\frac{4}{\pi}\sum_{n=0}^{+\infty}\frac{1}{2}\left\{\frac{e^{i\left(n+\frac{1}{2}\right)\pi t}\,(-1)^n}{2n+1} + \frac{e^{i\left(-n-1+\frac{1}{2}\right)\pi t}\,(-1)^{n+1}}{2(-n-1)+1}\right\}\\
&= 1-\frac{4}{\pi}\sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1}\,\cos\left(\left(n+\frac{1}{2}\right)\pi t\right).
\end{aligned}$$

Remark 3.2.2 It can be proved by using the Theory of Fourier series that

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{z\,\cosh z}\right\}(t) = 1+(-1)^{n+1} \qquad \text{for } t\in\,]2n-1,2n+1[, \quad n\in\mathbb{N}_0. \qquad \Diamond$$

Download free eBooks at bookboon.com

Example 3.2.3 *Find the inverse Laplace transform of* $\dfrac{1}{z^2\,\sinh z}$.

The singularities of the function $\dfrac{1}{z^2\,\sinh z}$ are

$$z = i\,n\,\pi, \qquad n \in \mathbb{Z},$$

where $z = 0$ for $n = 0$ is a triple pole, and all the other singularities are simple poles. We shall first compute the residua at these poles. We first get for $n = 0$,

$$\operatorname{res}\left(\frac{e^{zt}}{z^2\,\sinh z};0\right) = \frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}\left\{\frac{z\,e^{zt}}{\sinh z}\right\} = \frac{1}{2}\lim_{z\to 0}\frac{d}{dz}\left\{t\cdot\frac{z\,e^{zt}}{\sinh z}+\frac{\sinh z - z\,\cosh z}{\sinh^2 z}\,e^{zt}\right\}$$

$$= \frac{1}{2}\lim_{z\to 0}\left\{t^2\cdot\frac{z\,e^{zt}}{\sinh z}+2t\cdot\frac{\sinh z - z\,\cosh z}{\sinh^2 z}\cdot e^{zt}\right.$$

$$\left.+\frac{(\cosh z-\cosh z-z\sinh z)\sinh^2 z-2\sinh z\,\cosh z(\sinh z-z\cosh z)}{\sinh^4 z}\,e^{zt}\right\}$$

$$= \frac{1}{2}\,t^2\lim_{z\to 0}\frac{z}{\sinh z}+t\cdot\lim_{z\to 0}\frac{\sinh z - z\cosh z}{\sinh^2 z}+\frac{1}{2}\lim_{z\to 0}\frac{-z\sinh^2 z-2\sinh z\cdot\cosh z+2z\cosh^2 z}{\sinh^3 z}$$

$$= \frac{1}{2}\,t^2+t\lim_{z\to 0}\frac{\cosh z-\cosh z - z\sinh z}{2\sinh z\,\cosh z}$$

$$+\frac{1}{2}\lim_{z\to 0}\frac{-\sinh^2 z-2z\sinh z\,\cosh z-2\cosh^2 z-12\sinh^2 z+2\cosh^2+4z\sinh z\,\cosh z}{3\sinh^2 z\,\cosh z}$$

$$= \frac{1}{2}\,t^2+0+\frac{1}{2}\lim_{z\to 0}\frac{-3\sinh z+2z\cosh z}{3\sinh z\,\cosh z} = \frac{1}{2}\,t^2+\frac{1}{2}\left\{\frac{-3}{3}+\frac{2}{3}\right\} = \frac{1}{2}\,t^2-\frac{1}{6}.$$

The computation is simpler for $n \neq 0$,

$$\operatorname{res}\left(\frac{e^{zt}}{z^2\,\sinh z};in\pi\right) = \lim_{z\to in\pi}\frac{e^{zt}}{z^2\cosh z} = \frac{e^{in\pi t}}{-n^2\pi^2\cosh(in\pi)} = -\frac{(-1)^n}{n^2\pi^2}\,e^{in\pi t}.$$

Hence, by still an unjustified application of the residuum formula we get the following bet of the inverse Laplace transform,

$$\mathcal{L}^{-1}\left\{\frac{1}{z^2\sinh z}\right\}(t) \quad \text{“}=\text{”} \quad \frac{1}{2}\,t^2-\frac{1}{6}-\frac{1}{\pi^2}\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^2}\left\{e^{in\pi t}+e^{-in\pi t}\right\}$$

$$(9) \qquad = \quad \frac{1}{2}\,t^2-\frac{1}{6}-\frac{2}{\pi^2}\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^2}\cos n\pi t.$$

We shall now prove that (9) is indeed correct. Choose the path of integration as indicated on Figure 24, where $r_n = \left(n+\dfrac{1}{2}\right)\pi$. Then

$$\frac{1}{2\pi i}\oint_{C_n}\frac{e^{zt}}{z^2\sinh z}\,\mathrm{d}z = \frac{1}{2}2\pi i\int_{1-i\sqrt{r_n^2-1}}^{1+i\sqrt{r_n^2-1}}\frac{e^{zt}}{z^2\sinh z}\,\mathrm{d}z0\frac{1}{2\pi i}\int_{\operatorname{Arccos}\left(\frac{1}{r_n}\right)}^{2\pi-\operatorname{Arccos}\left(\frac{1}{r_n}\right)}\frac{\exp\left(r_n\,e^{i\Theta}t\right)\,i\,r_n\,e^{i\Theta}}{r_n^2\,e^{2i\Theta}\sinh\left(r_n\,e^{i\Theta}\right)}\,\mathrm{d}\Theta.$$

Download free eBooks at bookboon.com

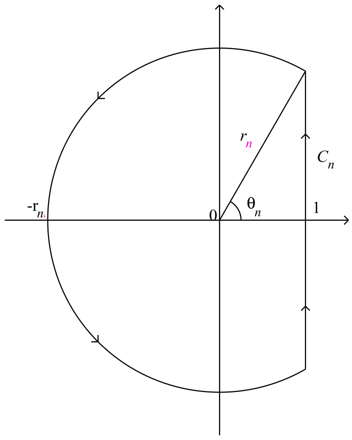


Figure 24: The path of integration in Example 3.2.3.

The left hand side of this equation converges for $n \to +\infty$ towards the sum of the right hand side of (9).

The former term on the right hand side converges towards $\mathcal{L}^{-1}\left\{\dfrac{1}{z^2 \sinh z}\right\}(t)$, and the latter term on the right hand side tends towards 0 for $n \to +\infty$, because we have the estimate

$$\left|\frac{1}{2\pi i}\int_{\text{Arccos}\left(\frac{1}{r_n}\right)}^{2\pi-\text{Arccos}\left(\frac{1}{r_n}\right)} \frac{\exp\left(r_n\, e^{i\Theta} t\right) i\, r_n\, e^{i\Theta}}{r_n^2\, e^{2i\Theta}\, \sinh\left(r_n\, e^{i\Theta}\right)} \mathrm{d}\Theta\right| \le \frac{1}{\pi}\int_0^{\pi} \frac{e^t}{r_n \cdot C}\, \mathrm{d}\Theta = \frac{e^t}{C\, r_n} \to 0$$

for every fixed t and $n \to +\infty$. Hence, we have proved that (9) is indeed the inverse Laplace transform of $\dfrac{1}{z^2 \sinh z}$.

Remark 3.2.3 It is possible to show that (9) represents a piecewise linear function. However, since this analysis is fairly difficult, it shall not be given here. $\Diamond$

Example 3.2.4 *Find the inverse Laplace transform of the function* $\dfrac{1}{z^2\left(1-e^{-z}\right)}$.

First notice that

$$\left|\frac{1}{z^2\left(1-e^{-z}\right)}\right| \le \frac{C}{|z|^2} \qquad \text{for } \Re z \ge 1,$$

so the inverse Laplace transform does exist, and it is given by the Bromwich integral

$$\mathcal{L}^{-1}\left\{\frac{1}{z^2\left(1-e^{-z}\right)}\right\}(t) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \frac{e^{zt}}{z^2\left(1-e^{-z}\right)}\, \mathrm{d}z.$$

The function has a triple pole for $z = 0$ and simple poles for $z = 2ip\pi$, $p \in \mathbb{Z} \setminus \{0\}$.

Download free eBooks at bookboon.com

We shall first compute the residua. We get for the simple poles, where $p \neq 0$,

$$\operatorname{res}\left(\frac{e^{zt}}{z^2\left(1-e^{-z}\right)};2ip\pi\right)=\left[\frac{e^{zt}}{z^2\,e^{-z}}\right]_{z=2ip\pi}=-\frac{1}{4p^2\pi^2}\,e^{2ip\pi t},$$

and for the triple pole, where $p = 0$,

$$\operatorname{res}\left(\frac{e^{zt}}{z^2\left(1-e^{-z}\right)};0\right)=\frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}\left\{\frac{z}{1-e^{-z}}\,e^{zt}\right\}.$$

We expand the factor $\dfrac{z}{1-e^{-z}}$ for small z in the following way,

$$\begin{aligned}\frac{z}{1-e^{-z}} &= \frac{z}{1-\left\{1-z+\frac{z^2}{2}-\frac{z^3}{6}+z^4\,g_1(z)\right\}}=\frac{1}{1-\frac{z^2}{2}+\frac{z^2}{6}-z^3\,g_1(z)}\\ &= 1+\left\{\frac{z}{2}-\frac{z^2}{6}+z^3\,g_1(z)\right\}+\left\{\frac{z}{2}-\frac{z^2}{6}+z^3\,g_1(z)\right\}^2+z^3\cdot g_2(z)\\ &= 1+\frac{z}{2}-\frac{z^2}{6}+\frac{z^2}{4}+z^3\,g_3(z)=1+\frac{z}{2}+\frac{z^2}{12}+z^3\,g_3(z),\end{aligned}$$

Click on the ad to read more

Download free eBooks at bookboon.com

hence, by insertion,

$$\begin{aligned}
\operatorname{res}\left(\frac{e^{zt}}{z^2\left(1-e^{-z}\right)};0\right) &= \frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}\left\{\left(1+\frac{z}{2}+\frac{z^2}{12}+z^3\,g_3(z)\right)e^{zt}\right\} \\
&= \frac{1}{2}\lim_{z\to 0}\frac{d}{dz}\left\{t\left(1+\frac{z}{2}+\frac{z^2}{12}+z^3\,g_3(z)\right)e^{zt}+\left(\frac{1}{2}+\frac{z}{6}+z^2\,g_4(z)\right)e^{zt}\right\} \\
&= \frac{1}{2}\lim_{z\to 0}\left\{t^2\left(1+z\cdot g_5(z)\right)e^{zt}+2t\left(\frac{1}{2}+z\,g_6(z)\right)e^{zt}+\left(\frac{1}{6}+z\,g_7(z)\right)e^{zt}\right\} \\
&= \frac{1}{2}\,t^2+\frac{1}{2}\,t+\frac{1}{12}.
\end{aligned}$$

If therefore the residuum formula holds, the

$$(10)\quad \mathcal{L}^{-1}\left\{\frac{1}{z^2\left(1-e^{-z}\right)}\right\}(t)=\frac{1}{2}\,t^2+\frac{1}{2}\,t+\frac{1}{2}-\frac{1}{2\pi^2}\sum_{p=1}^{+\infty}\frac{1}{p^2}\,\cos(2\pi\pi t).$$

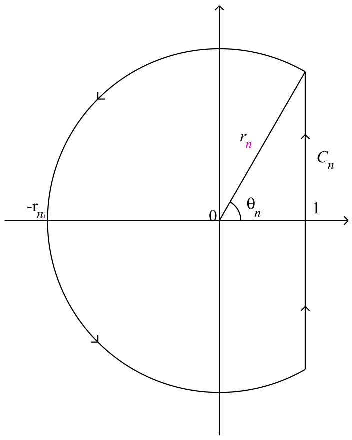


Figure 25: The path of integration in Example 3.2.4.

We shall now prove (10). We choose the well-known path of integration as indicated on Figure 25, where $r_n=(2n+1)\pi$. Then,

$$\begin{aligned}
\frac{1}{2\pi i}\oint_{C_n}\frac{e^{zt}}{z^2\left(1-e^{-z}\right)}\,\mathrm{d}z &= \frac{1}{2\pi i}\int_{1-i\sqrt{r_n^2-1}}^{1+i\sqrt{r_n^2-1}}\frac{e^{zt}}{z^2\left(1-e^{-z}\right)}\,\mathrm{d}z \\
&\quad +\frac{1}{2\pi i}\int_{\operatorname{Arccos}\left(\frac{1}{r_n}\right)}^{2\pi-\operatorname{Arccos}\left(\frac{1}{r_n}\right)}\frac{\exp\left(t\,r_n\,e^{i\Theta}\right)i\,r_n\,e^{i\Theta}}{r_n^2\,e^{2i\Theta}\left\{1-\exp\left(-r_n\,e^{i\Theta}\right)\right\}}\,\mathrm{d}\Theta,
\end{aligned}$$

where the former integral on the right hand side tends towards 0 for $n\to+\infty$, because we have the

Download free eBooks at bookboon.com

estimate

$$\left|\frac{1}{2\pi i}\int_{\text{Arccos}\left(\frac{1}{r_n}\right)}^{2\pi-\text{Arccos}\left(\frac{1}{r_n}\right)} \frac{\exp\left(t\, r_n\, e^{i\Theta}\right) i\, r_n\, e^{i\Theta}}{r_n^2\, e^{2i\Theta}\left\{1-\exp\left(-r_n\, e^{i\Theta}\right)\right\}}\,\mathrm{d}\Theta\right|$$

$$\leq \frac{1}{2\pi}\cdot 2\int_{\text{Arccos}\left(\frac{1}{r_n}\right)}^{\pi} \frac{\exp\left(t\, r_n\,\cos\Theta\right)}{r_n\,\left|1-\exp\left(-(2n+1)\pi\, e^{i\Theta}\right)\right|}\,\mathrm{d}\Theta$$

$$\leq \frac{1}{\pi}\cdot C\cdot\frac{1}{r_n}\cdot e^t\cdot\pi = \frac{C\, e^t}{r_n}\to 0 \qquad \text{for fixed } t \text{ and } n\to+\infty,$$

because $\exp\left(-(2n+1)\pi\, e^{i\Theta}\right) = 1$, if and only if $-(2n+1)\pi\, e^{i\Theta} = 2p\pi$, thus $e^{i\Theta} = \dfrac{2p}{2n+1}$, which can never be fulfilled, because $\left|e^{i\Theta}\right| = 1$, while $\left|\dfrac{2p}{2n+1}\right| \neq 1$ for all n and $p\in\mathbb{Z}$. The function $\left|1-\exp\left(-(/2n+1)\pi\, e^{i\Theta}\right)\right|$ is continuous in $\Theta\in[0,2\pi]$, so it has a minimum $\dfrac{1}{C}>0$, and the claim follows.

Summing up we have proved that (10) holds,

$$\mathcal{L}^{-1}\left\{\frac{1}{z^2\left(1-e^{-z}\right)}\right\}(t) = \frac{1}{2}\,t^2+\frac{1}{2}\,t+\frac{1}{2}-\frac{1}{2\pi^2}\sum_{p=1}^{+\infty}\frac{1}{p^2}\,\cos(2\pi\pi t). \qquad \Diamond$$

Example 3.2.5 *Given* $0<\lambda<a$. *Find the inverse Laplace transform of the function* $\dfrac{\sinh(\lambda z)}{z^2\,\cosh(az)}$.

It follows from the estimate

$$\left|\frac{\sinh(\lambda z)}{z^2\,\cosh(az)}\right| = \frac{1}{|z|^2}\cdot\frac{\left|e^{\lambda z}-e^{-\lambda z}\right|}{\left|e^{az}+e^{-az}\right|} \leq \frac{1}{|z|^2}\cdot\frac{2e^{\lambda\Re z}}{\frac{1}{2}\,e^{a\,\Re z}} \leq \frac{1}{|z|^2}$$

for $\Re z > k$, that the necessary and sufficient condition for the existence of the inverse Laplace transform is satisfied.

Then $\cosh(az)=0$ for $az = i\left(\dfrac{\pi}{2}+p\pi\right)$, thus for $z=\dfrac{i}{a}\left(\dfrac{\pi}{2}+p\pi\right)$. In particular,

$$\text{res}\left(\frac{\sinh(\lambda z)}{z^2\,\cosh(az)};\frac{i}{a}\left(\frac{\pi}{2}+p\pi\right)\right) = \left[\frac{\sinh(\lambda z)}{az^2\,\sinh(az)}\right]_{z=\frac{i}{a}\left(\frac{\pi}{2}+p\pi\right)}$$

$$= \frac{\sinh\left(i\cdot\frac{\lambda}{a}\left(\frac{\pi}{2}+p\pi\right)\right)}{a\left(-\frac{1}{a^2}\left\{\frac{\pi}{2}+p\pi\right\}^2\right)\sinh\left(i\left\{\frac{\pi}{2}+p\pi\right\}\right)} = \frac{\sin\left(\frac{\lambda}{a}\left\{\frac{\pi}{2}+p\pi\right\}\right)}{-\frac{\pi^2}{4a}\,(2p+1)^2\cdot(-1)^p}$$

$$= \frac{4a(-1)^{p+1}}{\pi^2(2p+1)^2}\,\sin\left(\frac{\lambda}{a}\,(2p+a)\frac{\pi}{2}\right),$$

and

$$\text{res}\left(\frac{\sinh(\lambda z)}{z^2\,\cosh(az)};0\right) = \lim_{z\to 0}\frac{\sinh\lambda z}{z}\cdot\frac{1}{\cosh(az)} = \frac{\lambda}{1} = \lambda,$$

Download free eBooks at bookboon.com

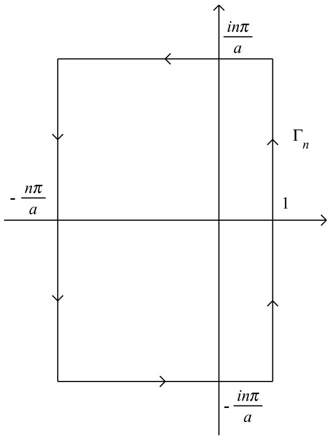


Figure 26: The path of integration in Example 3.2.5.

Choose the path of integration Γ_n as given on Figure 26. Then

$$\begin{aligned}\frac{1}{2\pi i}\oint_{\Gamma_n}\frac{\sinh(\lambda z)}{z^2\,\cosh(az)}\,\mathrm{d}z &= \lambda+\sum_{p=-n}^{n-1}\frac{4a(-1)^{p+1}}{\pi^2(2p+a)^2}\,\sin\left(\frac{\lambda}{a}\,(2p+1)\,\frac{\pi}{2}\right)\\ &= \lambda+\frac{8a}{\pi^2}\sum_{p=0}^{n-1}\frac{(-1)^{p+1}}{(2p+1)^2}\,\sin\left(\frac{\lambda}{a}\,(2p+1)\,\frac{\pi}{2}\right).\end{aligned}$$

On the other hand,

$$\begin{aligned}&\frac{1}{2\pi i}\oint_{\Gamma_n}\frac{\sinh(\lambda z)}{z^2\,\cosh(az)}\,\mathrm{d}z = \frac{1}{2\pi i}\int_{1-i\,\frac{n\pi}{a}}^{1+i\,\frac{n\pi}{a}}\frac{\sinh(\lambda z)}{z^2\,\cosh(az)}\,\mathrm{d}z\\ &\quad-\frac{1}{2\pi i}\int_{-\frac{n\pi}{a}}^{1}\frac{\sinh\left(\lambda\left\{x+i\,\frac{n\pi}{a}\right\}\right)}{\left(x+i\,\frac{n\pi}{a}\right)^2\cosh\left(a\left\{z+i\,\frac{n\pi}{a}\right\}\right)}\,\mathrm{d}x-\frac{1}{2\pi i}\int_{-\frac{n\pi}{a}}^{\frac{n\pi}{a}}\frac{\sinh\left(-\frac{n\pi}{a}+it\right)\lambda}{\left(-\frac{n\pi}{a}+i\,t\right)^2\cosh\left(-\frac{n\pi}{a}+i\,t\right)a}\,i\,\mathrm{d}t\\ &\quad+\frac{1}{2\pi i}\int_{-\frac{n\pi}{a}}^{1}\frac{\sinh\left(\lambda\left\{x-i\,\frac{n\pi}{a}\right\}\right)}{\left(x-i\,\frac{n\pi}{a}\right)^2\cosh\left(a\left\{x-i\,\frac{n\pi}{a}\right\}\right)}\,\mathrm{d}x.\end{aligned}$$

The first term on the right hand side tends according to the inversion theorem towards

$$\mathcal{L}^{-1}\left\{\frac{\sinh(\lambda z)}{z^2\,\cosh(az)}\right\}(t).$$

The second term is estimated in the following way,

$$\begin{aligned}\left|\frac{\sinh\left(\lambda\left\{x+i\,\frac{n\pi}{a}\right\}\right)}{\cosh\left(a\left\{x+i\,\frac{n\pi}{a}\right\}\right)}\right| &= \left|\frac{\sinh(\lambda x)\cos\left(\frac{\lambda}{a}\,n\pi\right)+i\,\cosh(\lambda x)\sin\left(\frac{\lambda}{a}\,n\pi\right)}{\cosh(ax)\cos(n\pi)+0}\right|\\ &\le \frac{|\sinh(\lambda x)|+\cosh(\lambda x)}{\cosh(ax)}\le 2\cdot\frac{e^{\lambda|x|}}{e^{a|x|}}=2\,e^{-(a-\lambda)|x|}<1,\end{aligned}$$

Download free eBooks at bookboon.com

and we conclude that

$$\left|-\frac{1}{2\pi i}\int_{-\frac{n\pi}{a}}^{1}\frac{\sinh\left(\lambda\left\{x+i\,\frac{n\pi}{a}\right\}\right)}{\left(x+i\,\frac{n\pi}{a}\right)^2\cosh\left(a\left\{x+i\,\frac{n\pi}{a}\right\}\right)}\,dx\right| \le \frac{2}{2\pi}\cdot\frac{1+\frac{n\pi}{a}}{\left(\frac{n\pi}{a}\right)^2}\to 0 \qquad \text{for } n\to+\infty.$$

The estimate of the fourth term is analogous.

Concerning the third term we get

$$\left|\frac{\sinh\left(\lambda\left\{-\frac{n\pi}{a}+i\,t\right\}\right)}{\cosh\left(a\left\{-\frac{n\pi}{a}+i\,t\right\}\right)}\right|^2 = \left|\frac{\sinh\left(-\frac{\lambda}{a}\,n\pi\right)\cos\lambda t+i\,\cosh\left(-\frac{\lambda}{a}\,n\pi\right)\sin\lambda t}{\cosh(n\pi)\cos(at)-i\,\sinh(n\pi)\sin(at)}\right|^2$$

$$= \frac{\sinh^2\left(\frac{\lambda}{a}\,n\pi\right)\cos^2\lambda t+\cosh^2\left(\frac{\lambda}{a}\,n\pi\right)\sin^2\lambda t}{\cosh^2(n\pi)\cos^2(at)+\sinh^2(n\pi)\sin^2(at)} = \frac{\cosh^2\left(\frac{\lambda}{a}\,n\pi\right)-\cos^2\lambda t}{\cosh^2(n\pi)-\sin^2(at)}$$

$$\le \frac{\cosh^2\left(\frac{\lambda}{a}\,n\pi\right)}{\cosh^2(n\pi)-1}\le C^2,$$

and we obtain the estimate

$$\left|-\frac{1}{2\pi i}\int_{-\frac{n\pi}{a}}^{\frac{n\pi}{a}}\frac{\sinh\left(\left\{-\frac{n\pi}{a}+it\right\}\lambda\right)}{\left(-\frac{n\pi}{a}+it\right)^2\cosh\left(-\frac{n\pi}{a}+it\right)a}\,i\,dt\right| \le \frac{1}{2\pi}\cdot C\cdot\frac{2\cdot\frac{n\pi}{a}}{\left(\frac{n\pi}{a}\right)^2} = \frac{Ca^2}{2\pi^2}\cdot\frac{1}{n}\to 0 \qquad \text{for } n\to+\infty.$$

Summing up, we get for $n\to+\infty$,

$$\mathcal{L}^{-1}\left\{\frac{\sinh(\lambda z)}{z^2\cosh(az)}\right\}(t)=\lambda+\frac{8a}{\pi^2}\sum_{n=0}^{+\infty}\frac{(-1)^{n+1}}{(2n+1)^2}\sin\left(\frac{\lambda}{a}\,(2n+1)\,\frac{\pi}{2}\right). \qquad \Diamond$$

Click on the ad to read more

Download free eBooks at bookboon.com

Example 3.2.6 *Given* $0 < \lambda < a$. *Find the inverse Laplace transform of* $\dfrac{\cosh(\lambda\sqrt{z})}{z \cdot \cosh(a\sqrt{z})}$.

First notice by using series expansions that the function in spite of the occurrence of the square root is analytic without branch cuts and a simple pole at $z = 0$ and either simple poles of removable singularities for $a\sqrt{z} = i\left(\frac{\pi}{2} + p\pi\right)$, $p \in \mathbb{Z}$, thus for

$$z = 0 \qquad \text{and} \qquad z = -\frac{\pi^2}{4a^2}(2n+1)^2, \qquad n \in \mathbb{N}_0,$$

Furthermore, it is not too hard to prove that

$$\left|\frac{\cosh(\lambda\sqrt{z})}{z \cdot \cosh(a\sqrt{z})}\right| \le \frac{C}{|z|} \qquad \text{for } \Re z > k.$$

First we compute the residua,

$$\operatorname{res}\left(\frac{\cosh(\lambda\sqrt{z})}{z \cdot \cosh(a\sqrt{z})} \cdot e^{zt}; 0\right) = 1,$$

and

$$\operatorname{res}\left(\frac{\cosh(\lambda\sqrt{z})}{z \cdot \cosh(a\sqrt{z})} \cdot e^{zt}; -\frac{\pi^2}{4a^2}(2n+1)^2\right) = \lim_{z \to -\frac{\pi^2}{4a^2}(2n+1)^2} \frac{\cosh(\lambda\sqrt{z}) \cdot e^{zt}}{z \cdot \sinh(a\sqrt{z}) \cdot a \cdot \frac{1}{2\sqrt{z}}}$$

$$= \frac{2}{a} \cdot \frac{\cosh\left(i\lambda \cdot \frac{\pi}{2a}(2n+1)\right)}{i \cdot \frac{\pi}{2a}(2n+1)\sinh\left(\frac{\pi}{2}(2n+1)\right)} \cdot \exp\left(-\frac{\pi^2}{4a^2}(2n+1)^2 t\right)$$

$$= \frac{4}{\pi} \cdot \frac{1}{2n+1} \cdot \frac{\cos\left(\frac{\lambda\pi}{2a}(2n+1)\right)}{i \cdot i \cdot \sin\left(\frac{\pi}{2} + n\pi\right)} \cdot \exp\left(-\frac{\pi^2}{4a^2}(2n+1)^2 t\right)$$

$$= \frac{4}{\pi} \cdot \frac{(-1)^{n+1}}{2n+1} \cdot \cos\left(\frac{\lambda}{a} \cdot \frac{\pi}{2}(2n+1)\right) \cdot \exp\left(-\frac{\pi^2}{4a^2}(2n+1)^2 t\right).$$

We choose the path of integration as given on Figure 27, where $r_n = \dfrac{\pi^2}{a^2} \cdot n^2$. Then, by Cauchy's residuum theorem

$$\frac{1}{2\pi i}\oint_{C_n} \frac{\cosh(\lambda\sqrt{z})}{z \cdot \cosh(a\sqrt{z})} e^{zt}\, dt = 1 + \frac{4}{\pi}\sum k = 0^{n-1} \frac{(-1)^{k+1}}{2k+1} \cos\left(\frac{\lambda}{a} \cdot \frac{<pio}{2}(2k+1)\right)\exp\left(-\frac{\pi^2}{4a^2}(2k+1)^2 t\right).$$

On the other hand, this expression is also equal to

$$\frac{1}{2\pi i}\int_{1-i\sqrt{r_n^2-1}}^{1+i\sqrt{r_n^2-1}} \frac{\cosh(\lambda\sqrt{z})}{z \cdot \cosh(a\sqrt{z})} \cdot e^{zt}\, dz + \frac{1}{2\pi i}\int_{\operatorname{Arccos}\left(\frac{1}{r_n}\right)}^{2\pi - \operatorname{Arccos}\left(\frac{1}{r_n}\right)} \frac{\cosh\left(\frac{\lambda\pi}{a}\, n\, e^{i\Theta/2}\right)\exp\left(t\, \frac{\pi^2}{a^2}\, n^2\, e^{i\Theta}\right)}{\frac{\pi^2}{a^2}\, n^2\, e^{i\Theta}\cosh\left(\pi\, n\, e^{i\Theta/2}\right)} \cdot i\, \frac{\pi^2}{a^2}\, n^2\, e^{i\Theta}\, d\Theta.$$

The former integral converges towards the wanted $\mathcal{L}^{-1}\left\{\dfrac{\cosh(\lambda\sqrt{z})}{z \cdot \cosh(a\sqrt{z})}\right\}(t)$.

Download free eBooks at bookboon.com

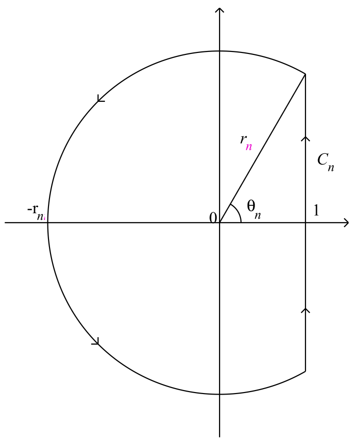


Figure 27: The path of integration in Example 3.2.6.

The latter integral is estimated in the following way,

$$\left|\frac{1}{2\pi i}\int_{\text{Arccos}\left(\frac{1}{r_n}\right)}^{2\pi-\text{Arccos}\left(\frac{1}{r_n}\right)} \frac{\cosh\left(\frac{\lambda\pi}{a}\, n\, e^{i\,\Theta/2}\right)\exp\left(t\,\frac{\pi^2}{a^2}\,n^2\,e^{i\Theta}\right)}{\frac{\pi^2}{a^2}\,n^2\,e^{i\Theta}\cosh\left(\pi\,n\,e^{i\Theta/2}\right)}\cdot i\,\frac{\pi^2}{a^2}\,n^2\,e^{i\Theta}\,d\Theta\right|$$

$$\leq \frac{1}{2\pi}\cdot 2\int_{\text{Arccos}\left(\frac{1}{r_n}\right)}^{\pi}\left|\frac{\cosh\left(\frac{\lambda\pi}{a}\cdot n\cdot\cos\frac{\Theta}{2}+i\,\frac{\lambda\pi}{a}\,\sin\frac{\Theta}{2}\right)}{\cosh\left(\pi n\,\cos\frac{\Theta}{2}+i\pi n\,\sin\frac{\Theta}{2}\right)}\right|\exp\left(t\cdot\frac{\pi^2}{a^2}\,n^2\,e^{i\Theta}\right)d\Theta.$$

Using that

$$|\cosh(x+iy)|^2=\cosh^2 x-\sin^2 y=\sinh^2 x+\cos^2 y,$$

we get

$$\left|\frac{\cosh\left(\frac{\lambda\pi}{a}\cdot n\cdot\cos\frac{\Theta}{2}+i\,\frac{\lambda\pi}{a}\,\sin\frac{\Theta}{2}\right)}{\cosh\left(\pi n\,\cos\frac{\Theta}{2}+i\pi n\,\sin\frac{\Theta}{2}\right)}\right|^2\leq\frac{\cosh^2\left(\frac{\lambda\pi}{a}\,n\,\cos\frac{\Theta}{2}\right)}{\sinh^2\left(\pi n\,\cos\frac{\Theta}{2}\right)+\cos^2\left(\pi n\,\sin\frac{\Theta}{2}\right)}.$$

The integral is estimated in the interval $\left[\text{Arccos}\left(\frac{1}{r_n}\right),\text{Arccos}\left(-\frac{\ln r_n}{r_n}\right)\right]$ in the following way,

$$\left|\frac{\cosh\left(\frac{\lambda\pi}{a}\cdot n\cdot\cos\frac{\Theta}{2}+i\,\frac{\lambda\pi}{a}\,\sin\frac{\Theta}{2}\right)}{\cosh\left(\pi n\,\cos\frac{\Theta}{2}+i\pi n\,\sin\frac{\Theta}{2}\right)}\right|\cdot\exp\left(t\cdot\frac{\pi^2}{a^2}\,n^2\,\cos\Theta\right)$$

$$\leq\frac{\cosh\left(\frac{\lambda}{a}\cdot\pi n\cdot\cos\left(\frac{1}{2}\,\text{Arccos}\left(\frac{1}{r_n}\right)\right)\right)}{\sinh\left(\pi n\,\cos\left(\frac{1}{2}\,\text{Arccos}\left(-\frac{\ln r_n}{r_n}\right)\right)\right)}\cdot e^t\to 0\qquad\text{for } n\to+\infty,$$

because $0<\dfrac{\lambda}{a}<1$ and

$$\cos\left(\frac{1}{2}\,\text{Arccos}\frac{1}{r_n}\right)\to\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}\quad\text{and}\quad\cos\left(\frac{1}{2}\,\text{Arccos}\left(-\frac{\ln r_n}{r_n}\right)\right)\to\frac{\sqrt{2}}{2}\qquad\text{for } n\to+\infty.$$

Download free eBooks at bookboon.com

In the interval $\left[\text{Arccos}\left(-\frac{\ln r_n}{r_n}\right), \pi\right]$ we estimate the integrand in the following way,

$$\left|\frac{\cosh\left(\frac{\lambda\pi}{a}\cdot n\cdot\cos\frac{\Theta}{2}+i\,\frac{\lambda\pi}{a}\,\sin\frac{\Theta}{2}\right)}{\cosh\left(\pi n\,\cos\frac{\Theta}{2}+i\pi n\,\sin\frac{\Theta}{2}\right)}\right|\cdot\exp\left(t\cdot\frac{\pi^2}{a^2}\,n^2\,\cos\Theta\right)$$

$$\leq C\cdot\exp(-t\cdot\ln r_n)\to 0 \qquad \text{for } n\to+\infty.$$

Summing up, the latter integral tends towards 0 for $n \to +\infty$, thus

$$\mathcal{L}^{-1}\left\{\frac{\cosh\left(\lambda\sqrt{z}\right)}{z\,\cosh\left(a\sqrt{z}\right)}\right\}(t)=1+\frac{4}{\pi}\sum_{n=0}^{+\infty}\frac{(-1)^{n+1}}{2n+1}\,\cos\left(\frac{\lambda}{a}\cdot\frac{\pi}{2}\,(2n+1)\right)\cdot\exp\left(-\frac{\pi^2}{4a^2}\,(2n+1)^2\,t\right). \qquad \diamond$$

Example 3.2.7 *Given $0 < \lambda < a$. Find the inverse Laplace transform of the function* $\dfrac{\cosh(\lambda z)}{z^3\,\cosh(az)}$.

We clearly have the estimate

$$\left|\frac{\cosh(\lambda z)}{z^3\,\cosh(az)}\right|\leq\frac{C}{|z|^3} \qquad \text{for } |\Re\,z|\geq k,$$

so the inverse Laplace transform exists.

We have a triple pole at $z = 0$ and simple poles at $z = \dfrac{i}{a}\left\{\dfrac{\pi}{2}+p\pi\right\}$, $p \in \mathbb{Z}$. The corresponding residua are

$$\text{res}\left(\frac{\cosh(\lambda z)}{z^3\,\cosh(az)}\cdot e^{zt};0\right)=\frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}\left\{\frac{\cosh(\lambda z)}{\cosh(az)}\cdot e^{zt}\right\}$$

$$=\frac{1}{2}\lim_{z\to 0}\frac{d}{dz}\left\{t\cdot\frac{\cosh(\lambda z)}{\cosh(az)}\cdot e^{zt}+\lambda\,\frac{\sinh(\lambda z)}{\cosh(az)}\,e^{zt}-a\cdot\frac{\cosh(\lambda z)\sinh(az)}{\cosh^2(az)}\,e^{zt}\right\}$$

$$=\frac{1}{2}\left[t^2\cdot\frac{\cosh(\lambda z)}{\cosh(az)}\cdot e^{zt}+2t\left\{\lambda\,\frac{\sinh(\lambda z)}{\cosh(az)}-a\,\frac{\cosh(\lambda z)\sinh(az)}{\cosh^2(az)}\right\}e^{zt}\right.$$

$$\left.+\lambda^2\,\frac{\cosh(\lambda z)}{\cosh(az)}\,e^{zt}+\sinh(\lambda z)\cdot\{\cdots\}-a^2\cdot\frac{\cosh(\lambda z)}{\cosh(az)}\cdot e^{zt}+\sinh(az)\cdot\{\cdots\}\right]_{z=0}$$

$$=\frac{1}{2}\left(t^2+\lambda^2-a^2\right),$$

and

$$\text{res}\left(\frac{\cosh(\lambda z)}{z^3\,\cosh(az)}\cdot e^{zt};\frac{i}{a}\cdot\left\{\frac{\pi}{2}+p\pi\right\}\right)=\lim_{z\to\frac{i}{a}\left(\frac{\pi}{a}+p\pi\right)}\frac{\cosh(\lambda z)\cdot e^{zt}}{z^3\cdot a\cdot\sinh(az)}$$

$$=\frac{\cosh\left(i\,\frac{\lambda}{a}\left\{\frac{\pi}{2}+p\pi\right\}\right)\exp\left(\frac{i}{a}\left\{\frac{\pi}{2}+p\pi\right\}t\right)}{-\frac{i}{a^3}\left\{\frac{\pi}{2}+\pi\right\}^3\cdot a\,\sinh\left(i\left\{\frac{\pi}{2}+p\pi\right\}\right)}$$

$$=\frac{8a^2(-1)^p}{\pi^3(2p+a)^3}\cdot\cos\left(\frac{\lambda}{a}\cdot\frac{\pi}{2}\,(2p+1)\right)\exp\left(\frac{i}{a}\cdot\frac{\pi}{2}\,(2p+1)t\right).$$

Download free eBooks at bookboon.com

When we pair the residua at conjugated poles, $\pm\frac{i}{a}\left\{\frac{\pi}{2}+p\pi\right\}$, we get the sum for each of these pairs,

$$\frac{16a^2(-1)^p}{\pi^3(2p+1)^3}\cos\left(\frac{\lambda}{a}\cdot\frac{\pi}{2}(2p+1)\right)\cos\left(\frac{\pi}{2a}(2p+1)t\right).$$

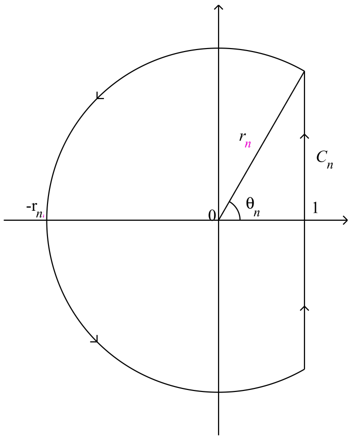


Figure 28: The path of integration in Example 3.2.7.

Click on the ad to read more

Download free eBooks at bookboon.com

Choose $r_n = \dfrac{n\pi}{a}$ and the usual path of integration as indicated on Figure 28. Then we get by Cauchy's residuum theorem

$$\frac{1}{2\pi i}\oint_{C_n} \frac{\cosh(\lambda z)e^{zt}}{z^3\,\cosh(az)}\,\mathrm{d}z = \frac{1}{2}\left(t^2+\lambda^2-1a^2\right)+\sum_{p=0}^{n-1}\frac{16a^2(-1)^p}{\pi^3(2p+1)^3}\,\cos\!\left(\frac{\pi}{2a}\,(2p+1)\lambda\right)\cos\!\left(\frac{\pi}{2a}\,(2p+1)t\right).$$

On the other hand, also

$$\frac{1}{2\pi i}\oint_{C_n} \frac{\cosh(\lambda z)e^{zt}}{z^3\,\cosh(az)}\,\mathrm{d}z$$

$$= \frac{1}{2\pi i}\int_{1-i\sqrt{r_n^2-1}}^{1+i\sqrt{r_n^2-1}} \frac{\cosh(\lambda z)e^{zt}}{z^3\,\cosh(az)}\,\mathrm{d}z + \frac{1}{2\pi i}\int_{\text{Arccos}\,\frac{1}{r_n}}^{2\pi-\text{Arccos}\,\frac{1}{r_n}} \frac{\cosh\left(\lambda r_n\,e^{i\Theta}\right)\exp\left(r_n\,e^{i\Theta}t\right)\,i\,r_n e^{i\Theta}}{r_n^3\,e^{3i\Theta}\cosh(a\,r_n\,e^{i\Theta})}\,\mathrm{d}\Theta.$$

The former integral on the right hand side of this equation converges towards

$$\mathcal{L}^{-1}\left\{\frac{\cosh(\lambda z)}{z^3\,\cosh(az)}\right\}(t) \qquad \text{for } n \to +\infty.$$

The latter integral is estimated in the following way,

$$\left|\frac{1}{2\pi i}\int_{\text{Arccos}\,\frac{1}{r_n}}^{2\pi-\text{Arccos}\,\frac{1}{r_n}} \frac{\cosh\left(\lambda r_n\,e^{i\Theta}\right)\exp\left(r_n\,e^{i\Theta}t\right)\,i\,r_n e^{i\Theta}}{r_n^3\,e^{3i\Theta}\cosh(a\,r_n\,e^{i\Theta})}\,\mathrm{d}\Theta\right|$$

$$\leq \frac{1}{2\pi}\cdot 2\int_0^{\pi}\frac{1}{r_n^2}\cdot 1\cdot e^t\,\mathrm{d}\Theta = \frac{1}{r_n^2}\cdot e^t \to 0 \qquad \text{for } n \to +\infty.$$

Summing up, we get by taking this limit,

$$\mathcal{L}^{-1}\left\{\frac{\cosh(\lambda z)}{z^3\cosh(az)}\right\}(t)=\frac{1}{2}\left(t^2+\lambda^2-a^2\right)+\frac{16a^2}{\pi^3}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^3}\,\cos\!\left(\frac{\pi}{2a}\,(2n+1)\lambda\right)\cos\!\left(\frac{\pi}{2a}\,(2n+1)t\right). \quad \Diamond$$

Example 3.2.8 *Consider the circuit on Figure 29, where the generator is specified by*

$$E(t) = (-1)^n\,E_0 \qquad \textit{for } t \in [n1, (n+1)a[, \quad n \in \mathbb{N}_0.$$

We assume that the current $I(0)$ is zero for $t = 0$. Find the current $I(t)$ at any later time $t > 0$.
Hint. *The result does not have a nice description.*

We first set up the governing differential equation

$$(11)\quad L\,\frac{dI}{dt} + RI = E(t), \qquad \text{where } I(0) = 0.$$

Download free eBooks at bookboon.com

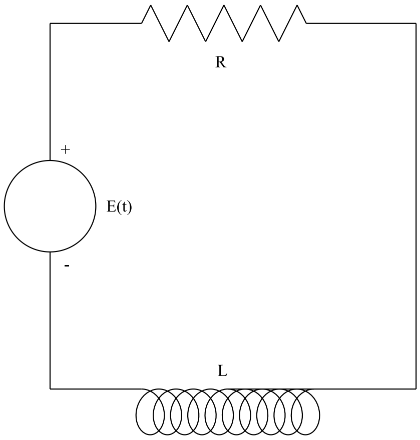


Figure 29: The circuit of Example 3.2.8.

Since $E(t)$ is periodic of period $2a$, it follows from the rule of periodicity that if $\Re z > 0$, then

$$\begin{aligned}
\mathcal{L}\{E(t)\}(z) &= \frac{1}{1-e^{-2az}} \int_0^{2a} e^{-zt}\,E(t)\,\mathrm{d}t = \frac{E_0\,e^{2az}}{e^{2az}-1}\left\{\int_0^a e^{-zt}\,\mathrm{d}t - \int_a^{2a} e^{-zt}\,\mathrm{d}t\right\} \\
&= \frac{E_0\,e^{2az}}{e^{2az}-1}\left\{\left[-\frac{1}{z}\,e^{-zt}\right]_0^a - \left[-\frac{1}{z}\,e^{-zt}\right]_a^{2a}\right\} = \frac{E_0\,e^{2az}}{e^{2az}-1}\cdot\frac{1}{z}\left\{-e^{-az}+1+e^{-2az}-e^{-az}\right\} \\
&= \frac{E_0}{z}\cdot\frac{1-2e^{az}+e^{2az}}{e^{2az}-1} = \frac{E_0}{z}\cdot\frac{\left(e^{az}-1\right)^2}{\left(e^{az}-1\right)\left(e^{az}+1\right)} = \frac{E_0}{z}\cdot\frac{e^{az}-1}{e^{az}+1} \\
&= \frac{E_0}{z}\cdot\frac{\exp\left(\frac{az}{2}\right)-\exp\left(-\frac{az}{2}\right)}{\exp\left(\frac{az}{2}\right)+\exp\left(-\frac{az}{2}\right)} = \frac{E_0}{z}\,\tanh\left(\frac{az}{2}\right),
\end{aligned}$$

so it follows by the Laplace transformation of (11) that

$$L\cdot z\cdot\mathcal{L}\{I(t)\}(z) + R\cdot\mathcal{L}\{I(t)\}(z) = \frac{E_0}{2}\cdot\tanh\left(\frac{az}{2}\right),$$

thus,

$$\mathcal{L}\{I(t)\}(z) = \frac{1}{Lz+R}\cdot\frac{E_0}{z}\,\tanh\left(\frac{az}{2}\right) = \frac{E_0}{L}\cdot\frac{\tanh\left(\frac{az}{2}\right)}{z\left(z+\frac{R}{L}\right)}.$$

Then use that

$$|\sinh z|^2 = \cosh^2 x - \cos^2 y, \qquad \text{and} \qquad |\cosh z|^2 = \cosh^2 x - \sin^2 y,$$

to get the estimate

$$(12)\quad \left|\tanh\left(\frac{az}{2}\right)\right|^2 = \frac{\cosh^2\left(\frac{ax}{2}\right)-\cos^2\left(\frac{ay}{2}\right)}{\cosh^2\left(\frac{ax}{2}\right)-\sin^2\left(\frac{ay}{2}\right)} \le \frac{\cosh^2\left(\frac{ax}{2}\right)}{\cosh^2\left(\frac{ax}{2}\right)-1} = 1+\frac{1}{\cosh^2\left(\frac{ax}{2}\right)-1}.$$

Download free eBooks at bookboon.com

We conclude that

$$I(t)=\frac{E_0}{L}\,\mathcal{L}^{-1}\left\{\frac{\tanh\left(\frac{az}{2}\right)}{z\left(z+\frac{R}{L}\right)}\right\}(t)=\frac{E_0}{2\pi Li}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{zt}\tanh\left(\frac{az}{2}\right)}{z\left(z+\frac{R}{L}\right)}\,\mathrm{d}z,\qquad \text{for } t\geq 0,$$

for some $\gamma>0$.

The singularities are given by $z=0$ and $z=-\dfrac{R}{L}$ and $\cosh\left(\dfrac{az}{2}\right)=0$, thus, $z=\dfrac{(2n+1)\pi}{a}\cdot i$, for $n\in\mathbb{Z}$.

The singularity at $z=0$ is removable, because

$$\lim_{z\to 0}\frac{e^{zt}\tanh\left(\frac{az}{2}\right)}{z\left(z+\frac{R}{L}\right)}=\frac{1}{\frac{R}{L}}\cdot\frac{a}{2}=\frac{aL}{2R}.$$

The singularity at $z=-\dfrac{R}{L}$ is real and simple, and

$$\operatorname{res}\left(\frac{\tanh\left(\frac{az}{2}\right)e^{zt}}{z\left(z+\frac{R}{L}\right)}\right)=\frac{\tanh\left(-\frac{aR}{2L}\right)}{-\frac{R}{L}}\exp\left(-\frac{R}{L}\,t\right)=\frac{L}{R}\,\exp\left(-\frac{R}{L}\,t\right)\tanh\left(\frac{aR}{2L}\right).$$

The singularities $z=\dfrac{(2n+1)\pi i}{a}$, $n\in\mathbb{Z}$, are all pure imaginary and simple, and we get

$$\begin{aligned}\operatorname{res}\left(\frac{e^{zt}\tanh\left(\frac{az}{2}\right)}{z\left(z+\frac{R}{L}\right)};\frac{(2n+1)\pi i}{a}\right)&=\lim_{z\to\frac{(2n+1)\pi i}{a}}\frac{e^{zt}}{z\left(z+\frac{R}{L}\right)}\cdot\frac{\sinh\left(\frac{az}{2}\right)}{\frac{a}{2}\,\sinh\left(\frac{az}{2}\right)}\\ &=\frac{2}{a}\cdot\frac{\exp\left(i\,\frac{(2n+1)\pi}{a}\,t\right)}{\frac{(2n+1)\pi i}{a}\left\{\frac{(2n+1)\pi i}{a}+\frac{R}{L}\right\}}.\end{aligned}$$

We put for convenience,

$$\varphi_n:=\operatorname{Arg}\left(\frac{R}{L}+i\,\frac{(2n+1)\pi}{a}\right)=\operatorname{Arctan}\left(\frac{(2n+1)L\pi}{Ra}\right).$$

Then

$$\begin{aligned}\operatorname{res}\left(\frac{e^{zt}\tanh\left(\frac{az}{2}\right)}{z\left(z+\frac{R}{L}\right)};\frac{(2n+1)\pi i}{a}\right)&=\frac{2}{i}\cdot\frac{\exp\left(i\left\{\frac{(2n+1)\pi}{a}\,t-\varphi_n\right\}\right)}{(2n+1)\pi\sqrt{\frac{R^2}{L^2}+\frac{(2n+1)^2\pi^2}{a^2}}}\\ &=\frac{2\sin\left(\frac{(2n+1)\pi}{a}\,t-\varphi_n\right)}{(2n+1)\pi\sqrt{\frac{R^2}{L^2}+\frac{(2n+1)^2\pi^2}{a^2}}}-i\cdot\frac{2\cos\left(\frac{(2n+1)\pi}{a}\,t-\varphi_n\right)}{(2n+1)\pi\sqrt{\frac{R^2}{L^2}+\frac{(2n+1)^2\pi^2}{a^2}}}.\end{aligned}$$

Notice that $\varphi_{-n}=-\varphi_{n-1}$, so

$$\cos\left(\frac{(2\{n-1\}+1)\pi}{a}\,t-\varphi_{n-1}\right)=\cos\left(\frac{(2\{-n\}+1)\pi}{a}\,t-\varphi_{-n}\right),$$

Download free eBooks at bookboon.com

hence a change of sign in the denominator implies that

$$\operatorname{res}\left(\frac{e^{zt}\tanh\left(\frac{az}{2}\right)}{z\left(z+\frac{R}{L}\right)};\frac{(2n+1)\pi i}{a}\right)+\operatorname{res}\left(\frac{e^{zt}\tanh\left(\frac{az}{2}\right)}{z\left(z+\frac{R}{L}\right)};-\frac{(2n+1)\pi i}{a}\right)$$

$$\frac{4\sin\left(\frac{(2n+1)\pi}{a}\,t-\varphi_n\right)}{(2n+1)\pi\sqrt{\frac{R^2}{L^2}+\frac{(2n+1)^2\pi^2}{a^2}}},\qquad n\in\mathbb{N}_0.$$

After a multiplication by $\dfrac{E_0}{L}$ it follows that the sum of the residua becomes

$$\text{(13)}\quad \frac{E_0}{R}\exp\left(-\frac{R}{L}\,t\right)\tanh\left(\frac{aR}{2L}\right)+\frac{4E_0}{\pi L}\sum_{n=0}^{+\infty}\frac{\sin\left(\frac{(2n+1)\pi}{a}\,t-\varphi_n\right)}{(2n+1)\sqrt{\frac{R^2}{L^2}+\frac{(2n+1)^2\pi^2}{a^2}}},$$

where $\varphi_n=\operatorname{Arctan}\left(\dfrac{(2n+1)L\pi}{Ra}\right)$, $n\in\mathbb{N}_0$. We notice that the denominator can be estimated by a polynomial of second degree in n, which implies that the series is convergent.

Click on the ad to read more

Download free eBooks at bookboon.com

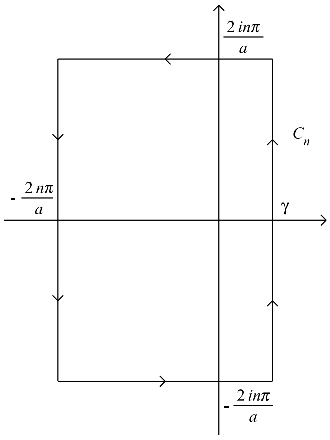


Figure 30: The path of integration in Example 3.2.8.

We shall still prove that $I(t)$ is given by (13). Choose the path of integration C_n as on Figure 30. Then

$$(14)\quad \lim_{n\to+\infty} \frac{1}{2\pi i}\oint_{C_n} \frac{e^{zt}\tanh\left(\frac{az}{2}\right)}{z\left(z+\frac{R}{L}\right)}\,\mathrm{d}z$$

is equal to (13). Let $x = -\dfrac{2n\pi}{a}$. Then we get for $\dfrac{2n\pi}{a} > \dfrac{R}{L}$ the estimate, cf. (12),

$$\left|\frac{1}{2\pi i}\oint_{C_n} \frac{e^{zt}\tanh\left(\frac{az}{2}\right)}{z\left(z+\frac{R}{L}\right)}\,\mathrm{d}z\right| \le \frac{1}{2\pi}\cdot\frac{\cosh^2\left(\frac{a}{a}\left\{-\frac{2n\pi}{a}\right\}\right)}{\cosh^2\left(\frac{a}{2}\left\{-\frac{2n\pi}{a}\right\}\right)-1}\cdot\frac{2\cdot\frac{2n\pi}{2}\,\exp\left(-\frac{2\pi n}{a}\,t\right)}{\frac{2n\pi}{a}\left\{\frac{2n\pi}{a}-\frac{R}{L}\right\}},$$

which clearly tends towards 0 for $n \to +\infty$.

If $y = \pm\dfrac{2n\pi}{a}\,i$, then it follows from (12) that

$$\left|\tanh\left(\frac{az}{2}\right)\right|^2 = \frac{\cosh^2\left(\frac{ax}{2}\right)-\cos^2(n\pi)}{\cosh^2\left(\frac{ax}{2}\right)-\sin^2(n\pi)} \le 1.$$

Since $x \in \left[-\frac{2n\pi}{a},\gamma\right]$, we have $|e^{zt}| \le e^{\gamma t}$, so the estimate of the integrals along the horizontal segments becomes

$$\left|\frac{1}{2\pi i}\int_{y=\pm\frac{2n\pi}{a}\,i} \frac{e^{zt}\tanh\left(\frac{az}{2}\right)}{z\left(z+\frac{R}{L}\right)}\,\mathrm{d}z\right| \le \frac{1}{2\pi}\cdot e^{\gamma t}\cdot 1\cdot\frac{1}{\frac{2n\pi}{a}}\cdot\frac{1}{\frac{2n\pi}{a}}\cdot\left(\frac{2n\pi}{a}+\gamma\right).$$

Download free eBooks at bookboon.com

In this analysis γ and t are fixed numbers, so we conclude that the line integrals along the three auxiliary line segments of C_n tend towards 0 for $n \to +\infty$, hence

$$\begin{aligned} I(t) &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{zt}\tanh\left(\frac{az}{2}\right)}{z\left(z+\frac{R}{L}\right)}\,\mathrm{d}z = \lim_{n\to+\infty}\frac{1}{2\pi i}\oint_{C_n}\frac{e^{zt}\tanh\left(\frac{az}{2}\right)}{z\left(z+\frac{R}{L}\right)}\,\mathrm{d}z \\ &= \frac{E_0}{R}\exp\left(-\frac{R}{L}t\right)\tanh\left(\frac{aR}{2L}\right)+\frac{4E_0}{\pi L}\sum_{n=0}^{+\infty}\frac{\sin\left(\frac{(2n+1)\pi}{a}t-\varphi_n\right)}{(2n+1)\sqrt{\frac{R^2}{L^2}+\frac{(2n+1)^2\pi^2}{a^2}}}, \end{aligned}$$

where we have put $\varphi_n = \operatorname{Arctan}\left(\dfrac{(2n+1)L\pi}{aR}\right)$, for $n \in \mathbb{N}_0$. $\Diamond$

Click on the ad to read more

Download free eBooks at bookboon.com

4 Appendices

4.1 Trigonometric formulæ

We repeat the formulæ known from e.g. *Ventus, Calculus 1-a, Functions in one Variable.* The *addition formulæ for trigonometric functions* are

(15) $\cos(x+y) = \cos x \cdot \cos y - \sin x \cdot \sin y,$

(16) $\cos(x-y) = \cos x \cdot \cos y + \sin x \cdot \sin y,$

(17) $\sin(x+y) = \sin x \cdot \cos y + \cos x \cdot \sin y,$

(18) $\sin(x-y) = \sin x \cdot \cos y - \cos x \cdot \sin y.$

Remark 4.1.1 One remembers these important rules by noting that $\cos x$ is even, and $\sin x$ is odd. Therefore, since $\cos(x \pm y)$ is even, the reduction must contain $\cos x \cdot \cos y$ (even times even) and $\sin x \cdot \sin y$ (odd times odd). Then we shall only remember the change of sign in front of $\sin x \cdot \sin y$.

Analogously, $\sin(x \pm y)$ is odd, so the reduction must contain $\sin x \cdot \cos y$ (odd times even) and $\cos x \cdot \sin y$ (even times odd). Here there is no change of sign. ♢

The antilogarithmic formulæ. These are derived from the *addition formulæ* above.

$$\sin x \cdot \sin y = \frac{1}{2}\{\cos(x-y) - \cos(x+y)\}, \qquad \text{even},$$

$$\cos x \cdot \cos y = \frac{1}{2}\{\cos(x-y) + \cos(x+y)\}, \qquad \text{even},$$

$$\sin x \cdot \cos y = \frac{1}{2}\{\sin(x-y) + \sin(x+y)\}, \qquad \text{odd}.$$

4.2 Integration of trigonometric polynomials

The task is to find the integral

$$\int \sin^m x \cdot \cos^n x \, dx, \qquad \text{for } m,\, n \in \mathbb{N}_0.$$

We shall in the following only consider one single term of the the form $\sin^m x \cdot \cos^n x$, where m and $n \in \mathbb{N}_0$, of a trigonometric polynomial, because we in general can find the result by linearity.

We define the *degree* of $\sin^m x \cdot \cos^n x$ as the sum $m+n$.

When we integrate such a single trigonometric product of degree $m+n$, we first must answer the following question: Is it of even or odd degree? These two possibilities are then again subdivided into to subcases, so we have four different variants of method, when we integrate a trigonometric polynomial.

Download free eBooks at bookboon.com

1) The degree $m+n$ is odd.
   a) $m=2p$ is even, and $n=2q+1$ is odd.
   b) $m=2p+1$ is odd, and $n=2q$ is odd.
2) The degree $m+n$ is even.
   a) $m=2p+1$ and $n=2q+1$ are both odd.
   b) $m=2p$ and $n=2q$ are both even.

We shall in the following go through the four possibilities.

1a) $m=2p$ is even and $n=2q+1$ is odd.

Use the substitution $u=\sin x$ (corresponding to $m=2p$ even) and write

$$\cos^{2q+1} x\,\mathrm{d}x = \left(1-\sin^2 x\right)^q \cos x\,\mathrm{d}x = \left(1-\sin^2 x\right)^q\,\mathrm{d}\sin x,$$

thus

$$\int \sin^{2p} x\cdot\cos^{2q+1} x\,\mathrm{d}x = \int \sin^{2p} x\left(1-\sin^2 x\right)^q\,\mathrm{d}\sin x = \int_{u=\sin x} u^{2p}\cdot\left(1-u^2\right)^q\,\mathrm{d}u,$$

and the problem is reduced to an integration of a polynomial, followed by a substitution.

1b) $m=2p+1$ odd and $n=2q$ even.

Apply the substitution $u=\cos x$ (corresponding to $n=2q$ even) and write

$$\sin^{2p+1} x\,\mathrm{d}x = \left(1-\cos^2 x\right)^p \cos x\,\mathrm{d}x = -\left(1-\cos^2 x\right)^p\,\mathrm{d}\cos x,$$

from which

$$\int \sin^{2p+1} x\cdot\cos^{2q} x\,\mathrm{d}x = -\int \left(1-\cos^2 x\right)^p\cdot\cos^{2q} x\,\mathrm{d}\cos x = -\int_{u=\cos x}\left(1-u^2\right)^p\cdot u^{2q}\,\mathrm{d}u,$$

and the problem is again reduced to an integration of a polynomial followed by a substitution.

2) When the degree $m+n$ is even, the trick is to use the double angle, using the formulæ

$$\sin^2 x = \frac{1}{2}(1-\cos 2x), \qquad \cos^2 x = \frac{1}{2}(1+\cos 2x), \qquad \sin x\cdot\cos x = \frac{1}{2}\sin 2x.$$

2a) $m=2p+1$ and $n=2q+1$ are both odd.

Rewrite the integrand in the following way,

$$\sin^{2p+1} x\cdot\cos^{2q+1} x = \left\{\frac{1}{2}(1-\cos 2x)\right\}^p\left\{\frac{1}{2}(1+\cos 2x)\right\}^q\cdot\frac{1}{2}\sin 2x.$$

This is a reduction to case 1b) above, so by the substitution $u=\cos 2x$ we get

$$\int \sin^{2p+1} x\cdot\cos^{2q+1} x\,\mathrm{d}x = -\frac{1}{2^{p+q+1}}\cdot\frac{1}{2}\int_{u=\cos 2x}(1-u)^p(1+u)^q\,\mathrm{d}u,$$

and the problem is again reduced to an integration of a polynomial followed by a substitution.

Download free eBooks at bookboon.com

2b) $m = 2p$ and $n = 2q$ are both even.

This is the most difficult one of the four cases. First rewrite the integrand in the following way,

$$\sin^{2p} x \cdot \cos^{2q} = \left\{ \frac{1}{2} (1 - \cos 2x) \right\}^p \left\{ \frac{1}{2} (1 + \cos 2x) \right\}^q .$$

The degree of the left hand side is $2p + 2q$ in the pair $(\cos x, \sin x)$, while the right hand side only has the degree $p + q$ in the pair $(\cos 2x, \sin 2x)$ with the double angle as new variable. The problem is that we at the same time by a multiplication get many terms on the right hand side of the equation, which then must be computed separately.

However, since the degree is halved, whenever 2b) is applied, the problem can be solved in a finite number of steps.

We shall illustrate the method of 2b) in the following example.

Need help with your dissertation?

Get in-depth feedback & advice from experts in your topic area. Find out what you can do to improve the quality of your dissertation!

Get Help Now

Go to www.helpmyassignment.co.uk for more info

Helpmyassignment

Click on the ad to read more

Download free eBooks at bookboon.com

Example 4.2.1 We shall compute the integral

$$\int \cos^6 x \, \mathrm{d}x.$$

The degree $0+6=6$ is even, and both $m=0$ and $n=6$ are even. Thus we are in case 2b). By using the double angle the integrand becomes

$$\cos^6 x = \left\{\frac{1}{2}(1+\cos 2x)\right\}^3 = \frac{1}{8}(1+3\,\cos 2x + 3\,\cos^2 2x + \cos^3 2x).$$

Integration of the first two terms is straightforward,

$$\frac{1}{8}\int (1+3\cos 2x)\,\mathrm{d}x = \frac{1}{8}\,x + \frac{3}{16}\,\sin 2x.$$

The third term is again of type 2b), so we transform it to the quadruple angle,

$$\frac{1}{8}\int 3\,\cos^2 2x\,\mathrm{d}x = \frac{3}{8}\int \frac{1}{2}(1+\cos 4x)\,\mathrm{d}x = \frac{3}{16}\,x + \frac{3}{64}\,\sin 4x.$$

The last term is of type 1a), so

$$\frac{1}{8}\int \cos^3 2x\,\mathrm{d}x = \frac{1}{8}\int \left(1-\sin^2 2x\right)\cdot\frac{1}{2}\,\mathrm{d}\sin 2x = \frac{1}{16}\,\sin 2x - \frac{1}{48}\,\sin^3 2x.$$

Summing up we get after a reduction,

$$\int \cos^6 x\,\mathrm{d}x = \frac{5}{16}\,x + \frac{1}{4}\,\sin 2x - \frac{1}{48}\,\sin^3 2x + \frac{3}{64}\,\sin 4x. \qquad \Diamond$$

Download free eBooks at bookboon.com

b!

	$f(t)$	$\mathcal{L}\{f\}(z)$	$\sigma(f)$
1	1	$\frac{1}{z}$	0
2	t^n	$\frac{n!}{z^{n+1}}$	0
3	e^{-at}	$\frac{1}{z+a}$	$-\Re a$
4	$\sin(at)$	$\frac{a}{z^2+a^2}$	$\lvert\Im a\rvert$
5	$\cos(at)$	$\frac{z}{z^2+a^2}$	$\lvert\Im a\rvert$
6	$\sinh(at)$	$\frac{a}{z^2-a^2}$	$\lvert\Re a\rvert$
7	$\cosh(at)$	$\frac{z}{z^2-a^2}$	$\lvert\Re a\rvert$

Table 1: The simplest Laplace transforms

4.3 Tables of some Laplace transforms and Fourier transforms

The simplest Laplace transforms were already derived in *Ventus, Complex Functions Theory a-4, The Laplace Transformation I.* These are given in Table 1.

We collect in the following tables the results from *Ventus, Complex Functions Theory a-5* where we always can use $\sigma(f) = 0$, so there is no need to specify $\sigma(f)$ in the tables. The first table is ordered according to the simplicity of the function $f(t)$, and the second one is ordered according to the simplicity of $\mathcal{L}\{f\}(z)$. Instead of $\sigma(f)$ we include a reference to where the function is handled in the text.

Download free eBooks at bookboon.com

	$f(t)$	$\mathcal{L}\{f\}(z)$	Reference
1	t^{α} for $\Re\,\alpha > -1$	$\dfrac{\Gamma(\alpha+1)}{z^{\alpha+1}}$	Complex Functions a-5
2	$\dfrac{1}{t+a}$ for $a>0$	$e^{az}\,\mathrm{Ei}(az)$	Complex Functions a-5
3	$\dfrac{1}{1+t^2}$	$\cos z \cdot \left\{\dfrac{\pi}{2} - \mathrm{Si}(z)\right\} - \sin z \cdot \mathrm{Ci}(z)$	Complex Functions a-5
4	$\ln t$	$-\dfrac{\gamma + \mathrm{Log}\, z}{z}$	Complex Functions a-5
5	$\dfrac{1}{\sqrt{\lvert t-1\rvert}}$	$\sqrt{\dfrac{\pi}{2}}\, e^{-z}\left\{1 - i \cdot \mathrm{erf}\left(i\sqrt{z}\right)\right\}$	Complex Functions a-5
6	$\exp\left(-t^2\right)$	$\dfrac{\sqrt{\pi}}{2}\exp\left(\dfrac{z^2}{2}\right)\mathrm{erfc}\left(\dfrac{z}{2}\right)$	Complex Function a-5
7	$t^{-\frac{3}{2}}\exp\left(-\dfrac{1}{4t}\right)$	$2\sqrt{\pi}\, e^{-\sqrt{z}}$	Complex Functions a-5
8	$\mathrm{erf}(t)$	$\dfrac{1}{z}\exp\left(\dfrac{z^2}{4}\right)\mathrm{erfc}\left(\dfrac{z}{2}\right)$	Complex Functions a-5
9	$\mathrm{erfc}(t)$	$\dfrac{1}{z}\left\{1 - \exp\left(\dfrac{z^2}{4}\right)\mathrm{erfc}\left(\dfrac{z}{2}\right)\right\}$	Complex Function a-5
10	$\mathrm{erfc}\left(\sqrt{t}\right)$	$\dfrac{1}{z\sqrt{z+1}}$	Complex Functions a-5
11	$\mathrm{erf}\left(\dfrac{1}{2\sqrt{t}}\right)$	$\dfrac{1-e^{-\sqrt{z}}}{z}$	Complex Functions a-5
12	$\mathrm{erfc}\left(\dfrac{1}{2\sqrt{t}}\right)$	$\dfrac{1}{z}\, e^{-\sqrt{z}}$	Complex Functions a-5
13	$\mathrm{Si}(t)$	$\dfrac{1}{z}\,\mathrm{Arctan}\,\dfrac{1}{z}$	Complex Functions a-5
14	$\mathrm{Ci}(t)$	$\dfrac{\mathrm{Log}\left(1+z^2\right)}{2z}$	Complex Functions a-5

Table 2: More advanced Laplace transforms

Download free eBooks at bookboon.com

	$f(t)$	$\mathcal{L}\{f\}(z)$	Reference
15	$\mathrm{Ei}(t)$	$\dfrac{\mathrm{Log}(1+z)}{z}$	Complex Functions a-5
16	$J_n(t)$ for $n \in \mathbb{N}_0$	$\dfrac{\left(\sqrt{z^2+1}-z\right)^n}{\sqrt{z^2+1}}$	Complex Functions a-5
17	$J_0\left(2\sqrt{t}\right)$	$\dfrac{1}{z}\exp\left(-\dfrac{1}{z}\right)$	Complex Functions a-5
18	$\dfrac{1}{\sqrt{t}}\,J_1\left(2\sqrt{t}\right)$	$1-\exp\left(-\dfrac{1}{z}\right)$	Complex Functions a-5

Table 3: More advanced Laplace transforms, continued

	$F(z)$	$\mathcal{L}^{-1}\{F\}(t)$	Reference
1	$\dfrac{1}{z}$	1	Complex Functions a-4
2	$\dfrac{1}{z+a}$	e^{-at}	Complex Functions a-4
3	z^{-n} for $n \in \mathbb{N}$	$\dfrac{1}{(n-1)!}\,t^{n-1}$	Complex Functions a-4
4	$z^{-\alpha}$, $\Re\,\alpha > 0$	$\dfrac{1}{\Gamma(\alpha)}\,t^{\alpha-1}$	Complex Functions a-5
5	$\dfrac{1}{z^2-a^2}, \quad a \neq 0$	$\dfrac{\sinh(at)}{a}$	Complex Functions a-4
6	$\dfrac{z}{z^2-a^2}$	$\cosh(at)$	Complex Functions a-4
7	$\dfrac{1}{z^2+a^2}, \quad a \neq 0$	$\dfrac{\sin(at)}{a}$	Complex Functions a-4
8	$\dfrac{z}{z^2+a^2}$	$\cos(at)$	Complex Functions a-4

Table 4: Table of inverse Laplace transforms

Download free eBooks at bookboon.com

	$F(z)$	$\mathcal{L}^{-1}\{F\}(t)$	Reference
9	$\dfrac{1}{z\sqrt{z+1}}$	$\operatorname{erf}\left(\sqrt{t}\right)$	Complex Functions a-5
10	$\dfrac{1}{\sqrt{z^2+1}}$	$J_0(t)$	{ Complex Functions a-4 and Complex Functions a-5
11	$\dfrac{\left(\sqrt{z^2+1}-z\right)^n}{\sqrt{z^2+1}}$ for $n \in \mathbb{N}_0$	$J_n(t)$	Complex Functions a-5
12	$1-\exp\left(-\dfrac{1}{z}\right)$	$\dfrac{1}{\sqrt{t}}\, J_1\left(2\sqrt{t}\right)$	Complex Functions a-5
13	$\dfrac{1}{z}\exp\left(-\dfrac{1}{z}\right)$	$J_0\left(2\sqrt{t}\right)$	Complex Functions a-5
14	$e^{-\sqrt{z}}$	$\dfrac{1}{2t\sqrt{\pi t}}\exp\left(-\dfrac{1}{4t}\right)$	Complex Functions a-5
15	$\dfrac{1}{z}\, e^{-\sqrt{z}}$	$\operatorname{erfc}\left(\dfrac{1}{2\sqrt{t}}\right)$	Complex Functions a-5
16	$\dfrac{1}{z}\left\{1-e^{-\sqrt{z}}\right\}$	$\operatorname{erf}\left(\dfrac{1}{2\sqrt{t}}\right)$	Complex Functions a-5
17	$\dfrac{\operatorname{Log} z}{z}$	$-\gamma-\ln t$	Complex Functions a-5
18	$\dfrac{1}{z}\operatorname{Log}(1+z)$	$\operatorname{Ei}(t)$	Complex Functions a-5
19	$\dfrac{1}{z}\operatorname{Log}\left(1+z^2\right)$	$2\operatorname{Ci}(t)$	Complex Functions a-5
20	$\dfrac{1}{z}\operatorname{Arctan}\dfrac{1}{z}$	$\operatorname{Si}(t)$	Complex Functions a-5
21	$\dfrac{1}{2}\operatorname{Log}\left(\dfrac{z+i}{z-i}\right)$	$2i\,\operatorname{Si}(t)$	Complex Functions a-5

Table 5: Table of inverse Laplace transforms, continued

Download free eBooks at bookboon.com

	$f(t)$	$\mathcal{F}\{f\}(\xi)$
1	$\chi_{[-T,T]}(x), \qquad T>0$	$2\,\dfrac{\sin T\xi}{\xi}$
2	$\left(1-\dfrac{\lvert x\rvert}{T}\right)\chi_{[-T,T]}(x), \qquad T>0$	$\dfrac{4}{T\xi^2}\,\sin^2\left(\dfrac{T\xi}{2}\right)$
3	$\dfrac{a}{x^2+a^2}, \qquad \Re\, a>0$	$\pi\, e^{-a\lvert\xi\rvert}$
4	$\dfrac{\sin(Tx)}{x}, \qquad T>0$	$\pi\,\chi_{[-T,T]}(\xi)$
5	$\cos(\omega\, x)\cdot\chi_{[-T,T]}(x), \qquad T>0$	$\dfrac{\sin(T(\xi-\omega))}{\xi-\omega}+\dfrac{\sin((T(\xi+\omega))}{\xi+\omega}$
6	$\sin(\omega\, x)\cdot\chi_{[-T,T]}(x), \qquad T>0$	$\dfrac{1}{i}\left\{\dfrac{\sin(T(\xi-\omega))}{\xi-\omega}-\dfrac{\sin(T(\xi+\omega))}{\xi+\omega}\right\}$
7	$e^{-a\lvert x\rvert}, \qquad \Re\, a>0$	$\dfrac{2a}{\xi^2+a^2}$
8	$e^{-ax}\,\chi_{\mathbb{R}_+}(x), \qquad \Re\, a>0$	$\dfrac{1}{a+i\xi}$
9	$e^{ax}\,\chi_{\mathbb{R}_-}(x), \qquad \Re\, a>0$	$\dfrac{1}{a-i\xi}$
10	$\exp\left(-ax^2\right), \qquad a>0$	$\sqrt{\dfrac{\pi}{a}}\cdot\exp\left(-\dfrac{\xi^2}{4a}\right)$
11	1	$2\pi\,\delta$
12	$x^n, \qquad n\in\mathbb{N}_0$	$2\pi\, i^n\,\delta^{(n)}$
13	$e^{ihx}, \qquad h\in\mathbb{R}$	$2\pi\,\delta_{(h)}$
14	$\cosh(hx), \qquad h\in\mathbb{R}$	$\pi\,\delta_{(h)}+\pi\,\delta_{(-h)}$
15	$\sin(hx), \qquad h\in\mathbb{R}$	$-i\,\pi\,\delta_{(h)}+i\,\pi\,\delta_{(-h)}$
16	δ	1
17	$\delta_{(h)}, \qquad h\in\mathbb{R}$	$e^{-ih\xi}$
18	$\delta^{(n)}, \qquad n\in\mathbb{N}_0$	$(i\xi)^n$

Table 6: Some Fourier transforms, $\mathcal{F}\{f\}(\xi)=\int_{-\infty}^{+\infty}e^{-ix\xi}\,f(x)\,\mathrm{d}x$.

Download free eBooks at bookboon.com

Index

Download free eBooks at bookboon.com