Real Functions in Several Variables: Volume XII

Vector Fields III

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Volume XII Vector Fields III Potentials Harmonic Functions Green's Identities

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Contents


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Preface

The topic of this series of books on *"Real Functions in Several Variables"* is very important in the description in e.g. *Mechanics* of the real 3-dimensional world that we live in. Therefore, we start from the very beginning, modelling this world by using the coordinates of $\mathbb{R}^3$ to describe e.g. a motion in space. There is, however, absolutely no reason to restrict ourselves to $\mathbb{R}^3$ alone. Some motions may be rectilinear, so only $\mathbb{R}$ is needed to describe their movements on a line segment. This opens up for also dealing with $\mathbb{R}^2$, when we consider plane motions. In more elaborate problems we need higher dimensional spaces. This may be the case in *Probability Theory* and *Statistics*. Therefore, we shall in general use $\mathbb{R}^n$ as our abstract model, and then restrict ourselves in examples mainly to $\mathbb{R}^2$ and $\mathbb{R}^3$.

For rectilinear motions the familiar *rectangular coordinate system* is the most convenient one to apply. However, as known from e.g. *Mechanics*, circular motions are also very important in the applications in engineering. It becomes natural alternatively to apply in $\mathbb{R}^2$ the so-called *polar coordinates* in the plane. They are convenient to describe a circle, where the rectangular coordinates usually give some nasty square roots, which are difficult to handle in practice.

Rectangular coordinates and polar coordinates are designed to model each their problems. They supplement each other, so difficult computations in one of these coordinate systems may be easy, and even trivial, in the other one. It is therefore important always in advance carefully to analyze the geometry of e.g. a domain, so we ask the question: Is this domain best described in rectangular or in polar coordinates?

Sometimes one may split a problem into two subproblems, where we apply rectangular coordinates in one of them and polar coordinates in the other one.

It should be mentioned that in *real life* (though not in these books) one cannot always split a problem into two subproblems as above. Then one is really in trouble, and more advanced mathematical methods should be applied instead. This is, however, outside the scope of the present series of books.

The idea of polar coordinates can be extended in two ways to $\mathbb{R}^3$. Either to *semi-polar* or *cylindric coordinates*, which are designed to describe a cylinder, or to *spherical coordinates*, which are excellent for describing spheres, where rectangular coordinates usually are doomed to fail. We use them already in daily life, when we specify a place on Earth by its longitude and latitude! It would be very awkward in this case to use rectangular coordinates instead, even if it is possible.

Concerning the contents, we begin this investigation by modelling point sets in an n-dimensional Euclidean space E^n by $\mathbb{R}^n$. There is a subtle difference between E^n and $\mathbb{R}^n$, although we often identify these two spaces. In E^n we use *geometrical methods* without a coordinate system, so the objects are independent of such a choice. In the coordinate space $\mathbb{R}^n$ we can use ordinary calculus, which in principle is not possible in E^n. In order to stress this point, we call E^n the "abstract space" (in the sense of calculus; not in the sense of geometry) as a warning to the reader. Also, whenever necessary, we use the colour black in the "abstract space", in order to stress that this expression is theoretical, while variables given in a chosen coordinate system and their related concepts are given the colours blue, red and green.

We also include the most basic of what mathematicians call *Topology*, which will be necessary in the following. We describe what we need by a function.

Then we proceed with limits and continuity of functions and define continuous curves and surfaces, with parameters from subsets of $\mathbb{R}$ and $\mathbb{R}^2$, resp..

Continue with (partial) differentiable functions, curves and surfaces, the chain rule and Taylor's formula for functions in several variables.

We deal with maxima and minima and extrema of functions in several variables over a domain in $\mathbb{R}^n$. This is a very important subject, so there are given many worked examples to illustrate the theory.

Then we turn to the problems of integration, where we specify four different types with increasing complexity, *plane integral, space integral, curve (or line) integral and surface integral.*

Finally, we consider *vector analysis*, where we deal with vector fields, Gauß's theorem and Stokes's theorem. All these subjects are very important in theoretical Physics.

The structure of this series of books is that each subject is *usually* (but not always) described by three successive chapters. In the first chapter a brief theoretical theory is given. The next chapter gives some practical guidelines of how to solve problems connected with the subject under consideration. Finally, some worked out examples are given, in many cases in several variants, because the standard solution method is seldom the only way, and it may even be clumsy compared with other possibilities.

I have as far as possible structured the examples according to the following scheme:

A *Awareness*, i.e. a short description of what is the problem.

D *Decision*, i.e. a reflection over what should be done with the problem.

I *Implementation*, i.e. where all the calculations are made.

C *Control*, i.e. a test of the result.

This is an ideal form of a general procedure of solution. It can be used in any situation and it is not linked to Mathematics alone. I learned it many years ago in the Theory of Telecommunication in a situation which did not contain Mathematics at all. The student is recommended to use it also in other disciplines.

From high school one is used to immediately to proceed to **I**. *Implementation.* However, examples and problems at university level, let alone situations in real life, are often so complicated that it in general will be a good investment also to spend some time on the first two points above in order to be absolutely certain of what to do in a particular case. Note that the first three points, **ADI**, can always be executed.

This is unfortunately not the case with **C** *Control*, because it from now on may be difficult, if possible, to check one's solution. It is only an extra securing whenever it is possible, but we cannot include it always in our solution form above.

I shall on purpose not use the logical signs. These should in general be avoided in Calculus as a shorthand, because they are often (too often, I would say) misused. Instead of $\wedge$ I shall either write "and", or a comma, and instead of $\vee$ I shall write "or". The arrows $\Rightarrow$ and $\Leftrightarrow$ are in particular misunderstood by the students, so they should be totally avoided. They are not telegram short hands, and from a logical point of view they usually do not make sense at all! Instead, write in a plain language what you mean or want to do. This is difficult in the beginning, but after some practice it becomes routine, and it will give more precise information.

When we deal with multiple integrals, one of the possible pedagogical ways of solving problems has been to colour variables, integrals and upper and lower bounds in blue, red and green, so the reader by the colour code can see in each integral what is the variable, and what are the parameters, which do not enter the integration under consideration. We shall of course build up a hierarchy of these colours, so the order of integration will always be defined. As already mentioned above we reserve the colour black for the theoretical expressions, where we cannot use ordinary calculus, because the symbols are only shorthand for a concept.

The author has been very grateful to his old friend and colleague, the late Per Wennerberg Karlsson, for many discussions of how to present these difficult topics on real functions in several variables, and for his permission to use his textbook as a template of this present series. Nevertheless, the author has felt it necessary to make quite a few changes compared with the old textbook, because we did not always agree, and some of the topics could also be explained in another way, and then of course the results of our discussions have here been put in writing for the first time.

The author also adds some calculations in MAPLE, which interact nicely with the theoretic text. Note, however, that when one applies MAPLE, one is forced first to make a geometrical analysis of the domain of integration, i.e. apply some of the techniques developed in the present books.

The theory and methods of these volumes on "Real Functions in Several Variables" are applied constantly in higher Mathematics, Mechanics and Engineering Sciences. It is of paramount importance for the calculations in *Probability Theory*, where one constantly integrate over some point set in space.

It is my hope that this text, these guidelines and these examples, of which many are treated in more ways to show that the solutions procedures are not unique, may be of some inspiration for the students who have just started their studies at the universities.

Finally, even if I have tried to write as careful as possible, I doubt that all errors have been removed. I hope that the reader will forgive me the unavoidable errors.

Leif Mejlbro
March 21, 2015

Introduction to volume XI, Vector Fields III; Potentials, Harmonic Functions and Green's Identities

This is the twelfth volume in the series of books on *Real Functions in Several Variables.*

It is the third volume on Vector Fields. It was necessary to split the material into three volumes because the material is so big. In the first volume we dealt with the *tangential line integral*, which e.g. can be used to describe the work of a particle when it is forced along a given curve by some force. It was natural to introduce the *gradient fields*, where the tangential line integral only depends on the initial and the terminal points of the curve and not of the curve itself. Such gradients fields are describing *conservative forces* in Physics.

Tangential line integrals are one-dimensional in nature. In case of two dimensions we consider the *flux* of a flow through a surface. When the surface $\partial\Omega$ is surrounding a three dimensional body Ω, this leads to *Gauß's theorem*, by which we can express the flux of a vector field $\mathbf{V}$ through $\partial\Omega$, which is a surface integral, by a space integral over Ω of the *divergence* of the vector field $\mathbf{V}$. This theorem works both ways. Sometimes, and most frequently, the surface integral is expressed as space integral, other times we express a space integral as a flux, i.e. a surface integral. Applications are obvious in *Electro-Magnetic Field Theory*, though other applications can also be found.

In this book we introduce the scalar potential H of a gradient field $\mathbf{V}$, i.e. if we know that $\mathbf{V}$ is a gradient field, then it can be written $\mathbf{V} = -\bigtriangledown H$, where the minus sign has conventionally been added. Similarly, if instead $\mathbf{V}$ is a rotational field, then there exists a vectorial potential $\mathbf{W}$, such that $\mathbf{V} = \bigtriangledown \times \mathbf{W} = \mathbf{rot}\mathbf{W}$. Given that $\mathbf{V}$ is a gradient field, we set up a solution formula for the corresponding scalar potential. Similarly, if we know that $\mathbf{V}$ is a rotational field, so its vectorial potential $\mathbf{W}$ exists, we set up a solution formula for $\mathbf{W}$. By using these ideas it is possible under mild assumptions, from the knowledge of $p = \text{div}\ \mathbf{V}$ and $\mathbf{P} = \mathbf{rot}\mathbf{V}$ to reconstruct the vector field $\mathbf{V}$.

We proceed by giving some applications in Physics and Electomagnetism, before we turn to *Poisson's equation* and *Laplace's equation.* The solutions of the latter equation are called *harmonic functions*, which in their two-dimensional version also occurs in *Complex Functions Theory*, because it is not hard to prove that the real part as well as the imaginary part of an analytic function are harmonic functions in 2 dimensions. Finally, we show Green's three identities.

38 Potentials

38.1 Definitions

It was previously shown that a gradient field is rotational free, i.e. $\bigtriangledown \times \bigtriangledown f = \mathbf{0}$, and also that a rotational field is divergence free, i.e. $\bigtriangledown \cdot (\bigtriangledown \times \mathbf{V}) = 0$. It is then quite natural to ask the following questions,

1) Is a rotational free vector field also a gradient field?

2) Is a divergence free vector field also a rotational field?

The answer is 'yes', if we add an extra assumption on the domain of $\mathbf{V}$, namely that it is *star shaped*. We have previously dealt with the first question, so we recall that if $\mathbf{V}$ is rotational free, $\mathbf{rot}\mathbf{V} = \bigtriangledown \times \mathbf{V} = \mathbf{0}$ in a star shaped domain, then there exists a *primitive* F, such that $\mathbf{V} = \bigtriangledown F$.

In the applications in Physics one by convention introduces a minus sign, so $F = -H$, and we get

$$\mathbf{V} = - \bigtriangledown H,$$

and the scalar field H is then called a *scalar potential* of $\mathbf{V}$. We note that if H and G both are scalar potentials of the same vector field $\mathbf{V}$, then there exists a constant $c \in \mathbb{R}$, such that

$$G(\mathbf{x}) = H(\mathbf{x}) + c.$$

In other words, every scalar potential of $\mathbf{V}$ has the form $H + c$, $c \in \mathbb{R}$, if H is a scalar potential.

There are several ways of calculating the (candidate of a) scalar potential. We list some of them in one of the following sections. We shall here for theoretical reasons focus on the following reconstruction formula, where we assume that we can use $\mathbf{0}$ as a *star point* in the *star shaped domain* Ω of $\mathbf{V}$, i.e. if $\mathbf{x} \in \Omega$, then also $\tau\mathbf{x} \in \Omega$ for all $\mathbf{x} \in [0,1]$, so all points of the line segment

$$[\mathbf{0};\mathbf{x}] := \{\tau\mathbf{x} \mid 0 \le \tau \le 1\} \subseteq \Omega \qquad \text{for all } \mathbf{x} \in \Omega.$$

Then a (candidate of a) *scalar potential* is given by

$$H_0(\mathbf{x}) := -\mathbf{x} \cdot \int_0^1 \mathbf{V}(\tau\mathbf{x})\,\mathrm{d}\tau \quad \left(= \int_0^1 \mathbf{V}(\tau\mathbf{x})\,\mathrm{d}\tau \cdot \mathbf{x} \right) \qquad \text{for } \mathbf{x} \in \Omega.$$

Before we prove this formula under the given assumption we shall give some *warnings* concerning the practiacl applications of this formula:

Even if Ω is star shaped with $\mathbf{0}$ as a star point, this formula does not automatically make the result H_0 a scalar potential. If $\mathbf{rot}\mathbf{V} \neq \mathbf{0}$, we know already that the scalar potential does not exist, because $\mathbf{V}$ is not a gradient field. And yet the formula can be applied! It only gives a wrong answer! Therefore, whenever one uses a solution formula like the above one should always afterwards check, if we indeed have $\mathbf{V} = - \bigtriangledown H_0$.

Another warning is the following. Although the solution formula apparently is straightforward to apply, it usually gives some very nasty calculations. One should instead try other methods, which are also applicable. It is mentioned here for theoretical reasons, because it gives a hint of how we should solve the other problem mentioned above.

We turn to the proof that H_0 given by

$$H_0(\mathbf{x}) := -\mathbf{x} \cdot \int_0^1 \mathbf{V}(\tau \mathbf{x})\, \mathrm{d}\tau \qquad \text{for } \mathbf{x} \in \Omega,$$

is a scalar potential, i.e. $\mathbf{V} = -\nabla H_0$, when Ω is star shaped, and $\mathbf{0}$ a star point, and $\mathbf{rot}\,\mathbf{V} = \mathbf{0}$. The latter condition is equivalent to

$$\frac{\partial V_i}{\partial x_j} = \frac{\partial V_j}{\partial x_i}, \qquad \text{for all } i,\, j \in \{1,\, 2,\, 3\}.$$

We shall use the notation

$$F(\mathbf{x}) := \mathbf{x} \cdot \int_0^1 \mathbf{V}(\tau \mathbf{x})\, \mathrm{d}\tau = \sum_{i=1}^3 x_i \int_0^1 V_i(\tau \mathbf{x})\, \mathrm{d}\tau = \sum_{i=1}^3 x_i \int_0^1 \mathbf{U}(\mathbf{x}, \tau)\, \mathrm{d}\tau,$$

where we for convenience later on have put

$$\mathbf{U}(\mathbf{x}, \tau) = \mathbf{V}(\tau \mathbf{x}), \qquad \text{with } \mathbf{U}(\mathbf{x}, 1) = \mathbf{V}(\mathbf{x}).$$

Due to the change of sign we shall prove that $\mathbf{V} = \nabla F$, so $H = -F$. We get by the *chain rule*,

$$\frac{\partial U_i}{\partial x_j} = \tau\, D_j V_i(\tau \mathbf{x}) \qquad \text{and} \qquad \frac{\partial U_i}{\partial \tau} = \sum_{j=1}^3 x_j\, D_j V_i(\tau \mathbf{x}),$$

where we have written $D_j V_i(\mathbf{y}) = \partial V_i / \partial y_j$, no matter that $\mathbf{y}$ is equal to $\tau\mathbf{x}$ in the following. Then we calculate the gradient of F by differentiating under the sign of integration by calculating each coordinate separately,

$$\begin{aligned}\frac{\partial F}{\partial x_j}(\mathbf{x}) &= \sum_{i=1}^{3} \left\{ \frac{\partial x_i}{\partial x_j} \int_0^1 U_i \,\mathrm{d}\tau + x_i \int_0^1 \frac{\partial U_i}{\partial x_j} \,\mathrm{d}\tau \right\} = \int_0^1 U_j \,\mathrm{d}\tau + \sum_{i=1}^{3} x_i \int_0^1 \tau\, D_j V_i(\tau\mathbf{x}) \,\mathrm{d}\tau \\ &= \int_0^1 U_j \,\mathrm{d}\tau + \sum_{i=1}^{3} x_i \int_0^1 \tau\, D_i V j(\tau\mathbf{x}) \,\mathrm{d}\tau \qquad D_j V_i = D_i V_j \text{ by assumption} \\ &= \int_0^1 U_j \,\mathrm{d}\tau + \int_0^1 \tau \frac{\partial U_j}{\partial \tau} \,\mathrm{d}\tau = \int_0^1 \frac{\partial}{\partial \tau} (\tau U_j) \,\mathrm{d}\tau = [\tau\, V_j(\tau\mathbf{x})]_{\tau=0}^1 = V_j(\mathbf{x}),\end{aligned}$$

from which we conclude that $\bigtriangledown F = \mathbf{V}$. Since $H = -F$, the claim follows.

We leave the scalar potential and turn to the *vectorial potentials*. We assume that $\mathbf{V}$ is a divergence free vector field, $\bigtriangledown \cdot \mathbf{V} = \operatorname{div} \mathbf{V} = 0$, in the star shaped domain Ω. Then the formula above for the scalar potential of a rotational free vector field gives us a hint that we may look for a formula of the form

$$\mathbf{W}_0(\mathbf{x}) = -\int_0^1 \tau\mathbf{x} \times \mathbf{V}(\tau\mathbf{x}) \,\mathrm{d}\tau = -\mathbf{x} \times \int_0^1 \tau\mathbf{V}(\tau\mathbf{x}) \,\mathrm{d}\tau \quad \left(= \int_0^1 \tau\mathbf{V}(\tau\mathbf{x}) \,\mathrm{d}\tau \times \mathbf{x} \right)$$

for $\mathbf{x} \in \Omega$, provided that $\bigtriangledown \cdot \mathbf{V} = 0$. We shall prove that this formula indeed is correct.

To shorten the notation we put

$$\mathbf{W}_0(\mathbf{x}) = -\mathbf{x} \times \mathbf{S}(\mathbf{x}), \qquad \text{where } \mathbf{S}(\mathbf{x}) := \int_0^1 \tau\mathbf{V}(\tau\mathbf{x}) \,\mathrm{d}\tau = \int_0^1 \tau\mathbf{U} \,\mathrm{d}\tau,$$

where we again write

$$\mathbf{U}(\mathbf{x}, \tau) := \mathbf{V}(\tau\mathbf{x}), \qquad \text{and note that } \mathbf{U}(\mathbf{x}, 1) = \mathbf{V}(\mathbf{x}).$$

Using the rules of calculation of nabla, listed in Section 36.2, we get

$$\begin{aligned}\bigtriangledown \times \mathbf{W}_0 &= \bigtriangledown \times (\mathbf{S} \times \mathbf{x}) = (\bigtriangledown \cdot \mathbf{x})\mathbf{S} + (\mathbf{x} \cdot \bigtriangledown)\mathbf{S} - (\bigtriangledown \cdot \mathbf{S})\mathbf{x} - (\mathbf{S} \cdot \bigtriangledown)\mathbf{x} \\ &= 3\mathbf{S} + (\mathbf{x} \cdot \bigtriangledown)\mathbf{S} - (\bigtriangledown \cdot \mathbf{S})\mathbf{x} - \mathbf{S} = 2\mathbf{S} + (\mathbf{x} \cdot \bigtriangledown)\mathbf{S} - (\bigtriangledown \cdot \mathbf{S})\mathbf{x}.\end{aligned}$$

As above we get by the chain rule for U_i, where $\mathbf{U} = (U_1, U_2, U_3)$,

$$\frac{\partial U_i}{\partial x_j} = \tau\, D_j V_i(\tau\mathbf{x}) \qquad \text{and} \qquad \frac{\partial U_i}{\partial \tau} = \sum_{j=1}^{3} x_j\, D_j V_i(\tau\mathbf{x}),$$

where $D_j V_i(\mathbf{y}) := \partial V_i / \partial y_j$. Since by assumption,

$$\sum_{i=1}^{3} \frac{\partial V_i}{\partial y_i}(\mathbf{y}) = \operatorname{div} \mathbf{V} = \bigtriangledown \cdot \mathbf{V} = 0,$$

we see that

$$\bigtriangledown \cdot \mathbf{S}(\mathbf{x}) = \int_0^1 \tau \sum_{i=1}^{3} \frac{\partial U_i}{\partial x_i} \,\mathrm{d}\tau = \int_0^1 \tau^2 \sum_{i=1}^{3} D_i V_i(\tau\mathbf{x}) \,\mathrm{d}\tau = 0,$$

and we have proved that $\mathbf{S}$ is also divergence free, $\bigtriangledown \cdot \mathbf{S} = 0$, so the result above is reduced to

$$\bigtriangledown \times \mathbf{W}_0 = 2\mathbf{S} + (\mathbf{x} \cdot \bigtriangledown)\mathbf{S}.$$

Finally, we calculate the i-th coordinate of the differential $(\mathbf{x} \cdot \bigtriangledown)\mathbf{S}$,

$$\begin{aligned}(\mathbf{x} \cdot \bigtriangledown)S_i(\mathbf{x}) &= \sum_{j=1}^3 x_j \int_0^1 \frac{\partial}{\partial x_j}(\tau U_i)\, \mathrm{d}\tau = \sum_{j=1}^3 x_j \int_0^1 \tau\, \frac{\partial U_i}{\partial x_j}\, \mathrm{d}\tau \\ &= \int_0^1 \left\{ \tau \sum_{j=1}^3 x_j \tau\, D_j V_i(\tau \mathbf{x}) \right\} \mathrm{d}\tau = \int_0^1 \left\{ \tau^2 \sum_{j=1}^3 x_j\, D_j V_j(\tau \mathbf{x}) \right\} \mathrm{d}\tau \\ &= \int_0^1 \tau^2\, \frac{\partial U_i}{\partial \tau}\, \mathrm{d}\tau = [\tau * 2U_i]_{\tau=0}^1 - \int_0^1 2\tau U_i\, \mathrm{d}\tau = V_i(\mathbf{x}) - 2S_i(\mathbf{x}).\end{aligned}$$

Then the i-th coordinate of $\bigtriangledown \times \mathbf{W}_0$ becomes

$$(\bigtriangledown \times \mathbf{W}_0)_i\,(\mathbf{x}) = 2S_i(\mathbf{x}) + (\mathbf{x} \cdot \bigtriangledown)\mathbf{S}(x) = 2S_i(\mathbf{x}) + V_i(\mathbf{x}) - 2S_i(\mathbf{x}) = V_i(\mathbf{x}),$$

so we conclude that under the assumptions above we indeed have $\bigtriangledown \times \mathbf{W}_0 = \mathbf{V}$.

We have proved under the assumptions that the domain Ω is star shaped and $\mathbf{0}$ a star point, that the formula

$$\mathbf{W}_0(\mathbf{x}) = -\mathbf{x} \times \int_0^1 \tau \mathbf{V}(\tau \mathbf{x})\, \mathrm{d}\tau = \int_0^1 \tau \mathbf{V}(\tau \mathbf{x})\, \mathrm{d}\tau \times \mathbf{x}$$

defines a vector field $\mathbf{W}_0$, for which

$$\mathbf{V} = \bigtriangledown \times \mathbf{W}_0 = \mathbf{rot} \mathbf{W}_0.$$

We call $\mathbf{W}_0$, derived in this way, a *vectorial potential* of the divergence free vector field $\mathbf{V}$.

Clearly, if a vector potential $\mathbf{W}_0$ of $\mathbf{V}$ exists in Ω, and $\bigtriangledown g$ is any gradient field in Ω, then since $\bigtriangledown \times \bigtriangledown g = \mathbf{0}$ we also get that $\mathbf{W}_0 + \bigtriangledown g$ is a vector potential of $\mathbf{V}$. In fact, a check shows by the linearity of **rot** that

$$\bigtriangledown \times (\mathbf{W}_0 + \bigtriangledown g) = \bigtriangledown \times \mathbf{W}_0 + \bigtriangledown \times \bigtriangledown g = \mathbf{V} + \mathbf{0} = \mathbf{V}.$$

Conversely, if both $\mathbf{W}_0$ and $\mathbf{W}_1$ are vector potentials of the same vector field $\mathbf{V}$, i.e.

$$\mathbf{V} = \bigtriangledown \times \mathbf{W}_0 = \bigtriangledown \times \mathbf{W}_1,$$

then we get by subtraction that

$$\bigtriangledown \times (\mathbf{W}_0 - \mathbf{W}_1) = \mathbf{0},$$

which shows that the vector field $\mathbf{W}_0 - \mathbf{W}_1$ is rotational free. Since we have assumed that the domain Ω is star shaped, it follows from the first part that there exists a *scalar potential* g, such that

$$\mathbf{W}_0 - \mathbf{W}_1 = -\bigtriangledown g, \qquad \text{i.e. by a rearrangement} \qquad \mathbf{W}_1 = \mathbf{W}_0 + \bigtriangledown g.$$

We collect the results above in the following

Theorem 38.1 Theorem of potentials. *Assume that* $\Omega \subseteq \mathbb{R}^3$ *is a star shaped domain of a* C^1 *vector field* $\mathbf{V} : \Omega \to \mathbb{R}^3$.

1) *If* $\bigtriangledown \times \mathbf{V} = \mathbf{rot}\mathbf{V} = \mathbf{0}$ *in all of* Ω*, then* $\mathbf{V}$ *is a* gradient field*, i.e. there exists a* scalar potential $H : \Omega \to \mathbb{R}$*, such that*

$$\mathbf{V} = -\bigtriangledown H \qquad in\ \Omega.$$

If H *is a scalar potential of* $\mathbf{V}$*, then every scalar potential of* $\mathbf{V}$ *has the form* $H + c$ *for some constant* c.

2) *If* $\bigtriangledown \cdot \mathbf{V} = \text{div}\ \mathbf{V} = 0$ *in all of* Ω*, then* $\mathbf{V}$ *is a rotational field, i.e.* $\mathbf{V}$ *can be written as a rotation of a* vectorial potential $\mathbf{W}$*, i.f.*

$$\mathbf{V} = \bigtriangledown \times \mathbf{W} = \mathbf{rot}\mathbf{W}.$$

If $\mathbf{W}$ *is a vectorial potential for* $\mathbf{V}$*, then to any other vectorial potential* $\mathbf{W}_1$ *of* $\mathbf{V}$ *there exists a function* g*, such that* $\mathbf{W}_1 = \mathbf{W} + \bigtriangledown g$.

We note the following application: Assume that $\mathbf{W}$ is a vectorial potential of $\mathbf{V}$, i.e. $\mathbf{V} = \bigtriangledown \times \mathbf{W}$, and $\mathbf{V}$ is divergence free. Let $\mathcal{F}$ be a C^1 (bounded) surface of closed piecewise C^1 booundary $\delta\mathcal{F}$. Then it follows from *Stokes's theorem* that

$$\int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n}\, \mathrm{d}S = \int_{\mathcal{F}} \mathbf{n} \cdot \bigtriangledown \times \mathbf{W}\, \mathrm{d}S = \oint_{\delta\mathcal{F}} \mathbf{W} \cdot \mathbf{t}\, \mathrm{d}s,$$

and we have proved

Theorem 38.2 *The flux of a divergence free vector field* $\mathbf{V}$*, which has the vectorial potential* $\mathbf{W}$*, is equal to the circulation of* $\mathbf{W}$ *along the boundary curve, i.e. if* $\mathbf{V} = \bigtriangledown \times \mathbf{W}$*, then*

$$\int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n}\, \mathrm{d}S = \oint_{\delta\mathcal{F}} \mathbf{W} \cdot \mathbf{t}\, \mathrm{d}s.$$

Concerning the reconstruction formula of the vectorial potential (always first check that $\bigtriangledown \mathbf{V} = 0$),

$$\mathbf{W}(\mathbf{x}) = -\mathbf{x} \times \int_0^1 \tau \mathbf{V}(\tau\mathbf{x})\, \mathrm{d}\tau = \int_0^1 \tau \mathbf{V}(\tau\mathbf{x})\, \mathrm{d}\tau \times \mathbf{x},$$

it should be noted that unlike the reconstruction formula for the scalar potential, where the path of integration can be modified, we are here forced to use integration along the line segment

$$[\mathbf{0}; \mathbf{x}] := \{\tau\mathbf{x} \mid 0 \le \tau \le 1\} \subseteq \Omega \qquad \text{for } \mathbf{x} \in \Omega.$$

If we change this path of integration without changing the integrand appropriately, we shall obtain a wrong result. However, in order not to make things too difficult in the computations when using this formula, we recommend that one first calculates the integral

$$\mathbf{U}(\mathbf{x}) := \int_0^1 \tau \mathbf{V}(\tau\mathbf{x})\, \mathrm{d}\tau \qquad \mathbf{x} \in \Omega,$$

and then the vector product,

$$\mathbf{W}(\mathbf{x}) = \mathbf{U}(\mathbf{x}) \times \mathbf{x}.$$

Example 38.1 We shall give a couple of very simple examples to illustrate the formula. The vector field

$$\mathbf{V}(x,y,z)=\left(3y^2z^2,xy,-xz\right)\qquad \text{for } (x,y,z)\in\mathbb{R}^3,$$

satisfies the condition

$$\text{div}\;\mathbf{V}=0+x-x=0,$$

so $\mathbf{V}$ is divergence free. Since $\mathbb{R}^3$ is trivially star shaped, vector potentials exist, and one of them can be calculated by the formula given above. We first compute

$$\mathbf{S}(x,y,z)=\int_0^1\tau\mathbf{V}(x\tau,y\tau,z\tau)\,\mathrm{d}\tau=\int_0^1\left(3y^2z^2\tau^5,xy\tau^3,-xz\tau^3\right)\,\mathrm{d}\tau=\left(\frac{1}{2}\,y^2z^2,\frac{1}{4}\,xy,-\frac{1}{4}\,xz\right).$$

Then a vector potential is given by

$$\begin{aligned}\mathbf{W}_0(x,y,z)&=\mathbf{S}(x,y,z)\times(x,y,z)=\left|\begin{array}{ccc}\mathbf{e}_1&\mathbf{e}_2&\mathbf{e}_3\\ \frac{1}{2}\,y^2z^2&\frac{1}{4}\,xy&-\frac{1}{4},xz\\ x&y&z\end{array}\right|\\&=\left(\frac{1}{2}\,xyz,-\frac{1}{4}\,x^2z-\frac{1}{2}\,y^2z^3,\frac{1}{2},y^3z^2-\frac{1}{2}\,x^2y\right).\qquad\Diamond\end{aligned}$$

Example 38.2 Even if Ω is not star shaped, a vector potential may exist. If e.g.

$$\mathbf{V}(x,y,z)=\left(\frac{y-z}{r},\frac{z-x}{r},\frac{x-y}{r}\right),\qquad\text{where } r=\sqrt{x^2+y^2+z^2}\neq 0,$$

then it is easy to check that div $\mathbf{V}=0$. (The calculations are left to the reader.) The domain $\Omega=\mathbb{R}^2\setminus\{\mathbf{0}\}$ is not star shaped, but it is easy to check (also left to the reader) that

$$\mathbf{U}(x,y,z):=(r,r,r)\qquad\text{for } r\neq 0,$$

is a vector potential, $\mathbf{V}=\mathbf{rot}\mathbf{U}=\bigtriangledown\times\mathbf{U}$. $\Diamond$

Example 38.3 If $\mathbf{V}_0$ is a *constant* vector field in $\mathbb{R}^3$, then we have trivially both

$$\text{div }\mathbf{V}_0=0\qquad\text{and}\qquad \mathbf{rot}\mathbf{V}_0=\mathbf{0},$$

so $\mathbf{V}_0$ has both scalar potentials and vector potentials. Using the formulæ above we get the scalar potential,

$$H_0(\mathbf{x})=-\mathbf{x}\cdot\int_0^1\mathbf{V}_0(\tau\mathbf{x})\,\mathrm{d}\tau=-\mathbf{x}\cdot\mathbf{V}_0\int_0^1\mathrm{d}\tau=-\mathbf{x}\cdot\mathbf{V}_0,$$

and the vector potential

$$\mathbf{W}_0(\mathbf{x})=\int_0^1\tau\mathbf{V}_0(\tau\mathbf{x})\,\mathrm{d}\tau\times\mathbf{x}=\int_0^1\tau\,\mathrm{d}\tau\,\mathbf{V}_0\times\mathbf{x}=\frac{1}{2}\,\mathbf{V}_0\times\mathbf{x}.\qquad\Diamond$$

38.2 A vector field given by its rotation and divergence

We shall in this section consider the following problem:

> Given the divergence $\mathrm{d}\mathbf{V}=\bigtriangledown\cdot\mathbf{V}=p$ and the rotation $\mathbf{rot}\mathbf{V}=\bigtriangledown\times\mathbf{V}=\mathbf{P}$ of an unknown vector field $\mathbf{V}$. What conditions should be put on the given scalar field p and vector field $\mathbf{P}$ in order to find a formula for the vector field $\mathbf{V}$ itself?

We shall give some sufficient conditions, namely that p and $\mathbf{P}$ are of class C^1, and that they both vanish outside a bounded domain $\Omega\subset\mathbb{R}^3$. These assumptions will in general be sufficient for the applications, because in Physics we may in most cases assume that the model is dealing with smooth functions, and also that the processes are bounded in space.

As usual we shall not give a strict proof; only sketch it quoting a result, which cannot be proved here.

Due to the linearity the problem can be split into two simpler problems which then are solved separately. We shall first find $\mathbf{V}_1$, when

$$\text{div }\mathbf{V}_1=\bigtriangledown\cdot\mathbf{V}_1=p,\qquad\text{and}\qquad \mathbf{rot}\mathbf{V}_1=\bigtriangledown\times\mathbf{V}_1=\mathbf{0}.$$

Here p is a scalar field, which is assumed to be 0 outside a bounded domain Ω. In mathematical terms, p is assumed to have *compact support* contained in the compact set $\overline{\Omega}\subset\mathbb{R}^3$. (In Euclidean spaces a *compact set* is a bounded closed set.)

Since $\mathbf{V}_1$ is supposed to be rotational free in $\mathbb{R}^3$, it is a *gradient field*, so there exists a scalar potential H, such that

$$\mathbf{V}_1 = - \bigtriangledown H.$$

Then by assumption,

$$\operatorname{div} \mathbf{V}_1 = \bigtriangledown \cdot \mathbf{V}_1 = - \bigtriangledown \cdot \bigtriangledown H = - \bigtriangledown^2 H = p,$$

so the scalar potential H must satisfy the so-called *Poisson equation*

$$\bigtriangledown^2 H = -p, \qquad \text{i.e.} \qquad \frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + \frac{\partial^2 H}{\partial z^2} = -p\,(x,y,z)\,.$$

Since p is zero outside the bounded set Ω, one can prove the following *solution formula for Poisson's equation*

$$H(x,y,z) = \frac{1}{4\pi} \int_\Omega \frac{p(\xi,\eta,\zeta)}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta.$$

Clearly, this space integral is improper, when $(x,y,z) \in \Omega$. However, it is easy to see that it is convergent. Let $(x,y,z) \in \Omega$ be fixed. By a translation of the coordinate system we may assume that $(x,y,z) = (0,0,0)$, in which case the denominator in spherical coordinates becomes the radius r, while the weight function $r^2 \sin\theta$ easily cancels the factor r in the denominator, when $r \to 0$. This can be done for every point of Ω, and since Ω is bounded and p is continuous, the improper integral is convergent. We shall not go further into the proof of that this really is the right solution.

Next we consider the similar problem, where instead

$$\operatorname{div} \mathbf{V}_2 = \bigtriangledown \cdot \mathbf{V}_2 = 0 \qquad \text{and} \qquad \mathbf{rot}\mathbf{V}_2 = \bigtriangledown \times \mathbf{V}_2 = \mathbf{P},$$

where $\mathbf{P}$ is a given divergence free (otherwise $\bigtriangledown \times \mathbf{V}_2 = \mathbf{P}$ would not make sense) vector field, which is $\mathbf{0}$ outside the bounded set Ω, so $\mathbf{P}$ has also compact support, contained in $\overline{\Omega}$.

Then $\mathbf{P}$ has a vector potential $\mathbf{W}$, i.f. $\mathbf{V}_2 = \bigtriangledown \times \mathbf{W}$. We assume that $\operatorname{div} \mathbf{W} = \bigtriangledown \cdot \mathbf{W} = 0$, which actually also will follow from the solution formula below.

Using the assumption above it follows that

$$\mathbf{P} = \bigtriangledown \times \mathbf{V} = \bigtriangledown \times (\bigtriangledown \times \mathbf{W}) = \bigtriangledown(\bigtriangledown \cdot \mathbf{W}) - \bigtriangledown^2 \mathbf{W} = - \bigtriangledown^2 \mathbf{W},$$

so $\mathbf{W}$ satisfies the *vectorial Poisson equation*

$$\bigtriangledown^2 \mathbf{W} = -\mathbf{P}, \qquad \text{which is equivalent to} \quad \begin{cases} \bigtriangledown^2 W_x = -P_x, \\ \bigtriangledown^2 W_y = -P_y, \\ \bigtriangledown^2 W_z = -P_z. \end{cases}$$

Each of these equations is a scalar Poisson equation, and since P_x, P_y and P_z are all zero outside the bounded set $\overline{\Omega}$, we can use the same solution formula as above, so we get

$$\mathbf{W}(x,y,z) = \frac{1}{4\pi} \int_\Omega \frac{\mathbf{P}(\xi,\eta,\zeta)}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta.$$

This particular $\mathbf{W}$ is therefore defined as a convergent improper space integral, and it is not hard to check that $\bigtriangledown \cdot \mathbf{W} = 0$ as required.

In the general case, where

$$\text{div}\,\mathbf{V} = \nabla \cdot \mathbf{V} = p \qquad \text{and} \qquad \mathbf{rot}\mathbf{V} = \nabla \times \mathbf{V} = \mathbf{P},$$

and where we assume that p and $\mathbf{P}$ vanish outside the bounded set Ω, we split $\mathbf{V}$ in the following way

$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2,$$

where

$$\text{div}\,\mathbf{V}_1 = \nabla \cdot \mathbf{V}_1 = p \qquad \text{and} \qquad \mathbf{rot}\mathbf{V}_1 = \nabla \times \mathbf{V}_1 = \mathbf{0},$$

and

$$\text{div}\,\mathbf{V}_2 = \nabla \cdot \mathbf{V}_2 = 0 \qquad \text{and} \qquad \mathbf{rot}\mathbf{V}_2 = \nabla \times \mathbf{V}_2 = \mathbf{P}.$$

Using the solution formulæ above we can find $\mathbf{V}_1$ and $\mathbf{V}_2$, hence also $\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2$. We have therefore justified the following

Theorem 38.3 Helmholtz'z theorem. *Let p and $\mathbf{P}$ be a scalar and a vector field, resp., of class C^1, which are vanishing outside a bounded set Ω. We define*

$$H(x,y,z) = \frac{1}{4\pi} \int_\Omega \frac{p(\xi,\eta,\zeta)}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta,$$

and

$$\mathbf{W}(x,y,z) = \frac{1}{4\pi} \int_\Omega \frac{\mathbf{P}(\xi,\eta,\zeta)}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta.$$

Then the vector field

$$\mathbf{V} := -\nabla H + \nabla \times \mathbf{W} = -\mathbf{grad}H + \mathbf{rot}\mathbf{W}$$

satisfies the differential equations

$$\text{div}\,\mathbf{V} = \nabla \cdot \mathbf{V} = p \qquad \textit{and} \qquad \mathbf{rot}\mathbf{V} = \nabla \times \mathbf{V} = \mathbf{P},$$

and the growth condition at infinity,

$$\mathbf{V}(\mathbf{x}) = \rightarrow \mathbf{0} \qquad \textit{for } \|\mathbf{x}\| \rightarrow +\infty.$$

This theorem is also called the *fundamental theorem of vector analysis.*

It should be noted that it in the omitted proof of this theorem is essential that we have $\mathbf{V}(\mathbf{x}) \rightarrow \mathbf{0}$ for $\mathbf{x} \rightarrow \infty$.

A consequence of the theorem above is that given the *four* equations (i.e. coordinate wise for **rot**)

$$\text{div}\,\mathbf{V} = p \qquad \text{and} \qquad \mathbf{rot}\mathbf{W} = \mathbf{P},$$

where p and $\mathbf{P}$ vanish outside a bounded domain Ω, then we can find all coordinate functions V_1, V_2 and V_3 of $\mathbf{V}$. Therefore, one cannot in general solve the following system of nine inhomogeneous equations,

$$\frac{\partial V_1}{\partial x_1} = w_{11}, \qquad \frac{\partial V_1}{\partial x_2} = w_{12}, \qquad \frac{\partial V_1}{\partial x_3} = w_{13},$$

$$\frac{\partial V_2}{\partial x_1} = w_{21}, \qquad \frac{\partial V_2}{\partial x_2} = w_{22}, \qquad \frac{\partial V_2}{\partial x_3} = w_{23},$$
$$\frac{\partial V_3}{\partial x_1} = w_{31}, \qquad \frac{\partial V_3}{\partial x_2} = w_{32}, \qquad \frac{\partial V_3}{\partial x_3} = w_{33},$$

for the nine arbitrary continuous functions w_{ij}, i, $j = 1, 2, 3$, even if they all vanish outside Ω. These nine functions must satisfy some (in fact, five) compatibility conditions. We shall not go further into this discussion here.

38.3 Some applications in Physics

Consider a fluid or a gas of density ϱ and velocity vector field $\mathbf{v}$. Let Ω be a (bounded) domain. Then the total mass in Ω is

$$M = \int_\Omega \varrho \, \mathrm{d}\Omega,$$

and the flux of mass through the surface $\partial\Omega$ (positive in the direction away from Ω), is given by

$$q = \int_{\partial\Omega} \varrho \mathbf{v} \cdot \mathbf{n} \, \mathrm{d}S.$$

Then the *law of conservation of mass* is written

$$q \, \mathrm{d}t = - \, \mathrm{d}M.$$

We get by a rearrangement and an application of *Gauß's theorem*,

$$0 = q + \frac{\mathrm{d}M}{\mathrm{d}t} = \int_{\partial\Omega} \varrho\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}S + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\varrho\,\mathrm{d}\Omega = \int_{\Omega}\left(\mathrm{div}\,(\varrho\mathbf{v}) + \frac{\partial\varrho}{\partial t}\right)\mathrm{d}\Omega.$$

This equation is true for every domain Ω, and since the integrand is continuous, it must be 0, and we have derived the so-called *continuity equation*

$$\mathrm{div}\,(\varrho\mathbf{v}) + \frac{\partial\varrho}{\partial t} = 0,$$

also written

$$\frac{\partial\varrho}{\partial t} + \bigtriangledown\cdot(\varrho\mathbf{v}) = 0.$$

If we restrict ourselves to the *stationary case*, i.e. $\dfrac{\partial\varrho}{\partial t} = 0$, then also $\bigtriangledown\cdot(\varrho\mathbf{v}) = 0$, so $\varrho\mathbf{v}$ is divergence free. We conclude in this case that there exists a vector potential for $\varrho\mathbf{v}$.

I the next special case we furthermore assume in this stationary case that the fluid is incompressible, which means that its density ϱ is constant. Then clearly $\bigtriangledown\cdot\mathbf{v} = 0$, so $\mathbf{v}$ is divergence free, and there exists a vector potential $\mathbf{A}$, such that $\mathbf{v} = \bigtriangledown\times\mathbf{A}$.

Returning to the general case it should be mentioned that the same mathematical structure can also be found in other situations in Physics. We list for convenience some of them below.

1) **Continuity equation**, i.e. the above,

$$\mathrm{div}\,(\varrho\mathbf{v}) + \frac{\partial\varrho}{\partial t} = 0, \qquad \text{also written } \bigtriangledown\cdot(\varrho\mathbf{v}) + \frac{\partial\varrho}{\partial t} = 0,$$

where ϱ is the density and $\mathbf{v}$ the velocity of a fluid gas.

2) **Conservation of energy**,

$$\mathrm{div}\,\mathbf{q} + \frac{\partial u}{\partial t} = 0, \qquad \text{also written } \bigtriangledown\cdot\mathbf{q} + \frac{\partial u}{\partial t} = 0,$$

where u denotes the energy density and $\mathbf{q}$ the density of the energy flow.

3) **Conservation of electric charge**,

$$\mathrm{div}\,\mathbf{J} + \frac{\partial\varrho}{\partial t} = 0, \qquad \text{also written } \bigtriangledown\cdot\mathbf{J} + \frac{\partial\varrho}{\partial t} = 0,$$

where $\mathbf{J}$ is the current density and ϱ the density of charge.

As an illustration we let $T(\mathbf{x},t)$ denote the *temperature field* in a body, in which heat is supposed to be the only transport of energy. Then by the law of conservation of energy above,

$$\bigtriangledown\cdot\mathbf{q} + \frac{\partial u}{\partial t} = 0,$$

where $\mathbf{q}$ denotes the density of heat flow, and u is the density of the inner energy.

Assume that the density of heat flow follows *Fourier's law*

$$\mathbf{q} = -\lambda\bigtriangledown T,$$

where λ is called the *conductivity of heat.*

Furthermore, assume that the *energy density* is given by the temperature alone, such that the following equation holds,

$$\frac{\partial u}{\partial t}=\frac{\mathrm{d}u}{\mathrm{d}T}\,\frac{\partial T}{\partial t}.$$

By convention we put

$$\frac{\mathrm{d}u}{\mathrm{d}T}=\varrho\,c, \qquad \text{thus} \qquad \frac{\partial u}{\partial t}=\varrho\,c\,\frac{\partial T}{\partial t},$$

where ϱ denotes the *density of mass*, and c denotes the *specific capacity of heat.*

Then we get by insertion,

$$\varrho\,c\,\frac{\partial T}{\partial t}=\frac{\partial u}{\partial t}=-\bigtriangledown\cdot\mathbf{q}=-\bigtriangledown\cdot(-\lambda\bigtriangledown T)=\bigtriangledown\lambda\cdot\bigtriangledown T+\lambda\bigtriangledown^2 T.$$

We shall restrict ourselves to the case where the conductivity of heat λ is a *constant.* Then $\bigtriangledown\lambda=\mathbf{0}$, and we get

$$\varrho\,c\,\frac{\partial T}{\partial t}=\lambda\bigtriangledown^2 T, \qquad \text{or by a rearrangement} \qquad \frac{\partial T}{\partial t}=\frac{\lambda}{\varrho\,c}\,\bigtriangledown^2 T.$$

This equation is called the *heat equation.* The same equation occurs in the case of diffusion, in which case it is called the *diffusion equation.* In civil engineering the same equation is known as *Fick's law.*

We shall turn to another physical situation. Consider a large mass of fluid of constant mass density ϱ_0, and let p denote the *pressure* in this fluid. Then

$$p(z)=p(0)+\varrho_0\,g\,z, \qquad \text{and clearly} \qquad \bigtriangledown p=\varrho_0\,g\,\mathbf{e}_z,$$

where g denotes the gravity constant, and the vertical coordinate z is chosen positive downwards in the same direction as the gravity.

Let a body Ω be immersed into the fluid. Then the fluid will affect an area element $\mathrm{d}S$ of the surface $\partial\Omega$ with a force $\mathrm{d}\mathbf{F}$ of the size $p\,\mathrm{d}S$ and of a direction perpendicular to the surface $\partial\Omega$ into the body. By our previously chosen convention we always understand the vector field $\mathbf{n}$ on the surface $\partial\Omega$ as the outgoing unit normal vector field, so we conclude that

$$\mathrm{d}\mathbf{F}=-p\,\mathbf{n}\,\mathrm{d}S \qquad \text{on } \partial\Omega.$$

Then the total force $\mathbf{F}$ acting on Ω is obtained by integrating $\mathrm{d}\mathbf{F}$ over the surface $\partial\Omega$. This integral can be calculated by using the *third variant of Gauß's theorem*, so

$$\mathbf{F}=\int_{\partial\Omega}(-p\,\mathbf{n})\,\mathrm{d}S=-\int_\Omega\bigtriangledown p\,\mathrm{d}\Omega=-\varrho_0\,g\,\mathbf{e}_z\int_\Omega\mathrm{d}\Omega=-\varrho_0\,\mathrm{vol}(\Omega)\,g\,\mathbf{e}_z.$$

We have thus above derived *Archimedes's theorem*, *The buoyancy of the fluid acting on the body Ω is equal to the gravitational force on the superseded fluid.*

The moment of the forces $\mathbf{M}$ with respect to origo $\mathcal{O}$ is obtained by integration of

$$\mathrm{d}\mathbf{M}=\mathbf{x}\times\,\mathrm{d}\mathbf{F}=-p\,\mathbf{x}\times\mathbf{n}\,\mathrm{d}S,$$

where we continue the calculations by applying the *second variant of Gauß's theorem*, and of course also that $\nabla \times \mathbf{x} = \mathbf{0}$,

$$\mathbf{M} = \int_{\partial\Omega} \mathbf{n} \times (p\mathbf{x})\,\mathrm{d}S = \int_{\Omega} \nabla \times (p\mathbf{x})\,\mathrm{d}\Omega = \int_{\Omega} \{(\nabla p) \times \mathbf{x} + p \nabla \times \mathbf{x}\}\,\mathrm{d}\Omega = \varrho_0\, g\, \mathbf{e}_z \times \int_{\Omega} \mathbf{x}\,\mathrm{d}\Omega.$$

If we here choose $\mathcal{O}$ as the centre of gravity of the superseded fluid, then $\int_{\Omega} \mathbf{x}\,\mathrm{d}\Omega = \mathbf{0}$, and $\mathbf{M} = \mathbf{0}$.

38.4 Examples from Electromagnetism

Example 38.4 We know from *Maxwell's equations*, cf. Section 35.3, that the electric field $\mathbf{E}$ and the magnetic flux density $\mathbf{B}$ satisfy the differential equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{and} \qquad \nabla \cdot \mathbf{B} = 0.$$

Since $\Omega = \mathbb{R}^3$ is star shaped and div $\mathbf{B} = 0$, we conclude that there exists a vector potential $\mathbf{A}$ for $\mathbf{B}$, so

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

This implies that the magnetic flux Φ through a surface $\mathcal{F}$ according to *Stokes's theorem* is equal to the circulation of the vector potential taken along the boundary curve $\delta\mathcal{F}$, i.e.

$$\int_{\mathcal{F}} \mathbf{B} \cdot \mathbf{n}\,\mathrm{d}S = \int_{\mathcal{F}} \mathbf{n} \cdot \nabla \times \mathbf{A}\,\mathrm{d}S = \oint_{\delta\mathcal{F}} \mathbf{A} \cdot \mathbf{t}\,\mathrm{d}s.$$

Then, by a rearrangement of the first equation above,

$$\mathbf{0} = \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E} + \frac{\partial}{\partial t}(\nabla \times \mathbf{A}) = \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right).$$

This result shows that $\mathbf{E} + \dfrac{\partial \mathbf{A}}{\partial t}$ is rotational free, hence a gradient field. Since the domain $\mathbb{R}^3$ is star shaped, there exists a scalar potential V, such that

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V, \qquad \text{i.e.} \qquad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla V.$$

The pair $(\mathbf{A}, V)$ is called the *set of electromagnetic potentials*. It is left to the reader to give the simple proof that a set of electromagnetic potentials $(\mathbf{A}, V)$ can be replaced by $\left(\mathbf{A} + \nabla g, V - \dfrac{\partial g}{\partial t}\right)$, where $g(x, y, z, t)$ is any C^1 function. $\Diamond$

Example 38.5 Let $\mathcal{K}$ denote a closed curve with a current I. We assume that $\mathcal{K}$ lies in a magnetic field of *constant* flux density $\mathbf{B}$. The magnetic field acts on the *vectorial curve element* $\mathbf{t}\,\mathrm{d}s$ with the so-called *Laplace force* $\mathrm{d}\mathbf{F} = I\,\mathbf{t} \times \mathbf{B}\,\mathrm{d}s$.

The total force $\mathbf{F}$ acting on $\mathcal{K}$ is given by the line integral

$$\mathbf{F} = \oint_{\mathcal{K}} I\,\mathbf{t} \times \mathbf{B}\,\mathrm{d}s = -\oint_{\mathcal{K}} I\,\mathbf{B} \times \mathbf{t}\,\mathrm{d}s = -I\,\mathbf{B} \times \oint_{\mathcal{K}} \mathbf{t}\,\mathrm{d}s = I\,\mathbf{B} \times \int_{\alpha}^{\beta} \mathbf{r}'(\tau)\,\mathrm{d}\tau = \mathbf{r}(\beta) - \mathbf{r}(\alpha) = \mathbf{0},$$

because $\mathcal{K}$ is closed, so the initial point $\mathbf{r}(\alpha)$ and the end point $\mathbf{r}(\beta)$ are coinciding, $\mathbf{r}(\alpha) = \mathbf{r}(\beta)$.

The moment of the forces $\mathbf{M}$ is obtained by integrating

$$\mathrm{d}\mathbf{M} = \mathbf{x} \times \mathrm{d}\mathbf{F}$$

along the curve $\mathcal{K}$. We note that since the resulting force $\mathbf{F} = \mathbf{0}$, the moment $\mathbf{M}$ is independent of the point $\mathcal{O}$ which is used when defining the moment.

We then choose a surface $\mathcal{F}$ which has the given curve $\mathcal{K}$ as its boundary, $\mathcal{K} = \delta\mathcal{F}$, and adjust the corresponding orientations. Using the rule of calculation of the double cross product of $\mathbf{x} \times (\mathbf{t} \times \mathbf{B})$, known from *Linear Algebra*, we find

$$\mathbf{M} = \oint_{\mathcal{K}} \mathbf{x} \times I(\mathbf{t} \times \mathbf{B})\,\mathrm{d}s = I \oint_{\mathcal{K}} \{(\mathbf{x} \cdot \mathbf{B})\mathbf{t} - (\mathbf{x} \cdot \mathbf{t})\mathbf{B}\}\,\mathrm{d}s = I \oint_{\mathcal{K}} (\mathbf{x} \cdot \mathbf{B})\mathbf{t}\,\mathrm{d}s - I \oint_{\mathcal{K}} \mathbf{x} \cdot \mathbf{t}\,\mathrm{d}s.$$

Then we apply the third version of *Stokes's theorem* with $f = \mathbf{x} \cdot \mathbf{B}$ on the first integral, and the first version of *Stokes's theorem* on the second integral. Since $\nabla \times \mathbf{x} = \mathbf{0}$, and since $\mathbf{B}$ is constant, so $\nabla(\mathbf{x} \cdot \mathbf{B}) = \mathbf{B}$, we get

$$\mathbf{M} = I \int_{\mathcal{F}} \mathbf{n} \times \nabla(\mathbf{x} \cdot \mathbf{B})\,\mathrm{d}S - I\,\mathbf{B} \int_{\mathcal{F}} \mathbf{n} \cdot \nabla \times \mathbf{x}\,\mathrm{d}S = I \int_{\mathcal{F}} \mathbf{n} \times \mathbf{B}\,\mathrm{d}S.$$

Finally, we have previously shown that

$$\int_{\mathcal{F}} \mathbf{n}\,\mathrm{d}S = \frac{1}{2}\oint_{\delta\mathcal{F}} \mathbf{x}\times\mathbf{t}\,\mathrm{d}s,$$

so

$$\mathbf{M} = I\int_{\mathcal{F}} \times\mathbf{B}\,\mathrm{d}S = \left(I\int_{\mathcal{F}} \mathbf{n}\,\mathrm{d}S\right)\times\mathbf{B} = \left(\frac{1}{2}\,I\oint_{\mathcal{K}} \mathbf{x}\times\mathbf{t}\,\mathrm{d}s\right)\times\mathbf{B}.$$

The factor

$$I\int_{\mathcal{F}} \mathbf{n}\,\mathrm{d}S = \frac{1}{2}\,I\oint_{\mathcal{K}} \mathbf{x}\times\mathbf{t}\,\mathrm{d}s,$$

which is given either as a surface integral or as a line integral, is called the *magnetic dipole moment* of $\mathcal{K}$. $\Diamond$

Example 38.6 *We consider the following equations for a stationary magnetic field,*

$$\nabla\times\mathbf{H} = \mathbf{H}, \qquad \mathbf{B} = \nabla\times\mathbf{A},$$

where $\mathbf{B}$ *is the magnetic flux density,* $\mathbf{H}$ *is the magnetic field intensity,* $\mathbf{A}$ *a magnetic vector potential, and* $\mathbf{J}$ *the electric flow density; the latter is only different from the zero vector in a bounded part of the space. We shall also assume that*

$$r^2\|\mathbf{H}\|, \quad r^2\|\mathbf{B}\|, \quad r^1\|\mathbf{A}\| \text{ are bounded and } \mathbf{B}\cdot\mathbf{H}\geq 0.$$

One can prove that we can attribute to the field the energy

$$W_M = \int \frac{1}{2}\,\mathbf{B}\cdot\mathbf{H}\,\mathrm{d}\Omega,$$

where we integrate over the whole space. Show by partial integration that this integral is convergent and that

$$W_M = \int \frac{1}{2}\,\mathbf{J}\cdot\mathbf{A}\,\mathrm{d}\Omega,$$

where we shall only integrate over the bounded part of the space, in which $\mathbf{J}\neq\mathbf{0}$.

A Nabla calculus and Electromagnetism.

D Show that $\int \frac{1}{2}\,\mathbf{B}\cdot\mathbf{H}\,\mathrm{d}\Omega$ and $\int \frac{1}{2}\,\mathbf{A}\cdot\mathbf{J}\,\mathrm{d}\Omega$ both exist. Then reduce $\frac{1}{2}\,\mathbf{B}\cdot\mathbf{H}-\frac{1}{2}\,\mathbf{A}\cdot\mathbf{J}$. (This dirty trick is equivalent to a partial integration).

I Formally, we must assume that all functions and vector functions are of class C^1 in all of $\mathbb{R}^3$ and that they are in particular finite in $\mathbf{0}$. If $\mathbf{J}=\mathbf{0}$ for $r\geq R_0$, then

$$\left|\int \frac{1}{2}\,\mathbf{A}\cdot\mathbf{J}\,\mathrm{d}\Omega\right| \leq \frac{1}{2}\int_{K(\mathbf{0};R_0)} \|\mathbf{A}\|\cdot\|\mathbf{J}\|\,\mathrm{d}\Omega \leq \frac{1}{2}\cdot\frac{4\pi R^3}{3}\max_{\|\mathbf{x}\|\leq R_0}\|\mathbf{A}(\mathbf{x})\|\cdot\max_{\|\mathbf{x}\leq R_0}\|\mathbf{J}(\mathbf{x})\| < +\infty.$$

When we use spherical coordinates we get for $R > 1$,

$$\begin{aligned}\left|\int_{K(\mathbf{0};R)\setminus K(\mathbf{0};1)} \frac{1}{2}\,\mathbf{B}\cdot\mathbf{H}\,\mathrm{d}\Omega\right| &\le \frac{1}{2}\int_{K(\mathbf{0};R)\setminus K(\mathbf{0};1)} r^2\|\mathbf{B}\|\cdot r^2\|\mathbf{H}\|\cdot\frac{1}{r^4}\,\mathrm{d}\Omega\\ &\le C\int_1^R \frac{1}{r^4}\,r^2\,\mathrm{d}r = C\left(1-\frac{1}{R}\right) < C,\end{aligned}$$

where C is independent of R, and it follows that the integral is convergent.

It follows from the definitions that

$$\frac{1}{2}\,\mathbf{B}\cdot\mathbf{H} - \frac{1}{2}\,\mathbf{A}\cdot\mathbf{J} = \frac{1}{2}\,(\bigtriangledown\times\mathbf{A})\cdot\mathbf{H} - \frac{1}{2}\,\mathbf{A}\cdot(\bigtriangledown\times\mathbf{H}) = \frac{1}{2}\,\bigtriangledown\cdot(\mathbf{A}\times\mathbf{H}),$$

hence by Gauß's theorem

$$\begin{aligned}(38.1)\quad &\int_{K(\mathbf{0};R)} \frac{1}{2}\,\mathbf{B}\cdot\mathbf{H}\,\mathrm{d}\Omega - \int_{K(\mathbf{0};R)} \frac{1}{2}\,\mathbf{A}\cdot\mathbf{J}\,\mathrm{d}\Omega\\ &\qquad = \frac{1}{2}\int_{K(\mathbf{0};R)} \bigtriangledown\cdot(\mathbf{A}\times\mathbf{H})\,\mathrm{d}\Omega = \int_{\partial K(\mathbf{0};R)} \mathbf{n}\cdot(\mathbf{A}\times\mathbf{H})\,\mathrm{d}S.\end{aligned}$$

Here we have the estimate

$$\begin{aligned}&\left|\int_{\partial K(\mathbf{0};R)} \mathbf{n}\cdot(\mathbf{A}\times\mathbf{H})\,\mathrm{d}S\right| \le \int_{\partial K(\mathbf{0};R)} \frac{R\|\mathbf{A}\|\cdot R^2\|\mathbf{H}\|}{R^3}\,\mathrm{d}S\\ &\qquad \le C\cdot\frac{1}{R^3}\,\mathrm{area}(\partial K(\mathbf{0};R)) = C_1\cdot\frac{R^2}{R^3} = C_1\cdot\frac{1}{R} \to 0 \qquad \text{for } R\to+\infty.\end{aligned}$$

By taking the limit $R \to +\infty$ we get from (38.1) that

$$\int_{\mathbb{R}^3} \frac{1}{2}\,\mathbf{B}\cdot\mathbf{H}\,\mathrm{d}\Omega - \int_{\mathbb{R}^3} \frac{1}{2}\,\mathbf{A}\cdot\mathbf{J}\,\mathrm{d}\Omega = 0,$$

hence by a rearrangement,

$$W_M = \int_{\mathbb{R}^3} \frac{1}{2}\,\mathbf{B}\cdot\mathbf{H}\,\mathrm{d}\Omega = \int_{\mathbb{R}^3} \frac{1}{2}\,\mathbf{A}\cdot\mathbf{J}\,\mathrm{d}\Omega = \int_{K(\mathbf{0};R)} \frac{1}{2}\,\mathbf{A}\cdot\mathbf{J}\,\mathrm{d}\Omega$$

as required

Example 38.7 *Given for a material which is not in an electric sense an ideal isolator,*

$$\nabla \cdot \mathbf{D} = \tilde{\varrho}, \qquad \nabla \cdot \mathbf{J} + \frac{\partial \tilde{\varrho}}{\partial t} = 0, \qquad \mathbf{D} = \alpha\, \mathbf{J},$$

where $\mathbf{D}$ *is the electric flux density,* $\mathbf{J}$ *is the flow density, and* $\tilde{\varrho}$ *is the charge density, while* α *is a scalar field, which describes the electric properties of the material, and* t *is the time. We further assume that we are in a* stationary *case and that we are* given *a current distribution, so* $\mathbf{J}$ *is a known vector field.*
Find an expression of $\tilde{\varrho}$.

A Nabla calculus and Electromagnetism.

D Analyze the equations, when $\mathbf{J}$ and α are given.

I We first derive that

$$\tilde{\varrho} = \nabla \cdot \mathbf{D} = \nabla \cdot (\alpha\, \mathbf{J}) = (\nabla \alpha) \cdot \mathbf{J} + \alpha \nabla \cdot \mathbf{J}.$$

Since we are in the stationary case, we have

$$\frac{\partial \tilde{\varrho}}{\partial t} = 0,$$

hence

$$\nabla \cdot \mathbf{J} + \frac{\partial \tilde{\varrho}}{\partial t} = 0$$

implies that $\nabla \cdot \mathbf{J} = 0$. Finally, by insertion,

$$\tilde{\varrho} = (\nabla \alpha) \cdot \mathbf{J}.$$

Example 38.8 *Considering potentials it can be proved that the electric field intensity* $\mathbf{E}$ *and the magnetic flux density* B *can be derived from a scalar potential* V *and a vector potential* A *in the following way:*

$$\mathbf{B} = \bigtriangledown \times \mathbf{A}, \qquad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \bigtriangledown V.$$

We also have equations of the same form with another set of potentials $(\tilde{\mathbf{A}}, \tilde{V})$*, provided that*

$$\tilde{V} = V - \frac{\partial g}{\partial t}, \qquad \tilde{\mathbf{A}} + \bigtriangledown g,$$

where g *is a scalar field. We note that the potentials are not uniquely determined, so it is natural to set up an* extra condition *on the potentials. One often applies the so-called* Lorentz condition

$$\bigtriangledown \cdot \mathbf{A} + \frac{\partial V}{\partial t} = 0.$$

1. *Derive the differential equation which the scalar field* g *must fulfil if one from any given set of potentials* $(\tilde{\mathbf{A}}, \tilde{V})$ *can create a set of potentials* $(\mathbf{A}, V)$*, which also satisfies the Lorentz condition.*

It turns up that one can solve this differential equation. We therefore assume in the following that the Lorentz condition is satisfied, and then consider the vector field

$$\mathbf{Z}(\mathbf{x}, t) = \int_{t_0}^{t} \mathbf{A}(\mathbf{x}, \tau)\, \mathrm{d}\tau,$$

where t_0 *is some constant.*

2. *Show that if we put* $V = -\bigtriangledown \cdot \mathbf{Z}$*, then the Lorentz condition is fulfilled.*

3. *Express the electromagnetic fields* $\mathbf{E}$ *and* $\mathbf{B}$ *by means of the vector field* $\mathbf{Z}$*.*

A Set of potentials satisfying the Lorentz condition.

D Insert into the equations. Note that the operator $\dfrac{\partial}{\partial t}$ commutes with the operators $\bigtriangledown$, $\bigtriangledown\cdot$ and $\bigtriangledown\times$.

I First assume that $(\mathbf{A}, V)$ is given, and let g be a scalar field. If

$$\tilde{V} = V - \frac{\partial g}{\partial t} \quad \text{and} \quad \tilde{\mathbf{A}} + \bigtriangledown g,$$

then

$$\bigtriangledown \times \tilde{\mathbf{A}} = \bigtriangledown \times \mathbf{A} + \bigtriangledown \times \bigtriangledown g = \mathbf{B} + \mathbf{0} = \mathbf{B},$$

and

$$-\frac{\partial \tilde{\mathbf{A}}}{\partial t} - \bigtriangledown \tilde{V} = -\frac{\partial \mathbf{A}}{\partial t} - \bigtriangledown V - \frac{\partial}{\partial t} \bigtriangledown g + \bigtriangledown \left(\frac{\partial g}{\partial t} \right) = \mathbf{E} + \mathbf{0} = \mathbf{E},$$

and we have proved that $(\mathbf{A}, V)$ and $(\tilde{\mathbf{A}}, \tilde{V})$ are both a set of potentials for $\mathbf{B}$ and $\mathbf{E}$.

1) It follows by a rearrangement that

$$V = \tilde{V} + \frac{\partial g}{\partial t} \quad \text{and} \quad \mathbf{A} = \tilde{\mathbf{A}} - \nabla g,$$

where the set of potentials $(\tilde{\mathbf{A}}, \tilde{V})$ is given. By insertion into the Lorentz condition we get

$$0 = \nabla \cdot \mathbf{A} + \frac{\partial V}{\partial t} = \nabla \cdot \tilde{\mathbf{A}} - \nabla \cdot \nabla g + \frac{\partial \tilde{V}}{\partial t} + \frac{\partial^2 g}{\partial t^2},$$

and we derive the requested differential equation

$$\nabla^2 g - \frac{\partial^2 g}{\partial t^2} = \nabla \cdot \tilde{\mathbf{A}} + \frac{\partial \tilde{V}}{\partial t}$$

where the right hand side is known. This is a classical inhomogeneous wave equation in three space variables and one time variable.

2) Assume that only the vector field $\mathbf{A}$ is given. Put

$$\mathbf{Z}(\mathbf{x}, t) = \int_{t_0}^{t} \mathbf{A}(\mathbf{x}, \tau)\, \mathrm{d}\tau, \quad \frac{\partial \mathbf{Z}}{\partial t} = \mathbf{A}, \quad \text{and} \quad V = - \nabla \cdot \mathbf{Z}.$$

Then

$$\nabla \cdot \mathbf{A} + \frac{\partial V}{\partial t} = \nabla \cdot \mathbf{A} - \frac{\partial}{\partial t}(\nabla \cdot \mathbf{Z}) = \nabla \cdot \mathbf{A} - \nabla \cdot \frac{\partial \mathbf{Z}}{\partial t} = \nabla \cdot \mathbf{A} - \nabla \cdot \mathbf{A} = 0,$$

and the Lorentz condition is fulfilled.

3) The set of potentials $(\mathbf{A}, V)$ above defines (expressed by $\mathbf{Z}$) the fields $\mathbf{B}$ and $\mathbf{E}$ by the formulæ

$$\mathbf{B} = \nabla \times \mathbf{A} = \nabla \times \frac{\partial \mathbf{Z}}{\partial t} = \frac{\partial}{\partial t}(\nabla \times \mathbf{Z}),$$

and

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla V = -\frac{\partial^2 \mathbf{Z}}{\partial t^2} + \nabla(\nabla \cdot \mathbf{Z}).$$

Example 38.9 *On the figure below we are given a normal cut in a* double wire *consisting of two identical, parallel, conductive strips of breadth b and distance a. In the strips are flowing two opposite equally distributed flows. Assuming that the two strips can be considered as infinitely thin and that the permeability μ has the same value everywhere one can show that the inductance of the wire per length $\mathcal{L}$ is given by*

$$\mathcal{L} = \frac{\mu}{\pi b^2} \int_0^b \left\{ \int_0^b \ln \frac{\sqrt{a^2 + (y - \tilde{y})^2}}{|y - \tilde{y}|} \, dy \right\} d\tilde{y}.$$

Consider this as an improper plane integral and find $\mathcal{L}$.

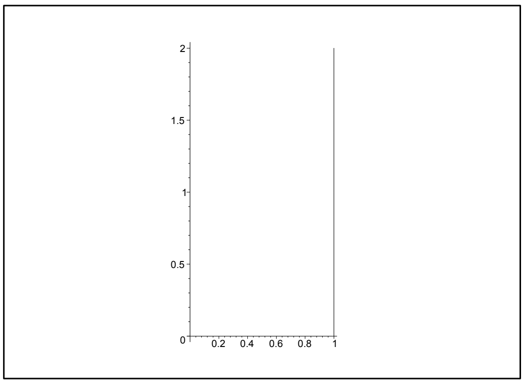

Figure 38.1: Double wire of distance a and length b.

A Improper plane integral.

D Split the integrand into two parts which each are integrated separately. There is no problem with the first of these integrands. Considering the second one we smooth out the singularity by the first integration.

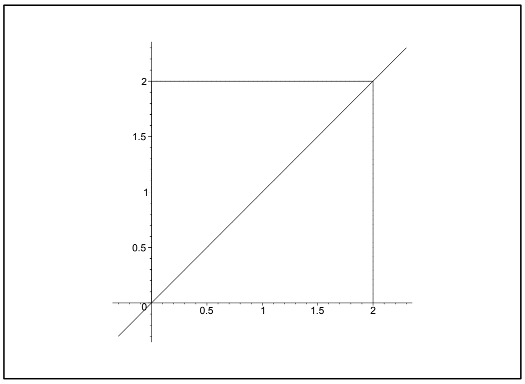

Figure 38.2: The domain of integration $B = [0, b] \times [0, b]$ for $b = 2$ in the $(y, \tilde{y})$-plane.

I Here, $B = [0, b] \times [0, b]$ in the $(y, \tilde{y})$-plane, and the integrand is not defined for $\tilde{y} = y$. We shall first find a primitive of

$$\ln\left(\frac{\sqrt{a^2 + (y - \tilde{y})^2}}{|y - \tilde{y}|}\right) = \frac{1}{2}\,\ln\left(a^2 + (y - \tilde{y})^2\right) - \frac{1}{2}\,\ln|y - \tilde{y}|$$

for $\tilde{y}$ fixed and $y \neq \tilde{y}$.

1) When $y \in [0, b]$, then $\frac{1}{2}\,\ln(a^2 + (y - \tilde{y})^2)$ has no singularity, so we get by a partial integration

$$\begin{aligned}
&\int \frac{1}{2}\,\ln\left(a^2 + (y - \tilde{y})^2\right)\,\mathrm{d}y \\
&\quad = \frac{1}{2}\,(y - \tilde{y})\,\ln\left(a^2 + (y - \tilde{y})^2\right) - \frac{1}{2}\int (y - \tilde{y}) \cdot \frac{2(y - \tilde{y})}{a^2 + (y - \tilde{y})^2}\,\mathrm{d}y \\
&\quad = \frac{1}{2}\,(y - \tilde{y})\,\ln\left(a^2 + (y - \tilde{y})^2\right) - \int \frac{a^2 + (y - \tilde{y})^2 - a^2}{a^2 + (y - \tilde{y})^2}\,\mathrm{d}y \\
&\quad = \frac{1}{2}\,(y - \tilde{y})\,\ln\left(a^2 + (y - \tilde{y})^2\right) - y + a\,\mathrm{Arctan}\left(\frac{y - \tilde{y}}{a}\right).
\end{aligned}$$

2) When $\tilde{y} < y \leq b$, we get by a partial integration

$$-\frac{1}{2}\int \ln|y - \tilde{y}|\,\mathrm{d}y = -\frac{1}{2}\int \ln(y - \tilde{y})\,\mathrm{d}y = -\frac{1}{2}\{(y - \tilde{y})\ln(y - \tilde{y}) - (y - \tilde{y})\},$$

which due to the order of magnitudes can be extended by 0 for $y = \tilde{y}$.

3) Similarly, we get for $0 \leq y < \tilde{y}$ that

$$-\frac{1}{2}\int \ln|y - \tilde{y}|\,\mathrm{d}y = -\frac{1}{2}\int \ln(\tilde{y} - y)\,\mathrm{d}y = -\frac{1}{2}\{(y - \tilde{y})\ln(\tilde{y} - y) - (y - \tilde{y})\},$$

which is also extended by 0 for $y = \tilde{y}$.

As a conclusion we get from 2) and 3) that

$$-\frac{1}{2}\int \ln|y - \tilde{y}|\,\mathrm{d}y = -\frac{1}{2}\,(y - \tilde{y})\ln|y - \tilde{y}| + \frac{1}{2}\,(y - \tilde{y}),$$

which by a continuous extension can be interpreted as 0 for $y = \tilde{y}$.

Thus, for fixed $\tilde{y} \in [0, b]$ we get for the inner integral,

$$
\begin{aligned}
&\int_0^b \ln\left(\frac{\sqrt{a^2+(y-\tilde{y})^2}}{|y-\tilde{y}|}\right) dy \\
&= \left[\frac{1}{2}(y-\tilde{y})\ln(a^2+(y-\tilde{y})^2)-y+a\operatorname{Arctan}\left(\frac{y-\tilde{y}}{a}\right)\right]_{y=0}^{b}+\left[-\frac{1}{2}(y-\tilde{y})\ln|y-\tilde{y}|+\frac{1}{2}(y-\tilde{y})\right]_{y=0}^{b} \\
&= -\frac{1}{2}(\tilde{y}-b)\ln((a^2+(\tilde{y}-b)^2)+\frac{1}{2}\tilde{y}\ln(a^2+\tilde{y}^2)-b-a\operatorname{Arctan}\left(\frac{\tilde{y}-b}{a}\right)+a\operatorname{Arctan}\left(\frac{\tilde{y}}{a}\right) \\
&\qquad +\frac{1}{2}(\tilde{y}-b)\ln(b-\tilde{y})-\frac{1}{2}\tilde{y}\ln\tilde{y}+\frac{1}{2}(b-\tilde{y})+\frac{1}{2}\tilde{y}^2,
\end{aligned}
$$

as the singularity has disappeared.

Then write t instead of $\tilde{y}$. Then

$$
\begin{aligned}
\mathcal{L} &= \frac{\mu}{\pi b^2}\int_0^b\left\{-\frac{1}{2}(t-b)\ln(a^2+(t-b)^2)\right\}\mathrm{d}t+\frac{\mu}{\pi b^2}\int_0^b\frac{1}{2}t\ln(a^2+t^2)\,\mathrm{d}t \\
&\quad +\frac{\mu}{\pi b^2}\int_0^b\frac{1}{2}(t-b)\ln|t-b|\,\mathrm{d}t-\frac{\mu}{\pi b^2}\int_0^b t\ln t\,\mathrm{d}t \\
&\quad -\frac{\mu}{\pi b^2}\,a\int_0^b\operatorname{Arctan}\left(\frac{t-b}{a}\right)\mathrm{d}t+\frac{\mu}{\pi b^2}\,a\int_0^b\operatorname{Arctan}\left(\frac{t}{a}\right)\mathrm{d}t-\frac{\mu}{\pi b^2}\int_0^b\frac{1}{2}\,b\,\mathrm{d}t.
\end{aligned}
$$

By some small calculations,

$$\int \tau \ln(k+\tau)\,\mathrm{d}\tau = \frac{1}{2}\int \ln(k+\tau^2)\,\mathrm{d}\tau^2 = \frac{1}{2}\left\{(k+\tau^2)\ln(k+\tau^2)-(k+\tau^2)\right\},$$

$$\int \tau \ln|\tau|\,\mathrm{d}\tau = \frac{1}{2}\tau^2\ln|\tau| - \frac{1}{2}\int \tau\,\mathrm{d}\tau = \frac{1}{2}\tau^2\ln|\tau| - \frac{1}{4}\,\tau^2,$$

$$\int \operatorname{Arctan}\left(\frac{\tau}{a}\right)\,\mathrm{d}\tau = \tau\cdot\operatorname{Arctan}\left(\frac{\tau}{a}\right) - \int \frac{\tau}{1+\left(\frac{\tau}{a}\right)^2}\cdot\frac{1}{a}\,\mathrm{d}\tau = \tau\,\operatorname{Arctan}\left(\frac{\tau}{a}\right) - \frac{a}{2}\,\ln\left(1+\left(\frac{\tau}{a}\right)^2\right),$$

hence by insertion and convenient choices of τ and k,

$$\begin{aligned}
\mathcal{L} &= \frac{\mu}{\pi b^2}\left\{\left[-\frac{1}{4}(a^2+(t-b)^2)\ln(a^2+(t-b)^2)+\frac{1}{4}(a^2(t-b)^2)\right]_{t=0}^{b}\right.\\
&\qquad +\left[\frac{1}{4}(a^2+t^2)\ln(a^2+t^2)-\frac{1}{4}(a^2+t^2)\right]_{t=0}^{b}\\
&\qquad +\left[\frac{1}{4}(t-b)^2\ln(b-t)-\frac{1}{8}(t-b)^2\right]_{t=0}^{b}+\left[-\frac{1}{4}t^2\ln t+\frac{1}{8}t^2\right]_{t\to 0}^{b}\\
&\qquad +\left[-a(t-b)\operatorname{Arctan}\left(\frac{t-b}{a}\right)+\frac{a^2}{2}\,\ln\left(1+\left(\frac{t-b}{a}\right)^2\right)\right]_{t=0}^{b}\\
&\qquad \left.+\left[at\operatorname{Arctan}\left(\frac{t}{a}\right)-\frac{a^2}{2}\ln\left\{1+\left(\frac{t}{a}\right)^2\right\}\right]_{t=0}^{b}-\frac{1}{2}b^2\right\}\\
&= \frac{\mu}{\pi b^2}\left\{-\frac{1}{4}a^2\ln(a^2)+\frac{1}{4}a^2 2+\frac{1}{4}(a^2+b^2)\ln(a^2+b^2)\right.\\
&\qquad -\frac{1}{4}(a^2+b^2)+\frac{1}{4}(a^2+b^2)\ln(a^2+b^2)-\frac{1}{4}(a^2+b^2)-\frac{1}{4}b^2\ln b+\frac{1}{8}b^2-\frac{1}{4}b^2\ln b\\
&\qquad +\frac{1}{8}b^2+ab\operatorname{Arctan}\left(\frac{b}{a}\right)-\frac{a^2}{2}\ln\left(\frac{a^2+b^2}{a^2}\right)\\
&\qquad \left.+ab\operatorname{Arctan}\left(\frac{b}{a}\right)-\frac{a^2}{2}\ln\left(\frac{a^2+b^2}{a^2}\right)-\frac{1}{2}b^2\right\}\\
&= \frac{\mu}{\pi b^2}\left\{-\frac{1}{2}a^2\ln a+\frac{1}{2}(a^2+b^2)\ln(a^2+b^2)-\frac{1}{2}b^2\ln b-a^2\ln(a^2+b^2)\right.\\
&\qquad \left.+a^2\ln a+2ab\operatorname{Arctan}\left(\frac{b}{a}\right)+\frac{1}{4}a^2-\frac{1}{2}(a^2+b^2)+\frac{1}{4}b^2-\frac{1}{2}b^2\right\}\\
&= \frac{\mu}{\pi b^2}\left\{\frac{1}{2}a^2\ln a-\frac{1}{2}(a^2-b^2)\ln(a^2+b^2)-\frac{1}{2}b^2\ln b+2ab\operatorname{Arctan}\left(\frac{b}{a}\right)-\frac{1}{4}(a^2-b^2)\right\}\\
&= \frac{\mu}{4\pi}\left\{2\left(\frac{a}{b}\right)^2\ln a-2\ln b+2\left[1-\left(\frac{a}{b}\right)^2\right]\ln\left[1+\left(\frac{a}{b}\right)^2\right]+8\,\frac{a}{b}\operatorname{Arctan}\left(\frac{b}{a}\right)+1-\left(\frac{a}{b}\right)^2\right\}.
\end{aligned}$$

Example 38.10 *Change the problem of* **Example 38.9** *in the following way: The two parallel strips are placed in the same plane. We assume that the strips may be considered as infinitely thin, that the flows are equally distributed and that the permeability μ is constant. The breadth is denoted by b and the distance by a, cf. the figure. It can be proved that the inductance per length $\mathcal{L}$ of this wire is given by*

$$\mathcal{L} = \frac{\mu}{\pi b^2} \int_{\frac{1}{2}a}^{\frac{1}{2}a+b} \left\{ \int_{\frac{1}{2}a}^{\frac{1}{2}a+b} \ln \frac{x+\tilde{x}}{|x-\tilde{x}|} \, \mathrm{d}x \right\} \mathrm{d}\tilde{x}.$$

We consider this as an improper plane integral and want to find $\mathcal{L}$. It will be convenient to apply the quotient $\alpha = \frac{a}{b}$ and introduce the new variables (ξ, η) by putting

$$x = \frac{1}{2}a + b\xi, \qquad \tilde{x} = \frac{1}{2}a + b\eta.$$

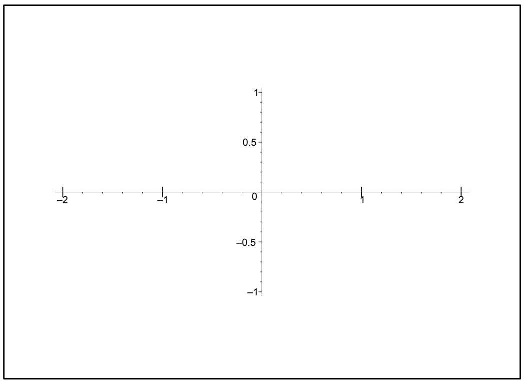

Figure 38.3: The parallel strips are represented by the intervals $[-2,-1]$ and $[1,2]$, corresponding to $a = 2$ and $b = 1$.

A Improper plane integral and Electromagnetism.

D Sketch the $(x, \tilde{x})$-domain and the (ξ, η)-domain and indicate where the integrand is not defined. Then transform the improper integral into the (ξ, η)-space.

I It follows from $\frac{a}{2} \le x = \frac{a}{2} + b\xi \le \frac{a}{2} + b$ that $0 \le \xi \le 1$, and similarly, $0 \le \eta \le 1$. Furthermore, $\tilde{x} = x$ corresponds to $\xi = \eta$. Finally,

$$\frac{x+\tilde{x}}{|x-\tilde{x}|} = \frac{\frac{a}{2} + b\xi + \frac{a}{2} + b\eta}{\left|\frac{a}{2} + b\xi - \frac{a}{2} - b\eta\right|} = \frac{a + b(\xi+\eta)}{b|\xi-\eta|} = \frac{\alpha + \xi + \eta}{|\xi - \eta|} > 1,$$

so the integrand is positive, and one does not need to be too careful in the computation of the improper plane integral: Either we get $+\infty$, or the right value.

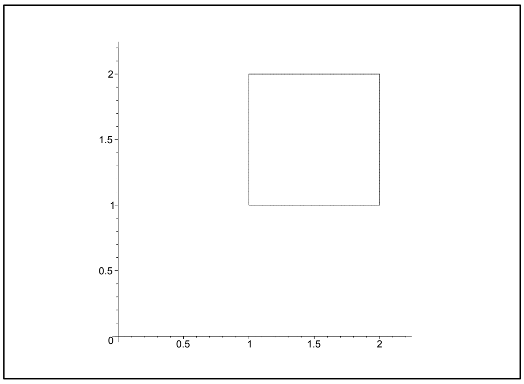

Figure 38.4: The domain in the $(x, \tilde{x})$-space.

We find

$$\begin{aligned}\mathcal{L} &= \frac{\mu}{\pi b^2}\int_{\frac{a}{2}} + \frac{a}{2} + b\left\{\int_{\frac{a}{2}}^{\frac{a}{2}+b} \ln\frac{x+\tilde{x}}{|x-\tilde{x}|}\,\mathrm{d}x\right\}\mathrm{d}\tilde{x} = \frac{\mu}{\pi b^2}\int_0^1\left\{\int_0^1 \ln\left(\frac{\alpha+\xi+\eta}{|\xi-\eta|}\right)\,b\,\mathrm{d}\xi\right\}b\,\mathrm{d}\eta \\ &= \frac{\mu}{\pi}\int_0^1\left\{\int_0^1 \ln(\alpha+\xi+\eta)\,\mathrm{d}\xi\right\}\mathrm{d}\eta - \frac{\mu}{\pi}\int_0^1\left\{\int_0^1 \ln|\xi-\eta|\,d\xi\right\}\mathrm{d}\eta.\end{aligned}$$

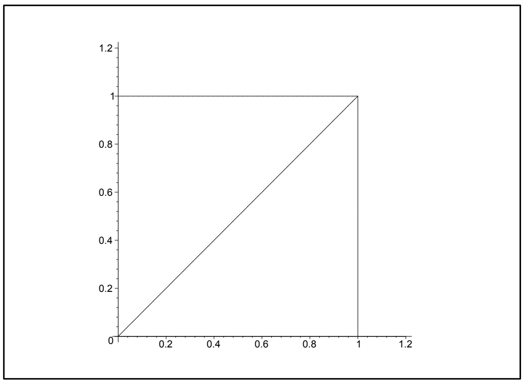

Figure 38.5: The domain in the (ξ, η)-plane.

Here,

$$\begin{aligned}
\int_0^1 \{\ln(\alpha+\xi+\eta)\,\mathrm{d}\xi\}\;\mathrm{d}\eta &= \int_0^1 [(\alpha+\xi+\eta)\ln(\alpha+\xi+\eta)-(\alpha+\xi+\eta)]_{\xi=0}^1\,\mathrm{d}\eta\\
&= \int_0^1 \{(\alpha+1+\eta)\ln(\alpha+1+\eta)-(\alpha+\eta)\ln(\alpha+\eta)-1\}\,\mathrm{d}\eta\\
&= \left[\frac{1}{2}(\alpha+1+\eta)^2\ln(\alpha+1+\eta)-\frac{1}{4}(\alpha+1+\eta)^2-\frac{1}{2}(\alpha+\eta)^2\ln(\alpha+\eta)+\frac{1}{4}(\alpha+\eta)^2\right]_{\eta=0}^1-1\\
&= \frac{1}{2}(\alpha+2)^2\ln(\alpha+2)-\frac{1}{4}(\alpha+2)^2-\frac{1}{2}(\alpha+1)^2\ln(\alpha+1)+\frac{1}{4}(\alpha+1)^2\\
&\qquad -\frac{1}{2}(\alpha+1)^2\ln(\alpha+1)+\frac{1}{4}(\alpha+1)^2+\frac{1}{2}\alpha^2\ln\alpha-\frac{1}{4}\alpha^2-1\\
&= \frac{1}{2}(\alpha+2)^2\ln(\alpha+2)-(\alpha+1)^2\ln(\alpha+1)+\frac{1}{2}\alpha^2\ln\alpha-\frac{3}{2}.
\end{aligned}$$

Then by a symmetric argument,

$$\begin{aligned}
\int_0^1\left\{\int_0^1 \ln|\xi-\eta|\,\mathrm{d}\xi\right\}\mathrm{d}\eta &= 2\int_0^1\left\{\int_0^\eta \ln(\eta-\xi)\,\mathrm{d}\xi\right\}\mathrm{d}\eta\\
&= 2\int_0^1 [(\xi-\eta)\ln(\eta-\xi)-(\xi-\eta)]_{\xi=0}^{\xi\to\eta}\,\mathrm{d}\eta = 2\int_0^1\{\eta\ln\eta+\eta\}\,\mathrm{d}\eta\\
&= 2\left[\frac{\eta^2}{2}\ln\eta-\frac{\eta^2}{4}+\frac{\eta^2}{2}\right]_{\eta\to 0}^1 = 2\cdot\frac{1}{4}=\frac{1}{2}.
\end{aligned}$$

By insertion,

$$\begin{aligned}\mathcal{L} &= \frac{\mu}{\pi}\left\{\frac{1}{2}(\alpha+2)^2\ln(\alpha+2)-(\alpha+1)^2\ln(\alpha+1)-\frac{3}{2}-\frac{1}{2}\right\}\\ &= \frac{\mu}{2\pi}\left\{(\alpha+2)^2\ln(\alpha+2)-2(\alpha+1)^2\ln(\alpha+1)+\alpha^2\ln\alpha-4\right\}.\end{aligned}$$

Example 38.11 *For an (infinitely) long conductive cylinder with an equally distributed current I where we assume that the permeability μ is constant in space, we get for the magnetic flux density that*

$$\mathbf{B}=\frac{\mu I}{2\pi a^2}\,\mathbf{V},$$

where $\mathbf{V}$ *is the vector field considered in* **Example 38.28**. *We have placed the coordinate system such that the axis of the cylinder is the z-axis, and we describe the cylinder by $\varrho \le a$. The magnetic field intensity* $\mathbf{H}$ *is equal to* $\mathbf{B}/\mu$.

1) Prove that Ampère's law is fulfilled for the considered circles.

2) Show by comparison with **Example 38.28** *that a magnetic vector potential $A\,\mathbf{e}_z$ is given by*

$$A=\begin{cases}\dfrac{\mu I}{4\pi}\left\{1-\left(\dfrac{\varrho}{a}\right)^2\right\}, & \varrho<a,\\[2ex] \dfrac{\mu I}{2\pi}\ln\dfrac{a}{\varrho}, & \varrho\ge a.\end{cases}$$

A Distribution of a current.

D Analyze Ampère's law. The last question is straightforward.

1) Let $\mathbf{H}$ denote the magnetic field intensity and $I(\mathcal{F})$ the electric flow through any surface $\mathcal{F}$. Then by Ampère's law,

$$\oint_{\partial\mathcal{F}}\mathbf{H}\cdot\mathbf{t}\,\mathrm{d}s=I(\mathcal{F}).$$

The flow is evenly distributed, so the flux density is

$$\mathbf{J}=\begin{cases}\dfrac{\mu I}{\pi a^2}\,\mathbf{e}_z & \text{for } \varrho\le a,\\[2ex] \mathbf{0}, & \text{for } \varrho\ge a,\end{cases}$$

because the area of a cross section of the wire is πa^2.

Now μ and I are constants, so when $\mathcal{F}$ is chosen as a circle ia a plane parallel to the xy-plane and of centrum of the z-axis and of radius ϱ, then

$$\oint_{\mathcal{K}}\mathbf{H}\cdot\mathbf{t}\,\mathrm{d}s=I(\mathcal{F})=\begin{cases}\dfrac{\mu I}{\pi a^2}\,\pi\varrho^2=\mu I\left(\dfrac{\varrho}{a}\right)^2, & \text{when } \varrho<a,\\[2ex] \dfrac{\mu I}{\pi a^2}\,\pi I, & \text{when } \varrho\ge a.\end{cases}$$

We have for comparison,

$$\oint_{\mathcal{K}} \frac{\mu I}{2\pi a^2}\,\mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s = \begin{cases} \dfrac{\mu I}{2\pi a^2}\,2\pi\varrho^2 = \mu I\left(\dfrac{\varrho}{a}\right)^2, & \text{when } \varrho < a,\\ \mu I, & \text{when } \varrho \ge a.\end{cases}$$

We conclude that

$$(38.2)\quad \oint_{\mathcal{K}}\left(\mathbf{H} - \frac{\mu I}{2\pi a^2}\,\mathbf{V}\right)\cdot\mathbf{t}\,ds = 0,$$

which is trivially satisfied for

$$\mathbf{H} = \frac{\mu I}{2\pi a^2}\,\mathbf{V}.$$

If we assume that $\mathbf{B} = \mathbf{H}$, we are almost finished. However, see also the following remark.

Remark. Since $\mathcal{K}$ is chosen among a very special set of curves we can strictly speaking not conclude the uniqueness. However, the existence is obvious. ♢

2) This is straightforward.

Example 38.12 *Consider a* double wire*, i.e. two parallel conductive cylinders. The direction of the generator is parallel to the z-axis. We denote the two domains in which the two cylinders intersect the (x, y)-plane by S_1 and S_2. We shall also assume the following:*
The flow density $\mathbf{J}$ of the conductors is parallel to the z-axis, the flows are I and $-I$, and the permeability μ is constant. It can be proved that we get a vector potential $(0, 0, A)$ by adding contributions from the two conductors and that the inductance per length $\mathcal{L}$ is given by

$$\mathcal{L}I^2 = \int_{S_1} JA\,\mathrm{d}S + \int_{S_2} JA\,\mathrm{d}S.$$

Show by applying the result of **Example 38.11** *and a mean value theorem for harmonic functions that if we consider a double wire consisting of two equal circular cylinders of radius a and distance c $(> 2a)$ between their axis and supporting equally distributed currents, that we have*

$$\mathcal{L} = \frac{\mu}{\pi}\left(\frac{1}{4} + \ln\frac{c}{a}\right).$$

A This is a fairly long example from Electromagnetism with a guideline.

D Sketch a figure. Add the vector potentials from **Example 38.11** in order to find J. Finally, compute $\mathcal{L}$ by showing that some convenient function is harmonic.

I Let S_1 be the disc of centrum $(0, 0)$, and S_2 the disc of centrum $(c, 0)$, both of radius a, where $c > 2a$.

We have according to **Example 38.11**,

$$A_1 = \begin{cases} \dfrac{\mu I}{4\pi}\left\{1 - \dfrac{x^2+y^2}{a^2}\right\} & \text{for } x^2+y^2 < a^2,\\[2ex] \dfrac{\mu I}{4\pi}\ln\left(\dfrac{a^2}{x^2+y^2}\right) & \text{for } x^2+y^2 \ge a^2,\end{cases}$$

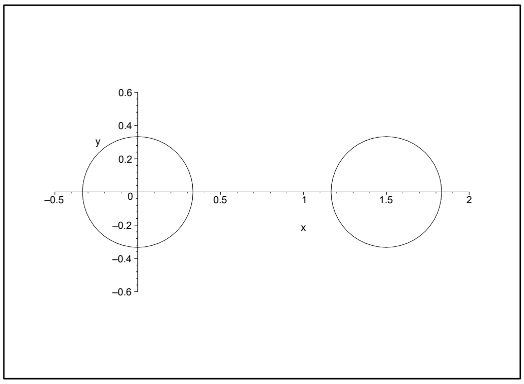

Figure 38.6: Cross section of the double wire.

and

$$A_2 = \begin{cases} -\dfrac{\mu I}{4\pi}\left\{1-\dfrac{(x-c)^2+y^2}{a^2}\right\} & \text{for } (x-c)^2+y^2<a^2, \\[2mm] -\dfrac{\mu I}{4\pi}\ln\left(\dfrac{a^2}{(x-c)^2+y^2}\right) & \text{for } (x-c)^2+y^2 \ge a^2. \end{cases}$$

Furthermore, $J_1 = \dfrac{I}{\pi a^2}$ and $J_2 = -\dfrac{I}{\pi a^2}$.
Therefore,

$$\begin{aligned} \mathcal{L} &= \frac{1}{I^2}\int_{S_1} JA\,\mathrm{d}S + \frac{1}{I^2}\int_{S_2} JA\,\mathrm{d}S \\ &= \frac{1}{I^2}\int_{S_1} J_1(A_1+A_2)\,\mathrm{d}S + \frac{1}{I^2}\int_{S_2} J_2(A_1+A_2)\,\mathrm{d}S \\ &= \frac{1}{I^2}\cdot\frac{I}{\pi a^2}\int_{S_1}\left\{\frac{\mu I}{4\pi}\left(1-\frac{x^2+y^2}{a^2}\right)-\frac{\mu I}{4\pi}\ln\left(\frac{a^2}{(x-c)^2+y^2}\right)\right\}\mathrm{d}S \\ &\quad +\frac{1}{I^2}\left(-\frac{I}{\pi a^2}\right)\int_{S_2}\left\{-\frac{\mu I}{4\pi}\left(1-\frac{(x-c)^2+y^2}{a^2}\right)+\frac{\mu I}{4\pi}\ln\left(\frac{a^2}{x^2+y^2}\right)\right\}\mathrm{d}S \\ &= \frac{1}{I^2}\cdot\frac{I}{\pi a^2}\cdot\frac{\mu I}{4\pi}\left\{2\int_{S_1}\left(1-\frac{x^2+y^2}{a^2}\right)\mathrm{d}S\right. \\ &\quad \left.+\int_{S_1}\left\{\ln\left(\frac{(x-c)^2+y^2}{a^2}\right)+\ln\left(\frac{(x+c=)^2+y^2}{a^2}\right)\right\}\mathrm{d}S\right\}. \end{aligned}$$

The function $f(x,y) = \ln(x^2+y^2)$ is harmonic. In fact,

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2} \qquad \text{and} \qquad \frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2},$$

hence

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{x^2+y^2} - \frac{4x^2}{(x^2+y^2)^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2} \qquad \text{and} \qquad \frac{\partial^2 f}{\partial y^2} = \frac{2x^2-2y^2}{(x^2+y^2)^2},$$

where the latter follows by either repeating the computation above or by exploiting the symmetry, i.e. by interchanging x and y. Then by adding the results,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0,$$

and we have proved that $f(x,y)$ is harmonic. Then

$$\ln\left(\frac{(x\pm c)^2+y^2}{a^2}\right)$$

is also harmonic and we conclude that

$$\int_{S_1}\left\{\ln\left(\frac{(x-c)^2+y^2}{a^2}\right)+\ln\left(\frac{(x+c)^2+y^2}{a^2}\right)\right\}\mathrm{d}S$$
$$= \text{area}(S_1)\cdot 2\ln\left(\frac{c^2}{a^2}\right) = 4\pi a^2 \ln\left(\frac{c}{a}\right).$$

Furthermore,

$$\int_{S_1}\left(1-\frac{x^2+y^2}{a^2}\right)\mathrm{d}S=\pi a^2-\frac{1}{a^2}\int_0^{2\pi}\left\{\int_0^a \varrho^2\cdot\varrho\,\mathrm{d}\varrho\right\}\mathrm{d}\varphi=\pi a^2-\frac{1}{a^2}\cdot 2\pi\left(\frac{a^4}{4}\right)=\frac{\pi}{2}\,a^2,$$

hence by insertion,

$$\mathcal{L}=\frac{1}{I^2}\cdot\frac{I}{\pi a^2}\cdot\frac{\mu I}{4\pi}\left\{2\cdot\frac{\pi}{2}\,a^2+4\pi a^2\ln\left(\frac{c}{a}\right)\right\}=\frac{\mu}{4\pi}\left\{1+4\ln\left(\frac{c}{a}\right)\right\}=\frac{\mu}{\pi}\left(\frac{1}{4}+\ln\frac{c}{a}\right)$$

as claimed above.

Example 38.13 *A conductive non-magnetic ball K of conductivity γ and radius a is rotating around a diameter of the angular velocity $\mho$ in an homogeneous magnetic field $\mathbf{B}$, which is perpendicular to the vector $\mho$. It can be proved that there is induced a current distribution in the ball with the density*

$$\mathbf{J}=\frac{1}{2}\,\gamma\,\mathbf{x}\times(\mathbf{B}\times\mho),$$

where $\mathbf{x}$ denotes the vector seen from the centrum of the ball. Find the Joule heat effect

$$P=\int_{\mathcal{K}}\frac{J^2}{\gamma}\,\mathrm{d}\Omega.$$

A A space integral from Electromagnetism.

D Introduce a convenient coordinate system. Compute $\mathbf{J}$ and then $J^2=\|\mathbf{J}\|^2$. Finally, find P.

I Let K denote the ball of centrum $(0,0,0)$, and assume that it is rotating around the z-axis. Thus for $(x,y,z)\in K\setminus\{(0,0,z)\}$,

$$\mho=\left(-\frac{y}{\sqrt{x^2+y^2}}\,,\,\frac{x}{\sqrt{x^2+y^2}}\,,\,0\right)\omega,$$

where we have put $\omega=\|\mho\|$. Hence, $\mho$ is perpendicular to $\mathbf{e}_z$ everywhere, Therefore, $\mathbf{B}=B\,\mathbf{e}_z$, and we have

$$\mathbf{B}\times\mho=\omega\begin{vmatrix}\mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z\\ 0 & 0 & B\\ -\frac{y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} & 0\end{vmatrix}=\frac{B\omega}{\sqrt{x^2+y^2}}\begin{vmatrix}\mathbf{e}_x & \mathbf{e}_y\\ -y & x\end{vmatrix}=\frac{B\omega}{\sqrt{x^2+y^2}}\,(x,y,0),$$

hence

$$\begin{aligned}\mathbf{J}&=\frac{1}{2}\,\gamma\,\mathbf{x}\times(\mathbf{B}\times\mho)=\frac{1}{2}\,\gamma\cdot\frac{B\omega}{\sqrt{x^2+y^2}}\begin{vmatrix}\mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z\\ x & y & z\\ x & y & 0\end{vmatrix}\\ &=\frac{1}{2}\,\frac{\gamma B\omega}{\sqrt{x^2+y^2}}\begin{vmatrix}\mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z\\ 0 & 0 & z\\ x & y & 0\end{vmatrix}=\frac{1}{2}\,\frac{\gamma B\omega}{\sqrt{x^2+y^2}}\,(-z)\begin{vmatrix}\mathbf{e}_x & \mathbf{e}_y\\ x & y\end{vmatrix}=-\frac{1}{2}\cdot\frac{\gamma B\omega z}{\sqrt{x^2+y^2}}\,(y,-x,0).\end{aligned}$$

Hence

$$\frac{J^2}{\gamma} = \frac{\|\mathbf{J}\|^2}{\gamma} = \frac{1}{4}\,\gamma\,B^2\omega^2z^2\cdot\frac{x^2+y^2}{x^2+y^2} = \frac{1}{4}\,\gamma\,B^2\omega^2z^2,$$

and thence

$$\begin{aligned} P &= \int_K \frac{J^2}{\gamma}\,\mathrm{d}\Omega = \frac{1}{4}\,\gamma B^2\omega^2\int_K z^2\,\mathrm{d}\Omega = \frac{1}{4}\,\gamma B^2\omega^2\int_{-a}^{a} z^2(a^2-z^2)\pi\,\mathrm{d}z \\ &= \frac{1}{2}\,\gamma B^2\omega^2\pi\int_0^a (a^2z^2-z^4)\,\mathrm{d}z = \frac{1}{2}\,\gamma\,B^2\omega^2\pi\left\{\frac{a^5}{3}-\frac{a^5}{5}\right\} = \frac{\pi}{15}\,\gamma\,B^2\omega^2a^5. \end{aligned}$$

38.5 Scalar and vector potentials

Example 38.14

A. *Given the* divergence free *vector field*

$$\mathbf{V}(x,y,z) = \left(3y^2z^2, xy, -xz\right), \qquad (x,y,z)\in\mathbb{R}^3.$$

Since $\mathbb{R}^3$ *is* star-shaped, $\mathbf{V}$ *has vector potentials* $\mathbf{W}$*. Find one of these, i.e. find* $\mathbf{W}$*, such that*

$$\mathbf{V} = \bigtriangledown\times\mathbf{W} = \mathbf{rot}\ \mathbf{W}.$$

D. The usual solution formula of textbooks looks here very harmless, but from my experience of teaching I can say that i is *not* that harmless. Therefore we shall give a thorough treatment of this example.

The method can briefly be described in the following way:

1) Has this exercise been solved earlier in the text?
2) If 'no', examine if $\mathbf{V}(x,y,z)$ is divergence free.
3) Calculate separately the auxiliary field $\mathbf{T}(\mathbf{x}) = \mathbf{V}(\mathbf{x})\times\mathbf{x}$.
4) Replace $\mathbf{x}$ by $\tau\,\mathbf{x}$. (This is the critical phase of the method).
5) Calculate $\mathbf{W}_0(\mathbf{x}) = \int_0^1 \mathbf{T}(\tau\,\mathbf{x})\,\mathrm{d}\tau$.
6) Check once more the preassumptions of the method.
7) Check the result.

I. 1) *We first check if we earlier have solve the problem.*

It happens quite often at an examination that one in the first part of a problem as a check are asked to calculate the rotation of e.g. the vector field

$$\mathbf{W}_1(x,y,z) = \left(xyz, -\frac{1}{2}\,y^2z^3, \frac{1}{2}\,y^3z^2\right).$$

This is of course given by

$$\nabla \times \mathbf{W}_1 = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xyz & -\dfrac{1}{2}y^2z^3 & \dfrac{1}{2}y^3z^2 \end{vmatrix}$$

$$= \left(\frac{3}{2}y^2z^2 + \frac{3}{2}y^2z^2, xy - 0, 0 - xz\right) = (3y^2z^2, xy, -xz) = \mathbf{V}.$$

Half an hour later one comes to another question in which one is asked to find a vector potential for $\mathbf{V}$. Some students may immediately see that $\mathbf{W}_1$ is a solution. Other poor souls have to go through the following steps:

2) *Examine if* $\mathbf{V}(x, y, z)$ *is divergence free.*

If not, there is no vector potential, and the problem is solved.

In the example under consideration we get

$$\text{div } \mathbf{V} = 0 + x - x = 0,$$

so we must carry on to the next point.

3) *Calculate the auxiliary field* $\mathbf{T}(\mathbf{x}) = \mathbf{V}(\mathbf{x}) = \times \mathbf{x}$.

Be very careful with the order of the factors. And do not yet introduce the parameter τ in the calculations. It will only confuse the overview.

In the case under consideration we get

$$\begin{aligned}
T(\mathbf{x}) &= \mathbf{V}(\mathbf{x}) \times \mathbf{x} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 3y^2z^2 & xy & -xz \\ x & y & z \end{vmatrix} \\
&= \left(xyz - (-xyz), -x^2z3y^2z^3, 3y^3z^2 - x^2y\right) \\
&= \left(2xyz, -x^2z - 3y^2z^3, 3y^3z^2 - x^2y\right).
\end{aligned}$$

4) *Replace x by $\tau\, x$, y by $\tau\, y$ and z by $\tau\, z$.*
This is the critical step where most errors are made. Be extremely scrupulous here. It may be useful to note that one in polynomials just add the factor τ^n to every term which contains n factors of the type x, y or z (they are considered as equal at this count).

In the case under consideration where $\mathbf{T}(\mathbf{x})$ is composed of polynomials we get

$$\mathbf{T}(\tau\,\mathbf{x}) = \left(2\tau^3xyz, -\tau^3x^2z - 3\tau^5y^2z^3, 3\tau^5y^3z^2 - t^4x^2y\right),$$

because xyz, x^2z and x^2y contain three factors, i.e. we shall multiply by τ^3, while y^2z^3 and y^3z^2 contain five factors, so here we add the factor τ^5.

5) *Calculate the candidate*

$$\mathbf{W}_0(\mathbf{x}) = \int_0^1 \mathbf{T}(\tau\,\mathbf{x})\,\mathrm{d}\tau$$

by a coordinate-wise integration after τ.
We have in the case under consideration

$$\begin{aligned}
\mathbf{W}_0(\mathbf{x}) &= \int_0^1 \mathbf{T}(\tau\,\mathbf{x})\,d\tau \\
&= \left(xyz\int_0^1 2\tau^3 d\tau,\ -x^2z\int_0^1 \tau^3\,\mathrm{d}\tau - y^2z^3\int_0^1 3\tau^5\,\mathrm{d}\tau, y^3z^2\int_0^1 3\tau^5\,\mathrm{d}\tau - x^2y\int_0^1 \tau^3\,\mathrm{d}\tau\right) \\
&= \left(\frac{1}{2}\,xyz, -\frac{1}{4}\,x^2z - \frac{1}{2}\,y^2z^3, \frac{1}{2}\,y^3z^2 - \frac{1}{4}\,x^2y\right).
\end{aligned}$$

The expressions are often so enormous that it pays to calculate each coordinate separately. We see that even in this simple case the equations are huge.

6) If our calculations have been correct – note that $\mathbf{0}$ must lie in the domain – then $\mathbf{W}_0(\mathbf{x})$ is a vector potential for $\mathbf{V}(\mathbf{x})$ in every sub-domain of Ω, which can be reached by a radial line in Ω from $\mathbf{0}$.

In the case under consideration we see that $\mathbb{R}^3$ satisfies this criterium, hence

$$\mathbf{W}_0(\mathbf{x}) = \left(\frac{1}{2}\,xyz,\ -\frac{1}{4}\,x^2z - \frac{1}{2}\,y^2z^3,\ \frac{1}{2}\,y^3z^2 - \frac{1}{4}\,x^2y\right)$$

is a vector potential in the whole of $\mathbb{R}^3$ (if our calculations were without errors, of course).

7) By a comparison we see that we have obtained two different solutions, $\mathbf{W}_0(\mathbf{x})$ in the latter case and $\mathbf{W}_1(\mathbf{x})$ in the former one. They are both correct, because their difference is only a gradient field. Nevertheless this phenomenon often causes some panic after the examination, because students are used to from high school that the solution is unique. It is *not* in problems of this type!

Since the method is so difficult it is highly *recommended that one always check every calculation in exercises of this type.* In the present case we get

$$\begin{aligned}\nabla \times \mathbf{W}_0 &= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{1}{2}\,xyz & -\dfrac{1}{4}\,x^2z - \dfrac{1}{2}\,y^2z^3 & \dfrac{1}{2}\,y^3z^2 - \dfrac{1}{4}\,x^2y \end{vmatrix} \\ &= \left(\frac{3}{2}\,y^2z^2 - \frac{1}{4}\,x^2 - \left\{-\frac{1}{4}\,x^2 - \frac{3}{2}\,y^2z^2\right\}, \frac{1}{2}\,xy - \left\{0 - \frac{1}{2}\,xy\right\}, -\frac{1}{2}\,xz - \frac{1}{2}\,xz\right) \\ &= \left(3y^2z^2, xy, -xz\right) = \mathbf{V}(x,y,z).\end{aligned}$$

We have now checked that $\mathbf{W}_0$ is a vector potential.

Remark 38.1 If a check does not give the right answer and div $\mathbf{V} = 0$, then we have made an error at some place, probably at 4) which would be the first place where I would search myself. This does not leave out the possibility of errors in the other steps as well. $\Diamond$

Example 38.15 *Prove in each of the following cases that the given vector field* $\mathbf{V} : \mathbb{R}^3 \to \mathbb{R}^3$ *is divergence free. The find a vector potential* $\mathbf{W} : \mathbb{R}^3 \to \mathbb{R}^3$*, such that* $\mathbf{V} = \nabla \times \mathbf{W}$*. (We may not necessarily consider the points where* $xyz = 0$*).*

1) $\mathbf{V}(x,y,z) = \left(\cosh(z^2), \cosh(x^2), \cosh(y^2)\right)$.

2) $\mathbf{V}(x,y,z) = (x^2y + z, xy^2 + z, -4xyz)$.

3) $\mathbf{V}(x,y,z) = (xz, yz, -z^2)$.

4) $\mathbf{V}(x,y,z) = \left(\dfrac{1}{1+y^2}, \dfrac{1}{1+z^2}, \dfrac{1}{1+x^2}\right)$.

5) $\mathbf{V}(x,y,z) = \left(\dfrac{\sin z}{z}, \dfrac{\sin x}{x}, \dfrac{\sin y}{y}\right)$.

6) $\mathbf{V}(x,y,z) = (\exp x, y\,\exp x, -2z\,\exp x)$.

A Vector potential.

D Clearly, the domain $\mathbb{R}^3$ is star shaped. First prove that the field is divergence free. Then compute

$$\mathbf{U}(\mathbf{x}) = \int_0^1 t\,\mathbf{V}(t\,\mathbf{x})\,\mathrm{d}t$$

and

$$\mathbf{W}(\mathbf{x}) = \mathbf{U}(\mathbf{x}) \times \mathbf{x} = -\mathbf{x} \times \int_0^1 t\,\mathbf{V}(t\,\mathbf{x})\,\mathrm{d}t = \int_0^1 \mathbf{V}(t\,\mathbf{x}) \times (t\,\mathbf{x})\,\mathrm{d}t.$$

Finally, check the result, i.e. show that

$$\nabla \times \mathbf{W} = \mathbf{V}.$$

I 1. Since each V_i does not depend on x_i, we clearly have that $\nabla \cdot \mathbf{V} = 0$.

Because of the symmetry it suffices to compute

$$\int_0^1 t\,\cosh\left((tu)^2\right)\,\mathrm{d}t = \int_0^1 t\,\cosh\left(t^2u^2\right)\,\mathrm{d}t = \frac{1}{2}\int_0^1 \cosh\left(\tau\,u^2\right)\,\mathrm{d}\tau = \frac{1}{2}\,\frac{\sinh(u^2)}{u^2},$$

where $\dfrac{\sinh\alpha}{\alpha}$ in general in the following is interpreted as 1, when $\alpha = 0$. Then continue either by a direct calculation or by a continuous extension, i.e. by going to the limit.

It follows from the above that

$$\mathbf{U}(\mathbf{x}) = \int_0^1 t\,\mathbf{V}(t\,\mathbf{x})\,\mathrm{d}t = \frac{1}{2}\left(\frac{\sinh(z^2)}{z^2}\,,\,\frac{\sinh(x^2)}{x^2}\,,\,\frac{\sinh(y^2)}{y^2}\right),$$

hence

$$\begin{aligned}\mathbf{W} &= \mathbf{U} \times \mathbf{x} = \frac{1}{2}\begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\sinh(z^2)}{z^2} & \dfrac{\sinh(x^2)}{x^2} & \dfrac{\sinh(y^2)}{y^2} \\ x & y & z \end{vmatrix} \\ &= \frac{1}{2}\left(z\cdot\frac{\sinh(x^2)}{x^2} - \frac{\sinh(y^2)}{y}\,,\,x\cdot\frac{\sinh(y^2)}{y^2} - \frac{\sinh(z^2)}{z}\,,\,y\cdot\frac{\sinh(z^2)}{z^2} - \frac{\sinh(x^2)}{x}\right).\end{aligned}$$

C TEST. We have

$$\begin{aligned}\nabla \times \mathbf{W} &= \frac{1}{2}\begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ W_1(x,y,z) & W_2(x,y,z) & W_3(x,y,z) \end{vmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \dfrac{\sinh(z^2)}{z^2} + 2\,\sinh(z^2) - \dfrac{\sinh(z^2)}{z^2} \\ \dfrac{\sinh(x^2)}{x^2} + 2\,\sinh(x^2) - \dfrac{\sinh(x^2)}{x^2} \\ \dfrac{\sinh(y^2)}{y^2} + 2\,\sinh(y^2) - \dfrac{\sinh(y^2)}{y^2} \end{pmatrix} \\ &= \left(\sinh(z^2)\,,\,\sinh(x^2)\,,\,\sinh(y^2)\right) = \mathbf{V}.\end{aligned}$$

The result is correct.

I 2. First compute

$$\operatorname{div} \mathbf{V} = \nabla \cdot \mathbf{V} = 2xy + 2xy - 4xy = 0,$$

thus the field is divergence free. Then

$$\begin{aligned}
\mathbf{U}(\mathbf{x}) &= \int_0^1 t\,\mathbf{V}(t\,\mathbf{x})\,\mathrm{d}t = \left(\int_0^1 t\{t^3x^2y+tz^2\}\,\mathrm{d}t, \int_0^1 t\{t^3xy^2+tz\}\,\mathrm{d}t, -4xyz\int_0^1 t\cdot t^3\,\mathrm{d}t\right) \\
&= \left(\frac{1}{5}\,x^2y+\frac{1}{3}\,z\,,\,\frac{1}{5}\,xy^2+\frac{1}{3}\,z\,,\,-\frac{4}{5}\,xyz\right),
\end{aligned}$$

and hence

$$\begin{aligned}
\mathbf{W} &= \mathbf{U}\times\mathbf{x} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{1}{5}\,x^2y+\frac{1}{3}\,z & \frac{1}{5}\,xy^2+\frac{1}{3}\,z & -\frac{4}{5}\,xyz \\ x & y & z \end{vmatrix} \\
&= \left(\frac{1}{5}\,xy^2z+\frac{1}{3}\,z^2+\frac{4}{5}\,xy^2z\,,\,-\frac{4}{5}\,x^2yz-\frac{1}{5}\,x^2yz-\frac{1}{3}\,z^2,\right. \\
&\qquad \left.\frac{1}{5}\,x^2y^2+\frac{1}{3}\,yz-\frac{1}{5}\,x^2y^2+\frac{1}{3}\,yz-\frac{1}{5}\,x^2y^2-\frac{1}{3}\,xz\right) \\
&= \left(xy^2z+\frac{1}{3}\,z^2\,,\,-x^2yz-\frac{1}{3}\,z^2\,,\,\frac{1}{3}\,yz-\frac{1}{3}\,xz\right) \\
&= z\left(xy^2+\frac{1}{3}\,z\,,\,-x^2y-\frac{1}{3}\,z\,,\,\frac{1}{3}\,y-\frac{1}{3}\,x\right).
\end{aligned}$$

C Check. Here

$$\nabla\times\mathbf{W} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy^2z+\frac{1}{3}\,z^2 & -x^2yz-\frac{1}{3}\,z^2 & \frac{1}{3}\,yz-\frac{1}{3}\,xz \end{vmatrix}$$

$$= \left(\frac{1}{3}\,z+x^2y+\frac{2}{3}\,z, xy^2+\frac{2}{3}\,z+\frac{1}{3}\,z, -2xyz-2xyz\right)$$

$$= (x^2y+z, xy^2+z, -4xyz) = \mathbf{V}.$$

Our result has proved to be correct.

I 3. Since

$$\text{div}\,\mathbf{V} = \nabla\cdot\mathbf{V} = z+z-2z = 0,$$

the field is divergence free.

Furthermore,

$$\mathbf{U}(\mathbf{x}) = \int_0^1 t\,\mathbf{V}(t\,\mathbf{x})\,\mathrm{d}t = \left(\int_0^1 t\cdot t^2xz\,\mathrm{d}t, \int_0^1 t\cdot t^2yz\,\mathrm{d}t, -\int_0^1 t\cdot t^2z^2\,\mathrm{d}t\right)$$

$$= \frac{1}{4}\,(xz,yz,-z^2) = \frac{1}{4}\,\mathbf{V}(x,y,z),$$

thus

$$\mathbf{W} = \mathbf{U}\times\mathbf{x} = \frac{1}{4}\begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ xz & yz & -z^2 \\ x & y & z \end{vmatrix} = \frac{z}{4}\begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ x & y & -z \\ x & y & z \end{vmatrix} = \frac{z}{4}\begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ x & y & -z \\ 0 & 0 & 2z \end{vmatrix}$$

$$= \frac{z^2}{2}\begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y \\ x & y \end{vmatrix} = \frac{1}{2}\,(yz^2,-xz^2,0).$$

C Check. We get

$$\nabla\times\mathbf{W} = \frac{1}{2}\begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz^2 & -xz^2 & 0 \end{vmatrix} = \frac{1}{2}\,(2xz,2yz,-z^2-z^2) = (xz,yz,-z^2) = \mathbf{V}.$$

We have checked our result.

I 4. Clearly, since each V_i is independent of x_i, we must have $\nabla \cdot \mathbf{V} = 0$.

Because of the symmetry it suffices to compute

$$\int_0^1 t \cdot \frac{1}{1+(tu)^2}\,\mathrm{d}t = \frac{1}{2}\int_0^1 \frac{1}{1+\tau\,u^2}\,\mathrm{d}\tau = \frac{1}{2}\,\frac{\ln(1+u^2)}{u^2},$$

where the result by continuous extension is interpreted as $\frac{1}{2}$ for $u = 0$. Hence

$$\mathbf{U}(\mathbf{x}) = \int_0^1 t\,\mathbf{V}(t\,\mathbf{x})\,\mathrm{d}t = \frac{1}{2}\left(\frac{\ln(1+y^2)}{y^2}, \frac{\ln(1+z^2)}{z^2}, \frac{\ln(1+x^2)}{x^2}\right),$$

and therefore

$$\mathbf{W} \;=\; \mathbf{U}\times\mathbf{x} = \frac{1}{2}\left|\begin{array}{ccc} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\ln(1+y^2)}{y^2} & \frac{\ln(1+z^2)}{z^2} & \frac{\ln(1+x^2)}{x^2} \\ x & y & z \end{array}\right|$$

$$= \frac{1}{2}\left(\frac{\ln(1+z^2)}{z} - y\,\frac{\ln(1+x^2)}{x^2}, \frac{\ln(1+x^2)}{x} - z\,\frac{\ln(1+y^2)}{y^2}, \frac{\ln(1+y^2)}{y} - x\,\frac{\ln(1+z^2)}{z^2}\right).$$

C CHECK. Here

$$\nabla\times\mathbf{W} \;=\; \frac{1}{2}\left|\begin{array}{ccc} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ W_1(x,y,z) & W_2(x,y,z) & W_3(x,y,z) \end{array}\right|$$

$$= \frac{1}{2}\begin{pmatrix} \frac{2}{1+y^2} - \frac{\ln(1+y^2)}{y^2} + \frac{\ln(1+y^2)}{y^2} \\ \frac{2}{1+z^2} - \frac{\ln(1+z^2)}{z^2} + \frac{\ln(1+z^2)}{z^2} \\ \frac{2}{1+x^2} - \frac{\ln(1+x^2)}{x^2} + \frac{\ln(1+x^2)}{x^2} \end{pmatrix} = \left(\frac{1}{1+y^2}\;\frac{1}{1+z^2}, \frac{1}{1+x^2}\right) = \mathbf{V}(x,y,z).$$

We have checked our result.

I 5. Interpret $\frac{\sin u}{u}$ as 1, when $u = 0$. Then V_i is independent of x_i (same index i in both places), and the field is clearly divergence free. Due to the symmetry it suffices to compute

$$\int_0^1 t \cdot \frac{\sin(tu)}{tu}\,\mathrm{d}t = \frac{1}{u}\int_0^1 \sin(tu)\,\mathrm{d}u = \frac{1-\cos u}{u^2} \qquad \text{for } u \neq 0,$$

and

$$\int_0^1 t\,\mathrm{d}t = \frac{1}{2} \qquad \text{for } u = 0,$$

where we interpret $\dfrac{1-\cos u}{u^2}$ as $\dfrac{1}{2}$, when $u=0$. This is in agreement with the continuous extension. Thus

$$\mathbf{U}(\mathbf{x}) = \int_0^1 t\,\mathbf{V}(t\,\mathbf{x})\,\mathrm{d}t = \left(\frac{1-\cos z}{z^2}\,,\,\frac{1-\cos x}{x^2}\,,\,\frac{1-\cos y}{y^2}\right),$$

and hence

$$\begin{aligned}\mathbf{W} &= \mathbf{U}\times\mathbf{x} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{1-\cos z}{z^2} & \dfrac{1-\cos x}{x^2} & \dfrac{1-\cos y}{y^2} \\ x & y & z \end{vmatrix} \\ &= \left(z\cdot\frac{1-\cos x}{x^2}-\frac{1-\cos y}{y}\,,\,x\cdot\frac{1-\cos y}{y^2}-\frac{1-\cos z}{z}\,,\,y\cdot\frac{1-\cos z}{z^2}-\frac{1-\cos x}{x}\right).\end{aligned}$$

C CHECK. It follows that

$$\begin{aligned}\nabla\times\mathbf{W} &= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ W_1(x,y,z) & W_2(x,y,z) & W_3(x,y,z) \end{vmatrix} \\ &= \begin{pmatrix} \dfrac{1-\cos z}{z^2}+\dfrac{\sin z}{z}-\dfrac{1-\cos z}{z^2} \\ \dfrac{1-\cos x}{x^2}+\dfrac{\sin x}{x}-\dfrac{1-\cos x}{x^2} \\ \dfrac{1-\cos y}{y^2}+\dfrac{\sin y}{y}-\dfrac{1-\cos y}{y^2} \end{pmatrix} = \left(\frac{\sin z}{z}\,,\,\frac{\sin x}{x}\,,\,\frac{\sin y}{y}\right) = \mathbf{V},\end{aligned}$$

and we have checked our result.

I 6. Since

$$\operatorname{div}\,\mathbf{V} = \nabla\cdot\mathbf{V} = \exp x + \exp x - 2\,\exp x = 0,$$

the field is divergence free.

Then

$$\begin{aligned}\mathbf{U}(x,y,z) &= \int_0^1 t\,(\exp(tx), ty\,\exp(tx), -2tz\,\exp(tx))\,\mathrm{d}t \\ &= \left(\int_0^1 t\,\exp(tx)\,\mathrm{d}t\,,\,y\int_0^1 t^2\exp(tx)\,\mathrm{d}t\,,\,-2z\int_0^1 t^2\exp(tx)\,\mathrm{d}t\right) \\ &= \left(\frac{1}{x^2}\int_0^x \tau\cdot\exp(\tau)\,\mathrm{d}\tau\,,\,\frac{y}{x^3}\int_0^x \tau^2\exp(\tau)\,\mathrm{d}\tau\,,\,-\frac{2z}{x^3}\int_0^x \tau^2\exp(\tau)\,\mathrm{d}\tau\right).\end{aligned}$$

A small computation gives

$$\int_0^x \tau\,\exp(\tau)\,\mathrm{d}\tau = (x-1)e^x + 1$$

and

$$\int_0^x \tau^2\,\exp(\tau)\,\mathrm{d}\tau = (x^2-2x+2)e^x - 2,$$

hence by insertion,

$$\mathbf{U}(x,y,z) = \left(\frac{(x-1)e^x+1}{x^2}\,,\; y\cdot\frac{(x^2-2x+2)e^x-2}{x^3}\,,\; -2z\cdot\frac{(x^2-2x+2)e^x-2}{x^3} \right).$$

Then

$$\begin{aligned}
\mathbf{W}(\mathbf{x}) &= \mathbf{U}(\mathbf{x})\times\mathbf{x} \\
&= \left|\begin{array}{ccc}
\mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\
\dfrac{(x-1)e^x+1}{x^2} & y\cdot\dfrac{(x^2-2x+2)e^x-2}{x^3} & -2z\cdot\dfrac{(x^2-2x+2)e^x-2}{x^3} \\
x & y & z
\end{array}\right| \\
&= \begin{pmatrix}
yz\cdot\dfrac{(x^2-2x+2)e^x-2}{x^3}+2yz\cdot\dfrac{(x^2-2x+2)e^x-2}{x^3} \\
-2z\cdot\dfrac{(x^2-2x+2)e^x-2}{x^2}+z\cdot\dfrac{(x-1)e^x+1}{x^2} \\
y\cdot\dfrac{(x-1)e^x+1}{x^2}-y\cdot\dfrac{(x^2-2x+2)e^x-2}{x^2}
\end{pmatrix} \\
&= \begin{pmatrix}
3yz\cdot\dfrac{(x^2-2x+2)e^x-2}{x^3} \\
-z\cdot\dfrac{(2x^2-5x+5)e^x-5}{x^2} \\
-y\cdot\dfrac{(x^2-3x+3)e^x-3}{x^2}
\end{pmatrix}.
\end{aligned}$$

C "CHECK". Even if the original expression of $\mathbf{V}(x,y,z)$ looks very simple, an insertion into the solution formula will give very difficult expressions with e.g. x^2 and x^3 in the denominator. We shall therefore not in this case test the result, i.e. compute

$$\nabla\times\mathbf{W} = \left|\begin{array}{ccc}
\mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\
\dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\
W_1(x,y,z) & W_2(x,y,z) & W_3(x,y,z)
\end{array}\right|.$$

Example 38.16 *Consider a vector field* $\mathbf{V} : A \to \mathbb{R}^2$*, where* A *is an open star shaped subset of the* (X, Y)*-plane. Furthermore, assume that the field* $\mathbf{V}$ *is divergence free.*

1) Prove that the vector field $\mathbf{e}_z \times \mathbf{V}$ *is rotation free and that there exists a scalar field* $W : A \to \mathbb{R}$*, such that* $W\,\mathbf{e}z$ *is a vector potential of* $\mathbf{V}$*.*

2) Prove that a level curve of W *is a field line of* $\mathbf{V}$*.*

A Vector potential.

D Analyze the text step by step and prove the claims in succession.

I 1) According to the assumption, $\mathbf{V} : A \to \mathbb{R}^2$ is a function of the variable (x, y), which satisfies

$$\operatorname{div} \mathbf{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} = 0.$$

Define a vector field $\tilde{\mathbf{V}}$ by

$$\tilde{\mathbf{V}}(x, y, z) = (V_1(x, y), V_2(x, y), 0)\,, \qquad (x, y, z) \in A \times \mathbb{R} = \tilde{A}.$$

Then $\tilde{A}$ is star shaped and $\tilde{\mathbf{V}}$ is also divergence free.

We shall in the following only write $\mathbf{V}$ instead of the more precise $\tilde{\mathbf{V}}$.

By one of the formulæ of differentiation of a product,

$$\begin{aligned}\bigtriangledown \times (\mathbf{e}_z \times \mathbf{V}) &= (\mathbf{V} \cdot \bigtriangledown)\mathbf{e}_z - \mathbf{V}(\bigtriangledown \cdot \mathbf{e}_z) - (\mathbf{e}_z \cdot \bigtriangledown)\mathbf{V} + \mathbf{e}_z(\bigtriangledown \cdot \mathbf{V}) \\ &= \mathbf{0} + \mathbf{0} - \frac{\partial}{\partial z}\mathbf{V} + \operatorname{div} \mathbf{V} \cdot \mathbf{e}_z = \mathbf{0},\end{aligned}$$

and the vector field $\mathbf{e}_z \times \mathbf{V}$ is rotation free.

Thus there exists a scalar field $\tilde{W} : A \times \mathbb{R} \to \mathbb{R}$, such that

$$\bigtriangledown \tilde{W} = \left(\frac{\partial \tilde{W}}{\partial x}, \frac{\partial \tilde{W}}{\partial y}, \frac{\partial \tilde{W}}{\partial z} \right) = \mathbf{e}_z \times \mathbf{V} = (-V_2, V_1, 0)\,.$$

Since $\mathbf{V}$ is independent of z, also $\tilde{W} = W$ must be independent of z, thus we can choose a scalar field $W : A \to \mathbb{R}^2$, such that

$$\bigtriangledown W = (-V_2, V_1, 0) = \mathbf{e}_z \times \mathbf{V}.$$

Furthermore,

$$\begin{aligned}\bigtriangledown \times (E\mathbf{e}_z) &= (\bigtriangledown W) \times \mathbf{e}_z + W \bigtriangledown \times \mathbf{e}_z = (\mathbf{e}_z \times \mathbf{V}) \times \mathbf{e}_z = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ -V_2 & V_1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y \\ -V_2 & V_1 \end{vmatrix} = (V_1, V_2, 0) = \mathbf{V}.\end{aligned}$$

(Alternatively we may twice apply the geometric interpretation of the cross product). This shows that $W\,\mathbf{e}_z$ is a vector potential for $\mathbf{V}$, and we have proved all the claims.

2) A level curve of W is given by

$$W(x,y)=c,$$

where the tangent field $\mathbf{U}(x,y)$ of the level curve satisfies

$$\triangledown W\cdot\mathbf{U}=(\mathbf{e}_z\times\mathbf{V})\cdot\mathbf{U}=0.$$

Clearly, this equation has the solution $\mathbf{U}=\mathbf{V}$, thus the level curve is also a field line of $\mathbf{V}$.

Example 38.17 *Let α be a constant, and let two vector fields on $\mathbb{R}^3$ be given in the following way:*

$$\mathbf{U}=(\triangledown f)\times(\triangledown g),\qquad \mathbf{W}=\alpha\,(f\triangledown g-f\triangledown f).$$

Show that one can choose α such that $\mathbf{W}$ is a vector potential for $\mathbf{U}$.
[Cf. **Example 36.12**.]

A Vector potential.

D Compute $\triangledown\times\mathbf{W}$ and compare with $\mathbf{U}=\triangledown f\times\triangledown g$.

I By the rules of calculations,

$$\begin{aligned}\triangledown\times\mathbf{W} &= \alpha\triangledown\times(f\triangledown g)-\alpha\triangledown\times(g\triangledown f)\\ &= \alpha\triangledown f\times\triangledown g+\alpha f\,(\triangledown\times\triangledown g)-\alpha\triangledown g\times\triangledown f-\alpha g\,(\triangledown\times\triangledown f)\\ &= \alpha\triangledown\times\triangledown g+\mathbf{0}+\alpha\triangledown f\times\triangledown g+\mathbf{0}\\ &= 2\alpha\triangledown f\times\triangledown g=2\alpha\,\mathbf{U}.\end{aligned}$$

We see that if $\alpha=\dfrac{1}{2}$, then $\mathbf{W}$ is a vector potential for $\mathbf{U}$.

Example 38.18 *Let* $\mathbf{V} : \mathbb{R}^3 \to \mathbb{R}^3$ *be a given vector field. Find in each of the cases below the following vector fields:*

$$\mathbf{S}(\mathbf{x}) = \int_0^1 \tau\, \mathbf{V}(\mathbf{x}\,\tau)\, \mathrm{d}\tau, \qquad \mathbf{U}(\mathbf{x}) = -\mathbf{x} \times \mathbf{S}(\mathbf{x}), \qquad \mathbf{W}(\mathbf{x}) = \nabla \times \mathbf{U}(\mathbf{x}).$$

1) $\mathbf{V}(x,y,z) = (x,y,z)$.

2) $\mathbf{V}(x,y,z) = \left(x^2, y^2, z^2\right)$.

3) $\mathbf{V}(x,y,z) = \left(4x^2, 0, 0\right)$.

4) $\mathbf{V}(x,y,z) = (0, \cos y, 0)$.

A The standard formula of computation of a vector potential applied on non-divergence free vector fields. The example shall illustrate what can go wrong when the assumptions are not fulfilled.

D First note that the given fields are not divergence free. Then just compute.

I 1) First note that div $\mathbf{V} = 3 \neq 0$, thus the vector potential does not exist.

We shall nevertheless compute the candidate of the "vector potential" according to the standard procedure. First,

$$\mathbf{S}(\mathbf{x}) = \int_0^1 \tau\, \mathbf{V}(\mathbf{x}\,\tau)\, \mathrm{d}\tau = \int_0^1 \tau\,(x\tau, y\tau, z\tau)\, \mathrm{d}\tau = (x,y,z) \int_0^1 \tau^2\, \mathrm{d}\tau = \frac{1}{3}\,(x,y,z).$$

Then

$$\mathbf{U}(\mathbf{x}) = -\mathbf{x} \times \mathbf{S}(\mathbf{x}) = \mathbf{S}(\mathbf{x}) \times \mathbf{x} = \frac{1}{3}\,\mathbf{x} \times \mathbf{x} = \mathbf{0},$$

and thus

$$\mathbf{W}(\mathbf{x}) = \bigtriangledown \times \mathbf{V}(\mathbf{x}) = \mathbf{0} \neq \mathbf{V}(\mathbf{x}).$$

2) Here

$$\text{div}\ \mathbf{V} = 2(x+y+z) \neq 0,$$

so the field is not divergence free.

By a direct computation,

$$\begin{aligned}\mathbf{S}(\mathbf{x}) &= \int_0^1 \tau\,\mathbf{V}(\mathbf{x}\,\tau)\,\mathrm{d}\tau = \int_0^1 \tau\left(x^2\tau^2, y^2\tau^2, z^2\tau^2\right)\,\mathrm{d}\tau \\ &= \left(x^2, y^2, z^2\right)\int_0^1 \tau^3\,\mathrm{d}\tau = \frac{1}{4}\,\left(x^2, y^2, z^2\right) = \frac{1}{4}\,\mathbf{V}.\end{aligned}$$

Then

$$\begin{aligned}\mathbf{U}(\mathbf{x}) &= -\mathbf{x} \times \mathbf{S}(\mathbf{x}) = \mathbf{S}(\mathbf{x}) \times \mathbf{x} = \frac{1}{4}\begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ x^2 & y^2 & z^2 \\ x & y & z \end{vmatrix} \\ &= \frac{1}{4}\,\left(y^2z - z^2y, z^2x - x^2z, x^2y - y^2x\right),\end{aligned}$$

hence,

$$\begin{aligned}\mathbf{W}(\mathbf{x}) &= \bigtriangledown \times \mathbf{U}(\mathbf{x}) = \frac{1}{4}\begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2z-z^2y & z^2x-x^2z & x^2y-y^2x \end{vmatrix} \\ &= \frac{1}{4}\begin{pmatrix} x^2 - 2yz - 2xz + x^2 \\ y^2 - 2yz - 2yz + y^2 \\ z^2 - 2xz - 2yz + z^2 \end{pmatrix} = \frac{1}{2}\,\left(x^2 - x(y+z), y^2 - y(x+z), z^2 - z(x+y)\right),\end{aligned}$$

which clearly is different from $\mathbf{V}(\mathbf{x})$.

3) Here div $\mathbf{V} = 8x \neq 0$, and the field is not divergence free.
By a direct computation,

$$\mathbf{S}(\mathbf{x}) = \int_0^1 \tau\,\mathbf{V}(\mathbf{x}\,\tau)\,\mathrm{d}\tau = \int_0^1 \tau(4x^2\tau^2, 0, 0)\,\mathrm{d}\tau = (x^2, 0, 0).$$

Then

$$\mathbf{U}(\mathbf{x}) = -\mathbf{x} \times \mathbf{S}(\mathbf{x}) = \mathbf{S}(\mathbf{x}) \times \mathbf{x} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ x^2 & 0 & 0 \\ x & y & z \end{vmatrix} = \left(0, -x^2 z, x^2 y\right),$$

hence

$$\mathbf{W}(\mathbf{x}) = \nabla \times \mathbf{U}(\mathbf{x}) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & -x^2 z & x^2 y \end{vmatrix} = (x^2 + x^2, -2xy, -2xz) = 2x\,(x, -y, -z),$$

which clearly is not equal to $\mathbf{V}(\mathbf{x})$.

4) It follows immediately that div $\mathbf{V} = -\sin y \neq 0$, so the field is not divergence free. Then by direct computation,

$$\mathbf{S}(\mathbf{x}) = \int_0^1 \tau\,\mathbf{V}(\mathbf{x}\,\tau)\,\mathrm{d}\tau = \int_0^1 \tau\,(0, \cos(y\,\tau), 0)\,\mathrm{d}\tau.$$

If $y = 0$, then

$$\mathbf{S}(x, 0, z) = \int_0^1 \tau\,(0,1,0)\,d\tau = \frac{1}{2}\,(0,1,0).$$

If $y \neq 0$, then

$$\int_0^1 \tau\,\cos(y\tau)\,d\tau = \frac{\sin y}{y} + \frac{1}{y^2}\,(\cos y - 1),$$

hence

$$\mathbf{S}(\mathbf{x}) = \begin{cases} \dfrac{1}{2}\,(0,1,0), & \text{for } y = 0, \\[2ex] \dfrac{y \sin y + \cos y - 1}{y^2}\,(0,1,0), & \text{for } y \neq 0. \end{cases}$$

Since the case $y = 0$ is obtained by taking the limit of the case $y \neq 0$, it suffices in the following only to consider $y \neq 0$. It follows from

$$-\mathbf{x} \times (0,1,0) = (0,1,0) \times \mathbf{x} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 0 & 1 & 0 \\ -x & y & z \end{vmatrix} = (z, 0, -x),$$

that

$$\begin{aligned} \mathbf{U}(\mathbf{x}) &= -\mathbf{x} \times \mathbf{S}(\mathbf{x}) = \frac{y \sin y + \cos y - 1}{y^2}\,(z, 0, -x) \\ &= \left(\frac{\sin y}{y} - \frac{1 - \cos y}{y^2}\right)(z, 0, -x)(z\,\varphi(y), 0, -x\,\varphi(y)), \end{aligned}$$

where we have put

$$\varphi(y)=\frac{\sin y}{y}-\frac{1-\cos y}{y^2}.$$

First calculate for $y\neq 0$,

$$\begin{aligned}\varphi'(y) &= \frac{\cos y}{y}-\frac{\sin y}{y^2}-\frac{\sin y}{y^2}+2\cdot\frac{1-\cos y}{y^3}=\frac{\cos y}{y}-2\cdot\frac{\sin y}{y^2}+2\cdot\frac{1-\cos y}{y^3}\\ &= \frac{y^2\cos y-2y\,\sin y+2-2\,\cos y}{y^3},\end{aligned}$$

where

$$\varphi'(0)=\lim_{y\to 0}\varphi'(0)=0.$$

Then

$$\begin{aligned}\mathbf{W}(\mathbf{x}) &= \nabla\times\mathbf{U}(\mathbf{x}=\begin{vmatrix}\mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ z\,\varphi(y) & 0 & -x\,\varphi(y)\end{vmatrix}=(-x\,\varphi'(y),\varphi(y)+\varphi(y),-z\,\varphi'(y))\\ &= (-x\,\varphi'(y),2\varphi(y),-z\,\varphi'(y)),\end{aligned}$$

which is different from $\mathbf{V}(x,y,z)$.

Example 38.19 *Let $\mathbf{V}:\mathbb{R}^3\to\mathbb{R}^3$ be a divergence free vector field. Show that the vector field*

$$\mathbf{W}(x,y,z)=\begin{pmatrix}-\int_\beta^y V_z(x,\eta,\gamma)\,\mathrm{d}\eta+\int_\gamma^z V_y(x,y,\zeta)\,\mathrm{d}\zeta\\ -\int_\gamma^z V_x(x,y,\zeta)\,\mathrm{d}\zeta\\ 0\end{pmatrix},$$

where β and γ are constants, is a vector potential for $\mathbf{V}$*.*

A Vector potential.

D Just test the given solution, i.e. show that $\nabla\times\mathbf{W}=\mathbf{V}$.

I Put $\mathbf{W}=(W_1,W_2,W_3)$. Then

$$\nabla\times\mathbf{W}=\begin{vmatrix}\mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ W_1 & W_2 & W_3\end{vmatrix}=\left(\frac{\partial W_3}{\partial y}-\frac{\partial W_2}{\partial z},\frac{\partial W_1}{\partial z}-\frac{\partial W_3}{\partial x},\frac{\partial W_2}{\partial x}-\frac{\partial W_1}{\partial y}\right).$$

Now,

$$W_1(x,y,z) = -\int_\beta^y V_x(x,\eta,\gamma)\,\mathrm{d}\eta + \int_\gamma^z V_y(x,y,\zeta)\,\mathrm{d}\zeta,$$

$$W_2(x,y,z) = -\int_\gamma^z V_x(x,y,\zeta)\,\mathrm{d}\zeta,$$

and $W_3(x,y,z)=0$, hence the first coordinate is

$$\frac{\partial W_3}{\partial y} - \frac{\partial W_2}{\partial z} = 0 + \frac{\partial}{\partial z}\int_\gamma^z V_x(x,y,\zeta)\,d\zeta = V_x(x,y,z),$$

and the second coordinate is

$$\begin{aligned}\frac{\partial W_1}{\partial z} - \frac{\partial W_3}{\partial x} &= -\frac{\partial}{\partial z}\int_\beta^y V_z(x,\eta,\gamma)\,d\eta + \frac{\partial}{\partial z}\int_\gamma^z V_y(x,y,\zeta)\,d\zeta - 0 \\ &= 0 + V_y(x,y,z) = V_y(x,y,z).\end{aligned}$$

Finally, we get for the third coordinate,

$$\begin{aligned}\frac{\partial W_2}{\partial x} - \frac{\partial W_1}{\partial y} &= -\int_\gamma^z \frac{\partial V_x}{\partial x}(x,y,\zeta)\,\mathrm{d}\zeta + V_z(x,y,\gamma) - \int_\gamma^z \frac{\partial V_y}{\partial y}(x,y,\zeta)\,\mathrm{d}\zeta \\ &= V_z(x,y,\gamma) - \int_\gamma^z \left\{\frac{\partial V_x}{\partial x}(x,y,\zeta) + \frac{\partial V_y}{\partial y}(x,y,\zeta)\right\}\mathrm{d}\zeta.\end{aligned}$$

From the assumption

$$\operatorname{div}\,\mathbf{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = 0,$$

follows by a rearrangement that the integrand is given by

$$-\frac{\partial V_x}{\partial x} - \frac{\partial V_y}{\partial y} = \frac{\partial V_z}{\partial z}.$$

Hence by insertion,

$$\frac{\partial W_2}{\partial x} - \frac{\partial W_1}{\partial y} = V_z(x,y,z) + \int_\gamma^z \frac{\partial V_z}{\partial z}(x,y,\zeta)\,\mathrm{d}\zeta = V_z(x,y,\gamma) + [V_z(x,y,\zeta)]_{\zeta=\gamma}^z = V_z(x,y,z).$$

Summarizing,

$$\nabla \times \mathbf{W} = \mathbf{V},$$

and we have proved that $\mathbf{W}$ is a vector potential for $\mathbf{V}$.

Remark. The formula of this example of a vector potential in $\mathbb{R}^3$ is far easier to apply than the usual procedure of solution given in most textbooks. $\Diamond$

Example 38.20 *Given the vector field*

$$\mathbf{V}(x,y,z) = \left(2x + x^2y, y - xy^2, 7z + 5z^3\right), \qquad (x,y,z) \in \mathbb{R}^3.$$

1. *Compute the divergence* $\nabla \cdot \mathbf{V}$ *and the rotation* $\nabla \times \mathbf{V}$.

2. *Check if there exists a vector field* $\mathbf{W} : \mathbb{R}^3 \to \mathbb{R}^3$*, such that* $\mathbf{V} = \nabla \times \mathbf{W}$.

Let $L = \{(x,y,z) \in \mathbb{R}^3 \mid x \geq 0,\ y \geq 0,\ x^2 + y^2 + z^2 \leq 9\}$.

3. *Find the flux of* $\mathbf{V}$ *through* ∂L.

Let $\mathcal{C}$ *denote the closed curve which is the intersection curve of* ∂L *and the plane* $z = 0$.

4. *Find the absolute value of the circulation* $\oint_{\mathcal{C}} \mathbf{V} \cdot \mathbf{t}\, ds$.

A Divergence, rotation, vector potential, flux, circulation.

D Follow the guidelines. Apply Gauß's theorem and Stokes's theorem.

I 1) We first get by straightforward calculations,

$$\nabla \cdot \mathbf{V} = \operatorname{div}\,\mathbf{V} = (2 + 2xy) + (1 - 2xy) + (7 + 15z^2) = 10 + 15z^2$$

and

$$\nabla \times \mathbf{V} = \mathbf{rot}\ \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x + x^2y & y - xy^2 & 7z + 5z^3 \end{vmatrix} = -\left(0, 0, x^2 + y^2\right).$$

2) Since div $\mathbf{V} \neq \mathbf{0}$, there does not exist a vector field $\mathbf{W}$, such that $\mathbf{V} = \bigtriangledown \times \mathbf{W}$, because $\bigtriangledown \cdot (\bigtriangledown \times \mathbf{V}) = 0$.

3) It follows from Gauss's theorem and 1) that

$$\begin{aligned}
\text{flux}(\partial L) &= \int_{\partial L} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S = \int_L \text{div}\,\mathbf{V}\,\mathrm{d}\Omega = \int_L (10+15z^2)\,\mathrm{d}\Omega \\
&= 10\,\text{vol}(L) + 15\int_L z^2\,\mathrm{d}\Omega = 10\cdot\frac{4\pi}{3}\cdot 3^3\cdot\frac{1}{4} + 15\int_{-3}^{3} z^2\cdot\frac{\pi}{4}]\,(9-z^2)\,\mathrm{d}z \\
&= 90\pi + \frac{15\pi}{4}\cdot 2\int_0^3 (9z^2-z^4)\,\mathrm{d}z = 90\pi + \frac{15\pi}{2}\left[3z^3 - \frac{1}{5}\,z^5\right]_0^3 \\
&= 90\pi + \frac{15\pi}{2}\left(3^4 - \frac{1}{5}\cdot 3^5\right) = 90\pi + \frac{15\pi}{2\cdot 5}\cdot 3^4\,(5-3) \\
&= 90\pi + 243\pi = 333\pi.
\end{aligned}$$

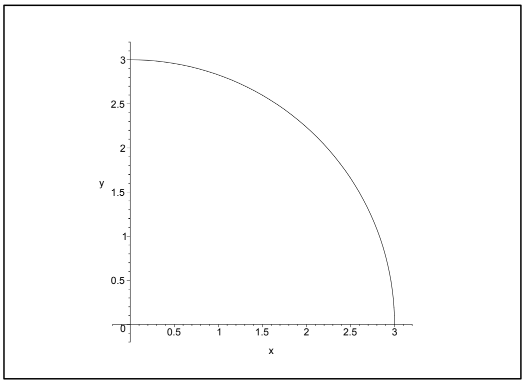

Figure 38.7: The curve $\mathcal{C}$ and the quarter disc B inside.

4) The curve $\mathcal{C}$ encircles the quarter disc B in the first quadrant of centrum $(0,0)$ and radius 3. Then by Stokes's theorem and 1),

$$\begin{aligned}
\left|\oint_{\mathcal{C}} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s\right| &= \left|\int_B \mathbf{rot}\ \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}x\,\mathrm{d}y\right| = \left|-\int_B (0,0,x^2+y^2)\cdot(0,0,1)\,\mathrm{d}x\,\mathrm{d}y\right| \\
&= \int_B \left(x^2+y^2\right)\,\mathrm{d}x\,\mathrm{d}y = \int_0^{\frac{\pi}{2}}\left\{\int_0^3 \varrho^2\cdot\varrho\,\mathrm{d}\varrho\right\}\,\mathrm{d}\varphi = \frac{\pi}{2}\cdot\left[\frac{\varrho^4}{4}\right]_0^3 = \frac{81\pi}{8}.
\end{aligned}$$

Example 38.21 *A surface of revolution $\mathcal{O}$ with the Z-axis as rotation axis is given in semi polar coordinates (ϱ, φ, z) by*

$$0 \le \varphi \le 2\pi, \qquad 0 \le \varrho \le a \quad \text{and} \quad z = a - \frac{\varrho^3}{a^2},$$

where $a \in \mathbb{R}_+$ is a given constant. The surface $\mathcal{O}$ is oriented, such that its unit normal vector $\mathbf{n}$ always has a negative z-coordinate.

1. *Sketch the meridian curve $\mathcal{M}$ of the surface.*

2. *Compute the surface integral*

$$\int_{\mathcal{O}} \left(\frac{a-z}{a}\right)^{\frac{2}{3}} \mathrm{d}S.$$

Furthermore, let there be given the vector fields

$$\mathbf{V}(x,y,z) = \left(\frac{y^2}{a^2+z^2} - 1, 1 - \frac{x^2}{a^2+z^2}, 1\right), \qquad (x,y,z) \in \mathbb{R}^3,$$

and

$$\mathbf{U}(x,y,z) = \left(3z - y, 2x + 3z, \frac{x^3+y^3}{a^2+z^2}\right), \qquad (x,y,z) \in \mathbb{R}^3.$$

3. *Prove the existence of a constant $\beta \in \mathbb{R}$, such that*

$$\mathbf{V} = \beta \ \nabla \times \mathbf{U},$$

and find β.

4. *Find the flux*

$$\int_{\mathcal{O}} \mathbf{V} \cdot \mathbf{n} \, \mathrm{d}S.$$

5. *Find a vector potential for $\mathbf{V}$.*

A Meridian curve; surface integral; flux; vector potential.

D There are many variants of calculations in this example.

I 1) The equation of the meridian curve is

$$z = a - \frac{\varrho^3}{a^2} = a\left\{1 - \left(\frac{\varrho}{a}\right)^3\right\}, \qquad \varrho \in [0,a].$$

2) We can compute the surface integral in several ways.

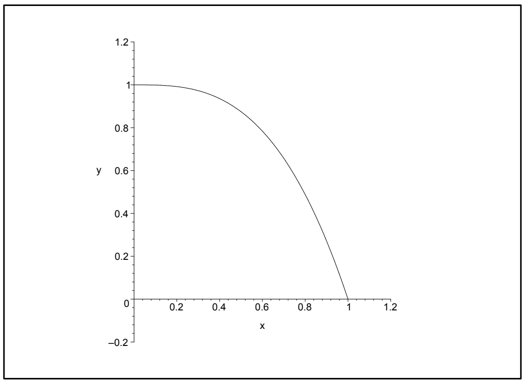

Figure 38.8: The meridian curve $\mathcal{M}$ for $a = 1$.

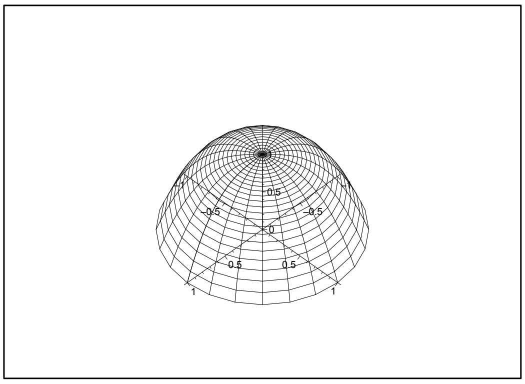

Figure 38.9: The surface $\mathcal{O}$ for $a = 1$.

a) When we use the reduction theorem of surface integrals we get

$$(38.3) \quad \int_{\mathcal{O}} \left(\frac{a-z}{a}\right)^{\frac{2}{3}} \mathrm{d}S = \int_E \left\{\frac{\varrho(\varphi,t)}{a}\right\}^2 \|\mathbf{N}(\varphi,t)\| \,\mathrm{d}\varphi\,\mathrm{d}t,$$

where we have used that

$$\frac{a-z}{a} = 1 - \left\{1 - \left(\frac{\varrho}{a}\right)^3\right\} = \left(\frac{\varrho}{a}\right)^3.$$

If we apply the parametric description

$$P(t) = t, \qquad Z(t) = a - \frac{1}{a^2}\,t^3, \qquad t \in [0,a],$$

we get

$$\mathbf{N}(t,\varphi) = P(t)\cdot(-Z'(t)\cos\varphi, -Z'(t)\sin\varphi, P'(t)) = t\left(\frac{3}{a^2}\,t^2\cos\varphi, \frac{3}{a^2}\,t^2\sin\varphi, 1\right),$$

hence,

$$\|\mathbf{N}(t,\varphi)\| = t\sqrt{1+\frac{9}{a^4}\,t^4}, \qquad t \in [0,a], \quad \varphi \in [0,2\pi].$$

Then by insertion into (38.3),

$$\begin{aligned}\int_{\mathcal{O}} \left\{\frac{a-z}{a}\right\}^{\frac{2}{3}} \mathrm{d}S &= \int_0^1 \left\{\int_0^{2\pi} \frac{t^2}{a^2}\cdot t\sqrt{1+9\left(\frac{t}{a}\right)^4}\,\mathrm{d}\varphi\right\} \mathrm{d}t \\ &= 2\pi\cdot\frac{a^2}{4}\int_0^a \left\{1+9\left(\frac{t}{a}\right)^4\right\}^{\frac{1}{2}}\cdot\frac{4t^3}{a^4}\,\mathrm{d}t = \frac{\pi a^2}{18}\int_{t=0}^a \left\{1+9\left(\frac{t}{a}\right)^4\right\}^{\frac{1}{2}}\mathrm{d}\left(1+9\left(\frac{t}{a}\right)^4\right) \\ &= \frac{\pi a^2}{18}\cdot\frac{2}{3}\left[\left(1+9\left(\frac{t}{a}\right)^4\right)^{\frac{3}{2}}\right]_{t=0}^a = \frac{\pi a^2}{27}\left\{10\sqrt{10}-1\right\}.\end{aligned}$$

b) ALTERNATIVELY insert directly into a standard formula:

$$\begin{aligned}\int_{\mathcal{O}} \left(\frac{a-z}{a}\right)^{\frac{2}{3}} \mathrm{d}S &= \int_{\mathcal{M}} 2\pi\left(\frac{a-z(\varrho)}{a}\right)^{\frac{2}{3}}\varrho\,\mathrm{d}s = 2\pi\int_{\mathcal{M}}\left(\frac{\varrho}{a}\right)^2\varrho\,\mathrm{d}s \\ &= 2\pi a\int_0^a\left(\frac{\varrho}{a}\right)^3\sqrt{1+9\left(\frac{\varrho}{a}\right)^4}\,\mathrm{d}\varrho = \frac{2\pi a^2}{36}\int_0^1\{1+9t\}^{\frac{1}{2}}\,\mathrm{d}(1+9t) \\ &= \frac{2\pi a^2}{36}\cdot\frac{2}{3}\left[(1+9t)^{\frac{3}{2}}\right]_0^1 = \frac{\pi a^2}{27}\left\{10\sqrt{10}-1\right\}.\end{aligned}$$

3) Clearly, $\mathbf{V}$ is divergence free.

Then by a straightforward calculation,

$$\nabla \times \mathbf{U} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3z - y & 2x + 3z & \frac{x^3+y^3}{a^2+z^2} \end{vmatrix} = \left(\frac{3y^2}{a^2+z^2} - 3, 3 - \frac{3x^2}{a^2+z^2}, 2+1 \right) = 3\mathbf{V},$$

hence

$$\mathbf{V} = \frac{1}{3} \nabla \times \mathbf{U},$$

so $\beta = \dfrac{1}{3}$, and $\dfrac{1}{3}\,\mathbf{U}$ is a vector potential for $\mathbf{V}$, cf. 5).

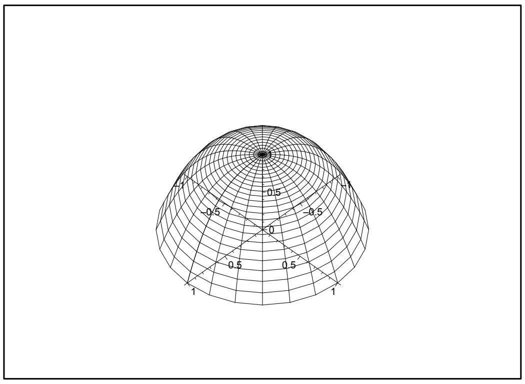

Figure 38.10: The body Ω for $a = 1$.

4) a) Let$B(\mathbf{0},a)$ denote the disc in the (X,Y)-plane of centrum $(0,0)$ and radius A. The union of the surfaces $\mathcal{O}$ and $B(\mathbf{0},a)$ surrounds a simple body Ω. Since $\mathbf{V}$ is divergence fret, the ingoing flux through $\mathcal{O}$ must be equal to the outgoing flux through $B(\mathbf{0},a)$, where $\mathbf{n} = (0,0,-1)$, hence the flux is

$$\begin{aligned} \int_{\mathcal{O}} \mathbf{V} \cdot \mathbf{n}\,\mathrm{d}S &= \int_{B(\mathbf{0},a)} \mathbf{V} \cdot \mathbf{n}\,\mathrm{d}S = \int_{B(\mathbf{0}),a)} \left(\frac{{y^2}^2}{a} - 1, 1 - \frac{x^2}{a^2}, 1 \right) \cdot (0,0,-1)\,\mathrm{d}S \\ &= -\int_{B(\mathbf{0},a)} \mathrm{d}S = - \text{ area } B(\mathbf{0},a) = -\pi a^2. \end{aligned}$$

b) Alternatively it follows from 3) and Stokes's theorem that

$$\int_{\mathcal{O}} \mathbf{V} \cdot \mathbf{n}\,\mathrm{d}S = \frac{1}{3} \int_{\mathcal{O}} (\nabla \times \mathbf{U}) \cdot \mathbf{n}\,\mathrm{d}S = \frac{1}{3} \oint_{\partial\mathcal{O}} \mathbf{U} \cdot \mathbf{t}\,\mathrm{d}s = \int_{B(\mathbf{0},a)} \mathbf{V} \cdot \mathbf{n}\,\mathrm{d}S = \cdots = -\pi a^2,$$

where the dots indicate that we proceed as above.

c) Alternatively we compute the line integral$\frac{1}{3} \oint_{\partial\mathcal{O}} \mathbf{U} \cdot \mathbf{t}\, a ds$. Here $\partial\mathcal{O}$ is the circle $\varrho = a$ in the plane $z = 0$ run through in a negative sense, because $\mathbf{n}$ has a negative z-component on $\mathcal{O}$. Thus a parametric description of $\partial\mathcal{O}$ is

$$(x,y,z) = a(\cos\varphi, -\sin\varphi, 0), \qquad \varphi \in [0,2\pi]$$

where

$$\mathbf{t} = -(\sin\varphi, \cos\varphi, 0), \qquad \mathrm{d}s = a\,\mathrm{d}\varphi,$$

thus

$$\begin{aligned}\int_{\mathcal{O}} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S &= \frac{1}{3}\oint_{\partial\mathcal{O}} \mathbf{U}\cdot\mathbf{t}\,\mathrm{d}s = \frac{1}{3}\oint_{\partial\mathcal{O}} \left(3z-y, 2x+3z, \frac{x^3+y^3}{a^2+z^2}\right)\cdot\mathbf{t}\,\mathrm{d}s \\ &= -\frac{a}{3}\int_0^{2\pi} \left(\sin\varphi, 2\cos\varphi, \cos^3\varphi - \sin^3\varphi\right)\cdot(\sin\varphi,\cos\varphi,0)a\,\mathrm{d}\varphi \\ &= -\frac{a^2}{3}\int_0^{2\pi}\{\sin^2\varphi + 2\cos^2\varphi\}\,\mathrm{d}\varphi = -\frac{a^2}{3}\left(\frac{3}{2}\cdot 2\pi\right) = -\pi a^2.\end{aligned}$$

d) ALTERNATIVELY there are also variants in which Green's theorem in the plane occurs. We shall only demonstrate one of them;

$$\begin{aligned}\int_{\mathcal{O}} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S &= \frac{1}{3}\oint_{\partial\mathcal{O}} \mathbf{t}\cdot\mathbf{U}\,\mathrm{d}s = -\frac{1}{3}\int_{B(\mathbf{0},a)}\left(\frac{\partial U_y}{\partial x} - \frac{\partial U_x}{\partial y}\right)\mathrm{d}S \\ &= -\frac{1}{3}\int_{B(\mathbf{0},a)}(2+1)\,\mathrm{d}S = -\text{ area } B(\mathbf{0},a) = -\pi a^2.\end{aligned}$$

5) Now, $\mathbf{W}_0$ is a vector potential for $\mathbf{V}$, if $\mathbf{V} = \nabla\times\mathbf{W}_0$. This is according to 3) fulfilled for

$$\mathbf{W}_0 = \frac{1}{3}\,\mathbf{U} = \frac{1}{3}\left(3z-y, 2x+3z, \frac{x^3+y^3}{a^2+z^2}\right).$$

ALTERNATIVELY (and far more difficult) we can find a vector potential $\mathbf{W}_0$ directly by means of the standard formula,

$$\mathbf{W}_0(\mathbf{x}) = -\mathbf{x}\times\int_0^1 \tau\,\mathbf{V}(\tau\,\mathbf{x})\,\mathrm{d}\tau.$$

Here

$$\int_0^1 \tau\,\mathbf{V}(\tau\,\mathbf{x})\,\mathrm{d}\tau = \left(\int_0^1 \tau\left\{\frac{\tau^2y^2}{a^2+\tau^2z^2}-1\right\}\mathrm{d}\tau, \int_0^1 \tau\left\{1-\frac{\tau^2x^2}{a^2+\tau^2z^2}\right\}d\tau, \int_0^1 \tau\,\mathrm{d}\tau\right).$$

We get by a calculation for $z \neq 0$,

$$\begin{aligned}\int_0^1 \tau\cdot\frac{\tau^2y^2}{a^2+\tau^2z^2}\,\mathrm{d}\tau &= \frac{y^2}{z^2}\int_0^1 \tau\cdot\frac{\tau^2z^2}{a^2+\tau^2z^2}\,\mathrm{d}\tau = \frac{y^2}{z^2}\int_0^1 \tau\left(1-\frac{1}{1+\dfrac{z^2}{a^2}\tau^2}\right)\mathrm{d}\tau \\ &= \frac{y^2}{z^2}\left[\frac{\tau^2}{2} - \frac{1}{2}\,\frac{a^2}{z^2}\,\ln\left\{1+\frac{z^2}{a^2}\,\tau^2\right\}\right]_{\tau=0}^1 = \frac{y^2}{2z^2} - \frac{a^2y^2}{2z^4}\,\ln\left(1+\frac{z^2}{a^2}\right).\end{aligned}$$

By taking the limit, or by a direct computation, we get

$$\int_0^1 \tau\cdot\frac{\tau^2y^2}{a^2+\tau^2z^2}\,\mathrm{d}\tau = \frac{y^2}{4a^2} \qquad \text{for } z=0.$$

Similarly

$$\int_0^1 \tau \cdot \frac{\tau^2 x^2}{a^2+\tau^2 z^2}\,\mathrm{d}\tau = \frac{x^2}{2z^2} - \frac{a^2x^2}{2z^4}\,\ln\left(1+\frac{z^2}{a^2}\right) \qquad \text{for } z \neq 0,$$

and

$$\int_0^1 \tau \cdot \frac{\tau^2 x^2}{a^2+\tau^2 z^2}\,\mathrm{d}\tau = \frac{x^2}{4a^2} \qquad \text{for } z = 0.$$

Due to the continuity it suffices in the following with the expressions for $z \neq 0$. Then

$$\int_0^1 \tau\,\mathbf{V}(\tau\,\mathbf{x})\,\mathrm{d}\tau = \frac{1}{2}\begin{pmatrix} -1+\dfrac{y^2}{z^2} - \dfrac{a^2y^2}{z^4}\,\ln\left\{1+\dfrac{z^2}{a^2}\right\} \\[2ex] 1-\dfrac{x^2}{z^2}+\dfrac{a^2x^2}{z^4}\,\ln\left\{1+\dfrac{z^2}{a^2}\right\}, \\[2ex] 1 \end{pmatrix}.$$

We then find $\mathbf{W}_0$ by

$$\begin{aligned}
\mathbf{W}_0(\mathbf{x}) &= \int_0^1 \tau\,\mathbf{V}(\tau\,\mathbf{x})\,\mathrm{d}\tau \times \mathbf{x} \\
&= \frac{1}{2}\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ -1+\dfrac{y^2}{z^2}-\dfrac{a^2y^2}{z^4}\ln\left(1+\dfrac{z^2}{a^2}\right) & 1-\dfrac{x^2}{z^2}+\dfrac{a^2x^2}{z^4}\ln\left(1+\dfrac{z^2}{a}\right) & 1 \\ x & y & z \end{vmatrix} \\
&= \frac{1}{2}\begin{pmatrix} z-y-\dfrac{x^2}{z}+\dfrac{a^2x^2}{z^3}\,\ln\left(1+\dfrac{z^2}{a^2}\right) \\[2ex] x+z-\dfrac{y^2}{z}+\dfrac{a^2y^2}{z^3}\,\ln\left(1+\dfrac{z^2}{a^2}\right) \\[2ex] -x-y+\dfrac{y^3+x^3}{z^2}-\dfrac{a^2}{z^4}\left(x^3+y^3\right)\ln\left(1+\dfrac{z^2}{a^2}\right) \end{pmatrix}, \qquad z \neq 0.
\end{aligned}$$

For $z = 0$ the result is obtained by taking the limit.

This horrible expression is of course not equal to $\dfrac{1}{3}\,\mathbf{U}$. On the other hand, a vector potential is not unique. Here we can only check our computations by insertion.

C Test. Put

$$\mathbf{W}_0 = (W_1, W_2, W_3) \quad \text{and} \quad \mathbf{V} = (V_1, V_2, V_3)\,.$$

Then

$$\begin{aligned}
\frac{\partial W_3}{\partial y}-\frac{\partial W_2}{\partial z} &= \frac{1}{2}\left\{-1+\frac{3y^2}{z^2}-\frac{3a^2y^2}{z^4}\ln\left(1+\frac{z^2}{a^2}\right)-1-\frac{y^2}{z^2}\right.\\
&\qquad\qquad \left.+\frac{3a^2y^2}{z^4}\ln\left(1+\frac{z^2}{a^2}\right)-\frac{a^2y^2}{z^3}\cdot\frac{2z}{1+\dfrac{z^2}{a^2}}\cdot\frac{1}{a^2}\right\}\\
&= \frac{1}{2}\left\{-2+\frac{2y^2}{z^2}-\frac{a^2y^2}{z^2}\cdot\frac{2}{a^2+z^2}\right\}\\
&= \frac{1}{2}\left\{-2+\frac{2y^2}{z^2(a^2+z^2)}\left(a^2+z^2-a^2\right)\right\}=V_1.
\end{aligned}$$

The computation of $\dfrac{\partial W_1}{\partial z}-\dfrac{\partial W_3}{\partial x}=V_2$ is similar, where we could apply the "asymmetry" (x and y are interchanged and we also change sign). Finally,

$$\frac{\partial W_2}{\partial x}-\frac{\partial W_1}{\partial y}=\frac{1}{2}\{1+1\}=1=V_3,$$

hence the found vector field $\mathbf{W}_0(\mathbf{x})$ is a vector potential for $\mathbf{V}$.

Example 38.22 *1) Find the divergence of the vector field*

$$\mathbf{V}(x,y,z) = (\sin y + \cos z, \sin z + \cos x, \sin x + \cos y), \qquad (x,y,z) \in \mathbb{R}^3.$$

2) Prove the existence of a constant α, such that **rot V** $= \alpha\mathbf{V}$*. Then find a vector potential for* **V***.*

A Divergence, rotation and vector potential.

D Just compute. In 2) one might get a better solution.

I 1) Clearly,

$$\text{div }\mathbf{V} = 0.$$

Then compute

$$\begin{aligned}\triangledown \times \mathbf{v} = \mathbf{rot}\ \mathbf{V} &= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \sin y + \cos z & \sin z + \cos x & \sin x + \cos y \end{vmatrix} \\ &= (-\sin y - \cos z, -\sin z - \cos x, -\sin x - \cos y) = -\mathbf{V}(x,y,z).\end{aligned}$$

2) It follows immediately that $\triangledown \times (-\mathbf{V}) = \mathbf{V}$, thus $-\mathbf{V}$ is according to the definition a vector potential for $\mathbf{V}$.

Alternatively, $\mathbf{V}$ is divergence free, thus there exists a vector potential. One of these is given by

$$\mathbf{W} = -\mathbf{x} \times \mathbf{S}(\mathbf{x}) = \mathbf{S}(\mathbf{x}) \times \mathbf{x},$$

where

$$\mathbf{S}(\mathbf{x}) = \int_0^1 \tau\, \mathbf{V}(\tau\,\mathbf{x})\,d\tau = \begin{pmatrix} \int_0^1 \tau\{\sin(\tau y) + \cos(\tau z)\}\,d\tau \\ \int_0^1 \tau\{\sin(\tau z) + \cos(\tau x)\}\,d\tau \\ \int_0^1 \tau\{\sin(\tau x) + \cos(\tau y)\}\,d\tau \end{pmatrix}.$$

By some small calculations we get

$$\int_0^1 \tau\,\sin\tau v\,d\tau = \begin{cases} \dfrac{1}{v^2}\{\sin v - v\cos v\} & \text{for } v \neq 0, \\ 0 & \text{for } v = 0, \end{cases}$$

and

$$\int_0^1 \tau\,\cos\tau v\,d\tau = \begin{cases} \dfrac{1}{v^2}\{\cos v - 1 + v\,\sin v\} & \text{for } v \neq 0, \\ \dfrac{1}{2} & \text{for } v = 0. \end{cases}$$

By insertion of these expressions into $\mathbf{S}(\mathbf{x})$ we get the rather complicated vector potential

$$
\begin{aligned}
\mathbf{W}(\mathbf{x}) &= \mathbf{S}(\mathbf{x}) \times \mathbf{x} \\
&= \begin{pmatrix}
\dfrac{1}{z}\{\sin z - z\cos z\} + \dfrac{z}{x^2}\{\cos x - 1 + x\sin x\} \\[2mm]
\dfrac{1}{x}\{\sin x - x\cos x\} + \dfrac{x}{y^2}\{\cos y - 1 + y\sin y\} \\[2mm]
\dfrac{1}{y}\{\sin y - y\cos y\} + \dfrac{y}{z^2}\{\cos z - 1 + z\sin z\}
\end{pmatrix} \\
&\qquad - \begin{pmatrix}
\dfrac{y}{x^2}\{\sin x - x\cos x\} + \dfrac{1}{y}\{\cos y - 1 + y\sin y\} \\[2mm]
\dfrac{z}{y^2}\{\sin y - y\cos y\} + \dfrac{1}{z}\{\cos z - 1 + z\sin z\} \\[2mm]
\dfrac{x}{z^2}\{\sin z - z\cos z\} + \dfrac{1}{x}\{\cos x - 1 + x\sin x\}
\end{pmatrix},
\end{aligned}
$$

with suitable interpretations when x, y or $z = 0$.

Example 38.23 *Consider the vector field*

$$\mathbf{V}(x,y,z) = (2x+3y, 2y+3x, -4z), \qquad (x,y,z) \in \mathbb{R}^3,$$

and the function

$$G(x,y,z) = \alpha x^2 + \beta y^2 + \gamma z^2 + \delta xy, \qquad (x,y,z) \in \mathbb{R}^3,$$

where α, β, γ, δ are constants.

1. *Show that one can choose the constants α, β, γ, δ such that $\mathbf{V} = \bigtriangledown G$.*

2. *Compute the tangential line integral*

$$\int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t}\, \mathrm{d}s,$$

where $\mathcal{K}$ is the broken line composed of the two line segments from $(2a,0,a)$ via $(2a,0,0)$ to $(a,0,0)$.

3. *Show that the vector field*

$$\mathbf{W}(x,y,z) = \left(2yz + xz - x^2, -2xz - yz + y^2, y^2 - x^2 + z^2\right), \qquad (x,y,z) \in \mathbb{R}^3,$$

is a vector potential for $\mathbf{V}$.

Let a be a positive constant, and let $\mathcal{F}$ be the oriented surface given by

$$x^2 + y^2 + z^2 = a^2, \qquad z \geq 0,$$

with the unit normal vector $\mathbf{n}$ pointing away from $(0,0,0)$.

4. *Find the flux*

$$\int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n}\, dS.$$

A Vector analysis, i.e. check the gradient field, tangential line integral, vector potential and flux.

D The examples can be solved in many ways, and I have probably not found all variants. Below we give the following variants:

1) We solve 1) in 5 variants.
2) We solve 2) in 2 variants.
3) We solve 3) in 2 variants.
4) We solve 4) in 4 variants and 1 subvariant (and there are more; we miss e.g. the calculations when $\mathcal{F}$ is a surface of revolution).

I 1) **First variant.** *A simple check.*

When we compute $\bigtriangledown G$ we get

$$\bigtriangledown G = (2\alpha x + \delta y, \delta x + 2\beta y, 2\gamma z).$$

Choose $\alpha = 1$, $\beta = 1$, $\gamma = -2$ and $\delta = 3$. Then

$$\bigtriangledown G = (2x+3y, 3y+2y, -4z) = \mathbf{V},$$

and $\mathbf{V} = \bigtriangledown G$ is a gradient field with the integral

$$G(x,y,z) = x^2 + y^2 - 2z^2 + 3xy.$$

Second variant. *Manipulation.*

We conclude from

$$\begin{aligned}\mathbf{V}\cdot\mathrm{d}\mathbf{x} &= (2x+3y)\,\mathrm{d}x+(2y+3x)\,\mathrm{d}y-4z\,\mathrm{d}z\\ &= \mathrm{d}\left(x^2\right)+\,\mathrm{d}\left(y^2\right)-\,\mathrm{d}\left(2z^2\right)+3\{y\,\mathrm{d}x+x\,\mathrm{d}y\}\\ &= \mathrm{d}\left(x^2+y^2-2z^2+3xy\right)\end{aligned}$$

that

$$G(x,y)=x^2+y^2-2z^2+3xy$$

is an integral of $\bigtriangledown G=\mathbf{V}$, and $\mathbf{V}$ is a gradient field.

Then by comparison, $\alpha=1$, $\beta=1$, $\gamma=-2$, $\delta=-3$.

Third variant. *Indefinite integration.*

Put

$$\omega=\mathbf{V}\cdot\,\mathrm{d}\mathbf{x}=(2x+3y)\,\mathrm{d}x+(2y+3x)\,\mathrm{d}y-4z\,\mathrm{d}z.$$

Then

$$F_1(x,y,z);=\int(2x+3y)dx=x^2+3xy,$$

thus

$$\omega-\,\mathrm{d}F_1=(2x+3y)\,\mathrm{d}x+(2y+3x)\,\mathrm{d}y-4z\,\mathrm{d}z-\{(2x+3y)\,\mathrm{d}x+3x\,\mathrm{d}y\}=2y\,\mathrm{d}y-4z\,\mathrm{d}z,$$

which is reduced to

$$\omega-\,\mathrm{d}\left(x^2+3xy\right)=\,\mathrm{d}\left(y^2-2z^2\right).$$

Then by a rearrangement,

$$\omega=\,\mathrm{d}\left(x^2+3xy\right)+\,\mathrm{d}\left(y^2-2z^2\right)=\,\mathrm{d}\left(x^2+y^2-2z^2+3xy\right),$$

and we conclude that

$$G(x,y,z)=x^2+y^2-2z^2+3xy$$

is an integral of $\mathbf{V}$, i.e. $\mathbf{V}=\bigtriangledown G$, and $\alpha=1$, $\beta=1$, $\gamma=-2$, $\delta=3$.

The latter two variants *assume* that we have proved that $\mathbf{V}$ is a gradient field. First note that

$$\mathbf{rot}\mathbf{V}=\left|\begin{array}{ccc}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 2x+3y & 3x+2y & -4z\end{array}\right|=(0-0,0-0,3-3)=\mathbf{0},$$

and $\mathbf{V}$ is rotation free. The domain $\mathbb{R}^3$ is simply connected (it is even convex), hence $\mathbf{V}$ is a gradient field.

Fourth variant. *Integration along a broken line.*

We get by a tangential line integration along the broken line

$$(0,0,0) \longrightarrow (x,0,0) \longrightarrow (x,y,0) \longrightarrow (x,y,z) \qquad \text{in } \mathbb{R}^3$$

that

$$\int_{\mathbf{0}}^{\mathbf{x}} \mathbf{V} \cdot \, \mathrm{d}\mathbf{x} = \int_0^x 2t \, \mathrm{d}t + \int_0^y (2t + 3x) \, \mathrm{d}t - \int_0^z 4t \, \mathrm{d}t = x^2 + y^2 + 3xy - 2z^2.$$

Since we already have proved that $\mathbf{V}$ is a gradient field, an integral is given by

$$G(x,y,z) = x^2 + y^2 - 2z^2 + 3xy, \qquad \bigtriangledown G = \mathbf{V},$$

and we get by comparison that $\alpha = 1$, $\beta = 1$, $\gamma = -2$, $\delta = 3$.

Fifth variant. *Radial integration.*

We have above proved that $\mathbf{V}$ is a gradient field. Therefore,

$$\begin{aligned}
G(x,y,z) &= (x,y,z) \cdot \int_0^1 \mathbf{V}(x\tau, y\tau, z\tau) \, \mathrm{d}\tau \\
&= (x,y,z) \cdot \int_0^1 ((2x+3y)\tau, (2y+3x)\tau, -4z\tau) \, \mathrm{d}\tau \\
&= (x,y,z) \cdot (2x+3y, 2y+3x, -4z) \int_0^1 \tau \, \mathrm{d}\tau \\
&= \frac{1}{2} \left\{ (2x^2 + 3xy) + (2y^2 + 3xy) - 4z^2 \right\} \\
&= x^2 + y^2 - 2z^2 + 3xy
\end{aligned}$$

is an integral of $\mathbf{V}$, i.e. $\bigtriangledown G = \mathbf{V}$, and we get by comparison that $\alpha = 1$, $\beta = 1$, $\gamma = -2$, $\delta = 3$.

2) **First variant.** *The gradient theorem.*

According to 1), the field $\mathbf{V}$ is a gradient field with the integral

$$G(x,y,z) = x^2 + y^2 - 2z^2 + 3xy.$$

Then by the gradient theorem,

$$\int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} \, \mathrm{d}s = G(a,0,0) - G(2a,0,a) = a^2 - (4a^2 - 2a^2) = -a^2.$$

Second variant. *Line integral.*

We have on the line segment from $(2a,0,a)$ to $(2a,0,0)$ that $x = 2a$ and $y = 0$, while z runs through the interval $[0,a]$ from a to 0 (the reverse direction).

On the line segment from $(2a,0,0)$ to $(a,0,0)$, the variable x runs through the interval $[a,2a]$ from $2a$ towards a, also in the reverse direction, while $y = 0$ and $z = 0$.

As a conclusion we get

$$\begin{aligned}
\int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} \, \mathrm{d}s &= \int_a^0 (-4t) \, \mathrm{d}t + \int_{2a}^a (2t + 3 \cdot 0) \, \mathrm{d}t = \left[-2t^2\right]_a^0 + \left[t^2\right]_{2a}^a \\
&= 2a^2 + (a^2 - 4a^2) = -a^2.
\end{aligned}$$

3) **First variant.** *Check.*

We shall only prove that $\mathbf{V} = \nabla \times \mathbf{W}$. We get

$$\begin{aligned}
\nabla \times \mathbf{W} &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\[2mm] \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\[2mm] 2yz + xz - x^2 & -2xz - yz + y^2 & y^2 - x^2 + z^2 \end{vmatrix} \\
&= (2y - (-2x - y), (2y + x) - (-2x), -2z - 2z) \\
&= (3y + 2x, 3x + 2y, -4z) = \mathbf{V},
\end{aligned}$$

and $\mathbf{W}$ is a vector potential for $\mathbf{V}$.

Second variant. *Insertion into a standard formula.*

The assumptions are that $\mathbb{R}^3$ is star shaped (obvious) and that $\mathbf{V}$ is divergence free. By a small computation,

$$\operatorname{div} \mathbf{V} = 2 + 2 - 4 = 0,$$

and it follows that $\mathbf{V}$ has a vector potential, which can be found by the formula

$$\mathbf{W}_0(\mathbf{x}) = -\mathbf{x} \times \int_0^1 \tau \, \mathbf{V}(\tau \mathbf{x}) \, d\tau = \left\{ \int_0^1 \tau \, \mathbf{V}(\tau \mathbf{x}) \, d\tau \right\} \times \mathbf{x}.$$

Since $\mathbf{V}$ is homogeneous of first degree,

$$\mathbf{V}(\tau\mathbf{x}) = \tau\,\mathbf{V}(\mathbf{x}),$$

it follows by insertion that

$$\begin{aligned}
\mathbf{W}_0(\mathbf{x}) &= \left\{\int_0^1 \tau\cdot\tau\,\mathrm{d}\tau\right\}\mathbf{V}(\mathbf{x})\times\mathbf{x} = \frac{1}{3}\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 2x+3y & 2y+3x & -4z \\ x & y & z \end{vmatrix} \\
&= \frac{1}{3}\left(z\{2y+3x\}+4zy, -4zx-z\{2x+3y\}, 2xy+3y^2-2xy-3x^2\right) \\
&= \frac{1}{3}\left(z\{3x+6y\}, -z\{6x+3y\}, 3y^2-3x^2\right) \\
&= \left(2yz+xz, -2xz-yz, y^2-x^2\right) \\
&= \mathbf{W}(\mathbf{x}) + \left(-x^2, y^2, z^2\right),
\end{aligned}$$

hence

$$\nabla\times\mathbf{W} = \nabla\times\mathbf{W}_0 + \nabla\times\left(-x^2, y^2, z^2\right) = \mathbf{V} + \mathbf{0} = \mathbf{0}.$$

We conclude that both $\mathbf{W}_0$ and $\mathbf{W}$ are vector potentials for $\mathbf{V}$.

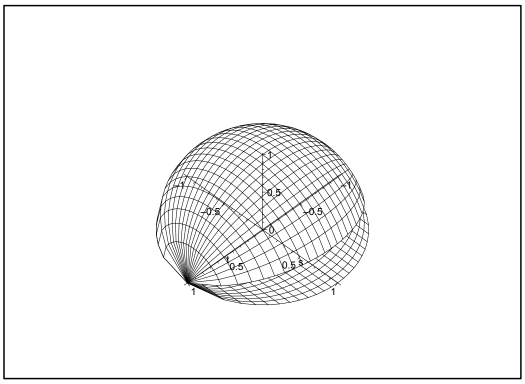

Figure 38.11: The half sphere $\mathcal{F}$ and the "bounding curve" $\delta\mathcal{F}$ for $a = 1$.

4) **First variant.** *Stokes's theorem.*

When the unit normal vector is pointing away from $(0,0,0)$, we get the natural orientation of the bounding curve (a circle in the XY-plane),

$$\delta\mathcal{F}: \quad \mathbf{r}(t) = a(\cos t, \sin t, 0), \qquad t\in[0,2\pi],$$

in its positive sense.

It follows from 3) that $\mathbf{V} = \bigtriangledown \times \mathbf{W}$, hence the flux is according to Stokes's theorem

$$\begin{aligned}\int_{\mathcal{F}} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S &= \int_{\mathcal{F}} (\bigtriangledown\times\mathbf{W})\cdot\mathbf{n}\,\mathrm{d}S = \int_{\delta\mathcal{F}} \mathbf{W}\cdot\mathbf{t}\,\mathrm{d}s \\ &= \int_0^{2\pi} (0-a^2\cos^2 t, 0-a^2\sin^2, a^2\sin^2 t - a^2\cos^2 t)\cdot a(-\sin t,\cos t,0)\,\mathrm{d}t \\ &= a^3\int_0^{2\pi}\{\cos^2 t\,\sin t - \sin^2 t\,\cos t\}\,\mathrm{d}t = a^3\left[-\frac{\cos^3 t}{3}-\frac{\sin^3 t}{3}\right]_0^{2\pi} = 0.\end{aligned}$$

Subvariant. From the second variant of 3) we also obtain that $\mathbf{V} = \bigtriangledown\times\mathbf{W}_0$. Now, $\mathbf{W}_0 = (0,0,\cdots)$ and $\mathbf{t} = (\cdots,\cdots,0)$ on $\delta\mathcal{F}$, so an application of Stokes's theorem shows that the flux is

$$\int_{\mathcal{F}} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S = \int_{\mathcal{F}} (\bigtriangledown\times\mathbf{W}_0)\cdot\mathbf{n}\,\mathrm{d}S = \int_{\delta\mathcal{F}} \mathbf{W}_0\cdot\mathbf{t}\,\mathrm{d}s = \int_{\delta\mathcal{F}} 0\,\mathrm{d}s = 0.$$

Second variant. *Surface integral, rectangular coordinates.*

The unit normal vector is

$$\mathbf{n} = \frac{1}{a}(x,y,z), \qquad \text{on } \mathcal{F},$$

hence the flux is

$$\begin{aligned}\int_{\mathcal{F}} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S &= \int_{\mathcal{F}} \frac{1}{a}(2x+3y,3x+2y,-4z)\cdot(x,y,z)\,\mathrm{d}S = \frac{1}{a}\int_{\mathcal{F}}\left(2x^2+3xy+2y^2-4z^2\right)\,\mathrm{d}S \\ &= \frac{2}{a}\int_{\mathcal{F}}\left(x^2+y^2-2z^2+\frac{3}{2}xy\right)\,\mathrm{d}S = \frac{2}{a}\int_{\mathcal{F}}(x^2+y^2-2z^2)\,\mathrm{d}S,\end{aligned}$$

because $\int_{\mathcal{F}} xy\,dS = 0$ of symmetric reasons.

It also follows by the symmetry that

$$\int_{\mathcal{F}} x^2\,\mathrm{d}S = \int_{\mathcal{F}} y^2\,\mathrm{d}S.$$

Let $\mathcal{F}_1$ be given by

$$x^2+y^2+z^2 = a^2, \qquad y \ge 0.$$

Then we get in exactly the same way,

$$\int_{\mathcal{F}} x^2\,\mathrm{d}S = \int_{\mathcal{F}_1} x^2\,\mathrm{d}S = \int_{\mathcal{F}_1} z^2\,\mathrm{d}S = \int_{\mathcal{F}} z^2\,\mathrm{d}S,$$

thus

$$\int_{\mathcal{F}} x^2\,\mathrm{d}S = \int_{\mathcal{F}} y^2\,\mathrm{d}S = \int_{\mathcal{F}} z^2\,\mathrm{d}S.$$

Hence by insertion,

$$\int_{\mathcal{F}} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S = \frac{2}{a}\left\{\int_{\mathcal{F}} x^2\,\mathrm{d}S + \int_{\mathcal{F}} y^2\,\mathrm{d}S - 2\int_{\mathcal{F}} z^2\,\mathrm{d}S\right\} = 0.$$

Third variant. *Surface integral, spherical coordinates.*

In spherical coordinates a parametric description of the surface is given by

$$\begin{cases} x = a\,\sin\theta\,\cos\varphi, \\ y = a\,\sin\theta\,\sin\varphi, \qquad \theta \in \left[0, \dfrac{\pi}{2}\right], \quad \varphi \in [0, 2\pi]. \\ z = a\,\cos\theta, \end{cases}$$

Thus the normal vector becomes

$$\begin{aligned} \mathbf{N}(\theta,\varphi) &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a\,\cos\theta\,\cos\varphi & a\,\cos\theta\,\sin\varphi & -a\,\sin\theta \\ -a\,\sin\theta\,\sin\varphi & a\,\sin\theta\,\cos\varphi & 0 \end{vmatrix} \\ &= a^2\left(\sin^2\theta\,\cos\varphi, \sin^2\theta\,\sin\varphi, \sin\theta\,\cos\theta\right) \\ &= a^2\sin\theta\left(\sin\theta\,\cos\varphi, \sin\theta\,\sin\varphi, \cos\theta\right), \end{aligned}$$

and we note that the z-component is positive, showing that we have obtained the right orientation. Then

$$\begin{aligned} &\mathbf{V}(\mathbf{x}(\theta,\varphi)) \cdot \mathbf{N}(\theta,\varphi) \\ &= (2a\sin\theta\cos\varphi + 3a\sin\theta\sin\varphi, 3a\sin\theta\cos\varphi + 2a\sin\theta\sin\varphi, -4a\cos\theta) \cdot \\ &\qquad \cdot (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)\,a^2\sin\theta \\ &= a^3\left\{2\sin^2\theta\cos^2\varphi + 3\sin^2\theta\sin\varphi\cos\varphi + 3\sin^2\theta\sin\varphi\cos\varphi \right. \\ &\qquad \left. +2\sin^2\theta\sin^2\varphi - 4\cos^2\theta\right\}\sin\theta \\ &= a^3\left\{2\sin^2\theta + 6\sin^2\theta\sin\varphi\cos\varphi - 4\cos^2\theta\right\}\sin\theta \\ &= a^3\left\{2 - 2\cos^2\theta - 4\cos^2\theta + 6\sin^2\theta\sin\varphi\cos\varphi\right\}\sin\theta \\ &= 2a^3\left\{1 - 3\cos^2\theta + 3\sin^2\theta\sin\varphi\cos\varphi\right\}\sin\theta. \end{aligned}$$

The flux is

$$\begin{aligned} \int_{\mathcal{F}} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S &= \int_E \mathbf{V}(\mathbf{x}(\theta,\varphi)) \cdot \mathbf{N}(\theta,\varphi)\,\mathrm{d}\theta\,\mathrm{d}\varphi \\ &= 2a^3\int_0^{2\pi}\left\{\int_0^{\frac{\pi}{2}}\left(1 - 3\cos^2\theta + 3\sin^2\theta\sin\varphi\cos\varphi\right)\sin\theta\,d\theta\right\}\mathrm{d}\varphi \\ &= 4\pi a^3\int_0^{\frac{\pi}{2}}\left\{1 - 3\cos^2\theta\right\}\sin\theta\,\mathrm{d}\theta = 4\pi a^3\left[-\cos\theta + \cos^3\theta\right]_0^{\frac{\pi}{2}} = 0. \end{aligned}$$

Fourth variant. *Gauß's theorem.*

First note that $\mathcal{F}$ does not surround any body Ω, so we cannot apply Gauß's theorem immediately. However, if we add the plane surface ("the bottom")

$$B = \{(x,y,0) \mid x^2+y^2 \le a^2\}$$

with the unit normal vector $\mathbf{n} = (0,0,-1)$, then the union $\mathcal{F} \cup B$ surrounds the half ball Ω. We found above that div $\mathbf{V} = 0$, so we conclude by Gauß's theorem that

$$\int_{\mathcal{F}} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S + \int_B \mathbf{V}\cdot(0,0,-1)\,\mathrm{d}S = \int_\Omega \operatorname{div}\mathbf{V}\,\mathrm{d}\Omega = \int_\Omega 0\,\mathrm{d}\Omega = 0,$$

hence by a rearrangement,

$$\int_{\mathcal{F}} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S = +\int_B \mathbf{V}\cdot(0,0,1)\,dS = \int_B 0\,\mathrm{d}S = 0,$$

where we have used that $\mathbf{V} = (2x+3y, 3x+2y, 0)$ on B.

Fifth variant. *The surface as a surface of revolution.*

According to the second variant we shall compute the surface integral

$$\int_{\mathbf{F}} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S = \frac{2}{a}\int_{\mathcal{F}} \left(x^2+y^2-2z^2\right)\mathrm{d}S.$$

It can of course be done by considering $\mathcal{F}$ as a surface of revolution. We shall leave this variant to the reader.

Example 38.24 *Given the two functions*

$$L(x,y) = e^y + y(e^x - e^{xy}), \qquad (x,y) \in \mathbb{R}^2,$$

$$M(x,y) = e^x + x(e^y - e^{xy}), \qquad (x,y) \in \mathbb{R}^2.$$

1) *Prove that the vector field* $\mathbf{V}(x,y) = (L(x,y), M(x,y))$, $(x,y) \in \mathbb{R}^2$, *is a gradient field, and find all integrals of* $\mathbf{V}$.

2) *Show that the vector field*

$$\mathbf{U}(x,y,z) = (z\,M(x,y), -z\,L(x,y), L(x,y)+M(x,y)), \qquad (x,y,z) \in \mathbb{R}^3,$$

is not *a gradient field, while there exists a vector potential for* $\mathbf{U}$. *(One shall* not *find such a vector potential).*

A Gradient field, vector potential.

D We shall prove in three ways that $\mathbf{V}$ is a gradient field. That $\mathbf{U}$ has a vector potential is shown by means of the necessary and sufficient conditions.

I 1) First note that $L(x,y)$ and $M(x,y)$ are of class C^∞ in all of $\mathbb{R}^2$.

First method. *Manipulation.*

By means of the rules of calculations we get by some manipulation,

$$\begin{aligned} L\,dx + M\,dy &= \{e^y + y(e^x - e^{xy})\}\,dx + \{e^x + x(e^y - e^{xy})\}\,dy \\ &= \{e^y\,dx + xe^y\,dy\} + \{ye^x\,dx + e^x\,dy\} - e^{xy}\{y\,dx + x\,dy\} \\ &= \{e^y\,dx + x\,d(e^y)\} + \{y\,d(e^x) + e^x\,dy\} - e^{xy}\,d(xy) \\ &= d(xe^y) + d(ye^x) - d(e^{xy}) = d(xe^y + ye^x - e^{xy}) \\ &= \nabla F \cdot (dx, dy). \end{aligned}$$

Hence $(L(x,y), M(x,y))$ is a gradient field and its integrals are given by

$$F(x,y) = xe^y + ye^x - e^{xy} + C, \qquad C \text{ arbitrary constant.}$$

Second method. *Indefinite integration.*

We first get

$$F_1(x,y) = \int L(x,y)\,dx = \int \{e^y + y(e^x - e^{xy})\}\,dx = xe^y + ye^x - e^{xy}.$$

Then by a check

$$\frac{\partial F_1}{\partial y} = xe^y + e^x - xe^{xy} = e^x + x(e^y - e^{xy}) = M(x,y),$$

which shows that $(L(x,y), M(x,y))$ is a gradient field and that its integrals are

$$F(x,y) = F_1(x,y) + C = xe^y + ye^x - e^{xy} + C, \qquad (x,y) \in \mathbb{R}^2,$$

where C is an arbitrary constant.

Third method. *Integration along a broken line followed by a check.*

When we integrate $L\,dx + M\,dy$ along the broken line

$$(0,0) \longrightarrow (x,0) \longrightarrow (x,y),$$

we get the *candidate*

$$\begin{aligned} F(x,y) &= \int_0^x L(t,0)\,\mathrm{d}t + \int_0^y M(x,t)\,\mathrm{d}t = \int_0^x \mathrm{d}t + \int_0^y \left\{e^x + x(e^t - e^{xt})\right\}\,\mathrm{d}t \\ &= x + \left[te^x + xe^t - e^{xt}\right]_0^y = x + ye^x + xe^y - e^{xy} - x + 1 \\ &= ye^x + xe^y - e^{xy} + 1. \end{aligned}$$

By *testing* (this is mandatory by this method) we get

$$\frac{\partial F}{\partial x} = ye^x + e^y - ye^{xy} = e^y + y(e^x - e^{xy}) = L(x,y),$$

$$\frac{\partial F}{\partial y} = e^x + xe^y - xe^{xy} = e^x + x(e^y - e^{xy}) = M(x,y).$$

It follows from the above that $(L(x,y), M(x,y))$ is a gradient field and that its integrals are

$$F(x,y) = xe^y + ye^x - e^{xy} + C, \qquad C \text{ an arbitrary constant.}$$

2) Now

$$\frac{\partial}{\partial z}\left\{z\,M(x,y)\right\} = M(x,y) = e^x + x(e^y - e^{xy})$$

and

$$\frac{\partial}{\partial x}\left\{L(x,y) + M(x,y)\right\} = ye^x - y^2e^{xy} + e^x - e^{xy} - xye^{xy} \neq \frac{\partial}{\partial z}\left\{z\,M(x,y)\right\},$$

so the necessary conditions for a gradient field are not satisfied, and $\mathbf{U}$ is not a gradient field.

Clearly, $\mathbf{U}$ is of class C^∞ in all of $\mathbb{R}^3$, and $\mathbb{R}^3$ is star shaped. (It is even convex.)
As (L,M) is a gradient field, we have in particular

$$\frac{\partial L}{\partial y} = \frac{\partial M}{\partial x},$$

thus

$$\operatorname{div}\,\mathbf{U} = z\,\frac{\partial M}{\partial x} - z\,\frac{\partial L}{\partial y} + 0 = z\left\{\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right\} = 0,$$

and $\mathbf{U}$ is divergence free and defined in a star shaped domain. Therefore $\mathbf{U}$ has a vector potential.

Remark. In principal the integrals of the formula of the vector potential can be computed. However, the result is very difficult to manage with a lot of exceptional cases. For this reason it is highly recommended always to find some other method before one tries to find the vector potential by means of the standard formulæ. ◊

Example 38.25 *Given the vector field*

$$\mathbf{V}(x,y,z) = (\cos y - \sin z, \cos z - \sin x, \cos x - \sin y), \qquad (x,y,z) \in \mathbb{R}^3.$$

1) Find the divergence $\nabla \cdot \mathbf{V}$ *and the rotation* $\nabla \times \mathbf{V}$.

2) Show the existence of a constant α, *such that* $\alpha\mathbf{V}$ *is a vector potential for* $\mathbf{V}$.

3) Let the curve $\mathcal{K}$ *be the boundary of the square of vertices* $(0,0,0)$, $(0,\pi,0)$, $(0,\pi,\pi)$ *and* $(0,0,\pi)$, *in the given succession. Find the circulation*

$$\oint_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t}\, \mathrm{d}s.$$

4) Let

$$\Omega = \left\{(x,y,z) \in \mathbb{R}^3 \mid 1 \le x^2+y^2+z^2 \le 4\right\}.$$

Find the flux of the vector field

$$\mathbf{U}(x,y,z) = (x,y,z) + \mathbf{V}(x,y,z), \qquad (x,y,z) \in \mathbb{R}^3,$$

through $\partial\Omega$, *when the unit normal vector* $\mathbf{n}$ *of* $\partial\Omega$ *is pointing away from* Ω.

A Divergence, rotation, vector potential, circulation, flux.

D Apply Stokes's theorem and Gauß's theorem.

I 1) It follows immediately that

$$\text{div } \mathbf{V} = \nabla \cdot \mathbf{V} = 0.$$

Then

$$\begin{aligned}
\textbf{rot } \mathbf{V} &= \nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \cos y - \sin z & \cos z - \sin x & \cos x - \sin y \end{vmatrix} \\
&= (-\cos y + \sin z, -\cos z + \sin x, -\cos x + \sin y) \\
&= -(\cos y - \sin z, \cos z - \sin x, \cos x - \sin y) = -\mathbf{V}.
\end{aligned}$$

2) If we choose $\alpha = -1$ in 1), then

$$\nabla \times (-\mathbf{V}) = \mathbf{V},$$

and it follows that $-\mathbf{V}$ is a vector potential for $\mathbf{V}$.

3) Here we give two variants.

a) *Stokes's theorem.* The square $\mathcal{K}$ lies in the YZ-plane, and the unit normal vector is in the chosen orientation given by

$$\mathbf{n} = (1,0,0).$$

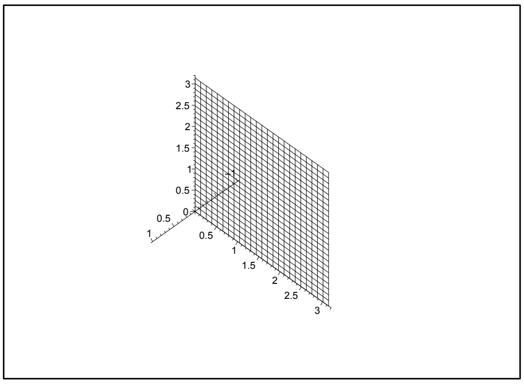

Figure 38.12: The square $\mathcal{K}$.

Then by 1) and Stokes's theorem,

$$\begin{aligned}
\oint_{\mathcal{K}} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s &= \int_{\tilde{\mathcal{K}}} \mathbf{n}\cdot\mathbf{rot}\;\mathbf{V}\,\mathrm{d}S = \int_{\tilde{\mathcal{K}}} (-\cos y + \sin z)\,\mathrm{d}S \\
&= \int_0^{\pi}\left\{\int_0^{\pi}(-\cos y)\,\mathrm{d}y\right\}\mathrm{d}z + \int_0^{\pi}\mathrm{d}y\cdot\int_0^{\pi}\sin z\,\mathrm{d}z \\
&= 0+\pi\cdot[-\cos z]_0^{\pi} = 2\pi.
\end{aligned}$$

b) *Straight forward computation of the line integral.* The curve $\mathcal{K}$ is composed of the curves

$$\begin{aligned}
\mathcal{K}_1 &: \quad \mathbf{r}_1(t) = (0,t,0), && t \in [0,\pi], \quad \mathbf{t} = (0,1,0),\\
\mathcal{K}_2 &: \quad \mathbf{r}_2(t) = (0,\pi,t), && t \in [0,\pi], \quad \mathbf{t} = (0,0,1),\\
\mathcal{K}_3 &: \quad \mathbf{r}_3(t) = (0,\pi-t,\pi), && t \in [0,\pi], \quad \mathbf{t} = (0,-1,0),\\
\mathcal{K}_4 &: \quad \mathbf{r}_4 = (0,0,\pi-t), && t \in [0,\pi], \quad \mathbf{t} = (0,0,-1).
\end{aligned}$$

Then by insertion,

$$\begin{aligned}
\int_{\mathcal{K}} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s &= \int_{\mathcal{K}_1} (\cos z - \sin x)\,\mathrm{d}s + \int_{\mathcal{K}_2} (\cos x - \sin y)\,\mathrm{d}s\\
&\quad + \int_{\mathcal{K}_3} (-\cos z + \sin x)\,\mathrm{d}s + \int_{\mathcal{K}_4} (-\cos x + \sin y)\,\mathrm{d}s\\
&= \int_0^\pi (1-0)\,\mathrm{d}t + \int_0^\pi (1-\sin\pi)\,\mathrm{d}t\\
&\quad + \int_0^\pi (-\cos\pi + \sin 0)\,\mathrm{d}t + \int_0^\pi (-1+\sin 0)\,\mathrm{d}t\\
&= \pi+\pi+\pi-\pi = 2\pi.
\end{aligned}$$

c) It follows from 1) and Gauß's theorem that

$$\begin{aligned}
\text{flux} &= \int_{\partial\Omega} \mathbf{U}\cdot\mathbf{n}\,\mathrm{d}S = \int_\Omega \operatorname{div}\,\mathbf{U}\,\mathrm{d}\Omega = \int_\Omega \{3 + \operatorname{div}\,\mathbf{V}\}\,\mathrm{d}\Omega\\
&= \int_\Omega (3+0)\,\mathrm{d}\Omega = 3\,\mathrm{vol}(\Omega) = 3\cdot\frac{4\pi}{3}\cdot\left(2^3-1^3\right) = 28\pi.
\end{aligned}$$

Example 38.26 **1.** *Find the rotation of the vector field*

$$\mathbf{U}(x,y,z) = (-yz, 0, xy), \qquad (x,y,z)\in\mathbb{R}^3,$$

and show that $\mathbf{U}$ *is not a gradient field.*
A space curve $\mathcal{K}$ *is given by the parametric description*

$$(x,y,z) = \mathbf{r}(t) = \left(\cos^3 t, 3\cos t, \sin^3 t\right), \qquad t\in\left[0,\frac{\pi}{2}\right].$$

2. *Compute the tangential line integral*

$$\int_{\mathcal{K}} \mathbf{U}\cdot\,\mathrm{d}\mathbf{x}.$$

3. *Find a function* $G(x,z)$, $(x,z)\in\mathbb{R}^2$, *such that the vector field*

$$\mathbf{W}(x,y,z) = (0, y\,G(x,z), 0), \qquad (x,y,z)\in\mathbb{R}^3,$$

is a vector potential for $\mathbf{U}$.

A Rotation; tangential line integral; vector potential.

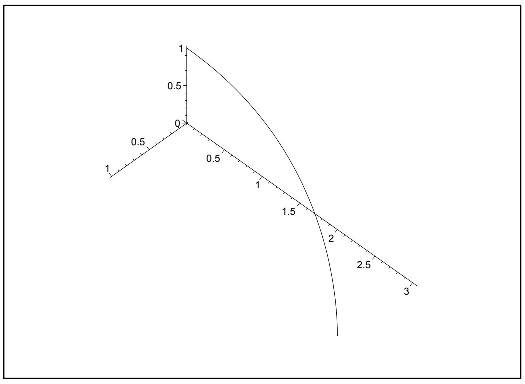

Figure 38.13: The space curve $\mathcal{K}$.

D Use the standard methods in the former two questions and check the conditions of a vector potential in the latter question.

I 1) It follows immediately that $\mathbf{U}$ is divergence free. Then

$$\mathbf{rot}\mathbf{U} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -yz & 0 & xy \end{vmatrix} = (x, -y-y, z) = (x, -2y, z).$$

It follows from $\mathbf{rot}\ \mathbf{U}(\mathbf{x}) \neq \mathbf{0}$ for $\mathbf{x} \neq \mathbf{0}$, that $\mathbf{U}$ is not a gradient field.

2) We get from

$$\mathbf{r}(t) = \left(\cos^3 t, 3\cos t, \sin^3 t\right)$$

that

$$\mathbf{r}'(t) = \left(-3\cos^2 t \sin t, -3\sin t, 3\sin^2 t \cos t\right),$$

and the tangential line integral is reduced to

$$\begin{aligned}
\int_{\mathcal{K}} \mathbf{U} \cdot \mathrm{d}\mathbf{x} &= 3 \cdot 3 \int_0^{\frac{\pi}{2}} (-\cos t \sin^3 t, 0, \cos^4 t) \cdot (-\cos^2 t \sin t, -\sin t, \sin^2 t \cos t)\, \mathrm{d}t \\
&= 9 \int_0^{\frac{\pi}{2}} \left\{\cos^3 t \sin^4 t + \cos^5 t \sin^2 t\right\} \mathrm{d}t \\
&= 9 \int_0^{\frac{\pi}{2}} \cos^3 t \sin^2\{\sin^2 t + \cos^2 t\}\, \mathrm{d}t = 9 \int_0^{\frac{\pi}{2}} \cos^3 t \sin^2 t\, \mathrm{d}t \\
&= 9 \int_0^{\frac{\pi}{2}} (1 - \sin^2 t) \sin^2 t \cdot \cos t\, \mathrm{d}t = 9 \int_0^1 (u^2 - u^4)\, \mathrm{d}u \\
&= 9\left(\frac{1}{3} - \frac{1}{5}\right) = 9 \cdot \frac{2}{15} = \frac{6}{5}.
\end{aligned}$$

3) If $\mathbf{W}(x,y,z) = (0, y\,G(x,z), 0)$, then

$$\mathbf{rot}\mathbf{W} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \\ 0 & y\,G(x,z) & 0 \end{vmatrix} = \left(-y\,G_z'(x,z), 0, y\,G_x'(x,z)\right)$$

is equal to $\mathbf{U}$ for

$$G_z'(x,z) = z \qquad \text{and} \qquad G_x'(x,z) = x,$$

hence by integration

$$G(x,z) = \frac{1}{2}\,z^2 + \varphi_1(x) = \frac{1}{2}\,x^2 + \varphi_2(z),$$

and by a rearrangement

$$\frac{1}{2}\,z^2 - \varphi_2(z) = \frac{1}{2}\,x^2 - \varphi_1(x) = \text{ constant},$$

so

$$G(x,z) = \frac{1}{2}\,x^2 + \frac{1}{2}\,z^2 + C, \qquad C \text{ arbitrary constant.}$$

Remark. Alternatively we should first note that $\mathbf{U}$ is divergence free in the star shaped (convex) domain $\mathbb{R}^3$ containing $\mathbf{0}$. This implies the existence of the vector potentials and that one of these can be found by the formula

$$\mathbf{W}_0(\mathbf{x}) = \int_0^1 \mathbf{T}(\tau\,\mathbf{x})\,\mathrm{d}\tau, \qquad \text{where } \mathbf{T}(\mathbf{x}) = \mathbf{U}(\mathbf{x}) \times \mathbf{x}.$$

First calculate

$$\mathbf{T}(\mathbf{x}) = \mathbf{U}(\mathbf{x}) \times \mathbf{x} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ -yz & 0 & xy \\ x & y & z \end{vmatrix} = \left(-xy^2, x^2y + yz^2, -y^2z\right).$$

All coordinates are precisely of degree 3, hence by an integration with respect to τ,

$$\mathbf{W}_0(\mathbf{x}) = \int_0^1 \mathbf{T}(\tau\,\mathbf{x})\,\mathrm{d}\tau = \mathbf{T}(\mathbf{x}) \int_0^1 \tau^3\,\mathrm{d}\tau = \left(-\frac{1}{4}\,xy^2, \frac{1}{4}\,(x^2+z^2)y, -\frac{1}{4}\,y^2z\right).$$

We see that $\mathbf{W}_0(\mathbf{x})$ is a vector potential for $\mathbf{U}(\mathbf{x})$. It is, however, not of the wanted type. $\Diamond$

Example 38.27 *Two vector fields* $\mathbf{V}$, $\mathbf{W} : \mathbb{R}^3 \to \mathbb{R}^3$ *are given by*

$$\mathbf{V}(x,y,z) = \left(e^y \sin z, x\, e^y \sin z, x\, e^y \cos z\right),$$

$$\mathbf{W}(x,y,z) = \left(x + 2x\, e^y \cos z, -2e^y \cos z, -z + z^3\right).$$

1) Find the divergence and the rotation of both vector fields.

2) Show that $\mathbf{V}$ *is a gradient field and find all its integrals.*

3) Compute the tangential line integral

$$\int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t}\, \mathrm{d}s,$$

where $\mathcal{K}$ *is the broken line composed of the three line segments: from* $(0,0,0)$ *to* $(1,0,0)$*, from* $(1,0,0)$ *to* $(1,2,0)$*, and from* $(1,2,0)$ *to* $\left(1,2,\frac{\pi}{2}\right)$.

4) Show the existence of a constant α*, such that* $\alpha\mathbf{W}$ *is a vector potential for* $\mathbf{V}$*; find* α.

5) Let $\mathcal{F}$ *be the sphere of centrum* $(0,0,0)$ *and radius* 3*. Find the flux*

$$\int_{\mathcal{F}} \mathbf{W} \cdot \mathbf{n}\, \mathrm{d}S,$$

where the unit normal vector $\mathbf{n}$ *is pointing away from the centrum of* $\mathcal{F}$.

A Divergence and rotation; gradient field and integrals; tangential line integral; vector potential; flux.

D It is in some sense better to go through the example in an other succession than the above. If we let 4) follow immediately after 1), then it becomes obvious. We give three variants of 2) and two variants of 3), while 5) is given in 3 variants.

I 1. We get by straightforward computations,

$$\begin{aligned}
\text{div}\,\mathbf{V} &= 0 + x\,e^y \sin z - x\,e^y \sin z = 0, \\
\mathbf{rot V} &= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^y \sin z & xe^y \sin z & xe^y \cos z \end{vmatrix} = \begin{pmatrix} xe^y \cos z - xe^y\ \cos z \\ e^y \cos z - e^y \cos z \\ e^y \sin z - e^y \sin z \end{pmatrix} \\
&= \mathbf{0}, \\
\text{div}\,\mathbf{W} &= 1 + 2e^y \cos z - 2e^y \cos z - 1 + 3z^2 = 3z^2, \\
\mathbf{rot W} &= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x + 2xe^y \cos z & -2e^y \cos z & -z + z^3 \end{vmatrix} \\
&= (0 - 2e^y \sin z, -2xe^y \sin z - 0, 0 - 2xe^y \cos z) \\
&= -2\,(e^y \sin z, xe^y \sin z, xe^y \cos z) = -2\mathbf{V}.
\end{aligned}$$

4. As $\mathbf{rot}\ \mathbf{W} = \bigtriangledown \times \mathbf{W} = -2\mathbf{V}$, we have $\bigtriangledown \times \left(-\frac{1}{2}\,\mathbf{W}\right) = \mathbf{V}$, and it follows immediately that $-\frac{1}{2}\,\mathbf{W}$ is a vector potential for $\mathbf{V}$, and that $\alpha = -\frac{1}{2}$.

2. As $\mathbf{rot}\ \mathbf{V} = \mathbf{0}$ and $\mathbb{R}^3$ is star shaped (it is even convex), $\mathbf{V}$ is a gradient field. Its integrals may be found in one of the following three ways:

First variant. *Indefinite integration.*

$$F_1(x,y,z) = \int V_x(x,y,z)\,\mathrm{d}x = \int e^y\ \sin z\,\mathrm{d}x = x\,e^y \sin z,$$

where

$$\bigtriangledown F_1 = (e^y \sin z, x\,e^y \sin z, x\,e^y \cos z) = \mathbf{V},$$

proving that F_1 is an integral of $\mathbf{V}$ and all integrals are given by

$$F(x,y,z) = x\,e^y \sin z + C, \qquad C \text{ arbitrary constant.}$$

Second variant. *Manipulation, using the rules of calculations.* It follows immediately from

$$\begin{aligned}
\mathbf{V} \cdot \mathrm{d}\mathbf{x} &= e^y \sin z\,\mathrm{d}x + xe^y \sin z\,\mathrm{d}y + xe^y \cos z\,\mathrm{d}z \\
&= e^y \cdot \sin z\,\mathrm{d}x + x \cdot \sin z \cdot\ \mathrm{d}\,(e^y) + xe^y\,\mathrm{d}(\sin z) \\
&= \mathrm{d}\,(xe^y \sin z) =\ \mathrm{d}\,(xe^y \sin z + C)\,,
\end{aligned}$$

that the integrals are given by

$$F(x,y,z) = x\,e^y \sin z + C, \qquad C \text{ arbitrary constant.}$$

Third variant. *Integration along a broken line*

$$(0,0,0) \longrightarrow (x,0,0) \longrightarrow (x,y,0) \longrightarrow (x,y,z).$$

This gives the candidates

$$\begin{aligned} F(x,y,z) &= C + \int_0^x 0\,\mathrm{d}t + \int_0^y 0\,\mathrm{d}t + \int_0^z x\,e^y \cos t\,\mathrm{d}t \\ &= x\,e^y \sin z + C. \end{aligned}$$

We have proved above that the integrals exist, so we conclude that these are the set of all integrals when the arbitrary constant $C \in \mathbb{R}$ varies.

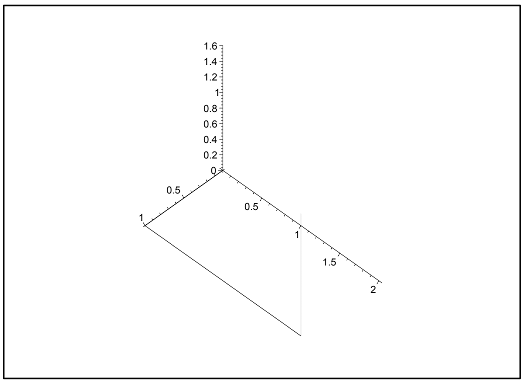

Figure 38.14: The curve $\mathcal{K}$.

3. We get by the *gradient theorem* that

$$\int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t}\,\mathrm{d}s = F\left(1, 2, \frac{\pi}{2}\right) - F(0,0,0) = 1 \cdot e^2 \cdot \sin\frac{\pi}{2} - 0 = e^2.$$

ALTERNATIVELY, write $\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_3$, where

$$\mathcal{K}_1: \quad (x,y,z) = (t,0,0), \quad t \in [0,1], \qquad \mathbf{t} = (1,0,0),$$

$$\mathcal{K}_2: \quad (x,y,z) = (1,t,0), \quad t \in [0,2], \qquad \mathbf{t} = (0,1,0),$$

$$\mathcal{K}_3: \quad (x,y,z) = (1,2,t), \quad t \in \left[0, \frac{\pi}{2}\right], \quad \mathbf{t} = (0,0,1),$$

thus

$$\int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t}\,\mathrm{d}s = \int_0^1 0\,\mathrm{d}t + \int_0^2 0\,\mathrm{d}t + \int_0^{\frac{\pi}{2}} 1 \cdot e^2 \cos t\,\mathrm{d}t = e^2.$$

4. This was answered previously.

5. The sphere $\mathcal{F}$ encloses the ball Ω, so it follows from Gauß's theorem and 1) that

$$\int_{\mathcal{F}} \mathbf{W} \cdot \mathbf{n}\,\mathrm{d}S = \int_{\Omega} \mathrm{div}\ \mathbf{W}\,\mathrm{d}\Omega = \int_{\Omega} 3z^2\,\mathrm{d}\Omega.$$

This space integral is then computed in three different ways:

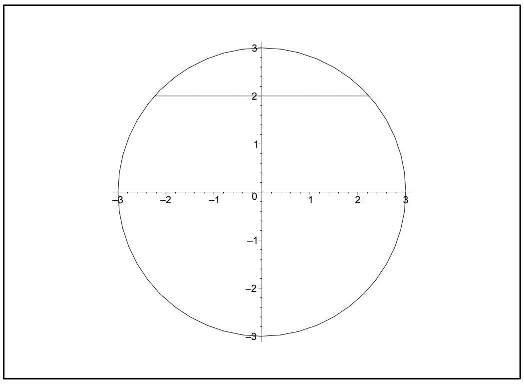

Figure 38.15: The intersection of the ball Ω with the (X,Z)-plane.

First variant. *The slicing method.* At height $z \in [-3,3]$ the ball is intersected into a disc $B(z)$ of radius $\varrho = \sqrt{9-z^2}$, thus

$$\begin{aligned}
\text{flux} &= \int_{\Omega} 3z^2 \,\mathrm{d}\Omega = \int_{-3}^{3} 3z^2 \left\{ \int_{B(z)} \mathrm{d}S \right\} \mathrm{d}z = \int_{-3}^{3} 3z^2 \,\text{area}\, B(z)\,\mathrm{d}z \\
&= \int_{-3}^{3} 3z^2 \cdot \pi \left(9-z^2\right)\,\mathrm{d}z = 3\pi \int_{-3}^{3} \left(9z^2 - z^4\right)\,\mathrm{d}z = 3\pi \left[3z^2 - \frac{z^5}{5}\right]_{-3}^{3} \\
&= 3\pi \cdot 2 \left(81 - \frac{3\cdot 81}{5}\right) = 6\pi \cdot 81 \cdot \left(1 - \frac{3}{5}\right) = \frac{972\pi}{5}.
\end{aligned}$$

Second variant. *The post method.* We get in polar coordinates

$$\begin{aligned}
\text{flux} &= \int_{\Omega} 3z^2 \,\mathrm{d}z = \int_{0}^{2\pi} \left\{ \int_0^3 \left(\int_{-\sqrt{9-\varrho^2}}^{\sqrt{9-\varrho^2}} 3z^2 \,\mathrm{d}z \right) \varrho \,\mathrm{d}\varrho \right\} \mathrm{d}\varphi \\
&= 2\pi \int_0^3 \left[z^3\right]_{-\sqrt{9-\varrho^2}}^{\sqrt{9-\varrho^2}} \varrho \,\mathrm{d}\varrho = 2\pi \int_0^3 \left(9-\varrho^2\right)^{\frac{3}{2}} \cdot 2\varrho \,\mathrm{d}\varrho \\
&= 2\pi \left[-\frac{2}{5} \left(9-\varrho^2\right)^{\frac{5}{2}} \right]_0^3 = \frac{4}{5}\,\pi \cdot 9^{\frac{5}{2}} = \frac{4\pi}{5} \cdot 3^5 = \frac{972\pi}{5}.
\end{aligned}$$

Third variant. *Spherical coordinates.* When we use spherical coordinates we get

$$\begin{aligned}
\text{flux} &= \int_{\Omega} 3z^2 \,\mathrm{d}z = \int_0^{2\pi} \left\{ \int_0^{\pi} \left(\int_0^3 3r^2 \cos^2\theta \cdot r^2 \sin\theta \,\mathrm{d}r \right) \mathrm{d}\theta \right\} \mathrm{d}\varphi \\
&= 2\pi \int_0^{\pi} 3\cos^2\theta \sin\theta \,\mathrm{d}\theta \cdot \int_0^3 r^5 \,\mathrm{d}r = 2\pi \left[-\cos^3\theta\right]_0^{\pi} \cdot \left[\frac{r^5}{5}\right]_0^3 \\
&= 2\pi \cdot 2 \cdot \frac{3^5}{5} = \frac{972\pi}{5}.
\end{aligned}$$

Remark. A direct computation of the flux by the definition alone looks impossible, because

$$\begin{aligned}
\mathbf{W}\cdot\mathbf{n} &= \left(x + 2xe^y \cos z, -2e^y \cos z, -z + z^3\right) \cdot \frac{1}{3}\,(x,y,z) \\
&= \frac{1}{3}\left\{x^2 + 2x^2 e^y \cos z - 2ye^y \cos z - z^2 + z^4\right\},
\end{aligned}$$

and what then? $\Diamond$

Example 38.28 *Given the vector field*

$$\mathbf{V}(x,y,z) = \begin{cases} (-y,x,0), & x^2+y^2<a^2, \\ \dfrac{a^2}{x^2+y^2}\,(-y,x,0), & x^2+y^2 \ge a^2. \end{cases}$$

1) Let $\mathcal{K}$ be the circle of constant values of the coordinates ϱ and z, and with a positive orientation with respect to the unit vector $\mathbf{e}_z$. Prove that

$$\oint_{\mathcal{K}} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s = \begin{cases} 2\pi\varrho^2, & \varrho<a, \\ 2\pi a^2, & \varrho \ge a. \end{cases}$$

2) Show that $(0,0,W)$, where

$$W(x,y,z) = \begin{cases} \dfrac{1}{2}\left(a^2-x^2-y^2\right), & x^2+y^2<a^2, \\ a^2\,\ln\dfrac{a}{\sqrt{x^2+y^2}}, & x^2+y^2\ge a^2, \end{cases}$$

is a vector potential for $\mathbf{V}$.

A Circulation along a curve; vector potential.

D Compute the circulation straight forward. (Consider if it is possible to use Stokes's theorem instead. Show that $\nabla \times (W\,\mathbf{e}_z) = \mathbf{V}$.

I 1) Since $\mathbf{V}$ does not depend on z, we may assume that $\mathcal{K}$ lies in the XY-plane. Then by Stokes'e theorem,

$$\oint_{\mathcal{K}} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s = \int_B \mathbf{e}_z\cdot\textbf{rot }\mathbf{V}\,\mathrm{d}S,$$

where B denoted the disc of radius ϱ. If $\varrho > a$, the right hand side does not look nice, so we compute instead the circulation by the definition. Let

$$\mathcal{K}: \quad (\varrho\,\cos\varphi, \varrho\,\sin\varphi), \qquad \varphi\in[0,2\pi].$$

Then

$$\mathbf{t}\,\mathrm{d}s = \varrho\,(-\sin\varphi,\cos\varphi)\,\mathrm{d}\varphi.$$

Then for $\varrho<a$,

$$\begin{aligned}\oint_{\mathcal{K}} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s &= \int_0^{2\pi} \varrho\,(-\sin\varphi,\cos\varphi)\cdot\varphi\,(-\sin\varphi,\cos\varphi)\,\mathrm{d}\varphi \\ &= \int_0^{2\pi} \varrho^2\left(\sin^2\varphi+\cos^2\varphi\right)\,\mathrm{d}\varphi \\ &= \varrho^2\int_0^{2\pi}\mathrm{d}\varphi = 2\pi\varrho^2 \qquad \text{for } \varrho<a,\end{aligned}$$

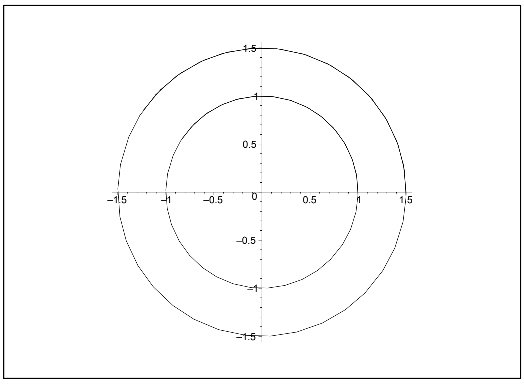

Figure 38.16: The curve $\mathcal{K}$ in the XY-plane for $a = 1$ and $\varrho = 1.5$.

and we have for $\varrho \geq a$

$$\begin{aligned}\oint_{\mathcal{K}} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s &= \int_0^{2\pi} \frac{a^2}{\varrho^2}\cdot\varrho\,(-\sin\varphi,\cos\varphi)\cdot\varrho\,(-\sin\varphi,\cos\varphi)\,\mathrm{d}\varphi \\ &= a^2\int_0^{2\pi}\mathrm{d}\varphi = 2\pi a^2, \qquad \text{for } \varrho \geq a,\end{aligned}$$

and the first claim has been proved.

2) If $x^2+y^2<a^2$, then

$$\nabla\times(W\,\mathbf{e}_z)=\begin{vmatrix}\mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 0 & \frac{1}{2}\left(a^2-\left(x^2+y^2\right)\right)\end{vmatrix}=(-y,x,0)=\mathbf{V},$$

and if $x^2+y^2>a^2$, then

$$\begin{aligned}\nabla\times(W\,\mathbf{e}_z) &= \begin{vmatrix}\mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 0 & a^2\ln\left(\dfrac{a}{\sqrt{x^2+y^2}}\right)\end{vmatrix} \\ &= \frac{a^2}{2}\left(-\frac{\partial}{\partial t}\left(x^2+y^2\right),\,\frac{\partial}{\partial x}\left(x^2+y^2\right),\,0\right)=\frac{a^2}{x^2+y^2}(-y,x,0)=\mathbf{V}.\end{aligned}$$

Due to the continuity from the inside and from the outside we get

$$\nabla\times(W\,\mathbf{e}_z)=(-y,x,0)=\mathbf{V}\qquad\text{for } x^2+y^2=a^2.$$

Hence we have proved that $W\,\mathbf{e}_z$ is a vector potential for $\mathbf{V}$.

39 Harmonic functions and Green's identities

39.1 Harmonic functions

We shall in this chapter prove some important consequences of *Gauß's theorem*. We shall start with noting that in connection with potentials – one of the subjects of Chapter 38 – we had to consider *Poisson's equation*

$$\bigtriangledown^2 w = p,$$

where $p(\mathbf{x})$ is a given function (a scalar field). Poisson's equation is one of the classical (inhomogeneous) elliptic partial differential equations. We mention without proof that in order to secure a unique solution in a given domain Ω we must also have given the boundary conditions on $\partial\Omega$, which in the following is assumed to consist of one or more surfaces, i.e. we exclude the case, in which $\partial\Omega$ contains isolated points. Then the above means that we require that the solution w of Poisson's equation is of class C^2 in the interior Ω° of Ω and continuous on $\overline{\Omega}$ (almost everywhere) with

$$w(\mathbf{x}) = q(\mathbf{x}) \qquad \text{for } \mathbf{x} \in \partial\Omega,$$

where $q(\mathbf{x})$ is a given function, continuous almost everywhere on the boundary $\partial\Omega$. We note the special case, when $\Omega = \mathbb{R}^n$ is the whole space. In this case $\partial\Omega = \emptyset$, and the Poisson equation does not have a unique solution.

In the initial study of the Poisson equation we assume for simplicity that we do not have any boundary condition, so we do not have uniqueness in this case. However, if w_1 and w_2 are two solutions of Poisson's equation, i.e.

$$\bigtriangledown^2 w_1 = p \qquad \text{and} \qquad \bigtriangledown^2 w_2 = p,$$

then it follows from the linearity of the differential operator $\bigtriangledown^2$ that

$$\bigtriangledown^2 (w_1 - w_2) = \bigtriangledown^2 w_1 - \bigtriangledown^2 w_2 = p - p = 0,$$

so the difference between any two solutions of Poisson's equation for the same given $p(\mathbf{x})$ must be a solution of the so-called *Laplace equation*,

$$\bigtriangledown^2 w = 0,$$

which is the classical homogeneous elliptic partial differential equation. It occurs in so many applications that it has been thoroughly studied during the last centuries, and there exists a large literature on this matter. Here we shall not go into details in the theory of linear partial differential operators, though we shall need some of the structure of the solutions, so we introduce

Definition 39.1 *An* harmonic function *in an open set* Ω° *is a solution of the Laplace equation in* Ω°*, i.e. the function* $w(\mathbf{x})$ *is harmonic in* Ω°*, if*

$$\bigtriangledown^2 w = 0 \qquad \textit{in } \Omega^\circ.$$

Every solution of Poisson's equation $\bigtriangledown^2 w = p$ in Ω° is then given by one particular solution of this equation plus an harmonic function.

In the plane $\mathbb{R}^2$ the harmonic functions are connected with the *Theory of Complex Functions*. We shall in this chapter mostly consider harmonic functions in ordinary space $\mathbb{R}^3$, so if nothing else is specified, we tacitly consider an open domain $\Omega^\circ \subseteq \mathbb{R}^3$.

Definition 39.2 *Given a body (containing interior points)* $\Omega \subset \mathbb{R}^3$ *with a piecewise smooth boundary* $\partial\Omega \neq \emptyset$. *Let* $\mathbf{n}$ *denote the outgoing unit normal vector field on the boundary* $\partial\Omega$. *Finally, let* f *be of class* C^1 *in a neighbourhood of (a part of) the boundary* $\partial\Omega$. *We define the* normal derivative *of* f *on (a part of) the boundary* $\partial\Omega$ *by*

$$\frac{\partial f}{\partial n} = \mathbf{n} \cdot \bigtriangledown f.$$

Clearly, the normal derivative of the function f indicates the change of f at the boundary point under consideration in the outgoing direction perpendicular to the boundary.

Let us assume that the function f is harmonic in an open set U, which contains the closure $\overline{\Omega}$ of the domain Ω, i.e. $\Omega^\circ \neq \emptyset$ and $\overline{\Omega} \subset U$. Since $\bigtriangledown^2 f = 0$, it follows immediately from *Gauß's theorem* that

$$\int_{\partial\Omega} \frac{\partial f}{\partial n}\,\mathrm{d}S = \int_{\partial\Omega} \mathbf{n} \cdot \bigtriangledown f\,\mathrm{d}S = \int_{\Omega} \bigtriangledown \cdot \bigtriangledown f\,\mathrm{d}\Omega = \int_{\Omega} \bigtriangledown^2 f\,\mathrm{d}\Omega = 0,$$

so we have proved the important

Lemma 39.1 *If* f *is harmonic in the open domain* $U \subseteq \mathbb{R}^3$, *then for every domain* Ω, *for which* $\overline{\Omega} \subset U$, *the surface integral over the boundary* $\partial\Omega$ *of the normal derivative of* f *is always 0,*

$$\int_{\partial\Omega} \frac{\partial f}{\partial n}\,\mathrm{d}S = 0.$$

39.2 Green's first identity

Let f and g be of class C^2 in an open set U, which contains the closure $\overline{\Omega}$ of some domain Ω. It follows from one of the rules of computation for $\bigtriangledown$ that

$$\bigtriangledown \cdot (g \bigtriangledown f) = g \bigtriangledown^2 f + \bigtriangledown g \cdot \bigtriangledown f,$$

so when we apply *Gauß's theorem* on the space integral of the right hand side, we get

$$\int_{\Omega} \left(g \bigtriangledown^2 f + \bigtriangledown g \cdot \bigtriangledown f\right)\mathrm{d}\Omega = \int_{\Omega} \bigtriangledown \cdot (g \bigtriangledown f)\,\mathrm{d}\Omega = \int_{\partial\Omega} \mathbf{n} \cdot g \bigtriangledown f\,\mathrm{d}S = \int_{\partial\Omega} g\,\frac{\partial f}{\partial n}\,\mathrm{d}S,$$

and we have proved

Theorem 39.1 Green's first identity. *Assume that* f, $g \in C^2(U)$, *where* $U \subseteq \mathbb{R}^3$ *is an open set, and let* Ω *be a domain in* $\mathbb{R}^3$, *such that* $\overline{\Omega} \subset U$ *and* $\partial\Omega \neq \emptyset$ *and consists of piecewise smooth surfaces. Then*

$$\int_{\Omega} \left(g \bigtriangledown^2 f + \bigtriangledown g \cdot \bigtriangledown f\right)\mathrm{d}\Omega = \int_{\partial\Omega} g\,\frac{\partial f}{\partial n}\,\mathrm{d}S.$$

A simple application of *Green's first identity* is a uniqueness theorem for the *Poisson equation.*

Theorem 39.2 Uniqueness theorem for the Poisson equation. *Let $\Omega \subset \mathbb{R}^3$ be an open set with a nonempty boundary $\partial\Omega$ consisting of piecewise smooth surfaces. Let p be a function in Ω, and let q be a function on the boundary $\partial\Omega$. If the boundary value problem for the Poisson equation has a solution w,*

$$\nabla^2 w = p \quad \text{in } \Omega, \qquad \text{and} \qquad w = q \quad \text{on } \partial\Omega,$$

then this solution is unique.

Remark 39.1 Note that the theorem does not say anything of the existence of a solution. If p or q are not continuous, such a solution may not exist. $\Diamond$

Proof. Let us assume that both w_1 and w_2 are solutions of this boundary value problem for the *Poisson equation.* If we put $f = w_1 - w_2$, it follows from the linearity of the problem that

$$\nabla^2 f = \nabla^2 w_1 - \nabla^2 w_2 = p - p = 0 \qquad \text{on } \Omega^\circ,$$

and

$$f = w_1 - w_2 = q - q = 0 \qquad \text{on } \partial\Omega.$$

When we put $g = f$ in *Green's first identity*, we get

$$\int_\Omega \left(f \nabla^2 f + \nabla f \cdot \nabla f\right) \mathrm{d}\Omega = 0 + \int_\Omega \| \nabla f\|^2 \,\mathrm{d}\Omega = \int_{\partial\Omega} f \,\frac{\partial f}{\partial n}\,\mathrm{d}S = 0,$$

because $\nabla^2 f = 0$ on Ω°, and $f = 0$ on $\partial\Omega$. Hence

$$\int_\Omega \| \nabla f\|^2 \,\mathrm{d}\Omega = 0.$$

The function $f = w_1 - w_2$ is of class C^2, so $\| \nabla f\|^2 \geq 0$ is continuous. The integral can only become zero, if $\| \nabla f\|^2 = 0$, which implies that $\nabla f = \mathbf{0}$, so f is constant. Since f is continuous on $\overline{\Omega}$, and $f = 0$ on $\partial\Omega$, we conclude that $f = w_1 - w_2 = 0$, and the uniqueness is proved. $\Box$

39.3 Green's second identity

Let f, $g \in C^2(U)$, where $U \subseteq \mathbb{R}^3$ is an open set, and let Ω be a set, such that $\overline{\Omega} \subset U$, and $\partial\Omega$ consists of piecewise smooth surfaces. Then we get by Green's first identity

$$\int_\Omega \left(g \nabla^2 f + \nabla g \cdot \nabla f\right) \mathrm{d}\Omega = \int_{\partial\Omega} g \,\frac{\partial f}{\partial n}\,\mathrm{d}S,$$

as well as

$$\int_\Omega \left(f \nabla^2 g + \nabla f \cdot \nabla g\right) \mathrm{d}\Omega = \int_{\partial\Omega} f \,\frac{\partial g}{\partial n}\,\mathrm{d}S,$$

hence by subtraction

$$\int_\Omega \left(g \nabla^2 f - f \nabla^2 g\right) \mathrm{d}\Omega = \int_{\partial\Omega} \left(g \,\frac{\partial f}{\partial n} - f \,\frac{\partial g}{\partial n}\right) \mathrm{d}S,$$

and we have proved

Theorem 39.3 Green's second identity. *Assume that $f, g \in C^2(U)$, where $U \subseteq \mathbb{R}^3$ is an open set, and let Ω be a domain in $\mathbb{R}^3$, such that $\overline{\Omega} \subset U$ and $\partial\Omega \neq \emptyset$ consists of piecewise smooth surfaces. Then*

$$\int_\Omega \left(g \bigtriangledown^2 f - f \bigtriangledown^2 g\right) \mathrm{d}\Omega = \int_{\partial\Omega} \left(g \frac{\partial f}{\partial n} - f \frac{\partial g}{\partial n}\right) \mathrm{d}S.$$

We shall use *Green's second identity* to prove the following

Theorem 39.4 Mean value theorem for harmonic functions in space. *Let f be harmonic in an open set U, and let $P \in U$ be a given point. Then the value $f(P)$ of f at the point P equal to the mean value of f over any ball or sphere of centrum P and contained in U. In the case of the sphere all points inside this sphere should also be contained in U.*

Proof. Let the point $P \in U$ be given. By translation we may assume that P is $\mathbf{0}$. Let f be harmonic in U, i.e.

$$\bigtriangledown^2 f = 0 \qquad \text{in } U,$$

and choose the radius $a > 0$, such that the closed ball $B[\mathbf{0}, a] \subset U$. The trick is to choose a convenient function g in U.

If we choose

$$g(x,y,z)=\frac{r^2}{6}=\frac{1}{6}\left(x^2+y^2+z^2\right), \qquad \text{then } \bigtriangledown^2 g=1.$$

Furthermore,

$$g(x,y,z)=\frac{a^2}{6} \qquad \text{for } r=a, \quad \text{i.e. for } (x,y,z)\in\partial B[\mathbf{0},a],$$

so g is constant on the sphere of centrum $\mathbf{0}$ and radius $a>0$. If we choose $\Omega=B[\mathbf{0},a]$, it follows from *Green's second identity* that

$$\begin{aligned}\int_\Omega\left(g\bigtriangledown^2 f-f\bigtriangledown^2 g\right)\mathrm{d}\Omega &= \int_\Omega g\bigtriangledown^2 g\,\mathrm{d}\Omega-\int_\Omega f\,\mathrm{d}\Omega=\int_{\partial\Omega}\left(g\,\frac{\partial f}{\partial n}-f\,\frac{\partial g}{\partial n}\right)\mathrm{d}S\\ &= \frac{a^2}{6}\int_{\partial\Omega}\frac{\partial f}{\partial n}\,\mathrm{d}S-\int_{\partial\Omega}f\,\frac{\partial g}{\partial n}\,\mathrm{d}S=-\int_{\partial\Omega}f\,\frac{\partial g}{\partial n}\,\mathrm{d}S,\end{aligned}$$

where we have used that $g=a^2/6$ is a constant on $\partial\Omega$, and Lemma 39.1. By a rearrangement,

$$\int_\Omega f\,\mathrm{d}\Omega=\int_\Omega g\bigtriangledown^2 f\,\mathrm{d}\Omega+\int_{\partial\Omega}f\,\frac{\partial g}{\partial n}\,\mathrm{d}\Omega=\int_{\partial\Omega}f\,\frac{\partial g}{\partial n}\,\mathrm{d}\Omega,$$

because $\bigtriangledown^2 f=0$ in Ω°. Since $\dfrac{\partial}{\partial n}=\dfrac{\partial}{\partial r}$ on the sphere $r=a$, we get

$$\frac{\partial g}{\partial n}=\frac{\partial g}{\partial r}=\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{6}\right)=\frac{r}{3}=\frac{a}{3} \qquad \text{on } \partial\Omega,$$

hence

$$\int_{B[\mathbf{0},a]}f\,\mathrm{d}\Omega=\frac{a}{3}\int_{\partial B[\mathbf{0},a]}f\,\mathrm{d}S.$$

We divide this result by the volume $\dfrac{4}{3}\,\pi\,a^3$ of $B[\mathbf{0},a]$,

$$\frac{1}{\frac{4}{3}\,\pi\,a^3}\int_{B[\mathbf{0},a]}f\,\mathrm{d}\Omega=\frac{1}{4\pi a^2}\int_{\partial B[\mathbf{0},a]}f\,\mathrm{d}S.$$

Since $4\pi a^2$ is the area of the sphere $\partial B[\mathbf{0},a]$, this shows that the mean value of the harmonic function f over the ball $B[\mathbf{0},a]$ is equal to the mean value of f over the sphere $\partial B[\mathbf{0},a]$.

We shall in the following exploit this remarkable identity. The given harmonic function f cannot be changed in *Green's second identity*. But g can, and again we shall choose g, such that g is constant on every sphere of centrum $\mathbf{0}$. We choose the harmonic function

$$g=\frac{1}{r} \quad \text{for } r\neq 0, \qquad \text{with } \bigtriangledown^2 g=0 \quad \text{(a simple check).}$$

We note that g is not defined at $\mathbf{0}$, so this time we must modify the set Ω. Let $0<b<a$, and define Ω as the shell

$$\Omega:=B(\mathbf{0},a)\setminus B[\mathbf{0},b]=\left\{(x,y,z)\mid b^2<x^2+y^2+z^2<a^2\right\}.$$

Then g is constant on the two surfaces of $\partial\Omega$, in fact,

$$g = \frac{1}{a} \quad \text{for } r = a \quad \text{(outer surface)}, \qquad \text{and} \qquad g = \frac{1}{b} \quad \text{for } r = b \quad \text{(inner surface)}.$$

Hence, we get by *Green's second identity*

$$0 = \int_{\Omega} \left(g \bigtriangledown^2 f - f \bigtriangledown^2 g \right) \mathrm{d}\Omega = \int_{\partial\Omega} g \, \frac{\partial f}{\partial n} \, \mathrm{d}S + \int_{\partial\Omega} f \, \frac{\partial g}{\partial n} \, \mathrm{d}S,$$

where we have used that f and g are harmonic, i.e. $\bigtriangledown^2 f = \bigtriangledown^2 g = 0$. Furthermore, by using that g is constant on the two surfaces of $\partial\Omega$, so its value can be put outside the integration sign, and an application of Lemma 39.1 gives

$$\int_{\partial\Omega} g \, \frac{\partial f}{\partial n} \, \mathrm{d}S = \frac{1}{a} \int_{|\mathbf{a}\|=a} \frac{\partial f}{\partial n} \, \mathrm{d}S + \frac{1}{b} \int_{|\mathbf{a}\|=b} \frac{\partial f}{\partial n} \, \mathrm{d}S = 0,$$

so finally,

$$0 = \int_{\partial\Omega} f \, \frac{\partial g}{\partial n} \, \mathrm{d}S = \int_{\|\mathbf{x}=a} f \, \frac{\partial g}{\partial n} \, \mathrm{d}S + \int_{\|\mathbf{x}=b} f \, \frac{\partial g}{\partial n} \, \mathrm{d}S.$$

Here we get

$$\text{for } r = a: \qquad \frac{\partial g}{\partial n} = \frac{\partial g}{\partial r} = \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\right) = -\frac{1}{r^2} = -\frac{1}{a^2},$$

$$\text{for } r = b: \qquad -\frac{\partial g}{\partial n} = -\frac{\partial g}{\partial r} = -\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\right) = \frac{1}{r^2} = \frac{1}{b^2},$$

so by insertion,

$$0 = -\frac{1}{a^2} \int_{\|\mathbf{x}\|=a} f \, \mathrm{d}S + \frac{1}{b^2} \int_{\|\mathbf{x}\|=b} f \, \mathrm{d}S.$$

Then divide by 4π to get the area of the surfaces in the denominators, and rearrange,

$$\frac{1}{4\pi a^2} \int_{\|\mathbf{x}\|=a} f \, \mathrm{d}S = \frac{1}{4\pi b^2} \int_{\|\mathbf{x}\|=b} f \, \mathrm{d}S, \qquad \text{for all } 0 < b < a.$$

Since $4\pi b^2$ is the area of the sphere of radius b, it follows that the mean value

$$\frac{1}{4\pi b^2} \int_{\|\mathbf{x}\|=b} f \, \mathrm{d}S, \qquad \text{for } b \in \,]0, a[,$$

is constant in b. Since f is continuous, we also have

$$f(\mathbf{x}) = f(\mathbf{0}) + \varepsilon(\mathbf{x}), \qquad \text{where } \varepsilon(\mathbf{x}) \to 0 \text{ for } \|\mathbf{x}\| \to 0,$$

so

$$\frac{1}{4\pi b^2} \int_{\|\mathbf{x}\|=b} f \, \mathrm{d}S = \frac{1}{4\pi b^2} \int_{\|\mathbf{x}\|=b} \{f(\mathbf{0}) + \varepsilon(\mathbf{x})\} \, \mathrm{d}S = f(\mathbf{0}) + \frac{1}{4\pi b^2} \int_{\|\mathbf{x}\|=b} \varepsilon(\mathbf{x}) \, \mathrm{d}S.$$

The latter integral is numerically smaller than $\|\varepsilon(\mathbf{x})\|$, so we get in the limit

$$\lim_{b\to 0+} \frac{1}{4\pi b^2} \int_{\|\mathbf{x}\|=b} f \, \mathrm{d}S = f(\mathbf{0}).$$

Summing up,

$$f(\mathbf{0}) = \frac{1}{4\pi a^2} \int_{\|\mathbf{x}\|=a} f \, \mathrm{d}S = \frac{1}{\frac{4}{3}\pi a^3} \int_{\|\mathbf{x}\|=a} f \, \mathrm{d}\Omega,$$

and the theorem is proved. □

Let f be harmonic in the set Ω. If f is not a constant, then f cannot have an extremum at an interior point $P \in \Omega^\circ$. In fact, assume e.g. that P is a maximum point. For convenience we translate the coordinate system, such that $P = \mathbf{0}$. Then choose $r > 0$, such that

$$f(\mathbf{x}) < f(\mathbf{0}), \qquad \text{for some points, for which } \|\mathbf{x}\| < r.$$

Then the mean value theorem is clearly violated, so the assumption that $\mathbf{0}$ was a maximum point was wrong. In particular, it follows that maxima and minima for harmonic functions in Ω are always to be found on the boundary $\partial\Omega$. This property is called the *maximum-minimum principle for harmonic functions*.

Green's identities above have only been stated for scalar fields. One may obtain similar formulæ for vector fields, where we start with the expressions $\triangledown \cdot (\mathbf{W} \times (\triangledown \times \mathbf{V}))$. The analogue vectorial version of e.g. *Green's second identity* is

$$\int_\Omega \{\mathbf{W} \cdot \triangledown \times (\triangledown \times \mathbf{V}) - \mathbf{V} \cdot \triangledown \times (\triangledown \times \mathbf{W})\} \, \mathrm{d}\Omega = \int_{\partial\Omega} \mathbf{n} \cdot \{\mathbf{V} \times (\triangledown \times \mathbf{W}) - \mathbf{W} \times (\triangledown \times \mathbf{V})\} \, \mathrm{d}S.$$

39.4 Green's third identity

Let Ω be a bounded set with $\mathbf{0}$ as an interior point, $\mathbf{0} \in \Omega^{\circ}$. Let f be a function of class C^2 in an open set $U \supset \overline{\Omega}$. We do not assume that f is harmonic, so in general, $\nabla^2 f \neq 0$. If we choose the auxiliary function $g = \dfrac{1}{r}$ for $r \neq 0$, then $\nabla^2 g = 0$, so g is harmonic.

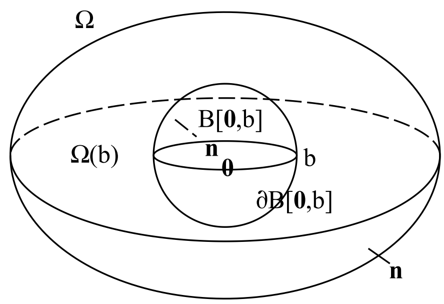

Figure 39.1: The set Ω and the subset $\Omega(b)$, where we have cut a solid ball $B[\mathbf{0}, b]$ out of Ω. We apply Green's second identity on $\Omega(b)$ with $g = \dfrac{1}{r}$ in the proof of Green's third identity.

Choose $b > 0$, such that $B := B[\mathbf{0}, b] \subset \Omega$. Then we apply *Green's second identity* on the set $\Omega(b) = \Omega \setminus B$, where we note that

$$\partial\Omega(b) = \partial\Omega \cup \partial B.$$

This gives

$$\begin{aligned}\int_{\Omega(b)} \left(g \nabla^2 f - f \nabla^2 g\right) \mathrm{d}\Omega &= \int_{\Omega(b)} \frac{\nabla^2 f}{r} \,\mathrm{d}\Omega = \int_{\partial\Omega(b)} \left(g \frac{\partial f}{\partial n} - f \frac{\partial g}{\partial n}\right) \mathrm{d}S \\ &= \int_{\partial\Omega} \left(\frac{1}{r} \frac{\partial f}{\partial n} - f \frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right) \mathrm{d}S + \int_{\partial B} \left(\frac{1}{r} \frac{\partial f}{\partial n} - f \frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right) \mathrm{d}S.\end{aligned}$$

The outgoing normal vector on the sphere ∂B, seen from $\Omega(b)$, points towards the centrum $\mathbf{0}$, i.e. for decreasing r, cf. Figure 39.1, so

$$\frac{\partial}{\partial n} = -\frac{\partial}{\partial r} \qquad \text{on } \partial B,$$

and therefore,

$$\int_{\partial B} \left(\frac{1}{r} \frac{\partial f}{\partial n} - f \frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right) \mathrm{d}S = \int_{\partial B} \left(-\frac{1}{r} \frac{\partial f}{\partial r} + f \frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right) \mathrm{d}S = -\frac{1}{b} \int_{\partial B} \frac{\partial f}{\partial r} \,\mathrm{d}S - \frac{1}{b^2} \int_{\partial B} f \,\mathrm{d}S,$$

where

$$-\frac{1}{b} \int_{\partial B} \frac{\partial f}{\partial r} \,\mathrm{d}S = -4\pi b \cdot \frac{1}{4\pi b^2} \int_{\partial B} \frac{\partial f}{\partial r} \,\mathrm{d}S \to 0 \cdot \frac{\partial f}{\partial r}(\mathbf{0}) = 0 \qquad \text{for } b \to 0+,$$

because the weighted integral is the mean value of $\dfrac{\partial f}{\partial r}$, which tends towards $\dfrac{\partial f}{\partial r}(\mathbf{0})$ for $b \to 0+$.

Similarly,

$$-\frac{1}{b^2}\int_{\partial B} f\,\mathrm{d}S = -4\pi \cdot \frac{1}{4\pi b^2}\int_{\partial B} f\,\mathrm{d}S \to -4\pi\, f(\mathbf{0}) \qquad \text{for } b \to 0+\,.$$

When we collect all results above it follows that

$$\int_{\Omega} \frac{\nabla^2 f}{r}\,\mathrm{d}\Omega = \lim_{b\to 0+}\int_{\Omega(b)} \frac{\nabla^2 f}{r}\,\mathrm{d}\Omega = \int_{\partial\Omega}\left(\frac{1}{r}\frac{\partial f}{\partial n} - f\,\frac{\partial}{\partial n}\left(\frac{2}{r}\right)\right)\mathrm{d}S - 4\pi\, f(\mathbf{0}),$$

hence by a rearrangement,

Theorem 39.5 Green's third identity. *Let Ω be a bounded set with $\mathbf{0} \in \Omega^{\circ}$ an interior point. Let $f \in C^2(U)$, where $U \supset \overline{\Omega}$ is an open set. Then*

$$f(\mathbf{0}) = -\frac{1}{4\pi}\int_{\Omega} \frac{\nabla^2 f}{r}\,\mathrm{d}\Omega + \frac{1}{\pi}\int_{\partial\Omega}\left(\frac{1}{r}\frac{\partial f}{\partial n} - f\,\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right)\mathrm{d}S.$$

It is possible from *Green's third identity* to derive a solution formula for the *Poisson equation*

$$\nabla^2 f = p \qquad \text{in } \mathbb{R}^3,$$

under some assumptions of the rapid decrease of $f(\mathbf{x})$ for $\|\mathbf{x}\| \to +\infty$. We shall not here give the precise requirements which secure that

$$\int_{\|\mathbf{x}\|=r}\left(\frac{1}{r}\frac{\partial f}{\partial n} - f\,\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right)\mathrm{d}S \to 0 \qquad \text{for } t \to +\infty,$$

where also $\nabla^2 = p$. We see, however, that if this condition is fulfilled, then we get by taking the limit $r \to +\infty$, that

$$f(\mathbf{0}) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\nabla^2 f}{r}\,\mathrm{d}\Omega = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{p(\mathbf{x})}{r}\,\mathrm{d}\Omega.$$

Since $\Omega = \mathbb{R}^3$ by this limit process, we get by a translation of the coordinate system that in general,

$$f(\mathbf{x}) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{p(\mathbf{u})}{\|\mathbf{x}-\mathbf{u}\|}\,\mathrm{d}\Omega, \qquad \text{for } \mathbf{x} \in \mathbb{R}^3.$$

The argument above is of course *not* a strict proof of the claim that this is indeed our solution formula for the *Poisson equation.* This can be proved, but not in the realm of this book, unless we should add quite a lot of other material.

39.5 Green's identities in the plane

In the plane case, i.e. in $\mathbb{R}^2$, we get some similar identities, which can also be derived by instead applying the theory of harmonic functions in the *Theory of Complex Functions*. This is not done here.

The first two of *Green's identities* are formally identical with the identities given previously in this chapter. The only difference is that we only use two coordinates (x, y) instead of three, (x, y, z). The proofs of these identities are identical with those given in the three-dimensional case. There is no need to repeat all this here.

The difference in structure can be seen in *Green's third identity in the plane*, which we here quote without being too precise with the assumptions. (Just take more or less the same assumptions as in the three-dimensional case.) Therefore, it is not formulated as a theorem. If $\mathbf{0} \in \omega^\circ$, then

$$f(\mathbf{0}) = -\frac{1}{2\pi} \int_\omega \nabla^2 f \ln \frac{1}{\varrho} \, \mathrm{d}S + \frac{1}{2\pi} \int_{\partial\omega} \left\{ \left(\ln \frac{1}{\varrho} \right) \frac{\partial f}{\partial n} - f \, \frac{\partial}{\partial n} \ln \frac{1}{\varrho} \right\} \mathrm{d}s.$$

Note that we have chosen the auxiliary function $g = \ln \dfrac{1}{\varrho}$, where $\varrho = \sqrt{x^2 + y^2}$.

39.6 Gradient, divergence and rotation in semi-polar and spherical coordinates

We shall in this section quote – without proofs – the formulæ for the *gradient*, the *divergence*, and the *rotation* in $\mathbb{R}^3$ in the three standard coordinate systems, the *rectangular* (x, y, z), the *semi-polar* $\varrho, \varphi, z)$ and the *spherical* (r, θ, φ), coordinate systems.

First recall that we in *rectangular coordinates* refer to the orthonormal system $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ of unit basis vectors, and we found that

$$\mathbf{grad} f = \triangledown f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \frac{\partial f}{\partial x}\,\mathbf{e}_x + \frac{\partial f}{\partial y}\,\mathbf{e}_y + \frac{\partial f}{\partial z}\,\mathbf{e}_z,$$

$$\text{div } \mathbf{V} = \triangledown \cdot \mathbf{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z},$$

$$\mathbf{rot V} = \triangledown \times \mathbf{V} = \left(\frac{\partial V_y}{\partial z} - \frac{\partial V_z}{\partial y}\right)\mathbf{e}_x + \left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial x}\right)\mathbf{e}_y + \left(\frac{\partial V_x}{\partial y} - \frac{\partial V_y}{\partial x}\right)\mathbf{e}_z.$$

We expect similar structures in semi-polar coordinates and spherical coordinates.

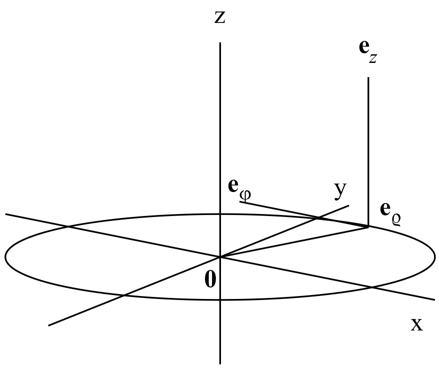

Figure 39.2: The orthonormal system $(\mathbf{e}_\varrho, \mathbf{e}_\varphi, \mathbf{e}_z)$ of unit vectors for semi-coordinates.

In *semi-polar coordinates* we also have an orthonormal system $(\mathbf{e}_\varrho, \mathbf{e}_\varphi, \mathbf{e}_z)$ of unit basis vectors, i.e. they are mutually perpendicular to each other, cf. Figure 39.2. Since $\|\mathbf{e}_\varphi\| = 1$, while the rectangular coordinates $x = \varrho \cos\varphi$ and $y = \varrho \sin\varphi$ includes a scaling with the factor ϱ, we may expect that ϱ enters the formulæ in semi-polar coordinates of the gradient, the divergence and the rotation. This is true, and it can be proved that we have

$$\mathbf{grad} f = \nabla f = \frac{\partial f}{\partial \varrho}\mathbf{e}_\varrho + \frac{1}{\varrho}\frac{\partial f}{\partial \varphi}\mathbf{e}_\varphi + \frac{\partial f}{\partial z}\mathbf{e}_z,$$

$$\operatorname{div}\,\mathbf{V} = \nabla\cdot\mathbf{V} = \frac{\partial V_r}{\partial \varrho} + \frac{V_\varrho}{\varrho} + \frac{1}{\varrho}\frac{\partial V_\varphi}{\partial \varphi} + \frac{\partial V_z}{\partial z},$$

$$\mathbf{rot}\mathbf{V} = \nabla\times\mathbf{V} = \left(\frac{1}{\varrho}\frac{\partial V_z}{\partial \varphi} - \frac{\partial V_\varphi}{\partial z}\right)\mathbf{e}_\varrho + \left(\frac{\partial V_\varrho}{\partial z} - \frac{\partial V_z}{\partial \varrho}\right)\mathbf{e}_\varphi + \left(\frac{\partial V_\varphi}{\partial \varrho} + \frac{V_\varphi}{\varrho} - \frac{1}{\varrho}\frac{\partial V_\varrho}{\partial \varphi}\right)\mathbf{e}_z.$$

Finally, we also have an orthonormal system $(\mathbf{r}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi)$ in spherical coordinates. Here, $\mathbf{e}_r$ at a point (x, y, z) is pointing in the direction specified, when only r varies, while θ and φ are kept fixed, and similarly for $\mathbf{e}_\theta$ and $\mathbf{r}_\varphi$. The formulæ are

$$\mathbf{grad} f = \nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\mathbf{e}_\varphi,$$

$$\operatorname{div}\,\mathbf{V} = \nabla\cdot\mathbf{V} = \frac{\partial V_\varphi}{\partial r} + \frac{2V_r}{r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_\theta\cot\theta}{r} + \frac{1}{r\,\sin\theta}\frac{\partial V_\varphi}{\partial \varphi},$$

$$\begin{aligned}\mathbf{rot}\mathbf{V} = \nabla\times\mathbf{V} \;\; = \;\; & \frac{1}{r}\left(\frac{\partial V_\varphi}{\partial \theta} + V_\varphi\cot\theta - \frac{1}{\sin\theta}\frac{\partial V_\theta}{\partial \varphi}\right)\mathbf{e}_r + \left(\frac{1}{r\sin\theta}\frac{\partial V_r}{\partial \varphi} - \frac{\partial V_\varphi}{\partial r} - \frac{V_\varphi}{r}\right)\mathbf{e}_\theta \\ & + \left(\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta}\right)\mathbf{e}_\varphi.\end{aligned}$$

It should be added that for a given function f the gradient ∇f is a geometrical object and as such independent of the chosen coordinate system. It is the *coordinates* which changes, when we choose another orthonormal basis.

39.7 Examples of applications of Green's identities

Example 39.1 *Consider a bounded domain $\Omega \subset \mathbb{R}^3$ with its boundary consisting of $m+1$ disjoint surfaces $\mathcal{F}_0$, $\mathcal{F}_1$, ..., $\mathcal{F}_m$, such that $\mathcal{F}_0$ surrounds all the others.*
We shall find a function w, which in Ω^0 fulfils Poisson's equation

$$\bigtriangledown^2 w = p,$$

and which has constant values on the surfaces $\mathcal{F}_0$, $\mathcal{F}_1$, ..., $\mathcal{F}_m$. Let Φ_i denote the flux of $\bigtriangledown w$ through $\mathcal{F}_i$, i.e.

$$\Phi_i = \int_{\mathcal{F}_i} \frac{\partial w}{\partial n}\,\mathrm{d}S.$$

1. *Let the function p be given, and assume that w is zero on $\mathcal{F}_0$, and for each $i \in \{1,\dots,m\}$ either Φ_i or the value of w is given.*

Show that w is uniquely determined.

Let Ω be an unbounded domain with its boundary consisting of m disjoint and bounded surfaces $\mathcal{F}_1$, ..., $\mathcal{F}_m$. Then the uniqueness theorem proved above also holds when the condition on $\mathcal{F}_0$ is replaced by the following:

There exist positive constants C_1, C_2, such that

$$\|\mathbf{x}\|\,|w(\mathbf{x})| \le C_1 \quad \textit{and} \quad \|\mathbf{x}\|^2 \|\bigtriangledown w(\mathbf{x})\| \le C_2 \qquad \textit{for all } \mathbf{x}\in\Omega.$$

2. *Prove this by considering $\Omega(R) = \Omega \cap \overline{K}(\mathbf{0};R)$ and then let R tend to plus infinity.*

A Uniqueness theorem for a mixed Dirichlet/Neumann problem for Poisson's equation.

D Assume that w and $\tilde{w}$ are solutions. Put $f = v - \tilde{w}$ and apply Green's first identity by choosing $g = f$ and applying that $f(\mathbf{x}) = 0$ on every $\mathcal{F}_i$.

In 2) we estimate the integrand in $\int_{\partial\Omega(R)} f\,\dfrac{\partial f}{\partial n}\,\mathrm{d}S$.

Remark 1. The example is dealing with a *uniqueness theorem* within a smaller class of functions than the mathematically most natural class. Therefore, one cannot expect that there actually exists a solution within this class. The problem is that the Neumann problem in some cases is difficult to treat. However, we can succeed if we have a boundary surface $\mathcal{F}_i$ with a Dirichlet condition instead, i.e. $f(\mathbf{x}) = \alpha_i$ on $\mathcal{F}_i$. The situation is worse if we are given the flux Φ_i on $\mathcal{F}_i$, because then we cannot in general conclude that $f(\mathbf{x})$ is equal to some (unknown) constant on $\mathcal{F}_i$. This is in general not the case, so we shall usually only expect to be able to show the uniqueness and not the existence of a solution within the given class of functions. ◊

I 1) First give the problem a mathematical description:

$$(39.1)\quad \begin{cases} \bigtriangledown^2 w = p, & \text{in } \Omega^0 \qquad\qquad \text{Poisson's equation} \\[2mm] \left.\begin{cases} w(\mathbf{x}) = 0, \ \text{in } \mathcal{F}_0 & i = 0 \\ w(\mathbf{x}) = \alpha_i, \ \text{in } \mathcal{F}_i & i \in \{i_1,\dots,i_k\} \end{cases}\right\} & \text{Dirichlet conditions} \\[2mm] \begin{cases} \int_{\mathcal{F}_i} \dfrac{\partial w}{\partial n}\,\mathrm{d}S = \Phi_i, & i \notin \{i_1,\dots,i_k\} \qquad \text{Neumann conditions} \\ w(\mathbf{x}) = \alpha_i \text{ in } \mathcal{F}_i & i \notin \{i_1,\dots,i_k\} \qquad \text{additional condition.} \end{cases} \end{cases}$$

It will be convenient to put

$$A=\bigcup_{j=1}^{k}\mathcal{F}_{i_j}\cup\mathcal{F}_0 \quad\text{and}\quad B=\partial\Omega\setminus A,\quad \text{hence } \partial\Omega=A\cup B,\ \text{disjunkt.}$$

Assume that w and $\tilde{w}$ are solutions of (39.1). By putting $f=w-\tilde{w}$ it follows by the linearity and the additional condition that f satisfies

$$(39.2)\quad \begin{cases} \bigtriangledown^2 f=0, & \\ f(\mathbf{x})=0 & \text{on } A\cup B=\partial\Omega, \\ \int_{\mathcal{F}_i}\dfrac{\partial f}{\partial n}\,\mathrm{d}S=0 & i\notin\{i_1,\dots,i_k\}. \end{cases}$$

Choose $g=f$ in Green's first identity. Then

$$\int_\Omega\{f\bigtriangledown^2 f+\bigtriangledown f\cdot\bigtriangledown f\}\,\mathrm{d}\Omega=\int_\Omega\{0+\|\bigtriangledown f\|^2\}\,\mathrm{d}\Omega=\int_{\partial\Omega}f\,\frac{\partial f}{\partial n}\,\mathrm{d}S=0,$$

because $f(\mathbf{x})=0$ on $\partial\Omega$, hence

$$\int_\Omega\|\bigtriangledown f\|^2\,\mathrm{d}\Omega=0.$$

Since $\| \bigtriangledown f\|^2$ is continuous and nonnegative, we must have $\bigtriangledown f = \mathbf{0}$, and we conclude that f is a constant. Now, f is continuous and zero on the boundary, so f must be identical zero, thus $w = \tilde{w}$, and we have proved the uniqueness in the bounded case.

REMARK 2. Note that the flux Φ_i through some of the surfaces $\mathcal{F}_i$ does not enter the argument at all, since we are only using the strong additional condition that $w(\mathbf{x}) = \alpha_i$ (the same though unknown constant) for $i \notin \{i_1, \dots, i_k\}$. Hence the problem is formally over-determined, since we do not apply all our information. (If this information is not in agreement with that we only get the zero solution, we clearly have a problem. This illustrates what is mathematically "wrong" with this example). $\Diamond$

2) Then consider the unbounded case with the additional growth conditions as a replacement of the missing surface $\mathcal{F}_0$.

When we take the intersection of Ω with the ball $\overline{K}(\mathbf{0};R)$, we get a bounded domain $\Omega(R)$. It is left to the reader to sketch the situation on a figure.

Then split the boundary of $\Omega(R)$ in the following way

$$A(R) = \overline{K}(\mathbf{0};R) \cap \bigcup_{j=1}^{k} \mathcal{F}_{i_j},$$

$$B(R) = \overline{K}(\mathbf{0};R) \cap \bigcup_{i \notin \{i_1,\dots,i_k\}} \mathcal{F}_i,$$

$$C(R) = \partial\overline{K}(\mathbf{0};R) \setminus \{A(R) \cup B(R)\},$$

where we have Dirichlet conditions on $\mathcal{F}_i$ for $i \in \{i_1, \dots, i_k\}$ and Neumann conditions on $\mathcal{F}_i$ for $i \notin \{i_1, \dots, i_k\}$. Apart from the fact that we do not know the behaviour on $C(R)$, the problem can with some modifications be written as in (39.1).

Let w and $\tilde{w}$ be solutions. We again put $f = w - \tilde{w}$. Then f satisfies (39.2) with the modifications that $A \cup B = \partial\Omega$ is replaced by $A(R) \cup B(R)\,[\subseteq \partial\Omega_R]$, and $\mathcal{F}_i$ is replaced by $\mathcal{F}_i \cap \overline{K}(\mathbf{0};R)$.

Choose as before $g = f$ in Green's first identity. Then

$$\int_{\Omega(R)} \| \bigtriangledown \|^2 \, \mathrm{d}\Omega = \int_{\partial\Omega(R)} f \frac{\partial f}{\partial n} \, \mathrm{d}S = \int_{A(R)} f \frac{\partial f}{\partial n} \, \mathrm{d}S + \int_{B(R)} f \frac{\partial f}{\partial b} \, \mathrm{d}S + \int_{C(R)} f \frac{\partial f}{\partial n} \, \mathrm{d}S.$$

Since f is zero on $A(R)$ and $B(R)$, this is reduced to

$$\int_{\Omega(R)} \| \bigtriangledown f\|^2 \, \mathrm{d}\Omega = \int_{C(R)} f \frac{\partial f}{\partial n} \, \mathrm{d}S,$$

which is not necessarily zero.

We note that according to the additional conditions it follows for $\mathbf{x} \in C(R)$ that

$$|f(\mathbf{x})| = |w(\mathbf{x}) - \tilde{w}(\mathbf{x})| \le |w(\mathbf{x})| + \|\tilde{w}(\mathbf{x})| \le \frac{2C_1}{\|\mathbf{x}\|} = \frac{2C_1}{R},$$

and

$$\left|\frac{\partial f}{\partial n}\right| \le \| \bigtriangledown f\| = \| \bigtriangledown w - \bigtriangledown \tilde{w}\| \le \| \bigtriangledown w\| + \| \bigtriangledown \tilde{w}\| \le \frac{2C_2}{R^2},$$

and we obtain the estimates

$$\begin{aligned}\left|\int_{C(R)} f\,\frac{\partial f}{\partial n}\,\mathrm{d}S\right| &\le \int_{C(R)} |f|\cdot\left|\frac{\partial f}{\partial n}\right|\mathrm{d}S \le \int_{C(R)} \frac{2C_1}{R}\cdot\frac{2C_2}{R^2}\,\mathrm{d}S \\ &= \frac{4C_1C_2}{R^3}\;\mathrm{area}(C(R)) \le \frac{4C_1C_2}{R^3}\;\mathrm{area}(\partial\overline{K}(\mathbf{0};R)) \\ &= \frac{4V_1C_2}{R^3}\cdot 4\pi R^2 = \frac{16\pi C_1C_2}{R} \to 0 \qquad \text{for } R\to+\infty,\end{aligned}$$

from which we conclude that

$$\int_{\Omega} \| \bigtriangledown f\|^2\,\mathrm{d}\Omega = \lim_{R\to\infty}\int_{\Omega(R)} \| \bigtriangledown f\|^2\,\mathrm{d}\Omega = \lim_{R\to+\infty}\int_{C(R)} f\,\frac{\partial f}{\partial n}\,\mathrm{d}S = 0.$$

Note that as $\| \bigtriangledown f\|^2 \ge 0$, we can take this limit to find the value of the improper integral

$$\int_{\Omega} \| \bigtriangledown f\|^2\,\mathrm{d}\Omega = 0.$$

Since $\| \bigtriangledown f\|^2 \ge 0$ is continuous we conclude as above that $\bigtriangledown f = 0$, i.e. f is a constant. Finally, it follows from the boundary value that $f(\mathbf{x}) = 0$ for $\mathbf{x} \in \Omega$, hence $w(\mathbf{x}) = \tilde{w}(\mathbf{x})$ in Ω, and we have proved the uniqueness.

Remark 3. As mentioned above this is not a proof of the existence. Consider as an extreme example the problem

$$\begin{cases} \bigtriangledown^2 w = p, & \text{Poisson equation} \\ \int_{\mathcal{F}} \frac{\partial w}{\partial n}\,\mathrm{d}S = \Phi & \text{Neumann problem on } \mathcal{F} \\ \|\mathbf{x}\|\cdot|w(\mathbf{x})| \le C_1 & \text{for } \mathbf{x}\in\Omega, \\ \|\mathbf{x}\|^{2+\varepsilon}\| \bigtriangledown w(\mathbf{x})\| \le C_2 & \text{for } \mathbf{x}\in\Omega \end{cases}$$

Apart from the fact that the exponent 2 has been changed to $2+\varepsilon$ of convergency reasons, this is a special case of 2) above.

When we integrate $\Omega(R)$ and choose $g = 1$ in Green's identity, we get that

$$\int_{\Omega(R)} \{1\cdot\bigtriangledown^2 w + \bigtriangledown 1\cdot\bigtriangledown f\}\,\mathrm{d}\Omega = \int_{\partial\Omega(R)} \frac{\partial w}{\partial n}\,\mathrm{d}S,$$

which is reduced to

$$\int_{\Omega(R)} p\,\mathrm{d}\Omega = \int_{C(R)} \frac{\partial w}{\partial n}\,\mathrm{d}S + \int_{\mathcal{F}\cap\Omega(R)} \frac{\partial w}{\partial n}\,\mathrm{d}S.$$

The former term on the right hand side is estimated by

$$\left|\int_{C(R)} \frac{\partial w}{\partial n}\,\mathrm{d}S\right| \le \int_{C(R)} \|\bigtriangledown w\|\,\mathrm{d}S \le \frac{C_2}{R^{2+\varepsilon}}\cdot 4\pi R^2 = \frac{4\pi C_2}{R^{\varepsilon}} \to 0 \qquad \text{for } R \to +\infty,$$

and the latter term clearly converges towards

$$\lim_{R\to+\infty} \int_{\mathcal{F}\cap\Omega(R)} \frac{\partial w}{\partial n}\,\mathrm{d}S = \int_{\mathcal{F}} \frac{\partial w}{\partial n}\,\mathrm{d}S = \Phi,$$

and we get the compatibility condition

$$\int_{\Omega} p\,\mathrm{d}\Omega = \Phi,$$

proving that p and Φ are not independent of each other.

Note that if we also have a Dirichlet condition and the improper integral $\int_{\Omega} p\,\mathrm{d}\Omega$ is convergent, then the unknown flux through the Dirichlet boundary forces that the compatibility condition is fulfilled. $\Diamond$

Remark 4. The example has been formulated from a physical point of view. In general, the corresponding mathematical problem in the bounded case is described as follows:

$$\begin{cases} \bigtriangledown^2 w = p, & \text{in } \Omega^0, & \text{Poisson,} \\ w(\mathbf{x}) = 0, & \text{in } \mathcal{F}_0, & \text{Dirichlet,} \\ w(\mathbf{x}) = \alpha_i, & \text{in } \mathcal{F}_i \text{ for } i \in \{i_1,\dots,i_k\}, & \text{Dirichlet,} \\ \dfrac{\partial w}{\partial n} = h_i(\mathbf{x}), & \text{in } \mathcal{F}_i \text{ for } i \notin \{i_1,\dots,i_k\}, & \text{Neumann,} \end{cases}$$

or similarly $\left(\text{e.g. } \dfrac{\partial w}{\partial n} = h_0(\mathbf{x}) \text{ in } \mathcal{F}_0\right)$.

If the boundary conditions are only of Neumann type, we must add a compatibility condition:

$$\int_{\Omega} p\,\mathrm{d}\Omega = \int_{\partial\Omega} h\,\mathrm{d}S,$$

where $h(\mathbf{x}) = h_i(\mathbf{x})$ on $\mathcal{F}_i$.

Note that we do *not* assume that w is constant on the Neumann boundaries.

Assuming that p and h are nice functions we can prove that we have both an existence and a uniqueness theorem for the problem. For the pure Dirichlet problem the proof is classical known. However, if just one Neumann boundary occurs, the proof becomes very difficult. One shall e.g. apply *Hopf's maximum principle:* In a connected domain Ω a non-constant harmonic function w only attains its maximum values (if they exist) on the boundary $\partial\Omega$, and we have at such a maximum point

$$\frac{\partial w}{\partial n} > 0. \qquad \Diamond$$

Example 39.2 *Let Ω be a domain in the space for a given non-constant function*

$g : \Omega \to [0, +\infty[$.

We shall find a function w, which satisfies

$$(39.3) \quad \nabla^2 w + \lambda\, g\, w = 0 \quad \text{on } \Omega^\circ, \qquad w = 0 \quad \text{on } \partial\Omega,$$

where λ is a constant. It can be proved that a nontrivial solution w in general only exists for some values of λ, the so-called eigenvalues.

1. *Show by applying Green's first identity that the eigenvalues are positive.*

Assume that w and W are solutions of (39.2) for different *eigenvalues, such that*

$$\left.\begin{array}{l} \nabla^2 w + \lambda\, g\, w = 0 \\ \nabla^2 W + \Lambda\, g\, W = 0 \end{array}\right\} \quad \text{on } \Omega^\circ, \qquad \left.\begin{array}{r} w = 0 \\ W = 0 \end{array}\right\} \quad \text{on } \partial\Omega, \qquad \lambda \neq \Lambda.$$

2. *Show by applying Green's second identity that*

$$\int_\Omega g(\mathbf{x}\, w(\mathbf{x})\, W(\mathbf{x})\, \mathrm{d}\Omega = 0.$$

We say that the functions w and W are orthogonal, *and g is called a* weight function.

A Eigenvalue problem; Green's first and second formulæ.

D Follow the guidelines.

I 1) Choose $g = g = w$ in Green's first formula. Then $w = 0$ on $\partial\Omega$ and

$$(39.4) \quad \int_\Omega \left\{ w\, \nabla^2\, w + \| \nabla f \|^2 \right\} \mathrm{d}\Omega = \int_{\partial\Omega} w\, \frac{\partial w}{\partial n}\, \mathrm{d}S = 0.$$

We have by (39.3 that $\nabla^2 w = -\lambda\, g\, w$, hence by a rearrangement of (39.4),

$$\int_\Omega \| \nabla w \|^2\, \mathrm{d}\Omega = -\int_\Omega w\, \nabla^2\, w\, \mathrm{d}\Omega = +\lambda \int_\Omega g \cdot w^2\, \mathrm{d}\Omega.$$

since w is a non-trivial solution, we must have that $\nabla w \neq 0$ (w is not a constant), and

$$\int_\Omega \| \nabla w \|^2\, \mathrm{d}\Omega > 0 \quad \text{and} \quad \int_\Omega g \cdot w^2\, \mathrm{d}\Omega > 0,$$

hence

$$\lambda = \frac{\int_\Omega \| \nabla w \|^2\, \mathrm{d}\Omega}{\int_\Omega g \cdot w^2\, \mathrm{d}\Omega}$$

is defined and positive. It follows that every eigenvalue λ is positive.

2) Let w and W be non-trivial solutions for different eigenvalues λ and Λ. Then apply Green's second identity, using that w and W are zero on $\partial\Omega$,

$$\int_\Omega \{w \bigtriangledown^2 W - W \bigtriangledown^2 w\} \, d\Omega = \int_{\partial\Omega} \left\{ w \frac{\partial W}{\partial n} - W \frac{\partial w}{\partial n} \right\} dS = 0.$$

Hence,

$$\int_\Omega w \bigtriangledown^2 W \, d\Omega - \int_\Omega W \bigtriangledown^2 w, \, d\Omega = 0.$$

Since

$$\bigtriangledown^2 W = -\Lambda \, g \, W \qquad \text{and} \qquad \bigtriangledown^2 w = -\lambda \, g \, w,$$

it follows by insertion that

$$0 = -\int_\Omega w \, \Lambda \, g \, W \, d\Omega + \int_\Omega W \, \lambda \, g \, w \, d\Omega = (\lambda - \Lambda) \int_\Omega g \, w \, W \, d\Omega.$$

As $\lambda \neq \Lambda$, this implies

$$\int_\Omega g(\mathbf{x}) \, w(\mathbf{x}) \, W(\mathbf{x}) \, d\Omega = 0$$

as requested.

Example 39.3 *Let m be a constant. consider a function $f : \mathbb{R}^3 \to \mathbb{R}$, which satisfies*

$$f(\tau\,\mathbf{x}) = \tau^m\, f(\mathbf{x})$$

for every $\mathbf{x}$ and for those values of τ, for which τ^m is defined. We say that such a function is homogeneous *of degree m.*

1) Show that if f is also differentiable, then

$$\mathbf{x} \cdot \bigtriangledown f(\mathbf{x}) = m\, f(\mathbf{x}).$$

2) Show that if f furthermore is harmonic, then

$$\int_{K(\mathbf{x};a)} \| \bigtriangledown f\|^2 \,\mathrm{d}\Omega = \frac{m}{a} \int_{\partial \overline{K}(\mathbf{0};a)} f^2 \,\mathrm{d}S.$$

A Homogeneous functions of degree m.

D The first question follows by differentiation of the definition with respect to τ. In the second question we apply Green's first identity.

I 1) When we differentiate $f(\tau\,\mathbf{x}) = \tau^m\, f(\mathbf{x})$ with respect to τ, we get

$$m\,\tau^{m-1}\, f(\mathbf{x}) = \frac{\mathrm{d}}{\mathrm{d}\tau}\, f(\tau\,\mathbf{x}) = \mathbf{x} \cdot \bigtriangledown f(\tau\,\mathbf{x}).$$

Put $\tau = 1$. Then

$$\mathbf{x} \cdot \bigtriangledown f(\mathbf{x}) = m\, f(\mathbf{x}).$$

2) By Green's first identity,

$$\int_\Omega (g \bigtriangledown^2 f + \bigtriangledown g \cdot \bigtriangledown f)\,\mathrm{d}\Omega = \int_{\partial\Omega} g\, \frac{\partial f}{\partial n}\,\mathrm{d}S.$$

Choose $\Omega = K(\mathbf{0};a)$ and $g = f$. Since f is harmonic, $\bigtriangledown^2 f = 0$, it follows that

$$\int_{K(\mathbf{0};a)} \| \bigtriangledown f\|^2 \,\mathrm{d}\Omega = \int_{\partial \overline{K}(\mathbf{0};a)} f\, \frac{\partial f}{\partial n}\,\mathrm{d}S.$$

We have on the sphere that $\mathbf{x} = a\,\mathbf{n}$, hence by 1),

$$\frac{\partial f}{\partial n} = \mathbf{n} \cdot \bigtriangledown f(\mathbf{x}) = \frac{1}{a}\, \mathbf{x} \cdot \bigtriangledown f(\mathbf{x}) = \frac{m}{a}\, f(\mathbf{x}).$$

$$\int_{K(\mathbf{0};a)} \| \bigtriangledown f\|^2 \,\mathrm{d}\Omega = \frac{m}{a} \int_{\partial \overline{K}(\mathbf{0};a)} f^2 \,\mathrm{d}S.$$

Remark. We strongly exploit that Ω is a ball of centrum $\mathbf{0}$. $\Diamond$

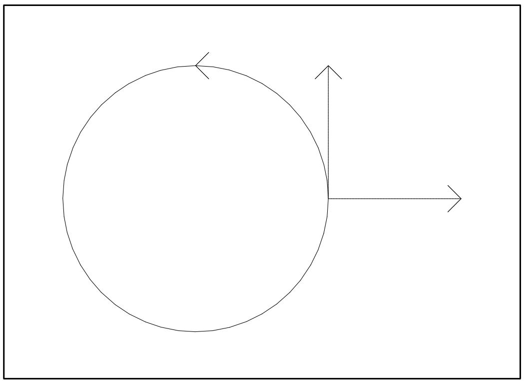

Figure 39.3: The unit circle ∂E and its unit tangent $\mathbf{t}$ and unit normal $\mathbf{n}$.

39.8 Overview of Green's theorems in the plane

For completeness we here include a small section, where Gauß's and Stokes's theorems are "translated" to the 2-dimensional case. In this case they are called *Green's theorems in the plane*. They are applicable at the most strange places in the technical sciences, so they ought to be more emphasized in the textbooks.
Conventions for Green's theorems in the plane:

Let ∂E be a closed and plane curve without double points. The *orbital direction* for ∂E is always chosen such that the domain E to the *left* of ∂E, is *bounded*.

The *direction of the tangent* is determined by the orbital direction, so $\mathbf{t}$ is unique. The *normal* is always *directed away* from E, so $\mathbf{n}$ is also uniquely determined.

Theorem 39.6 Green's theorems in the plane:

1) Gauß-like:

$$\oint_{\partial E} (V_x n_x + V_y n_y)\,\mathrm{d}s = \int_E \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} \right) \mathrm{d}S,$$

2) Stokes-like:

$$\oint_{\partial E} (V_x t_x + V_y t_y)\,\mathrm{d}s = \int_E \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) \mathrm{d}S.$$

Note that

$$V_x n_x + V_y n_y = \mathbf{V}\cdot\mathbf{n} \quad \text{and} \quad V_x t_x + V_y t_y = \mathbf{V}\cdot\mathbf{t}.$$

39.9 Miscellaneous examples

We collect in this section some examples which are relevant and yet could not be included in the previous chapters.

Example 39.4 *The plane domain on the figure S is the union of three rectangles, and it is symmetric with respect to the y-axis.*

1. *Find the barycentre for each of the three rectangles.*

2. *Find the barycentre B for S.*

The dotted line is indicating the y-axis, and the x-axis is put through B.

3. *Compute the* axial moment

$$I_a = \int_S y^2 \, dS.$$

4. *Compute I_a and the area of S for $a = \sqrt{3}$ cm. The moment is given in four decimals.*

A Barycentre and axial moment.

D Find the barycentre and compute the plane integral.

I Assume that S is covered homogeneously. Choose the y-axis as the axis of symmetry, and the lower edge of the figure as the x-axis. Then all three barycentres lie on the y-axis.

1) Put $S = S_1 \cup S_2 \cup S_3$, where S_1 is the upper, S_2 the middle and S_3 the lower rectangle. Clearly, of symmetric reasons,

$$\begin{aligned} y_1 &= a + 5a + \frac{1}{2} \cdot 2a = 7a, && \text{area}(S_1) = 8a^2, \\ y_2 &= a + \frac{1}{2} \cdot 5a = \frac{7}{2}\, a, && \text{area}(S_2) = 5a^2, \\ y_3 &= \frac{1}{2}\, a, && \text{area}(S_3) = 7a^2, \end{aligned}$$

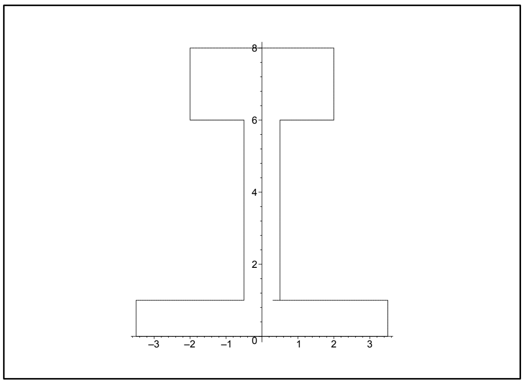

Figure 39.4: The domain S where the dotted line is replaced by the y-axis and where the lower rectangle has the dimensions 7×1, the rectangle in the middle has the dimensions 1×5 and the upper rectangle has the dimensions 4×2. We have as usual put $a = 1$.

where y_i denotes the ordinate of the corresponding barycentre. We see in particular that

$$\text{area}(S) = (8+5+7)a^2 = 20a^2.$$

2) Let y denote the ordinate of B. Then

$$y \cdot \text{area}(S) = y_1 \cdot \text{area}(S_1) + y_2 \cdot \text{area}(S_2) + y_3 \cdot \text{area}(S_3),$$

hence

$$y = \frac{a}{20}\left(7\cdot 8 + \frac{7}{2}\cdot 5 + \frac{1}{2}\cdot 7\right) = \frac{7a}{20}\left(8 + \frac{5}{2} + \frac{1}{2}\right) = \frac{77}{20}\,a.$$

3) Then put the x-axis through B. We get

$$S_1 = [-2a, 2a] \times \left[6a - \frac{77}{20}\,a, 8a - \frac{77}{20}\,a\right] = [-2a, 2a] \times \left[\frac{43}{20}\,a, \frac{83}{20}\,a\right],$$

$$S_2 = \left[-\frac{1}{2}\,a, \frac{1}{2}\,a\right] \times \left[a - \frac{77}{20}\,a, 6a - \frac{77}{20}\,a\right] = \left[-\frac{1}{2}\,a, \frac{1}{2}\,a\right] \times \left[-\frac{57}{20}\,a, \frac{43}{20}\,a\right],$$

$$S_3 = \left[-\frac{7}{2}\,a, \frac{7}{2}\,a\right] \times \left[-\frac{77}{20}\,a, -\frac{57}{20}\,a\right],$$

and the axial moment becomes

$$\begin{aligned}
I_a &= \int_S y^2\,\mathrm{d}S = \int_{S_1} y^2\,\mathrm{d}S + \int_{S_2} y^2\,\mathrm{d}S + \int_{S_3} y^3\,\mathrm{d}S \\
&= 4a\left[\frac{y^3}{3}\right]_{\frac{43}{20}\,a}^{\frac{83}{20}\,a} + a\left[\frac{y^3}{3}\right]_{-\frac{57}{20}\,a}^{\frac{43}{20}\,a} + 7a\left[\frac{y^3}{3}\right]_{-\frac{77}{20}\,a}^{-\frac{57}{20}\,a} \\
&= \frac{a^4}{3}\left\{4\left(\frac{83}{20}\right)^3 - 4\left(\frac{43}{20}\right)^3 + \left(\frac{43}{20}\right)^3 + \left(\frac{57}{20}\right)^3 - 7\left(\frac{57}{20}\right)^3 + 7\left(\frac{77}{20}\right)^3\right\} \\
&= \frac{a^4}{3\cdot 20^3}\left\{4\cdot 83^3 - 4\cdot 43^3 + 43^3 + 57^3 - 7\cdot 57^3 + 7\cdot 77^3\right\} \\
&= \frac{a^4}{24000}\left\{4\cdot 83^3 + 7\cdot 77^3 - 3\cdot 43^3 - 6\cdot 57^3\right\} = \frac{4133200}{24000}\,a^4 = \frac{10333}{60}\,a^4.
\end{aligned}$$

4) We get for $a = \sqrt{3}$ cm,

$$\text{area}(S) = 20\cdot 3 = 60\ \text{cm}^2,$$

and

$$I_a = \frac{10333}{60}\cdot 9 = \frac{30999}{20} \approx 1550\ \text{cm}^4.$$

Example 39.5 *Consider for every $a \in \mathbb{R}_+$ the set*

$$L_a = \{(x,y,z) \mid x^2+y^2 \le az \le a^2\}.$$

1. *Find the volume of L_a.*

2. *Compute the space integral $\int_{L_a}\left(x^2+y^2+z^2\right)\, d\Omega$.*

Let the vector field $\mathbf{V}:\mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$\mathbf{V}(x,y,z) = \left(y^2x, z^2y, x^2z\right).$$

3. *Find the flux of $\mathbf{V}$ through the boundary ∂L_a.*

Let $\mathcal{F}_a$ denote that part of ∂L_a, which is given by $x^2+y^2 = az$ and $z \le a$, and let $\mathbf{n}$ denote the outward unit normal vector field of the surface $\mathcal{F}_a$.

4. *Find the flux*

$$\int_{\mathcal{F}_a} \mathbf{n}\cdot \mathbf{rot}\ \mathbf{V}\,\mathrm{d}S.$$

Furthermore, let the vector field $\mathbf{W}:\mathbb{R}^3\to\mathbb{R}^3$ be given by

$$\mathbf{W}(x,y,z) = \left(xz^2, yx^2, zy^2\right),$$

and put $\mathbf{U} = \mathbf{V}+\mathbf{W}$.

5. *Show that the vector field $\mathbf{U}$ is a gradient field and find all its integrals.*

A Volume; mass; flux; gradient field.

D Sketch L_a. Then follow the guidelines. Apply Gauß's theorem, and possibly also Stokes's theorem. Finally, show that $\mathbf{U}\cdot d\mathbf{x}$ is a total differential.

I 1) The set L_a is intersected at the height $z \in [0,a]$ in a disc of area $\pi(x^2+y^2) = \pi a z$, so we get by the slicing method that

$$\mathrm{vol}(L_a) = \int_0^a a\pi z\,\mathrm{d}z = \frac{\pi}{2}\,a^3.$$

2) Put $B_a = \{(x,y) \mid x^2+y^2 \le a^2\}$. Then we get the integral

$$\begin{aligned}
\int_{L_a}(x^2+y^2+z^2)\,\mathrm{d}\Omega &= \int_{B_a}\left\{\int_{\frac{x^2+y^2}{a}}^a (x^2+y^2+z^2)\,\mathrm{d}z\right\}\mathrm{d}x\,\mathrm{d}y\\
&= \int_{B_a}\left[(x^2+y^2)z+\frac{1}{3}\,z^3\right]_{z=(x^2+y^2)/a}^a \mathrm{d}x\,\mathrm{d}y\\
&= \int_{B_a}\left\{a(x^2+y^2)+\frac{1}{3}\,a^3-\frac{1}{a}\,(x^2+y^2)^2-\frac{1}{3a^3}\,(x^2+y^2)^3\right\}\mathrm{d}x\,\mathrm{d}y\\
&= 2\pi\int_0^a\left\{a\varrho^2+\frac{1}{3}\,a^3-\frac{1}{a}\,\varrho^4-\frac{1}{3a^3}\,\varrho^6\right\}\varrho\,\mathrm{d}\varrho\\
&= 2\pi\int_0^a\left\{\frac{1}{3}\,a^3\varrho+a\,\varrho^3-\frac{1}{a}\,\varrho^5-\frac{1}{3a^3}\,\varrho^7\right\}\mathrm{d}\varrho\\
&= 2\pi\left\{\frac{1}{6}\,a^5+\frac{1}{4}\,a^5-\frac{1}{6}\,a^5-\frac{1}{24}\,a^5\right\} = 2\pi\cdot\frac{5}{24}\,a^5 = \frac{5\pi}{12}\,a^5.
\end{aligned}$$

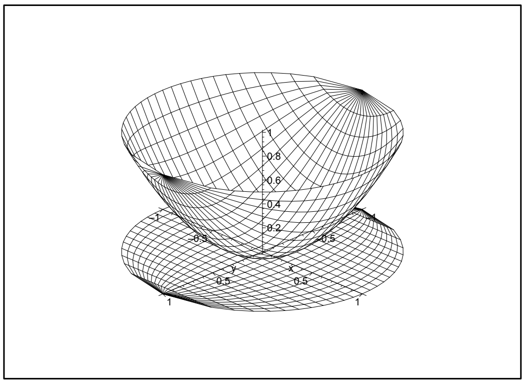

Figure 39.5: The body L_a and its projection onto the (x, y)-plane for $a = 1$.

3) Now, div $\mathbf{V} = y^2 + z^2 + x^2$, so by Gauß's theorem and 2) the flux becomes

$$\text{flux}\,(\partial L_a) = \int_{L_a} \text{div}\; \mathbf{V}\, \mathrm{d}\Omega = \int_{L_a} (x^2 + y^2 + z^2)\, \mathrm{d}\Omega = \frac{5\pi}{12}\, a^5.$$

4) By Stokes's theorem we get

$$\int_{\mathcal{F}_a} (\mathbf{rot} \mathbf{V}) \cdot \mathbf{n}\, \mathrm{d}S = \int_{\partial \mathcal{F}_a} \mathbf{V} \cdot \mathbf{t}\, \mathrm{d}s,$$

where (cf. the figure)

$$\partial \mathcal{F}_a = \{(x, y, a) \mid x^2 + y^2 = a^2\} = \{(a \cos t, a \sin t, a) \mid t \in [0, 2\pi]\}.$$

Hence along $\partial \mathcal{F}_a$,

$$\mathbf{V}(t) = \left(a^3 \cos t\, \sin^2 t, a^3 \sin t, a^3 \cos^2 t\right), \quad t \in [0, 2\pi].$$

When we consult the figure we see that the orientation is pointing in the wrong direction, so in order to obtain an outward normal we must multiply by a factor -1:

$$\int_{\mathcal{F}_a} (\mathbf{rot}\;\, \mathbf{V}) \cdot \mathbf{n}\, \mathrm{d}S = - \int_{\partial \mathcal{F}_a} \mathbf{V} \cdot \mathbf{t}\, \mathrm{d}s = - \int_0^{2\pi} \mathbf{V} \cdot (-a \sin t, a \cos t, 0)\, \mathrm{d}t$$

$$= \int_0^{2\pi} \left\{ +a^4 \cos t \cdot \sin^3 t - a^4 \sin t \cdot \cos t + 0 \right\}\, \mathrm{d}t = \left[\frac{a^4}{4}\, \sin^4 t - \frac{a^4}{2}\, \sin^2 t \right]_0^{2\pi} = 0,$$

which shows that there has been no need to consider whether the orientation was correct.

Alternatively it follows by a straightforward calculation that

$$\mathbf{rot} \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 & z^2 y & x^2 z \end{vmatrix} = (-2yz, -2xz, -2xy).$$

By using the parametric description $(x,y,z)=\left(u,v,\dfrac{u^2+v^2}{a}\right)$ of the surface we get

$$\frac{\partial \mathbf{r}}{\partial u}=\left(1,0,\frac{2u}{a}\right),\qquad \frac{\partial \mathbf{r}}{\partial v}=\left(0,1,\frac{2v}{a}\right),$$

hence

$$\mathbf{N}_1(u,v)=\left|\begin{array}{ccc} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 1 & 0 & \dfrac{2u}{a} \\ 0 & 1 & \dfrac{2v}{a}\end{array}\right| = \left(-\frac{2u}{a},-\frac{2v}{a},1\right).$$

Since $\mathbf{n}$ is the outward normal field of $\mathcal{F}_a$, it follows by inspection of the figure that the z-coordinate of $\mathbf{n}$ must be negative. We therefore choose

$$\mathbf{N}(u,v)=-\mathbf{N}_1(u,v)=\frac{1}{a}\,(2u,2v,-a).$$

Finally, by using a symmetric argument in the computation of the integrals,

$$\begin{aligned}\text{flux}(\mathcal{F}_a) &= \int_{B_a}\mathbf{rot}\mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S = -\frac{1}{a^2}\int_{B_a}\left(2v(u^2+v^2),2u(u^2+v^2),2auv\right)\cdot(-2u,-2v,a)\,\mathrm{d}u\,\mathrm{d}v \\ &= \frac{1}{a^2}\int_{B_a}\left\{4uv(u^2+v^2)+4uv(u^2+v^2)-2a^2uv\right\}\,\mathrm{d}u\,\mathrm{d}v=0.\end{aligned}$$

5) First compute the sum

$$\mathbf{U}=\mathbf{V}+\mathbf{W}=\left(y^2x,z^2y,x^2z\right)+\left(xz^2,yx^2,zy^2\right)=\left(x(y^2+z^2),y(x^2+z^2),z(x^2+y^2)\right).$$

This implies

$$\begin{aligned}\mathbf{U}\cdot\mathrm{d}\mathbf{x} &= x(y^2+z^2)\,\mathrm{d}x+y(x^2+z^2)\,\mathrm{d}y+z(x^2+y^2)\,\mathrm{d}z \\ &= \frac{1}{2}\left\{(y^2+z^2)\,\mathrm{d}\left(x^2\right)+(x^2+z^2)\,\mathrm{d}\left(y^2\right)+(x^2+y^2)\,\mathrm{d}\left(z^2\right)\right\} \\ &= \mathrm{d}\left\{\frac{1}{2}\left(x^2y^2+x^2z^2+y^2z^2\right)\right\},\end{aligned}$$

and we conclude that $\mathbf{U}$ is a gradient field with its integrals given by

$$F(x,y,z)=\frac{1}{2}\left(x^2y^2+y^2z^2+z^2x^2\right)+C,\qquad C\in\mathbb{R}.$$

Example 39.6 *Consider the vector field*

$$\mathbf{V}(x,y) = \left(\frac{xy^2(1+xy)}{1+x^2y^2}, \frac{x^2y(1+xy)}{1+x^2y^2} \right), \qquad (x,y) \in \mathbb{R}^2.$$

1) Show that $\mathbf{V}$ *is a gradient field and find its integral* $F : \mathbb{R}^2 \to \mathbb{R}$*, which is* 0 *at the point* $(0,0)$*.*

2) Write F *as a composite function:*

$$F(x,y) = f(u), \qquad u = g(x,y).$$

3) Find the maximum and the minimum of F *on the set*

$$A = \{(x,y) \mid |x| + |y| \le 2\}$$

by e.g. finding the range $g(A)$ *by a geometric consideration.*

A Gradient field, integral. Maximum and minimum.

D First find an integral. This is here done in three different ways.

I 1) **First variant.** We get by a small manipulation,

$$\begin{aligned}\omega &= \mathbf{V}\cdot \mathrm{d}\mathbf{x} = \frac{xy^2(1+xy)}{1+x^2y^2}\,\mathrm{d}x + \frac{x^2y(1+xy)}{1+x^2y^2}\,\mathrm{d}y \\ &= \frac{xy(1+xy)}{1+x^2y^2}\,(y\,\mathrm{d}x + x\,\mathrm{d}y) = \frac{1+x^2y^2+xy-1}{1+x^2+y^2}\,\mathrm{d}(xy) \\ &= \left(1+\frac{xy}{1+(xy)^2} - \frac{1}{1+(xy)^2}\right)\mathrm{d}(xy) \\ &= \mathrm{d}\left\{xy + \frac{1}{2}\ln\left(1+x^2y^2\right) - \operatorname{Arctan}(xy)\right\},\end{aligned}$$

proving that $\mathbf{V}$ is a gradient field with the integrals

$$F_C(x,y) = xy + \frac{1}{2}\ln\left(1+x^2y^2\right) - \operatorname{Arctan}(xy) + C, \qquad C\in\mathbb{R}.$$

That particular integral which is 0 at $(0,0)$, corresponds to $C=0$, thus

$$F(x,y) = xy + \frac{1}{2}\ln\left(1+x^2y^2\right) - \operatorname{Arctan}(xy).$$

Second variant. When be integrate along a broken line from $(0,0)$, we get

$$\begin{aligned}F(x,y) &= \int_0^x 0\,\mathrm{d}t + \int_0^y \frac{x^2t(1+xt)}{1+x^2t^2}\,\mathrm{d}t = \int_0^{xy}\frac{u(1+u)}{1+u^2}\,\mathrm{d}u \\ &= \int_0^{xy}\frac{1+u^2+u-1}{1+u^2}\,\mathrm{d}u = \int_0^{xy}\left\{1+\frac{u}{1+u^2}-\frac{1}{1+u^2}\right\}\mathrm{d}u \\ &= xy + \frac{1}{2}\ln\left(1+x^2y^2\right) - \operatorname{Arctan}(xy).\end{aligned}$$

C We shall here test the candidate,

$$\begin{aligned}\mathrm{d}F &= \left(y+\frac{1}{2}\,\frac{2xy^2}{1+x^2y^2} - \frac{2xy^2}{1+x^2y^2} - \frac{y}{1+x^2y^2}\right)\mathrm{d}x \\ &\qquad + \left(x+\frac{1}{2}\,\frac{2x^2y}{1+x^2y^2} - \frac{x}{1+x^2y^2}\right)\mathrm{d}y \\ &= \frac{y(1+x^2y^2)+xy^2-y}{1+x^2y^2}\,\mathrm{d}x + \frac{x(1+x^2y^2)+x^2y-x}{1+x^2y^2}\,\mathrm{d}y \\ &= \frac{xy^2(1+xy)}{1+x^2y^2}\,\mathrm{d}x + \frac{x^2y(1+xy)}{1+x^2y^2}\,\mathrm{d}y = \mathbf{V}\cdot\,\mathrm{d}\mathbf{x}.\end{aligned}$$

Third variant. We get for y arbitrary,

$$\begin{aligned}F(x,y) &= \int \frac{xy^2(1+xy)}{1+x^2y^2}\,\mathrm{d}x = \int \frac{xy(1+xy)}{1+(xy)^2}\,\mathrm{d}(xy) = \cdots \\ &= xy + \frac{1}{2}\ln\left(1+x^2y^2\right) - \operatorname{Arctan}(xy),\end{aligned}$$

where the computations follow the same pattern as in the **Second variant**.

C Since $\mathrm{d}F = \omega = \mathbf{V}\cdot\,\mathrm{d}\mathbf{x}$, it follows that F is an integral, and since $F(0,0)=0$, the required integral is precisely

$$F(x,y) = xy + \frac{1}{2}\ln\left(1+x^2y^2\right) - \operatorname{Arctan}(xy).$$

2) If we put $u = g(x, y) = xy$, then

$$F(x, y) = f(u) = u + \frac{1}{2} \ln\left(1 + u^2\right) - \text{ Arctan } u.$$

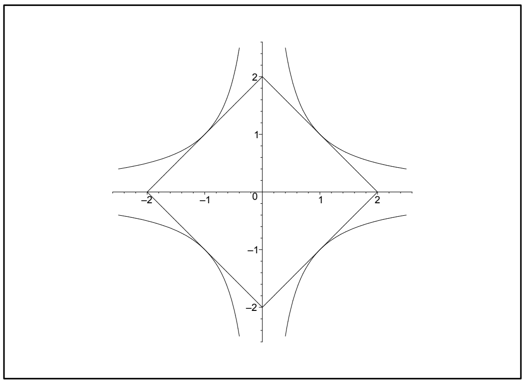

Figure 39.6: The domain A and the extremal curves $u = xy = \pm 1$.

3) Since $F(x, y)$ is of class C^∞, and A is closed and bounded, it follows from the second main theorem for continuous functions that F has both a maximum and a minimum on A.
It follows from

$$f(u) = u + \frac{1}{2} \ln\left(1 + u^2\right) - \text{ Arctan } u$$

that (cf. the integrand of the **Second variant**)

$$f'(u) = \frac{u(1 + u)}{1 + u^2}, \qquad u = xy,$$

which is zero for either $u = 0$ or $u = -1$. Furthermore, f is increasing for $u \in]-\infty, -1[$, decreasing for $u \in]-1, 0[$, and increasing for $u \in]0, +\infty[$.

For $u = xy = 0$ we get

$$F(x, y) = F(0, y) = F(x, 0) = 0.$$

For $u = xy = -1$ we get

$$f(-1) = -1 + \frac{1}{2} \ln 2 + \frac{\pi}{4} > 0.$$

In A these correspond to the points $(-1, 1)$ and $(1, -1)$.

Let $u = xy = +1$. This corresponds to the points $(1, 1)$ and $(-1, -1)$ in A. In this case we get the values

$$f(1) = 1 + \frac{1}{2} \ln 2 - \frac{\pi}{4} > f(-1).$$

We conclude from $u \in [-1,1]$ for $(x,y) \in A$ that the maximum is

$$f(1,1) = f(-1,1) = 1 - \frac{\pi}{4} + \frac{1}{2}\ln 2,$$

and the minimum is

$$f(x,0) = f(0,y) = 0.$$

ALTERNATIVELY we may find the possible stationary points follows by an examination of the boundary.

The possible stationary points satisfy $\mathbf{V}(x,y) = \mathbf{0}$, thus

$$\frac{xy(1+xy)}{1+x^2y^2}\,(y,x) = (0,0).$$

We thus get three possibilities:

$$x = 0, \qquad y = 0, \quad \text{or} \quad xy_{-}1.$$

In the *interior* of A we get

$$\{(x,0) \mid x \in]-2,2[\} \quad \text{and} \quad \{(0,y) \mid y \in]-2,2[\},$$

because the hyperbola $xy = -1$ only intersects A in the boundary points $(1,1)$ and $(-1,1)$ in ∂A.

The boundary is symmetric with respect to $(0,0)$. As $F(-x,-y) = F(x,y)$, it suffices to consider the following boundary curves

$$x + y = 2, \quad x \in [0,2], \quad \text{and} \quad x - y = 2, \quad x \in [0,2].$$

a) If $x + y = 2$, i.e. $y = -x + 2$, $x \in [0,2]$, we find the restriction

$$\begin{aligned} h_1(x) &= F(x, 2-x) \\ &= (2x - x^2) + \frac{1}{2}\ln\left(1 + \left(2x - x^2\right)^2\right) - \operatorname{Arctan}\left(2x - x^2\right) \end{aligned}$$

where

$$\begin{aligned} h_1'(x) &= 2 - 2x + \frac{1}{2}\cdot\frac{2(2x-x^2)\cdot(2-2x)}{1+(2x-x^2)^2} - \frac{2-2x}{1+(2x-x^2)^2} \\ &= \frac{2(1-x)}{1+(2x-x^2)^2}\cdot\left\{1 + (2x-x^2)^2 + (2x-x^2) - 1\right\} \\ &= \frac{2(1-x)}{1+(2x-x^2)^2}\cdot x(2-x)\{x(2-x)+1\}. \end{aligned}$$

When $x \in]0,2[$ this is zero for $x = 1$, corresponding to

$$F(1,1) = F(-1,-1) = 1 + \frac{1}{2}\ln 2 - \frac{\pi}{4}.$$

b) If $x - y = 2$, i.e. $y = x - 2$, $x \in [0,2]$, then the restriction is given by

$$h_2(x) = F(x, x-2) = x^2 - 2x + \frac{1}{2}\ln\left(1 + \left(x^2 - 2x\right)^2\right) - \operatorname{Arctan}\left(x^2 - 2x\right)$$

where

$$\begin{aligned} h_2'(x) &= 2x-2+\frac{1}{2}\cdot\frac{2(x^2-2x)\cdot(2x-2)}{1+(x^2-2x)^2}-\frac{2x-2}{1+(x^2-2x)^2} \\ &= \frac{2(x-1)}{1+(x^2-2x)^2}\left\{1+(x^2-2x)^2+(x^2-2x)-1\right\} \\ &= \frac{2(x-1)}{1+(x^2-2x)^2}\cdot x(x-2)\cdot(x-1)^2. \end{aligned}$$

In $x \in]0,2[$, this is zero for $x=1$. We get for $x=1$,

$$F(1,-1)=F(-1,1)=-1+\frac{1}{2}\ln 2+\frac{\pi}{4}.$$

Finally, at the stationary points,

$$F(0,y)=F(x,0)=0.$$

By a numerical comparison of the possible extremum values it follows that the maximum is

$$F(1,1)=F(-1,-1)=1+\frac{1}{2}\ln 2-\frac{\pi}{4},$$

and the minimum is

$$F(x,0)=F(0,y)=0.$$

Example 39.7 *Given a C^1-function $U(\mathbf{x})$, $\mathbf{x} \in A$, where $A \subseteq \mathbb{R}^3$, and consider a curve such that $\mathbf{x}$ is a function in time t. The curve is determined by the differential equation*

$$\mathbf{x}''(t) + \bigtriangledown U(\mathbf{x}(t)) = 0,$$

where $'$ denotes differentiation with respect to t.
Prove by using the chain rule that

$$\frac{1}{2}\,\|\mathbf{x}'\|^2 + U = C,$$

where C is a constant. (This differential equation is called a first integral *of the above because the order is reduced by 1).*
In Mechanics, $\mathbf{x}(t)$ can be interpreted as the path of a particle in a field of the potential U; then the two differential equations express Newton's second law and the energy theorem.

A Derivation of the first integral.

D When we analyze the desired result, we see that here occurs $\|\mathbf{x}'\|^2 = \|\mathbf{x}'\| \cdot \|\mathbf{x}'\|$, which roughly speaking means that we must have "something like $\mathbf{x}^2$". Hence, the idea must be to take the dot product between the first differential equation and $\mathbf{x}'(t)$ follows by an integration over the parameter interval $I = [t_0, t]$.

I By following the analysis above we get

$$\begin{aligned}
0 &= \int_I \{\mathbf{x}''(t) + \bigtriangledown U(\mathbf{x}(t))\} \cdot \mathbf{x}'(t)\,\mathrm{d}t = \int_I \mathbf{x}''(t) \cdot \mathbf{x}'(t)\,\mathrm{d}t + \int_I \bigtriangledown U(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\,\mathrm{d}t \\
&= \int_I \sum_{i=1}^3 x_i''(t)\,c_i'(t)\,\mathrm{d}t + \int_I \sum_{i=1}^3 \frac{\partial U}{\partial x_i} \cdot \frac{\mathrm{d}x_i}{\mathrm{d}t}\,\mathrm{d}t = \sum_{i=1}^3 \frac{1}{2}\left[(x_i'(\tau))\right]_{\tau=t_0}^t + \int_{t_0}^t \mathrm{d}U(\mathbf{x}(\tau)) \\
&= \frac{1}{2}\,\|\mathbf{x}'(t)\| - c_1 + U(\mathbf{x}(t)) - c_2,
\end{aligned}$$

hence by a rearrangement,

$$\frac{1}{2}\,\|\mathbf{x}'(t)\|^2 + U(\mathbf{x}(t)) = C \qquad \text{(constant in } t\text{)}.$$

40 Formulæ

Some of the following formulæ can be assumed to be known from high school. It is highly recommended that one *learns most of these formulæ in this appendix by heart.*

40.1 Squares etc.

The following simple formulæ occur very frequently in the most different situations.

$$
\begin{array}{ll}
(a+b)^2 = a^2+b^2+2ab, & a^2+b^2+2ab = (a+b)^2, \\
(a-b)^2 = a^2+b^2-2ab, & a^2+b^2-2ab = (a-b)^2, \\
(a+b)(a-b) = a^2-b^2, & a^2-b^2 = (a+b)(a-b), \\
(a+b)^2 = (a-b)^2+4ab, & (a-b)^2 = (a+b)^2-4ab.
\end{array}
$$

40.2 Powers etc.

Logarithm:

$$
\begin{array}{rcl}
\ln|xy| = & \ln|x|+\ln|y|, & x,y\neq 0, \\
\ln\left|\dfrac{x}{y}\right| = & \ln|x|-\ln|y|, & x,y\neq 0, \\
\ln|x^r| = & r\ln|x|, & x\neq 0.
\end{array}
$$

Power function, fixed exponent:

$$(xy)^r = x^r\cdot y^r, x,y>0 \quad \text{(extensions for some } r\text{)},$$

$$\left(\frac{x}{y}\right)^r = \frac{x^r}{y^r}, x,y>0 \quad \text{(extensions for some } r\text{)}.$$

Exponential, fixed base:

$$a^x\cdot a^y = a^{x+y}, \quad a>0 \quad \text{(extensions for some } x,\, y\text{)},$$
$$(a^x)^y = a^{xy}, a>0 \quad \text{(extensions for some } x,\, y\text{)},$$

$$a^{-x} = \frac{1}{a^x}, a>0, \quad \text{(extensions for some } x\text{)},$$

$$\sqrt[n]{a} = a^{1/n}, a\geq 0, \qquad n\in\mathbb{N}.$$

Square root:

$$\sqrt{x^2} = |x|, \qquad x\in\mathbb{R}.$$

Remark 40.1 It happens quite frequently that students make errors when they try to apply these rules. They must be mastered! In particular, as one of my friends once put it: "If you can master the square root, you can master everything in mathematics!" Notice that this innocent looking square root is one of the most difficult operations in Calculus. Do not forget the *absolute value!* ♢

40.3 Differentiation

Here are given the well-known rules of differentiation together with some rearrangements which sometimes may be easier to use:

$$\{f(x) \pm g(x)\}' = f'(x) \pm g'(x),$$

$$\{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x) = f(x)g(x)\left\{\frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}\right\},$$

where the latter rearrangement presupposes that $f(x) \neq 0$ and $g(x) \neq 0$.
If $g(x) \neq 0$, we get the usual formula known from high school

$$\left\{\frac{f(x)}{g(x)}\right\}' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

It is often more convenient to compute this expression in the following way:

$$\left\{\frac{f(x)}{g(x)}\right\} = \frac{d}{dx}\left\{f(x) \cdot \frac{1}{g(x)}\right\} = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2} = \frac{f(x)}{g(x)}\left\{\frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}\right\},$$

where the former expression often is *much easier* to use in practice than the usual formula from high school, and where the latter expression again presupposes that $f(x) \neq 0$ and $g(x) \neq 0$. Under these assumptions we see that the formulæ above can be written

$$\frac{\{f(x)g(x)\}'}{f(x)g(x)} = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)},$$

$$\frac{\{f(x)/g(x)\}'}{f(x)/g(x)} = \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}.$$

Since

$$\frac{d}{dx}\ln|f(x)| = \frac{f'(x)}{f(x)}, \qquad f(x) \neq 0,$$

we also name these the *logarithmic derivatives.*

Finally, we mention the rule of **differentiation of a composite function**

$$\{f(\varphi(x))\}' = f'(\varphi(x)) \cdot \varphi'(x).$$

We first differentiate the function itself; then the insides. This rule is a 1-dimensional version of the so-called *Chain rule.*

40.4 Special derivatives.

Power like:

$$\frac{d}{dx}(x^\alpha) = \alpha \cdot x^{\alpha-1}, \qquad \text{for } x > 0, \;\; \text{(extensions for some } \alpha).$$

$$\frac{d}{dx}\ln|x| = \frac{1}{x}, \qquad \text{for } x \neq 0.$$

Exponential like:

$$\frac{d}{dx}\exp x = \exp x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\left(a^x\right) = \ln a \cdot a^x, \qquad \text{for } x \in \mathbb{R} \text{ and } a > 0.$$

Trigonometric:

$$\frac{d}{dx}\sin x = \cos x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\cos x = -\sin x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\tan x = 1 + \tan^2 x = \frac{1}{\cos^2 x}, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, p \in \mathbb{Z},$$

$$\frac{d}{dx}\cot x = -(1 + \cot^2 x) = -\frac{1}{\sin^2 x}, \qquad \text{for } x \neq p\pi, p \in \mathbb{Z}.$$

Hyperbolic:

$$\frac{d}{dx}\sinh x = \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\cosh x = \sinh x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\tanh x = 1 - \tanh^2 x = \frac{1}{\cosh^2 x}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\coth x = 1 - \coth^2 x = -\frac{1}{\sinh^2 x}, \qquad \text{for } x \neq 0.$$

Inverse trigonometric:

$$\frac{d}{dx}\,\text{Arcsin}\; x = \frac{1}{\sqrt{1-x^2}}, \qquad \text{for } x \in]-1, 1[,$$

$$\frac{d}{dx}\,\text{Arccos}\; x = -\frac{1}{\sqrt{1-x^2}}, \qquad \text{for } x \in]-1, 1[,$$

$$\frac{d}{dx}\,\text{Arctan}\; x = \frac{1}{1+x^2}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\,\text{Arccot}\; x = \frac{1}{1+x^2}, \qquad \text{for } x \in \mathbb{R}.$$

Inverse hyperbolic:

$$\frac{d}{dx}\,\text{Arsinh}\; x = \frac{1}{\sqrt{x^2+1}}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\,\text{Arcosh}\; x = \frac{1}{\sqrt{x^2-1}}, \qquad \text{for } x \in]1, +\infty[,$$

$$\frac{d}{dx}\,\text{Artanh}\; x = \frac{1}{1-x^2}, \qquad \text{for } |x| < 1,$$

$$\frac{d}{dx}\,\text{Arcoth}\; x = \frac{1}{1-x^2}, \qquad \text{for } |x| > 1.$$

Remark 40.2 The derivative of the trigonometric and the hyperbolic functions are to some extent exponential like. The derivatives of the inverse trigonometric and inverse hyperbolic functions are power like, because we include the logarithm in this class. $\Diamond$

40.5 Integration

The most obvious rules are dealing with linearity

$$\int \{f(x) + \lambda g(x)\}\, dx = \int f(x)\, dx + \lambda \int g(x)\, dx, \qquad \text{where } \lambda \in \mathbb{R} \text{ is a constant,}$$

and with the fact that differentiation and integration are "inverses to each other", i.e. modulo some arbitrary constant $c \in \mathbb{R}$, which often tacitly is missing,

$$\int f'(x)\, dx = f(x).$$

If we in the latter formula replace $f(x)$ by the product $f(x)g(x)$, we get by reading from the right to the left and then differentiating the product,

$$f(x)g(x) = \int \{f(x)g(x)\}'\, dx = \int f'(x)g(x)\, dx + \int f(x)g'(x)\, dx.$$

Hence, by a rearrangement

The rule of partial integration:

$$\int f'(x)g(x)\, dx = f(x)g(x) - \int f(x)g'(x)\, dx.$$

The differentiation is moved from one factor of the integrand to the other one by changing the sign and adding the term $f(x)g(x)$.

Remark 40.3 This technique was earlier used a lot, but is almost forgotten these days. It must be revived, because MAPLE and pocket calculators apparently do not know it. It is possible to construct examples where these devices cannot give the exact solution, unless you first perform a partial integration yourself. $\diamond$

Remark 40.4 This method can also be used when we estimate integrals which cannot be directly calculated, because the antiderivative is not contained in e.g. the catalogue of MAPLE. The idea is by a succession of partial integrations to make the new integrand smaller. $\diamond$

Integration by substitution:

If the integrand has the special structure $f(\varphi(x)) \cdot \varphi'(x)$, then one can change the variable to $y = \varphi(x)$:

$$\int f(\varphi(x)) \cdot \varphi'(x)\, dx = \text{“} \int f(\varphi(x))\, d\varphi(x)\text{''} = \int_{y=\varphi(x)} f(y)\, dy.$$

Integration by a monotonous substitution:

If $\varphi(y)$ is a *monotonous* function, which maps the y-interval *one-to-one* onto the x-interval, then

$$\int f(x)\,dx = \int_{y=\varphi^{-1}(x)} f(\varphi(y))\varphi'(y)\,dy.$$

Remark 40.5 This rule is usually used when we have some "ugly" term in the integrand $f(x)$. The idea is to put this ugly term equal to $y = \varphi^{-1}(x)$. When e.g. x occurs in $f(x)$ in the form $\sqrt{x}$, we put $y = \varphi^{-1}(x) = \sqrt{x}$, hence $x = \varphi(y) = y^2$ and $\varphi'(y) = 2y$. ♢

40.6 Special antiderivatives

Power like:

$$\int \frac{1}{x}\,dx = \ln|x|, \qquad \text{for } x \neq 0. \quad \text{(Do not forget the numerical value!)}$$

$$\int x^\alpha\,dx = \frac{1}{\alpha+1}x^{\alpha+1}, \qquad \text{for } \alpha \neq -1,$$

$$\int \frac{1}{1+x^2}\,dx = \text{ Arctan } x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{1-x^2}\,dx = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|, \qquad \text{for } x \neq \pm 1,$$

$$\int \frac{1}{1-x^2}\,dx = \text{ Artanh } x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{1-x^2}\,dx = \text{ Arcoth } x, \qquad \text{for } |x| > 1,$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \text{ Arcsin } x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = -\text{ Arccos } x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \text{ Arsinh } x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \ln(x+\sqrt{x^2+1}), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{x}{\sqrt{x^2-1}}\,dx = \sqrt{x^2-1}, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \text{ Arcosh } x, \qquad \text{for } x > 1,$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \ln|x+\sqrt{x^2-1}|, \qquad \text{for } x > 1 \text{ eller } x < -1.$$

There is an error in the programs of the pocket calculators TI-92 and TI-89. The *numerical signs* are missing. It is obvious that $\sqrt{x^2-1} < |x|$ so if $x < -1$, then $x + \sqrt{x^2-1} < 0$. Since you cannot take the logarithm of a negative number, these pocket calculators will give an error message.

Exponential like:

$$\int \exp x \, dx = \exp x, \qquad \text{for } x \in \mathbb{R},$$

$$\int a^x \, dx = \frac{1}{\ln a} \cdot a^x, \qquad \text{for } x \in \mathbb{R}, \text{ and } a > 0, a \neq 1.$$

Trigonometric:

$$\int \sin x \, dx = -\cos x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \cos x \, dx = \sin x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \tan x \, dx = -\ln|\cos x|, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \cot x \, dx = \ln|\sin x|, \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos x} \, dx = \frac{1}{2} \ln\left(\frac{1+\sin x}{1-\sin x}\right), \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin x} \, dx = \frac{1}{2} \ln\left(\frac{1-\cos x}{1+\cos x}\right), \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos^2 x} \, dx = \tan x, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin^2 x} \, dx = -\cot x, \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z}.$$

Hyperbolic:

$$\int \sinh x \, dx = \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \cosh x \, dx = \sinh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \tanh x \, dx = \ln \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \coth x \, dx = \ln|\sinh x|, \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh x} \, dx = \operatorname{Arctan}(\sinh x), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\cosh x} \, dx = 2 \operatorname{Arctan}(e^x), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh x} \, dx = \frac{1}{2} \ln\left(\frac{\cosh x - 1}{\cosh x + 1}\right), \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\sinh x}\,dx = \ln\left|\frac{e^x-1}{e^x+1}\right|, \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh^2 x}\,dx = \tanh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh^2 x}\,dx = -\coth x, \qquad \text{for } x \neq 0.$$

40.7 Trigonometric formulæ

The trigonometric formulæ are closely connected with circular movements. Thus $(\cos u, \sin u)$ are the coordinates of a point P on the unit circle corresponding to the angle u, cf. figure A.1. This geometrical interpretation is used from time to time.

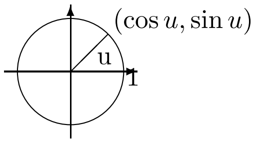

Figure 40.1: The unit circle and the trigonometric functions.

The fundamental trigonometric relation:

$$\cos^2 u + \sin^2 u = 1, \qquad \text{for } u \in \mathbb{R}.$$

Using the previous geometric interpretation this means according to *Pythagoras's theorem*, that the point P with the coordinates $(\cos u, \sin u)$ always has distance 1 from the origo $(0,0)$, i.e. it is lying on the boundary of the circle of centre $(0,0)$ and radius $\sqrt{1} = 1$.

Connection to the complex exponential function:

The *complex exponential* is for imaginary arguments defined by

$$\exp(\mathrm{i}\,u) := \cos u + \mathrm{i}\;\sin u.$$

It can be checked that the usual functional equation for exp is still valid for complex arguments. In other word: The definition above is extremely conveniently chosen.

By using the definition for $\exp(\mathrm{i}\,u)$ and $\exp(-\mathrm{i}\,u)$ it is easily seen that

$$\cos u = \frac{1}{2}(\exp(\mathrm{i}\,u) + \exp(-\mathrm{i}\,u)),$$

$$\sin u = \frac{1}{2i}(\exp(\mathrm{i}\,u) - \exp(-\mathrm{i}\,u)),$$

.

Moivre's formula: We get by expressing $\exp(\mathrm{i}nu)$ in two different ways:

$$\exp(\mathrm{i}nu) = \cos nu + \mathrm{i}\,\sin nu = (\cos u + \mathrm{i}\,\sin u)^n.$$

Example 40.1 If we e.g. put $n = 3$ into Moivre's formula, we obtain the following typical application,

$$\begin{aligned}\cos(3u) + \mathrm{i}\,\sin(3u) &= (\cos u + \mathrm{i}\,\sin u)^3\\ &= \cos^3 u + 3\mathrm{i}\,\cos^2 u \cdot \sin u + 3\mathrm{i}^2 \cos u \cdot \sin^2 u + \mathrm{i}^3 \sin^3 u\\ &= \{\cos^3 u - 3\cos u \cdot \sin^2 u\} + \mathrm{i}\{3\cos^2 u \cdot \sin u - \sin^3 u\}\\ &= \{4\cos^3 u - 3\cos u\} + \mathrm{i}\{3\sin u - 4\sin^3 u\}\end{aligned}$$

When this is split into the real- and imaginary parts we obtain

$$\cos 3u = 4\cos^3 u - 3\cos u, \qquad \sin 3u = 3\sin u - 4\sin^3 u. \quad \diamond$$

Addition formulæ:

$$\sin(u+v) = \sin u\,\cos v + \cos u\,\sin v,$$

$$\sin(u-v) = \sin u\,\cos v - \cos u\,\sin v,$$

$$\cos(u+v) = \cos u\,\cos v - \sin u\,\sin v,$$

$$\cos(u-v) = \cos u\,\cos v + \sin u\,\sin v.$$

Products of trigonometric functions to a sum:

$$\sin u\,\cos v = \frac{1}{2}\sin(u+v) + \frac{1}{2}\sin(u-v),$$

$$\cos u\,\sin v = \frac{1}{2}\sin(u+v) - \frac{1}{2}\sin(u-v),$$

$$\sin u\,\sin v = \frac{1}{2}\cos(u-v) - \frac{1}{2}\cos(u+v),$$

$$\cos u\,\cos v = \frac{1}{2}\cos(u-v) + \frac{1}{2}\cos(u+v).$$

Sums of trigonometric functions to a product:

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right),$$

$$\sin u - \sin v = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right),$$

$$\cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right),$$

$$\cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right).$$

Formulæ of halving and doubling the angle:

$$\sin 2u = 2\sin u\,\cos u,$$

$$\cos 2u = \cos^2 u - \sin^2 u = 2\,\cos^2 u - 1 = 1 - 2\,\sin^2 u,$$

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}} \qquad \text{followed by a discussion of the sign,}$$

$$\cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}} \qquad \text{followed by a discussion of the sign,}$$

40.8 Hyperbolic formulæ

These are very much like the trigonometric formulæ, and if one knows a little of Complex Function Theory it is realized that they are actually identical. The structure of this section is therefore the same as for the trigonometric formulæ. The reader should compare the two sections concerning similarities and differences.

The fundamental relation:

$$\cosh^2 x - \sinh^2 x = 1.$$

Definitions:

$$\cosh x = \frac{1}{2}\left(\exp(x) + \exp(-x)\right), \qquad \sinh x = \frac{1}{2}\left(\exp(x) - \exp(-x)\right).$$

"Moivre's formula":

$$\exp(x) = \cosh x + \sinh x.$$

This is trivial and only rarely used. It has been included to show the analogy.

Addition formulæ:

$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y),$$

$$\sinh(x-y) = \sinh(x)\cosh(y) - \cosh(x)\sinh(y),$$

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y),$$

$$\cosh(x-y) = \cosh(x)\cosh(y) - \sinh(x)\sinh(y).$$

Formulæ of halving and doubling the argument:

$$\sinh(2x) = 2\sinh(x)\cosh(x),$$

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x) = 2\cosh^2(x) - 1 = 2\sinh^2(x) + 1,$$

$$\sinh\left(\frac{x}{2}\right) = \pm\sqrt{\frac{\cosh(x)-1}{2}} \qquad \text{followed by a discussion of the sign,}$$

$$\cosh\left(\frac{x}{2}\right) = \sqrt{\frac{\cosh(x)+1}{2}}.$$

Inverse hyperbolic functions:

$$\operatorname{Arsinh}(x) = \ln\left(x + \sqrt{x^2+1}\right), \qquad x \in \mathbb{R},$$

$$\operatorname{Arcosh}(x) = \ln\left(x + \sqrt{x^2-1}\right), \qquad x \geq 1,$$

$$\operatorname{Artanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \qquad |x| < 1,$$

$$\operatorname{Arcoth}(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right), \qquad |x| > 1.$$

40.9 Complex transformation formulæ

$$\cos(\mathrm{i}x) = \cosh(x), \qquad \cosh(\mathrm{i}x) = \cos(x),$$

$$\sin(\mathrm{i}x) = \ \mathrm{i}\sinh(x), \qquad \sinh(\mathrm{i}x) = \ \mathrm{i}\sin x.$$

40.10 Taylor expansions

The generalized binomial coefficients are defined by

$$\binom{\alpha}{n} := \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{1\cdot 2\cdots n},$$

with n factors in the numerator and the denominator, supplied with

$$\binom{\alpha}{0} := 1.$$

The Taylor expansions for *standard functions* are divided into *power like* (the radius of convergency is finite, i.e. = 1 for the standard series) and*exponential like* (the radius of convergency is infinite).
Power like:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1,$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x| < 1,$$

$$(1+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j, \qquad n \in \mathbb{N}, x \in \mathbb{R},$$

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \qquad \alpha \in \mathbb{R}\setminus\mathbb{N}, |x| < 1,$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n}, \qquad |x| < 1,$$

$$\mathrm{Arctan}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \qquad |x| < 1.$$

Exponential like:

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \qquad x \in \mathbb{R}$$

$$\exp(-x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} x^n, \qquad x \in \mathbb{R}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}, \qquad x \in \mathbb{R},$$

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}, \qquad x \in \mathbb{R},$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}, \qquad x \in \mathbb{R},$$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}, \qquad x \in \mathbb{R}.$$

40.11 Magnitudes of functions

We often have to compare functions for $x \to 0+$, or for $x \to \infty$. The simplest type of functions are therefore arranged in an hierarchy:

1) logarithms,

2) power functions,

3) exponential functions,

4) faculty functions.

When $x \to \infty$, a function from a higher class will always dominate a function form a lower class. More precisely:

A) A *power function* dominates a *logarithm* for $x \to \infty$:

$$\frac{(\ln x)^\beta}{x^\alpha} \to 0 \qquad \text{for } x \to \infty, \quad \alpha,\ \beta > 0.$$

B) An *exponential* dominates a *power function* for $x \to \infty$:

$$\frac{x^\alpha}{a^x} \to 0 \qquad \text{for } x \to \infty, \quad \alpha,\ a > 1.$$

C) The *faculty function* dominates an *exponential* for $n \to \infty$:

$$\frac{a^n}{n!} \to 0, \qquad n \to \infty, \quad n \in \mathbb{N}, \quad a > 0.$$

D) When $x \to 0+$ we also have that a *power function* dominates the *logarithm:*

$$x^\alpha \ln x \to 0-, \qquad \text{for } x \to 0+, \quad \alpha > 0.$$

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