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Real Functions in Several Variables: Volume XI

Vector Fields II

Leif Mejlbro

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Volume XI Vector Fields II

Stokes's Theorem Nabla Calculus

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Stokes's Theorem Nabla Calculus
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Contents

Preface

The topic of this series of books on *"Real Functions in Several Variables"* is very important in the description in e.g. *Mechanics* of the real 3-dimensional world that we live in. Therefore, we start from the very beginning, modelling this world by using the coordinates of $\mathbb{R}^3$ to describe e.g. a motion in space. There is, however, absolutely no reason to restrict ourselves to $\mathbb{R}^3$ alone. Some motions may be rectilinear, so only $\mathbb{R}$ is needed to describe their movements on a line segment. This opens up for also dealing with $\mathbb{R}^2$, when we consider plane motions. In more elaborate problems we need higher dimensional spaces. This may be the case in *Probability Theory* and *Statistics*. Therefore, we shall in general use $\mathbb{R}^n$ as our abstract model, and then restrict ourselves in examples mainly to $\mathbb{R}^2$ and $\mathbb{R}^3$.

For rectilinear motions the familiar *rectangular coordinate system* is the most convenient one to apply. However, as known from e.g. *Mechanics*, circular motions are also very important in the applications in engineering. It becomes natural alternatively to apply in $\mathbb{R}^2$ the so-called *polar coordinates* in the plane. They are convenient to describe a circle, where the rectangular coordinates usually give some nasty square roots, which are difficult to handle in practice.

Rectangular coordinates and polar coordinates are designed to model each their problems. They supplement each other, so difficult computations in one of these coordinate systems may be easy, and even trivial, in the other one. It is therefore important always in advance carefully to analyze the geometry of e.g. a domain, so we ask the question: Is this domain best described in rectangular or in polar coordinates?

Sometimes one may split a problem into two subproblems, where we apply rectangular coordinates in one of them and polar coordinates in the other one.

It should be mentioned that in *real life* (though not in these books) one cannot always split a problem into two subproblems as above. Then one is really in trouble, and more advanced mathematical methods should be applied instead. This is, however, outside the scope of the present series of books.

The idea of polar coordinates can be extended in two ways to $\mathbb{R}^3$. Either to *semi-polar* or *cylindric coordinates*, which are designed to describe a cylinder, or to *spherical coordinates*, which are excellent for describing spheres, where rectangular coordinates usually are doomed to fail. We use them already in daily life, when we specify a place on Earth by its longitude and latitude! It would be very awkward in this case to use rectangular coordinates instead, even if it is possible.

Concerning the contents, we begin this investigation by modelling point sets in an n-dimensional Euclidean space E^n by $\mathbb{R}^n$. There is a subtle difference between E^n and $\mathbb{R}^n$, although we often identify these two spaces. In E^n we use *geometrical methods* without a coordinate system, so the objects are independent of such a choice. In the coordinate space $\mathbb{R}^n$ we can use ordinary calculus, which in principle is not possible in E^n. In order to stress this point, we call E^n the "abstract space" (in the sense of calculus; not in the sense of geometry) as a warning to the reader. Also, whenever necessary, we use the colour black in the "abstract space", in order to stress that this expression is theoretical, while variables given in a chosen coordinate system and their related concepts are given the colours blue, red and green.

We also include the most basic of what mathematicians call *Topology*, which will be necessary in the following. We describe what we need by a function.

Then we proceed with limits and continuity of functions and define continuous curves and surfaces, with parameters from subsets of $\mathbb{R}$ and $\mathbb{R}^2$, resp..

Continue with (partial) differentiable functions, curves and surfaces, the chain rule and Taylor's formula for functions in several variables.

We deal with maxima and minima and extrema of functions in several variables over a domain in $\mathbb{R}^n$. This is a very important subject, so there are given many worked examples to illustrate the theory.

Then we turn to the problems of integration, where we specify four different types with increasing complexity, *plane integral, space integral, curve (or line) integral and surface integral*.

Finally, we consider *vector analysis*, where we deal with vector fields, Gauß's theorem and Stokes's theorem. All these subjects are very important in theoretical Physics.

The structure of this series of books is that each subject is *usually* (but not always) described by three successive chapters. In the first chapter a brief theoretical theory is given. The next chapter gives some practical guidelines of how to solve problems connected with the subject under consideration. Finally, some worked out examples are given, in many cases in several variants, because the standard solution method is seldom the only way, and it may even be clumsy compared with other possibilities.

I have as far as possible structured the examples according to the following scheme:

A *Awareness*, i.e. a short description of what is the problem.

D *Decision*, i.e. a reflection over what should be done with the problem.

I *Implementation*, i.e. where all the calculations are made.

C *Control*, i.e. a test of the result.

This is an ideal form of a general procedure of solution. It can be used in any situation and it is not linked to Mathematics alone. I learned it many years ago in the Theory of Telecommunication in a situation which did not contain Mathematics at all. The student is recommended to use it also in other disciplines.

From high school one is used to immediately to proceed to **I**. *Implementation*. However, examples and problems at university level, let alone situations in real life, are often so complicated that it in general will be a good investment also to spend some time on the first two points above in order to be absolutely certain of what to do in a particular case. Note that the first three points, **ADI**, can always be executed.

This is unfortunately not the case with **C** *Control*, because it from now on may be difficult, if possible, to check one's solution. It is only an extra securing whenever it is possible, but we cannot include it always in our solution form above.

I shall on purpose not use the logical signs. These should in general be avoided in Calculus as a shorthand, because they are often (too often, I would say) misused. Instead of $\wedge$ I shall either write "and", or a comma, and instead of $\vee$ I shall write "or". The arrows $\Rightarrow$ and $\Leftrightarrow$ are in particular misunderstood by the students, so they should be totally avoided. They are not telegram short hands, and from a logical point of view they usually do not make sense at all! Instead, write in a plain language what you mean or want to do. This is difficult in the beginning, but after some practice it becomes routine, and it will give more precise information.

When we deal with multiple integrals, one of the possible pedagogical ways of solving problems has been to colour variables, integrals and upper and lower bounds in blue, red and green, so the reader by the colour code can see in each integral what is the variable, and what are the parameters, which do not enter the integration under consideration. We shall of course build up a hierarchy of these colours, so the order of integration will always be defined. As already mentioned above we reserve the colour black for the theoretical expressions, where we cannot use ordinary calculus, because the symbols are only shorthand for a concept.

The author has been very grateful to his old friend and colleague, the late Per Wennerberg Karlsson, for many discussions of how to present these difficult topics on real functions in several variables, and for his permission to use his textbook as a template of this present series. Nevertheless, the author has felt it necessary to make quite a few changes compared with the old textbook, because we did not always agree, and some of the topics could also be explained in another way, and then of course the results of our discussions have here been put in writing for the first time.

The author also adds some calculations in MAPLE, which interact nicely with the theoretic text. Note, however, that when one applies MAPLE, one is forced first to make a geometrical analysis of the domain of integration, i.e. apply some of the techniques developed in the present books.

The theory and methods of these volumes on "Real Functions in Several Variables" are applied constantly in higher Mathematics, Mechanics and Engineering Sciences. It is of paramount importance for the calculations in *Probability Theory*, where one constantly integrate over some point set in space.

It is my hope that this text, these guidelines and these examples, of which many are treated in more ways to show that the solutions procedures are not unique, may be of some inspiration for the students who have just started their studies at the universities.

Finally, even if I have tried to write as careful as possible, I doubt that all errors have been removed. I hope that the reader will forgive me the unavoidable errors.

Leif Mejlbro
March 21, 2015

Introduction to volume XI, Vector Fields II; Stokes's Theorem; Nabla calculus

This is the eleventh volume in the series of books on *Real Functions in Several Variables*.

It is also the second volume on Vector Fields. It was necessary to split the material into three volumes because the material is too big for one volume, and even these three volumes are large. In the first volume we dealt with the *tangential line integral*, which e.g. can be used to describe the work of a particle when it is forced along a given curve by some force. It was natural to introduce the *gradient fields*, where the tangential line integral only depends on the initial and the terminal points of the curve and not of the curve itself. Such gradients fields are describing *conservative forces* in Physics.

Tangential line integrals are one-dimensional in nature. In case of two dimensions we consider the *flux* of a flow through a surface. When the surface $\partial\Omega$ is surrounding a three dimensional body Ω, this leads to *Gauß's theorem*, by which we can express the flux of a vector field $\mathbf{V}$ through $\partial\Omega$, which is a surface integral, by a space integral over Ω of the *divergence* of the vector field $\mathbf{V}$. This theorem works both ways. Sometimes, and most frequently, the surface integral is expressed as space integral, other times we express a space integral as a flux, i.e. a surface integral. Applications are obvious in *Electro-Magnetic Field Theory*, though other applications can also be found.

In this volume we shall study *Stokes's theorem*. By using a more advanced mathematical formalism from modern *Differential Geometry* it is possible to show that *Gauß's theorem* and *Stokes's theorem* as presented here in Volume X and Volume XI can be considered as special cases of the same *general Gauß's theorem*. This is difficult to see in the terminology chosen here. However, the pattern is similar in both cases. An integration in n dimensions over a domain Ω is transformed into an integration in $n-1$ dimensions over the intrinsic boundary $\delta\Omega$, and the integrand is changed appropriately during this transformation.

Gauß's and Stokes's theorems have always been considered as extremely difficult to understand for the student. Therefore we have included a section on *Maxwell's equations* from Physics, where these two theorems are constantly been applied. We also give lots of examples of worked out problems.

We also include a chapter on nabla calculus, which in three dimensions uses a formalism with the cross product and the dot product known from *Linear Algebra*. Some formulæ become easier to comprehend than the traditional ones using the notations **grad**, div and **rot**, of some authors also written **curl**. This is actually the first step towards the unification of *Gauß's* and *Stokes's theorems* in the *general Gauß's theorem* mentioned above. However, we shall not go into the full generality in n dimensions. The following Volume XII is the third one concerning these vector fields. Here we shall conclude with introducing vector potentials, harmonic functions and Green's theorems.

35 Rotation of a vector field; Stokes's theorem

35.1 Rotation of a vector field in $\mathbb{R}^3$

We considered in Volume X the *flux* of a vector field $\mathbf{V}$ through a closed surface $\mathcal{F}$ in the sense that $\mathcal{F} = \partial\Omega$ is the boundary of a three dimensional body Ω. We shall in this chapter instead consider a closed curve $\mathcal{K}$ in the space $\mathbb{R}^3$ (i.e. coinciding initial and final point of the curve), which then can be considered as the boundary curve $\delta\mathcal{G}$ of some surface $\mathcal{G}$ in $\mathbb{R}^3$. Note that $\delta\mathcal{G}$ means the *intrinsic boundary* of $\mathcal{G}$, because in $\mathbb{R}^3$ we have $\partial\mathcal{G} = \mathcal{G}$ for every continuous and piecewise C^1 two dimensional surface $\mathcal{G}$.

Remark 35.1 When we compare with *Gauß's theorem,* we see that we here replace the three dimensional body Ω with a two dimensional surface $\mathcal{F}$, and the boundary surface $\partial\Omega$ is replaced with an in principle one dimensional closed curve $\mathcal{K} = \delta\mathcal{G}$. At the first glance the reader may feel a little uneasy, because there are many piecewise C^1 surfaces $\mathcal{G}$ which satisfy the requirement that its boundary curve is equal to the given curve $\mathcal{K}$, so $\mathcal{G}$ is not uniquely determined, in contrast to the body Ω in Gauß's theorem. This is probably the reason for why *Stokes's theorem* below intuitively is felt to be "difficult". We shall in this chapter try to explain what lies behind. $\diamond$

The idea is that given a vector field $\mathbf{V}$ and a surface $\mathcal{F}$ of the closed (intrinsic) boundary curve $\delta\mathcal{F}$, then it should be possible to express the circulation of $\mathbf{V}$ along the closed curve $\delta\mathcal{F}$, i.f.

$$\oint_{\delta\mathcal{F}} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s,$$

as a *flux* of some vector field $\mathbf{W}$, depending only on $\mathbf{V}$, through the surface $\mathcal{F}$, i.e.

$$(35.1) \qquad \oint_{\delta\mathcal{F}} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s = \int_{\mathcal{F}} \mathbf{W}\cdot\mathbf{n}\,\mathrm{d}S,$$

where we shall find the relationship between $\mathbf{W}$ and $\mathbf{V}$.

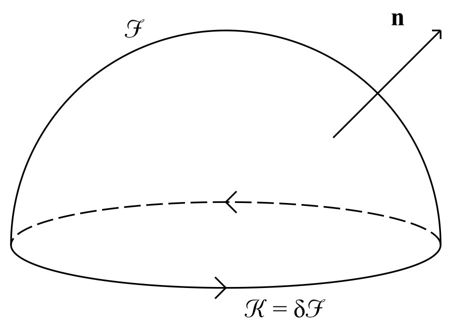

Figure 35.1: Coupled orientation of a surface $\mathcal{F}$ and its intrinsic boundary curve $\delta\mathcal{F}$.

The first problem is of course that there are two orientations of the intrinsic boundary curve $\delta\mathcal{F}$ and two ways to define the unit normal field $\mathbf{n}$ on $\mathcal{F}$. We therefore need to describe how these two ways of orientation are coupled.

1) If the continuous normal vector field $\mathbf{n}$ is given on $\mathcal{F}$, then the orientation of $\delta\mathcal{F}$ is determined in the following way. Choose an $\mathbf{n}$ close to the boundary curve $\delta\mathcal{F}$. Grasp this $\mathbf{n}$ with your right hand, such that your thumb is pointing in the same direction as $\mathbf{n}$. Then the other four fingers will indicate the orientation of $\delta\mathcal{F}$, cf. Figure 35.1.

2) If instead the orientation of $\delta\mathcal{F}$ is given, then put your right hand, such that your four fingers are pointing in the direction of the tangent of $\delta\mathcal{F}$. Then your thumb will indicate the direction of the unit normal vector field $\mathbf{n}$ on $\mathcal{F}$ in the neighbourhood of $\delta\mathcal{F}$. Finally, extend $\mathbf{n}$ by continuity to all of $\mathcal{F}$.

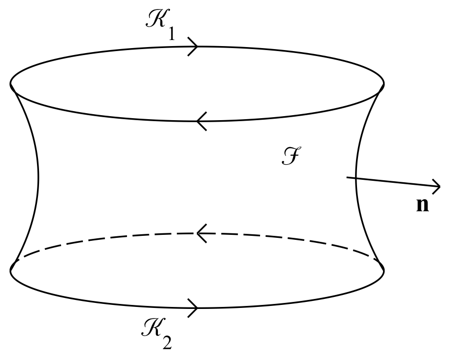

Figure 35.2: The coupling between $\mathbf{n}$ on $\mathcal{F}$ and the orientations of the two (intrinsically) closed boundary curves $\mathcal{K}_1$ and $\mathcal{K}_2$, when the intrinsic boundary has two components.

A connected surface $\mathcal{F}$ may have several intrinsic boundary curves. If $\mathbf{n}$ is a continuous normal vector field on $\mathcal{F}$, then procedure 2) above defines the orientations on all boundary curves. An example is shown on Figure 35.2, where we note that we have a sense of that "$\mathcal{K}_1$ and $\mathcal{K}_2$ are given opposite orientations". This is, however, due to the chosen convention.

Remark 35.2 The northern hemisphere of the Earth satisfies the convention described above. If you are on the North Pole and let your right thumb point along the axis of rotation away from the surface, then the other four fingers will indicate the direction of the rotation of the Earth, i.e. eastwards. Clearly, this rule does not apply on the South Pole. $\diamond$

Let us return to the problem of determining $\mathbf{W}$ in (35.1), i.e.

$$\oint_{\delta\mathcal{F}} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s = \int_{\mathcal{F}} \mathbf{W}\cdot\mathbf{n}\,\mathrm{d}S,$$

where we assume that the orientations of $\delta\mathcal{F}$ and $\mathcal{F}$ are linked together as described above. Let us first start with the simple case, where $\mathcal{F}$ is a plane domain E in the (x, y)-plane, cf. Figure 35.3.

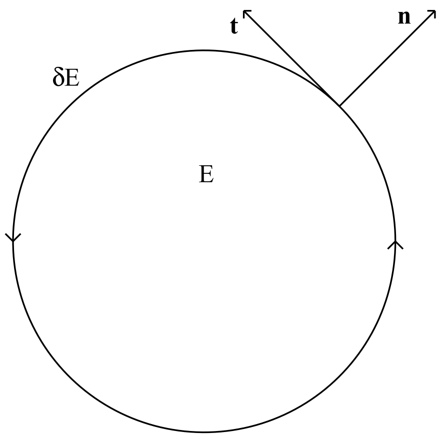

Figure 35.3: Analysis of the circulation along a closed curve δE surrounding a plane set E.

Note that we have changed the notation on Figure 35.3. Since δE is given the positive orientation in the plane, the normal vector field in the whole $\mathbb{R}^3$ space is represented by the normed vector $\mathbf{e}_z$, which points towards you from the paper. If $\mathbf{t}$ is the unit tangent field along δE, and $\mathbf{n}$ is the normal field to the curve *in the plane*, then it is well-known that

$$\mathbf{n} = \mathbf{t} \times \mathbf{e}_z, \qquad \text{i.e.} \qquad n_x = t_y \quad \text{and} \quad n_y = -t_x.$$

The circulation is given by

$$C = \int_{\partial E} (V_x\, t_x + V_y\, t_y)\, \mathrm{d}s = \int_{\partial E} (-V_x\, n_y + V_y\, n_x)\, \mathrm{d}s = \int_{\partial E} (V_y, -V_x) \cdot (n_x, n_y)\, \mathrm{d}s.$$

Then use Gauß's theorem for a plane vector field, cf. Volume X, to conclude that this is equal to

$$= \int_E \mathrm{div}\,(V_y, -V_x)\, \mathrm{d}S = \int_E \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) \mathrm{d}S.$$

It follows that we have proved that the circulation is

$$\oint_{\partial E} (V_x\, t_x + V_y\, t_y)\, \mathrm{d}s = \int_E \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) \mathrm{d}S = \int_E \mathbf{W} \cdot \mathbf{e}_z\, \mathrm{d}S = \int_E W_z\, \mathrm{d}S,$$

because in this setup $\mathbf{e}_z$ is the unit normal field.

If instead E is lying in the (y,z)-plane with $\mathbf{e}_x$ as its normal vector field, then a similar analysis shows that

$$\oint_{\partial E} (V_y\, t_y + V_z\, t_z)\, \mathrm{d}s = \int_E \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) \mathrm{d}S = \int_E W_x\, \mathrm{d}S,$$

and when E lies in the (z,x)-plane in this order (defining the orientation from z towards x), then $\mathbf{e}_y$ is its normal vector, so we obtain by similar calculations

$$\oint_{\partial E} (V_z\, t_z + V_x\, t_x)\, \mathrm{d}s = \int_E \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right) \mathrm{d}S = \int_E W_y\, \mathrm{d}S.$$

Note that if we instead had chosen the (x,z)-plane with the orientation from x towards z, then the result would change its sign, but the same would the normal vector field, when it satisfies the right convention, so we get the same result.

Before we proceed we first collect the above in a convenient definition of the *rotation*, because the expression of $\mathbf{W}$ derived above occurs over and over again in these calculations.

Definition 35.1 *Let* $\mathbf{V}$ *be a differentiable vector field in the ordinary space* $\mathbb{R}^3$*. Then we define the* rotation *of* V*, written* $\mathbf{rot}\mathbf{V}$ *by*

$$\mathbf{rot}\mathbf{V} := \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\mathbf{e}_x + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\mathbf{e}_y + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\mathbf{e}_z,$$

where $\mathbf{e}_x$*,* $\mathbf{e}_y$ *and* $\mathbf{e}_z$ *are orthonormal unit vectors.*

In some books, **rot** is instead written **curl**.

There is an easier way to remember this formula, namely as a formal determinant, using a similar determinant as when we calculate the cross product of two vectors, namely

$$\mathbf{rot}\mathbf{V} = \bigtriangledown \times \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \\ V_x & V_y & V_z \end{vmatrix}.$$

This structure invites one to also use the notation $\bigtriangledown\times$ instead of **rot**. This formal determinant is calculating by developing the determinant after the first row and then always let the differential operator work on the function.

35.2 Stokes's theorem

We proved in Section 35.1 that the only candidate $\mathbf{W}$ of the solution of the problem

$$\oint_{\partial \mathcal{F}} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s = \int_{\mathcal{F}} \mathbf{W}\cdot\mathbf{n}\,\mathrm{d}S,$$

in some special cases is $\mathbf{W} = \mathbf{rot}\mathbf{V}$. It can be proved that this is true in general. As usual in these matters the proof of this statement is too complicated to be brought here, so following the usual style of these books we shall only formulate without given a correct proof the important

Theorem 35.1 Stokes's theorem. *Let $\mathcal{F}$ be an oriented, piecewise C^1-surface with a continuous normal vector field* $\mathbf{n}$*, defined almost everywhere.*

1) Assume that its boundary curve $\delta\mathcal{F}$ is a closed and piecewise C^1-curve without double points, and with a tangent almost everywhere.

2) Let the orientations of $\mathcal{F}$ and $\delta\mathcal{F}$ be linked by the right hand convention as described above.

3) Let $\mathbf{V}: A \to \mathbb{R}^3$ be a C^1-vector field, where $\mathcal{F} \subseteq A \subseteq \mathbb{R}^3$.

Then

$$\oint_{\partial \mathcal{F}} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s = \int_{\mathcal{F}} \mathbf{n}\cdot\mathbf{rot}\mathbf{V}\,\mathrm{d}S.$$

As in the case of *Gauß's theorem* there is also a two dimensional version of this theorem. This was more or less proved in Section 35.1, when we derived the possible structure of *Stokes's theorem* in the three dimensional case. This two dimensional version is also called *Green's theorem in the plane.*

Theorem 35.2 Green's theorem in the plane. *Let $E \subset \mathbb{R}^2$ be a bounded plane set with a boundary ∂E which is a piecewise C^1-curve with no double points with a tangent field almost everywhere. Let the unit normal vector field* $\mathbf{n}$ *on ∂E always point away from E. Then*

$$\oint_{\partial E} \{V_x\,\mathrm{d}x + V_y\,\mathrm{d}y\} = \int_E \left\{\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right\}\mathrm{d}S.$$

Stokes's theorem (or *Green's theorem*) can be used in two ways. We mention the procedure in one of them below. The task is to find the circulation of a C^1-vector field $\mathbf{V}$ along a closed curve $\mathcal{K}$, then we shall choose a surface $\mathcal{F}$, such that

1) The surface $\mathcal{F}$ has $\mathcal{K}$ as its boundary curve, $\delta\mathcal{F} = \mathcal{K}$.

2) Depending on the structure of $\mathbf{rot}\mathbf{V}$, which we calculate at this step, we should furthermore choose $\mathcal{F}$, such that the surface integral

$$\int_{\mathcal{F}} \mathbf{n} \cdot \mathbf{rot}\mathbf{V}\, \mathrm{d}S$$

becomes easy to handle. This is not always an easy task.

We say that a vector field $\mathbf{V}$ is *rotational free*, if $\mathbf{rot}\mathbf{V} = \mathbf{0}$ in all the domain of $\mathbf{V}$

We have the following important simple results

Theorem 35.3 *A C^1-gradient field, $\mathbf{grad} f$, is always rotational free.*

Proof. In fact, $f \in C^2$, so the order of differentiation can be interchanged in the following. We get straightforward

$$\begin{aligned} \mathbf{rot}(\mathbf{grad} f) &= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix} = \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}, \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) \\ &= (0,0,0) = \mathbf{0}. \qquad \Diamond \end{aligned}$$

One can prove the following converse statement, which is not formulated in its full generality

Theorem 35.4 *If $\mathbf{v}$ is a rotational free vector field in a starshaped domain, then $\mathbf{V} = \mathbf{grad} f$ is a gradient field, i.e.*

$$\mathbf{rot}\mathbf{V} = \mathbf{0} \quad \textit{implies the existence of } f, \textit{ such that } \mathbf{V} = \mathbf{grad} f.$$

Proof. Just calculate,

$$\begin{aligned} \mathrm{div}\,(\mathbf{rot}\mathbf{V}) &= \frac{\partial}{\partial x}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) + \frac{\partial}{\partial y}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right) + \frac{\partial}{\partial z}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) \\ &= \left(\frac{\partial^2 V_z}{\partial x \partial y} - \frac{\partial^2 V_y}{\partial x \partial z} \right) + \left(\frac{\partial^2 V_x}{\partial y \partial z} - \frac{\partial^2 V_z}{\partial y \partial x} \right) + \left(\frac{\partial^2 V_y}{\partial z \partial x} - \frac{\partial^2 V_x}{\partial z \partial y} \right) \\ &= = \left(\frac{\partial^2 V_x}{\partial y \partial z} - \frac{\partial^2 V_x}{\partial z \partial y} \right) + \left(-\frac{\partial^2 V_y}{\partial x \partial z} + \frac{\partial^2 V_y}{\partial z \partial x} \right) + \left(\frac{\partial^2 V_z}{\partial x \partial y} - \frac{\partial^2 V_z}{\partial y \partial x} \right) = 0. \qquad \Diamond \end{aligned}$$

We also have a theorem, which states when a divergence free vector field is a rotational field. We postpone the formulation of this result to a later section, where it fits into the context.

Remark 35.3 As mentioned previously we also write

$$\mathbf{grad} f = \bigtriangledown f, \qquad \text{div}\,\mathbf{V} = \bigtriangledown \cdot \mathbf{V} \quad \text{and} \quad \mathbf{rot}\mathbf{V} = \bigtriangledown \times \mathbf{V},$$

where nabla, $\bigtriangledown$, is the three dimensional differential operator

$$\bigtriangledown = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right).$$

Then the results above are also written

$$\bigtriangledown \times \bigtriangledown f = \mathbf{0} \qquad \text{and} \qquad \bigtriangledown \cdot (\bigtriangledown \times \mathbf{V}) = 0,$$

which should remind one of similar rules in *Linear Algebra*, when we are dealing with vectors in $\mathbb{R}^3$.

We shall here include some simple examples, which illustrate the theorems above. In a later section we supply these with others which may go into various other directions. $\Diamond$

Example 35.1 We shall find the circulation C of the given vector field

$$\mathbf{V}(x,y,z) = \left(z^2x, x^2y, y^2z\right), \qquad \text{for } (x,y,z) \in \mathbb{R}^3,$$

along the oriented curve $\mathcal{K}$ on Figure 35.4, consisting of three circular arcs, all of centrum $\mathbf{0}$ and radius $a > 0$, and which furthermore lie in the planes given by $z = 0$, $y = -x$ and $x = y\sqrt{3}$.

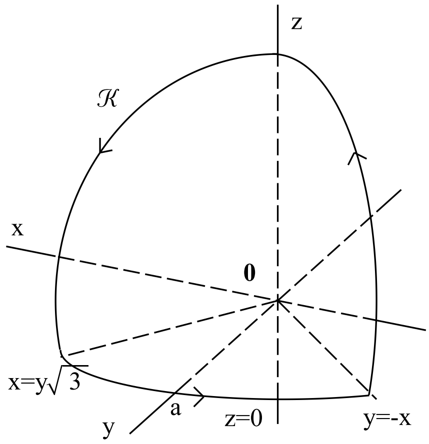

Figure 35.4: The curve $\mathcal{K}$ in Example 35.1.

If we should directly calculate the circulation by using the definition using line integrals, then we should deal with three different line integrals. It is, of course, possible, However, it is easier here to apply *Stokes's theorem*, because all three curves lie on the same spherical surface of centrum $\mathbf{0}$ and radius a. Therefore, we choose $\mathcal{F}$ as the spherical triangle on this sphere, bounded by the curve $\mathcal{K}$. The orientation of $\mathcal{K}$ and the right hand convention forces the unit normal vector field always to point away from $\mathbf{0}$, so the unit normal vector field is given by

$$\mathbf{n} = \frac{1}{a}(x,y,z), \qquad \text{for } (x,y,z) \in \mathcal{F}, \qquad \text{i.e. } x^2+y^2+z^2 = a^2.$$

We first compute $\mathbf{rot}\mathbf{V}$ to see if it has a simple structure,

$$\mathbf{rot}\mathbf{V}(x,y,z) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z^2 x & z^2 y & y^2 z \end{vmatrix} = (2yz - 0, 2zx + 0, 2xy - 0) = 2(yz, zx, xy),$$

so restricted to $\mathcal{F}$ we get

$$\mathbf{n} \cdot \mathbf{rot}\mathbf{V} = \frac{1}{a}\,(x,y,z) \cdot 2(yz,zx,xy) = \frac{6}{a}\,xyz, \qquad \text{where } x^2+y^2+z^2 = a^2.$$

The surface $\mathcal{F}$ defined above is in spherical coordinates given by

$$\mathcal{F}: \qquad \left\{(r,\varphi,\theta) \mid r=a,\, \varphi \in \left[\frac{\pi}{6},\frac{3\pi}{4}\right],\, \theta \in \left[0,\frac{\pi}{2}\right]\right\},$$

cf. also Figure 35.4, so we finally get by Stokes's theorem,

$$\begin{aligned} C &= \oint_{\mathcal{K}} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}s = \int_{\mathcal{F}} \mathbf{n}\cdot\mathbf{V}\,\mathrm{d}S = \frac{6}{a}\int_{\mathcal{F}} xyz\,\mathrm{d}S \\ &= \frac{6}{a}\int_{\frac{\pi}{6}}^{\frac{3\pi}{4}} \left\{\int_0^{\frac{\pi}{2}} \left(a^3\cos\theta\sin^2\theta\cos\varphi\sin\varphi\right) a^2\sin\theta\,\mathrm{d}\theta\right\}\mathrm{d}\varphi \\ &= 6a^4\left[\frac{1}{4}\sin^4\theta\right]_0^{\frac{\pi}{2}}\cdot\left[\frac{1}{2}\sin^2\varphi\right]_{\frac{\pi}{6}}^{\frac{3\pi}{4}} \\ &= \frac{3}{4}a^4\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{3}{16}a^4. \qquad \Diamond \end{aligned}$$

Example 35.2 Clearly, $\mathbf{V}(x,y,z) = (x,y,z)$ is a *rotational free* vector field, $\mathbf{rot}\mathbf{V} = \mathbf{0}$, so it follows from *Stokes's theorem* that the circulation along any closed curve $\mathcal{K}$ is

$$\oint_{\mathcal{K}} \mathbf{x}\cdot\mathbf{t}\,\mathrm{d}s = 0. \qquad \Diamond$$

Example 35.3 We cannot in advance predict the direction of $\mathbf{rot}\mathbf{V}$ for given $\mathbf{V}$. We shall in this example show that in some cases $\mathbf{rot}\mathbf{V}$ and $\mathbf{V}$ may be parallel to each other, and in other cases they are perpendicular to each other. To illustrate the first case we choose the vector field

$$\mathbf{V}(x,y,z) = (\cos y, \cos x, \sin y - \sin x), \qquad \text{for } (x,y,z)\in\mathbb{R}^3.$$

Then

$$\mathbf{rot}\mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \cos y & \cos x & \sin y - \sin x \end{vmatrix} = (\cos y, \cos x, \sin y - \sin x) = \mathbf{V},$$

so $\mathbf{rot}\mathbf{V} = \mathbf{V}$, and the rotation of $\mathbf{V}$ is even equal to $\mathbf{V}$ itself, and they are trivially parallel.

Concerning the other statement, we choose the vector field

$$\mathbf{U}(x,y,z) = (r,r,r), \qquad \text{where } r = \sqrt{x^2+y^2+z^2}.$$

We note that, when $r \neq 0$, then

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \qquad \frac{\partial r}{\partial y} = \frac{y}{r}, \qquad \frac{\partial r}{\partial z} = \frac{z}{r},$$

so we get for $r \neq 0$,

$$\mathbf{rot}\mathbf{U} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ r & r & r \end{vmatrix} = \left(\frac{y-z}{r}, \frac{z-x}{r}, \frac{x-y}{r} \right),$$

and it follows that

$$\mathbf{U} \cdot \mathbf{rot}\mathbf{U} = (y-z) + (z-x) + (x-y) = 0,$$

which shows that $\mathbf{U} \perp \mathbf{rot}\mathbf{U}$. In other words, $\mathbf{U}$ and $\mathbf{rot}\mathbf{U}$ are perpendicular to each other, and the claim is proved. $\Diamond$

Example 35.4 We shall in this example demonstrate how we may choose the surface $\mathcal{F}$ with a given closed curve $\mathcal{K}$ as its intrinsic boundary $\delta\mathcal{F} = \mathcal{K}$, when we apply *Stokes's theorem.*

1) *Specification of the curve* $\mathcal{K}$. We let the curve $\mathcal{K}$ be defined as the intersection of the circular cylindric surface of the equation

$$x^2 + y^2 = ax,$$

with the surface, given by the equation

$$z = \sqrt{4a^2 - ax}.$$

The latter is one half of a parabolic cylindric surface.
We choose the orientation on $\mathcal{K}$, such that the direction of the z-axis is the direction of the bounded part of the circular cylindric surface above of boundary $\mathcal{K}$.

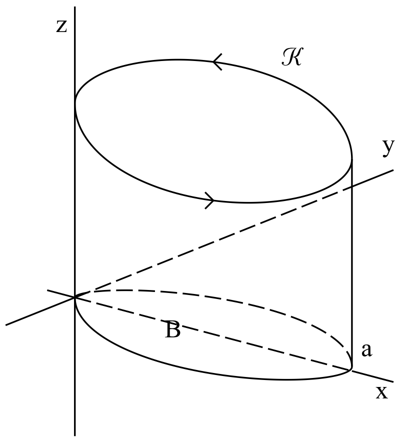

Figure 35.5: The curve $\mathcal{K}$ in Example 35.4.

2) *Specification of the vector field* $\mathbf{V}$ *and its rotation.* We choose

$$\mathbf{V}(x,y,z) = \left(3xy, 2x^2, -yz\right) \qquad \text{for } (x,y,z) \in \mathbb{R}^3,$$

and we shall find the circulation C of $\mathbf{V}$ along the curve $\mathcal{K}$ above with the chosen orientation. Since we shall apply *Stokes's theorem*, we start by calculating the rotation of $\mathbf{V}$,

$$\mathbf{rot}\,\mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3xy & 2x^2 & -yz \end{vmatrix} = (-z, 0, x).$$

3) *Choice of surface $\mathcal{F}$, such that $\delta\mathcal{F} = \mathcal{K}$.* First note that the projection of $\mathcal{K}$ onto the (x,y)-plane is the circle of centrum $\left(\frac{a}{2}, 0\right)$ and radius $\frac{a}{2}$, so it cuts the plane into two open domains, of which only the bounded domain B is of interest for us, so

$$B := \left\{ (x,y) \mid \left(x - \frac{a}{2}\right)^2 + y^2 \le \left(\frac{a}{2}\right)^2 \right\},$$

so the surface $\mathcal{F}$, we are going to choose, should be parametrized over B.

Since $\mathcal{K}$ is the intersection of two surfaces, it lies on both of them, so it is obvious to choose

$$\mathcal{F}_1 : \qquad z = Z(x,y) := \sqrt{4a^2 - ax}, \qquad \text{for } (x,y) \in B.$$

However, since $\mathcal{K}$ is defined by

$$x^2 + y^2 = ax, \qquad \text{and} \qquad z^2 = 4a^2 - ax, \quad z > 0,$$

it follows by addition that $\mathcal{K}$ also lies on the surface

$$\mathcal{F}_2 : \qquad x^2 + y^2 + z^2 = 4a^2, \qquad z > 0,$$

which is a part of a sphere.

a) First consider $\mathcal{F}_1$, which is the graph of the function $Z(x,y) = \sqrt{4a^2 - ax}$. It therefore follows from Section 13.5 that its field of normal vectors is given by

$$\mathbf{N}(x,y) = \mathbf{r}'_x \times \mathbf{r}'_y = \left(-Z'_x, -Z'_y, 1\right) = \left(\frac{a}{2z}, 0, 1\right),$$

which has a positive z-component, so the orientation is correct. It therefore follows from *Stokes's theorem* that the circulation is

$$\begin{aligned} C &= \int_{\mathcal{F}_1} \mathbf{n} \cdot \mathbf{rot}\,\mathbf{V}\,\mathrm{d}S = \int_B \mathbf{N} \cdot \mathbf{rot}\,\mathbf{V}\,\mathrm{d}x\,\mathrm{d}y = \int_B \left(\frac{a}{2z}, 0, 1\right) \cdot (-z, 0, x)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_B \left(-\frac{a}{2} + x\right)\,\mathrm{d}x\,\mathrm{d}y = -\frac{a}{2}\,\pi\left(\frac{a}{2}\right)^2 + \int_B x\,\mathrm{d}x\,\mathrm{d}y, \end{aligned}$$

where

$$\begin{aligned}\int_B x\,\mathrm{d}x\,\mathrm{d}y &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left\{\int_0^{a\cos\varphi} r\cdot\cos\varphi\cdot r\,\mathrm{d}r\right\}\mathrm{d}\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos^4\varphi\cdot a^3}{3}\,\mathrm{d}\varphi \\ &= \frac{a^3}{3\cdot 4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(1+2\cos 2\varphi+\cos^2 2\varphi\right)\mathrm{d}\varphi = \frac{a^3}{12}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{3}{2}+\frac{\cos 4\varphi}{2}\right)\mathrm{d}\varphi \\ &= \frac{a^3}{12}\cdot\frac{3}{2}\cdot\pi = \frac{a^3}{8}\,\pi,\end{aligned}$$

so summing up,

$$C = -\frac{a}{2}\,\pi\left(\frac{a}{2}\right)^2+\frac{a^3}{8}\,\pi = 0.$$

b) Then consider $\mathcal{F}_2$, i.e. the part of the sphere $x^2+y^2+z^2=4a^2$, which lies above B. Then,

$$\mathbf{n} = \frac{1}{2a}\,(x,y,z),$$

because the normal vector is always pointing away from $\mathbf{0}$, and furthermore,

$$\mathbf{n}\cdot\mathbf{rot}\mathbf{V} = \frac{1}{2a}\,(x,y,z)\cdot(-z,0,x) = 0,$$

and it follows trivially that

$$C = \int_{\mathcal{F}_2}\mathbf{n}\cdot\mathbf{rot}\mathbf{V}\,\mathrm{d}S = 0. \qquad \Diamond$$

Example 35.5 When we apply *Green's theorem in the plane* on the vector fields $\mathbf{V}_1 = (0, x)$ and $\mathbf{V}_2 = (-y, 0)$ along a closed curve ∂E, which is the boundary of the domain E, we get

$$\oint_{\partial E} V_{x,1}\,\mathrm{d}x + V_{y,1}\,\mathrm{d}y = \oint_{\partial E} x\,\mathrm{d}y = \int_E \left(\frac{\partial V_{y,1}}{\partial x} - \frac{\partial V_{x,1}}{\partial y}\right)\mathrm{d}S = \int_E \mathrm{d}S = \text{area}(E),$$

and

$$\oint_{\partial E} V_{x,2}\,\mathrm{d}x + V_{y,2}\,\mathrm{d}y = \oint_{\partial E} (-y)\,\mathrm{d}y = \int_E \left(\frac{\partial V_{y,2}}{\partial x} - \frac{\partial V_{x,2}}{\partial y}\right)\mathrm{d}S = \int_E \mathrm{d}S = \text{area}(E),$$

hence

$$\text{area}(E) = \oint_{\partial E} x\,\mathrm{d}y = \oint_{\partial E} (-y)\,\mathrm{d}x = \frac{1}{2}\oint_{\partial E} x\,\mathrm{d}y - y\,\mathrm{d}x,$$

where ∂E has been given the positive orientation in the plane.

Formulæ of this type are applied in e.g. the theory of materials of magnetic hysteresis. $\Diamond$

35.3 Maxwell's equations

Gauß's and *Stokes's* equations are in particular applied in the theory of electro-magnetic fields. We shall here briefly sketch the connection.

35.3.1 The electrostatic field

We consider the space $\mathbb{R}^3$ containing a set Ω of electrically charged particles. The force on any one particle is proportional to the strength of its own charge. The collection of all charged particles defines a vector field, the *electrostatic field* $\mathbf{E}$ in $\mathbb{R}^3$, where the vector $\mathbf{E}(x, y, z)$ at one particular point of the coordinates (x, y, z) is defined as the ratio of the force (a vector) on a test particle at this point to the strength of charge (a scalar) of the test particle.

Consider for the time being the electrostatic field $\mathbf{E}_q$ created by one single particle of charge q, which we may assume lies at origo $\mathcal{O}:\ (0, 0, 0)$. Then experience has shown that the force at a point (x, y, z) in absolute value is inverse proportional to the square of the distance to $(0, 0, 0)$, and proportional to the charge q. We get for $r > 0$

$$|\mathbf{E}_q(x, y, z)| = \frac{1}{\varepsilon}\,\frac{|q|}{r^2}, \qquad \text{where } r^2 = x^2 + y^2 + z^2.$$

The constant ε, which for convenience has been put into the denominator, is characteristic of the medium and is called the *dielectric constant* of the medium. The direction of the vector field $\mathbf{E}_q$ is given by

$$\pm\left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right), \qquad \text{where } r = \sqrt{x^2 + y^2 + z^2}.$$

Here + is chosen if $q > 0$, and - when $q < 0$, because the force is repulsive (away from $\mathbf{0}$), when $q > 0$, and attractive (towards $\mathbf{0}$), when $q < 0$, when applied to a test particle of charge $+1$. Hence,

$$\mathbf{E}_q(x, y, z) = \frac{q}{\varepsilon r^3}(x, y, z), \qquad \text{for } (x, y, z) \neq (0, 0, 0).$$

This is a conservative vector field of the same structure as the *Coulomb vector field* considered previously in Section 33.3.2, so we get immediately that if Ω is an open set containing $\mathbf{0}$, then – cf. the calculations of Section 33.3.2,

$$\int_{\partial\Omega} \mathbf{E}_q \cdot \mathbf{n}\, \mathrm{d}S = \frac{q}{\varepsilon} \cdot 4\pi = \frac{4\pi}{\varepsilon}\, q,$$

because the only difference from the Coulomb vector field is the constant factor $\frac{q}{\varepsilon}$. We note that we in Section 33.3.2 applied *Gauß's theorem* to derive this result.

We have derived for a single particle that the flux through any closed surface surrounding $\mathbf{0}$ is equal to $4\pi/\varepsilon$ times the charge q. It should be equal to div $\mathbf{E}_q$, but it is not, because $\mathbf{E}_q$ is not defined at $\mathbf{0}$. We shall now repair this. If the open set Ω contains many points of charge q_n, say, then we should add all these contributions. It is, however, customary instead to replace their contributions by a smooth distribution of charge of *density* $\varrho(x, y, z, t)$. Then the vector field $\mathbf{E}$ is also smooth, and we get by *Gauß's theorem* the usual connection between the flux through a closed surface and the divergence of $\mathbf{E}$. In particular, when this density is independent of time, and Ω is a small axiparallel parallelepipedum of edge lengths $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$, and containing the point $(x, y, z) \in \Omega$ under consideration, then the net flux out of this infinitesimal volume element Ω is

$$\oint_{\partial\Omega} \mathbf{E} \cdot \mathrm{d}\mathbf{A} = \oint_{\partial\Omega} \mathbf{E} \cdot \mathbf{n}\, \mathrm{d}S \approx \operatorname{div} \mathbf{E}(x, y, z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z,$$

where we have used the (mathematical) *mean value theorem* on the right hand side. Hence, by taking the limit diam $\Omega \to 0$,

$$\operatorname{div} \mathbf{E} = \frac{4\pi}{\varepsilon}\, \varrho.$$

This is called *Gauß's law* in Physics. It is also the first of *Maxwell's four equations.* The constant 4π is the area of the unit sphere, and it occurred only in the derivation, so instead one often writes

$$\varepsilon_0 = \frac{\varepsilon}{4\pi}$$

in books on electro-magnetic field theory. We shall here use both notations.

Gauß's law is usually given in two equivalent versions. The one above, which we have just derived, is called *Gauß's law as a differential equation*, because div is a differential operator,

$$\operatorname{div} \mathbf{E} = \frac{\varrho}{\varepsilon_0} \qquad \left(= \frac{4\pi}{\varepsilon}\, \varrho \right).$$

The other version is *Gauß's law as an integral equation*,

$$\int_{\partial E} \mathbf{E} \cdot \mathrm{d}\mathbf{A} = \frac{1}{\varepsilon_0} \int_{\Omega} \varrho\, \mathrm{d}\Omega, \qquad \text{where } d\mathbf{A} := \mathbf{n}\, \mathrm{d}S.$$

This was actually used in the derivation of Gauß's law as a differential equation, and it follows immediately by insertion and an application of Gauß's (mathematical) theorem, that if the first law is given, then the second one follows. So the two versions are equivalent.

When the *dielectric constant* ε is not a constant in the medium, it is often better instead to use the so-called *displacement field* $\mathbf{D}$, which is given by the two equations,

$$\operatorname{div} \mathbf{D} = 4\pi\varrho, \qquad \text{and} \qquad \mathbf{D} = \varepsilon \mathbf{E}.$$

35.3.2 The magnostatic field

Ferromagnetic materials *behave* as if they were charged with a "magnetic fluid" analogous to the electrical fluid, so one would expect that the magnetic "charge" would be described by a *magnetic vector field* $\mathbf{H}$, just like we obtained the electrical vector field $\mathbf{E}$. In analogy with the dielectric constant ε above we would expect another constant μ, called the *permeability*, such that the displacement vector $\mathbf{D} = \varepsilon\mathbf{E}$ is analogous to the *magnetic induction* $\mathbf{B} = \mu\mathbf{H}$. This construction, however, requires the existence of a unit positive magnetic charge, and no such magnetic charge (a so-called *monopole*) has ever been found. So we must use another procedure.

Since it does not look like that a magnetic monopole exists, we instead of an analogue of the equations of the displacement field above, introduce the following equation for the *induction field* $\mathbf{B}$,

$$\text{div}\ \mathbf{B} = 0.$$

This describes precisely that there is no magnetic monopole. This is actually *Gauß's law for magnetism* formulated as a differential equation,

$$\text{div}\ \mathbf{B} = 0.$$

Then let Ω be a body of surface $\partial\Omega$. According to Gauß's (mathematical) theorem from Chapter 33 the magnetic flux of $\mathbf{B}$ through the closed surface $\partial\Omega$ is

$$\int_{\partial\Omega} \mathbf{B}\cdot\mathbf{n}\,\mathrm{d}S = \int_{\Omega} \text{div}\ \mathbf{B}\,\mathrm{d}\Omega = 0,$$

and we have derived *Gauß's law for magnetism as an integral equation*,

$$\int_{\partial\Omega} \mathbf{B}\cdot\,\mathrm{d}\mathbf{A} = 0, \qquad \mathrm{d}\mathbf{A} := \mathbf{n}\,\mathrm{d}S.$$

This is the second one of Maxwell's equations in its two equivalent formulations.

We saw above that $\mathbf{B}$ is a divergence free vector field, div $\mathbf{B} = 0$.

Let $\mathbf{V}$ be any C^1 divergence free vector field in $\mathbb{R}^3$, and let $\Omega \subset \mathbb{R}^2$ be a domain, such that its boundary $\partial\Omega$ is cut into two surfaces, $\mathcal{F}_1$ and $\mathcal{F}_2$, by a closed C^1 curve $\mathcal{K}$ without double points, cf. Figure 35.6. Choose the normal vector field $\mathbf{n}$, such that it points into Ω on the surface $\mathcal{F}_1$, and away from Ω on the surface $\mathcal{F}_2$. Let Φ_1 and Φ_2 denote the fluxes on $_1$ and $\mathcal{F}_2$, resp.. Then it follows from Gauß's theorem that the outgoing flux (seen from Ω) is

$$\Phi_2 + (-\Phi_1) = \int_{\mathcal{F}_2} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S + \int_{\mathcal{F}_1} \mathbf{V}\cdot(-\mathbf{n})\,\mathrm{d}S = \int_{\Omega} \text{div}\ \mathbf{V}\,\mathrm{d}\Omega = 0,$$

so we get by a rearrangement that $\Phi_2 = \Phi_1$. It therefore follows that we can talk about the flux of $\mathbf{V}$ passing through the loop formed by the closed curve $\mathcal{K}$, because the flux is the same on all (admissible) bounded surfaces having $\mathcal{K}$ as its boundary curve.

This applies in particular to the magnetic field $\mathbf{B}$, and it is therefore possible to talk about a *magnetic flux* surrounded by a closed curve $\mathcal{K}$.

Consider a current I through a closed curve $\mathcal{K}$ as above. Then it produces a magnetic field, which is governed by the *Ampère-Laplace law*,

$$\mathbf{B} = K\,I \oint_{\mathcal{K}} \frac{1}{r^2}\,\mathbf{t}\times\mathbf{u}\,\mathrm{d}s,$$

where K is a constant only depending on the chosen physical units, and $\mathbf{t}$ is he unit tangent field on the curve $\mathcal{K}$, and $\mathbf{u}$ is the unit vector which points from the point on the curve towards the point P under consideration.

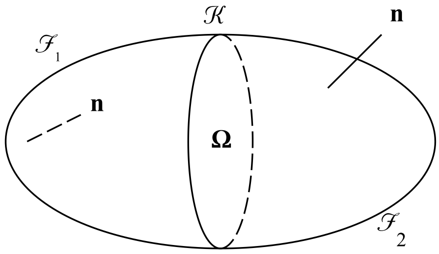

Figure 35.6: When div $\mathbf{V} = 0$, then the flux of $\mathbf{V}$ through a surface depends only on its boundary curve $\mathcal{K}$.

The law above was obtained *after many experiments and measurements on closed curves of different shapes*. It is therefore a mathematical model which gives a good mathematical description of the magnetic field. Note in particular that it is not derived from some mathematical theorem.

It is customary to write $K = \mu_0/4\pi$, where μ_0 is called the *magnetic permeability of vacuum.* Thus we shall in the following write the *Ampère-Laplace law* in the following way,

$$\mathbf{B} = \frac{\mu_0}{4\pi}\, I \oint_{\mathcal{K}} \frac{1}{r^2}\,\mathbf{t}\times\mathbf{u}\,\mathrm{d}s.$$

It follows in particular that a magnetic field is produced when we move electric charges.

If we stretch $\mathcal{K}$ towards an "infinite loop" we get in the limit the magnetic field produced by an infinite rectilinear current I. In fact, choose e.g. $\mathcal{K}_n$ in the (x,z)-space, such that $\mathcal{K}_n$ is composed of the line segments

$$\{0\}\times[-n,n], \qquad [0,n]\times\{n\}, \qquad \{n\}\times[-n,n], \qquad [0,n]\times\{-n\},$$

cf. Figure 35.7.

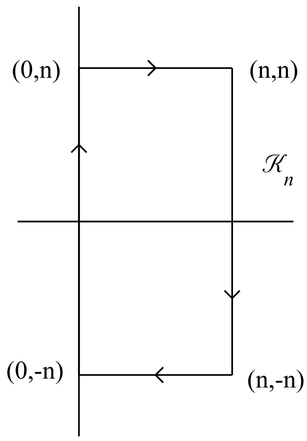

Figure 35.7: The curve $\mathcal{K}_n$.

The line integrals along the three latter line segments are all of size $\sim n^{-1}$, so their contributions tend to zero, when $n \to +\infty$. Therefore, the magnetic field generated by a current $I > 0$ along the infinitely thin z-axis is s point $(x,y,0)$ in the (x,y)-plane given by

$$\mathbf{B} = \frac{\mu_0}{4\pi}\, I \int_{-\infty}^{+\infty} \frac{1}{r^2}\,(0,0,1)\times\left(\frac{x}{r},\frac{y}{r},\frac{z}{r}\right)\,\mathrm{d}z = \frac{\mu_0}{4\pi}\, I\,(-y,x,0)\int_{-\infty}^{+\infty}\frac{1}{r^3}\,\mathrm{d}z,$$

where

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{x^2+y^2}\cdot\sqrt{1+\left(\frac{z}{\sqrt{x^2+y^2}}\right)^2}.$$

If we put

$$z = \sqrt{x^2+y^2}\cdot\sinh t, \qquad \mathrm{d}z = \sqrt{x^2+y^2}\cdot\cosh t\,\mathrm{d}t,$$

we get by this substitution,

$$\mathbf{B} = \frac{\mu_0}{4\pi}\, I\,(-y,x,0)\,\frac{2}{x^2+y^2} = \frac{\mu_0}{2\pi r}\, I\,\left(-\frac{y}{r},\frac{x}{r},0\right) = \frac{\mu_0 I}{2\pi r}\,\mathbf{t} = \frac{\mu_0}{2\pi\,(x^2+y^2)}\,(-y,x,0),$$

where $r = \sqrt{x^2+y^2}$ and $\|\mathbf{t}\| = \|(-y,x,0)/r\| = 1$.

Then we compute the circulation of $\mathbf{B}$ around a circular path of radius $r > 0$. It follows from the above that the magnetic field $\mathbf{B}$ is tangent to this circular path, so

$$\mathbf{B}\cdot\mathbf{t}\,\mathrm{d}s = B\,\mathrm{d}s, \qquad \text{where} \qquad B := \|\mathbf{B}\| = \frac{\mu_0 I}{2\pi r},$$

which is constant in magnitude. The circle C of radius r has the length $\ell(C) = 2\pi r$, so the *magnetic circulation* $\Lambda_{\mathbf{B}}$ is

$$\Lambda_{\mathbf{B}} = \oint_C \mathbf{B}\cdot\mathbf{t}\,\mathrm{d}s = \oint_C B\,\mathrm{d}s = B\int_C \mathrm{d}s = B\ell(C) = 2\pi r B = 2\pi r\cdot\frac{\mu_0 I}{2\pi r} = \mu_0 I.$$

This shows that the magnetic circulation $\Lambda_{\mathbf{B}}$ (also called the *magnetic force*, mmf) is proportional to the electric current I, and it is independent of the radius $r > 0$. *Locally* $\mathbf{B}$ is a gradient field,

$$\mathbf{B} = \frac{\mu_0 I}{2\pi}\,\triangledown\,\mathrm{Arctan}\left(\frac{y}{x}\right) \qquad \text{for } x > 0 \text{ or } x < 0,$$

or

$$\mathbf{B} = \frac{\mu_0 I}{2\pi}\left(\frac{\pi}{2} - \triangledown\mathrm{Arccot}\left(\frac{x}{y}\right)\right) \qquad \text{for } y > 0 \text{ or } y < 0,$$

i.e. a gradient field in each of the four open half-planes. If we therefore consider any closed loop C winding once aroung the infinite current I along the z-axis, then we can deform C into a circle in the plane, supplied with some additional loops, which lie in one of the four half-planes above or can be projected into one of them. In particular, these additional loops do not surround the z-axes, so their total contribution is 0. This argument shows that the magnetic circulation corresponding to a rectilinear current along the infinitely thin z-axis is independent of the closed path, as long as it does not go through points on the z-axis.

We have in the simple example of Figure 35.8 illustrated the technique. It follows in this particular case that

$$\begin{aligned}\oint_C \mathbf{B}\cdot\mathbf{t}\,\mathrm{d}s &= \int_{C_1}\mathbf{B}\cdot\mathbf{t}\,\mathrm{d}s + \int_{C_2}\mathbf{B}\cdot\mathbf{t}\,\mathrm{d}s + \left(\int_{\Gamma_1}\mathbf{B}\cdot\mathbf{t}\,\mathrm{d}s + \int_{\Gamma_2}\mathbf{B}\cdot\mathbf{t}\,\mathrm{d}s\right)\\ &= \int_{C_1+\Gamma_1}\mathbf{B}\cdot\mathbf{t}\,\mathrm{d}s + \int_{C_2+\Gamma_2}\mathbf{B}\cdot\mathbf{t}\,\mathrm{d}s = \mu_0\,I + 0 = \mu_0\,I.\end{aligned}$$

We say that a closed curve C links a current I, if the path of I traverses every bounded C^1-surface $\mathcal{F}$, which has C as its intrinsic boundary.

Using a similar, though more sophisticated technique it is possible to prove that the result is true for any shape of the path of the current, or even currents, so we have derived

Theorem 35.5 Ampère's law for the magnetic field. *The circulation of the magnetic field* $\mathbf{B}$ *along a closed path* C*, which links the currents* I_1, I_2, $\dots$, *is*

$$\Lambda_{\mathbf{B}} = \oint_C \mathbf{B}\cdot\mathbf{t}\,\mathrm{d}s = \mu_0\,(I_1 + I_2 + \cdots) = \mu_0\,I,$$

where $I = I_1 + I_2 + \cdots$ *denotes the total current linked to the path* C.

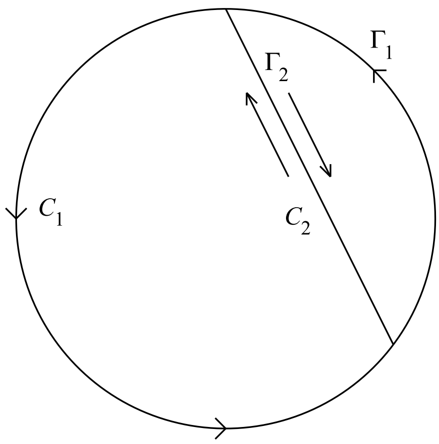

Figure 35.8: By adding the line integral of $\bigtriangledown F$ along $\Gamma = \gamma_1 + \Gamma_2$, we just add 0. On the other hand, the line integrals along C_2 and Γ_2 cancel each other, because they have the same integrand and opposite orientations. Similarly for other closed plane curves as well as closed space curves.

In the next step we shall establish a connection between the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$. The electric field is defined as the force per unit charge, hence the tangential line integral along a curve C is equal to the work done when we move onte unit of charge along the curve C.

When C is closed, this tangential line integral becomes the circulation of the electric field,

$$V_{\mathbf{E},C} = \oint_C \mathbf{E} \cdot \mathbf{t} \, \mathrm{d}s.$$

Then assume that given an electric conductor which forms a closed path C. We place it in a region in which a magnetic field exists.

Let $\mathcal{F}$ be any (admissible) bunded surface which has the closed curve C as its intrinsic boundary. If the magnetic flux $\Phi_{\mathbf{B}} = \int_{\mathcal{F}} \mathbf{B} \cdot \mathbf{n} \, \mathrm{d}S$ through $\mathcal{F}$ *varies with time*, a current is observed in circuit, while the flux is varying, and this current again produces an electric field $\mathbf{E}$, which is called the *induced electric field.*

Also, in this case the physicists have been forced to rely on experiments to set up a model. Measurements of this induced electric field have shown that it depends on the rate of change $\mathrm{d}\Phi_{\mathbf{E}}/\,\mathrm{d}t$. One observed also that the greater the rate of change of the flux, the larger the induced electrical field. Also, the direction in which the induced electrical field acts depends on whether the magnetic flux is increasing or decreasing. One can use the right-hand rule to determine the direction of the act of the induced electric field. If the right-hand thumb points in the direction of the magnetic field, then the induced electrical field acts in the opposite/same direction as the fingers, when the flux increases/decreases. A simple analysis shows that if $\mathrm{d}\Phi_{\mathbf{E}}/\,\mathrm{d}t$ is positive, then the induced electrical field $V_{\mathbf{E}} = \oint_C \mathbf{E} \cdot \mathbf{t} \, \mathrm{d}s$ acts in the negative sense, and *vice versa*, so they have always opposite signs.

More detailed measurements in experiments have shown that if we choose the physical units right, then a mathematical model is as simple as

$$V_{\mathbf{E}} = \oint_C \mathbf{E} \cdot \mathbf{t} \, \mathrm{d}s = -\frac{\mathrm{d}\Phi_{\mathbf{E}}}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{F}} \mathbf{B} \cdot \mathbf{n} \, \mathrm{d}S,$$

and we can – under the assumption of the chosen model above – formulate the following

Theorem 35.6 The Faraday-Henry law of electromagnetic induction. *Let* **B** *be a dynamic magnetic field. Then an electric field* **E** *is induced in any closed circuit. The induced electrical field* **E** *is equal to the negative of the time rate of the magnetic flux through the circuit, i.e.*

$$\oint_C \mathbf{E}\cdot\mathbf{t}\,\mathrm{d}s = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{F}} \mathbf{B}\cdot\mathbf{n}\,\mathrm{d}S.$$

When we apply *Stokes's* (mathematical) *theorem* on the left hand side of the *Faraday-Henry law* of electromagnetic induction, where we use the same surface $\mathcal{F}$ in the resulting surface integral, we get

$$\int_{\mathcal{F}} \mathbf{rot}\mathbf{E}\cdot\mathbf{n}\,\mathrm{d}S = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{F}} \mathbf{B}\cdot\mathbf{n}\,\mathrm{d}S, \quad \text{i.e.} \quad \int_{\mathcal{F}} \left(\mathbf{rot}\mathbf{E} + \frac{\partial \mathbf{B}}{\partial t}\right)\cdot\mathbf{n}\,\mathrm{d}S = 0.$$

This relation must hold for every (admissable) surface $\mathcal{F}$, so using the usual argument we easily conclude that we must have

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{rot}\mathbf{E} = \mathbf{0}, \qquad \text{i.e.} \qquad \mathbf{rot}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

Conversely, this equation immediately implies Theorem 35.6, so the two results are equivalent.

This result in its two versions is also called *Maxwell's third equation.*

Finally, by inspecting the first three of Maxwell's equations above we get the feeling that there should also exist a result containing the circulation of the magnetic field $\mathbf{B}$ along a closed curve C, i.e. $\oint_C \mathbf{B}\cdot\mathbf{t}\,\mathrm{d}s$, when the field is not static. At this step this is only a hunch guided by some vague sense of symmetry, but we shall see in the following that our guess is actually giving us the right equation.

Let us start by analyzing the *Faraday-Henry law*, Theorem 35.6, above, i.e.

$$\oint_C \mathbf{E}\cdot\mathbf{t}\,\mathrm{d}s = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{F}} \mathbf{B}\cdot\mathbf{n}\,\mathrm{d}S.$$

A symmetric statement, where $\mathbf{E}$ and $\mathbf{B}$ have been interchanged, should therefore contain the time rate of change of flux of the electric, i.e. we may expect that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{F}} \mathbf{E}\cdot\mathbf{n}\,\mathrm{d}S$$

enters. However, when we look at the static case, i.e. *Ampère's law*, Theorem 35.5, which should hold in the limit, we get instead

$$\oint_C \mathbf{B}\cdot\mathbf{t}\,\mathrm{d}s = \mu_0\, I,$$

which apparently has nothing to do with a time rate of change of flux, so something must be missing.

The trick is the following: Consider some closed surface $\mathcal{F}$ and cut it by some closed oriented curve C on $\mathcal{F}$ into two. We let $\mathcal{F}_C$ denote the part of $\mathcal{F}$, which is given the right-hand orientation induced by the orientation of C. We keep $\mathcal{F}$ fixed and let C shrink to a point. Then in some obvious sense $\mathcal{F}_C \to \mathcal{F}$, and clearly,

$$\lim_{C\to 0}\oint_C \mathbf{B}\cdot\mathbf{t}\,\mathrm{d}s = 0.$$

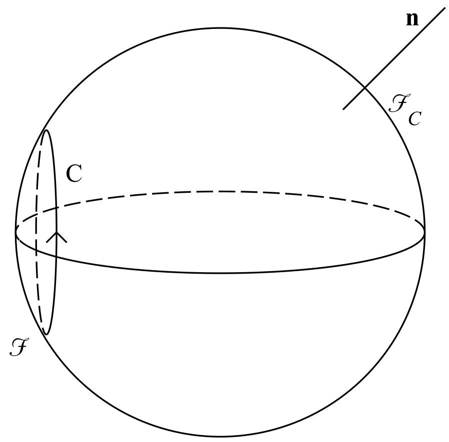

Figure 35.9: The closed surface $\mathcal{F}$ is cut into two by the closed oriented curve C, where we choose the part $\mathcal{F}_C$, for which the normal vector field induced by the orientation of C is pointing outwards from $\mathcal{F} \supset \mathcal{F}_C$.

Let $q(t)$ in the dynamic case denote the net charge inside the bounded closed surface $\mathcal{F}$ at a given time t. Then the net outgoing charge flux passing through $\mathcal{F}$ per unit time is a current $I_{\mathcal{F}}$, so

$$I_{\mathcal{F}} = -\frac{\mathrm{d}q}{\mathrm{d}t},$$

with a minus sign, because when $q(t)$ decreases in t, then the current $I_{\mathcal{F}}$ is leaving $\mathcal{F}$ in its positive direction.

Given the dynamic electric field $\mathbf{E}(t)$, the total charge within $\mathcal{F}$ is expressed by the surface integral

$$q(t) = \varepsilon_0 \int_{\mathcal{F}} \mathbf{E}(t) \cdot \mathbf{n}\,\mathrm{d}S,$$

hence

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \varepsilon_0 \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{F}} \mathbf{E}(t) \cdot \mathbf{n}\,\mathrm{d}S = \varepsilon_0 \int_{\mathcal{F}} \frac{\partial \mathbf{E}}{\partial t} \cdot \mathbf{n}\,\mathrm{d}S \qquad (= -I_{\mathcal{F}})\,.$$

It therefore follows that

$$I_{\mathcal{F}} + \varepsilon_0 \int_{\mathcal{F}} \mathbf{E}(t) \cdot \mathbf{n}\,\mathrm{d}S = 0.$$

Then let $\mathcal{G} \subset \mathcal{F}$ be any sub-surface of the bounded closed surface $\mathcal{F}$ with the closed curve C as its intrinsic boundary, and then replace I in *Ampère's law*

$$\oint_C \mathbf{B} \cdot \mathbf{t}\,\mathrm{d}s = \mu_0\, I$$

by

$$I = I_{\mathcal{G}} + \varepsilon_0 \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{G}} \mathbf{E}(t) \cdot \mathbf{n}\,\mathrm{d}S.$$

By this qualified *guess* we replace *Ampère's law* by

$$\oint_C \mathbf{B}(t) \cdot \mathbf{t}\,\mathrm{d}s = \mu_0 \left\{ I_{\mathcal{G}} + \varepsilon_0 \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{G}} \mathbf{E}(t) \cdot \mathbf{n}\,\mathrm{d}S \right\}.$$

We see that we keep the structure of *Ampère's law* and at the same time obtain that if C shrinks towards a point, then we indeed get 0 as requested.

We coin the above model in the following

Theorem 35.7 The Ampère-Maxwell law. *Let $\mathbf{E}(t)$ be a dynamic electric field with the corresponding current density $\mathbf{J}$, and let $\mathcal{F}$ be an oriented surface with the oriented closed curve C as its intrinsic boundary. Then the current I through $\mathcal{F}$ is given by*

$$I = \int_{\mathcal{F}} \mathbf{J}\,\mathrm{d}S.$$

Furthermore, a dynamic magnetic field $\mathbf{B}$ is induced, and

$$\oint_C \mathbf{B} \cdot \mathbf{t}\,\mathrm{d}s = \mu_0 \left\{ I + \varepsilon_0 \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{F}} \mathbf{E}(t) \cdot \mathbf{n}\,\mathrm{d}S \right\} = \mu_0 \left\{ \int_{\mathcal{F}} \mathbf{J}\,\mathrm{d}S + \varepsilon_0 \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{F}} \mathbf{E}(t) \cdot \mathbf{n}\,\mathrm{d}S \right\}.$$

This is the fourth of the so-called *Maxwell's equations*.

Then note that according to *Stokes's theorem*, the left hand side is also equal to

$$\begin{aligned}\int_{\mathcal{F}} \mathbf{rot}\mathbf{B}\cdot\mathbf{n}\,\mathrm{d}S &= \left(\oint_C \mathbf{B}\cdot\mathbf{t}\,\mathrm{d}s\right) = \mu_0\left\{I+\varepsilon_0\,\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{F}}\mathbf{E}(t)\cdot\mathbf{n}\,\mathrm{d}S\right\}\\ &= \mu_0\left\{\int_{\mathcal{F}}\mathbf{J}\,\mathrm{d}S+\varepsilon_0\,\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{F}}\mathbf{E}(t)\cdot\mathbf{n}\,\mathrm{d}S\right\}.\end{aligned}$$

Hence, by a rearrangement,

$$0=\int_{\mathcal{F}}\left\{\mathbf{rot}\mathbf{B}-\mu_0\left(\mathbf{J}+\varepsilon_0\,\frac{\partial\mathbf{E}}{\partial t}\right)\right\}\cdot\mathbf{n}\,\mathrm{d}S,$$

This equation holds for every choice of an admissible surface $\mathcal{F}$. This is only possible, if the first factor of the integrand is 0,

$$\mathbf{rot}\mathbf{B}-\mu_0\left(\mathbf{J}+\varepsilon_0\,\frac{\partial\mathbf{E}}{\partial t}\right)=0,$$

and we have shown the following pointwise result,

$$\mathbf{rot}\mathbf{B}=\mu_0\left(\mathbf{J}+\varepsilon_0\,\frac{\partial\mathbf{E}}{\partial t}\right).$$

Conversely, it is obvious that Theorem 35.7 follows from this pointwise equation, so the two results are equivalent.

Note that Theorem 35.7 so far *is only a model derived under the assumption* that we can extend the static Ampère's law

$$\oint_C \mathbf{B}(t)\cdot\mathbf{t}\,\mathrm{d}s=\mu_0\,I$$

to the dynamic case by replacing I by adding a term concerning the time rate of the flux of the electric field,

$$I+\varepsilon_0\,\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{F}}\mathbf{E}(t)\cdot\mathbf{n}\,\mathrm{d}S.$$

It has therefore afterwards been necessary through the years to verify this model experimentally, and it turned up that it really describes the actual situation found in nature.

35.3.3 Summary of Maxwell's equations

It is customary to collect all Maxwell's equations in a separate section, so this is also done here. First we make some preparations. The notion of an *electromagnetic field* is characterized by a vector consisting of the *electric field* $\mathbf{E}$ and the *magnetic field* $\mathbf{B}$. They produce a force on an electric charge q, which is given by

$$\mathbf{F}=q(\mathbf{E}+\mathbf{v}\times\mathbf{B}),$$

where the charge q is moving with the velocity $\mathbf{v}$. This mathematical model has been verified by many experiments and measurements. It does not follow from mathematics alone.

Then we shall use the *nabla notation*, i.e.

$$\mathbf{grad}\mathbf{V} = \triangledown \mathbf{V}, \qquad \text{div}\,\mathbf{V} = \triangledown \cdot \mathbf{V}, \qquad \text{and} \qquad \mathbf{rot}\mathbf{V} = \triangledown \times \mathbf{V},$$

suggesting the strong connection with Linear Algebra and Geometry in the Euclidean space $E_3 \sim \mathbb{R}^3$. In some books **rot** is written **curl** instead.

We then turn to Maxwell's equations.

Maxwell's first law = Gaußs law.

1) As integrals, where q denotes the density of the charge,

$$\int_{\partial\Omega} \mathbf{E} \cdot \mathbf{n} \, \mathrm{d}S = \int_{\Omega} \nabla \cdot \mathbf{E} \, \mathrm{d}V = \frac{1}{\varepsilon_0} \int_{\Omega} \varrho \, \mathrm{d}V.$$

2) Pointwise,

$$\nabla \cdot \mathbf{E} = \frac{\varrho}{\varepsilon_0}.$$

Maxwell's second law = Gauß's law on magnetism. We assume that no magnetic monopole exists.

1) As a surface integral, where $\mathcal{F}$ is a closed surface,

$$\int_{\mathcal{F}} \mathbf{B} \cdot \mathbf{n} \, \mathrm{d}S = 0.$$

2) Pointwise,

$$\nabla \cdot \mathbf{B} = 0.$$

Maxwell's third law = Faraday-Henry's law of electromagnetic induction.

1) As integrals, where C is a closed curve, and $\mathcal{F}$ is any surface having C as its intrinsic boundary,

$$\oint_C \mathbf{E} \cdot \mathbf{t} \, \mathrm{d}s = -\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{F}} \mathbf{B} \cdot \mathbf{n} \, \mathrm{d}S.$$

2) Pointwise,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

Maxwell's fourth law = Ampère-Maxwell's law. Let **J** denote the *vectorial current density*, i.e. the current I through a surface $\mathcal{F}$ is given by

$$I = \int_{\mathcal{F}} \mathbf{J} \cdot \mathbf{n} \, \mathrm{d}S.$$

1) As integrals,

$$\oint_C \mathbf{B} \cdot \mathbf{t} \, \mathrm{d}s = \mu_0 \int_{\mathcal{F}} \mathbf{J} \cdot \mathbf{n} \, \mathrm{d}S + \mu_0 \varepsilon_0 \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{F}} \mathbf{E} \cdot \mathbf{n} \, \mathrm{d}S.$$

2) Pointwise,

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon \frac{\partial \mathbf{E}}{\partial t}.$$

Maxwell's four equations together with the above mentioned law of the force on a particle of charge q and of velocity $\mathbf{v}$,

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}),$$

form the basis of the electromagnetic theory. In the description above some effort has been made to explain, when a result follows from mathematics, in particular from Gauß's and Stokes's theorems, and when some experimental measurements have been necessary to set up the right mathematical model. In particular, in the derivation of *Ampère-Maxwell's law* we first made some reasonable mathematical assumptions, though in the beginning were not justified. By using some mathematics we deduced a possible model, and finally this model was indeed verified through the years by lots of measurements on experiments.

35.4 Procedure for calculation of the rotation of a vector field and application of Stokes's theorem

Let us first mention the most practical way of calculating the *rotation* of a 3-dimensional C^1 vector field by means of a formal determinant,

$$\mathbf{rot}\ \mathbf{V} = \bigtriangledown \times \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}, \qquad \mathbf{V} = (V_x, V_y, V_z)\,.$$

Note that this rule is only valid *when rectangular coordinates are used!*

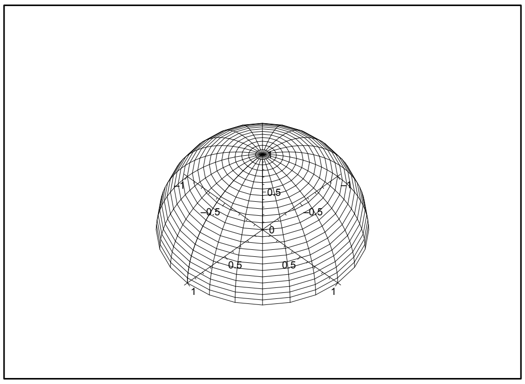

Figure 35.10: As an example we choose again the surface of the upper unit half sphere $\mathcal{F}$, where the boundary curve $\delta\mathcal{F}$ is the unit circle in the (x, y)-plane.

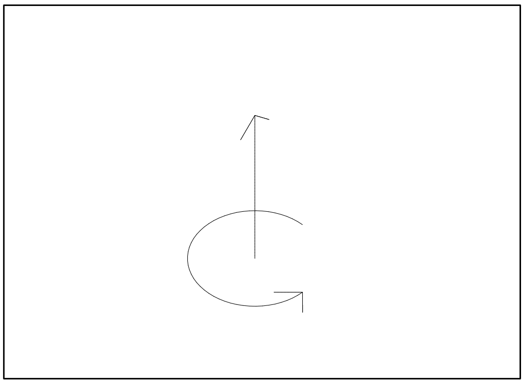

Figure 35.11: The connection between a vector and the corresponding curl which fixes the orientation.

The fixing of the *orientation of a surface* (i.e. the *direction of the normal vector*), when the *direction of the run through of the boundary curve*, is given by:

- **The convention of orientation:** Let the *normal vector field* on the surface $\mathcal{F}$ be supplied with a *curl* around the foot of the normal vector. The *direction* of the curl is by continuity fixed by the *direction* of the run through of the boundary curve. If you put your right hand along the normal vector with the four fingers in the direction of the curl, then your thumb will point in the direction of the normal. Cf. also Figure 35.11.

Whenever *circulation of a vector field* is mentioned, one should think of an application of **Stokes's theorem:**

$$\oint_{\delta\mathcal{F}} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s = \int_{\mathcal{F}} \mathbf{n}\cdot\mathbf{rot}\;\mathbf{V}\,\mathrm{d}S,$$

where the left hand side is the circulation of $\mathbf{V}$ (1 dimension), and the right hand side is a flux of the vector field $\mathbf{rot}\;\mathbf{V}$ (2 dimensions).

In exercises the student will often in an earlier question have calculated $\mathbf{rot}\;\mathbf{V}$, so the task is reduced to the choice of a convenient surface $\mathcal{F}$ for the given closed (boundary) curve $\delta\mathcal{F}$. As long as the student is learning these ideas, the surface $\mathcal{F}$ will typically either be flat or a part of a sphere.

Remark 35.4 Stokes's theorem can also be applied from the right to the left. If e.g. one shall find the *flux* $\int_{\mathcal{F}} \mathbf{n}\cdot\mathbf{U}\,\mathrm{d}S$ of a *rotational field* $\mathbf{U} = \mathbf{rot}\;\mathbf{V}$, and $\mathbf{V}$ is *fairly easy to find*, then

$$\int_{\mathcal{F}} \mathbf{n}\cdot\mathbf{U}\,\mathrm{d}S = \int_{\mathcal{F}} \mathbf{n}\cdot\mathbf{rot}\;\mathbf{V}\,\mathrm{d}S = \oint_{\delta\mathcal{F}} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s. \qquad \Diamond$$

35.5 Examples of calculation of rotation of a vector field and application of Stokes's theorem

35.5.1 Examples of divergence and rotation of a vector field

Example 35.6 *Find coordinate expressions of the vector fields* **rot**(**rot V**) *and* **grad**(div **V**). Then prove the formula

$$\mathbf{grad}(\text{div } \mathbf{V}) - \mathbf{rot}(\mathbf{rot}\ \mathbf{V}) = (\text{div}(\mathbf{grad}\ V_x), \text{div}(\mathbf{grad}\ V_y), \text{div}(\mathbf{grad}\ V_z)).$$

A Calculations using nabla.

D The results can be obtained by very mechanical calculations. It is a matter of taste whether one prefers the notation above or

$$\mathbf{grad} = \bigtriangledown, \qquad \text{div} = \bigtriangledown\cdot, \qquad \mathbf{rot} = \bigtriangledown\times.$$

We shall here use the latter, thereby keeping the formal connection to the geometric relationships that the operations are describing.

I Let **V** ne a vector field of class C^2. Then

$$\mathbf{rot}\mathbf{V} = \bigtriangledown \times \mathbf{V} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\mathbf{e}_x + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\mathbf{e}_y + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\mathbf{e}_z.$$

By repeating this pattern we get for the double rotation that

$$\begin{aligned}\mathbf{rot}(\mathbf{rot}\ \mathbf{V}) &= \bigtriangledown\times(\bigtriangledown\times\mathbf{V})\\ &= \left(\frac{\partial}{\partial y}\left\{\frac{\partial V_y}{\partial x}-\frac{\partial V_x}{\partial y}\right\}-\frac{\partial}{\partial z}\left\{\frac{\partial V_x}{\partial z}-\frac{\partial V_z}{\partial x}\right\}\right)\mathbf{e}_x\\ &\quad+\left(\frac{\partial}{\partial z}\left\{\frac{\partial V_z}{\partial y}-\frac{\partial V_y}{\partial z}\right\}-\frac{\partial}{\partial x}\left\{\frac{\partial V_y}{\partial x}-\frac{\partial V_x}{\partial y}\right\}\right)\mathbf{e}_y\\ &\quad+\left(\frac{\partial}{\partial x}\left\{\frac{\partial V_x}{\partial z}-\frac{\partial V_z}{\partial x}\right\}-\frac{\partial}{\partial y}\left\{\frac{\partial V_z}{\partial y}-\frac{\partial V_y}{\partial z}\right\}\right)\mathbf{e}_z,\end{aligned}$$

thus

$$\begin{aligned}&\mathbf{rot}(\mathbf{rot}\ \mathbf{V}) = \bigtriangledown\times(\bigtriangledown\mathbf{V})\\ &\quad=\left(-\frac{\partial^2 V_x}{\partial y^2}-\frac{\partial^2 V_x}{\partial z^2}+\frac{\partial^2 V_z}{\partial x\partial z}+\frac{\partial^2 V_y}{\partial x\partial y}\right)\mathbf{e}_x\\ &\qquad+\left(-\frac{\partial^2 V_y}{\partial x^2}-\frac{\partial^2 V_y}{\partial z^2}+\frac{\partial^2 V_y}{\partial z^2}+\frac{\partial^2 V_x}{\partial x\partial y}+\frac{\partial^2 V_z}{\partial y\partial z}\right)\mathbf{e}_y\\ &\qquad+\left(-\frac{\partial^2 V_z}{\partial x^2}-\frac{\partial^2 V_z}{\partial y^2}+\frac{\partial^2 V_y}{\partial y\partial z}+\frac{\partial^2 V_z}{\partial x\partial z}\right)\mathbf{e}_z\\ &\quad=-\left(\bigtriangledown^2 V_x,\bigtriangledown^2 V_y,\bigtriangledown^2 V_z\right)\\ &\qquad+\frac{\partial}{\partial x}\left\{\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}\right\}\mathbf{e}_x\\ &\qquad+\frac{\partial}{\partial y}\left\{\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}\right\}\mathbf{e}_y\\ &\qquad+\frac{\partial}{\partial z}\left\{\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}\right\}\mathbf{e}_z\\ &\quad=-\bigtriangledown^2\mathbf{V}+\bigtriangledown(\bigtriangledown\cdot\mathbf{V})\\ &\quad=-(\mathrm{div}(\mathbf{grad}\ V_x),\mathrm{div}(\mathbf{grad}\ V_y),\mathrm{div}(\mathbf{grad}\ V_z))+\mathbf{grad}(\mathrm{div}\ \mathbf{V}),\end{aligned}$$

and the formula follows by a rearrangement.

REMARK 1. Note that the formula can also be written

$$\bigtriangledown(\bigtriangledown\cdot\mathbf{V})-\bigtriangledown\times(\bigtriangledown\times\mathbf{V})=\bigtriangledown^2\mathbf{V}.\quad\Diamond$$

REMARK 2. We note for completeness

$$\begin{aligned}\bigtriangledown(\bigtriangledown\cdot\mathbf{V}) &= \bigtriangledown\left(\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}\right)\\ &=\left(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_y}{\partial x\partial y}+\frac{\partial^2 V_z}{\partial x\partial z}\right)\mathbf{e}_x+\left(\frac{\partial^2 V_x}{\partial x\partial y}+\frac{\partial^2 V_y}{\partial y^2}+\frac{\partial^2 V_z}{\partial y\partial z}\right)\mathbf{e}_y\\ &\quad+\left(\frac{\partial^2 V_x}{\partial x\partial z}+\frac{\partial^2 V_y}{\partial y\partial z}+\frac{\partial^2 V_z}{\partial z^2}\right)\mathbf{e}_z.\qquad\Diamond\end{aligned}$$

Example 35.7 *Find div* **V** *and* **rot V** *for each of the following vector fields on* $\mathbb{R}^3$.

1) $\mathbf{V}(x,y,z) = (xz, -y^2, 2x^2y)$.

2) $\mathbf{V}(x,y,z) = (z+\sin y, -z+\cos y, 0)$.

3) $\mathbf{V}(x,y,z) = (e^{xy}, \cos(xy), \cos(xz^2))$.

4) $\mathbf{V}(x,y,z) = (x^2+yz, y^2+xz, z^2+xy)$.

5) $\mathbf{V}(x,y,z) = (x+\text{Arctan } y, 3x-z, 2^{yz})$.

6) $\mathbf{V}(x,y,z) = (xz^3, -2x^2yz, 2yz^4)$.

7) $\mathbf{V}(x,y,z) = (\sinh(xyz), z, x)$.

8) $\mathbf{V}(\text{Arctan } z, \text{Arctan } x, \text{Arctan } y)$.

A This is just a simple exercise in finding the divergence and the rotation.

D Insert into the formulæ

$$\text{div } \mathbf{V} = \bigtriangledown \cdot \mathbf{V} \quad \text{and} \quad \mathbf{rot}\ \mathbf{V} = \bigtriangledown \times \mathbf{V}.$$

I 1) We get for $\mathbf{V} = (xz, -y^2, 2x^2y)$,

$$\text{div } \mathbf{V} = z - 2y + 0 = z - 2y$$

and

$$\mathbf{rot}\ \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xz & -y^2 & 2x^2y \end{vmatrix} = (2x^2, x-4xy, 0).$$

2) We get for $\mathbf{V} = (z+\sin y, -z+\cos y, 0)$,

$$\text{div } \mathbf{V} = 0 - \sin y + 0 = -\sin y$$

and

$$\mathbf{rot}\ \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z+\sin y & -z+\cos y & 0 \end{vmatrix} = (1, 1, -\cos y).$$

3) We get for $\mathbf{V} = (e^{xy}, \cos(xy), \cos(xz^2))$,

$$\text{div } \mathbf{V} = y\, e^{xy} - x\, \sin(xy) - 2xz\, \sin(xz^2)$$

and

$$\mathbf{rot}\ \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^{xy} & \cos(xy) & \cos(xz^2) \end{vmatrix} = (0, z^2 \sin(xz^2), -y\ \sin(xy) - x\, e^{xy}).$$

4) We get for $\mathbf{V} = (x^2 + yz, y^2 + xz, z^2 + xy)$,

$$\text{div}\ \mathbf{V} = 2(x+y+z)$$

and

$$\mathbf{rot}\ \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2+yz & y^2+xz & z^2+xy \end{vmatrix} = (x-x, y-y, z-z) = (0,0,0) = \mathbf{0}.$$

5) When

$$\mathbf{V} = (x + \text{Arctan}\ y, 3x - z, e^{yz}) = \left(x + \text{Arctan}\ y, 3x - z, e^{\ln 2 \cdot yz}\right)$$

we find

$$\text{div}\ \mathbf{V} = 1 + 0 + \ln 2 \cdot y \cdot e^{\ln 2 \cdot yz} = 1 + \ln 2 \cdot y \cdot 2^{yz}$$

and

$$\mathbf{rot}\ \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+\text{Arctan}\ y & 3x-z & e^{\ln 2 \cdot yz} \end{vmatrix} = \left(\ln 2 \cdot z \cdot 2^{yz} + 1, 0, 3 - \frac{1}{1+y^2}\right).$$

6) If $\mathbf{V} = (xz^3, -2x^2yz, 2yz^4)$, then

$$\text{div}\ \mathbf{V} = z^3 - 2x^2z + 8yz^3$$

and

$$\mathbf{rot}\ \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix} = (2z^4 + 2x^2y, 3xz^2, -4xyz).$$

7) If $\mathbf{V} = (\sinh(xyz), z, x)$, then

$$\text{div } \mathbf{V} = yz \cosh(xyz)$$

and

$$\textbf{rot } \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \sinh(xyz) & z & x \end{vmatrix} = (-1, xy \cosh(xyz) - 1, -xz \cos(xyz)).$$

8) If $\mathbf{V} = (\text{Arctan } z, \text{Arctan } x, \text{Arctan } y)$ then

$$\text{div } \mathbf{V} = 0$$

and

$$\textbf{rot } \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \text{Arctan } z & \text{Arctan } x & \text{Arctan } y \end{vmatrix} = \left(\frac{1}{1+y^2}, \frac{1}{1+z^2}, \frac{1}{1+x^2} \right).$$

Example 35.8 *Find the divergence and the rotation of the vector field (the so-called* Coulomb vector field*),*

$$\mathbf{V}(x,y,z) = \frac{1}{r^3}\,(x,y,z), \qquad (x,y,z) \neq (0,0,0), \qquad r = \sqrt{x^2+y^2+z^2}.$$

[*Cf.* **Example 33.14**]

A Divergence and rotation.

D Compute div $\mathbf{V} = \bigtriangledown \cdot \mathbf{V}$ and **rot** $\mathbf{V} = \bigtriangledown \times \mathbf{V}$.

I First note that

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \qquad \frac{\partial r}{\partial y} = \frac{y}{r} \quad \text{og} \quad \frac{\partial r}{\partial z} = \frac{z}{r}.$$

These are easy rules of calculations, by which

$$\begin{aligned}\text{div}\,\mathbf{V} &= \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right)\\ &= \left(\frac{1}{r^3} - \frac{3x}{r^4}\,\frac{\partial}{\partial x}\right) + \left(\frac{1}{r^3} - \frac{3y}{r^4}\,\frac{\partial r}{\partial y}\right) + \left(\frac{1}{r^3} - \frac{3z}{r^4}\,\frac{\partial r}{\partial z}\right)\\ &= \frac{3}{r^3} - \frac{3}{r^4}\cdot\frac{1}{r}\,(x^2+y^2+z^2) = \frac{3}{r^3} - \frac{3}{r^3} = 0,\end{aligned}$$

and

$$\begin{aligned}\textbf{rot}\ \mathbf{V} &= \begin{vmatrix} \mathbf{e}x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{r^3} & \frac{y}{r^3} & \frac{z}{r^3} \end{vmatrix}\\ &= \left\{ z\,\frac{\partial}{\partial y}\left(\frac{1}{r^3}\right) - y\,\frac{\partial}{\partial z}\left(\frac{1}{r^3}\right)\right\}\mathbf{e}_x + \left\{ x\,\frac{\partial}{\partial z}\left(\frac{1}{r^3}\right) - z\,\frac{\partial}{\partial x}\left(\frac{1}{r^3}\right)\right\}\mathbf{e}_y\\ &\qquad + \left\{ y\,\frac{\partial}{\partial x}\left(\frac{1}{r^3}\right) - x\,\frac{\partial}{\partial y}\left(\frac{1}{r^3}\right)\right\}\mathbf{e}_z\\ &= -\frac{3}{r^4}\left(z\,\frac{\partial r}{\partial y} - y\frac{\partial r}{\partial z}, x\,\frac{\partial r}{\partial z} - z\,\frac{\partial r}{\partial x}, y\,\frac{\partial r}{\partial x} - x\,\frac{\partial r}{\partial y}\right)\\ &= -\frac{3}{r^4}\left(z\cdot\frac{y}{r} - y\cdot\frac{z}{r}\, x\cdot\frac{z}{r} - z\cdot\frac{x}{r}, y\cdot\frac{x}{r} - x\cdot\frac{y}{r}\right) = (0,0,0).\end{aligned}$$

REMARK. A variant is

$$\mathbf{rot}\;\mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{x}{r^3} & \dfrac{y}{r^3} & \dfrac{z}{r^3} \end{vmatrix} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x}\left(\dfrac{1}{r^3}\right) & \dfrac{\partial}{\partial y}\left(\dfrac{1}{r^3}\right) & \dfrac{\partial}{\partial z}\left(\dfrac{1}{r^3}\right) \\ x & y & z \end{vmatrix}$$

$$= -\frac{3}{r^4}\begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial r}{\partial x} & \dfrac{\partial r}{\partial y} & \dfrac{\partial r}{\partial z} \\ x & y & z \end{vmatrix} = -\frac{3}{r^5}\begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ x & y & x \\ x & y & z \end{vmatrix} = -\frac{3}{r^5}\,\mathbf{x}\times\mathbf{x} = \mathbf{0}. \qquad \Diamond$$

Example 35.9 *Choose the constants α and β, such that the vector field*

$$\mathbf{V}(x,y,z) = (xyz)^{\beta}\,\left(x^{\alpha},y^{\alpha},z^{\alpha}\right), \qquad (x,y,z)\in\mathbb{R}_+^3,$$

has zero rotation.

A Rotation free vector field.

D Compute **rot V**.

I We get by a calculation,

$$\mathbf{rot}\;\mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^{\alpha+\beta}y^{\beta}z^{\beta} & x^{\beta}y^{\alpha+\beta} & x^{\beta}y^{\beta}z^{\alpha+\beta} \end{vmatrix}$$

$$= \beta\left(x^{\beta}y^{\beta-1}z^{\alpha+\beta} - x^{\beta}y^{\alpha+\beta}z^{\beta-1}\right)\mathbf{e}_x + \beta\left(x^{\alpha+\beta}y^{\beta}z^{\beta-1} - x^{\beta-1}y^{\beta}z^{\alpha+\beta}\right)\mathbf{e}_y + \beta\left(x^{\beta-1}y^{\alpha+\beta}z^{\beta} - x^{\alpha+\beta}y^{\beta-1}z^{\beta}\right)\mathbf{e}_z$$

$$= \beta(xyz)^{\beta}\left(y^{-1}z^{\alpha} - y^{\beta}z^{-1}, x^{\alpha}z^{-1} - x^{-1}z^{\alpha}, x^{-1}y^{\alpha} - x^{\alpha}y^{-1}\right).$$

If $\beta = 0$, then the factor outside the vector is 0, and the vector field becomes rotation free in $\mathbb{R}_+^3$. This corresponds to the vector field

$$\mathbf{V}(x,y,z) = \left(x^{\alpha},y^{\alpha},z^{\alpha}\right), \qquad \alpha\in\mathbb{R},$$

where the condition $(x,y,z)\in\mathbb{R}_+^3$ assures that the vector is always defined.

The second possibility is that the vector is

$$\left(y^{-1}z^{\alpha} - y^{\alpha}z^{-1}, x^{\alpha}z^{-1} - x^{-1}z^{\alpha}, x^{-1}y^{\alpha} - x^{\alpha}y^{-1}\right) = \mathbf{0}.$$

This gives the condition $\alpha = -1$, in which case the vector field becomes

$$\mathbf{V}(x,y,z) = \left(x^{\beta-1}y^{\beta}z^{\beta}, x^{\beta}y^{\beta-1}z^{\beta}, x^{\beta}y^{\beta}z^{\beta-1}\right) = (xyz)^{\beta}\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$$

which is also free of rotation.

35.5.2 General examples

Example 35.10

A. *Find the circulation C of the vector field*

$$\mathbf{V}(x,y,z) = \left(z^2x, x^2y, y^2z\right), \qquad (x,y,z) \in \mathbb{R}^3,$$

along the curve $\mathcal{K}$ on the figure which is composed of three circle arcs of centre $\mathbf{0}$ *and radius a, and which lie in the planes $z=0$, $y=-x$ and $x = y\sqrt{3}$, respectively.*

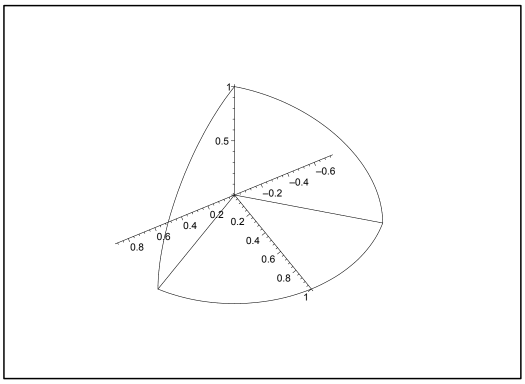

Figure 35.12: The curve $\mathcal{K}$ for $a = 1$.

D. A circulation along a closed curve can either be calculated by using its definition as an ordinary *line integral*, or it can be transformed by means of *Stokes's theorem* to a surface integral. Most students are at their first encounter with this problem inclined to preferring the line integral, because it should now be better known, even when applications of *Stokes's theorem* very often give much simpler calculations. For that reason we shall here demonstrate both variants.

I 1. *The circulation as a line integral.*

Let us first give the parametric representations of the three arcs, which $\mathcal{K}$ can be composed into in a natural way:

$$\mathcal{K}_1: \quad (x,y,z) = a(\cos\varphi, \sin\varphi, 0), \qquad \varphi \in \left[\frac{\pi}{6}, \frac{3\pi}{4}\right],$$

$$\mathcal{K}_2: \quad (x,y,z) = a\left(-\frac{1}{\sqrt{2}}\cos\theta, \frac{1}{\sqrt{2}}\cos\theta, \sin\theta\right), \quad \theta \in \left[0, \frac{\pi}{2}\right],$$

$$\mathcal{K}_3: \quad (x,y,z) = a\left(\frac{\sqrt{3}}{2}\sin\theta\, \frac{1}{2}\sin\theta, \cos\theta\right), \qquad \theta \in \left[0, \frac{\pi}{2}\right].$$

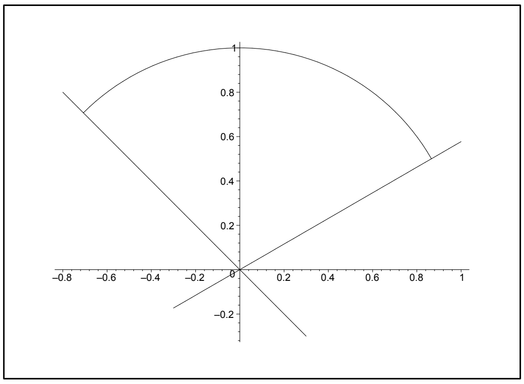

Figure 35.13: The projection onto the (x, y)-plane for $a = 1$.

Check here that all three curves are circular arcs of radius a and centre $\mathbf{0}$. Then check that the initial and the end points are the right ones, i.e. that the direction of the run through is correct. (Here this is left to the reader).

In order to avoid too many complications in the calculations we rewrite the integrand in the following way,

$$\begin{aligned}\mathbf{V}\cdot \mathrm{d}\mathbf{x} &= z^2 x\,\mathrm{d}x + x^2 y\,\mathrm{d}y + y^2 z\,\mathrm{d}z \\ &= \frac{1}{2}\left\{z^2\,\mathrm{d}\left(x^2\right) + x^2\,\mathrm{d}\left(y^2\right) + y^2\,\mathrm{d}\left(z^2\right)\right\}.\end{aligned}$$

By insertion we get the following confusing picture of formulæ, where there are lots of possibilities of making errors,

$$\begin{aligned}
C &= \frac{1}{2}\int_{\mathcal{K}_1} + \frac{1}{2}\int_{\mathcal{K}_2} + \frac{1}{2}\int_{\mathcal{K}_3} \left(z^2\,\mathrm{d}\left(x^2\right) + x^2\,\mathrm{d}\left(y^2\right) + y^2\,\mathrm{d}\left(z^2\right)\right) \\
&= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{3\pi}{4}} \left\{0\cdot\frac{\mathrm{d}(x^2)}{\mathrm{d}\varphi} + a^2\cos^2\varphi\cdot 2a^2\,\sin\varphi\cdot\cos\varphi + y^2\cdot 0\right\}\mathrm{d}\varphi \\
&\quad + \frac{1}{2}\int_0^{\frac{\pi}{2}} \left\{a^2\sin^2\theta\cdot\frac{a^2}{2}\,(-2\cos\theta\sin\theta) + \frac{a^2}{2}\,\cos^2\theta\cdot\frac{a^2}{2}\,(-2\cos\theta\sin\theta)\right. \\
&\qquad\quad \left. + \frac{a^2}{2}\,\cos^2\theta\cdot a^2\cdot 2\sin\theta\cos\theta\right\}\mathrm{d}\theta \\
&\quad + \frac{1}{2}\int_0^{\frac{\pi}{2}} \left\{a^2\cos^2\theta\cdot\frac{3a^2}{4}\cdot 2\cos\theta\sin\theta + \frac{3a^2}{4}\,\sin^2\theta\cdot\frac{a^2}{4}\cdot 2\cos\theta\,\sin\theta\right. \\
&\qquad\quad \left. + \frac{a^2}{4}\,\sin^2\theta\cdot a^2(-2\sin\theta\cos\theta)\right\}\mathrm{d}\theta \\
&= a^4\int_{\frac{\pi}{6}}^{\frac{3\pi}{4}} \cos^3\varphi\,\sin\varphi\,\mathrm{d}\varphi + \frac{a^4}{2}\int_0^{\frac{\pi}{2}} \left(-\sin^3\theta\cos\theta - \frac{1}{2}\,\cos^3\theta\sin\theta + \cos^3\theta\sin\theta\right)\mathrm{d}\theta \\
&\quad + \frac{a^4}{2}\int_0^{\frac{\pi}{2}} \left\{\frac{3}{2}\,\cos^3\theta\sin\theta + \frac{3}{8}\,\sin^3\theta\cos\theta - \frac{1}{2}\,\sin^3\theta\cos\theta\right\}\mathrm{d}\theta,
\end{aligned}$$

i.e.

$$\begin{aligned}
C &= a^4\left[-\frac{1}{4}\,\cos^4\varphi\right]_{\frac{\pi}{6}}^{\frac{3\pi}{4}} + \frac{a^4}{2}\int_0^{\frac{\pi}{2}} \left\{\left(-1+\frac{3}{8}-\frac{1}{2}\right)\sin^3\theta\cos\theta\;\mathrm{d}\theta\right. \\
&= \frac{a^4}{4}\left\{\left(\frac{\sqrt{3}}{2}\right)^4 - \left(-\frac{1}{\sqrt{2}}\right)^4\right\} + \frac{a^4}{2}\int_0^{\frac{\pi}{2}} \left\{-\frac{9}{8}\,\sin^3\theta\cos\theta + 2\cos^3\theta\sin\theta\right\}\mathrm{d}\theta \\
&= \frac{a^4}{4}\left(\frac{9}{16}-\frac{1}{4}\right) + \frac{a^4}{2}\left[-\frac{9}{8}\cdot\frac{1}{4}\,\sin^4\theta - 2\cdot\frac{1}{4}\,\cos^4\theta\right]_0^{\frac{\pi}{2}} \\
&= \frac{5a^4}{64} + \frac{a^4}{2}\left(-\frac{9}{32} - 0 + 0 + \frac{1}{2}\right) = \frac{a^4}{64}\,(5-9+16) = \frac{12a^4}{64} = \frac{3a^4}{16}.
\end{aligned}$$

The calculations can be carried out, though they are far from simple.

I 2. *Circulation by means of Stokes's theorem.*

We shall now demonstrate that the calculations by an application of Stokes's theorem in this case is far simpler. First note that since all sub-curves are circular arcs of radius a and centre $\mathbf{0}$, they all lie on the sphere of radius a and centre $\mathbf{0}$. They bound a part $\mathcal{F}$ of the sphere, where the orientation of $\mathcal{K}$ forces the normal vector $\mathbf{n}$ on $\mathcal{F}$ to point away from $\mathbf{0}$ (the right hand convention), i.e.

$$\mathbf{n} = \frac{1}{a}\,(x,y,z), \qquad \text{where } a = \sqrt{x^2+y^2+z^2}.$$

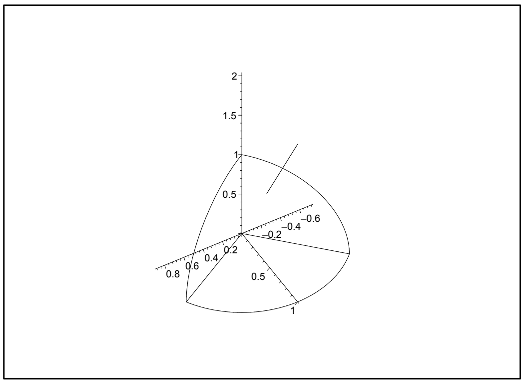

Figure 35.14: The surface $\mathcal{F}$ for $a = 1$ with a single normal vector $\mathbf{n}$.

Furthermore,

$$\mathbf{rot}\ \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z^2x & x^2y & y^2z \end{vmatrix} = (2yz, 2zx, 2xy).$$

It is on the sphere most natural to use *spherical* coordinates. Then by Stokes's theorem and a reduction,

$$\begin{aligned} C &= \int_{\mathcal{K}} \mathbf{V}\cdot d\mathbf{x} = \int_{\mathcal{F}} \mathbf{n}\cdot \mathbf{rot}\ \mathbf{V}\,\mathrm{d}S \\ &= \int_{\mathcal{F}} \frac{1}{a}(x,y,z)\cdot(2yz,2zx,2xy)\,\mathrm{d}S = \frac{6}{a}\int_{\mathcal{F}} xyz\,\mathrm{d}S \\ &= \frac{6}{a}\int_{\frac{\pi}{6}}^{\frac{3\pi}{4}} \left\{ \int_0^{\frac{\pi}{2}} a\ \sin\theta\cos\varphi\cdot a\ \sin\theta\sin\varphi\cdot a\ \cos\theta\, a^2\sin\theta\,\mathrm{d}\theta \right\} \mathrm{d}\varphi \\ &= 6a^4\int_{\frac{\pi}{6}}^{\frac{3\pi}{4}} \cos\varphi\cdot\sin\varphi\,\mathrm{d}\varphi\cdot\int_0^{\frac{\pi}{2}} \sin^3\theta\,\cos\theta\,\mathrm{d}\theta \\ &= 6a^4\left[\frac{1}{2}\sin^2\varphi\right]_{\frac{\pi}{6}}^{\frac{3\pi}{4}}\cdot\left[\frac{1}{4}\sin^4\theta\right]_0^{\frac{\pi}{2}} \\ &= 6a^4\cdot\frac{1}{2}\left\{\frac{1}{2}-\frac{1}{4}\right\}\cdot\frac{1}{4} = \frac{6a^4}{32} = \frac{3a^4}{16}, \end{aligned}$$

i.e. far easier calculations than by the line integral. ♢

Example 35.11

A. *An oriented curve $\mathcal{K}$ is given as the intersection curve between*

- *the cylindric surface* $x^2 + y^2 = ax$,

and

- *half of the parabolic cylindric surface*

$$z = \sqrt{4a^2 - ax}.$$

The orientation of $\mathcal{K}$ obeys the right hand convention with the positive z-axis. Find the circulation *along $\mathcal{K}$ of the vector field*

$$\mathbf{V}(x, y, z) = (3xy, 2x^2, -yz), \qquad (x, y, z) \in \mathbb{R}^3.$$

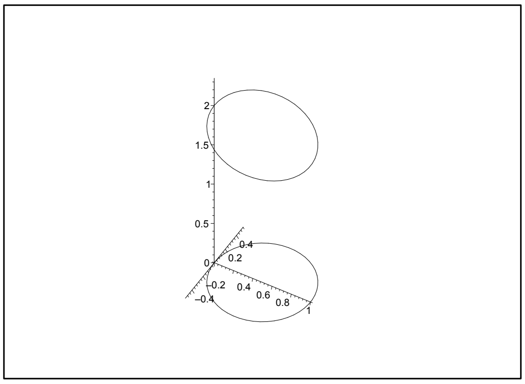

Figure 35.15: The curve $\mathcal{K}$ and its projection onto the (x, y)-plane.

D. We shall here demonstrate *three* variants:

1) *The line integral* (the calculations are worse than those of Example 35.10).
2) *Stokes's theorem* by the surface

$$\mathcal{F}_1: \qquad z = \sqrt{4a^2 - ax}, \qquad (x, y) \in B,$$

3) *Stokes' sætning* by the surface

$$\mathcal{F}_2: \qquad x^2 + y^2 + z^2 = (2a)^2, \qquad (x, y) \in B, \quad z > 0.$$

I 1. As a general warning we shall first give the solution calculated as a *line integral.*

When $\mathcal{K}$ is projected onto the (x, y)-plane we get the curve $\mathcal{L}$ of the parametric representation

$$(x, y, z) = (\varrho \cos\varphi, \varrho \sin\varphi, 0) = a\left(\cos^2\varphi, \cos\varphi \sin\varphi, 0\right), \quad \varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

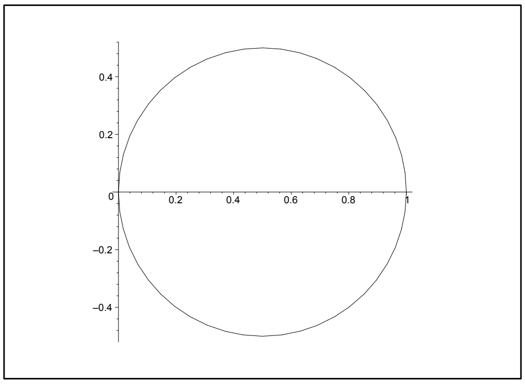

Figure 35.16: The parametric domain B is the disc in the (x, y)-plane of centre $\left(\frac{a}{2}, 0\right)$ and radius $\frac{a}{2}$. Here we have chosen $a = 1$.

Now $z = \sqrt{4a^2 - ax}$, so $\mathcal{K}$ has the parametric representation

$$\mathcal{K}: \quad \mathbf{r}(\varphi) = (x, y, z) = a\left(\cos^2\varphi, \cos\varphi\,\sin\varphi, \sqrt{4 - \cos^2\varphi}\right), \qquad \varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Then we get the values of the vector field along the curve $\mathcal{K}$,

$$\begin{aligned}\mathbf{V}(x, y, z) &= \left(3xy, 2x^2, -yz\right)\\ &= \left(3a\,\cos^2\varphi \cdot a\,\cos\varphi\,\sin\varphi, 2a^2\cos^4\varphi, -a\,\cos\varphi\,\sin\varphi \cdot a\sqrt{4 - \cos^2\varphi}\right)\\ &= a^2\left(3\cos^3\varphi\,\sin\varphi, 2\cos^4\varphi, -\cos\varphi \cdot \sin\varphi \cdot \sqrt{4 - \cos^2\varphi}\right).\end{aligned}$$

Hence,

$$\mathbf{r}'(\varphi) = a\left(-2\sin\varphi\cdot\cos\varphi, \cos^2\varphi - \sin^2\varphi, \frac{\cos\varphi\cdot\sin\varphi}{\sqrt{4-\cos^2\varphi}}\right).$$

In order to avoid too big calculations we first calculate the *integrand* separately,

$$\begin{aligned}
&\mathbf{V}\cdot\mathbf{r}'(\varphi)\\
&= a^3\left(3\cos^3\varphi\,\sin\varphi\cdot(-2\sin\varphi\,\cos\varphi)+2\cos^4\varphi\cdot\left(\cos^2\varphi-\sin^2\varphi\right)-\cos^2\varphi\,\sin^2\varphi\right)\\
&= a^3\left(3\cos^2\varphi\,\cos\varphi\,\sin\varphi\cdot(-\sin 2\varphi)+2(\cos^2\varphi)^2\cdot\cos 2\varphi-\frac{1}{4}\left(2\cos\varphi\,\sin\varphi\right)^2\right)\\
&= a^3\left\{3\,\frac{1+\cos 2\varphi}{2}\,\frac{\sin 2\varphi}{2}\cdot(-\sin 2\varphi)+2\left(\frac{1+\cos 2\varphi}{2}\right)^2\cos 2\varphi-\frac{1}{4}\,\sin^2 2\varphi\right\}\\
&= \frac{a^3}{4}\left\{-3(1+\cos 2\varphi)\sin^2 2\varphi+2(1+\cos 2\varphi)^2\cos 2\varphi-\sin^2 2\varphi\right\}\\
&= \frac{a^3}{4}\left\{-3\sin^2 2\varphi-3\cos 2\varphi\,\sin^2 2\varphi+2(1+2\cos 2\varphi+\cos^2 2\varphi)\,\cos 2\varphi-\sin^2 2\varphi\right\}\\
&= \frac{a^3}{4}\left\{-4\sin^2 2\varphi-3\cos 2\varphi\,\sin^2 2\varphi+2\cos 2\varphi+4\cos^2 2\varphi+2\cos^2 2\varphi\cdot\cos 2\varphi\right\}\\
&= \frac{a^3}{4}\left\{4\left(\cos^2 2\varphi-\sin^2 2\varphi\right)-5\cos 2\varphi\,\sin^2 2\varphi+2\cos 2\varphi\left(\sin^2 2\varphi+\cos^2 2\varphi\right)+2\cos 2\varphi\right\},
\end{aligned}$$

i.e.

$$\mathbf{V}\cdot\mathbf{r}'(\varphi) = \frac{a^3}{4}\left\{4\cos 4\varphi - 5\cos 2\varphi\,\sin^2 2\varphi + 4\cos 2\varphi\right\},$$

which is a fairly tough calculation. At the same time we see why we do not immediately insert the expression into the integral.

However, after this reduction the circulation becomes easy to calculate,

$$\begin{aligned}
C &= \oint_{\mathcal{K}}\mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathbf{V}\cdot\mathbf{r}'(\varphi)\,\mathrm{d}\varphi\\
&= \frac{a^3}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left\{4\cos 4\varphi-5\cos 2\varphi\,\sin^2 2\varphi+4\cos 2\varphi\right\}\,\mathrm{d}\varphi\\
&= \frac{a^3}{4}\left\{[\sin 4\varphi]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}-5\left[\frac{1}{2}\cdot\frac{1}{3}\,\sin^3 2\varphi\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}+[2\,\sin 2\varphi]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right\}\\
&= 0,
\end{aligned}$$

and we see that after so much trouble we only get zero as our result.

I. As an introduction to the applications of *Stokes's theorem* we first calculate

$$\mathbf{rot}\ \mathbf{V} = \begin{vmatrix}\mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 3xy & 2x^2 & -yz\end{vmatrix} = (-z, 0, 4x-3x) = (-z, 0, x),$$

which looks promising, because **rot V** is much simpler than **V**.

Then we shall choose a surface which has $\mathcal{K}$ as its boundary curve. This surface is of course not unique.

I 2. The most obvious possibility is to choose the parabolic cylindric surface

$$\mathcal{F}_1 : \quad z = \sqrt{4a^2 - ax}, \qquad (x,y) \in B,$$

where the parametric domain B is best described in *polar* coordinates,

$$B = \left\{ (\varrho, \varphi) \;\middle|\; \varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right], \, 0 \le \varrho \le a \cos\varphi \right\},$$

while (x, y, z) still denotes the *rectangular* coordinates.

Because $\mathcal{F}_1$ is given as a *graph*, the parametric representation is

$$\mathbf{r}(x,y) = (x,y,z) = \left(x, y, \sqrt{4a^2 - ax} \right), \qquad (x,y) \in B,$$

from which we get the tangential vector fields

$$\mathbf{r}'_x = \left(1, 0, \frac{1}{2} \frac{-a}{\sqrt{4a^2 - ax}} \right) = \left(1, 0, -\frac{a}{2z} \right), \quad \text{and} \quad \mathbf{r}'_y = (0,1,0),$$

and the corresponding normal vector field

$$\mathbf{N}(x,y) = \mathbf{r}'_x \times \mathbf{r}'_y = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 1 & 0 & -\dfrac{a}{2z} \\ 0 & 1 & 0 \end{vmatrix} = \left(\frac{a}{2z}, 0, 1 \right).$$

We see that $\mathbf{N}(x,y)$ and the orientation of $\mathcal{K}$ satisfy the right hand convention $\mathbf{N} \cdot \mathbf{e}_z = 1$, so we have the correct orientation.

An application of *Stokes's theorem* then gives

$$\begin{aligned}
C &= \oint_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} \, \mathrm{d}s = \int_{\mathcal{F}_1} \mathbf{n} \cdot \mathbf{rot}\ \mathbf{V} \, \mathrm{d}S = \int_B \left(\frac{a}{2z}, 0, 1 \right) \cdot (-z, 0, x) \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_B \left(-\frac{1}{2} a + x \right) dx\, dy = -\frac{a}{2} \operatorname{area}(B) + \int_B x \, \mathrm{d}x \, \mathrm{d}y \\
&= -\frac{a}{2} \cdot \pi \left(\frac{a}{2} \right)^2 + \int_{-\frac{\pi}{2}}^{\pi} 2 \left\{ \int_0^{a \cos\varphi} \varrho \cos\varphi \cdot \varrho \, \mathrm{d}\varrho \right\} \mathrm{d}\varphi \\
&= -\frac{\pi a^3}{8} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi \cdot \left[\frac{1}{3} \varrho^3 \right]_0^{a \cos\varphi} \mathrm{d}\varphi = -\frac{\pi a^3}{8} + \frac{a^3}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 \varphi \, \mathrm{d}\varphi \\
&= -\frac{\pi a^3}{8} + \frac{2a^3}{3} \int_0^{\frac{\pi}{2}} \left(\frac{1 + \cos 2\varphi}{2} \right)^2 \mathrm{d}\varphi \\
&= -\frac{\pi a^3}{8} + \frac{2a^3}{3} \cdot \frac{1}{4} \int_0^{\frac{\pi}{2}} \left(1 + 2\cos 2\varphi + \cos^2 2\varphi \right) \mathrm{d}\varphi \\
&= -\frac{\pi a^3}{8} + \frac{a^3}{6} \left\{ \frac{\pi}{2} + 0 + \frac{\pi}{4} \right\} = 0.
\end{aligned}$$

I 3. The calculations of **I 2.** were much simpler than the calculations of **I 1.**, although they still gave some trigonometric problems. The question is now, if it is possible to choose another surface $\mathcal{F}$ with $\mathcal{K}$ as its boundary curve, such that the calculations become even more easy. We shall now show that this is possible, though far from obvious.

First we note that since K satisfies the two conditions

$$\text{a) } x^2+y^2=ax \quad \text{and} \quad \text{b) } z^2=4a^2-ax,$$

the parametric representation of $\mathcal{F}$ must also satisfy

$$\left(x^2+y^2\right)+z^2=ax+\left(4a^2-ax\right)=4a^2=(2a)^2, \qquad z>0,$$

i.e. $\mathcal{K}$ lies on the half sphere of centre $\mathbf{0}$ and radius $2a$ and $z>0$. We choose $\mathcal{F}_2$ as that part of this half sphere which lies above the parametric domain B:

$$\mathcal{F}_2: \quad x^2+y^2+z^2=(2a)^2, \qquad (x,y)\in B, \quad z\geq 0.$$

The sphere of centre $\mathbf{0}$ and radius $2a$ has the unit normal vector

$$\mathbf{n}=\frac{1}{2a}\,(x,y,z),$$

hence we have on $\mathcal{F}_2$

$$\mathbf{n}\cdot\textbf{rot }\mathbf{V}=\frac{1}{2a}\,(x,y,z)\cdot(-z,0,x)=0.$$

By an application of *Stokes's theorem* the calculation of the circulation is now reduced to a triviality,

$$C = \oint_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t}\, \mathrm{d}s = \int_{\mathcal{F}_2} \mathbf{n} \cdot \mathbf{rot}\ \mathbf{V}\, \mathrm{d}S = \int_{\mathcal{F}_2} 0\, \mathrm{d}S = 0,$$

in which we shall not even insert a parametric representation followed by some reduction theorem! ♢

35.5.3 Examples of applications of Stokes's theorem

Example 35.12 *Apply in each of the following cases Stokes's theorem to find the circulation of the given vector field* $\mathbf{V} : \mathbb{R}^3 \to \mathbb{R}^3$ *along the given closed curve* $\mathcal{K}$*, we one shall indicate the direction of the curve on a figure.*

1) The circulation of the vector field

$$\mathbf{V}(x, y, z) = (y \sinh(xy) + z^2, x \sinh(xy) + z^2 + x, 2x^2 + 2y^2)$$

along the closed curve $\mathcal{K}$ *given by* $x^2 + y^2 = 1$*,* $z = 1$*.*

2) The circulation of the vector field

$$\mathbf{V}(x, y, z) = (y^2 + z^4, (x - a)^2 + z^4, x^2 + y^2)$$

along the closed curve $\mathcal{K}$ *given by*

$$x^2 + y^2 = bx, \quad z^4 = a^2 - x^2 - y^2, \quad \textit{where } b < a \textit{ and } z > 0.$$

3) The circulation of the vector field

$$\mathbf{V}(x, y, z) = (y, x - yz, x^2)$$

along the closed curve $\mathcal{K}$ *given by*

$$x^2 + y^2 = 1, \qquad z = 4 - 2x^2 - y^2.$$

4) The circulation of the vector field

$$\mathbf{V}(x, y, z) = (yz - 2y, xz + 4x, xy)$$

along the closed curve $\mathcal{K}$ *given by*

$$\varrho = 1 + \cos\varphi, \quad z = \sqrt{4 - \varrho^2} \quad \textit{for } \varphi \in [-\pi, \pi].$$

5) The circulation of the vector field

$$\mathbf{V}(x, y, z) = (y^2 - 2xy, 2xy, 2az + 3a^2)$$

along the closed curve $\mathcal{K}$ *given by*

$$x^2 + y^2 = ax \qquad z = a - \sqrt{x^2 + y^2}.$$

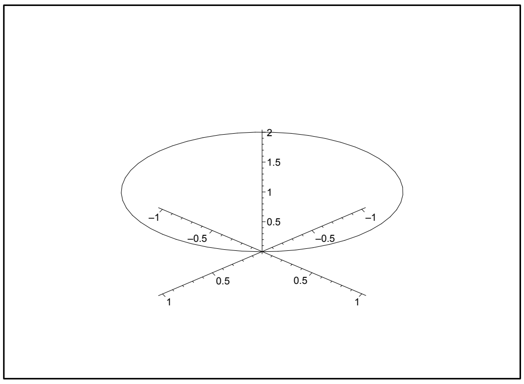

Figure 35.17: The curve $\mathcal{K}$ of **Example 35.12.1**.

6) The circulation of the vector field $\mathbf{V}(x,y,z) = (z,x,y)$ *along the boundary of the triangle* $\mathcal{K}$ *of vertices* $(0,0,1)$, $(0,1,0)$ *and* $(1,0,0)$.

7) The circulation of the vector field $\mathbf{V}(x,y,z) = (y,z,x)$ *along the closed curve* $\mathcal{K}$ *given by*

$$x^2+y^2+z^2=a^2, \qquad z=y-x.$$

8) The circulation of the vector field $\mathbf{V}(x,y,z) = (y+\sin z, x, x\cos z)$ *along the closed curve* $\mathcal{K}$ *given by*

$$x^2+y^2+z^2=1, \qquad z=x.$$

9) The circulation of the vector field $\mathbf{V}(x,y,z) = (z^2, ax+z^2, 2x^2+2y^2)$ *along the closed curve* $\mathcal{K}$ *given by*

$$x^2+y^2=a^2, \qquad z=a.$$

10) The circulation of the vector field

$$\mathbf{V}(x,y,z) = (-y(x^2+2z^2), x(x^2+4y^2+2z^2), z^3)$$

along the closed curve $\mathcal{K}$ *given by*

$$x^2+y^2=a^2, \qquad z=a.$$

A Circulation of vector fields.

D Sketch the curve and choose an orientation of it. Compute **rot V**, and choose the surface $\mathcal{F}$. Finally, apply Stokes's theorem.

I 1) The most obvious choice of the surface is

$$\mathcal{F} = \{(x,y,1) \mid x^2+y^2 \le 1\}$$

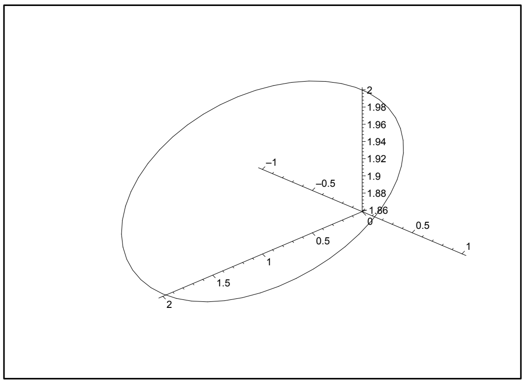

Figure 35.18: The curve $\mathcal{K}$ of **Example 35.12.2** for $a = 4$ and $b = 2$.

where the orientation is given by the normal vector $\mathbf{n} = (0, 0, 1)$. Hence

$$\mathbf{n} \cdot \mathbf{rot}\ \mathbf{V} = \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} = \sinh(xy) + xy\ \cosh(xy) + 1 - \sinh(xy) - xy\ \cosh(xy) = 1.$$

Then according to Stokes's theorem the circulation is

$$\int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t}\, \mathrm{d}s = \int_{\mathcal{F}} \mathbf{n} \cdot \mathbf{rot}\ \mathbf{V}\, \mathrm{d}S = 1 \cdot \text{area}(\mathcal{F}) = \pi.$$

2) If we choose the orientation of $\mathcal{K}$, such that the projection of the curve onto the XY-plane has a positive orientation, it is quite natural to choose the corresponding surface

$$\mathcal{F} = \left\{ (x, y, \sqrt[4]{z^2 - x^2 - y^2}) \mid x^2 + y^2 \leq bx \right\},$$

with the normal vector $\mathbf{n} \cdot \mathbf{e}_z > 0$. We first find

$$\mathbf{rot}\ \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 + z^4 & (x-a)^2 + z^4 & x^2 + y^2 \end{vmatrix} = (2y - 4z^3, 4z^3 - 2x, 2(x-a) - 2y).$$

Then calculate the normal vector of the surface $\mathcal{F}$,

$$\begin{aligned} \mathbf{N}(x,y) &= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 1 & 0 & -\frac{1}{2}\,x(\sqrt[4]{a^2 - x^2 - y^2})^{-3} \\ 0 & 1 & -\frac{1}{2}\,y(\sqrt[4]{a^2 - x^2 - y^2})^{-3} \end{vmatrix} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 1 & 0 & -\frac{1}{2}\,\frac{x}{z^3} \\ 0 & 1 & -\frac{1}{2}\,\frac{y}{z^3} \end{vmatrix} \\ &= \left(\frac{1}{2}\,\frac{x}{z^3}, \frac{1}{2}\,\frac{y}{z^3}, 1 \right), \end{aligned}$$

hence

$$\begin{aligned}\mathbf{n}\cdot\mathbf{rot}\;\mathbf{V} &= \frac{1}{\|\mathbf{N}(x,y)\|}\left\{\frac{1}{2}\,\frac{x}{z^3}(2y-4z^3)+\frac{1}{2}\,\frac{y}{z^3}(4z^3-2x)+2(x-a)-2y\right\}\\ &= \frac{1}{\|\mathbf{N}(x,y)\|}\left\{\frac{xy}{z^3}-2x+2y-\frac{yx}{z^3}+2x-2a-2y\right\} = -\frac{2a}{\|\mathbf{N}(x,y)\|}.\end{aligned}$$

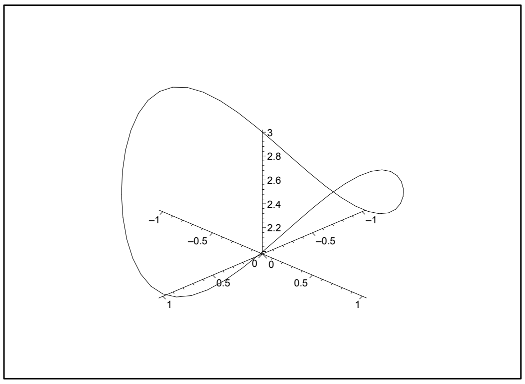

Figure 35.19: The curve $\mathcal{K}$ of **Example 35.12.3**

Choose the parameter domain

$$B = \{(x,y) \mid x^2+y^2 \mid bx\}.$$

According to Stokes's theorem the circulation of $\mathbf{V}$ along the curve $\mathcal{K}$ is given by

$$\begin{aligned}\int_{\mathcal{K}} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s &= \int_{\mathcal{F}} \mathbf{n}\cdot\mathbf{rot}\ \mathbf{V}\,\mathrm{d}S = \int_B \frac{-2a}{\|\mathbf{N}(x,y)\|}\,\|\mathbf{N}(x,y)\|\,\mathrm{d}x\,\mathrm{d}y \\ &= -2a\,\text{area}(B) = -2a\cdot\pi\left(\frac{b}{2}\right)^2 = -\frac{\pi}{2}\,ab^2.\end{aligned}$$

3) Here we choose the surface

$$\mathcal{F} = \{(x,y,2-2x^2-y^2) \mid x^2+y^2 \le 1\},$$

where the boundary curve $\mathcal{K}$ is oriented such that it is positive in the XY-plane. Then $\mathbf{n}\cdot\mathbf{e}_z > 0$ for the normal vector on $\mathcal{F}$.

Then by computing,

$$\mathbf{rot}\ \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & x-yz & x^2 \end{vmatrix} = (y,-2x,1-1) = (y,-2x,0).$$

and

$$\mathbf{N}(x,y) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 1 & 0 & -4x \\ 0 & 1 & -2y \end{vmatrix} = (4x,2y,1),$$

hence

$$\mathbf{n}\cdot\mathbf{rot}\ \mathbf{V} = \frac{1}{\|\mathbf{N}(x,y)\|}\,(2x,2y,1)\cdot(y,-2x,0) = 0.$$

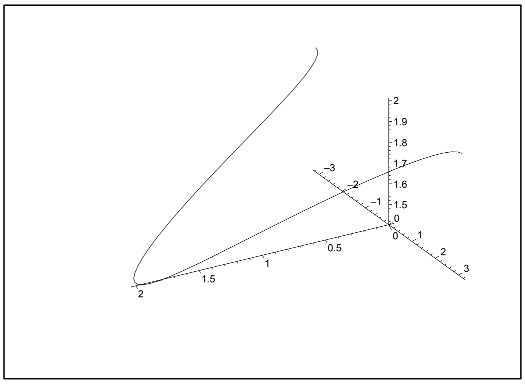

Figure 35.20: The curve $\mathcal{K}$ of **Example 35.12.4**

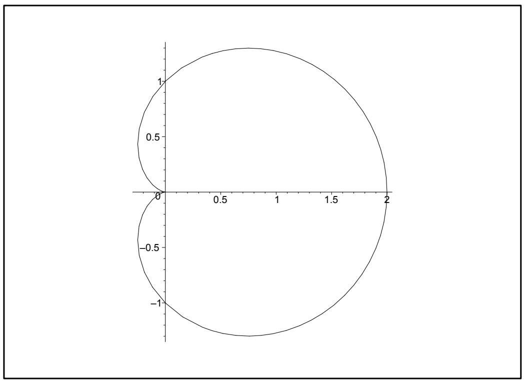

Figure 35.21: The parameter domain B of **Example 35.12.4**.

Then it is easy to find the circulation,

$$\int_{\mathcal{K}} \mathbf{t} \cdot \mathbf{V}\,\mathrm{d}s = \int_{\mathcal{F}} \mathbf{n} \cdot \mathbf{rot}\ \mathbf{V}\,\mathrm{d}S = 0.$$

4) Choose the surface which is given in semi polar coordinates by

$$\mathcal{F} = \left\{(\varrho, \varphi, \sqrt{4-\varrho^2}) \mid 0 \le \varrho \le 1 + \cos\varphi,\ \varphi \in [-\pi, \pi]\right\} = \{(\varrho, \varphi, \sqrt{4-\varrho^2}) \mid (\varrho, \varphi) \in B\},$$

where the parameter domain

$$B = \{(\varrho, \varphi) \mid 0 \le \varrho \le 1 + \cos\varphi,\ \varphi \in [-\pi, \pi]\}$$

lies inside the cardioid.
Choose the orientation of $\mathcal{K}$ such that the projection of $\mathcal{K}$ onto the cardioid is run through in

the the positive sense of the plane. Then by a computation,

$$\mathbf{rot}\ \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz-2y & xz+4x & xy \end{vmatrix} = (x-x, y-y, z+4-z+2) = (0,0,6).$$

The surface $\mathcal{F}$ is described in rectangular coordinates (though in polar parameters) by

$$(x,y,z) = (\varrho\cos\varphi, \varrho\sin\varphi, \sqrt{4-\varrho^2}).$$

This rectangular description is necessary when we compute the normal vector by the usual method,

$$\mathbf{N}(\varrho,\varphi) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \cos\varphi & \sin\varphi & -\dfrac{\varrho}{\sqrt{4-\varrho^2}} \\ -\varrho\sin\varphi & \varrho\cos\varphi & 0 \end{vmatrix} = \left(\frac{\varrho^2\cos\varphi}{\sqrt{4-\varrho^2}}, \frac{\varrho^2\sin\varphi}{\sqrt{4-\varrho^2}}, \varrho \right),$$

hence

$$\mathbf{n}\cdot\mathbf{rot}\ \mathbf{V} = \frac{6\varrho}{\|\mathbf{B}(\varrho,\varphi)\|}.$$

The circulation along $\mathcal{K}$ is

$$\begin{aligned} \int_{\mathcal{F}} \frac{6\varrho}{\|\mathbf{N}(\varrho,\varphi)\|}\,\mathrm{d}S &= \int_B 6\varrho\,\mathrm{d}\varrho\,\mathrm{d}\varphi = \int_{-\pi}^{\pi}\left\{\int_0^{1+\cos\varphi} 6\varrho\,\mathrm{d}\varrho\right\}\mathrm{d}\varphi \\ &= \int_{-\pi}^{\pi} 3(1+\cos\varphi)^2\,\mathrm{d}\varphi = 3\int_{-\pi}^{\pi}(1+2\cos\varphi+\cos^2\varphi)\,\mathrm{d}\varphi \\ &= 3\int_{-\pi}^{\pi}\left\{1+\cos^2\varphi\right\}\,\mathrm{d}\varphi + 0 = \frac{3}{2}\cdot 3\cdot 2\pi = 9\pi. \end{aligned}$$

5) Choose the surface

$$\mathcal{F} = \{(x,y,a-\sqrt{x^2+y^2}) \mid x^2+y^2 \le ax\} = \{(x,y,a-\sqrt{x^2+y^2}) \mid (x,y)\in B\},$$

where the parameter domain B is described in polar coordinates of the plane,

$$B = \left\{(\varrho,\varphi) \;\middle|\; \varrho \le a\cos\varphi,\ \varphi\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right\}.$$

Then

$$\mathbf{rot}\ \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2-2xy & 2xy & 2az+3a^2 \end{vmatrix} = (0,0,2y-2y+2x) = (0,0,2x),$$

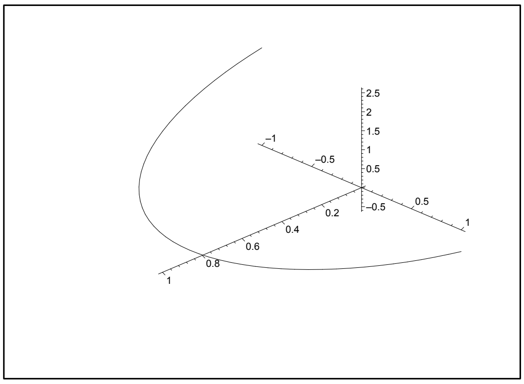

Figure 35.22: The curve $\mathcal{K}$ of **Example 35.12.5** for $a = 1$.

and

$$\mathbf{N}(x,y) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 1 & 0 & -\dfrac{x}{\sqrt{x^2+y^2}} \\ 0 & 1 & -\dfrac{y}{\sqrt{x^2+y^2}} \end{vmatrix} = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, 1 \right).$$

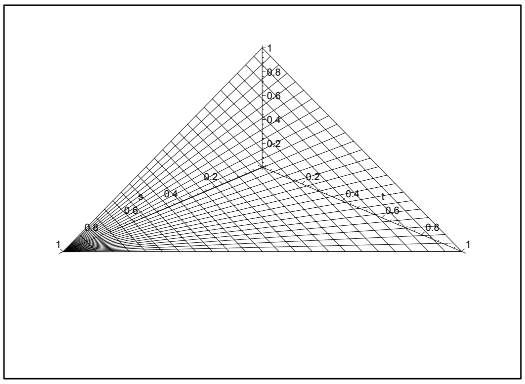

Figure 35.23: The surface $\mathcal{F}$ of **Example 35.12.6**.

The circulation af $\mathbf{V}$ along $\mathcal{K}$ is then by Stokes's theorem,

$$\int_{\mathcal{K}} \mathbf{t}\cdot\mathbf{V}\,\mathrm{d}s = \int_{\mathcal{F}} \mathbf{n}\cdot\mathbf{V}\,\mathrm{d}S = \int_B \mathbf{N}\cdot\mathbf{V}\,\mathrm{d}x\,\mathrm{d}y = \int_B 2x\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{\int_0^{a\,\cos\varphi} 2\varrho\cos\varphi\cdot\varrho\,\mathrm{d}\varrho\right\}\mathrm{d}\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{2}{3}\,\varrho^3\right]_0^{a\,\cos\varphi} \cos\varphi\,\mathrm{d}\varphi$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2}{3}\,a^3\cos^4\varphi\,d\varphi = \frac{2}{3}\,a^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1+\cos 2\varphi}{2}\right)^2 \mathrm{d}\varphi$$

$$= \frac{1}{6}\,a^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1+2\cos 2\varphi+\frac{1+\cos 4\varphi}{2}\right)\mathrm{d}\varphi = \frac{1}{6}\,a^3\cdot\frac{3}{2}\,\pi = \frac{\pi}{4}\,a^3.$$

6) First calculate

$$\mathbf{rot}\;\mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = (1,1,1).$$

We choose naturally the surface $\mathcal{F}$ in the following way

$$\mathcal{F} = \{(x,y,1-x-y) \mid 0 \le x \le 1,\, 0 \le y \le 1-x\}$$

with the normal vector

$$\mathbf{N}(x,y) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = (1,1,1).$$

The circulation of the vector field along $\mathcal{K}$ is then

$$\int_{\mathcal{K}} \mathbf{t}\cdot\mathbf{V}\,\mathrm{d}s = \int_{\mathcal{F}} \mathbf{n}\cdot\mathbf{rot}\;\mathbf{V}\,\mathrm{d}S = \int_B \mathbf{N}\cdot\mathbf{rot}\;\mathbf{V}\,\mathrm{d}x\,\mathrm{d}y = 3\,\mathrm{area}(B) = \frac{3}{2}.$$

7) Choose $\mathcal{F}$ as the plane surface (a disc)

$$\mathcal{F} = \{(x, y, y-x) \mid x^2 + y^2 + (y-x)^2 \le a^2\}$$

with $\mathbf{N} \cdot \mathbf{e}_z > 0$ by the chosen orientation. We get

$$\mathbf{rot}\ \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = (-1, -1, -1)$$

and

$$\mathbf{N}(x, y) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{vmatrix} = (1, -1, 1),$$

hence

$$\mathbf{N} \cdot \mathbf{rot}\ \mathbf{V} = (-1, -1, -1) \cdot (1, -1, 1) = -1.$$

The projection B of $\mathcal{F}$ onto the XY-plane is given by

$$a^2 \ge x^2 + y^2 + (y-x)^2 = x^2 + y^2 - \frac{1}{2}(x-y)^2 + \frac{3}{2}(x-y)^2 = \left(\frac{x+y}{\sqrt{2}}\right)^2 + 3\left(\frac{x-y}{\sqrt{2}}\right)^2,$$

which describes the interior of an elliptic disc with the directions of the axes

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \quad \text{and} \quad \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

and the half axes a and $\dfrac{a}{\sqrt{3}}$.

The circulation is

$$\int_{\mathcal{K}} \mathbf{t} \cdot \mathbf{V}\, \mathrm{d}s = \int_{\mathcal{F}} \mathbf{n} \cdot \mathbf{rot}\ \mathbf{V}\, \mathrm{d}S = \int_B \mathbf{N} \cdot \mathbf{rot}\ \mathbf{V}\, \mathrm{d}x\, \mathrm{d}y = -\operatorname{area}(B) = -\pi \cdot \frac{a^2}{\sqrt{3}}.$$

8) Since

$$\mathbf{rot}\mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y + \sin z & x & x\cos z \end{vmatrix} = (0, \cos z - \cos z, 1 - 1) = (0, 0, 0),$$

the circulation is trivially 0 by Stokes's theorem.

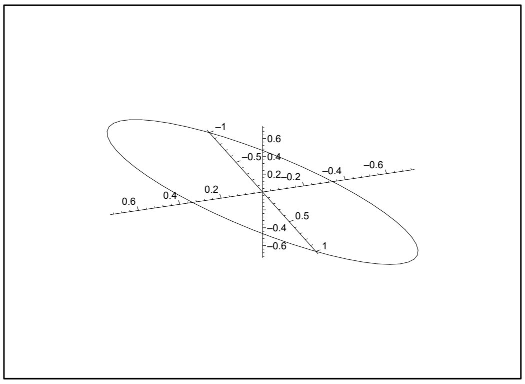

Figure 35.24: The curve $\mathcal{K}$ of **Example 35.12.8**.

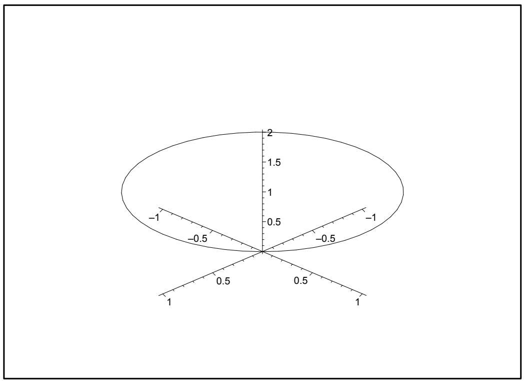

Figure 35.25: The curve $\mathcal{K}$ of **Example 35.12.9** and **Example 35.12.10** for $a = 1$.

9) Here

$$\mathbf{rot}\ \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & ax+z^2 & 2x^2+2y^2 \end{vmatrix} = (4y-2z, 2z-4x, a).$$

We have $\mathbf{n} = (0,0,1)$ in the chosen orientation of $\mathcal{K}$, so the circulation becomes

$$\int_{\mathcal{K}} \mathbf{t}\cdot\mathbf{V}\,ds = \int_{\mathcal{F}} \mathbf{n}\cdot\mathbf{rot}\ \mathbf{V}\,dS = \int_B a\,dx\,dy = a\cdot\text{area}(B) = \pi a^3.$$

ALTERNATIVELY it is here also easy to compute the circulation as a line integral. We choose the parametric description

$$(x,y,z) = (a\,\cos\varphi, a\,\sin\varphi, a), \qquad \varphi\in[0,2\pi],$$

for $\mathcal{K}$. Then we get the tangent vector field

$$\mathbf{t}\,\mathrm{d}s = (-a\,\sin\varphi, a\,\cos\varphi, 0)\,\mathrm{d}\varphi,$$

hence

$$\begin{aligned}\int_{\mathcal{K}} \mathbf{t}\cdot\mathbf{V}\,\mathrm{d}s &= \int_0^{2\pi} (-a\,\sin\varphi, a\,\cos\varphi, 0)\cdot(a^2, a^2\cos\varphi + a^2, 2a^2)\,\mathrm{d}\varphi \\ &= a^3\int_0^{2\pi}(-\sin\varphi + \cos^2\varphi + \cos\varphi + 0)\,\mathrm{d}\varphi \\ &= a^3\int_0^{2\pi}\cos^2\varphi\,\mathrm{d}\varphi = a^3\int_0^{2\pi}\sin^2\varphi\,\mathrm{d}\varphi \\ &= a^3\int_0^{2\pi}\frac{\cos^2\varphi+\sin^2\varphi}{2}\,\mathrm{d}\varphi = \frac{a^3}{2}\int_0^{2\pi}\mathrm{d}\varphi = \pi a^3.\end{aligned}$$

10) The surface $\mathcal{F}$ is the same as in **Example 35.12.9**, so we can reuse $\mathbf{n} = \mathbf{N} = (0,0,1)$ and

$$\mathbf{n}\cdot\mathbf{rot}\ \mathbf{V} = \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} = 3x^2+4y^2+2z^2+x^2+2z^2 = 4(x^2+y^2+z^2).$$

Since $z = a$ on $\mathcal{F}$, the circulation becomes

$$\begin{aligned}\int_{\mathcal{K}} \mathbf{t}\cdot\mathbf{V}\,\mathrm{d}s &= \int_{\mathcal{F}} \mathbf{n}\cdot\mathbf{rot}\ \mathbf{V}\,\mathrm{d}S = 4\int_B (x^2+y^2+a^2)\,\mathrm{d}x\,\mathrm{d}y \\ &= 4\int_0^{2\pi}\left\{\int_0^a \varrho^2\cdot\varrho\,\mathrm{d}\varrho\right\}\mathrm{d}\varphi + 4a^2\cdot\pi a^2 = 2\pi a^4 + 4\pi a^4 = 6\pi a^4.\end{aligned}$$

Example 35.13 *In each of the following cases apply Stokes's theorem to compute the flux*

$$\int_{\mathcal{F}} \mathbf{n}\cdot\mathbf{rot}\ \mathbf{V}\,\mathrm{d}S$$

of the rotation of the given vector field $\mathbf{V}:\mathbb{R}^3\to\mathbb{R}^3$ *through the surface* $\mathcal{F}$*, where we shall choose an orientation, which is indicated on a figure.*

1) *The flux of* $\mathbf{V}(x,y,z) = (y^2, x-2xz, -xy)$ *through the surface* $\mathcal{F}$ *given by* $z = \sqrt{a^2-x^2-y^2}$ *for* $x^2+y^2 \le a^2$.

2) *The flux of* $\mathbf{V}(x,y,z) = (2y^3, x^2+yz, x)$ *through the triangle* $\mathcal{F}$ *with the vertices* $(1,0,0)$, $(0,1,0)$ *and* $(0,0,1)$.

3) *The flux of* $\mathbf{V}(x,y,z) = (y+z^2, z\ln(1-x^2+y^2), \mathrm{Arctan}(xyz))$ *through the surface* $\mathcal{F}$ *given by* $z = 1-x^2-y^2$ *for* $x^2+y^2\le 1$.

A Flux computed by means of Stokes's theorem.

D Sketch the surface $\mathcal{F}$ and the boundary curve $\mathcal{K}$ and choose an orientation. (It has not been possible for me to sketch the orientation of the figures). Finally, exploit that the flux according to Stokes's theorem is given by

$$(35.2)\quad \int_{\mathcal{F}} (\nabla\times\mathbf{V})\cdot\mathbf{n}\,\mathrm{d}S = \int_{\mathcal{K}} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s, \qquad \text{where } \mathcal{K} = \text{“}\partial\mathcal{F}\text{”}.$$

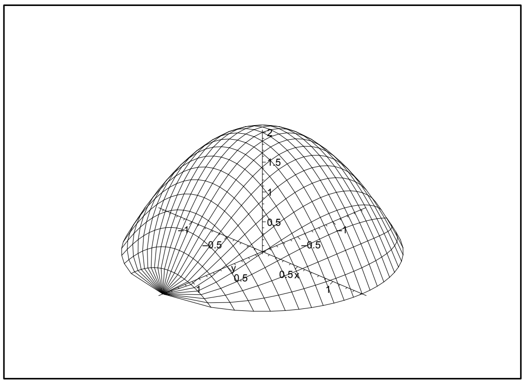

Figure 35.26: The surface $\mathcal{F}$ of **Example 35.13.1** for $a = \sqrt{2}$.

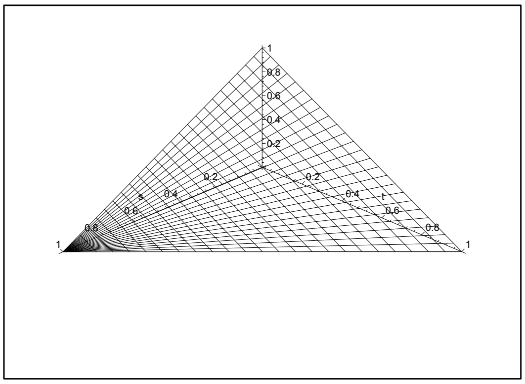

Figure 35.27: The surface $\mathcal{F}$ of **Example 35.13.2**

I 1) The boundary curve is the circle in the XY-plane of centrum $(0,0)$ and radius a. Choose the parametric description

$$(x,y,z) = (a\,\cos\varphi, a\,\sin\varphi, 0), \qquad \varphi \in [0,2\pi],$$

for $\mathcal{K}$ in $\mathbb{R}^3$ corresponding to a positive orientation. Since $\mathbf{t} = (-\sin\varphi, \cos\varphi, 0)$ and $ds = a\,d\varphi$, the flux is according to (35.2) given by

$$\begin{aligned}\int_{\mathcal{K}} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s &= \int_0^{2\pi} (a^2\sin^2\varphi, a\cos\varphi, -a^2\sin\varphi\,\cos\varphi)\cdot(-\sin\varphi,\cos\varphi,0)a\,\mathrm{d}\varphi \\ &= \int_0^{2\pi}\left\{-a^3\sin^3\varphi + a^2\cos^2\varphi + 0\right\}\mathrm{d}\varphi = 0 + a^2\cdot\frac{1}{2}\cdot 2\pi + 0 = \pi a^2.\end{aligned}$$

2) The boundary curve is the boundary of the triangle with e.g. the parametric description

$$\begin{cases} \mathcal{K}_1: \quad (x,y,z) = (1-t,t,0), \quad t\in[0,1], \\ \mathcal{K}_2: \quad (x,y,z) = (0,1-t,t), \quad t\in[0,1], \\ \mathcal{K}_3: \quad (x,y,z) = (t,0,1-t), \quad t\in[0,1], \end{cases}$$

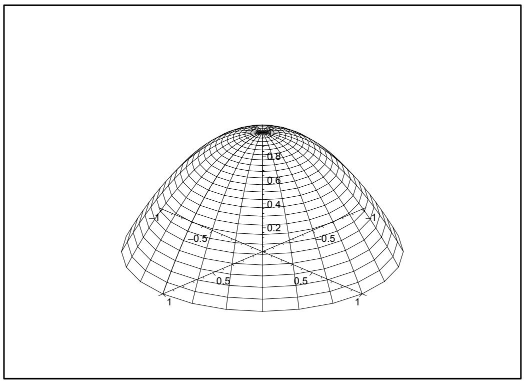

Figure 35.28: The surface $\mathcal{F}$ of **Example 35.13.3**.

where $\mathrm{d}s = \sqrt{2}\,\mathrm{d}t$ on each of the three sub-curves. According to (35.2) the flux is given by

$$\begin{aligned}
\int_{\mathcal{F}} (\nabla \times \mathbf{V}) \cdot \mathbf{n}\,\mathrm{d}S &= \int_{\mathcal{K}_1} \mathbf{V} \cdot \mathbf{t}\,\mathrm{d}s + \int_{\mathcal{K}_2} \mathbf{V} \cdot \mathbf{t}\,\mathrm{d}s + \int_{\mathcal{K}_3} \mathbf{V} \cdot \mathbf{t}\,\mathrm{d}s \\
&= \int_0^1 (2t^3, (1-t)^2, 1-t) \cdot (-1,1,0)\,\mathrm{d}t \\
&\qquad + \int_0^1 (2(1-t)^3, (1-t)t, 0) \cdot (0,-1,1)\,\mathrm{d}t \\
&\qquad + \int_0^1 (0, t^2, t) \cdot (1,0,-1)\,\mathrm{d}t \\
&= \int_0^2 \{-2t^3 + (1-t)^2\}\,\mathrm{d}t + \int_0^1 \{-(1-t)t\}\,\mathrm{d}t + \int_0^1 (-t)\,\mathrm{d}t \\
&= \int_0^1 \{-2t^3 + (1-t)^2 +)1-t^2)^2 - (1-t) - t\}\,\mathrm{d}t \\
&= \int_0^1 \{-2t^3 + 2(t-1)^2 - 1\}\,\mathrm{d}t = \left[-\frac{1}{2}\,t^4 + \frac{2}{3}\,(t-1)^3 - t\right]_0^1 \\
&= -\frac{1}{2} - 1 + \frac{2}{3} = \frac{2}{3} - \frac{3}{2} = -\frac{5}{6}.
\end{aligned}$$

3) The boundary curve $\mathcal{K}$ is the unit circle in the XY-plane. Choose the orientation corresponding to the parametric description

$$(x,y,z) = (\cos\varphi, \sin\varphi, 0), \qquad \varphi \in [0, 2\pi],$$

for $\mathcal{K}$. Then

$$\mathbf{t} = (-\sin\varphi, \cos\varphi, 0) \qquad \text{and} \qquad \mathrm{d}s = da\varphi,$$

and the flux through $\mathcal{F}$ is then according to (35.13) given by

$$\int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t}\,\mathrm{d}s = \int_0^{2\pi} (\sin\varphi, 0, 0) \cdot (-\sin\varphi, \cos\varphi, 0)\,\mathrm{d}\varphi = -\int_0^{2\pi} \sin^2\varphi\,\mathrm{d}\varphi = -\frac{1}{2} \cdot 2\pi = -\pi.$$

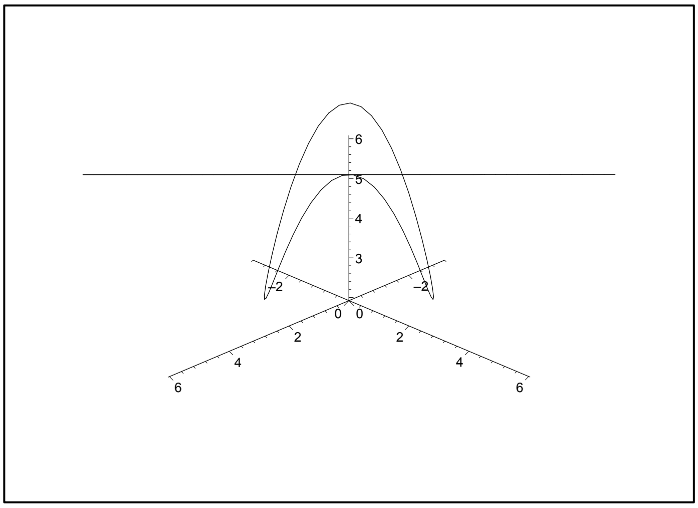

Figure 35.29: The space curve $\mathcal{K}$ and its tangent at $\mathbf{r}\left(\frac{\pi}{4}\right)$.

Example 35.14 *A space curve $\mathcal{K}$ is given by the parametric description*

$$\mathbf{r}(t) = (2\cos t, 2\sin t, 4 + 2\sin(2t)), \qquad t \in \mathbb{R}.$$

1. *Find a parametric description of the tangent of $\mathcal{K}$ at the point $\mathbf{r}\left(\frac{\pi}{4}\right)$.*

2. *Show that $\mathcal{K}$ lies on the surface $\mathcal{F}$, given by the equation $z = 4 + xy$.*

Let $\mathcal{K}_1$ be the restriction of $\mathcal{K}$ corresponding to the parameter interval $[0, 2\pi]$, where its orientation is corresponding to increasing t. Furthermore, we have given the vector field

$$\mathbf{V}(x, y, z) = (y, x, y^2 + 2z), \qquad (x, y, z) \in \mathbb{R}^3.$$

3. *Find the circulation of the vector field along the curve $\mathcal{K}_1$.*

A Space curve; circulation along a closed curve.

D Find $\mathbf{r}'(t)$ and the tangent corresponding to $t = \frac{\pi}{4}$.

Put $(x, y, z) = \mathbf{r}(t)$ into the equation of $\mathcal{F}$.

Try to apply Stokes's theorem. Alternatively, compute directly the circulation.

I 1) It follows from

$$\mathbf{r}'(t) = (-2\sin t, 2\cos t, 4\cos 2t)$$

that

$$\mathbf{r}\left(\frac{\pi}{4}\right) = \left(\frac{2}{\sqrt{2}}, \frac{2}{\sqrt{2}}, 4+2\right) = (\sqrt{2}, \sqrt{2}, 6)$$

and

$$\mathbf{r}'\left(\frac{\pi}{4}\right) = \left(-\frac{2}{\sqrt{2}}, \frac{2}{\sqrt{2}}, 0\right) = (-\sqrt{2}, \sqrt{2}, 0),$$

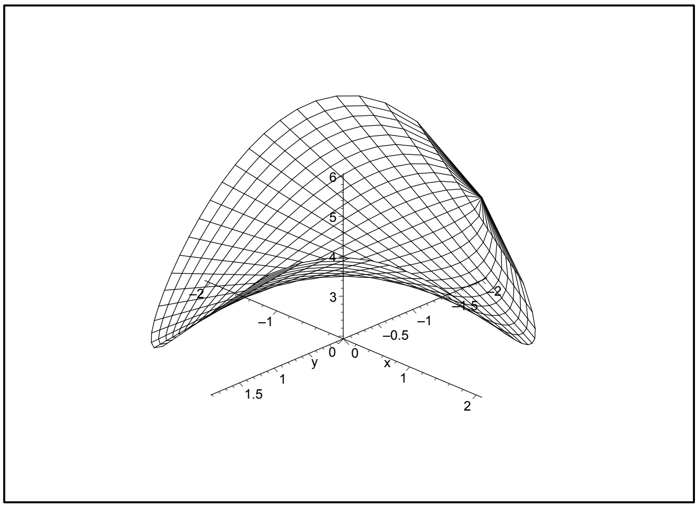

Figure 35.30: The surface $\mathcal{F}$ of equation $z = 4 + xy$ for $x^2 + y^2 \leq 4$

hence the equation of the tangent is

$$(x,y,z) = (\sqrt{2} - \sqrt{2}\,u, \sqrt{2} + \sqrt{2}\,u, 6) = (\sqrt{2}(1-u), \sqrt{2}(1+u), 6), \qquad u \in \mathbb{R}.$$

2) Since

$$4 + x(t)y(t) = 4 + 4\cos t \cdot \sin t = 4 + 2\sin 2t = z,$$

the curve $\mathcal{K}$ lies on the surface $\mathcal{F}$.

3) It follows from Stokes's theorem that

$$\int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t}\, \mathrm{d}s = \int_{\mathcal{F}} \mathbf{n} \cdot \mathbf{rot}\ \mathbf{V}\, \mathrm{d}S,$$

where

$$\mathbf{rot}\ \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & x & y^2 + 2z \end{vmatrix} = (2y, -1, 0).$$

Since $z = 4 + xy$, $x^2 + y^2 \le 4$, we get for the surface $\mathcal{F}$ that

$$\frac{\partial z}{\partial x} = y \qquad \text{and} \qquad \frac{\partial z}{\partial y} = x,$$

hence

$$\mathbf{N}(x, y) = (-y, -x, 1).$$

ALTERNATIVELY, $(x, y, z) = (u, v, 4 + uv)$, thus

$$\mathbf{N}(u, v) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 1 & 0 & v \\ 0 & 1 & u \end{vmatrix} = (-v, -u, 1) = (-y, -x, 1).$$

If we put $B = \{(x, y) \mid x^2 + y^2 \le 4\}$, then

$$\begin{aligned} \int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t}\, \mathrm{d}s &= \int_{\mathcal{F}} \mathbf{n} \cdot \mathbf{rot}\ \mathbf{V}\, \mathrm{d}S = \int_B \mathbf{N} \cdot \mathbf{rot}\ \mathbf{V}\, \mathrm{d}x\, \mathrm{d}y \\ &= \int_B (-2y^2 + x + 0)\, \mathrm{d}x\, \mathrm{d}y = -2 \int_0^{2\pi} \left\{ \int_0^2 (\varrho^2 \sin^2 \varphi + 0)\, \varrho\, \mathrm{d}\varrho \right\} \mathrm{d}\varphi \\ &= -2 \left[\frac{2\pi}{2} \right] \cdot \left[\frac{\varrho^4}{4} \right]_0^2 = -2\pi \cdot \frac{16}{4} = -8\pi. \end{aligned}$$

ALTERNATIVELY a direct computation gives

$$\begin{aligned} \int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t}\, \mathrm{d}s &= \int_0^{2\pi} \mathbf{V} \cdot \mathbf{r}'(t)\, \mathrm{d}t \\ &= \int_0^{2\pi} (2\sin t, 2\cos t, 4\sin^2 t + 8 + 4\sin 2t) \cdot (-\sin t, 2\cos t, 4\cos 2t)\, \mathrm{d}t \\ &= \int_0^{2\pi} \{-4\sin^2 t + 4\cos^2 t + 16\sin^2 t \cos 2t + 32\cos 2t + 16\sin 2t \cos 2t\}\, \mathrm{d}t \\ &= 16 \int_0^{2\pi} \sin^2 t \cos 2t\, \mathrm{d}t = 16 \int_0^{2\pi} \sin^2 t (2\cos^2 t - 1)\, \mathrm{d}t \\ &= 8 \int_0^{2\pi} \sin^2 2t\, \mathrm{d}t - 16\pi = 8 \cdot \frac{2\pi}{2} - 16\pi = -8\pi. \end{aligned}$$

Example 35.15 *Let α be a constant, and consider the vector field*

$$\mathbf{V}(x,y,z) = (\alpha x^2+xz+yz, \alpha y^2-xz-yz, \alpha(x^2-y^2+z^2)), \quad (x,y,z) \in \mathbb{R}^3.$$

1. *Find div* $\mathbf{V}$.

2. *Show that* $\mathbf{V}$ *is not a gradient field in* $\mathbb{R}^3$ *for any choice of* α.

Let $\mathcal{K}$ *denote the circle given by* $x^2+y^2=a^2$, $z=a$.

3. *Find the circulation of* $\mathbf{V}$ *along* $\mathcal{K}$*; indicate the chosen orientation.*

Let the domain $\Omega \subset \mathbb{R}^3$ *be given by* $x^2+y^2 \le a^2$, $y \ge 0$, $0 \le z \le a$.

4. *Find the flux of* $\mathbf{V}$ *through* $\partial\Omega$.

A Divergence, circulation and flux.

D Compute div $\mathbf{V}$. Check $\dfrac{\partial V_i}{\partial x_j}$ for some i and j. Find the circulation, e.g. by Stokes'a theorem. Finally, apply Gauß's theorem to find the flux.

I 1) The divergence is

$$\text{div}\,\mathbf{V} = 2\alpha x + z + 2\alpha y - z + 2\alpha z = 2\alpha(x+y+z).$$

2) It follows from

$$\frac{\partial V_1}{\partial y} = z \quad \text{and} \quad \frac{\partial V_2}{\partial z} = -z,$$

that

$$\frac{\partial V_1}{\partial y} \neq \frac{\partial V_2}{\partial x} \qquad \text{for } z \neq 0.$$

The surface $z=0$ does not contain inner points, thus $\mathbf{V}$ is not a gradient field for any value of α.

3) It follows by the definition of the circulation that

$$\begin{aligned}
&\int_{\mathcal{K}} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s \\
&= \int_0^{2\pi} \left(\alpha a^2\cos^2 t + a^2\cos^2 t + a^2\sin t, \alpha a^2\sin^2 t - a^2\cos t - a^2\sin t, \alpha a^2(\cos^2 t - \sin^2 + a)\right)\cdot \\
&\qquad\qquad \cdot(-\sin t, \cos t, 0)\,a\,\mathrm{d}t \\
&= a^2\int_0^{2\pi}\left\{-\alpha\cos^2 t\sin t - \cos t\sin t - \sin^2 t + \alpha\sin^2 t\cos t - \cos^2 t - \sin t\cos t + 0\right\}\,\mathrm{d}t \\
&= a^3\int_0^{2\pi}\left(-\sin^2 t - \cos^2 t\right)\,\mathrm{d}t = -2\pi a^2.
\end{aligned}$$

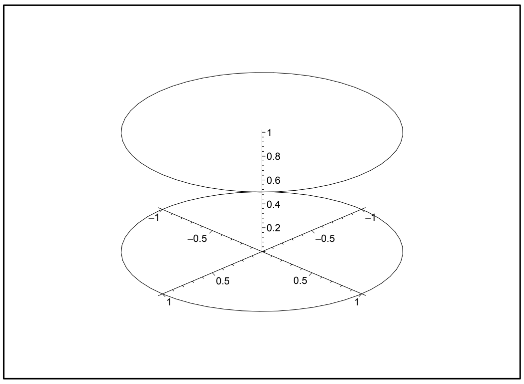

Figure 35.31: The curve $\mathcal{K}$ and its projection onto the (X, Y)-plane for $a = 1$.

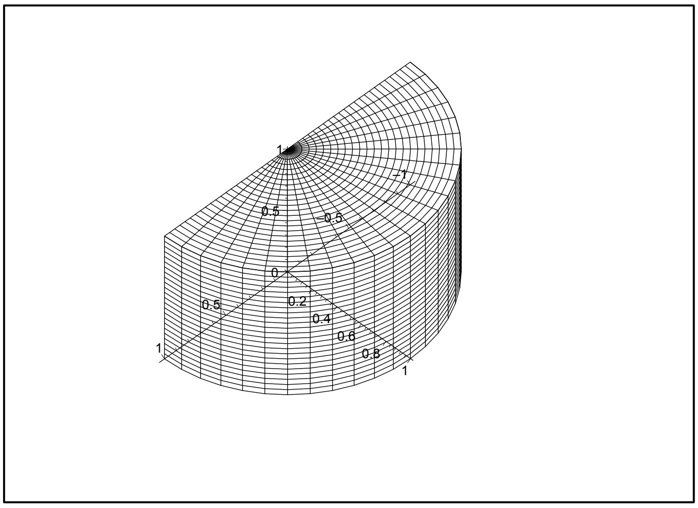

Figure 35.32: The body Ω for $a = 1$.

ALTERNATIVELY,

$$\mathbf{rot}\ \mathbf{V}\cdot\mathbf{e}_z = \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ xz+yz & -xz-yz \end{vmatrix} = -z-z=-2z.$$

Choose $\mathcal{F}$ as the disc $x^2+y^2\le a^2$, $z=a$. Then we get by Stokes's theorem that

$$\oint_{\mathcal{K}} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s = \int_{\mathcal{F}} \mathbf{rot}\ \mathbf{V}\cdot\mathbf{e}_z\,\mathrm{d}S = \int_{\mathcal{F}}(-2a)\,dS = -2a\ \text{area}(\mathcal{F}) = -2\pi a^3.$$

4) When we apply Gauß's theorem and 1), it follows that the flux is given by

$$\begin{aligned}
\int_{\partial\Omega}\mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S &= \int_{\Omega}\mathrm{div}\ \mathbf{V}\,\mathrm{d}\Omega = 2\alpha\int_{\Omega}(x+y+z)\,\mathrm{d}\Omega = 2\alpha\int_{\Omega}(y+z)\,\mathrm{d}\Omega \\
&= 2\alpha a\int_{-a}^{a}\left\{\int_0^{\sqrt{a^2-x^2}} y\,\mathrm{d}y\right\}\mathrm{d}x + 2\alpha\cdot\frac{\pi a^2}{2}\int_0^a z\,\mathrm{d}z \\
&= \alpha a\cdot 2\int_0^a (a^2-x^2)\,\mathrm{d}x + \alpha\pi a^2\cdot\frac{a^2}{2} = 2\alpha a\left(a^3-\frac{1}{3}a^3\right)+\frac{1}{2}\alpha\pi a^4 \\
&= \frac{4}{3}\alpha a^4+\frac{1}{2}\alpha\pi a^4 = \alpha a^4\left(\frac{4}{3}+\frac{\pi}{2}\right).
\end{aligned}$$

Example 35.16 *Consider the space curve $\mathcal{K}$ given by the parametric description*

$$(x,y,z) = \left(1+\cos t, \sin t, 2\sin\frac{t}{2}\right), \qquad t \in [0,2\pi].$$

1. *Find a parametric description of the tangent of $\mathcal{K}$ at the point corresponding to $t = \frac{\pi}{2}$.*

2. *Show that $\mathcal{K}$ lies on a sphere of centrum at $(0,0,0)$, and find an equation of the sphere.*

Furthermore, consider the surface $\mathcal{F}$ given by the parametric description

$$(x,y,z) = \left(1+\cos t, \sin t, 2u\sin\frac{t}{2}\right), \qquad (t,u) \in [0,2\pi]\times[0,1],$$

and the vector field $\mathbf{V}(x,y,z) = (x,y,z)$, $(x,y,z)\in\mathbb{R}^3$.

3. *Find the area of $\mathcal{F}$.*

4. *Find the circulation of the vector field along the curve $\mathcal{K}$.*

A Space curve, surface area, circulation of a vector field.

D 1) First calculate $\mathbf{r}'(t)$.

2) Show that $x^2+y^2+z^2 = r^2 > 0$ and find $r > 0$.

3) Compute the surface area.

4) Apply Stokes's theorem. Alternatively the circulation is computed directly as a line integral.

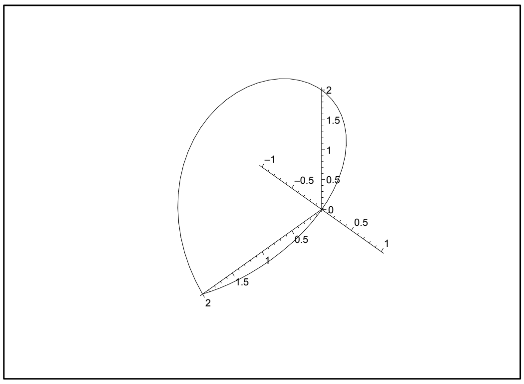

Figure 35.33: The curve $\mathcal{K}$.

I 1) We find

$$\mathbf{r}'(t) = \left(-\sin t, \cos t, \cos\frac{t}{2}\right), \qquad \mathbf{r}'\left(\frac{\pi}{2}\right) = \left(-1, 0, \frac{1}{\sqrt{2}}\right).$$

Now, $\mathbf{r}\left(\frac{\pi}{2}\right) = (1,1,\sqrt{2})$, hence a parametric description of the tangent is given by

$$(1,1,\sqrt{2}) + u\left(-1,0,\frac{1}{\sqrt{2}}\right), \qquad u\in\mathbb{R}.$$

2) Since

$$\begin{aligned} x(t)^2 + y(t)^2 + z(t)^2 &= (1+\cos t)^2 + \sin^2 t + 4\sin^2 \frac{t}{2} \\ &= 1 + 2\cos t + \cos^2 t + \sin^2 t + 2(1-\cos t) \\ &= 1+1+2 = 4 = 2^2, \end{aligned}$$

it follows that $\mathcal{K}$ lies on the sphere of centrum $(0,0,0)$ and radius 2.

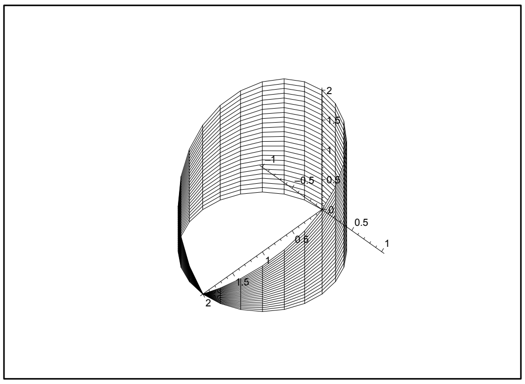

Figure 35.34: The surface $\mathcal{F}$.

3) It follows from

$$\frac{\partial \mathbf{r}}{\partial t} = \left(-\sin t, \cos t, u\,\cos\frac{t}{2}\right) \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial u} = \left(0, 0, 2\sin\frac{t}{2}\right)$$

that

$$\mathbf{N}(t,u) = \left| \begin{array}{ccc} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ -\sin t & \cos t & u\,\cos\dfrac{t}{2} \\ 0 & 0 & 2\sin\dfrac{t}{2} \end{array} \right| = 2\sin\frac{t}{2}\,(\cos t, \sin t, 0),$$

hence

$$\|\mathbf{N}(t,u)\| = \left|2\sin\frac{t}{2}\right| \cdot 1 = 2\sin\frac{t}{2}, \qquad t \in [0, 2\pi], \quad u \in [0,1].$$

Hence

$$\text{area}(\mathcal{F}) = \int_0^1 \left\{ \int_0^{2\pi} 2\sin\frac{t}{2}\,dt \right\} = \left[-4\cos\frac{t}{2}\right]_0^{2\pi} = 4(1+1) = 8.$$

4) Since $\mathbf{rot}\ \mathbf{V} = \mathbf{0}$, it follows by Stokes's theorem no matter how we choose the surface $\mathcal{F}_1$ with boundary curve $\mathcal{K}$ that

$$\oint_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t}\,\mathrm{d}t = \int_{\mathcal{F}_1} \mathbf{n} \cdot \mathbf{rot}\ \mathbf{V}\,\mathrm{d}S = 0.$$

ALTERNATIVELY we get by the definition that

$$\oint_{\mathcal{K}} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}t = \int_0^{2\pi} \mathbf{V}\cdot\mathbf{r}'(t)\,\mathrm{d}t = \int_0^{2\pi} \mathbf{r}(t)\cdot\mathbf{r}'(t)\,\mathrm{d}t$$

$$= \int_0^{2\pi} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\mathbf{r}(t)\|^2\right)\,\mathrm{d}t = \frac{1}{2}\left(\|\mathbf{r}(2\pi)\|^2 - \|\mathbf{r}(0)\|^2\right) = 0,$$

because the curve is closed.

ALTERNATIVELY it is possible though extremely tedious to insert the parametric description and then reduce.

Example 35.17 *Given the vector field*

$$\mathbf{V}(x,y,z) = \left(y^3 - xz^2, -x^3 + yz^2, z^3\right), \qquad (x,y,z) \in \mathbb{R}^3.$$

1. *Find the divergence $\bigtriangledown \cdot \mathbf{V}$ and the rotation $\bigtriangledown \times \mathbf{V}$.*

Let a be a positive constant, and let L denote the half spherical shell given by

$$z \geq 0, \qquad a^2 \leq x^2 + y^2 + z^2 \leq 3a^2.$$

2. *Find the flux of $\mathbf{V}$ through ∂L.*

Let $\mathcal{C}$ be the circle in the plane $z = a$ of centrum $(0,0,a)$ and radius a.

3. *Find the absolute value of the circulation $\oint_{\mathcal{C}} \mathbf{V} \cdot \mathbf{t}\, ds$.*

4. *Check if there exists a vector field $\mathbf{W} : \mathbb{R}^3 \to \mathbb{R}^3$, such that*

$$\mathbf{V} = \bigtriangledown \times \mathbf{W},$$

in the whole space.

5. *Check if there exists a scalar field $F : \mathbb{R}^3 \to \mathbb{R}$, such that $\mathbf{V} = \bigtriangledown F$ in the whole space.*

A Divergence, rotation, flux, circulation, vector potential, gradient field.

D Apply Gauß's theorem and Stokes's theorem, whenever it is possible.

I 1) We get by some very simple calculations that

$$\text{div } \mathbf{V} = \bigtriangledown \cdot \mathbf{V} = -z^2 + z^2 + 3z^2 = 3z^2,$$

and

$$\bigtriangledown \times \mathbf{V} = \mathbf{rot}\ \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^3 - xz^2 & -x^3 + yz^2 & z^3 \end{vmatrix} = (-2yz, -2xz, -3zx^2 - 3y^2).$$

2) It follows from Gauß's theorem and the result of 1) that

$$\int_{\partial L} \mathbf{V} \cdot \mathbf{n}\, \mathrm{d}S = \int_L \text{div } \mathbf{V}\, \mathrm{d}\Omega = \int_L 3z^2\, \mathrm{d}\Omega.$$

At height $z \in [0,a]$ the body L is cut into an annulus of the area

$$\pi(3a^2 - z^2) - \pi(a^2 - z^2) = 2\pi a^2.$$

At height $z \in [a, \sqrt{3}\, a]$ the body L is cut into a circle of area

$$\pi(3a^2 - z^2).$$

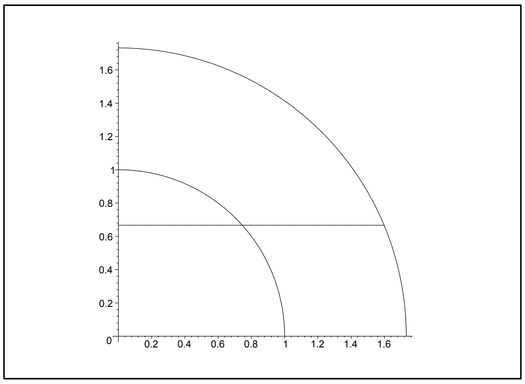

Figure 35.35: The meridian cut for $a = 1$ with the cut at height z.

Hence by insertion,

$$\begin{aligned}\int_{\partial L} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S &= \int_L 3z^2\,\mathrm{d}\Omega = \int_0^a 3z^2\cdot 2\pi a^2\,\mathrm{d}z + \int_a^{\sqrt{3}a} 3z^2\pi(3a^2-z^2)\,\mathrm{d}z \\ &= 6\pi a^2\int_0^a z^2\,\mathrm{d}z + 9\pi a^2\int_a^{\sqrt{3}a} z^2\,\mathrm{d}z - 3\pi\int_a^{\sqrt{3}a} z^4\,\mathrm{d}z \\ &= 9\pi a^2\int_0^{\sqrt{3}a} z^2\,\mathrm{d}z - 3\pi a^2\int_0^a z^2\,\mathrm{d}z - 3\pi\int_a^{\sqrt{3}a} z^4\,\mathrm{d}z \\ &= 3\pi a^2\cdot 3\sqrt{3}\,a^3 - \pi a^2\cdot a^3 - \frac{3}{5}\,\pi(9\sqrt{3}-1)a \\ &= \pi a^5\left(9\sqrt{3}-1-\frac{27}{5}\sqrt{3}+\frac{3}{5}\right) = \frac{\pi}{5}\,(18\sqrt{3}-2)a^5.\end{aligned}$$

3) Put $B = \{(x,y) \mid x^2+y^2 \le a^2\}$. Then we get by Stokes's theorem and the result of 1) that

$$\begin{aligned}\left|\oint_{\mathcal{C}} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s\right| &= \left|\int_B \mathbf{rot}\ \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}x\,\mathrm{d}y\right| = \left|\int_B \left(-2ya,-2xa,-3x^2-3y^2\right)\cdot(0,0,1)\,\mathrm{d}x\,\mathrm{d}y\right| \\ &= 3\int_B (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y = 3\cdot 2\pi\int_0^a \varrho^2\cdot\varrho\,\mathrm{d}\varrho = 6\pi\cdot\frac{a^4}{4} = \frac{3\pi a^4}{2}.\end{aligned}$$

4) Since div $\mathbf{V} = 3z^2 \neq 0$ for $z \neq 0$, there exists no vector potential $\mathbf{W}$ of $\mathbf{V}$ in all of the space.

5) It follows from

$$\frac{\partial V_1}{\partial y} = 3y^2 \quad\text{and}\quad \frac{\partial V_2}{\partial x} = -3x^2,$$

that

$$\frac{\partial V_1}{\partial y} \neq \frac{\partial V_2}{\partial x} \qquad \text{for } (x,y) \neq (0,0),$$

and $\mathbf{V}$ is not a gradient field, and there exists no integral F of $\mathbf{V}$.

Example 35.18 *Given the vector fields*

$$\mathbf{U}(x,y,z) = (z^2 + y\cos x, x^2 + \sin x, y^2), \qquad \mathbf{V}(x,y,z) = (y,z,x),$$

in the space $\mathbb{R}^3$.

1) Find the divergence and the rotation of both vector fields.

2) Find the flux of $\mathbf{U}$ *through the surface of the cube*

$$\{(x,y,z) \in \mathbb{R}^3 \mid 0 \le x \le 1,\, 0 \le y \le 1,\, 0 \le z \le 1\}.$$

3) Let a *be a positive constant. Find the circulation of* $\mathbf{V}$ *along the circle in the* (X,Z)*-plane of centrum at* $(a,0,2a)$ *and radius* a*. Choose an orientation of the circle.*

4) Find a vector potential for $\mathbf{V}$.

A Vector analysis.

D Apply Gauß's theorem and Stokes's theorem.

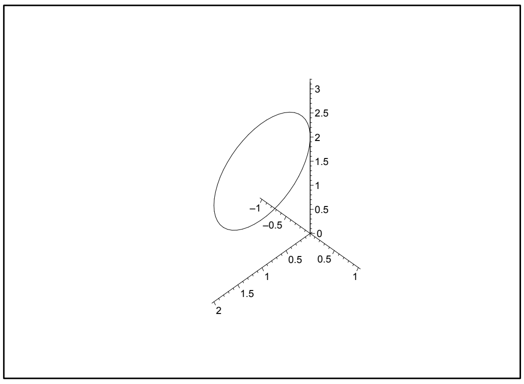

Figure 35.36: The circle of 3).

I 1) By simple calculations,

$$\text{div}\,\mathbf{U} = -y\sin x, \qquad \text{div}\,\mathbf{V} = 0,$$

$$\textbf{rot}\ \mathbf{U} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z^2 + y\cos x & x^2 + \sin x & y^2 \end{vmatrix} = (2y, 2z, 2x) = 2\mathbf{V},$$

$$\textbf{rot}\ \mathbf{V} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = (-1,-1,-1).$$

In particular, $\mathbf{V} = \nabla \times \left(\frac{1}{2}\,\mathbf{U}\right)$, thus $\frac{1}{2}\,\mathbf{U}$ is a vector potential of $\mathbf{V}$ (cf. 4)).

2) According to Gauß's theorem the flux of $\mathbf{U}$ through ∂T is given by

$$\begin{aligned}\int_{\partial T} \mathbf{U}\cdot\mathbf{n}\,\mathrm{d}S &= \int_T \operatorname{div}\,\mathbf{U}\,\mathrm{d}\Omega = \int_0^1 \left\{\int_0^1 \left\{\int_0^1 (-y\,\sin x)\,\mathrm{d}x\right\}\mathrm{d}y\right\}\mathrm{d}z \\ &= 1\cdot\left[\frac{y^2}{2}\right]_0^1 \cdot [\cos x]_0^1 = -\frac{1}{2}\,(1-\cos 1).\end{aligned}$$

3) According to Stokes's theorem,

$$\oint_{\partial A} \mathbf{t}\cdot\mathbf{V}\,\mathrm{d}s = \int_A \mathbf{n}\cdot\mathbf{rot}\mathbf{V}\,\mathrm{d}S = \int_A (0,-1,0)\cdot(-1,-1,-1)\,\mathrm{d}S = \text{ area}(A) = \pi a^2.$$

4) According to the result of 1), the field $\frac{1}{2}\,\mathbf{U}$ is a vector potential of $\mathbf{V}$.

Example 35.19 *Let $p \in \mathbb{R}$ and $b \in \mathbb{R}_+$ be constants. consider the circle $\mathcal{K}$ given by $x^2 + y^2 = b^2$, $z = p$; the circle is given the orientation which forms a right hand turn with the Z-axis. Furthermore, consider the vector field*

$$\mathbf{W}(x,y,z) = \left(\frac{yz}{\sqrt{x^2+y^2}}, \frac{-xz}{\sqrt{x^2+y^2}}, \sqrt{x^2+y^2} \right), \qquad (x,y) \neq (0,0).$$

Denote the circulation of $\mathbf{W}$ along the oriented circle $\mathcal{K}$ by $C(b,p)$.

1) Show that $C(b,p) = -2\pi pb$.

2) Let $\mathbf{V} = \mathbf{rot}\ \mathbf{W}$. Show that

$$\mathbf{V}(x,y,z) = \frac{1}{\sqrt{x^2+y^2}}\,(x+y, y-x, -z), \qquad (x,y) \neq (0,0).$$

3) Show that $\mathbf{W}$ is not a gradient field.

4) Show that $\mathbf{W}$ has zero divergence.

5) Let $\mathcal{O}$ be the surface of revolution which is given in semi-polar coordinates by

$$\varrho \in [a, 2a], \qquad \varphi \in [0, 2\pi], \qquad z = 2a - \frac{\varrho^2}{a},$$

where $a > 0$ is a positive constant.
Find the flux

$$\int_{\mathcal{O}} \mathbf{V} \cdot \mathbf{n}\, \mathrm{d}S,$$

where we also shall choose an orientation of $\mathcal{O}$.

A Circulation, rotation, gradient field, divergence, flux.

D The circulation can be computed in various ways. The computation of the flux has also some variants.

I 1) We have two variants.

First variant. *The definition of the circulation as a line integral.*
We use the following parametric description of the circle $\mathcal{K}$,

$$\mathbf{r}(t) = (x,y,z) = (b \cdot \cos t, b \cdot \sin t, p), \qquad t \in [0, 2\pi].$$

Then

$$\mathbf{r}'(t) = b\,(-\sin t, \cos t, 0),$$

and the circulation is according to the definition given by

$$\begin{aligned} C(b,p) &= \int_{\mathcal{K}} \mathbf{W} \cdot \mathbf{t}\, \mathrm{d}s = \int_0^{2\pi} \left(\frac{b \sin t \cdot p}{b}, -\frac{b \cos t \cdot p}{b}, b \right) \cdot b\,(-\sin t, \cos t, 0)\, \mathrm{d}t \\ &= -pb \int_0^{2\pi} \left\{ \sin^2 t + \cos^2 t + 0 \right\} \mathrm{d}t = -2\pi pb. \end{aligned}$$

Second variant. *Stokes's theorem.*

An application of Stokes's theorem gives

$$\int_{\mathcal{K}} \mathbf{W} \cdot \mathbf{t}\, \mathrm{d}s = \int_{\mathcal{F}} \mathbf{rot}\ \mathbf{W} \cdot \mathbf{n}\, \mathrm{d}S,$$

where $\mathcal{F}$ is the disc at height $z = p$ and radius b, and where the unit normal vector is parallel to the Z-axis.

The unit normal is trivially $\mathbf{n} = (0,0,1)$. Then by 2),

$$\mathbf{rot}\ \mathbf{W} = \mathbf{V}.$$

When we apply the expression of $\mathbf{V}$, we obtain in polar coordinates

$$\int_{\mathcal{K}} \mathbf{W} \cdot \mathbf{t}\, \mathrm{d}s = \int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n}\, \mathrm{d}S = \int_{\mathcal{F}} \left(-\frac{p}{\sqrt{x^2+y^2}} \right) \mathrm{d}S = -p \int_0^{2\pi} \left\{ \int_0^b \frac{1}{\varrho} \cdot \varrho\, \mathrm{d}\varrho \right\} \mathrm{d}\varphi = -2\pi p b.$$

2) Let $(x,y) \neq (0,0)$. Then

$$\mathbf{V} = \mathbf{rot}\ \mathbf{W} = \bigtriangledown \times \mathbf{W} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{yz}{\sqrt{x^2+y^2}} & \dfrac{-xz}{\sqrt{x^2+y^2}} & \sqrt{x^2+y^2} \end{vmatrix}$$

$$= \begin{pmatrix} \dfrac{y}{\sqrt{x^2+y^2}} - \left(-\dfrac{x}{\sqrt{x^2+y^2}} \right) \\ \dfrac{y}{\sqrt{x^2+y^2}} - \dfrac{x}{\sqrt{x^2+y^2}} \\ -\dfrac{z}{\sqrt{x^2+y^2}} + \dfrac{x^2 z}{(\sqrt{x^2+y^2})^3} - \dfrac{z}{\sqrt{x^2+y^2}} + \dfrac{y^2 z}{(\sqrt{x^2+y^2})^3} \end{pmatrix}$$

$$= \frac{1}{\sqrt{x^2+y^2}} \begin{pmatrix} y+x \\ y-x \\ -2z + \dfrac{x^2 z + y^2 z}{x^2+y^2} \end{pmatrix} = \frac{1}{\sqrt{x^2+y^2}}\,(x+y, y-x, -z).$$

3) Suppose $\mathbf{W}$ was a gradient field, $\bigtriangledown F$. Then

$$\mathbf{V} = \bigtriangledown \times \mathbf{W} = \bigtriangledown \times \bigtriangledown F = \mathbf{0}.$$

But $\mathbf{V} \neq \mathbf{0}$, thus we conclude that $\mathbf{W}$ is not a gradient field.

4) By just computing,

$$\begin{aligned} \operatorname{div} \mathbf{W} &= \frac{\partial}{\partial x}\left(\frac{yz}{\sqrt{x^2+y^2}} \right) - \frac{\partial}{\partial y}\left(\frac{xz}{\sqrt{x^2+y^2}} \right) + \frac{\partial}{\partial z}\left(\sqrt{x^2+y^2} \right) \\ &= -\frac{xyz}{(x^2+y^2)^{3/2}} - \left(-\frac{xyz}{(x^2+y^2)^{3/2}} \right) + 0 = 0. \end{aligned}$$

Alternatively it follows that *if* $\mathbf{U}$ is defined by

$$3\mathbf{U} = \frac{z^2}{\sqrt{x^2+y^2}}\,(-x,-y,0) + \sqrt{x^2+y^2}\,(-y.x.z), \qquad (x,y) \neq (0,0),$$

then

$$\mathbf{W} = \mathbf{rot}\;\mathbf{U} = \bigtriangledown \times \mathbf{U},$$

and hence

$$\text{div}\;\mathbf{W} = \bigtriangledown \cdot (\bigtriangledown \times \mathbf{U}) = 0.$$

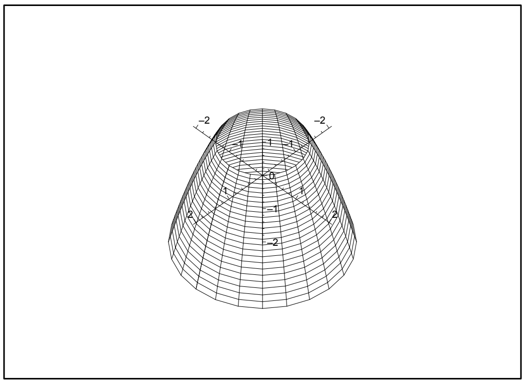

Figure 35.37: The surface $\mathcal{O}$. The upper boundary curve $\mathcal{K}_1$ is oriented as a left hand screw, while the lower boundary curve $\mathcal{K}_2$ is oriented as a right hand screw. Hence the normal vector field on $\mathcal{O}$ is everywhere pointing away from the Z-axis.

5) Choose the orientation on $\mathcal{O}$ as described in the caption of the figure. Then $\delta\mathcal{O} = \mathcal{K}_2 - \mathcal{K}_1$, where the minus sign in front of $\mathcal{K}_1$ means that this circle is run through in the opposite direction of the usual one, i.e. as a left handed screw.

There are two variants.

1. variant. *Stokes's theorem combined with 1).*

We get by Stokes's theorem,

$$\begin{aligned}\int_{\mathcal{O}} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S &= \int_{\mathbf{O}} (\bigtriangledown \times \mathbf{W})\cdot\mathbf{n}\,\mathrm{d}S = \oint_{\delta\mathcal{O}} \mathbf{W}\cdot\mathbf{t}\,\mathrm{d}s = -\int_{\mathcal{K}_1} \mathbf{W}\cdot\mathbf{t}\,\mathrm{d}s + \int_{\mathcal{K}_2} \mathbf{W}\cdot\mathbf{t}\,\mathrm{d}s \\ &= -C(a,a) + C(2a,-2a) = +2\pi a\cdot a + (+2\pi\cdot 2a\cdot 2a) = 10\pi a^2.\end{aligned}$$

Second variant. *Surface integral.*

The meridian curve has the equation

$$z = 2a - \frac{\varrho^2}{a},$$

so we conclude that the tangent vector is $\left(1,-\dfrac{2\varrho}{a}\right)$. Hence the normal vector $\mathbf{N}=(2\varrho,a)$, and thus the unit normal vector

$$\mathbf{n}=\frac{1}{\sqrt{a^2+4\varrho^2}}\,(2\varrho,a).$$

Then the outgoing unit normal vector field of the surface $\mathcal{O}$ is

$$\mathbf{n}(\varrho,\varphi)=\frac{1}{\sqrt{a^2+4\varrho^2}}\,(2\varrho\cos\varphi,2\varrho\sin\varphi,a).$$

We have on $\mathcal{O}$,

$$\begin{aligned}\mathbf{V}&=\frac{1}{\sqrt{x^2+y^2}}\,(x+y,y-x,-z)\\&=\frac{1}{\varrho}\left(\varrho\cos\varphi+\varrho\sin\varphi,\varrho\sin\varphi-\varrho\cos\varphi,-2a+\frac{\varrho^2}{a}\right)\\&=\left(\cos\varphi+\sin\varphi,\sin\varphi-\cos\varphi,\frac{\varrho}{a}-2\,\frac{a}{\varrho}\right),\end{aligned}$$

hence the integrand over $\mathcal{O}$ is written

$$\begin{aligned}f(x,y,z)&=\mathbf{V}\cdot\mathbf{n}=\frac{1}{\sqrt{a^2+4\varrho^2}}\left\{2\varrho\left(\cos^2\varphi+\cos\varphi\sin\varphi+\sin^2\varphi-\cos\varphi\sin\varphi\right)+\varrho-2\,\frac{a^2}{\varrho}\right\}\\&=\frac{1}{\sqrt{a^2+4\varrho^2}}\left\{3\varrho-2\,\frac{a^2}{\varrho}\right\}.\end{aligned}$$

Finally, by inserting into a known formula of the surface integral over surfaces of revolution, we get

$$\begin{aligned}\int_{\mathcal{O}} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S &= \int_a^{2a}\left\{\int_0^{2\pi}\frac{1}{\sqrt{a^2+4\varrho^2}}\left\{3\varrho-2\,\frac{a^2}{\varrho}\right\}\mathrm{d}\varphi\right\}\varrho\sqrt{1+\frac{4\varrho^2}{a^2}}\,\mathrm{d}\varrho \\ &= \frac{2\pi}{a}\int_a^{2a}\left(3\varrho^2-2a^2\right)\mathrm{d}\varrho = \frac{2\pi}{a}\left[\varrho^3-2a^2\varrho\right]_a^{2a} \\ &= \frac{2\pi}{a}\left\{8a^3-4a^3-a^3+2a^3\right\} = 10\pi a^2.\end{aligned}$$

Example 35.20 *Let $\mathcal{F}$ be one eighth of a sphere given by*

$$x^2+y^2+z^2=a^2, \qquad x\ge 0, \quad y\ge 0, \quad z\ge 0,$$

where a is a positive constant. Thus the boundary curve $\delta\mathcal{F}$ is composed of three circular arcs. Also, consider the vector field

$$\mathbf{V}(x,y,z)=(ay+yz,-ax+zx,z^2-2xy), \qquad (x,y,z)\in\mathbb{R}^3.$$

1) Find the rotation $\bigtriangledown\times\mathbf{V}$.

2) Show that $\mathbf{V}$ is not a gradient field.

3) Find the circulation

$$\oint_{\delta\mathcal{F}}\mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s,$$

where we choose an orientation of $\delta\mathcal{F}$.

A Rotation, circulation, Stokes's theorem.

D Sketch a figure. Apply Stokes's theorem.

I 1) The rotation is

$$\begin{aligned}\mathbf{rot}\;\mathbf{V} &= \bigtriangledown\times\mathbf{V} = \begin{vmatrix}\mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ ay+yz & -ax+zx & z^2-2xy\end{vmatrix} \\ &= (-2x-x,y+2y,-a+zaz)=(3x,3y,-2a).\end{aligned}$$

2) It follows from $\mathbf{rot}\;\mathbf{V}\neq\mathbf{0}$ that $\mathbf{V}$ is not a gradient field.

3) Choose the orientation of $\delta\mathcal{F}$ as described on the figure. Then the unit normal vector field on $\mathcal{F}$ is pointing outwards, i.e.

$$\mathbf{n}=\frac{1}{a}\,(x,y,z), \qquad \text{for } (x,y,z)\in\mathcal{F}.$$

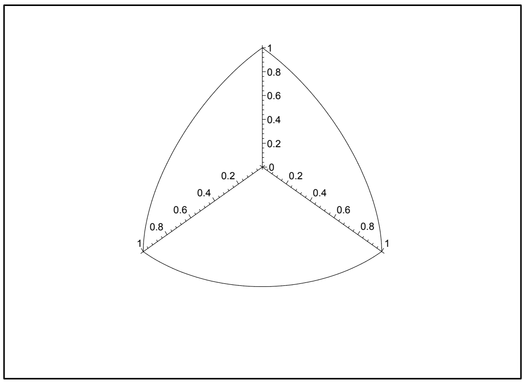

Figure 35.38: The surface $\mathcal{F}$ and the boundary curve $\delta\mathcal{F}$ for $a = 1$. On the surface $\mathcal{F}$ the unit normal vector field is always directed away from $(0, 0, 0)$, and the curve $\delta\mathcal{F}$ is oriented correspondingly, i.e. from the X-axis towards the y-axis, then towards the z-axis and finally back to the x-axis.

When we apply Stokes's theorem we conclude that the circulation along $\delta\mathcal{F}$ is

$$\begin{aligned}\oint_{\delta\mathcal{F}} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s &= \int_{\mathcal{F}} \mathbf{n}\cdot\mathbf{rot}\ \mathbf{V}\,\mathrm{d}S = \int_{\mathcal{F}} \frac{1}{a}\,(x,y,z)\cdot(-3x,3y,-2a)\,\mathrm{d}S \\ &= \frac{1}{a}\int_{\mathcal{F}} (-3x^2+3y^2-2az)\,\mathrm{d}S = -2\int_{\mathcal{F}} z\,\mathrm{d}S,\end{aligned}$$

where it follows by the symmetry that

$$\int_{\mathcal{F}} x^2\,\mathrm{d}S = \int_{\mathcal{F}} y^2\,\mathrm{d}S.$$

The following computations can be given in various variants.

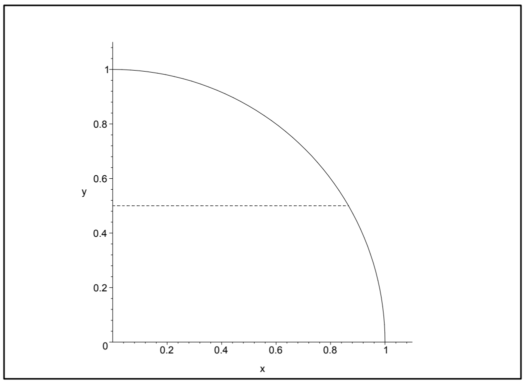

Figure 35.39: The meridian cut of $\mathcal{F}$ for $a = 1$. We have at height z that $\varrho = \sqrt{a^2 - z^2}$.

First variant. If we first (i.e. innermost) at height z and denote the circle by ℓ, then

$$\oint_{\delta\mathcal{F}} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s = -2\int_{\mathcal{F}} z\,\mathrm{d}S = -2\int_{\ell} z\cdot\frac{\pi}{2}\cdot\sqrt{a^2-z^2}\,\mathrm{d}s$$
$$= -\pi\int_{\ell} z\sqrt{a^2-z^2}\,\mathrm{d}s \quad \left(-\pi\int_{\ell} z\varrho\,\mathrm{d}s\right).$$

Using the parametric description

$$\varrho = a\,\cos\varphi, \qquad z = a\,\sin\varphi,$$

of ℓ we get $ds = a\,\mathrm{d}\varphi$, and the computations continue as follows,

$$\oint_{\delta\mathcal{F}} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s = -\pi\int_0^{\frac{\pi}{2}} a\,\sin\varphi\cdot a\,\cos\varphi\cdot a\,\mathrm{d}\varphi = -\pi a^3\int_0^{\frac{\pi}{2}}\sin\varphi\,\cos\varphi\,\mathrm{d}\varphi$$
$$= -\pi a^3\left[\frac{\sin^2\varphi}{2}\right]_0^{\frac{\pi}{2}} = -\frac{\pi}{2}\,a^3.$$

Second variant. The surface $\mathcal{F}$ is described in spherical coordinates by

$$\begin{cases} x = a\,\sin\theta\,\cos\varphi, \\ y = a\,\sin\theta\,\sin\varphi, \\ z = a\,\cos\theta, \end{cases} \qquad \begin{cases} \theta\in\left[0,\frac{\pi}{2}\right], \\ \varphi\in\left[0,\frac{\pi}{2}\right], \end{cases} \qquad \text{weight: } a^2\sin\theta,$$

hence by insertion,

$$\oint_{\delta\mathcal{F}} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s = -\int_{\mathcal{F}} 2z\,\mathrm{d}S = -2\int_0^{\frac{\pi}{2}}\left\{\int_0^{\frac{\pi}{2}} a\cos\theta\cdot a^2\sin\theta\,\mathrm{d}\theta\right\}\mathrm{d}\varphi$$
$$= -2\cdot\frac{\pi}{2}\cdot a^3\left[\frac{\sin^2\varphi}{2}\right]_0^{\frac{\pi}{2}} = -\frac{\pi}{2}\,a^3.$$

Third variant. *Direct computation of the line integrals without the use of Stokes's theorem.*

First note that the boundary curve $\delta\mathcal{F}$ is composed of the subcurves:

Γ_1: $(x,y,z) = (a\,\cos\varphi, a\,\sin\varphi, 0)$, $\varphi\in\left[0,\frac{\pi}{2}\right]$, with the unit tangent vector $\mathbf{t} = (-\sin\varphi, \cos\varphi, 0)$, and the line element $\mathrm{d}s = a\,\mathrm{d}\varphi$.

Γ_2: $(x,y,z) = (0, a\,\cos\varphi, a\,\sin\varphi)$, $\varphi\in\left[0,\frac{\pi}{2}\right]$, with the unit tangent vector $\mathbf{t} = (0, -\sin\varphi, \cos\varphi)$, and the line element $\mathrm{d}s = a\,\mathrm{d}\varphi$,

Γ_3: $(x,y,z) = (a\,\sin\varphi, 0, a\,\cos\varphi)$, $\varphi\in\left[0,\frac{\pi}{2}\right]$, with the unit tangent vector $\mathbf{t} = (\cos\varphi, 0, -\sin\varphi)$ and the line element $\mathrm{d}s = a\,\mathrm{d}\varphi$.

We get by insertion,

$$\oint_{\delta\mathcal{F}} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s = \int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma_3}(1t + yz, -ax + zx, z^2 - 2xy)\cdot\mathbf{t}\,\mathrm{d}s.$$

The integrals are computed separately,

$$\begin{aligned}
\int_{\Gamma_1} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s &= \int_0^{\frac{\pi}{2}} \left(a^2\sin\varphi, -a^2\cos\varphi, -2a^2\sin\varphi\cos\varphi\right)\cdot(-\sin\varphi,\cos\varphi,0)\,a\,\mathrm{d}\varphi \\
&= \int_0^{\frac{\pi}{2}} a^3\left(-\sin^2\varphi-\cos^2\varphi\right)\,\mathrm{d}\varphi = -\frac{\pi}{2}\,a^3, \\
\int_{\Gamma_2} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s &= \int_0^{\frac{\pi}{2}} \left(a^2\cos\varphi+a^2\sin\varphi\cos\varphi, 0, a^2\sin^2\varphi\right)\cdot(0,-\sin\varphi,\cos\varphi)\,a\,\mathrm{d}\varphi \\
&= \int_0^{\frac{\pi}{2}} a^3\sin^2\varphi\cos\varphi\,\mathrm{d}\varphi = {}^3\left[\frac{\sin^3\varphi}{3}\right]_0^{\frac{\pi}{2}} = \frac{1}{3}\,a^3, \\
\int_{\Gamma_3} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s &= \int_0^{\frac{\pi}{2}} \left(0,-a^2\sin\varphi+a^2\sin\varphi\cos\varphi, a^2\cos^2\varphi\right)\cdot(\cos\varphi,0,-\sin\varphi)\,a\,\mathrm{d}\varphi \\
&= \int_0^{\frac{\pi}{2}} \left(-a^3\cos^2\varphi\sin\varphi\right)\,\mathrm{d}\varphi = a^3\left[\frac{\cos^3\varphi}{3}\right]_0^{\frac{\pi}{2}} = -\frac{1}{3}\,a^3.
\end{aligned}$$

Summarizing,

$$\oint_{\delta\mathcal{F}} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s = -\frac{\pi}{2}\,a^3 + \frac{1}{3}\,a^3 - \frac{1}{3}\,a^3 = -\frac{\pi}{2}\,a^3.$$

Example 35.21 *Consider the vector field*

$$\mathbf{V}(x,y,z) = (xz, yz + xz, 2xz - yz), \qquad (x,y,z) \in \mathbb{R}^3.$$

1. *Find the divergence* $\nabla \cdot \mathbf{V}$ *and the rotation* $\nabla \times \mathbf{V}$.

Let A *denote the half ball given by*

$$x^2 + y^2 + z^2 \le c^2, \qquad z \ge 0,$$

where c *is a positive constant, and let* $\mathbf{n}$ *be the outwards unit normal vector of the surface* ∂A.

2. *Find the flux*

$$\Phi = \int_{\partial A} \mathbf{V} \cdot \mathbf{n} \, \mathrm{d}S.$$

3. *The surface* ∂A *is the union of a disc* $\mathcal{F}_1$ *and a half sphere* $\mathcal{F}_2$. *Find the fluxes*

$$\Phi_1 = \int_{\mathcal{F}_1} \mathbf{V} \cdot \mathbf{n} \, \mathrm{d}S \quad \text{and} \quad \Phi_2 = \int_{\mathcal{F}_2} \mathbf{V} \cdot \mathbf{n} \, \mathrm{d}S.$$

Let $\mathcal{K}$ *denote a circle in the plane of equation* $z = b$. *We denote the centrum of the circle by* (x_0, y_0, b), *and its radius is called* a.

4. *Choose an orientation of the circle* $\mathcal{K}$. *Then find the circulation*

$$C = \oint_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} \, \mathrm{d}s.$$

A Divergence, rotation, flux, circulation.

D Follow the guidelines which give the simplest variant.

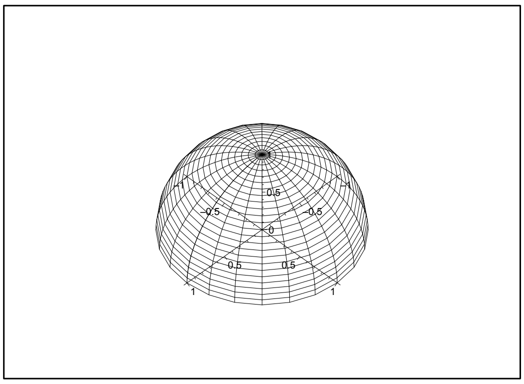

Figure 35.40: The half ball A for $c = 1$.

I 1) By just computing we get

$$\text{div}\ \mathbf{V} = \bigtriangledown \cdot \mathbf{V} = z + z + 2x - y = 2x - y + 2z$$

and

$$\begin{aligned}\mathbf{rot}\ \mathbf{V} &= \bigtriangledown \times \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xz & yz + xz & 2xz - yz \end{vmatrix} \\ &= (-z - y - x, x - 2x, z) = (-x - y - z, -x, z).\end{aligned}$$

2) Then by Gauß's theorem,

$$\begin{aligned}\Phi &= \int_{\partial A} \mathbf{V} \cdot \mathbf{n}\, \mathrm{d}S = \int_A \text{div}\ \mathbf{V}\, \mathrm{d}\Omega = \int_A (2x - y + 2z)\, \mathrm{d}\Omega \\ &= 0 + 0 + 2\int_A z\, \mathrm{d}\Omega = 2 \cdot \frac{\pi}{4}\, c^4 = \frac{\pi}{2}\, c^4.\end{aligned}$$

3) Now $\mathbf{n} = (0,0,-1)$ on $\mathcal{F}_1$, where also $z = 0$. Hence

$$\Phi_1 = \int_{\mathcal{F}_1} (0,0,0) \cdot \mathbf{n}\, \mathrm{d}S = 0.$$

Then apply the result of 2) and that $\Phi = \Phi_1 + \Phi_2$, to get

$$\Phi_2 = \Phi - \Phi_1 = \frac{\pi}{2}\, c^4.$$

4) Choose the orientation such that the projection onto the (X,Y)-plane has a positive orientation. Then the corresponding unit normal vector is $\mathbf{n} = (0,0,1)$.

By Stokes's theorem, the circulation along $\mathcal{K}$ (which encircles the disc B) is given by

$$\begin{aligned}C &= \oint_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t}\, \mathrm{d}s = \int_B \mathbf{n} \cdot \bigtriangledown \times \mathbf{V}\, \mathrm{d}S = \int_B (0,0,1) \cdot (-x - y - b, -x, -b)\, \mathrm{d}S \\ &= -b\int_B \mathrm{d}S = -b \cdot \text{area}(B) = -b \cdot \pi \cdot a^2.\end{aligned}$$

36 Nabla calculus

From time to time we have previously in $\mathbb{R}^3$ used the notations

$$\mathbf{grad} = \nabla, \qquad \text{div} = \nabla \cdot \quad \text{and} \quad \mathbf{rot} = \mathbf{curl} = \nabla \times .$$

In this section we shall more systematically use nabla (∇) instead of **grad**, div, **rot** and **curl**. It turns up that this change of notation is very convenient, because the formulæ containing ∇ will be very similar to those already known from *Linear Algebra* in the three-dimensional Euclidean space $E_3 \sim \mathbb{R}^3$.

Remark 36.1 The gradient, divergence etc. were introduced in the middle of the 19th century. Clearly, one needed a shorthand for these new operations, and one chose to put delta, Δ, upside down to get ∇. Concerning the name of this new operator one first tried to spell delta backwards, "atled", but this suggestion was never successful. It probably resembled "Amled =Hamlet" too much. At that time, however, one had started to excavate the ruins of ancient Iraq, and on some of the clay tablets one also found pictures of harps. These harps had the same shape as ∇, so this new symbol was afterwards called "nabla", which is the Assyrian word for harp. One may still find a remnant of it in David's 57th psalm, verse 9, where the word "nevel" is used. This is an ancient Hebrew word for a harp, which no longer is commonly understood. $\Diamond$

36.1 The vectorial differential operator ∇

The vectorial differential operator ∇ is defined in all spaces $\mathbb{R}^n$ in a rectilinear coordinate system by

$$\nabla = \mathbf{e}_1 \frac{\partial}{\partial x_1} + \cdots + \mathbf{e}_n \frac{\partial}{\partial x_n} = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right).$$

In most applications, however, we restrict ourselves to $E_3 \sim \mathbb{R}^3$, where we write

$$\nabla = \mathbf{e}_1 \frac{\partial}{\partial x} + \mathbf{e}_2 \frac{\partial}{\partial y} + \frac{\partial}{\partial z} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right).$$

We collect the definitions and rules, which we have met previously.

1) **Gradient of a scalar field** $f : \Omega \to \mathbb{R}$, where $\Omega \subseteq \mathbb{R}$, and $f \in C^1(\Omega)$,

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right).$$

2) **Divergence of a vector field** $\mathbf{V} : \Omega \to \mathbb{R}^n$, $\Omega \subseteq \mathbb{R}^n$, and $\mathbf{V} \in C^1(\Omega) \times \cdots \times C^1(\Omega)$,

$$\nabla \cdot \mathbf{V} = \frac{\partial V_1}{\partial x_1} + \cdots + \frac{\partial V_n}{\partial x_n}.$$

3) **Rotation of a vector field** $\mathbf{V} : \Omega \to \mathbb{R}$, $\Omega \subseteq \mathbb{R}^3$, and $\mathbf{V} \in C^1(\Omega)^3$. Note that rotation *is only introduced in* $\mathbb{R}^3$.

$$\nabla \times \mathbf{V} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}, \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}, \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix},$$

The formal determinant is interpreted in the following way as if we were dealing with an ordinary determinant. The only difference is that we *always* calculate downwards and let the differential operators in the second line act on the functions in the third line. Note that if $\bigtriangledown$ is replaced by a vector field $\mathbf{U}$, then we just get the usual determinant formula for $\mathbf{U} \times \mathbf{V}$.

4) **Differential of a scalar field** $f : \Omega \to \mathbb{R}$, where $\Omega \subseteq \mathbb{R}^n$ and $f \in C^1(\Omega)$. We introduce the infinitesimal vector

$$\mathrm{d}\mathbf{x} = (\,\mathrm{d}x_1, \dots, \mathrm{d}x_n)\,.$$

Then

$$\mathrm{d}f = (\,\mathrm{d}\mathbf{x} \cdot \bigtriangledown)f = \bigtriangledown f \cdot \mathrm{d}\mathbf{x} = \frac{\partial f}{\partial x_1}\,\mathrm{d}x_1 + \cdots \frac{\partial f}{\partial x_n}\,\mathrm{d}x_n = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\,\mathrm{d}x_j.$$

5) **Differential of a vector field** $\mathbf{V} : \Omega \to \mathbb{R}$, where $\Omega \subseteq \mathbb{R}^n$ and $\mathbf{V} \in C^1(\Omega)^n$. This is similar to the case of a scalar field above, so

$$\mathrm{d}\mathbf{V} = (\,\mathrm{d}\mathbf{x} \cdot \bigtriangledown)\mathbf{V} = (\bigtriangledown V_1 \cdot \mathrm{d}\mathbf{x},\, \dots\,, \bigtriangledown V_n \cdot \mathrm{d}\mathbf{x}) = (\,\mathrm{d}V_1, \dots\,, \mathrm{d}V_n)\,.$$

It follows from these definitions that all these differential operators are linear.

We shall in the following let α and β denote scalars, f, $g \in C^1(\Omega)$ real functions, $\Omega \subseteq \mathbb{R}^n$ and $\mathbf{U}$, $\mathbf{V} \in C^1/\Omega)^n$ vector fields. Finally, $\mathrm{d}\mathbf{x} = (\mathrm{d}x_1, \dots, \mathrm{d}x_n)$ denotes the infinitesimal vector. Then the linearities goe as follows:

1) **Gradient**

$$\nabla(\alpha f + \beta g) = \alpha \nabla f + \beta \nabla g.$$

2) **Divergence**

$$\nabla \cdot (\alpha \mathbf{U} + \beta \mathbf{V}) = \alpha \nabla \cdot \mathbf{U} + \beta \nabla \cdot \mathbf{V}.$$

3) **Rotation** (only in $\mathbb{R}^3$)

$$\nabla \times (\alpha \mathbf{U} + \beta \mathbf{V}) = \alpha \nabla \times \mathbf{U} + \beta \nabla \times \mathbf{V}.$$

4) **Differential of a scalar field**

$$\mathrm{d}(\alpha f + \beta g) = \alpha \,\mathrm{d}f + \beta \,\mathrm{d}g = \alpha \nabla f \cdot \mathrm{d}\mathbf{x} + \beta \nabla g \cdot \mathrm{d}\mathbf{x}.$$

5) **Differential of vector fields**

$$\mathrm{d}(\alpha \mathbf{U} + \beta \mathbf{V}) = (\mathrm{d}\mathbf{x} \cdot \nabla)(\alpha \mathbf{U} + \beta \mathbf{V}) = \alpha \,\mathrm{d}\mathbf{U} + \beta \,\mathrm{d}\mathbf{V}.$$

36.2 Differentiation of products

Let f, g be C^1 functions and let $\mathbf{U}$, $\mathbf{V}$ be C^1 vector fields on the open set Ω. Then fg and $\mathbf{U} \cdot \mathbf{V}$ are C^1 functions, i.e. scalar fields, on Ω, so we can form the gradient and the differential of them. The rules are

1) **Gradient of** fg.

$$\nabla(fg) = g \nabla f + f \nabla g.$$

this is proved by considering each coordinate separately,

$$\frac{\partial}{\partial x_j}(fg) = g \,\frac{\partial f}{\partial x_j} + f \,\frac{\partial g}{\partial x_j}.$$

We note that if $g = f$, then the formula above reduces to

$$\nabla \left(f^2\right) = 2f \nabla f.$$

2) Differential of fg.

$$\mathrm{d}(fg) = \mathrm{d}\mathbf{x} \cdot \nabla(fg) = f \,\mathrm{d}\mathbf{x} \cdot \nabla f + f \,\mathrm{d}\mathbf{x} \cdot \nabla g = g \,\mathrm{d}f + f \,\mathrm{d}g.$$

3) **Gradient of** $\mathbf{U} \cdot \mathbf{V}$. In general, this formula becomes messy, so it is not given here. However, we shall later return to the special case, when we are dealing with three dimensions.

Then we consider the product $f\mathbf{U}$, which is a vector field. We get only one possibility.

1) **Divergence of $f\mathbf{U}$,**

$$\text{div }(f\mathbf{U}) = \triangledown \cdot (f\mathbf{U}) = (\triangledown f) \cdot \mathbf{U} + f \triangledown \cdot \mathbf{U} = \mathbf{grad} f \cdot \mathbf{U} + f \text{div }\mathbf{U}.$$

We leave the straightforward proof to the reader. Note here that the formula with nabla is easier to remember than the interpretation

$$\text{div }(f\mathbf{U}) = \mathbf{grad} f \cdot \mathbf{U} + f \text{div }\mathbf{U}.$$

In the remaining formulæ we assume that we consider the three dimensional case, so we can form the vector product. Then the possibilities of differentiation of a product, using these differential operators, are

$$\mathbf{rot}(f\mathbf{U}) = \triangledown \times (f\mathbf{U}), \quad \text{div }(\mathbf{U} \times \mathbf{V}) = \triangledown \cdot (\mathbf{U} \times \mathbf{V}), \quad \mathbf{rot}(\mathbf{U} \times \mathbf{V}) = \triangledown \times (\mathbf{U} \times \mathbf{V}),$$

and

$$\mathbf{grad}(\mathbf{U} \cdot \mathbf{V}) = \triangledown(\mathbf{U} \cdot \mathbf{V}),$$

supplied with the differentials $\mathrm{d}(f\mathbf{U})$, $\mathrm{d}(\mathbf{U} \cdot \mathbf{V})$ and $\mathrm{d}(\mathbf{U} \times \mathbf{V})$. Since the differentials follow from the rules $\mathrm{d}f = \mathrm{d}\mathbf{x} \cdot \triangledown f$ and $\mathrm{d}\mathbf{U} = (\mathrm{d}\mathbf{x} \cdot \triangledown)\mathbf{U}$ above, we shall not further deal with the differentials.

The rules are

1) **Rotation of $f\mathbf{U}$,** $n = 3$.

$$\mathbf{rot}(f\mathbf{U}) = \triangledown \times (f\mathbf{U}) = (\triangledown f) \times \mathbf{U} + f \triangledown \times \mathbf{U} = \mathbf{grad} f \times \mathbf{U} + f\, \mathbf{rot}\, \mathbf{U}.$$

We note that $\triangledown \times$ behaves like a derivation.

2) **Divergence of $\mathbf{U} \times \mathbf{V}$,** $n = 3$.

$$\text{div }(\mathbf{U} \times \mathbf{V}) = \triangledown \cdot (\mathbf{U} \times \mathbf{V}) = (\triangledown \times \mathbf{U}) \cdot \mathbf{V} - (\triangledown \times \mathbf{V}) \cdot \mathbf{U} = \mathbf{rot}\mathbf{U} \cdot \mathbf{V} - \mathbf{rot}\mathbf{V} \cdot \mathbf{U}.$$

The structure is almost the same as for a derivation. The minus sign is caused by the fact that the vector product is anticommutative, $\mathbf{U} \times \mathbf{V} = -\mathbf{V} \times \mathbf{U}$. This follows from the proof, which again is left to the reader.

3) **Rotation of $\mathbf{U} \times \mathbf{V}$,** $n = 3$. The complicated formula below relies on the rule for the double vector product in $\mathbb{R}^3$,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}, \qquad \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3,$$

known from *Linear Algebra*. We quote again without proof the result

$$\mathbf{rot}(\mathbf{U} \times \mathbf{V}) = \triangledown \times (\mathbf{U} \times \mathbf{V}) = (\mathbf{V} \cdot \triangledown)\mathbf{U} - \mathbf{V}(\triangledown \cdot \mathbf{V}) - (\mathbf{U} \cdot \triangledown)\mathbf{V} + \mathbf{U}(\triangledown \cdot \mathbf{V}).$$

4) **Gradient of $\mathbf{U} \cdot \mathbf{V}$,** $n = 3$.

$$\triangledown(\mathbf{U} \cdot \mathbf{V}) = (\mathbf{V} \cdot \triangledown)\mathbf{U} + \mathbf{V} \times (\triangledown \times \mathbf{U}) + (\mathbf{U} \cdot \triangledown)\mathbf{V} + \mathbf{U} \times (\triangledown \times \mathbf{V}).$$

We shall not prove this rule either, although the proof is not too hard. We note that the expected terms are

$$(\mathbf{V} \cdot \triangledown)\mathbf{U} + (\mathbf{U} \cdot \triangledown)\mathbf{V},$$

and then we have the two additional correction terms

$$\mathbf{V} \times (\triangledown \times \mathbf{U}) + \mathbf{U} \times (\triangledown \times \mathbf{V}) = \mathbf{V} \times \mathbf{rot}\mathbf{U} + \mathbf{U} \times \mathbf{rot}\mathbf{V},$$

which involve the rotations of $\mathbf{U}$ and $\mathbf{V}$. Again we skip the proof and only mention that it uses the formula for $\triangledown \times (\mathbf{U} \times \mathbf{V})$.

In the special case, where $\mathbf{V} = \mathbf{U}$, we get

$$\mathbf{grad}\left(\|\mathbf{U}\|^2\right) = \triangledown\left(\|\mathbf{U}\|^2\right) = \triangledown(\mathbf{U} \cdot \mathbf{U}) = 2\mathbf{U} \times (\triangledown \mathbf{U}) = 2\mathbf{U} \times \mathbf{rot}\mathbf{U}.$$

It should be mentioned that if another variable is involved, e.g. $t\,(\neq x,\, y,\, z)$, then we have the usual rules

$$\frac{\partial}{\partial t}(\mathbf{U} \cdot \mathbf{V}) = \frac{\partial \mathbf{U}}{\partial t} \cdot \frac{\partial \mathbf{V}}{\partial t}, \qquad \text{and} \qquad \frac{\partial}{\partial t}(\mathbf{U} \times \mathbf{V}) = \frac{\partial \mathbf{U}}{\partial t} \times \frac{\partial \mathbf{V}}{\partial t}.$$

36.3 Differentiation of second order

The three vectorial differential operators of first order,

$\mathbf{grad} = \triangledown$, (from 1 dimension to 3), div $= \triangledown\cdot$, (from 3 dimensions to 1),

$\mathbf{rot} = \triangledown\times$, (from 3 dimensions to 3),

can be combined to form differential operators of second order. Note that whenever $\mathbf{rot} = \triangledown\times$ is involved, then we tacitly require that the dimension is 3. There are five possibilities:

1) **Rotation of gradient**, $n = 3$. From 1 via 3 to 3 dimensions,

$$\mathbf{rot}(\mathbf{grad} f) = \triangledown \times \triangledown f = \mathbf{0},$$

because $\triangledown \times \triangledown$ is the zero operator,

2) **Divergens of rotation**, $n = 3$. From 3 via 3 to 1 dimension,

$$\text{div}\,(\mathbf{rot}\mathbf{U}) = \triangledown \cdot (\triangledown \times \mathbf{U}) = 0.$$

This follows straightforward by some tedious calculations.

3) **Divergence of a gradient, all** $n \in \mathbb{N}$. From 1 via n to 1 dimension.

$$\triangledown^2 f = \triangledown \cdot \triangledown f = \frac{\partial^2 f}{\partial x_1^2} + \cdot + \frac{\partial^2 f}{\partial x_n^2}.$$

This is the so-called *Laplace operator*, which is also denoted by Δ, thus

$$\Delta = \triangledown^2 = \triangledown \cdot \triangledown = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

We note that Δ can also act on a vector field,

$$\triangledown^2 \mathbf{U} = \triangledown^2 \left(U_1 \mathbf{e}_1 + \cdots + U_n \mathbf{e}_n\right) = \left(\triangledown^2 U_1\right) \mathbf{e}_1 + \cdots + \left(\triangledown^2 V_n\right) \mathbf{e}_n.$$

We also mention the following formula for the Laplace operator acting on a product of two functions,

$$\triangledown^2 (fg) = f \triangledown^2 g + 2 \triangledown f \cdot \triangledown g + g \triangledown^2 f.$$

4) **Double rotation**, $n = 3$, From 3 via 3 to 3 dimensions.

$$\nabla \times (\nabla \times \mathbf{U}) = \nabla(\nabla \cdot \mathbf{U}) - \nabla \cdot \nabla \mathbf{U},$$

which can also be written

$$\mathbf{rot}(\mathbf{rot}\mathbf{U}) = \mathbf{grad}(\text{div } \mathbf{U}) - \Delta \mathbf{U},$$

where

$$\Delta = \nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is the *Laplace operator*.

5) **Gradient of divergence, general** $n \in \mathbb{N}$. From n via 1 to n dimensions.

$$\nabla(\nabla \cdot \mathbf{U}) = \mathbf{grad}(\text{div } \mathbf{U}) = \mathbf{grad}\left(\sum_{j=1}^{n} \frac{\partial U_j}{\partial x_j}\right),$$

which cannot be further simplified.

36.4 Nabla applied on x

In the applications one often needs to consider, $\bigtriangledown$, $\bigtriangledown\cdot$, or $\bigtriangledown\times$, applied to expressions in $\mathbf{x}$. We list some of the formulæ below.

1) *Formulæ valid for all* $n \in \mathbb{N}$ *and all* $\mathbf{x} \in \mathbb{R}^n$,

$$\text{div}\;\mathbf{x} = \bigtriangledown \cdot \mathbf{x} = n,$$

$$\Delta\mathbf{x} = \bigtriangledown^2\mathbf{x} = 0,$$

$$\Delta(\mathbf{x}\cdot\mathbf{x}) = \Delta\left(\|\mathbf{x}\|^2\right) = 2n,$$

$$(\mathbf{U}\cdot\bigtriangledown)\mathbf{x} = \mathbf{U},$$

$$\mathbf{grad}(\mathbf{a}\cdot\mathbf{x}) = \bigtriangledown(\mathbf{a}\cdot\mathbf{x}) = \mathbf{a}, \qquad \text{constant } \mathbf{a} \in \mathbb{R}^n.$$

2) *Formulæ valid for all* $n \in \mathbb{N}$ *and for* $\mathbf{x} \neq \mathbf{0}$.
If $f(\mathbf{x}) = F(\|\mathbf{x}\|)$ only depends on $\|\mathbf{x}\|$, then

$$\bigtriangledown f(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|}\,F(\|\mathbf{x}\|).$$

3) *Formulæ for all* $n \geq 3$ *and* $\mathbf{x} \neq \mathbf{0}$.

$$\bigtriangledown\left(\|\mathbf{x}|^{2-n}\right) = 0 \qquad \text{for } \mathbf{x} \neq \mathbf{0}.$$

4) *If* $n = 3$*, then*

$$\mathbf{rot}\mathbf{x} = \bigtriangledown \times \mathbf{x} = \mathbf{0}, \qquad \text{for all } \mathbf{x} in \mathbb{R}^3,$$

and

$$\Delta\left(\frac{1}{\|\mathbf{x}\|}\right) = \bigtriangledown^2\left(\frac{1}{\|\mathbf{x}\|}\right) = 0 \qquad \text{for all } \mathbf{x} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}.$$

5) *When* $n = 2$*, then*

$$\bigtriangledown^2(\ln\|\mathbf{x}\|) = 0.$$

As an application we consider a rotating body in space of angle velocity ω with respect to an axis through the origin. Then the velocity at the point $\mathbf{x}$ is $\mathbf{v} = \omega \times \mathbf{x}$, and we find

$$\text{div}\;\mathbf{v} = \bigtriangledown\cdot\mathbf{v} = \bigtriangledown\cdot(\omega\times\mathbf{x}) = (\bigtriangledown\times\omega)\cdot\mathbf{x} - (\bigtriangledown\times\mathbf{x})\cdot\omega = 0,$$

and

$$\mathbf{rot}\mathbf{v} = \bigtriangledown\times\mathbf{v} = \bigtriangledown\times(\omega\times\mathbf{x}) = (\mathbf{x}\cdot\bigtriangledown)\omega - \mathbf{x}(\bigtriangledown\cdot\omega) - (\omega\cdot\bigtriangledown)\mathbf{x} + \omega(\bigtriangledown\cdot\mathbf{x}) = -\omega + 3\omega = 2\omega.$$

36.5 The integral theorems

We shall here see what happens when we apply the nabla calculus in the cases of Gauß's and Stokes's theorems. This will give us an inspiration to derive new formulæ, which are also valid.

We first recall *Gauß's theorem*

$$\int_\Omega \operatorname{div} \mathbf{V} = \int_{\partial\Omega} \mathbf{n}\cdot\mathbf{V}\,\mathrm{d}S.$$

When we replace the operator div with the symbol $\bigtriangledown\cdot$, we get

$$\int_\Omega \bigtriangledown\cdot\mathbf{V}\,\mathrm{d}\Omega = \int_{\partial\Omega} \mathbf{n}\cdot\mathbf{V}\,\mathrm{d}S,$$

which shows the structure of the theorem. When we go from the left to the right, the body Ω is replaced by the surface $\partial\Omega$, and the operator $\bigtriangledown\cdot$ is replaced by an inner product with the unit normal vector field, $\mathbf{n}\cdot$.

This gives us the hint that we might try to replace $\bigtriangledown\cdot$ by the other differential operator $\mathbf{rot} = \bigtriangledown\times$, and then of course write $\mathbf{n}\times$ instead of $\mathbf{n}\cdot$. This suggests the formula

$$\int_\Omega \times\mathbf{V}\,\mathrm{d}\Omega = \int_{\partial\Omega} \mathbf{n}\times\mathbf{V}\,\mathrm{d}S.$$

It can be proved that this is indeed true, so we have obtained the following formula, written with **rot** instead of $\bigtriangledown\times$,

$$\int_{\mathbf{rot}\mathbf{V}} \mathrm{d}\Omega = \int_{\partial\Omega} \mathbf{n}\times\mathbf{V}\,\mathrm{d}S,$$

where $\mathbf{n}$ as usual denotes the outgoing unit normal vector field on the surface $\partial\Omega$.

If we instead replace $\bigtriangledown\cdot$ by $\bigtriangledown$ alone (and the vector field $\mathbf{V}$ by a function f), then we see that we may expect a formula of the form

$$\int_\Omega \bigtriangledown f\,\mathrm{d}\Omega = \int_{\partial\Omega} \mathbf{n}f\,\mathrm{d}S, \quad \text{i.e.} \quad \int_\Omega \mathbf{grad} f\,\mathrm{d}\Omega = \int_{\partial\Omega} \mathbf{n}f\,\mathrm{d}S.$$

Again this formula can be proved to be correct. Note that the result here is a vector field.

We have above given three versions of *Gauß's theorem.*

Then let us turn to *Stokes's theorem*,

$$\int_{\mathcal{F}} \mathbf{n}\cdot\mathbf{rot}\mathbf{V}\,\mathrm{d}S = \oint_{\delta\mathcal{F}} \mathbf{t}\cdot\mathbf{V}\,\mathrm{d}s,$$

which using the nabla notation is written

$$\int_{\mathcal{F}} \mathbf{n}\cdot\bigtriangledown\times\mathbf{V}\,\mathrm{d}S = \int_{\mathcal{F}} (\mathbf{n}\cdot\bigtriangledown)\times\mathbf{V}\,\mathrm{d}S = \oint_{\delta\mathcal{F}} \mathbf{t}\cdot\mathbf{V}\,\mathrm{d}s.$$

When we replace $\mathbf{t}\cdot\mathbf{V}$ by $\mathbf{t}\times\mathbf{V}$, we must apparently by analogy replace $(\mathbf{n}\cdot\bigtriangledown)$ bu $(\mathbf{n}\times\bigtriangledown)$ to see what happens. In this way we see that we may expect the following formula, only derived by analogy,

$$\int_{\mathcal{F}} (\mathbf{n}\times\bigtriangledown)\times\mathbf{V}\,\mathrm{d}S = \oint_{\delta\mathcal{F}} \mathbf{t}\cdot\mathbf{V}\,\mathrm{d}s.$$

Once we have got a hunch of the formula, it is not difficult to prove that it is indeed correct. As usual we shall not do it here.

Finally, let us see what we should expect, when the dot has disappeared on the right hand side of the original Stokes's theorem. Clearly, we now must consider a function f times the unit tangent vector field $\mathbf{n}$ as the integrand, because we have not defined pointwise multiplication of two vector fields. So we derive that

$$\oint_{\delta\mathcal{F}} \mathbf{t} f \, \mathrm{d}s$$

should be expected to occur on the right hand side.

By analogy, the integrand on the left hand side must contain $\mathbf{n}$ and $\times$ and $\bigtriangledown f$. This is only possible in the following way,

$$\int_{\mathcal{F}} \mathbf{n} \times \bigtriangledown f \, \mathrm{d}S = \oint_{\delta\mathcal{F}} \mathbf{t} f \, \mathrm{d}s.$$

Again it can be proved that this formula indeed is valid.

Summarizing we have got three versions of *Gauß's theorem*,

1. $$\int_{\Omega} \bigtriangledown \cdot \mathbf{V} \, \mathrm{d}\Omega = \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{V} \, \mathrm{d}S,$$
2. $$\int_{\Omega} \bigtriangledown \times \mathbf{V} \, \mathrm{d}\Omega = \int_{\partial\Omega} \mathbf{n} \times \mathbf{V} \, \mathrm{d}S,$$
3. $$\int_{\Omega} \bigtriangledown f \, \mathrm{d}\Omega = \int_{\partial\Omega} \mathbf{n} f \, \mathrm{d}S,$$

where $\mathbf{n}$ is the outgoing unit normal vector field of the surface $\partial\Omega$, and three versions of *Stokes's theorem*,

1. $$\int_{\mathcal{F}} \mathbf{n} \cdot \bigtriangledown \times \mathbf{V} \, \mathrm{d}S = \oint_{\delta\mathcal{F}} \mathbf{t} \cdot \mathbf{V} \, \mathrm{d}s,$$
2. $$\int_{\mathcal{F}} (\mathbf{n} \times \bigtriangledown) \times \mathbf{V} \, \mathrm{d}S = \oint_{\delta\mathcal{F}} \mathbf{t} \times \mathbf{V} \, \mathrm{d}s,$$
3. $$\int_{\mathcal{F}} \mathbf{n} \times \bigtriangledown f \, \mathrm{d}S = \oint_{\delta\mathcal{F}} \mathbf{t} f \, \mathrm{d}s,$$

where $\mathbf{t}$ is the unit tangent field of $\delta\mathcal{F}$, and $\mathbf{n}$ is the unit normal vector field on $\mathcal{F}$, corresponding to the orientation of $\delta\mathcal{F}$.

As simple applications we first use the second variant of *Gauß's theorem* above with $\mathbf{V} = \bigtriangledown f$. We get in this case

$$\int_{\partial\Omega} \mathbf{n} \times \bigtriangledown f \, \mathrm{d}S = \int_{\Omega} \bigtriangledown \times \bigtriangledown f \, \mathrm{d}\Omega = \mathbf{0}.$$

Then we – also with $\mathbf{V} = \bigtriangledown f$ – apply the third version of Stokes's theorem. Since $\delta\mathcal{F} = \emptyset$, we get trivially

$$\int_{\partial\Omega} \mathbf{n} \times \bigtriangledown f \, \mathrm{d}S = \oint_{\emptyset} \mathbf{t} f \, \mathrm{d}s = \mathbf{0}.$$

Then we choose $\mathbf{V} = \mathbf{x}$ in the second version of Stokes's theorem,

$$\int_{\mathcal{F}} (\mathbf{n} \times \bigtriangledown) \times \mathbf{x}\, \mathrm{d}S = \oint_{\delta\mathcal{F}} \mathbf{t} \times \mathbf{x}\, \mathrm{d}s.$$

The integrand of the integrand of the left hand side is calculated,

$$(\mathbf{n} \times \bigtriangledown) \times \mathbf{x} = \bigtriangledown(\mathbf{x} \cdot \mathbf{n}) - \mathbf{n}(\bigtriangledown \cdot \mathbf{x}) = \mathbf{n} - 3\mathbf{n} = -2\mathbf{n},$$

and since $\mathbf{t} \times \mathbf{x} = -\mathbf{x} \times \mathbf{t}$, we conclude by insertion and reduction that

$$\int_{\mathcal{F}} \mathbf{n}\, \mathrm{d}S = \frac{1}{2} \oint_{\delta\mathcal{F}} \mathbf{x} \times \mathbf{t}\, \mathrm{d}s.$$

This expression is sometimes called the *vectorial area*, though this term may be misleading. In particular, every closed surface $\mathcal{F}$ has the vectorial surface $\mathbf{0}$, because $\delta\mathcal{F} = \emptyset$.

36.6 Partial integration

The principle of partial integration is in its simplest form of functions in one real variable derived from the rule of differentiation,

$$\frac{\mathrm{d}}{\mathrm{d}x}(fg) = f\,\frac{\mathrm{d}g}{\mathrm{d}x} + g\,\frac{\mathrm{d}f}{\mathrm{d}x},$$

from which by a rearrangement

$$f\,\frac{\mathrm{d}g}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(fg) - g\,\frac{\mathrm{d}f}{\mathrm{d}x}.$$

Hence by an integration,

$$\int f\,\frac{\mathrm{d}g}{\mathrm{d}x}\,\mathrm{d}x = f(x)g(x) - \int g\,\frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}x.$$

We can derive some similar rules for some of the differentiations introduced in Section 36.2 by using Gauß's and Stokes's theorem.

1) **Gradient of a product of functions and Gauß's theorem**

$$\bigtriangledown(fg) = g \bigtriangledown f + f \bigtriangledown g, \qquad \text{or by a rearrangement} \qquad f \bigtriangledown g = \bigtriangledown(fg) - g \bigtriangledown f,$$

from which by the third version of Gauß's theorem,

$$\int_\Omega f \bigtriangledown g\,\mathrm{d}\Omega = \int_\Omega \bigtriangledown(fg)\,\mathrm{d}\Omega - \int_\Omega f \bigtriangledown f\,\mathrm{d}\Omega = \int_{\partial\Omega} \mathbf{n}fg\,\mathrm{d}S - \int_\Omega g \bigtriangledown f\,\mathrm{d}\Omega.$$

Note that we recover the third version of Gauß's theorem by choosing $f = 1$, so this formula may be considered as an extension of this third version.
Choosing $g = f$ we get by a rearrangement,

$$\int_\Omega f \bigtriangledown f\,\mathrm{d}\Omega = \frac{1}{2}\int_{\partial\Omega} \mathbf{n}f^2\,\mathrm{d}S.$$

When furthermore $f = 1$, we just get the known result that $\int_{\partial\Omega} \mathbf{n}\,\mathrm{d}S = \mathbf{0}$ for all closed surfaces $\partial\Omega$.

2) **Divergence of a product of a function and a vector field and Gauß's theorem**

$$\bigtriangledown \cdot (f\mathbf{U}) = (\bigtriangledown f)\cdot\mathbf{U} + f \bigtriangledown \cdot\mathbf{U}, \qquad \text{or by a rearrangement} \qquad f \bigtriangledown \cdot\mathbf{U} = \bigtriangledown \cdot (f\mathbf{U}) - (\bigtriangledown f)\cdot\mathbf{U},$$

so by an integration over the three dimensional body Ω followed by the first variant of Gauß's theorem,

$$\int_\Omega f \bigtriangledown \cdot\mathbf{U}\,\mathrm{d}\Omega = \int_\Omega \bigtriangledown \cdot (f\mathbf{U})\,\mathrm{d}\Omega - \int_\Omega \bigtriangledown f\cdot\mathbf{U}\,\mathrm{d}\Omega = \int_{\partial\Omega} \mathbf{n}\cdot f\mathbf{U}\,\mathrm{d}S - \int_\Omega \bigtriangledown f\cdot\mathbf{U}\,\mathrm{d}\Omega.$$

When $f = 1$, this new formula degenerates to the first version of Gauß's theorem, so

$$\int_\Omega f \bigtriangledown \cdot\mathbf{U}\,\mathrm{d}\Omega = \int_{\partial\Omega} \mathbf{n}\cdot f\mathbf{U}\,\mathrm{d}S - \int_\Omega \bigtriangledown f\cdot\mathbf{U}\,\mathrm{d}\Omega,$$

may be considered as an extension of this first version of Gauß's theorem.

3) **Gradient of a product of functions and Stokes's theorem**
The formula

$$f \bigtriangledown g = \bigtriangledown(fg) - g \bigtriangledown f$$

is multiplied from the left by $\mathbf{n}\times$, so

$$f\mathbf{n} \times \bigtriangledown g = \mathbf{n} \times \bigtriangledown(fg) - g\mathbf{n} \times \bigtriangledown f.$$

Let $\mathcal{F}$ be a surface of boundary $\delta\mathcal{F}$. Then we get by the third version of Stokes's theorem that

$$\int_{\mathcal{F}} f\mathbf{n} \times \bigtriangledown g \, \mathrm{d}S = \int_{\mathcal{F}} \mathbf{n} \times \bigtriangledown(fg) \, \mathrm{d}S - \int_{\mathcal{F}} g\mathbf{n} \times \bigtriangledown f \, \mathrm{d}S = \oint_{\delta\mathcal{F}} fg\mathbf{t} \, \mathrm{d}s - \int_{\mathcal{F}} g\mathbf{n} \times \bigtriangledown f \, \mathrm{d}S,$$

which is an extension of the third version of Stokes's theorem. This is again obtained by putting $f = 1$.

4) **Rotation of a product of a function and a vector field and Stokes's theorem**,

$$f \bigtriangledown \times\mathbf{U} = \bigtriangledown \times (f\mathbf{U}) - \bigtriangledown f \times \mathbf{U}.$$

When this equation is multiplied from the left by $\mathbf{n}\cdot$, we get

$$\mathbf{n} \cdot f \bigtriangledown \times\mathbf{U} = \mathbf{n} \cdot \bigtriangledown \times (f\mathbf{u}) - \mathbf{n} \cdot \bigtriangledown f \times \mathbf{U}.$$

Let $\mathcal{F}$ be a surface of boundary $\delta\mathcal{F}$. Then it follows from the first version of Stokes's theorem that

$$\int_{\mathcal{F}} \mathbf{n} \cdot f \bigtriangledown \times\mathbf{U} \, \mathrm{d}S = \int_{\mathcal{F}} \mathbf{n} \cdot \bigtriangledown \times (f\mathbf{U}) \, \mathrm{d}S - \int_{\mathcal{F}} \mathbf{n} \cdot \bigtriangledown f \times \mathbf{U} \, \mathrm{d}S = \oint_{\delta\mathcal{F}} \mathbf{t} \cdot f\mathbf{U} \, \mathrm{d}s - \int_{\mathcal{F}} \mathbf{n} \cdot \bigtriangledown f \times \mathbf{U} \, \mathrm{d}S,$$

which can be considered as an extension of the first version of Stokes's theorem, because this is recovered, when we put $f = 1$.

36.7 Overview of Nabla calculus

In complicated cases it may be useful to apply the abstract theory of *nabla calculus* instead of the enormous calculations with coordinates.

The theory may look a little confusing the first time one sees it, until one realizes that there are three different products in the vector analysis in $\mathbb{R}^3$:

scalar multiplication:	no special notation	$\alpha\,\mathbf{V}$,	vector,
inner product *(dot product)* :	dot	$\mathbf{U} \cdot \mathbf{V}$,	scalar,
vector product *(cross product)* :	cross	$\mathbf{U} \times \mathbf{V}$,	vector.

These are transferred to a nabla notation,

$$\bigtriangledown = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right),$$

by the relations

no special notation	$\mathbf{grad} f = \bigtriangledown f,$
dot	$\operatorname{div} \mathbf{V} = \bigtriangledown \cdot \mathbf{V},$
cross	$\mathbf{rot} \mathbf{V} = \bigtriangledown \times \mathbf{V}$

When we apply the correspondence $\bigtriangledown \sim \mathbf{n}$ we get **Gauß's theorem** in three versions:

$$\int_{\partial\Omega} \mathbf{n}\, f \,\mathrm{d}S = \int_\Omega \bigtriangledown f \,\mathrm{d}\Omega = \int_\Omega \mathbf{grad} f \,\mathrm{d}\Omega, \qquad \text{no special notation}$$

$$\int_{\partial\Omega} \mathbf{n} \cdot \mathbf{V} \,\mathrm{d}S = \int_\Omega \bigtriangledown \cdot \mathbf{V} \,\mathrm{d}\Omega = \int_\Omega \operatorname{div} \mathbf{V} \,\mathrm{d}\Omega, \qquad \text{dot product, usual version}$$

$$\int_{\partial\Omega} \mathbf{n} \times \mathbf{V} \,\mathrm{d}S = \int_\Omega \bigtriangledown \times \mathbf{V} \,\mathrm{d}\Omega = \int_\Omega \mathbf{rot}\mathbf{V} \,\mathrm{d}\Omega, \qquad \text{cross product.}$$

When we apply the correspondence $(\mathbf{n}, \bigtriangledown, \times) \sim \mathbf{t}$, where $\cdot$ and $\times$ are always put in a meaningful connection, we obtain **Stokes's theorem** in three versions:

$$\oint_{\delta\mathcal{F}} \mathbf{t}\, f \,\mathrm{d}s = \int_{\mathcal{F}} \mathbf{n} \times \bigtriangledown f \,\mathrm{d}S = \int_{\mathcal{F}} \mathbf{n} \times \mathbf{grad} f \,\mathrm{d}S, \qquad \text{no special notation}$$

$$\oint_{\delta\mathcal{F}} \mathbf{t} \cdot \mathbf{V} \,\mathrm{d}s = \int_{\mathcal{F}} \mathbf{n} \cdot (\bigtriangledown \times \mathbf{V}) \,\mathrm{d}S = \int_{\mathbf{F}} \mathbf{n} \cdot \mathbf{rot}\mathbf{V} \,\mathrm{d}S, \qquad \text{dot product, usual version}$$

$$\oint_{\delta\mathcal{F}} \mathbf{t} \times \mathbf{V} \,\mathrm{d}s = \int_{\mathcal{F}} (\mathbf{n} \times \bigtriangledown) \times \mathbf{V} \,\mathrm{d}S, \qquad \text{cross product.}$$

36.8 Overview of partial integration in higher dimensions

The simplest case is described by

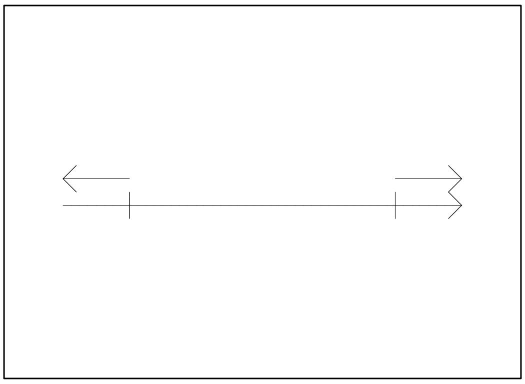

Figure 36.1: The interval $[a, b]$, where one goes from 1 dimension to 0 dimensions.

1) **Main theorem of the differential and integral calculus**, 1 dimension:

$$\int_a^b \frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}x = [f(x)]_a^b = f(b) - f(a),$$

i.e. a 1-dimensional integral is transformed into a 0-dimensional "boundary integral". Note that in the right end point, where the outer normal is equal to the direction of the axis, we use the sign $+$, and in the left end point, where the outer normal is opposite to the direction of the axis, we use the sign $-$.

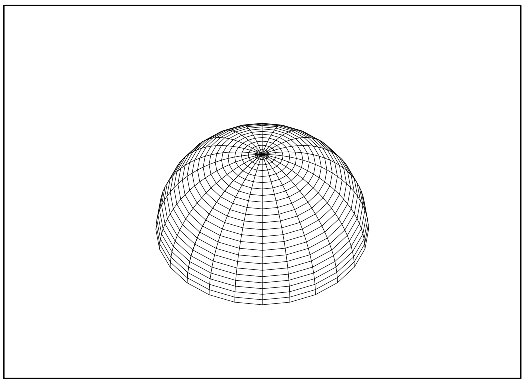

Figure 36.2: A surface $\mathcal{F}$ with its limiting curve $\delta\mathcal{F}$.

2) **Stokes's theorem**, 2 dimensions:

$$\int_{\mathcal{F}} \mathbf{n} \cdot \mathbf{rot}\mathbf{V}\,\mathrm{d}S = \oint_{\delta\mathcal{F}} \mathbf{t} \cdot \mathbf{B}\,\mathrm{d}s,$$

i.e. a 2-dimensional surface integral is transferred into a 1-dimensional "boundary integral".

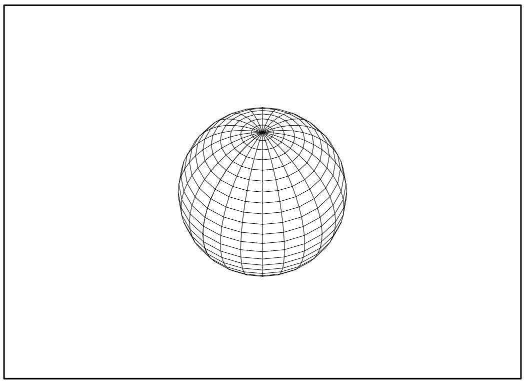

Figure 36.3: A domain in space Ω with surface $\partial\Omega$. The unit normal vector field is everywhere directed away from Ω.

3) **Gauß's theorem**, 3 dimensions:

$$\int_{\Omega} \operatorname{div} \mathbf{V} \, d\Omega = \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{V} \, dS,$$

i.e. a 3-dimensional space integral is transferred into a 2-dimensional "boundary integral".

The more advanced applications use the nabla calculus, cf. the previous sections in this chapter. Here we have several special cases, because there in e.g. in $\mathbb{R}^3$ exist three types of product (scalar multiplication, dot and cross product) and three types of differentiation (gradient, divergence, rotation). The basic idea is that if the integrand is of the form $\Phi \otimes \mathcal{D}\Psi$, where $\otimes$ stands for any of the possible products, and $\mathcal{D}$ stands for any of the possible differentiations, then we get a new integrand (by a partial integration) of a simpler form $-\tilde{\mathcal{D}}\Phi\tilde{\otimes}\Psi$, where $(\tilde{\otimes}, \tilde{\mathcal{D}})$ does not have to be equal to $(\otimes, \mathcal{D})$. The price for this is that we get an additional boundary integral.

36.9 Examples in nabla calculus

Example 36.1 *Let* $\mathbf{V}$ *denote a vector field, which is both divergence free and rotation free, and let* $\mathbf{e}$ *be a fixed unit vector. We consider also the following fields,*

$$F = -\mathbf{e}\cdot\mathbf{V}, \qquad \mathbf{W} = \mathbf{V}\times\mathbf{e}, \qquad \mathbf{U} = -\bigtriangledown F, \qquad \mathbf{T} = \bigtriangledown\times\mathbf{W}.$$

1) show that

$$\bigtriangledown\times(\mathbf{V}\times\mathbf{x}) = \mathbf{V} + \bigtriangledown(\mathbf{V}\cdot\mathbf{x}).$$

2) Show that $\mathbf{T}$ *is the same vector field as* $\mathbf{U}$*, and that this field also is both divergence free and rotation free.*

A Nabla calculus.

D Just exploit the assumptions,

$$\text{div}\,\mathbf{V} = \bigtriangledown\cdot\mathbf{V} = 0 \quad \text{and} \quad \textbf{rot}\,\mathbf{V} = \bigtriangledown\times\mathbf{V} = \mathbf{0},$$

and the rules of differentiation of products.

I 1) We shall use the following well-known rule of calculation

$$\bigtriangledown\times(\mathbf{V}\times\mathbf{W}) = (\mathbf{W}\cdot\bigtriangledown)\mathbf{V} - \mathbf{W}(\bigtriangledown\cdot\mathbf{V}) - (\mathbf{V}\cdot\bigtriangledown)\mathbf{W} + \mathbf{V}(\bigtriangledown\cdot\mathbf{W})$$

with $\mathbf{W} = \mathbf{x}$, thus

$$\begin{aligned}\bigtriangledown\times(\mathbf{V}\times\mathbf{x}) &= (\mathbf{x}\cdot\bigtriangledown)\mathbf{V} - \mathbf{x}(\bigtriangledown\cdot\mathbf{V}) - (\mathbf{V}\cdot\bigtriangledown)\mathbf{x} + \mathbf{V}\,(\bigtriangledown\cdot\mathbf{x})\\ &= (\mathbf{x}\cdot\bigtriangledown)\mathbf{V} - \mathbf{0} - (\mathbf{V}\cdot\bigtriangledown)\mathbf{x} + 3\mathbf{V}\\ &= \mathbf{V} + (\mathbf{x}\cdot\bigtriangledown)\mathbf{V} - (\mathbf{V}\cdot\bigtriangledown)\mathbf{x} + 2(\mathbf{V}\cdot\bigtriangledown)\mathbf{x}\\ &= \mathbf{V} + (\mathbf{x}\cdot\bigtriangledown)\mathbf{V} + (\mathbf{V}\cdot\bigtriangledown)\mathbf{x}\\ &= \mathbf{V} + \bigtriangledown(\mathbf{V}\cdot\mathbf{x}),\end{aligned}$$

where we have used that

$$(\mathbf{V}\cdot\bigtriangledown)\mathbf{x} = \left\{V_1\frac{\partial}{\partial x} + V_2\frac{\partial}{\partial y} + V_3\frac{\partial}{\partial z}\right\}(x,y,z) = (V_1,V_2,V_3) = \mathbf{V},$$

and that

$$\bigtriangledown(\mathbf{V}\cdot\mathbf{x}) = (\mathbf{V}\cdot\bigtriangledown)\mathbf{x} + (\mathbf{x}\cdot\bigtriangledown)\mathbf{V}.$$

2) Consider in particular $\mathbf{T}$ and put $\mathbf{W} = \mathbf{x}$. Then

$$\begin{aligned}\mathbf{T} &= \bigtriangledown\times\mathbf{W} = \bigtriangledown\times(\mathbf{V}\times\mathbf{e})\\ &= (\mathbf{e}\cdot\bigtriangledown)\mathbf{V} - \mathbf{e}(\bigtriangledown\cdot\mathbf{V}) - (\mathbf{V}\cdot\bigtriangledown)\mathbf{e} + \mathbf{V}(\bigtriangledown\cdot\mathbf{e})\\ &= (\mathbf{e}\cdot\bigtriangledown)\mathbf{V} - \mathbf{0} - \mathbf{0} + \mathbf{0} = (\mathbf{e}\cdot\bigtriangledown)\mathbf{V}\\ &= (\mathbf{e}\cdot\bigtriangledown)\mathbf{V} + (\mathbf{V}\cdot\bigtriangledown)\mathbf{e} = \bigtriangledown(\mathbf{e}\cdot\mathbf{V}) = -\bigtriangledown F = \mathbf{U},\end{aligned}$$

and the first claim is proved.

Since $\mathbf{T} = \mathbf{U} = -\bigtriangledown F$ is a gradient field, is is rotation free,

$$\bigtriangledown\times\mathbf{T} = -\bigtriangledown\times\bigtriangledown F = \mathbf{0}.$$

Since $\mathbf{T} = \mathbf{U} = \bigtriangledown\times\mathbf{W}$ is a rotation field, it is divergence free:

$$\bigtriangledown\cdot\mathbf{T} = \bigtriangledown\cdot\bigtriangledown\times\mathbf{W} = \mathbf{0}.$$

Example 36.2 *Let f be a C^1-function in r $(= \sqrt{x^2+y^2+z^2})$. We shall also (cf. the short hand notation in connection with the chain rule) consider f as a composed function $f(r(x,y,z))$, where $(x,y,z) \neq (0,0,0)$.*

1) Express ∇f by the derivative f' and $\mathbf{x}$.

2) Then set up formulæ for $\nabla \times (\mathbf{x}\, f)$ and for $\nabla \cdot (\mathbf{x}\, f)$.

3) Find the integer n, for which $\nabla \cdot (r^n\, \mathbf{x}) = 0$.

A Nabla calculus.

D Just follow the guidelines.

I We shall of course always assume that $r \neq 0$. Then

$$\nabla r = \left(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z}\right) = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) = \frac{1}{r}\,\mathbf{x}.$$

1) We get by the chain rule,

$$\nabla f = \left(f'(r)\,\frac{\partial r}{\partial x}, f'(r)\,\frac{\partial r}{\partial y}, f'(t)\,\frac{\partial r}{\partial z}\right) = f'(t)\,\nabla r = \frac{f'(r)}{r}\,\mathbf{x}.$$

2) A direct computation gives

$$\nabla \times (\mathbf{x}\, f) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x\,f(r) & y\,f(r) & z\,f(r) \end{vmatrix}$$

$$= \left(z\,\frac{f'(r)}{r}\,y - y\,\frac{f'(r)}{r}\,z, x\,\frac{f'(r)}{r}\,z - z\,\frac{f'(r)}{r}\,x, y\,\frac{f'(r)}{r}\,x - x\,\frac{f'(r)}{r}\,y \right) = (0,0,0).$$

A variant is

$$\begin{aligned} \nabla \times (\mathbf{x}\, f) &= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x\,f(r) & y\,f(r) & z\,f(r) \end{vmatrix} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial f(r)}{\partial x} & \dfrac{\partial f(r)}{\partial y} & \dfrac{\partial f(r)}{\partial z} \\ x & y & z \end{vmatrix} \\ &= \nabla f \times \mathbf{x} = \frac{f'(r)}{r}\,\mathbf{x} \times \mathbf{x} = \mathbf{0}. \end{aligned}$$

Furthermore,

$$\begin{aligned} \nabla \cdot (\mathbf{x}\, f) &= \left(f(r) + x\,\frac{\partial f}{\partial x} \right) + \left(f(r) + y\,\frac{\partial f}{\partial y} \right) + \left(f(r) + z\,\frac{\partial f}{\partial z} \right) \\ &= 3\,f(r) + \mathbf{x} \cdot \nabla f = 3f(r) + \frac{f'(r)}{r}\,\mathbf{x} \cdot \mathbf{x} \\ &= 3f(r) + r\,f'(r). \end{aligned}$$

3) Choose $f(r) = r^n$. Then it follows from the above,

$$\nabla \cdot (r^n\,\mathbf{x}) = 3r^n + n\,r^{n-1} = (3+n)r^n.$$

When $r \neq 0$, this is equal to 0 for $n = -3$.
Remark. In general, $\nabla \cdot (\mathbf{x}\,f(r)) = 0$ generates the differential equation

$$r\,f'(r) + 3f(r) = 0.$$

Then by separation of the variables,

$$\frac{f'(r)}{f(r)} \left[= \frac{\ln|f(r)|}{dr} \right] = -\frac{3}{r},$$

and the complete solution is obtained by an integration,

$$f(r) = C \cdot r^{-3}, \qquad r \neq 0, \quad \text{where } C \in \mathbb{R}. \qquad \Diamond$$

Example 36.3 *Let* $\mathbf{a}$ *be a constant vector, and let* f *be a* C^1*-function in one variable. We define*

$$g(\mathbf{x}) = f(\mathbf{a} \cdot \mathbf{x}).$$

1) Express the gradient ∇g *by the derivative* f'.
(Use one of the special cases of the chain rule).

2) Let also $\mathbf{V}$ *be a gradient field, and let* $k = 3$. *Show that the vector* $\nabla \times (g\mathbf{V})$ *is perpendicular to both* $\mathbf{a}$ *and* $\mathbf{V}$.

A Nabla calculus.

D Just compute.

I 1) If $\mathbf{a} = (a_1, \dots, a_k)$ and $\mathbf{x} = (x_1, \dots, x_k)$, then

$$g(\mathbf{x}) = f(\mathbf{a} \cdot \mathbf{x}) = f\left(\sum_{j=1}^{k} a_j\, x_j\right),$$

hence

$$\frac{\partial g}{\partial x_j} = f'(\mathbf{a} \cdot \mathbf{x})\, a_j,$$

hence

$$\nabla g = f'(\mathbf{a} \cdot \mathbf{x})\, \mathbf{a}.$$

2) If $\mathbf{V}$ is a gradient field, then there exists a function F, such that $\mathbf{V} = \nabla F$. Hence,

$$\begin{aligned} \nabla \times (g\,\mathbf{V}) &= (\nabla g) \times \mathbf{V} + g\ \nabla \times \mathbf{V} \\ &= f'(\mathbf{a} \cdot \mathbf{x})\, \mathbf{a} \times \mathbf{V} + f(\mathbf{a} \cdot \mathbf{x})\ \nabla \times (\nabla F) \\ &= f'(\mathbf{a} \cdot \mathbf{x})\, \mathbf{a} \times \mathbf{V} + \mathbf{0} \\ &= f'(\mathbf{a} \cdot \mathbf{V}, \end{aligned}$$

which shows that $\nabla \times (g\,\mathbf{V})$ is perpendicular on both $\mathbf{a}$ and $\mathbf{V}$.

Example 36.4 *Show the formula*

$$2(\bigtriangledown f)\cdot(\bigtriangledown\times(f\,\mathbf{V}))=(\bigtriangledown\times\mathbf{V})\cdot\bigtriangledown(f^2).$$

A Nabla calculus.

D Just compute.

I We get straight away,

$$\begin{aligned}
&2(\bigtriangledown f)\cdot(\bigtriangledown\times(f\,\mathbf{V}))\\
&=2\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)\cdot\left(\frac{\partial}{\partial y}(fV_z)-\frac{\partial}{\partial z}(fV_y),\frac{\partial}{\partial z}(fV_x)-\frac{\partial}{\partial x}(fV_z),\frac{\partial}{\partial x}(fV_y)-\frac{\partial}{\partial y}(fV_x)\right)\\
&=2\,\frac{\partial f}{\partial x}\left\{\frac{\partial f}{\partial y}\,V_z+f\,\frac{\partial V_z}{\partial y}-\frac{\partial f}{\partial z}\,V_y-f\,\frac{\partial V_y}{\partial z}\right\}\\
&\qquad+2\,\frac{\partial f}{\partial y}\left\{\frac{\partial f}{\partial z}\,V_x+f\,\frac{\partial V_x}{\partial z}-\frac{\partial f}{\partial x}\,V_z-f\,\frac{\partial V_z}{\partial x}\right\}\\
&\qquad+2\,\frac{\partial f}{\partial z}\left\{\frac{\partial f}{\partial x}\,V_y+f\,\frac{\partial V_y}{\partial x}-\frac{\partial f}{\partial y}\,V_x-f\,\frac{\partial V_x}{\partial y}\right\}\\
&=\frac{\partial(f^2)}{\partial x}\left\{\frac{\partial V_z}{\partial y}-\frac{\partial V_y}{\partial z}\right\}+\frac{\partial(f^2)}{\partial y}\left\{\frac{\partial V_x}{\partial z}-\frac{\partial V_z}{\partial x}\right\}+\frac{\partial(f^2)}{\partial z}\left\{\frac{\partial V_y}{\partial x}-\frac{\partial V_x}{\partial y}\right\}\\
&=\bigtriangledown(f^2)\cdot(\bigtriangledown\times\mathbf{V}).
\end{aligned}$$

Example 36.5 *Let* $\mathbf{V}$ *be a* C^1 *vector field in the set* $A\subseteq\mathbb{R}^3$*. Show that if there exists a* C^1 *function* $g:A\to\mathbb{R}\setminus\{0\}$*, such that* $g\,\mathbf{V}$ *is a gradient field in* A*, then*

$$\mathbf{V}\cdot(\bigtriangledown\times\mathbf{V})=0$$

in the set A*.*

A Nabla calculus.

D Start by analyzing the assumption. Compute $\bigtriangledown\times\mathbf{V}$ by means of the rules of calculations.

I The assumption assures that there exists a C^2 function F, such that

$$g\,\mathbf{V}=\bigtriangledown F,\quad\text{i.e.}\quad\mathbf{V}=\frac{1}{g}\,\bigtriangledown F=h\,\bigtriangledown F,$$

where $h:A\to\mathbb{R}\setminus\{0\}$ is C^1, because $g(\mathbf{x})\neq 0$. Then

$$\begin{aligned}
\bigtriangledown\times\mathbf{V}&=\bigtriangledown\times(h\,\bigtriangledown F)\\
&=(\bigtriangledown h)\times\bigtriangledown F+h\,\bigtriangledown\times\bigtriangledown F\\
&=(\bigtriangledown h)\times\bigtriangledown F,&&\text{[the rotation of a gradient is }\mathbf{0}\text{]}.
\end{aligned}$$

Now, ∇F is perpendicular to $(\nabla h) \times (\nabla F)$, hence

$$\mathbf{V} \cdot (\nabla \times \mathbf{V}) = h \nabla F \cdot \{(\nabla h) \times \nabla F\} = 0.$$

Example 36.6 *Let α be a constant. Find $\nabla(r^\alpha)$ and $\nabla^2(r^\alpha)$.*

A Nabla calculus.

D Just compute.

I When $r \neq 0$, then

$$\nabla r = \frac{1}{r}(x, y, z),$$

hence by the chain rule,

$$\nabla(r^\alpha) = \alpha r^{\alpha-1} \nabla r = \alpha r^{\alpha-2}(x, y, z) = \alpha r^{\alpha-2} \mathbf{x}.$$

By taking the divergence we get

$$\begin{aligned}\nabla^2(r^\alpha) &= \nabla \cdot \nabla(r^\alpha) = \nabla \cdot \{\alpha r^{\alpha-2}(x, y, z)\} \\ &= \alpha(\alpha-2) r^{\alpha-4}(x, y, z) \cdot (x, y, z) + 3\alpha r^{\alpha-2} \\ &= \alpha(\alpha-2) r^{\alpha-4} \cdot r^2 + 3\alpha r^{\alpha-2} \\ &= \alpha(\alpha+1) r^{\alpha-2}.\end{aligned}$$

Example 36.7 *Let* $\mathbf{e}$ *be a constant unit vector. Show that*

$$\mathbf{e} \cdot \{\bigtriangledown(\mathbf{V} \cdot \mathbf{e}) - \bigtriangledown \times (\mathbf{V} \times \mathbf{e})\} = \bigtriangledown \cdot \mathbf{V}.$$

A Nabla calculus.

D Just compute.

I We get by means of the rules of calculation,

$$\begin{aligned}
&\mathbf{e} \cdot \{\bigtriangledown(\mathbf{V} \cdot \mathbf{e}) - \bigtriangledown \times (\mathbf{V} \times \mathbf{e})\} \\
&\qquad = \mathbf{e} \cdot \{[(\mathbf{e} \cdot \bigtriangledown)\mathbf{V} + \mathbf{e} \times (\bigtriangledown \times \mathbf{V}) + (\mathbf{V} \cdot \bigtriangledown)\mathbf{e} + \mathbf{V} \times (\bigtriangledown \times \mathbf{e})] - \bigtriangledown \times (\mathbf{V} \times \mathbf{e})\} \\
&\qquad = \mathbf{e} \cdot \{(\mathbf{e} \cdot \bigtriangledown)\mathbf{V} + [\mathbf{e} \times (\bigtriangledown \times \mathbf{V})] - \bigtriangledown \times (\mathbf{V} \times \mathbf{e})\} \\
&\qquad = \mathbf{e} \cdot \{(\mathbf{e} \cdot \bigtriangledown)\mathbf{V} - \bigtriangledown \times (\mathbf{V} \times \mathbf{e})\} + \mathbf{e} \cdot [\mathbf{e} \times (\bigtriangledown \times \mathbf{V})] \\
&\qquad = \mathbf{e} \cdot \{(\mathbf{e} \cdot \bigtriangledown)\mathbf{V} - [(\mathbf{e} \cdot \bigtriangledown)\mathbf{V} - \mathbf{e}(\bigtriangledown \cdot \mathbf{V}) - (\mathbf{V} \cdot \bigtriangledown)\mathbf{e} + \mathbf{V}(\bigtriangledown \cdot \mathbf{e})]\} + 0 \\
&\qquad = \mathbf{e} \cdot \mathbf{e}(\bigtriangledown \cdot \mathbf{V}) + 0 \\
&\qquad = \bigtriangledown \cdot \mathbf{V}.
\end{aligned}$$

This formula can of course also be written in the form

$$\mathbf{e} \cdot \{\mathbf{grad}\,\mathrm{div}\;\mathbf{e}) - \mathbf{rot}(\mathbf{V} \times \mathbf{e})\} = \mathrm{div}\;\mathbf{V}.$$

Example 36.8 *Let* α *be a constant vector. For* $\mathbf{x} \neq \mathbf{0}$ *we consider the fields*

$$U(\mathbf{x}) = \frac{\mathbf{a} \cdot \mathbf{x}}{\|\mathbf{x}\|^3}, \qquad \mathbf{W}(\mathbf{x}) = \frac{\mathbf{a} \times \mathbf{x}}{\|\mathbf{x}\|^3}.$$

Show that

$$\bigtriangledown \times \mathbf{W} = - \bigtriangledown U.$$

A Nabla calculus.

D Just compute by using the rulse of calculation and the result of **Example 36.6**.

I Clearly, U and $\mathbf{W}$ are C^∞ for $\mathbf{x} \neq \mathbf{0}$. Put $r = \|\mathbf{x}\|$. Then by **Example 36.6**,

$$\bigtriangledown(r^\alpha) = \alpha\, r^{\alpha-2}\,\mathbf{x} \qquad \text{for } \mathbf{x} \neq \mathbf{0}.$$

Then we shall use the following result from Linear Algebra,

$$\mathbf{x} \times (\mathbf{a} \times \mathbf{x}) = (\mathbf{x} \cdot \mathbf{x})\mathbf{a} - (\mathbf{a} \cdot \mathbf{x})\mathbf{x} = r^2\,\mathbf{a} - (\mathbf{a} \cdot \mathbf{x})\,\mathbf{x}.$$

Applying these preparations we get

$$\begin{aligned}
\nabla \times \mathbf{W} &= \nabla \times \left\{ r^{-3}(\mathbf{a} \times \mathbf{x}) \right\} && \text{definition of } \mathbf{W} \\
&= (\nabla r^{-3}) \times (\mathbf{a} \times \mathbf{x} + r^{-3} \nabla \times (\mathbf{a} \times \mathbf{x}) && \text{rule of calculation} \\
&= -3r^{-5}\mathbf{x} \times (\mathbf{a} \times \mathbf{x}) + r^{-3} \nabla \times (\mathbf{a} \times \mathbf{x}) && \textbf{Example 36.6} \\
&= -3r^{-5}\{r^2\mathbf{a} - (\mathbf{a} \cdot \mathbf{x})\mathbf{x}\} + r^{-3} \nabla \times (\mathbf{a} \times \mathbf{x}) && \text{Linear Algebra} \\
&= -\frac{3}{r^3}\,\mathbf{a} + \frac{3\mathbf{a} \cdot \mathbf{x}}{r^5}\,\mathbf{x} + \frac{1}{r^3}\{0 - 0 - (\mathbf{a} \cdot \nabla)\mathbf{x} + \mathbf{a}(\nabla \cdot \mathbf{x})\} && \text{rule of computation} \\
&= -\frac{3}{r^3}\,\mathbf{a} + \frac{3}{r^5}\,(\mathbf{a} \cdot \mathbf{x})\mathbf{x} + \frac{1}{r^3}\,(-\mathbf{a} + 3\mathbf{a}) && \text{computation} \\
&= -\frac{1}{r^3}\,\mathbf{a} + \frac{3}{r^5}\,(\mathbf{a} \cdot \mathbf{x})\,\mathbf{x}, && \text{reduction,}
\end{aligned}$$

and

$$\begin{aligned}
\nabla U &= \nabla \left(r^{-3}(\mathbf{a} \cdot \mathbf{x}) \right) && \text{definition of } U \\
&= (\mathbf{a} \cdot \mathbf{x}) \nabla \left(r^{-3} \right) + \frac{1}{r^3} \nabla (\mathbf{a} \cdot \mathbf{x}) && \text{rule of calculation} \\
&= (\mathbf{a} \cdot \mathbf{x}) \cdot \left\{ -\frac{3}{r^5}\,\mathbf{x} \right\} + \frac{1}{r^3}\,\mathbf{a} && \textbf{Example 36.6} \text{ and } \nabla (\mathbf{a} \cdot \mathbf{x}) = \mathbf{a}.
\end{aligned}$$

It follows by a comparison of these two expressions that

$$\nabla \times \mathbf{W} = - \nabla U.$$

This can also be written

$$\textbf{rot } \mathbf{W} = -\textbf{grad } U,$$

where U and $\mathbf{W}$ are given above.

Example 36.9 *Consider the composite vector function*

$$\mathbf{V}(\mathbf{x}) = \mathbf{U}(w), \qquad w = f(\mathbf{x}).$$

Find an expression for $\nabla \cdot \mathbf{V}$ *and* $\nabla \times \mathbf{V}$.

A Nabla calculus.

D Just compute.

I In general,

$$\frac{\partial V_j}{\partial x_i} = \frac{\partial (U_j \circ f)}{\partial x_i} = \frac{dU_j}{dw} \cdot \frac{\partial f}{\partial x_i}, \qquad w = f(\mathbf{x}).$$

When we change notation $(x_1, x_2, x_3) = (x, y, z)$, it follows that

$$\nabla \cdot \mathbf{V} = \sum_{i=1}^{3} U_i'(w)\,\frac{\partial f}{\partial x_i} = \nabla f \cdot \mathbf{U}'(f(\mathbf{x})),$$

and

$$\begin{aligned}\nabla \times \mathbf{V} &= \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}, \frac{\partial V_x}{ddz} - \frac{\partial V_z}{\partial x}, \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) \\ &= \begin{pmatrix} U_z'(w)\,\dfrac{\partial f}{\partial y} - U_y'(w)\,\dfrac{\partial f}{\partial z} \\[2ex] U_x'(w)\,\dfrac{\partial f}{\partial z} - U_z'(w)\,\dfrac{\partial f}{\partial x} \\[2ex] U_y'(w)\,\dfrac{\partial f}{\partial x} - U_x'(w)\,\dfrac{\partial f}{\partial y} \end{pmatrix} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\[1ex] \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \\[1ex] U_x' & U_y' & U_z' \end{vmatrix} = \nabla f \times \mathbf{U}'(f(\mathbf{x})).\end{aligned}$$

ALTERNATIVELY, a more sophisticated reasoning is the following,

$$\nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\[1ex] \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\[1ex] U_x \circ f & U_y \circ f & U_z \circ f \end{vmatrix} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\[1ex] \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \\[1ex] U_x' \circ f & U_y' \circ f & U'z \circ f \end{vmatrix} = \nabla f \times \mathbf{U}'(f(\mathbf{x})).$$

Summarizing we obtain the results

$$\nabla \cdot (\mathbf{U} \circ f(\mathbf{x})) = \nabla f(\mathbf{x}) \cdot \mathbf{U} - (f(\mathbf{x})) \qquad \text{and} \qquad \nabla \times (\mathbf{U} \circ f(\mathbf{x})) = \nabla f(\mathbf{x}) \times \mathbf{U}'(f(\mathbf{x})).$$

Example 36.10 *Given a C^1 vector field $\mathbf{V}$ and a C^2 scalar field f with the following property: The vector $\mathbf{V}$ is at each point (x,y,z) perpendicular to the level surface of f through the point (x,y,z). Prove that $\mathbf{V} \cdot (\nabla \times \mathbf{V}) = 0$.*

A Nabla calculus.

D Analyze the assumption. Then find a relation between f and $\mathbf{V}$. Finally, compute $\mathbf{V} \cdot (\nabla \times \mathbf{V})$.

I Since both ∇f and $\mathbf{V}$ are perpendicular to the level surface, they are proportional at each point. Hence, there exists a function g, such that (usually)

$$(36.1)\quad \mathbf{V}(x,y,z) = g(x,y,z)\ \nabla\, f(x,y,z).$$

When $\nabla f \neq \mathbf{0}$, then clearly g is os class C^1. Thus, when $\nabla f \neq \mathbf{0}$, then

$$\nabla \times \mathbf{V} = \nabla \times (g\ \nabla\, f) = (\nabla g) \times (\nabla f) + g\,(\nabla \times \nabla f) = (\nabla g) \times (\nabla f) + \mathbf{0}.$$

Since ∇f is perpendicular to $\nabla g \times \nabla f$, we get

$$\mathbf{V} \cdot (\nabla \times \mathbf{V}) = g\ \nabla\, f \cdot \{\nabla g \times \nabla f\} = 0.$$

If $\nabla f = 0$, then (36.1) does not necessary hold. However, if (36.1) holds, the relation is trivial.

Now assume that (36.1) does not hold, i.e. $\mathbf{V}(x,y,z) \neq \mathbf{0}$ and $\nabla f(x,y,z) = \mathbf{0}$. We shall then use a continuity argument:

Since f has level surfaces, we must have $\nabla f \neq \mathbf{0}$ arbitrarily close to (x,y,z), and it follows from the above that $\mathbf{V}\cdot(\nabla\times\mathbf{V}) = 0$ at these points. This relation is continuous, so it follows by a continuous extension that $\mathbf{V}\cdot\{\nabla\times\mathbf{V}\} = 0$ also is valid at points, where $\nabla f(x,y,z) = \mathbf{0}$.

Example 36.11 *Show by means of* Gauß's theorem *that for any* closed *surface* $\mathcal{F}$*,*

$$\int_{\mathcal{F}} \mathbf{n}\,dS = \mathbf{0}.$$

A Gauß's theorem in its operator version.

D Insert the obvious into Gauß's theorem in its operator version.

I Let $\mathcal{F}$ be the boundary of the domain Ω. Then by Gauß's theorem in its operator version,

$$\int_{\Omega} \nabla\,\Box\,d\Omega = \int_{\partial\Omega} \mathbf{n}\,\Box\,\mathrm{d}S = \int_{\mathbf{F}} \mathbf{n}\,\Box\,\mathrm{d}S.$$

If we replace $\Box$ with 1, it follows that

$$\int_{\mathcal{F}} \mathbf{n}\,\mathrm{d}S = \int_{\Omega} \nabla 1\,\mathrm{d}\Omega = \int_{\Omega} \mathbf{0}\,\mathrm{d}\Omega = \mathbf{0}.$$

Example 36.12 *Find the divergence of the vector field*

$$\mathbf{V} = (\nabla f)\times(\nabla g).$$

[Cf. **Example 38.17***.]*

A Nabla calculus.

D Just compute.

I The rotation of a gradient is $\mathbf{0}$, i.e. every gradient field is rotation free. Hence

$$\nabla\cdot(\nabla f\times\nabla g) = (\nabla\times\nabla f)\cdot\nabla g - (\nabla\times\nabla g)\cdot\nabla f = 0 - 0 = 0.$$

Example 36.13 *Consider the vector field* $\mathbf{V}:\mathbb{R}^3\to\mathbb{R}^3$ *given by*

$$\mathbf{V}(x,y,z) = f(x,y)\,\mathbf{e}_z,$$

which also satisfies

$$\nabla\times(\nabla\times\mathbf{V}) = \alpha\,\mathbf{V},$$

where α *is a constant. Find a differential equation which has the function* f *as one of its solutions.*

A Double rotation.

D Compute the left hand side.

I It follows from $\mathbf{V}(x,y,z) = f(x,y)\,\mathbf{e}_z$ that

$$\nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & f(x,y) \end{vmatrix} = \left(f_y', -f_x', 0\right)$$

and

$$\nabla \times (\nabla \times \mathbf{V}) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_y' & -f_x' & 0 \end{vmatrix} = \left(0\,,\,0\,,-f_{xx}'' - f_{yy}''\right) = (0,0,\nabla^2 f),$$

and

$$\alpha\,\mathbf{V} = (0,0,\alpha\,f(x,y)).$$

Then from $\nabla \times (\nabla \times \mathbf{V}) = \alpha\,\mathbf{V}$,

$$-\nabla^2 f = \alpha\,f \quad \text{or} \quad \nabla^2 f + \alpha\,f = \Delta f + \alpha\,f = 0.$$

Example 36.14 *Let V denote the volume of a domain Ω in space with the outwards unit normal vector field $\mathbf{n}$, and let $\mathbf{a}$ be a constant vector. Find*

$$\frac{1}{V}\int_{\partial\Omega} \mathbf{n} \times (\mathbf{x} \times \mathbf{a})\,\mathrm{d}S.$$

A Nabla calculus.

D Apply a variant Gauß's theorem and use the nabla calculations.

I By a variant of Gauß's theorem,

$$\int_{\partial\Omega} \mathbf{n} \times \mathbf{V}\,\mathrm{d}S = \int_{\Omega} \nabla \times \mathbf{V}\,\mathrm{d}\Omega.$$

Put $\mathbf{V} = \mathbf{x} \times \mathbf{a}$. Then

$$\frac{1}{V}\int_{\partial\Omega} \mathbf{n} \times (\mathbf{x} \times \mathbf{a})\,\mathrm{d}S = \frac{1}{V}\int_{\Omega} \nabla \times (\mathbf{x} \times \mathbf{a})\,\mathrm{d}S.$$

Then by a rule of calculation,

$$\begin{aligned} \nabla \times (\mathbf{x} \times \nabla a) &= (\mathbf{a}\cdot\nabla)\mathbf{x} - \mathbf{a}(\nabla\cdot\mathbf{x}) - (\mathbf{x}\cdot\nabla)\mathbf{a} + \mathbf{x}(\nabla\cdot\mathbf{a}) \\ &= (\mathbf{a}\cdot\nabla)\mathbf{x} - \mathbf{a}(\nabla\cdot\mathbf{x}) - \mathbf{0} + \mathbf{0} \\ &= \left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}\right)(x,y,z) - \mathbf{a}\cdot(1+1+1) \\ &= \mathbf{a} - 3\mathbf{a} = -2\mathbf{a}, \end{aligned}$$

which is a constant. Hence by insertion,

$$\frac{1}{V}\int_{\partial\Omega}\mathbf{n}\times(\mathbf{x}\times\mathbf{a})\,\mathrm{d}S=\frac{1}{V}\int_{V}(-2\mathbf{a}\,\mathrm{d}S=-2\mathbf{a}.$$

ADDITION. For completeness we here prove the variant of Gauß's theorem, which is applied above. First note that the usual version of Gauß's theorem can be written

$$\int_{\partial\Omega}\mathbf{n}\cdot\mathbf{W}\,\mathrm{d}S=\int_{\Omega}\bigtriangledown\cdot\mathbf{W}\,\mathrm{d}\Omega.$$

Choose $\mathbf{W}=\mathbf{V}\times\mathbf{b}$, where $\mathbf{b}$ is any constant vector. Then

$$(36.2)\quad \int_{\partial\Omega}\mathbf{n}\cdot(\mathbf{V}\times\mathbf{b})\,\mathrm{d}S=\int_{\Omega}\bigtriangledown\cdot(\mathbf{V}\times\mathbf{b})\,\mathrm{d}\Omega.$$

The geometric interpretation of $\mathbf{n}\cdot(\mathbf{V}\times\mathbf{b})$ is that it is equal to the (signed) volume of the parallelepiped defined by the vectors $\mathbf{n}$, $\mathbf{V}$ and $\mathbf{b}$. (This simple result is also known from Linear Algebra).

The same interpretation is true for $(\mathbf{n}\times\mathbf{V})\cdot\mathbf{b}$ (with the same sign, because the sequence of the vectors is not changed), thus

$$\mathbf{n}\cdot(\mathbf{V}\times\mathbf{b})=(\mathbf{n}\times\mathbf{V})\cdot\mathbf{b}.$$

Since $\mathbf{b}$ is constant, it follows by a rule of calculation,

$$\bigtriangledown\cdot(\mathbf{V}\times\mathbf{b})=(\bigtriangledown\times\mathbf{V})\cdot\mathbf{b}-(\bigtriangledown\times\mathbf{b}\cdot\mathbf{V})=(\bigtriangledown\times\mathbf{V})\cdot\mathbf{b}.$$

By inserting these two results into (36.2), we get

$$\int_{\partial\Omega}(\mathbf{n}\times\mathbf{V})\cdot\mathbf{b}\,\mathrm{d}S=\int_{\Omega}(\bigtriangledown\times\mathbf{V})\cdot\mathbf{b}\,\mathrm{d}\Omega.$$

Since $\mathbf{b}$ is a constant vector, it follows by a rearrangement that

$$\left\{\int_{\partial\Omega}\mathbf{n}\times\mathbf{V}\,\mathrm{d}S-\int_{\Omega}\bigtriangledown\times\mathbf{V}\,\mathrm{d}\Omega\right\}\cdot\mathbf{b}=0.$$

Since $\mathbf{0}$ is the only vector, which is perpendicular to all vectors, the first factor must be $\mathbf{0}$, and we get by another rearrangement.

$$\int_{\partial\Omega}\mathbf{n}\times\mathbf{V}\,\mathrm{d}S=\int_{\Omega}\bigtriangledown\times\mathbf{V}\,\mathrm{d}\Omega,$$

and the variant of Gauß's theorem has been proved.

Example 36.15 *Let* $\mathbf{V}$, $\mathbf{W}$ *be vector fields in the space which also depend on time* t *and satisfy the equations*

$$\nabla \times \mathbf{V} = \alpha \, \frac{\partial \mathbf{W}}{\partial t}, \qquad \nabla \times \mathbf{W} = -\beta \, \frac{\partial \mathbf{V}}{\partial t},$$

where α *and* β *are constants. Show that the vector field*

$$\mathbf{U} = \beta \mathbf{V} \times \frac{\partial \mathbf{V}}{\partial t} + \alpha \mathbf{W} \times \frac{\partial \mathbf{W}}{\partial t}$$

and the scalar field

$$f = \beta \mathbf{V} \cdot \nabla \times \mathbf{V} + \alpha \mathbf{W} \cdot \nabla \times \mathbf{W}$$

satisfy the differential equation

$$\nabla \cdot \mathbf{U} + \frac{\partial f}{\partial t} = 0.$$

(An equation of this type is often called a continuity equation *or a* preservation theorem*).*

A Continuity equation.

D Nabla calculus.

I Since $\dfrac{\partial}{\partial t}$ is a differentiation with respect to a "parameter", where can interchange $\dfrac{\partial}{\partial t}$ with any of the operators ∇, $\nabla \cdot$ and $\nabla \times$. Hence

$$\begin{aligned}
\frac{\partial f}{\partial t} &= \beta \, \frac{\partial \mathbf{V}}{\partial t} \cdot (\nabla \times \mathbf{V}) + \beta \mathbf{V} \cdot \left(\nabla \times \frac{\partial \mathbf{V}}{\partial t} \right) + \alpha \, \frac{\partial \mathbf{W}}{\partial t} \cdot (\nabla \times \mathbf{W}) + \alpha \mathbf{W} \cdot \left(\nabla \times \frac{\partial \mathbf{W}}{\partial t} \right) \\
&= -(\nabla \times \mathbf{W}) \cdot (\nabla \times \mathbf{V}) + \beta \mathbf{V} \cdot \left(\nabla \times \frac{\partial \mathbf{V}}{\partial t} \right) + (\nabla \times \mathbf{V}) \cdot (\nabla \times \mathbf{W}) + \alpha \mathbf{W} \cdot \left(\nabla \times \frac{\partial \mathbf{W}}{\partial t} \right) \\
&= \beta \mathbf{V} \cdot \left(\nabla \times \frac{\partial \mathbf{V}}{\partial t} \right) + \alpha \mathbf{W} \cdot \left(\nabla \times \frac{\partial \mathbf{W}}{\partial t} \right).
\end{aligned}$$

By using this rule of calculation we get similarly,

$$\begin{aligned}
\nabla \cdot \mathbf{U} &= \beta \, \nabla \cdot \left(\mathbf{V} \times \frac{\partial \mathbf{V}}{\partial t} \right) + \alpha \, \nabla \cdot \left(\mathbf{W} \times \frac{\partial \mathbf{W}}{\partial t} \right) \\
&= \beta (\nabla \times \mathbf{V}) \cdot \frac{\partial \mathbf{V}}{\partial t} - \beta \left(\nabla \times \frac{\partial \mathbf{V}}{\partial t} \right) \cdot \mathbf{V} + \alpha (\nabla \times \mathbf{W}) \cdot \frac{\partial \mathbf{W}}{\partial t} - \alpha \left(\nabla \times \frac{\partial \mathbf{W}}{\partial t} \right) \cdot \mathbf{W} \\
&= -(\nabla \times \mathbf{V}) \cdot (\nabla \times \mathbf{W}) - \beta \mathbf{V} \cdot \left(\nabla \times \frac{\partial \mathbf{V}}{\partial t} \right) + (\nabla \times \mathbf{V}) \cdot (\nabla \times \mathbf{W}) - \alpha \mathbf{W} \cdot \left(\nabla \times \frac{\partial \mathbf{W}}{\partial t} \right) \\
&= -\beta \mathbf{V} \cdot \left(\nabla \times \frac{\partial \mathbf{V}}{\partial t} \right) - \alpha \mathbf{W} \cdot \left(\nabla \times \frac{\partial \mathbf{W}}{\partial t} \right)
\end{aligned}$$

Finally, by adding these expressions we get

$$\nabla \cdot \mathbf{U} + \frac{\partial f}{\partial t} = 0.$$

37 Formulæ

Some of the following formulæ can be assumed to be known from high school. It is highly recommended that one *learns most of these formulæ in this appendix by heart.*

37.1 Squares etc.

The following simple formulæ occur very frequently in the most different situations.

$$\begin{array}{ll} (a+b)^2 = a^2+b^2+2ab, & a^2+b^2+2ab = (a+b)^2, \\ (a-b)^2 = a^2+b^2-2ab, & a^2+b^2-2ab = (a-b)^2, \\ (a+b)(a-b) = a^2-b^2, & a^2-b^2 = (a+b)(a-b), \\ (a+b)^2 = (a-b)^2+4ab, & (a-b)^2 = (a+b)^2-4ab. \end{array}$$

37.2 Powers etc.

Logarithm:

$$\begin{array}{rcll} \ln|xy| &=& \ln|x|+\ln|y|, & x,y\neq 0, \\ \ln\left|\dfrac{x}{y}\right| &=& \ln|x|-\ln|y|, & x,y\neq 0, \\ \ln|x^r| &=& r\ln|x|, & x\neq 0. \end{array}$$

Power function, fixed exponent:

$$(xy)^r = x^r\cdot y^r, x,y>0 \quad \text{(extensions for some } r\text{)},$$

$$\left(\frac{x}{y}\right)^r = \frac{x^r}{y^r}, x,y>0 \qquad \text{(extensions for some } r\text{)}.$$

Exponential, fixed base:

$$a^x\cdot a^y = a^{x+y}, \quad a>0 \quad \text{(extensions for some } x,\, y\text{)},$$
$$(a^x)^y = a^{xy}, a>0 \qquad \text{(extensions for some } x,\, y\text{)},$$

$$a^{-x} = \frac{1}{a^x}, a>0, \qquad \text{(extensions for some } x\text{)},$$

$$\sqrt[n]{a} = a^{1/n}, a\geq 0, \qquad n\in\mathbb{N}.$$

Square root:

$$\sqrt{x^2} = |x|, \qquad x\in\mathbb{R}.$$

Remark 37.1 It happens quite frequently that students make errors when they try to apply these rules. They must be mastered! In particular, as one of my friends once put it: "If you can master the square root, you can master everything in mathematics!" Notice that this innocent looking square root is one of the most difficult operations in Calculus. Do not forget the *absolute value!* ◊

37.3 Differentiation

Here are given the well-known rules of differentiation together with some rearrangements which sometimes may be easier to use:

$$\{f(x) \pm g(x)\}' = f'(x) \pm g'(x),$$

$$\{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x) = f(x)g(x)\left\{\frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}\right\},$$

where the latter rearrangement presupposes that $f(x) \neq 0$ and $g(x) \neq 0$.
If $g(x) \neq 0$, we get the usual formula known from high school

$$\left\{\frac{f(x)}{g(x)}\right\}' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

It is often more convenient to compute this expression in the following way:

$$\left\{\frac{f(x)}{g(x)}\right\} = \frac{d}{dx}\left\{f(x) \cdot \frac{1}{g(x)}\right\} = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2} = \frac{f(x)}{g(x)}\left\{\frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}\right\},$$

where the former expression often is *much easier* to use in practice than the usual formula from high school, and where the latter expression again presupposes that $f(x) \neq 0$ and $g(x) \neq 0$. Under these assumptions we see that the formulæ above can be written

$$\frac{\{f(x)g(x)\}'}{f(x)g(x)} = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)},$$

$$\frac{\{f(x)/g(x)\}'}{f(x)/g(x)} = \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}.$$

Since

$$\frac{d}{dx}\ln|f(x)| = \frac{f'(x)}{f(x)}, \qquad f(x) \neq 0,$$

we also name these the *logarithmic derivatives.*

Finally, we mention the rule of **differentiation of a composite function**

$$\{f(\varphi(x))\}' = f'(\varphi(x)) \cdot \varphi'(x).$$

We first differentiate the function itself; then the insides. This rule is a 1-dimensional version of the so-called *Chain rule.*

37.4 Special derivatives.

Power like:

$$\frac{d}{dx}(x^\alpha) = \alpha \cdot x^{\alpha-1}, \qquad \text{for } x > 0, \quad \text{(extensions for some } \alpha).$$

$$\frac{d}{dx}\ln|x| = \frac{1}{x}, \qquad \text{for } x \neq 0.$$

Exponential like:

$$\frac{d}{dx}\exp x = \exp x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\left(a^x\right) = \ln a \cdot a^x, \qquad \text{for } x \in \mathbb{R} \text{ and } a > 0.$$

Trigonometric:

$$\frac{d}{dx}\sin x = \cos x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\cos x = -\sin x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\tan x = 1 + \tan^2 x = \frac{1}{\cos^2 x}, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, p \in \mathbb{Z},$$

$$\frac{d}{dx}\cot x = -\left(1 + \cot^2 x\right) = -\frac{1}{\sin^2 x}, \qquad \text{for } x \neq p\pi, p \in \mathbb{Z}.$$

Hyperbolic:

$$\frac{d}{dx}\sinh x = \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\cosh x = \sinh x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\tanh x = 1 - \tanh^2 x = \frac{1}{\cosh^2 x}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\coth x = 1 - \coth^2 x = -\frac{1}{\sinh^2 x}, \qquad \text{for } x \neq 0.$$

Inverse trigonometric:

$$\frac{d}{dx}\text{ Arcsin } x = \frac{1}{\sqrt{1-x^2}}, \qquad \text{for } x \in \,]-1, 1\,[,$$

$$\frac{d}{dx}\text{ Arccos } x = -\frac{1}{\sqrt{1-x^2}}, \qquad \text{for } x \in \,]-1, 1\,[,$$

$$\frac{d}{dx}\text{ Arctan } x = \frac{1}{1+x^2}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\text{ Arccot } x = \frac{1}{1+x^2}, \qquad \text{for } x \in \mathbb{R}.$$

Inverse hyperbolic:

$$\frac{d}{dx}\text{ Arsinh } x = \frac{1}{\sqrt{x^2+1}}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\text{ Arcosh } x = \frac{1}{\sqrt{x^2-1}}, \qquad \text{for } x \in \,]1, +\infty\,[,$$

$$\frac{d}{dx}\text{ Artanh } x = \frac{1}{1-x^2}, \qquad \text{for } |x| < 1,$$

$$\frac{d}{dx}\text{ Arcoth } x = \frac{1}{1-x^2}, \qquad \text{for } |x| > 1.$$

Remark 37.2 The derivative of the trigonometric and the hyperbolic functions are to some extent exponential like. The derivatives of the inverse trigonometric and inverse hyperbolic functions are power like, because we include the logarithm in this class. $\Diamond$

37.5 Integration

The most obvious rules are dealing with linearity

$$\int \{f(x) + \lambda g(x)\}\, dx = \int f(x)\, dx + \lambda \int g(x)\, dx, \qquad \text{where } \lambda \in \mathbb{R} \text{ is a constant,}$$

and with the fact that differentiation and integration are "inverses to each other", i.e. modulo some arbitrary constant $c \in \mathbb{R}$, which often tacitly is missing,

$$\int f'(x)\, dx = f(x).$$

If we in the latter formula replace $f(x)$ by the product $f(x)g(x)$, we get by reading from the right to the left and then differentiating the product,

$$f(x)g(x) = \int \{f(x)g(x)\}'\, dx = \int f'(x)g(x)\, dx + \int f(x)g'(x)\, dx.$$

Hence, by a rearrangement

The rule of partial integration:

$$\int f'(x)g(x)\, dx = f(x)g(x) - \int f(x)g'(x)\, dx.$$

The differentiation is moved from one factor of the integrand to the other one by changing the sign and adding the term $f(x)g(x)$.

Remark 37.3 This technique was earlier used a lot, but is almost forgotten these days. It must be revived, because MAPLE and pocket calculators apparently do not know it. It is possible to construct examples where these devices cannot give the exact solution, unless you first perform a partial integration yourself. $\Diamond$

Remark 37.4 This method can also be used when we estimate integrals which cannot be directly calculated, because the antiderivative is not contained in e.g. the catalogue of MAPLE. The idea is by a succession of partial integrations to make the new integrand smaller. $\Diamond$

Integration by substitution:

If the integrand has the special structure $f(\varphi(x)) \cdot \varphi'(x)$, then one can change the variable to $y = \varphi(x)$:

$$\int f(\varphi(x)) \cdot \varphi'(x)\, dx = \text{“}\int f(\varphi(x))\, d\varphi(x)\text{''} = \int_{y=\varphi(x)} f(y)\, dy.$$

Integration by a monotonous substitution:

If $\varphi(y)$ is a *monotonous* function, which maps the y-interval *one-to-one* onto the x-interval, then

$$\int f(x)\,dx = \int_{y=\varphi^{-1}(x)} f(\varphi(y))\varphi'(y)\,dy.$$

Remark 37.5 This rule is usually used when we have some "ugly" term in the integrand $f(x)$. The idea is to put this ugly term equal to $y = \varphi^{-1}(x)$. When e.g. x occurs in $f(x)$ in the form $\sqrt{x}$, we put $y = \varphi^{-1}(x) = \sqrt{x}$, hence $x = \varphi(y) = y^2$ and $\varphi'(y) = 2y$. $\Diamond$

37.6 Special antiderivatives

Power like:

$$\int \frac{1}{x}\,dx = \ln|x|, \qquad \text{for } x \neq 0. \quad \text{(Do not forget the numerical value!)}$$

$$\int x^{\alpha}\,dx = \frac{1}{\alpha+1}x^{\alpha+1}, \qquad \text{for } \alpha \neq -1,$$

$$\int \frac{1}{1+x^2}\,dx = \text{ Arctan } x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{1-x^2}\,dx = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|, \qquad \text{for } x \neq \pm 1,$$

$$\int \frac{1}{1-x^2}\,dx = \text{ Artanh } x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{1-x^2}\,dx = \text{ Arcoth } x, \qquad \text{for } |x| > 1,$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \text{ Arcsin } x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = -\text{ Arccos } x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \text{ Arsinh } x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \ln(x+\sqrt{x^2+1}), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{x}{\sqrt{x^2-1}}\,dx = \sqrt{x^2-1}, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \text{ Arcosh } x, \qquad \text{for } x > 1,$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \ln|x+\sqrt{x^2-1}|, \qquad \text{for } x > 1 \text{ eller } x < -1.$$

There is an error in the programs of the pocket calculators TI-92 and TI-89. The *numerical signs* are missing. It is obvious that $\sqrt{x^2-1} < |x|$ so if $x < -1$, then $x + \sqrt{x^2-1} < 0$. Since you cannot take the logarithm of a negative number, these pocket calculators will give an error message.

Exponential like:

$$\int \exp x\,dx = \exp x, \qquad \text{for } x \in \mathbb{R},$$

$$\int a^x\,dx = \frac{1}{\ln a} \cdot a^x, \qquad \text{for } x \in \mathbb{R}, \text{ and } a > 0, a \neq 1.$$

Trigonometric:

$$\int \sin x\,dx = -\cos x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \cos x\,dx = \sin x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \tan x\,dx = -\ln|\cos x|, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \cot x\,dx = \ln|\sin x|, \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos x}\,dx = \frac{1}{2}\ln\left(\frac{1+\sin x}{1-\sin x}\right), \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin x}\,dx = \frac{1}{2}\ln\left(\frac{1-\cos x}{1+\cos x}\right), \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos^2 x}\,dx = \tan x, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin^2 x}\,dx = -\cot x, \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z}.$$

Hyperbolic:

$$\int \sinh x\,dx = \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \cosh x\,dx = \sinh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \tanh x\,dx = \ln\cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \coth x\,dx = \ln|\sinh x|, \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh x}\,dx = \text{Arctan}(\sinh x), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\cosh x}\,dx = 2\,\text{Arctan}(e^x), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh x}\,dx = \frac{1}{2}\ln\left(\frac{\cosh x - 1}{\cosh x + 1}\right), \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\sinh x}\,dx = \ln\left|\frac{e^x-1}{e^x+1}\right|, \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh^2 x}\,dx = \tanh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh^2 x}\,dx = -\coth x, \qquad \text{for } x \neq 0.$$

37.7 Trigonometric formulæ

The trigonometric formulæ are closely connected with circular movements. Thus $(\cos u, \sin u)$ are the coordinates of a point P on the unit circle corresponding to the angle u, cf. figure A.1. This geometrical interpretation is used from time to time.

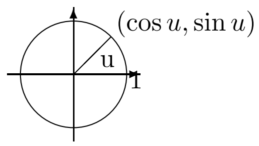

Figure 37.1: The unit circle and the trigonometric functions.

The fundamental trigonometric relation:

$$\cos^2 u + \sin^2 u = 1, \qquad \text{for } u \in \mathbb{R}.$$

Using the previous geometric interpretation this means according to *Pythagoras's theorem*, that the point P with the coordinates $(\cos u, \sin u)$ always has distance 1 from the origo $(0,0)$, i.e. it is lying on the boundary of the circle of centre $(0,0)$ and radius $\sqrt{1} = 1$.

Connection to the complex exponential function:

The *complex exponential* is for imaginary arguments defined by

$$\exp(\mathrm{i}\,u) := \cos u + \mathrm{i}\;\sin u.$$

It can be checked that the usual functional equation for exp is still valid for complex arguments. In other word: The definition above is extremely conveniently chosen.

By using the definition for $\exp(\mathrm{i}\,u)$ and $\exp(-\mathrm{i}\,u)$ it is easily seen that

$$\cos u = \frac{1}{2}(\exp(\mathrm{i}\,u) + \exp(-\mathrm{i}\,u)),$$

$$\sin u = \frac{1}{2i}(\exp(\mathrm{i}\,u) - \exp(-\mathrm{i}\,u)),$$

.

Moivre's formula: We get by expressing $\exp(\mathrm{i}nu)$ in two different ways:

$$\exp(\mathrm{i}nu) = \cos nu + \mathrm{i}\,\sin nu = (\cos u + \mathrm{i}\,\sin u)^n.$$

Example 37.1 If we e.g. put $n = 3$ into Moivre's formula, we obtain the following typical application,

$$\begin{aligned}\cos(3u) + \mathrm{i}\,\sin(3u) &= (\cos u + \mathrm{i}\,\sin u)^3\\ &= \cos^3 u + 3\mathrm{i}\,\cos^2 u \cdot \sin u + 3\mathrm{i}^2 \cos u \cdot \sin^2 u + \mathrm{i}^3 \sin^3 u\\ &= \{\cos^3 u - 3\cos u \cdot \sin^2 u\} + \mathrm{i}\{3\cos^2 u \cdot \sin u - \sin^3 u\}\\ &= \{4\cos^3 u - 3\cos u\} + \mathrm{i}\{3\sin u - 4\sin^3 u\}\end{aligned}$$

When this is split into the real- and imaginary parts we obtain

$$\cos 3u = 4\cos^3 u - 3\cos u, \qquad \sin 3u = 3\sin u - 4\sin^3 u. \quad \diamond$$

Addition formulæ:

$$\sin(u+v) = \sin u\, \cos v + \cos u\, \sin v,$$

$$\sin(u-v) = \sin u\, \cos v - \cos u\, \sin v,$$

$$\cos(u+v) = \cos u\, \cos v - \sin u\, \sin v,$$

$$\cos(u-v) = \cos u\, \cos v + \sin u\, \sin v.$$

Products of trigonometric functions to a sum:

$$\sin u\, \cos v = \frac{1}{2}\sin(u+v) + \frac{1}{2}\sin(u-v),$$

$$\cos u\, \sin v = \frac{1}{2}\sin(u+v) - \frac{1}{2}\sin(u-v),$$

$$\sin u\, \sin v = \frac{1}{2}\cos(u-v) - \frac{1}{2}\cos(u+v),$$

$$\cos u\, \cos v = \frac{1}{2}\cos(u-v) + \frac{1}{2}\cos(u+v).$$

Sums of trigonometric functions to a product:

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right),$$

$$\sin u - \sin v = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right),$$

$$\cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right),$$

$$\cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right).$$

Formulæ of halving and doubling the angle:

$$\sin 2u = 2\sin u\, \cos u,$$

$$\cos 2u = \cos^2 u - \sin^2 u = 2\,\cos^2 u - 1 = 1 - 2\,\sin^2 u,$$

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}} \qquad \text{followed by a discussion of the sign,}$$

$$\cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}} \qquad \text{followed by a discussion of the sign,}$$

37.8 Hyperbolic formulæ

These are very much like the trigonometric formulæ, and if one knows a little of Complex Function Theory it is realized that they are actually identical. The structure of this section is therefore the same as for the trigonometric formulæ. The reader should compare the two sections concerning similarities and differences.

The fundamental relation:

$$\cosh^2 x - \sinh^2 x = 1.$$

Definitions:

$$\cosh x = \frac{1}{2}\left(\exp(x) + \exp(-x)\right), \qquad \sinh x = \frac{1}{2}\left(\exp(x) - \exp(-x)\right).$$

"Moivre's formula":

$$\exp(x) = \cosh x + \sinh x.$$

This is trivial and only rarely used. It has been included to show the analogy.

Addition formulæ:

$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y),$$

$$\sinh(x-y) = \sinh(x)\cosh(y) - \cosh(x)\sinh(y),$$

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y),$$

$$\cosh(x-y) = \cosh(x)\cosh(y) - \sinh(x)\sinh(y).$$

Formulæ of halving and doubling the argument:

$$\sinh(2x) = 2\sinh(x)\cosh(x),$$

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x) = 2\cosh^2(x) - 1 = 2\sinh^2(x) + 1,$$

$$\sinh\left(\frac{x}{2}\right) = \pm\sqrt{\frac{\cosh(x) - 1}{2}} \qquad \text{followed by a discussion of the sign,}$$

$$\cosh\left(\frac{x}{2}\right) = \sqrt{\frac{\cosh(x) + 1}{2}}.$$

Inverse hyperbolic functions:

$$\operatorname{Arsinh}(x) = \ln\left(x + \sqrt{x^2+1}\right), \qquad x \in \mathbb{R},$$

$$\operatorname{Arcosh}(x) = \ln\left(x + \sqrt{x^2-1}\right), \qquad x \geq 1,$$

$$\operatorname{Artanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \qquad |x| < 1,$$

$$\operatorname{Arcoth}(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right), \qquad |x| > 1.$$

37.9 Complex transformation formulæ

$$\cos(\mathrm{i}x) = \cosh(x), \qquad \cosh(\mathrm{i}x) = \cos(x),$$

$$\sin(\mathrm{i}x) = \mathrm{i}\sinh(x), \qquad \sinh(\mathrm{i}x) = \mathrm{i}\sin x.$$

37.10 Taylor expansions

The generalized binomial coefficients are defined by

$$\binom{\alpha}{n} := \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{1\cdot 2\cdots n},$$

with n factors in the numerator and the denominator, supplied with

$$\binom{\alpha}{0} := 1.$$

The Taylor expansions for *standard functions* are divided into *power like* (the radius of convergency is finite, i.e. $= 1$ for the standard series) and*exponential like* (the radius of convergency is infinite).
Power like:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1,$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x| < 1,$$

$$(1+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j, \qquad n \in \mathbb{N}, x \in \mathbb{R},$$

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \qquad \alpha \in \mathbb{R} \setminus \mathbb{N}, |x| < 1,$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \qquad |x| < 1,$$

$$\operatorname{Arctan}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \qquad |x| < 1.$$

Exponential like:

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \qquad x \in \mathbb{R}$$

$$\exp(-x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} x^n, \qquad x \in \mathbb{R}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}, \qquad x \in \mathbb{R},$$

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}, \qquad x \in \mathbb{R},$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}, \qquad x \in \mathbb{R},$$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}, \qquad x \in \mathbb{R}.$$

37.11 Magnitudes of functions

We often have to compare functions for $x \to 0+$, or for $x \to \infty$. The simplest type of functions are therefore arranged in an hierarchy:

1) logarithms,

2) power functions,

3) exponential functions,

4) faculty functions.

When $x \to \infty$, a function from a higher class will always dominate a function form a lower class. More precisely:

A) A *power function* dominates a *logarithm* for $x \to \infty$:

$$\frac{(\ln x)^{\beta}}{x^{\alpha}} \to 0 \qquad \text{for } x \to \infty, \quad \alpha,\, \beta > 0.$$

B) An *exponential* dominates a *power function* for $x \to \infty$:

$$\frac{x^{\alpha}}{a^{x}} \to 0 \qquad \text{for } x \to \infty, \quad \alpha,\, a > 1.$$

C) The *faculty function* dominates an *exponential* for $n \to \infty$:

$$\frac{a^n}{n!} \to 0, \qquad n \to \infty, \quad n \in \mathbb{N}, \quad a > 0.$$

D) When $x \to 0+$ we also have that a *power function* dominates the *logarithm:*

$$x^{\alpha} \ln x \to 0-, \qquad \text{for } x \to 0+, \quad \alpha > 0.$$

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