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Real Functions in Several Variables: Volume X

Vector Fields I

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Volume X Vector Fields I Tangential Line Integral and

Gradient Fields Gauß's Theorem

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Preface

The topic of this series of books on *"Real Functions in Several Variables"* is very important in the description in e.g. *Mechanics* of the real 3-dimensional world that we live in. Therefore, we start from the very beginning, modelling this world by using the coordinates of $\mathbb{R}^3$ to describe e.g. a motion in space. There is, however, absolutely no reason to restrict ourselves to $\mathbb{R}^3$ alone. Some motions may be rectilinear, so only $\mathbb{R}$ is needed to describe their movements on a line segment. This opens up for also dealing with $\mathbb{R}^2$, when we consider plane motions. In more elaborate problems we need higher dimensional spaces. This may be the case in *Probability Theory* and *Statistics*. Therefore, we shall in general use $\mathbb{R}^n$ as our abstract model, and then restrict ourselves in examples mainly to $\mathbb{R}^2$ and $\mathbb{R}^3$.

For rectilinear motions the familiar *rectangular coordinate system* is the most convenient one to apply. However, as known from e.g. *Mechanics*, circular motions are also very important in the applications in engineering. It becomes natural alternatively to apply in $\mathbb{R}^2$ the so-called *polar coordinates* in the plane. They are convenient to describe a circle, where the rectangular coordinates usually give some nasty square roots, which are difficult to handle in practice.

Rectangular coordinates and polar coordinates are designed to model each their problems. They supplement each other, so difficult computations in one of these coordinate systems may be easy, and even trivial, in the other one. It is therefore important always in advance carefully to analyze the geometry of e.g. a domain, so we ask the question: Is this domain best described in rectangular or in polar coordinates?

Sometimes one may split a problem into two subproblems, where we apply rectangular coordinates in one of them and polar coordinates in the other one.

It should be mentioned that in *real life* (though not in these books) one cannot always split a problem into two subproblems as above. Then one is really in trouble, and more advanced mathematical methods should be applied instead. This is, however, outside the scope of the present series of books.

The idea of polar coordinates can be extended in two ways to $\mathbb{R}^3$. Either to *semi-polar* or *cylindric coordinates*, which are designed to describe a cylinder, or to *spherical coordinates*, which are excellent for describing spheres, where rectangular coordinates usually are doomed to fail. We use them already in daily life, when we specify a place on Earth by its longitude and latitude! It would be very awkward in this case to use rectangular coordinates instead, even if it is possible.

Concerning the contents, we begin this investigation by modelling point sets in an n-dimensional Euclidean space E^n by $\mathbb{R}^n$. There is a subtle difference between E^n and $\mathbb{R}^n$, although we often identify these two spaces. In E^n we use *geometrical methods* without a coordinate system, so the objects are independent of such a choice. In the coordinate space $\mathbb{R}^n$ we can use ordinary calculus, which in principle is not possible in E^n. In order to stress this point, we call E^n the "abstract space" (in the sense of calculus; not in the sense of geometry) as a warning to the reader. Also, whenever necessary, we use the colour black in the "abstract space", in order to stress that this expression is theoretical, while variables given in a chosen coordinate system and their related concepts are given the colours blue, red and green.

We also include the most basic of what mathematicians call *Topology*, which will be necessary in the following. We describe what we need by a function.

Then we proceed with limits and continuity of functions and define continuous curves and surfaces, with parameters from subsets of $\mathbb{R}$ and $\mathbb{R}^2$, resp..

Continue with (partial) differentiable functions, curves and surfaces, the chain rule and Taylor's formula for functions in several variables.

We deal with maxima and minima and extrema of functions in several variables over a domain in $\mathbb{R}^n$. This is a very important subject, so there are given many worked examples to illustrate the theory.

Then we turn to the problems of integration, where we specify four different types with increasing complexity, *plane integral, space integral, curve (or line) integral and surface integral*.

Finally, we consider *vector analysis*, where we deal with vector fields, Gauß's theorem and Stokes's theorem. All these subjects are very important in theoretical Physics.

The structure of this series of books is that each subject is *usually* (but not always) described by three successive chapters. In the first chapter a brief theoretical theory is given. The next chapter gives some practical guidelines of how to solve problems connected with the subject under consideration. Finally, some worked out examples are given, in many cases in several variants, because the standard solution method is seldom the only way, and it may even be clumsy compared with other possibilities.

I have as far as possible structured the examples according to the following scheme:

A *Awareness*, i.e. a short description of what is the problem.

D *Decision*, i.e. a reflection over what should be done with the problem.

I *Implementation*, i.e. where all the calculations are made.

C *Control*, i.e. a test of the result.

This is an ideal form of a general procedure of solution. It can be used in any situation and it is not linked to Mathematics alone. I learned it many years ago in the Theory of Telecommunication in a situation which did not contain Mathematics at all. The student is recommended to use it also in other disciplines.

From high school one is used to immediately to proceed to **I**. *Implementation*. However, examples and problems at university level, let alone situations in real life, are often so complicated that it in general will be a good investment also to spend some time on the first two points above in order to be absolutely certain of what to do in a particular case. Note that the first three points, **ADI**, can always be executed.

This is unfortunately not the case with **C** *Control*, because it from now on may be difficult, if possible, to check one's solution. It is only an extra securing whenever it is possible, but we cannot include it always in our solution form above.

I shall on purpose not use the logical signs. These should in general be avoided in Calculus as a shorthand, because they are often (too often, I would say) misused. Instead of $\wedge$ I shall either write "and", or a comma, and instead of $\vee$ I shall write "or". The arrows $\Rightarrow$ and $\Leftrightarrow$ are in particular misunderstood by the students, so they should be totally avoided. They are not telegram short hands, and from a logical point of view they usually do not make sense at all! Instead, write in a plain language what you mean or want to do. This is difficult in the beginning, but after some practice it becomes routine, and it will give more precise information.

When we deal with multiple integrals, one of the possible pedagogical ways of solving problems has been to colour variables, integrals and upper and lower bounds in blue, red and green, so the reader by the colour code can see in each integral what is the variable, and what are the parameters, which do not enter the integration under consideration. We shall of course build up a hierarchy of these colours, so the order of integration will always be defined. As already mentioned above we reserve the colour black for the theoretical expressions, where we cannot use ordinary calculus, because the symbols are only shorthand for a concept.

The author has been very grateful to his old friend and colleague, the late Per Wennerberg Karlsson, for many discussions of how to present these difficult topics on real functions in several variables, and for his permission to use his textbook as a template of this present series. Nevertheless, the author has felt it necessary to make quite a few changes compared with the old textbook, because we did not always agree, and some of the topics could also be explained in another way, and then of course the results of our discussions have here been put in writing for the first time.

The author also adds some calculations in MAPLE, which interact nicely with the theoretic text. Note, however, that when one applies MAPLE, one is forced first to make a geometrical analysis of the domain of integration, i.e. apply some of the techniques developed in the present books.

The theory and methods of these volumes on "Real Functions in Several Variables" are applied constantly in higher Mathematics, Mechanics and Engineering Sciences. It is of paramount importance for the calculations in *Probability Theory*, where one constantly integrate over some point set in space.

It is my hope that this text, these guidelines and these examples, of which many are treated in more ways to show that the solutions procedures are not unique, may be of some inspiration for the students who have just started their studies at the universities.

Finally, even if I have tried to write as careful as possible, I doubt that all errors have been removed. I hope that the reader will forgive me the unavoidable errors.

Leif Mejlbro
March 21, 2015

Introduction to volume X, Vector Fields I; Gauß's Theorem

This is the tenth volume in the series of books on *Real Functions in Several Variables*.

It is the first volume on Vector Fields. It was necessary to split the material into three volumes because the material is very big. In this first volume we deal with the *tangential line integral*, which e.g. can be used to describe the work of a particle when it is forced along a given curve by some force. It is here natural to introduce the *gradient fields*, where the tangential line integral only depends on the initial and the terminal points of the curve and not of the curve itself. Such gradients fields are describing *conservative forces* in Physics.

Tangential line integrals are one-dimensional in nature. In case of two dimensions we consider the *flux* of a flow through a surface. When the surface $\partial\Omega$ is surrounding a three dimensional body Ω, this leads to *Gauß's theorem*, by which we can express the flux of a vector field $\mathbf{V}$ through $\partial\Omega$, which is a surface integral, by a space integral over Ω of the *divergence* of the vector field $\mathbf{V}$. This theorem works both ways. Sometimes, and most frequently, the surface integral is expressed as space integral, other times we express a space integral as a flux, i.e. a surface integral. Applications are obvious in *Electro-Magnetic Field Theory*, though other applications can also be found.

The present volume should be followed by reading *Volume XI, Vector Fields II*, in which we define the rotation of a vector field $\mathbf{V}$ in the ordinary three dimensional space $\mathbb{R}^3$ and then describe *Stokes's theorem*. We shall also consider the so-called *nabla calculus*, which more or less shows that the theorems mentioned above follow the same abstract structure.

Gauß's and Stokes's theorems have always been considered as extremely difficult to understand for the reader. Therefore we have given lots of examples of worked out problems.

32 Tangential line integrals

32.1 Introduction

We shall in this book introduce the analogues of the *differential and integral calculus* for functions in one variable, extending the theory to vector fields. Since we are dealing with *fields*, we give ordinary functions the name *scalar fields*.

The main issue will be to extend the following equivalent rules for a function $F : [a, b] \to \mathbb{R}$, where we assume that its derivative $F' : [a, b] \to \mathbb{R}$ exists and is continuous (of course half tangents at the endpoints). The first one is

$$\int_a^b F'(x)\,\mathrm{d}x = F(b) - F(a).$$

In this one-dimensional version this well-known formula can also be interpreted in the following way. To the left the interval of integration $[a, b] \subset \mathbb{R}$ has the boundary $\partial[a, b] = \{a, b\}$ consisting of the two endpoints, a and b. Therefore, when we move from left to right, the ordinary integration of the derivative $F'(x)$ over the interval $[a, b]$ is replaced by the right hand side, where we in some sense (to be defined later on) "integrate" the function $F(x)$ itself (without being differentiated) over the two boundary points $\partial[a, b] = \{a, b\}$. This is a geometrical/topological idea combined with measure theory. We shall deal with the problem of how to generalize the above to all the various forms of integrals, which we have already met, i.e. to line, plane, space and surface integrals.

The second rule, which we want to generalize to functions or vector fields in several variables, is, given F as above,

$$F(x) = F(a) + \int_a^x F'(\xi)\,\mathrm{d}\xi.$$

In this case we may expect some reconstruction formulæ of a scalar or vector field, given its derivatives. We may of course also expect some difficulties in this process, because for the time being it is not obvious how the partial derivatives of $F(\mathbf{x})$ (a function in several variables) should enter the right hand side of the generalization of the equation above.

To ease matters, we shall only specify the domains and the order of differentiability needed of the scalar or vector fields under consideration in important definitions and theorems. Otherwise, when these properties are not explicitly described, we shall tacitly assume that $F(\mathbf{x})$, or $\mathbf{F}(\mathbf{x})$, is of class C^∞, so it is always allowed to interchange the order of differentiation. Also, in these cases, the domain will always be a nice one.

Since this chapter in particular is supporting physical theories, we shall in most cases only consider domains which lie in either $\mathbb{R}^3$ or $\mathbb{R}^2$.

32.2 The tangential line integral. Gradient fields.

The tangential line integral is introduced in Physics, when we shall calculate e.g the work, which a force executes on a particle bound to a fixed curve. Let $\mathbf{V}$ denote the *force* (given as a field in the space), and let $\mathcal{F}$ be a given curve in space of a given parametric description, so we can determine its *tangent vector field* $\mathbf{t}$. If $\mathrm{d}s$ denotes the infinitesimal length element on $\mathcal{K}$, then the infinitesimal work done by $\mathbf{V}$ on a unit particle at $\mathbf{x} \in \mathcal{K}$ must be $\mathbf{V} \cdot \mathbf{t}\,\mathrm{d}s$, cf. Figure 32.1.

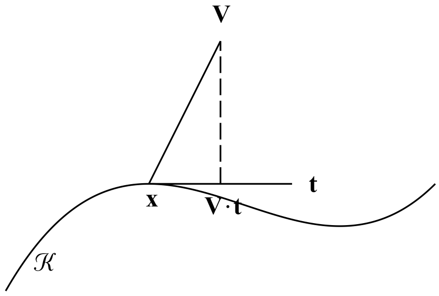

Figure 32.1: Geometrical analysis of the tangential line integral. Here $\mathbf{t}$ is the unit tangent vector field to the curve $\mathcal{K}$, and $\mathbf{V}$ is a vector field, where we are going to integrate the dot product $\mathbf{V}\cdot\mathbf{t}$ along $\mathcal{K}$.

We get the total work done of $\mathbf{V}$ on this unit particle by integrating along the curve $\mathcal{K}$, a process we denote by anyone of the symbols

$$\int_{\mathcal{K}} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s, \qquad \text{or} \qquad \int_{\mathcal{K}} \mathbf{V}\cdot\mathrm{d}\mathbf{x}, \qquad \text{or} \qquad \int_{\mathcal{K}} \mathbf{V}(\mathbf{x})\cdot\mathrm{d}\mathbf{x},$$

depending on the context. Note the appearance of the dot product.

If $\mathbf{V}$ instead denotes an electrical field, then the tangential line integral along $\mathcal{K}$ is equal to the difference in potential between the end point and the initial point, provided that we can neglect the contribution from inductance.

Assume that the curve $\mathcal{K}$ has the parametric description $\mathbf{x} = \mathbf{r}(\tau)$, where $\mathbf{r} : [\alpha, \beta] \to \mathbb{R}^n$ is a C^1 vector field. If furthermore, $\mathbf{r}'(\tau) \neq \mathbf{0}$, then the unit tangent vector field is given by

$$\mathbf{t} = \frac{\mathbf{r}'(\tau)}{\|\mathbf{r}'(\tau)\|}, \qquad \text{and} \qquad \mathrm{d}s = \|\mathbf{r}'(\tau)\|\,\mathrm{d}\tau, \qquad \text{hence} \qquad \mathrm{d}\mathbf{x} = \mathbf{r}'(\tau)\,\mathrm{d}\tau.$$

Since $\|\mathbf{r}'(\tau)\|$ is cancelled by this process, we may allow that we in some points have $\mathbf{r}'(\tau) = \mathbf{0}$, as long as this set is small. We quote without proof,

Theorem 32.1 Reduction of a tangential line integral. *Assume that $\mathcal{K}$ is an oriented continuous and piecewise C^1 curve in the domain $A \subseteq \mathbb{R}^n$, given by the parametric description $\mathbf{r} : [\alpha, \beta] \to \mathbb{R}^n$, where $\mathbf{r}$ is injective almost everywhere, and where $\mathbf{r}' \neq \mathbf{0}$ also almost everywhere.*
Let $\mathbf{V} : A \to \mathbb{R}^n$ be a C^0 vector field. Then we have the following reduction of the tangential line integral of $\mathbf{V}$ along $\mathcal{K}$,

$$\int_{\mathcal{K}} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s = \int_{\alpha}^{\beta} \mathbf{V}(\mathbf{r}(\tau))\cdot\mathbf{r}'(\tau)\,\mathrm{d}\tau.$$

The abstract integral in blue is to the left, and the ordinary 1-dimensional integral (in black), which can be calculated, is to the right. We note that we introduce a compensating factor to the integrand in the dot product to the right.

Clearly, the value of the integral changes its sign, when the orientation of the curve is reversed, or, if the particle is moved in the opposite direction.

The tangential line integral is also called the *current of the vector field along the curve.*

Example 32.1 The following simple example is only illustrating the methods. It will probably never be met in practice.
Given the vector field

$$\mathbf{V}(x,y,z) = \left(2x, e^{-a}+z, yz\right), \qquad \text{for } (x,y,z) \in \mathbb{R}^3.$$

We shall show how we find its current along the curve $\mathcal{K}$ of the parametric description

$$\mathcal{K}: \quad (x,y,z) = \mathbf{r}(\tau) = \left(\ln\tau, \tau^3, \frac{1}{\tau}\right), \qquad \text{for } \tau \in [1,2].$$

We first calculate

$$\mathbf{r}'(\tau) = \left(\frac{1}{\tau}, 3\tau^2, -\frac{1}{\tau^2}\right) \qquad \text{and} \qquad \mathbf{V}(\mathbf{r}(\tau)) = \left(2\ln t, \frac{2}{\tau}, t^2\right).$$

Then the current C of $\mathbf{V}$ along $\mathcal{K}$ is given by

$$\begin{aligned} C \quad &:= \int_{\mathcal{K}} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s = \int_1^2 \mathbf{V}(\mathbf{r}(\tau))\cdot\mathbf{r}'(\tau)\,\mathrm{d}\tau \\ &= \int_1^2 \left\{2\,\frac{\ln\tau}{\tau} + 6\tau - 1\right\}\mathrm{d}\tau = \left[(\ln\tau)^2 + 3\tau^2 - \tau\right]_{\tau=1}^2 = (\ln 2)^2 + 8. \qquad \diamond \end{aligned}$$

When we use rectangular coordinates in $\mathbb{R}^3$ we also write

$$\mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s = \mathbf{V}(\mathbf{x})\cdot\mathrm{d}\mathbf{x} = (V_x, V_y, V_z)\cdot(\mathrm{d}x, \mathrm{d}y, \mathrm{d}z) = V_x\,\mathrm{d}x + V_y\,\mathrm{d}y + V_z\,\mathrm{d}z,$$

where we have put $\mathbf{V} = (V_x, V_y, V_z)$ and $(\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$ in rectangular coordinates. In this case the result of Theorem 32.1 is written

$$\int_{\mathcal{K}} V_x\,\mathrm{d}x + V_y\,\mathrm{d}y + V_z\,\mathrm{d}z = \int_\alpha^\beta \left\{V_x\,\frac{\mathrm{d}x}{\mathrm{d}\tau} + V_y\,\frac{\mathrm{d}y}{\mathrm{d}\tau} + V_z\,\frac{\mathrm{d}z}{\mathrm{d}\tau}\right\}\mathrm{d}\tau,$$

and similarly for rectangular coordinates in the general space $\mathbb{R}^n$.

An important special case, is when $\mathcal{K}$ is a *closed* curve, i.e. its endpoints coincide. In this case the tangential line integral is called the *circulation* of the vector field $\mathbf{V}$ alont $\mathcal{K}$, and it is denoted

$$\oint_{\mathcal{K}} \mathbf{V}(\mathbf{x})\cdot\mathrm{d}\mathbf{x}, \qquad \text{or e.g.} \quad \oint_{\mathcal{K}} V_x\,\mathrm{d}x + V_y\,\mathrm{d}y + V_z\,\mathrm{d}z.$$

The shall below consider the important vector fields $\mathbf{V}$ (the *gradient fields*), for which the circulation is 0, no matter the choice of an admissible curve $\mathcal{K}$ in the definition of the circulation. But first we include a small exercise,

Example 32.2 Consider again the vector field

$$\mathbf{V}(x,y,z) = \left(2x, e^{-a} + z, yz\right), \qquad \text{for } (x,y,z) \in \mathbb{R}^3,$$

from Example 32.1, and let $\mathcal{K}$ be the circle given by the parametric description

$$\mathcal{K}: \quad (x,y,z) = \mathbf{r}(\tau) = (1, \cos\tau, \sin\tau), \qquad \tau \in [0, 2\pi[.$$

Then

$$\mathbf{r}'(\tau) = (0, -\sin\tau, \cos\tau), \qquad \text{and} \qquad \mathbf{V}(\mathbf{r}(\tau)) = \left(2, \frac{1}{e} + \sin\tau, \cos\tau\sin\tau\right),$$

and the circulation becomes

$$C := \int_0^{2\pi} \mathbf{V}(\mathbf{r}(\tau)) \cdot \mathbf{r}'(\tau)\,\mathrm{d}\tau = \int_0^{2\pi} \left\{0 - \frac{\sin\tau}{e} - \sin^2\tau + \cos^2\tau\right\} \mathrm{d}\tau = -\pi \qquad \Diamond$$

We then introduce the *gradient fields*, i.e. vector fields $\mathbf{V}$, for which there exists a C^1 function F, such that

$$\mathbf{V} = \triangledown F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right).$$

We get by the *chain rule* (cf. Section 9.2) that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\{F(\mathbf{r}(\tau))\} = \triangledown F(\mathbf{r}(\tau)) \cdot \mathbf{r}'(\tau),$$

so by the reduction theorem and then by inserting this equation from the right to the left,

$$\begin{aligned}\int_{\mathcal{K}} \triangledown F(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} &= \int_\alpha^\beta \triangledown F(\mathbf{r}(\tau)) \cdot \mathbf{r}'(\tau)\,\mathrm{d}\tau = \int_\alpha^\beta \frac{\mathrm{d}}{\mathrm{d}\tau}\{F(\mathbf{r}(\tau))\}\,\mathrm{d}\tau \\ &= [F(\mathbf{r}(\tau))]_{\tau=\alpha}^{\beta} = F(\mathbf{r}(\beta)) - F(\mathbf{r}(\alpha)).\end{aligned}$$

In other words, for gradient fields the value of the tangential line integral along $\mathcal{K}$ only depends on the endpoints and not on the permitted curve joining the endpoints. Thus, if $\mathcal{K}_1$ and $\mathcal{K}_2$ are two permitted curves between the same endpoints, then

$$\int_{\mathcal{F}_1} \triangledown F(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} = \int_{\mathcal{F}_2} \triangledown F(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} = F(\text{final point}) - F(\text{initial point}).$$

This result is coined in the following theorem (as usual without its full proof)

Theorem 32.2 The gradient integral theorem. *Given a C^1 function $F : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^2$, and let $\mathbf{a}$, $\mathbf{b} \in A$. then*

$$\int_{\mathcal{K}} \triangledown F(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} = F(\mathbf{b}) - F(\mathbf{a})$$

for every continuous and piecewise C^1 curve $\mathcal{K}$ lying in A with initial point $\mathbf{a} \in A$ and final point $\mathbf{b} \in A$.

The reader who is familiar with the *Theory of Complex Functions* will in case of $n = 2$ recognize this as connected with analytic functions. In Physics, the gradient field $\triangledown F$ in $\mathbb{R}^2$ and $\mathbb{R}^3$ is interpreted as a *conservative vector field.*

We shall now prove the important *circulation theorem.*

If we choose $\mathcal{K}$ as any permitted curve in A from a point $\mathbf{a} \in A$ to another point $\mathbf{x} \in A$, and the gradient field $\triangledown F$ is given in A, then we get by a rearrangement of the result of Theorem 32.2,

$$F(\mathbf{x}) = F(\mathbf{a}) + \int_{\mathbf{a}}^{\mathbf{x}} \triangledown F(\mathbf{u}) \cdot \mathrm{d}\mathbf{u},$$

so we can reconstruct $F(\mathbf{x})$, using our knowledge of $\mathbf{V}(\mathbf{u}) = \triangledown F(\mathbf{u})$. Note that we are strictly speaking only given that $\mathbf{V}(\mathbf{u})$ is a gradient field, so to begin with we only know the existence of the function F, so the right formulation of the above would be that

$$F(\mathbf{x}) = F(\mathbf{a}) + \int_{\mathbf{a}}^{\mathbf{x}} \mathbf{V}(\mathbf{u}) \cdot \mathrm{d}\mathbf{u},$$

where it is given that $\mathbf{V}$ is a gradient field.

We note that if furthermore the endpoints coincide, $\mathbf{a} = \mathbf{b}$, the curve $\mathcal{K}$ is closed, so the circulation is for gradient fields,

$$C = \oint_{\mathcal{K}} \triangledown F(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} = 0,$$

and we have proved that the circulation of a gradient field along any closed curve is always 0.

Then we prove the opposite, namely that if the circulation of $\mathbf{V}$ along every closed curve in the open domain A of $\mathbf{V}$ is zero, then $\mathbf{V}$ is a gradient field. The idea is of course to construct the function F and then prove that it is indeed a primitive of $\mathbf{V}$.

We choose a fixed point $\mathbf{a} \in A$, i.e. the open domain of $\mathbf{V}$, and we let $\mathbf{x} \in A$ be any other (variable) point in A. Since by assumption

$$\oint_{\mathcal{K}} \mathbf{V}(\mathbf{u}) \cdot \mathrm{d}\mathbf{u} = 0$$

for every closed (permitted) curve $\mathcal{K}$ in A, it follows that the tangential line integral from $\mathbf{a}$ to $\mathbf{x}$ is independent of the integration path from $\mathbf{a}$ to $\mathbf{x}$.

In fact, let $\mathcal{K}_1$ and $\mathcal{K}_2$ be any two paths from $\mathbf{a}$ to $\mathbf{x}$, and let $-\mathcal{K}_2$ denote the path from $\mathbf{x}$ to $\mathbf{a}$ of $\mathcal{K}_2$ in the reversed direction. Then the concatenated curve $\mathcal{K} := \mathcal{K}_1 - \mathcal{K}_2$ is closed, so by splitting the integral,

$$0 = \oint_{\mathcal{K}} \mathbf{V}(\mathbf{u}) \cdot \mathrm{d}\mathbf{u} = \int_{\mathcal{K}_1} \mathbf{V}(\mathbf{u}) \cdot \mathrm{d}\mathbf{u} - \int_{\mathcal{K}_2} \mathbf{V}(\mathbf{u}) \cdot \mathrm{d}\mathbf{u},$$

and it follows by a rearrangement, that the value of the integral of the differential form $\mathbf{V}(\mathbf{u}) \cdot \mathrm{d}\mathbf{u}$ does not depend on the path from $\mathbf{a}$ to $\mathbf{x}$.

We can therefore unambiguously define the function

$$F(\mathbf{x}) := \int_{\mathbf{a}}^{\mathbf{x}} \mathbf{V}(\mathbf{u}) \cdot \mathrm{d}\mathbf{u},$$

where we can choose any (permitted) integration path from $\mathbf{a}$ to $\mathbf{x}$.

The increase of this function is the difference

$$\Delta F = F(\mathbf{x}+\mathbf{h}) - F(\mathbf{x}) = \int_{\mathbf{x}}^{\mathbf{x}+\mathbf{h}} \mathbf{V}(\mathbf{u}) \cdot \mathrm{d}\mathbf{u}.$$

Since A was assumed to be an open domain, and $\mathbf{x} \in A$, we can choose $r > 0$, such that $\mathbf{x}+\mathbf{h} \in A$, whenever $\|\mathbf{h}\| < r$. Then the whole line segment $[\mathbf{x};\mathbf{x}+\mathbf{h}]$ lies in A, whenever $0 < \|\mathbf{x}\| < r$, which we assume in the following. When we integrate along this line segment, it follows from the *mean value theorem*, cf. e.g. Section 9.5 or Section 20.2, that there exist numbers $\theta_1, \dots, \theta_n \in]0,1[$, such that

$$\Delta F = \int_0^1 \mathbf{V}(\mathbf{x}+\tau\mathbf{h}) \cdot \mathbf{h} \, \mathrm{d}\tau = \sum_{i=1}^n h_i \int_0^1 V_i(\mathbf{x}+\tau\mathbf{h}) \, \mathrm{d}\tau - \sum_{i=1}^n h_i V_i\left(\mathbf{x}+\theta_i\mathbf{h}\right).$$

When we add and subtract the right term, $\mathbf{h} \cdot \mathbf{V}(\mathbf{x})$, then

$$\Delta F = \mathbf{h} \cdot \mathbf{V}(\mathbf{x}) = \sum_{i=1}^n h_i V_i\left(\mathbf{x}+\theta_i\mathbf{h}\right) - \sum_{i=1}^n h_i V_i(\mathbf{x}) = \mathbf{h} \cdot \mathbf{V}(\mathbf{x}) + \sum_{i=1}^n h_i \left\{ V_i\left(\mathbf{x}+\theta_i\mathbf{h}\right) - V_i(\mathbf{x}) \right\}.$$

Since $\mathbf{V}$ is continuous, and all $\theta_i \in]0,1[$, it follows that

$$\sum_{i=1}^{n} h_i \left\{ V_i\left(\mathbf{x}+\theta_i \mathbf{h}\right) - V_i(\mathbf{x}) \right\} = \|\mathbf{h}\| \sum_{i=1}^{n} \frac{h_i}{\|\mathbf{h}\|} \left\{ V_i\left(\mathbf{x}+\theta_i \mathbf{h}\right) - V_i(\mathbf{x}) \right\} = \|\mathbf{h}\|\varepsilon(\mathbf{h}),$$

where

$$\varepsilon(\mathbf{h}) = \sum_{i=1}^{n} \frac{h_i}{\|\mathbf{h}\|} \left\{ V_i\left(\mathbf{x}+\theta_i \mathbf{h}\right) - V_i(\mathbf{x}) \right\} \to 0 \qquad \text{for } \mathbf{h} \to \mathbf{0},$$

because $|h_i/\|\mathbf{x}\|| \leq 1$ is bounded, and because $\mathbf{V}$ is continuous. We therefore conclude that the constructed function F is differentiable of the gradient $\triangledown F = \mathbf{V}$.

Hence, we have proved

Theorem 32.3 The circulation theorem. *A C^0 vector field $\mathbf{V}$ on A is a* gradient field*, if and only if the circulation is 0 for every closed permitted curve $\mathcal{K}$ contained in A,*

$$\oint_{\mathcal{K}} \mathbf{V} \cdot \mathrm{d}\mathbf{x} = 0.$$

In practice it is only possible to use Theorem 32.3 to prove that a given vector field is *not* a gradient field. The vector field in Example 32.2 is therefore not a gradient field, because we have found a closed curve, along which the circulation is $-\pi \neq 0$.

The circulation does not always have to be zero in important applications. If e.g. $\mathbf{H}$ denotes a *magnetic field*, an $\mathcal{K}$ is a closed curve, then *Ampère's law* says that the circulation of $\mathbf{H}$ along $\mathcal{K}$ is given by

$$\oint_{\mathcal{K}} \mathbf{H} \cdot \mathbf{t}\, \mathrm{d}s = I,$$

where I is the current, which is linked by the closed curve $\mathcal{K}$. Since in general, $I \neq 0$, this means that the magnetic field is not a gradient field.

We shall then derive some other criteria which assure that a given C^1 vector field $\mathbf{V}$ is a gradient field. Assume to begin with that $\mathbf{V}$ is a gradient field. Then there exists a scalar field F, such that $\mathbf{V} = \triangledown F$. We then call the scalar field F a *primitive* of the vector field $\mathbf{V}$, or of the differential form $\mathbf{V}(\mathbf{x}) \cdot \mathbf{x}$.

Clearly, if $\mathbf{V}$ has the primitive F, then all primitives o $\mathbf{V}$ are given by $F + c$, where $c \in \mathbb{R}$ is an arbitrary constant.

Let us furthermore assume that $\mathbf{V}$ is a C^1 gradient field (and not just C^0) with the C^2 primitive F. Then we have in coordinates

$$\left(V_1, V_2, \dots, V_n\right) = \left(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \dots, \frac{\partial F}{\partial x_n} \right), \qquad \text{or} \qquad V_i = \frac{\partial F}{\partial x_i}, \quad i = 1, \dots, n,$$

so interchanging the order of differentiation, which we may, because $F \in C^2$,

$$\frac{\partial V_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\partial F}{\partial x_i} \right) = \frac{\partial^2 F}{\partial x_j \partial x_i} = \frac{\partial^2 F}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial F}{\partial x_j} \right) = \frac{\partial V_j}{\partial x_i}.$$

So whenever $\mathbf{V}$ is a C^1 vector field, a necessary condition for $\mathbf{V}$ being a gradient field is that

$$\frac{\partial V_i}{\partial x_j} = \frac{\partial V_j}{\partial x_i} \qquad \text{for all } i,\, j \in \{1, \dots, n\}. \tag{32.1}$$

Whenever (32.1) holds, we call

$$\mathbf{V} \cdot \mathrm{d}\mathbf{x} = V_1\,\mathrm{d}x_1 + \cdots + V_n\,\mathrm{d}x_n$$

a *closed differential form*. Thus the differential of a C^2 gradient field is always a closed differential form.

Unfortunately, this necessary condition is not sufficient. We need an extra condition on the domain A of $\mathbf{V}$, namely that A is *simply connected*, cf. Section 5.9.

Simply connected domains are easy to describe in $\mathbb{R}^2$. Let $A \subseteq \mathbb{R}^2$ be a connected plane set. Every closed bounded curve $\mathcal{K}$ in $\mathbb{R}^2$ divides the plane into three mutually disjoint sets, the curve $\mathcal{K}$ itself, the outer and unbounded open set B_1, and the inner and bounded open set B_2. We say that A is *simply connected*, if for every closed curve $\mathcal{K}$ in A, the inner bounded set B_2 by this division is contained in A, thus $B_2 \subset A$. This is very easy to visualize on a figure. The typical example of a connected plane set, which is not simply connected, is $\mathbb{R}^2 \setminus \{\mathbf{0}$, because if we as $\mathcal{K}$ choose the unit circle, then the point $\mathbf{0}$ lies inside $\mathcal{K}$ and not in $A = \mathbb{R}^2 \setminus \{\mathbf{0}\}$.

In higher dimensions simply connected sets are more difficult to visualize. For instant the set $\mathbb{R}^3 \setminus \{\mathbf{0}\}$ is simply connected. The problem is of cause that we, opposite to the plane case, cannot define precisely what lies inside a closed curve. However, at broad class of connected sets is consisting of simply connected sets, and the members are also easy to visualize, namely the *star-shaped domains*. The open domain A is called *star-shaped*, if there is a point $\mathbf{a} \in A$, such that for every other $\mathbf{x} \in A$ the straight line segment from $\mathbf{a}$ to $\mathbf{x}$ lies entirely in A, i.e.

$$[\mathbf{a}; \mathbf{x}] := \{(1-\lambda)\mathbf{a} + \lambda\frown \mid \lambda \in [0,1]\} \subseteq A.$$

We shall only formulate the following theorem for star-shaped sets, because the proof here is fairly simple, and we note that it is also true in general for simply connected sets,

Theorem 32.4 The primitive of a gradient field. *Let $A \subseteq \mathbb{R}^n$ be a star-shaped open domain, and assume that $\mathbf{V} : A \to \mathbb{R}^n$ is a C^1 vector field, which fulfils the condition*

$$\frac{\partial V_i}{\partial x_j}(\mathbf{x}) = \frac{\partial V_j}{\partial x_i}(\mathbf{x}) \qquad \text{for all } \mathbf{x} \in A \text{ and for all } i,\, j \in \{1, \dots, n\}.$$

Then $\mathbf{V}$ is a gradient field, and a C^2 scalar primitive is defined by

$$F(\mathbf{x}) := \int_{\mathbf{a}}^{\mathbf{x}} \mathbf{V}(\mathbf{u}) \cdot \mathrm{d}\mathbf{u}, \qquad \text{for all } \mathbf{x} \in A,$$

where $\mathbf{a} \in A$ is fixed, and where we integrate along any continuous and piecewise C^1 curve lying in A and going from $\mathbf{a}$ to $\mathbf{x}$.
Every primitive of $\mathbf{V}$ is of the form $F + c$, where $c \in \mathbb{R}$ is a constant.

Proof. Given the assumptions of Theorem 32.4. Since A is star-shaped, we choose the point $\mathbf{a} \in A$, such that any other point $\mathbf{x} \in A$ can be "seen from $\mathbf{a}$ by a straight line segment lying totally in A. Using, if necessary, a translation, we may assume that $\mathbf{a} = \mathbf{0}$. We then *define*

$$F(\mathbf{x}) := \mathbf{x} \cdot \int_0^1 \mathbf{V}(\tau\mathbf{x})\, \mathrm{d}\tau, \qquad \mathbf{x} \in A,$$

where the path integral here is another way to write the line integral from $\mathbf{a} = \mathbf{0}$ to $\mathbf{x}$ alont the straight line segment $[\mathbf{0}; \mathbf{x}] \subseteq A$. Then

$$F(\mathbf{x}) = \sum_{i=1}^n x_j \int_0^1 V_i(\tau\mathbf{x})\, \mathrm{d}\tau = \sum_{i=1}^n \int_0^1 U_i(\mathbf{x}, \tau)\, \mathrm{d}\tau,$$

where we for technical reasons later on have put

$$\mathbf{U}(\mathbf{x}, \tau) := \mathbf{V}(\tau\mathbf{x}).$$

It follows from the chain rule that

$$\frac{\partial U_i}{\partial x_j} = \tau D_j V_i(\tau\mathbf{x}) \qquad \text{and} \qquad \frac{\partial U_i}{\partial \tau} = \sum_{i=1}^n x_j D_j V_i(\tau\mathbf{x}),$$

where D_jV_i denotes the derivative of the function V_i with respect to the j-th variable ($y_j = \tau x_j$). Finally,

$$\begin{aligned}
\frac{\partial F}{\partial x_j}(\mathbf{x}) &= \sum_{i=1}^{n}\left\{\frac{\partial x_i}{\partial x_j}\int_0^1 U_i\,\mathrm{d}\tau + x_i\int_0^1 \frac{\partial U_i}{\partial x_j}\,\mathrm{d}\tau\right\} = \int_0^1 U_j\,\mathrm{d}\tau + \sum_{i=1}^{n} x_i\int_0^1 \tau D_jV_i(\tau\mathbf{x})\,\mathrm{d}\tau \\
&= \int_0^1 U_j\,\mathrm{d}\tau + \int_0^1\left\{\tau\sum_{i=1}^{n} x_iD_iV_j(\tau\mathbf{x})\,\mathrm{d}\tau\right\} = \int_0^1 U_j\,\mathrm{d}\tau + \int_0^1 \tau\,\frac{\partial U_j}{\partial \tau}\,\mathrm{d}\tau = \int_0^1 \frac{\partial}{\partial \tau}\left\{\tau U_j\right\}\,\mathrm{d}\tau \\
&= \left[\tau V_j(\tau\mathbf{x})\right]_0^1 = V_j(\mathbf{x}),
\end{aligned}$$

and we have proved that $\bigtriangledown F = \mathbf{V}$, so $\mathbf{V}$ is indeed a gradient field. ♢

Example 32.3 The proof of Theorem 32.4 gives a concrete solution formula, once the assumptions have been checked. Namely, calculate the line integral of the differential form $\mathbf{V}(\mathbf{x})\cdot\mathrm{d}\mathbf{x}$ along the straight line segment $[\mathbf{a};\mathbf{x}]$. We shall demonstrate this method on the vector field

$$\mathbf{V}(x,y,z) = \left(y^2+z, 2xy+2yz^2, 2y^2z+x\right), \qquad \text{for } (x,y,z)\in\mathbb{R}^3,$$

where we have the coordinate functions

$$V_x(x,y,z) = y^2+z, \qquad V_y(x,y,z) = 2xy+2yz^2, \qquad V_z(x,y,z) = 2y^2z+x.$$

We first check

$$\frac{\partial V_x}{\partial y} = 2y = \frac{\partial V_y}{\partial x}, \qquad \frac{\partial V_x}{ddz} = 1 = \frac{\partial V_z}{\partial x}, \qquad \frac{\partial V_y}{\partial z} = 4yz = \frac{\partial V_z}{\partial y},$$

so $\mathbf{V}\cdot\mathrm{d}\mathbf{x}$ is a closed differential form. Since $A = \mathbb{R}^3$ is trivially star-shaped, it follows from Theorem 32.4 that $\mathbf{V}(\mathbf{x})$ is a gradient field.

According to the theorem, one possible solution formula is

$$F(\mathbf{x}) = \mathbf{x}\cdot\int_0^1 \mathbf{V}(\tau\mathbf{x})\,\mathrm{d}\tau, \qquad \mathbf{x}\in A,$$

where

$$\begin{aligned}
\int_0^1 \mathbf{V}(\tau x,\tau y,\tau z)\,\mathrm{d}\tau &= \int_0^1 \left(\tau^2y^2+\tau z, 2\tau^2xy+2\tau^3yz^2, 2\tau^3y^2z+\tau x\right)\,\mathrm{d}\tau \\
&= \left(\frac{1}{2}\,y^2+\frac{1}{2}\,z, \frac{2}{3}\,xy+\frac{1}{2}\,yz^2, \frac{1}{2}\,y^2z+\frac{1}{2}\,x\right),
\end{aligned}$$

so a primitive is given by

$$\begin{aligned}
F(x,y,z) &= (x,y,z)\cdot\int_0^1 \mathbf{V}(\tau x,\tau y,\tau z)\,\mathrm{d}\tau \\
&= \frac{1}{3}\,xy^2+\frac{1}{2}\,xz+\frac{2}{3}\,xy^2+\frac{1}{2}\,y^2z^2+\frac{1}{2}\,y^2z^2+\frac{1}{2}\,xz = xz+xy^2+y^2z^2.
\end{aligned}$$

Check;

$$\frac{\partial F}{\partial x} = z+y^2 = V_x, \qquad \frac{\partial F}{\partial y} = 2xy+2yz^2 = V_y, \qquad \frac{\partial F}{\partial z} = x+2y^2z = V_z,$$

so $F(x,y,z) = xz+xy^2+y^2z^2$ is indeed a primitive of $\mathbf{V}$. We then get all primitives by adding an arbitrary constant $c\in\mathbb{R}$. ♢

The method of radial integration, as in Example 32.3, often requires some hard calculations. We note, however, that we may choose other and more reasonable integration paths. A commonly used method is integration along a continuous step line, where each of the steps is parallel to one of the coordinate axes. When we describe this method we assume for convenience that we integrate from $\mathbf{0}$. If $\mathbf{a} \to \mathbf{b}$ designates that we integrate along the straight line segment between $\mathbf{a}$ and $\mathbf{b}$, then the idea is – whenever possible – to use the following paths of integration,

1) In $\mathbb{R}^2$: $\qquad (0,0) \to (x,0) \to (x,y)$.

2) In $\mathbb{R}^3$: $\qquad (0,0,0) \to (x,0,0) \to (x,y,0) \to (x,y,z)$,

We see that each arrow represents an integration along an axiparallel line segment. More explicitly,

1) In $\mathbb{R}^2$, the vector field is $\mathbf{V}(x,y) = (V_x(x,y), V_y(x,y))$, and the line integration from $(0,0)$ can be written

$$F(x,y) = \int_0^x V_x(\tau,0)\,\mathrm{d}\tau + \int_0^y V_y(x,\tau)\,\mathrm{d}\tau,$$

because $\mathbf{V}(x,y)\cdot(\,\mathrm{d}x,\,\mathrm{d}y) = V_x(x,0)\,\mathrm{d}x$ on the line segment from $(0,0)$ to $(x,0)$, since here $\mathrm{d}y = 0$, and $\mathbf{V}(x,y)\cdot(\,\mathrm{d}x,\,\mathrm{d}y) = V_y(x,y)\,\mathrm{d}y$ on the line segment from $(x,0)$ to (x,y), because here $\mathrm{d}x = 0$.

2) In $\mathbb{R}^3$ the vector field is $\mathbf{V}(x,y,z) = (V_x(x,y,z), V_y(x,y,z), V_z(x,y,z))$, so the analogue solution formula becomes

$$F(x,y,z) = \int_0^x V_x(\tau,0,0)\,\mathrm{d}\tau + \int_0^y V_y(x,\tau,0)\,\mathrm{d}\tau + \int_0^z V_z(x,y,\tau)\,\mathrm{d}\tau.$$

In some cases this step line does not lie in A, but one may modify this construction to obtain this property by choosing another axiparallel step line. It should be easy for the reader to carry out the necessary modification in such cases.

The advantage of this method is that all usual variables, except for one, are constants in each of the subintegrals. If we in particular integrate from $\mathbf{0}$, then we get lots of zeros in the integrands, so some of the terms may even disappear. We shall see this phenomenon in Example 32.4 below.

It may occur in some cases that we cannot find $F(\mathbf{x})$ everywhere in A by only using a simple step line as above, though we may get a result in a nonempty subset $B \subset A$. Then it is legal just to check by differentiation, if we indeed have $\bigtriangledown F = \mathbf{V}$ in all of A, and that solves the problem.

Example 32.4 We consider again the gradient field from Example 32.3 above, (no need to check once more that it is a gradient field),

$$\mathbf{V}(x,y,z) = \left(y^2+z,\, 2xy+2yz^2,\, 2y^2z+x\right), \qquad \text{for } (x,y,z) \in \mathbb{R}^3,$$

where we have the coordinate functions

$$V_x(x,y,z) = y^2+z, \qquad V_y(x,y,z) = 2xy+2yz^2, \qquad V_z(x,y,z) = 2y^2z+x.$$

Then by the method of step lines,

$$\begin{aligned} F(x,y,z) &= \int_0^x V_x(\tau,0,0)\,\mathrm{d}\tau + \int_0^y V_y(x,\tau,0)\,\mathrm{d}\tau + \int_0^z V_z(x,y,\tau)\,\mathrm{d}\tau \\ &= \int_0^x 0\,\mathrm{d}\tau + \int_0^y 2x\tau\,\mathrm{d}\tau + \int_0^z \left(2y^2\tau + x\right)\,\mathrm{d}\tau \\ &= 0 + \left[x\tau^2\right]_0^y + \left[y^2\tau^2 + x\tau\right]_0^z = xy^2 + y^2z^2 + xz, \end{aligned}$$

which is calculated with less effort than in the method of Example 32.3. $\Diamond$

A third method is to manipulate with the differential form $\mathbf{V}(\mathbf{x})\cdot \mathrm{d}\mathbf{x}$ by using the rules of computation of differentials in the "unusual direction" finally getting $\mathrm{d}F(\mathbf{x})$, where $F(\mathbf{x})$ is the wanted primitive. This method requires some skill, though it is also the most elegant one, because if one succeeds, then there is no need to check the assumptions of Theorem 32.4.

Example 32.5 Consider again from the two previous examples

$$\mathbf{V}(x,y,z) = \left(y^2+z, 2xy+2yz^2, 2y^2z+x\right), \qquad \text{for } (x,y,z)\in\mathbb{R}^3,$$

where we have the coordinate functions

$$V_x(x,y,z) = y^2+z, \qquad V_y(x,y,z) = 2xy+2yz^2, \qquad V_z(x,y,z) = 2y^2z+x.$$

Then the corresponding closed differential form is

$$\mathbf{V}(x,y,z)\cdot(\,\mathrm{d}x,\,\mathrm{d}y,\,\mathrm{d}x) = \left(y^2+z\right)\,\mathrm{d}x + \left(2xy+2yz^2\right)\,\mathrm{d}y + \left(2y^z+x\right)\,\mathrm{d}z.$$

The strategy is to split all the terms and then pair them, so that they can stepwise be included as the differential of some function. When we deal with polynomials we may also collect terms of the same (general) degree. In general, if e.g. we have a function $\varphi(y)$ in y alone as a factor of $\mathrm{d}y$, then use that $\varphi(y)\,\mathrm{d}y = \mathrm{d}\Phi(y)$, where $\Phi'(y)=\varphi(y)$. Similarly for the other variables.

In the present case we get, using these methods,

$$\begin{aligned}\mathbf{V}(x,y,z)\cdot(\,\mathrm{d}x,\,\mathrm{d}y,\,\mathrm{d}z) &= y^2\,\mathrm{d}x+z\,\mathrm{d}x+2xy\,\mathrm{d}y+2yz^2\,\mathrm{d}y+2y^2z\,\mathrm{d}z+x\,\mathrm{d}z\\ &= y^2\,\mathrm{d}x+z\,\mathrm{d}x+x\,\mathrm{d}\left(y^2\right)+z^2\,\mathrm{d}\left(y^2\right)+y^2\,\mathrm{d}\left(z^2\right)+x\,\mathrm{d}z\\ &= \left\{y^2\,\mathrm{d}x+x\,\mathrm{d}\left(y^2\right)\right\}+\{z\,\mathrm{d}x+x\,\mathrm{d}z\}+\left\{z^2\,\mathrm{d}\left(y^2\right)+y^2\,\mathrm{d}\left(z^2\right)\right\}\\ &= \mathrm{d}\left(xy^2\right)+d(xz)+d\left(y^2z^2\right)=\mathrm{d}\left(xy^2+xz+y^2z^2\right),\end{aligned}$$

from which we conclude that $\mathbf{V}$ has a primitive, so it is a gradient field, and that modulo a constant this primitive is given by

$$F(x,y,z)=xy^2+xz+y^2z^2.$$

We note again that this method has the advantage that if it succeeds, then it is not necessary to check the assumptions of Theorem 32.4. $\Diamond$

Example 32.6 Consider the vector field

$$\mathbf{V}(x,y)=(V_x(x,y),V_y(x,y))=\left(\frac{x}{\sqrt{x^2+y^2}},\frac{y}{\sqrt{x^2+y^2}}\right),\qquad\text{for }(x,y)\neq(0,0),$$

where the domain $A=\mathbb{R}^2\setminus\{(0,0)\}$ is not simply connected.

However, using the differential form we immediately get

$$V_x(x,y)\,\mathrm{d}x+V_y(x,,y)\,\mathrm{d}y=\frac{x}{\sqrt{x^2+y^2}}\,\mathrm{d}x+\frac{y}{\sqrt{x^2+y^2}}\,\mathrm{d}y=\frac{1}{2}\,\frac{\mathrm{d}\left(x^2+y^2\right)}{\sqrt{x^2+y^2}}=\mathrm{d}\left(\sqrt{x^2+y^2}\right),$$

so $\mathbf{V}$ is a gradient field in A, and all its primitives are given by

$$F(x,y)=\sqrt{x^2+y^2}+c,\qquad\text{where }c\in\mathbb{R}\text{ is an arbitrary constant.}$$

Alternatively, we first note that

$$\frac{\partial V_x}{\partial y}=\frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2}}\right)=-\frac{1}{2}\,\frac{x}{x^2+y^2}\,\frac{y}{\sqrt{x^2+y^2}}=\frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2}}\right)=\frac{\partial V_y}{\partial x},$$

so $\mathbf{V}(x,y)\cdot(\,\mathrm{d}x,\,\mathrm{d}y)$ is closed, and $\mathbf{V}(x,y)$ is a gradient field in every star-shaped domain contained in A.

One may choose the right half plane $x>0$ as our subdomain. Here we can use the step line,

$$(1,0)\to(x,0)\to(x,y),\qquad x>0,$$

so by the solution formula,

$$\begin{aligned}F(x,y) &= \int_1^x V_x(\tau,0)\,\mathrm{d}\tau+\int_0^y V_y(x,\tau)\,\mathrm{d}\tau\\ &= \int_1^x \mathrm{d}\tau+\int_0^y\frac{\tau}{\sqrt{x^2+\tau^2}}\,\mathrm{d}\tau=x-1+\left[\sqrt{x^2+\tau^2}\right]_0^y\\ &= x-1+\sqrt{x^2+y^2}-x=\sqrt{x^2+y^2}-1\qquad\text{for }x>0.\end{aligned}$$

However, $F(x,y)=\sqrt{x^2+y^2}-1$ is C^1 in $\mathbb{R}^2\setminus\{(0,0)\}$, and

$$\frac{\partial F}{\partial x}(x,y)=\frac{x}{\sqrt{x^2+y^2}}=V_x(x,y),\qquad \frac{\partial F}{\partial y}(x,y)=\frac{y}{\sqrt{x^2+y^2}}=V_y(x,y),$$

so we have checked that the result, which was only derived for $x>0$ also holds in all of $\mathbb{R}^2\setminus\{(0,0)\}$.

Note also that we get all primitives by adding an arbitrary constant, so it is no error above that we get $\sqrt{x^2+y^2}$ in the first method, and $\sqrt{x^2+y^2}-1$ in the second one. $\Diamond$

32.3 Tangential line integrals in Physics

Consider a unit particle which moves along a curve $\mathcal{K}$ under the action of a force $\mathbf{F}(\mathbf{x})$. Then the *work* done by this force is given by the tangential line integral

$$W=\int_{\mathcal{K}}\mathbf{F}(\mathbf{x})\cdot\,\mathrm{d}\mathbf{x}.$$

If $\mathbf{F}=-\bigtriangledown E_p$ is a *gradient field*, then the work is independent of the path, so

$$\int_{\mathcal{K}}\mathbf{F}(\mathbf{x})\cdot\,\mathrm{d}\mathbf{x}=E_p(A)-E_p(B),$$

where A is the initial point of $\mathcal{K}$ and B is the final point, The function E_p with the conventional minus sign in front of it, is the *potential energy*.

A force $\mathbf{F}$, which is also a gradient field, is in *Physics* called a *conservative force*.

The tangential line integrals are especially used in *Electro-magnetic Field Theory*. An electric field $\mathcal{E}=\mathcal{E}(\mathbf{x},t)$, where t is the time variable, describes the force per unit charge, so when one unit of charge is moved along the curve $\mathcal{K}$, then the work done by $\mathcal{E}(\mathbf{x},t)$ is equal to the tangential line integral

$$W=\int_{\mathcal{K}}\mathcal{E}(\mathbf{x},t)\cdot\,\mathrm{d}\mathbf{x}.$$

If $\mathcal{K}$ is closed, we get the circulation of the electric field along $\mathcal{K}$. This is also called the *electromotive force* (emf) applied to the closed path $\mathcal{K}$,

$$\mathrm{emf}=\oint_{\mathcal{K}}\mathcal{E}(\mathbf{x},t)\cdot\,\mathrm{d}\mathbf{x},$$

although this is not a force, but an *energy*.

If $\mathcal{E}(\mathbf{x})$ is time-independent, we call it a *static electric field*. In this case the circulation along a closed curve $\mathcal{K}$ is always zero,

$$\mathrm{emf}=\oint_{\mathcal{K}}\mathcal{E}(\mathbf{x})\cdot\,\mathrm{d}\mathbf{x}=0,$$

so $\mathcal{E}(\mathbf{x})$ is in this case a gradient field.

We have previously also mentioned *Ampère's law*, where the magnetic field $\mathbf{H}$ in general is not a gradient field.

The physical examples above are just the simplest ones of the applications of the tangential line integrals in *Physics*. We shall later introduce the more powerful *Gauß's* and *Stokes's theorems* and see some applications of them.

32.4 Overview of the theorems and methods concerning tangential line integrals and gradient fields

The **current** of a *vector field* $\mathbf{V}$ along a *curve* $\mathcal{K}$ of parametric representation $\mathbf{r}(t)$ is defined by:

$$\int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t}\, \mathrm{d}s = \int_{\mathcal{K}} \mathbf{V} \cdot \mathrm{d}\mathbf{x} = \int_{\alpha}^{\beta} \mathbf{V}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, \mathrm{d}t,$$

where we have identified

$$\mathbf{x} = \mathbf{r}(t) \quad \text{and} \quad \mathrm{d}\mathbf{x} = \mathbf{r}'(t)\, ta.$$

It can in some cases be identified as an electric current along wire, represented by the curve.

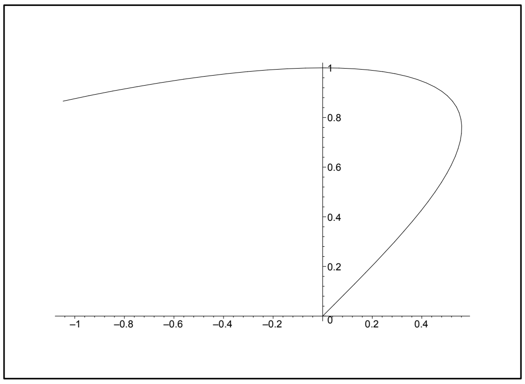

Figure 32.2: Example of a plane curve $\mathcal{K}$ with initial point $(0, 0)$.

There are here two important special cases:

1) **The gradient integral theorem:**

$$\int_{\mathcal{K}} \nabla F(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} = F(\mathbf{b}) - F(\mathbf{a}),$$

no matter how the curve $\mathcal{K}$ from $\mathbf{a}$ to $\mathbf{b}$ is chosen.

2) **Circulation**, i.e. $\mathcal{K}$ is a *closed* curve.

Whenever the word *"circulation"* occurs in an example, always think of *Stokes's theorem*,

$$\oint_{\delta\mathcal{F}} \mathbf{V} \cdot \mathbf{t}\, \mathrm{d}s = \int_{\mathcal{F}} \mathbf{n} \cdot \mathbf{rot}\ \mathbf{V}\, \mathrm{d}S,$$

and see if it applies, cf. Chapter 35.

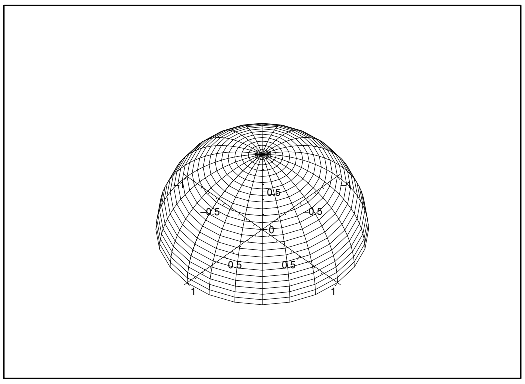

Figure 32.3: The half sphere $\mathcal{F}$ gives a typical example, when we shall apply Stokes's theorem.

We shall here only consider the **gradient integral theorem**, because the *circulation* will be treated separately later.

A necessary condition (which is not sufficient). The "cross derivatives" agree,

$$\frac{\partial V_i}{\partial x_j} = \frac{\partial V_j}{\partial x_i} \qquad \text{for all } i,\, j = 1, \dots, k.$$

A trap: Even if the necessary conditions are all fulfilled, the field $\mathbf{V}$ is *not* always a gradient field, although many readers believe it.

A sufficient condition (which is not necessary). The "cross derivatives" agree:

$$\frac{\partial V_i}{\partial x_j} = \frac{\partial V_j}{\partial x_i} \qquad \text{for all } i,\, j = 1, \dots, k,$$

and

the domain A is *star shaped.*

Remark 32.1 Even when $\mathbf{V}$ is a gradient field, the corresponding domain A does *not* have to be star shaped. $\diamond$

Concerning the calculations in practice we refer to Section 32.2:

1) Indefinite integration,

2) Method of inspection,

3) Integration along a curve consisting of lines parallel with one of the axes,

4) Radial integration.

The *radial integration* cannot be recommended as a standard procedure.

In some cases a differential form can be simplified by removing a gradient field:

$$\mathbf{V} = \bigtriangledown F + \mathbf{U},$$

or more conveniently,

$$\mathbf{V} \cdot \mathrm{d}\mathbf{x} = V_x\,\mathrm{d}x + V_y\,\mathrm{d}y + V_z\,\mathrm{d}z = \mathrm{d}F + U_x\,\mathrm{d}x + U_y\,\mathrm{d}y + U_z\,\mathrm{d}z,$$

where $\mathbf{U}$ ought to be simpler than $\mathbf{V}$.
If so, then

$$\int_{\mathcal{K}} \mathbf{V} \cdot \mathrm{d}\mathbf{x} = F(\mathbf{b}) - F(\mathbf{a}) + \int_{\mathcal{K}} \mathbf{U} \cdot \mathrm{d}\mathbf{x}.$$

This method is e.g. used in Thermodynamics, where the vector field usually is *not* a gradient field.

In these reductions one can take advantage of the well-known *rules of calculus for differentials*:

$$\alpha\,\mathrm{d}f + \mathrm{d}g = \mathrm{d}(\alpha\, f + g), \qquad \alpha \text{ constant}$$

$$f\,\mathrm{d}g + g\,\mathrm{d}f = \mathrm{d}(fg),$$

$$f\,\mathrm{d}g - g\,\mathrm{d}f = f^2\,\mathrm{d}\left(\frac{g}{f}\right), \qquad f \neq 0,$$

$$F'(f)\,\mathrm{d}f = \mathrm{d}(F \circ f).$$

32.5 Examples of tangential line integrals

Example 32.7 *Calculate in each of the following cases the tangential line integral*

$$\int_{\mathcal{K}} \mathbf{V}(\mathbf{x}) \cdot \mathrm{d}\mathbf{x}$$

of the vector field $\mathbf{V}$ *along the plane curve* $\mathcal{K}$. *This curve will either be given by a parametric description or by an equation. First sketch the curve.*

1) *The vector field* $\mathbf{V}(x,y) = (x^2+y^2, x^2-y^2)$ *along the curve* $\mathcal{K}$ *given by* $y = 1-|1-x|$ *for* $x \in [0,2]$.

2) *The vector field* $\mathbf{V}(x,y) = (x^2-2xy, y^2-2xy)$ *along the curve* $\mathcal{K}$ *given by* $y = x^2$ *for* $x \in [-1,1]$.

3) *The vector field* $\mathbf{V}(x,y) = (2a-y, x)$ *along the curve* $\mathcal{K}$ *given by* $\mathbf{r}(t) = a(t-\sin t, 1-\cos t)$ *for* $t \in [0, 2\pi]$.

4) *The vector field* $\mathbf{V}(x,y) = \left(\dfrac{x+y}{x^2+y^2}, \dfrac{y-x}{x^2+y^2}\right)$ *along the curve* $\mathcal{K}$ *given by* $x^2+y^2 = a^2$ *and run through in the positive orientation of the plane.*

5) *The vector field* $\mathbf{V}(x,y) = (x^2-y^2, -(x+y))$ *along the curve* $\mathcal{K}$ *given by* $\mathbf{r}(t) = (a\,\cos t, b\,\sin t)$ *for* $t \in \left[0, \dfrac{\pi}{2}\right]$.

6) *The vector field* $\mathbf{V}(x,y) = (x^2-y^2, -(x+y))$ *along the curve* $\mathcal{K}$ *given by* $\mathbf{r}(t) = (a(1-t), b\,t)$ *for* $t \in [0,1]$.

7) *The vector field* $\mathbf{V}(x,y) = (-y^3, x^3)$ *along the curve* $\mathcal{K}$ *given by* $\mathbf{r}(t) = (1+\cos t, \sin t)$ *for* $t \in \left[\dfrac{\pi}{2}, \pi\right]$.

8) *The vector field* $\mathbf{V}(x,y) = \left(-y^2, a^2 \sinh \dfrac{x}{a}\right)$ *along the curve* $\mathcal{K}$ *given by* $y = a\,\cosh \dfrac{x}{a}$ *for* $x \in [a, 2a]$.

A Tangential line integrals.

D First sketch the curve. Then compute the tangential line integral.

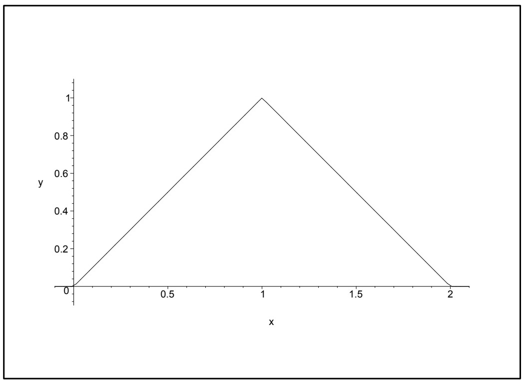

Figure 32.4: The curve $\mathcal{K}$ of **Example 32.7.1**.

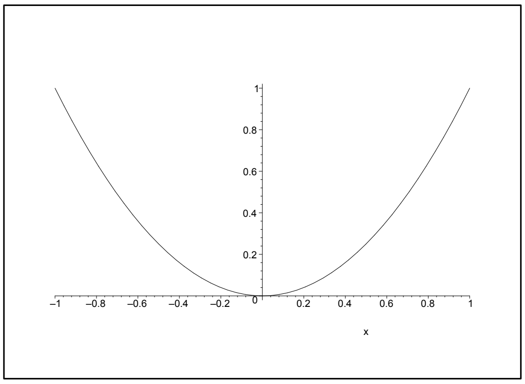

Figure 32.5: The curve $\mathcal{K}$ of **Example 32.7.2**.

I 1) Here the parametric description of the curve can also be written

$$y = \begin{cases} x & \text{for } x \in [0,1], \\ 2-x & \text{for } x \in [1,2]. \end{cases}$$

This gives the following calculation of the tangential line integral

$$\begin{aligned}
\int_{\mathcal{K}} \mathbf{V}(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} &= \int_{\mathcal{K}} \left\{ \left(x^2+y^2\right) \mathrm{d}x + \left(x^2-y^2\right) \mathrm{d}y \right\} \\
&= \int_0^1 \left\{ \left(x^2+x^2\right) \mathrm{d}x + \left(x^2-x^2\right) \mathrm{d}x \right\} \\
&\quad + \int_1^2 \left\{ \left(x^2+(2-x)^2\right) \mathrm{d}x + \left(x^2-(2-x)^2\right)(-\mathrm{d}x) \right\} \\
&= \int_0^1 2x^2\,\mathrm{d}x + \int_1^2 2\,(2-x)^2\,\mathrm{d}x = \frac{2}{3}\left[x^3\right]_0^1 + \frac{2}{3}\left[(x-2)^3\right]_1^2 \\
&= \frac{2}{3}+\frac{2}{3} = \frac{4}{3}.
\end{aligned}$$

2) Here

$$\begin{aligned}
\int_{\mathcal{K}} \mathbf{V}(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} &= \int_{\mathcal{K}} \left\{ \left(x^2-2xy\right) \mathrm{d}x + \left(y^2-2xy\right) \mathrm{d}y \right\} \\
&= \int_{-1}^1 \left\{ \left(x^2-2x^3\right) \mathrm{d}x + \left(x^4-2x^3\right) \cdot 2x\,\mathrm{d}x \right\} \\
&= \int_{-1}^1 \left(x^2-2x^3+2x^5-4x^4\right) \mathrm{d}x \\
&= \int_{-1}^1 \left(x^2-4x^4\right) \mathrm{d}x + 0 = 2\left[\frac{1}{3}\,x^3 - \frac{4}{5}\,x^5\right]_0^1 \\
&= 2\left(\frac{1}{3}-\frac{4}{5}\right) = \frac{2}{15}\,(5-12) = -\frac{14}{15}.
\end{aligned}$$

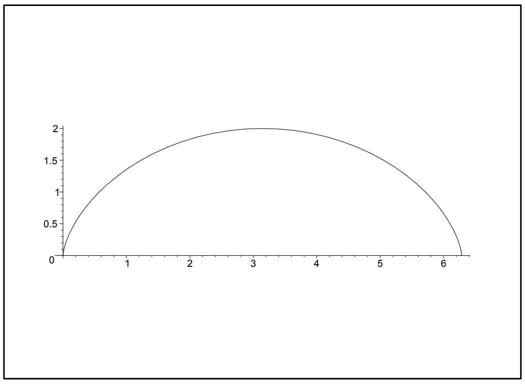

Figure 32.6: The curve $\mathcal{K}$ of **Example 32.7.3** for $a = 1$.

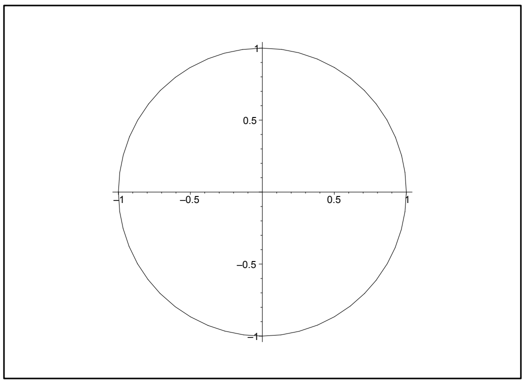

Figure 32.7: The curve $\mathcal{K}$ of **Example 32.7.4** for $a = 1$.

3) Similarly we get

$$\begin{aligned}\int_{\mathcal{K}} \mathbf{V}(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} &= \int_{\mathcal{K}} \{(2a - y)\,\mathrm{d}x + x\,\mathrm{d}y\} \\ &= \int_0^{2\pi} \{(2a - a(1-\cos t))a(1-\cos t) + a(t-\sin t)a\sin t\}\,\mathrm{d}t \\ &= a^2 \int_0^{2\pi} \{(1+\cos t)(1-\cos t) + (t-\sin t)\sin t\}\,\mathrm{d}t \\ &= a^2 \int_0^{2\pi} \{1-\cos^2 t + t\sin t - \sin^2 t\} dt = a^2 \int_0^{2\pi} t\sin t\,\mathrm{d}t \\ &= a^2[-t\cos t + \sin t]_0^{2\pi} = -2\pi a^2.\end{aligned}$$

4) We split the curve $\mathcal{K}$ into two pieces, $\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$, where $\mathcal{K}_1$ lies in the upper half plane, and $\mathcal{K}_2$ lies in the lower half plane, i.e. $y > 0$ inside $\mathcal{K}_1$, and $y < 0$ inside $\mathcal{K}_2$. Then we get the

tangential line integral

$$
\begin{aligned}
\int_{\mathcal{K}} \mathbf{V}(\mathbf{x}) \cdot d\mathbf{x} &= \int_{\mathcal{K}} \left(\frac{x+y}{x^2+y^2}\, dx + \frac{y-x}{x^2+y^2}\, dy \right) \\
&= \int_{\mathcal{K}} \frac{1}{x^2+y^2} \, \frac{1}{2} \, d\left(x^2+y^2\right) + \int_{\mathcal{K}} \frac{1}{x^2+y^2}\,(y\, dx - x\, dy) \\
&= \int_{\mathcal{K}} \frac{1}{2}\, d \ln\left(x^2+y^2\right) + \int_{\mathcal{K}_1} \frac{1}{1+\left(\frac{x}{y}\right)^2} \left(\frac{1}{y}\, dx + x\left(-\frac{1}{y^2}\right) dy \right) \\
&\qquad + \int_{\mathcal{K}_2} \frac{1}{1+\left(\frac{x}{y}\right)^2} \left(\frac{1}{y}\, dx + x\left(-\frac{1}{y^2}\right) dy \right) \\
&= 0 + \int_{\mathcal{K}_1} \frac{1}{1+\left(\frac{x}{y}\right)^2}\, d\left(\frac{x}{y}\right) + \int_{\mathcal{K}_2} \frac{1}{1+\left(\frac{x}{y}\right)^2}\, d\left(\frac{x}{y}\right) \\
&= \int_{\mathcal{K}_1} d\text{Arctan}\left(\frac{x}{y}\right) + \int_{\mathcal{K}_2} d\text{Arctan}\left(\frac{x}{y}\right) \\
&= [\text{Arctan}\, t]_{+\infty}^{-\infty} + [\text{Arctan}\, t]_{+\infty}^{-\infty} = -\pi - \pi = -2\pi.
\end{aligned}
$$

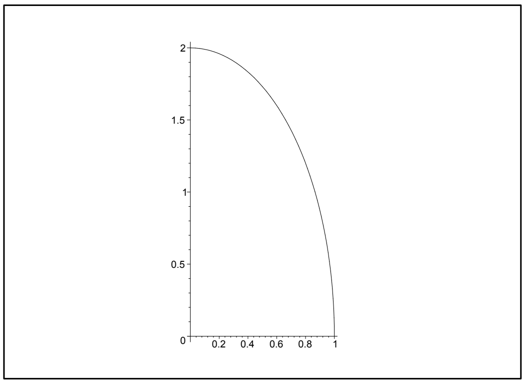

Figure 32.8: The curve $\mathcal{K}$ of **Example 32.7.5** for $a = 1$ and $b = 2$.

ALTERNATIVELY we get by using the parametric description

$$(x, y) = a\,(\cos t, \sin t), \qquad t \in [0, 2\pi],$$

that

$$\begin{aligned}\int_{\mathcal{K}} \mathbf{V}(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} &= \int_{\mathcal{K}} \left(\frac{x+y}{x^2+y^2}\,\mathrm{d}x + \frac{y-x}{x^2+y^2}\,\mathrm{d}y \right) \\ &= \int_0^{2\pi} \frac{a^2}{a^2}\{(\cos t + \sin t)(-\sin t) + (\sin t - \cos t)\cos t\}\,\mathrm{d}t \\ &= \int_0^{2\pi} \{-\cos t \cdot \sin t - \sin^2 t + \cos t \cdot \sin t - \cos^2 t\}\,\mathrm{d}t \\ &= -\int_0^{2\pi} dt = -2\pi.\end{aligned}$$

5) Here

$$\begin{aligned}\int_{\mathcal{K}} \mathbf{V}(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} &= \int_{\mathcal{K}} \{(x^2 - y^2)\,\mathrm{d}x - (x+y)\,\mathrm{d}y\} \\ &= \int_0^{\frac{\pi}{2}} \{(a^2\cos^2 t - b^2 \sin^2 t)(-a\sin t) - (a\cos t + b\sin t) b\cos t\}\,\mathrm{d}t \\ &= \int_0^{\frac{\pi}{2}} \{-a[(a^2+b^2)\cos^2 t - b^2]\sin t - ab\cos^2 t - b\sin t\,\cos t\}\,\mathrm{d}t \\ &= \left[+a(a^2+b^2)\frac{1}{3}\cos^3 t - ab^2\cos t - \frac{ab}{2}(t + \frac{1}{2}\sin 2t) - \frac{1}{2}b^2\sin^2 t \right]_0^{\frac{\pi}{2}} \\ &= -\frac{ab}{2} \cdot \frac{\pi}{2} - \frac{b^2}{2} - \frac{a(a^2+b^2)}{3} + ab^2 = \frac{a}{3}(2b^2 - a^2) - \frac{b}{4}(2b + a\pi).\end{aligned}$$

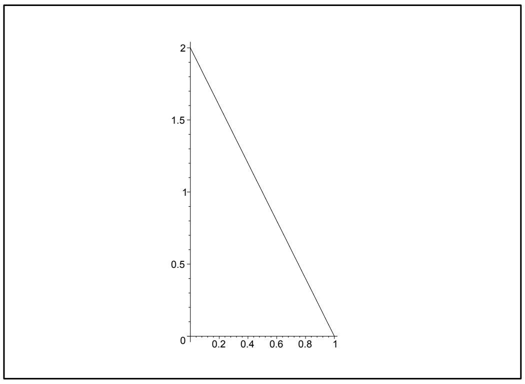

Figure 32.9: The curve $\mathcal{K}$ of **Example 32.7.6** for $a = 1$ and $b = 2$.

6) Here

$$\begin{aligned}
\int_{\mathcal{V}} \mathbf{V}(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} &= \int_{\mathcal{K}} \{(x^2-y^2)\,\mathrm{d}x - (x+y)\,\mathrm{d}y\} \\
&= \int_0^1 \{[a^2(1-t)^2 - b^2t^2](-a) - [a-at+bt]\cdot b\}\,\mathrm{d}t \\
&= \int_0^1 \{-a^3(t-1)^2 + ab^2t^2 + b(a-b)t - ab\}\,\mathrm{d}t \\
&= \left[-\frac{a^3}{3}(t-1)^3 + \frac{ab^2}{3}t^3 + \frac{1}{2}b(a-b)t^2 - abt\right]_0^1 \\
&= \frac{ab^2}{3} + \frac{1}{2}(a-b)b - ab - \frac{a^3}{3} \\
&= \frac{a}{3}(b^2-a^2) - \frac{b}{2}(a+b).
\end{aligned}$$

Remark. The vector field $\mathbf{V}(\mathbf{x})$ is the same as that in **Example 32.7.5** and in **Example 32.7.6**. Furthermore, the curves of these two examples have the same initial point and end point. Nevertheless the two tangential line integrals give different results. We shall later be interested in those vector fields $\mathbf{V}(\mathbf{x})$, for which the tangential line integral *only* depends on the initial and end points of the curve $\mathcal{K}$. (In Physics such vector fields correspond to the so-called conservative forces.) We have here an example in which this ideal property is *not* satisfied. ♢

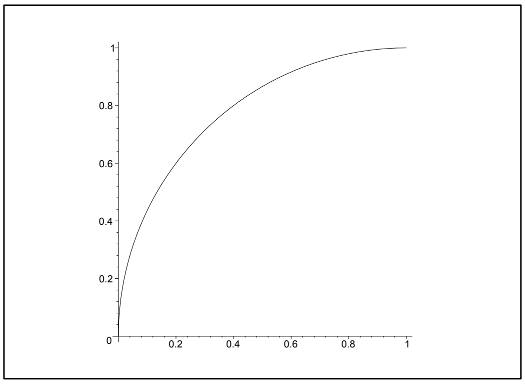

Figure 32.10: The curve $\mathcal{K}$ of **Example 32.7.7**.

7) We get

$$\int_{\mathcal{K}} \mathbf{V}(\mathbf{x})\cdot d\mathbf{x} = \int_{\mathcal{K}} \{-y^3\,dx + x^3\,dy\}$$
$$= \int_{\frac{\pi}{2}}^{\pi} \{-\sin^3 t\cdot(-\sin t) + (1+\cos t)^3\cos t\}\,dt$$
$$= \int_{\frac{\pi}{2}}^{\pi} \{\sin^4 t + \cos t + 3\cos^2 t + 3\cos^3 t + \cos^4 t\}\,dt$$
$$= \int_{\frac{\pi}{2}}^{\pi} \left\{\sin^4 t + \cos^4 t + \left(2\cos^2 t\cdot\sin^2 t - \frac{1}{2}\sin^2 2t\right) + \cos t + \frac{3}{2} + \frac{3}{2}\cos 2t + 3\cos^3 y\right\}\,dt$$
$$= \int_{\frac{\pi}{2}}^{\pi} \left\{(\sin^2 t + \cos^2 t)^2 - \frac{1}{4} + \frac{1}{4}\cos 4t + \cos t + \frac{3}{2}\cos 2t + 3\cos t - 3\sin^2 t\,\cos t\right\}\,dt$$
$$= \left[t - \frac{t}{4} + \frac{1}{16}\sin 4t + \sin t + \frac{3}{2}t + \frac{3}{4}\sin 2t + 3\sin t - \sin^3 t\right]_{\frac{\pi}{2}}^{\pi}$$
$$= \left(1 - \frac{1}{4} + \frac{3}{2}\right)\frac{\pi}{2} - 4 + 1 = \frac{9\pi}{8} - 3.$$

8) We get

$$\int_{\mathcal{K}} \mathbf{V}(\mathbf{x})\cdot d\mathbf{x} = \int_{\mathcal{K}} \left\{-y^2\,dx + a^2\sinh\frac{x}{a}\,dy\right\}$$
$$= \int_a^{2a} \left\{-a^2\cosh^2\frac{x}{a}\,dx + a^2\sinh\frac{x}{a}\cdot\sinh\frac{x}{a}\,dx\right\}$$
$$= -a^2\int_a^{2a}\left\{\cosh^2\left(\frac{x}{a}\right) - \sinh^2\left(\frac{x}{a}\right)\right\}\,dx = -a^3.$$

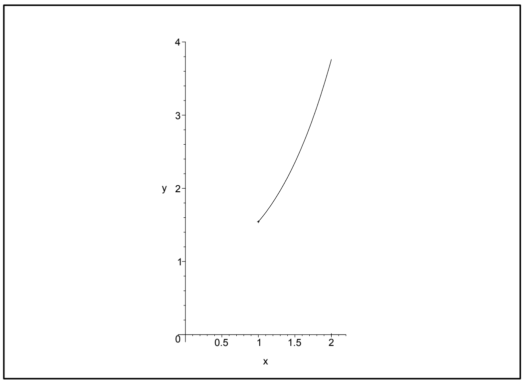

Figure 32.11: The curve $\mathcal{K}$ of **Example 32.7.8** for $a = 1$.

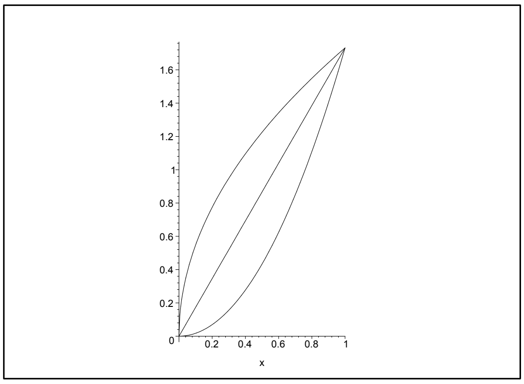

Figure 32.12: The curves $y = \sqrt{3x}$, $y = \sqrt{3}\,x$ and $y = \sqrt{3}\,x^2$.

Example 32.8 *Compute the tangential line integral of the vector field*

$$\mathbf{V}(x,y) = (2xy, x^6y^2)$$

along the curve $\mathcal{K}$ given by $y = a\,x^b$, $x \in [0,1]$. Then find a such that the line integral becomes independent of b.

A Tangential line integral.

D Just use the standard method.

I We calculate the line integral

$$\begin{aligned}\int_{\mathcal{K}} \mathbf{V}(x,y)\cdot \mathrm{d}\mathbf{x} &= \int_{\mathcal{K}} 2xy\,dx + x^6y^2\,\mathrm{d}y = \int_0^1 \left\{2x\,a\,x^b + x^6a^2x^{2b}\cdot ab\,x^{b-1}\right\}\mathrm{d}x \\ &= \int_0^1 \left\{2a\,x^{b+1} + a^3b\,x^{3b+5}\right\}\mathrm{d}x = \frac{2a}{b+2} + \frac{a^3b}{3(b+2)} = \frac{a(a^2b+6)}{3(b+2)}.\end{aligned}$$

Assume that this result is independent of b. Then $b+2$ must be proportional to a^2b+6, so $a^2 = 3$.

According to the convention $a > 0$, hence $a = \sqrt{3}$. By choosing this a we get

$$\int_{\mathcal{K}} \mathbf{V}(x,y) \cdot \mathrm{d}\mathbf{x} = \frac{\sqrt{3}(3b+6)}{3(b+2)} = \sqrt{3},$$

which is independent of b.

Example 32.9 *Calculate in each of the following cases the tangential line integral*

$$\int_{\mathcal{K}} \mathbf{V}(\mathbf{x}) \cdot \mathrm{d}\mathbf{x}$$

of the vector field $\mathbf{V}$ *along the space curve* $\mathcal{K}$*, which is given by the parametric description*

$$\mathcal{K} = \left\{ \mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} = \mathbf{r}(t),\, t \in I \right\}.$$

1) *The vector field is* $\mathbf{V}(x,y,z) = (y^2 - z^2, 2yz, -x^2)$*, and the curve* $\mathcal{K}$ *is given by* $\mathbf{r}(t) = (t, t^2, t^3)$ *for* $t \in I$*.*

2) *The vector field is* $\mathbf{V}(x,y,z) = \left(\dfrac{1}{x+z}\,,\, y+z\,,\, \dfrac{2}{x+y+z} \right)$*, and the curve* $\mathcal{K}$ *is given by* $\mathbf{r}(t) = (t, t^2, t^3)$ *for* $t \in [1,2]$*.*

3) *The vector field is* $\mathbf{V}(x,y,z) = (3x^2 - 6yz, 2y + 3xz, 1 - 4xyz^2)$*, and the curve* $\mathcal{K}$ *is given by* $\mathbf{r}(t) = (t, t^2, t^3)$ *for* $t \in [0,1]$*.*

4) *The vector field is* $\mathbf{V}(x,y,z) = (3x^2 - 6yz, 2y + 3xz, 1 - 4xyz^2)$*, and the curve* $\mathcal{K}$ *is given by* $\mathbf{r}(t) = (t, t, t)$ *for* $t \in [0,1]$*.*

5) *The vector field is* $\mathbf{V}(x,y,z) = (3x^2 - 6yz, 2y + 3xz, 1 - 4xyz^2)$*, and the curve* $\mathcal{K}$ *is given by*

$$\mathbf{r}(t) = \begin{cases} (0,0,t), & \text{for } t \in [0,1], \\ (0, t-1, 1), & \text{for } t \in [1,2], \\ (t-2, 1, 1), & \text{for } t \in [2,3]. \end{cases}$$

6) *The vector field is* $\mathbf{V}(x,y,z) = (x, y, xz - y)$*, and the curve* $\mathcal{K}$ *is given by* $\mathbf{r}(t) = (t, 2t, 4t)$ *for* $t \in [0,1]$*.*

7) *The vector field is* $\mathbf{V}(x,y,z) = (2x + yz, 2y + xz, 2z + xy)$*, and the curve* $\mathcal{K}$ *is given by*

$$\mathbf{r}(t) = (a(\cosh t)\cos t, a(\cosh t)\sin t, at) \qquad \text{for } t \in [0, 2\pi].$$

8) *The vector field is* $\mathbf{V}(x,y,z) = (y^2 - z^2, 2yz, -x^2)$*, and the curve* $\mathcal{K}$ *is given by* $\mathbf{r}(t) = (t, t, t)$ *for* $t \in [0,1]$*.*

A Tangential line integrals in space.

D Insert the parametric descriptions and calculate the tangential line integral. Note that **Example 32.9.7** is a gradient field, so it is in this case possible to find the integral directly.

I 1) We get

$$\begin{aligned} \int_{\mathcal{K}} \mathbf{V}(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} &= \int_{\mathcal{K}} \left\{ (y^2 - z^2)\,\mathrm{d}x + 2yz\,\mathrm{d}y - x^2\,\mathrm{d}z \right\} \\ &= \int_0^1 \left\{ (t^4 - t^6) + 2t^2 \cdot t^3 \cdot 2t - t^3 \cdot 3t^2 \right\} \mathrm{d}t \\ &= \int_0^1 \{3t^6 - 2t^4\} dt = \frac{3}{7} - \frac{2}{5} = \frac{1}{35}. \end{aligned}$$

2) Here

$$\begin{aligned}\int_{\mathcal{K}} \mathbf{V}(\mathbf{x})\cdot \mathrm{d}\mathbf{x} &= \int_{\mathcal{K}}\left\{\frac{1}{x+z}\,\mathrm{d}x+(y+z)\,\mathrm{d}y+\frac{2}{x+y+z}\,\mathrm{d}z\right\}\\ &= \int_1^2\left\{\frac{1}{t+t^3}+(t+t^3)+\frac{6t^2}{2t+t^3}\right\}\mathrm{d}t\\ &= \int_1^2\left\{\frac{1}{t}-\frac{t}{1+t^2}+\frac{6t}{2+t^2}+t+t^3\right\}\mathrm{d}t\\ &= \left[\ln t-\frac{1}{2}\ln(1+t^2)+3\ln(2+t^2)+\frac{t^2}{2}+\frac{t^4}{4}\right]_1^2\\ &= \ln 2-\frac{1}{2}\ln 5+3\ln 6-\frac{1}{2}\ln 2-3\ln 3+\frac{4}{2}+\frac{16}{4}-\frac{1}{2}-\frac{1}{4}\\ &= \frac{9}{2}\ln 2+\frac{1}{2}\ln 5+\frac{21}{4}=\frac{21}{4}+\frac{1}{2}\ln\frac{512}{5}.\end{aligned}$$

3) First note that for any curve,

$$\int_{\mathcal{K}} \mathbf{V}(\mathbf{x})\cdot d\mathbf{x}=\int_{\mathcal{K}}\{(3x^2-6yz)\,\mathrm{d}x+(2y+3xz)\,\mathrm{d}y+(1-4xyz^2)\,\mathrm{d}z\}$$

$$(32.2)\qquad =\int_{\mathcal{K}}\mathrm{d}(x^3+y^2+z)-\int_{\mathcal{K}}z\{6y\,\mathrm{d}x-3x\,\mathrm{d}y+4xyz\,\mathrm{d}z\}.$$

Such a rearrangement can also be used with success in **Example 32.9.3, Example 32.9.4** and **Example 32.9.5**.

When we apply (32.2), we get

$$\begin{aligned}\int_{\mathcal{K}} \mathbf{V}(\mathbf{x})\cdot \mathrm{d}(\mathbf{x}) &= \left[x^3+y^2+z\right]_{(0,0,0)}^{(1,1,1)}-\int_0^1 t^3\{6t^2-3t\cdot 2t+t^6\cdot 3t^2\}dt\\ &= 3-\int_0^1 12\,t^{11}dt=3-1=2.\end{aligned}$$

Alternatively, it follows by a direct insertion that

$$\begin{aligned}\int_{\mathcal{K}} \mathbf{V}(\mathbf{x})\cdot \mathrm{d}\mathbf{x} &= \int_{\mathcal{K}}\{(3x^2-6yz)\,\mathrm{d}x+(2y+3xz)\,\mathrm{d}y+(1-4xyz^2)\,\mathrm{d}z\}\\ &= \int_0^1\{(3t^2-6t^2\cdot t^3)+(2t^2+3t\cdot t^3)2t+(1-4t\cdot t^2\cdot t^6)3t^2\}\,\mathrm{d}t\\ &= \int_0^1\{3t^2-6t^5+4t^3+6t^5+3t^2-12t^{11}\}\,\mathrm{d}t\\ &= \int_0^1(6t^2+4t^3-12t^{11})dt=\left[2t^3+t^4-t^{12}\right]_0^1=2.\end{aligned}$$

4) The vector field is the same as in **Example 32.9.3**. We get by (32.2),

$$\begin{aligned}\int_{\mathcal{K}} \mathbf{V}(\mathbf{x})\cdot \mathrm{d}\mathbf{x} &= \left[x^3+y^2+z\right]_{(0,0,0)}^{(1,1,1)}-\int_0^1(6t^2-3t^2+4t^4)\,\mathrm{d}t\\ &= 3-\int_0^1(3t^2+4t^4)\,\mathrm{d}t=3-1-\frac{4}{5}=\frac{6}{5}.\end{aligned}$$

ALTERNATIVELY, it follows by a direct insertion that

$$\begin{aligned}\int_{\mathcal{K}} \mathbf{V}(\mathbf{x})\cdot \mathrm{d}\mathbf{x} &= \int_0^1 \{(3t^2-6t^2)+(2t+3t^2)+1-4t^4\}\,\mathrm{d}t \\ &= \int_0^1 (1+2t-4t^4)dt = 1+1-\frac{4}{5} = \frac{6}{5}.\end{aligned}$$

5) The vector field is the same as in **Example 32.9.3**. When we apply (32.2) and just check that $\mathbf{r}(t)$ is a continuous curve, we get

$$\begin{aligned}\int_{\mathcal{K}} \mathbf{V}(\mathbf{x})\cdot \mathrm{d}\mathbf{x} &= \left[x^3+y^2+z\right]_{(0,0,0)}^{(1,1,1)} - [\int_{\mathcal{K}} z\{6y\,\mathrm{d}x-3x\,\mathrm{d}y+4xyz\,\mathrm{d}z\} \\ &= 3-\int_0^1 0\,\mathrm{d}t - \int_1 +20\,\mathrm{d}t - \int_2^3 1\cdot 6\,\mathrm{d}t = 3-6=-3.\end{aligned}$$

ALTERNATIVELY, it follows by direct insertion that

$$\begin{aligned}\int_{\mathcal{K}} \mathbf{V}(\mathbf{x})\cdot \mathrm{d}\mathbf{x} &= \int_{\mathcal{K}} \{(3x^2-6yz)\,\mathrm{d}x+(2y+3xz)\,\mathrm{d}y+(1-4xyz^2)\,\mathrm{d}z\} \\ &= \int_0^1 (1-4\cdot 0)\,\mathrm{d}t + \int_1^2 \{2(t-1)+0\}\,\mathrm{d}t + \int_2^3 \{3(t-2)^2-6\}\,\mathrm{d}t \\ &= [t]_0^1 + 2\left[\frac{1}{2}(t-1)^2\right]_1^2 + 3\left[\frac{1}{3}(t-2)^3-2t\right]_2^3 \\ &= 1+1+1-3\cdot 2\cdot 3+3\cdot 2\cdot 2 = 3(1-6+4) = -3.\end{aligned}$$

6) Here we get by insertion,

$$\begin{aligned}\int_{\mathcal{K}} \mathbf{V}(\mathbf{x})\cdot \mathrm{d}\mathbf{x} &= \int_{\mathcal{K}} \{x\,\mathrm{d}x + y\,\mathrm{d}y + (xz-y)\,\mathrm{d}z\} \\ &= \int_0^1 \{t+2t\cdot 2+(t\cdot 4t-2t)\cdot 4\}\,\mathrm{d}t \\ &= \int_0^1 (t+4t+16t^2-8t)dt = \int_0^1 (16t^2-3t)\,\mathrm{d}t \\ &= \frac{16}{3}-\frac{3}{2} = \frac{32-9}{6} = \frac{23}{6}.\end{aligned}$$

7) It follows immediately that

$$\begin{aligned}\int_{\mathcal{K}} \mathbf{V}(\mathbf{x})\cdot \mathrm{d}\mathbf{x} &= \int_{\mathcal{K}} \{(2x+yz)\,\mathrm{d}x+(2y+xz)\,\mathrm{d}y+(2zxy)\,\mathrm{d}z\} \\ &= \int_{\mathcal{K}} \{\,\mathrm{d}(x^2+y^2+z^2)+(yz\,\mathrm{d}x+xz\,\mathrm{d}y+xy\,\mathrm{d}z)\} \\ &= \int_{\mathcal{K}} \mathrm{d}(x^2+y^2+z^2+xyz) = \left[x^2+y^2+z^2+xyz\right]_{(x,y,z)=(a,0,0)}^{a(\cosh 2\pi,0,2\pi)} \\ &= a^2\cosh^2 2\pi + 4a^2\pi^2 - a^2 = a^2(4\pi^2+\sinh^2 2\pi).\end{aligned}$$

ALTERNATIVELY, we get by the parametric description

$$\mathbf{r}(t) = a(\cosh t\cdot\cos t, \cosh t\cdot\sin t, t), \quad t\in[0,2\pi],$$

that

$$\mathbf{r}'(t) = a(\sinh t\cdot\cos t-\cosh t\cdot\sin t, \sinh t\cdot\sin t+\cosh t\cdot\cos t, 1),$$

thus

$$\begin{aligned}
\int_{\mathcal{K}} \mathbf{V}(\mathbf{x}) \cdot d\mathbf{x} &= \int_{\mathcal{K}} \{(2x+yz)\,dx + (2y+xz)\,dy + (2z+xy)\,dz\} \\
&= \int_0^{2\pi} (2a\cosh t\,\cos t + a^2 t\,\cosh t\,\sin t)a(\sinh t\,\cos t - \cosh t\,\sin t)\,dt \\
&\quad + \int_0^{2\pi} (2a\cosh t\,\sin t + a^2\cosh t\,\cos t)a(\sinh t\,\sin t + \cosh t\,\cos t)\,dt \\
&\quad + \int_0^{2\pi} (2at + a^2\cosh^2 t \cdot \cos t \cdot \sin t)a\,dt \\
&= a^2 \cdot (\cdots) + a^3 \cdot (\cdots).
\end{aligned}$$

Then the easiest method is to reduce and use that

$$\cos t = \frac{1}{2}\left(e^{it} + e^{-it}\right), \qquad \sin t = \frac{1}{2i}\left(e^{it} - e^{-it}\right),$$

and similarly for $\cosh t$ and $\sinh t$. We finally obtain the result by a partial integration.

The variants above are somewhat sophisticated, so we proceed here by first calculating the coefficient of a^2:

$$\begin{aligned}&\int_0^{2\pi} 2\cosh t\cdot t(\sinh t\cdot\cos t-\cosh t\cdot\sin t)\,\mathrm{d}t\\&\qquad+\int_0^{2\pi} 2\cosh t\cdot\sin t(\sinh t\cdot\sin t+\cosh t\cdot\cos t)\,\mathrm{d}t+\int_0^{2\pi}2t\,\mathrm{d}t\\&\qquad=2\int_0^{2\pi}\cosh t\cdot\sinh t\,\mathrm{d}t+\int_0^{2\pi}2t\,\mathrm{d}t=\left[\sinh^2 t+t^2\right]_0^{2\pi}=4\pi^2+\sinh^2 2\pi.\end{aligned}$$

Then we find the coefficient of a^3:

$$\begin{aligned}&\int_0^{2\pi} t\{\cosh t\,\sinh t\,\sin t\,\cos t-\cosh^2 t\,\sin^2\}\,\mathrm{d}t\\&\qquad+\int_0^{2\pi} t\{\cosh t\,\sinh t\,\sin t\,\cos t+\cosh^2 t\,\cos^2 t\}\,\mathrm{d}t+\int_0^{2\pi}\cosh^2 t\,\cos t\,\sin t\,\mathrm{d}t\\&\qquad=\int_0^{2\pi} t(\cosh t\,\sinh t\,\sin 2t+\cosh^2 t\,\cos 2t)\,\mathrm{d}t+\frac{1}{2}\int_0^{2\pi}\cosh^2 t\,\sin 2t\,\mathrm{d}t.\end{aligned}$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{2}\cosh^2 t\cdot\sin 2t\right\}=\cosh t\cdot\sinh t\cdot\sin 2t+\cosh^2 t\cdot\cos 2t,$$

so the whole expression can then be written

$$\begin{aligned}&\int_0^{2\pi}\left\{t\,\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\cosh^2 t\,\sin 2t\right)+\frac{\mathrm{d}t}{\mathrm{d}t}\cdot\left(\frac{1}{2}\cosh^2 t\,\sin 2t\right)\right\}\,\mathrm{d}t\\&\qquad=\int_0^{2\pi}\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{t}{2}\cosh^2 t\,\sin 2t\right)\,\mathrm{d}t=\left[\frac{t}{2}\cosh^2 t\cdot\sin 2t\right]_0^{[2\pi}=0.\end{aligned}$$

As a conclusion we get

$$\int_{\mathcal{K}}\mathbf{V}(\mathbf{x})\cdot\mathrm{d}\mathbf{x}=a^2(4\pi^2+\sinh^2 2\pi)+0\cdot a^3=a^2(4\pi^2+\sinh^2 2\pi).$$

8) Here we get [cf. also **Example 32.9.1**, where the vector field is the same]

$$\begin{aligned}\int_{\mathcal{K}}\mathbf{V}(\mathbf{x})\cdot\mathrm{d}\mathbf{x}&=\int_{\mathcal{K}}\{(y^2-z^2)\,\mathrm{d}x+2yz\,\mathrm{d}y-x^2\,\mathrm{d}z\}\\&=\int_0^1\{(t^2-t^2)+2t^2-t^2\}\,\mathrm{d}t=\int_0^1 t^2\,\mathrm{d}t=\frac{1}{3}.\end{aligned}$$

Example 32.10 *Calculate in each of the following cases the tangential line integral of the given vector field* $\mathbf{V}$ *along the given curve* $\mathcal{K}$.

1) *The vector field is* $\mathbf{V}(x,y) = (x+y, x-y)$*, and the curve* $\mathcal{K}$ *is the ellipse of centrum* $(0,0)$ *and half axes* a, b*, run through in the positive orientation of the plane.*

2) *The vector field is* $\mathbf{V}(x,y) = \left(\frac{1}{|x|+|y|}, \frac{1}{|x|+|y|}\right)$*, and the curve* $\mathcal{K}$ *is the square defined by its vertices*

$$(1,0), \quad (0,1), \quad (-1,0), \quad (0,-1),$$

in the positive orientation of the plane.

3) *The vector field is* $\mathbf{V}(x,y) = (x^2-y, y^2+x)$*, and the curve* $\mathcal{K}$ *is the line segment from* $(0,1)$ *to* $(1,2)$.

4) *The vector field is* $\mathbf{V}(x,y) = (x^2-y^2, y^2+x)$*, and the curve* $\mathcal{K}$ *is the broken line from* $(0,1)$ *over* $(1,1)$ *to* $(1,2)$.

5) *The vector field is* $\mathbf{V}(x,y) = (x^2-y, y^2+x)$*, and the curve* $\mathcal{K}$ *is that part of the parabola of equation* $y = 1+x^2$*, which has the initial point* $(0,1)$ *and the final point* $(1,2)$.

6) *The vector field is* $\mathbf{V}(x,y,z) = (yz\,,\, xz\,,\, x(y+1))$*, and the curve* $\mathcal{K}$ *is the triangle given by its vertices*

$$(0,0,0), \qquad (1,1,1), \qquad (-1,1,-1),$$

and run through as defined by the given sequence.

7) *The vector field is* $\mathbf{V}(x,y,z) = (\sin y, \sin z, \sin x)$*, and the curve* $\mathcal{K}$ *is the line segment from* $(0,0,0)$ *to* (π,π,π).

8) *The vector field is* $\mathbf{V}(x,y,z) = (z\,,\,x\,,\,-y)$*, and the curve* $\mathcal{K}$ *is the quarter circle from* $(a,0,0)$ *to* $(0,0,a)$ *followed by another quarter circle from* $(0,0,a)$ *to* $(0.a.0)$*, both of centrum* $(0,0,0)$.

A Tangential line integrals in the 2-dimensional and the 3-dimensional space.

D Sketch in the 2-dimensional case the curve $\mathcal{K}$. Then check if any part of $\mathbf{V}(\mathbf{x})\cdot \mathrm{d}\mathbf{x}$ can be sorted out as a total differential. Finally, insert the parametric description and calculate.

I 1) As $\mathcal{K}$ is a closed curve, we get

$$\int_{\mathcal{K}} \mathbf{V}(\mathbf{x})\cdot \mathrm{d}\mathbf{x} = \int_{\mathcal{K}} \{(x+y)\,\mathrm{d}x + (x-y)\,\mathrm{d}y\} = \int_{\mathcal{K}} \mathrm{d}\left(\frac{1}{2}\,x^2 + xy - \frac{1}{2}\,y^2\right) = 0,$$

because $\mathbf{V}\cdot \mathrm{d}\mathbf{x}$ is a total differential.

ALTERNATIVELY, $\mathcal{K}$ has e.g. the parametric description

$$(x,y) = \mathbf{r}(t) = (a\,\cos t, b\,\sin t), \qquad t \in [0, 2\pi],$$

hence

$$\mathbf{r}'(t) = (-a\,\sin t, b\,\cos t).$$

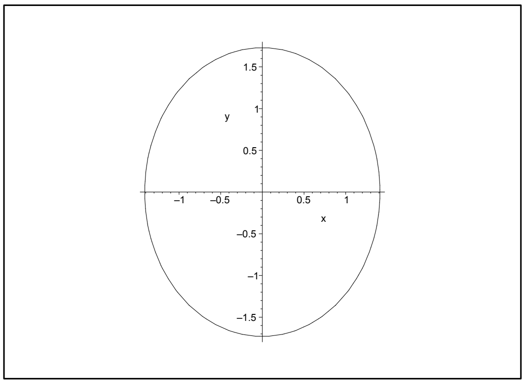

Figure 32.13: A possible curve $\mathcal{K}$ in **Example 32.10.1**.

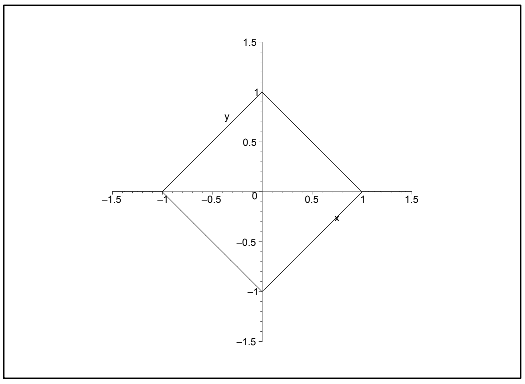

Figure 32.14: The curve of **Example 32.10.2**.

Then by insertion,

$$\int_{\mathcal{K}} \mathbf{V}\cdot d\mathbf{x} = \int_{\mathcal{K}} \{(x+y)\,dx + (x-y)\,dy\}$$

$$= \int_0^{2\pi} \{(a\cos t + b\sin t)(-a\sin t) + (a\cos t - b\sin t)b\cos t\}\,dt$$

$$= \int_0^{2\pi} \{-a^2\cos t\,\sin t - ab\sin^2 t + ab\cos^2 t - v^2\sin t\,\cos t\}\,dt$$

$$= \int_0^{2\pi} \left\{ab\cos 2t - \frac{1}{2}(a^2+b^2)\sin 2t\right\}\,dt = 0.$$

2) Since $|x|+|y|=1$ on $\mathcal{K}$, we have

$$\int_{\mathcal{K}} \mathbf{V}\cdot d\mathbf{x} = \int_{\mathcal{K}} \frac{1}{|x|+|y|}\,(dx+dy) = \int_{\mathcal{K}} 1\,d(x+y) = 0.$$

Alternatively, and more difficult we can use the parametric description of $\mathcal{K}$ given by

$$\mathbf{r}(t) = \begin{cases} (1-t,t), & t \in [0,1], \\ (1-t,2-t), & t \in [1,2], \\ (t-3,2-t), & t \in [2,3], \\ (t-3,t-4), & t \in [3,4]. \end{cases}$$

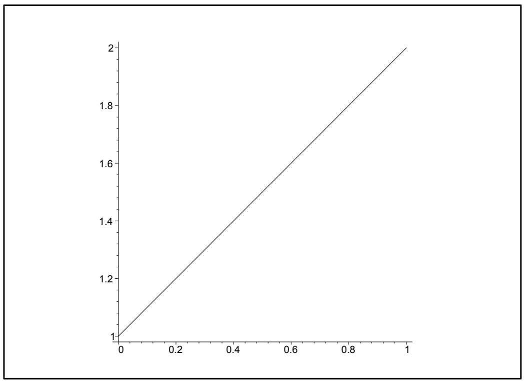

Figure 32.15: The curve $\mathcal{K}$ of **Example 32.10.3**

Then

$$\mathbf{r}'(t) = \begin{cases} (-1,1), & t \in]0,1[, \\ (-1,-1), & t \in]1,2[, \\ (1,-1), & t \in]2,3[, \\ (1,1), & t \in]3,4[. \end{cases}$$

Since $|x| + |y| = 1$ on $\mathcal{K}$, we get

$$\begin{aligned} \int_{\mathcal{K}} \mathbf{V} \cdot \mathrm{d}\mathbf{x} &= \int_{\mathcal{K}} (\,\mathrm{d}x + \,\mathrm{d}y) \\ &= \int_0^1 (-1+1)\,\mathrm{d}t + \int_1^2 (-1-1)\,\mathrm{d}t + \int_2^3 (1-1)\,\mathrm{d}t + \int_3^4 (1+1)\,\mathrm{d}t \\ &= 0 - 2 + 0 + 2 = 0. \end{aligned}$$

3) First note that

$$\begin{aligned} \int_{\mathcal{K}} \mathbf{V} \cdot \mathrm{d}\mathbf{x} &= \int_{\mathcal{K}} \{(x^2-y)\,\mathrm{d}x + (y^2+x)\,\mathrm{d}y\} = \frac{1}{3}\int_{\mathcal{K}} \mathrm{d}(x^3+y^3) + \int_{\mathcal{K}} (-y\,\mathrm{d}x + x\,\mathrm{d}y) \\ &= \frac{1}{3}(8+1-1) + \int_{\mathcal{K}} (-y\,\mathrm{d}x + x\,\mathrm{d}y), \end{aligned}$$

so

$$\int_{\mathcal{K}} \mathbf{V} \cdot \mathrm{d}\mathbf{x} = \frac{8}{3} + \int_{\mathcal{K}} (-y\,\mathrm{d}x + x\,\mathrm{d}y) \tag{32.3}$$

$$= \int_{\mathcal{K}} \{(x^2-y)\,\mathrm{d}x + (y^2+x)\,\mathrm{d}y\}. \tag{32.4}$$

Then we calculate **Example 32.10.3, Example 32.10.4** and **Example 32.10.5** in the two variants corresponding to (32.3) and (32.4), respectively.

A parametric description of $\mathcal{K}$ is e.g.

$$\mathbf{r}(t) = (t, 1+t), \qquad t \in [0,1],$$

and accordingly, $\mathbf{r}'(t) = (1,1)$.

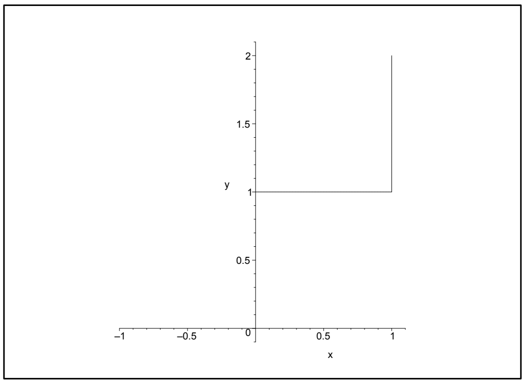

Figure 32.16: The curve $\mathcal{K}$ of **Example 32.10.4**.

Then by (32.3),

$$\int_{\mathcal{K}} \mathbf{V}\cdot \mathrm{d}\mathbf{x} = \frac{8}{3} + \int_0^1 \{-(1+t)+t\}\,\mathrm{d}t = \frac{8}{3} - 1 = \frac{5}{3}.$$

ALTERNATIVELY, we get by (32.4) that

$$\begin{aligned}\int_{\mathcal{K}} \mathbf{V}\cdot \mathrm{d}\mathbf{x} &= \int_0^1 \{t^2-(1+t)+(1+t)^2+t\}\,\mathrm{d}t = \int_0^1 \{(1+t)^2+t^2-1\}\,\mathrm{d}t \\ &= \left[\frac{1}{3}(1+t)^3+\frac{1}{3}t^3-t\right]_0^1 = \frac{8+1-1}{3}-1=\frac{5}{3}.\end{aligned}$$

4) It follows from (32.3) that

$$\int_{\mathcal{K}} \mathbf{V}\cdot \mathrm{d}\mathbf{x} = \frac{8}{3} + \int_0^1 (-1)\,\mathrm{d}x + \int_1^2 1\,\mathrm{d}y = \frac{8}{3} - 1 + 1 = \frac{8}{3}.$$

ALTERNATIVELY, we get by (32.4),

$$\begin{aligned}\int_{\mathcal{K}} \mathbf{V}\cdot \mathrm{d}\mathbf{x} &= \int_0^1 (x^2-1)\,\mathrm{d}x + \int_1^2 (y^2+1)\,\mathrm{d}y = \left[\frac{1}{3}x^3-x\right]_0^1 + \left[\frac{1}{3}y^3+y\right]_1^2 \\ &= \frac{1}{3}-1+\frac{8}{3}+2-\frac{1}{3}-1=\frac{8}{3}.\end{aligned}$$

5) By (32.3),

$$\begin{aligned}\int_{\mathcal{K}} \mathbf{V}\cdot \mathrm{d}\mathbf{x} &= \frac{8}{3} + \int_0^1 \{(-1-x^2)+x\cdot 2x\}\,\mathrm{d}x \\ &= \frac{8}{3} + \int_0^1 (x^2-1)\,\mathrm{d}x = \frac{8}{3} + \left[\frac{1}{3}x^3-x\right]_0^1 = \frac{8}{3}+\frac{1}{3}-1=2.\end{aligned}$$

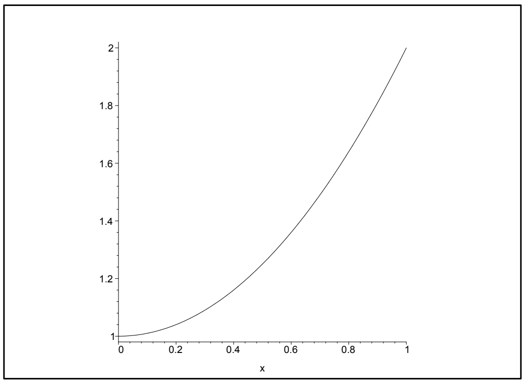

Figure 32.17: The curve $\mathcal{K}$ of **Example 32.10.5**.

ALTERNATIVELY, by (32.4),

$$\begin{aligned}\int_{\mathcal{K}} \mathbf{V}\cdot \mathrm{d}\mathbf{x} &= \int_{\mathcal{K}} \{x^2 - x^2 - 1 + [(x^2+1)^2 + x]\cdot 2x\}\,\mathrm{d}x \\ &= \int_{\mathcal{K}} \{2x^5 + 4x^3 + 2x^2 + 2x - 1\}\,\mathrm{d}x \\ &= \left[\frac{1}{3}x^6 + x^3 + \frac{2}{3}x^3 + x^2 - x\right]_0^1 = \frac{1}{3} + 1 + \frac{2}{3} + 1 - 1 = 2.\end{aligned}$$

6) Here a parametric description is e.g. given by

$$\mathbf{r}(t) = \begin{cases} (t,t,t), & t \in [0,1], \\ (3-2t\,,\,1\,,\,3-2t), & t \in [1,2], \\ (t-3,\,,-t+3\,,\,t-3), & t \in [2,3], \end{cases}$$

hence

$$\mathbf{r}'(t) = \begin{cases} (1,1,1), & t \in]0,1[, \\ (-2,0,-2), & t \in]1,2[, \\ (1,-1,1), & t \in]2,3[. \end{cases}$$

FIRST VARIANT. We get by direct insertion,

$$\begin{aligned}\int_{\mathcal{K}} \mathbf{V}\cdot \mathrm{d}\mathbf{x} &= \int_{\mathcal{K}} \{yz\,\mathrm{d}x + xz\,\mathrm{d}y + x(y+1)\,\mathrm{d}z\} \\ &= \int_0^1 (t^2+t^2+t^2+t)\,\mathrm{d}t + \int_1^2 \{\cdot(3-2t)\cdot(-2)+(3-2t)\cdot 2\cdot(-2)\}\,\mathrm{d}t \\ &\quad + \int_2^3 \{(-t+3)(t-3)\cdot 1 + (t-3)^2\cdot(-1) + (t-3)(-t+3)\cdot 1 + t - 3\}\,\mathrm{d}t \\ &= \int_0^1 (3t^2+t)dt - 6\int_1^2 (3-2t)dt - \int_2^3 \{3(t-3)^2 - (t-3)\}\,\mathrm{d}t \\ &= \left[t^3 + \frac{1}{2}t^2\right]_0^1 + 6\left[t^2 - 3t\right]_1^2 - \left[(t-3)^3 - \frac{1}{2}(t-3)^2\right]_2^3 \\ &= 1 + \frac{1}{2} + 6(4-6-1+3) + \left(-1 - \frac{1}{2}\right) = 0.\end{aligned}$$

2. variant. Reduction by removing a total differential.
As

$$\mathbf{V}\cdot d\mathbf{x} = yz\,dx + xz\,dy + xy\,dz + x\,dz = d(xyz) + x\,dz,$$

and as $\mathcal{K}$ is a closed curve, we have $\int_{\mathcal{K}} d(xyz) = 0$, so the calculations are simplified by removing $d(xyz)$:

$$\begin{aligned}\int_{\mathcal{K}} \mathbf{V}\cdot d\mathbf{x} &= \int_{\mathcal{K}} d(xyz) + \int_{\mathcal{K}} x\,dz = 0 + \int_{\mathcal{K}} x\,dz \\ &= \int_0^1 t\,dt + \int_1^2 (3-2t)\cdot(-2)\,dt + \int_2^3 (t-3)\,dt \\ &= \left[\frac{1}{2}\,t^2\right]_0^1 + \int_1^2 (4t-6)\,dt + \left[\frac{1}{2}\,(t-3)^2\right]_2^3 \\ &= \frac{1}{2} + \left[2t^2-6t\right]_1^2 - \frac{1}{2} = 8-12-2+6 = 0.\end{aligned}$$

Remark. The expressions would have been even simpler, if we did not insist on that the parametric intervals $[0,1]$, $[1,2]$, $[2,3]$ should follow each other. Instead one can split $\mathcal{K}$ into three subcurves

$$\begin{aligned}&\mathcal{K}_1: \quad \mathbf{r}_1(t) = (t,t,t), && t\in[0,1],\\ &\mathcal{K}_2: \quad \mathbf{r}_2(t) = (1-2t,1,1-2t), && t\in[0,1],\\ &\mathcal{K}_3: \quad \mathbf{r}_3(t) = (t-1,1-t,t-1), && t\in[0,1],\end{aligned}$$

where

$$\begin{aligned}
\mathcal{K}_1 &: \quad \mathbf{r}_1'(t) = (1,1,1), \qquad t \in]0,1[;\\
\mathcal{K}_2 &: \quad \mathbf{r}_2'(t) = (-2,0,2), \quad t \in]0,1[;\\
\mathcal{K}_3 &: \quad \mathbf{r}_3'(t) = (1,-1,1). \quad t \in]0,1[.
\end{aligned}$$

We obtain that the three line integrals can be joined like in the second variant above:

$$\begin{aligned}
\int_{\mathcal{K}} \mathbf{V} \cdot \mathrm{d}\mathbf{x} &= \cdots = \int_{\mathcal{K}} x\,\mathrm{d}z = \int_{\mathcal{K}_1} x\,\mathrm{d}z + \int_{\mathcal{K}_2} x\,\mathrm{d}z + \int_{\mathcal{K}_3} x\,\mathrm{d}z\\
&= \int_0^1 t\,\mathrm{d}t + \int_0^1 (1-2t)\cdot(-2)\,\mathrm{d}t + \int_0^1 (t-1)\,\mathrm{d}t\\
&= 3\int_0^1 (2t-1)dt = 3\left[t^2-t\right]_0^1 = 0. \qquad \diamond
\end{aligned}$$

7) The most obvious parametric description is here

$$\mathbf{r}(t) = t\,(1,1,1), \quad \text{with } \mathbf{r}'(t) = (1,1,1), \qquad t \in [0,\pi].$$

Thus we can put $x = y = z = t$ everywhere. Then

$$\int_{\mathcal{K}} \mathbf{V} \cdot \mathrm{d}\mathbf{x} = \int_{\mathcal{K}} \{\sin y\,\mathrm{d}x + \sin z\,\mathrm{d}y + \sin x\,\mathrm{d}z\} = \int_0^\pi 3\sin t\,\mathrm{d}t = [-3\cos t]_0^\pi = 6.$$

8) If we call the two curve segments $\mathcal{K}_1$ and $\mathcal{K}_2$, then the most obvious parametric description is

$$\begin{aligned}
\mathcal{K}_1 &: \quad a\,(\cos t, 0, \sin t), \quad t \in \left[0,\frac{\pi}{2}\right],\\
\mathcal{K}_2 &: \quad a\,(0, \sin t, \cos t), \quad t \in \left[0,\frac{\pi}{2}\right].
\end{aligned}$$

Then by insertion,

$$\begin{aligned}
\int_{\mathcal{K}} \mathbf{V} \cdot \mathrm{d}\mathbf{x} &= \int_{\mathcal{K}} (z\,\mathrm{d}x + x\,\mathrm{d}y - y\,\mathrm{d}z)\\
&= a^2 \int_0^{\frac{\pi}{2}} (\sin t \cdot (-\sin t))\,\mathrm{d}t + a^2 \int_0^{\frac{\pi}{2}} (-\sin t)\cdot(-\sin t)\,\mathrm{d}t\\
&= -a^2 \int_0^{\frac{\pi}{2}} \sin^2 t\,dt + a^2 \int_0^{\frac{\pi}{2}} \sin^2 t\,\mathrm{d}t = 0.
\end{aligned}$$

Example 32.11 *Find in each of the following cases a function*

$$\mathbf{\Phi}(x,y) = \int_{\mathcal{K}} \mathbf{V}(\tilde{\mathbf{x}}) \cdot \mathrm{d}\tilde{\mathbf{x}},$$

to the given vector field $\mathbf{V} : A \to \mathbb{R}^2$*, where* $\mathcal{K}$ *is the broken line which runs from* $(0,0)$ *over* $(x,0)$ *to* (x,y)*. Check if* $\mathbf{\Phi}$ *is defined in all of* A*, and calculate finally the gradient* $\bigtriangledown\mathbf{\Phi}$*.*

1) The vector field $\mathbf{V}(x,y) = (y^2 - 2xy, -x^2 + 2xy)$ *is defined in* $A = \mathbb{R}^2$*.*

2) The vector field $\mathbf{V}(x,y) = \left(\dfrac{1}{\sqrt{y^2 - x^2 + 1}}\,,\, x \right)$ *is defined in*

$$A = \{(x,y) \mid -\sqrt{1+y^2} < x < \sqrt{1+y^2}\}.$$

3) The vector field $\mathbf{V}(x,y) = \left(\dfrac{x}{\sqrt{1-x^2-y^2}}\,,\, \dfrac{y}{\sqrt{a-x^2-y^2}} \right)$ *is defined in the disc* A *given by* $x^2 + y^2 < 1$*.*

4) The vector field $\mathbf{V}(x,y) = \left(\dfrac{x-1}{\sqrt{(x-1)^2+y^2}}\,,\, \dfrac{y}{\sqrt{(x-1)^2+y^2}} \right)$ *in the set* A *given by* $(x,y) \neq (1,0)$*.*

5) The vector field $\mathbf{V}(x,y) = (\cos y, \cos x)$ *is defined in* $A = \mathbb{R}^2$*.*

6) The vector field $\mathbf{V}(x,y) = (\cos(xy), 0)$ *is defined in* $A = \mathbb{R}^2$*.*

7) The vector field $\mathbf{V}(x,y) = (x^2 + y^2\,,\, xy)$ *is defined in* $A = \mathbb{R}^2$*.*

8) The vector field $\mathbf{V}(x,y) = (x^2 + y^2\,,\, 2xy)$ *is defined in* $A = \mathbb{R}^2$*.*

A Tangential line integrals.

D Remove, whenever possible, total differentials. Integrate along a broken line. Finally, compute the gradient $\bigtriangledown\mathbf{\Phi}$.

I 1) We get by inspection,

$$\begin{aligned} \mathbf{\Phi}(x,y) &= \int_{\mathcal{K}} \mathbf{V}(\tilde{\mathbf{x}}) \cdot \mathrm{d}\tilde{\mathbf{x}} = \int_{\mathcal{K}} \{(\tilde{y}^2 - 2\tilde{x}\tilde{y})\,\mathrm{d}\tilde{x} + (-\tilde{x}^2 + 2\tilde{x}\tilde{y})\,\mathrm{d}\tilde{y}\} \\ &= \int_{\mathcal{K}} \mathrm{d}(\tilde{x}\tilde{y}^2 - \tilde{x}^2\tilde{y}) = xy^2 - x^2 y \qquad (= xy(y-x)). \end{aligned}$$

Alternatively,

$$\mathbf{\Phi}(x,y) = \int_{\mathcal{K}} \{(\tilde{y}^2 - 2\tilde{x}\tilde{y})\,\mathrm{d}\tilde{x} + (-\tilde{x}^2 + 2\tilde{x}\tilde{y})\,\mathrm{d}\tilde{y}\} = \int_0^x 0\,\mathrm{d}t + \int_0^y (-x^2 + 2xt)\,\mathrm{d}t = xy^2 - x^2 y.$$

Finally,

$$\bigtriangledown\mathbf{\Phi} = (y^2 - 2xy, 2xy - x^2) = \mathbf{V}(x,y),$$

and $\mathbf{\Phi}$ is defined in all of $A = \mathbb{R}^2$.

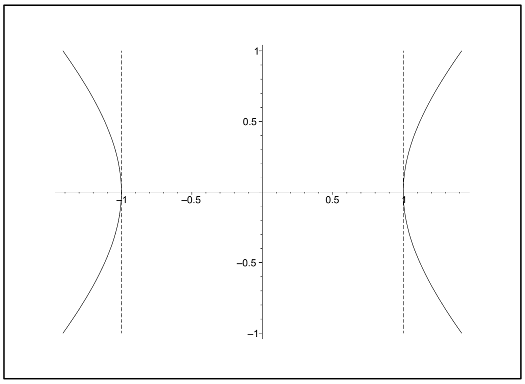

Figure 32.18: The domains of $\mathbf{V}(x,y)$ and $\mathbf{\Phi}(x,y)$.

2) The domain A for $\mathbf{V}(x,y)$ lies between the two hyperbolic branches given by $x^2-y^2=1$. The domain $\tilde{A}$ of $\mathbf{\Phi}(x,y)$ is smaller, in fact only the points lying between the two vertical lines $x=\pm 1$, because we can only reach these by curves of the type $\mathcal{K}$. (The curve $\mathcal{K}$ must never leave A, because we require that $\mathbf{V}$ is defined).

We get for $(x,y)\in\tilde{A}$,

$$(32.5)\quad \mathbf{\Phi}(x,y)=\int_0^x \frac{1}{\sqrt{1-t^2}}\,\mathrm{d}t+\int_0^y x\,\mathrm{d}t = \ \text{Arcsin}\, x + xy.$$

The function $\mathbf{\Phi}$ is only defined in $\tilde{A}$. In this subset of A we get

$$\nabla\mathbf{\Phi}(x,y)=\left(\frac{1}{\sqrt{1-x^2}}+y\,,\,x\right)\neq\mathbf{V}(x,y).$$

In particular, $\mathbf{V}(x,y)$ is not a gradient field.

Remark. Formula (32.5) is a mindless insertion into one of the solution formulæ for this type of problems. It cannot be applied here because the *assumptions* of it are *not* fulfilled. $\diamondsuit$

3) Here we get

$$\begin{aligned}\mathbf{\Phi}(x,y) &= \int_{\mathcal{K}}\left\{\frac{x}{\sqrt{1-x^2-y^2}}\,\mathrm{d}x+\frac{y}{\sqrt{1-x^2-y^2}}\,\mathrm{d}y\right\}\\ &= \int_{\mathcal{K}}\mathrm{d}\left(-\sqrt{1-x^2-y^2}\right)=1-\sqrt{1-x^2-y^2}.\end{aligned}$$

Alternatively we get for $x^2+y^2<1$ by an integration along the broken line that

$$\begin{aligned}\mathbf{\Phi}(x,y) &= \int_0^x\frac{t}{\sqrt{1-t^2}}\,\mathrm{d}t+\int_0^y\frac{t}{\sqrt{1-x^2-t^2}}\,\mathrm{d}t=\left[-\sqrt{1-t^2}\right]_0^x+\left[-\sqrt{1-x^2-t^2}\right]_0^y\\ &= 1-\sqrt{1-x^2}+\sqrt{1-x^2}-\sqrt{1-x^2-y^2}=1-\sqrt{1-x^2-y^2}.\end{aligned}$$

It follows immediately that

$$\nabla\mathbf{\Phi}(x,y)=\left(\frac{x}{\sqrt{1-x^2-y^2}}\,,\,\frac{y}{\sqrt{1-x^2-y^2}}\right)=\mathbf{V}(x,y),$$

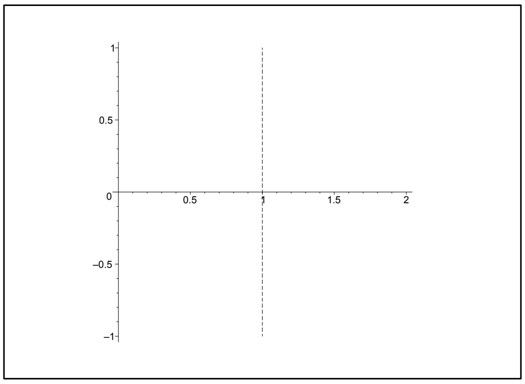

Figure 32.19: The domain $\tilde{A}$ of $\mathbf{\Phi}$ lies to the left of the dotted line $x = 1$.

and that $\mathbf{\Phi}$ is defined in all of A.

4) In this case we have for any curve $\mathcal{K}$ from $(0,0)$ in A that

$$\begin{aligned}\mathbf{\Phi}(x,y) &= \int_{\mathcal{K}} \left\{ \frac{x-1}{\sqrt{(x-1)^2+y^2}}\,\mathrm{d}x + \frac{y}{\sqrt{(x-1)^2+y^2}}\,\mathrm{d}y \right\} \\ &= \int_{\mathcal{K}} \mathrm{d}\left(\sqrt{(x-1)^2+y^2}\right) = \sqrt{(x-1)^2+y^2} - 1.\end{aligned}$$

If we only integrate along curves $\mathcal{K}$ of this type, then we can only reach points in

$$\tilde{A} = \{(x,y) \mid x < 1,\, y \in \mathbb{R}\}.$$

By integration along a broken line in this domain,

$$\begin{aligned}
\mathbf{\Phi}(x,y) &= \int_0^x \frac{t-1}{\sqrt{(t-1)^2+0^2}}\,\mathrm{d}t + \int_0^y \frac{t}{\sqrt{(x-1)^2+t^2}}\,\mathrm{d}t \\
&= \int_0^x \frac{t-1}{|t-1|}\,\mathrm{d}t + \left[\sqrt{(x-1)^2+t^2}\right]_0^y \\
&= \int_0^x (-1)\,\mathrm{d}t + \left[\sqrt{(x-1)^2+y^2} - \sqrt{(x-1)^2}\right] \qquad \text{(because } t < x < 1\text{)} \\
&= -x - |x-1| + \sqrt{(x-1)^2+y^2} = x-(1-x)+\sqrt{(x-1)^2+y^2} \\
&= \sqrt{(x-1)^2+y^2} - 1.
\end{aligned}$$

It follows that $\triangledown\mathbf{\Phi} = \mathbf{V}$ and that $\mathbf{\Phi}$ can be extended to all of A.

5) When we integrate along the broken line

$$(0,0) \longrightarrow (x,0) \longrightarrow (x,y)$$

we get

$$\mathbf{\Phi}(x,y) = \int_{\mathcal{K}} \mathbf{V}\cdot\mathrm{d}\mathbf{x} = \int_0^x \cos 0\,\mathrm{d}t + \int_0^y \cos x\,\mathrm{d}t = x + y\,\cos x,$$

which is defined in all of $\mathbb{R}^2$. Here,

$$\triangledown\mathbf{\Phi}(x,y) = (1 - y\,\sin x, \cos x) \neq \mathbf{V}.$$

It is seen that $\mathbf{V}$ is not a gradient field.

6) When we integrate along the broken line

$$(0,0) \longrightarrow (x,0) \longrightarrow (x,y)$$

we get in all of $\mathbb{R}^2$,

$$\mathbf{\Phi}(x,y) = \int_{\mathcal{K}} \mathbf{V}\cdot\mathrm{d}\mathbf{x} = \int_0^x \cos(t\cdot 0)\,\mathrm{d}t + 0 = x,$$

where $\triangledown\mathbf{\Phi} = (1,0) \neq \mathbf{V}$, so $\mathbf{V}$ is not a gradient field.

7) When we integrate along the broken line

$$(0,0) \longrightarrow (x,0) \longrightarrow (x,y)$$

we get in all of $\mathbb{R}^2$,

$$\mathbf{\Phi}(x,y) = \int_{\mathcal{K}} \mathbf{V}\cdot\mathrm{d}\mathbf{x} = \int_0^x (t^2+0^2)dt + \int_0^y xt\,\mathrm{d}t = x^3 + \frac{1}{2}\,xy^2,$$

where

$$\triangledown\mathbf{\Phi} = \left(3x^2 + \frac{1}{2}\,y^2, xy\right) \neq \mathbf{V}(x,y).$$

It follows that $\mathbf{V}$ is not a gradient field.

8) When we integrate along the broken line

$$(0,0) \longrightarrow (x,0) \longrightarrow (x,y)$$

we get in $\mathbb{R}^2$

$$\mathbf{\Phi}(x,y) = \int_{\mathcal{K}} \mathbf{V} \cdot \mathrm{d}\mathbf{x} = \int_0^x (t^2+0^2)\,\mathrm{d}t + \int_0^y 2xt\,\mathrm{d}t = \frac{x^3}{3} + xy^2,$$

where

$$\nabla \mathbf{\Phi} = (x^2+y^2, 2xy) = \mathbf{V}(x,y).$$

In this case $\mathbf{V}(x,y)$ is a gradient field.

Example 32.12 *Calculate in each of the following cases the tangential line integral of the given vector field* $\mathbf{V} : \mathbb{R}^2 \to \mathbb{R}^2$ *along the described curve* $\mathcal{K}$.

1) *The vector field* $\mathbf{V}(x,y) = \nabla\left(\frac{1}{2}\,x^2 + xy - \frac{1}{2}\,y^2\right)$ *along the ellipse* $\mathcal{K}$ *of centrum* $(0,0)$ *and half axes* a, b, *in the positive orientation of the plane.*

2) *The vector field* $\mathbf{V}(x,y) = \nabla\left(x^4 + \ln(1+y)\right)$ *along the arc of the parabola* $\mathcal{K}$ *given by* $y = x^2$, $x \in [-1,3]$.

3) *The vector field* $\mathbf{V}(x,y) = \nabla(x+2y-\exp(xy))$ *along the broken line* $\mathcal{K}$, *which goes from* $(2,0)$ *over* $(1,2)$ *to* $(0,1)$.

A Line integral of a gradient field.

D As $\mathbf{V}(x,y) = \nabla F$, the tangential line integral is only depending on the initial point and the end point,

$$\int_{\mathcal{K}} \mathbf{V} \cdot \mathrm{d}\mathbf{x} = F(\mathbf{x}_s) - F(\mathbf{x}_b).$$

I 1) The ellipse is a closed curve, so

$$\int_{\mathcal{K}} \mathbf{V} \cdot \mathrm{d}\mathbf{x} = 0.$$

2) The initial point is $(-1,1)$, and the end point is $(3,9)$, hence

$$\int_{\mathcal{K}} \mathbf{V} \cdot \mathrm{d}\mathbf{x} = \left[x^4 + \ln(1+y)\right]_{(-1,1)}^{(3,9)} = 81 + \ln 10 - 1 - \ln 2 = 80 + \ln 5.$$

3) The initial point is $(2,0)$, and the end point is $(0,1)$, hence

$$\int_{\mathcal{K}} \mathbf{V} \cdot \mathrm{d}\mathbf{x} = [x+2y-\exp(xy)]_{(2,0)}^{(0,1)} = 0+2-1-2-0+1 = 0.$$

Example 32.13 *Calculate in each of the following cases the tangential line integral of the given vector field* $\mathbf{V} : \mathbb{R}^3 \to \mathbb{R}^3$ *along the described curve* $\mathcal{K}$.

1) *The vector field* $\nabla(x^2 + yz)$ *along the curve* $\mathcal{K}$, *given by*

$$\mathbf{r}(t) = (\cos t, \sin t, \sin(2t)), \qquad t \in [0, 2\pi].$$

2) *The vector field* $\nabla(\cos(xyz))$ *along the line segment* $\mathcal{K}$ *from* $\left(\pi, \frac{1}{2}, 0\right)$ *to* $\left(\frac{1}{2}, \pi, -1\right)$.

3) *The vector field* $\nabla(\exp x + \ln(1 + |yz|)$ *along the broken line* $\mathcal{K}$, *which goes from* $(0, 1, 1)$, *via* $(\pi, -3, 2)$ *to* $(1, \sqrt{3}, -\sqrt{3})$.

A Tangential line integrals of gradient fields.

D Use that

$$\int_{\mathcal{K}} \nabla F \cdot \mathrm{d}\mathbf{x} = F(\text{end point}) - F(\text{initial point}),$$

is independent of the path of integration.

Since the absolute value occurs in **Example 32.13.3**, we shall here be very careful.

I 1) As $\mathcal{K}$ is a closed curve (i.e. the initial point $(1, 0, 0)$ is equal to the end point), it follows that

$$\int_{\mathcal{K}} \mathbf{V} \cdot \mathrm{d}\mathbf{x} = 0.$$

2) Since $F(x,y,z) = \cos(xyz)$, and the initial point and the end point are given, we have

$$\int_{\mathcal{K}} \mathbf{V}\cdot \mathrm{d}\mathbf{x} = \cos\left(\frac{1}{2}\cdot\pi\cdot(-1)\right) = -\cos\left(\pi\cdot\frac{1}{2}\cdot 0\right) = -1.$$

3) First note that

$$F(x,y,z) = \begin{cases} \exp x + \ln(1+yz) & \text{for } yz > 0, \\ \exp x + \ln(1-yz) & \text{for } yz < 0, \end{cases}$$

so we must be very careful, whenever the curve $\mathcal{K}$ intersects one of the planes $y=0$ or $z=0$. In case of the first curves this can occur, because the parametric description is

$$t(0,1,1) + (1-t)(\pi,-3,2) = ((1-t)\pi, 4t-3, 2-t), \quad t\in[0,1],$$

and the same is true for the second curve, because it has the parametric description

$$t(\pi,-3,2) + (1-t)(1,\sqrt{3},-\sqrt{3}) = (1+t(\pi-1), \sqrt{3}-t(3+\sqrt{3}), -\sqrt{3}+t(2+\sqrt{3})),$$

for $t\in[0,1]$.

The former curve intersects the plane $y=0$ for $t=\dfrac{3}{4}$, and the latter curve intersects both the plane $y=0$ and the plane $z=0$. The point is, however, that in everyone of these intersection points the dubious term $\ln(1+|xy|)=0$, so they are of no importance. Hence we can conclude that

$$\begin{aligned}\int_{\mathcal{K}} \mathbf{V}\cdot \mathrm{d}\mathbf{x} &= [\exp x + \ln(1+|yz|]_{(0,1,1)}^{(1,\sqrt{3},-\sqrt{3})} \\ &= e + \ln 4 - 1 - \ln 2 = e - 1 + \ln 2.\end{aligned}$$

Remark. Always be very careful when either the absolute value or the square root occur. One should at least give a note on them. $\Diamond$

Example 32.14 *Given the vector field*

$$\mathbf{V}(x,y) = \left(\frac{2y}{2x+y}, \frac{y}{2x+y} + \ln(2x+y) \right).$$

1. *Sketch the domain of* $\mathbf{V}$*, and explain why* $\mathbf{V}$ *is a gradient field.*

2. *Find every integral of* $\mathbf{V}$*.*

Let $\mathcal{K}$ *be the curve given by*

$$(x,y) = (2t^2, t), \qquad 1 \le t \le 2.$$

3. *Compute the value of the tangential line integral*

$$\int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} \, \mathrm{d}s.$$

Let F *be the integral of* $\mathbf{V}$*, for which* $F(1,1) = 0$*.*

4. *Find an equation of the tangent at the en point* $(1,1)$ *of that level curve for* F*, which goes through the point* $(1,1)$*.*

A Gradient field, integrals, tangential line integral, level curve.

D Follow the guidelines.

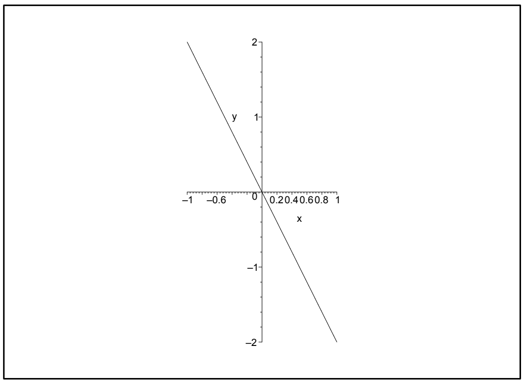

Figure 32.20: The domain is the open half plane above the oblique line.

I 1) Clearly, $\mathbf{V}(x,y)$ is defined in the domain where $2x + y > 0$, cf. the figure.

As
$$\frac{\partial V_1}{\partial y} = \frac{2}{2x+y} - \frac{2y}{(2x+y)^2},$$
and
$$\frac{\partial V_2}{\partial x} = -\frac{2y}{(2x+y)^2} + \frac{2}{2x+y} = \frac{\partial V_1}{\partial y},$$
it follows that $V_1 \, \mathrm{d}x + V_2 \, \mathrm{d}y$ is a closed differential form. Since the domain is simply connected, the differential form is even exact, and $\mathbf{V}$ is a gradient field.

2) Since

$$F_1(x,y) = \int \frac{2y}{2x+y}\,\mathrm{d}x = y\int \frac{2\,\mathrm{d}x}{2x+y} = y\,\ln(2x+y), \qquad 2x+y>0,$$

where

$$\bigtriangledown F_1 = \left(\frac{2y}{2x+y}, \frac{y}{2x+y} + \ln(2x+y)\right) = \mathbf{V}(x,y),$$

all integrals are given by

$$F(x,y) = y\,\ln(2x+y) + C, \qquad C \in \mathbb{R}.$$

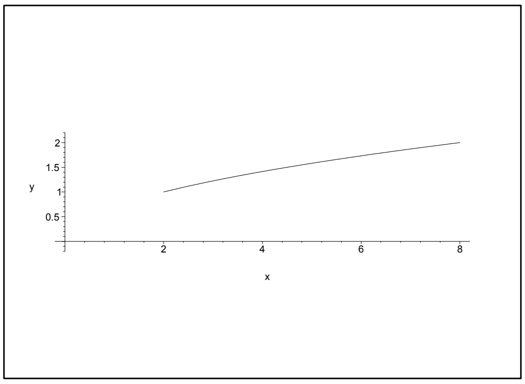

Figure 32.21: The curve $\mathcal{K}$.

3) We get by the reduction theorem for tangential line integrals that

$$\begin{aligned}
\int_{\mathcal{K}} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s &= \int_1^2 \mathbf{V}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,\mathrm{d}t \\
&= \int_1^2 \left(\frac{2t}{4t^2+t}, \frac{t}{4t^2+t} + \ln(4t^2+t)\right)\cdot(4t,1)\,\mathrm{d}t \\
&= \int_1^2 \left\{\frac{8t}{4t+1} + \frac{1}{4t+1} + \ln(4t^2+t)\right\}\,\mathrm{d}t \\
&= \int_1^2 \left\{2 - \frac{1}{4t+1} + \ln t + \ln(4t+1)\right\}\,\mathrm{d}t \\
&= 2 - \frac{1}{4}[\ln(4t+1)]_1^2 + [t\ln t - t]_1^2 + [t\ln(4t+1)]_1^2 - \int_1^2 \frac{4t}{4t+1}\,\mathrm{d}t \\
&= 2 - \frac{1}{4}[\ln(4t+1)]_1^2 + 2\ln 2 - 1 + 2\ln 9 - \ln 5 - 1 + \frac{1}{4}[\ln(4t+1)]_1^2 \\
&= 2\ln 2 + 4\ln 3 - \ln 5 = \ln\frac{324}{5}.
\end{aligned}$$

4) It follows from $F(1,1) = \ln 3 + C = 0$ that $C = -\ln 3$, so

$$F(x,y) = y\,\ln(2x+y) - \ln 3.$$

However, we shall not need the exact value of $C = -\ln 3$ in the following.

The normal of the level curve is $\bigtriangledown F = \mathbf{V}$, hence

$$\mathbf{V}(1,1) = \left(\frac{2}{3}, \frac{1}{3} + \ln 3\right),$$

and the direction of the tangent is e.g.

$$\mathbf{v} = \left(\frac{1}{3} + \ln 3, -\frac{2}{3}\right),$$

and we get a parametric description of the tangent,

$$(x(t), y(t)) = (1,1) = t\left(\frac{1}{3} + \ln 3, -\frac{2}{3}\right), \qquad t \in \mathbb{R}.$$

If we instead want an equation of the tangent, then one possibility is given by

$$0 = \mathbf{V} \cdot (x-1, y-1) = \frac{2}{3}\,x + \left(\frac{1}{3} + \ln 3\right) y - 1 - \ln 3.$$

33 Flux and divergence of a vector field. Gauß's theorem

33.1 Flux

Let $\mathbf{V}$ be a C^0 vector field in a domain in $\mathbb{R}^3$, and let $\mathcal{F}$ be a C^1 surface in this domain. Then we can define a continuous unit vector field $\mathbf{n}$ of unit normal vectors to $\mathcal{F}$.

The *flux* of $\mathbf{V}$ through $\mathcal{F}$ with respect to this vector field of normals (there are locally two possibilities of the orientation of $\mathbf{n}$) is defined as the surface integral of $\mathbf{V}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})$ over $\mathcal{F}$, also denoted

$$\int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n} \, \mathrm{d}S, \qquad \text{or} \qquad \int_{\mathcal{F}} \mathbf{V}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) \, \mathrm{d}S, \qquad \text{or} \qquad \int_{\mathcal{F}} \mathbf{V} \cdot \, \mathrm{d}\mathbf{S},$$

where we have put $\mathrm{d}\mathbf{S} := \mathbf{n} \, \mathrm{d}S$ for the *vectorial element of area.*

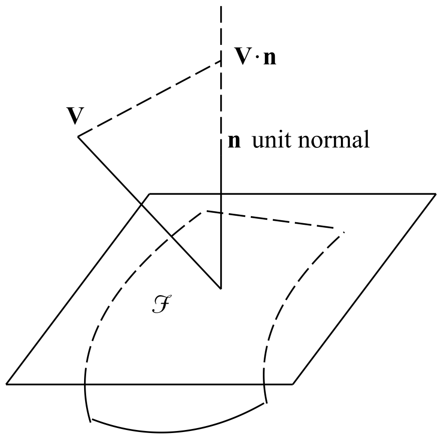

Figure 33.1: Illustration of the flux. We integrate the dot product $\mathbf{V} \cdot \mathbf{n}$ over the surface $\mathcal{F}$.

The flux describes the flow of the vector field $\mathbf{V}$ through the surface $\mathcal{F}$ in the direction of the normal vector field $\mathbf{n}$. It is obvious that if we replace $\mathbf{n}$ by the opposite normal vector field $-\mathbf{n}$, then the flux changes its sign.

We mention a couple of examples from Physics. If e.g. $\mathbf{V} = \mathbf{J}$ is the density of an electric current, then $\int_{\mathcal{F}} \mathbf{J} \cdot \mathbf{n} \, \mathrm{d}S$ is the electric current passing through $\mathcal{F}$ (measured in the direction of $\mathbf{n}$). If instead $\mathbf{V} = \mathbf{B}$ is a magnetic field, then $\int \mathcal{F} \mathbf{B} \cdot \mathbf{n} \, \mathrm{d}S$ is the magnetic flux through the surface $\mathcal{F}$ (also measured in the direction of $\mathbf{n}$).

Then we shall see how the flux in practice is calculated, when we introduce coordinates. Let $\mathbf{r} : E \to \mathbb{R}^3$ be a (rectangular) C^1 parametric description of the surface $\mathcal{F}$ in the parameters $(u, v) \in E \subseteq \mathbb{R}^2$. We have previously shown that

$$\mathbf{N}(u, v) := \mathbf{r}'_u(u, v) \times \mathbf{r}'_v(u, v)$$

is a normal to $\mathcal{F}$, provided that $\mathbf{N}(u, v) \neq \mathbf{0}$. Hence, the unit normal field $\mathbf{n}$ is for $\mathbf{N} \neq \mathbf{0}$ given by

$$\mathbf{n} := \frac{\mathbf{N}}{\|\mathbf{N}\|}, \qquad \text{and} \qquad \mathrm{d}S = \|\mathbf{N}\| \, \mathrm{d}u \, \mathrm{d}v.$$

It follows immediately that the vectorial area element is

$$\mathbf{n}\,\mathrm{d}S = \mathbf{N}\,\mathrm{d}u\,\mathrm{d}v.$$

We note that as usual that we may allow $\mathbf{N}(u,v) = \mathbf{0}$ in the applications as long as this only takes place in a null set, i.e. we only require that $\mathbf{N}(u,v) \neq \mathbf{0}$ almost everywhere. Then we have (again, the proof is omitted),

Theorem 33.1 Reduction of a flux of a vector field. *Let $A \subseteq \mathbb{R}^3$ be an open domain, and let $\mathbf{V} : A \to \mathbb{R}^3$ be a C^0 vector field. Finally let $\mathcal{F} \subset A$ be a continuous and piecewise C^1 surface of the parametric description $\mathbf{r} : E \to \mathbb{R}^3$, where we assume that E is a closed and bounded domain in the (u,v)-plane, that $\mathbf{r}$ is injective almost everywhere, and where the normal vector field $\mathbf{N}(u,v) \neq \mathbf{0}$ almost everywhere.*
Then the flux of $\mathbf{V}$ through $\mathcal{F}$ in the direction of the normal vector field $\mathbf{N}$ can be calculated by the following plane integral over E,

$$\int_{\mathcal{F}} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S = \int_E \mathbf{V}(\mathbf{r}(u,v))\cdot\mathbf{N}(u,v)\,\mathrm{d}u\,\mathrm{d}v.$$

This is very important, so we include a couple of examples to exercise the method.

Example 33.1 We shall find the flux Φ of the vector field

$$\mathbf{V}(x,y,z) = \left(yz, -xz, x^2+y^2\right), \qquad \text{for } (x,y,z) \in \mathbb{R}^3,$$

through the surface $\mathcal{F}$, given by the parametric description

$$\mathbf{r}(u,v) = (u\,\sin v, u\,\cos v, uv), \qquad \text{for } 0 \le u \le 1 \text{ and } 0 \le v \le u.$$

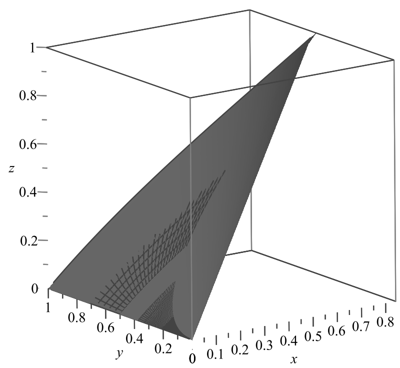

Figure 33.2: The surface $\mathcal{F}$ of Example 33.1

Since

$$\mathbf{r}'_u(u,v) = (\sin v, \cos v, v), \qquad \text{and} \qquad \mathbf{r}'_v(u,v) = (u\,\cos v, -u\,\sin v, u),$$

we get

$$\mathbf{N}(u,v) = \begin{vmatrix} \sin v & \cos v & v \\ u\,\cos v & -u\,sinv & v \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{vmatrix} = u(\cos v + v\,\sin v, v, \cos v - \sin v, -1),$$

so $\mathbf{N}(0,0) = \mathbf{0}$ and $\mathbf{N}(u,v) \neq \mathbf{0}$ elsewhere in E.

The vector field $\mathbf{V}$ restricted to $\mathcal{F}$ is described in the parameters (u,v) as

$$\mathbf{V}(\mathbf{r}(u,v)) = \left(u^2 v \cos v, -u^2 v \sin v, u^2\right) = u^2(v\cos v, -v\sin v, 1),$$

so

$$\mathbf{V}(\mathbf{r}(u,v)) \cdot \mathbf{N}(u,v) = u^3\left(v\cos^2 v + v^2 \sin v \cos v - v^2 \sin v \cos v + v^2 \sin^2 v - 1\right) = u^3\left(v^2 - 1\right),$$

and the flux of $\mathbf{V}$ through $\mathcal{F}$ is given by

$$\begin{aligned}\Phi &= \int_E \mathbf{V}(\mathbf{r}(u,v)) \cdot \mathbf{N}(u,v)\,\mathrm{d}u\,\mathrm{d}v = \int_E u^3\left(v^2-1\right)\,\mathrm{d}u\,\mathrm{d}v \\ &= \int_0^1 \left\{\int_0^u (v-a)\,\mathrm{d}v\right\} u^3\,\mathrm{d}u = \int_0^1 \left\{\frac{1}{2}u^5 - u^4\right\}\mathrm{d}u = \frac{1}{12} - \frac{1}{5} = -\frac{7}{60}, \qquad \Diamond\end{aligned}$$

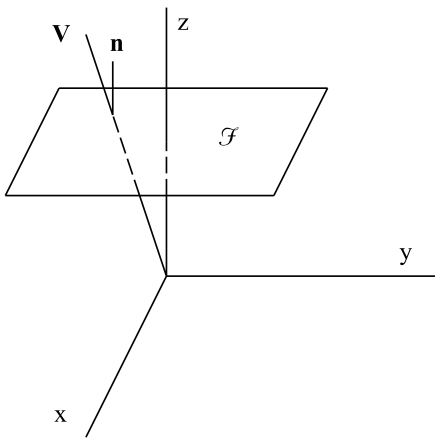

Figure 33.3: The flux of the Coulomb field through $\mathcal{F}$ in Example 33.2

Example 33.2 The *Coulomb field* or *Newton field* is defined by

$$\mathbf{V}_k(x,y,z) := k\,\frac{(x,y,z)}{(x^2+y^2+z^2)^{3/2}} \qquad \text{for } (x,y,z) \neq (0,0,0),$$

where $k \neq 0$ is some given constant. Its direction is always directed away from $\mathbf{0}$, and

$$\|\mathbf{V}\| = \frac{|k|}{\|\mathbf{x}\|^2}.$$

We choose $k = 1$ in the following and write $\mathbf{V}$ instead of $\mathbf{V}_1$. We shall find the flux of $\mathbf{V}$ through the surface $\mathcal{F}$, which is the following square at height $a > 0$, and given by

$$\mathcal{F}: \qquad [-a,a]\times[-a,a]\times\{a\}, \qquad \text{with the unit normal } \mathbf{n} = (0,0,1).$$

The flux is

$$\begin{aligned}\Phi &:= \int_{\mathcal{F}} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S = \int_{-a}^{a}\left\{\int_{-a}^{a}\left[\frac{(x,y,z)\cdot(0,0,1)}{(x^2+y^2+z^2)^{3/2}}\right]_{z=a}\mathrm{d}x\right\}\mathrm{d}y \\ &= \int_{-a}^{a}\left\{\int_{-a}^{a}\frac{a}{(x^2+y^2+a^2)^{3/2}}\,\mathrm{d}x\right\}\mathrm{d}y.\end{aligned}$$

It follows from the symmetry that this plane integral is 8 times the plane integral restricted to the triangle T on Figure 33.4. Since the two variables (x, y) only occur in the integrand in the form of $x^2 + y^2$, it is natural to change to polar coordinates, $\varrho^2 = x^2 + y^2$. This means that we should use polar coordinates on the triangle T. This is given in polar coordinates by

$$T: \qquad 0 \le \varphi \le \frac{\pi}{4}, \qquad \text{and} \qquad 0 \le \varrho \le \frac{a}{\cos\varphi}.$$

Then the flux becomes, where we remember the weight function ϱ,

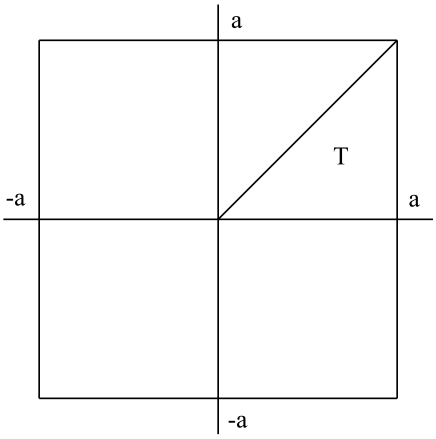

Figure 33.4: The restricted domain T of Example 33.2

$$\begin{aligned}
\varphi &= 8\int_T \frac{a}{(x^2+y^2+a^2)^{3/2}}\,\mathrm{d}x\,\mathrm{d}y = 8\int_0^{\frac{\pi}{4}} \left\{ \int_0^{\frac{a}{\cos\varphi}} \frac{a\varrho}{(a^2+\varrho^2)^{3/2}}\,\mathrm{d}\varrho \right\} \mathrm{d}\varphi \\
&= 8\int_0^{\frac{\pi}{4}} \left[\frac{a}{\sqrt{a^2+\varrho^2}} \right]_{\frac{a}{\cos\varphi}}^{0} \mathrm{d}\varphi = 8\int_0^{\frac{\pi}{4}} \left\{ 1 - \frac{|\cos\varphi|}{\sqrt{1+\cos^2\varphi}} \right\} \mathrm{d}\varphi \\
&= 8\left(\frac{\pi}{4} - \int_0^{\frac{\pi}{4}} \frac{\cos\varphi}{\sqrt{2-\sin^2\varphi}}\,\mathrm{d}\varphi \right) = 2\pi - 8\int_0^{\frac{\pi}{4}} \frac{\cos\varphi}{\sqrt{2-\sin^2\varphi}}\,\mathrm{d}\varphi.
\end{aligned}$$

If we change variable from φ to ψ by putting $\sin\varphi = \sqrt{2}\sin\psi$, then we get for $\varphi = \dfrac{\pi}{4}$ that $\sin\psi = \dfrac{1}{2}$, so $\psi = \dfrac{\pi}{6}$. The ψ-interval becomes $\psi \in \left[0, \dfrac{\pi}{6}\right]$, and the flux is

$$\begin{aligned}
\Phi &= 2\pi - 8\int_0^{\frac{\pi}{4}} \frac{\cos\varphi}{\sqrt{2-\sin^2\varphi}}\,d\varphi = 2\pi - 8\int_{\varphi=0}^{\frac{\pi}{4}} \frac{\sqrt{2}\,d\sin\psi}{\sqrt{2-2\sin^2\psi}} \\
&= 2\pi - 8\int_{\psi=0}^{\frac{\pi}{6}} \frac{\cos\psi}{\sqrt{1-\sin^2\psi}}\,d\psi = 2\pi - 8\int_0^{\frac{\pi}{6}} \frac{\cos\psi}{\cos\psi}\,d\psi = 2\pi - \frac{8\pi}{6} = \frac{2\pi}{3}. \qquad \lozenge
\end{aligned}$$

33.2 Divergence and Gauß's theorem

We shall in this section consider a closed continuous and piecewise C^1 surface $\mathcal{F}$, which we also assume to be the boundary of a bounded domain $\Omega \subset \mathbb{R}^3$ in space. Furthermore, we assume that the unit normal vector field $\mathbf{n}$ on $\mathcal{F}$ is defined almost everywhere. We shall for a bounded surface which is the boundary of some domain Ω, i.e. $\mathcal{F} = \partial\Omega$, always assume that $\mathbf{n}$ is oriented away from Ω.

Let $\mathbf{V}$ be a C^1 vector field defined on an open domain containing the closure $\overline{\Omega}$. Then the *flux* of $\mathbf{V}$ through $\mathcal{F} = \partial\Omega$, i.e.

$$\Phi := \int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n} \, \mathrm{d}S,$$

is interpreted as the flow of something, which flows out of Ω through the boundary surface $\mathcal{F} = \partial\Omega$. Clearly, this is of interest in Physics, so we shall here analyze this situation more closely in order to obtain an alternative way of calculating the flux. It is obvious that we in some sense must take into account what is created inside Ω, so we can expect that the result is a space integral of some integrand derived by a mathematical process from $\mathbf{V}$.

We shall start the analysis with a very simple case, where we assume that Ω is an axiparallel parallelepipedum. Using if necessary a translation we may assume that Ω is described by

$$\omega = [0, a] \times [0, b] \times [0, c], \qquad \text{for some } a,\, b,\, c > 0.$$

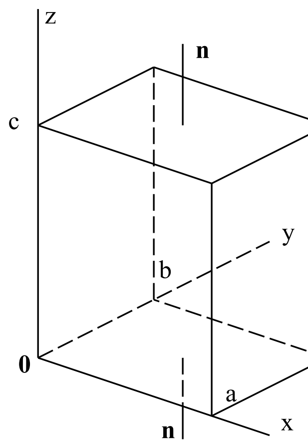

Figure 33.5: The flux of $\mathbf{V}$ through the two parallel surfaces of height $z = 0$ and $z = c$ of the parallelepipedum ω. Note that the field $\mathbf{n}$ is pointing in opposite directions on the two surfaces.

We shall first consider the flux through the surfaces of ω of height $z = 0$ and $z = c$. We clearly have $\mathbf{n} = (0, 0, -1)$ on the surface, for which $z = 0$, and $\mathbf{n} = (0, 0, 1)$ on the plane surface, for which $z = c$, cf. Figure 33.5. Hence, the combined contribution to the flux from these two opposite surfaces is

$$\begin{aligned}\Phi_z &= \int_0^a \left\{ \int_0^b V_z(x,y,c)\,\mathrm{d}y \right\} + \int_0^a \left\{ \int_0^b (-1)V_z(x,y,0)\,\mathrm{d}y \right\} \\ &= \int_0^a \left\{ \int_0^b [V_z(x,y,z)]_{z=0}^{z=c}\,\mathrm{d}y \right\}\mathrm{d}x = \int_0^a \left\{ \int_0^b \left\{ \int_0^c \frac{\partial V_z}{\partial z}(x,y,z) \right\}\mathrm{d}y \right\}\mathrm{d}x \\ &= \int_\omega \frac{\partial V_z}{\partial z}(x,y,z)\,\mathrm{d}\Omega.\end{aligned}$$

Similarly,

$$\Phi_y = \int_\omega dfrac \partial V_y \partial y(x,y,z)\,\mathrm{d}\Omega \quad \text{and} \quad \Phi_z = \int_\omega dfrac \partial V_z \partial z(x,y,z)\,\mathrm{d}\Omega,$$

so the total flux Φ of $\mathbf{V}$ through the boundary surface of ω is

$$\Phi = \Phi_x + \varphi_y + \Phi_z = \int_\omega \left\{ \frac{V_x}{\partial x} + \frac{V_y}{\partial y} + \frac{V_z}{\partial z} \right\} da\omega.$$

This result is valid for every axiparallel parallelepipedum.

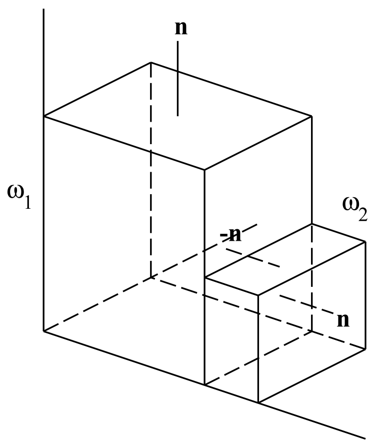

Figure 33.6: The flux of $\mathbf{V}$ through the surface of the union of two axiparallel parallelepipeda. The contribution to the total flux is cancelled on the common surface, because the only change in the integrand is the unit normal, which changes its sign.

When we calculate the total flux of two adjacent axiparallel parallelepipeda, we see that on a common surface the total contribution to the flux is zero, because the only change in the integrand is the sign of the unit normal vector field, which is + on the surface of ω_1 and - on the surface of ω_2.

By iterating this result we see in general, that if Ω is a union of finitely many axiparallel parallelepipeda, then the total contribution from every "inner surface" must be zero, i.e. when there are two smaller parallelepipeda ω_i and ω_j with a common surface. These "inner surfaces" also disappear in $\partial\Omega$, so we can replace $\bigcup_{j=1}^k \partial\omega_j$ with $\partial \bigcup_{j=1}^k \omega_j = \partial\Omega$.

The argument above makes it *plausible* that when we approximate Ω from the inside with finite unions $\bigcup_{j=1}^{n} \omega_j$ of such parallelepipeda, and let $n \to +\infty$, then we may *expect* that the flux of $\mathbf{V}$ through a general $\partial\Omega$ is given by

$$\Phi := \int_{\partial\Omega} \mathbf{V} \cdot \mathbf{n} \, \mathrm{d}S = \int_{\Omega} \left\{ \frac{V_x}{\partial x} + \frac{V_y}{\partial y} + \frac{V_z}{\partial z} \right\} \mathrm{d}\Omega,$$

and although the above is far from a correct proof, this is true under the assumptions we shall state below in Theorem 33.2. However, since the integrand on the right hand side keeps occurring in many cases, we first shorten the notation by giving it a name, which in the general $\mathbb{R}^n$ is coined by the following definition.

Definition 33.1 *Let* $\mathbf{V} = (V_1, \dots, V_n)$ *be a* C^1 *vector field in an open domain* $\Omega \subseteq \mathbb{R}^n$*. We define the* divergence *of* $\mathbf{V}$ *by*

$$\operatorname{div} \mathbf{V} := \frac{\partial V_1}{\partial x_1} + \cdots + \frac{\partial V_n}{\partial x_n}.$$

It is not possible here to give the proof of the following important theorem, because the closed surface $\mathcal{F} = \partial\Omega$ may not be nice, and even if it is, the approximation of Ω from the inside by $\Omega_k = \bigcup_{j=1}^{k} \omega_j$ as described above in general gives a boundary $\partial\Omega_k$ which is difficult to handle in comparison with the surface integral over Ω. We therefore just quote the following very important theorem.

Theorem 33.2 Gauß's theorem in $\mathbb{R}^3$. *Given a C^1 vector field $\mathbf{V}$ on a domain $A \subseteq \mathbb{R}^3$. Let $\Omega \subseteq A$ be closed and bounded, where we assume that the boundary $\partial\Omega$ is the union of closed continuous and piecewise C^1 surfaces, each with a unit normal vector field pointing away from Ω almost everywhere. Then the flux of $\mathbf{V}$ through $\partial\Omega$ out of Ω is given by*

$$\Phi := \int_{\partial\Omega} \mathbf{V} \cdot \mathbf{n}\, dS = \int_{\Omega} \text{div}\ \mathbf{V}\, d\Omega.$$

Consider again the axiparallel parallelepipedum ω, which was used to make this theorem plausible. Then clearly the normal field does not exist on the edges, but these edges are just nullsets. This is what we mean by the formulation of the theorem above.

It is important to note that the formula

$$\int_{\partial\Omega} \mathbf{V} \cdot \mathbf{n}\, \mathrm{d}S = \int_{\Omega} \text{div}\ \mathbf{V}\, \mathrm{d}\Omega$$

can be read and applied in both directions. At first it may seem strange that we reformulate a two dimensional surface integral to the left as a three dimensional space integral on the right hand side of the equation. However, the examples later on will show that the calculations often become easier in the space integral than in the surface integral.

The other situation may of course also occur. We shall give some examples in Section 33.5.

Note also, that the geometrical analysis is important. One should e.g. always check that the unit vector field $\mathbf{n}$ is *pointing away from* Ω.

Theorem 33.2 is formulated for sets and vector fields in $\mathbb{R}^3$. There exists a similar result in $\mathbb{R}^2$, which is given in the next theorem.

Theorem 33.3 Gauß's theorem in $\mathbb{R}^2$. *Consider a plane C^1 vector field $\mathbf{V}$ on a plane domain $A \subseteq \mathbb{R}^2$. Let $E \subset A$ be a closed and bounded plane domain, where the boundary ∂E is the union of closed continuous and piecewise C^1 curves, each with a unit normal vector field pointing away from E almost everywhere. Then the flux of $\mathbf{V}$ through E is given by*

$$\Phi := \int_{\partial E} \left(V_x n_x + V_y n_y\right)\, ds = \int_E \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y}\right)\, dS,$$

where $\mathbf{V} = (V_x.V_y)$ and $\mathbf{n} = (n_x, n_y)$ in rectangular coordinates.

If div $\mathbf{V} = 0$, then we say that the vector field $\mathbf{V}$ is *divergence free.* Divergence free vector fields are important in the applications in e.g. Physics, though not all relevant vector fields are divergence free.

Example 33.3 *Area and volume formulæ.* Let us first consider $\mathbb{R}^2$. If we consider thee vector field $\mathbf{V}(x, y) = (x, y)$, then the divergence is given by

$$\text{div}\ \mathbf{V} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} = 2.$$

When we apply Gauß's theorem in two dimensions we get the *area formula*,

$$\frac{1}{2} \int_{\partial E} \mathbf{x} \cdot \mathbf{n}\, \mathrm{d}s = \frac{1}{2} \int_E \text{div}\ \mathbf{V}\, \mathrm{d}x\, \mathrm{d}y = \int_E \mathrm{d}x\, \mathrm{d}y = \text{area}(E).$$

In $\mathbb{R}^3$ the vector field $\mathbf{V}(x,y,z) = (x,y,z)$ has the divergence

$$\text{div}\ \mathbf{V} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3,$$

so it follows from Gauß's theorem in three dimensions that we have the following *volume formula,*

$$\frac{1}{3}\int_{\partial\Omega} \mathbf{x}\cdot\mathbf{n}\,\mathrm{d}S = \frac{1}{3}\int_{\Omega} \text{div}\ \mathbf{V}\,\mathrm{d}\Omega = \text{vol}(\Omega). \qquad \Diamond$$

Example 33.4 If $\mathbf{a}$ is a constant vector field, then trivially div $\mathbf{a} = 0$, so the flux through any closed boundary surface $\mathcal{F} = \partial\Omega$ is zero, because we get from Gauß's theorem that

$$\Phi := \int_{\partial\Omega} \mathbf{a}\cdot\mathbf{n}\,\mathrm{d}S = \int_{\Omega} \text{div}\ \mathbf{a}\,\mathrm{d}\Omega = 0. \qquad \Diamond$$

Example 33.5 Let a, b, c be positive constants. We shall find the flux of Φ of the vector field

$$\mathbf{V}(x,y,z) = (y,x,z+c) \qquad \text{for } (x,y,z) \in \mathbb{R}^3,$$

through the surface of the upper half of the massive ellipsoid, given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1 \qquad \text{and} \qquad z \ge 0.$$

It follows immediately that

$$\text{div}\ \mathbf{V} = \frac{\partial y}{\partial x} + \frac{\partial x}{\partial y} + \frac{\partial(z+c)}{\partial z} = 1,$$

so the flux is

$$\Phi = \int_{\partial\Omega} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S = \int_{\Omega} \text{div}\ \mathbf{V}\,\mathrm{d}\Omega = \text{vol}(\mathbf{V}) = \frac{1}{2}\cdot\frac{4\pi}{3}\,abc = \frac{2\pi}{3}\,abc. \qquad \Diamond$$

33.3 Applications in Physics

We shall in this section give some applications of Gauß's theorem in Physics, demonstrating that this theory is indeed important in Physics.

33.3.1 Magnetic flux

The density of the C^1 *magnetic flux* $\mathbf{B}$ satisfies for every domain Ω,

$$\int_{dd\Omega} \mathbf{B}\cdot\mathbf{n}\,\mathrm{d}S = 0,$$

which is the integral formulation of one of *Maxwell's equations.* By an application of Gauß's theorem we get

$$\int_{\Omega} \text{div}\ \mathbf{B}\,\mathrm{d}\Omega = 0 \qquad \text{for every (measurable) set } \Omega \subseteq \mathbb{R}^3.$$

Since div $\mathbf{B}$ is *continuous* (in fact, $\mathbf{B} \in C^1$), this implies that

$$\text{div } \mathbf{B} = 0,$$

which is the differential formulation of the same of Maxwell's equations as above.

In fact, if div $\mathbf{B}(\mathbf{x}_0) > 0$, then due to the continuity also div $\mathbf{B}(\mathbf{x}) > 0$, whenever $\|\mathbf{x} - \mathbf{x}_0\| < \delta$ for $\delta > 0$ sufficiently small. This would imply that $\int_{\Omega_\delta} \text{div } \mathbf{B} \, d\Omega > 0$, where $\Omega_\delta = \{\mathbf{x} \mid \|\mathbf{x} - \mathbf{x}_0\| < \delta\}$, which is a contradiction, so we conclude that div $\mathbf{B} = 0$.

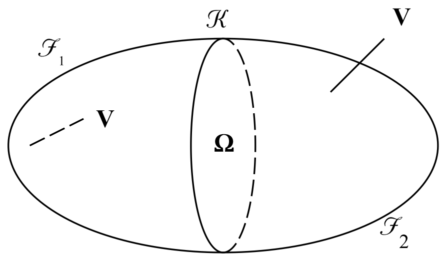

Figure 33.7: For divergence free vector fields the flux inwards through $\mathcal{F}_1$ is equal to the flux outwards through $\mathcal{F}_2$.

Then let $\mathbf{V}$ be a divergence free C^1 vector field. Consider a domain Ω, such that the surface boundary $\partial\Omega$ is cut into two subsurfaces $\mathcal{F}_1$ and $\mathcal{F}_2$ as indicated on Figure 33.7 by a closed curve $\mathcal{K}$.

Let Φ_1 the the flux *into* Ω through $\mathcal{F}_1$, and Φ_2 the flux *out* of Ω through $\mathcal{F}_2$. It follows fromGauß's theorem that the flux out of Ω through the whole of $\partial\Omega = \mathcal{F}_1 \cup \mathcal{F}_2$ is given by

$$\Phi = \Phi_2 - \Phi_1 = \int_{\mathcal{F}_2} \mathbf{V} \cdot \mathbf{n} \, dS + \int_{\mathcal{F}_1} \mathbf{V} \cdot (-\mathbf{n}) \, dS = \int_{\Omega} \text{div } \mathbf{V} \, d\Omega = 0.$$

In other words, the flux of a divergence free vector field through a surface of fixed boundary curve $\mathcal{K}$ only depends on this closed curve $\mathcal{K}$ and not of the shape of the surface, which has $\mathcal{K}$ as boundary curve.

We have already derived that the density of the *magnetic flux* is divergence free. Therefore, according to the result above we can now talk of the *magnetic flux being surrounded by a closed curve.*

33.3.2 Coulomb vector field

Then we return to the *Coulomb vector field*, already considered in Example 33.2. We shall for convenience choose $k = 1$, so the Coulomb field is here

$$\mathbf{V}(x, y, z) = \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}, \qquad \text{for } (x, y, z) \neq (0, 0, 0).$$

We shall first prove that $\mathbf{V}$ is divergence free, $\operatorname{div}\mathbf{V} = 0$. It follows from straight forward differentiation that

$$\frac{\partial V_x}{\partial x} = \frac{\partial}{\partial x}\left\{x\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}\right\} = \left(x^2+y^2+z^2\right)^{-\frac{3}{2}} - 3x^2\left(x^2+y^2+z^2\right)^{-\frac{5}{2}},$$

and similarly, due to the symmetry,

$$\frac{\partial V_y}{\partial y} = \frac{\partial}{\partial y}\left\{x\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}\right\} = \left(x^2+y^2+z^2\right)^{-\frac{3}{2}} - 3y^2\left(x^2+y^2+z^2\right)^{-\frac{5}{2}},$$

$$\frac{\partial V_z}{\partial z} = \frac{\partial}{\partial z}\left\{x\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}\right\} = \left(x^2+y^2+z^2\right)^{-\frac{3}{2}} - 3z^2\left(x^2+y^2+z^2\right)^{-\frac{5}{2}},$$

hence, by adding these three equations,

$$\operatorname{div}\mathbf{V} = \frac{V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = 3\left(x^2+y^2+z^2\right)^{-\frac{3}{2}} - 3\left(x^2+y^2+z^2\right)^{1-\frac{5}{2}} = 0.$$

Using Gauß's theorem we conclude that the flux through any closed boundary surface $\partial\Omega$ of the Coulomb field is zero, *provided that* $\mathbf{0} \notin \Omega$*!*

Then assume that $\mathbf{0} \in \Omega^\circ$ (an interior point of Ω). We put

$$B_a = B[\mathbf{0}, a] = \left\{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \le a^2\right\}, \qquad a > 0,$$

where $a > 0$ is chosen so small that $B_a \subset \Omega^\circ$. When we apply Gauß's theorem on the set $\Omega \setminus B_a$, where $\mathbf{0} \notin \Omega \setminus B_a$, then we get by the previous result that the flux through $\partial(\Omega \setminus B_a) = \partial\Omega \cup \partial B_a$ is

$$\int_{\partial\Omega} \mathbf{V} \cdot \mathbf{n}\, \mathrm{d}S + \int_{\partial B_a} \mathbf{V} \cdot (-\mathbf{n})\, \mathrm{d}S = \int_{\Omega \setminus B_a} \operatorname{div} \mathbf{V}\, \mathrm{d}\Omega = 0,$$

cf. Figure 33.8.

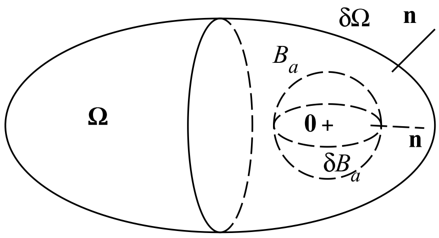

Figure 33.8: Analysis of Gauß's theorem applied to the set $\Omega \setminus B_a$.

We note that $-\mathbf{n}$ on ∂B_a is the unit normal vector field pointing away from the solid set $\Omega \setminus B_a$. Also, $x^2+y^2+z^2 = a^2$ on ∂B_a, so the Coulomb field is on ∂B_a given by

$$\mathbf{V}_{\partial B_a} = \frac{a\mathbf{n}}{a^3} = \frac{\mathbf{n}}{a^2}, \qquad \text{for } x^2+y^2+z^2 = a^2.$$

Combining the results and comments above we conclude that when $\mathbf{0} \in \Omega$, then the flux of the Coulomb field through $\partial\Omega$ is given by

$$\begin{aligned}\int_{\partial\Omega} \mathbf{V} \cdot \mathbf{n}\, \mathrm{d}S &= \int_{\partial\Omega} \mathbf{V} \cdot \mathbf{n}\, \mathrm{d}S + \int_{\partial B_a} \mathbf{V} \cdot (-\mathbf{n})\, \mathrm{d}S + \int_{\partial B_a} \mathbf{V} \cdot \mathbf{n}\, \mathrm{d}S = 0 + \int \partial B_a \mathbf{V} \cdot \mathbf{n}\, \mathrm{d}S \\ &= \int_{\partial B_a} \frac{\mathbf{n}}{a^2} \cdot \mathbf{n}\, \mathrm{d}S = \frac{1}{a^2} \int_{\partial B_a} \mathrm{d}S = \frac{1}{a^2} \text{area}\,(\partial B_a) = 4\pi.\end{aligned}$$

In other words, we have proved that for any solid body Ω with a reasonable ssurface $dd\Omega$, the flux of the Coulomb vector field $\mathbf{V}$ through $\partial\Omega$ is given by

$$\int_{\partial\Omega} \mathbf{V} \cdot \mathbf{n}\, \mathrm{d}S = \begin{cases} 0 & \text{if } \mathbf{0} \notin \overline{\Omega}, \\ 4\pi & \text{if } \mathbf{0} \in \Omega^\circ. \end{cases}$$

We do not consider the case, when $\mathbf{0} \in \partial\Omega$.

33.3.3 Continuity equation

Consider a fluid or gas of density ϱ and velocity field $\mathbf{v}$. Then the mass in a domain Ω is given by

$$M = \int_{\Omega} \varrho \, \mathrm{d}\Omega,$$

and the flow of mass through the surface $\partial\Omega$ away from Ω is given by the flux

$$q := \int_{\partial\Omega} \varrho \, \mathbf{v} \cdot \mathbf{n} \, \mathrm{d}S.$$

The *law of conservation of mass* is then infintesimally expressed in the following way,

$$q \, \mathrm{d}t = - \, \mathrm{d}M.$$

Then we get by an application of Gauß's theorem,

$$0 = q + \frac{daM}{\mathrm{d}t} = \int_{\partial\Omega} \varrho \, \mathbf{v} \cdot \mathbf{n} \, \mathrm{d}S + \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} \varrho \, \mathrm{d}\Omega = \int_{\Omega} \left\{ \mathrm{div}(\varrho \, \mathbf{v}) + \frac{\partial \varrho}{\partial t} \right\} \mathrm{d}\Omega.$$

Assuming that ϱ and $\mathbf{v}$ are of class C^1, we see that the integrand $\mathrm{div}(\varrho \, \mathbf{v}) + \dfrac{\partial \varrho}{\partial t}$ is continuous, and we have previously seen that if f is a continuous function satisfying

$$\int_{\Omega} f \, \mathrm{d}\Omega = 0 \qquad \text{for all subsets of } \Omega,$$

then $f \equiv 0$. We have therefore proved the *continuity equation*

$$\mathrm{div}(\varrho \, \mathbf{v}) + \frac{\partial \varrho}{\partial t} = 0.$$

This equation can also be found in other physical disciplines – the mathematical proof above is the same and only the physical interpretations are different. If for instance u denotes the *energy density*, and $\mathbf{q}$ the *density of the energy flow*, then the *conservation of the energy* is expressed by the similar equation

$$\mathrm{div} \; \mathbf{q} + \frac{\partial u}{\partial t} = 0.$$

Similarly, if $\mathbf{J}$ denotes the *density of a current* and ϱ the *density of the charge*, then the *law of conservation of the electric charge* is written

$$\mathrm{div} \; \mathbf{J} + \frac{\partial \varrho}{\partial t} = 0.$$

All these results stem from an application of Gauß's theorem.

33.4 Procedures for flux and divergence of a vector field; Gauß's theorem

33.4.1 Procedure for calculation of a flux

The **flux** Φ of a vector field $\mathbf{V}$ through a surface $\mathcal{F}$ is given by a *surface integral* (cf. Chapter 27) in the following way:

If $\mathbf{x} = \mathbf{r}(u,v)$, $(u,v) \in E$ is a *parametric representation* of the surface $\mathcal{F}$ with a given continuous unit normal vector field $\mathbf{n}$, then the flux is given by

$$\Phi_{\mathcal{F}}(\mathbf{V}) = \int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n}\, \mathrm{d}S = \int_{\mathcal{F}} \mathbf{V}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})\, \mathrm{d}S = \int_E \mathbf{V}(\mathbf{r}(u,v)) \cdot \mathbf{N}(u,v)\, \mathrm{d}u\, \mathrm{d}v.$$

It is the amount of the vector field which "flows through the surface in the direction of the normal vector" (e.g. per time unit).

Typically there are two different ways in which the flux can be calculated.

Standard procedure.

In principle this can always be applied, but it is often very cumbersome.

1) Divide if necessary $\mathcal{F}$ into convenient sub-surfaces $\mathcal{F}_1, \dots, \mathcal{F}_k$ each having its own unit normal vector field $\mathbf{n}_1, \dots, \mathbf{n}_k$.

2) Check, whether $\mathcal{F}$ (or $\mathcal{F}_j$) is *"flat"*, and if it is not too difficult to calculate $\int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n}\, \mathrm{d}S$ as an ordinary *plane integral*, because $\mathcal{F}$ is lying in a plane set.

3) If $\mathcal{F}$ is not flat, we calculate the normal vector corresponding to the specific parametric representation in the variables (u,v),

$$\mathbf{N}(u,v) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{vmatrix}.$$

   Note that $\mathbf{N}(u,v)$ no longer is a *unit* normal vector field. Stated roughly, we build the weight function into the new normal vector field $\mathbf{N}(u,v)$.

4) Calculate the *plane integral* over the *parametric domain* E,

$$\Phi_{\mathcal{F}}(\mathbf{V}) = \int_E \mathbf{V}(\mathbf{r}(u,v)) \cdot \mathbf{N}(u,v)\, \mathrm{d}u\, \mathrm{d}v.$$

33.4.2 Application of Gauß's theorem

The method can in principle always be applied when the surface is "closed", i.e. one adds a surface with *two* numerically equal normal vector fields, which are pointing in the opposite directions, $\mathbf{n}$ and $-\mathbf{n}$, such that one surrounds a 3-dimensional domain Ω with outgoing normal, and an additional surface integral, which hopefully should be easy to calculate.

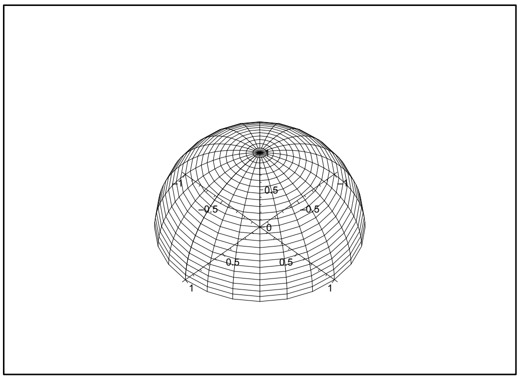

Figure 33.9: The surface of the unit half sphere is closed by adding the unit disc in the (x, y)-plane with a normal vector pointing downwards (this gives us the closed upper unit half sphere with normal vectors pointing outwards) and the unit disc in the (x, y)-plane with the normal vector pointing upwards.

1) Check that $\mathcal{F} = \partial\Omega$ is *closed*, i.e. this surface surrounds a 3-dimensional *body* Ω.

2) Quote *Gauß's theorem* and reduce the surface integral to a *space integral*,

$$\int_{\partial\Omega} \mathbf{n} \cdot \mathbf{V}\, \mathrm{d}S = \int_{\Omega} \operatorname{div} \mathbf{V}\, \mathrm{d}\Omega.$$

3) Calculate the space integral $\int_{\Omega} \operatorname{div} \mathbf{V}\, \mathrm{d}\Omega$ by applying one of the methods from Chapter 24.

Remark 33.1 Usually one would not call it a reduction to go from 2 dimensions to 3 dimensions; but note that the surface $\mathcal{F}$ of dimension 2 may have a fairly complicated geometry, while we in principle always end up with rectangular coordinates i 3 dimensions, which here may be considered as a simpler situation. $\Diamond$

33.5 Examples of flux and divergence of a vector field; Gauß's theorem

33.5.1 Examples of calculation of the flux

Example 33.6

A. *Find the flux* Φ_2 *of the vector field*

$$\mathbf{V}(x,y,z) = \left(x^2+y^2, z^2, y^2\right), \qquad (x,y,z) \in \mathbb{R}^3,$$

through the surface $\mathcal{F}$ *defined by*

$$\mathbf{r}(u,v) = (u+v, u-v, u+2v), \qquad u^2+v^2 \le 4.$$

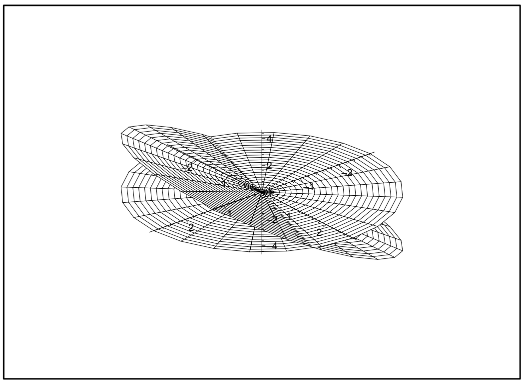

Figure 33.10: The surface $\mathcal{F}$ and its projection onto the (x,y)-plane.

D. We see that the surface $\mathcal{F}$ lies in a plane, but because this plane is oblique, it is very difficult to exploit its flat structure. Instead we analyze the reduction formula

$$\int_{\mathcal{F}} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S = \int_E \mathbf{V}(\mathbf{r}(u,v))\cdot\mathbf{N}(u,v)\,\mathrm{d}u\,\mathrm{d}v,$$

where the abstract *surface* integral is rewritten as an abstract *plane* integral. By inspecting the right hand side it is seen that we shall

1) identify the parametric domain E,
2) find the normal vector $\mathbf{N}(u,v)$ for the surface $\mathcal{F}$, corresponding to the parameters (u,v),
3) express $\mathbf{V}(\mathbf{r}(u,v))$ on the surface $\mathcal{F}$ as a function in the parameters (u,v).

I. 1) *The parametric domain* is the disc of centre $(0,0)$ and radius 2,

$$E = \{(u,v) \mid u^2+v^2 \le 4 = 2^2\}.$$

2) *The normal vector.* It follows from the parametric representation of the surface that

$$\frac{\partial \mathbf{r}}{\partial u} = (1,1,1) \qquad \text{and} \qquad \frac{\partial \mathbf{r}}{\partial \mathbf{v}} = (1,-1,2).$$

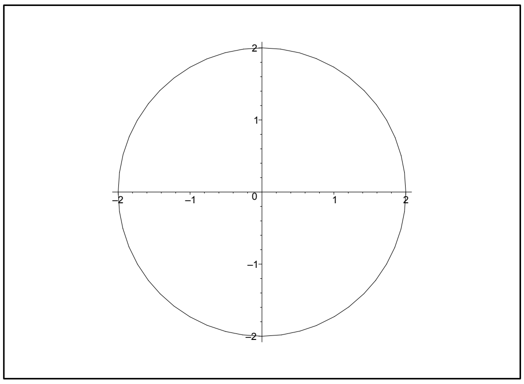

Figure 33.11: The parametric domain E is a disc of centre $(0,0)$ and radius 2.

Hence, the normal vector is

$$\mathbf{N}(u,v)=\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}=\left|\begin{array}{ccc}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\\ 1 & 1 & 1\\ 1 & -1 & 2\end{array}\right|=(3,-1,-2).$$

3) *The restriction of the vector field* to the surface is given by

$$\begin{aligned}\mathbf{V}(x,y,z) &= \left(x^2,y^2,z^2\right)\\ &= \left((u+v)^2+(u-v)^2,(y+2v)^2,(u-v)^2\right)\\ &= \left(2\left(u^2+v^2\right),(u+2v)^2,(u-v)^2\right).\end{aligned}$$

4) *The integrand* is according to 2 and 3,

$$\begin{aligned}\mathbf{V}\cdot\mathbf{N} &= (3,-1,-2)\cdot\left(2\left(u^2+v^2\right),(u+2v)^2,(u-v)^2\right)\\ &= 6\left(u^2+v^2\right)-(u+2v)^2-2(u-v)^2\\ &= 6u^2+6v^2-\left(u^2+4uv+4v^2\right)-\left(2u^2-4uv+2v^2\right)\\ &= 3u^2.\end{aligned}$$

5) By *insertion* of 4 into the reduction formula we get by also using 1,

$$\Phi_2=\int_{\mathcal{F}}\mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S=\int_E\mathbf{V}(\mathbf{r}(u,v))\cdot\mathbf{N}(u,v)\,\mathrm{d}u\,\mathrm{d}v=3\int_E u^2\,\mathrm{d}u\,\mathrm{d}v.$$

Since the parametric domain E is a *disc*, it is easiest to reduce it in *polar* coordinates,

$$u=\varrho\cos\varphi,\quad v=\varrho\sin\varphi,\qquad 0\le\varrho\le 2,\quad 0\le\varphi\le 2\pi.$$

Hence we get the result

$$\begin{aligned}\Phi_2 &= 3\int_E u^2\,\mathrm{d}u\,\mathrm{d}v=3\int_0^{2\pi}\left\{\int_0^2\varrho^2\cos^2\varphi\cdot\varrho\,\mathrm{d}\varrho\right\}\mathrm{d}\varphi\\ &= 3\int_0^{2\pi}\cos^2\varphi\,\mathrm{d}\varphi\cdot\int_0^2\varrho^3\,\mathrm{d}\varrho=3\cdot\pi\cdot\left[\frac{1}{4}\varrho^4\right]_0^2\\ &= 12\pi.\qquad\Diamond\end{aligned}$$

Example 33.7

A. *Let a, b, $c>0$, be constants, and let*

$$\mathbf{V}(x,y,z)=(y,x,z+c),\qquad (x,y,z)\in\mathbb{R}^3.$$

Find the flux Φ_3 of $\mathbf{V}$ through the half ellipsoidal surface

$$\mathcal{F}_1=\left\{(x,y,z)\;\middle|\;\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1,\,z\ge 0\right\}$$

where the normal is directed u upwards, $\mathbf{n}\cdot\mathbf{e}_z\ge 0$, and the flux Φ_4 of $\mathbf{V}$ through the projection $\mathcal{F}_2$ of $\mathcal{F}_1$ onto the (x,y)-plane,

$$\mathcal{F}_2=\left\{(x,y,z)\;\middle|\;\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2\le 1\right\},\qquad \mathbf{n}=(0,0,1).$$

D. Summing up we see that $\mathcal{F}_1$ and $\mathcal{F}_2$ surround a spatial domain Ω. The flux Φ_3 represents e.g. the energy which flows out of Ω through $\mathcal{F}_1$, and Φ_4 represents the energy which flows into Ω through $\mathcal{F}_2$. Hence, the difference $\Phi_3-\Phi_4$ represents the energy which is created by $\mathbf{V}$ in Ω.

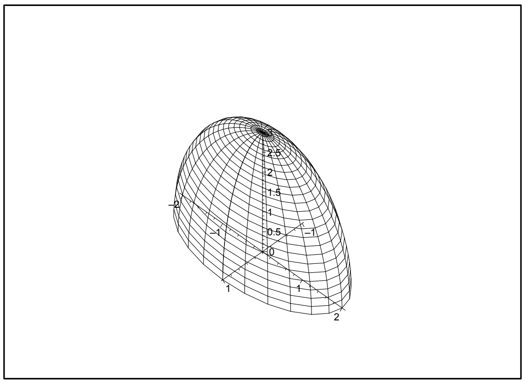

Figure 33.12: The half ellipsoidal surface $\mathcal{F}_1$ for $a = 1$, $b = 2$ and $c = 3$. The surface $\mathcal{F}_2$ is hidden below $\mathcal{F}_1$ in the (x, y)-plane.

I 1. Consider first

$$\mathcal{F}_1 = \left\{ (x,y,z) \;\middle|\; \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1,\, z \geq 0 \right\}, \qquad \mathbf{n} \cdot \mathbf{e}_3 \geq 0.$$

The easiest method, which can be found in some textbooks, is to use *spherical* coordinates (left to the reader). We shall here as an *alternative* apply *rectangular* coordinates instead. Then we can consider $\mathcal{F}_1$ as the *graph* of the function

$$z = f(x,y) = c\sqrt{1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2}, \qquad \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leq 1.$$

Then the hidden parametric representation is given by

$$\mathbf{r}(x,y) = \left(x, y, c\sqrt{1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2} \right), \qquad \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leq 1.$$

This parametric representation is differentiable when

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 < 1,$$

i.e. when $z > 0$. If so, we get

$$\frac{\partial \mathbf{r}}{\partial x} = \left(1, 0, -\frac{c}{a^2} \frac{x}{\sqrt{1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2}} \right) = \left(1, 0, -\frac{c^2}{a^2} \cdot \frac{x}{z} \right),$$

and analogously

$$\frac{\partial \mathbf{r}}{\partial y} = \left(0, 1, -\frac{c}{b^2} \frac{y}{\sqrt{1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2}} \right) = \left(0, 1, -\frac{c^2}{b^2} \cdot \frac{y}{z} \right),$$

where we have used that $z = c\sqrt{1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2}$ in order not to be overburdened with a square root in the following. (It is always possible to substitute back again, if necessary). Then

$$\mathbf{N}(x,y) = \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & 0 & -\frac{c^2}{a^2}\,\frac{x}{z} \\ 0 & 1 & -\frac{c^2}{b^2}\,\frac{y}{z} \end{vmatrix} = \left(\frac{c^2}{a^2}\cdot\frac{x}{z}, \frac{c^2}{b^2}\cdot\frac{y}{z}, 1\right).$$

Now $\mathbf{N}\cdot\mathbf{e}_3 = 1 > 0$, so $\mathbf{N}(x,y)$ is pointing in the right direction.

The integrand is then calculated,

$$\mathbf{V}\cdot\mathbf{N} = (y, x, z+c)\cdot\left(\frac{c^2}{a^2}\cdot\frac{x}{z}, \frac{c^2}{b^2}\cdot\frac{y}{z}, 1\right) = c^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\frac{xy}{z}+z+c.$$

The domain of integration is the ellipse in the (x,y)-plane

$$E = \left\{(x,y) \;\middle|\; \left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2 \le 1\right\}.$$

Hence, the flux is equal to the *improper* plane integral

$$\begin{aligned}\Phi_3 &= \int_{\mathcal{F}_1} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S \\ &= \int_E \left\{ c\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\cdot\frac{xy}{\sqrt{1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2}} + c\sqrt{1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2}+c\right\}\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Then note that we have e.g.

$$\int_x \sqrt{1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2}\,\mathrm{d}x = -a^2\sqrt{1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2},$$

i.e. if we integrate over an interval of the form $[0,k]$ (where the integrand is ≥ 0) or over $[-k,0]$ (where the integrand is ≤ 0), then we get *finite* values in both cases, i.e. the improper integral is convergent.

If we put $k = a\sqrt{1-\left(\frac{y}{b}\right)^2}$, it follows of symmetric reasons that

$$\begin{aligned}&\int_E c\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\frac{xy}{\sqrt{1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2}}\,\mathrm{d}S \\ &= \lim_{\varepsilon\to 0+}\int_{-b}^{b} c\left(\frac{1}{a^2}+\frac{1}{b^2}\right) y\left\{\int_{-a\sqrt{1-\left(\frac{y}{b}\right)^2}+\varepsilon}^{a\sqrt{1-\left(\frac{y}{b}\right)^2}-\varepsilon} \frac{x\,\mathrm{d}x}{\sqrt{1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2}}\right\} \\ &= \lim_{\varepsilon\to 0+} c\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\int_{-b}^{b} y\cdot 0\,\mathrm{d}y = 0.\end{aligned}$$

The expression of the flux is therefore reduced to

$$\begin{aligned}\Phi_3 &= 0+\int_E (z+c)\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_E c\sqrt{1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2}\,\mathrm{d}x\,\mathrm{d}y + c\cdot\text{area}(E)\\ &= c\int_E \sqrt{1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2}\,\mathrm{d}x\,\mathrm{d}y + c\cdot\pi ab.\end{aligned}$$

The purpose of the following *elaborated* variant is to straighten up the ellipse by the change of variables

$$u=\frac{x}{a},\quad v=\frac{y}{b},\quad \text{i.e.}\quad x=a\,u,\quad y=b\,v.$$

The corresponding Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} a & 0\\ 0 & b\end{vmatrix}=ab>0.$$

By the transformation formula the parametric domain E is mapped into the unit disc B in the (u,v)-plane, hence

$$\begin{aligned}
\Phi_3 &= \pi abc + c\int_E \sqrt{1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2}\,\mathrm{d}x\,\mathrm{d}y \\
&= \pi abc + c\int_B \sqrt{1-u^2-v^2}\,\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,\mathrm{d}u\,\mathrm{d}v \\
&= \pi abc + ab\cdot c\int_B \sqrt{1-u^2-v^2}\,\mathrm{d}u\,\mathrm{d}v \\
&= abc\left\{\pi+\int_0^{2\pi}\left\{\int_0^1 \sqrt{1-\varrho^2}\cdot\varrho\,\mathrm{d}\varrho\right\}\mathrm{d}\varphi\right\} \\
&= abc\left\{\pi+2\pi\int_0^1 \sqrt{1-t}\cdot\frac{1}{2}\,\mathrm{d}t\right\} \\
&= \pi abc\left\{1+\left[-\frac{2}{3}(1-t)^{\frac{3}{2}}\right]_0^1\right\} \\
&= \pi abc\cdot\left(1+\frac{2}{3}\right)=\frac{5}{3}\,\pi abc.
\end{aligned}$$

No matter whether one is using spherical or rectangular coordinates, it is very difficult to find Φ_3, and there are lots of pit falls (as seen above we get e.g. an improper surface integral in the rectangular version).

I 2. Next look at

$$\mathcal{F}_2 = E = \left\{(x,y,z)\;\middle|\;\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2\le 1,\, z=0\right\},\qquad \mathbf{n}=(0,0,1).$$

The restriction of $\mathbf{V}$ to E is obtained by putting $z=0$, i.e.

$$\mathbf{V}(x,y,0)=(y,x,c).$$

The unit normal vector is $\mathbf{n}=(0,0,1)$, so the integrand becomes

$$\mathbf{V}(x,y,0)\cdot\mathbf{n}=(y,x,c)\cdot(0,0,1)=c.$$

We conclude by using the reduction theorem on the simple calculation

$$\Phi_4=\int_E \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S=c\int_E \mathrm{d}S=c\cdot\mathrm{area}(E)=c\cdot\pi ab=\pi abc.$$

I 3. Finally we have (cf. Figure 33.13),

The flux *out of* $\partial\Omega$ of $\mathbf{V}$ is according to **I 1.** and **I 2.** given by

$$\Phi_3-\Phi_4=\frac{5}{3}\,\pi\,abc-\pi\,abc=\frac{2}{3}\,\pi\,abc,$$

where we use $-\Phi_4$, because Φ_4 indicates the flux *into* Ω through $\mathcal{F}_2$.

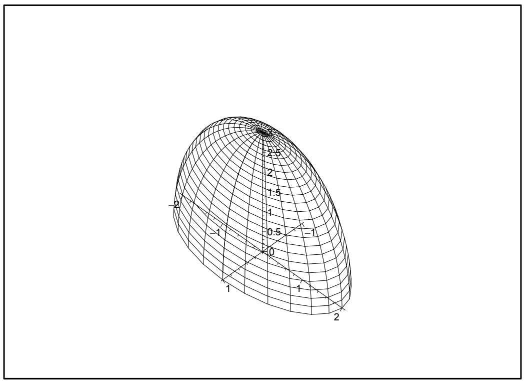

Figure 33.13: The domain Ω.

Let us then **alternatively** show the same result by means of *Gauß's theorem*.

We first realize that $\mathcal{F}_1$ and $\mathcal{F}_2$ surround a spatial domain

$$\Omega = \left\{(x,y,z) \;\middle|\; \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \le 1,\, z \ge 0\right\}.$$

From $\mathbf{V}(x,y,z) = (y,x,z+c)$ we then get

$$\operatorname{div} \mathbf{V} = 0+0+1 = 1.$$

The flux out through $\partial\Omega$ (the normal of direction away from the domain) is then according to *Gauß's theorem*,

$$\Phi = \int_{\partial\Omega} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S = \int_{\Omega} \operatorname{div} \mathbf{V}\,\mathrm{d}\Omega = \int_{\Omega} \mathrm{d}\Omega = \operatorname{vol}(\Omega) = \frac{1}{2}\left(\frac{4\pi}{3}\,abc\right) = \frac{2\pi}{3}\,abc.$$

By comparison we see that this is exactly $\Phi_3 - \Phi_3$ as we claimed.

Summarizing, Φ_3 in **I 1.** was difficult to compute, while Φ_4 in **I 2.** and Φ in **I 3.** were easy. Since $\Phi_3 - \Phi_4 = \Phi$, we might have calculated Φ_3 by computing the easy right hand side of

$$\Phi_3 = \Phi_4 + \Phi,$$

i.e. expressed in integrals,

$$(33.1)\quad \int_{\mathcal{F}_1} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S = \int_{\mathcal{F}_2} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S + \int_{\Omega} \operatorname{div} \mathbf{V}\,\mathrm{d}\Omega,$$

or put in other words: an ugly surface integral (the left hand side) is expressed as the sum of a simple surface integral (here even a plane integral) and a simple spatial integral (the right hand side).

This technique can often be applied when one shall calculate the flux through a more or less complicated surface $\mathcal{F}_1$.

1) First draw a figure, thereby realizing where $\mathcal{F}_1$ is placed in the space.

2) Then add a nice surface $\mathcal{F}_2$, such that $\mathcal{F}_1 \cup \mathcal{F}_2$ becomes the boundary of a spatial body Ω. Check in particular that the normal vector on $\mathcal{F}_2$ is always pointing *away* from the domain Ω).

3) Calculate the right hand side of (33.1), thereby finding the flux through $\mathcal{F}_1$.

Remark 33.2 We shall later in Example 33.9 give some comments which will give us an even more easy version of calculation. $\Diamond$

Example 33.8

A. *Let the surface* $\mathcal{F}$ *be the square*

$$\mathcal{F} = \{(x,y,z) \mid |x| \le a,\ |y| \le a,\ z = a\}, \qquad a > 0,$$

at the height a *with the unit normal vector* $\mathbf{n} = (0,0,1)$ *pointing upwards.*

Find the flux through $\mathcal{F}$ *of the* Coulomb field

$$\mathbf{V}(x,y,z) = \frac{(x,y,z)}{\left(x^2+y^2+z^2\right)^{3/2}}, \qquad (x,y,z) \neq (0,0,0).$$

(Concerning the Coulomb field see also Example 33.9).

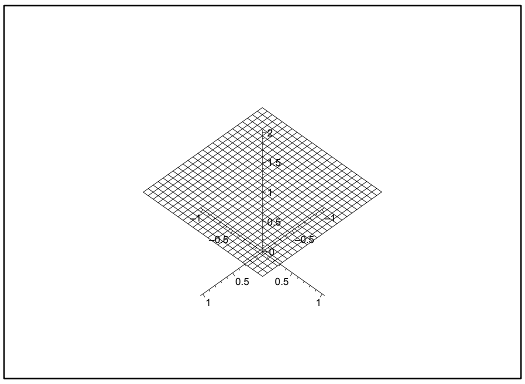

Figure 33.14: The surface $\mathcal{F}$ for $a = 1$.

D. Using *rectangular* coordinates we get from the reduction theorem that

$$\begin{aligned}\Phi_5 &= \int_{\mathcal{F}} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S = \int_{-a}^{a}\left\{\int_{-a}^{a} \frac{(x,y,a)\cdot(0,0,1)}{\left(x^2+y^2+a^2\right)^{3/2}}\,\mathrm{d}x\right\}\mathrm{d}y \\ &= \int_{-a}^{a}\left\{\int_{0}^{a} \frac{a}{\left(x^2+y^2+a^2\right)^{3/2}}\,\mathrm{d}x\right\}\mathrm{d}y = 4a\int_{0}^{a}\left\{\int_{0}^{a} \frac{1}{\left(x^2+y^2+a^2\right)^{3/2}}\,\mathrm{d}x\right\}\mathrm{d}y,\end{aligned}$$

where we have used that the integrand is even in both x and y, and that the domain is symmetric.

So far, so good, but from now on the calculations become really tough. The reason is that the integrand invites to the application of *polar* coordinates, while the domain is better described in *rectangular* coordinates. The mixture of these two coordinate systems will always cause some difficulties.

For pedagogical reasons we shall here show both variants, first the *rectangular version*, which is extremely difficult, and afterwards the *polar version*, which is "only" difficult. This exercise will show that one cannot just restrict oneself to rely on the rectangular method alone!

I 1. *Rectangular variant.* Calculate directly

$$\Phi_5 = 4 \int_0^a \left\{ \int_0^a \frac{a}{\left(x^2+y^2+a^2\right)^{3/2}}\, \mathrm{d}x \right\} \mathrm{d}y.$$

First we note by a partial integration of an auxiliary function that

$$\begin{aligned} \int_0^a 1 \cdot \left(t^2+c^2\right)^\alpha \, \mathrm{d}t &= \left[t \cdot \left(t^2+c^2\right)^\alpha\right]_0^a - \int_0^a t \cdot \alpha \left(t^2+c^2\right)^{\alpha-1} \cdot 2t \, \mathrm{d}t \\ &= a\left(a^2+c^2\right)^\alpha - 2\alpha \int_0^a \left(t^2+c^2-c^2\right)\left(t^2+c^2\right)^{\alpha-1} \mathrm{d}t \\ &= a\left(a^2+c^2\right)^\alpha - 2\alpha \int_0^a \left(t^2+c^2\right)^\alpha \mathrm{d}t + 2\alpha c^2 \int_0^a \left(t^2+c^2\right)^{\alpha-1} \mathrm{d}t. \end{aligned}$$

When $\alpha \neq 0$ and $c > 0$, we get by a rearrangement

$$(33.2) \quad \int_0^a \left(t^2+c^2\right)^{\alpha-1} \mathrm{d}t = \frac{1+2\alpha}{2\alpha c^2} \int_0^a \left(t^2+c^2\right)^\alpha \mathrm{d}t - \frac{a\left(a^2+c^2\right)^\alpha}{2\alpha c^2}.$$

Choosing $t = x$ and $\alpha = -\frac{1}{2}$ and $c^2 = y^2 + a^2$ in (33.2) and multiplying by a, we get the inner integral in Φ_5: Since $1 + 2\alpha = 0$ we have

$$\int_0^a \frac{a}{\left(x^2+y^2+a^2\right)^{3/2}} \, \mathrm{d}x = -\frac{a^2\left(a^2+y^2+a^2\right)^{-1/2}}{2\left(-\frac{1}{2}\right)\left(a^2+y^2\right)} = \frac{a^2}{\left(y^2+a^2\right)\sqrt{y^2+2a^2}},$$

which gives by insertion

$$\Phi_5 = 4 \int_0^a \frac{a^2}{\left(y^2+a^2\right)\sqrt{y^2+2a^2}} \, \mathrm{d}y.$$

So far we can still use the pocket calculator TI-89, but from now on it denies to calculate the exact value! Therefore, we must from now on continue by using the old-fashioned, though well tested methods from the time before the pocket calculators.

When we consider the dimensions we see that $y \sim a$, hence a convenient substitution must be $y = a\,u$. Then

$$\begin{aligned} \Phi_5 &= 4 \int_0^a \frac{a^2}{\left(y^2+a^2\right)\sqrt{y^2+2a^2}} \, \mathrm{d}y = 4 \int_0^1 \frac{a^2}{\left(a^2u^2+a^2\right)\sqrt{a^2u^2+2a^2}} \cdot a \, \mathrm{d}u \\ &= 4 \int_0^1 \frac{1}{\left(u^2+1\right)\sqrt{u^2+2}} \, \mathrm{d}u = 4 \int_0^1 \frac{1}{\left(u^2+1\right)\sqrt{\left(u^2+1\right)+1}} \, \mathrm{d}u, \end{aligned}$$

where it should be surprising that Φ_5 is *independent of* a.

The following circumscription is governed by the following general principle:

- *Whenever the square of two terms is involved then it should be rewritten as 1 plus/minus something which has "something to do" with the other terms in the integrand.*

The circumscription indicates that we should try the *monotonous* substitution

$$t = u^2 + 1, \quad u = \sqrt{t-1}, \quad \mathrm{d}u = \frac{1}{2}\,\frac{1}{\sqrt{t-1}}\,\mathrm{d}t, \qquad t \in [1,2].$$

By this substitution we get

$$\Phi_5 = 4\int_0^1 \frac{1}{(u^2+1)\sqrt{(u^2+1)+1}}\,\mathrm{d}u = 4\int_1^2 \frac{1}{t\sqrt{t+1}}\cdot\frac{1}{2}\,\frac{1}{\sqrt{t-1}}\,\mathrm{d}t = 2\int_1^2 \frac{\mathrm{d}t}{t\sqrt{t^2-1}}.$$

The structure $\sqrt{t^2-1}$ looks like

$$\sqrt{\cosh^2 w - 1} = \sqrt{\sinh^2 w} = |\sinh w|,$$

which is a means to get rid of the square root. We therefore try another substitution,

$$t = \cosh w, \quad w = \ln(t + \sqrt{t^2-1}), \quad \mathrm{d}t = \sinh w\,\mathrm{d}w, \quad w \in [0, \ln(2+\sqrt{3})].$$

Since we have $\sinh w \geq 0$ in this interval, we get

$$\begin{aligned}
\Phi_5 &= 2\int_1^2 \frac{\mathrm{d}t}{t\sqrt{t^2-1}} = 2\int_0^{\ln(2+\sqrt{3})} \frac{\sinh w}{\cosh w \cdot \sinh w}\,\mathrm{d}w \\
&= 2\int_0^{\ln(2+\sqrt{3})} \frac{\mathrm{d}w}{\cosh w} = 2\int_0^{\ln(2+\sqrt{3})} \frac{2}{e^w + e^{-w}}\,\mathrm{d}w \\
&= 4\int_0^{\ln(2+\sqrt{3})} \frac{e^w}{1+(e^w)^2}\,\mathrm{d}w = 4\,\left[\operatorname{Arctan}(e^w)\right]_0^{\ln(2+\sqrt{3})} \\
&= 4\left\{\operatorname{Arctan}(2+\sqrt{3})) - \frac{\pi}{4}\right\} = 4\operatorname{Arctan}(2+\sqrt{3}) - \pi.
\end{aligned}$$

Our troubles in the rectangular case are not over. How can we find $\operatorname{Arctan}(2+\sqrt{3})$ without using a pocket calculator?

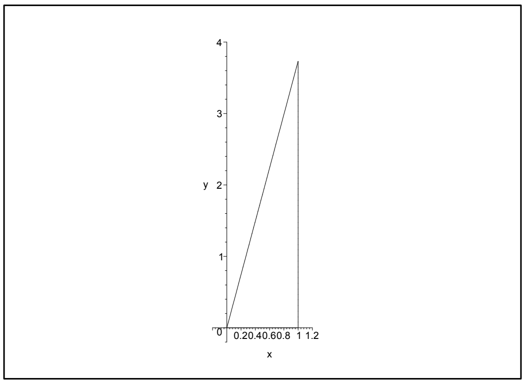

Figure 33.15: The rectangular triangle with the opposite side $= 2+\sqrt{3}$, so $\varphi = \operatorname{Arctan}(2+\sqrt{3})$ is the nearby angle.

Geometrically $\varphi = \operatorname{Arctan}(2+\sqrt{3})$ is that angle in the rectangular triangle on the figure, which is $> \frac{\pi}{4}$.

The hypothenuse can be found by Pythagoras' theorem,

$$r^2 = (2+\sqrt{3})^2 + 1^2 = 4+3+4\sqrt{3}+1 = 8+4\sqrt{3} = 4(2+\sqrt{3}),$$

i.e. $r = 2\sqrt{2+\sqrt{3}}$. From $\varphi > \frac{\pi}{4}$, follows that $\psi = \frac{\pi}{2} - \varphi < \frac{\pi}{4}$, and

$$\cos\psi = \frac{1}{r}\,(2+\sqrt{3}) = \frac{1}{2}\sqrt{2+\sqrt{3}}.$$

We shall get rid of the square root by squaring, so we try

$$\cos 2\psi = 2\cos^2\psi - 1 = 2\cdot\frac{1}{4}\,(2+\sqrt{3}) - 1 = 1 + \frac{1}{2}\sqrt{3} - 1 = \frac{\sqrt{3}}{2}.$$

Since we have been so careful to show that $0 < \psi < \frac{\pi}{4}$, it follows that $0 < 2\psi < \frac{\pi}{2}$, hence

$$2\psi = \operatorname{Arccos}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}, \qquad \text{i.e. } \psi = \frac{\pi}{12}.$$

Then

$$\operatorname{Arctan}(2+\sqrt{3}) = \varphi = \frac{\pi}{2} - \psi = \frac{\pi}{2} - \frac{\pi}{12} = \frac{5\pi}{12}.$$

By a final insertion we get that the flux is

$$\Phi_5 = 4\operatorname{Arctan}(2+\sqrt{3}) - \pi = 4\cdot\frac{5\pi}{12} - \pi = \frac{5\pi}{3} - \pi = \frac{2\pi}{3}.$$

Remark 33.3 It is obvious why this variant is never seen in ordinary textbooks. The morale is that even if something can be done, it does not always have to, and we should of course have avoided this variant. It should, however, be added that the pocket calculator finally will find that

$$\operatorname{Arctan}(2+\sqrt{3}) = \frac{5\pi}{12}. \qquad \Diamond$$

I 2. *Polar variant.* We shall start from the very beginning by

$$\Phi_5 = 4\int_0^a \left\{ \int_0^a \frac{a}{(x^2+y^2+a^2)^{3/2}}\,dx \right\} dy.$$

The domain $[0,n]^2$ is not fit for a polar description, but if we note that the integrand is symmetrical about the line $y = x$, then this symmetry gives that

$$\Phi_5 = 2\cdot 4\int_T \frac{a}{(x^2+y^2+a^2)^{3/2}}\,dx\,dy = 8\int_T \frac{a}{(x^2+y^2+a^2)^{3/2}}\,dx\,dy,$$

where the triangle T is bounded by $y = 0$ in the right half-plane (corresponding in polar coordinates to $\varphi = 0$), the line $y = x$ (corresponding to $\varphi = \frac{\pi}{4}$) and $x = \varrho\cos\varphi = a$, i.e.

$$\varrho = \frac{a}{\cos\varphi}.$$

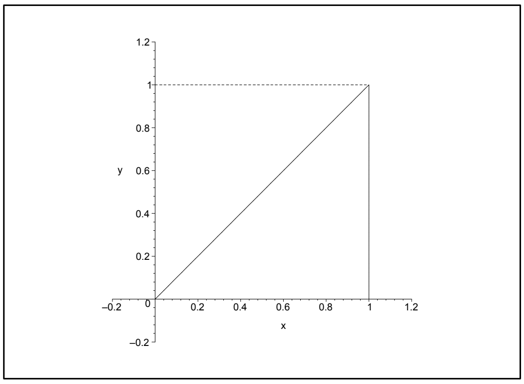

Figure 33.16: The domain T is the lower triangle and $a = 1$.

Therefore, a *polar* description of T is

$$T = \left\{ (\varrho, \varphi) \;\middle|\; 0 \le \varphi \le \frac{\pi}{4},\, 0 \le \varrho \le \frac{a}{\cos\varphi} \right\}.$$

Then the rest follows from the usual reduction theorems,

$$\Phi_5 = 8\int_T \frac{a}{(x^2+y^2+a^2)^{3/2}}\,\mathrm{d}S = 8\int_0^{\frac{\pi}{4}}\left\{\int_0^{\frac{a}{\cos\varphi}} \frac{a}{(\varrho^2+a^2)^{3/2}}\cdot\varrho\,\mathrm{d}\varrho\right\}\mathrm{d}\varphi.$$

The inner integral is calculated by using the substitution

$$t=\varrho, \qquad \mathrm{d}t = 2\varrho\,\mathrm{d}\varrho, \qquad \text{i.e. } \varrho\,\mathrm{d}\varrho = \frac{1}{2}\,\mathrm{d}t,$$

hence

$$\begin{aligned}
8\int_0^{\frac{a}{\cos\varphi}} \frac{a}{(\varrho^2+a^2)^{3/2}}\,\varrho\,\mathrm{d}\varrho &= 4\int_0^{\frac{4}{\cos^2\varphi}} \frac{a}{(t+a^2)^{3/2}}\,\mathrm{d}t \\
&= 4\left[-\frac{2a}{(t+a^2)^{1/2}}\right]^{\frac{a^2}{\cos^2\varphi}} = 8\left\{1-\frac{a}{\sqrt{\dfrac{a^2}{\cos^2\varphi}+a^2}}\right\} \\
&= 8\left\{1-\frac{a|\cos\varphi|}{a\sqrt{1+\cos^2\varphi}}\right\} = 8\left\{1-\frac{|\cos\varphi|}{\sqrt{1+\cos^2\varphi}}\right\}.
\end{aligned}$$

Since $|\cos\varphi| = \cos\varphi$ for $0\le\varphi\le\frac{\pi}{4}$, we get by an insertion and an application of the substitution

$$u=\sin\varphi, \qquad \mathrm{d}u = \cos\varphi\,\mathrm{d}\varphi, \qquad \cos^2\varphi = 1-\sin^2\varphi = 1-u^2,$$

that

$$\begin{aligned}
\Phi_5 &= 8\int_0^{\frac{\pi}{4}}\left\{1-\frac{1}{\sqrt{1+\cos^2\varphi}}\cdot\cos\varphi\right\}\mathrm{d}\varphi \\
&= 8\cdot\frac{\pi}{4} - 8\int_0^{\frac{1}{\sqrt{2}}}\frac{\mathrm{d}u}{\sqrt{1+(1-u^2)}} = 2\pi - 8\int_0^{\frac{1}{\sqrt{2}}}\frac{\mathrm{d}u}{\sqrt{2-u^2}} \\
&= 2\pi - 8\int_0^{\frac{1}{\sqrt{2}}}\frac{1}{\sqrt{1-\left(\dfrac{u}{\sqrt{2}}\right)^2}}\cdot\frac{1}{\sqrt{2}}\,\mathrm{d}u = 2\pi - 8\int_0^{\frac{1}{2}}\frac{\mathrm{d}v}{\sqrt{1-v^2}} \\
&= 2\pi - 8\,[\text{Arcsin } v]_0^{\frac{1}{2}} = 2\pi - 8\cdot\frac{\pi}{6} = \frac{2\pi}{3}.
\end{aligned}$$

Remark 33.4 It should be admitted that the polar version also contains some difficulties, though they are not as bad as in the rectangular version. ◊

Example 33.9
The Coulomb vector field (cf. Example 33.8),

$$\mathbf{V}(x,y,z)=\frac{(x,y,z)}{\left(x^2+y^2+z^2\right)^{3/2}}, \qquad \text{for } (x,y,z)\neq(0,0,0),$$

satisfies (where one absolutely should *not* put everything in the same fraction with the same denominator, unless one wants to obscure everything)

$$\begin{aligned}\frac{\partial V_x}{\partial x} &= \frac{\partial}{\partial x}\left\{\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}}\right\}=\frac{1}{\left(x^2+y^2+z^2\right)^{3/2}}-\frac{3x^2}{\left(x^2+y^2+z^2\right)^{5/2}},\\ \frac{\partial V_y}{\partial y} &= \frac{\partial}{\partial y}\left\{\frac{y}{\left(x^2+y^2+z^2\right)^{3/2}}\right\}=\frac{1}{\left(x^2+y^2+z^2\right)^{3/2}}-\frac{3y^2}{\left(x^2+y^2+z^2\right)^{5/2}},\\ \frac{\partial V_z}{\partial z} &= \frac{\partial}{\partial z}\left\{\frac{z}{\left(x^2+y^2+z^2\right)^{3/2}}\right\}=\frac{1}{\left(x^2+y^2+z^2\right)^{3/2}}-\frac{3z^2}{\left(x^2+y^2+z^2\right)^{5/2}},\end{aligned}$$

from which we get by adding these expressions,

$$\operatorname{div}\mathbf{V}=\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}=\frac{3}{\left(x^2+y^2+z^2\right)^{3/2}}-\frac{3\left(x^2+y^2+z^2\right)}{\left(x^2+y^2+z^2\right)^{5/2}}=0,$$

i.e. $\mathbf{V}$ is divergence free so we can use the results of Section 33.3.2 for domains Ω, which do *not* contain the point $(0,0,0)$.

A. *Let Ω be any spatial domain with $(0,0,0)$ as an inner point. Find the flux of the Coulomb field through $\partial\Omega$, i.e. find $\int_{\partial\Omega}\mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S$.*

D. Since div $\mathbf{V}$ is *not* defined in $(0,0,0)$, we cannot apply Gauß's theorem directly. But since $(0,0,0)$ is an inner point, there exists a ball

$$K=K(\mathbf{0};r)\subset\Omega,$$

totally contained in Ω. If we cut K out of Ω, we get a domain $\tilde{\Omega}=\Omega\setminus K$, in which div $\mathbf{V}$ is defined everywhere and equal to 0. According to Section 33.3.2 the surface $\partial\Omega$ can be deformed into ∂K, and then the flux through ∂K can be calculated as an ordinary surface integral. (The singular point $(0,0,0)$ lies in K, so we cannot apply Gauß's theorem in the latter calculation).

I. We are just missing one thing. Since both $\partial\Omega$ and ∂K are closed surfaces, neither of them has a boundary curve, so we get formally

$$\delta(\partial\Omega)=\emptyset=\delta(\partial K).$$

Alternatively there is flowing just as much into $\tilde{\Omega}$ through ∂K as out of $\tilde{\Omega}$ through $\partial\Omega$, because the flow is balanced.

Thus we have proved by using Gauß's theorem that

$$\int_{\partial\Omega}\mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S=\int_{\partial K}\mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S.$$

The right hand side is calculated as a usual surface integral, where it this time is worthwhile to keep the *abstract* formulation as long as possible.

1) On ∂K we have $r^2 = x^2 + y^2 + z^2$, i.e. $r = \left(x^2+y^2+z^2\right)^{1/2}$, and

$$\mathbf{n}(x,y,z) = \frac{1}{r}\,(x,y,z).$$

2) The vector field is then rewritten in the following way

$$\mathbf{V}(x,y,z) = \frac{(x,y,z)}{(x^2+y^2+z^2)^{3/2}} = \frac{1}{r^3}\,(x,y,z) = \frac{1}{r^2}\cdot\frac{1}{r}\,(x,y,z) = \frac{1}{r^2}\,\mathbf{n},$$

where we have used 1).

3) Since $\mathbf{n}\cdot\mathbf{n} = \|\mathbf{n}\|^2 = 1$, we get by an insertion of 2) that the flux is given by

$$\begin{aligned}\int_{\partial\Omega}\mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S &= \int_{\partial K}\mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S = \int_{\partial K}\frac{1}{r^2}\,\mathbf{n}\cdot\mathbf{n}\,\mathrm{d}S \\ &= \frac{1}{r^2}\int_{\partial K}\mathrm{d}S = \frac{1}{r^2}\,\mathrm{area}(\partial K) = \frac{1}{r^2}\cdot 4\pi r^2 = 4\pi,\end{aligned}$$

because the area of a sphere of radius r is given by $4\pi r^2$.

The result can be applied in an improved version of the horrible Example 33.8. Let $\Omega = K(\mathbf{0};r)$, where $r > \sqrt{3}\,a$, and let T be the cube of centre $\mathbf{0}$ and edge length $2a$. Then the flux through ∂T is equal to the flux through the sphere $\partial\Omega$, i.e. according to the above,

$$\int_{\partial T}\mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S = \int_{\partial\Omega}\mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S = 4\pi.$$

On the other hand, ∂T is disintegrated in a natural way into six squares of the same congruent form: They appear from each other by a convenient revolution around one of the axis. The Coulomb field is due to its symmetry invariant (apart from a change of letters) by these revolutions, so the flux is the same through every one of the six squares. If we choose one of these. e.g.

$$\mathcal{F} = \{(x,y,z) \mid -a \le x \le a,\ -a \le y \le a,\ z = a\} = [-a,a]\times[-a,a]\times\{a\},$$

then

$$4\pi = \int_{\partial T}\mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S = 6\int_{\cap F}\mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S,$$

from which

$$\int_{\mathcal{F}}\mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S = \frac{1}{6}\cdot 4\pi = \frac{2\pi}{3}.$$

Obviously this method is far easier in its calculations than the method applied in Example 33.8. ◊

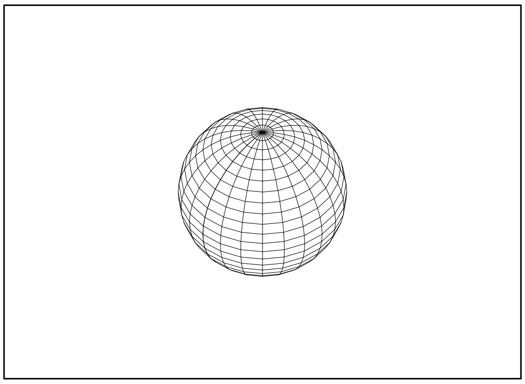

Figure 33.17: In most calculus courses Ω is typically a ball.

Example 33.10 Assume that Ω e.g. represents a subsoil water reservoir, which is polluted by some fluid or gas of *density* $\varrho = \varrho(x, y, z, t)$ and *velocity vector* $\mathbf{v} = \mathbf{v}(x, y, z, t)$.
The *mass* of the polluting agent in Ω at time t is given by

$$M = M(t) = \int_{\Omega} \varrho(x, y, z, t) \, \mathrm{d}\Omega = \int_{\Omega} \varrho \, \mathrm{d}\Omega.$$

The *change of mass* in time is then obviously equal to

$$(33.3) \quad \frac{\mathrm{d}M}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} \varrho(x, y, z, t) \, \mathrm{d}\Omega = \int_{\Omega} \frac{\partial \varrho}{\partial t} \, \mathrm{d}\Omega.$$

This change must be equal to the flow of mass *into* Ω by the vector field

$$\varrho\,\mathbf{v} = \varrho(x,y,z,t)\,\mathbf{v}(x,y,z,t).$$

Since $-\mathbf{n}$ points *into* Ω, this amount is according to *Gauß's theorem* equal to

$$-q = -\int_{\partial\Omega} \varrho\,\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}S = -\int_{\Omega} \operatorname{div}(\varrho\,\mathbf{v})\,\mathrm{d}\Omega.$$

When this expression is equated to (33.3), we get after a rearrangement that

$$0 = q + \frac{\mathrm{d}M}{\mathrm{d}t} = \int_{\Omega} \operatorname{div}(\varrho\,\mathbf{v})\,\mathrm{d}\Omega + \int_{\Omega} \frac{\partial\varrho}{\partial t}\,\mathrm{d}\Omega = \int_{\Omega} \left\{\operatorname{div}(\varrho\,\mathbf{v}) + \frac{\partial\varrho}{\partial t}\right\}\mathrm{d}\Omega.$$

This is true for *every* domain Ω. Assuming that the integrand is *continuous* (what it always is in practical applications), it must be 0 everywhere. In fact, if the integrand e.g was positive in a point, then it had due to the continuity also to be positive in an open domain Ω_1, and then the integral over Ω_1 becomes positive too, contradicting the assumption.

Thus we have once more derived the *continuity equation*

$$\operatorname{div}(\varrho\,\mathbf{v}) + \frac{\partial\varrho}{\partial t} = 0,$$

which the density and the velocity vector field of the pollution vector field must satisfy.

Remark 33.5 Here the divergence is referring to the *spatial* variables and *not* to the time variable t. Hence, the continuity equation is written in all details in the following way

$$\frac{\partial}{\partial x}\,(\varrho\,v_x) + \frac{\partial}{\partial y}\,(\varrho\,v_y) + \frac{\partial}{\partial z}\,(\varrho\,v_z) + \frac{\partial\varrho}{\partial t} = 0.$$

Furthermore it should be noted that there is a *big difference* here between the application of $\dfrac{\mathrm{d}}{\mathrm{d}t}$ and $\dfrac{\partial}{\partial t}$. $\Diamond$

Example 33.11 *Find in each of the following cases the flux of the given vector field through the described oriented surface $\mathcal{F}$.*

1) *The flux of* $\mathbf{V}(x,y,z) = (z, x, -3y^2z)$ *through the surface* $\mathcal{F}$ *given by* $x^2+y^2 = 16$ *for* $x \geq 0$, $y \geq 0$ *and* $z \in [0,5]$, *where the normal vector* $\mathbf{n}$ *is pointing away from the* Z*-axis.*

2) *The flux of* $\mathbf{V}(x,y,z) = (\cos x, 0, \cos x + \cos y)$ *through the surface* $\mathcal{F}$ *given by* $(x,y) \in [0,\pi] \times \left[0, \frac{\pi}{2}\right]$ *and* $z = 0$, *and where* $\mathbf{n} = \mathbf{e}_z$.

3) *The flux of* $\mathbf{V}(x,y,z) = (xy, z^2, 2yz)$ *through the surface* $\mathcal{F}$ *given by* $x^2+y^2+z^2 = a^2$, *and* $x \geq 0$, $y \geq 0$, $z \geq 0$, *and where* $\mathbf{n}$ *is pointing away from origo.*

4) *The flux of* $\mathbf{V}(x,y,z) = (x+y, x-y, y^2+z)$ *through the surface* $\mathcal{F}$ *given by* $x^2+y^2 \leq 1$ *and* $z = xy$, *and where* $\mathbf{n} \cdot \mathbf{e}_z > 0$.

5) *The flux of*
$$\mathbf{V}(x,y,z) = \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}\,(x,y,z),$$
through the surface $\mathcal{F}$ *given by* $\varrho \leq a$ *and* $z = h$, *and where* $\mathbf{n} = \mathbf{e}_z$.
[Cf. **Example 33.14**].

6) *The flux of*
$$\mathbf{V}(x,y,z) = \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}\,(x,y,z),$$
through the surface $\mathcal{F}$ *given by* $\varrho = a$ *and* $z \in [-h,h]$, *and where* $\mathbf{n}$ *is pointing away from the* Z*-axis.*
[Cf. **Example 33.14**].

7) *The flux of* $\mathbf{V}(x,y,z) = (y, x, x+y+z)$ *through the surface* $\mathcal{F}$ *given by the parametric description*
$$\mathbf{r}(u,v) = (u\,\cos v, u\,\sin v, hv), \qquad u \in [0,1], \quad v \in [0,2\pi].$$

8) *The flux of* $\mathbf{V}(x,y,z) = (y, -x, z^2)$ *through the surface* $\mathcal{F}$ *given by the parametric description*
$$\mathbf{r}(u,v) = \left(\sqrt{u}\,\cos v, \sqrt{u}\,\sin v, v^{\frac{3}{2}}\right), \qquad 1 \leq u \leq 2, \quad 0 \leq v \leq u.$$

9) *The flux of* $\mathbf{V}(x,y,z) = (yz, -xz, hz)$ *through the surface* $\mathcal{F}$ *given by the parametric description*
$$\mathbf{r}(u,v) = (u\,\cos v, u\,\sin v, hv), \qquad u \in [0,1], \quad v \in [0,2\pi].$$

A Flux of a vector field through a surface.

D Sketch whenever possible the surface. If the surface is only described in words, set up a parametric description. Compute the normal vector $\mathbf{N}$ (possibly the normed normal vector $\mathbf{n}$) and check the orientation. Finally, find the flux.

I 1) The surface is in semi polar coordinates described by
$$\varrho = a, \quad \varphi \in \left[0, \frac{\pi}{2}\right], \qquad z \in [0,5],$$
and the surface is a cylinder with the parameter domain
$$E = \left\{(\varphi, z) \;\middle|\; \varphi \in \left[0, \frac{\pi}{2}\right],\, z \in [0,5]\right\}.$$

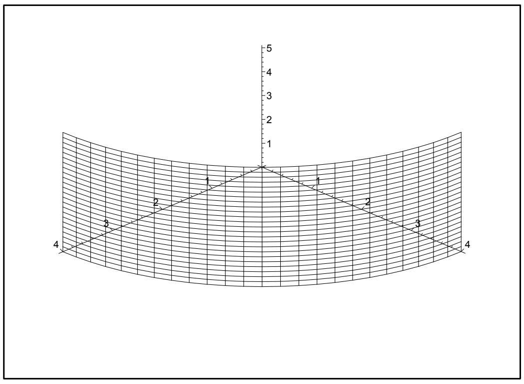

Figure 33.18: The surface $\mathcal{F}$ of **Example 33.11.1**.

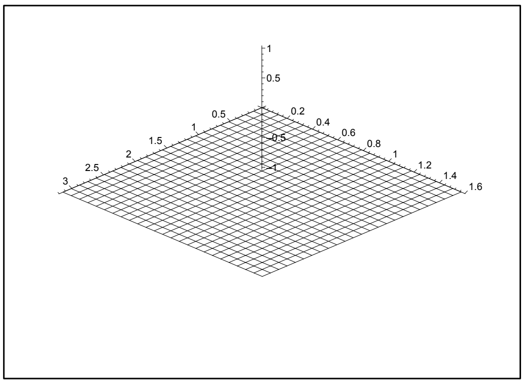

Figure 33.19: The surface $\mathcal{F}$ of **Example 33.11.2**.

The unit normal vector is

$$\mathbf{n} = (\cos\varphi, \sin\varphi, 0),$$

and the area element is

$$\mathrm{d}S = \mathrm{d}s\,\mathrm{d}z = 4\,\mathrm{d}\varphi\,\mathrm{d}z.$$

Hence we get the flux

$$\begin{aligned}
\int_{\mathcal{F}} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S &= \int_E \{z\,\cos\varphi + 4\cos\varphi\cdot\sin\varphi\}\cdot 4\,\mathrm{d}\varphi\,\mathrm{d}z \\
&= 4\int_0^{\frac{\pi}{2}} \left\{\int_0^5 (z\cos\varphi + 4\,\sin\varphi\cdot\cos\varphi)\,\mathrm{d}z\right\}\mathrm{d}\varphi \\
&= 4\int_0^{\frac{\pi}{2}} \left\{\frac{25}{2}\,\cos\varphi + 20\,\sin\varphi\,\cos\varphi\right\}\mathrm{d}\varphi = 4\cdot\frac{25}{2} + 4\cdot 20\cdot\frac{1}{2} \\
&= 90.
\end{aligned}$$

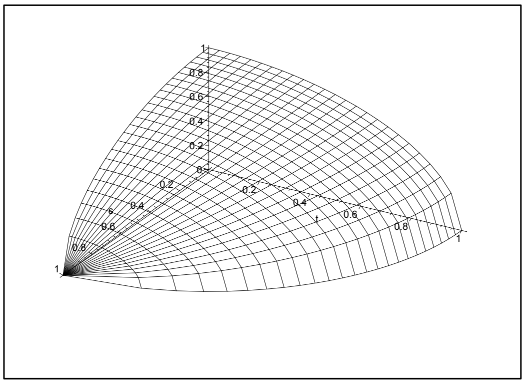

Figure 33.20: The surface $\mathcal{F}$ of **Example 33.11.3** for $a = 1$.

2) In this case the flux is

$$\begin{aligned}\int_{\mathcal{F}} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S &= \int_0^{\pi}\left\{\int_0^{\frac{\pi}{2}}(\cos x+\cos y)\,\mathrm{d}y\right\}\mathrm{d}x \\ &= \int_0^{\pi}\left\{\frac{\pi}{2}\cos x+1\right\}\mathrm{d}x = 0+1\cdot\pi=\pi.\end{aligned}$$

3) The surface is a subset of the sphere of centrum $(0,0,0)$ and radius a, lying in the first octant.

Choosing *rectangular coordinates* we find the area element on $\mathcal{F}$,

$$\mathrm{d}S = \frac{a}{\sqrt{a^2-x^2-y^2}}\,\mathrm{d}x\,\mathrm{d}y \qquad \left(= \text{“}\frac{a}{z}\,\mathrm{d}x\,\mathrm{d}y\text{”}\right),$$

and the unit normal vector is

$$\mathbf{n} = \frac{1}{a}(x,y,z) = \frac{1}{a}\left(x,y,\sqrt{a^2-x^2-y^2}\right), \qquad (x,y)\in E,$$

where the parameter domain is

$$E = \left\{(x,y) \mid 0 \le x \le a,\, 0 \le y \le \sqrt{a^2-x^2}\right\}.$$

Then the flux of the vector field $\mathbf{V}(x,y,z) = (xy, z^2, 2yz)$ through $\mathcal{F}$ is

$$\begin{aligned}
\int_{\mathcal{F}} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S &= \int_{\mathcal{F}} (xy,z^2,2yz)\cdot\frac{1}{a}(x,y,z)\,\mathrm{d}S \\
&= \frac{1}{a}\int_{\mathcal{F}}\left\{x^2y+yz^2+2yz^2\right\}\mathrm{d}S = \frac{1}{a}\int_{\mathcal{F}} y(x^2+3z^2)\,\mathrm{d}S \\
&= \frac{1}{a}\int_E a\left\{\frac{yx^2}{\sqrt{a^2-x^2-y^2}} + 3y\sqrt{a^2-x^2-y^2}\right\}\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^a\left\{\int_0^{\sqrt{a^2-x^2}}\left\{\frac{x^2}{\sqrt{a^2-x^2-y^2}} + 3\sqrt{a^2-x^2-y^2}\right\} y\,\mathrm{d}y\right\}\mathrm{d}x \\
&= \frac{1}{2}\int_0^a\left\{\int_0^{a^2-x^2}\left(\frac{x^2}{\sqrt{a^2-x^2-t^2}} + 3\sqrt{a^2-x^2-t}\right)\mathrm{d}t\right\}\mathrm{d}x \\
&= \frac{1}{2}\int_0^a\left[-2x^2\sqrt{a^2-x^2-t^2} - 3\cdot\frac{2}{3}\left(\sqrt{a^2-x^2t}\right)^3\right]_{t=0}^{a^2-x^2}\mathrm{d}x \\
&= \int_0^a\left\{x^2\sqrt{a^2-x^2} + (a^2-x^2)\sqrt{a^2-x^2}\right\}\mathrm{d}x \\
&= a^2\int_0^a \sqrt{a^2-x^2}\,dx = a^2\cdot\frac{\pi}{4}\cdot a^2 = \frac{\pi a^4}{4}.
\end{aligned}$$

ALTERNATIVELY, the area element on $\mathcal{F}$ is given in *polar coordinates* by

$$\mathrm{d}S = a^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi, \qquad \theta\in\left[0,\frac{\pi}{2}\right], \quad \varphi\in\left[0,\frac{\pi}{2}\right],$$

thus the parameter domain is

$$E = \left\{(\theta,\varphi)\;\middle|\; 0\le\theta\le\frac{\pi}{2},\, 0\le\varphi\le\frac{\pi}{2}\right\} = \left[0,\frac{\pi}{2}\right]\times\left[0,\frac{\pi}{2}\right].$$

As

$$(x,y,z) = a\,(\sin\theta\,\cos\varphi, \sin\theta\,\sin\varphi, \cos\theta),$$

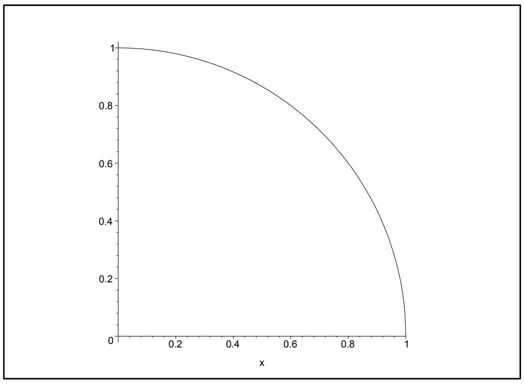

Figure 33.21: The parameter domain of **Example 33.11.3** for $a = 1$.

the unit normal vector is

$$\mathbf{n} = \frac{1}{a}(x, y, z) = (\sin\theta\,\cos\varphi, \sin\theta\,\sin\varphi, \cos\theta).$$

The flux of the vector field

$$\mathbf{V}(x, y, z) = (xy, z^2, 2yz)$$

through the surface $\mathcal{F}$ is

$$\begin{aligned}
\int_{\mathcal{F}} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S &= \int_{\mathcal{F}} (xy, z^2, 2yz)\cdot\frac{1}{a}(x, y, z)\,\mathrm{d}S \\
&= \frac{1}{2}\int_{\mathcal{F}} \left\{x^2y + yz^2 + 2yz^2\right\}\mathrm{d}S = \frac{1}{a}\int_{\mathcal{F}} y(x^2+3z^2)\,\mathrm{d}S \\
&= \frac{1}{a}\int_E a\sin\theta\,\sin\varphi\cdot a^2\left\{\sin^2\theta\,\cos^2\varphi + 3\cos^2\theta\right\}\cdot a^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&= a^4\int_0^{\frac{\pi}{2}}\left\{\int_0^{\frac{\pi}{2}}\sin^2\theta\left(\sin^2\theta\,\cos^2\varphi + 3\cos^2\theta\right)\sin\varphi\,\mathrm{d}\varphi\right\}\mathrm{d}\theta \\
&= a^4\int_0^{\frac{\pi}{2}}\sin^2\theta\left[-\frac{1}{3}\sin^2\theta\cos^3\varphi - \frac{1}{3}\cos^2\theta\cos\varphi\right]_{\varphi=0}^{\frac{\pi}{2}}\mathrm{d}\theta \\
&= a^4\int_0^{\frac{\pi}{2}}\sin^2\theta\left(\frac{1}{3}\sin^2\theta + 3\cos 2\theta\right)\mathrm{d}\theta.
\end{aligned}$$

We compute the integrand by introducing the double angle,

$$\begin{aligned}
&\int^{\theta}\left(3\cos^2\theta+\frac{1}{3}\sin^2\theta\right)\mathrm{d}\theta\\
&=\frac{1}{2}\,(1-\cos 2\theta)\left\{\frac{3}{2}\,(1+\cos 2\theta)+\frac{1}{6}\,(1-\cos 2\theta)\right\}\\
&=\frac{1}{12}\,(1-\cos 2\theta)\{9(1+\cos 2\theta)+(1-\cos 2\theta)\}\\
&=\frac{1}{12}(1-\cos 2\theta)(10+8\,\cos 2\theta)=\frac{1}{6}\,(1-\cos 2\theta)(5+4\,\cos 2\theta)\\
&=\frac{1}{6}\,(5-\cos 2\theta-4\cos^2 2\theta)=\frac{1}{6}\{5-\cos 2\theta-2(1+\cos 4\theta)\}\\
&=\frac{1}{6}\,(3-\cos 2\theta-2\cos 4\theta)=\frac{1}{2}-\frac{1}{6}\,\cos 2\theta-\frac{1}{3}\,\cos 4\theta.
\end{aligned}$$

The flux is obtained by insertion,

$$\begin{aligned}
\int_{\mathcal{F}}\mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S &= a^4\int_0^{\frac{\pi}{2}}\sin^2\theta\left(\frac{1}{3}\,\sin^2\theta+3\,\cos^2\theta\right)\,\mathrm{d}\theta\\
&= a^4\int_0^{\frac{\pi}{2}}\left\{\frac{1}{2}-\frac{1}{6}\,\cos 2\theta-\frac{1}{3}\,\cos 4\theta\right\}\,\mathrm{d}\theta\\
&= a^4\cdot\frac{1}{2}\cdot\frac{\pi}{2}-a^4\cdot\frac{1}{6}\cdot\frac{1}{2}[\sin 2\theta]_0^{\frac{\pi}{2}}-a^4\cdot 13\cdot\frac{1}{4}[\sin 4\theta]_0^{\frac{\pi}{2}}=\frac{\pi a^4}{4}.
\end{aligned}$$

4) Let $E=\{(x,y)\mid x^2+y^2\le 1\}$ be the unit disc. Then a parametric description of the surface $\mathcal{F}$ is given by

$$\{(x,y,xy)\mid (x,y)\in E\},$$

where the normal vector is

$$\mathbf{N}(x,y)=\left|\begin{array}{ccc}\mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z\\ 1 & 0 & y\\ 0 & 1 & x\end{array}\right|=(-y,-x,1),$$

and clearly, $\mathbf{N}\cdot\mathbf{e}_z=1>0$.

Then the flux of the vector field

$$\mathbf{V}(x,y,z)=(x+y,x-y,y^2+z)$$

through $\mathcal{F}$ is given by

$$\begin{aligned}
\int_{\mathcal{F}}\mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S &= \int_E\mathbf{V}\cdot\mathbf{N}\,\mathrm{d}x\,\mathrm{d}y=\int_E(x+y,x-y,y^2+xy)\cdot(-y,-x,1)\,\mathrm{d}x\,\mathrm{d}y\\
&= \int_E\left\{-xy-y^2-x^2+xy+y^2+xy\right\}\,\mathrm{d}x\,\mathrm{d}y=\int_E(xy-x^2)\,\mathrm{d}x\,\mathrm{d}y\\
&= \int_0^{2\pi}\left\{\int_0^1\varrho^2(\cos\varphi\cdot\sin\varphi-\cos^2\varphi)\varrho\,\mathrm{d}\varrho\right\}\,\mathrm{d}\varphi\\
&= \frac{1}{4}\int_0^{2\pi}(\cos\varphi\cdot\sin\varphi-\cos^2\varphi)\,d\varphi\\
&= 0-\frac{1}{4}\cdot 2\pi\cdot\frac{1}{2}=-\frac{\pi}{4}.
\end{aligned}$$

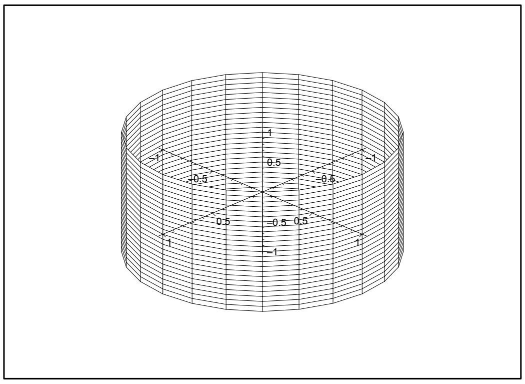

Figure 33.22: The surface $\mathcal{F}$ of **Example 33.11.6** for $a = 1$ og $h = 1$.

5) The surface $\mathcal{F}$ is a disc parallel to the XY-plane at the height h. We choose

$$E = \{(x, y) \mid x^2 + y^2 = \varrho^2 \le a^2\}.$$

as the parameter domain. Then the flux through $\mathcal{F}$ is

$$\begin{aligned}\int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n}\, \mathrm{d}S &= \int_E \frac{h}{(x^2+y^2+h^2)^{\frac{3}{2}}}\, \mathrm{d}x\, \mathrm{d}y = h \int_0^{2\pi} \left\{ \int_0^a \frac{1}{(\varrho^2+h^2)^{\frac{3}{2}}}\, \varrho\, \mathrm{d}\varrho \right\} \mathrm{d}\varphi \\ &= h \cdot 2\pi \left[\frac{1}{2}(-2) \frac{1}{\sqrt{\varrho^2+h^2}} \right]_{\varrho=0}^{a} = 2\pi h \left(\frac{1}{\sqrt{h^2}} - \frac{1}{\sqrt{a^2+h^2}} \right) \\ &= 2\pi \left(1 - \frac{h}{\sqrt{a^2+h^2}} \right).\end{aligned}$$

6) In this case $\mathcal{F}$ is a cylindric surface which is given in semi polar coordinates by the parametric description

$$\{(a, \varphi, z) \mid \varphi \in [0, 2\pi],\ z \in [-h, h]\},$$

and the parameter domain becomes

$$E = \{(\varphi, z) \mid \varphi \in [0, 2\pi],\ z \in [-h, h]\} = [0, 2\pi] \times [-h, h].$$

The unit normal vector pointing away from the Z-axis is

$$\mathbf{n} = (\cos\varphi, \sin\varphi, 0),$$

and the area element on $\mathcal{F}$ is

$$\mathrm{d}S = \mathrm{d}s\, \mathrm{d}z = a\, \mathrm{d}\varphi\, \mathrm{d}z,$$

thus the flux through $\mathcal{F}$ is

$$\begin{aligned}\int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n}\, \mathrm{d}S &= \int_E \frac{a}{(a^2+z^2)^{\frac{3}{2}}} (\cos^2\varphi + \sin^2\varphi + 0)\, a\, \mathrm{d}\varphi\, \mathrm{d}z \\ &= a^2 \cdot 2\pi \int_{-h}^{h} \frac{1}{(a^2+z^2)^{\frac{3}{2}}}\, \mathrm{d}z = 4\pi a^2 \int_0^h \frac{1}{(a^2+z^2)^{\frac{3}{2}}}\, \mathrm{d}z.\end{aligned}$$

It is natural here to introduce the substitution

$$z = a\,\sinh t, \qquad dz = a\,\cosh t\,dt, \qquad t = \text{ Arsinh}\left(\frac{z}{a}\right).$$

Then we get the flux through the surface

$$\begin{aligned}\int_{\mathcal{F}} \mathbf{V}\cdot\mathbf{n}\,dS &= 4\pi a^2 \int_0^{\text{Arsinh}\left(\frac{h}{a}\right)} \frac{a\,\cosh t}{a^3\cosh^3 t}\,dt = 4\pi \int_0^{\text{Arsinh}\left(\frac{h}{a}\right)} \frac{dt}{\cosh^2 t}\\ &= 4\pi[\tanh t]_0^{\text{Arsinh}\left(\frac{h}{a}\right)} = 4\pi\left[\frac{\sinh t}{\sqrt{1+\sinh^2 t}}\right]_0^{\text{Arsinh}\left(\frac{h}{a}\right)} = 4\pi\cdot\frac{\frac{h}{a}}{\sqrt{1+\frac{h^2}{a^2}}}\\ &= \frac{4\pi h}{\sqrt{a^2+h^2}}.\end{aligned}$$

Remark. The field of **Example 33.11.5** and **Example 33.11.6** is the so-called *Coulomb field*, cf. **Section 33.3.2**. It is tempting to combine the results of **Example 33.11.5** and **Example 33.11.6** to find the flux of the Coulomb field through the surface of the whole cylinder. Since $\mathbf{n} = -\mathbf{e}_z$, when we consider the surface of **Example 33.11.5** at height $-h$, it follows that

$$\text{flux} = 2\pi\left(1-\frac{h}{\sqrt{a^2+h^2}}\right)+\frac{4\pi h}{\sqrt{a^2+h^2}}-2\pi\left(\frac{-h}{\sqrt{h^2}}-\frac{(-h)}{\sqrt{a^2+h^2}}\right) = 4\pi. \qquad \Diamond$$

7) Here

$$\mathbf{N}(u,v)=\begin{vmatrix}\mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z\\ \cos v & \sin v & 0\\ -u\,\sin v & u\,\cos v & h\end{vmatrix}=(h\,\sin v,-h\,\cos v,u),$$

so the flux of the vector field $(y,x,x+y+z)$ through $\mathcal{F}$ is

$$\begin{aligned}\int_{\mathcal{F}}\mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S&=\int_E\mathbf{V}\cdot\mathbf{N}(u,v)\,\mathrm{d}u\,\mathrm{d}v\\
&=\int_E(u\sin v,u\cos v,u(\cos v+\sin v)+hv)\cdot(h\sin v,-h\cos v,u)\,\mathrm{d}u\,\mathrm{d}v\\
&=\int_E(hu\sin^2-hu\cos^2 v+u^2(\cos v+\sin v)+huv)\,\mathrm{d}u\,\mathrm{d}v\\
&=\int_E hu(-\cos 2v)\,\mathrm{d}u\,\mathrm{d}v+\int_E u^2(\cos v+\sin v)\,\mathrm{d}u\,\mathrm{d}v+h\int_E uv\,\mathrm{d}u\,\mathrm{d}v\\
&=0+0+h\int_0^1 u\,\mathrm{d}u\int_0^{2\pi}v\,\mathrm{d}v=h\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot 4\pi^2=h\pi^2.\end{aligned}$$

8) The normal vector of the surface $\mathcal{F}$ of the parametric description

$$\mathbf{r}(u,v)\left(\sqrt{u}\,\cos v,\sqrt{u}\,\sin v,f^{3/2}\right),\qquad 1\le u\le 2,\quad 0\le v\le u,$$

is

$$\begin{aligned}\mathbf{N}(u,v)&=\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}=\begin{vmatrix}\mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z\\ \frac{1}{2\sqrt{u}}\cos v & \frac{1}{2\sqrt{u}}\sin v & 0\\ -\sqrt{u}\,\sin v & \sqrt{u}\,\cos v & \frac{3}{2}\sqrt{v}\end{vmatrix}\\
&=\left(\frac{3}{4}\sqrt{\frac{v}{u}}\,\sin v,-\frac{3}{4}\sqrt{\frac{v}{u}}\,\cos v,\frac{1}{2}\right).\end{aligned}$$

The flux of $\mathbf{V}(x,y,z)=(y,-x,z^2)$ through $\mathcal{F}$ is

$$\begin{aligned}\int_{\mathcal{F}}\mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S&=\int_E\mathbf{V}(u,v)\cdot\mathbf{N}(u,v)\,\mathrm{d}u\,\mathrm{d}v\\
&=\int_E\left(\sqrt{u}\sin v,-\sqrt{u}\cos v,v^3\right)\cdot\left(\frac{3}{4}\sqrt{\frac{v}{u}}\sin v,-\frac{3}{4}\sqrt{\frac{v}{u}}\cos v,\frac{1}{2}\right)\,\mathrm{d}u\,\mathrm{d}v\\
&=\int_E\left\{\frac{3}{4}\sqrt{v}\sin^2 v+\frac{3}{4}\sqrt{v}\cos^2 v+\frac{1}{2}v^3\right\}\,\mathrm{d}u\,\mathrm{d}v\\
&=\int_E\left\{\frac{3}{4}\sqrt{v}+\frac{1}{2}v^3\right\}\,\mathrm{d}u\,\mathrm{d}v=\int_1^2\left\{\int_0^u\left(\frac{3}{4}v^{\frac{1}{2}}+\frac{1}{2}v^3\right)\,\mathrm{d}v\right\}\,\mathrm{d}u\\
&=\int_1^2\left[\frac{3}{4}\cdot\frac{2}{3}v^{\frac{3}{2}}+\frac{1}{8}v^4\right]_0^u\,\mathrm{d}u=\int_1^2\left(\frac{1}{2}u^{\frac{3}{2}}+\frac{1}{8}u^4\right)\,\mathrm{d}u\\
&=\left[\frac{1}{2}\cdot\frac{2}{5}u^{\frac{5}{2}}+\frac{1}{40}u^5\right]_1^2=\frac{1}{5}(\sqrt{2})^5+\frac{1}{40}\cdot 2^5-\frac{1}{5}-\frac{1}{40}\\
&=\frac{1}{40}(8\cdot 4\sqrt{2}+32-8-1)=\frac{1}{40}(32\sqrt{2}+23)=\frac{4\sqrt{2}}{5}+\frac{23}{40}.\end{aligned}$$

9) Here we have [cf. **Example 33.11.7**]

$$\mathbf{N}(u,v) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \cos v & \sin v & 0 \\ -u\,\sin v & u\,\cos v & h \end{vmatrix} = (h\,\sin v, -h\,\cos v, u),$$

and the flux of the vector field $(yz, -xz, hz)$ through the surface $\mathcal{F}$ becomes

$$\begin{aligned} \int_{\mathcal{F}} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S &= \int_E \mathbf{V}\cdot\mathbf{N}(u,v)\,\mathrm{d}u\,\mathrm{d}v \\ &= \int_E (uhv\sin v, -uhv\cos v, h^2 v)\cdot(h\sin v, -\cos v, u)\,\mathrm{d}u\,\mathrm{d}v \\ &= h\int_E (uh\sin^2 v + uh\cos^2 v + huv)\,\mathrm{d}u\,\mathrm{d}v = h^2\int_E u(1+v)\,\mathrm{d}u\,\mathrm{d}v \\ &= h^2\int_0^1 u\,\mathrm{d}u\cdot\int_0^{2\pi}(v+1)\,\mathrm{d}v = h^2\cdot\frac{1}{2}\left[\frac{v^2}{2}+v\right]_0^{2\pi} = \frac{h^2}{2}\cdot\{2\pi^2+2\pi\} = h^2\pi(\pi+1). \end{aligned}$$

33.5.2 Examples of application of Gauß's theorem

Example 33.12 *Find in each of the following cases the flux of the given vector field* $\mathbf{V}$ *through the surface of the given set* Ω *in the space.*

1) *The vector field* $\mathbf{V}(x,y,z) = (5xz, y^2-2yz, 2yz)$*, defined in the domain* Ω *by* $x^2+y^2 \le a^2$*,* $y \ge 0$*,* $0 \le z \le b$*.*

2) *The vector field* $\mathbf{V}(x,y,z) = \left(2x-\sqrt{1+z^2}, x^2y, -xz^2\right)$*, defined in the cube* $\Omega = [0,1]\times[0,1]\times[0,1]$*.*

3) *The vector field* $\mathbf{V}(x,y,z) = (x^2+y^2, y^2+z^2, z^2+x^2)$ *given in the domain* Ω *defined by* $x^2+y^2+z^2 \le a^2$ *and* $z \ge 0$*.*

4) *The vector field* $\mathbf{V}(x,y,z) = \left(2x+\sqrt[3]{y^2+z^2}, y-\cosh(xz), y^2+2z\right)$*, defined in the solid ball*
$$\Omega = \overline{K}((3,-1,2);3).$$

5) *The vector field* $\mathbf{V}(x,y,z) = (-x+\cos z, -xy, 3z+e^y)$*, defined in the domain* Ω *given by* $x \in [0,3]$*,* $y \in [0,2]$*,* $z \in [0,y^2]$*.*

6) *The vector field* ∇T*, where* $T(x,y,z) = x^2+y^2+z^2$ *is defined in the domain* Ω *given by* $x^2+y^2 \le 2$ *and* $z \in [0,2]$*.*

7) *The vector field* $\mathbf{V}(x,y,z) = (x^3+xy^2, 4yz^2-2x^2y, -z^3)$*, defined in the solid ball given by*
$$x^2+y^2+z^2 \le a^2.$$

8) *The vector field* $\mathbf{V}(x,y,z) = (2x, 3y, -z)$*, defined in the ellipsoid* Ω*, given by*
$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \le 1.$$

A Flux out of a body in space.

D Apply Gauß's theorem of divergence.

I According to Gauß's theorem the flux is given by
$$\int_{\partial\Omega} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S = \int_\Omega \operatorname{div}\,\mathbf{V}\,\mathrm{d}\Omega.$$

1) Since
$$\operatorname{div}\,\mathbf{V} = 5z+2y-2z+2y = 3z+4y,$$
the flux is
$$\int_\Omega \operatorname{div}\,\mathbf{V}\,\mathrm{d}\Omega = \int_0^b 3z\,\mathrm{d}z\cdot\frac{1}{2}\,\pi\,a^2 + 4\int_0^b\int_0^\pi\left\{\int_0^a \varrho\sin\varphi\cdot\varrho\,\mathrm{d}\varrho\right\}\mathrm{d}\varphi\,\mathrm{d}z = \frac{3}{4}\,\pi\,a^2b^2+\frac{8}{3}\,a^3b.$$

2) Since
$$\operatorname{div}\,\mathbf{V} = 2+x^2-2xz,$$
the flux is
$$\int_\Omega \operatorname{div}\mathbf{V}\,\mathrm{d}\Omega = 2+\int_\Omega x^2\,\mathrm{d}\Omega - \int_\Omega 2xz\,\mathrm{d}\Omega = 2+\frac{1}{3}-\frac{1}{2} = \frac{11}{6}.$$

3) Here

$$\operatorname{div} \mathbf{V} = 2x + 2y + 2z.$$

It follows by the symmetry that

$$\int_\Omega 2x \, d\Omega = \int_\Omega 2y \, d\Omega = 0.$$

We obtain the flux by an application of Gauß's theorem, the argument of symmetry above and semi polar coordinate,

$$\begin{aligned}\int_\Omega \operatorname{div} \mathbf{V} \, d\Omega &= \int_\Omega 2x \, d\Omega + \int_\Omega 2y \, d\Omega + \int_\Omega 2z \, d\Omega = \int_\Omega 2z \, d\Omega \\ &= \int_0^{2\pi} \left\{ \int_0^a \left\{ \int_0^{\sqrt{a^2-\varrho^2}} 2z \, dz \right\} \varrho \, d\varrho \right\} d\varphi \\ &= 2\pi \int_0^a (a^2 - \varrho^2)\varrho \, d\varrho = 2\pi \left[\frac{a^2}{2}\varrho^2 - \frac{\varrho^4}{4} \right]_0^a = 2\pi \cdot \frac{a^4}{4} = \frac{\pi a^4}{2}.\end{aligned}$$

4) Since

$$\operatorname{div} \mathbf{V} = 2 + 1 + 2 = 5,$$

the flux is

$$\int_\Omega \operatorname{div} \mathbf{V} \, d\Omega = 5 \operatorname{vol}(\overline{K}((3,-1,2);3)) = 5 \cdot \frac{4\pi}{3} \cdot 3^3 = 180\pi.$$

5) Since

$$\operatorname{div} \mathbf{V} = -1 - x + 3 = 2 - x,$$

the flux is given by

$$\begin{aligned}\int_\Omega \operatorname{div} \mathbf{V} \, d\Omega &= \int_\Omega (2-x) \, d\Omega = \int_0^3 (2-x) \left\{ \int_0^2 \left\{ \int_0^{y^2} dz \right\} dy \right\} dx \\ &= \left[2x - \frac{x^2}{2} \right]_0^3 \cdot \int_0^2 y^2 \, dy = \left(6 - \frac{9}{2} \right) \cdot \left[\frac{y^3}{3} \right]_0^2 = \frac{3}{2} \cdot \frac{8}{3} = 4.\end{aligned}$$

6) Since

$$\operatorname{div} \mathbf{V} = \Delta(x^2 + y^2 + z^2) = 2 + 2 + 2 = 6,$$

the flux is given by

$$\int_\Omega \operatorname{div} \mathbf{V} \, d\Omega = 6 \operatorname{vol}(\Omega) = 6 \cdot \pi \cdot (\sqrt{2})^2 \cdot 2 = 24\pi.$$

7) Here,

$$\operatorname{div} \mathbf{V} = 3x^2 + y^2 + 4z^2 - 2x^2 - 3z^2 = x^2 + y^2 + z^2.$$

The flux is easiest computed in spherical coordinates,

$$\int_\Omega \operatorname{div} \mathbf{V} \, d\Omega = \int_0^{2\pi} \left\{ \int_0^\pi \left\{ \int_0^a r^2 \cdot r \sin\theta \, dr \right\} d\theta \right\} d\varphi = 2\pi \left[\frac{r^5}{5} \right]_0^1 \cdot [-\cos\theta]_0^\pi = \frac{4}{5}\pi a^5.$$

8) From

$$\mathrm{div}\ \mathbf{V} = 2 + 3 - 1 = 4,$$

follows that the flux is

$$\int_\Omega \mathrm{div}\ \mathbf{V}\,\mathrm{d}\Omega = 4\,\mathrm{vol}(\Omega) = 4 \cdot \frac{4\pi}{3}\,abc = \frac{16}{3}\,\pi\,abc.$$

Example 33.13 *Find in each of the following cases the flux of the given vector field* $\mathbf{V}$ *through the surface of the described body of revolution* Ω.

1) The vector field is $\mathbf{V}(x,y,z) = (y^2+z^4, (x-a)^2+z^4, x^2+y^2)$, *and the meridian cut of* Ω *is given by* $\varrho \le a$ *and* $0 \le z \le \sqrt[4]{a^2-\varrho^2}$.

2) The vector field is

$$\mathbf{V}(x,y,z) = (x^2-2xy, 2y^2+6x^2z^2, 2z-2xz-2yz),$$

and the meridian cut of Ω *is given by* $0 \le z \le 1$ *and* $\varrho \le e^{-z}$.

3) The vector field is $\mathbf{V}(x,y,z))(x^2-xz, y^2-yz, z^2)$, *and the meridian cut of* Ω *is given by* $\varrho \le \sqrt{\ln z}$ *and* $z \in [e, e^2]$.

4) The vector field is $\mathbf{V}(x,y,z) = (2x+2y, 2y+z, z+2x)$, *and the meridian cut of* Ω *is given by*

$$\varrho \le a, \qquad \frac{\varrho^2-a^2}{a} \le z \le \sqrt{a^2-\varrho}.$$

A Flux through the surface of a body of revolution.

D Sketch if possible the meridian cut. Calculate div $\mathbf{V}$ and apply Gauß's theorem.

I 1) From div $\mathbf{V} = 0$, follows trivially that the flux is

$$\int_\Omega \mathrm{div}\ \mathbf{V}\,\mathrm{d}\Omega = 0,$$

and we do not have to think about the body of revolution at all.

2) We conclude from

$$\mathrm{div}\ \mathbf{V} = 2x - 2y + 4y + 2 - 2x - 2y = 2,$$

that the flux is

$$\int_\Omega \mathrm{div}\ \mathbf{V}\,\mathrm{d}\Omega = 2\,\mathrm{vol}(\Omega) = 2\int_0^1 \pi\,e^{-2z}\,\mathrm{d}z = \pi\left(1-e^{-2}\right).$$

3) Here,

$$\mathrm{div}\ \mathbf{V} = 2x - z + 2y - z + 2z = 2x + 2y.$$

If we put

$$B(z) = \{(x,y) \mid x^2+y^2 \le \ln z\}, \qquad z \in [e, e^2],$$

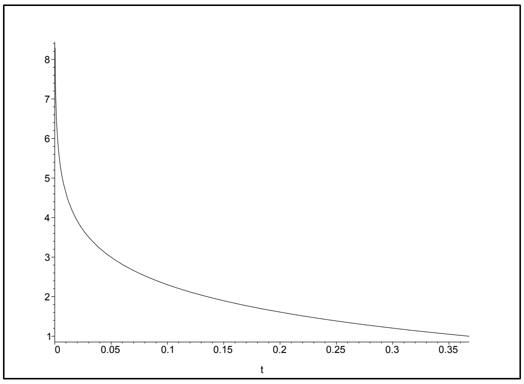

Figure 33.23: The meridian cut of i **Example 33.13.2**.

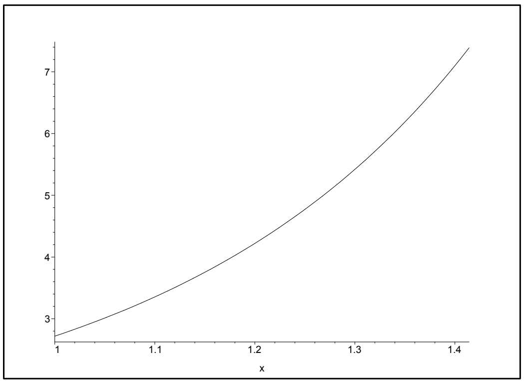

Figure 33.24: The meridian cut of **Example 33.13.3**.

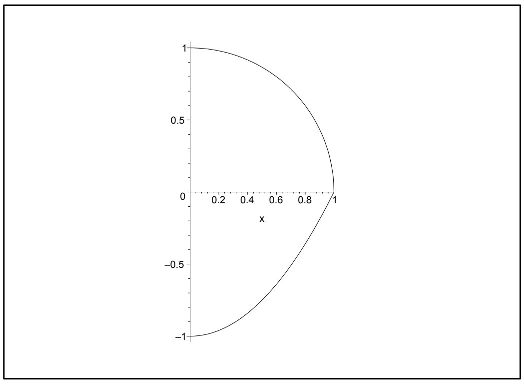

Figure 33.25: The meridian cut of **Example 33.13.4**.

then the flux is

$$\int_\Omega \operatorname{div} \mathbf{V}\, \mathrm{d}\Omega = \int_\Omega (2x+2y)\, \mathrm{d}\Omega = \int_e^{e^2} \left\{ \int_{B(z)} (2x+2y)\, \mathrm{d}x\, \mathrm{d}y \right\} dz = 0,$$

because it follows from the symmetry that

$$\int_{B(z)} x\, \mathrm{d}x\, \mathrm{d}y = \int_{B(z)} y\, \mathrm{d}x\, \mathrm{d}y = 0.$$

4) It follows from the equations of the meridian cut that when $z > 0$ we have the quarter of a circle, and when $z < 0$ we get an arc of a parabola. It is natural to split the cut of Ω_0 correspondingly in Ω_1 (for $z > 0$) and Ω_2 (for $z < 0$).

Since

$$\text{div}\, \mathbf{V} = 2+2+1 = 5,$$

we get by Gauß's theorem that the flux is

$$\begin{aligned}
\text{flux} &= \int_{\partial\Omega} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S = \int_{\Omega} \text{div}\,\mathbf{V}\,\mathrm{d}\Omega = 5\,\text{vol}(\Omega) = 5\,\text{vol}(\Omega_1) + 5\,\text{vol}(\Omega_2) \\
&= 5\cdot\frac{1}{2}\cdot\frac{4\pi}{3}\,a^3 + 5\int_{-a}^{0} \pi\varrho(z)^2\,\mathrm{d}z = \frac{10\pi}{3}\,a^3 + 5\pi\int_{-a}^{0}(az+a^2)\,\mathrm{d}z \\
&= \frac{10\pi}{3}\,a^3 + 5\pi\left[\frac{az^2}{2}+a^2z\right]_{-a}^{0} = \frac{10\pi}{3}\,a^3 + 5\pi\left(-\frac{a^3}{2}+a^3\right) \\
&= 5\pi a^3\left(\frac{2}{3}+\frac{1}{2}\right) = 5\pi a^3\cdot\frac{7}{6} = \frac{35}{6}\,\pi a^3.
\end{aligned}$$

Example 33.14 *Let Ω denote the cylinder given by $z \in [-h,h]$, $\varrho \in [0,a]$, $\varphi \in [0,2\pi]$. Find the flux through the surface $\partial\Omega$ of the* Coulomb vector field

$$\mathbf{V}(x,y,z) = \frac{1}{r^3}\,(x,y,z), \qquad (x,y,z) \neq (0,0,0), \qquad r = \sqrt{x^2+y^2+z^2}.$$

[*Cf.* **Example 33.11.5**, **Example 33.11.6** *and* **Example 35.8**].

A Flux through the surface of a body.

D Think of how to treat the singularity at $(0,0,0)$ before we can apply Gauß's theorem. Find the flux.

I When $(x,y,z) \neq (0,0,0)$, we get [cf. **Example 35.8**]

$$\frac{\partial V_1}{\partial x} = \frac{1}{r^3} - \frac{3}{r^5}\,x^2, \quad \frac{\partial V_2}{\partial y} = \frac{1}{r^3} - \frac{3}{r^5}\,y^2, \quad \frac{\partial V_3}{\partial z} = \frac{1}{r^3} - \frac{3}{r^5}\,z^2,$$

hence

$$\text{div}\,\mathbf{V} = \frac{3}{r^3} - \frac{3}{r^5}\,(x^2+y^2+z^2) = \frac{3}{r^3} - \frac{3}{r^5}\,r^2 = 0.$$

One could therefore be misled to "conclude" that the flux is 0, "because $(0,0,0)$ is a null set"; but this is not true.

Let $R \in]0, \min\{a,h\}[$. An application of Gauß's theorem shows that the flux through the surface of $\Omega \setminus K(\mathbf{0};R)$ is

$$\int_{\Omega\setminus K(\mathbf{0};R)} \text{div}\,\mathbf{V}\,\mathrm{d}\Omega = 0,$$

because $(0,0,0) \notin \Omega \setminus K(\mathbf{0};R)$. Hence, the flux is

$$\begin{aligned}\int_{\partial\Omega} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S &= \left\{\int_{\partial\Omega} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S - \int_{\partial K(\mathbf{0};R)} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S\right\} + \int_{\partial K(\mathbf{0};R)} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S \\ &= \int_{\Omega\setminus K(\mathbf{0};R)} \operatorname{div}\,\mathbf{V}\,\mathrm{d}\Omega + \int_{\partial K(\mathbf{0};R)} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S = \int_{\partial K(\mathbf{0};R)} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S.\end{aligned}$$

On the boundary $\partial K(\mathbf{0};R)$ the outer unit normal vector is given in rectangular coordinates by $\mathbf{n} = \dfrac{1}{R}\,(x,y,z)$, thus

$$\mathbf{V}\cdot\mathbf{n} = \frac{1}{R^3}\,(x,y,z)\cdot\frac{1}{R}\,(x,y,z) = \frac{1}{R^2}.$$

The area element is given in polar coordinates by

$$\mathrm{d}S = R^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi.$$

Then the flux through $\partial\Omega$ is given by

$$\int_{\partial\Omega} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S = \int_{\partial K(\mathbf{0};R)} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S = \int_0^{2\pi}\left\{\int_0^{\pi}\frac{1}{R^2}\cdot R^2\sin\theta\,\mathrm{d}\theta\right\}\mathrm{d}\varphi = 2\pi[-\cos\theta]_0^{\pi} = 4\pi,$$

Example 33.15 *We shall find the flux Φ of the vector field*

$$\mathbf{V}(x,y,z) = (e^y+\cosh z, e^x+\sinh z, x^2z^2), \qquad (x,y,z)\in\mathbb{R}^3,$$

through the oriented half sphere $\mathcal{F}$ given by

$$x^2+y^2+z^2-2az=0, \qquad z\le a, \quad \mathbf{n}\cdot\mathbf{e}_z \ge 0.$$

It turns up that the integration over $\mathcal{F}$ is rather difficult, while on the other hand the expression of div $\mathbf{V}$ is fairly simple. One will therefore try to arrange the calculations such that it becomes possible to apply Gauß's theorem.

1) Construct a closed surface by adding an oriented dist $\mathcal{F}_1$ to $\mathcal{F}$. Sketch the meridian half plane.

2) Find the flux Φ_1 of the vector field $\mathbf{V}$ through $\mathcal{F}_1$.

3) Apply Gauß's theorem on the body Ω of the boundary $\partial\Omega = \mathcal{F}\cup\mathcal{F}_1$, and then find Φ.

A Computation of the flux of a vector field through a surface where a direct calculation becomes very difficult.

D Apply the guidelines, i.e. add a surface $\mathcal{F}_1$, such that $\mathcal{F}\cup\mathcal{F}_1$ surrounds a body, on which Gauß's theorem can be applied. Hence, something is added and then subtracted again, and then one uses Gauß's theorem.

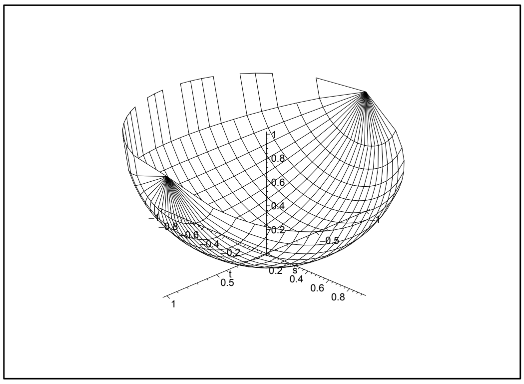

Figure 33.26: The surface of **Example 33.15** for $a = 1$.

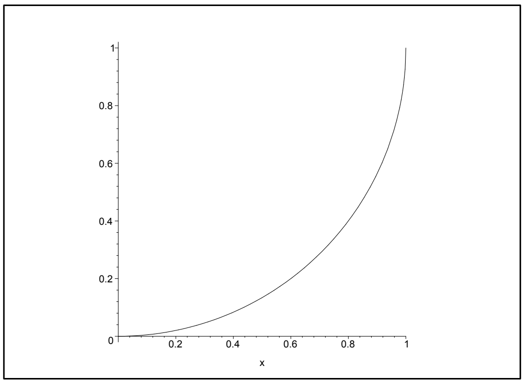

Figure 33.27: The meridian curve of **Example 33.15** for $a = 1$.

I 1) When we add a^2 to both sides of the equation of the half sphere, we obtain

$$a^2 = x^2 + y^2 + z^2 - 2az + a^2 = \varrho^2 + (z-a)^2.$$

It follows from the condition $\mathbf{n} \cdot \mathbf{e}_z \geq 0$ that the curve in the meridian half plane of $\mathcal{F}$ is the quarter of a circle of centrum $(0, a)$ and radius a,

$$\varrho^2 + (z-a)^2 = a^2, \qquad z \leq a, \quad \varrho \geq 0.$$

Note that the normal vector has an upwards pointing component.
The disc ("the lid"), which shall be added is of course the disc in the plane $z = a$ of centrum $(0, 0, a)$ and radius a.

2) The flux of $\mathbf{V}$ through $\mathcal{F}_1$ of normal $\mathbf{e}_z$ is

$$\int_{\mathcal{F}_1} \mathbf{V} \cdot \mathbf{n} \, \mathrm{d}S = \int_{\mathcal{F}_1} x^2 a^2 \, \mathrm{d}S = a^2 \int_0^{2\pi} \cos^2 \varphi \left\{ \int_0^a \varrho^2 \cdot \varrho \, \mathrm{d}\varrho \right\} \mathrm{d}\varphi = a^2 \pi \cdot \left[\frac{\varrho^4}{4} \right]_0^a = \frac{\pi}{4} a^6.$$

3) Let Ω be the domain which is surrounded by $\mathcal{F}_1 \cup (-\mathcal{F})$, where $-\mathcal{F}$ indicates that we have reversed the orientation, such that the normal is pointing away from Ω) on both $\mathcal{F}_1$ and $-\mathcal{F}$.

Then

$$\mathrm{div}\; \mathbf{V} = 0 + 0 + 2x^2 z = 2x^z = 2x^2(z-a) + 2ax^2,$$

so it follows from Gauß's theorem that

$$-\int_{\mathcal{F}} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S+\int_{\mathcal{F}_1} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S=-\int_{\mathcal{F}} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S+\frac{\pi}{4}\,a^6=\int_{\Omega} \operatorname{div}\mathbf{V}\,\mathrm{d}\Omega,$$

hence by a rearrangement,

$$\begin{aligned}\Phi &= \int_{\mathcal{F}} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S=\frac{\pi}{4}\,a^6-\int_{\Omega} \operatorname{div}\mathbf{V}\,\mathrm{d}\Omega=\frac{\pi}{4}\,a^6-\int_{\Omega} 2ax^2\,\mathrm{d}\Omega-\int_{\Omega} 2x^2(z-a)\,\mathrm{d}\Omega\\ &= \frac{\pi}{4}\,a^6-a\int_{\Omega}\mathrm{d}\Omega+\int_{\Omega}(x^2+y^2)(a-z)\,\mathrm{d}\Omega,\end{aligned}$$

where we have used the symmetry in x and y in the domain of integration in the latter equality.

By the transformation $z \curvearrowright a-z$ the solid half ball Ω is mapped into the solid half ball

$$\Omega_1=\{(x,y,z)\mid x^2+y^2+z^2\le a^2,\ z\ge 0\},$$

so

$$\Phi=\frac{\pi}{4}\,a^6-a\int_{\Omega_1}(x^2+y^2)\delta\Omega+\int_{\Omega_1}(x^2+y^2)z\,d\Omega.$$

When we use the slicing method, we see that Ω_1 at height $z\in[0,a]$ is cut into the circle

$$B(z)=\{(x,y,z)\mid x^2+y^2\le a^2-z^2\}=\{(x,y,z)\mid \varrho\le\sqrt{a^2-z^2}\},\qquad z\in[0,a]\text{ fixed},$$

hence

$$\begin{aligned}a\int_{\Omega_1}(x^2+y^2)\,\mathrm{d}\Omega &= a\int_0^a\left\{\int_{B(z)}(x^2+y^2)\,\mathrm{d}S\right\}\mathrm{d}z\\ &= a\int_0^a\left\{\int_0^{2\pi}\left[\int_0^{\sqrt{a^2-z^2}}\varrho^2\cdot\varrho\,\mathrm{d}\varrho\right]\mathrm{d}\varphi\right\}\mathrm{d}z=2\pi a\int_0^a\left[\frac{\varrho^4}{4}\right]_0^{\sqrt{a^2-z^2}}\mathrm{d}z\\ &= \frac{\pi}{2}\,a\int_0^a(a^2-z^2)^2\,\mathrm{d}z=\frac{\pi}{2}\,a\int_0^a(z^4-2a^2z^2+a^4)\,\mathrm{d}z\\ &= \frac{\pi}{2}\,a\left[\frac{z^5}{5}-\frac{2a^2}{3}\,z^3+a^4z\right]_0^a=\frac{\pi}{2}\,a\left\{\frac{a^5}{5}-\frac{2}{3}\,a^5+a^5\right\}\\ &= \frac{\pi}{2}\,a^6\cdot\left(\frac{1}{5}-\frac{2}{3}+1\right)=\frac{4\pi}{15}\,a^6,\end{aligned}$$

and by some reuse of previous results,

$$\begin{aligned}\int_{\Omega_1}(x^2+y^2)z\,\mathrm{d}\Omega &= \int_0^a z\left\{\int_{B(z)}(x^2+y^2)\,\mathrm{d}S\right\}\mathrm{d}z\\ &= \frac{\pi}{2}\int_0^a(z^2-a^2)^2\cdot z\,\mathrm{d}z=\frac{\pi}{4}\left[\frac{1}{3}(z^2-a^2)^3\right]_0^a=\frac{\pi}{12}\,a^6.\end{aligned}$$

Finally, we get by insertion that

$$\begin{aligned}\Phi &= \frac{\pi}{4}\,a^6-a\int_{\Omega_1}(x^2+y^2)\,\mathrm{d}\Omega+\int_{\Omega_1}(x^2+y^2)z\,\mathrm{d}\Omega\\ &= \frac{\pi}{4}\,a^6-\frac{4\pi}{15}\,a^6+\frac{\pi}{12}\,a^6=\frac{\pi a^6}{60}\,(15-16+5)=\frac{\pi a^6}{15}.\end{aligned}$$

Example 33.16 *Let a set* $\Omega \subset \mathbb{R}^3$ *and a vector field* $\mathbf{V} : \mathbb{R}^3 \to \mathbb{R}^3$ *be given in the following way,*

$$\Omega = \left\{ (x,y,z) \;\middle|\; \frac{x^2+y^2-a^2}{a} \le z \le \sqrt{a^2-x^2-y^2} \right\},$$

$$\mathbf{V}(x,y,z) = (2x+2y, 2y+z, z+2x).$$

The boundary $\partial\Omega$ *is oriented such that the normal vector is always pointing away from the body. By* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *we denote the subsets of* $\partial\Omega$*, for which* $z \ge 0$*, and* $z \le 0$*, respectively. Find the fluxes of* $\mathbf{V}$ *through* $\mathcal{F}_1$ *and* $\mathcal{F}_2$*, respectively.*

A Flux through surfaces.

D Apply both rectangular and polar coordinates. Check Gauß's theorem. This cannot be applied directly. It can, however, come into play by a small extra argument.

Finally, calculate the fluxes.

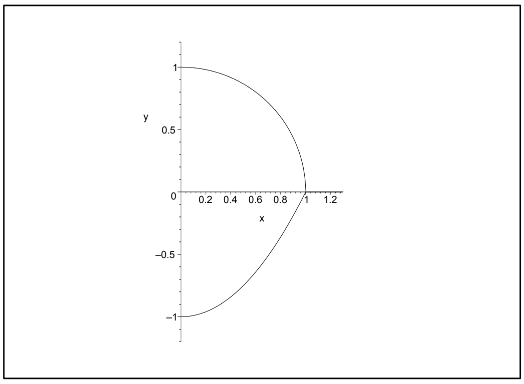

Figure 33.28: The cut of the meridian half plane for $a = 1$.

I By using semi polar coordinates we obtain that

$$az \ge \varrho^2 - a^2 \quad \text{og} \quad z^2 + \varrho^2 \le a^2,$$

and the meridian half plane becomes like shown on the figure.

As

$$\begin{aligned}
\text{vol}(\Omega) &= \text{vol}(\Omega_1) + \text{vol}(\Omega_2) = \frac{1}{2}\cdot\frac{4\pi}{3}\,a^3 + \int_{-a}^{0} \pi\varrho(z)^2\,\mathrm{d}z = \frac{2\pi}{3}\,a^3 + \pi\int_{-a}^{0} a(a+z)\,\mathrm{d}z \\
&= \frac{2\pi}{3}\,a^3 + \frac{\pi}{2}\,a\int_0^a 2t\,\mathrm{d}t = \frac{2\pi}{3}\,a^3 + \frac{\pi}{2}\,a^3 = \frac{7\pi}{6}\,a^3,
\end{aligned}$$

and $\text{div}\,\mathbf{V} = 2+2+1 = 5$, it follows from Gauß's theorem that

$$\text{flux}(\mathcal{F}) = \text{flux}(\mathcal{F}_1) + \text{flux}(\mathcal{F}_2) = \int_{\mathcal{F}} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S = \int_{\Omega} \text{div}\,\mathbf{V}\,\mathrm{d}\Omega = 5\,\text{vol}(\Omega) = \frac{35\pi}{6}\,a^3.$$

The parametric description of $\mathcal{F}_1$ is chosen as

$$\mathbf{r}(u,v) = \left(u, v, \sqrt{a^2-u^2-v^2}\right), \qquad u^2+v^2 \le a^2,$$

and then

$$\frac{\partial \mathbf{r}}{\partial u} = \left(1, 0, -\frac{u}{\sqrt{a^2-u^2-v^2}}\right) \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial v} = \left(0, 1, -\frac{v}{\sqrt{a^2-u^2-v^2}}\right),$$

from which we get the normal vector

$$\mathbf{N}(u,v) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 1 & 0 & -\dfrac{u}{\sqrt{a^2-u^2-v^2}} \\ 0 & 1 & -\dfrac{v}{\sqrt{a^2-u^2-v^2}} \end{vmatrix} = \frac{1}{\sqrt{a^2-u^2-v^2}} \left(u, v, \sqrt{a^2-u^2-v^2}\right),$$

which is clearly pointing away from the body, because the Z-coordinate is $+1$.

If we put $B = \{(u,v) \mid u^2+v^2 < a^2\}$, it follows from $(x,y,z) = \left(u, v, \sqrt{a^2-u^2-v^2}\right)$ that

$$\begin{aligned}
\text{flux}(\mathcal{F}_1) &= \int_{\mathcal{F}_1} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S = \int_B \mathbf{V}(u,v)\cdot\mathbf{N}(u,v)\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_B \left(2u+2v, 2v+\sqrt{a^2-u^2-v^2}, \sqrt{a^2-u^2-v^2}+2u\right) \\
&\qquad\qquad \cdot\frac{1}{\sqrt{a^2-u^2-v^2}}\left(u, v, \sqrt{a^2-u^2-v^2}\right)\mathrm{d}u\,\mathrm{d}v \\
&= \int_B \frac{1}{\sqrt{a^2-u^2-v^2}}\left\{2u^2+2uv+2v^2+v\sqrt{a^2-u^2-v^2}\right\} \\
&\qquad\qquad +(a^2-u^2-v^2)+2u\sqrt{a^2-u^2-v^2}\}\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_B \frac{a^2+u^2+v^2}{\sqrt{a^2-u^2-v^2}}\,\mathrm{d}u\,\mathrm{d}v + 0 = \int_0^{2\pi}\left\{\int_0^a \frac{a^2+\varrho^2}{\sqrt{a^2-\varrho^2}}\cdot\varrho\,\mathrm{d}\varrho\right\}\mathrm{d}\varphi = \pi\int_0^{a^2}\frac{a^2+t}{\sqrt{a^2-t}}\,\mathrm{d}t \\
&= \pi\int_0^{a^2}\left\{\frac{2a^2}{\sqrt{a^2-t}} - \sqrt{a^2-t}\right\}\mathrm{d}t = \pi\left[-4a^2\sqrt{a^2-t}+\frac{2}{3}\left(\sqrt{a^2-t}\right)^3\right]_0^{a^2} \\
&= \pi\left\{4a^2\sqrt{a^2}-\frac{2}{3}a^3\right\} = \frac{10\pi}{3}a^3.
\end{aligned}$$

Hence

$$\text{flux}(\mathcal{F}_2) = \text{flux}(\mathcal{F}) - \text{flux}(\mathcal{F}_1) = \frac{35\pi}{6}a^3 - \frac{10\pi}{3}a^3 = \frac{5}{2}\pi a^3,$$

and thus

$$\text{flux}(\mathcal{F}_1) = \frac{10\pi}{3}a^3 \quad \text{and} \quad \text{flux}(\mathcal{F}_2) = \frac{5\pi}{2}a^3.$$

ALTERNATIVELY, $\mathcal{F}_2$ is given by the parametric description

$$\mathbf{r} = (x, y, z) = \left(u, v, \frac{1}{a}\left(u^2 + v^2 - a^2\right) \right), \qquad (u, v) \in B,$$

hence

$$\frac{\partial \mathbf{r}}{\partial u} = \left(1, 0, \frac{2u}{a}\right) \qquad \text{and} \qquad \frac{\partial \mathbf{r}}{\partial v} = \left(0, 1, \frac{2v}{a}\right)$$

and hence

$$\mathbf{N}_1(u, v) = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 1 & 0 & \dfrac{2u}{a} \\ 0 & 1 & \dfrac{2v}{a} \end{vmatrix} = \left(-\frac{2u}{a}, -\frac{2v}{a}, 1 \right).$$

This normal vector is pointing inwards, so we are forced to choose

$$\mathbf{N}(u, v) = -\mathbf{N}_1(u, v) = \left(\frac{2u}{a}, \frac{2v}{a}, -1 \right).$$

Then

$$\begin{aligned}
\text{flux}(\mathcal{F}_2) &= \int_{\mathcal{F}_2} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S = \int_B \mathbf{V}(u,v)\cdot\mathbf{N}(u,v)\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_B \left(2u+2v, 2v+\frac{1}{a}\left(u^2+v^2-a^2\right), \frac{1}{a}\left(u^2+v^2-a^2\right)\right)\cdot\left(\frac{2u}{a},\frac{2v}{a},-1\right)\mathrm{d}u\,\mathrm{d}v \\
&= \int_B \left\{\frac{4u^2}{a}+\frac{4uv}{a}+\frac{4v^2}{a}+\frac{2v}{a}(u^2+v^2-a^2)-\frac{1}{a}\left(u^2+v^2-a^2\right)\right\}\mathrm{d}u\,\mathrm{d}v \\
&= \frac{1}{a}\int_B \left\{4u^2+4v^2-u^2-v^2+a^2\right\}\mathrm{d}u\,\mathrm{d}v + 0 \\
&= \frac{a^2}{2}\,\text{area}(B)+\frac{3}{a}\int_B (u^2+v^2)\,\mathrm{d}u\,\mathrm{d}v = a\cdot\pi a^2+\frac{3}{a}\cdot 2\pi\int_0^a \varrho^2\cdot\varrho\,\mathrm{d}\varrho \\
&= \pi a^3+\frac{6\pi}{a}\cdot\frac{a^4}{4} = \frac{5\pi}{2}\,a^3,
\end{aligned}$$

in accordance with the previous found result.

Example 33.17 *Let K be the solid ball $\overline{}(\mathbf{x}_0;a)$, and let $\mathbf{V}$ be a C^1 vector field on A, where $A \supset K$. Prove the following claims by using partial integration, Gauß's divergence theorem and the formula*

$$\mathbf{x} = \frac{1}{2}\,\nabla(\mathbf{x}\cdot\mathbf{x}).$$

1) If the divergence of $\mathbf{V}$ is a constant p, then

$$\int_K (\mathbf{x}-\mathbf{x}_0)\cdot\mathbf{V}(\mathbf{x})\,d\Omega = \frac{4}{15}\,a^5 p.$$

2) If the rotation of $\mathbf{V}$ is a constant vector $\mathbf{P}$, then

$$\int_K (\mathbf{x}-\mathbf{x}_0)\times\mathbf{V}(\mathbf{x})\,d\Omega = \frac{4}{15}\,a^5\,\mathbf{P}.$$

A Generalized partial integration.

D Follow the guidelines.

I 1) It follows from

$$\mathbf{x}-\mathbf{x}_0 = \frac{1}{2}\,\nabla\left((\mathbf{x}-\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)\right) = \frac{1}{2}\,\nabla\left(\|\mathbf{x}-\mathbf{x}_0\|^2\right),$$

and $f(\mathbf{x}) = \|\mathbf{x}-\mathbf{x}_0\|^2$ that

$$\begin{aligned}
\int_K (\mathbf{x}-\mathbf{x}_0)\cdot\mathbf{V}(\mathbf{x})\,\mathrm{d}\Omega &= \frac{1}{2}\int_K \nabla\left(\|\mathbf{x}-\mathbf{x}_0\|^2\right)\cdot\mathbf{V}(\mathbf{x})\,\mathrm{d}\Omega \\
&= \frac{1}{2}\int_{\partial K} \mathbf{n}\cdot\mathbf{V}(\mathbf{x})\,\|\mathbf{x}-\mathbf{x}_0\|^2\,\mathrm{d}S - \frac{1}{2}\int_\Omega \|\mathbf{x}-\mathbf{x}_0\|^2\ \nabla\cdot\mathbf{V}\,\mathrm{d}\Omega \\
&= \frac{1}{2}\,a^2\int_{\partial K} \mathbf{n}\cdot\mathbf{V}(\mathbf{x})\,\mathrm{d}S - \frac{1}{2}\,p\int_\Omega \|\mathbf{x}-\mathbf{x}_0\|^2\,\mathrm{d}\Omega \\
&= \frac{1}{2}\,a^2\int_\Omega \nabla\cdot\mathbf{V}(\mathbf{x})\,\mathrm{d}\Omega - \frac{1}{2}\,p\int_0^a\left\{\int_0^{2\pi}\left(\int_0^\pi r^2\cdot r^2\sin\theta\,\mathrm{d}\theta\right)\mathrm{d}\varphi\right\}\mathrm{d}r \\
&= \frac{1}{2}\,pa^2\cdot\mathrm{vol}(\Omega) - \frac{1}{2}\,p\int_0^a r^r\,\mathrm{d}r\cdot 2\pi\cdot\int_0^\pi \sin\theta\,\mathrm{d}\theta \\
&= \frac{1}{2}\,pa^2\cdot\frac{4\pi}{3}\,a^3 - \frac{1}{2}\,p\cdot\frac{a^5}{5}\cdot 2\pi\cdot 2 = \frac{pa^5\pi}{15}\cdot\{10-6\} = \frac{4}{15}\,a^5\pi p.
\end{aligned}$$

2) We can then replace $\cdot$ by $\times$, hence

$$\begin{aligned}
\int_K (\mathbf{x}-\mathbf{x}_0)\times\mathbf{V}(\mathbf{x})\,\mathrm{d}\Omega &= \frac{1}{2}\int_K \nabla\left(\|\mathbf{x}-\mathbf{x}_0\|^2\right)\times\mathbf{V}(\mathbf{x})\,\mathrm{d}\Omega \\
&= \frac{1}{2}\int_{\partial K} \mathbf{n}\times\mathbf{V}(\mathbf{x})\,\|\mathbf{x}-\mathbf{x}_0\|^2\,\mathrm{d}S - \frac{1}{2}\int_\Omega \|\mathbf{x}-\mathbf{x}_0\|^2\ \nabla\times\mathbf{V}\,\mathrm{d}\Omega \\
&= \frac{1}{2}\,a^2\int_\Omega \nabla\times\mathbf{V}(\mathbf{x})\,\mathrm{d}\Omega - \frac{1}{2}\,\mathbf{P}\int_0^a\left\{\int_0^{2\pi}\left(\int_0^\pi r^2\cdot r^2\sin\theta\,\mathrm{d}\theta\right)\mathrm{d}\varphi\right\}\mathrm{d}r \\
&= \frac{1}{2}\,a^2\mathbf{P}\cdot\mathrm{vol}(\Omega) - \frac{1}{2}\,\mathbf{P}\int_0^a r^4\,\mathrm{d}r\cdot 2\pi\cdot\int_0^\pi \sin\theta\,\mathrm{d}\theta \\
&= \left\{\frac{1}{2}\,a^2\cdot\frac{4\pi}{3}\,a^3 - \frac{1}{2}\,a^5\cdot 2\pi\cdot 2\right\}\mathbf{P} = \frac{4}{15}\,a^5\pi\,\mathbf{P}.
\end{aligned}$$

Example 33.18 *Let a be a positive constant. We let T denote the subset of*

$$T_1 = \left\{(x,y,z) \in \mathbb{R}^3 \mid z \geq 0,\, x^2+y^2+z^2 \leq 9a^2\right\},$$

which also lies outside the set

$$T_2 = \left\{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+(z-a)^1 < a^2\right\},$$

hence $T = T_1 \setminus T_2$.

1) Explain why T is given in spherical coordinates by

$$\theta \in \left[0, \frac{\pi}{2}\right], \quad \varphi \in [0,2\pi], \quad r \in [2a\,\cos\theta, 3a].$$

2) Find the mass of T when the density of mass on T is $\mu(x,y,z) = \dfrac{z}{a^4}$.

3) Find the flux of the vector field

$$\mathbf{V}(x,y,z) = \left(xz+4xy, yz-2y^2, x^2y^2\right), \qquad (x,y,z) \in \mathbb{R}^3,$$

through ∂T.

4) Find the volume of the subset $T^\star$ of T, which is given by the inequalities

$$x \geq 0, \qquad y \geq 0, z \geq \sqrt{x^2+y^2}.$$

A Spherical coordinates, mass, flux, volume.

D Sketch the meridian half plane; compute a space integral; apply Gauß's theorem; once again, consider the meridian half plane.

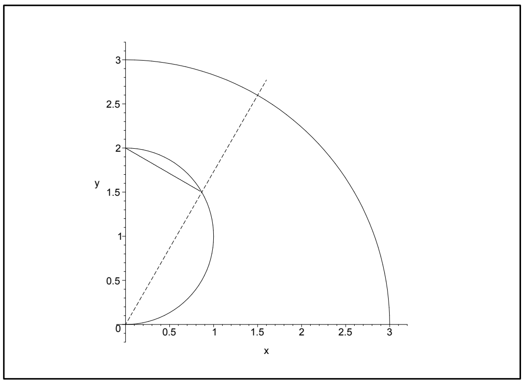

Figure 33.29: The meridian half plane for T, when $a = 1$. The angle between the Z-axis and the dotted radius is θ. The two dotted lines are perpendicular to each other.

I 1) When we consider the meridian half plane, it follows immediately that

$$\theta \in \left[0, \frac{\pi}{2}\right] \quad \text{and} \quad \varphi \in [0,2\pi].$$

It only remains to prove that the meridian cut of ∂T_2 has the equation

$$r = 2a\,\cos\theta.$$

Draw a radius and the perpendicular line on this as shown by the dotted lines on the figure. Together with the line segment $[0, 2a]$ on the Y-axis these form a rectangular triangle. The angle between the Z-axis and the dotted radius is θ, and the hypothenuse (the line segment on the Z-axis) is $2a$. Hence, the closest of the smaller sides (i.e. placed up to ∂T_2) must have the length $2a\cos\theta$. This proves that the equation of ∂T_2 is

$$r = 2a\,\cos\theta.$$

It then follows that $r \in [2a\,\cos\theta, 3a]$ in T.

2) We have in spherical coordinates

$$\mu(x,y,z) = \frac{z}{a^4} = \frac{r}{a^4}\,\cos\theta,$$

hence the mass is given by

$$\begin{aligned} M &= \int_T \mu\,\mathrm{d}\Omega = \int_0^{\frac{\pi}{2}} \left\{ \int_0^{2\pi} \left\{ \int_{2a\,\cos\theta}^{3a} \frac{1}{a^4}\,r\,\cos\theta \cdot r^2 \sin\theta\,\mathrm{d}r \right\} \mathrm{d}\varphi \right\} \mathrm{d}\theta \\ &= \frac{2\pi}{a^4} \int_0^{\frac{\pi}{2}} \cos\theta \cdot \sin\theta \left[\frac{r^4}{4} \right]_{2a\,\cos\theta}^{3a} \mathrm{d}\theta = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \left(81 - 16\cos^4\theta\right) \cos\theta\,\sin\theta\,\mathrm{d}\theta \\ &= \frac{\pi}{2} \left[-\frac{81}{2}\,\cos^2\theta + \frac{16}{6}\,\cos^6\theta \right]_0^{\frac{\pi}{2}} = \frac{\pi}{2} \left(\frac{81}{2} - \frac{16}{6} \right) = \frac{\pi}{12}\,(243 - 16) = \frac{117\pi}{12}. \end{aligned}$$

3) From

$$\text{div } \mathbf{V} = z + 4y + z - 4y + 0 = 2z,$$

follows by Gauß's theorem and 2) that the flux is

$$\int_{\partial T} \mathbf{V} \cdot \mathbf{n} \, \mathrm{d}S = \int_T \text{div } \mathbf{V} \, \mathrm{d}\Omega = \int_T 2z \, \mathrm{d}\Omega = 2a^4 \int_T \mu \, \mathrm{d}\Omega = \frac{227\pi}{6} \, a^4.$$

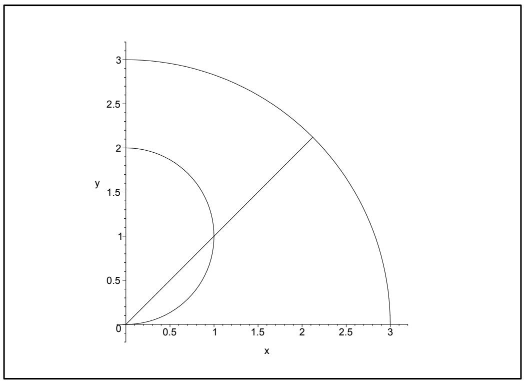

Figure 33.30: The meridian cut of $T^\star$ is the domain between the two circular arcs lying above the line $z = \varrho$.

4) By analyzing the meridian half plane once more we see that $T^\star$ is given by

$$\theta \in \left[0, \frac{\pi}{4}\right], \quad \varphi \in \left[+, \frac{\pi}{2}\right], \quad r \in [2a \, \cos\theta, 3a],$$

hence the volume is

$$\begin{aligned}
\text{vol}(T^\star) &= \int_0^{\frac{\pi}{4}} \left\{ \int_0^{\frac{\pi}{2}} \left\{ \int_{2a \, \cos\theta}^{3a} r^2 \sin\theta \, \mathrm{d}r \right\} \mathrm{d}\varphi \right\} \mathrm{d}\theta = \frac{\pi}{2} \int_0^{\frac{\pi}{4}} \sin\theta \cdot \left[\frac{1}{3} r^3\right]_{2a \, \cos\theta}^{3a} \mathrm{d}\theta \\
&= \frac{\pi}{6} \, a^3 \int_0^{\frac{\pi}{4}} \left(27 - 8 \, \cos^3\theta\right) \sin\theta \, \mathrm{d}\theta = \frac{\pi}{6} \, a^3 \, \left[-27\cos\theta + 2\cos^4\theta\right]_0^{\frac{\pi}{4}} \\
&= \frac{\pi}{6} \, a^3 \left(-\frac{27}{\sqrt{2}} + \frac{2}{4} + 27 - 2 \right) = \frac{\pi}{12} \, (51 - 27\sqrt{2}) \, a^3.
\end{aligned}$$

Example 33.19 *Let a be a positive constant and consider the function*

$$f(x,y,z) = a^2x^2 + a^3y + z^4, \qquad (x,y,z) \in \mathbb{R}^3.$$

1) Find the gradient $\mathbf{V} = \nabla f$ and the tangential line integral

$$\int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t}\, \mathrm{d}s,$$

where $\mathcal{K}$ is the line segment from $(0,0,a)$ to $(2a,3a,0)$.

2) Find the flux of $\mathbf{V}$ through the surface of the half sphere given by

$$x^2+y^2+z^2 \le a^2 \quad \text{and} \quad z \ge 0.$$

A Gradient; tangential line integral; flux.

D Apply Gauß's theorem in 2).

I 1) The gradient is

$$\mathbf{V} = \nabla f = \left(2a^2x, a^3, 4z^3\right).$$

Since $\mathbf{V}$ is a gradient field, $\mathbf{V} = \nabla f$, we get

$$\int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t}\, \mathrm{d}s = f(2a,3a,0) - f(0,0,a) = \left(a^2 \cdot 4a^2 + a^3 \cdot 3a\right) - a^4 = 6a^4.$$

2) Then by Gauß's theorem,

$$\begin{aligned}\text{flux}(\partial L) &= \int_{\partial L} \mathbf{V} \cdot \mathbf{n}\, \mathrm{d}S = \int_L \operatorname{div} \mathbf{V}\, \mathrm{d}\Omega = \int_L \left(2a^2 + 12z^2\right) \mathrm{d}\Omega \\ &= 2a^2 \cdot \frac{1}{2} \cdot \frac{4\pi}{3}\, a^3 + 12 \int_L z^2\, \mathrm{d}\Omega = \frac{4\pi}{3}\, a^5 + 12 \int_0^a z^2 \cdot \pi\left(a^2 - z^2\right) \mathrm{d}z \\ &= \frac{4\pi}{3}\, a^5 + 12\pi \left[a^2 \cdot \frac{1}{3}\, z^3 - \frac{1}{5}\, z^5\right]_0^a = \frac{4\pi}{3}\, a^5 + \frac{24}{15}\, \pi\, a^5 = \frac{44\pi}{15}\, a^5.\end{aligned}$$

Example 33.20 *Given the tetrahedron*

$$T = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \le x,\, 0 \le y,\, 4-x-2y \le z \le 8-2x-4y\}.$$

and the vector field

$$\mathbf{V}(x,y,z) = \left(z\cos x + 3yz,\, x^2y + x\sinh z,\, \frac{1}{2}\,z^2\sin x + 3x^2 - 5y^2\right), \quad (x,y,z) \in \mathbb{R}^3.$$

Find the flux of $\mathbf{V}$ *through* ∂T.

A Flux of a vector field through a closed surface.

D Apply Gauß's theorem.

I It follows from

$$\operatorname{div}\mathbf{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} = -z\sin x + x^3 + \frac{1}{2}\cdot 2z\sin x = x^2,$$

by Gauß's theorem that the flux of $\mathbf{V}$ through ∂T is given by

$$(33.4) \quad \int_{\partial T} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S = \int_T \operatorname{div}\mathbf{V}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_T x^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

The bounds of the tetrahedron give the estimates

$$4-x-2y \le z \le 8-2x-4y = 2(4-x-2y),$$

hence $4-x-2y \ge 0$, and thus $0 \le x \le 4-2y$ and $0 \le y \le 2$. By a reduction of (33.4) we then get

$$\begin{aligned}
\int_{\partial T} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S &= \int_T x^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_0^2 \left\{\int_0^{4-2y}\left(\int_{4-x-2y} 8-2x-4y x^2\,\mathrm{d}z\right)\mathrm{d}x\right\}\mathrm{d}y \\
&= \int_0^2 \left\{\int_0^{4-2y} x^2(4-x-2y)\,\mathrm{d}x\right\}\mathrm{d}y = \int_0^2 \left\{\int_0^{4-2y} (4x^2 - x^3 - 2yx^2)\,\mathrm{d}x\right\}\mathrm{d}y \\
&= \int_0^2 \left[\frac{4}{3}\,x^3 - \frac{1}{4}\,x^4 - \frac{2}{3}\,yx^3\right]_{x=0}^{4-2y}\mathrm{d}y \\
&= \int_0^2 \left\{\frac{4}{3}(2\{2-y\})^3 - \frac{1}{4}\,(2\{2-y\})^4 - \frac{2}{3}\,y\,(2\{2-y\})^3\right\}\mathrm{d}y \\
&= \int_0^2 \left\{\frac{32}{3}\,(2-y)^3 - \frac{16}{4}\,(2-y)^4 - \frac{16}{3}\,y\,(2-y)^3\right\}\mathrm{d}y \\
&= \left(\frac{16}{3} - \frac{16}{4}\right)\int_0^2 (2-y)^4\,\mathrm{d}y = \frac{16}{12}\int_0^2 t^4\,\mathrm{d}t = \frac{4}{3}\left[\frac{1}{5}\,t^5\right]_0^2 = \frac{128}{15}.
\end{aligned}$$

Example 33.21 *Given the vector field*

$$\mathbf{V}(x,y,z)=\left(4x+3y^3,9xy^2+z,y\right),\qquad (x,y,z)\in\mathbb{R}^3.$$

1) Find div V *and* **rot V**.

2) Show that **V** *is a gradient field and find all its integrals.*

3) Compute the tangential line integral

$$\int_{\mathcal{K}}\mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s=\int_{\mathcal{K}}(4x+3y^3)\,a\,\mathrm{d}x+(9xy^2+z)\,\mathrm{d}y+y\,\mathrm{d}z,$$

where $\mathcal{K}$ *denotes the line segment from the point* $(0,0,0)$ *to the point* $(1,1,1)$.

4) Find the flux of **V** *through the unit sphere* $x^2+y^2+z^2=1$ *with a normal vector pointing away from the ball.*

A Vector analysis.

D Follow the guidelines

I 1) We get by direct computations

$$\text{div }\mathbf{V}=4+18xy^2,$$

and

$$\textbf{rot }\mathbf{V}=\begin{vmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 4x+3y^3 & 9xy^2+z & y\end{vmatrix}=\left(1-1,0-0,9y^2-9y^2\right)=(0,0,0),$$

and we note that **V** is rotation free.

2) Since the field is rotation free and the domain is simply connected, we conclude that **V** is a gradient field. Then by calculating the differential form,

$$\begin{aligned}\mathbf{V}\cdot(\,\mathrm{d}x,\,\mathrm{d}y,\,\mathrm{d}z) &= (4x+3y^3)\,\mathrm{d}x+(9xy^2+z)\,\mathrm{d}y+y\,\mathrm{d}z\\ &= 4x\,\mathrm{d}x+3\left(y^3\,\mathrm{d}x+x\cdot 3y^2\,\mathrm{d}y\right)+(z\,\mathrm{d}y+y\,\mathrm{d}z)\\ &= \mathrm{d}\left(2x^2+3xy^3+yx\right),\end{aligned}$$

and it follows once more that **V** is a gradient field with all its integrals given by

$$F(x,y,z)=2x^2+3xy^3+yz+C,\qquad C\in\mathbb{R}.$$

3) We have proved that **V** is a gradient field with an integral F. Then it follows that

$$\begin{aligned}\int_{\mathcal{K}}\mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s &= \int_{\mathcal{K}}(4x+3y^3)\,\mathrm{d}x+(9xy^2+z)\,\mathrm{d}y+y\,\mathrm{d}z\\ &= [F(x,y,z)]_{(0,0,0)}^{(1,1,1)}=\left[2x^2+3xy^3+yz\right]_{(0,0,0)}^{(1,1,1)}=2+3+1=6.\end{aligned}$$

4) An application of Gauß's theorem gives

$$\int_{\partial\Omega}\mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S=\int_{\Omega}\text{div }\mathbf{V}\,\mathrm{d}\Omega=\int_{\Omega}(4+18xy^2)\,\mathrm{d}\Omega=4\,\text{vol}(\Omega)+0=4\cdot\frac{4\pi}{3}=\frac{16\pi}{3},$$

because $\int_{\Omega}18xy^2\,\mathrm{d}\Omega=0$ of symmetric reasons. The integrand is odd in x, and the body is symmetric with respect to the (Y,Z)-plane.

Example 33.22 *A body of revolution L with the Z-axis as axis of rotation is given in semi polar coordinates (ϱ, φ, z) given by the inequalities*

$$0 \le \varphi \le 2\pi, \qquad -a \le z \le a, \qquad 0 \le \varrho \le a - \frac{z^2}{a},$$

where $a \in \mathbb{R}_+$ is some given constant.

1. *Calculate the space integral*

$$I = \int_L z^2 \, \mathrm{d}\Omega.$$

Given the vector field

$$\mathbf{V}(x,y,z) = \left(\cos x, y \, \sin x, z^3\right), \qquad (x,y,z) \in \mathbb{R}^3.$$

2. *Find the flux*

$$\int_{\partial L} \mathbf{V} \cdot \mathbf{n} \, \mathrm{d}S,$$

where the unit normal vector $\mathbf{n}$ is pointing away from the body.

A Space integral and flux in semi polar coordinates.

D Slice up the body; apply Gauß's theorem.

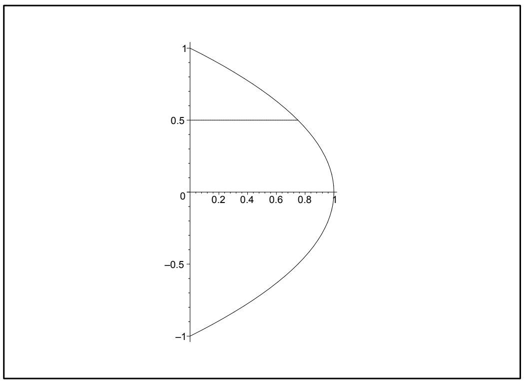

Figure 33.31: The meridian curve when $a = 1$.

I 1) It follows from the rearrangement

$$\varrho = a\left\{1 - \left(\frac{z}{a}\right)^2\right\}$$

that the meridian curve is an arc of a parabola.

The space integral is computed by the method of slicing,

$$\begin{aligned}
I &= \int_L z^2\,\mathrm{d}\Omega = \pi\int_{-a}^{a}\left(a-\frac{z^2}{a}\right)^2 z^2\,\mathrm{d}z = 2\pi\int_0^a\left(a^2-2z^2+\frac{z^4}{a^2}\right)z^2\,\mathrm{d}z \\
&= 2\pi\int_0^a\left\{a^2z^2-2z^4+\frac{z^6}{a^2}\right\}\mathrm{d}z = 2\pi\left[\frac{a^2}{3}\,z^3-\frac{2}{5}\,z^5+\frac{z^7}{7a^2}\right]_0^a \\
&= 2\pi a^5\left(\frac{1}{3}-\frac{2}{5}+\frac{1}{7}\right) = \frac{2\pi a^5}{105}\,(35-42+15) = \frac{16\pi a^5}{105}.
\end{aligned}$$

2) The flux is according to Gauß's theorem given by

$$\begin{aligned}
\int_{\partial L}\mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S &= \int_\Omega \operatorname{div}\mathbf{V}\,\mathrm{d}\Omega = \int_\Omega\left\{-\sin x+\sin x+sz^2\right\}\mathrm{d}\Omega \\
&= 3\int_\Omega z^2\,\mathrm{d}\Omega = 3I = \frac{16\pi a^5}{35},
\end{aligned}$$

where we have used the result of 1).

Example 33.23 *Given the vector field*

$$\mathbf{V}(x,y,z) = (3xz^2 - x^3, 3yz^2 - y^3, 3z(x^2+y^2)), \qquad (x,y,z) \in \mathbb{R}^3,$$

and the constant $a \in \mathbb{R}_+$.

1. *Show that* $\mathbf{V}$ *is a gradient field and find all its integrals.*

Let $\mathcal{K}$ *be the curve which is composed of the quarter circle of centrum at* $(0,0,0)$ *and runs from* $(a,0,0)$ *to* $(0,a,0)$*, and the line segment from* $(0,a,0)$ *to* $(0,a,2a)$.

2. *Find the tangential line integral*

$$\int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t}\, \mathrm{d}s.$$

3. *Find the flux of* $\mathbf{V}$ *through the surface of the ball of centrum* $(0,0,0)$ *and radius* a.

A Vector analysis.

D Each question can be answered in several ways. We shall here demonstrate some of the variants.

I 1) First note that $\mathbf{V}$ is of class C^∞.

First variant. Prove directly by some manipulation that the differential form $\mathbf{V} \cdot d\mathbf{x}$ can be written as dF where F then by the definition is an integral. Do this by pairing terms which are similar to each other.

$$\begin{aligned}\mathbf{V}\cdot \mathrm{d}\mathbf{x} &= (3xz^2 - x^3)\,\mathrm{d}x + (3yz^2 - y^3)\,\mathrm{d}y + 3z(x^2+y^2)\,\mathrm{d}z \\ &= \frac{3}{2}\,z^2\,\mathrm{d}(x^2) - \frac{1}{4}\,\mathrm{d}(x^4) + \frac{3}{2}\,z^2\,\mathrm{d}(y^2) - \frac{1}{4}\,\mathrm{d}(y^4) + \frac{3}{2}(x^2+y^2)\,\mathrm{d}(z^2) \\ &= \mathrm{d}\left(\frac{3}{2}\,(x^2+y^2)z^2 - \frac{1}{4}\,x^4 - \frac{1}{4}\,y^4\right).\end{aligned}$$

It follows immediately from this result that $\mathbf{V}$ is a gradient field and that all integrals are given by

$$F(x,y,z) = \frac{3}{2}\,(x^2+y^2)z^2 - \frac{1}{4}\,x^4 - \frac{1}{4}\,y^4 + C,$$

where C is an arbitrary constant.

Second variant. Clearly, $\mathbb{R}^3$ is simply connected. Furthermore,

$$\frac{\partial L}{\partial y} = 0, \qquad \frac{\partial M}{\partial x} = 0, \qquad \text{hence } \frac{\partial L}{\partial y} = \frac{\partial M}{\partial x},$$

$$\frac{\partial L}{\partial z} = 6xz, \qquad \frac{ddN}{\partial x} = 6xz, \qquad \text{hence } \frac{\partial L}{\partial z} = \frac{\partial N}{\partial x},$$

$$\frac{\partial M}{\partial z} = 6yz, \qquad \frac{\partial N}{\partial y} = 6yz, \qquad \text{hence} \frac{\partial M}{\partial z} = \frac{\partial N}{\partial y}.$$

Since all the "mixed derivatives" are equal, it follows that $\mathbf{V} \cdot d\mathbf{x}$ is closed and hence exact. This means that $\mathbf{V}$ is a gradient field and the integrals of $\mathbf{V}$ exist.

In this variant we shall find the integrals by using line integrals. There are two sub-varants:

a) *Integration along the broken line*

$$(0,0,0) \longrightarrow (x,0,0) \longrightarrow (x,y,0) \longrightarrow (x,y,z).$$

In this case,

$$\begin{aligned} F_0(x,y,z) &= \int_0^x (-t^3)\,\mathrm{d}t + \int_0^y (-t^3)\,\mathrm{d}t + \int_0^z 3t(x^2+y^2)\,\mathrm{d}t \\ &= \frac{3}{2}\,(x^2+y^2)z^2 - \frac{1}{4}\,(x^4+y^4). \end{aligned}$$

The integrals are

$$F(x,y,z) = \frac{3}{2}\,(x^2+y^2)z^2 - \frac{1}{4}(x^4+y^4) + C,$$

where C is an arbitrary constant.

b) Radial integration along $(0,0,0) \longrightarrow (x,y,z)$.
The coordinates of $\mathbf{V}$ are homogeneous of degree 3. Hence,

$$\begin{aligned} F_0(x,y,z) &= (x,y,z)\cdot\left((3xz^2-x^3)\int_0^1 t^3\,\mathrm{d}t, (3yz^2-y^3)\int_0^1 t^3\,\mathrm{d}t, 3z(x^2+y^2)\int_0^1 t^3\,\mathrm{d}t\right) \\ &= \frac{1}{4}\,(x,y,z)\cdot(3xz^2-x^3, 3yz^2-y^3, 3z(x^2+y^2)) \\ &= \frac{1}{4}\,\left\{3x^2z^2 - x^4 + 3y^2z^2 - y^4 + 3z^2(x^2+y^2)\right\} \\ &= \frac{3}{2}\,(x^2+y^2)z^2 - \frac{1}{4}\,(x^4+y^4). \end{aligned}$$

The integrals are

$$F(x,y,z) = \frac{3}{2}\,(x^2+y^2)z^2 - \frac{1}{4}\,(x^4+y^4) + C,$$

where C is an arbitrary constant.

Third variant. Start by one of the variants 2a) and 2b) above without proving in advance that $\mathbf{V}$ is a gradient field. The *possible* candidates of the integrals are

$$F(x,y,z) = \frac{3}{2}\,(x^2+y^2)z^2 - \frac{1}{4}\,(x^4+y^4) + C.$$

Check these!:

$$\nabla F(x,y,z) = (3xz^2-x^3, 3yz^2-y^3, 3z(x^2+y^2)) = \mathbf{V}(x,y,z).$$

This shows that $\mathbf{V}$ is a gradient field and its integrals are given by

$$F(x,y,z) = \frac{3}{2}\,(x^2+y^2)z^2 - \frac{1}{4}\,(x^4+y^4) + C,$$

where C is an arbitrary constant.

Fourth variant. *Improper integration.*
First put

$$\omega = \mathbf{V}\cdot\,\mathrm{d}\mathbf{x} = (3xz^2-x^3)\,\mathrm{d}x + (3yz^2-y^3)\,\mathrm{d}y + 3z(x^2+y^2)\,\mathrm{d}z.$$

By an improper integration of the first term on the right hand side we get

$$F_1(x,y,z) = \int^x (3tz^2 - t^3)\,\mathrm{d}t = \frac{3}{2}\,x^2z^2 - \frac{1}{4}\,x^4.$$

The differential is

$$\mathrm{d}F_1 = (3xz^2 - x^3)\,\mathrm{d}x + 3x^2z\,\mathrm{d}z,$$

hence

$$\omega - \,\mathrm{d}F_1 = (3yz^2 - y^3)\,\mathrm{d}y + 3zy^2\,\mathrm{d}z,$$

which neither contains x nor $\mathrm{d}x$.

When we repeat this procedure on $\omega - \,\mathrm{d}F_1$ we get

$$F_2(y,z) = \int^y (3tz^2 - t^3)\,\mathrm{d}t = \frac{3}{2}\,y^2z^2 - \frac{1}{4}\,y^4$$

with the differential

$$\mathrm{d}F_2 = (3yz^2 - y^3)\,\mathrm{d}y + 3zy^2\,\mathrm{d}z = \omega - \,\mathrm{d}F_1.$$

Then by a rearrangement,

$$\omega = \mathbf{V}\cdot\,\mathrm{d}\mathbf{x} = \,\mathrm{d}F_1 + \,\mathrm{d}F_2 = \,\mathrm{d}\left(\frac{3}{2}\,x^2z^2 - \frac{1}{4}\,x^4 + \frac{3}{2}\,y^2z^2 - \frac{1}{4}\,y^4\right),$$

proving that $\mathbf{V}$ is a gradient field with the integrals

$$F(x,y,z) = \frac{3}{2}\,(x^2+y^2)z^2 - \frac{1}{4}\,(x^4+y^4) + C,$$

C being an arbitrary constant.

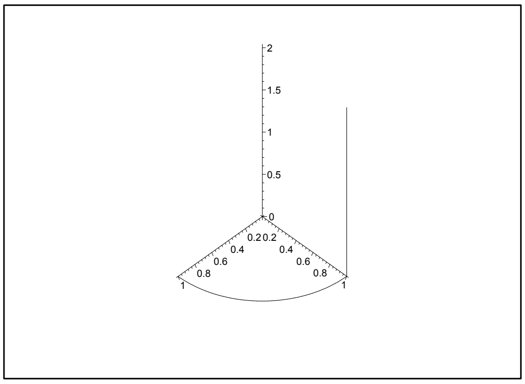

Figure 33.32: The curve $\mathcal{K}$ for $a = 1$.

2) Here we have two variants.

First variant. Since $\mathbf{V}$ is a gradient field with the integral

$$F_0(x,y,z) = \frac{3}{2}\,(x^2+y^2)z^2 - \frac{1}{4}\,(x^4+y^4),$$

and $\mathcal{K}$ is a connected curve, we have

$$\begin{aligned}\int_{\mathcal{K}} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s &= F_0(0,a,2a) - F_0(a,0,0)\\ &= \frac{3}{2}\,(0^2+a^2)\cdot 4a^2 - \frac{1}{4}\,(0^4+a^4) + \frac{1}{4}\,(a^4+0^4) = 6a^4.\end{aligned}$$

Second variant. The definition of a tangential line integral.

The curve $\mathcal{K}$ is composed of the two sub-curves

$$\mathcal{K}_1: \quad (x(t),y(t),z(t)) = a(\cos t, \sin t, 0), \qquad t \in \left[0,\frac{\pi}{2}\right],$$

$$\mathcal{K}_2: \quad (x(t),y(t),z(t)) = a(0,1,t), \qquad t \in [0,2].$$

First calculate

$$\begin{aligned}\int_{\mathcal{K}_1} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s &= \int_0^{\frac{\pi}{2}} a^3\left(-\cos^3 t, -\sin^3 t, 0\right)\cdot a(-\sin t,\cos t,0)\,\mathrm{d}t\\ &= a^4\int_0^{\frac{\pi}{2}}\left\{\cos^3 t\cdot\sin t-\sin^3 t\cdot\cos t\right\}\,\mathrm{d}t\\ &= \frac{a^4}{4}\left[-\cos^4 t-\sin^4 t\right]_0^{\frac{\pi}{2}} = \frac{a^4}{4}\{-1+1\}=0,\end{aligned}$$

and

$$\begin{aligned}\int_{\mathcal{K}_2} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s &= \int_0^2 a^3\left(0,3t^2-1,3t\left(0^2+1^2\right)\right)\cdot a(0,0,1)\,\mathrm{d}t\\ &= a^4\int_0^2 3t\,\mathrm{d}t=\frac{3}{2}\,a^4\cdot 4=6a^4.\end{aligned}$$

Summarizing we get

$$\int_{\mathcal{K}} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s=\int_{\mathcal{K}_1} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s+\int_{\mathcal{K}_2} \mathbf{V}\cdot\mathbf{t}\,\mathrm{d}s=0+6a^4=6a^4.$$

3) This problem can also be solved in various ways.

First variant. According to Gauß's theorem,

$$\text{flux} = \int_{\overline{K}(\mathbf{0};a)} \operatorname{div}\mathbf{V}\,\mathrm{d}\Omega=\int_{\overline{K}(\mathbf{0};a)} 6z^2\,\mathrm{d}\Omega,$$

because

$$\operatorname{div}\mathbf{V}=3z^2-3x^2+3z^2-3y^2+3(x^2+y^2)=6z^2.$$

The calculation of this integral is most probably performed in one of the following sub-variants, although there exist some other (and more difficult) ways of calculation.

a) *Partition of $\overline{K}(\mathbf{0};a)$ into slices parallel to the XY-plane.*
By using this slicing method we get

$$\begin{aligned}\text{flux} &= \int_{\overline{K}(\mathbf{0};a)} 6z^2\,\mathrm{d}\Omega=\int_{-a}^{a}\left\{\int_{\overline{K}((0,0);\sqrt{a^2-z^2})} 6z^2\,\mathrm{d}x\,\mathrm{d}y\right\}\mathrm{d}z\\ &= \int_{-a}^{a} 6z^2\,\text{area}(K(0,0);\sqrt{a^2-z^2})\,\mathrm{d}z=\int_{-a}^{a} 6z^2\pi(a^2-z^2)\,\mathrm{d}z\\ &= 12\pi\int_0^a (a^2z^2-z^4)\,\mathrm{d}z=12\pi\left[\frac{1}{3}\,a^2z^3-\frac{1}{5}\,z^5\right]_0^a=12\pi a^5\cdot\frac{2}{15}=\frac{8\pi}{5}\,a^5.\end{aligned}$$

b) *Calculation in spherical coordinates:*

$$\begin{aligned}\text{flux} &= \int_{\overline{K}(\mathbf{0};a)} 6z^2\,\mathrm{d}\Omega=\int_0^{2\pi}\left\{\int_0^{\pi}\left(\int_0^a 6r^2\cos^2\theta\cdot r^2\sin\theta\,\mathrm{d}r\right)\mathrm{d}\theta\right\}\mathrm{d}\varphi\\ &= 2\pi\int_0^{\pi} 6\cos^2\theta\cdot\sin\theta\,\mathrm{d}\theta\cdot\int_0^a r^4\,\mathrm{d}r=2\pi\left[2(-\cos^3\theta)\right]_0^{\pi}\cdot\frac{a^5}{5}\\ &= \frac{4\pi}{5}\,a^5(1+1)=\frac{8\pi}{5}\,a^5.\end{aligned}$$

Second variant. Direct application of the definition.

Put $\mathcal{F} = \partial K(\mathbf{0};a)$. Then the unit normal vector field on $\mathcal{F}$ is given by

$$\mathbf{n} = \frac{1}{a}\,(x,y,z).$$

By insertion into the definition,

$$\begin{aligned}\text{flux} &= \int_{\mathcal{F}} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S = \frac{1}{a}\int_{\mathcal{K}} \left\{3x^2z^2 - x^4 + 3y^2z^2 - y^4 + 3z^2(x^2+y^2)\right\}\,\mathrm{d}S \\ &= \frac{1}{a}\int_{\mathcal{F}} \left\{6z^2(x^2+y^2) - x^4 - y^4\right\}\mathrm{d}S.\end{aligned}$$

We shall in the following calculate this surface integral in two different ways. Notice that there are many other possibilities. In both of these two sub-variants we shall need the following:

Calculations:

$$(33.5)\quad \int_0^{2\pi} \left(\cos^4\varphi + \sin^4\varphi\right)\,\mathrm{d}\varphi$$

$$\begin{aligned}&= \int_0^{2\pi} \left(\cos^4\varphi+\sin^4\varphi+2\sin^2\varphi\,\cos^2\varphi-2\sin^2\varphi\,\cos^2\varphi\right)\,\mathrm{d}\varphi \\ &= \int_0^{2\pi} \left\{\left(\cos^2\varphi+\sin^2\varphi\right)^2 - \frac{1}{2}\,\sin^2 2\varphi\right\}\,\mathrm{d}\varphi \\ &= \int_0^{2\pi} \left\{1 - \frac{1}{2}\cdot\frac{1}{2}\,(1-\cos 4\varphi)\right\}\,\mathrm{d}\varphi = \frac{3}{4}\cdot 2\pi = \frac{3\pi}{2}.\end{aligned}$$

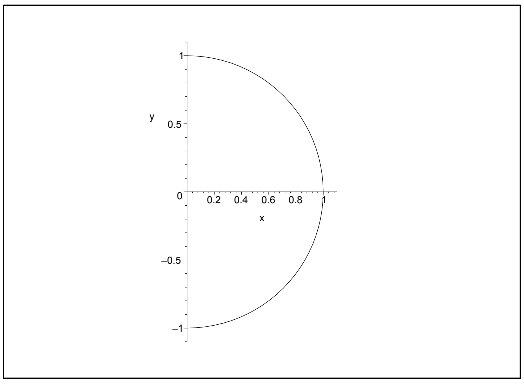

Figure 33.33: The meridian curve $\mathcal{M}$.

a) Consider the surface $\mathcal{F}$ as a surface of revolution with the meridian curve

$$\mathcal{M}:\quad \varrho(z) = \sqrt{a^2-z^2}, \qquad z\in[-a,a],$$

thus

$$x(z) = \sqrt{a^2-z^2}\,\cos\varphi, \quad y = \sqrt{a^2-z^2}\,\sin\varphi, \quad z=z,$$

and the weight function

$$\sqrt{\{\varrho'(z)\}^2+1} = \sqrt{1+\frac{z^2}{a^2-z^2}} = \sqrt{\frac{a^2}{a^2-z^2}} = \frac{a}{\sqrt{a^2-z^2}}.$$

By insertion into a suitable formula we get

$$\begin{aligned}
\text{flux} &= \frac{1}{a}\int_{\mathcal{F}} \left\{6z^2\left(x^2+y^2\right)-x^4-y^4\right\}\,\mathrm{d}S \\
&= \frac{1}{a}\int_{-a}^{a}\left\{\int_0^{2\pi}\{6z^2([a^2-z^2]\cos^2\varphi+[a^2-z^2]\sin^2\varphi)\}\right. \\
&\qquad\left. -(a^2-z^2)^2(\cos^4\varphi+\sin^4\varphi)\,\mathrm{d}\varphi\right\}\frac{\sqrt{a^2-z^2}\cdot a}{\sqrt{a^2-z^2}}\,\mathrm{d}z \\
&= \int_{-a}^{a}\left\{\int_0^{2\pi}\{6z^2(a^2-z^2)-(a^2-z^2)^2(\cos^4\varphi+\sin^4\varphi)\}\,\mathrm{d}\varphi\right\}\mathrm{d}z \\
&= \int_{-a}^{a}\left\{2\pi\cdot 6z^2(a^2-z^2)-\frac{3\pi}{2}(a^2-z^2)^2\right\}\mathrm{d}z \qquad (\text{by } (33.5)) \\
&= 12\pi\int_{-a}^{a}(a^2z^2-z^4)\,\mathrm{d}z-\frac{3\pi}{2}\int_{-a}^{a}(a^4-2a^2z^2+z^4)\,\mathrm{d}z \\
&= 2\cdot 12\pi\left[\frac{a^2}{3}z^3-\frac{1}{5}z^5\right]_0^a-2\cdot\frac{3\pi}{2}\left[a^4z-\frac{2}{3}a^2z^3+\frac{1}{5}z^5\right]_0^a \\
&= 24\pi a^5\cdot\frac{2}{15}-3\pi a^5\left(1-\frac{2}{3}+\frac{1}{5}\right)=\pi a^5\cdot\left(\frac{16}{5}-1-\frac{3}{5}\right)=\frac{8\pi a^5}{5}.
\end{aligned}$$

b) Alternatively it follows by the symmetry that the flux through

$$\mathcal{F}_+ = \{(x,y,z)\in\mathcal{F}\mid z\ge 0\}$$

is equal to the flux through $\mathcal{F}\setminus\mathcal{F}_+$, thus

$$\text{flux} = \frac{2}{a}\int_{\mathcal{F}_+}\{6z^2(x^2+y^2)-x^4-y^4\}\,\mathrm{d}S.$$

The surface $\mathcal{F}_+$ is the graph of

$$z=\sqrt{a^2-x^2-y^2},\qquad (x,y)\in B=\{(x,y)\mid x^2+y^2\le a^2\},$$

and the normal vector is

$$\mathbf{N}(x,y)=\left(-\frac{\partial z}{\partial x},-\frac{\partial z}{\partial y},1\right)=\left(\frac{x}{\sqrt{a^2-x^2-y^2}},\frac{y}{\sqrt{a^2-x^2-y^2}},1\right),$$

hence

$$\|\mathbf{N}(x,y)\|=\frac{a}{\sqrt{a^2-x^2-y^2}}.$$

Then by

i) reduction of the surface integral to a plane integral,
ii) reduction in polar coordinates,
iii) application of the calculation (33.5),
iv) the change of variable $t=\sqrt{a^2-r^2}$, i.e.

$$r^2=a^2-t^2\quad\text{and}\quad \mathrm{d}t=-\frac{r}{\sqrt{a^2-r^2}}\,\mathrm{d}r,$$

we finally get

$$
\begin{aligned}
\text{flux} &= \frac{2}{a}\int_B \{6(a^2-x^2-y^2)(x^2+y^2)-x^4-y^4\}\,\frac{a}{\sqrt{a^2-x^2-y^2}}\,dx\,dy \\
&= 2\int_0^{2\pi}\left\{\int_0^a \{6(a^2-r^2)r^2-r^4(\cos^4\varphi+\sin^4\varphi)\}\,\frac{r}{\sqrt{a^2-r^2}}\,dr\right\}d\varphi \\
&= 2\int_0^a \left\{12\pi(a^2-r^2)r^2-\frac{3\pi}{2}\,r^4\right\}\frac{r}{\sqrt{a^2-r^2}}\,dr \qquad \text{(by (33.5))} \\
&= \pi\int_0^a \{24t^2(a^2-t^2)-3(a^2-t^2)^2\}\,dt \\
&= \pi\int_0^a \{24a^2t^2-24t^4-3a^4+6a^2t^2-3t^4\}\,dt \\
&= \pi a^5\left\{8-\frac{24}{5}-3+2-\frac{3}{5}\right\} = \pi a^5\left\{7-\frac{27}{5}\right\} \\
&= \pi a^5\cdot\frac{35-27}{5} = \frac{8\pi a^5}{5}.
\end{aligned}
$$

Example 33.24 *Let a be a positive constant. Consider the set*

$$A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \le a^2,\, 0 \le y,\, -y \le x \le y,\, |z| \le 2a\}.$$

1) Describe A in semi polar coordinates (ϱ, φ, z).

2) Compute the space integrals

$$I = \int_A x\,\mathrm{d}\Omega, \qquad J = \int_A y\,\mathrm{d}\Omega, \qquad K = \int_A z^2\,\mathrm{d}\Omega.$$

3) Find the flux of the vector field

$$\mathbf{V}(x,y,z) = \left(3xz^2+\cosh y,\, z^2e^x,\, z^3-3axz+\sinh y\right), \quad (x,y,z) \in \mathbb{R}^3,$$

through the surface ∂A with its normal vector pointing outwards.

A Space integrals; flux.

D The first two problems are solved by the reduction theorems. In 3) we apply Gauß's theorem.

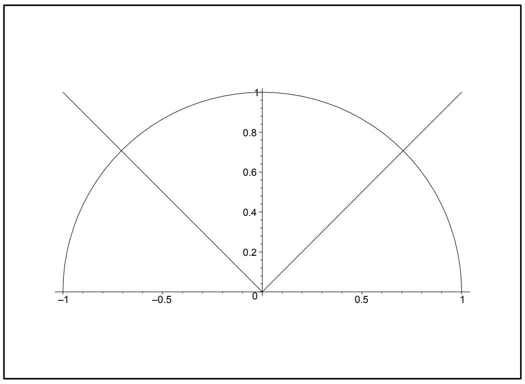

Figure 33.34: The domain B for $a = 1$ lies inside the upper angular space and inside the half circle.

I 1) Clearly, A is a cylinder with a quarter disc B in the (X,Y)-plane as generating surface. Hence A is described in semi polar coordinates by

$$A = \left\{(\varrho,\varphi,z)\;\middle|\; 0 \le \varrho \le a,\, \frac{\pi}{4} \le \varphi \le \frac{3\pi}{4},\, -2a \le z \le 2a\right\}.$$

2) By an argument of symmetry (first integrate with respect to x) we get

$$I = \int_A x\,\mathrm{d}\Omega = 0.$$

Alternatively,

$$\begin{aligned} I &= \int_A x\,\mathrm{d}\Omega = \int_{-2a}^{2a}\left\{\int_0^a\left(\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \varrho\cos\varphi\cdot\varrho\,\mathrm{d}\varphi\right)\mathrm{d}\varrho\right\}\mathrm{d}z \\ &= 4a\int_0^a \varrho^2\,\mathrm{d}\varrho \cdot \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\cos\varphi\,\mathrm{d}\varphi = 4a\cdot\frac{a^3}{3}\left[\sin\varphi\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = 0. \end{aligned}$$

Furthermore,

$$\begin{aligned} J &= \int_A y \, d\Omega = \int_{-2a}^{2a} \left\{ \int_0^a \left(\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \varrho \sin\varphi \cdot \varrho \, d\varphi \right) d\varrho \right\} dz \\ &= 4a \cdot \frac{a^3}{3} \left[-\cos\varphi \right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = \frac{4a^4}{3} \cdot \left\{ \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right\} = \frac{4\sqrt{2}a^4}{3}. \end{aligned}$$

Finally, by the slicing method,

$$K = \int_A z^2 \, d\Omega = \int_{-2a}^{2a} z^2 \, \text{area}(B) \, dz = \frac{1}{4} \cdot \pi a^2 \left[\frac{z^3}{3} \right]_{-2a}^{2a} = \frac{1}{4} \pi a^2 \cdot 2 \cdot \frac{8a^3}{3} = \frac{4\pi a^5}{3}.$$

3) By an application of Gauß's theorem,

$$\text{flux} = \int_A \text{div} \, \mathbf{V} \, d\Omega = \int_A \left\{ 3z^2 + 0 + 3z^2 - 3az \right\} d\Omega = 6K - 3aI = 8\pi a^5,$$

where we have inserted the values of K and I found in 2).

Example 33.25 *Consider the function*

$$F(x,y,z) = x^4 + x\,e^y \sin z, \qquad (x,y,z) \in \mathbb{R}^3,$$

and the vector field $\mathbf{V} = \bigtriangledown F$.

1) *Find the divergence* $\bigtriangledown \cdot \mathbf{V}$ *and the rotation* $\bigtriangledown \times \mathbf{V}$.

2) *Check if* $\mathbf{V}$ *has a vector potential.*

3) *Find the flux of* $\mathbf{V}$ *through* ∂A, *where* A *is the half ball given by the inequalities*

$$x^2 + y^2 + z^2 \le 9, \qquad z \le 0.$$

4) *Find the flux of* $\mathbf{V}$ *through the surface* $\mathcal{F}$ *given by*

$$x^2 + y^2 + z^2 = 9, \qquad z \le 0.$$

Show the orientation of $\mathcal{F}$ *on a figure. (Hint: Use that the surface* $\mathcal{F}$ *is a subset of the surface* ∂A *of 3).*

A Divergence, rotation, flux.

D Find $\mathbf{V}$. Use the rules of calculations and finally also Gauß's theorem.

I 1) First calculate

$$\mathbf{V} = \bigtriangledown F = \left(4x^3 + e^y \sin z, xe^y \sin z, xe^y \cos z\right).$$

Then

$$\bigtriangledown \cdot \mathbf{V} = \bigtriangledown \cdot \bigtriangledown F = \Delta F = 12x^2 + xe^y \sin z - xe^y \sin z = 12x^2$$

and

$$\bigtriangledown \times \mathbf{V} = \bigtriangledown \times \bigtriangledown F = \mathbf{0},$$

which is obvious because $\mathbf{V}$ is a gradient field and thence rotation free.

2) Since $\mathbf{V}$ is not divergence free in any open domain, $\mathbf{V}$ does not have a vector potential.

3) We get by Gauß's theorem, an argument of symmetry and using spherical coordinates,

$$\begin{aligned}
\text{flux}(\partial A) &= \int_{\partial A} \mathbf{V} \cdot \mathbf{n}\, \mathrm{d}S = \int_A \bigtriangledown \cdot \mathbf{V}\, \mathrm{d}\Omega = 12\int_A x^2\, \mathrm{d}\Omega = 12\int_A y^2\, \mathrm{d}\Omega \\
&= 6\int_A (x^2+y^2)\, \mathrm{d}\Omega = 6\int_0^{2\pi} \left\{ \int_{\frac{\pi}{2}}^{\pi} \left(\int_0^3 r^2 \sin^2\theta \cdot r^2 \sin\theta\, \mathrm{d}r \right) \mathrm{d}\theta \right\} \mathrm{d}\varphi \\
&= 6 \cdot 2\pi \int_{\frac{\pi}{2}}^{\pi} \left(1-\cos^2\theta\right) \sin\theta\, \mathrm{d}\theta \cdot \int_0^3 r^4\, \mathrm{d}r \\
&= \frac{12\pi}{5} \cdot 3^5 \cdot \left[-\cos\theta + \frac{1}{3}\cos^3\theta \right]_{\frac{\pi}{2}}^{\pi} = \frac{12\pi}{5} \cdot 3^5 \cdot \frac{2}{3} = \frac{1944\pi}{5}.
\end{aligned}$$

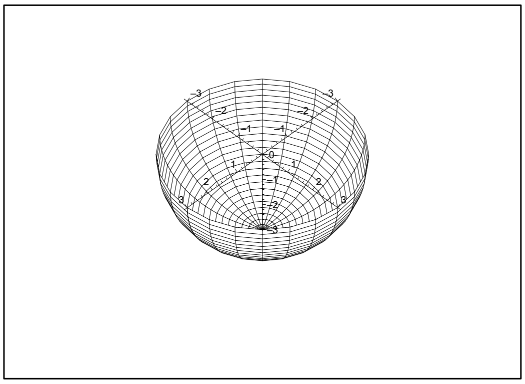

Figure 33.35: The body A.

4) Let $\mathcal{G}$ denote the disc in the (X,Y)-plane with the unit normal vector field pointing upwards, and let $\mathcal{F}$ denote the half sphere with the unit normal vector field pointing downward. Then according to 3),

$$\text{flux}(\partial A) = \text{flux}(\mathcal{F}) + \text{flux}(\mathcal{G}) = \frac{1944\pi}{5}.$$

Since $\mathbf{n} = (0,0,1)$ on $\mathcal{G}$, it follows by a rearrangement that

$$\begin{aligned}\text{flux}(\mathcal{F}) &= \frac{1944\pi}{5} - \text{flux}(\mathcal{G}) = \frac{1944\pi}{5} - \int_{\mathcal{G}} [xe^y \cos z]_{z=0}\, \mathrm{d}S = \frac{1944\pi}{5} - \int_{\mathcal{G}} xe^y\, \mathrm{d}S \\ &= \frac{1944\pi}{5} - \int_{-3}^{3} e^y \left\{ \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} x\, \mathrm{d}x \right\} \mathrm{d}y = \frac{1944\pi}{5} - 0 = \frac{1944\pi}{5},\end{aligned}$$

where we for symmetric reasons calculate the plane integral over the disc in rectangular coordinates.

Example 33.26 *The set* $\Omega \subset \mathbb{R}^3$ *is given in semi polar coordinates* (ϱ, φ, z) *by the inequalities*

$$-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}, \qquad 0 \le z \le h, \qquad 0 \le \varrho \le a\left(1 - \frac{z}{h}\right),$$

where a *and* h *are positive constants.*
Also given the vector field

$$\mathbf{U}(x,y,z) = \left(x^3 z + 2y\cos x, y^3 z + y^2 \sin x, x^2 y^2\right), \qquad (x,y,z) \in \mathbb{R}^3.$$

1) Find the divergence $\bigtriangledown \cdot \mathbf{U}$.

2) Find the flux Φ *of the vector field* $\mathbf{U}$ *through the surface* $\partial\Omega$.

A Vector field, flux.

D Sketch a figure. Apply Gauß's theorem.

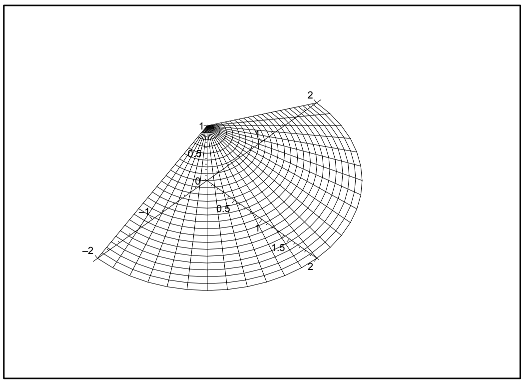

Figure 33.36: The body Ω for $a = 2$ and $h = 1$.

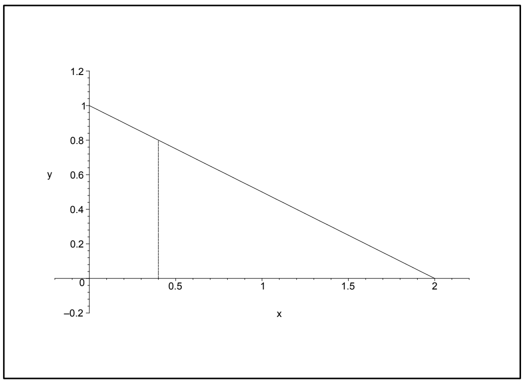

Figure 33.37: The meridian cut of Ω for $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $a = 2$, $h = 1$.

I We see that Ω is (half of) a cone (of revolution) with the top point $(0,0,h)$ and a half disc in the (X,Y)-plane as its basis.

1) The divergence is

$$\operatorname{div}\mathbf{U} = \nabla\cdot\mathbf{U} = (3x^2z - 2y\sin x) + (3y^2z + 2y\sin x) + 0 = 3z(x^2+y^2).$$

2) By applying Gauß's theorem and reducing in semi polar coordinates we conclude that the flux is

$$\begin{aligned}\Phi &= \int_\Omega \operatorname{div}\mathbf{U}\,d\Omega = \int_\Omega 3z(x^2+y^2)\,d\Omega = \int_0^h \left\{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\int_0^{a(1-\frac{z}{h})} 3z\varrho^2\cdot\varrho\,d\varrho\right) d\varphi\right\} dz \\ &= 3\pi\int_0^h z\left(\int_0^{a(1-\frac{z}{h})} \varrho^3\,d\varrho\right) dz = 3\pi\cdot\frac{1}{120}\,a^4\,h^2 = \frac{\pi}{40}\,a^4\,h^2.\end{aligned}$$

An ALTERNATIVE calculation is

$$\begin{aligned}\Phi &= \int_0^h \pi\cdot 3z\cdot\frac{1}{4}\,a^4\left(1-\frac{z}{h}\right)^4 dz = \frac{3\pi}{4}\,a^4\,h\int_0^h \left\{1-\left(1-\frac{z}{h}\right)\right\}\left(1-\frac{z}{h}\right)^4 dz \\ &= \frac{3\pi}{4}\,a^4\,h\int_0^h \left\{\left(1-\frac{z}{h}\right)^4 - \left(1-\frac{z}{h}\right)^5\right\} dz = \frac{3\pi}{4}\,a^4\,h^2\int_0^1 \left\{\zeta^4-\zeta^5\right\} d\zeta \\ &= \frac{3\pi}{4}\,a^4\,h^2\cdot\left(\frac{1}{5}-\frac{1}{6}\right) = \frac{\pi}{40}\,a^4\,h^2.\end{aligned}$$

Example 33.27 *Find the divergence and the rotation of the vector field*

$$\mathbf{V}(x,y,z)=\left(2x+xy,7x-\frac{1}{2}\,y^2,3z\right),\qquad (x,y,z)\in\mathbb{R}^3,$$

and find the flux of $\mathbf{V}$ *through the unit sphere* $x^2+y^2+z^2=1$*, where the normal vector is pointing outwards.*

A Divergence, rotation and flux).

D Apply Gauß's theorem.

I The divergence is

$$\text{div }\mathbf{V}=2+y-y+3=5.$$

The rotation is

$$\mathbf{rot}\ \mathbf{V}=\begin{vmatrix}\mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2x+xy & 7x-\frac{1}{2}\,y^2 & 3z\end{vmatrix}=\begin{vmatrix}\mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ xy & 7x & 0\end{vmatrix}=(0,0,7-x).$$

By Gauß's theorem the flux through the surface $\mathcal{F}$ of the unit sphere is given by

$$\int_{\mathcal{F}}\mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S=\int_{\Omega}\text{div }\mathbf{V}\,\mathrm{d}\Omega=\int_{\Omega}5\,\mathrm{d}\Omega=5\,\text{vol}(\Omega)=5\cdot\frac{4\pi}{4}\cdot 1^3=\frac{20\pi}{3}.$$

Example 33.28 .

1) Find the volume of the body of revolution

$$A=\left\{(x,y,z)\in\mathbb{R}^3\;\middle|\;\frac{1}{2}\,x^2+\frac{1}{2}\,y^2-1\le z\le 1\right\}.$$

2) Find the flux of the vector field

$$\mathbf{V}(x,y,z)=\left(y^2+x,xz^2-yx^2,x^2z\right),\qquad (x,y,z)\in\mathbb{R}^3,$$

through ∂A*, where the unit normal vector is always pointing away from the body.*

A Volume and flux.

D Sketch a section of A in the meridian half plane. Apply the method of slicing by finding the volume. The flux is found by means of Gauß's theorem.

I 1) It follows from the sketch of the meridian half plane that the domain is described in semi polar coordinates by

$$0\le\varrho\le\sqrt{2z+2},\qquad -1\le z\le 1,$$

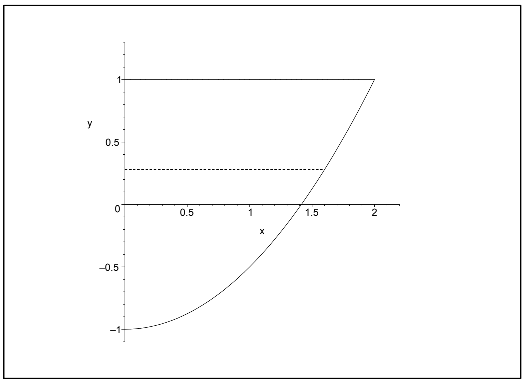

Figure 33.38: The meridian cut for A. The boundary curve has the equation $z = \frac{1}{2}\,\varrho^2 - 1$.

and that the body of revolution is a subset of a paraboloid of revolution.

THE SLICING METHOD. The paraboloid of revolution is intersected by a plane at the height $z \in]-1,1]$ (the dotted line on the figure) in a circle of area

$$\pi \cdot \varrho(z)^2 = 2\pi(z+1).$$

Thus the volume of the body of revolution is

$$\mathrm{vol}(A) = \int_{-1}^{1} 2\pi(z+1)\,\mathrm{d}z = \left[\pi(z+1)^2\right]_{-1}^{1} = 4\pi.$$

2) According to Gauss's theorem, the flux of $\mathbf{V}$ through ∂A is given by

$$\int_{\partial A} \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S = \int_A \mathrm{div}\,\mathbf{V}\,\mathrm{d}\Omega = \int_A \left\{1 - x^2 + x^2\right\}\,\mathrm{d}\Omega = \mathrm{vol}(A) = 4\pi.$$

34 Formulæ

Some of the following formulæ can be assumed to be known from high school. It is highly recommended that one *learns most of these formulæ in this appendix by heart.*

34.1 Squares etc.

The following simple formulæ occur very frequently in the most different situations.

$$
\begin{array}{ll}
(a+b)^2 = a^2+b^2+2ab, & a^2+b^2+2ab = (a+b)^2, \\
(a-b)^2 = a^2+b^2-2ab, & a^2+b^2-2ab = (a-b)^2, \\
(a+b)(a-b) = a^2-b^2, & a^2-b^2 = (a+b)(a-b), \\
(a+b)^2 = (a-b)^2+4ab, & (a-b)^2 = (a+b)^2-4ab.
\end{array}
$$

34.2 Powers etc.

Logarithm:

$$
\begin{array}{rcl}
\ln|xy| = & \ln|x|+\ln|y|, & x,y \neq 0, \\
\ln\left|\dfrac{x}{y}\right| = & \ln|x|-\ln|y|, & x,y \neq 0, \\
\ln|x^r| = & r\ln|x|, & x \neq 0.
\end{array}
$$

Power function, fixed exponent:

$$(xy)^r = x^r \cdot y^r, x,y>0 \quad \text{(extensions for some } r\text{)},$$

$$\left(\frac{x}{y}\right)^r = \frac{x^r}{y^r}, x,y>0 \quad \text{(extensions for some } r\text{)}.$$

Exponential, fixed base:

$$a^x \cdot a^y = a^{x+y}, \quad a>0 \quad \text{(extensions for some } x,\, y\text{)},$$
$$(a^x)^y = a^{xy}, a>0 \quad \text{(extensions for some } x,\, y\text{)},$$

$$a^{-x} = \frac{1}{a^x}, a>0, \quad \text{(extensions for some } x\text{)},$$

$$\sqrt[n]{a} = a^{1/n}, a \geq 0, \qquad n \in \mathbb{N}.$$

Square root:

$$\sqrt{x^2} = |x|, \qquad x \in \mathbb{R}.$$

Remark 34.1 It happens quite frequently that students make errors when they try to apply these rules. They must be mastered! In particular, as one of my friends once put it: “If you can master the square root, you can master everything in mathematics!” Notice that this innocent looking square root is one of the most difficult operations in Calculus. Do not forget the *absolute value!* $\Diamond$

34.3 Differentiation

Here are given the well-known rules of differentiation together with some rearrangements which sometimes may be easier to use:

$$\{f(x) \pm g(x)\}' = f'(x) \pm g'(x),$$

$$\{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x) = f(x)g(x)\left\{\frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}\right\},$$

where the latter rearrangement presupposes that $f(x) \neq 0$ and $g(x) \neq 0$.
If $g(x) \neq 0$, we get the usual formula known from high school

$$\left\{\frac{f(x)}{g(x)}\right\}' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

It is often more convenient to compute this expression in the following way:

$$\left\{\frac{f(x)}{g(x)}\right\} = \frac{d}{dx}\left\{f(x) \cdot \frac{1}{g(x)}\right\} = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2} = \frac{f(x)}{g(x)}\left\{\frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}\right\},$$

where the former expression often is *much easier* to use in practice than the usual formula from high school, and where the latter expression again presupposes that $f(x) \neq 0$ and $g(x) \neq 0$. Under these assumptions we see that the formulæ above can be written

$$\frac{\{f(x)g(x)\}'}{f(x)g(x)} = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)},$$

$$\frac{\{f(x)/g(x)\}'}{f(x)/g(x)} = \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}.$$

Since

$$\frac{d}{dx}\ln|f(x)| = \frac{f'(x)}{f(x)}, \qquad f(x) \neq 0,$$

we also name these the *logarithmic derivatives*.

Finally, we mention the rule of **differentiation of a composite function**

$$\{f(\varphi(x))\}' = f'(\varphi(x)) \cdot \varphi'(x).$$

We first differentiate the function itself; then the insides. This rule is a 1-dimensional version of the so-called *Chain rule*.

34.4 Special derivatives.

Power like:

$$\frac{d}{dx}(x^\alpha) = \alpha \cdot x^{\alpha-1}, \qquad \text{for } x > 0, \ \text{(extensions for some } \alpha).$$

$$\frac{d}{dx}\ln|x| = \frac{1}{x}, \qquad \text{for } x \neq 0.$$

Exponential like:

$$\frac{d}{dx}\exp x = \exp x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}(a^x) = \ln a \cdot a^x, \qquad \text{for } x \in \mathbb{R} \text{ and } a > 0.$$

Trigonometric:

$$\frac{d}{dx}\sin x = \cos x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\cos x = -\sin x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\tan x = 1 + \tan^2 x = \frac{1}{\cos^2 x}, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, p \in \mathbb{Z},$$

$$\frac{d}{dx}\cot x = -(1 + \cot^2 x) = -\frac{1}{\sin^2 x}, \qquad \text{for } x \neq p\pi, p \in \mathbb{Z}.$$

Hyperbolic:

$$\frac{d}{dx}\sinh x = \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\cosh x = \sinh x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\tanh x = 1 - \tanh^2 x = \frac{1}{\cosh^2 x}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\coth x = 1 - \coth^2 x = -\frac{1}{\sinh^2 x}, \qquad \text{for } x \neq 0.$$

Inverse trigonometric:

$$\frac{d}{dx}\text{ Arcsin } x = \frac{1}{\sqrt{1-x^2}}, \qquad \text{for } x \in]-1, 1[,$$

$$\frac{d}{dx}\text{ Arccos } x = -\frac{1}{\sqrt{1-x^2}}, \qquad \text{for } x \in]-1, 1[,$$

$$\frac{d}{dx}\text{ Arctan } x = \frac{1}{1+x^2}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\text{ Arccot } x = \frac{1}{1+x^2}, \qquad \text{for } x \in \mathbb{R}.$$

Inverse hyperbolic:

$$\frac{d}{dx}\text{ Arsinh } x = \frac{1}{\sqrt{x^2+1}}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\text{ Arcosh } x = \frac{1}{\sqrt{x^2-1}}, \qquad \text{for } x \in]1, +\infty[,$$

$$\frac{d}{dx}\text{ Artanh } x = \frac{1}{1-x^2}, \qquad \text{for } |x| < 1,$$

$$\frac{d}{dx}\text{ Arcoth } x = \frac{1}{1-x^2}, \qquad \text{for } |x| > 1.$$

Remark 34.2 The derivative of the trigonometric and the hyperbolic functions are to some extent exponential like. The derivatives of the inverse trigonometric and inverse hyperbolic functions are power like, because we include the logarithm in this class. ◊

34.5 Integration

The most obvious rules are dealing with linearity

$$\int \{f(x)+\lambda g(x)\}\,dx = \int f(x)\,dx + \lambda \int g(x)\,dx, \qquad \text{where } \lambda \in \mathbb{R} \text{ is a constant,}$$

and with the fact that differentiation and integration are "inverses to each other", i.e. modulo some arbitrary constant $c \in \mathbb{R}$, which often tacitly is missing,

$$\int f'(x)\,dx = f(x).$$

If we in the latter formula replace $f(x)$ by the product $f(x)g(x)$, we get by reading from the right to the left and then differentiating the product,

$$f(x)g(x) = \int \{f(x)g(x)\}'\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx.$$

Hence, by a rearrangement

The rule of partial integration:

$$\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx.$$

The differentiation is moved from one factor of the integrand to the other one by changing the sign and adding the term $f(x)g(x)$.

Remark 34.3 This technique was earlier used a lot, but is almost forgotten these days. It must be revived, because MAPLE and pocket calculators apparently do not know it. It is possible to construct examples where these devices cannot give the exact solution, unless you first perform a partial integration yourself. $\Diamond$

Remark 34.4 This method can also be used when we estimate integrals which cannot be directly calculated, because the antiderivative is not contained in e.g. the catalogue of MAPLE. The idea is by a succession of partial integrations to make the new integrand smaller. $\Diamond$

Integration by substitution:

If the integrand has the special structure $f(\varphi(x))\cdot\varphi'(x)$, then one can change the variable to $y=\varphi(x)$:

$$\int f(\varphi(x))\cdot\varphi'(x)\,dx = \text{“}\int f(\varphi(x))\,d\varphi(x)\text{''} = \int_{y=\varphi(x)} f(y)\,dy.$$

Integration by a monotonous substitution:

If $\varphi(y)$ is a *monotonous* function, which maps the y-interval *one-to-one* onto the x-interval, then

$$\int f(x)\,dx = \int_{y=\varphi^{-1}(x)} f(\varphi(y))\varphi'(y)\,dy.$$

Remark 34.5 This rule is usually used when we have some "ugly" term in the integrand $f(x)$. The idea is to put this ugly term equal to $y = \varphi^{-1}(x)$. When e.g. x occurs in $f(x)$ in the form $\sqrt{x}$, we put $y = \varphi^{-1}(x) = \sqrt{x}$, hence $x = \varphi(y) = y^2$ and $\varphi'(y) = 2y$. $\Diamond$

34.6 Special antiderivatives

Power like:

$$\int \frac{1}{x}\,dx = \ln|x|, \qquad \text{for } x \neq 0. \quad \text{(Do not forget the numerical value!)}$$

$$\int x^{\alpha}\,dx = \frac{1}{\alpha+1}x^{\alpha+1}, \qquad \text{for } \alpha \neq -1,$$

$$\int \frac{1}{1+x^2}\,dx = \text{ Arctan } x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{1-x^2}\,dx = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|, \qquad \text{for } x \neq \pm 1,$$

$$\int \frac{1}{1-x^2}\,dx = \text{ Artanh } x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{1-x^2}\,dx = \text{ Arcoth } x, \qquad \text{for } |x| > 1,$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \text{ Arcsin } x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = -\text{ Arccos } x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \text{ Arsinh } x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \ln(x+\sqrt{x^2+1}), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{x}{\sqrt{x^2-1}}\,dx = \sqrt{x^2-1}, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \text{ Arcosh } x, \qquad \text{for } x > 1,$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \ln|x+\sqrt{x^2-1}|, \qquad \text{for } x > 1 \text{ eller } x < -1.$$

There is an error in the programs of the pocket calculators TI-92 and TI-89. The *numerical signs* are missing. It is obvious that $\sqrt{x^2-1} < |x|$ so if $x < -1$, then $x + \sqrt{x^2-1} < 0$. Since you cannot take the logarithm of a negative number, these pocket calculators will give an error message.

Exponential like:

$$\int \exp x \, dx = \exp x, \qquad \text{for } x \in \mathbb{R},$$

$$\int a^x \, dx = \frac{1}{\ln a} \cdot a^x, \qquad \text{for } x \in \mathbb{R}, \text{ and } a > 0, a \neq 1.$$

Trigonometric:

$$\int \sin x \, dx = -\cos x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \cos x \, dx = \sin x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \tan x \, dx = -\ln|\cos x|, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \cot x \, dx = \ln|\sin x|, \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos x} \, dx = \frac{1}{2} \ln\left(\frac{1+\sin x}{1-\sin x}\right), \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin x} \, dx = \frac{1}{2} \ln\left(\frac{1-\cos x}{1+\cos x}\right), \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos^2 x} \, dx = \tan x, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin^2 x} \, dx = -\cot x, \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z}.$$

Hyperbolic:

$$\int \sinh x \, dx = \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \cosh x \, dx = \sinh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \tanh x \, dx = \ln \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \coth x \, dx = \ln|\sinh x|, \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh x} \, dx = \operatorname{Arctan}(\sinh x), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\cosh x} \, dx = 2 \operatorname{Arctan}(e^x), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh x} \, dx = \frac{1}{2} \ln\left(\frac{\cosh x - 1}{\cosh x + 1}\right), \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\sinh x}\, dx = \ln\left|\frac{e^x-1}{e^x+1}\right|, \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh^2 x}\, dx = \tanh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh^2 x}\, dx = -\coth x, \qquad \text{for } x \neq 0.$$

34.7 Trigonometric formulæ

The trigonometric formulæ are closely connected with circular movements. Thus $(\cos u, \sin u)$ are the coordinates of a point P on the unit circle corresponding to the angle u, cf. figure A.1. This geometrical interpretation is used from time to time.

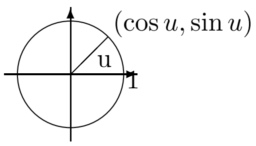

Figure 34.1: The unit circle and the trigonometric functions.

The fundamental trigonometric relation:

$$\cos^2 u + \sin^2 u = 1, \qquad \text{for } u \in \mathbb{R}.$$

Using the previous geometric interpretation this means according to *Pythagoras's theorem*, that the point P with the coordinates $(\cos u, \sin u)$ always has distance 1 from the origo $(0,0)$, i.e. it is lying on the boundary of the circle of centre $(0,0)$ and radius $\sqrt{1} = 1$.

Connection to the complex exponential function:

The *complex exponential* is for imaginary arguments defined by

$$\exp(\mathrm{i}\, u) := \cos u + \mathrm{i}\ \sin u.$$

It can be checked that the usual functional equation for exp is still valid for complex arguments. In other word: The definition above is extremely conveniently chosen.

By using the definition for $\exp(\mathrm{i}\, u)$ and $\exp(-\mathrm{i}\, u)$ it is easily seen that

$$\cos u = \frac{1}{2}(\exp(\mathrm{i}\, u) + \exp(-\mathrm{i}\, u)),$$

$$\sin u = \frac{1}{2i}(\exp(\mathrm{i}\, u) - \exp(-\mathrm{i}\, u)),$$

.

Moivre's formula: We get by expressing $\exp(\mathrm{i}nu)$ in two different ways:

$$\exp(\mathrm{i}nu) = \cos nu + \mathrm{i}\sin nu = (\cos u + \mathrm{i}\sin u)^n.$$

Example 34.1 If we e.g. put $n = 3$ into Moivre's formula, we obtain the following typical application,

$$\begin{aligned}\cos(3u) + \mathrm{i}\sin(3u) &= (\cos u + \mathrm{i}\sin u)^3\\ &= \cos^3 u + 3\mathrm{i}\cos^2 u\cdot\sin u + 3\mathrm{i}^2\cos u\cdot\sin^2 u + \mathrm{i}^3\sin^3 u\\ &= \{\cos^3 u - 3\cos u\cdot\sin^2 u\} + \mathrm{i}\{3\cos^2 u\cdot\sin u - \sin^3 u\}\\ &= \{4\cos^3 u - 3\cos u\} + \mathrm{i}\{3\sin u - 4\sin^3 u\}\end{aligned}$$

When this is split into the real- and imaginary parts we obtain

$$\cos 3u = 4\cos^3 u - 3\cos u, \qquad \sin 3u = 3\sin u - 4\sin^3 u. \quad \diamond$$

Addition formulæ:

$$\sin(u+v) = \sin u\,\cos v + \cos u\,\sin v,$$

$$\sin(u-v) = \sin u\,\cos v - \cos u\,\sin v,$$

$$\cos(u+v) = \cos u\,\cos v - \sin u\,\sin v,$$

$$\cos(u-v) = \cos u\,\cos v + \sin u\,\sin v.$$

Products of trigonometric functions to a sum:

$$\sin u\,\cos v = \frac{1}{2}\sin(u+v) + \frac{1}{2}\sin(u-v),$$

$$\cos u\,\sin v = \frac{1}{2}\sin(u+v) - \frac{1}{2}\sin(u-v),$$

$$\sin u\,\sin v = \frac{1}{2}\cos(u-v) - \frac{1}{2}\cos(u+v),$$

$$\cos u\,\cos v = \frac{1}{2}\cos(u-v) + \frac{1}{2}\cos(u+v).$$

Sums of trigonometric functions to a product:

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right),$$

$$\sin u - \sin v = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right),$$

$$\cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right),$$

$$\cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right).$$

Formulæ of halving and doubling the angle:

$$\sin 2u = 2\sin u\,\cos u,$$

$$\cos 2u = \cos^2 u - \sin^2 u = 2\cos^2 u - 1 = 1 - 2\sin^2 u,$$

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}} \qquad \text{followed by a discussion of the sign,}$$

$$\cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}} \qquad \text{followed by a discussion of the sign,}$$

34.8 Hyperbolic formulæ

These are very much like the trigonometric formulæ, and if one knows a little of Complex Function Theory it is realized that they are actually identical. The structure of this section is therefore the same as for the trigonometric formulæ. The reader should compare the two sections concerning similarities and differences.

The fundamental relation:

$$\cosh^2 x - \sinh^2 x = 1.$$

Definitions:

$$\cosh x = \frac{1}{2}\left(\exp(x) + \exp(-x)\right), \qquad \sinh x = \frac{1}{2}\left(\exp(x) - \exp(-x)\right).$$

"Moivre's formula":

$$\exp(x) = \cosh x + \sinh x.$$

This is trivial and only rarely used. It has been included to show the analogy.

Addition formulæ:

$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y),$$

$$\sinh(x-y) = \sinh(x)\cosh(y) - \cosh(x)\sinh(y),$$

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y),$$

$$\cosh(x-y) = \cosh(x)\cosh(y) - \sinh(x)\sinh(y).$$

Formulæ of halving and doubling the argument:

$$\sinh(2x) = 2\sinh(x)\cosh(x),$$

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x) = 2\cosh^2(x) - 1 = 2\sinh^2(x) + 1,$$

$$\sinh\left(\frac{x}{2}\right) = \pm\sqrt{\frac{\cosh(x)-1}{2}} \qquad \text{followed by a discussion of the sign,}$$

$$\cosh\left(\frac{x}{2}\right) = \sqrt{\frac{\cosh(x)+1}{2}}.$$

Inverse hyperbolic functions:

$$\text{Arsinh}(x) = \ln\left(x + \sqrt{x^2+1}\right), \qquad x \in \mathbb{R},$$

$$\text{Arcosh}(x) = \ln\left(x + \sqrt{x^2-1}\right), \qquad x \geq 1,$$

$$\text{Artanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \qquad |x| < 1,$$

$$\text{Arcoth}(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right), \qquad |x| > 1.$$

34.9 Complex transformation formulæ

$$\cos(\mathrm{i}x) = \cosh(x), \qquad \cosh(\mathrm{i}x) = \cos(x),$$
$$\sin(\mathrm{i}x) = \mathrm{i}\,\sinh(x), \qquad \sinh(\mathrm{i}x) = \mathrm{i}\,\sin x.$$

34.10 Taylor expansions

The generalized binomial coefficients are defined by

$$\binom{\alpha}{n} := \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{1\cdot 2\cdots n},$$

with n factors in the numerator and the denominator, supplied with

$$\binom{\alpha}{0} := 1.$$

The Taylor expansions for *standard functions* are divided into *power like* (the radius of convergency is finite, i.e. $= 1$ for the standard series) and*exponential like* (the radius of convergency is infinite).
Power like:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1,$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x| < 1,$$

$$(1+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j, \qquad n \in \mathbb{N}, x \in \mathbb{R},$$

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \qquad \alpha \in \mathbb{R}\setminus\mathbb{N}, |x| < 1,$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n}, \qquad |x| < 1,$$

$$\operatorname{Arctan}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \qquad |x| < 1.$$

Exponential like:

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \qquad x \in \mathbb{R}$$

$$\exp(-x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} x^n, \qquad x \in \mathbb{R}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}, \qquad x \in \mathbb{R},$$

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}, \qquad x \in \mathbb{R},$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}, \qquad x \in \mathbb{R},$$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}, \qquad x \in \mathbb{R}.$$

34.11 Magnitudes of functions

We often have to compare functions for $x \to 0+$, or for $x \to \infty$. The simplest type of functions are therefore arranged in an hierarchy:

1) logarithms,

2) power functions,

3) exponential functions,

4) faculty functions.

When $x \to \infty$, a function from a higher class will always dominate a function form a lower class. More precisely:

A) A *power function* dominates a *logarithm* for $x \to \infty$:

$$\frac{(\ln x)^\beta}{x^\alpha} \to 0 \qquad \text{for } x \to \infty, \quad \alpha,\, \beta > 0.$$

B) An *exponential* dominates a *power function* for $x \to \infty$:

$$\frac{x^\alpha}{a^x} \to 0 \qquad \text{for } x \to \infty, \quad \alpha,\, a > 1.$$

C) The *faculty function* dominates an *exponential* for $n \to \infty$:

$$\frac{a^n}{n!} \to 0, \qquad n \to \infty, \quad n \in \mathbb{N}, \quad a > 0.$$

D) When $x \to 0+$ we also have that a *power function* dominates the *logarithm:*

$$x^\alpha \ln x \to 0-, \qquad \text{for } x \to 0+, \quad \alpha > 0.$$

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