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# Real Functions in Several Variables: Volume VII

## Space Integrals

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Real Functions in Several Variables: Volume VII Space Integrals
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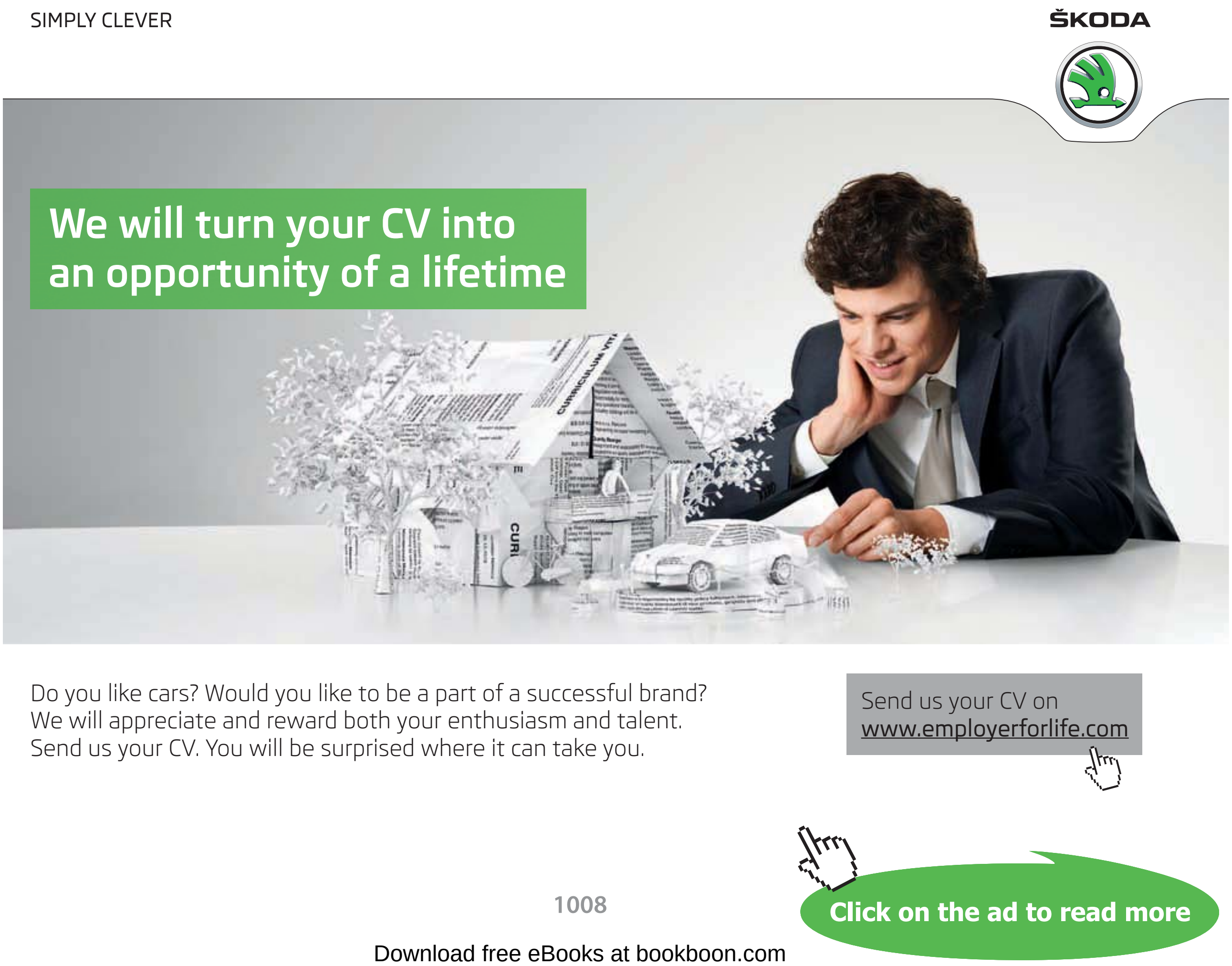
# Contents



# Preface

The topic of this series of books on *"Real Functions in Several Variables"* is very important in the description in e.g. *Mechanics* of the real 3-dimensional world that we live in. Therefore, we start from the very beginning, modelling this world by using the coordinates of $\mathbb{R}^3$ to describe e.g. a motion in space. There is, however, absolutely no reason to restrict ourselves to $\mathbb{R}^3$ alone. Some motions may be rectilinear, so only $\mathbb{R}$ is needed to describe their movements on a line segment. This opens up for also dealing with $\mathbb{R}^2$, when we consider plane motions. In more elaborate problems we need higher dimensional spaces. This may be the case in *Probability Theory* and *Statistics*. Therefore, we shall in general use $\mathbb{R}^n$ as our abstract model, and then restrict ourselves in examples mainly to $\mathbb{R}^2$ and $\mathbb{R}^3$.

For rectilinear motions the familiar *rectangular coordinate system* is the most convenient one to apply. However, as known from e.g. *Mechanics*, circular motions are also very important in the applications in engineering. It becomes natural alternatively to apply in $\mathbb{R}^2$ the so-called *polar coordinates* in the plane. They are convenient to describe a circle, where the rectangular coordinates usually give some nasty square roots, which are difficult to handle in practice.

Rectangular coordinates and polar coordinates are designed to model each their problems. They supplement each other, so difficult computations in one of these coordinate systems may be easy, and even trivial, in the other one. It is therefore important always in advance carefully to analyze the geometry of e.g. a domain, so we ask the question: Is this domain best described in rectangular or in polar coordinates?

Sometimes one may split a problem into two subproblems, where we apply rectangular coordinates in one of them and polar coordinates in the other one.

It should be mentioned that in *real life* (though not in these books) one cannot always split a problem into two subproblems as above. Then one is really in trouble, and more advanced mathematical methods should be applied instead. This is, however, outside the scope of the present series of books.

The idea of polar coordinates can be extended in two ways to $\mathbb{R}^3$. Either to *semi-polar* or *cylindric coordinates*, which are designed to describe a cylinder, or to *spherical coordinates*, which are excellent for describing spheres, where rectangular coordinates usually are doomed to fail. We use them already in daily life, when we specify a place on Earth by its longitude and latitude! It would be very awkward in this case to use rectangular coordinates instead, even if it is possible.

Concerning the contents, we begin this investigation by modelling point sets in an $n$-dimensional Euclidean space $E^n$ by $\mathbb{R}^n$. There is a subtle difference between $E^n$ and $\mathbb{R}^n$, although we often identify these two spaces. In $E^n$ we use *geometrical methods* without a coordinate system, so the objects are independent of such a choice. In the coordinate space $\mathbb{R}^n$ we can use ordinary calculus, which in principle is not possible in $E^n$. In order to stress this point, we call $E^n$ the "abstract space" (in the sense of calculus; not in the sense of geometry) as a warning to the reader. Also, whenever necessary, we use the colour black in the "abstract space", in order to stress that this expression is theoretical, while variables given in a chosen coordinate system and their related concepts are given the colours blue, red and green.

We also include the most basic of what mathematicians call *Topology*, which will be necessary in the following. We describe what we need by a function.

Then we proceed with limits and continuity of functions and define continuous curves and surfaces, with parameters from subsets of $\mathbb{R}$ and $\mathbb{R}^2$, resp..

Continue with (partial) differentiable functions, curves and surfaces, the chain rule and Taylor's formula for functions in several variables.

We deal with maxima and minima and extrema of functions in several variables over a domain in $\mathbb{R}^n$. This is a very important subject, so there are given many worked examples to illustrate the theory.

Then we turn to the problems of integration, where we specify four different types with increasing complexity, *plane integral, space integral, curve (or line) integral and surface integral*.

Finally, we consider *vector analysis*, where we deal with vector fields, Gauß's theorem and Stokes's theorem. All these subjects are very important in theoretical Physics.

The structure of this series of books is that each subject is *usually* (but not always) described by three successive chapters. In the first chapter a brief theoretical theory is given. The next chapter gives some practical guidelines of how to solve problems connected with the subject under consideration. Finally, some worked out examples are given, in many cases in several variants, because the standard solution method is seldom the only way, and it may even be clumsy compared with other possibilities.

I have as far as possible structured the examples according to the following scheme:

**A** *Awareness*, i.e. a short description of what is the problem.

**D** *Decision*, i.e. a reflection over what should be done with the problem.

**I** *Implementation*, i.e. where all the calculations are made.

**C** *Control*, i.e. a test of the result.

This is an ideal form of a general procedure of solution. It can be used in any situation and it is not linked to Mathematics alone. I learned it many years ago in the Theory of Telecommunication in a situation which did not contain Mathematics at all. The student is recommended to use it also in other disciplines.

From high school one is used to immediately to proceed to **I**. *Implementation*. However, examples and problems at university level, let alone situations in real life, are often so complicated that it in general will be a good investment also to spend some time on the first two points above in order to be absolutely certain of what to do in a particular case. Note that the first three points, **ADI**, can always be executed.

This is unfortunately not the case with **C** *Control*, because it from now on may be difficult, if possible, to check one's solution. It is only an extra securing whenever it is possible, but we cannot include it always in our solution form above.

I shall on purpose not use the logical signs. These should in general be avoided in Calculus as a shorthand, because they are often (too often, I would say) misused. Instead of $\wedge$ I shall either write "and", or a comma, and instead of $\vee$ I shall write "or". The arrows $\Rightarrow$ and $\Leftrightarrow$ are in particular misunderstood by the students, so they should be totally avoided. They are not telegram short hands, and from a logical point of view they usually do not make sense at all! Instead, write in a plain language what you mean or want to do. This is difficult in the beginning, but after some practice it becomes routine, and it will give more precise information.

When we deal with multiple integrals, one of the possible pedagogical ways of solving problems has been to colour variables, integrals and upper and lower bounds in blue, red and green, so the reader by the colour code can see in each integral what is the variable, and what are the parameters, which do not enter the integration under consideration. We shall of course build up a hierarchy of these colours, so the order of integration will always be defined. As already mentioned above we reserve the colour black for the theoretical expressions, where we cannot use ordinary calculus, because the symbols are only shorthand for a concept.

The author has been very grateful to his old friend and colleague, the late Per Wennerberg Karlsson, for many discussions of how to present these difficult topics on real functions in several variables, and for his permission to use his textbook as a template of this present series. Nevertheless, the author has felt it necessary to make quite a few changes compared with the old textbook, because we did not always agree, and some of the topics could also be explained in another way, and then of course the results of our discussions have here been put in writing for the first time.

The author also adds some calculations in MAPLE, which interact nicely with the theoretic text. Note, however, that when one applies MAPLE, one is forced first to make a geometrical analysis of the domain of integration, i.e. apply some of the techniques developed in the present books.

The theory and methods of these volumes on “Real Functions in Several Variables” are applied constantly in higher Mathematics, Mechanics and Engineering Sciences. It is of paramount importance for the calculations in *Probability Theory*, where one constantly integrate over some point set in space.

It is my hope that this text, these guidelines and these examples, of which many are treated in more ways to show that the solutions procedures are not unique, may be of some inspiration for the students who have just started their studies at the universities.

Finally, even if I have tried to write as careful as possible, I doubt that all errors have been removed. I hope that the reader will forgive me the unavoidable errors.

Leif Mejlbro
March 21, 2015

# Introduction to volume VII, The space integral

This is the seventh volume in the series of books on *Real Functions in Several Variables*.

We continue the investigation of how to integrate a real function in several variables in three dimensions, i.e. how we can reduce abstract *space integrals*. We start with the reduction theorems in rectangular coordinates. This is just an extension of the theory of the plane integral in rectangular coordinates. There are some small complications in the Geometry, when we have to visualize sets $A$ in $\mathbb{R}^3$, but in principle, it is the same theory.

In the next chapter we use the theory of the reduction of the plane integral in polar coordinates to the reduction of a space integral in semi-polar coordinates. The reduction takes place in a parameter space, where we introduce a weight function and then integrate as in the case of the rectangular coordinates with respect to the semi-polar coordinates in the parameter domain.

The same idea is used in Chapter 24, where we reduce in spherical coordinates. We add a weight function as a factor to the integrand and then integrate as in the rectangular case with respect to the spherical coordinates in the parameter space, which must not be confused with the body itself.

Since we here are dealing with three dimensions, there are lots of variants, which cannot all be coined as theorems. However, once the coordinate system has been chosen, and we have identified the corresponding weight function, then the problem is always reduced to a geometric analysis of the body under consideration.

We add some examples of how to find a volume, the centre of gravity of a body, or the moment of inertia.

# 22 The space integral in rectangular coordinates

## 22.1 Introduction

The extension of the plane integral to the *space integral* follows the pattern known from the plane integral. First we must define a *volume element* $\mathrm{d}\Omega$ in $\mathbb{R}^3$. As a guideline we see in rectangular coordinates $(x, y, z)$ that it is most reasonable that we interpret

$$\mathrm{d}\Omega = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

because $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ represents the volume of an axiparallel infinitesiml box of edge lengths $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$. Given a continuous function $f_A \to \mathbb{R}$, where $A \subset \mathbb{R}^3$ is a closed and bounded set in $\mathbb{R}^3$. Then the space integral of $f$ over $A$ is denoted by the (abstract) symbol

$$\int_A f(x, y, z)\,\mathrm{d}\Omega,$$

where the green colour indicates that we are dealing with an abstract integral in three dimensions.

When $f(x, y, z) \equiv 1$, then the space integral is the *volume* of $A$,

$$\mathrm{vol}(A) = \int_A 1\,\mathrm{d}\Omega = \int_A \mathrm{d}\Omega,$$

because we, roughly speaking, fill up $A$ with mutually disjoint infinitesimal boxes and then add all their volumes, which intuitively give us the volume of $A$. Actually, this construction is not quite correct, but close to. It represents the idea of the volume and can immediately be extended to the idea of the more general space integral by multiplying each infinitesimal volume with the value of $f$ at some point in this box, and then add the results. By letting $\mathrm{d}\Omega \to 0$ in some controlled sense (which is not obvious here) we obtain the (idea of the) space integral.

We shall not be concerned with correcting the intuitive abstract considerations above. Instead we shall – without proofs – in the next section quote some *reduction theorem* of an abstract space integral, so it in practice becomes possible to calculate its value.

## 22.2 Overview of setting up of a line, a plane, a surface or a space integral

We start with a general section which explains how one analyzes the setting up of an integral in order to obtain a reduction formula, which can be applied in practice. The material is covering all types of integrals considered in this series of books. For clarity we shall not use colours in this section.

When we set up an integral, we have two possible approaches:

1) A geometrical analysis.

2) Measure theory (i.e. concerning integration).

In elementary books on Calculus the *geometrical analysis* is dominating, although there may occur examples, where the *measure theory* plays a bigger role than the geometry. The latter is often the case when we apply the transformation theorems. It may also occur when we shall choose between semi-polar and spherical coordinates. We shall in the following not consider these exceptions, so usually we start with a *geometrical analysis* of the domain of integration.

This analysis is depending on

**1a.** Dimension.

**1b.** Choice of coordinates.

The classical coordinates are:

**Dimension 2:**

**Rectangular:** $(x, y)$, or $(u, v)$, $(\varrho, \varphi)$ or similarly in the parametric domain.

**Polar:** $x = \varrho \cos\varphi$, $y = \varrho \sin\varphi$.

**General:** $x = X(u, v)$, $y = Y(u, v)$, injective almost everywhere.

**Dimension 3:** **Rectangular:** $(x, y, z)$, or $(u, v, w)$, $(\varrho, \varphi, z)$, $(r, \theta, \varphi)$ or similarly in the *parametric domain.*

**Semi-polar:** $z = \varrho \cos\varphi$, $y = \varrho \sin\varphi$, $z = z$,

**Spherical:** $x = r \sin\theta \cos\varphi$, $y = r \sin\theta \sin\varphi$, $z = r \cos\theta$.

**General:** $x = X(u, v, w)$, $y = Y(u, v, w)$, $z = Z(u, v, w)$, injective almost everywhere.

These are our building stones in the ongoing geometrical analysis.

**Remark 22.1** If the example does not contain any hint, then choose among the *rectangular, polar, semi-polar* or *spherical* coordinates. On the other hand, if more general coordinates are needed in an exercise, then these will always be given, usually in their *inverse* form:

dimension 2 : $\quad u = U(x,y), \qquad v = V(x,y),$

dimension 3 : $\quad u = U(x,y,z), \quad v = V(x,y,z), \quad w = W(x,y,z).$

If they are given in their inverse form, we start by solving them with respect to $x$, $y$ (and $z$), cf. Chapter 17. $\Diamond$

**Solution strategy:**
Choose the coordinates, such that

**a)** the geometry of the *parametric domain* becomes simple,

**b)** the *integrand* becomes simple.

This is an order of priority, so the *geometry* is most important in a), while the *measure theory* is dominating in b). Both cases can be found in elementary textbooks, though most of the examples are of the type given in a).

In the following we set up an overview guided by the dimension of the classical coordinates. In each case the structure will be given by:

**i) Formula**, where the right hand side is the reduced expression.

**ii) Geometry**, where the domain is compared with the parametric domain.

**iii) Measure theory**, which briefly describes the weight function.

**iv) Possible comments.**

It is seen that we are aiming at the reduction of a given abstract integral to a *rectangular* integration over a convenient *parametric domain.*

**Dimension 1.**

Characteristics: There is only one variable of integration $t$.

We have two cases, **a) An ordinary integral** and **b) A line integral**.

**a) The ordinary integral.**

| | |
|---|---|
| Formula: | $\int_a^b f(t)\,\mathrm{d}t$. |
| Geometry: | The domain of integration $=$ the parametric interval. |
| Measure theory: | Weight $= 1$. |
| Comment: | Basic form known from high school and previous courses in Calculus. |

**b) The line integral.**

Formula: $\int_{\mathcal{K}} f(x,y,z)\,\mathrm{d}s = \int_a^b f(\mathbf{r}(t))\,\|\mathbf{r}'(t)\|\,\mathrm{d}t$, where $\mathcal{K}:\ (x,y,z) = \mathbf{r}(t)$.

Geometry: Curve $\neq$ parametric interval.

Measure theory: Weight $= \|\mathbf{r}'(t)\|$.

Comments: By the transform given above b) is transferred back to the basic form a). Note that there is a *hidden square root* in the weight function, so pocket calculators cannot always be successfully applied. Note that the curve is embedded in the *space* $\mathbb{R}^3$.

**Dimension 2.**

Characteristics: There are *two* variables of integration, e.g. $(u,v)$.
Here we have four cases:

**a) Rectangular plane integral.**

**b) Polar plane integral.**

**c) General transform.**

**d) Surface integral.**

These cases are treated one by one in the following.

**a) Rectangular plane integral.**

Formula: $\int_A f(x,y)\,\mathrm{d}x\,\mathrm{d}y$.

Geometry: domain of integration $=$ parametric domain.

Measure theory: Weight $= 1$.

Comment: Basic form.

**b) Polar plane integral.**

Formula: $\int_A f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_B f(\varrho\cos\varphi, \varrho\sin\varphi)\,\varrho\,\mathrm{d}\varrho\,\mathrm{d}\varphi$.

Geometry: domain of integration $\neq$ parametric domain.

Measure theory: Weight $= \varrho$.

Comment: Note that b) is reduced to a).

**c) General.**

Formula: $\int_A f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_B f(x(u,v),y(u,v)) \left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\,\mathrm{d}v.$

Geometry: domain of integration $\neq$ parametric domain.

Measure theory: Weight $= \left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|.$

**d) Surface integral.**

Formula: $\int_A f(x,y,z)\,\mathrm{d}S = \int_B f(\mathbf{r}(u,v))\,\|\mathbf{N}(u,v)\|\,\mathrm{d}u\,\mathrm{d}v,$

where the surface $\mathcal{F}$ is given by (x,y,z) = $\mathbf{r}$(u,v).

Geometry: Surface $\neq$ parametric domain.

Measure theory: Weight $= \|\mathbf{N}(u,v)\|.$

Comment: Note that for *rotational surfaces* we usually apply the *semi-polar* coordinates with the weight function $\varrho$, where $\varrho$ is a function of one parameter $t$.
Note altso that the surface is embedded in $\mathbb{R}^3$.

**Dimension 3.**

Characteristics: There are *three* variables of integration, e.g. $(u,v,w)$.

Here we have five cases, of which only four usually are treated in elementary courses in Calculus.

**a) Rectangular space integral.**

Formula: $\int_A f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$

Geometry: domain of integration $=$ parametric domain.

Measure theory: Weight $= 1.$

**b) Semi-polar space integral.**

Formula: $\int_A f(x,y,z)\,\mathrm{d}\Omega = \int_B f(\varrho\cos\varphi,\varrho\sin\varphi,z)\,\varrho\,\mathrm{d}\varrho\,\mathrm{d}\varphi\,\mathrm{d}z.$

Geometry: domain of integration $\neq$ parametric domain.

Measure theory: Weight $= \varrho.$

Comment: The method is in particular applied on
rotational bodies.

**c) Spherical space integral.**

Formula: $\int_A f(x,y,z)\,\mathrm{d}\Omega = \int_B f(r\,\sin\theta\cos\varphi, r\,\sin\theta\sin\varphi, r\,\cos\theta)\,r^2\sin\theta\;\mathrm{d}r\,\mathrm{d}\theta\,d\varphi.$

Geometry: domain of integration $\neq$ parametric domain.

Measure theory: Weight $= r^2\sin\theta$.

Comment: The method is usually applied on *spherical shells*.

**d) General.**

Formula:

$$\int_A f(x,y,z)\,\mathrm{d}\Omega = \int_B f(x(u,v,w),y(u,v,w),z(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w.$$

Geometry: domain of integration $\neq$ parametric domain.

Measure theory: Weight $= \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|$.

e) **Curved space integral.**

One may come across integration over a curved 3-dimensional space in the Theory of Relativity (embedded in $\mathbb{R}^4$) in Physics. This is of course analogous to the surface integral embedded in $\mathbb{R}^3$. Usually it does not occur in elementary textbooks of Calculus.

## 22.3 Reduction theorems in rectangular coordinates

The idea of the reduction theorems is to reduce an abstract (green) space integral to an abstract (blue) plane integral and an ordinary one dimensional integral (either in black or in red). Then the abstract (blue plane integral is reduced further by the methods already given in Chapter 20, so whenever convenient, we may even consider a triple integral.

There are two variants of the reduction theorem for space integrals in rectangular coordinates.

1) Either the order of integration is first (e.g.) *vertically* with respect to $z$ (the inner integral), and then the outer integral is an abstract (blue) plane integral.

2) Or we start with (the inner integration) an abstract (blue) plane integration for each fixed $z$, and then integrate (black) with respect to $z$.

vs

**Theorem 22.1** First reduction theorem for the space integral in rectangular coordinates.
*Let $A \subset \mathbb{R}^3$, and let $f : A \to \mathbb{R}$ be a continuous function. We assume that there is a bounded and closed set $B \subset \mathbb{R}^2$ and two functions*

$$Z_1,\, Z_2 \in C^0(B) \qquad \textit{and} \qquad Z_1(x,y) < Z_2(x,y) \quad \textit{for } (x,y) \in B^\circ .$$

*Then the abstract space integral is reduced in the following way,*

$$\int_A f(x,y,z)\, d\Omega = \int_B \left\{ \int_{Z_1(x,y)}^{Z_2(x,y)} f(x,y,z)\, dz \right\} dS.$$

**Theorem 22.2** Second reduction theorem for the space integral in rectangular coordinates.
*Let $A \subset \mathbb{R}^3$, and let $f : A \to \mathbb{R}$ be a continuous function. Assume that $A$ lies in the horizontal slab defined by $a \le z \le b$, and that for every fixed $z \in [a,b]$the set*

$$B(z) := \left\{ (x,y) \in \mathbb{R}^2 \mid (x,y,z) \in A \right\}, \qquad z \in [a,b],$$

*is bounded and closed. Then the abstract space integral is reduced in the following way,*

$$\int_A f(x,y,z)\, \mathrm{d}\Omega = \int_a^b \left\{ \int_{B(z)} f(x,y,z)\, \mathrm{d}S \right\} \mathrm{d}z.$$

We mention the following reduction theorem.

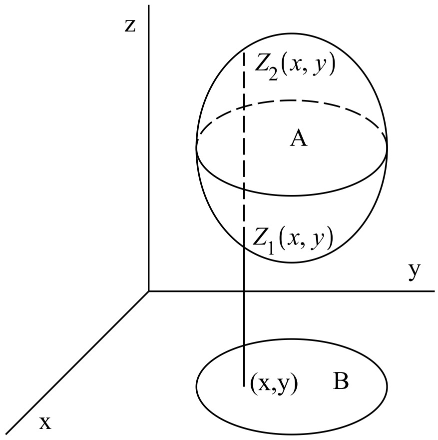



Figure 22.1: Illustration of Theorem 22.1. For fixed $(x, y) \in B^\circ$ the vertical line $\{(x, y, z) \mid z \in\}$ cuts $A$ in an interval $[Z_1(x, y), Z_2(x, y)]$, over which $f(x, y, z)$ is integrated with respect to $z$. Collect the result $F(x, y)$ as the value of the new function $F$, and then integrate $F(x, y)$ over $B$ as in Chapter 20.

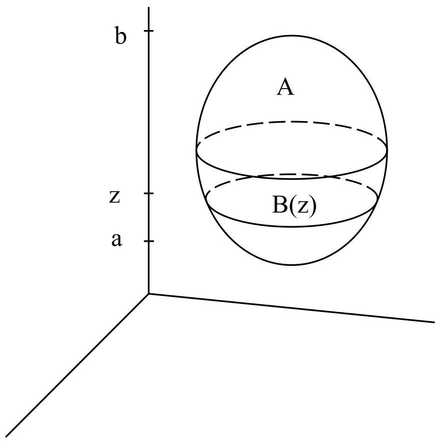



Figure 22.2: Illustration of Theorem 22.1. For fixed $z \in [a, b]$ we cut $A$ in a plane domain $B(z)$. First perform a plane integration over $B(z)$. This defines an ordinary function $F(z)$, which is then integrated over the interval $[a, b]$.

**Theorem 22.3** Reduction of a space integral as a triple integral *Let the closed and bounded domain $A$ have the following special structure,*

$$A = \left\{(x, y, z) \in \mathbb{R}^3 \mid a \le x \le b,\ Y_1(x) \le y \le Y_2(x),\ Z_1(x, y) \le z \le Z_2(x, y)\right\},$$

*where $Y_1$, $Y_2$, $Z_1$ and $Z_2$ are continuous functions in their respective domains. Then the abstract space integral is reduced in the following way as a triple integral,*

$$\int_A f(x, y, z)\, d\Omega = \int_a^b \left\{ \int_{Y_1(x)}^{Y_2(x)} \left\{ \int_{Z_1(x,y)}^{Z_2(x,y)} f(x, y, z)\, dz \right\} dy \right\} dx.$$

The bounds $Z_1$ and $Z_2$ of $z$ depend on both $x$ and $y$, while the bounds $Y_1$, $Y_2$ of $y$ only depend on $x$. In Theorem 22.3 we have used the colour code

green — blue — black — red

to illustrate the order of integration. We go backwards. First wi integrate with respect to the red variable, then with respect to the black one, and finally, with respect to the blue variable.

Note that the order of $x$, $y$, $z$ may be changed everywhere in the theorems above, causing only an interchange in letter.

## 22.4 Procedure for reduction of space integral in rectangular coordinates

We shall here more explicitly describe the procedure, when we reduce an abstract space integral in rectangular coordinates. The methods are similar to those given in Section 20.3. The only new is that the dimension 3 (a number) can be divided as a sum of integers in three different ways:

1) The method of vertical posts: $3 = 2 + 1$,

2) The method of cutting into slabs: $3 = 1 + 2$,

3) The triple integral: $3 = 1 + 1 + 1$.

These three cases are treated separately in the following.

**The method of posts** I this case it follows from a figure that one of the variables, e.g. $z$, lies between the graphs of two $C^0$ functions $Z_1(x, y)$ and $Z_2(x, y)$ in the other two variables $(x, y)$. Furthermore, these variables lie in a specified domain $B$ in the $(x, y)$ plane, $(x, y) \in B$. The graphs of the two functions are surfaces, which cut $A$ from the cylinder over $B$.

**Procedure.**

1) Write the set $A$ in the form

$$A = \{(x, y, z) \mid (x, y) \in B,\, Z_1(x, y) \le z \le Z_2(x, y)\} \subseteq \mathbb{R}^3.$$

We identify the set $B \subseteq \mathbb{R}^2$ in the $(x, y)$ plane and the functions $Z_1(x, y)$ and $Z_2(x, y)$. Then we set up the reduction formula

$$\int_A f(x, y, z)\, \mathrm{d}\Omega = \int_B \left\{ \int_{Z_1(x,y)}^{Z_2(x,y)} f(x, y,z)\, \mathrm{d}z \right\} \mathrm{d}S.$$

The colour code is the usual one. The green integral is the abstract space integral. The blue integral is the abstract plane integral of lower dimension, while the red and innermost integral is a usual integral, which can be calculated by elementary methods.

2) For fixed $(x, y) \in B$ we first integrate with respect to $z$, i.e. along a vertical post,

$$\varphi(x, y) := \int_{Z_1(x,y)}^{Z_2(x,y)} f(x, y,z)\, \mathrm{d}z,$$

where the right hand side is a usual integral.

3) The abstract space integral is then by insertion reduced to a simpler abstract plane integral,

$$\int_A f(x, y, z)\, \mathrm{d}\Omega = \int_B \varphi(x, y)\, \mathrm{d}S.$$

Finally, the right hand side is calculated by one of the methods given in Chapter 20; either rectangular or polar.

**The method of cutting $A$ into slabs.** This method is in particular applied, when the integrand only depends on one of the three variables, e.g. $z$, though it can also be applied in other cases. The idea is that $A$ at height $z$ is cut into a slab $B(z)$, so we first integrate with respect to $(x, y) \in B(z)$, and afterwards with respect to $z$. Roughly speaking, one first find the total “mass” in the slab $B(z)$, and then we collect all these “masses” by integrating with respect to the “parameter” $z$.

**Procedure.**

1) Write the set $A$ in the form

$$A = \{(x, y, z) \mid a \le z \le b, (x, y) \in B(z)\},$$

so we identify (e.g. by analyzing a figure) the cut (intersection of $A$ with the plane at height $z$), or slab $B(z)$ for every relevant $z$.

Then set up the reduction formula

$$\int_A f(x, y, z)\, d\Omega = \int_a^b \left\{ \int_{B(z)} f(x, y, z)\, dS \right\} dz.$$

Here the green integral is the abstract space integral. The inner blue integral is the abstract plane integral of lower dimension, and the outmost black integral is an ordinary integral.

2) For fixed $z \in [a, b]$ we calculate the abstract plane integral

$$\psi(z) := \int_{B(z)} f(x, y, z)\, dS$$

by one of the methods from Chapter 20, in either rectangular or polar coordinates.

3) By insertion of the result the abstract space integral is reduced to an ordinary integral in one variable,

$$\int_A f(x, y, z)\, d\Omega = \int_a^b \psi(z)\, dz.$$

Remark. If the integrand only depends on $z$, the 2) above is reduced to

$$\psi(z) := f(z) \cdot \text{area } B(z),$$

so 3) can be written

$$\int_A f(z)\, d\Omega = \int_a^b f(z) \cdot \text{area } B(z)\, dz,$$

where area $B(z)$ quite often can be found by an alternative simple geometrical argument.

**Triple integral.** This is a special case of the method of posts above, because we assume that the domain $B$ is also bounded by graphs of functions, this time in one variable.

**Procedure.**

1) Write the set $A$ in the form,

$$A = \{(x, y, z) \mid a \le x \le b, Y_1(x) \le y \le Y_2(x), Z_1(x, y) \le z \le Z_2(x, y)\}.$$

This is the most difficult point of this process, so one should always afterwards check if the bounds $Y_1(x)$, $Y_2(x)$, $Z_1(x, y)$, $Z_2(x, y)$, are right. Usually, errors occur at this step of the process.

2) Set up the reduction formuula

$$\int_A f(x,y,z)\,\mathrm{d}\Omega = \int_a^b \left\{ \int_{Y_1(x)}^{Y_2(x)} \left\{ \int_{Z_1(x,y)}^{Z_2(x,y)} f(x,y,z)\,\mathrm{d}z \right\} \mathrm{d}y \right\} \mathrm{d}x.$$

3) For fixed $(x, y)$ we calculate the innermost integral,

$$f(x,y) = \int_{Z_1(x,y)}^{Z_2(x,y)} f(x,y,z)\,\mathrm{d}z.$$

After this calculation, the black $z$ should have disappeared (check!) If not, we have made an error.

4) After insertion we calculate for fixed $x$ the middle integral,

$$h(x) := \int_{Y_1(x)}^{Y_2(x)} g(x,y)\,\mathrm{d}y.$$

Check, that the red variable $y$ has disappeared.

5) Finally, insert the result and calculate the outer integral,

$$\int_A f(x,y,z)\,\mathrm{d}\Omega = \int_a^b h(x)\,\mathrm{d}x.$$

## 22.5 Examples of space integrals in rectangular coordinates

**A.** *Calculate the space space integral,*

$$I = \int_A (3+y-z)\,x\,\mathrm{d}\Omega,$$

*where*

$$A = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in B,\, 0 \le z \le 2y\},$$

*and $B$ is the upper triangle on the figure.*

**D** Apply Theorem 22.1.

It follows that

$$(22.1)\quad I = \int_A (3+y-z)x\,\mathrm{d}\Omega = \int_B \left\{ \int_0^{2y} (3+y-z)x\,\mathrm{d}z \right\} \mathrm{d}S.$$

In the inner integral, $x$ and $y$ are considered as constants, so

$$\begin{aligned} \int_0^{2y} (3+y-z)x\,\mathrm{d}z &= x\int_0^{2y} (3+y-z)\,\mathrm{d}z = x\left[(3+y)z - \frac{1}{2}z^2\right]_{z=0}^{2y} \\ &= x\left\{(3+y)\cdot 2y - 2y^2\right\} = x\cdot 2y\{3+y-y\} = 6xy. \end{aligned}$$

By insertion into (22.1), we reduce to an abstract plane integral over $B$, i.e. of lower dimension,

$$I = \int_B 6xy\,\mathrm{d}S = 6\int_B xy\,\mathrm{d}S.$$

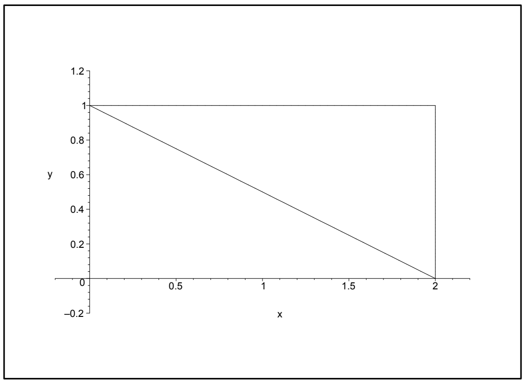



Figure 22.3: The domain $B$, i.e. the perpendicular projection of the body $A$ onto the $(x, y)$ plane.

We have previously in Section 20.4 found that

$$\int_B xy \, \mathrm{d}S = \frac{5}{6},$$

so

$$I = \int_A (3 + y - z)x \, \mathrm{d}\Omega = 6 \int_B xy \, \mathrm{d}S = 6 \cdot \frac{5}{6} = 5.$$

**A.** *Calculate the space integral*

$$\int_A (x+2y+z)\exp\left(z^4\right)\, d\Omega,$$

*where*

$$A = \{(x,y,z) \mid z \in [2,0],\ (x,y) \in B(z)\}$$

*and where the cut at the height $z$ is given by*

$$B(z) = [0,z] \times \left[0, \frac{z}{2}\right], \qquad z \in \,]0,2].$$

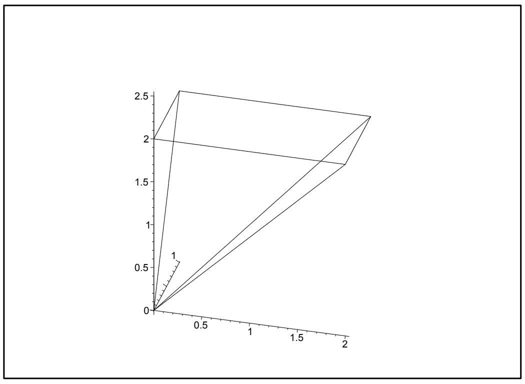



Figure 22.4: Legemet $A$.

**D.** Apply Theorem 22.2.

**I.** We get by insertion into Theorem 22.2,

$$\text{(22.2)}\quad I = \int_A (x+2y+z)\exp\left(z^4\right)\, \mathrm{d}\Omega = \int_0^2 \exp\left(z^4\right)\left\{\int_{B(z)} (x+2y+z)\,\mathrm{d}S\right\}\mathrm{d}z.$$

Then we reduce for every fixed $z$ the innermost *abstract* plane integral,

$$\begin{aligned}\int_{B(z)} (x+2y+z)\,\mathrm{d}S &= \int_{B(z)} x\,\mathrm{d}S + \int_{B(z)} 2y\,\mathrm{d}S + z\int_{B(z)} \mathrm{d}S \\ &= \int_0^z x\,\mathrm{d}x \cdot \int_0^{\frac{z}{2}} \mathrm{d}y + \int_0^z \mathrm{d}x \cdot \int_0^{\frac{z}{2}} 2y\,\mathrm{d}y + z\cdot \mathrm{area}B(z) \\ &= \frac{z^2}{2}\cdot\frac{z}{2} + z\cdot\frac{z^2}{4} + z\cdot\left\{z\cdot\frac{1}{2}\,z\right\} = z^3.\end{aligned}$$

We insert this result into (22.2) and apply the substitution

$$t = z^4, \qquad \mathrm{d}t = 4z^3\,\mathrm{d}z, \quad \text{i.e.} \quad z^3\,\mathrm{d}z = \frac{1}{4}\,\mathrm{d}t,$$

to get the result

$$I = \int_0^2 \exp\left(z^4\right)\cdot z^3\,dz = \int_0^{2^4} e^t\cdot\frac{1}{4}\,dt = \frac{1}{4}\left(e^{16}-1\right).$$

**Example 22.1** *Calculate in each of the following cases the given space integral over a point set*

$$A = \{(x,y,z) \mid (x,y) \in B, \quad Z_1(x,y) \le z \le Z_2(x,y)\}.$$

1) *The space integral* $\int_A xy^2 z \, d\Omega$*, where the plane point set* $B$ *is given by* $x \ge 0$, $y \ge 0$ *and* $x+y \le 1$*, and where* $Z_1(x,y) = 0$ *and* $Z_2(x,y) = 2 - x - y$*.*

2) *The space integral* $\int_A xy^2 z^3 \, d\Omega$*, where the plane point set* $B$ *is given by* $0 \le x \le y \le 1$*, and where* $Z_1(x,y) = 0$ *and* $Z_2(x,y) = xy$*.*

3) *The space integral* $\int_A z \, d\Omega$*, where the plane point set* $B$ *is given by* $0 \le x \le 6$ *and* $2-x \le y \le 3-\dfrac{x}{2}$*, and where* $Z_1(x,y) = 0$ *and* $Z_2 = \sqrt{16-y^2}$*.*

4) *The space integral* $\int_A y \, d\Omega$*, where the plane point set* $B$ *is given by* $-2 \le y \le 1$ *and* $y^2 \le x \le 2-y$*, and where* $Z_1(x,y) = 0$ *and* $Z_2(x,y) = 4 - 2x - 2y$*.*

5) *The space integral* $\int_A \dfrac{1}{x^2y^2z^2} \, d\Omega$*, where the plane point set* $B$ *is given by* $1 \le x \le \sqrt{3}$ *and* $\dfrac{1}{1+x^2} \le y \le 1$*, and where* $Z_1(x,y) = \dfrac{1}{1+x^2}$ *and* $Z_2(x,y) = 1 + x^2$*.*

6) *The space integral* $\int_A yz \, d\Omega$*, where the plane point set* $B$ *is given by* $0 \le x \le 1$ *and* $0 \le y \le x$*, and where* $Z_1(x,y) = 0$ *and* $Z_2(x,y) = 2 - 2x$*.*

   [Cf. **Example 22.2.6**.]

7) *The space integral* $\int_A xz \, d\Omega$*, where the plane point set* $B$ *is given by* $0 \le x \le 1$ *and* $0 \le y \le 1$*, and where* $Z_1(x,y) = 0$ *and* $Z_2(x,y) = 1 - y$*.*

   [Cf. **Example 22.2.7**.]

8) *The space integral* $\int_A z \, d\Omega$*, where the plane point set* $B$ *is given by* $\sqrt{x^2+y^2} \le 2$*, and where* $Z_1(x,y) = 0$ *and* $Z_2(x,y) = 2 - \sqrt{x^2+y^2}$*.*

   [Cf. **Example 22.2.8**]

**A** Space integral in rectangular coordinates.

**D** Apply the first theorem of reduction.

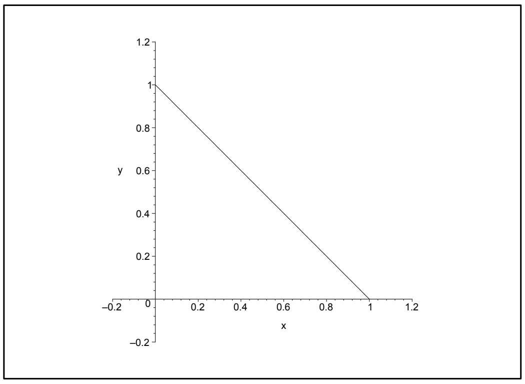



Figure 22.5: The domain $B$ of **Example 22.1.1**.

**I** 1) By the first theorem of reduction,

$$\begin{aligned}
\int_A xy^2 z\,\mathrm{d}\Omega &= \int_B xy^2\left\{\int_0^{2-x-y} z\,\mathrm{d}z\right\}\mathrm{d}S = \frac{1}{2}\int_B x^2y(2-x-y)^2\,\mathrm{d}S \\
&= \frac{1}{2}\int_B x^2y\left\{(2-x)^2-2(2-x)y+y^2\right\}\,\mathrm{d}S \\
&= \frac{1}{2}\int_0^1 x^2\left\{\int_0^{1-x}\left[(2-x)^2y-2(2-x)y^2+y^3\right]\,\mathrm{d}y\right\}\mathrm{d}x \\
&= \frac{1}{2}\int_0^1 x^2\left[\frac{1}{2}(2-x)^2y^2-\frac{2}{3}(2-x)y^3+\frac{1}{4}y^4\right]_{y=0}^{1-x}\mathrm{d}x \\
&= \frac{1}{24}\int_0^1 x^2\left\{6(2-x)^2(1-x)^2-8(2-x)(1-x)^3+3(1-x)^4\right\}\,\mathrm{d}x \\
&= \frac{1}{24}\int_0^1 x^2(1-x)^2\left\{6(4-4x+x^2)-8(2-3x+x^2)+3(1-2x+x^2)\right\}\,\mathrm{d}x \\
&= \frac{1}{24}\int_0^1 (x^4-2x^3+x^2)(x^2-6x+11)\,\mathrm{d}x \\
&= \frac{1}{24}\int_0^1 \left\{x^6-8x^5+24x^4-28x^3+11x^2\right\}\,\mathrm{d}x \\
&= \frac{1}{24}\left[\frac{1}{7}x^7-\frac{4}{3}x^6+\frac{24}{5}x^5-7x^4+\frac{11}{3}x^3\right]_0^1 \\
&= \frac{1}{24}\left(\frac{1}{7}-\frac{4}{3}+\frac{24}{5}-7+\frac{11}{3}\right) = \frac{1}{24}\left(\frac{1}{7}+2+\frac{1}{3}+5-\frac{1}{5}-7\right) \\
&= \frac{1}{24}\left(\frac{1}{3}-\frac{1}{5}+\frac{1}{7}\right) = \frac{1}{24}\left(\frac{2}{15}+\frac{1}{7}\right) = \frac{29}{24\cdot 105} = \frac{29}{2520}.
\end{aligned}$$

MAPLE. This is of course very easy for MAPLE. We use the commands,

with(Student[MultivariateCalculus]):

$$\frac{1}{2}\cdot \text{MultiInt}\left(x^2\cdot y\cdot\left((2-x)^2-2(2-x)\cdot y+y^2\right), y=0..1-x, x=0..1\right)$$

$$\frac{29}{2520}$$

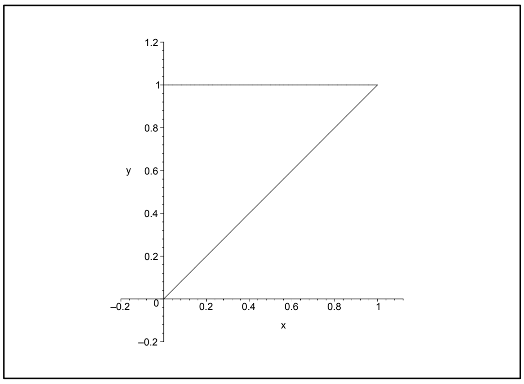



Figure 22.6: The domain $B$ of **Eksempel 22.1.2**.

2) By the first theorem of reduction,

$$\begin{aligned}\int_A xy^2z^3\,\mathrm{d}\Omega &= \int_B xy^2\left\{\int_0^{xy} z^3\,\mathrm{d}z\right\}\mathrm{d}S = \frac{1}{4}\int_B xy\left[z^4\right]_{z=0}^{xy}\mathrm{d}S = \frac{1}{4}\int_B x^5y^6\,\mathrm{d}S\\ &= \frac{1}{4}\int_0^1 y^6\left\{\int_0^y x^5\,\mathrm{d}x\right\}\mathrm{d}y = \frac{1}{24}\int_0^1 y^{12}\,dy = \frac{1}{24\cdot 13} = \frac{1}{312}.\end{aligned}$$

MAPLE. This is of course very easy for MAPLE. We use the commands,

with(Student[MultivariateCalculus]):

$$\frac{1}{4}\cdot \text{MultiInt}\left(x^5\cdot y^6, x=0..y, y=0..1\right)$$

$$\frac{1}{312}$$

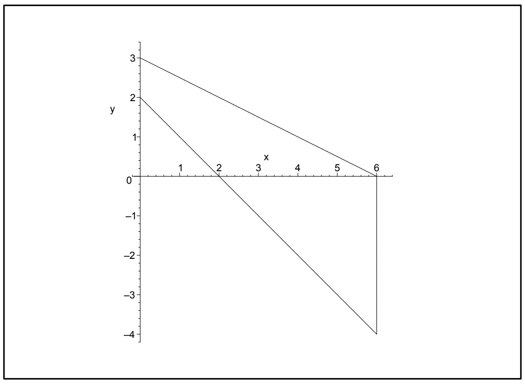



Figure 22.7: The domain $B$ of **Example 22.1.3**.

3) By the theorem of reduction,

$$\begin{aligned}\int_A z\,\mathrm{d}\Omega &= \int_B\left\{\int_0^{\sqrt{16-y^2}} z\,\mathrm{d}z\right\}\mathrm{d}S = \frac{1}{2}\int_B (16-y^2)\,\mathrm{d}S\\ &= \frac{1}{2}\int_0^6\left\{\int_{2-x}^{3-\frac{x}{2}}(16-y^2)\,\mathrm{d}y\right\}\mathrm{d}x = \frac{1}{2}\int_0^6\left[16y-\frac{1}{3}y^3\right]_{y=2-x}^{3-\frac{x}{2}}\mathrm{d}x\\ &= \frac{1}{2}\int_0^6\left\{16\left(3-\frac{x}{2}\right)-\frac{1}{3}\left(3-\frac{x}{2}\right)^3-16(2-x)+\frac{1}{3}(2-x)^3\right\}\mathrm{d}x\\ &= \frac{1}{2}\int_0^6\left\{16+8x+\frac{1}{24}(x-6)^2-\frac{1}{3}(x-2)^3\right\}\mathrm{d}x\\ &= \frac{1}{2}\left[16x+4x^2+\frac{1}{96}(x-6)^4-\frac{1}{12}(x-2)^4\right]_0^6\\ &= \frac{1}{2}\left\{96+144+0-\frac{4^4}{12}-\frac{6^4}{96}+\frac{2^4}{12}\right\} = \frac{1}{2}\left\{240-\frac{64}{3}-\frac{216}{16}+\frac{4}{3}\right\}\\ &= \frac{1}{2}\left\{220-\frac{27}{2}\right\} = \frac{413}{4}.\end{aligned}$$

MAPLE. This is very easy for MAPLE. We use the commands,

with(Student[MultivariateCalculus]):

$$\frac{1}{2} \cdot \text{MultiInt}\left(16 - y^2, y = 2 - x..3 - 0.5x, x = 0..6\right)$$

$$\frac{413}{4}$$

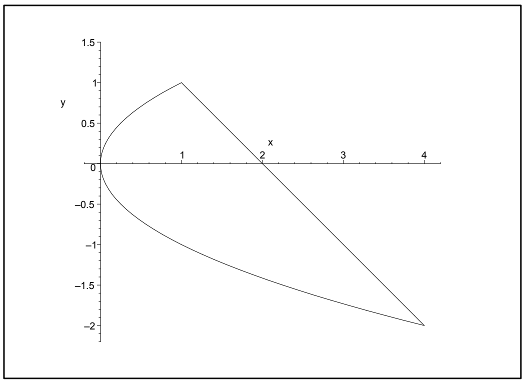



Figure 22.8: The domain $B$ of **Example 22.1.4**.

4) By the theorem of reduction,

$$\begin{aligned}
\int_A y \, d\Omega &= \int_B y \left\{ \int_0^{4-2x-2y} dz \right\} dS = \int_B y(4-2x-2y)\, dS \\
&= \int_{-2}^1 y \left\{ \int_{y^2}^{2-y} (4-2x-2y)\, dx \right\} dy = \int_{-2}^1 y \left[ 4x - x^2 - 2xy \right]_{x=y^2}^{2-y} dy \\
&= \int_{-2}^1 y \left\{ 4(2-y) - (2-y)^2 - 2y(2-y) - 4y^2 + y^4 + 2y^3 \right\} dy \\
&= \int_{-2}^1 y \left\{ 8 - 4y - 4 + 4y - y^2 - 4y + 2y^2 - 4y^2 + 2y^3 + y^4 \right\} dy \\
&= \int_{-2}^1 (y^5 + 2y^4 - 3y^3 - 4y^2 + 4y)\, dy = \left[ \frac{1}{6} y^6 + \frac{2}{5} y^5 - \frac{3}{4} y^4 - \frac{4}{3} y^3 + 2y^2 \right]_{-2}^1 \\
&= \frac{1}{6} + \frac{2}{5} - \frac{3}{4} - \frac{4}{3} + 2 - \frac{2^6}{6} + \frac{2}{5} \cdot 2^5 + \frac{3}{4} \cdot 2^4 - \frac{4}{3} \cdot 2^3 - 8 = -\frac{81}{20}.
\end{aligned}$$

MAPLE. This is very easy for MAPLE. We use the commands,

with(Student[MultivariateCalculus]):

$$\text{MultiInt}\left(y \cdot (4 - 2x - 2y), x = y^2..2 - y, y = -2..1\right)$$

$$-\frac{81}{20}$$

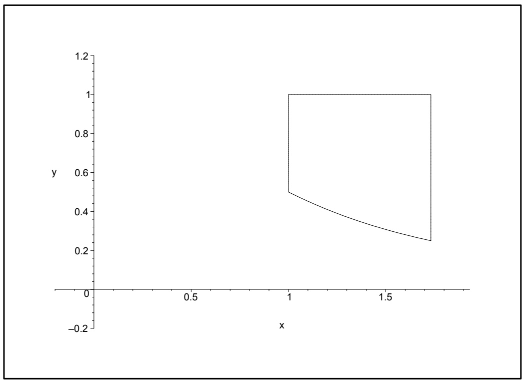



Figure 22.9: The domain $B$ of **Example 22.1.5**.

5) First note that the $z$-integral does not depend on $y$. When we exploit this observation we get by the theorem of reduction,

$$\begin{aligned}
\int_A \frac{1}{x^2y^2z^2}\,d\Omega &= \int_1^{\sqrt{3}} \frac{1}{x^2}\left\{\int_{\frac{1}{1+x^2}}^{1} \frac{1}{y^2}\left(\int_{\frac{1}{1+x^2}}^{1+x^2} \frac{1}{z^2}\,dz\right)dy\right\}dx \\
&= \int_1^{\sqrt{3}} \frac{1}{x^2}\left[-\frac{1}{y}\right]_{\frac{1}{1+x^2}}^{1} \cdot \left[-\frac{1}{z}\right]_{\frac{1}{1+x^2}}^{1+x^2} dx \\
&= \int_1^{\sqrt{3}} \frac{1}{x^2}\left(1+x^2-1\right)\cdot\left(1+x^2-\frac{1}{1+x^2}\right)dx \\
&= \int_1^{\sqrt{3}} \left(1+x^2-\frac{1}{1+x^2}\right)dx = \left[\frac{x^3}{3}+x-\ \text{Arctan}\ x\right]_1^{\sqrt{3}} \\
&= \frac{3\sqrt{3}}{3}+\sqrt{3}-\ \text{Arctan}\ \sqrt{3}-\frac{1}{3}-1+\ \text{Arctan}\ 1 \\
&= 2\sqrt{3}-\frac{4}{3}-\frac{\pi}{3}+\frac{\pi}{4} = 2\sqrt{3}-\frac{4}{3}-\frac{\pi}{12}.
\end{aligned}$$

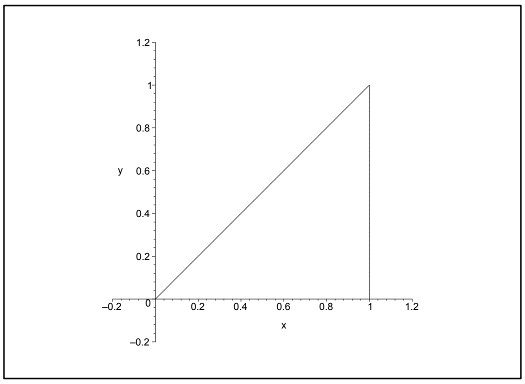



Figure 22.10: The domain $B$ of **Example 22.1.6**.

MAPLE. This shows that MAPLE can also handle triple integrals. We use the commands,

with(Student[MultivariateCalculus]):

$$\frac{1}{2}\cdot \text{MultiInt}\left(\frac{1}{x^2\cdot y^2\cdot z^2}, z=\frac{1}{1+x^2}..1+x^2, y=\frac{1}{1+x^2}..1, x=0..\sqrt{3}\right)$$

$$-\frac{1}{12}\pi-\frac{4}{3}+2\sqrt{3}$$

6) By the theorem of reduction,

$$\begin{aligned}\int_A yz\,\mathrm{d}\Omega &= \int_0^1\left\{\int_0^x y\left(\int_0^{2-2x} z\,\mathrm{d}z\right)\mathrm{d}y\right\}\mathrm{d}x = \int_0^1\left[\frac{y^2}{2}\right]_0^x\cdot\left[\frac{z^2}{2}\right]_0^{2-2x}\mathrm{d}x\\
&= \frac{1}{4}\int_0^1 x^2\cdot(2-2x)^2\,\mathrm{d}x = \int_0^1 x^2(1-x^2)\,\mathrm{d}x = \int_0^1 x^2(x^2-2x+1)\,\mathrm{d}x\\
&= \int_0^1 (x^4-2x^3+x^2)\,\mathrm{d}x = \frac{1}{5}-\frac{2}{4}+\frac{1}{3} = \frac{6-15+10}{30} = \frac{1}{30}.\end{aligned}$$

MAPLE. This is easy for MAPLE. We use the commands,

with(Student[MultivariateCalculus]):

MultiInt $(y \cdot z, z = 0..2 - 2x, y = 0..x, x = 0..1)$

$$\frac{1}{30}$$

REMARK. The domain is also described by

$$0 \le z \le 2, \qquad 0 \le y \le x \le 1 - \frac{z}{2},$$

cf. **Example 22.2.6**. The two examples therefore give the same result. $\Diamond$

7) Here, $B = [0,1] \times [0,1]$, It therefore follows by the theorem of reduction that

$$\int_A xz\,d\Omega = \int_0^1 x\,dx \cdot \int_0^1 \left\{ \int_0^{1-y} z\,dz \right\} dy = \frac{1}{2}\int_0^1 \frac{1}{2}(1-y)^2\,dy = \frac{1}{4}\int_0^1 t^2\,dt = \frac{1}{12}.$$

8) Here, $B$ is the closed disc of centrum $(0,0)$ and radius 2. By using the theorem of reduction in semi-polar coordinates,

$$\begin{aligned}\int_A z\,d\Omega &= 2\pi \int_0^2 \left\{ \int_0^{2-\varrho} z\,dz \right\} \varrho\,d\varrho = \pi \int_0^2 (2-\varrho)^2 \varrho\,d\varrho = \pi \int_0^2 (\varrho^3 - 4\varrho^2 + 4\varrho)\,d\varrho \\ &= \pi \left[ \frac{\varrho^4}{4} - \frac{4}{3}\varrho^3 + 2\varrho^2 \right]_0^2 = \pi \left\{ 4 - \frac{32}{3} + 8 \right\} = \frac{4\pi}{3}.\end{aligned}$$

**Example 22.2** *Calculate in each of the following cases the given space integral over a point set*

$$A = \{(x,y,z) \mid \alpha \le z \le \beta, \quad (x,y) \in B(z)\}.$$

1) *The space integral* $\int_A z^2 \, d\Omega$, *where* $B(z)$ *is given by* $|x| \le z$ *and* $|y| \le 2z$ *for* $z \in [0,1]$.

2) *The space integral* $\int_A xz \, d\Omega$, *where* $B(z)$ *is given by* $0 \le x$, $0 \le y$ *and* $x+y \le z^2$ *for* $z \in [0,1]$.

3) *The space integral* $\int_A xy^2 z \, d\Omega$, *where* $B(z)$ *is given by* $0 \le x$, $0 \le y$ *and* $x+y \le 2-z$ *for* $z \in [1,2]$.

4) *The space integral* $\int_A \frac{1}{xy^2} \, d\Omega$, *where* $B(z)$ *is given by* $1 \le x \le z$ *and* $z \le y \le z$ *for* $z \in [1,3]$.

5) *The space integral* $\int_A \left(\frac{\sin z}{z}\right)^2 d\Omega$, *where* $B(z)$ *is given by* $|x|+|y| \le |z|$ *for* $z \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

6) *The space integral* $\int_A yz \, d\Omega$, *where* $B(z)$ *is given by* $0 \le y \le x \le 1 - \frac{z}{2}$ *for* $z \in [0,2]$.
[Cf. **Example 22.1.6**.]

7) *The space integral* $\int_A xz \, d\Omega$, *where* $B(z)$ *is given by* $0 \le x \le 1$ *and* $0 \le y \le 1-z$ *for* $z \in [0,1]$.
[Cf. **Example 22.1.7**.]

8) *The space integral* $\int_A z \, d\Omega$, *where* $B(z)$ *is given by* $x^2+y^2 \le (2-z)^2$ *for* $z \in [0,2]$.
[Cf. **Example 22.1.8**.]

**A** Space integrals in rectangular coordinates, where the domain is sliced, $B(z)$, with the height $z$ as parameter.

**D** Whenever it is necessary, sketch $B(z)$. Then apply the second theorem of reduction.

**I** 1) Here,

$$B(z) = \{(x,y) \mid -z \le x \le z, \, -2z \le y \le 2z\} = [-z,z] \times [-2z,2z],$$

which is a rectangle for every $z \in ]0,1]$ of the area

$$\text{area}\{B(z)\} = 8z^2.$$

We get by reduction,

$$\int_A z^2 \, d\Omega = \int_0^1 z^2 \left\{ \int_{B(z)} dS \right\} dz = \int_0^1 z^2 \cdot \text{area}\{B(z)\} \, dz = \int_0^1 8z^4 \, dz = \frac{8}{5}.$$

2) Here

$$B(z) = \{(x,t) \mid 0 \le x, \, 0 \le y, \, x+y \le z^2\}$$

is a triangle for every $z \in ]0,1]$, namely the lower triangle of the square $[0,z^2] \times [0,z^2]$, when this is cut by a diagonal from the upper left corner to the lower right corner.
We get by the theorem of reduction,

$$\begin{aligned}\int_A xz \, d\Omega &= \int_0^1 z \left\{ \int_{B(z)} x \, dS \right\} dz = \int_0^1 z \left\{ \int_0^{z^2} x \left[ \int_0^{z^2-x} dy \right] dx \right\} dz \\ &= \int_0^1 z \left\{ \int_0^{z^2} (xz^2 - x^2) dx \right\} dz = \int_0^1 z \left[ \frac{1}{2} x^2 z^2 - \frac{1}{3} x^3 \right]_0^{z^2} dz = \int_0^1 \frac{1}{6} z^7 dz = \frac{1}{48}.\end{aligned}$$

MAPLE. This is easy for MAPLE. We use the commands,

with(Student[MultivariateCalculus]):

$$\text{MultiInt}\left(x\cdot z, y=0..z^2-x, x=0..z^2, z=0..1\right)$$

$$\frac{1}{48}$$

3) Here

$$B(z)=\{(x,y)\mid 0\le x,\ 0\le y,\ x+y\le 2-z\}$$

is a triangle for every $z\in[1.2[$, namely the lower triangle of the square $[0,2-z]\times[0,2-z]$, when this is cut by a diagonal from the upper left corner to the lower right corner.
Then by the theorem of reduction,

$$\begin{aligned}\int_A xy^2z\,\mathrm{d}\Omega &= \int_1^2 z\left\{\int_{B(z)} xy^2\,\mathrm{d}S\right\}\mathrm{d}z=\int_1^2 z\left\{\int_0^{2-z} y^2\left[\int_0^{2-z-y} x\,\mathrm{d}x\right]\mathrm{d}y\right\}\mathrm{d}z\\
&= \frac{1}{2}\int_1^2 z\left\{\int_0^{2-z} y^2(z-2+y)^2\,\mathrm{d}y\right\}\mathrm{d}z\\
&= \frac{1}{2}\int_1^2 \{(z-2)+2\}\left\{\int_0^{2-z}\left[(z-2)^2y^2+2(z-2)y^3+y^4\right]\mathrm{d}y\right\}\mathrm{d}z\\
&= \frac{1}{2}\int_1^2 \{(z-2)+2\}\left[\frac{1}{3}(z-2)^2y^3+\frac{1}{2}(z-2)y^4+\frac{1}{5}y^5\right]_{y=0}^{2-z}\mathrm{d}z\\
&= \frac{1}{2}\int_1^2 \{(z-2)+2)\}\cdot\left(-\frac{1}{3}+\frac{1}{2}-\frac{1}{5}\right)(z-2)^5\,\mathrm{d}z\\
&= -\frac{1}{60}\int_1^2\left\{(z-2)^6+2(z-2)^5\right\}\mathrm{d}z\\
&= -\frac{1}{60}\left[\frac{1}{7}(z-2)^7+\frac{1}{3}(z-2)^6\right]_1^2=-\frac{1}{60}\left(\frac{1}{7}-\frac{1}{3}\right)=\frac{4}{3\cdot 7\cdot 60}=\frac{1}{315}.\end{aligned}$$

MAPLE. This is easy for MAPLE. We use the commands,

with(Student[MultivariateCalculus]):

$$\text{MultiInt}\left(y\cdot y^2\cdot z, x=0..2-z-y, y=0..2-z, z=1..2\right)$$

$$\frac{1}{315}$$

4) Here

$$B(z)=\{(x,y)\mid 1\le x\le z,\ z\le y\le 2z\},\qquad z\in[1,3],$$

which is sketched on the figure.
We get by the theorem of reduction,

$$\begin{aligned}\int_A \frac{1}{xy^2}\,\mathrm{d}\Omega &= \int_1^3\left\{\int_{B(z)}\frac{1}{xy^2}\,\mathrm{d}S\right\}\mathrm{d}z=\int_1^3\left\{\int_1^z\frac{1}{x}\left[\int_z^{2z}\frac{1}{y^2}\,\mathrm{d}y\right]\mathrm{d}x\right\}\mathrm{d}z\\
&= \int_1^3\left\{\int_1^z\frac{1}{x}\left[-\frac{1}{y}\right]_z^{2z}\mathrm{d}x\right\}\mathrm{d}z=\int_1^3\frac{1}{2z}\left[\ln x\right]_{x=1}^z\,\mathrm{d}z=\frac{1}{4}\left[(\ln z)^2\right]_1^3=\frac{1}{4}(\ln 3)^2.\end{aligned}$$

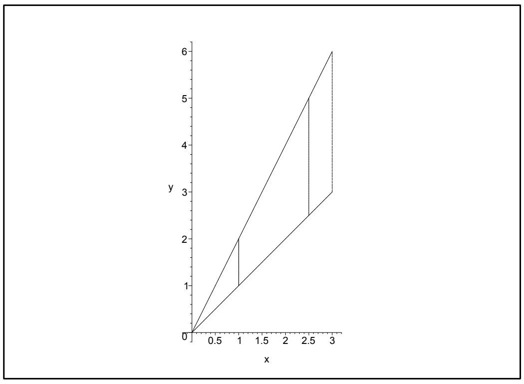



Figure 22.11: The domain $B(z)$ of **Example 22.2.4**.

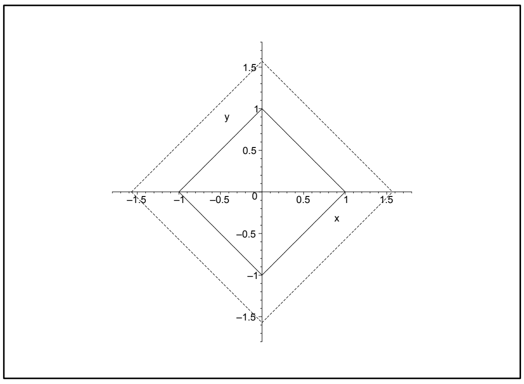



Figure 22.12: The domain $B(z)$ of **Example 22.2.5**.

MAPLE. This is easy for MAPLE. We use the commands,

with(Student[MultivariateCalculus]):

$$\text{MultiInt}\left(\frac{1}{x \cdot y^2}, y = z..2z, x = 1..z, z = 1..3\right)$$

$$\frac{1}{4}(\ln(3))^2$$

5) By a continuous extension the integrand is put equal to 1 for $z = 0$. Note that $B(z)$ is a square of edge length $\sqrt{2}|z|$, hence of the area

$$\text{area}\{B(z)\} = 2z^2.$$

Then by the theorem of reduction,

$$\begin{aligned}
\int_A \left(\frac{\sin z}{z}\right)^2 d\Omega &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\sin z}{z}\right)^2 \left\{\int_{B(z)} dS\right\} dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\sin z}{z}\right) \text{area}\{B(z)\}\, dz \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\sin z}{z}\right)^2 \cdot 2z^2\, dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\sin^2 z\, dz \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - \cos 2z)\, dz = \pi - \left[\frac{1}{2}\sin 2z\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \pi.
\end{aligned}$$

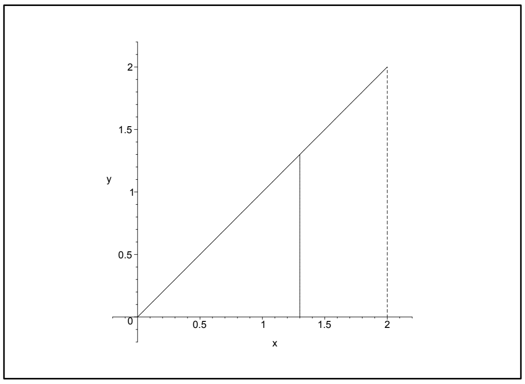



Figure 22.13: The domain $B(z)$ of **Example 22.2.6**.

6) Here

$$B(z)=\left\{(x,y)\;\middle|\;0\le y\le x\le 1-\frac{z}{2}\right\}$$

is a triangle for every $z\in[0,2[$.

We get by the second theorem of reduction,

$$\begin{aligned}\int_A yz\,\mathrm{d}\Omega &= \int_0^2 z\left\{\int_{B(z)} y\,\mathrm{d}S\right\}\mathrm{d}z=\int_0^2 z\left\{\int_0^{1-\frac{z}{2}}\left(\int_0^x y\,\mathrm{d}y\right)\mathrm{d}x\right\}\mathrm{d}z\\ &= \int_0^2 z\left\{\int_0^{1-\frac{z}{2}}\frac{1}{2}\,x^2\,\mathrm{d}x\right\}\mathrm{d}z=\frac{1}{6}\int_0^2 z\left(-\frac{z}{2}\right)^3\mathrm{d}z\\ &= \frac{1}{6}\int_0^2 z\left(1-\frac{3}{2}\,z+\frac{3}{4}\,z^2-\frac{1}{8}\,z^3\right)\mathrm{d}z=\frac{1}{6}\int_0^2\left(z-\frac{3}{2}\,z^2+\frac{3}{4}\,z^3-\frac{1}{8}\,z^4\right)\mathrm{d}z\\ &= \frac{1}{6}\left[\frac{1}{2}\,z^2-\frac{1}{2}\,z^3+\frac{3}{16}\,z^4-\frac{1}{40}\,z^5\right]_0^2=\frac{1}{6}\left(\frac{4}{2}-\frac{8}{4}+\frac{3}{16}\cdot 16-\frac{1}{40}\cdot 32\right)\\ &= \frac{1}{6}\left(2-4+3-\frac{4}{5}\right)=\frac{1}{6}\left(1-\frac{4}{5}\right)=\frac{1}{30}.\end{aligned}$$

MAPLE. This is easy for MAPLE. We use the commands,

with(Student[MultivariateCalculus]):

$$\text{MultiInt}\left(y\cdot z, y=0..x, x=0..1-\frac{z}{2}, z=0..2\right)$$

$$\frac{1}{30}$$

REMARK. The domain is also described by

$$0\le x\le 1,\quad 0\le y\le x,\quad 0\le z\le 2-2x,$$

cf. **Example 22.1.6**, and we have computed the integral in two different ways (and luckily obtained the same result). ♢

7) The have the same integrand and the same domain as in **Example 22.1.7**, so we must get the same result. The only difference is that we here cut the domain into slices, while we in i **Example 22.1** used the "method of vertical posts".

We get by the theorem of reduction,

$$\begin{aligned}\int_A xz\,\mathrm{d}\Omega &= \int_0^1 x\,\mathrm{d}x \cdot \int_0^1 z\left\{\int_0^{1-z} \mathrm{d}y\right\}\mathrm{d}z = \frac{1}{2}\int_0^1 z(1-z)\,\mathrm{d}z \\ &= \frac{1}{2}\int_0^1 \left\{z-z^2\right\}\mathrm{d}z = \frac{1}{2}\left\{\frac{1}{2}-\frac{1}{3}\right\} = \frac{1}{2}\cdot\frac{1}{6} = \frac{1}{12}.\end{aligned}$$

8) We have the same integrand and the same set as in **Example 22.1.8**, so we must get the same result. The only difference is that we here cut the domain into slices, while we in **Example 22.1** used the "method of vertical posts". Also note that we use polar coordinates in each slice, so the example should actually be moved to **Example 24.2**.

We get by the theorem of reduction in semi-polar coordinates and the change of variables $u = 2 - z$ that

$$\begin{aligned}\int_A z\,\mathrm{d}\Omega &= \int_0^2 z\cdot\pi\left(2-z^2\right)\mathrm{d}z = \pi\int_0^2 (2-u)u^2\,\mathrm{d}u = \pi\int_0^2 \left(2u^2-u^3\right)\mathrm{d}u = \pi\left[\frac{2}{3}u^3-\frac{1}{4}u^4\right] \\ &= \pi\left\{\frac{16}{3}-\frac{16}{4}\right\} = \frac{16}{12}\pi = \frac{4\pi}{3}.\end{aligned}$$

**Example 22.3** *Let $A$ be the tetrahedron of the vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. Compute in each of the following cases the space integral*

$$\int_A f(x,y,z)\,\mathrm{d}\Omega,$$

*where*

*1)* $f(x,y,z) = x+y+z$,

*2)* $f(x,y,z) = \cos(x+y+z)$,

*3)* $f(x,y,z) = \exp(x+y+z)$,

*4)* $f(x,y,z) = (1+x+y+z)^{-3}$,

*5)* $f(x,y,z) = x^2+y^2+z^2$,

*6)* $f(x,y,z) = xy-yz$.

**A** Space integrals over a tetrahedron.

**D** Consider the tetrahedron as a cone with $(0,0,0)$ as its top point in the first four questions, where the natural variable is $x+y+z$. Therefore, first analyze this special case. Calculate the space integral with respect to this variable. Alternatively, calculate the triple integral. There is also the possibility of some arguments of symmetry.

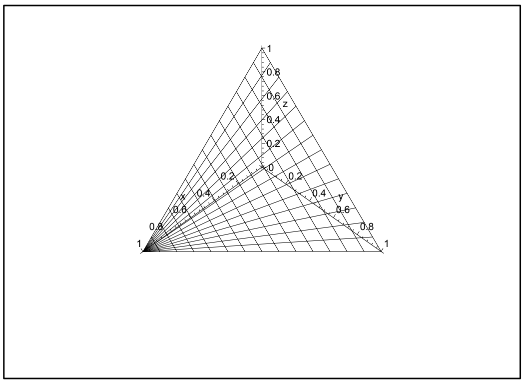



Figure 22.14: The tetrahedron of the vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

**I** Preparations. The distance from $(0,0,0)$ to the plane $x+y+z=1$ is

$$\sqrt{\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2} = \frac{1}{\sqrt{3}}.$$

Hence we can consider the tetrahedron as a cone of height $h = \dfrac{1}{\sqrt{3}}$ and with the surface where $x+y+z=1$ as its base. the area of this base is

$$\frac{1}{2}\,|(1,0,0)-(0,1,0)|\cdot\left|(0,0,1)-\left(\frac{1}{2},\frac{1}{2},0\right)\right| = \frac{1}{2}\,\sqrt{2}\cdot\sqrt{\frac{1}{4}+\frac{1}{4}+1} = \frac{1}{2}\,\sqrt{2}\cdot\sqrt{\frac{3}{2}} = \frac{\sqrt{3}}{2}.$$

Intersect the tetrahedron by the plane $x + y + z = t$, $t \in [0,1]$, parallel to the base. Then the distance from the new triangle $B(t)$ to the top point $(0,0,0)$ is $\frac{t}{\sqrt{3}}$, thus the area of this triangle $B(t)$ is due to the similarity given by

$$\operatorname{area}(B(t)) = \left(\frac{1/\sqrt{3}}{1/\sqrt{3}}\right)^2 t^2\,\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}\,t^2, \qquad t \in [0,1].$$

If the integrand $f(x,y,z) = g(x+y+z)$ is a function in $t = x+y+z$, we even get the simpler formula

$$(22.3)\quad \int_A f(x,y,z)\,d\Omega = \frac{1}{\sqrt{3}}\int_0^1 g(t)\,\operatorname{area}(B(t))\,dt = \frac{1}{2}\int_0^1 t^2 g(t)\,dt.$$

Clearly, (22.3) can be applied in the first four questions.

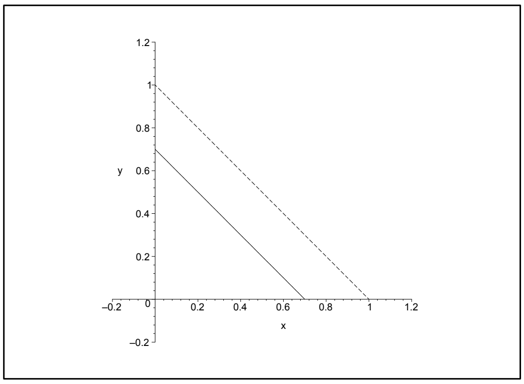



Figure 22.15: The projection of $B'(z)$ onto the $XY$-plane.

1) From $f(x,y,z) = x+y+z = t = g(t)$ and (22.3) follows that

$$\int_A (x+y+z)\,d\Omega = \frac{1}{2}\int_0^1 t^2 g(t)\,dt = \frac{1}{2}\int_0^1 t^3\,dt = \frac{1}{8}.$$

ALTERNATIVELY the plane $z$ = constant, $z \in [0,1[$, intersects the tetrahedron in a set, the projection of which onto the $XY$-plane is

$$B'(z) = \{(x,y) \mid x \geq 0,\ y \geq 0,\ x+y \leq 1-z\}.$$

Hence by more traditional calculations,

$$\int_A (x+y+z)\,d\Omega = \int_0^1 z \cdot \operatorname{area}(B'(z))\,dz + \int_0^1 \left\{\int_{B'(z)} (x+y)\,dS\right\} dz.$$

It follows from the symmetry that

$$\int_{B'(z)} x\,dS = \int_{B'(z)} y\,dS,$$

hence

$$\int_A (x+y+z)\,\mathrm{d}\Omega = \int_0^1 z\cdot\frac{1}{2}(1-z)^2\,\mathrm{d}z+2\int_0^1\left\{\int_{B'(z)} x\,\mathrm{d}S\right\}\mathrm{d}z$$
$$=\frac{1}{2}\int_0^1(1-t)t^2\,\mathrm{d}t+2\int_0^1\left\{\int_0^{1-z} x\left\{\int_0^{1-x-z}\mathrm{d}y\right\}\mathrm{d}x\right\}\mathrm{d}z$$
$$=\frac{1}{2}\left[\frac{1}{3}t^3-\frac{1}{4}t^4\right]_0^1+2\int_0^1\left\{\int_0^{1-z} x[(1-z)-x]\,\mathrm{d}x\right\}\mathrm{d}z$$
$$=\frac{1}{24}+2\int_0^1\left[\frac{1}{2}x^2(1-z)-\frac{1}{3}x^3\right]_0^{1-z}\mathrm{d}z$$
$$=\frac{1}{24}+\frac{2}{6}\int_0^1(1-z)^3\,\mathrm{d}z=\frac{1}{24}+\frac{1}{12}\left]-(1-z)^4\right]_0^1=\frac{1}{24}+\frac{1}{12}=\frac{1}{8}.$$

2) It follows from $f(x,y,z)=\cos(x+y+z)=\cos t=g(t)$ and (22.3) that

$$\int_A \cos(x+y+z)\,\mathrm{d}\Omega = \frac{1}{2}\int_0^1 t^2\cos t\,\mathrm{d}t=\frac{1}{2}\left[t^2\sin t+2t\cos t-2\sin t\right]_0^1$$
$$=\frac{1}{2}\sin 1+\cos 1-\sin 1=\cos 1-\frac{1}{2}\sin 1.$$

Alternatively, we get by more traditional calculations, where we use the same set $B'(z)$ as in 1),

$$\int_A \cos(x+y+z)\,\mathrm{d}\Omega=\int_0^1\left\{\int_{B'(z)}\cos(x+y+z)\,\mathrm{d}S\right\}\mathrm{d}z$$
$$=\int_0^1\left\{\int_0^{1-z}\left\{\int_0^{1-z-x}\cos(x+y+z)\,\mathrm{d}y\right\}\mathrm{d}x\right\}\mathrm{d}z$$
$$=\int_0^1\left\{\int_0^{1-z}[\sin(x+y+z)]_{y=0}^{1-z-x}\,\mathrm{d}x\right\}dz=\int_0^1\left\{\int_0^{1-z}\{\sin 1-\sin(x+z)\}dx\right\}\mathrm{d}z$$
$$=\sin 1\cdot\int_0^1(1-z)\,\mathrm{d}z+\int_0^1[\cos(x+z)]_{x=0}^{1-z}\,\mathrm{d}z=\frac{1}{2}\sin 1+\int_0^1\{\cos 1-\cos z\}\,\mathrm{d}z$$
$$=\frac{1}{2}\sin 1+\cos 1-\sin 1=\cos 1-\frac{1}{2}\sin 1.$$

3) It follows from $f(x,y,z)=\exp(x+y+z)=e^t=g(t)$ and (22.3) that

$$\int_A \exp(x+y+z)\,\mathrm{d}\Omega = \frac{1}{2}\int_0^1 t^2e^t\,\mathrm{d}t=\frac{1}{2}\left[t^2e^t-2e^t+2e^t\right]_0^1$$
$$=\frac{1}{2}(e-2e+2e-2)=\frac{1}{2}(e-2).$$

ALTERNATIVELY, by traditional computations,

$$\int_A \exp(x+y+z)\,d\Omega = \int_0^1 \left\{ \int_0^{1-z} \left\{ \int_0^{1-z-x} \exp(x+y+z)\,dy \right\} dx \right\} dz$$

$$= \int_0^1 \left\{ \int_0^{1-z} [\exp(x+y+z)]_{y=0}^{1-z-x}\,dx \right\} dz = \int_0^1 \left\{ \int_0^{1-z} \left(e - e^{x+z}\right) dx \right\} dz$$

$$= e\int_0^1 \left\{ \int_0^{1-z} dx \right\} dz - \int_0^1 \left\{ \int_0^{1-z} e^{x+z}\,dx \right\} dz$$

$$= e\int_0^1 (1-z)\,dz - \int_0^1 \left[e^{x+z}\right]_{x=0}^{1-z} dz = \frac{1}{2}\,e - \int_0^1 (e - e^z)\,dz$$

$$= \frac{1}{2}\,e - e + [e^z]_0^1 = \frac{e}{2} - 1 = \frac{1}{2}\,(e-2).$$

4) From $f(x,y,z) = (1+x+y+z)^{-3} = (1+t)^{-3} = g(t)$ and (22.3) follows that

$$\int_A (1+x+y+z)^{-3}\,d\Omega = \frac{1}{2}\int_0^1 \frac{t^2}{(1+t)^3}\,dt = \frac{1}{2}\int_0^1 \left\{ \frac{1}{(t+1)^3} - \frac{2}{(t+1)^2} + \frac{1}{t+1} \right\} dt$$

$$= \frac{1}{2}\left[ -\frac{1}{2}\cdot\frac{1}{(t+1)^2} + \frac{2}{t+1} + \ln(t+1) \right]_0^1 = \frac{1}{2}\left\{ -\frac{1}{2}\cdot\frac{1}{4} + \frac{2}{2} + \ln 2 + \frac{1}{2} - 2 \right\}$$

$$= \frac{1}{2}\left\{ \ln 2 - \frac{5}{8} \right\} = \frac{1}{2}\,\ln 2 - \frac{5}{16}.$$

ALTERNATIVELY, by traditional calculations,

$$\int_A (1+x+y+z)^{-3}\,d\Omega = \int_0^1 \left\{\int_0^{1-z}\left\{\int_0^{1-z-x}(1+x+y+z)^{-3}\,dy\right\}dx\right\}dz$$

$$= \int_0^1 \left\{\int_0^{1-z}\left[-\frac{1}{2}(1+x+y+z)^{-2}\right]_{y=0}^{1-z-x} dx\right\}dz$$

$$= \frac{1}{2}\int_0^1 \left\{\int_0^{1-z}\left\{(1+x+z)^{-2}-\frac{1}{4}\right\}dx\right\}dz$$

$$= \frac{1}{2}\int_0^1 \left[-(1+x+z)^{-1}\right]_{x=0}^{1-z} dz - \frac{1}{8}\int_0^1\left\{\int_0^{1-z}dx\right\}dz$$

$$= \frac{1}{2}\int_0^1\left\{\frac{1}{1+z}-\frac{1}{2}\right\}dz - \frac{1}{16} = \frac{1}{2}\ln 2 - \frac{1}{4} - \frac{1}{16} = \frac{1}{2}\ln 2 - \frac{5}{16}.$$

5) In this case we can no longer apply (22.3). We note by symmetry that

$$\int_{B'(z)} x^2\,dS = \int_{B'(z)} y^2\,dS,$$

hence by traditional calculations,

$$\int_A (x^2+y^2+z^2)\,d\Omega = \int_0^1 z^2\,\text{area}(B'(z))\,dz + 2\int_0^1\left\{\int_{B'(z)} x^2\,dS\right\}dz$$

$$= \frac{1}{2}\int_0^1 z^2(1-z)^2\,dz + 2\int_0^1\left\{\int_0^{1-z} x^2\left\{\int_0^{1-z-x}dy\right\}dx\right\}dz$$

$$= \frac{1}{2}\int_0^1 (z^2-2z^3+z^4)\,dz + 2\int_0^1\left\{\int_0^{1-z} x^2(1-z-x)\,dx\right\}dz$$

$$= \frac{1}{2}\left[\frac{1}{3}z^3 1 - \frac{2}{4}z^4 + \frac{1}{5}z^5\right]_0^1 + 2\int_0^1\left[\frac{1}{3}x^3(1-z) - \frac{1}{4}x^4\right]_{x=0}^{1-z} dz$$

$$= \frac{1}{2}\left(\frac{1}{3}-\frac{1}{2}+\frac{1}{5}\right) + 2\cdot\frac{1}{12}\int_0^1 (1-z)^4\,dz$$

$$= \frac{1}{60}(10-15+6) + \frac{1}{6}\left[-\frac{1}{5}(1-z)^5\right]_0^1 = \frac{1}{60}+\frac{1}{30} = \frac{1}{20}.$$

6) Put

$$B''(y) = \{(x,z) \mid 0 \le x,\ 0 \le z,\ x+z \le 1-y\}.$$

It follows from symmetric reasons that

$$\int_{B''(y)} xy\,dS = \int_{B''(y)} yz\,dS.$$

Hence

$$\int_A (xy-yz)\,d\Omega = \int_0^1\left\{\int_{B''(y)}(xy-yz)\,dS\right\}dy = 0.$$

ALTERNATIVELY we get by traditional calculations,

$$\begin{aligned}\int_A (xy-yz)\,d\Omega &= \int_0^1 \left\{\int_{B'(z)} (x-z)y\,dS\right\} dz \\ &= \int_0^1 \left\{\int_0^{1-z} (x-z)\left\{\int_0^{1-(x+z)} y\,dy\right\} dx\right\} dz \\ &= \frac{1}{2}\int_0^1 \left\{\int_0^{1-z} (x-z)[1-(x+z)]^2\,dx\right\} dz \\ &= \frac{1}{2}\int_0^1 \left\{\int_0^{1-z} (x+z)[(x+z)-1]^2\,dx\right\} dz - \int_0^1 z\left\{\int_0^{1-z} [(x+z)-1]^2\,dx\right\} dz.\end{aligned}$$

In order to avoid too complicated expressions we compute the two double integrals one by one:

$$\begin{aligned}\frac{1}{2}\int_0^1 \left\{\int_0^{1-z} (x+z)[(x+z)-1]^2\,dx\right\} dz &= \frac{1}{2}\int_0^1 \left\{\int_0^{1-z} \{(x+z)^3-2(x+z)^2+(x+z)\}\,dx\right\} dz \\ &= \frac{1}{2}\int_0^1 \left[\frac{1}{4}(x+z)^4-\frac{2}{3}(x+z)^3+\frac{1}{2}(x+z)^2\right]_{x=0}^{1-z} dz \\ &= \frac{1}{2}\int_0^1 \left\{\frac{1}{4}-\frac{2}{3}+\frac{1}{2}-\frac{1}{4}z^4+\frac{2}{3}z^3-\frac{1}{2}z^2\right\} dz \\ &= \frac{1}{24}+\frac{1}{2}\left[-\frac{1}{20}+\frac{1}{6}-\frac{1}{6}\right] = \frac{1}{24}-\frac{1}{40} = \frac{5-3}{120} = \frac{1}{60},\end{aligned}$$

and

$$\begin{aligned}\int_0^1 z\left\{\int_0^{1-z} [(x+z)-1]^2\,dx\right\} dz &= \int_0^1 z\left[\frac{1}{3}(x+z-1)^3\right]_{x=0}^{1-z} dz \\ &= \frac{1}{3}\int_0^1 z(1-z)^3\,dz = \frac{1}{3}\int_0^1 (1-t)t^3\,dt = \frac{1}{3}\left[\frac{1}{4}t^4-\frac{1}{5}t^5\right]_0^1 = \frac{1}{60}.\end{aligned}$$

Finally, we get by insertion,

$$\int_A (xy-yz)\,d\Omega = \frac{1}{60}-\frac{1}{60} = 0.$$

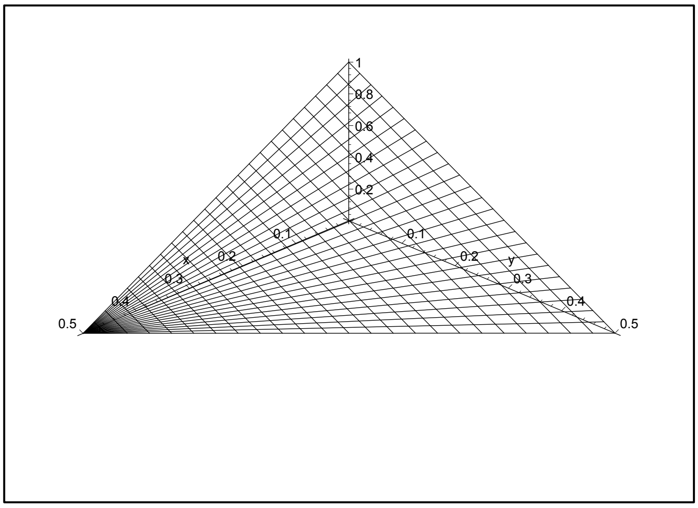



Figure 22.16: The tetrahedron $A$ with its projection $B$ onto the $(x, y)$-plane.

**Example 22.4** *Let $B$ be the triangle which is bounded by the $X$-axis and the $Y$-axis and the line of the equation $x + y = \dfrac{1}{2}$. Furthermore, let $A$ be the tetrahedron bounded by the three coordinate planes and the plane of the equation $2x + 2y + z = 1$. Compute the integrals*

$$\int_B (1 - 2x - 2y)\,\mathrm{d}S \quad \textit{and} \quad \int_A (x + y + z)\,\mathrm{d}\Omega.$$

**A** Plane integral and space integral.

**D** Sketch $B$ and $A$. Then compute the integrals.

**I** It follows immediately that $B$ is that surfaces of $A$, which lies in the i $(x, y)$-plane.

First calculate the plane integral (it is actually the volume of the tetrahedron $A$),

$$\int_B (1 - 2x - 2y)\,\mathrm{d}S = \int_0^{\frac{1}{2}} \left\{ \int_0^{\frac{1}{2}-x} (1 - 2x - 2y)\,\mathrm{d}y \right\} \mathrm{d}x$$
$$= \int_0^{\frac{1}{2}} \left\{ (1-2x)\left(\frac{1}{2} - x\right) - \left(\frac{1}{2} - x\right)^2 \right\} \mathrm{d}x = \int_0^{\frac{1}{2}} \left(\frac{1}{2} - x\right)^2 dx$$
$$= \left[ \frac{1}{3}\left(x - \frac{1}{2}\right)^3 \right]_0^{\frac{1}{2}} = 0 - \frac{1}{3}\left(-\frac{1}{2}\right)^3 = \frac{1}{24}.$$

Then calculate the space integral,

$$\int_A (x+y+z)\,\mathrm{d}\Omega = \int_B \left\{ \int_0^{1-2x-2y} (x+y+z)\,\mathrm{d}z \right\} \mathrm{d}S$$
$$= \int_B \left\{ (x+y)(1-2x-2y) + \frac{1}{2}(1-2x-2y)^2 \right\} \mathrm{d}S$$
$$= \frac{1}{2}\int_B (1-2x-2y)\{2x+2y+(1-2x-2y)\}\,\mathrm{d}S$$
$$= \frac{1}{2}\int_B (1-2x-2y)\,\mathrm{d}S = \frac{1}{2}\cdot\frac{1}{24} = \frac{1}{48},$$

where we have inserted the value of the plane integral.

**Example 22.5** *Given the tetrahedron*

$$T = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \le x,\, 0 \le y.\, 0 \le z,\, z + 2x + 4y \le 8\}.$$

*Calculate the space integral*

$$\int_T x \, d\Omega.$$

**A** Space integral.

**D** First find the base of $T$ in the $(x,y)$-plane.

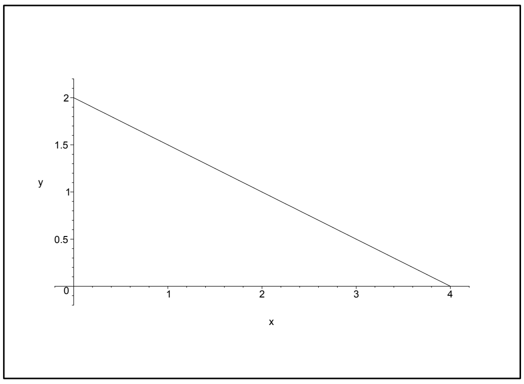



Figure 22.17: The base $B$ of $T$ in the plane $z = 0$.

**I** The base $B$ is given by

$$0 \le x, \qquad 0 \le y, \qquad 2x + 4y \le 8,$$

i.e.

$$B = \{(x,y) \mid 0 \le x,\, 0 \le y,\, x + 2y \le 4\}.$$

Then we get the space integral

$$\begin{aligned}
\int_T x \, d\Omega &= \int_0^4 \left\{ \int_0^{\frac{1}{2}(4-x)} x \cdot (8 - 2x - 4y) \, dy \right\} dx \\
&= -\frac{1}{8} \int_0^4 x \left[ (8 - 2x - 4y)^2 \right]_{y=0}^{\frac{1}{2}(4-x)} dx = \frac{1}{8} \int_0^4 x(8-2x)^2 \, dx \\
&= \frac{4}{8} \int_0^4 \{(x-4)+4\}(x-4)^2 \, dx = \frac{1}{2} \left[ \frac{1}{4}(x-4)^4 + \frac{4}{3}(x-4)^3 \right]_0^4 \\
&= \frac{1}{2} \left\{ -\frac{1}{4} \cdot 4^4 + \frac{4}{3} \cdot 4^3 \right\} = \frac{4^3}{2} \left( \frac{4}{3} - 1 \right) = \frac{32}{3}.
\end{aligned}$$

ALTERNATIVELY, start by integrating with respect to $x$. Then

$$\begin{aligned}\int_T x\,\mathrm{d}\Omega &= \int_0^2 \left\{ \int_0^{4-2y} (8x-2x^2-4xy)\,\mathrm{d}x \right\} \mathrm{d}y = \int_0^2 \left[ 4x^2 - \frac{2}{3}\,x^3 - 2x^2y \right]_{x=0}^{4-2y} \mathrm{d}y \\ &= \int_0^2 \left\{ (4-2y)\cdot(4-2y)^2 - \frac{2}{3}\,(4-2y)^3 \right\} \mathrm{d}y = \frac{1}{3}\int_0^2 (4-2y)^3\,\mathrm{d}y = \frac{8}{3}\int_0^2 (2-y)^3\,\mathrm{d}y \\ &= \frac{8}{3}\int_0^2 t^3\,\mathrm{d}t = \frac{8}{3}\left[\frac{t^4}{4}\right]_0^2 = \frac{32}{3}.\end{aligned}$$

MAPLE. In MAPLE this is easy.

with(Student[MultivariateCalculus])

MultiInt $\left(8x - 2x^2 - 4x \cdot y, x = 0..4-2y, y = 0..2\right)$

$$\frac{32}{3}$$

**Example 22.6** *Let $a$ be a positive constant, and let*

$$A = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in B,\ \sqrt{ax} \le z \le \sqrt{ax+y^2}\},$$

*where*

$$B = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le a,\ -x \le y \le 2x\}.$$

*Calculate the space integral*

$$\int_A xyz\,\mathrm{d}\Omega.$$

**A** Space integral.

**D** Reduce the integral by first integrating with respect to $z$.

**I** When we reduce as a triple integral, we get

$$\begin{aligned}\int_A xyz\,\mathrm{d}\Omega &= \int_0^a \left\{ \int_{-x}^{2x} \left( \int_{\sqrt{ax}}^{\sqrt{ax+y^2}} xyz\,\mathrm{d}z \right) \mathrm{d}y \right\} \mathrm{d}x = \int_0^a x \left\{ \int_{-x}^{2x} y \left[ \frac{1}{2}\,z^2 \right]_{\sqrt{ax}}^{\sqrt{ax+y^2}} \mathrm{d}y \right\} \mathrm{d}x \\ &= \frac{1}{2}\int_0^a x \left\{ \int_{-x}^{2x} y^3\,\mathrm{d}y \right\} \mathrm{d}x = \frac{1}{2}\int_0^a x \left[ \frac{1}{4}\,y^4 \right]_{-x}^{2x} \mathrm{d}x = \frac{1}{8}\int_0^a x\left\{2^4-1\right\} x^4\,\mathrm{d}x \\ &= \frac{15}{8}\int_0^a x^5\,\mathrm{d}x = \frac{15}{8}\cdot\frac{a^6}{6} = \frac{5}{16}\,a^6.\end{aligned}$$

MAPLE. In MAPLE this is easy.

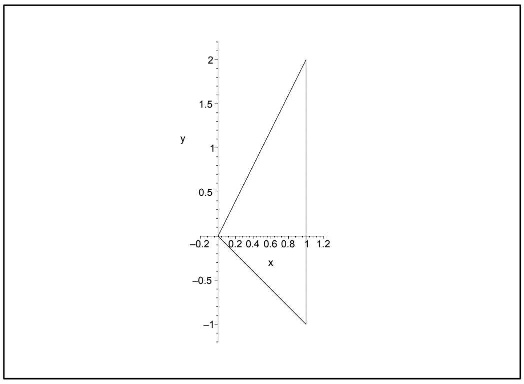



Figure 22.18: The domain $B$ for $a = 1$.

with(Student[MultivariateCalculus])

$$\text{MultiInt}\left(x \cdot y \cdot z, z = \sqrt{a \cdot x}..\sqrt{a \cdot x + y^2}, y = -x..2x\right)$$

$$\frac{5}{16}\,a^6$$

# 23 The space integral in semi-polar coordinates

## 23.1 Reduction theorem in semi-polar coordinates

In Chapter 22 we obtained some reduction theorems in rectangular coordinates. We shall here derive a reduction theorem in semi-polar coordinates. Let us consider the polar coordinates in the 2-dimensional plan. Then we found in Chapter 20 that the area of a small, almost rectangular domain in polar coordinates is $\Delta S = \varrho \Delta\varrho \Delta\varphi$, cf. Figure 23.1.

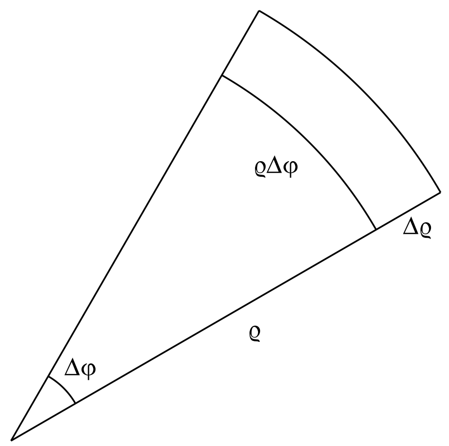



Figure 23.1: Analysis of the area element in polar coordinates.

The $z$-axis is perpendicular to the $(x, y)$-plane, so a small body of height $\Delta z$ and of base above the small plane set of area $\Delta S$ would therefore have the volume

$$\Delta\Omega = \Delta S \cdot \Delta z = \varrho \Delta\varrho \Delta\varphi \Delta z.$$

One would therefore guess that the volume element is in semi-polar coordinates given by

$$\mathrm{d}\Omega = \varrho \,\mathrm{d}\varrho \,\mathrm{d}z \,\mathrm{d}\varphi,$$

and it can actually be proved that this is indeed the case.

The space integral in semi-polar coordinates is in particular useful, when we integrate over a *rotational body* with the $z$-axis as its axis of revolution. To see this we first in general consider the following situation. For every fixed $\varphi$ we define a half-plane, called the *meridian half-plane*. It is characterized by its two coordinates, $(\varrho, z) \in \overline{\mathbb{R}}_+ \times \mathbb{R}$, which together with the chosen $\varphi$ form the semi-polar coordinates.

For every fixed $\varphi$ this meridian half-plan cut the body $A \subset \mathbb{R}^3$ in a plane point set, which we denote $B(\varphi)$. If $f : A \to \mathbb{R}$ is a continuous function on the closed and bounded set $A \subset \mathbb{R}^3$, then the idea is first to integrate $f$ over the set $B(\varphi)$ [where we must not forget some weight function to be derived later on], which defines a function $F(\varphi)$, which only depends on $\varphi$. So in order to calculate the space integral of $f$ over $A$ we first find $F(\varphi)$, and then integrate $F(\varphi)$ with respect to $\varphi$.

Then note that *if* $A$ is a *rotational body* with respect to the $z$-axis, then all $B(\varphi) = B$ are equal, so we can in this case expect some simplifications.

First recall that the rectangular coordinates $(x, y, z)$ in semi-polar coordinates are described by

$$(x, y, z) = (\varrho \cos \varphi, \varrho \sin \varphi, z).$$

After these remarks we formulate without proof the following

**Theorem 23.1** The reduction theorem of the space integral in semi-polar coordinates. *Assume that $A \subset \mathbb{R}^3$ is a closed and bounded set, and that $f : A \to \mathbb{R}$ is a continuous function. If $A$ is described in semi-polar coordinates by its corresponding parameter set*

$$\tilde{A} = \{(\varrho, \varphi, z) \mid \alpha \le \varphi \le \beta, \, (\varrho, z) \in B(\varphi), \, 0 < \beta - \alpha \le 2\pi\},$$

*where $B(\varphi)$ for every $\varphi \in [\alpha, \beta]$ is a closed and bounded plane set in the meridian half-plane, then the space integral of $f$ over $A$ is reduced in the following way in semi-polar coordinates,*

$$(23.1) \qquad \int_A f(x, y, z) \, \mathrm{d}\Omega = \int_\alpha^\beta \left\{ \int_{B(\varphi)} f(\varrho \cos \varphi, \varrho \sin \varphi, z) \varrho \, \mathrm{d}\varrho \, \mathrm{d}z \right\} \mathrm{d}\varphi.$$

So we just insert

$$x = \varrho \cos \varphi, \quad y = \varrho \sin \varphi, \qquad \text{and add the weight function } \varrho \text{ as a factor,}$$

to get the integrand right.

In case of a *rotational body*, where $B(\varphi) = B$ for all $\varphi$, formula (23.1) is also written

$$\int_A f(x, y, z) \, \mathrm{d}\Omega = \int_B \left\{ \int_\alpha^\beta f(\varrho \cos\varphi, \varrho \sin\varphi, z) \, \mathrm{d}\varphi \right\} \varrho \, \mathrm{d}\varrho \, \mathrm{d}z.$$

One should of course we aware of that the body $A$ itself is *not* equal to its corresponding parameter domain $\tilde{A}$, so these two sets must never be confused. Note in particular that we after the introduction of the weight function $\varrho$ in the parameter space $\tilde{A}$, we integrate here as if we had rectangular coordinates. This is *not* the case in the body $A$ itself.

There are of course other variants of the reduction theorem above, but they are difficult to formulate in general, and in practice one will never doubt what should be done, so there is no need to formulate such results. Only Theorem 23.1 above is nice, because it refers to the meridian half-plane, which can be visualized.

## 23.2 Procedures for reduction of space integral in semi-polar coordinates

This method resembles the *"disc" method*. The difference is that the domain this time is cut like a *pie* with respect to the $z$-axis.
The formal reduction formula is

$$\int_A f(x, y, z) \, \mathrm{d}\Omega = \int_\alpha^\beta \left\{ \int_{B(\varphi)} f(\varrho \cos \varphi, \varrho \sin \varphi, z) \, \varrho \, \mathrm{d}\varrho \, \mathrm{d}z \right\} \mathrm{d}\varphi,$$

where we have used *semi-polar* coordinates

$$x = \varrho \cos \varphi, \qquad y = \varrho \sin \varphi, \qquad z = z,$$

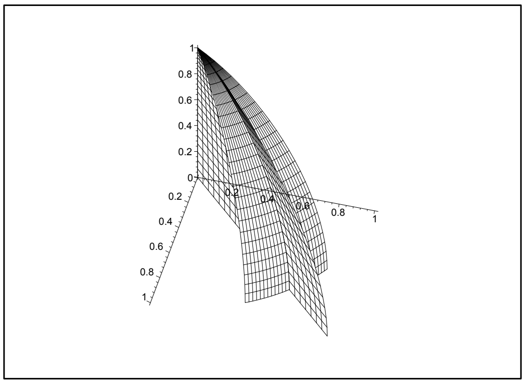


Figure 23.2: A fixed angle $\varphi$ determines a half-plane through $A$ with the $z$-axis as axis. This half-plane is then transferred into the meridian half-plane.

and the corresponding *volume element*

$$d\Omega = \varrho \, d\varrho \, dz \, d\varphi.$$

The pie cut is $B(\varphi)$, which for every fixed $\varphi \in [\alpha, \beta]$ represents a domain in the meridian half-plane, i.e. in the $(\varrho, z)$-plane.

**Procedure:**

1) Sketch a figure (at least in the meridian half-plane).

2) Describe the domain $A$ in *semi-polar* coordinates by its parameter domain

$$\tilde{A} = \{(\varrho, \varphi, z) \mid \alpha \leq \varphi \leq \beta, \, (\varrho, z) \in B(\varphi)\}.$$

Identify the *meridian cut* $B(\varphi)$ for every fixed $\varphi \in [\alpha, \beta]$.

3) Keep $\varphi \in [\alpha, \beta]$ fixed and calculate the abstract (inner) plane integral

$$\Phi(\varphi) := \int_{B(\varphi)} f(\varrho \cos \varphi, \varrho \sin \varphi, z) \, \varrho \, d\varrho \, dz,$$

by applying one of the methods from Chapter 20. Here one must *not* forget the weight function $\varrho$ which should be put as an extra factor in the integrand.

4) Insert and calculate finally the *ordinary integral* on the right hand side,

$$\int_A f(x, y, z) \, d\Omega = \int_\alpha^\beta \Phi(\varphi) \, d\varphi.$$

**Remark 23.1** If $\Omega$ is a *rotational domain* then $B(\varphi) = B$ is independent of $\varphi$. In this case we get a better procedure by interchanging the order of integration,

$$\int_A f(x, y, z) \, d\Omega = \int_B \left\{ \int_\alpha^\beta f(\varrho \cos \varphi, \varrho \sin \varphi, z) \, d\varphi \right\} \varrho \, d\varrho \, dz,$$

hence one starts in this case by calculating the inner *ordinary integral*. $\Diamond$

## 23.3 Examples of space integrals in semi-polar coordinates

**A.** *Calculate the space integral*

$$I = \int_A x^2 yz \, d\Omega,$$

*where*

$$A = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \le a^2,\ y \ge 0,\ 0 \le z \le h\}.$$

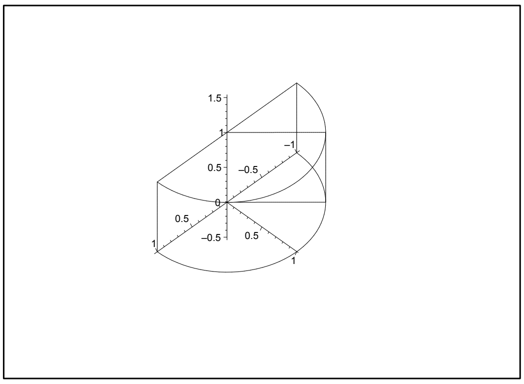



Figure 23.3: The domain (body) $A$ for $a = h = 1$ with a cut $B(\varphi)$.

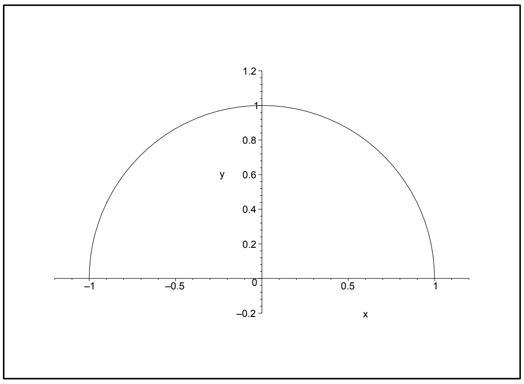



Figure 23.4: Projection of $A$ onto the $(x, y)$-plane $(a = 1)$.

**D.** It is possible here to calculate in rectangular coordinates, but the calculations are far from easy to perform, so we shall skip this variant here. Instead we shall try the *semi-polar* version (where we must *not* forget the weight function $\varrho$).

When we use semi-polar coordinates $A$ is represented by the parameter domain

$$\tilde{A} = \{(\varrho, \varphi, z) \mid 0 \le \varrho \le a,\ 0 \le \varphi \le \pi,\ 0 \le z \le h\}.$$

We shall go through a couple of variants of the reduction.

**I 1.** For fixed $\varphi$ the set $A$ is cut into $B(\varphi) = [0,a] \times [0,h]$. Therefore, we get the following reduction, where $\varphi$ is the "outer" variable,

$$\begin{aligned} I &= \int_0^{\pi} \left\{ \int_{B(\varphi)} z \cdot \varrho^2 \cos^2\varphi \cdot \varrho \sin\varphi \cdot \varrho \, \mathrm{d}\varrho \right\} \mathrm{d}\varphi \\ &= \int_0^{\pi} \cos^2\varphi \, \sin\varphi \, \mathrm{d}\varphi \cdot \int_0^h z \, \mathrm{d}z \cdot \int_0^a \varrho^4 \, \mathrm{d}\varrho \\ &= \left[ -\frac{1}{3}\cos^3\varphi \right]_0^{\pi} \cdot \left[ \frac{1}{2} z^2 \right]_0^h \cdot \left[ \frac{1}{5} \varrho^5 \right]_0^a = \frac{2}{3} \cdot \frac{1}{2} h^2 \cdot \frac{1}{5} a^5 = \frac{1}{15} h^2 a^5. \end{aligned}$$

The calculations in MAPLE arewith(Student[MultivariateCalculus])

$$\text{MultiInt}\left(\cos(t)^2 \cdot \sin(t) \cdot z \cdot r^4, t = 0..\pi,\ z = 0..h,\ r = 0..a\right)$$

$$\frac{1}{15} h^2 a^5$$

**I 2.** If instead we integrate innermost with respect to $z$, then we get with $B$ as the half disc,

$$\begin{aligned} I &= \int_B x^2 y \left\{ \int_0^h z\,\mathrm{d}z \right\} \mathrm{d}S \\ &= \left[\frac{1}{2}\,z^2\right]_0^h \cdot \int_0^\pi \left\{ \int_0^a \varrho^2 \cos^2\varphi \cdot \varrho \sin\varphi \cdot \varrho\,\mathrm{d}\varrho \right\} \mathrm{d}\varphi \\ &= \frac{1}{2}\,h^2 \int_0^\pi \cos^2\varphi\,\sin\varphi\,\mathrm{d}\varphi \cdot \int_0^a \varrho^4\,\mathrm{d}\varrho \\ &= \frac{1}{2}\,h^2 \left[-\frac{1}{3}\cos^3\varphi\right]_0^\pi \cdot \frac{a^5}{5} = \frac{1}{15}\,h^2 a^5. \end{aligned}$$

**C.** *Weak control* (Check of dimension). We get from

$$x \sim a, \quad y \sim a, \quad z \sim h, \qquad \int \cdots\,\mathrm{d}\Omega = \int \cdots\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \sim a \cdot a \cdot h = a^2 h$$

that

$$I = \int_A x^2 y^2 z\,\mathrm{d}\Omega \sim a^2 \cdot a^2 \cdot h \cdot (a^2 h) = h^2 a^5,$$

so the result *must* have the form constant$\cdot h^2 a^5$. If this is not the case then we have made an error. Note, however, that even if we get $c \cdot h^2 a^2$, we may not have found the right constant $c$, so the method gives only a weak control.

**A.** *Calculate the space integral*

$$I = \int_A xy^2 z\,\mathrm{d}\Omega,$$

*where*

$$A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 \le a^2,\ x \ge 0,\ \sqrt{x^2+y^2} \le z \le a\}.$$

When we consider the dimensions, we see that $x$, $y$, $z \sim a$ and $\int \cdots\,\mathrm{d}\Omega \sim a^3$, so

$$I = \int_A xy^2 z\,\mathrm{d}\Omega \sim \cdot a^2 \cdot a \cdot a^3 = a^7.$$

Hence the result *must* have the form constant$\cdot a^7$.

**D.** The shape of $A$ (which is a part of a rotational body) invites to the application of *semi-polar* coordinates (*do not forget the* weight function $\varrho$!), where $A$ is represented by the parameter domain

$$\tilde{A} = \left\{ (\varrho, \varphi, z) \;\middle|\; 0 \le \varrho \le a,\ -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2},\ \varrho \le z \le a \right\}.$$

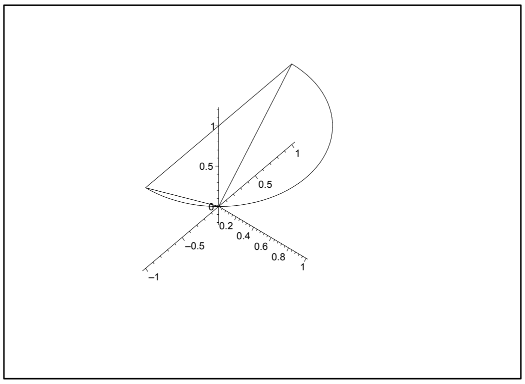



Figure 23.5: The body $A$ for $a = 1$.

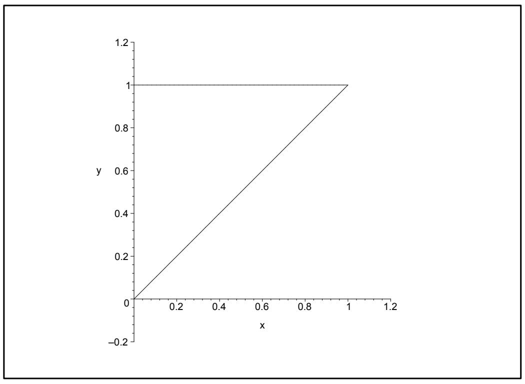



Figure 23.6: The cut in the meridian half-plane for $a = 1$.

**I.** The cut $B(\varphi)$, which is rotated around the $z$-axis, must be independent of $\varphi$. We get in the meridian half-plane

$$B(\varphi) = \{(\varrho, z) \mid 0 \le z \le a,\, 0 \le \varrho \le z\}.$$

Since the $\varphi$-integral can be separated from the rest, it follows from Theorem 23.1 that

$$\begin{aligned} I &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ \int_{B(\varphi)} \varrho \cos\varphi \cdot \varrho^2 \sin^2\varphi \cdot z \cdot \varrho \, d\varrho \, dz \right\} d\varphi \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\varphi \cdot \cos\varphi \, d\varphi \cdot \int_{B(\varphi)} \varrho^4 z \, d\varrho \, dz \\ &= \left[ \frac{1}{3} \sin^3\varphi \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdot \int_0^a z \left\{ \int_0^z \varrho^4 \, d\varrho \right\} dz = \frac{2}{3} \cdot \int_0^a z \cdot \frac{1}{5} z^5 \, dz \\ &= \frac{2}{15} \int_0^a z^6 \, dz = \frac{2}{15} \cdot \frac{1}{7} a^7 = \frac{2}{105} a^7. \end{aligned}$$

The calculations in MAPLE arewith(Student[MultivariateCalculus])

$$\text{MultiInt}\left(\sin(t)^2 \cdot \cos(t) \cdot z \cdot r^4, t = -\frac{\pi}{2}..\frac{\pi}{2}, \; r = 0..z, \; z = 0..a\right)$$

$$\frac{2}{105}\, a^7$$

**C.** A weak control shows that the result has the form constant$\cdot a^7$ as foreseen in **A**.

**Example 23.1** *Compute in each of the following cases the given space integral over a point set $A$, which in semi-polar coordinates is bounded by*

$$\alpha \le \varphi \le \beta \quad \text{and} \quad (\varrho, z) \in B(\varphi).$$

*One shall first from the given description of the domain of integration find $\alpha$, $\beta$ and $B(\varphi)$.*

*1) The space integral $\int_A \sqrt{x^2+y^2}\,\mathrm{d}\Omega$, where the domain of integration $A$ is given by*

$$\sqrt{x^2+y^2} \le z \le 1.$$

*2) The space integral $\int_A \ln(1+x^2+y^2)\,\mathrm{d}\Omega$, where the domain of integration $A$ is given by*

$$\frac{1}{2}\left(x^2+x^2\right) \le z \le 2.$$

*3) The space integral $\int_A (x+y^2)z\,\mathrm{d}\Omega$, where the domain of integration $A$ is given by*

$$x^2+y^2 \le 1 \quad \text{and} \quad x^2+y^2 \le z \le \sqrt{2-x^2-y^2}.$$

*4) The space integral $\int_A (x^2+y^2)\,\mathrm{d}\Omega$, where the domain of integration $A$ is given by*

$$\frac{x^2+y^2}{a} \le z \le h.$$

*5) The space integral $\int_A xy\,\mathrm{d}\Omega$, where the domain of integration $A$ is given by the conditions*

$$x \ge 0, \quad y \ge 0 \quad \text{and} \quad \frac{x^2+y^2}{a} \le z \le h.$$

*6) The space integral $\int_A xz\,\mathrm{d}\Omega$, where the domain of integration $A$ is given by*

$$x^2+y^2 \le 2x \quad \text{and} \quad 0 \le z \le \sqrt{x^2+y^2}.$$

*7) The space integral $\int_A (z^2+y^2)\,\mathrm{d}\omega$, where the domain of integration $A$ is given by*

$$0 \le z \le h - \frac{h}{a}\sqrt{x^2+y^2}.$$

*8) The space integral $\int_A (x^2+y^2)\,\mathrm{d}\Omega$, where the domain of integration $A$ is given by*

$$x^2+y^2 \le 3 \quad \text{and} \quad 0 \le z \le \sqrt{1+x^2+y^2}.$$

*9) The space integral $\int_A xy\,\mathrm{d}\Omega$, where the domain of integration $A$ is given by*

$$x^2+y^2 \le 3 \quad \text{and} \quad 0 \le z \le \sqrt{1+x^2+y^2}.$$

*10) The space integral $\int_A (x^2z+z^3)\,\mathrm{d}\Omega$, where the domain of integration $A$ is given by*

$$0 \le z \le \sqrt{a^2-x^2-y^2}.$$

11) *The space integral* $\int_A |y| z \, \mathrm{d}\Omega$*, where the domain of integration* $A$ *is given by*

$$x^2 + y^2 \le ax \quad \text{and} \quad 0 \le z \le \frac{x^2}{a}.$$

12) *The space integral* $\int_A xz \, \mathrm{d}\Omega$*, where the domain of integration is one half cone of revolution of vertex* $(0,0,h)$ *and its base in the plane* $z = 0$ *given by*

$$x^2 + y^2 \le a^2 \qquad \text{for } x \ge 0.$$

13) *The space integral* $\int_A z \, \mathrm{d}\Omega$*, where the domain of integration* $A$ *is given by* $x^2 + y^2 \le (2-z)^2$ *for* $z \in [0,2]$.

*[This is also* **Example 22.2.8***, so we may compare the results. Cf. also* **Example 22.1.8***.]*

**A** Space integrals in semi-polar coordinates.

**D** Find the interval $[\alpha, \beta]$ for $\varphi$. Describe $B(\varphi)$ in semi-polar coordinates and sketch if necessary $B(\varphi)$ in the meridian half plane. Finally, compute the space integral by using the theorem of reduction in semi-polar coordinates.

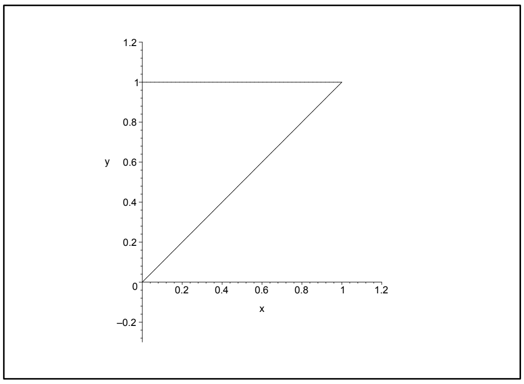



Figure 23.7: The meridian cut $B(\varphi)$ in **Example 23.1.1**.

**I** 1) Here $\varphi \in [0, 2\pi]$ and

$$B(\varphi) = \{(\varrho, z) \mid 0 \le \varrho \le 1, \, \varrho \le z \le 1\} = \{(\varrho, z) \mid 0 \le z \le 1, \, 0 \le \varrho \le z\}.$$

Then by the reduction theorem,

$$\int_A \sqrt{x^2+y^2} \, \mathrm{d}\Omega = \int_0^{2\pi} \left\{ \int_{B(\varphi)} \varrho \cdot \varrho \, \mathrm{d}\varrho \, \mathrm{d}z \right\} \mathrm{d}\varphi$$

$$= 2\pi \int_0^1 \left\{ \int_0^z \varrho^2 \, \mathrm{d}\varrho \right\} \mathrm{d}z = \frac{2\pi}{3} \int_0^1 z^3 \, dz = \frac{\pi}{6}.$$

The calculations in MAPLE are

with(Student[MultivariateCalculus])

$2\pi \cdot \text{MultiInt}\left(r^2, r = 0..z, \, z = 0..1\right)$

$\frac{1}{6}\pi$

2) Here $\varphi \in [0, 2\pi]$, and

$$B(\varphi) = \left\{ (\varrho, z) \;\middle|\; 0 \le \varrho \le 2, \, \frac{1}{2}\,\varrho^2 \le z \le 2 \right\} = \{(\varrho, z) \mid 0 \le z \le 2, \, 0 \le \sqrt{2z}\}$$

which does not depend on $\varphi$. Then by the reduction theorem,

$$\int_A \ln(1+x^2+y^2)\,d\Omega = \int_0^{2\pi}\left\{\int_{B(\varphi)} \ln(1+\varrho^2)\cdot\varrho\,d\varrho\,dz\right\}d\varphi$$

$$= 2\pi\int_0^2\left\{\int_0^{\sqrt{2z}}\ln(1+\varrho^2)\,\varrho\,d\varrho\right\}dz = 2\pi\int_0^2\left[\frac{1}{2}\left\{(1+\varrho^2)\ln(1+\varrho^2)-\varrho^2\right\}\right]_{\varrho=0}^{\sqrt{2z}}dz$$

$$= \pi\int_0^2\{(1+2z)\ln(1+2z)-2z\}\,dz = \frac{\pi}{2}\int_0^4(1+t)\ln(1+t)\,dt - \pi\left[z^2\right]_{z=0}^2$$

$$= \frac{\pi}{2}\left[\frac{1}{2}(1+t)^2\ln(1+t)-\frac{1}{4}(1+t)^2\right]_0^4 - 4\pi = \frac{\pi}{2}\left\{\frac{25}{2}\ln 5 - \frac{25}{4}+\frac{1}{4}\right\} - 4\pi$$

$$= \pi\left\{\frac{25}{4}\ln 5 - 7\right\}.$$

The calculations in MAPLE are

with(Student[MultivariateCalculus])

$$2\pi * \text{MultiInt}\left(\ln\left(1+r^2\right), r=0..\sqrt{2z}, z=0..2\right)$$

$$2\pi\left(-\frac{7}{2}+\frac{25}{8}\ln(5)\right)$$

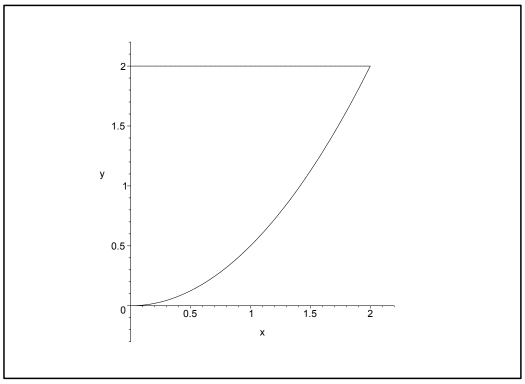



Figure 23.8: The meridian cut $B = B(\varphi)$ of **Example 23.1.2**.

3) Here $\varphi \in [0, 2\pi]$, and $B(\varphi)$ does not depend on $\varphi$,

$$B = B(\varphi) = \left\{(\varrho, z) \mid 0 \le \varrho \le 1,\ \varrho^2 \le z \le \sqrt{2-\varrho^2}\right\}.$$

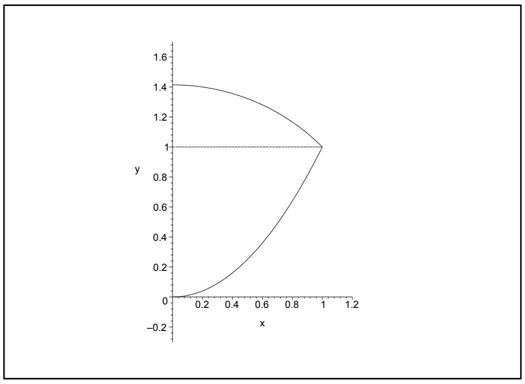



Figure 23.9: The meridian cut $B(\varphi)$ of **Example 23.1.3**.

It follows by the symmetry that

$$\begin{aligned}\int_A (x+y^2)z\,\mathrm{d}\Omega &= \int_A xz\,\mathrm{d}\Omega + \int_A y^2 z\,\mathrm{d}\Omega = 0 + \int_A y^2 z\,\mathrm{d}\Omega \\ &= \int_0^{2\pi}\left\{\int_{B(\varphi)} \varrho^2\sin^2\varphi\cdot z\cdot\varrho\,\mathrm{d}\varrho\,\mathrm{d}z\right\}\mathrm{d}z = \left\{\int_0^{2\pi}\sin^2\varphi\,\mathrm{d}\varphi\right\}\cdot\left\{\int_B z\varrho^2\,\mathrm{d}\varrho\,\mathrm{d}z\right\} \\ &= \pi\int_0^1 \varrho^3\left\{\int_{\varrho^2}^{\sqrt{2-\varrho^2}} z\,\mathrm{d}z\right\}\mathrm{d}\varrho = \frac{\pi}{2}\int_0^1 \varrho^3\left[z^3\right]_{z=\varrho^2}^{\sqrt{2-\varrho^2}}\mathrm{d}\varrho = \frac{\pi}{2}\int_0^1\varrho^3\left(2-\varrho^2-\varrho^4\right)\mathrm{d}\varrho \\ &= \frac{\pi}{2}\int_0^1\left\{2\varrho^3-\varrho^5-\varrho^7\right\}\mathrm{d}\varrho = \frac{\pi}{2}\left(\frac{1}{2}-\frac{1}{6}-\frac{1}{8}\right) = \frac{5\pi}{48}.\end{aligned}$$

The calculations in MAPLE are

with(Student[MultivariateCalculus])

$$\pi\cdot\text{MultiInt}\left(r^3\cdot z, z = r^2..\sqrt{2-r^2}, r = 0..1\right)$$

$$\frac{5}{48}\pi$$

4) Here $\varphi\in[0,2\pi]$, and $B(\varphi)$ does not depend on $\varphi$,

$$B = B(\varphi) = \left\{(\varrho,z)\;\middle|\;\frac{\varrho^2}{a}\le z\le h\right\} = \{(\varrho,z)\mid 0\le z\le h,\,0\le\varrho\le\sqrt{az}\}.$$

Then by the reduction theorem,

$$\begin{aligned}\int_A (x^2+y^2)\,\mathrm{d}\Omega &= \int_0^{2\pi}\left\{\int_B \varrho^2\cdot\varrho\,\mathrm{d}\varrho\,\mathrm{d}z\right\}\mathrm{d}\varphi = 2\pi\int_0^h\left\{\int_0^{\sqrt{az}}\varrho^3\,\mathrm{d}\varrho\right\}\mathrm{d}z \\ &= \frac{2\pi}{4}\int_0^h a^2z^2\,\mathrm{d}z = \frac{\pi a^2h^3}{6}.\end{aligned}$$

The calculations in MAPLE are

with(Student[MultivariateCalculus])

$$2\pi \cdot \text{MultiInt}\left(r^3, r = 0..\sqrt{a \cdot z},\ z = 0..h\right)$$

$$\frac{1}{6}\,a^2h^3$$

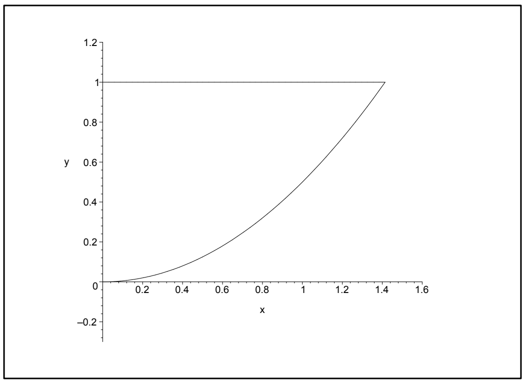



Figure 23.10: The meridian cut $B(\varphi)$ for $a = 2$ and $h = 1$ in **Example 23.1.4** and **Example 23.1.5**.

5) Here $\varphi \in \left[0, \frac{\pi}{2}\right]$. Note that $B = B(\varphi)$ is the same set as in 4),

$$B = B(\varphi) = \{(\varrho, z) \mid 0 \le z \le h,\ 0 \le \varrho \le \sqrt{az}\}.$$

Then by the reduction theorem,

$$\begin{aligned}\int_A xy\,d\Omega &= \int_0^{\frac{\pi}{2}} \left\{ \int_B \varrho^2 \cos\varphi \cdot \sin\varphi \cdot \varrho\,\mathrm{d}\varrho\,\mathrm{d}z \right\} \mathrm{d}\varphi \\ &= \int_0^{\frac{\pi}{2}} \cos\varphi \cdot \sin\varphi\,\mathrm{d}\varphi \cdot \int_0^h \left\{ \int_0^{\sqrt{az}} \varrho^3\,\mathrm{d}\varrho \right\} \mathrm{d}z = \left[\frac{\sin^2\varphi}{2}\right]_0^{\frac{\pi}{2}} \cdot \frac{1}{4} \int_0^h a^2 z^2\,\mathrm{d}z \\ &= \frac{a^2h^3}{24}.\end{aligned}$$

The calculations in MAPLE are

with(Student[MultivariateCalculus])

$$\text{MultiInt}\left(\cos(t) \cdot \sin(t) \cdot r^3, 1 = 0..\frac{\pi}{2}, r = 0..\sqrt{a \cdot z},\ z = 0..h\right)$$

$$\frac{1}{24}\,a^2h^3$$

6) It follows from $x^2 + y^2 \le 2x$ that

$$\varrho \le 2\cos\varphi, \qquad \varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

corresponding to the disc $(x-1)^2+y^2 \le 1$ in the $XY$-plane. Furthermore,

$$B(\varphi)=\{(\varrho,z)\mid 0\le\varrho\le 2\cos\varphi,\, 0\le z\le\varrho\},$$

which depends on $\varphi$. The domain of integration $A$ is obtained by removing the open cone $z>\sqrt{x^2+y^2}$ from the half infinite $(x-1)^2+y^2\le 1$.

We get by using the reduction theorem,

$$\begin{aligned}
\int_A xz\,\mathrm{d}\Omega &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left\{\int_{B(\varphi)} \varrho\cos\varphi\cdot z\cdot\varrho\,\mathrm{d}\varrho\,\mathrm{d}z\right\}\mathrm{d}\varphi\\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\varphi\left\{\int_0^{2\cos\varphi}\varrho^2\left[\int_0^{\varrho} z\,\mathrm{d}z\right]\mathrm{d}\varrho\right\}\mathrm{d}\varphi=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left\{\int_0^{2\cos\varphi}\varrho^4\,\mathrm{d}\varrho\right\}\mathrm{d}\varphi\\
&= \frac{1}{2\cdot 5}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\varphi\left[\varrho^5\right]_{\varrho=0}^{2\cos\varphi}\mathrm{d}\varphi=\frac{1}{5}\int_0^{\frac{\pi}{2}} 32\cdot\cos^6\varphi\,\mathrm{d}\varphi\\
&= \frac{4}{5}\int_0^{\frac{\pi}{2}}(\cos 2\varphi+1)^3\,\mathrm{d}\varphi=\frac{2}{5}\int_0^{\pi}(\cos t+1)^3\,\mathrm{d}t\\
&= \frac{2}{5}\int_0^{\pi}\left\{\cos^3 t+3\cos^2 t+3\cos t+1\right\}\mathrm{d}t\\
&= \frac{2}{5}\int_0^{\pi}\left\{(1-\sin^2 t)\cos t+\frac{3}{2}(\cos 2t+1)+3\cos t+1\right\}\mathrm{d}t\\
&= \frac{2}{5}\left[-\frac{1}{3}\sin^3 t+4\sin t+\frac{3}{4}\sin 2t+\frac{5}{2}t\right]_0^{\pi}=\pi.
\end{aligned}$$

The calculations in MAPLE are

with(Student[MultivariateCalculus])

$$\text{MultiInt}\left(\cos(t)\cdot r^2\cdot z, z=0..r, r=0..2\cos(t),\, t=-\frac{\pi}{2}..\frac{\pi}{2}\right)$$

$$\pi$$

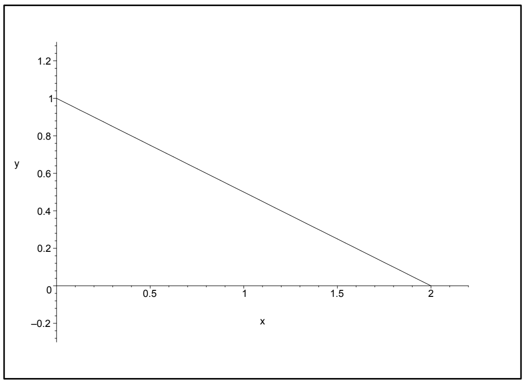



Figure 23.11: The meridian cut $B = B(\varphi)$ for $a = 2$ and $h = 1$ in **Example 23.1.7**.

7) Here $\varphi \in [0, 2\pi]$, and $\varrho \in [0, a]$, and $B = B(\varphi)$ does not depend on $\varphi$,

$$B = B(\varphi) = \left\{(\varrho, z) \;\middle|\; 0 \le \varrho \le a,\, 0 \le z \le h - \frac{h}{a}\,\varrho\right\}.$$

We get by the reduction theorem,

$$\begin{aligned}
\int_A (z^2+y^2)\,d\Omega &= \int_0^{2\pi}\left\{\int_B z^2\cdot\varrho\,d\varrho\,dz\right\}d\varphi + \int_0^{2\pi}\left\{\int_B \varrho^2\sin^2\varphi\cdot\varrho\,d\varrho\,dz\right\}d\varphi \\
&= 2\pi\int_0^a \varrho\left\{\int_0^{h(1-\frac{\varrho}{a})} z^2\,dz\right\}d\varrho + \int_0^{2\pi}\sin^2\varphi\,d\varphi\cdot\int_0^a \varrho^3\left\{\int_0^{h(1-\frac{\varrho}{a})} dz\right\}d\varrho \\
&= \frac{2\pi}{3}\int_0^a \varrho\cdot h^3\left(1-\frac{\varrho}{a}\right)^3 d\varrho + \pi\int_0^a \varrho^3\cdot h\left(1-\frac{\varrho}{a}\right)d\varrho \\
&= \frac{2\pi h^3}{3}\cdot a^2\int_0^a\left\{1-\left(1-\frac{\varrho}{a}\right)\right\}\left(1-\frac{\varrho}{a}\right)^3\frac{1}{a}\,d\varrho + \pi h a^4\int_0^a\left(\frac{\varrho}{a}\right)^3\cdot\left(1-\frac{\varrho}{a}\right)\frac{1}{a}\,d\varrho \\
&= \frac{2\pi h^3}{3}\cdot a^2\int_0^1 (1-t)t^3\,dt + \pi h a^4\int_0^1 t^3(1-t)\,dt = \pi h a^2\left(\frac{2}{3}h^2+a^2\right)\int_0^1 (t^3-t^4)\,dt \\
&= \frac{\pi h a^2}{20}\left(\frac{2}{3}h^2+a^2\right).
\end{aligned}$$

The calculations in MAPLE arewith(Student[MultivariateCalculus])

$$2\pi\cdot\text{MultiInt}\left(r\cdot z^2, r=0..h\cdot\left(1-\frac{r}{a}\right), r=0..a\right)+\text{MultiInt}\left(\sin(t)^2\cdot r^3, z=0..h\cdot\left(1-\frac{r}{a}\right), r=0..a\right)$$

$$\frac{1}{30}\pi a^2h^3 + \frac{1}{20}\pi h a^4$$

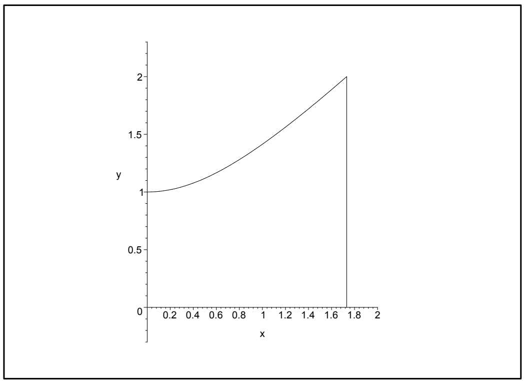



Figure 23.12: The meridian cut $B = B(\varphi)$ in **Example 23.1.8** and **Example 23.1.9**.

8) Here $\varphi \in [0, 2\pi]$ and $0 \le \varrho \le \sqrt{3}$, and

$$B = B(\varphi) = \{(\varrho, z) \mid 0 \le \varrho \le \sqrt{3},\, 0 \le z \le \sqrt{1+\varrho^2}\},$$

which is independent of $\varphi$.
By the reduction theorem,

$$\int_A (x^2+y^2)\,\mathrm{d}\Omega = \int_0^{2\pi} \left\{ \int_B \varrho^2 \cdot \varrho \,\mathrm{d}\varrho \,\mathrm{d}z \right\} \mathrm{d}\varphi = 2\pi \int_0^{\sqrt{3}} \varrho^3 \left\{ \int_0^{\sqrt{1+\varrho^2}} \mathrm{d}z \right\} \mathrm{d}\varrho$$

$$= 2\pi \int_0^{\sqrt{3}} \varrho^2 \sqrt{1+\varrho^2}\,\mathrm{d}\varrho = \pi \int_0^3 t\sqrt{1+t}\,\mathrm{d}t = \pi \int_0^3 \left\{ (1+t)^{\frac{3}{2}} - (1+t)^{\frac{1}{2}} \right\} \mathrm{d}t$$

$$= \pi \left[ \frac{2}{5}(1+t)^{\frac{2}{5}} - \frac{2}{3}(1+t)^{\frac{3}{2}} \right]_0^3 = 2\pi \left( \frac{1}{5}\left\{ 4^{\frac{5}{2}} - 1 \right\} - \frac{1}{3}\left\{ 4^{\frac{3}{2}} - 1 \right\} \right)$$

$$= 2\pi \left( \frac{31}{5} - \frac{7}{3} \right) = \frac{116\pi}{15}.$$

The calculations in MAPLE are

with(Student[MultivariateCalculus])

$$2\pi \cdot \text{MultiInt}\left( r^3, z = 0..\sqrt{1+r^2},\, r = 0..\sqrt{3} \right)$$

$$\frac{116}{15}\,\pi$$

9) The domain of integration is the same as in **Example 23.1.8**, so $\varphi \in [0, 2\pi]$, and

$$B = B(\varphi) = \{(\varrho, z) \mid 0 \le \varrho \le \sqrt{3},\, 0 \le z \le \sqrt{1+\varrho^2}\}.$$

Now $A$ is symmetric with respect to e.g. the plane $y = 0$, so

$$\int_A xy\,\mathrm{d}\Omega = 0.$$

ALTERNATIVELY we have the following calculation

$$\int_A xy\,d\Omega = \int_0^{2\pi}\left\{\int_B \varrho^2 \sin\varphi\cdot\cos\varphi\cdot\varrho\,d\varrho\,dz\right\}d\varphi = \int_0^{2\pi}\sin\varphi\cdot\cos\varphi\,d\varphi\cdot\int_B \varrho^3\,d\varrho\,dz = 0,$$

where we have used that $B$ does not depend on $\varphi$ and also that

$$\int_0^{2\pi}\sin\varphi\cdot\cos\varphi\,d\varphi = \left[\frac{\sin^2\varphi}{2}\right]_0^{2\pi} = 0.$$

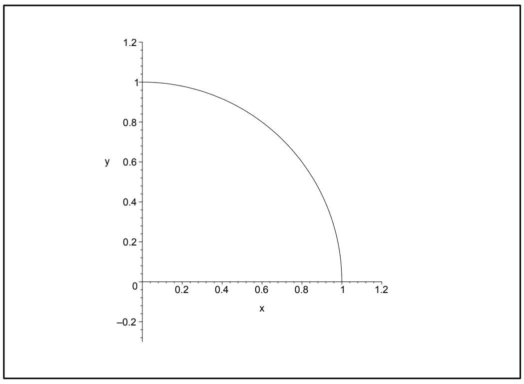



Figure 23.13: The meridian cut $B = B(\varphi)$ in **Example 23.1.10**.

10) Here $A$ is the half ball in the half space $z \geq 0$ of centrum $(0,0,0)$ and radius $a$, thus $\varphi \in [0, 2\pi]$, and $B(\varphi)$ does not depend on $\varphi$,

$$B = B(\varphi) = \{(\varrho, z) \mid 0 \leq \varrho \leq a,\ 0 \leq z \leq \sqrt{a^2-\varrho^2}\}.$$

By the reduction theorem,

$$\int_A (x^2z+z^3)\,d\Omega = \int_0^{2\pi}\left\{\int_B (\varrho^2\cos^2\varphi\cdot z+z^3)\varrho\,d\varrho\,dz\right\}d\varphi$$

$$= \int_0^{2\pi}\cos^2\varphi\,d\varphi\cdot\int_0^a \varrho^3\left\{\int_0^{\sqrt{a^2-\varrho^2}} z\,dz\right\}d\varrho + 2\pi\int_0^a \varrho\left\{\int_0^{\sqrt{a^2-\varrho^2}} z^3\,dz\right\}d\varrho$$

$$= \pi\cdot 12\int_0^a (a^2\varrho^3-\varrho^5)\,d\varrho + \frac{2\pi}{4}\int_0^a \varrho(a^2-\varrho^2)^2\,d\varrho$$

$$= \frac{\pi}{2}\left[\frac{a^2}{4}\varrho^4 - \frac{1}{6}\varrho^6\right]_0^a + \frac{\pi}{2}\cdot\frac{1}{2}\int_0^{a^2}(a^2-t)^2\,dt$$

$$= \frac{\pi}{2}\left(\frac{a^6}{4}-\frac{a^6}{6}\right) + \frac{\pi}{12}\left[-(a^2-t)^3\right]_0^{a^2} = \frac{\pi a^6}{24} + \frac{\pi a^6}{12} = \frac{\pi a^6}{8}.$$

The calculations in MAPLE arewith(Student[MultivariateCalculus])

$$\text{MultiInt}\left((r^2\cdot\cos(t)^2\cdot z + z^3)\cdot r, t = 0..2\pi, z = 0..\sqrt{a^2-r^2}, r = 0..a\right)$$

$$\frac{1}{8}\pi a^6$$

11) Here $x^2+y^2 \le ax$, hence $\varrho \le a\cos\varphi$, and $0 \le z \le \dfrac{1}{a}\,\varrho^2\cos^2\varphi$, and $\varphi \in \left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$, so

$$B(\varphi)=\left\{(\varrho,z)\;\middle|\; 0\le\varrho\le a\cos\varphi,\; 0\le z\le \frac{1}{a}\,\varrho^2\cos^2\varphi\right\}.$$

Clearly, $B(\varphi)$ depends on $\varphi$, so we can only conclude that any meridian curve for fixed $\varphi$ is a parabola in the $PZ$-plane, and there is no need to sketch it.

The set $A$ is symmetric with respect to the plane $y=0$, so by the reduction theorem,

$$\begin{aligned}
\int_A |y|z\,d\Omega &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left\{\int_{B(\varphi)} \varrho|\sin\varphi|\cdot z\cdot\varrho\,d\varrho\,dz\right\}d\varphi = 2\int_0^{\frac{\pi}{2}}\left\{\int_{B(\varphi)} \varrho\sin\varphi\cdot z\cdot\varrho\,d\varrho\,dz\right\}d\varphi\\
&= 2\int_0^{\frac{\pi}{2}}\left\{\int_0^{a\cos\varphi}\varrho^2\sin\varphi\left\{\int_0^{\frac{1}{a}\varrho^2\cos^2\varphi} z\,dz\right\}d\varrho\right\}d\varphi\\
&= 2\int_0^{\frac{\pi}{2}}\sin\varphi\left\{\int_0^{a\cos\varphi}\varrho^2\left[\frac{z^2}{2}\right]_{z=0}^{\frac{1}{a}\varrho^2\cos^2\varphi}d\varrho\right\}d\varphi\\
&= \int_0^{\frac{\pi}{2}}\sin\varphi\left\{\int_0^{a\cos\varphi}\frac{1}{a^2}\,\varrho^6\cos^4\varphi\,d\varrho\right\}d\varphi\\
&= \frac{1}{a^2}\int_0^{\frac{\pi}{2}}\sin\varphi\cdot\cos^4\varphi\left\{\int_0^{a\cos\varphi}\varrho^6\,d\varrho\right\}d\varphi\\
&= \frac{1}{7a^2}\int_0^{\frac{\pi}{2}}\sin\varphi\cdot\cos^4\varphi\cdot a^7\cos^7\varphi\,d\varphi = \frac{a^5}{7}\left[-\frac{\cos^{12}\varphi}{12}\right]_0^{\frac{\pi}{2}} = \frac{a^5}{84}.
\end{aligned}$$

The calculations in MAPLE are

with(Student[MultivariateCalculus])

$$2 \cdot \text{MultiInt}\left(r^2 \cdot \sin(t) \cdot z, z = 0..\frac{1}{a} \cdot r^2 \cdot \cos(t)^2, r = 0..a \cdot \cos(t), t = 0..\frac{\pi}{2}\right)$$

$$\frac{1}{84}\, a^5$$

12) Here $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and we have for any fixes $\varphi$ that

$$B(\varphi) = \left\{(\varrho, z) \;\middle|\; 0 \le x \le h,\, 0 \le \varrho \le a\left(1 - \frac{z}{h}\right)\right\}.$$

The meridian cut does not depend on $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Note that it is identical with the meridian cut in **Example 23.1.7**.

We get by the reduction theorem in semi-polar coordinates,

$$\begin{aligned}
\int_A xz\,\mathrm{d}\Omega &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ \int_{B(\varphi)} \varrho\cos\varphi \cdot z \cdot \varrho\,\mathrm{d}\varrho\,\mathrm{d}z \right\} \mathrm{d}\varrho \\
&= \int_{-\frac{\pi}{2}}^{\frac{\varphi}{2}} \cos\varphi \left\{ \int_0^h z \left( \int_0^{a(1-\frac{z}{h})} \varrho^2\,\mathrm{d}\varrho \right) \mathrm{d}z \right\} \mathrm{d}\varphi = [\sin\varphi]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^h z \left[\frac{1}{3}\,\varrho^3\right]_0^{a(1-\frac{z}{h})} \mathrm{d}z \\
&= 2 \cdot \frac{a^3}{3} \int_0^h z\left(1 - \frac{z}{h}\right)^3 \mathrm{d}z = \frac{2}{3}\,a^3 \int_0^h z\left(1 - \frac{3}{h}\,z + \frac{3}{h^2}\,z^2 - \frac{1}{h^3}\,z^3\right) \mathrm{d}z \\
&= \frac{2}{3}\,a^3 \int_0^h \left(z - \frac{3}{h}\,z^2 + \frac{3}{h^2}\,z^3 - \frac{1}{h^3}\,z^4\right) \mathrm{d}z = \frac{2}{3}\,a^3 \left[\frac{1}{2}\,z^2 - \frac{1}{h}\,z^3 + \frac{3}{4h^2}\,z^4 - \frac{1}{5h^2}\,z^5\right]_0^h \\
&= \frac{2}{3}\,a^3h^2\left(\frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5}\right) = \frac{2}{3}\left(\frac{1}{4} - \frac{1}{5}\right) a^3h^2 = \frac{1}{30}\,a^3h^2.
\end{aligned}$$

The calculations in MAPLE arewith(Student[MultivariateCalculus])

$$\text{MultiInt}\left(\cos(t) \cdot z \cdot r^2, r = 0..a \cdot \left(1 - \frac{z}{h}\right), z = 0..h, t = -\frac{\pi}{2}..\frac{\pi}{2}\right)$$

$$\frac{1}{30}\, a^3 h^2$$

13) In this case we integrate over the same set as in **Example 22.1.8**. Then by the reduction theorem in semi-polar coordinates followed by the change of variables $u = 2 - z$,

$$\begin{aligned}
\int_A z\,\mathrm{d}\Omega &= \int_0^2 z \cdot \pi(2-z)^2\,\mathrm{d}z = \pi \int_0^2 (2-u)u^2\,\mathrm{d}u = \pi \int_0^2 (2u^2 - u^3)\,\mathrm{d}u \\
&= \pi\left[\frac{2}{3}\,u^3 - \frac{1}{4}\,u^4\right]_0^2 = \pi\left[\frac{16}{3} - \frac{16}{4}\right] = \frac{16}{12}\,\pi = \frac{4\pi}{3}.
\end{aligned}$$

**Example 23.2** *Consider two balls and their intersection*

$$\Omega_1 = \overline{K}((0,0,0);a), \qquad \Omega_2 = \overline{K}\left((0,0,a);\frac{a}{2}\right), \qquad \Omega = \Omega_1 \cap \Omega_2.$$

*1) Sketch the three point sets by means of a meridian half plane, and describe the position of the intersection circle* $\partial\Omega_1 \cap \partial\Omega_2$.

*2) Find the volume of* $\Omega$.

*3) Compute the space integral*

$$\int_\Omega (2-xy)\,\mathrm{d}\Omega.$$

**A** Space integrals.

**D** Follow the given guidelines.

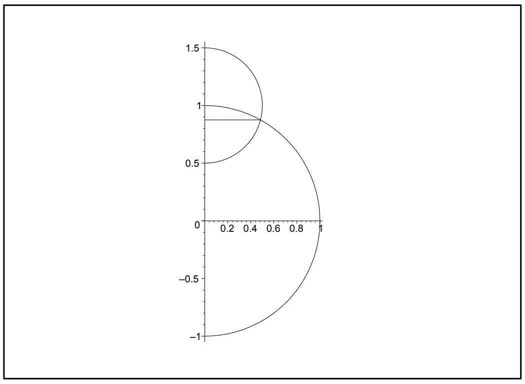



Figure 23.14: The situation in the meridian half plane for $a=1$.

**I** 1) The two circles cut each other at height $z \in \left]\frac{a}{2},a\right[$. Then by Pythagoras's theorem,

$$r^2 = a^2 - z^2 = \left(\frac{a}{2}\right)^2 - (a-z)^2 = -\frac{3}{4}\,a^2 + 2az - z^2.$$

A reduction gives $2az = \frac{7}{4}\,a^2$, thus $z = \frac{7}{8}\,a$, which indicates the whereabouts of the plane, in which the intersection circle $\partial\Omega_1 \cap \partial\Omega_2$ lies.

2) Then split $\Omega = \omega_1 \cup \omega_2$ into its two natural subregions, where $\omega_1$ lies above the plane $z = \frac{7}{8}\,a$, and $\omega_2$ lies below the same plane. We use in each of the subregions $\omega_1$ and $\omega_2$ the "method of slices", where each slice is parallel to the $(x,y)$-plane. By translating the subregion $\omega_2$ in a

convenient way we finally get

$$\begin{aligned}
\operatorname{vol}(\Omega) &= \operatorname{vol}(\omega_1)+\operatorname{vol}(\omega_2) = \int_{\frac{7}{8}a}^{a} \pi\left(a^2-z^2\right)\,\mathrm{d}z + \int_{-\frac{1}{2}a}^{-\frac{1}{8}a} \pi\left(\frac{a^2}{4}-z^2\right)\,\mathrm{d}z \\
&= \pi\left[a^2 z-\frac{1}{3}z^3\right]_{\frac{7}{8}a}^{a} + \pi\left[\frac{a^2}{4}z-\frac{1}{3}z^3\right]_{\frac{1}{8}a}^{\frac{1}{2}a} \\
&= \pi a^3\left\{\left(1-\frac{1}{3}-\frac{7}{8}+\frac{1}{3}\cdot\left(\frac{7}{8}\right)^3\right)+\left(\frac{1}{4}\cdot\frac{1}{2}-\frac{1}{3}\cdot\frac{1}{8}-\frac{1}{4}\cdot\frac{1}{8}+\frac{1}{3}\cdot\left(\frac{1}{8}\right)^3\right)\right\} \\
&= \frac{\pi a^3}{8}\left\{1-\frac{8}{3}+\frac{7}{3}\cdot\left(\frac{7}{8}\right)^2+1-\frac{1}{3}-\frac{1}{4}+\frac{1}{3}\cdot\frac{1}{8^2}\right\} \\
&= \frac{\pi a^3}{24}\left\{-3-\frac{3}{4}+\frac{1}{64}(343+1)\right\} = \frac{\pi a^3}{24}\left\{-\frac{15}{4}+\frac{43}{8}\right\} = \frac{13}{192}\,\pi a^3.
\end{aligned}$$

3) Of symmetric reasons, $\int_\Omega xy\,\mathrm{d}\Omega = 0$, hence

$$\int_\Omega (2-xy)\,\mathrm{d}\Omega = 2\cdot\operatorname{vol}(\Omega) = \frac{13}{96}\,\pi a^3.$$

**Example 23.3** *Given a curve $\mathcal{K}$ in the $(z,x)$-plane of the equation*

$$x = \cos z, \qquad z \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

*The curve $\mathcal{K}$ is rotated once around the $z$-axis in the $(x,y,z)$-space, creating the surface of revolution $\mathcal{F}$. Let $A$ denote the bounded domain in the $(x,y,z)$-space with $\mathcal{F}$ as its boundary surface.*

*1) Find the volume of $A$.*

*2) Compute the space integral*

$$\int_A \sqrt{x^2+y^2}\,\mathrm{d}\Omega.$$

**A** Body of revolution an space integral.

**D** Sketch a figure and then just compute.

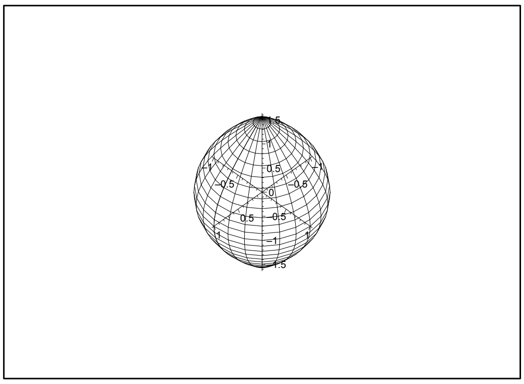


Figure 23.15: The domain $A$ with the boundary surface $\mathcal{F}$.

**I** 1) The domain $A$ is the spindle shaped body on the figure.
We get by slicing the body,

$$\operatorname{vol}(A) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi\cos^2 z\,\mathrm{d}z = 2\pi \int_0^{\frac{\pi}{2}} \frac{1+\cos 2z}{2}\,\mathrm{d}z = 2\pi\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi^2}{2}.$$

2) If we put

$$B_z = \{(x,y) \mid \sqrt{x^2+y^2} \le \cos z\}, \qquad z \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

then

$$\begin{aligned}
\int_A \sqrt{x^2+y^2}\,\mathrm{d}\Omega &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{\int_{B_z} \sqrt{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y\right\}\mathrm{d}z = 2\int_0^{\frac{\pi}{2}} \left\{2\pi\int_0^{\cos z} \varrho\cdot\varrho\,\mathrm{d}\varrho\right\}\mathrm{d}z \\
&= 4\pi \int_0^{\frac{\pi}{2}} \left[\frac{\varrho^3}{3}\right]_0^{\cos z}\mathrm{d}z = \frac{4\pi}{3}\int_0^{\frac{\pi}{2}} \cos^3 z\,\mathrm{d}z \\
&= \frac{4\pi}{3}\int_0^{\frac{\pi}{2}} (1-\sin^2 z)\cos z\,\mathrm{d}z = \frac{4\pi}{3}\left[\sin z - \frac{1}{3}\sin^3 z\right]_0^{\frac{\pi}{2}} = \frac{8\pi}{9}.
\end{aligned}$$

**Example 23.4** *Let $c$ be a positive constant. Consider the half ball $A$ given by the inequalities*

$$x^2+y^2+z^2 \le c^2, \qquad z \ge 0.$$

*1) Compute the space integral*

$$J = \int_A z\,\mathrm{d}\Omega.$$

*2) Show that both the space integrals $\int_A x\,\mathrm{d}\Omega$ and $\int_A y\,\mathrm{d}\Omega$ are zero.*

**A** Space integrals.

**D** Apply the slicing method and convenient symmetric arguments.
Alternatively, reduce in

1) spherical coordinates,
2) semi-polar coordinates,
3) rectangular coordinates.

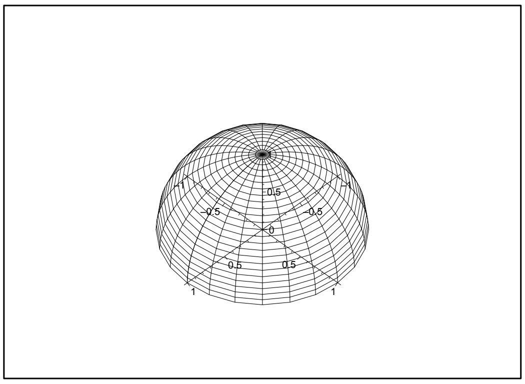


Figure 23.16: The half ball $A$ for $c = 1$.

**I** 1) **First variant.** *The slicing method.*
At the height $z$ the body $A$ is cut into a disc $B(z)$ of radius $\sqrt{c^2-z^2}$, hence of area $(c^2-z^2)\pi$. Then we get by the slicing method,

$$J = \int_A z\,\mathrm{d}\Omega = \int_0^c z\ \mathrm{area}(B(z))\,\mathrm{d}z = \pi\int_0^c \{c^2z-z^3\}\,\mathrm{d}z = \pi\left[c^2\cdot\frac{z^2}{2}-\frac{z^4}{4}\right]_0^c = \frac{\pi}{4}\,c^4.$$

**Second variant.** *Spherical coordinates.*
The set $A$ is in spherical coordinates described by

$$\begin{cases} x = r\sin\theta\cos\varphi, \\ y = r\sin\theta\sin\varphi, \\ z = r\cos\theta, \end{cases} \qquad \begin{cases} \varphi\in[0,2\pi], \\ \theta\in\left[0,\dfrac{\pi}{2}\right], \\ r\in[0,c], \end{cases}$$

and $d\Omega = r^2 \sin\theta \, dr \, d\theta \, d\varphi$. Thus we get by reduction

$$\begin{aligned} J &= \int_A z \, d\Omega = \int_0^{2\pi} \left\{ \int_0^{\frac{\pi}{2}} \left( \int_0^c r\cos\theta \cdot r^2 \sin\theta \, dr \right) d\theta \right\} d\varphi \\ &= 2\pi \cdot \left[ \frac{1}{2} \sin^2\theta \right]_0^{\frac{\pi}{2}} \cdot \left[ \frac{r^4}{4} \right]_0^c = 2\pi \cdot \frac{1}{2} \cdot \frac{1}{4} c^4 = \frac{\pi}{4} c^4. \end{aligned}$$

**Third variant.** *Semi-polar coordinates.*

In semi-polar coordinates $A$ is described by

$$\begin{cases} x = \varrho\cos\varphi, \\ y = \varrho\sin\varphi, \\ z = z, \end{cases} \qquad \begin{cases} \varphi \in [0, 2\pi], \\ z \in [0, c], \\ \varrho \in \left[0, \sqrt{c^2 - z^2}\right], \end{cases}$$

and $d\Omega = \varrho \, d\varrho \, d\varphi \, dz$. We therefore get by reduction

$$\begin{aligned} J &= \int_A z \, d\Omega = \int_0^{2\pi} \left\{ \int_0^c \left( \int_0^{\sqrt{c^2-z^2}} z \cdot \varrho \, d\varrho \right) dz \right\} d\varphi \\ &= 2\pi \int_0^c z \left[ \frac{1}{2} \varrho^2 \right]_0^{\sqrt{c^2-z^2}} dz = \pi \int_0^c \left( c^2 z - z^3 \right) dz = \frac{\pi}{4} c^4. \end{aligned}$$

**Fourth variant.** *Rectangular coordinates.*

Here $A$ is described by

$$0 \le z \le c, \quad |x| \le \sqrt{c^2 - z^2}, \quad |y| \le \sqrt{c^2 - z^2 - x^2},$$

hence

$$\begin{aligned} J &= \int_A z \, d\Omega = \int_0^c z \left\{ \int_{-\sqrt{c^2-z^2}}^{\sqrt{c^2-z^2}} \left( \int_{-\sqrt{c^2-z^2-x^2}}^{\sqrt{c^2-z^2-x^2}} dy \right) dx \right\} dz \\ &= 2 \int_0^c z \left\{ \int_{-\sqrt{c^2-x^2}}^{\sqrt{c^2-z^2}} \sqrt{c^2 - z^2 - x^2} \, dx \right\} dz. \end{aligned}$$

We then get by the substitution $x = \sqrt{c^2 - z^2} \cdot t$,

$$\begin{aligned} J &= 4 \int_0^c z \left\{ \int_0^{\sqrt{c^2-z^2}} \sqrt{(c^2 - z^2) - x^2} \, dx \right\} dz \\ &= 4 \int_0^c z \left\{ \int_0^1 \left( \sqrt{c^2 - z^2} \right)^2 \cdot \sqrt{1 - t^2} \, dt \right\} dz \\ &= 4 \int_0^c z(c^2 - z^2) \, dz \cdot \int_0^1 \sqrt{1 - t^2} \, dt = c^4 \cdot \frac{\pi}{4}, \end{aligned}$$

where there are lots of similar variants.

2) **First variant.** *A symmetric argument.*

The set $A$ is symmetric with respect to the planes $y = 0$ and $x = 0$, and the integrand $x$, resp. $y$, is an odd function. Hence,

$$\int_A x \, d\Omega = 0 \quad \text{and} \quad \int_A y \, d\Omega = 0.$$

**Second variant.** *Spherical coordinates.*

By insertion,

$$\begin{aligned}\int_A x\,\mathrm{d}\Omega &= \int_0^{2\pi}\left\{\int_0^{\frac{\pi}{2}}\left(\int_0^c r\sin\theta\cos\varphi\cdot r^2\sin\theta\,\mathrm{d}r\right)\mathrm{d}\theta\right\}\mathrm{d}\varphi \\ &= \int_0^{2\pi}\cos\varphi\,\mathrm{d}\varphi\cdot\int_0^{\frac{\pi}{2}}\sin^2\theta\,\mathrm{d}\theta\cdot\int_0^c r^3\,\mathrm{d}r = [\sin\varphi]_0^{2\pi}\cdot\frac{\pi}{4}\cdot\frac{c^4}{4} = 0,\end{aligned}$$

and similarly.

**Third and fourth variant.** These are similar to the previous semi-polar and rectangular cases.

# 24 The space integral in spherical coordinates

## 24.1 Reduction theorem in spherical coordinates

We shall finally introduce the *spherical coordinates* in the reduction of space integrals. We shall here only sketch the idea and then quote the theorem without proof.

When we considered the semi-polar coordinates, we used rectangular coordinates $(\varrho, z)$ in the *meridian half-plane*, and the weight function is $\varrho\,\mathrm{d}\varrho\,\mathrm{d}z$ in $\mathbb{R}^3$, when we use semi-polar coordinates.

If we now instead use *polar coordinates* $(r, \theta)$ in the meridian half-plane, where $\theta$ for convenience *is measured from the $z$-axis, and the usual orientation is changed to the opposite one*, so we always have $\theta \in [0, \pi]$), then – cf. Chapter 20 – we shall replace the area element $\mathrm{d}\varrho\,\mathrm{d}z$ with $r\,\mathrm{d}r\,\mathrm{d}\theta$. Since $\varrho = r\sin\theta$ and $z = r\cos\theta$, because $\theta$ is measured from the $z$-axis, we therefore see that we have the formal calculation of the volume element in spherical coordinates,

$$\varrho\,\mathrm{d}\varrho\,\mathrm{d}z\,\mathrm{d}\varphi = \{\varrho\}\{\,\mathrm{d}\varrho\,\mathrm{d}z\}\,\mathrm{d}\varphi = \{r\sin\theta\}\{r\,\mathrm{d}r\,\mathrm{d}\theta\}\,\mathrm{d}\varphi = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi,$$

which also turns up to be the right volume for infinitesimal small bodies. In particular, we see that the *weight function* here is $r^2\sin\theta$.

In semi-polar coordinates we fixed $\varphi$ in order to get the meridian half-plane. This meridian half-plane then cuts $A$ in some set $B(\varphi)$, over which we integrate with respect to the rectangular coordinates $(\varrho, z)$. When we instead use spherical coordinates, this domain of integration is transformed into another domain of integration $\tilde{B}(\varphi)$ in the variables $(r, \theta)$. This may seem very abstract and strange, but in the applications one will never doubt, what to do. We shall use the notation $\tilde{B}(\varphi)$ in the formulation of the reduction theorem below.

**Theorem 24.1** Reduction theorem for a space integral in spherical coordinates. *Let $A \subset \mathbb{R}^3$ be a closed and bounded set, and assume that $A$ in spherical coordinates is described by its parameter domain in the form*

$$\tilde{A} = \left\{(r, \theta, \varphi) \mid \alpha \le \varphi \le \beta,\ (r, \theta) \in \tilde{B}(\varphi)\right\},$$

*where the constants $\alpha$ and $\beta$ satisfy $0 < \beta - \alpha \le 2\pi$, and where we for each fixed $\varphi$ have given the domain of integration $\tilde{B}(\varphi)$, as described above.*

*If $f : A \to \mathbb{R}$ is continuous, then the space integral of $f$ over $A$ is reduced in the following way,*

$$\int_A f(x, y, z)\,\mathrm{d}\Omega = \int_\alpha^\beta \left\{\int_{\tilde{B}\varphi} f(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta) r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\right\}\mathrm{d}\varphi,$$

This is not the only version of the reduction theorem. We mention among others the following. With some more information on the description of $A$ in spherical coordinates we also have a version with a triple integral.

**Theorem 24.2** Triple integral for a space integral in spherical coordinates. *Let $A \subset \mathbb{R}^3$ be a closed and bounded set, and assume that $A$ in spherical coordinates is described by its parameter domain in the form*

$$\tilde{A} = \{(r,\theta,\varphi) \mid \alpha \le \varphi \le \beta,\, \Theta_1(\varphi) \le \theta \le \Theta_2(\varphi),\, R_1(\theta,\varphi) \le r \le R_2(\theta,\varphi)\}\,,$$

*where the constants $\alpha$ and $\beta$ satisfy $0 < \beta - \alpha \le 2\pi$.If $f : A \to \mathbb{R}$ is continuous, then the space integral of $f$ over $A$ is reduced in the following way,*

$$\int_A f(x,y,z)\,\mathrm{d}\Omega = \int_\alpha^\beta \left\{ \int_{\Theta_1(\varphi)}^{\Theta_2(\varphi)} \left\{ \int_{R_1(\theta,\varphi)}^{R_2(\theta,\varphi)} f(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta) r^2\,\mathrm{d}r \right\} \sin\theta\,\mathrm{d}\theta \right\} \mathrm{d}\varphi.$$

## 24.2 Procedures for reduction of space integral in spherical coordinates

We use here the *spherical* coordinates

$$x = r\,\sin\theta\cos\varphi, \qquad y = r\,\sin\theta\sin\varphi, \qquad z = r\,\cos\theta,$$

with the corresponding *volume element*

$$\mathrm{d}\Omega = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi,$$

i.e. the *weight function* is $r^2\sin\theta$.

The reduction formula is not easy to comprehend,

$$\int_A f(x,y,z)\,\mathrm{d}\Omega = \int_\alpha^\beta \left\{ \int_{B^\star(\varphi)} f(r\,\sin\theta\cos\varphi, r\,\sin\theta\sin\varphi, r\,\cos\theta)\, r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta \right\} \mathrm{d}\varphi.$$

**Procedure:**

1) If $A$ is a reasonable subset of a ball, then write in *spherical* coordinates

   $$A = \{(r,\theta,\varphi) \mid \alpha \le \varphi \le \beta, (r,\theta) \in B^\star(\varphi)\},$$

   where $B^\star(\varphi)$ is the *meridian cut* for fixed $\varphi \in [\alpha,\beta]$, expressed in the *spherical* coordinates. Identify and sketch $B^\star(\varphi)$.

   **Remark 24.1** This is the most difficult part of this version. $\Diamond$

2) Keep $\varphi \in [\alpha,\beta]$ fixed and apply the methods from Chapter 20 in the calculation of the *abstract (inner) plane integral*,

   $$H(\varphi) := \int_{B^\star(\varphi)} f(r\,\sin\theta\cos\varphi, r\,\sin\theta\sin\varphi, r\,\cos\theta)\, r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta.$$

   Do not forget the weight function $r^2\sin\theta$ here as a factor of the integrand.

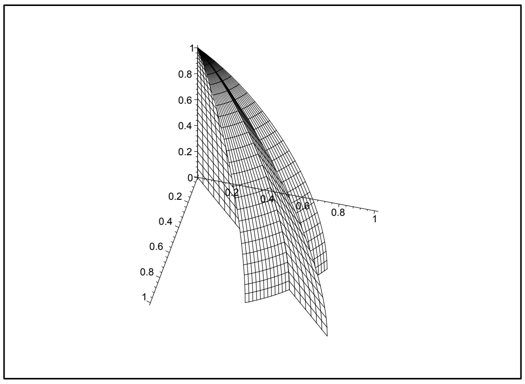



Figure 24.1: The meridian cut $B^\star(\varphi)$ for fixed $\varphi$.

3) Insert the result and calculate the ordinary integral on the right hand side,

$$\int_A f(x,y,z)\,\mathrm{d}\Omega = \int_\alpha^\beta H(\varphi)\,\mathrm{d}\varphi.$$

**Remark 24.2** If $\Omega$ is a *rotational domain*, then $B^\star(\varphi) = B^\star$ is independent of $\varphi$. In this case one gets simpler calculations by interchanging the order of integration

$$\int_A f(x,y,z)\,\mathrm{d}\Omega = \int_{B^\star}\left\{\int_\alpha^\beta f(r\,\sin\theta\,\cos\varphi, r\,\sin\theta\,\sin\varphi, r\,\cos\theta)\,\mathrm{d}\varphi\right\} r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta.$$

Notice that the inner integral (which is calculated first)

$$\int_\alpha^\beta f(r\,\sin\theta\,\cos\varphi, r\,\sin\theta\,\sin\varphi, r\,\cos\theta)\,\mathrm{d}\varphi$$

becomes much simpler, because the weight $r^2\sin\theta$ only is added as a factor *after* the calculation of this integral. $\Diamond$

## 24.3 Examples of space integrals in spherical coordinates

**Example 24.1**

**A.** *Let $A$ be an upper half sphere of radius $2a$, from which we have removed a cylinder of radius $a$ and then halved the resulting domain by the plane $x+y=0$. We shall only consider that part for which $x+y \geq 0$. Calculate the space integral*

$$\int_A xz \, d\Omega.$$

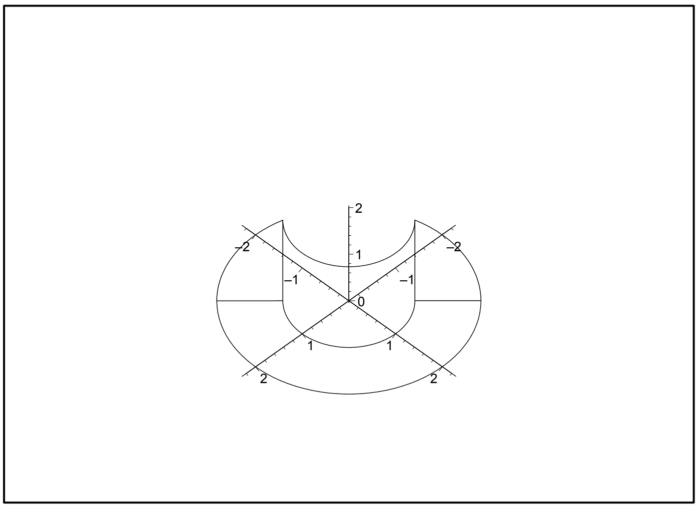


Figure 24.2: The domain $A$ for $a = 1$ in the $(x, y, z)$-space.

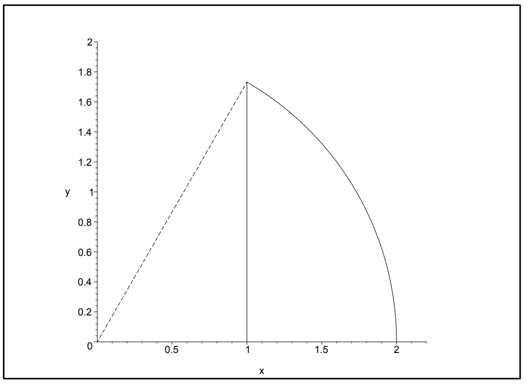



Figure 24.3: The cut in the meridian half-plane for $a = 1$, i.e. in the $(\varrho, z)$-half-plane.

When we consider the dimensions (i.e. a rough overview) we get

$$x \sim a, \quad y \sim a, \quad z \sim a, \quad \int_A \cdots \, d\Omega \sim a^3,$$

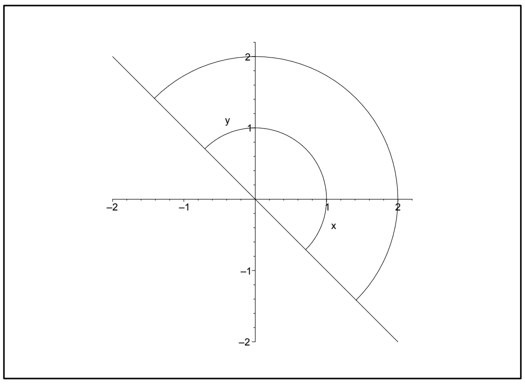



Figure 24.4: The projection of $A$ onto the $(x,y)$-plane for $a = 1$.

from which $\int_A xz\,d\Omega \sim a^5$, and thus

$$\int_A xz\,d\Omega = \text{ constant} \cdot a^5.$$

**D.** The geometrical structure of revolution and the sphere indicates that one either should apply **I 1.** *semi-polar coordinates* or **I 2.** *spherical* coordinates. We shall in the following go through both possibilities for comparison.

**I 1.** In *semi-polar* coordinates the domain $A$ is represented by

$$\tilde{A}=\left\{(\varrho,\varphi,z)\;\middle|\; a\le\varrho\le 2a,-\frac{\pi}{4}\le\varphi\le\frac{3\pi}{4},\,0\le z\le\sqrt{4a^2-\varrho^2}\right\}.$$

Hence by the reduction theorem (where the weight function is $\varrho$),

$$\begin{aligned}
I &= \int_A xz\,\mathrm{d}\Omega=\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}\left\{\int_a^{2a}\left\{\int_0^{\sqrt{4a^2-\varrho^2}}\varrho\cos\varphi\cdot z\,\mathrm{d}z\right\}\varrho\,\mathrm{d}\varrho\right\}\mathrm{d}\varphi\\
&= \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}\cos\varphi\,\mathrm{d}\varphi\cdot\int_a^{2a}\varrho^2\left\{\int_0^{\sqrt{4a^2-\varrho^2}}z\,\mathrm{d}z\right\}\mathrm{d}\varrho\\
&= [\sin\varphi]_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}\cdot\int_a^{2a}\varrho^2\left[\frac{1}{2}z^2\right]_0^{\sqrt{4a^2-\varrho^2}}\mathrm{d}\varrho=\sqrt{2}\cdot\frac{1}{2}\int_a^{2a}\varrho^2\left(4a^2-\varrho^2\right)\mathrm{d}\varrho\\
&= \frac{\sqrt{2}}{2}\int_a^{2a}\left(4a^2\varrho^2-\varrho^4\right)\mathrm{d}\varrho=\frac{\sqrt{2}}{2}\left[\frac{4}{3}a^2\varrho^3-\frac{1}{5}\varrho^5\right]_a^{2a}\\
&= \frac{\sqrt{2}}{2}\left\{\left(\frac{4}{4}3\,a^2\cdot 8a^3-\frac{32}{5}a^5\right)-\left(\frac{4}{3}a^2\cdot a^3-\frac{1}{5}a^5\right)\right\}\\
&= \frac{\sqrt{2}}{2}a^5\left\{\frac{32}{3}-\frac{32}{5}-\frac{4}{3}+\frac{1}{5}\right\}\\
&= \frac{\sqrt{2}}{2}a^5\cdot\left\{\frac{28}{3}-\frac{31}{5}\right\}=\frac{47\sqrt{2}\,a^5}{30}.
\end{aligned}$$

**I 2.** If we instead choose *spherical* coordinates then

$$x=r\,\sin\theta\,\cos\varphi,\qquad y=r\,\sin\theta\,\sin\varphi,\qquad z=r\,\cos\theta,$$

where $\theta$ is *measured from the $z$-axis* (and *not* from the $(x,y)$-plane, which one might expect), and the weight function is $r^2\sin\theta$, and the domain $A$ is represented by the parametric space

$$\hat{A}=\left\{(r,\varphi,\theta)\;\middle|\; -\frac{\pi}{4}\le\varphi\le\frac{3\pi}{4},\,\frac{\pi}{6}\le\theta\le\frac{\pi}{2},\,\frac{a}{\sin\theta}\le r\le 2a\right\},$$

where the vertical bounding line for $B_0$ is described by $r\,\sin\theta=a$, so the lower bound for $r$ is $\dfrac{a}{\sin\theta}\le r$.

Then we get by the reduction theorem

$$\begin{aligned}
\int_A xz\,\mathrm{d}\Omega &= \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \left\{ \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left\{ \int_{\frac{a}{\sin\theta}}^{2a} r\sin\theta\cos\varphi\cdot r\cos\theta\, r^2 \sin\theta\,\mathrm{d}r \right\} \mathrm{d}\theta \right\} \mathrm{d}\varphi \\
&= \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \cos\varphi\,\mathrm{d}\varphi \cdot \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^2\theta\,\cos\theta \left\{ \int_{\frac{a}{\sin\theta}}^{2a} r^4\,\mathrm{d}r \right\} \mathrm{d}\theta \\
&= [\sin\varphi]_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \cdot \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^2\theta\cdot\cos\theta \left[\frac{1}{5}\,r^5\right]_{\frac{a}{\sin\theta}}^{2a} \mathrm{d}\theta \\
&= \sqrt{2}\cdot 5 \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^2\theta\,\cos\theta\cdot\left\{32\,a^5 - \frac{a^5}{\sin^5\theta}\right\} \mathrm{d}\theta \\
&= \frac{\sqrt{2}}{5}\,a^5 \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left\{32\,\sin^2\theta - \frac{1}{\sin^3\theta}\right\} \cos\theta\,\mathrm{d}\theta \\
&= \frac{\sqrt{2}}{5}\,a^5 \left[\frac{32}{3}\,\sin^3\theta + \frac{1}{2}\,\frac{1}{\sin^2\theta}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\
&= \frac{\sqrt{2}}{5}\,a^5 \left\{\left(\frac{32}{3}+\frac{1}{2}\right) - \left(\frac{32}{3}\cdot\frac{1}{8}+\frac{1}{2}\cdot 4\right)\right\} \\
&= \frac{\sqrt{2}}{5}\,a^5 \left\{\frac{32}{3}+\frac{1}{2}-\frac{4}{3}-2\right\} = \frac{47\sqrt{2}}{60}\,a^5.
\end{aligned}$$

**C.** We see in both variants that the result is $\sim a^5$, so we get a weak control, cf. the examination of the dimensions in **A**.In MAPLE we use the following commands,

with(Student[MultivariateCalculus])

$$\mathrm{MultiInt}\left(r^2\cdot\sin(v)\cdot\cos(t)\cdot r\cdot r^2\cdot\sin(v), r=\frac{a}{\sin(v)}..2a, v=\frac{\pi}{6}..\frac{\pi}{2}, t=-\frac{\pi}{4}..\frac{3pi}{4}\right)$$

$$\frac{47}{30}\,a^5\sqrt{2}$$

**Example 24.2** *Calculate in each of the following cases the given space integral over a point set A, which in spherical coordinates is bounded by*

$$\alpha \le \varphi \le \beta \quad \textit{and} \quad (r,\theta) \in B^*(\varphi);$$

*1) The space integral* $\int_A \sqrt{x^2+y^2}\,d\Omega$, *where the domain of integration A is given by*

$$x^2+y^2+z^2 \le 2.$$

*2) The space integral* $\int_A (x^2+y^2+z^2)^2\,d\Omega$, *where the domain of integration A is given by*

$$x^2+y^2+z^2 \le 1.$$

*3) The space integral* $\int_A xyz\,d\Omega$, *where the domain of integration A is given by*

$$x^2+y^2+z^2 \le 1, \quad x \ge 0, \quad y \ge 0, \quad z \ge 0.$$

*4) The space integral* $\int_A (x^2+y^2+z^2)^{-\frac{3}{2}}\,d\Omega$, *where the domain of integration A is given by*

$$a^2 \le x^2+y^2+z^2 \le b^2, \qquad \textit{where } b > a.$$

*5) The space integral* $\int_A (x^2z+z^3)\,d\Omega$, *where the domain of integration A is given by*

$$x^2+y^2+z^2 \le a^2 \quad \textit{and} \quad z \ge 0.$$

*6) The space integral* $\int_A \frac{y}{z^2}\,d\Omega$, *where the domain of integration A is given by*

$$x^2+y^2+z^2 \le (2a)^2, \quad a \le z, \quad 0 \le y \le x.$$

**A** Space integrals in spherical coordinates.

**D** Identify the point set. Sketch if necessary the meridian cut. Finally, compute the space integral by reduction in spherical coordinates.

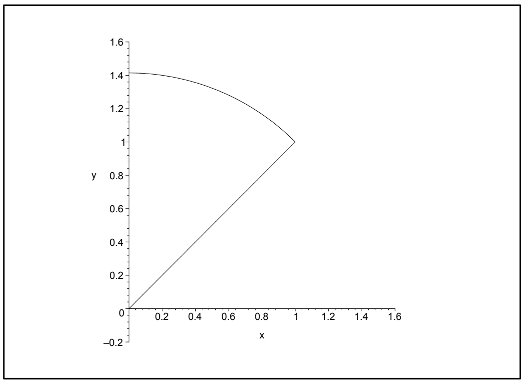



Figure 24.5: The meridian cut $B^*$ in **Example 24.2.1**.

**I** 1) It is obvious that $A$ is a conic slice of the ball of centrum $(0,0,0)$ and radius $\sqrt{2}$. Thus $0 \le \varphi \le 2\pi$, and the meridian cut

$$B^* = B^*(\varphi) = \left\{(r,\theta) \;\middle|\; 0 \le r \le \sqrt{2},\, 0 \le \theta \le \frac{\pi}{4}\right\}$$

does not depend on $\varphi$ Then by the reduction theorem in spherical coordinates,

$$\begin{aligned}\int_A \sqrt{x^2+y^2}\,\mathrm{d}\Omega &= \int_0^{2\pi}\left\{\int_{B^*(\varphi)} r\sin\theta \cdot r^2 \sin\theta \,\mathrm{d}r\,\mathrm{d}\theta\right\}\mathrm{d}\varphi = 2\pi\int_0^{\sqrt{2}} r^3\,\mathrm{d}r \cdot \int_0^{\frac{\pi}{4}} \sin^2\theta\,\mathrm{d}\theta \\ &= 2\pi\left[\frac{r^4}{4}\right]_0^{\sqrt{2}} \int_0^{\frac{\pi}{4}} \frac{1-\cos 2\theta}{2}\,\mathrm{d}\theta = 2\pi \cdot \frac{4}{4}\cdot\frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}} \\ &= \pi\left(\frac{\pi}{4}-\frac{1}{2}\right) = \frac{\pi^2}{4} - \frac{\pi}{2}.\end{aligned}$$

In MAPLE we use the following commands,

with(Student[MultivariateCalculus])

$$\text{MultiInt}\left(r\cdot\sin(v)\cdot r^2\cdot\sin(v), t = 0..2\pi, r = 0..\sqrt{2}, v = 0..\frac{\pi}{4}\right)$$

$$-\frac{1}{2}\pi + \frac{1}{4}\pi^2$$

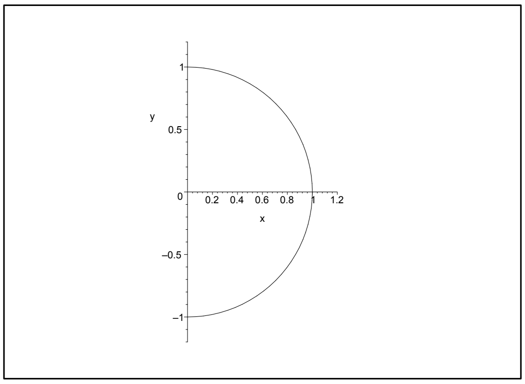



Figure 24.6: The meridian cut $B^*$ in **Example 24.2.2**.

2) The set $A$ is the unit ball, so $\varphi \in [0, 2\pi]$, and $B^* = B^*(\varphi)$ is the unit half circle in the right half plane which does not depend on $\varphi$,

$$B^* = B^*(\varphi) = \{(r, \theta) \mid 0 \leq r \leq 1,\, 0 \leq \theta \leq \pi\}.$$

Then by the reduction theorem in spherical coordinates,

$$\int_A (x^2 + y^2 + z^2)^2 \, d\Omega = 2\pi \int_{B^*} r^4 \cdot r^2 \sin\theta \, dr \, d\theta = 2\pi \int_0^1 r^6 \cdot \int_0^\pi \sin\theta \, d\theta = \frac{4\pi}{7}.$$

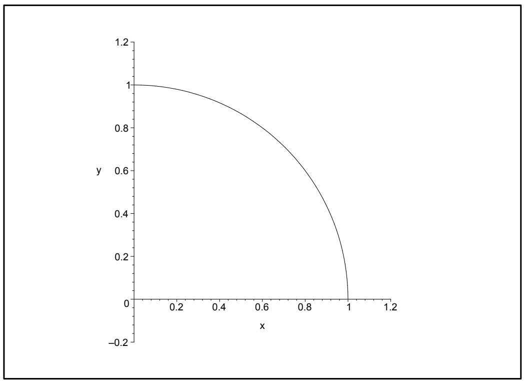



Figure 24.7: The meridian cut $B^*(\varphi)$, $\varphi \in \left[0, \frac{\pi}{2}\right]$ in **Example 24.2.3**.

3) The domain of integration is that part of the unit ball which lies in the first octant, thus $0 \leq \varphi \leq \frac{\pi}{2}$ and

$$B^*(\varphi) = \left\{ (r, \theta) \;\middle|\; 0 \leq \theta \leq \frac{\pi}{2} \right\} \qquad \text{for } 0 \leq \varphi \leq \frac{\pi}{2}.$$

By the reduction theorem in spherical coordinates,

$$\int_A xyz\,\mathrm{d}\Omega = \int_0^{\frac{\pi}{2}}\left\{\int_{B^*} r^3\sin^2\theta\cos\theta\cdot\sin\varphi\,\cos\varphi\cdot r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\right\}\mathrm{d}\varphi$$

$$= \int_0^{\frac{\pi}{2}}\sin\varphi\cdot\cos\varphi\,\mathrm{d}\varphi\cdot\int_0^1 r^5\,\mathrm{d}r\cdot\int_0^{\frac{\pi}{2}}\sin^3\theta\,\cos\theta\,\mathrm{d}\theta$$

$$= \left[\frac{1}{2}\sin^2\varphi\right]_0^{\frac{\pi}{2}}\cdot\left[\frac{r^2}{6}\right]_0^1\cdot\left[\frac{1}{4}\sin^4\theta\right]_0^{\frac{\pi}{2}} = \frac{1}{2}\cdot\frac{1}{6}\cdot\frac{1}{4} = \frac{1}{48}.$$

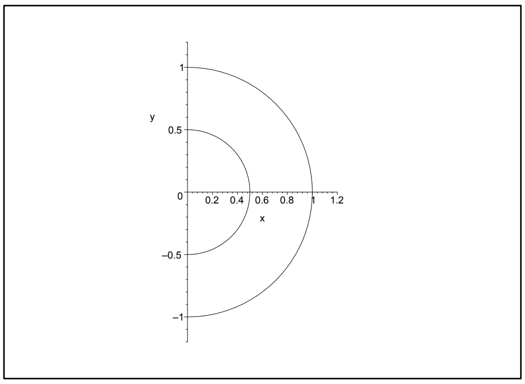



Figure 24.8: The meridian cut $B^*$ in **Example 24.2.4** for $a = \frac{1}{2}$ and $b = 1$.

In MAPLE we use the following commands,

with(Student[MultivariateCalculus])

$$\text{MultiInt}\left(r^3\cdot\sin(v)^2\cdot\cos(v)\cdot\sin(t)\cdot\cos(t)\cdot r^2\cdot\sin(v), t = 0..\frac{\pi}{2}, r = 0..1, v = 0..\frac{\pi}{2}\right)$$

$$\frac{1}{48}$$

4) Here $A$ is a shell, so $\varphi \in [0, 2\pi]$, and

$$B^* = B^*(\varphi) = \{(r,\theta) \mid a \le r \le b,\ 0 \le \theta \le \pi\}$$

does not depend on $\varphi$.
By the reduction theorem in spherical coordinates,

$$\int_A (x^2+y^2+z^2)^{-\frac{3}{2}}\,\mathrm{d}\Omega = \int_0^{2\pi}\left\{\int_{B^*} r^{-3}\,r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\right\}\mathrm{d}\varphi$$

$$= 2\pi\int_1^b\frac{1}{r}\,\mathrm{d}r\cdot\int_0^\pi\sin\theta\,\mathrm{d}\theta = 2\pi[\ln r]_a^b\cdot[-\cos\theta]_0^\pi = 4\pi\ln\left(\frac{b}{a}\right).$$

5) Here $A$ is that part of the ball of centrum $(0,0,0)$ and radius $a$, which lies in the upper half space, thus $0 \le \varphi \le 2\pi$, and

$$B^* = B^*(\varphi) = \left\{(r,\theta)\;\middle|\; 0 \le r \le a,\ 0 \le \theta \le \frac{\pi}{2}\right\}.$$

By the reduction theorem in spherical coordinates,

$$\begin{aligned}
\int_A (x^2z+z^3)\,d\Omega &= \int_0^{2\pi}\left\{\int_{B^*}(r^2\sin^2\theta\cos^2\varphi\, r\cos\theta + r^3\cos^3\theta)r^2\sin\theta\,dr\,d\theta\right\}d\varphi \\
&= \int_0^{2\pi}\left\{\int_0^a r^5\left\{\int_0^{\frac{\pi}{2}}(\cos^2\varphi\,\sin^2\theta\,\cos\theta+\cos^3\theta)\sin\theta\,d\theta\right\}dr\right\}d\varphi \\
&= \left[\frac{r^6}{6}\right]_0^a\int_0^{2\pi}\left\{\int_0^{\frac{\pi}{2}}\{\cos^2\varphi(\cos\theta-\cos^3\theta)+\cos^3\theta\}\sin\theta\,d\theta\right\}d\varphi \\
&= \frac{a^6}{6}\int_0^{2\pi}\left\{\int_0^{\frac{\pi}{2}}\{\cos^2\varphi\,\cos\theta+\sin^2\varphi\,\cos^3\theta\}\sin\theta\,d\theta\right\}d\varphi \\
&= \frac{a^6}{5}\int_0^{2\pi}\left[-\cos^2\varphi\cdot\frac{1}{2}\cos^2\theta-\sin^2\varphi\cdot\frac{1}{4}\cos^4\theta\right]_{\theta=0}^{\frac{\pi}{2}}d\varphi \\
&= \frac{a^6}{24}\int_0^{2\pi}\{2\cos^2\varphi+\sin^2\varphi\}\,d\varphi = \frac{a^6}{24}\int_0^{2\pi}\left\{\frac{3}{2}+\frac{1}{2}\cos 2\varphi\right\}d\varphi \\
&= \frac{a^6}{24}\cdot\frac{3}{2}\cdot 2\pi = \frac{a^6\pi}{8}.
\end{aligned}$$

In MAPLE we use the following commands,

with(Student[MultivariateCalculus])

$$\text{MultiInt}\left((r^2\cdot\sin(v)^2\cdot\cos(t)^2\cdot r\cdot\cos(v)+r^3\cdot\cos(v)^3)\cdot r^2\cdot\sin(v), t=0..2pi, r=0..a, v=0..\frac{\pi}{2}\right)$$

$$\frac{1}{8}\pi a^6$$

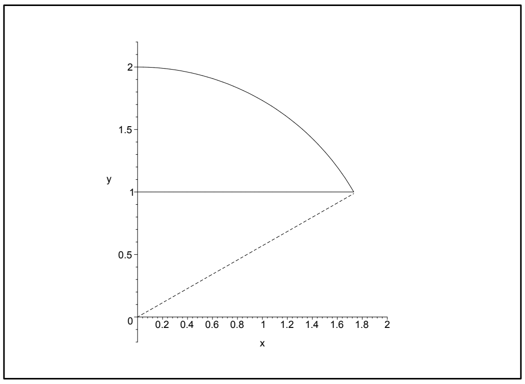



Figure 24.9: The meridian cut $B^*(\varphi)$ for $\varphi\in\left[0,\frac{\pi}{4}\right]$ and $a=1$ in **Example 24.2.6**.

6) The domain of integration is in spherical coordinates described by $0\le\varphi\le\frac{\pi}{4}$ (from the request $0\le y\le x$) and

$$\begin{aligned}
B^*(\varphi) &= \left\{(r,\theta)\;\middle|\; 0\le\theta\le\frac{\pi}{3},\ a\le r\cos\theta,\ r\le 2a\right\} \\
&= \left\{(r,\theta)\;\middle|\; 0\le\theta\le\frac{\pi}{3},\ \frac{a}{\cos\theta}\le r\le 2a\right\},
\end{aligned}$$

for $\varphi \in \left[0, \dfrac{\pi}{4}\right]$. We see that $B^* = B^*(\varphi)$ does not change in this $\varphi$-interval, hence by a reduction in spherical coordinates,

$$\begin{aligned}
\int_A \frac{y}{z^2}\,d\Omega &= \int_0^{\frac{\pi}{4}} \left\{ \int_{B^*} \frac{r\sin\theta\sin\varphi}{r^2\cos^2\theta}\cdot r^2\sin\theta\,dr\,d\theta \right\} d\varphi \\
&= \int_0^{\frac{\pi}{4}} \left\{ \int_0^{\frac{\pi}{3}} \left( \int_{\frac{a}{\cos\theta}}^{2a} r\cdot\frac{\sin^2\theta}{\cos^2\theta}\cdot\sin\varphi\,dr \right) d\theta \right\} d\varphi \\
&= [-\cos\varphi]_0^{\frac{\pi}{4}} \cdot \int_0^{\frac{\pi}{3}} \frac{\sin^2\theta}{\cos^2\theta}\left[\frac{1}{2}r^2\right]_{\frac{a}{\cos\theta}}^{2a} d\theta \\
&= \left(1-\frac{1}{\sqrt{2}}\right)\int_0^{\frac{\pi}{3}} \frac{\sin^2\theta}{\cos^2\theta}\left(2a^2-\frac{a^2}{2}\,\frac{1}{\cos^2\theta}\right)d\theta \\
&= \left(1-\frac{1}{\sqrt{2}}\right)a^2\left\{2\int_0^{\frac{\pi}{3}}\frac{1-\cos^2\theta}{\cos^2\theta}\,d\theta-\frac{1}{2}\int_0^{\frac{\pi}{3}}\tan^2\theta\cdot\frac{1}{\cos^2\theta}\,d\theta\right\} \\
&= \left(1-\frac{1}{\sqrt{2}}\right)a^2\left\{2[\tan\theta-\theta]_0^{\frac{\pi}{3}}-\frac{1}{2}\left[\frac{1}{3}\tan^3\theta\right]_0^{\frac{\pi}{3}}\right\} \\
&= \left(1-\frac{1}{\sqrt{2}}\right)\left\{2\left(\sqrt{3}-\frac{\pi}{3}\right)-\frac{1}{6}\cdot 3\sqrt{3}\right\} = \left(1-\frac{\sqrt{2}}{2}\right)\left(\frac{3}{2}\sqrt{3}-\frac{2}{3}\pi\right)a^2.
\end{aligned}$$

In MAPLE we use the following commands,

with(Student[MultivariateCalculus])

$$\operatorname{MultiInt}\left(\frac{r\cdot\sin(v)\cdot\sin(t)}{r^2\cdot\cos(v)^2}\cdot r^2\cdot\sin(v), r=\frac{a}{\cos(v)}..2a, v=0..\frac{\pi}{3}, t=0..\frac{\pi}{4}\right)$$

$$-\frac{2}{3}a^2\pi+\frac{3}{2}a^2\sqrt{3}+\frac{1}{3}\sqrt{2}\,a^2\pi-\frac{3}{4}\sqrt{2}\,a^2\sqrt{3}$$

**Example 24.3** *Let $a$ denote a positive constant. Let $K_0$ denote the closed half ball of centrum $(0,0,0)$, of radius $2a$, and where $z\geq 0$. Finally, let lad $K_1$ denote the open ball of centrum $(0,0,a)$ and radius $a$. We define a closed body of revolution $A$ by removing $K_1$ from $K_0$. Thus $A=K_0\setminus K_1$. Let $B$ denote a meridian cut in $A$.*

*1) Sketch $B$, and explain why $A$ in spherical coordinates $(r,\theta,\varphi)$ is given by*

$$\varphi\in[0,2\pi],\qquad \theta\in\left[0,\frac{\pi}{2}\right],\qquad r\in[2a\,\cos\theta,2a].$$

*2) Calculate the space integral $\int_A z^2\,\mathrm{d}\Omega$.*

**A** Space integral in spherical coordinates.

**D** Analyze geometrically the meridian half plane (add a line perpendicular to the radius vector). Then use the spherical reduction of the space integral.

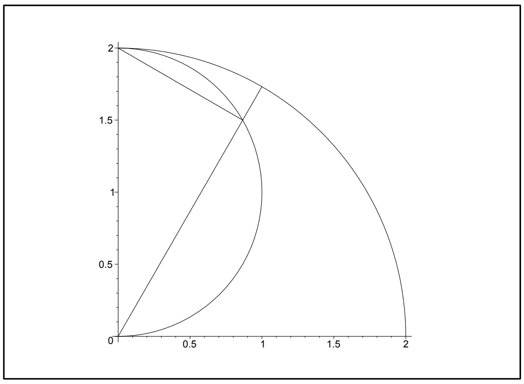



Figure 24.10: The meridian cut for $a=1$ with a radius vector and a perpendicular line.

**I** 1) The figure shows that we have a rectangular triangle with the hypothenuse of length $2a$ along the $Z$-axis and the angle $\theta$ between radius vector and the $Z$-axis. Then a geometrical consideration shows that the distance from origo to the intersection point with the circle of radius $a$ and centrum $(0,a)$ is give by $2a\,\cos\theta$. This gives us the lower limit for $r$, thus

$$r\in[2a\,\cos\theta,2a].$$

The domains of the other coordinates are obvious.

2) The integrand is written in spherical coordinates in the following way,

$$f(x,y,z) = z^2 = r^2\cos^2\theta.$$

Then by the reduction theorem for space integrals in spherical coordinates,

$$\begin{aligned}
\int_A z^2\,\mathrm{d}\Omega &= 2\pi\int_0^{\frac{\pi}{2}}\left\{\int_{2a\cos\theta}^{2a} r^2\cos^2\theta\cdot r^2\sin\theta\,\mathrm{d}r\right\}\mathrm{d}\theta \\
&= 2\pi\int_0^{\frac{\pi}{2}}\cos^2\theta\sin\theta\cdot\left[\frac{r^5}{5}\right]_{2a\cos\theta}^{2a}\mathrm{d}\theta \\
&= \frac{64\pi a^5}{5}\int_0^{\frac{\pi}{2}}\left\{\cos^2-\cos^7\theta\right\}\sin\theta\,d\theta = \frac{64\pi a^5}{5}\left[\frac{1}{8}\cos^8\theta-\frac{1}{3}\cos^3\theta\right]_0^{\frac{\pi}{2}} \\
&= \frac{64\pi a^5}{5}\cdot\frac{8-3}{8\cdot 3} = \frac{8\pi a^5}{3}.
\end{aligned}$$

In MAPLE we use the following commands,

with(Student[MultivariateCalculus])

$$\text{MultiInt}\left(r^2\cdot\cos(v)^2\cdot r^2\cdot\sin(v), r=2a\cdot\cos(v)..2a, v=0..\frac{\pi}{2}\right)$$

$$\frac{8}{3}\pi a^3$$

**Example 24.4** *Let*

$$A = \{(x,y,z)\in\mathbb{R}^3 \mid z\ge 0,\; x^2+y^2+z^2\le 4,\; \frac{1}{3}(x^2+y^2)\le z^2\le 3(x^2+y^2)\}.$$

*1) Sketch the curve of the intersection with the $(x,z)$-plane.*

*2) Compute the space integral*

$$\int_A z\,\mathrm{d}\Omega.$$

*(The best method here is to use spherical coordinates).*

**A** Space integral in spherical coordinates.

**D** Follow the guidelines of the text.

**I** 1) It $y=0$, then we get the limitations $z\ge 0$, $x^2+z^2\le 2^2$ and $\frac{1}{3}x^2\le z^2\le 3x^2$, thus

$$\frac{|x|}{\sqrt{3}}\le z\le\sqrt{3}\,|x|.$$

The intersection curve is given in spherical coordinates $(r,\theta)$ by

$$\left\{(r\theta)\;\middle|\; r\in[0,2],\,\theta\in\left[\frac{\pi}{6},\frac{\pi}{3}\right]\cup\left[-\frac{\pi}{3},-\frac{\pi}{6}\right]\right\},$$

where $\theta$ is measured from the $Z$-axis and positive towards the $X$-axis.

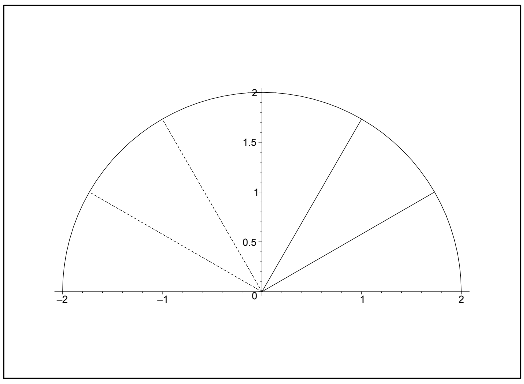



Figure 24.11: The intersection curve with the $(x, z)$-plane. It follows from the symmetry that we are only interested in the sector of the first quadrant.

2) The space integral is then calculated by reduction in spherical coordinates,

$$\begin{aligned}\int_A z\,\mathrm{d}\Omega &= \int_0^{2\pi}\left\{\int_0^2\left\{\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} r\,\cos\theta\cdot r^2\sin\theta\,\mathrm{d}\theta\right\}\mathrm{d}r\right\}\mathrm{d}\varphi \\ &= 2\pi\int_0^2 r^3\,\mathrm{d}r\cdot\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\sin\theta\cdot\cos\theta\,\mathrm{d}\theta = 2\pi\left[\frac{r^4}{4}\right]_0^2\left[\frac{\sin^2\theta}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\ &= 2\pi\cdot\frac{16}{4}\cdot\frac{1}{2}\left(\frac{3}{4}-\frac{1}{4}\right) = 2\pi\cdot 4\cdot\frac{1}{2}\cdot\frac{1}{2} = 2\pi.\end{aligned}$$

**Example 24.5** *Let $a$ and $c$ be positive constants, and let $A$ denote the half shell given by the inequalities*

$$a^2 \le x^2+y^2+z^2 \le 4a^2, \qquad z \ge 0,$$

*Calculate the space integral*

$$\int_A \frac{z}{c^2+x^2+y^2+z^2}\,\mathrm{d}\Omega.$$

**A** Space integral.

**D** We give here four variants:

1) Reduction in spherical coordinates.
2) Reduction in semi-polar coordinates.
3) Reduction by the slicing method.
4) Reduction in rectangular coordinate.

These methods are here numbered according to their increasing difficulty. The fourth variant is possible, but it is not worth here to produce all the steps involved, because the method cannot be recommended i this particular case.

**I First variant.** *Spherical coordinates.*

The set $A$ is described in spherical coordinates by

$$\left\{(r,\varphi,\theta)\;\middle|\; r\in[a,2a],\,\varphi\in[0,2\pi],\,\theta\in\left[0,\frac{\pi}{2}\right]\right\},$$

hence by the reduction of the space integral,

$$\int_A \frac{z}{c^2+x^2+y^2+z^2}\,\mathrm{d}\Omega = \int_0^{2\pi}\left\{\int_0^{\frac{\pi}{2}}\left(\int_a^{2a}\frac{r\cos\theta}{c^2+r^2}\cdot r^2\sin\theta\,\mathrm{d}r\right)\mathrm{d}\theta\right\}\mathrm{d}\varphi$$

$$= 2\pi\int_0^{\frac{\pi}{2}}\cos\theta\,\sin\theta\,\mathrm{d}\theta\cdot\int_a^{2a}\frac{r^2}{c^2+r^2}\cdot r\,\mathrm{d}r \qquad [t=r^2]$$

$$= 2\pi\left[\frac{\sin^2\theta}{2}\right]_0^{\frac{\pi}{2}}\cdot\int_{a^2}^{4a^2}\frac{t+c^2-c^2}{c^2+t}\cdot\frac{1}{2}\,\mathrm{d}t$$

$$= \frac{\pi}{2}\int_{a^2}^{4a^2}\left\{1-\frac{c^2}{c^2+t}\right\}\mathrm{d}t = \frac{\pi}{2}\left[t-c^2\ln\left(c^2+t\right)\right]_{t=a^2}^{4a^2} = \frac{\pi}{2}\left\{3a^2-c^2\ln\left(\frac{4a^2+c^2}{a^2+c^2}\right)\right\}.$$

**2. variant.** *Semi-polar coordinates.*

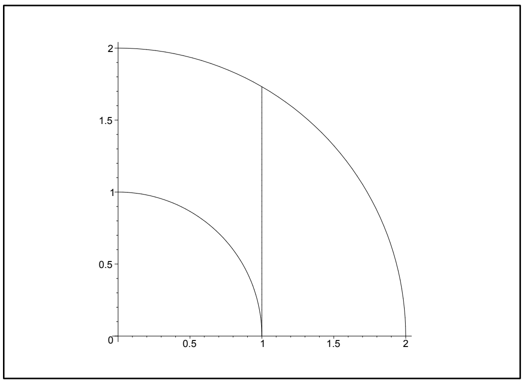



Figure 24.12: The meridian cut for $a = 1$ with the line $x = a = 1$.

We must here split the investigation into two according to whether $\varrho \in [0, a[$ or $\varrho \in [a, 2a]$, cf. the figure.

That part $A_1$ of $A$, which is given by $\varrho \in [0, a[$, is described in semi-polar coordinates by

$$\left\{(\varrho, \varphi, z) \mid \varrho \in [0, a[,\ \varphi \in [0, 2\pi],\ \sqrt{a^2 - \varrho^2} \le z \le \sqrt{4a^2 - \varrho^2}\right\}.$$

That part $A_2$ of $A$, which is given by $\varrho \in [a, 2a]$, is described in semi-polar coordinates by

$$\left\{(\varrho, \varphi, z) \mid \varrho \in [a, 2a],\ \varphi \in [0, 2\pi],\ 0 \le z \le \sqrt{4a^2 - \varrho^2}\right\}.$$

Then by reduction in semi-polar coordinates,

$$\begin{aligned}
\int_A \frac{z}{c^2 + x^2 + y^2 + z^2}\, d\Omega &= \int_{A_1} \frac{z}{c^2+x^2+y^2+z^2}\, d\Omega + \int_{A_2} \frac{z}{c^2+x^2+y^2+z^2}\, d\Omega \\
&= \int_0^{2\pi} \left\{ \int_0^a \left( \int_{\sqrt{a^2-\varrho^2}}^{\sqrt{4a^2-\varrho^2}} \frac{z}{c^2+\varrho^2+z^2}\, \varrho\, dz \right) d\varrho \right\} d\varphi \\
&\qquad + \int_0^{2\pi} \left\{ \int_1^{2a} \left( \int_0^{\sqrt{4a^2-\varrho^2}} \frac{z}{c^2+\varrho^2+z^2}\, \varrho\, dz \right) d\varrho \right\} d\varphi \\
&= 2\pi \int_0^a \left[ \frac{1}{2} \ln\left(c^2+\varrho^2+z^2\right) \right]_{z=\sqrt{a^2-\varrho^2}}^{\sqrt{4a^2-\varrho^2}} \varrho\, d\varrho + 2\pi \int_a^{2a} \left[ \frac{1}{2} \ln\left(c^2+\varrho^2+z^2\right) \right]_{z=0}^{\sqrt{4a^2-\varrho}} \varrho\, d\varrho \\
&= \pi \int_0^a \left\{ \ln\left(c^2+4a^2\right) - \ln\left(c^2+a^2\right) \right\} \varrho\, d\varrho + \pi \int_a^{2a} \left\{ \ln\left(c^2+4a^2\right) - \ln\left(c^2+\varrho^2\right) \right\} \varrho\, d\varrho,
\end{aligned}$$

thus

$$\int_A \frac{z}{c^2+x^2+y^2+z^2}\,d\Omega$$
$$= \pi\left\{\ln\left(c^2+4a^2\right)-\ln\left(c^2+a^2\right)\right\}\cdot\frac{a^2}{2}+\pi\ln\left(c^2+4a^2\right)\cdot\frac{1}{2}\left\{4a^2-a^2\right\},$$
$$-\frac{\pi}{2}\int_{a^2}^{4a^2}\ln\left(c^2+t\right)\,dt \qquad \text{(putting } t=\varrho^2\text{)}$$
$$=\frac{\pi}{2}\cdot 4a^2\ln\left(c^2+4a^2\right)-\frac{\pi}{2}a^2\ln\left(c^2+a^2\right)-\frac{\pi}{2}\left[\left(c^2+t\right)\ln\left(c^2+t\right)-t\right]_{t=a^2}^{4a^2}$$
$$=\frac{\pi}{2}\cdot 4a^2\ln\left(c^2+4a^2\right)-\frac{\pi}{2}a^2\ln\left(c^2+a^2\right)-\frac{\pi}{2}\left(c^2+4a^2\right)\ln\left(c^2+4a^2\right)$$
$$+\frac{\pi}{2}\cdot 4a^2+\frac{\pi}{2}\left(c^2+a^2\right)\ln\left(c^2+a^2\right)-\frac{\pi}{2}\cdot a^2$$
$$=\frac{\pi}{2}\cdot\left\{3a^2-c^2\ln\left(\frac{c^2+4a^2}{c^2+a^2}\right)\right\}.$$

**Third variant.** *The slicing method.*

The plane at height $z = [0, a[$ intersects $A$ in an annulus $B(z)$, which is described in polar coordinates by

$$\left\{(\varrho,\varphi)\mid \varphi\in[0,2\pi],\ \sqrt{a^2-z^2}\le\varrho\le\sqrt{4a^2-z^2}\right\}.$$

The plane at height $z\in[a,2a]$ intersects $A$ in a disc $B(z)$, which is described in polar coordinates by

$$\left\{(\varrho,\varphi)\mid \varphi\in[0,2\pi],\ 0\le\varrho\le\sqrt{4a^2-z^2}\right\}.$$

If we first integrate over $B(z)$ and then with respect to $z$, we get the following reduction,

$$\int_A \frac{z}{c^2+x^2+y^2+z^2}\,d\Omega$$
$$=\int_0^a\left\{\int_{B(z)}\frac{z}{c^2+x^2+y^2+z^2}\,dS\right\}dz+\int_a^{2a}\left\{\int_{B(z)}\frac{z}{c^2+x^2+y^2+z^2}\,dS\right\}dz$$
$$=\int_0^a\left\{\int_0^{2\pi}\left(\int_{\sqrt{a^2-z^2}}^{\sqrt{4a^2-z^2}}\frac{z}{c^2+z^2+\varrho^2}\,\varrho\,d\varrho\right)d\varphi\right\}dz$$
$$+\int_a^{2a}\left\{\int_0^{2\pi}\left(\int_0^{\sqrt{4a^2-z^2}}\frac{z}{c^2+z^2+\varrho^2}\,\varrho\,d\varrho\right)d\varphi\right\}dz,$$

hence

$$
\begin{aligned}
&\int_A \frac{z}{c^2+x^2+y^2+z^2}\,\mathrm{d}\Omega \\
&\quad = 2\pi \int_0^a \left[\frac{1}{2}\,\ln\left(c^2+z^2+\varrho^2\right)\right]_{\varrho=\sqrt{a^2-z^2}}^{\sqrt{4a^2-z^2}} z\,\mathrm{d}z + 2\pi \int_a^{2a} \left[\frac{1}{2}\,\ln\left(c^2+z^2+\varrho^2\right)\right]_{\varrho=0}^{\sqrt{4a^2-z^2}} z\,\mathrm{d}z \\
&\quad = \pi \int_0^a \left\{\ln\left(c^2+4a^2\right)-\ln\left(c^2+a^2\right)\right\} z\,\mathrm{d}z + \pi \int_a^{2a} \left\{\ln\left(c^2+4a^2\right)-\ln\left(c^2+z^2\right)\right\}\,\mathrm{d}z \\
&\quad = \pi\cdot\frac{a^2}{2}\,\ln\left(\frac{c^2+4a^2}{c^2+a^2}\right) + \pi\,\ln\left(c^2+4a^2\right)\cdot\left[\frac{z^2}{2}\right]_a^{2a} - \frac{\pi}{2}\int_{a^2}^{4a^2} \ln\left(c^2+t\right)\,\mathrm{d}t \qquad (\text{“}t=z^2\text{”}) \\
&\quad = \frac{\pi}{2}\cdot a^2\ln\left(\frac{c^2+4a^2}{c^2+a^2}\right) + \frac{\pi}{2}\cdot 3a^2\ln\left(c^2+4a^2\right) - \frac{\pi}{2}\left[\left(c^2+t\right)\ln\left(c^2+t\right)-t\right]_{t=a^2}^{4a^2} \\
&\quad = 2\pi a^2\ln\left(c^2+4a^2\right) - \frac{\pi}{2}\,a^2\ln\left(c^2+a^2\right) \\
&\qquad\qquad -\frac{\pi}{2}(c^2+4a^2)\ln(c^2+4a^2) + \frac{\pi}{2}(c^2+{}^2)\ln(c^2+a^2) + \frac{\pi}{2}\cdot 3a^2 \\
&\quad = \frac{\pi}{2}\left\{3a^2 - c^2\ln\left(\frac{c^2+4a^2}{c^2+a^2}\right)\right\}.
\end{aligned}
$$

**Fourth variant.** *Rectangular coordinates.*

This is a very difficult variant, which I have only been through once. The computations here are only sketchy just to scare people away, because it cannot be recommended.

Let

$$A_0 = \{(x,y,z) \mid a^2 \le x^2+y^2+z^2 \le 4a^2,\, x \ge 0,\, y \ge 0,\, z \ge 0\}$$

be that part of $A$, which lies in the first octant. Then by an argument of symmetry on the integrand we conclude that

$$\int_A \frac{z}{c^2+x^2+y^2+z^2}\,d\Omega = 4\int_{A_0} \frac{z}{c^2+x^2+y^2+z^2}\,d\Omega.$$

When $x \in [0,a[$ is fixed, the corresponding plane intersects the set $A_0$ in a domain $B(x)$, which is given in rectangular coordinates by

$$\{(y,z) \mid y \in [0,\sqrt{a^2-x^2}],\, \sqrt{a^2-x^2-y^2} \le z \le \sqrt{4a^2-x^2-y^2}\}$$
$$\cup\{(y,z) \mid y \in ]\sqrt{a^2-x^2},\sqrt{4a^2-x^2}],\, 0 \le z \le \sqrt{4a^2-x^2-y^2}\}.$$

Remark. We see that the description in polar coordinates would be easier here, but I shall here demonstrate how bad things can be if one only uses rectangular coordinates. $\Diamond$

Similarly, $A_0$ is cut for $x \in [a,2a]$ into a quarter disc

$$\{(y,z) \mid y \in [0,\sqrt{4a^2-x^2}],\, 0 \le z \le \sqrt{4a^2-x^2-y^2}\}.$$

Then by reduction in rectangular coordinates

$$\begin{aligned}
\int_A \frac{z}{c^2+x^2+y^2+z^2}\,d\Omega &= 4\int_{A_0} \frac{z}{c^2+x^2+y^2+z^2}\,d\Omega \\
&= 4\int_0^a \left\{\int_0^{\sqrt{a^2-x^2}} \left(\int_{\sqrt{a^2-x^2-y^2}}^{\sqrt{4a^2-x^2-y^2}} \frac{z}{c^2+x^2+y^2+z^2}\,dz\right) dy\right\} dx \\
&\quad +4\int_0^a \left\{\int_{\sqrt{a^2-x^2}}^{\sqrt{4a^2-x^2}} \left(\int_0^{\sqrt{4a^2-x^2-y^2}} \frac{z}{c^2+x^2+y^2+z^2}\,dz\right) dy\right\} dx \\
&\quad +4\int_a^{2a} \left\{\int_0^{\sqrt{4a^2-x^2}} \left(\int_0^{\sqrt{4a^2-x^2-y^2}} \frac{z}{c^2+x^2+y^2+z^2}\,dz\right) dy\right\} dx \\
&= 2\int_0^a \left\{\int_0^{\sqrt{a^2-x^2}} \left\{\ln\left(c^2+4a^2\right)-\ln\left(c^2+a^2\right)\right\} dy\right\} dx \\
&\quad +2\int_0^a \left\{\int_{\sqrt{a^2-x^2}}^{\sqrt{4a^2-x^2}} \left\{\ln\left(c^2+4a^2\right)-\ln\left(c^2+x^2+y^2\right)\right\} dy\right\} dx \\
&\quad +2\int_a^{2a} \left\{\int_0^{\sqrt{4a^2-x^2}} \left\{\ln\left(c^2+4a^2\right)-\ln\left(c^2+x^2+y^2\right)\right\} dy\right\} dx.
\end{aligned}$$

The former of these integrals is easy to compute, because it is a constant integrated over a quarter circle,

$$2\int_0^a \left\{\int_0^{\sqrt{a^2-x^2}} \left\{\ln\left(c^2+4a^2\right)-\ln\left(c^2+a^2\right)\right\} dy\right\} dx = \frac{\pi}{2}\,a^2\ln\left(\frac{c^2+4a^2}{c^2+a^2}\right).$$

The following two integrals are very difficult, if one only sticks to rectangular coordinates. But even in polar coordinates each of these two integrals are very difficult to compute, though nothing in comparison with the rectangular variant.

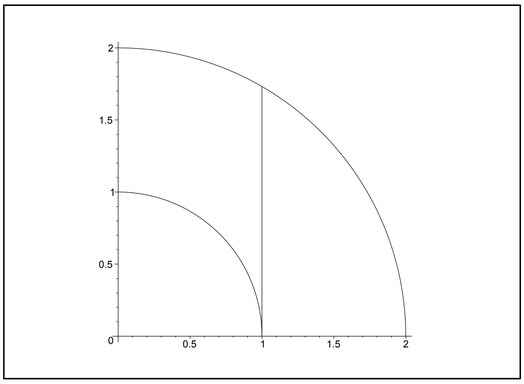



Figure 24.13: The domains $B_1$ and $B_2$ in the meridian half plane.

We shall of course start with a geometric analysis, because the integrand is the same in both cases. We can therefore join the two integrations over one single one over the domain $B_1 \cup B_2$, which again is more suitable for a polar description:

$$\left\{(\varrho,\varphi) \;\middle|\; \varrho \in [a,2a],\, \varphi \in \left[0,\frac{\pi}{2}\right]\right\}.$$

By using this trick we get by insertion,

$$\int_A \frac{z}{c^2+x^2+y^2+z^2}\,\mathrm{d}\Omega$$
$$= \frac{\pi}{2}\,a^2\,\ln\left(\frac{c^2+4a^2}{c^2+a^2}\right) + 2\int_0^{\frac{\pi}{2}}\left(\int_a^{2a}\left\{\ln\left(c^2+4a^2\right)-\ln\left(c^2+\varrho^2\right)\right\}\varrho\,\mathrm{d}\varrho\right)\mathrm{d}\varphi,$$

and the following computations are reduced to variants of those from the second and the third variant.

Remark. To my knowledge the full computation in rectangular coordinates without any trick has only been carried through once. We also tried to use MAPLE in an earlier version, at that did not work at all. The reason is that one has to apply a dirty rectangular trick at some place, which cannot be foreseen by the computer. ♢

**Example 24.6** *Let $a$ be a positive constant, and let*

$$A = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; x^2+y^2+z^2 \le a^2,\; z \le \sqrt{\frac{x^2+y^2}{3}} \right\}.$$

*1) Sketch a meridian half plane, and explain why $A$ is given in spherical coordinates $(r,\theta,\varphi)$ by*

$$r \in [0,a], \qquad \theta \in \left[\frac{\pi}{3},\pi\right], \qquad \varphi \in [0,2\pi].$$

*2) Compute the space integral*

$$\int_A (x^2+z^2)\,\mathrm{d}\Omega.$$

**A** Space integral in spherical coordinate.

**D** The space integral is here calculated in four variants.

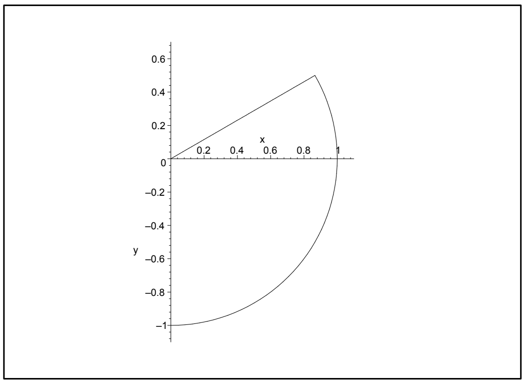



Figure 24.14: The meridian cut $A^\star$ for $a=1$.

**I** 1) In the meridian half plane the cut $A^\star$ has the line $z = \dfrac{1}{\sqrt{3}}\,\varrho$ as an upper bound, corresponding to $\theta \in \left[\dfrac{\pi}{3},\pi\right]$. The other variables are not restricted further, so $A$ is given in spherical coordinates by

$$r \in [0,a], \qquad \theta \in \left[\frac{\pi}{3},\pi\right], \qquad \varphi \in [0,2\pi].$$

2) The space integral is here computed in four variants.

**First variant.** *Direct insertion:*

$$\int_A (x^2+z^2)\,d\Omega = \int_0^{2\pi}\left\{\int_{\frac{\pi}{3}}^{\pi}\left(\int_0^a \left(r^2\sin^2\theta\cos^2\varphi + r^2\cos^2\theta\right) r^2\sin\theta\,dr\right)\,d\theta\right\}\,d\varphi$$

$$= \frac{a^5}{5}\int_0^{2\pi}\left\{\int_{\frac{\pi}{3}}^{\pi\left\{(1-\cos^2\theta)\cos^2\varphi+\cos^2\theta\right\}\sin\theta}\right\}\,d\varphi$$

$$= \frac{a^5}{5}\int_0^{2\pi}\left\{\left[-\cos\theta+\frac{1}{3}\cos^3\theta\right]_{\frac{\pi}{3}}^{\pi}\cos^2\varphi + \left[-\frac{1}{3}\cos^3\theta\right]_{\frac{\pi}{3}}^{\pi}\right\}\,d\varphi$$

$$= \frac{a^5}{5}\int_0^{2\pi}\left\{\left(1-\frac{1}{3}+\frac{1}{2}-\frac{1}{24}\right)\cos^2\varphi + \left(\frac{1}{3}+\frac{1}{24}\right)\right\}\,d\varphi$$

$$= \frac{a^5}{5}\left\{\left(\frac{3}{2}-\frac{9}{24}\right)\pi + \frac{9}{24}\cdot 2\pi\right\} = \frac{a^5}{5}\,\pi\left\{\frac{3}{2}-\frac{3}{8}+\frac{3}{4}\right\}$$

$$= \frac{3\pi}{5}\,a^5\left\{\frac{1}{2}+\frac{1}{8}\right\} = \frac{3\pi}{5}\cdot a^5\cdot\frac{5}{8} = \frac{3\pi}{8}\,a^5.$$

**Second variant.** *A small reduction:*
It follows from $x^2+z^2=r^2-y^2$ that

$$\int_A (x^2+z^2)\,d\Omega = \int_A (r^2-y^2)\,d\Omega$$
$$= \int_0^{2\pi}\left\{\int_{\frac{\pi}{3}}^{\pi}\left(\int_0^a r^2\left(1-\sin^2\theta\sin^2\varphi\right) r^2\sin\theta\,dr\right)d\theta\right\}d\varphi$$
$$= \frac{a^5}{5}\int_{\frac{\pi}{3}}^{\pi}\left\{2\pi\sin\theta-\pi\sin^3\theta\right\}d\theta$$
$$= \pi\cdot\frac{a^5}{5}\left\{[-2\cos\theta]_{\frac{\pi}{3}}^{\pi}-\int_{\frac{\pi}{3}}^{\pi}\left(1-\cos^2\theta\right)\sin\theta\,d\theta\right\}$$
$$= \pi\cdot\frac{a^5}{5}\left\{2+1+\left[\cos\theta-\frac{1}{3}\cos^3\theta\right]_{\frac{\pi}{3}}^{\pi}\right\} = \pi\cdot\frac{a^5}{5}\cdot\left\{3+\left(-1+\frac{1}{3}-\frac{1}{2}+\frac{1}{24}\right)\right\}$$
$$= \pi\cdot\frac{a^5}{5}\left(\frac{3}{2}+\frac{3}{8}\right) = \pi\cdot\frac{a^5}{5}\cdot\frac{15}{8} = \frac{3\pi}{8}a^5.$$

**Third variant.** *A symmetric argument:*
For symmetric reasons,

$$\int_A (x^2+z^2)\,d\Omega = \int_A (y^2+z^2)\,d\Omega = \frac{1}{2}\int_A \left\{(x^2+y^2+z^2)+z^2\right\}d\Omega$$
$$= \frac{1}{2}\int_0^{2\pi}\left\{\int_{\frac{\pi}{3}}^{\pi}\left(\int_0^a \left(r^2+r^2\cos^2\theta\right) r^2\sin\theta\,dr\right)d\theta\right\}d\varphi$$
$$= \frac{1}{2}\cdot 2\pi\int_{\frac{\pi}{3}}^{\pi}(1+\cos^2\theta)\sin\theta\,d\theta\cdot\int_0^a r^4\,dr = \pi\left[-\cos\theta-\frac{1}{3}\cos^3\theta\right]_{\frac{\pi}{3}}^{\pi}\cdot\frac{a^5}{5}$$
$$= \pi\cdot\frac{a^5}{5}\left\{1+\frac{1}{3}+\frac{1}{2}+\frac{1}{3\cdot 8}\right\} = \pi\cdot\frac{a^5}{5}\left(\frac{3}{2}+\frac{3}{8}\right) = \pi\cdot\frac{a^5}{5}\cdot\frac{15}{8} = \frac{3\pi}{8}a^5.$$

**Fourth variant.** *The slicing method.*
At the height $z\in\,]-a,0]$ the body $A$ is cut into a disc $D_z$ given by

$$0\le\varrho\le\sqrt{a^2-z^2}.$$

If instead $z\in\left]0,\frac{a}{2}\right[$, then $A$ is cut into an annulus $D_z$ given by

$$\sqrt{3}\,z\le\varrho\le\sqrt{a^2-z^2}.$$

For symmetric reasons we have for any $z\in\left]-a,\frac{a}{2}\right[$ that

$$\int_{D_z}(x^2+z^2)\,dS = \int_{D_z}(y^2+z^2)\,dS = \int_{D_z}\left\{\frac{1}{2}(x^2+y^2)+z^2\right\}dS.$$

Then we get

$$\int_A (x^2+z^2)\,d\Omega$$

$$= \int_{-a}^{0}\left\{\int_{D_z}\left\{\frac{1}{2}(x^2+y^2)+z^2\right\}dS\right\}dz+\int_0^{\frac{a}{2}}\left\{\int_{D_z}\left\{\frac{1}{2}(x^2+y^2)+z^2\right\}dS\right\}dz$$

$$= \int_{-a}^{0}\left\{\int_0^{2\pi}\left(\int_0^{\sqrt{a^2-z^2}}\frac{1}{2}\,\varrho^2\cdot\varrho\,d\varrho\right)d\varphi+z^2\,\text{area}(D_z)\right\}dz$$

$$+\int_0^{\frac{a}{2}}\left\{\int_0^{2\pi}\left(\int_{\sqrt{3}\,z}^{\sqrt{a^2-z^2}}\frac{1}{2}\,\varrho^2\cdot\varrho\,d\varrho\right)d\varphi+z^2\,\text{area}(D_z)\right\}dz,$$

i.e.

$$\int_A (x^2+z^2)\,d\Omega$$

$$= \int_{-a}^{0}\left\{2\pi\left[\frac{1}{8}\,\varrho^4\right]_0^{\sqrt{a^2-z^2}}+z^2\pi\left(a^2-z^2\right)\right\}dz$$

$$+\int_0^{\frac{a}{2}}\left\{2\pi\left[\frac{1}{8}\,\varrho^4\right]_{\sqrt{3}\,z}^{\sqrt{a^2-z^2}}+z^2\pi\left\{(a^2-z^2)-3z^2\right\}\right\}dz$$

$$= \frac{\pi}{4}\int_{-a}^{0}\left\{(a^2-z^2)^2+4z^2(a^2-z^2)\right\}dz$$

$$+\frac{\pi}{4}\int_0^{\frac{a}{2}}\left\{(a^2-z^2)^2-9z^4+4z^2(a^2-4z^2)\right\}dz$$

$$= \frac{\pi}{2}\int_{-a}^{0}\left\{a^4-2a^2z^2+z^4+4a^2z^2-4z^4\right\}dz$$

$$+\frac{\pi}{4}\int_0^{\frac{a}{2}}\left\{a^4-2a^2z^2+z^4-9z^4+4a^2z^2-16z^4\right\}dz$$

$$= \frac{\pi}{4}\int_{-a}^{0}\left\{a^4+2a^2z^2-3z^4\right\}dz+\frac{\pi}{4}\int_0^{\frac{a}{2}}\left\{a^4+2a^2z^2-24z^4\right\}dz$$

$$= \frac{\pi}{4}\left\{\left[a^4z+\frac{2}{3}\,a^2z^3-\frac{3}{5}\,z^5\right]_{-a}^{0}+\left[a^4z+\frac{2}{3}\,a^2z^3-\frac{24}{5}\,z^5\right]_0^{\frac{a}{2}}\right\}$$

$$= \frac{\pi}{4}\left\{a^5+\frac{2}{3}\,a^5-\frac{3}{5}\,a^5+\frac{1}{2}\,a^5-\frac{24}{5\cdot 32}\,a^5\right\}$$

$$= \frac{\pi}{4}\,a^5\left\{1+\frac{2}{3}-\frac{3}{5}+\frac{1}{2}+\frac{1}{12}-\frac{3}{20}\right\}=\frac{\pi}{4}\,a^5\left\{\frac{3}{2}+\frac{9}{12}-\frac{3}{4}\right\}=\frac{3\pi}{8}\,a^5.$$

## 24.4 Examples of volumes

**Example 24.7** *Set up a formula for the volume of the ellipsoid by applying that an ellipse of half axes $a$ and $b$ has the area $\pi ab$.*

**A** Volume of an ellipsoid found by a space integral.

**D** Use the slicing method and describe the ellipse for every $z$, and continue by computing the corresponding space integral.

**I** Let the ellipsoid be given by the inequality

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1.$$

For any fixed $z\in[-c,c]$ let $B(z)$ denote the ellipse (in $(x,y)$-coordinates) given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1-\frac{z^2}{c^2}.$$

When $z\in\,]-c,c[$, this describes an ellipse with the half axes

$$a\sqrt{1-\frac{z^2}{c^2}}\quad\text{and}\quad b\sqrt{1-\frac{z^2}{c^2}}.$$

Then by the slicing method,

$$\begin{aligned}\text{vol}(B) &= \int_{-c}^{c}\left\{\int_{B(z)}\mathrm{d}x\,\mathrm{d}y\right\}\mathrm{d}z=\int_{-c}^{c}\text{area}(B(z))\,\mathrm{d}z\\ &= \int_{-c}^{c}\pi a\sqrt{1-\frac{z^2}{c^2}}\cdot b\sqrt{1-\frac{z^2}{c^2}}\,\mathrm{d}z=\pi ab\int_{-c}^{c}\left(1-\frac{z^2}{c^2}\right)\mathrm{d}z\\ &= 2\pi abc\int_0^1(1-t^2)\,\mathrm{d}t=2\pi abc\left[t-\frac{1}{3}\,t^3\right]_0^1=\frac{4\pi}{3}\,abc.\end{aligned}$$

**C** As a weak control we know that for the solid ball of radius $r=a=b=c$ we get the well-known volume $\frac{4\pi}{3}\,r^3$.

**Example 24.8** *Let $B$ be a closed domain in the $(x, y)$-plane, and let $P_0$ be a point of $z$-coordinate $h$. Draw linear segments through $P_0$ and the points of $B$. The union of these are making up a (solid) cone. One calls $B$ the base of the cone and $h$ is the height of the cone.*

*1) A plane of constant $z \in [0, h]$ intersects the cone in a plane domain $B(z)$. Show that the area of $B(z)$ is equal to the area of $B$ multiplied by the factor $\left(1 - \frac{z}{h}\right)^2$. (Consider e.g. elements of area which correspond to each other by the segments mentioned above).*

*2) Prove that the volume of the cone is $\frac{1}{3}\, hA$, where $A$ is the area of the base $B$.*

*3) Prove that the $z$-coordinate of the centre of gravity is given by $\frac{1}{4}\, h$.*

**A** The volume of a cone found by a space integral.

**D** Follow the guidelines given above.

**I** 1) By considering a rectangular element of area in $B$ we see by using similar triangles that every length in the corresponding element of area in $B(z)$ is diminished by the factor

$$\frac{h - z}{h} = 1 - \frac{z}{h}.$$

The element of area is determined by two lengths ("length" and "breadth"), so the area is reduced by the factor $\left(1-\frac{z}{h}\right)^2$, i.e.

$$\text{area}(B(z)) = \left(1-\frac{z}{h}\right)^2 \text{area } B = \left(1-\frac{z}{h}\right)^2 A.$$

2) Using the result from 1) we get by the slicing method,

$$\begin{aligned}\text{vol}(K) &= \int_K \mathrm{d}\Omega = \int_0^h \left\{\int_{B(z)} \mathrm{d}x\,\mathrm{d}y\right\} \mathrm{d}z = \int_0^h \text{area}((B(z))\,\mathrm{d}z \\ &= \int_0^h \left(1-\frac{z}{h}\right)^2 A\,\mathrm{d}z = Ah\left[-\frac{1}{3}\left(1-\frac{z}{h}\right)^3\right]_0^h = \frac{1}{3}\,hA.\end{aligned}$$

3) Let the cone be homogeneously coated (density $\mu > 0$). Then the mass is

$$M = \mu\,\text{vol}(K) = \frac{1}{3}\,\mu hA.$$

The $z$-coordinate $\zeta$ of the centre of gravity is given by

$$M \cdot \zeta = \mu \int_K z\,d\Omega,$$

thus

$$\begin{aligned}\zeta &= \frac{\mu}{M}\int_K z\,\mathrm{d}\Omega = \frac{\mu}{\frac{1}{3}\,\mu hA}\int_0^h z\cdot\text{area}(B(z))\,\mathrm{d}z = \frac{3}{hA}\int_0^h z\left(1-\frac{z}{h}\right)^2 A\,\mathrm{d}z \\ &= 3\int_0^h \frac{z}{h}\left(1-\frac{z}{h}\right)^2 \mathrm{d}z = 3h\int_0^1 (1-t)t^2\,\mathrm{d}t = 3h\int_0^1 (t^2-t^3)\,\mathrm{d}t \\ &= 3h\left[\frac{1}{3}\,t^3-\frac{1}{4}\,t^4\right]_0^1 = 3h\left(\frac{1}{3}-\frac{1}{4}\right) = \frac{h}{4}.\end{aligned}$$

**Example 24.9** *Find the volume of the point set*

$$\Omega = \{(x,y,z) \mid x^2+y^2 \le a^2,\, |x| \le a+y,\, 0 \le z \le x^2+y^2\}.$$

*Then compute the space integral*

$$\int_\Omega (xy+1)\,\mathrm{d}\Omega.$$

**A** Volume and space integral.

**D** Sketch the projection $B$ of $\Omega$ onto the $(x,y)$-plane. Find the volume and the space integral.

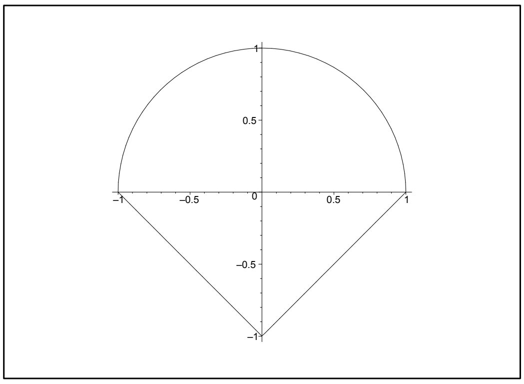



Figure 24.15: The projection $B$ of $\Omega$ for $a = 1$.

**I** The volume is

$$\begin{aligned}
\mathrm{vol}(\Omega) &= \int_B (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y = \int_0^\pi \left\{\int_0^a \varrho^2\cdot\varrho\,\mathrm{d}\varrho\right\}\mathrm{d}\varphi + \int_{-a}^0 \left\{\int_{-a-y}^{a+y} (x^2+y^2)\,\mathrm{d}x\right\}\mathrm{d}y \\
&= \pi\cdot\frac{a^4}{4} + \int_{-a}^0 \left[\frac{1}{3}x^3 + y^2 x\right]_{x=-a-y}^{a+y}\mathrm{d}y = \frac{\pi a^4}{4} + \int_{-a}^0 \left\{\frac{2}{3}(a+y)^3 + 2y^2(a+y)\right\}\mathrm{d}y \\
&= \frac{\pi a^4}{4} + \left[\frac{1}{6}(a+y)^4 + \frac{2}{3}ay^3 + \frac{1}{2}y^4\right]_{-a}^0 = \frac{\pi a^4}{4} + \frac{1}{6}a^4 + \frac{2}{3}a^4 - \frac{1}{2}a^4 \\
&= a^4\left(\frac{\pi}{4} + \frac{1}{6} + \frac{2}{3} - \frac{1}{2}\right) = a^4\left(\frac{\pi}{4}+\frac{1}{3}\right).
\end{aligned}$$

Due to the symmetry with respect to the $Y$-axis, we get for the space integral that

$$\int_\Omega (xy+1)\,\mathrm{d}\Omega = \int_B xy(x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y + \mathrm{vol}(\Omega) = 0 + \mathrm{vol}(\Omega) = a^4\left(\frac{\pi}{4}+\frac{1}{3}\right).$$

**Example 24.10** *Let $\mathcal{C}$ denote the cylindric surface the generators of which are parallel to the $Z$-axis and the intersection curve of which with the $(x, y)$-plane has the equation $y^2 = x$. Find the volume of the point set $\Omega$, which is bounded by*

*1) the cylindric surface $\mathcal{C}$,*

*2) the $(x, y)$plane, and*

*3) the plane of the equation $2x + 2y + z = 4$.*

**A** Volume.

**D** Sketch $\Omega$, or at least the projection $D$ of $\Omega$ onto the $(x, y)$-plane.

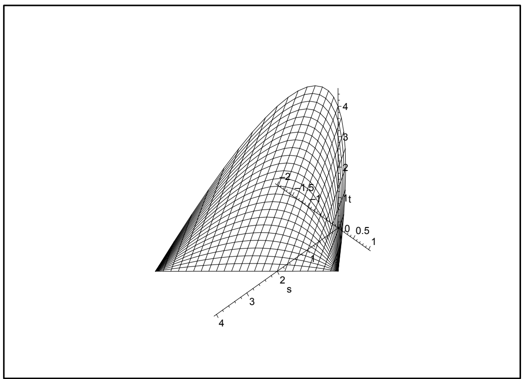



Figure 24.16: The body $\Omega$.

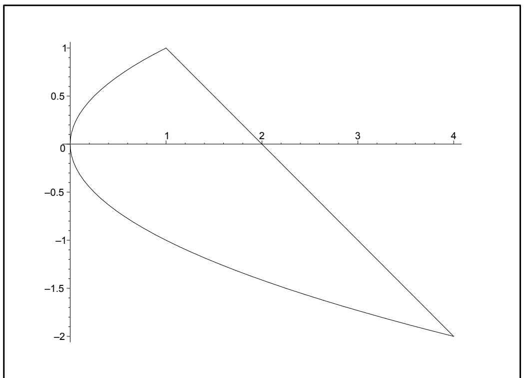



Figure 24.17: The projection $D$ of $\Omega$ onto the $(x, y)$-plane.

**I** It follows from

$$D = \{(x, y) \mid -2 \leq y \leq 1,\, y^2 \leq x \leq 2 - y\},$$

and

$$0 \leq z \leq 4 - 2x - 2y = 2(2 - x - y),$$

that

$$
\begin{aligned}
\operatorname{vol}(\Omega) &= \int_D 2(2-x-y)\,\mathrm{d}x\,\mathrm{d}y = \int_{-2}^{1}\left\{\int_{y^2}^{2-y} 2(2-x-y)\,\mathrm{d}x\right\}\mathrm{d}y \\
&= \int_{-2}^{1}\left[-(2-x-y)^2\right]_{x=y^2}^{2-y}\mathrm{d}y = \int_{-2}^{1}\left(2-y-y^2\right)^2\mathrm{d}y \\
&= \int_{-2}^{1}(y+2)^2(y-1)^2\,\mathrm{d}y = \int_{-\frac{3}{2}}^{\frac{3}{2}}\left(t+\frac{3}{2}\right)^2\left(t-\frac{3}{2}\right)^2\mathrm{d}t = 2\int_0^{\frac{3}{2}}\left(t^2-\frac{9}{4}\right)^2\mathrm{d}t \\
&= 2\int_0^{\frac{3}{2}}\left(t^4-\frac{9}{2}t^2+\frac{81}{16}\right)\mathrm{d}t = 2\left\{\frac{1}{5}\left(\frac{3}{2}\right)^5-\frac{3}{2}\left(\frac{3}{2}\right)^3+\frac{81}{16}\cdot\frac{3}{2}\right\} \\
&= \frac{2}{32}\left\{\frac{1}{5}\cdot 3^5-2\cdot 3^4+3^5\right\} = \frac{3^4}{16}\cdot\left\{\frac{3}{5}+1\right\} = \frac{81}{16}\cdot\frac{8}{5} = \frac{81}{10}.
\end{aligned}
$$

**Example 24.11** *Let*

$$f(x,y) = \ln\left(2 - 2x^2 - 3y^2\right) + 2 - 4x^2 - 6y^2,$$

*and let $B$ be that part of the $(x,y)$-plane, in which $f(x,y) \geq 0$. Let $L$ denote the point set in the space which is given by*

$$(x,y) \in B, \qquad 0 \leq z \leq f(x,y).$$

*Find the volume of $L$ by the slicing method.*

**A** Volume.

**D** Consider $f(x,y) = \ln(2 - 2x^2 - 3y^2) + 2 - 4x^2 - 6y^2$ as a function in one single variable.

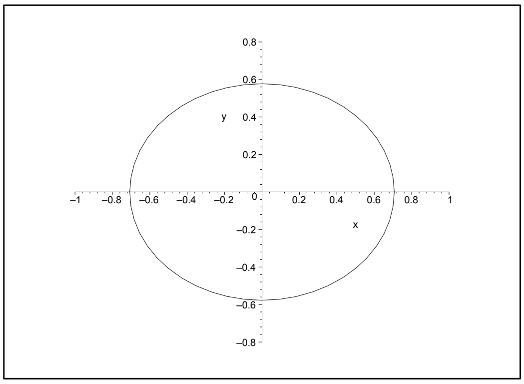



Figure 24.18: The domain $B$, in which $f(x,y) \geq 0$.

**I** Since $f(x,y) = 0$, for $2x^2 + 3y^2 = 1$, the domain $B$ is bounded by the ellipse

$$\left(\frac{x}{\frac{1}{\sqrt{2}}}\right)^2 + \left(\frac{y}{\frac{1}{\sqrt{3}}}\right)^2 = 1.$$

This ellipse has the half axes $\dfrac{1}{\sqrt{2}}$ and $\dfrac{1}{\sqrt{3}}$ and the area $\dfrac{\pi}{\sqrt{6}}$.

The function $f(x,y)$ is in reality only a function in $t = 1 - 2x^2 - 3y^2$, $t \in [0,1]$, since we have by this substitution

$$z = f(x,y) = F(t) = \ln(1+t) + 2t, \qquad t \in [0,1].$$

When $t \in [0,1]$ is fixed, then $2x^2 + 3y^2 \leq 1 - t$ describes an elliptic disc $A_t$ of area

$$\text{area}\,(A_t) = \frac{\pi}{\sqrt{6}}\,(1-t),$$

thus we get the volume by the slicing method,

$$\begin{aligned}\text{vol}(L) &= \int_0^1 \text{area}\,(A_t) \cdot \frac{\mathrm{d}z}{\mathrm{d}t}\,\mathrm{d}t = \int_0^1 \frac{\pi}{\sqrt{6}}\,(1-t) \cdot \left\{\frac{1}{1+t} + 2\right\}\mathrm{d}t \\ &= \frac{\pi}{\sqrt{6}} \int_0^1 \left\{\frac{2-(1+t)}{1+t} + 2 - 2t\right\}\mathrm{d}t = \frac{\pi}{\sqrt{6}}\left[\ln(1+t) - t + 2t - t^2\right]_0^1 = \frac{\pi}{\sqrt{6}}\ln 2.\end{aligned}$$

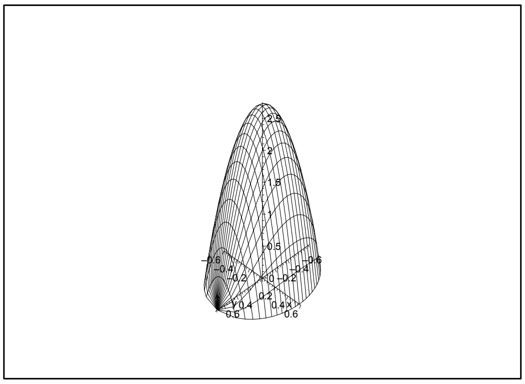


Figure 24.19: The body $L$.

**Example 24.12** *Find the volume of the point set*

$$Q = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \le 2,\, 0 \le z \le 4-\left(x^2+y^4\right)^2\}.$$

**A** Volume.

**D** Sketch the set and just compute.

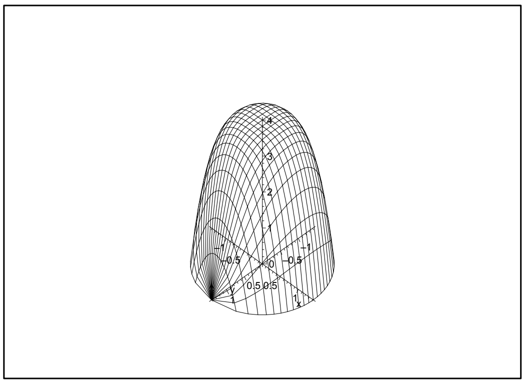


Figure 24.20: The point set $Q$.

**I** The set $Q$ is cut at height $z \in [0,4]$ in a disc of radius $\sqrt[4]{4-z}$. Then by the slicing method,

$$\operatorname{vol}(Q) = \int_0^4 \pi\sqrt{4-z}\,\mathrm{d}z = \pi\left[-\frac{2}{3}\,\left(\sqrt{4-z}\right)^3\right]_0^4 = \frac{2}{3}\,\pi\cdot(\sqrt{4})^3 = \frac{16}{3}\,\pi.$$

ALTERNATIVELY we first integrate with respect to $z$,

$$\begin{aligned}\operatorname{vol}(Q) &= \int_{\overline{K}(\mathbf{0};\sqrt{2})} \{4-(x^2+y^2)^2\}\,\mathrm{d}x\,\mathrm{d}y = 4\,\text{area}\left(\overline{K}(\mathbf{0};\sqrt{2})\right) - 2\pi\int_0^{\sqrt{2}} \varrho^4\cdot\varrho\,\mathrm{d}\varrho \\ &= 4\cdot 2\pi - 2\pi\left\{\frac{(\sqrt{2})^6}{6}\right\} = 8\pi - \frac{8\pi}{3} = \frac{16\pi}{3}.\end{aligned}$$

**Example 24.13** *Let $B(a)$ denote the bounded point set in the plane which is bounded by the parabola $y = x^2$ and the line $y = a$. Let $B$ denote the unbounded point set which is defined by the inequalities $y \geq x^2$. Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ be given by*

$$f(x,y) = |x| \exp\left(x^2 - 2y\right),$$

*and put*

$$A(a) = \{(x,y,z) \mid (x,y) \in B(a),\, 0 \leq z \leq f(x,y)\}.$$

*1) Find the volume of $A(a)$.*

*2) Prove that the improper plane integral*

$$\int_B f(x,y)\,\mathrm{d}S$$

*is convergent, and find its value.*

**A** Volume and improper plane integral.

**D** Sketch $B(a)$ and $B$; find vol $A(a)$, and compute the improper plane integral.

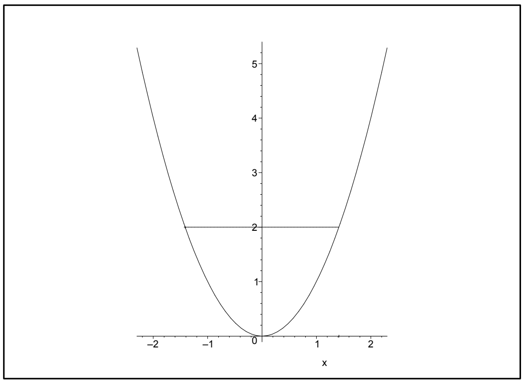



Figure 24.21: The parabola with the truncation at $y = a = 2$.

**I** 1) We get by direct computation,

$$\begin{aligned}\text{vol}(A(a)) &= \int_{B(a)} f(x,y)\,\mathrm{d}S = 2\int_0^a \left\{\int_0^{\sqrt{y}} x \cdot e^{x^2} e^{-2y}\,\mathrm{d}x\right\}\mathrm{d}y \\ &= 2\int_0^a e^{-2y}\left[\frac{1}{2}\,e^{x^2}\right]_{x=0}^{\sqrt{y}}\mathrm{d}y = \int_0^a e^{-2y}\left(e^y - 1\right)\mathrm{d}y \\ &= \int_0^a \left(e^{-y} - e^{-2y}\right)\mathrm{d}y = \left[-e^{-y} + \frac{1}{2}\,e^{-2y}\right]_0^a = \frac{1}{2} - e^{-a} + \frac{1}{2}\,e^{-2a}.\end{aligned}$$

2) Since $f(x,y) \geq 0$, we get

$$\int_B f(x,y)\,\mathrm{d}S = \lim_{a\to+\infty}\int_{B(a)} f(x,y)\,\mathrm{d}S = \lim_{a\to+\infty}\left\{\frac{1}{2} - e^{-a} + \frac{1}{2}\,e^{-2a}\right\} = \frac{1}{2}.$$

## 24.5 Examples of moments of inertia and centres of gravity

**Example 24.14** *Given the solid ellipsoid*

$$\Omega = \left\{ (x,y,z) \;\middle|\; \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \le 1 \right\}.$$

*1) Compute the space integral*

$$\int_{\Omega} \sqrt{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2}\, d\Omega.$$

*2) Let $\Omega$ be homogeneously coated by a mass, where $M$ denotes the total mass. Find the moment of inertia $I_x$ of $\Omega$ with respect to the $X$-axis expressed by $a$, $b$, $c$ and $M$.*

**A** Space integral; moment of inertia.

**D** Follow the guidelines.

**I** 1) By putting

$$(x,y,z) = (au, bv, cw), \qquad u^2+v^2+w^2 \le 1,$$

and then applying spherical coordinates in the $(u,v,w)$-space we get

$$\int_{\Omega} \sqrt{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2}\, d\Omega = abc \int_{\overline{K}(\mathbf{0};1)} \sqrt{u^2+v^2+w^2}\, du\, dv\, dw$$

$$= abc \int_0^{2\pi} \left\{ \int_0^{\pi} \left\{ \int_0^1 \varrho \cdot \varrho^2 \sin\theta\, d\varrho \right\} d\theta \right\} d\varphi = abc \cdot 2\pi \cdot 2 \cdot \frac{1}{4} = abc\pi.$$

2) It is well-known that the volume is $\text{vol}(\Omega) = \dfrac{4\pi}{3}\,abc$, hence the mass can be written $M = \dfrac{4\pi}{3}\,abc \cdot \mu$, from which we get the density $\mu = \dfrac{3M}{4\pi abc}$.
Due to the symmetry, the moment of inertia with respect to the $X$-axis is given by

$$\begin{aligned} I_x &= \mu \int_\Omega (y^2+z^2)\,\mathrm{d}\Omega = \mu \int_\Omega y^2\,\mathrm{d}\Omega + \mu \int_\Omega z^2\,\mathrm{d}\Omega \\ &= \mu b^2(abc) \int_{\overline{K}(\mathbf{0};1)} v^2\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w + \mu c^2(abc) \int_{\overline{K}(\mathbf{0};1)} w^2\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w \\ &= \mu abc(b^2+c^2) \int_{\overline{K}(\mathbf{0};1)} u^2\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w = \mu(b^2+c^2)abc \int_{-1}^{1} u^2\pi(1-u^2)\,\mathrm{d}u \\ &= 2\mu\pi abc(b^2+c^2) \int_0^1 (u^2-u^4)\,\mathrm{d}u = 2\cdot \frac{3M}{4\pi abc}\,abc(b^2+c^2)\left(\frac{1}{3}-\frac{1}{5}\right) \\ &= \frac{3}{2}\,M\,(b^2+c^2)\cdot\frac{2}{15} = \frac{1}{5}\,M(b^2+c^2). \end{aligned}$$

**Example 24.15** *Find the centre of gravity for the part of the intersection of the ball of centrum* $(0,0,0)$ *and of radius* $a>0$ *in the first octant, i.e. given by the inequalities*

$$x \ge 0, \quad y \ge 0, \quad z \ge 0, \quad x^2+y^2+z^2 \le a^2.$$

**A** Centre of gravity.

**D** First reduce to the case $a=1$. Find $\text{vol}(\Omega)$. Compute

$$\xi = \frac{1}{\text{vol}(\Omega)} \int_\Omega x\,\mathrm{d}\Omega.$$

It follows by the symmetry that $\xi = \eta = \zeta$.

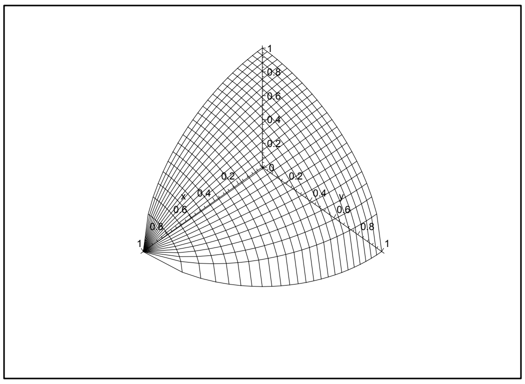



Figure 24.22: The set $\Omega$ for $a=1$.

**I** We may of geometrical reasons assume that $a=1$, thus

$$\Omega = \{(x,y,z) \mid x\ge 0,\, y \ge 0,\, z\ge 0,\, x^2+y^2+z^2 \le 1\}.$$

If $(\xi, \eta, \zeta)$ denotes the centre of gravity for $\Omega$, then $(a\xi, a\eta, a\zeta)$ is the centre of gravity for the initial set of radius $a$.

It follows clearly by the symmetry that $\xi = \eta = \zeta$.

Finally,

$$\text{vol}(\Omega) = \frac{1}{8} \cdot \frac{4\pi}{3} \cdot 1^3 = \frac{\pi}{6}.$$

It follows that

$$\begin{aligned}\xi &= \frac{1}{\text{vol}(\Omega)} = \frac{6}{\pi} \int_0^1 x \left\{ \int_{y^2+z^2 \le 1-x^2} \mathrm{d}y\,\mathrm{d}z \right\} \mathrm{d}x \\ &= \frac{6}{\pi} \int_0^1 x \cdot \frac{1}{4}\pi \left(1-x^2\right) \mathrm{d}x = \frac{3}{2} \int_0^1 \left(x - x^3\right) \mathrm{d}x = \frac{3}{2}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{3}{8}.\end{aligned}$$

Therefore, if $a = 1$, then

$$(\xi, \eta, \zeta) = \left(\frac{3}{8}, \frac{3}{8}, \frac{3}{8}\right).$$

We get for a general $a > 0$,

$$(\xi, \eta, \zeta) = \left(\frac{3}{8}\,a, \frac{3}{8}\,a, \frac{3}{8}\,a\right).$$

**Example 24.16** *Let $R$ denote a positive constant. Consider the point set*

$$T=\{(x,y,z)\in\mathbb{R}^3 \mid 0\le z,\; x^2+y^2+z^2\le R^2,\; x^2+y^2\le 3z^2\}.$$

*1) Explain why $T$ is given in spherical coordinates by*

$$r\in[0,R],\qquad \theta\in\left[0,\frac{\pi}{3}\right],\qquad \varphi\in[0,2\pi].$$

*2) Compute the space integrals $\int_T 1\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ and $\int_T z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$.*

*3) Find the coordinates of the centre of gravity of $T$.*

*4) Find the area of the boundary surface of $T$.*

**A** Spherical coordinates, space integrals, centre of gravity and surface area.

**D** First make a sketch in the meridian half plane.

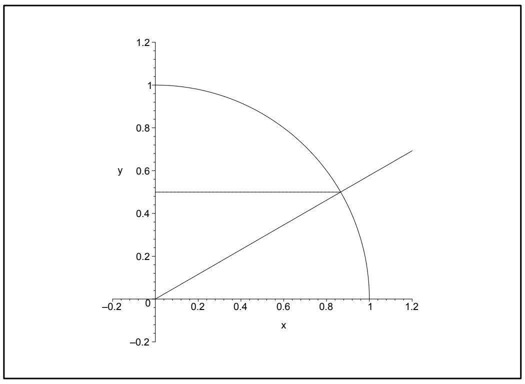



Figure 24.23: The sketch in the meridian half plane for $R=1$.

**I** 1) The sketch of the meridian half plane shows that

$$z\ge 0,\quad r^2\le R^2,\quad \varrho^2\le 3z^2,\quad r^2=\varrho^2+z^2,$$

in spherical coordinates is expressed by

$$r\in[0,R],\qquad \theta\in\left[0,\frac{\pi}{3}\right],\qquad \varphi\in[0,2\pi].$$

2) The volume is

$$\begin{aligned}
\mathrm{vol}(T) &= \int_T 1\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=\int_0^{\frac{R}{2}}\pi\cdot 3z^2\,\mathrm{d}z+\int_{\frac{R}{2}}^{R}\pi\left(R^2-z^2\right)\,\mathrm{d}z\\
&= \pi\left(\frac{R}{2}\right)^3+\pi\left[R^2z-\frac{1}{3}\,z^3\right]_{\frac{R}{2}}^{R}=\frac{\pi}{8}\,R^3+\pi\left\{R^3-\frac{R^3}{3}-\frac{R^3}{2}+\frac{1}{24}\,R^3\right\}\\
&= \pi R^3\left\{\frac{1}{8}+1-\frac{1}{3}-\frac{1}{2}+\frac{1}{24}\right\}=\frac{\pi R^3}{24}\{3+24-8-12+1\}=\frac{\pi R^3}{3}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_T z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_0^{\frac{R}{2}} z\cdot\pi\cdot 3z^2\,\mathrm{d}z + \int_{\frac{R}{2}}^R z\cdot\pi\left(R^2-z^2\right)\,\mathrm{d}z \\
&= \frac{3\pi}{4}\cdot\left(\frac{R}{2}\right)^4 + \pi\left[\frac{R^2}{2}\,z^2-\frac{1}{4}\,z^4\right]_{\frac{R}{2}}^R \\
&= \frac{3\pi}{64}\,R^4+\pi\left\{\frac{R^4}{2}-\frac{R^4}{4}-\frac{R^4}{8}+\frac{R^4}{64}\right\} \\
&= \frac{\pi R^4}{64}\left\{3+32-16-8+1\right\} = \frac{12\pi R^4}{64} = \frac{3\pi}{16}\,R^4.
\end{aligned}$$

3) Of symmetric reasons the centre of gravity must lie on the $Z$-axis, so $\xi=\eta=0$, and

$$\zeta = \frac{1}{\mathrm{vol}(T)}\int_T z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{3}{\pi R^3}\cdot\frac{3\pi}{16}\,R^4 = \frac{9}{16}\,R,$$

where we have used the results of 2).

4) The boundary curve $\mathcal{M}$ in the meridian half plane is now split up into

$$\mathcal{M}_1:\quad \varrho=\sqrt{3}\cdot z,\quad \mathrm{d}s=\sqrt{1+3}\,\mathrm{d}z = 2\,\mathrm{d}z,\quad z\in\left[0,\frac{R}{2}\right],$$

$$\mathcal{M}_2:\quad \varrho=\sqrt{R^2-z^2},\quad \mathrm{d}s=\frac{R}{\sqrt{R^2-z^2}}\,\mathrm{d}z,\qquad z\in\left[\frac{R}{2},R\right],$$

so the surface area becomes

$$\begin{aligned}
2\pi\int_{\mathcal{M}} P\,\mathrm{d}s &= 2\pi\int_0^{\frac{R}{2}}\sqrt{3}\cdot z\cdot 2\,\mathrm{d}z + 2\pi\int_{\frac{R}{2}}^R\sqrt{R^2-z^2}\cdot\frac{R}{\sqrt{R^2-z^2}}\,\mathrm{d}z \\
&= 2\pi\sqrt{3}\,\left[z^2\right]_0^{\frac{R}{2}}+2\pi R\cdot\frac{R}{2} = 2\sqrt{3}\,\pi\cdot\frac{R^2}{4}+\pi R^2 = \left(1+\frac{\sqrt{3}}{2}\right)\pi\,R^2.
\end{aligned}$$

**Example 24.17** *Let $\Omega$ denote that part of the closed ball $\overline{K}(\mathbf{0};a)$, which lies above the $(x,y)$-plane and inside a cylindric surface with its generators parallel to the $Z$-axes through the curve in the $(x,y)$-plane given by the equation*

$$\varrho = a\sqrt{\cos(2\varphi)}, \qquad -\frac{\pi}{4} \le \varphi \le \frac{\pi}{4}.$$

*1) Find the volume of $\overline{\Omega}$.*

*2) Find the $z$-coordinate of the centre of gravity for $\overline{\Omega}$.*

**A** Volume; centre of gravity.

**D** Sketch $\Omega$ and compute $\text{vol}(\Omega)$. Find the centre of gravity.

**I** 1) Since $z = +\sqrt{a^2-\varrho^2}$ on the shell, we get

$$\begin{aligned}
\text{vol}(\Omega) &= \int_B \sqrt{a^2-\varrho^2}\,\mathrm{d}S = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left\{ \int_0^{a\sqrt{\cos 2\varphi}} \sqrt{a^2-\varrho^2}\cdot\varrho\,\mathrm{d}\varrho \right\} \mathrm{d}\varphi \\
&= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[ -\frac{1}{2}\cdot\frac{1}{3}\left(a^2-\varrho^2\right)^{\frac{3}{2}} \right]_{\varrho=0}^{a\sqrt{\cos 2\varphi}} \mathrm{d}\varphi = \frac{1}{3}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left\{ a^3 - a^3(1-\cos 2\varphi)^{\frac{3}{2}} \right\} \mathrm{d}\varphi \\
&= 2\cdot\frac{a^3}{3}\int_0^{\frac{\pi}{4}} \left\{ 1-\left(2\sin^2\varphi\right)^{\frac{3}{2}} \right\} \mathrm{d}\varphi = \frac{2}{3}a^3\int_0^{\frac{\pi}{4}} \left\{ 1-2\sqrt{2}\sin^3\varphi \right\} \mathrm{d}\varphi \\
&= \frac{2}{3}a^3\cdot\frac{\pi}{4} - \frac{2}{3}a^3\cdot 2\sqrt{2}\int_0^{\frac{\pi}{4}} \left(1-\cos^2\varphi\right)\sin\varphi\,\mathrm{d}\varphi \\
&= \frac{\pi}{6}a^3 + \frac{4\sqrt{2}}{3}a^3\left[\cos\varphi - \frac{1}{3}\cos^3\varphi\right]_0^{\frac{\pi}{4}} = \frac{\pi}{6}a^3 + \frac{4\sqrt{2}}{3}a^3\left(\frac{1}{\sqrt{2}} - \frac{1}{6\sqrt{2}} - \frac{2}{3}\right) \\
&= \frac{\pi}{6}a^3 + \frac{4}{3}\cdot\frac{5}{6}a^3 - \frac{8\sqrt{2}}{9}a^3 = a^3\left(\frac{\pi}{6} + \frac{10}{9} - \frac{8\sqrt{2}}{9}\right).
\end{aligned}$$

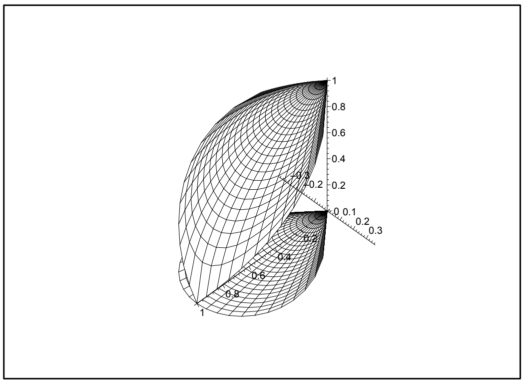



Figure 24.24: The domain $\Omega$ for $a = 1$.

2) We have due to the symmetry,

$$\int_\Omega y\,\mathrm{d}\Omega = \int_B y\sqrt{a^2-\varrho^2}\,\mathrm{d}S = 0.$$

Furthermore,

$$
\begin{aligned}
\int_\Omega z\,\mathrm{d}\Omega &= \int_B \left\{\int_0^{\sqrt{a^2-\varrho^2}} z\,\mathrm{d}z\right\} \mathrm{d}S = \frac{1}{2}\int_B \left(a^2-\varrho^2\right)\,\mathrm{d}S \\
&= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left\{\int_0^{a\sqrt{\cos 2\varphi}} \left(a^2-\varrho^2\right)\,\varrho\,\mathrm{d}\varrho\right\} \mathrm{d}\varphi = \int_0^{\frac{\pi}{4}} \left[\frac{1}{2}\,a^2\varrho^2 - \frac{1}{4}\,\varrho^4\right]_0^{a\sqrt{\cos 2\varphi}} \mathrm{d}\varphi \\
&= \frac{a^4}{4}\int_0^{\frac{\pi}{4}} \left\{2\cos 2\varphi - \cos^2 2\varphi\right\} \mathrm{d}\varphi \\
&= \frac{a^4}{4}\left[\sin 2\varphi\right]_0^{\frac{\pi}{4}} - \frac{a^4}{4}\cdot\frac{1}{2}\int_0^{\frac{\pi}{4}} \{1+\cos 4\varphi\}\,\mathrm{d}\varphi = \frac{a^4}{4} - \frac{a^4\pi}{32} = \frac{a^4}{32}\,(8-\pi).
\end{aligned}
$$

Finally, we get by interchanging the order of integration that

$$\begin{aligned}
\int_\Omega x\,\mathrm{d}\Omega &= \int_B x\sqrt{a^2-\varrho^2}\,\mathrm{d}S = 2\int_0^{\frac{\pi}{4}} \cos\varphi \left\{\int_0^{a\sqrt{\cos 2\varphi}} \sqrt{a^2-\varrho^2}\cdot\varrho^2\,\mathrm{d}\varrho\right\}\mathrm{d}\varphi \\
&= 2a^4 \int_0^{\frac{\pi}{4}} \cos\varphi \left\{\int_0^{\sqrt{\cos 2\varphi}} t^2\sqrt{1-t^2}\,\mathrm{d}t\right\}\mathrm{d}\varphi \\
&= 2a^4 \int_0^1 t^2\sqrt{1-t^2}\left\{\int_0^{\frac{1}{2}\,\mathrm{Arccos}(t^2)} \cos\varphi\,\mathrm{d}\varphi\right\}\mathrm{d}t \\
&= 2a^4 \int_0^1 t^2\sqrt{1-t^2}\,\sin\left(\frac{1}{2}\,\mathrm{Arccos}(t^2)\right)\mathrm{d}t \\
&= 2a^4 \int_0^1 t^2\sqrt{1-t^2}\cdot\sqrt{1-\cos\left(2\cdot\frac{1}{2}\,\mathrm{Arccos}\,(t^2)\right)}\,\mathrm{d}t \\
&= 2a^4 \int_0^1 t^2\sqrt{1-t^2}\cdot\sqrt{1-t^2}\,\mathrm{d}t = 2a^4\int_0^1 t^2(1-t^2)\,\mathrm{d}t = 2a^4\left\{\frac{1}{3}-\frac{1}{5}\right\} = \frac{4}{15}\,a^4.
\end{aligned}$$

The centre of gravity is

$$\xi = \frac{1}{\mathrm{vol}(\Omega)}\int_\Omega \mathbf{x}\,\mathrm{d}\Omega = \frac{a}{\frac{\pi}{6}+\frac{10}{9}-\frac{8\sqrt{2}}{9}}\left(\frac{4}{15},0,\frac{8-\pi}{32}\right).$$

# 25 Formulæ

Some of the following formulæ can be assumed to be known from high school. It is highly recommended that one *learns most of these formulæ in this appendix by heart.*

## 25.1 Squares etc.

The following simple formulæ occur very frequently in the most different situations.

$$
\begin{array}{ll}
(a+b)^2 = a^2+b^2+2ab, & a^2+b^2+2ab = (a+b)^2, \\
(a-b)^2 = a^2+b^2-2ab, & a^2+b^2-2ab = (a-b)^2, \\
(a+b)(a-b) = a^2-b^2, & a^2-b^2 = (a+b)(a-b), \\
(a+b)^2 = (a-b)^2+4ab, & (a-b)^2 = (a+b)^2-4ab.
\end{array}
$$

## 25.2 Powers etc.

**Logarithm:**

$$
\begin{array}{rcll}
\ln|xy| &=& \ln|x|+\ln|y|, & x,y\neq 0, \\
\ln\left|\dfrac{x}{y}\right| &=& \ln|x|-\ln|y|, & x,y\neq 0, \\
\ln|x^r| &=& r\ln|x|, & x\neq 0.
\end{array}
$$

**Power function, fixed exponent:**

$$(xy)^r = x^r\cdot y^r, x,y>0 \quad \text{(extensions for some } r\text{)},$$

$$\left(\frac{x}{y}\right)^r = \frac{x^r}{y^r}, x,y>0 \quad \text{(extensions for some } r\text{)}.$$

**Exponential, fixed base:**

$$a^x\cdot a^y = a^{x+y}, \quad a>0 \quad \text{(extensions for some } x,\ y\text{)},$$
$$(a^x)^y = a^{xy}, a>0 \quad \text{(extensions for some } x,\ y\text{)},$$

$$a^{-x} = \frac{1}{a^x}, a>0, \quad \text{(extensions for some } x\text{)},$$

$$\sqrt[n]{a} = a^{1/n}, a\geq 0, \qquad n\in\mathbb{N}.$$

**Square root:**

$$\sqrt{x^2} = |x|, \qquad x\in\mathbb{R}.$$

**Remark 25.1** It happens quite frequently that students make errors when they try to apply these rules. They must be mastered! In particular, as one of my friends once put it: "If you can master the square root, you can master everything in mathematics!" Notice that this innocent looking square root is one of the most difficult operations in Calculus. Do not forget the *absolute value!* ♢

## 25.3 Differentiation

Here are given the well-known rules of differentiation together with some rearrangements which sometimes may be easier to use:

$$\{f(x) \pm g(x)\}' = f'(x) \pm g'(x),$$

$$\{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x) = f(x)g(x)\left\{\frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}\right\},$$

where the latter rearrangement presupposes that $f(x) \neq 0$ and $g(x) \neq 0$.
If $g(x) \neq 0$, we get the usual formula known from high school

$$\left\{\frac{f(x)}{g(x)}\right\}' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

It is often more convenient to compute this expression in the following way:

$$\left\{\frac{f(x)}{g(x)}\right\} = \frac{d}{dx}\left\{f(x) \cdot \frac{1}{g(x)}\right\} = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2} = \frac{f(x)}{g(x)}\left\{\frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}\right\},$$

where the former expression often is *much easier* to use in practice than the usual formula from high school, and where the latter expression again presupposes that $f(x) \neq 0$ and $g(x) \neq 0$. Under these assumptions we see that the formulæ above can be written

$$\frac{\{f(x)g(x)\}'}{f(x)g(x)} = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)},$$

$$\frac{\{f(x)/g(x)\}'}{f(x)/g(x)} = \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}.$$

Since

$$\frac{d}{dx}\ln|f(x)| = \frac{f'(x)}{f(x)}, \qquad f(x) \neq 0,$$

we also name these the *logarithmic derivatives.*

Finally, we mention the rule of **differentiation of a composite function**

$$\{f(\varphi(x))\}' = f'(\varphi(x)) \cdot \varphi'(x).$$

We first differentiate the function itself; then the insides. This rule is a 1-dimensional version of the so-called *Chain rule.*

## 25.4 Special derivatives.

**Power like:**

$$\frac{d}{dx}(x^\alpha) = \alpha \cdot x^{\alpha-1}, \qquad \text{for } x > 0, \ \text{(extensions for some } \alpha\text{)}.$$

$$\frac{d}{dx}\ln|x| = \frac{1}{x}, \qquad \text{for } x \neq 0.$$

**Exponential like:**

$$\frac{d}{dx}\exp x = \exp x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\left(a^x\right) = \ln a \cdot a^x, \qquad \text{for } x \in \mathbb{R} \text{ and } a > 0.$$

**Trigonometric:**

$$\frac{d}{dx}\sin x = \cos x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\cos x = -\sin x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\tan x = 1 + \tan^2 x = \frac{1}{\cos^2 x}, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, p \in \mathbb{Z},$$

$$\frac{d}{dx}\cot x = -\left(1 + \cot^2 x\right) = -\frac{1}{\sin^2 x}, \qquad \text{for } x \neq p\pi, p \in \mathbb{Z}.$$

**Hyperbolic:**

$$\frac{d}{dx}\sinh x = \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\cosh x = \sinh x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\tanh x = 1 - \tanh^2 x = \frac{1}{\cosh^2 x}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\coth x = 1 - \coth^2 x = -\frac{1}{\sinh^2 x}, \qquad \text{for } x \neq 0.$$

**Inverse trigonometric:**

$$\frac{d}{dx}\text{ Arcsin } x = \frac{1}{\sqrt{1-x^2}}, \qquad \text{for } x \in \,]-1,1\,[,$$

$$\frac{d}{dx}\text{ Arccos } x = -\frac{1}{\sqrt{1-x^2}}, \qquad \text{for } x \in \,]-1,1\,[,$$

$$\frac{d}{dx}\text{ Arctan } x = \frac{1}{1+x^2}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\text{ Arccot } x = \frac{1}{1+x^2}, \qquad \text{for } x \in \mathbb{R}.$$

**Inverse hyperbolic:**

$$\frac{d}{dx}\text{ Arsinh } x = \frac{1}{\sqrt{x^2+1}}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\text{ Arcosh } x = \frac{1}{\sqrt{x^2-1}}, \qquad \text{for } x \in \,]1,+\infty\,[,$$

$$\frac{d}{dx}\text{ Artanh } x = \frac{1}{1-x^2}, \qquad \text{for } |x| < 1,$$

$$\frac{d}{dx}\text{ Arcoth } x = \frac{1}{1-x^2}, \qquad \text{for } |x| > 1.$$

**Remark 25.2** The derivative of the trigonometric and the hyperbolic functions are to some extent exponential like. The derivatives of the inverse trigonometric and inverse hyperbolic functions are power like, because we include the logarithm in this class. ◊

## 25.5 Integration

The most obvious rules are dealing with linearity

$$\int \{f(x) + \lambda g(x)\}\, dx = \int f(x)\, dx + \lambda \int g(x)\, dx, \qquad \text{where } \lambda \in \mathbb{R} \text{ is a constant,}$$

and with the fact that differentiation and integration are "inverses to each other", i.e. modulo some arbitrary constant $c \in \mathbb{R}$, which often tacitly is missing,

$$\int f'(x)\, dx = f(x).$$

If we in the latter formula replace $f(x)$ by the product $f(x)g(x)$, we get by reading from the right to the left and then differentiating the product,

$$f(x)g(x) = \int \{f(x)g(x)\}'\, dx = \int f'(x)g(x)\, dx + \int f(x)g'(x)\, dx.$$

Hence, by a rearrangement

**The rule of partial integration:**

$$\int f'(x)g(x)\, dx = f(x)g(x) - \int f(x)g'(x)\, dx.$$

The differentiation is moved from one factor of the integrand to the other one by changing the sign and adding the term $f(x)g(x)$.

**Remark 25.3** This technique was earlier used a lot, but is almost forgotten these days. It must be revived, because MAPLE and pocket calculators apparently do not know it. It is possible to construct examples where these devices cannot give the exact solution, unless you first perform a partial integration yourself. $\lozenge$

**Remark 25.4** This method can also be used when we estimate integrals which cannot be directly calculated, because the antiderivative is not contained in e.g. the catalogue of MAPLE. The idea is by a succession of partial integrations to make the new integrand smaller. $\lozenge$

**Integration by substitution:**

If the integrand has the special structure $f(\varphi(x)) \cdot \varphi'(x)$, then one can change the variable to $y = \varphi(x)$:

$$\int f(\varphi(x)) \cdot \varphi'(x)\, dx = \text{“} \int f(\varphi(x))\, d\varphi(x)\text{''} = \int_{y=\varphi(x)} f(y)\, dy.$$

**Integration by a monotonous substitution**:

If $\varphi(y)$ is a *monotonous* function, which maps the $y$-interval *one-to-one* onto the $x$-interval, then

$$\int f(x)\,dx = \int_{y=\varphi^{-1}(x)} f(\varphi(y))\varphi'(y)\,dy.$$

**Remark 25.5** This rule is usually used when we have some "ugly" term in the integrand $f(x)$. The idea is to put this ugly term equal to $y = \varphi^{-1}(x)$. When e.g. $x$ occurs in $f(x)$ in the form $\sqrt{x}$, we put $y = \varphi^{-1}(x) = \sqrt{x}$, hence $x = \varphi(y) = y^2$ and $\varphi'(y) = 2y$. $\Diamond$

## 25.6 Special antiderivatives

**Power like:**

$$\int \frac{1}{x}\,dx = \ln|x|, \qquad \text{for } x \neq 0. \quad \text{(Do not forget the numerical value!)}$$

$$\int x^\alpha\,dx = \frac{1}{\alpha+1}x^{\alpha+1}, \qquad \text{for } \alpha \neq -1,$$

$$\int \frac{1}{1+x^2}\,dx = \text{ Arctan } x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{1-x^2}\,dx = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|, \qquad \text{for } x \neq \pm 1,$$

$$\int \frac{1}{1-x^2}\,dx = \text{ Artanh } x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{1-x^2}\,dx = \text{ Arcoth } x, \qquad \text{for } |x| > 1,$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \text{ Arcsin } x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = -\text{ Arccos } x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \text{ Arsinh } x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \ln(x+\sqrt{x^2+1}), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{x}{\sqrt{x^2-1}}\,dx = \sqrt{x^2-1}, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \text{ Arcosh } x, \qquad \text{for } x > 1,$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \ln|x+\sqrt{x^2-1}|, \qquad \text{for } x > 1 \text{ eller } x < -1.$$

There is an error in the programs of the pocket calculators TI-92 and TI-89. The *numerical signs* are missing. It is obvious that $\sqrt{x^2-1} < |x|$ so if $x < -1$, then $x + \sqrt{x^2-1} < 0$. Since you cannot take the logarithm of a negative number, these pocket calculators will give an error message.

**Exponential like:**

$$\int \exp x\, dx = \exp x, \qquad \text{for } x \in \mathbb{R},$$

$$\int a^x\, dx = \frac{1}{\ln a} \cdot a^x, \qquad \text{for } x \in \mathbb{R}, \text{ and } a > 0, a \neq 1.$$

**Trigonometric:**

$$\int \sin x\, dx = -\cos x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \cos x\, dx = \sin x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \tan x\, dx = -\ln|\cos x|, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \cot x\, dx = \ln|\sin x|, \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos x}\, dx = \frac{1}{2} \ln\left(\frac{1+\sin x}{1-\sin x}\right), \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin x}\, dx = \frac{1}{2} \ln\left(\frac{1-\cos x}{1+\cos x}\right), \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos^2 x}\, dx = \tan x, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin^2 x}\, dx = -\cot x, \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z}.$$

**Hyperbolic:**

$$\int \sinh x\, dx = \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \cosh x\, dx = \sinh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \tanh x\, dx = \ln \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \coth x\, dx = \ln|\sinh x|, \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh x}\, dx = \text{Arctan}(\sinh x), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\cosh x}\, dx = 2\,\text{Arctan}(e^x), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh x}\, dx = \frac{1}{2} \ln\left(\frac{\cosh x - 1}{\cosh x + 1}\right), \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\sinh x}\, dx = \ln\left|\frac{e^x-1}{e^x+1}\right|, \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh^2 x}\, dx = \tanh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh^2 x}\, dx = -\coth x, \qquad \text{for } x \neq 0.$$

## 25.7 Trigonometric formulæ

The trigonometric formulæ are closely connected with circular movements. Thus $(\cos u, \sin u)$ are the coordinates of a point $P$ on the unit circle corresponding to the angle $u$, cf. figure A.1. This geometrical interpretation is used from time to time.

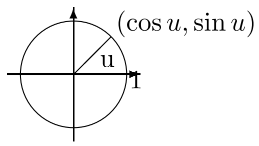



Figure 25.1: The unit circle and the trigonometric functions.

**The fundamental trigonometric relation**:

$$\cos^2 u + \sin^2 u = 1, \qquad \text{for } u \in \mathbb{R}.$$

Using the previous geometric interpretation this means according to *Pythagoras's theorem*, that the point $P$ with the coordinates $(\cos u, \sin u)$ always has distance 1 from the origo $(0,0)$, i.e. it is lying on the boundary of the circle of centre $(0,0)$ and radius $\sqrt{1} = 1$.

**Connection to the complex exponential function**:

The *complex exponential* is for imaginary arguments defined by

$$\exp(\mathrm{i}\, u) := \cos u + \mathrm{i}\ \sin u.$$

It can be checked that the usual functional equation for exp is still valid for complex arguments. In other word: The definition above is extremely conveniently chosen.

By using the definition for $\exp(\mathrm{i}\, u)$ and $\exp(-\mathrm{i}\, u)$ it is easily seen that

$$\cos u = \frac{1}{2}(\exp(\mathrm{i}\, u) + \exp(-\mathrm{i}\, u)),$$

$$\sin u = \frac{1}{2i}(\exp(\mathrm{i}\, u) - \exp(-\mathrm{i}\, u)),$$

.

**Moivre's formula**: We get by expressing $\exp(\mathrm{i}nu)$ in two different ways:

$$\exp(\mathrm{i}nu) = \cos nu + \mathrm{i}\,\sin nu = (\cos u + \mathrm{i}\,\sin u)^n.$$

**Example 25.1** If we e.g. put $n = 3$ into Moivre's formula, we obtain the following typical application,

$$\begin{aligned}\cos(3u) + \mathrm{i}\,\sin(3u) &= (\cos u + \mathrm{i}\,\sin u)^3 \\ &= \cos^3 u + 3\mathrm{i}\,\cos^2 u \cdot \sin u + 3\mathrm{i}^2 \cos u \cdot \sin^2 u + \mathrm{i}^3 \sin^3 u \\ &= \{\cos^3 u - 3\cos u \cdot \sin^2 u\} + \mathrm{i}\{3\cos^2 u \cdot \sin u - \sin^3 u\} \\ &= \{4\cos^3 u - 3\cos u\} + \mathrm{i}\{3\sin u - 4\sin^3 u\}\end{aligned}$$

When this is split into the real- and imaginary parts we obtain

$$\cos 3u = 4\cos^3 u - 3\cos u, \qquad \sin 3u = 3\sin u - 4\sin^3 u. \quad \Diamond$$

**Addition formulæ**:

$$\sin(u+v) = \sin u\,\cos v + \cos u\,\sin v,$$

$$\sin(u-v) = \sin u\,\cos v - \cos u\,\sin v,$$

$$\cos(u+v) = \cos u\,\cos v - \sin u\,\sin v,$$

$$\cos(u-v) = \cos u\,\cos v + \sin u\,\sin v.$$

**Products of trigonometric functions to a sum**:

$$\sin u\,\cos v = \frac{1}{2}\sin(u+v) + \frac{1}{2}\sin(u-v),$$

$$\cos u\,\sin v = \frac{1}{2}\sin(u+v) - \frac{1}{2}\sin(u-v),$$

$$\sin u\,\sin v = \frac{1}{2}\cos(u-v) - \frac{1}{2}\cos(u+v),$$

$$\cos u\,\cos v = \frac{1}{2}\cos(u-v) + \frac{1}{2}\cos(u+v).$$

**Sums of trigonometric functions to a product**:

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right),$$

$$\sin u - \sin v = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right),$$

$$\cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right),$$

$$\cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right).$$

**Formulæ of halving and doubling the angle:**

$$\sin 2u = 2\sin u\,\cos u,$$

$$\cos 2u = \cos^2 u - \sin^2 u = 2\,\cos^2 u - 1 = 1 - 2\,\sin^2 u,$$

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}} \qquad \text{followed by a discussion of the sign,}$$

$$\cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}} \qquad \text{followed by a discussion of the sign,}$$

## 25.8 Hyperbolic formulæ

These are very much like the trigonometric formulæ, and if one knows a little of Complex Function Theory it is realized that they are actually identical. The structure of this section is therefore the same as for the trigonometric formulæ. The reader should compare the two sections concerning similarities and differences.

**The fundamental relation**:

$$\cosh^2 x - \sinh^2 x = 1.$$

**Definitions**:

$$\cosh x = \frac{1}{2}\left(\exp(x) + \exp(-x)\right), \qquad \sinh x = \frac{1}{2}\left(\exp(x) - \exp(-x)\right).$$

**"Moivre's formula"**:

$$\exp(x) = \cosh x + \sinh x.$$

This is trivial and only rarely used. It has been included to show the analogy.

**Addition formulæ**:

$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y),$$

$$\sinh(x-y) = \sinh(x)\cosh(y) - \cosh(x)\sinh(y),$$

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y),$$

$$\cosh(x-y) = \cosh(x)\cosh(y) - \sinh(x)\sinh(y).$$

**Formulæ of halving and doubling the argument**:

$$\sinh(2x) = 2\sinh(x)\cosh(x),$$

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x) = 2\cosh^2(x) - 1 = 2\sinh^2(x) + 1,$$

$$\sinh\left(\frac{x}{2}\right) = \pm\sqrt{\frac{\cosh(x)-1}{2}} \qquad \text{followed by a discussion of the sign,}$$

$$\cosh\left(\frac{x}{2}\right) = \sqrt{\frac{\cosh(x)+1}{2}}.$$

**Inverse hyperbolic functions**:

$$\text{Arsinh}(x) = \ln\left(x + \sqrt{x^2+1}\right), \qquad x \in \mathbb{R},$$

$$\text{Arcosh}(x) = \ln\left(x + \sqrt{x^2-1}\right), \qquad x \geq 1,$$

$$\text{Artanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \qquad |x| < 1,$$

$$\text{Arcoth}(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right), \qquad |x| > 1.$$

## 25.9 Complex transformation formulæ

$$\cos(\mathrm{i}x) = \cosh(x), \qquad \cosh(\mathrm{i}x) = \cos(x),$$

$$\sin(\mathrm{i}x) = \mathrm{i}\sinh(x), \qquad \sinh(\mathrm{i}x) = \mathrm{i}\sin x.$$

## 25.10 Taylor expansions

The generalized binomial coefficients are defined by

$$\binom{\alpha}{n} := \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{1\cdot 2\cdots n},$$

with $n$ factors in the numerator and the denominator, supplied with

$$\binom{\alpha}{0} := 1.$$

The Taylor expansions for *standard functions* are divided into *power like* (the radius of convergency is finite, i.e. $= 1$ for the standard series) and*exponential like* (the radius of convergency is infinite).
**Power like**:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1,$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x| < 1,$$

$$(1+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j, \qquad n \in \mathbb{N}, x \in \mathbb{R},$$

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \qquad \alpha \in \mathbb{R}\setminus\mathbb{N}, |x| < 1,$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n}, \qquad |x| < 1,$$

$$\mathrm{Arctan}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \qquad |x| < 1.$$

**Exponential like**:

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \qquad x \in \mathbb{R}$$

$$\exp(-x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} x^n, \qquad x \in \mathbb{R}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}, \qquad x \in \mathbb{R},$$

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}, \qquad x \in \mathbb{R},$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}, \qquad x \in \mathbb{R},$$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}, \qquad x \in \mathbb{R}.$$

## 25.11 Magnitudes of functions

We often have to compare functions for $x \to 0+$, or for $x \to \infty$. The simplest type of functions are therefore arranged in an hierarchy:

1) logarithms,

2) power functions,

3) exponential functions,

4) faculty functions.

When $x \to \infty$, a function from a higher class will always dominate a function form a lower class. More precisely:

**A)** A *power function* dominates a *logarithm* for $x \to \infty$:

$$\frac{(\ln x)^\beta}{x^\alpha} \to 0 \qquad \text{for } x \to \infty, \quad \alpha,\, \beta > 0.$$

**B)** An *exponential* dominates a *power function* for $x \to \infty$:

$$\frac{x^\alpha}{a^x} \to 0 \qquad \text{for } x \to \infty, \quad \alpha,\, a > 1.$$

**C)** The *faculty function* dominates an *exponential* for $n \to \infty$:

$$\frac{a^n}{n!} \to 0, \qquad n \to \infty, \quad n \in \mathbb{N}, \quad a > 0.$$

**D)** When $x \to 0+$ we also have that a *power function* dominates the *logarithm:*

$$x^\alpha \ln x \to 0-, \qquad \text{for } x \to 0+, \quad \alpha > 0.$$

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