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Real Functions in Several Variables: Volume V

The range of a function Extrema of a Function in Several...

Leif Mejlbro

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Real Functions in Several Variables: Volume V
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Contents

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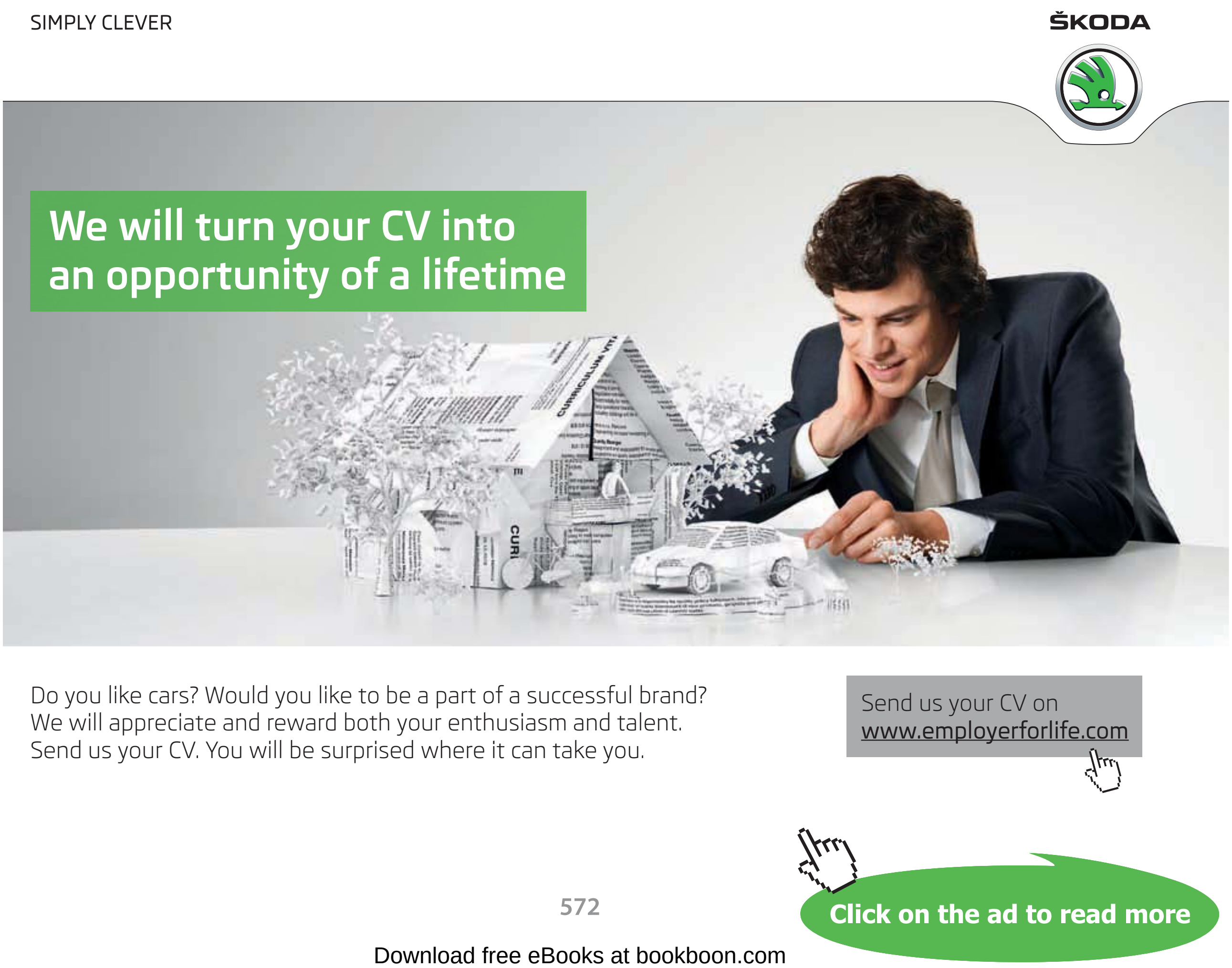

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Preface

The topic of this series of books on *"Real Functions in Several Variables"* is very important in the description in e.g. *Mechanics* of the real 3-dimensional world that we live in. Therefore, we start from the very beginning, modelling this world by using the coordinates of $\mathbb{R}^3$ to describe e.g. a motion in space. There is, however, absolutely no reason to restrict ourselves to $\mathbb{R}^3$ alone. Some motions may be rectilinear, so only $\mathbb{R}$ is needed to describe their movements on a line segment. This opens up for also dealing with $\mathbb{R}^2$, when we consider plane motions. In more elaborate problems we need higher dimensional spaces. This may be the case in *Probability Theory* and *Statistics*. Therefore, we shall in general use $\mathbb{R}^n$ as our abstract model, and then restrict ourselves in examples mainly to $\mathbb{R}^2$ and $\mathbb{R}^3$.

For rectilinear motions the familiar *rectangular coordinate system* is the most convenient one to apply. However, as known from e.g. *Mechanics*, circular motions are also very important in the applications in engineering. It becomes natural alternatively to apply in $\mathbb{R}^2$ the so-called *polar coordinates* in the plane. They are convenient to describe a circle, where the rectangular coordinates usually give some nasty square roots, which are difficult to handle in practice.

Rectangular coordinates and polar coordinates are designed to model each their problems. They supplement each other, so difficult computations in one of these coordinate systems may be easy, and even trivial, in the other one. It is therefore important always in advance carefully to analyze the geometry of e.g. a domain, so we ask the question: Is this domain best described in rectangular or in polar coordinates?

Sometimes one may split a problem into two subproblems, where we apply rectangular coordinates in one of them and polar coordinates in the other one.

It should be mentioned that in *real life* (though not in these books) one cannot always split a problem into two subproblems as above. Then one is really in trouble, and more advanced mathematical methods should be applied instead. This is, however, outside the scope of the present series of books.

The idea of polar coordinates can be extended in two ways to $\mathbb{R}^3$. Either to *semi-polar* or *cylindric coordinates*, which are designed to describe a cylinder, or to *spherical coordinates*, which are excellent for describing spheres, where rectangular coordinates usually are doomed to fail. We use them already in daily life, when we specify a place on Earth by its longitude and latitude! It would be very awkward in this case to use rectangular coordinates instead, even if it is possible.

Concerning the contents, we begin this investigation by modelling point sets in an n-dimensional Euclidean space E^n by $\mathbb{R}^n$. There is a subtle difference between E^n and $\mathbb{R}^n$, although we often identify these two spaces. In E^n we use *geometrical methods* without a coordinate system, so the objects are independent of such a choice. In the coordinate space $\mathbb{R}^n$ we can use ordinary calculus, which in principle is not possible in E^n. In order to stress this point, we call E^n the "abstract space" (in the sense of calculus; not in the sense of geometry) as a warning to the reader. Also, whenever necessary, we use the colour black in the "abstract space", in order to stress that this expression is theoretical, while variables given in a chosen coordinate system and their related concepts are given the colours blue, red and green.

We also include the most basic of what mathematicians call *Topology*, which will be necessary in the following. We describe what we need by a function.

Then we proceed with limits and continuity of functions and define continuous curves and surfaces, with parameters from subsets of $\mathbb{R}$ and $\mathbb{R}^2$, resp..

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Continue with (partial) differentiable functions, curves and surfaces, the chain rule and Taylor's formula for functions in several variables.

We deal with maxima and minima and extrema of functions in several variables over a domain in $\mathbb{R}^n$. This is a very important subject, so there are given many worked examples to illustrate the theory.

Then we turn to the problems of integration, where we specify four different types with increasing complexity, *plane integral, space integral, curve (or line) integral and surface integral*.

Finally, we consider *vector analysis*, where we deal with vector fields, Gauß's theorem and Stokes's theorem. All these subjects are very important in theoretical Physics.

The structure of this series of books is that each subject is *usually* (but not always) described by three successive chapters. In the first chapter a brief theoretical theory is given. The next chapter gives some practical guidelines of how to solve problems connected with the subject under consideration. Finally, some worked out examples are given, in many cases in several variants, because the standard solution method is seldom the only way, and it may even be clumsy compared with other possibilities.

I have as far as possible structured the examples according to the following scheme:

A *Awareness*, i.e. a short description of what is the problem.

D *Decision*, i.e. a reflection over what should be done with the problem.

I *Implementation*, i.e. where all the calculations are made.

C *Control*, i.e. a test of the result.

This is an ideal form of a general procedure of solution. It can be used in any situation and it is not linked to Mathematics alone. I learned it many years ago in the Theory of Telecommunication in a situation which did not contain Mathematics at all. The student is recommended to use it also in other disciplines.

From high school one is used to immediately to proceed to **I**. *Implementation*. However, examples and problems at university level, let alone situations in real life, are often so complicated that it in general will be a good investment also to spend some time on the first two points above in order to be absolutely certain of what to do in a particular case. Note that the first three points, **ADI**, can always be executed.

This is unfortunately not the case with **C** *Control*, because it from now on may be difficult, if possible, to check one's solution. It is only an extra securing whenever it is possible, but we cannot include it always in our solution form above.

I shall on purpose not use the logical signs. These should in general be avoided in Calculus as a shorthand, because they are often (too often, I would say) misused. Instead of $\wedge$ I shall either write "and", or a comma, and instead of $\vee$ I shall write "or". The arrows $\Rightarrow$ and $\Leftrightarrow$ are in particular misunderstood by the students, so they should be totally avoided. They are not telegram short hands, and from a logical point of view they usually do not make sense at all! Instead, write in a plain language what you mean or want to do. This is difficult in the beginning, but after some practice it becomes routine, and it will give more precise information.

When we deal with multiple integrals, one of the possible pedagogical ways of solving problems has been to colour variables, integrals and upper and lower bounds in blue, red and green, so the reader by the colour code can see in each integral what is the variable, and what are the parameters, which

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do not enter the integration under consideration. We shall of course build up a hierarchy of these colours, so the order of integration will always be defined. As already mentioned above we reserve the colour black for the theoretical expressions, where we cannot use ordinary calculus, because the symbols are only shorthand for a concept.

The author has been very grateful to his old friend and colleague, the late Per Wennerberg Karlsson, for many discussions of how to present these difficult topics on real functions in several variables, and for his permission to use his textbook as a template of this present series. Nevertheless, the author has felt it necessary to make quite a few changes compared with the old textbook, because we did not always agree, and some of the topics could also be explained in another way, and then of course the results of our discussions have here been put in writing for the first time.

The author also adds some calculations in MAPLE, which interact nicely with the theoretic text. Note, however, that when one applies MAPLE, one is forced first to make a geometrical analysis of the domain of integration, i.e. apply some of the techniques developed in the present books.

The theory and methods of these volumes on "Real Functions in Several Variables" are applied constantly in higher Mathematics, Mechanics and Engineering Sciences. It is of paramount importance for the calculations in *Probability Theory*, where one constantly integrate over some point set in space.

It is my hope that this text, these guidelines and these examples, of which many are treated in more ways to show that the solutions procedures are not unique, may be of some inspiration for the students who have just started their studies at the universities.

Finally, even if I have tried to write as careful as possible, I doubt that all errors have been removed. I hope that the reader will forgive me the unavoidable errors.

Leif Mejlbro
March 21, 2015

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Introduction to volume V,
The range of a function, Extrema of a Function in Several Variables

This is the fifth volume in the series of books on *Real Functions in Several Variables*. Its topic is dealing with the range of a functions, its global and local extrema.

Let $f : A \to \mathbb{R}$ be a continuous function, where $A \subseteq \mathbb{R}^m$. We show that extrema of f can only exist at either points in the interior of A, where f is not differentiable – also called exceptional points – or at the so-called stationary points, i.e. points in the interior of A, where the gradient is $\mathbf{0}$ – or at the points of the boundary of A also lying in A, i.e. in $A \cap \partial A$. This eases the task a lot, though there may still be problems.

One of the problems is that points of extrema, i.e. where f attains its maximum or minimum, do not exist in general. However, if A is closed and bounded in $\mathbb{R}^m$, then we prove that we always have both a global maximum and a global minimum.

As usual the number of practical computations increase factorially with the dimension, so in practice only the cases of two or three space variables are manageable. Even an innocent looking problem like finding extrema for a second order polynomial in m variables over some closed and bounded set $A \subset \mathbb{R}^m$ may turn out to be a computational mess, if m is "large". The author has experienced this once for $m = 7$, and had to create an alternative solution method than the standard procedures given in this book.

However, in $\mathbb{R}^2$ and $\mathbb{R}^3$ the calculations are in general moderate. In some cases MAPLE could help a lot. On the other hand, the focus in this book has been to emphasize the possible methods, so most of the examples are fairly simple, and it would seem to be too much to apply MAPLE on them.

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16 The range of a function

16.1 Introduction

We shall in this chapter study the range of a function in n variables. We shall assume, unless stated otherwise, that the function is continuous. This assumption immediately invites us to apply the main theorems of continuous functions, which already have been quoted in Section 5.11.

We shall start with

Theorem 16.1 The first main theorem of continuous functions. *Assume that $A \subseteq \mathbb{R}^n$ is connected, and that $\mathbf{f} : A \to \mathbb{R}^m$ is continuous. Then the range $\mathbf{f}(A)$ is also connected.*

In the special case, when $m = 1$, the range $\mathbf{f}(A) \subseteq \mathbb{R}$ becomes an interval. Depending on the definition of $\mathbf{f}$ and A this range can be any type of interval, closed, half-open or open. We cannot derive more from Theorem 16.1.

If $\mathbf{f} : A \to \mathbb{R}$ is continuous, while A is not connected, then we use that A can be decompose into connected subsets,

$$A = A_1 \cup \cdots \cup A_k, \qquad \text{or} \qquad A = A_1 \cup \cdots \cup A_k \cup \cdots,$$

where all the A_k are connected sets which are mutually disjoint. Using Theorem 16.1 above, each subrange $f(A_j)$ is an interval, so in a general analysis we may without loss of generality from the very beginning restrict ourselves to the case, where the domain A of f is connected.

The real axis $\mathbb{R}$ is ordered by the ordinary order relation $\leq$, and since A is connected, hence $f(A) = I$ an interval, we can introduce the following definition.

Definition 16.1 *Let $A \subset \mathbb{R}^m$ be a connected set, and let $f : A \to \mathbb{R}$ be a continuous function.*

1) If there exists a point $\mathbf{a} \in A$, such that

$$f(\mathbf{a}) \leq f(\mathbf{x}) \qquad \textit{for all } \mathbf{x} \in A,$$

then the image $f(\mathbf{a})$ of the point $\mathbf{a}$ is called the (global) minimum *of f on A.*

2) If there exists a point $\mathbf{b} \in A$, such that

$$f(\mathbf{x}) \leq f(\mathbf{b}) \qquad \textit{for all } \mathbf{x} \in A,$$

then the image $f(\mathbf{b})$ of the point $\mathbf{b}$ is called the (global) maximum *of f on A.*

If f is continuous on the connected set A, and f has both a minimum $f(\mathbf{a})$ and a maximum $f(\mathbf{b})$ on A, then it follows from the above that the range is the closed interval

$$f(A) = [f(\mathbf{a}), f(\mathbf{b})].$$

Then we turn to

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Theorem 16.2 The second main theorem of continuous functions. *Assume that $A \subseteq \mathbb{R}^m$ is a bounded and closed set. If $\mathbf{f} : A \to \mathbb{R}^k$ is continuous, then the range $\mathbf{f}(A) \subseteq \mathbb{R}^k$ is also a bounded and closed set.*

Remark. Subsets in $\mathbb{R}^m$, which are both bounded and closed are also called *compact sets*. The old-fashioned term is used here, because “compact” is still not recognized everywhere. $\Diamond$

It follows from Theorem 16.2 that if $f : A \to \mathbb{R}$ is continuous, and A is bounded and closed, then the range $f(A)$ is also bounded and closed, though not necessarily connected, if A is not connected. So $f(A) \subseteq I$, where I is the smallest bounded and closed interval, which contains $f(A)$. In particular, the two end points of I *must* belong to the range $f(A)$, because otherwise we could find a smaller interval containing $f(A)$. This means that f has both a minimum and a maximum, whenever $f : A \to \mathbb{R}$ is continuous on the bounded and closed set A into the set of real numbers $\mathbb{R}$.

In the applications we may also be interested in *local* minima and maxima. A collective word for minima and maxima is *extrema*. We shall in the following sections more closely study first the global extrema, and then the local extrema.

Since the reader may feel this topic difficult, some examples in the text have been worked out in all details, while the more standard treatment of examples is given in Chapter 17, because otherwise the volume would be overwhelming.

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16.2 Global extrema of a continuous function

16.2.1 A necessary condition

When we shall find the smallest and largest value of a continuous function $f : A \to \mathbb{R}$, the strategy is to split the domain A of f into four subdomains, and then consider the possibility of extrema in each of them. In particular, it will turn up, that one of these subsets will never contain extrama. The four sets are listed below:

1) The set A_s of *stationary points*. A point $\mathbf{u} \in A^\circ$ (the interior of A) is called a *stationary point* of f, if f is differentiable at $\mathbf{u}$, and $\bigtriangledown f(\mathbf{u}) = \mathbf{0}$.

2) The set A_e of *exceptional points*. A point $\mathbf{u} \in A^\circ$ iis called an *exceptional point* of f, if either f is not differentiable at $\mathbf{u}$, or it is too difficult to check if it is differentiable at $\mathbf{u}$.

3) The set ∂A of *boundary points*. This is just the ordinary boundary of the set A. It was introduced in Section 1.5.1.

4) The set A_r of remaining points in A°. This means that if $\mathbf{u} \in A_r$, then f is differentiable at $\mathbf{u}$ and $\bigtriangledown f(\mathbf{u}) \neq \mathbf{0}$, so $\mathbf{u}$ is neither an exceptional nor a stationary point, and since $\mathbf{u} \in A^\circ$, it is not a boundary point either.

Clearly, $A \subseteq A_s \cup A_e \cup \partial A \cup A_r$. Every point of A lies in one of the four subsets, while there may be boundary points $\mathbf{u} \in \partial A$, which do not belong to A.

Let us assume the $\mathbf{u} \in A_r$, so $\mathbf{u} \in A^\circ$ and f is differentiable with $\bigtriangledown f(\mathbf{u}) \neq \mathbf{0}$. We let $\mathbf{e}$ denote the unit vector in the direction of the gradient og f at $\mathbf{u}$, i.e.

$$\mathbf{e} := \frac{\bigtriangledown f(\mathbf{u})}{\| \bigtriangledown f(\mathbf{u})\|}.$$

Then introduce the function

$$F(t) := f(\mathbf{u} + t\mathbf{e}), \qquad \text{where } \mathbf{u} + t\mathbf{e} \in A \text{ for } |t| < \delta, \text{ and } F(0) = f(\mathbf{u}).$$

We get by the *chain rule*,

$$F'(0) = \mathbf{e} \cdot \bigtriangledown f(\mathbf{u}) = \| \bigtriangledown f(\mathbf{u})\| > 0,$$

so when we take the restriction of f to the line segment $\{\mathbf{u} + t\mathbf{e} \mid |t| < \delta\}$, this restriction $(= F(t))$ is increasing in a neighbourhood of $\mathbf{u}$. Therefore, on this line segment, $f(\mathbf{u})$ can neither be a minimum nor a maximum.

In other words, this simple argument shows that the set A_r does not contain any extremum, and we have proved

Theorem 16.3 A necessary condition for global extrema. *Assume that a function $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^m$, has a global extremum at a point $\mathbf{u} \in A$. Then*

$$\mathbf{u} \in A_s \cup A_e \cup \partial A,$$

i.e. $\mathbf{u}$ is either a stationary point, or an exceptional point, or a boundary point.

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Theorem 16.3 does not say anything about the *existence* of global extrema. It only gives some hints of where to search for *possible* global extrema.

The set of stationary points A_s are found by solving the vector equation

$$\nabla f(\mathbf{u}) = \mathbf{0},$$

which we split into a system of m (in general nonlinear) equations in the m unknown coordinates. We shall discuss this later on.

The set of exceptional points is in principle easy to spot, because it consists of the points, where f either is not differentiable, or where it is very difficult to prove whether it is differentiable or not. In most of the simple applications, however, the set of exceptional points is either empty, or contains only a finite number of points. If e.g. the square root occurs in the definition of f, then A_e may contain even curves, so one cannot rule out A_e from the beginning.

Finally, concerning the investigation of the values of f on the boundary, we shall *usually* reduce the problem to an $m-1$-dimensional case, because it is *usually* possible to eliminate one of the variables on the boundary. This means that the restriction to ∂A is equivalent to a new problem with a new continuous function $f_1 : A_1 \to \mathbb{R}$ on a closed and bounded set $A_1 \subset \mathbb{R}^{m-1}$ in a lower dimensional space, and so we proceed.

In principle, this method should be possible, but .... If the dimension m is large - even for moderate m this phenomenon occurs - the number of special cases, which require an inspection, may be overwhelming. The author was once asked to find the extrema of a squared function on a closed and bounded set in $\mathbb{R}^8$. There were no exceptional points, and the possible stationary point was outside the set A, so "only" the boundary ∂A remained. It turned up that it was consisting of $\sim 7!$ special cases! The problem was solved in the end, but not by using the "standard procedure" described here.

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16.2.2 The case of a closed and bounded domain of f

We shall then take a closer look on the problem. To ease matters, we shall assume the $f : A \to \mathbb{R}$ is continuous on a closed and bounded domain $A \subset \mathbb{R}^m$, in which case it follows from the second main theorem, cf. Theorem 16.2, page 580, that f has both a global maximum and a global minimum on A. It follows from the analysis in Section 16.2.1 that each of them belongs to one of the following subsets of $\mathbb{R}$,

$$T_s = \{f(\mathbf{u}) \in \mathbb{R} \mid \mathbf{u} \in A^\circ \text{ is a stationary point, } \bigtriangledown f(\mathbf{u}) = \mathbf{0}\},$$

$$T_e = \{f(\mathbf{u}) \in \mathbb{R} \mid f \text{ is not diffentiable at } \mathbf{u} \in A^\circ\},$$

$$T_b = \{f(\mathbf{u}) \mid \mathbf{u} \in \partial A\}.$$

Usually A_s and A_e only contain a finitely many points from A°, if any, so we just insert these points and compare the sizes of their values.

Also, *usually* the restriction of the function to the boundary ∂A is in practice reduced to a function in $m-1$ variables, so in principle we have a new situation of a continuous function $f_1 : A_1 \to \mathbb{R}$, where $A_1 \subset \mathbb{R}^{m-1}$ is closed and bounded. Then the investigation starts from the beginning, where we must find stationary and exceptional points in A_1 for this new function f_1, and we also get a new boundary ∂A_1.

In this way we proceed $m-1$ times, until we get the restriction written as a continuous function $f_{m-1} : A_{m-1} \to \mathbb{R}$, where $A_{m-1} \subset \mathbb{R}$ is 1-dimensional and closed and bounded, and the problem is reduced to a high school problem.

Needless to say, that concerning global extrema on a closed and bounded set $A \subset \mathbb{R}^m$, the investigation of the boundary is *usually* the biggest task.

The use of the word *"usually"* above does not imply that it is *always* so. One may construct extremum problems where either the stationary points or the exceptional points require a lot of work.

In order to get some feeling of this theory we shall in the following start with only considering $m = 2$, so $f : A \to \mathbb{R}$ is from now on a continuous function on a closed and bounded plane set $A \subset \mathbb{R}^2$, where we use the rectangular coordinates $(x, y) \in A$.

Let $(x, y) \in A^\circ$ be a point, where f is differentiable, *If* is a *stationary point*, then we must have $\bigtriangledown f(x, y) = \mathbf{0}$, i.e. in coordinates,

$$f'_x(x, y) = 0 \qquad \text{and} \qquad f'_y(x, y) = 0.$$

In order to find all stationary points we shall solve this system of (usually nonlinear) two equations in the two variables x and y. There is no standard method for doing this, and the problem is in general hard to solve. However, a couple of guidelines may be useful.

1) If one or both of the left hand sides of the equations can be factorized, then we can reduce the problem considerably. In fact, the left hand side is zero, if and only if at least one of its factors is zero, so we split the investigation into a number of simpler problems, putting each of the factors equal to zero and then solving the simpler systems. Due to this potential possibility *one should never multiply the factors on the left hand side*, when they occur from the beginning. By doing this one shall lose some information.

2) Another possibility occurs, when we can eliminate one of the variables, x or y. In this case we obtain one (usually nonlinear) equation in only one variable. This is solved by some known procedure, e.g. by a factorization, by guessing a root, by a graphical consideration, or by an application of the *Newton-Raphson iteration formula*.

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Concerning the investigation of the values of the function f on the boundary $\partial A \subset \mathbb{R}^2$, where A is closed and bounded, we note that *in most of the applications in practice*, the boundary ∂A is a closed piecewise C^1-curve, or a union of such piecewise C^1-curves. The simplest case occurs of course, when ∂A is a closed curve, given by a parametric description (cf. e.g. Volume IV in this series),

$$(x, y) = (X(t), Y(t)) \qquad \text{for } t \in [a, b],$$

where $X(t)$ and $Y(t)$ are given functions. In this case the restriction of f to the boundary ∂A is described by the (new) ordinary function

$$g(t) := f(X(t), Y(t)), \qquad \text{for } t \in [a, b],$$

in one variable. This method is also applied, when ∂A is broken up into pieces where we can use a parametric description, although this piece does not have to be a closed curve in the plane.

It should here be added that the reader should never believe that the methods described above are the only possibilities of methods in these extremum problems. If we by some other method, e.g. by inspection, can find the range $[c, d]$, there is of course no need at all to go through all the procedures described above, because then we already have that

$$\min_{\mathbf{u} \in A} f(\mathbf{u}) = c \qquad \text{and} \qquad \max_{\mathbf{u} \in A} f(\mathbf{u}) = d.$$

The exception is of course, when we also want to know where these extrema are attained.

In order to show how the theory above is applied in practice in $\mathbb{R}^2$ we proceed with some worked out examples.

Example 16.1

A *Let A be a closed and bounded (i.e. compact) subset of the plane where the boundary ∂A is a closed curve of the parametric representation*

$$\mathbf{r}(t) = \left(4t^{\frac{1}{3}}(1-t)^{\frac{2}{3}}, 4t^{\frac{2}{3}}(1-t)^{\frac{1}{3}}\right), \qquad t \in [0, 1].$$

Find the maximum and minimum in A for the C^∞-function

$$f(x, y) = x^3 + y^3 - 3xy, \qquad (x, t) \in A.$$

D *Standard procedure:*

1) Sketch the *domain A* and apply the *second main theorem for continuous functions*, from which we conclude the existence of a maximum and a minimum.
2) Identify the *exceptional points* in A°, if any, and calculate the values $f(x, y)$ in these points.
3) Set up the equations for the *stationary points*; find these – which quite often is a fairly difficult task, because the system of equations is usually nonlinear. Finally, compute the values $f(x, y)$ in all stationary points.
4) Examine the function on the boundary, i.e. restrict the function $f(x, y)$ to the boundary and repeat the investigation above to a set which is of lower dimension. Then find the maximum and minimum on the boundary.

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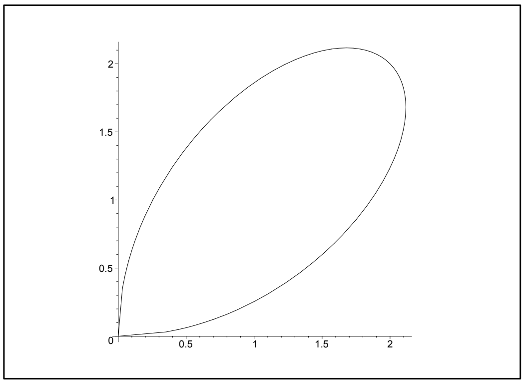

Figure 16.1: The closed and bounded domain A.

5) Collect all the *candidates* for a maximum and a minimum found previously in 2)–4). Then the maximum S and the minimum M are found by a simple numerical comparison.

Remark 16.1 Note that by using this method there is no need to use the complicated (r, s, t)-method, which will be described later and which should only be applied when we shall find *local* extrema in the plane. Here we are dealing with *global* maxima and minima in a set A. $\Diamond$

Remark 16.2 Sometimes it is alternatively easy to identify the *level curves* $f(x, y) = c$ for the function f. In such a case, sketch a convenient number of the level curves, from which it may be easy to find the largest and the smallest constant c, for which the corresponding level curve has points in common with the set A. Then these values of c are automatically the maximum S, resp. the minimum M for f on A.

Note, however, that this alternative method is demanding some experience before one can use it as a standard method of solution. It has once been used with success by a brilliant student at an examination. $\Diamond$

I The level curves $f(x, y) = x^3 + y^3 - 3xy = c$ do not look to promising, so we stick to the standard procedure.

1) The domain A has already been sketched. Since A is *closed and bounded*, and $f(x, y)$ is *continuous* on A, it follows from the *second main theorem for continuous functions* that the function f has a maximum and a minimum on the set A.
2) Since f is of class C^∞ in A°, there are no *exceptional points.*
3) The *stationary points* satisfy the two equations

$$\frac{\partial f}{\partial x} = 3x^2 - 3y = 0, \quad \text{i.e.} \quad y = x^2,$$

$$\frac{\partial f}{\partial y} = 3y^2 - 3x = 0, \quad \text{i.e.} \quad x = y^2.$$

When we look at the *graph* we obtain the two solutions:

$$(0, 0) \in \partial A \qquad \text{and} \qquad (1, 1) \in A^\circ.$$

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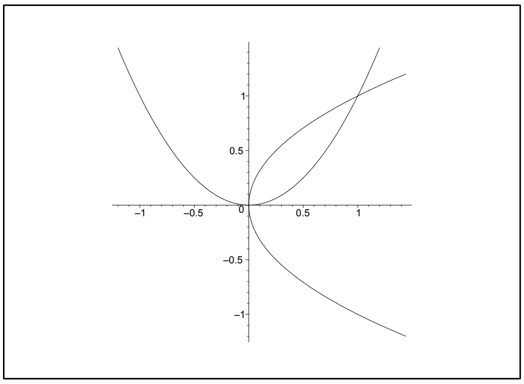


Figure 16.2: The stationary points are the intersections between the curves $y = x^2$ and $x = y^2$.

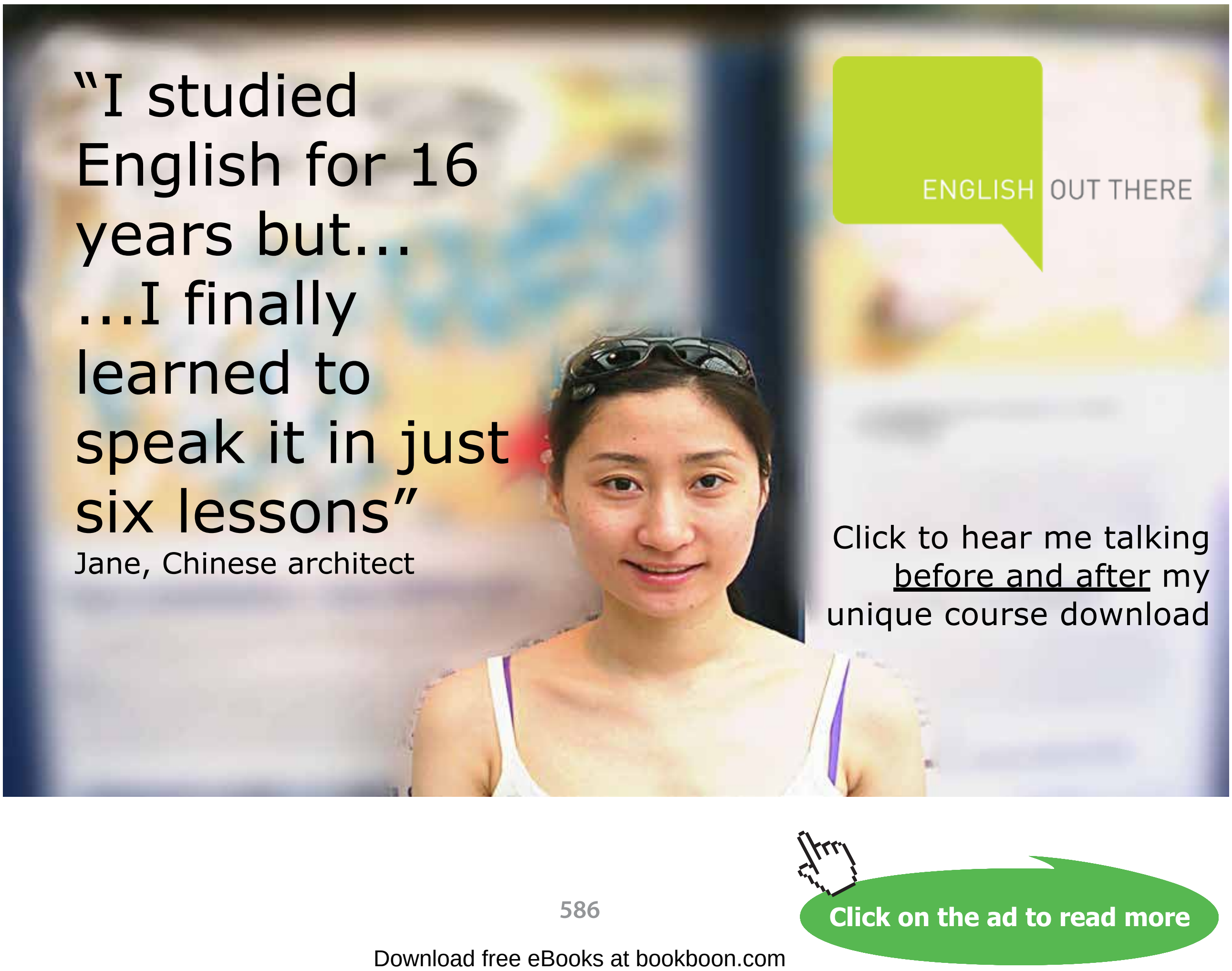
Alternatively one inserts $y = x^2$ into the second equation

$$0 = y^2 - x = x^4 - x = x(x^3 - 1) = x(x-1)(x^2 + x + 1).$$

Here $x^2 + x + 1$ has only *complex* roots, hence the only *real* roots are $x = 0$ (with $y = x^2 = 0$) and $x = 1$ (with $y = x^2 = 1$), corresponding to

$$(0, 0) \in \partial A \qquad \text{and} \qquad (1, 1) \in A^\circ.$$

Since $(0, 0)$ is a *boundary point*, we see that $(1, 1) \in A^\circ$ is the *only stationary point* for f in A°.



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We transfer the value

$$f(1,1)=1+1-3=-1.$$

to the collection of all values in 5) below.

4) *The Boundary.* When we apply the parametric representation

$$(x,y)=\mathbf{r}(t), \qquad t\in[0,1],$$

we get the *restriction* to the boundary

$$\begin{aligned} g(t) &= f(\mathbf{r}(t))=f\left(4t^{\frac{1}{3}}(1-t)^{\frac{2}{3}},4t^{\frac{2}{3}}(1-t)^{\frac{1}{3}}\right)\\ &= \left\{64t(1-t)^2\right\}+\left\{64t^2(1-t)\right\}-3\cdot\left\{4t^{\frac{1}{3}}(1-t)^{\frac{2}{3}}\right\}\cdot\left\{4t^{\frac{2}{3}}(1-t)^{\frac{1}{3}}\right\}\\ &= 64\,t(1-t)^2+64\,t^2(1-t)-48\,t(1-t)\\ &= 16\,t(1-t)\{4(1-t)+4t-3\}=16\,t(1-t), \qquad\qquad t\in[0,1]. \end{aligned}$$

We have finally reduced the problem to a problem known from high school

$$g'(t)=12(2t-1)=0 \qquad \text{for } t=\frac{1}{2},$$

corresponding to

$$g\left(\frac{1}{2}\right)=f\left(4\cdot 2^{-\frac{1}{3}}\cdot 2^{-\frac{2}{3}},4\cdot 2^{-\frac{2}{3}}\cdot 2^{-\frac{1}{3}}\right)=f(2,2)=4.$$

At the end points of the interval, $t=0$ and $t=1$, we get

$$g(0)=g(1)=f(0,0)=0.$$

5) We collect all the candidates:

exceptional points:	None,	[from 2)]
Stationary point:	$f(1,1)=-1$,	[from 3)]
Boundary points:	$f(0,0)=0$ and $f(2,2)=4$,	[from 4)].

By a *numerical comparison* we get

- The *minimum* is $f(1,1)=-1$ (a stationary point),
- The *maximum* is $f(2,2)=4$ (a boundary point).

6) A typical addition: Since A is *connected*, and f is *continuous*, it also follows from the *first main theorem for continuous functions*, that the range is an interval (i.e. connected), hence

$$f(A)=[M,S]=[-1,4]. \qquad \Diamond$$

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Example 16.2

A. *Find maximum and minimum of the C^∞-function*

$$f(x,y) = x^4 + 4x^2y^2 + y^4 - 4x^3 - 4y^3$$

in the set A given by $x^2 + y^2 \le 4 = 2^2$.

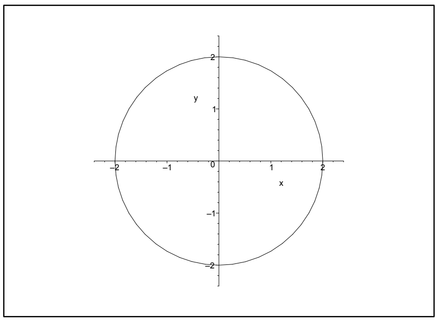

Figure 16.3: The domain A.

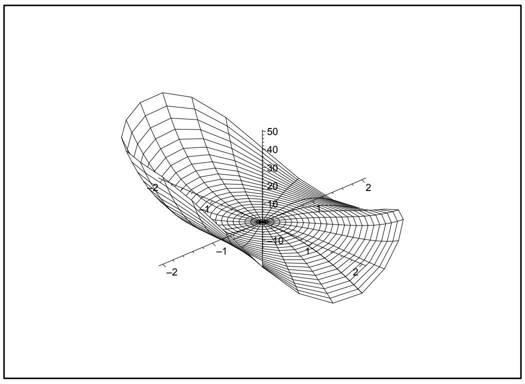

Figure 16.4: The graph of $f(x,y)$ over A. Note that a consideration of the graph does not give any hint.

D. Even if the rewriting of the function

$$f(x,y) = (x^2+y^2)^2 + 2x^2y^2 - 4(x^3+y^3)$$

looks reasonably nice it is still not tempting to apply an analysis of the level curves $f(x,y) = c$, so we shall again use the standard method as described in the previous example, to which we refer for the description.

I. 1) The domain A has been sketched already. Since A is *closed and bounded*, and $f(x,y)$ is *continuous* on A, it follows from the *second main theorem for continuous functions* that $f(x,y)$ has a maximum and a minimum on A.

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While we are dealing with theoretical considerations we may aside mention that since A is obviously *connected*, it follows from the *first main theorem for continuous functions* that the range is connected, i.e. an interval, which necessarily is given by

$$f(A) = [M, S].$$

2) Since $f(x, y)$ is of class C^∞, there is *no exceptional point*.
3) The *stationary points* (if any) satisfies the system of equations

$$0 = \frac{\partial f}{\partial x} = 4x^3 + 8xy^2 - 12x^2 = 4x(x^2 + 2y^2 - 3x),$$

$$0 = \frac{\partial f}{\partial y} = 8x^2 y + 4y^3 - 12y^2 = 4y(2x^2 + y^2 - 3y).$$

Note that it is *extremely important* to factorize the expressions as much as possible in order to solve the system. In fact, when this is done, we can reduce the system to

$$\frac{\partial f}{\partial x} = 0: \qquad x = 0 \quad \text{or} \quad x^2 + 2y^2 - 3x = 0,$$

$$\frac{\partial f}{\partial y} = 0: \qquad y = 0 \quad \text{or} \quad 2x^2 + y^2 - 3y = 0.$$

These conditions are now paired in $2 \cdot 2 = 4$ ways which are handled one by one.

a) When $x = 0$ and $y = 0$, we get $(0, 0) \in A^\circ$, i.e. $(0, 0)$ is a stationary point with the value of the function

$$f(0, 0) = 0.$$

b) When $x = 0$ and $2x^2 + y^2 - 3y = 0$, we get

$$0 + y^2 - 3y = y(y - 3) = 0, \qquad \text{hence } y = 0 \text{ or } y = 3.$$

Thus, we have two possibilities: $(0, 0) \in A^\circ$, which has already been found previously, and $(0, 3) \notin A$, so this point does not participate in the competition. We therefore do not get further points in this case.

c) When $y = 0$ and $x^2 + 2y^2 - 3x = 0$, we get by an interchange of letters $(x, y) \to (y, x)$ that the candidates are $(0, 0) \in A^\circ$ [found previously] and $(3, 0) \notin A$. Hence we get no further point in this case.

d) It still remains the last possibility

$$x^2 + 2y^2 - 3x = 0 \quad \text{and} \quad 2x^2 + y^2 - 3y = 0.$$

From the rewriting (cf. e.g. Linear Algebra)

$$\left(x - \frac{3}{2}\right)^2 + 2y^2 = \left(\frac{3}{2}\right)^2 \quad \text{and} \quad 2x^2 + \left(y - \frac{3}{2}\right)^2 = \left(\frac{3}{2}\right)^2$$

it is seen that the stationary points are the intersections of the two ellipses. It follows from the *symmetry* that the points must lie on the line $y = x$. By eliminating y we get

$$0 = x^2 + 2y^2 - 3x = 3x^2 - 3x = 3x(x - 1).$$

Hence we get either $x = 0$, corresponding to $(0, 0) \in A^\circ$ [found previously] or $x = 1$ corresponding to $(1, 1) \in A^\circ$, which is a new candidate with the value

$$f(1, 1) = 1 + 4 + 1 - 4 - 4 = -2.$$

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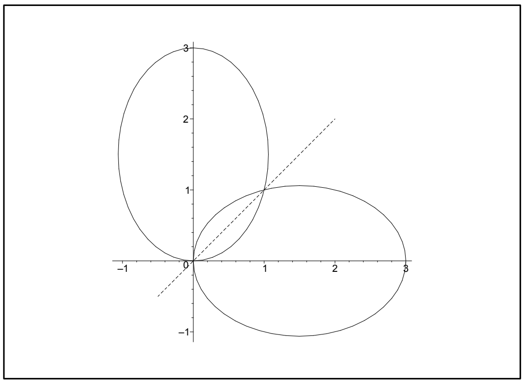

Figure 16.5: The ellipses $x^2 + 2y^2 - 3x = 0$ and $2x^2 + y^2 - 3y = 0$ and the line of symmetry $y = x$.

Summarizing we get the *stationary points* $(0,0)$ and $(1,1)$ with the corresponding values of the function

$$f(0,0) = 0 \qquad \text{and} \qquad f(1,1) = -2.$$

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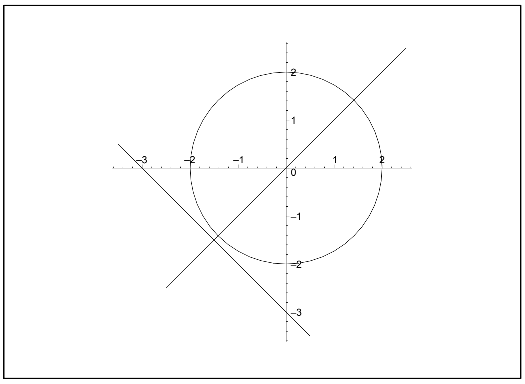

Figure 16.6: The intersections of the circle and the lines $x = 0$, $y = 0$, $y = x$ and $x + y + 3 = 0$.

4) The *boundary.* The simplest version is the following *alternative* to the standard procedure: A parametric representation of the boundary curve is

$$(x, y) = \mathbf{r}(\varphi) = (2\,\cos\varphi, 2\,\sin\varphi), \qquad \varphi \in [0, 2\pi], \quad \text{(possibly } \varphi \in \mathbb{R}),$$

where we note that

$$(16.1) \quad \left(\frac{\mathrm{d}x}{\mathrm{d}\varphi}, \frac{\mathrm{d}y}{\mathrm{d}\varphi}\right) = \mathbf{r}'(\varphi) = (-2\,\sin\varphi, 2\,\cos\varphi) = (-y, x).$$

If we put $g(\varphi) = f(\mathbf{r}(\varphi))$, where

$$f(x, y) = x^4 + 4x^2y^2 + y^4 - 4x^3 - 4y^3,$$

then we get by the *chain rule*, that the maximum and the minimum on the boundary should be searched among the points on the *boundary*

$$x^2 + y^2 = 4,$$

for which (apply (16.1)),

$$\begin{aligned}
0 &= g'(\varphi) = \frac{\partial f}{\partial x} \cdot \frac{\mathrm{d}x}{da\varphi} + \frac{\partial f}{\partial y} \cdot \frac{\mathrm{d}y}{\mathrm{d}\varphi} \\
&= \left\{4x^3+8xy^2-12x^2\right\} \cdot (-y) + \left\{8x^2y+4y^3-12y^2\right\} x \\
&= 4x\left\{x^2 + 2y^2 - 3x\right\}(-y) + 4y\left\{2x^2 + y^2 - 3y\right\} x \\
&= 4xy\left\{-x^2 - 2y^2 + 3x + 2x^2 + y^2 - 3y\right\} \\
&= 4xy\left\{x^2 - y^2 + 3(x - y)\right\} \\
&= 4xy(x - y)\{3 + x + y\}.
\end{aligned}$$

Hence we shall find the intersections between the *circle* $x^2 + y^2 = 4 = 2^2$ and the *lines*

$$x = 0, \qquad y = 0, \qquad y = x \quad \text{and} \quad x + y + 3 = 0.$$

It follows immediately that these intersections are

$$(2, 0), \quad (\sqrt{2}, \sqrt{2}), \quad (0, 2), \quad (-2, 0), \quad (-\sqrt{2}, -\sqrt{2}), \quad (0, -2).$$

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We note the values

$$f(2,0) = f(0,2) = 16 - 32 = -16,$$

$$f(-2,0) = f(0,-2) = 16 + 32 = 48,$$

$$f(\sqrt{2},\sqrt{2}) = 6 \cdot 4 - 2 \cdot 4 \cdot 2\sqrt{2} = 24 - 16\sqrt{2},$$

$$f(-\sqrt{2},-\sqrt{2}) = 24 + 16\sqrt{2}.$$

5) Summarizing we shall compare *numerically*

exceptional points: none,

stationary points: $f(0,0) = 0, \quad f(1,1) = -2,$

boundary points: $f(2,0) = f(0,2) = -16,$

$f(-2,0) = f(0,-2) = 48,$

$f(\sqrt{2},\sqrt{2}) = 24 - 16\sqrt{2},$

$f(-\sqrt{2},-\sqrt{2}) = 24 + 16\sqrt{2}.$

Since $16\sqrt{2} < 16 \cdot \dfrac{3}{2} = 24$, it follows that

the minimum is $\quad M = f(2,0) = f(0,2) = -16,$

the maximum is $\quad S = f(-2,0) = f(0,-2) = 48,$

and that both the minimum and the maximum are lying on the boundary.

6) Finally, we get from 1) that due to the first main theorem for continuous functions the range is the interval

$$f(A) = [M,S] = [-16,48]. \qquad \diamond$$

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Example 16.3

A. *Find maximum and minimum for the function*

$$f(x,y) = \sqrt{x^2+16y^2} - y^4$$

in the set

$$A = \{(x,y) \mid x^2+36y^2 \le 81\}.$$

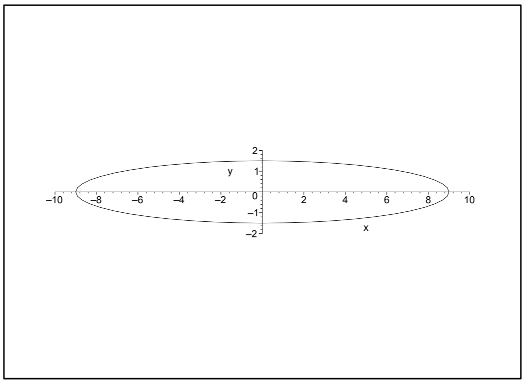

Figure 16.7: The closed and bounded domain A.

D. In this case one might find the level curves $f(x,y) = c$, which by using that

$$a^2 - b^2 = (a+b)(a-b)$$

can be rewritten as

$$x^2 = \left(y^4+c\right)^2 - 16y^2 = \left(y^4+4y+c\right)\left(y^4-4y+c\right).$$

This expression still looks too difficult to analyze, so we shall again stick to the standard procedure as described in the first example.

I. 1) Using some Linear Algebra, the set A is written as

$$\left(\frac{x}{9}\right)^2 + \left(\frac{y}{\frac{3}{2}}\right)^2 \le 1,$$

which shows that at A is a closed ellipsoidal disc, cf. the figure.

Since the set A is *closed and bounded*, and even *connected*, and $f(x,y)$ is *continuous* on A, it follows from the *second main theorem for continuous functions* that f has a minimum M and a maximum S on A. It follows furthermore from the *first main theorem for continuous functions* that the range is connected, i.e. an interval, which necessarily is

$$f(A) = [M,S].$$

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2) Since the square root is *not* differentiable at 0, it follows that $(0,0)$ is an *exceptional point!* We make a note for 5) of the value

$$f(0,0)=0.$$

3) The *stationary points* in $A^{\circ}\setminus\{(0,0)\}$, if any, must satisfy the system of equations

$$\frac{\partial f}{\partial x}=\frac{x}{\sqrt{x^2+16y^2}}=0 \quad \text{and} \quad \frac{\partial f}{\partial y}=\frac{16y}{\sqrt{x^2+16y^2}}-4y^3=0.$$

The first equation is only fulfilled for $x=0$. Thus any stationary point must lie on the y-axis.

Since $(0,0)$ is an exceptional point, we must have $y\neq 0$ for any stationary point. When we put $x=0$ into the second equation, we get (NB: $\sqrt{y^2}=|y|$)

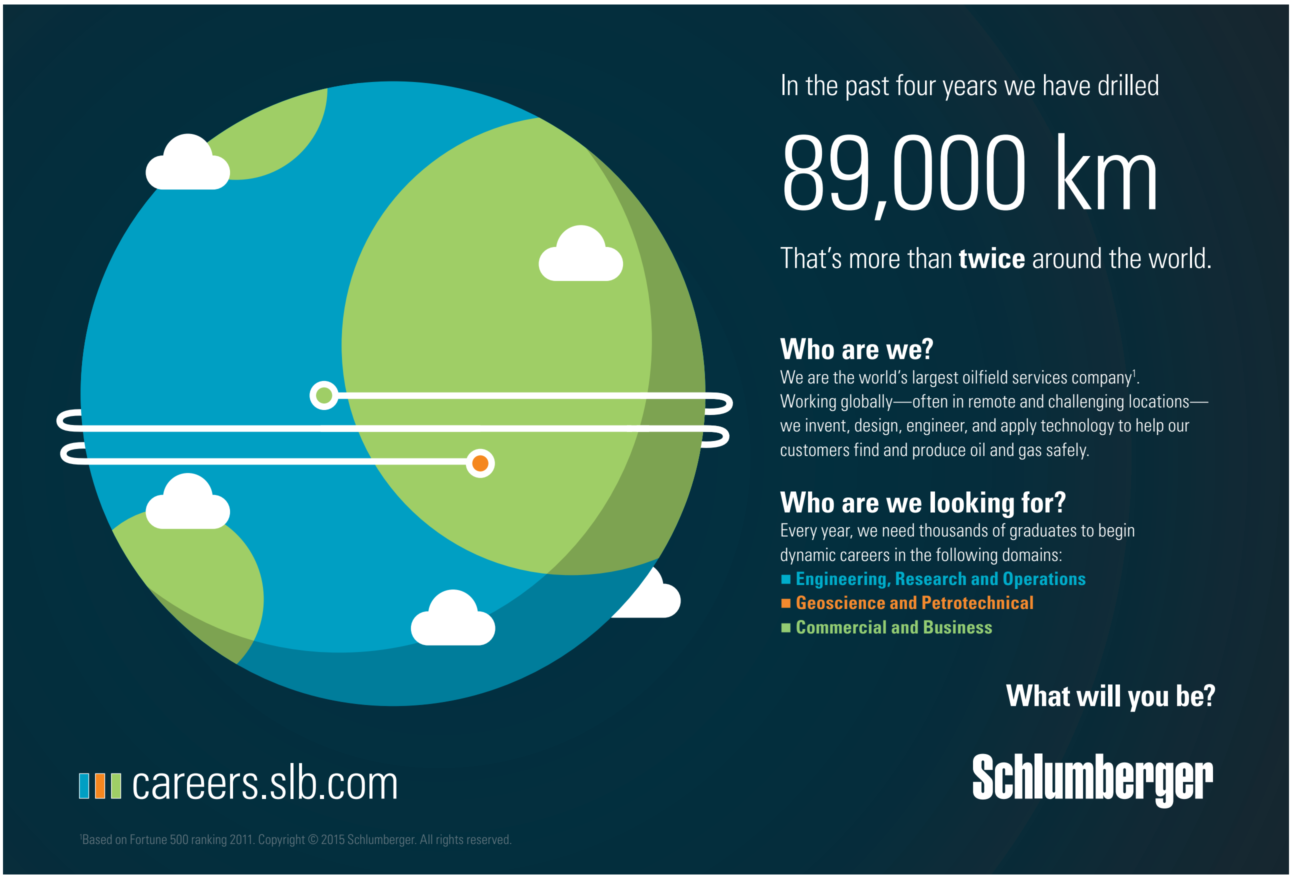
$$0=\frac{16y}{\sqrt{16y^2}}-4y^3=4y\left\{\frac{1}{|y|}-y^2\right\}=4\,\frac{y}{|y|}\,\left\{1-|y|^3\right\}.$$

Since $y\neq 0$, we must have $|y|=1$, i.e. $y=\pm 1$. Hence the *stationary points* are $(0,1)$ and $(0,-1)$. We make a note for 5) of the value

$$f(0,1)=f(0,-1)=\sqrt{16}-1=3.$$

4) The *boundary*. On the boundary we get $x^2+36y^2=81$, i.e.

$$x^2=81-36y^2.$$



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Since $f(x,y)$ *only* contains x in the form x^2, we can use this equation to eliminate x^2 when we write down the restriction,

$$\begin{aligned} f(y) &= \sqrt{x^2+16y^2}-y^4=\sqrt{81-36y^2+16y^2}-y^4 \\ &= \sqrt{81-20y^2}-y^4 \qquad y\in\left[-\frac{3}{2},\frac{3}{2}\right]. \end{aligned}$$

It follows immediately that $g(y)$ is *decreasing* in the new variable $t=y^2\in\left[0,\frac{9}{4}\right]$, hence the maximum on the boundary is

$$g(0)=f(-9,0)=f(9,0)=9,$$

and the minimum on the boundary is

$$g\left(\pm\frac{3}{2}\right)=f\left(0,\frac{3}{2}\right)=f\left(0,-\frac{3}{2}\right)=\sqrt{16\cdot\frac{9}{4}}-\frac{81}{16}=6-\frac{81}{16}=\frac{15}{16}.$$

5) A *numerical comparison* of

$$\begin{array}{ll} \text{exceptional point:} & f(0,0)=0, \\ \text{stationary points:} & f(0,1)=f(0,-1)=3, \\ \text{boundary points:} & f\left(0,\frac{3}{2}\right)=f\left(0,-\frac{3}{2}\right)=\frac{15}{16}, \\ & f(-9,0)=f(9,0)=9, \end{array}$$

gives

$$\begin{array}{lll} \text{maximum:} & f(-9,0)=f(9,0)=9, & \text{(boundary points)}, \\ \text{minimum:} & f(0,0)=0, & \text{(exceptional point)}. \end{array}$$

6) According to 1) the range is given by

$$f(A)=[M,S]=[0,9],$$

where we have used the *first main theorem for continuous functions.* $\diamond$

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Example 16.4

A. *Consider the function*

$$f(x,y) = x + 3y - 2\ln(1+4xy)$$

defined on the triangle A with its vertices $(1,0)$, $(4,0)$ and $(1,1)$. Find the maximum and minimum of $f(x,y)$ on A.

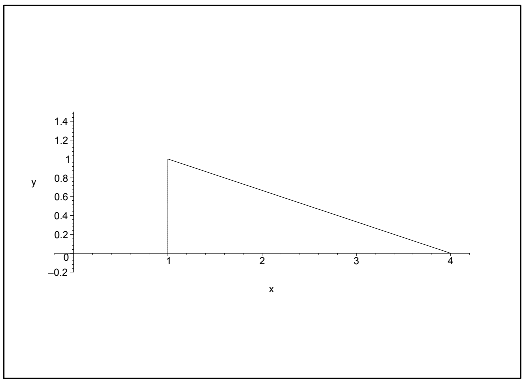

Figure 16.8: The closed and bounded domain A.

D. Here it is far too difficult directly to find the level curves, so we apply the standard procedure as described previously.

I. 1) We first sketch A. Since $f(x,y)$ is *continuous* on the *closed and bounded* triangle A (note in particular that $1+4xy>0$), it follows from the *second main theorem for continuous functions* that $f(x,y)$ has both a maximum S and a minimum M on A. Since A is also connected, it follows from the *first main theorem for continuous functions* that the range is connected, i.e. an interval, and we have necessarily

$$f(A) = [M,S].$$

2) Since f everywhere in A° is of class C^∞, it follows that $f(x,y)$ has *no exceptional point.*

3) The *stationary points*, if any, must satisfy the equations

$$\frac{\partial f}{\partial x} = 1 - \frac{8y}{1+4xy} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 3 - \frac{8x}{1+4xy} = 0,$$

i.e.

$$8y = 1+4xy \qquad \text{and} \qquad 8x = 3(1+4xy).$$

When $1+4xy>0$ is eliminated we get $8x = 3\cdot 8y$, from which $x = 3y$, which is a condition that the stationary points necessarily must satisfy.

By insertion of $x=3y$ we get

$$8y = 1+4xy = 1+12y^2,$$

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which is rewritten as

$$0 = 12y^2 - 8y + 1 = 12\left(y - \frac{1}{6}\right)\left(y - \frac{1}{2}\right).$$

From this we either get $y = \frac{1}{6}$, corresponding to $x = 3 \cdot \frac{1}{6} = \frac{1}{2}$, i.e. $\left(\frac{1}{2}, \frac{1}{6}\right) \notin A$, or $y = \frac{1}{2}$, corresponding to

$$\left(\frac{3}{2}, \frac{1}{2}\right) \in A^\circ.$$

We only find one stationary point $\left(\frac{3}{2}, \frac{1}{2}\right)$. We make a note of the value for 5) below,

$$f\left(\frac{3}{2}, \frac{1}{2}\right) = \frac{3}{2} + \frac{3}{2} - 2\ln\left(1 + 4 \cdot \frac{3}{2} \cdot 12\right) = 3 - 2\ln 4 = 3 - 4\ln 2.$$

4) The *investigation of the boundary* is divided into three cases:

a) On the line $x = 1$, $y \in [0, 1]$, we get the restriction

$$g_1(y) = 1 + 3y - 2\ln(1 + 4y),$$

where

$$g_1'(y) = 3 - \frac{8}{1 + 4y} = 0 \quad \text{for } 1 + 4y = \frac{8}{3}, \text{ i.e. } y = \frac{5}{12} \in [0, 1],$$

corresponding to

$$f\left(1, \frac{5}{12}\right) = g_1\left(\frac{5}{12}\right) = 1 + \frac{5}{4} - 2\ln\left(1 + \frac{5}{3}\right) = \frac{9}{4} - 2\ln\left(\frac{8}{3}\right).$$

NB: We must *not* forget the endpoints of the line:

$$\begin{aligned} f(1,0) &= g_1(0) = 1 + 0 - 2\ln(1 + 4 \cdot 0) = 1, \\ f(1,1) &= g_1(1) = 1 + 3 - 2\ln(1 + 4 \cdot 1) = 4 - 2\ln 5. \end{aligned}$$

b) On the line $y = 0$, $x \in [1, 4]$, we get the restriction

$$g_2(x) = x - 2\ln(1 + 4 \cdot x \cdot 0) = x,$$

which obviously is increasing. Therefore we shall only make a note on the values at the endpoints,

$$f(1,0) = 1 \qquad \text{and} \qquad f(4,0) = 4.$$

c) On the line $x + 3y = 4$, i.e. $x = 4 - 3y$, $y \in [0, 1]$, the restriction is given by

$$g_3(y) = 4 - 2\ln(1 + 4(4 - 3y)y) = 4 - 2\ln(1 + 16y - 12y^2).$$

Here we get

$$g_3'(y) = -\frac{2}{1+16y-12y^2}(16 - 24y) = 0 \qquad \text{for } y = \frac{2}{3} \in [0, 1],$$

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corresponding to $x = 4 - 3 \cdot \dfrac{2}{3} = 2$. The interesting point is $\left(2, \dfrac{2}{3}\right) \in \partial A$ with the value

$$
\begin{aligned}
f\left(2, \frac{2}{3}\right) &= g_3\left(\frac{2}{3}\right) = 4 - 2\ln\left(1 + 16 \cdot \frac{2}{3} - 12 \cdot \frac{4}{9}\right) \\
&= 4 - 2\ln\left(1 + \frac{32}{3} - \frac{16}{3}\right) = 4 - 2\ln\frac{19}{3}.
\end{aligned}
$$

We have already earlier treated the two endpoints.

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5) Finally we shall compare *numerically*

exceptional points: none,

stationary point: $f\left(\frac{3}{2},\frac{1}{2}\right) = 3 - 4\ln 2 \approx 0.23,$

boundary a): $f\left(1,\frac{5}{12}\right) = \frac{9}{4} - 2\ln\left(\frac{8}{3}\right) \approx 0.29,$

$f(1,1) = 4 - 2\ln 5 \approx 0.79,$

$f(1,0) = 1,$

boundary b): $f(4,0) = 4,$

boundary c): $f\left(2,\frac{2}{3}\right) = 4 - 2\ln\frac{19}{3} \approx 0.31.$

By a comparison we see that

the maximum is $S = f(0,4) = 4,$ (boundary point),

the minimum is $M = f\left(\frac{3}{2},\frac{1}{2}\right) = 3 - 4\ln 2,$ (stationary point).

Remark 16.3 Note that the comparison is made approximatively, while the result is given in an exact form. $\Diamond$

6) According to 1) we finally get by the *first main theorem for continuous functions* that the range is

$$f(A) = [M,S] = [3 - 4\ln 2, 4]. \qquad \Diamond$$

16.2.3 The case of a bounded but not closed domain of f

Consider a continuous function $f : A \to \mathbb{R}$, where the domain $A \subset \mathbb{R}^m$ is bounded, but not closed. Then we cannot apply the second main theorem and we cannot conclude that f has extrema on A. Indeed, it is easy to give examples, when this is not the case.

When we analyze this case, we first check if f has a *continuous extension* $\overline{f} : \overline{A} \to \mathbb{R}$ to the closure $\overline{A}$ of A, i.e. $\overline{f}$ is continuous, and $\overline{f}(\mathbf{u}) = f(\mathbf{u})$ for all $\mathbf{u} \in A$. In this case we just apply the methods described previously in Section 16.2.2, because due to the second main theorem, $\overline{f}$ has both a minimum and a maximum on $\overline{A}$, so we find these and then check, if they are also attained at points in A.

We elaborate a little on this. Assume that the continuous extension $\overline{f} : \overline{A} \to \mathbb{R}$ exists, and let the (global) extrema be

$$\overline{f}(\mathbf{u}) = \overline{f}_{\min}, \qquad \text{and} \qquad \overline{f}(\mathbf{v}) = \overline{f}_{\max}.$$

Clearly, if $\mathbf{u} \in A$, then $\overline{f}_{\min}$ is also a global minimum for f, because A is a smaller set than $\overline{A}$.

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Similarly, if $\mathbf{v} \in A$, then $\overline{f}_{\max}$ is also a global maximum for f.

Then we turn to the case, when $\mathbf{u} \in \overline{A} \setminus A$. Just note that due to the continuity of $\overline{f}$ we can to every $\varepsilon > 0$ find a point $\mathbf{u}_\varepsilon \in A$, such that

$$\overline{f}(\mathbf{u}) = \overline{f}_{\min} < f\left(\mathbf{u}_\varepsilon\right) < \overline{f}_{\min} + \varepsilon,$$

which shows that we inside A can get as close to the value $\overline{f}_{\min}$ as we want without ever reaching this value.

Similarly, if $\mathbf{v} \in \overline{A} \setminus A$, where we to every $\varepsilon > 0$ can find $\mathbf{v}_\varepsilon \in A$, such that

$$\overline{f}_{\max} - \varepsilon < f\left(\mathbf{v}_\varepsilon\right) < \overline{f}_{\max} = f(\mathbf{v}),$$

which shows that we inside A can get as close to the value $\overline{f}_{\max}$ as we want without ever reaching this value.

Example 16.5

Let A be the open *triangle*

$$A = \{(x, y) \mid 0 < x < 1, \, -x < y < 4x\},$$

and let the function $f(x, y)$ on A be given by

$$f(x, y) = 2xy + 3\ln(1 - x), \qquad (x, y) \in A.$$

Find the range $f(A)$.

A.

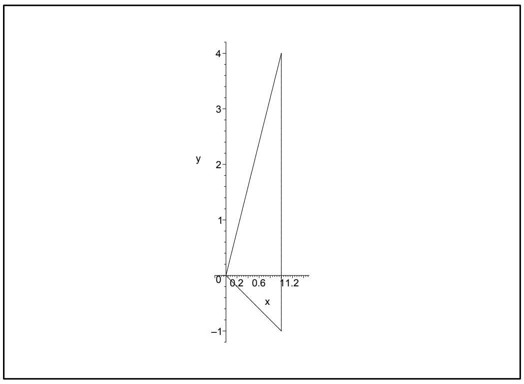

Figure 16.9: The open and bounded domain A.

D. Here it is possible to find the *level curves*. In fact, since $x > 0$ in A, we get that

$$f(x, y) = 2xy + 3\ln(1 - x) = c, \qquad (x, y) \in A,$$

is equivalent to

$$y = \varphi_c(x) = \frac{c}{2x} - \frac{3}{2} \cdot \frac{\ln(1 - x)}{x}.$$

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Although the expression looks very complicated, it is actually possible to analyze these level curves. The reader is referred to section **I 2** below which, however, may be considered a bit advanced for a common use.

We therefore start with the *standard procedure* in section **I 1** with some necessary modifications. First we exploit the theoretical main theorems as much as possible. Then we extend f to the parts of the boundary where it is possible, and we discuss what happens at the boundary points where such a continuous extension of f is not possible.

We see that both methods have a common theoretical start, which we here call section **I**.

I. Since $f(x,y)$ is *continuous* on the *connected* set A, it follows from the *first main theorem for continuous functions* that the range $f(A)$ is *connected*, i.e. an interval.

Since A is bounded, though *not closed*, we cannot apply the *second main theorem for continuous functions*. We shall first check whether $f(x,y)$ has a continuous extension to (parts of) the boundary of A or not.

It follows immediately that $f(x,y)$ can be continuously extended to the lines

$$y = 4x \quad \text{and} \quad y = -x, \qquad x \in [0,1[,$$

with the same formal expression of the function, i.e. the extension is given by

$$\overline{f}(x,y) = 2xy + 3\ln(1-x) \qquad \text{for } 0 \le x < 1, \quad -x \le y \le 4x.$$

On the other hand, we cannot extend to the vertical line $x = 1$, because

$$\lim_{x \to 1-} f(x,y) = 2y + 3 \lim_{x \to 1-} \ln(1-x) = -\infty.$$

However, we see that the *lower bound* is $-\infty$, so $f(A)$ must be a semi-infinite, i.e. either $]-\infty, a[$ or $]-\infty, a]$, because the theorems do not assure that the upper bound a actually belongs to $f(A)$. This question can only be decided by an explicit analysis.

It follows that we shall only search the *maximum* in

$$B = \{(x,y) \mid 0 \le x < 1, \, -x \le y \le 4x\}.$$

Since we also have $f(x,y) \to -\infty$ for $x \to 1-$, in B, there exists an $\varepsilon \in]0,1[$, such that

$$\overline{f}(x,y) < S \qquad \text{for } (x,y) \in \overline{B} \text{ and } 1 - \varepsilon \le x < 1.$$

The maximum S is therefore attained in the *closed and bounded* and truncated domain

$$B_\varepsilon = \{(x,y) \mid 0 \le x \le 1-\varepsilon, \, -x \le y \le 4x\},$$

where we of course assume that S exists and $S < +\infty$.

This follows, however, from the *second main theorem for continuous functions*, applied on B_ε.

Since we only want to find the maximum, the standard procedure is hereafter the same as for closed and bounded domains. The only modification is that we shall not go through an investigation of the boundary on the line $x = 1 - \varepsilon$.

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I 1. *Standard procedure.*

1) We have already sketched a figure and quoted and applied the second main theorem.
2) Since $f(x, y)$ belongs to the class C^{∞} in A, there is *no exceptional point.*
3) The *stationary points* in A, if any, must satisfy the equations

$$\frac{\partial f}{\partial x} = 2y - \frac{3}{1-x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 2x = 0.$$

It follows from the latter equation that $x = 0$; but since $x > 0$ in A, we see that we have *no stationary point* in A for the function f.

4) *Modified investigation of the boundary.*

a) For $y = 4x$ we get the restriction

$$g_1(x) = 8x^2 + 3\ln(1-x), \qquad \text{for } x \in [0, 1[,$$

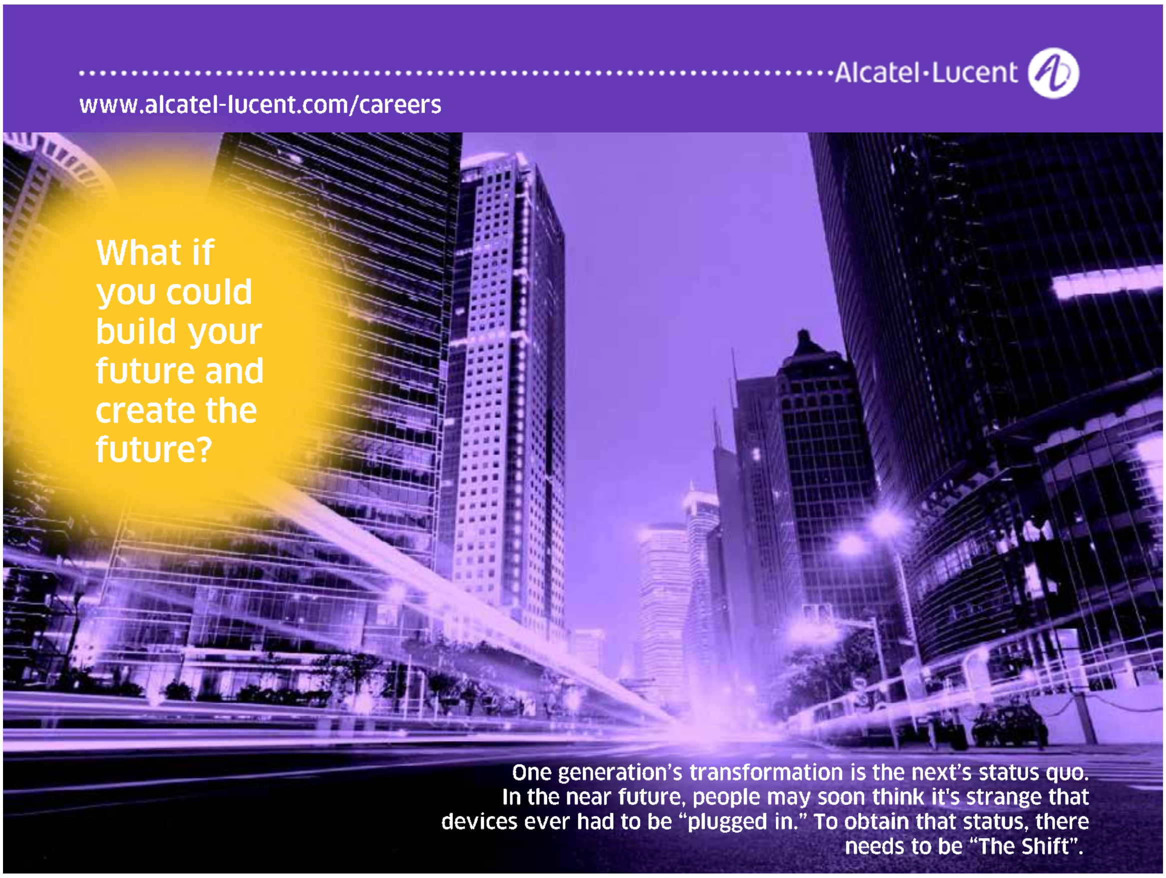
where

$$g_1'(x) = 16x - \frac{3}{1-x}.$$

Hence, $g_1'(x) = 0$ for

$$0 = 16x^2 - 16x + 3 = (4x-3)(4x-1),$$

i.e. for $x = \dfrac{1}{4}$ or $x = \dfrac{3}{4}$.



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When we apply high school calculus it is seen that the maximum is either attained for $x = 0$, corresponding to $g_1(0) = \overline{f}(0,0) =$, or for $x = \dfrac{3}{4}$, corresponding to

$$\begin{aligned} g_1\left(\frac{3}{4}\right) &= \overline{f}\left(\frac{3}{4},3\right) = \frac{8\cdot 9}{16} + 3\ln\left(1-\frac{3}{4}\right) = \frac{9}{2} - 6\ln 2 \\ &\geq 4,5 - 6\cdot 0,7 = 0,3 > 0. \end{aligned}$$

b) For $y = -x$ we get the restriction

$$g_2(x) = -2x^2 + 3\ln(1-x), \qquad \text{for } x \in [0,1[,$$

where

$$g_2'(x) = -4x - \frac{3}{1-x} < 0.$$

Hence, $g_2(x)$ is *decreasing*. The maximum on this line is therefore $g_2(0,0) = \overline{f}(0,0) = 0$.

c) *Numerical comparison.* When we compare the values of the candidates above it follows that the maximum in B is

$$\overline{f}\left(\frac{3}{4},3\right) = \frac{9}{2} - 6\,\ln 2 > 0.$$

This value is *only* attained at the boundary point $\left(\dfrac{3}{4},3\right)$, so

- $\overline{f}(B) = \left]-\infty, \dfrac{9}{2} - 6\,\ln 2\right]$,

and

- $f(A) = \left]-\infty, \dfrac{9}{2} - 6\,\ln 2\right[$,

because A is obtained by removing all boundary points from B.

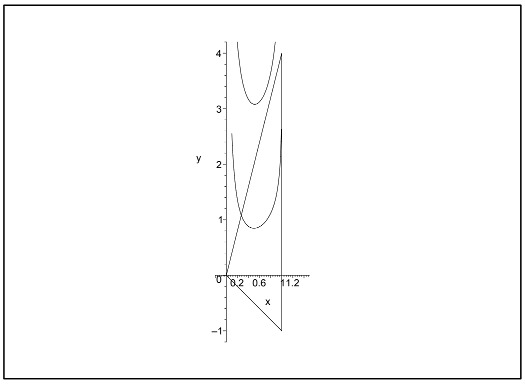

Figure 16.10: The level curves for $c = \dfrac{1}{2}$ (below) and $c = 1$ (above).

I 2. *The method of level curves.* The level curve

$$y = \varphi_c(x) = \frac{c}{2x} - \frac{3}{2}\cdot\frac{\ln(1-x)}{x}$$

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is defined in the strip $0 < x < 1$ as the graph of a function. If $c \neq 0$, then both $x = 0$ and $x = 1$ are asymptotes. It follows that

$$\lim_{x \to 1-} \varphi_c(x) = +\infty \qquad \text{for all } c \in \mathbb{R},$$

and that

$$\lim_{x \to 0+} \varphi_c(x) = +\infty \qquad \text{for } c > 0,$$

and

$$\lim_{x \to 0+} \varphi_c(x) = -\infty \qquad \text{for } c < 0.$$

The curves are characterized by $f(x, y)$ being constant c along $y = \varphi_c(x)$. We have sketched two level curves on the figure (where $c > 0$), from which it is seen that the curves "move upwards", when c increases.

Hence we are looking for the biggest c, for which $y = \varphi_c(x)$ just is contacting the boundary of B without intersecting B. This is not possible for $c < 0$, and at the same time we get the line $y = -x$ excluded. Thus the maximum can *only* lie on the line $y = 4x$. Since $y = \varphi_c(x)$ only *touches* this line, the following two conditions must be fulfilled:

1) The curves must go through the *same point*, i.e. $y = 4x = \varphi_c(x)$, or

$$4x = -\frac{1}{2x}\{-c + 3\ln(1 - x)\},$$

from which

$$-c + 3\ln(1 - x) = -8x^2.$$

2) The curves must have the *same slope* at this point, i.e.

$$4 = \varphi_c'(x) = \frac{1}{2x^2}\left\{-c + 3\ln(1 - x) + \frac{3x}{1 - x}\right\}.$$

The ugly terms $-c + 3\ln(1 - x)$ in 2) can be eliminated by applying 1), hence

$$8x^2 = -8x^2 + \frac{3x}{1 - x},$$

which is rewritten as

$$0 = 16x^2(1 - x) - 3x = x\{16x - 16x^2 - 3\} = -x(4x - 1)(4x - 3).$$

From this we get the solutions $x = 0$, $x = \frac{1}{4}$ and $x = \frac{3}{4}$, and since $y = 4x$, we finally get the *candidates*

$$(0, 0), \qquad \left(\frac{1}{4}, 1\right), \qquad \left(\frac{3}{4}, 3\right),$$

with the corresponding function values for the extended function,

$$\overline{f}(0, 0) = 0, \quad \overline{f}\left(\frac{1}{4}, 1\right) = \frac{1}{2} - 3\ln\frac{4}{3}, \quad \overline{f}\left(\frac{3}{4}, 3\right) = \frac{9}{2} - 6\ln 2.$$

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By a numerical comparison we get that the maximum is attained at the point $\left(\frac{3}{4}, 3\right)$. Hence we conclude that

$$\overline{f}(B) = -\left]-\infty, \frac{9}{2} - 6\,\ln 2\right].$$

Finally, when we remove the boundary points from B, we obtains as previously that

$$f(A) = \left]-\infty, \frac{9}{2} - 6\,\ln 2\right[. \qquad \Diamond$$

If $f : A \to \mathbb{R}$ does not have a continuous extension to the closure $\overline{A}$, then either f is bounded on A, or unbounded on A. We first give an example, when f is bounded on A without having a continuous extension to all of $\overline{A}$.

Example 16.6 A nasty example which usually is not given in any textbook, is given by the following. It also illustrates that the usual division into cases in most textbooks is not exhaustive.

Let $A = K(\mathbf{0}; 1)$ be the open unit disc, and consider the function

$$f(x,y) = \left(x^2+y^2\right)\,\cos\left(\frac{1}{1-x^2-y^2}\right), \qquad x^2+y^2<1.$$

Then $f(x,y)$ is bounded *on A,*

$$|f(x,y)| \le x^2+y^2 < 1 \qquad \text{for } (x,y) \in A,$$

and we see that $f(x,y)$ has no continuous extension to any point on the boundary.

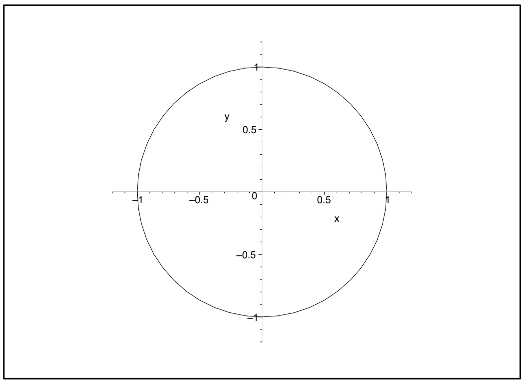

Figure 16.11: The set A is the open unit disc.

Then note that

1) $f(x,y) = 1 - \dfrac{1}{2p\pi}$ for $x^2+y^2 = 1 - \dfrac{1}{2p\pi}$, $p \in \mathbb{N}$,

2) $f(x,y) = -1 + \dfrac{1}{(2p+1)\pi}$ for $x^2+y^2 = 1 - \dfrac{1}{(2p+1)\pi}$, $\quad p \in \mathbb{N}$,

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from which we conclude that f has neither a maximum nor a minimum in the open set A.

However, since $f(x, y)$ is *continuous* on the *connected* set A, it follows from the *first main theorem for continuous functions* that $f(A)$ also is connected, i.e. an interval.

According to 1) the function $f(x, y)$ attains values smaller than 1, though we can get as close to 1 as we wish.

According to 2) the function $f(x, y)$ attains values bigger than -1, though we can get as close to -1 as we wish.

Hence we conclude that the range is given by

$$f(A) =]-1, 1[. \qquad \Diamond$$

Then turn to the case, when f is unbounded on A. Then either we can find a sequence $\mathbf{u}_n \in A$, such that $f(\mathbf{u}_n) \to -\infty$ for $n \to +\infty$, or a sequence $\mathbf{v}_n \in A$, such that $f(\mathbf{v}_n) \to +\infty$ for $n \to +\infty$. In order to show, what we have in mind, we include the following example.

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Example 16.7 Let A denote the *open* triangle on Figure 16.12, i.e. A is described by the inequalities $0 < x < 1$ and $-x < y < 4x$.

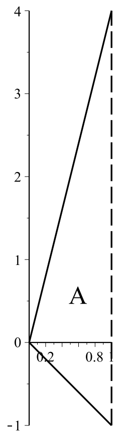

Figure 16.12: The domain A is the open triangle. The function can be continuously extended to the solid line segments.

Consider the function on A defined by

$$f(x,y) = 2xy + 3\ln(1-x) \qquad \text{for } (x,y) \in A.$$

First note that since A is connected, and f is continuous on A, it follows from the first main theorem for continuous functions that the range of f is connected, hence an interval.

Due to the logarithmic term, $f(x,y) \to -\infty$, when $(x,y) \in A$ and $x \to 1-$. This shows that f does not have a global minimum, and that the range has $-\infty$ as its left endpoint.

Then f has no *exceptional points* in A, because f is differentiable everywhere in the open set A.

If there are *stationary points*, these must satisfy the equations

$$\frac{\partial f}{\partial x} = 2y + \frac{-3}{1-x} = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial y} = 2x = 0.$$

It follows immediately that $x = 0$ and then by insertion that $y = \dfrac{3}{2}$, so the only candidate of a stationary point is $\left(0, \dfrac{3}{2}\right)$. Since this point lies outside A, there are no stationary points either.

Summarizing, there are no exceptional points, no stationary points, and no boundary points in A, so f cannot have any extremum.

Then a simple estimate shows that

$$f(x,y) = 2xy + 3\ln(1-x) \le 2 \cdot 1 \cdot 4 + 3 \cdot 0 = 11,$$

so f is bounded from above. Therefore, $f(A) =]-\infty, c[$ for some finite c. This value c can only be attained for the continuous extension $\overline{f}$ to the parts of the boundary (indicated by solid lines), which is not lying on the vertical line $x = 1$.

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We note for later use that at the vertex $f(0,0) = 0$.

Fix $0 < x < 1$ and let $y \in [-x, 4x]$ vary, i.e. we consider the restriction

$$g_x(y) = 2xy + 3\ln(1-x) \le 8x^2 + 3\ln(1-x) \qquad \text{for } x \in [-x, 4x],$$

where the equality sign is obtained for $y = 4x$. This means that the value c is attained at the half open line segment $y = 4x$, $0 \le x < 1$. Therefore, we introduce

$$\varphi(x) := 8x^2 + 3\ln(1-x), \qquad \text{for } 0 \le x < 1.$$

We get by a differentiation,

$$\varphi'(x) = 16x - \frac{3}{1-x} = \frac{-16x^2 + 16x - 3}{1-x} = \frac{(4x-3)(1-4x)}{1-x},$$

which is 0 for $x = \frac{1}{4}$ or $x = \frac{3}{4}$. The values of the extended function are therefore

$$\varphi\left(\frac{1}{4}\right) = \overline{f}\left(\frac{1}{4}, 1\right) = 8 \cdot \left(\frac{1}{4}\right)^2 + 3\ln\left(1 - \frac{1}{4}\right) = \frac{1}{2} - 3\ln\frac{4}{3} \approx -0.36,$$

and

$$\varphi\left(\frac{3}{4}\right) = \overline{f}\left(\frac{3}{4}, 3\right) = 8 \cdot \left(\frac{3}{4}\right)^2 + 3\ln\left(1 - \frac{3}{4}\right) = \frac{9}{2} - 3\ln 4 \approx 0.34.$$

Finally, we must not forget the endpoint $x = 0$, where as mentioned above,

$$\overline{f}(0) = 0.$$

By a simple numerical comparison we see that

$$c = \overline{f}\left(\frac{3}{4}, 3\right) = \frac{9}{2} - 3\ln 4 = \frac{9}{2} - 6\ln 2,$$

so the range of the function f is

$$f(A) = \left]-\infty, \frac{9}{2} - 6\ln 2\right[. \qquad \Diamond$$

16.2.4 The case of an unbounded domain of f

To avoid unnecessary irrelevant complications we shall for simplicity in this section only consider continuous functions $f : \mathbb{R}^m \to \mathbb{R}$, defined on all of $\mathbb{R}^m$. The topic is difficult to describe in all details, so we shall only give a couple of guidelines concerning the question of existence of extrema in this case, where these guidelines *may* be useful in the applications.

1. If we can define a restriction of the function (to e.g. a simple curve, or just a straight line), such that the values $f(\mathbf{u})$ of this restriction tend to $+\infty$, when $\|\mathbf{u}\| \to +\infty$, then f has no global maximum. Similarly, if $f(\mathbf{u})$ of this restriction tend to $-\infty$, when $\|\mathbf{u}\| \to +\infty$, then f has no global minimum.

We shall illustrate this by the following example.

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Example 16.8 Consider the function

$$f(x,y) = x^3y^2 - x^2y^3 + y^4 - \sin(xy), \qquad \text{for } (x,y) \in \mathbb{R}^2.$$

The function is continuous in the connected set $\mathbb{R}^2$, so by the first main theorem the range is connected, hence an interval.

When we take the restriction to the line $y = -x$, we get

$$f(x,-x) = 2x^5 + x^4 + \sin\left(x^2\right) \to \begin{cases} +\infty & \text{for } x \to +\infty, \\ \\ -\infty & \text{for } x \to -\infty. \end{cases}$$

We conclude that the range is $f\left(\mathbb{R}^2\right) = \mathbb{R}$. $\Diamond$

Let B denote the set of all stationary points and all exceptional points joined together, and let the restriction of f to the set B have the global maximum $f_{\max}^B$ and the global minimum $f_{\min}^B$.

Usually (in the cases met in practice) B only consists of a finite number of points, so it is easy by a numerical comparison to find the numbers $f_{\max}^B$ and $f_{\min}^B$.

Using the assumption that the domain is $\mathbb{R}^m$ it follows that the boundary is the empty set. Then it is easy to conclude the following

2a If there is a point $\mathbf{u} \in \mathbb{R}^m$, such that $f(\mathbf{u}) > f_{\max}^B$, then f has no global maximum.

2b If there is a point $\mathbf{u} \in \mathbb{R}^m$, such that $f(\mathbf{u}) < f_{\min}^B$, then f has no global minimum.

Finally, we assume that the limit

$$\lim_{\|\mathbf{u}\| \to +\infty} f(\mathbf{u}) = L$$

exists, where we also allow the infinite values, $L = \pm\infty$. Then we get

3a If $L < f_{\min}^B$, then f has no global minimum. Its global maximum is $f_{\max}^B$, and its range is $\left]L, f_{\max}^B\right]$.

3b If $f_{\min}^B \le L \le f_{\max}^B$, then $f_{\min}^B$ is the global minimum, and $f_{\max}^B$ is the global maximum, and the range i $\left[f_{\min}^B, f_{\max}^B\right]$.

3c If $f_{\max}^B < L$, then $f_{\min}^B$ is a global minimum. There is no global maximum, and the range is $\left[f_{\min}^B, L\right[$.

Example 16.9 Consider the function

$$f(x,y) = x^4 + 4x^2y^2 + y^4 - 4x^3 - 4y^3 \qquad \text{for } (x,y) \in \mathbb{R}^2.$$

Since f is a polynomial in two variables, there are no exceptional points.

The possible stationary points are solutions of the system of equations

$$\frac{\partial f}{\partial x} = 4x^3 + 8xy^2 - 12x^2 = 4x\left(x^2 + 2y^2 - 3x\right) = 0,$$

$$\frac{\partial f}{\partial y} = 8x^2y + 4y^3 - 12y^2 = 4y\left(2x^2 + y^2 - 3y\right) = 0.$$

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The possible stationary points are the solutions of either

$$x = 0 \quad \text{and} \quad y = 0, \qquad \text{i.e. } (0,0),$$

or

$$x = 0 \quad \text{and} \quad 2x^2 + y^2 - 3y = 0, \qquad \text{i.e. } (0,0) \text{ or } (0,3),$$

or

$$x^2 + 2y^2 - 3x = 0 \quad \text{and} \quad y = 0, \qquad \text{i.e. } (0,0) \text{ or } (3,0),$$

or

$$x^2 + 2y^2 - 3x = 0 \quad \text{and} \quad 2x^2 + y^2 - 3y = 0.$$

In the latter case we get by subtraktion,

$$y^2 - x^2 + 3y - 3x = (y-x)(y+x+3) = 0,$$

so either $y = x$ or $y = -x-3$.

In the former case we get by insertion $3x^2 - 3x = 0$, so either $x = 1$ or $x = 0$, and the stationary points are $(0,0)$ and $(1,1)$.

In case of $y = -x-3$ we get by insertion,

$$0 = x^2 + 2y^2 - 3x = x^2 + 2(x+3)^2 - 3x = 3x^2 + 9x + 18 = 3\left(x^2 + 3x + 6\right),$$

which has only complex roots.

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Summing up we have found the stationary points $B = \{(0,0),(1,1),(3,0),(0,3)\}$, and the values in these points are

$$f(0,0,) = 0, \qquad f(1,1) = -2, \qquad f(3,0) = f(0,3) = -27.$$

Hence, in the chosen notation,

$$f_{\min}^{B} = -27 \qquad \text{and} \qquad f_{\max}^{B} = 0.$$

Considering the limit $\|\mathbf{u}\| \to +\infty$, we see that

$$f(x,y) = x^4 + 4x^2y^2 + y^4 - 4x^3 - 4y^3 = \left(x^2+y^2\right)^2 + 2x^2y^2 - 4\left(x^3+y^3\right),$$

so we get in polar coordinates $x = \varrho\cos\varphi$ and $y = \varrho\sin\varphi$, (note that polar coordinates are natural to use, when we shall take the limit $\|\mathbf{u}\| \to +\infty$)

$$\begin{aligned} f(\varrho\cos\varphi, \varrho\sin\varphi) &= \varrho^4\left(1+\frac{1}{2}\sin^2 2\varphi\right) - 4\varrho^3\left(\cos^3\varphi + \sin^3\varphi\right) \\ &\geq \varrho^4 - 8\varrho^3 = \varrho^4\left(1-\frac{8}{\varrho}\right) \to +\infty \qquad \text{for } \varrho \to +\infty, \end{aligned}$$

and we have proved that L exists and that $L = +\infty$.

Then we conclude from the above that the global minimum is $f_{\min} = f_{\min}^{B} - 27$, and that the global maximum does not exist, and finally that the range is $f\left(\mathbb{R}^2\right) = [-27,+\infty[$. $\Diamond$

16.3 Local extrema of a continuous function

16.3.1 Local extrema in general

We investigated in the previous section the global range of a given continuous function $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^m$. We shall in this section consider the *local* properties of f, i.e. if f has a *local extremum* at a given point $\mathbf{x}_0 \in A$.

We introduce the *increment* of a function. This is defined as

$$\Delta f\left(= \Delta f\left(\mathbf{x};\mathbf{x}_0\right)\right) = f(\mathbf{x}) - f\left(\mathbf{x}_0\right),$$

where $\mathbf{x}_0 \in A$ is the point under consideration, and $\mathbf{x} \in A$ is the variation from $\mathbf{x}_0$.

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Definition 16.2 *Given a continuous function $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^m$, and a fixed point $\mathbf{x}_0 \in A$.*

1) Maximum. *If there is a $\delta > 0$, such that for all $\mathbf{x} \in A$ satisfying $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$,*

$$f(\mathbf{x}) \le f(\mathbf{x}_0), \qquad \textit{i.e.}\ \Delta f \le 0,$$

we call $f(\mathbf{x}_0)$ a maximum *at $\mathbf{x}_0 \in A$.*

2) Proper maximum. *If there is a $\delta > 0$, such that for all $\mathbf{x} \in A$ satisfying $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$,*

$$f(\mathbf{x}) < f(\mathbf{x}_0), \qquad \textit{i.e.}\ \Delta f < 0,$$

we call $f(\mathbf{x}_0)$ a proper maximum *at $\mathbf{x}_0 \in A$.*

3) Minimum. *If there is a $\delta > 0$, such that for all $\mathbf{x} \in A$ satisfying $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$,*

$$f(\mathbf{x}) \ge f(\mathbf{x}_0), \qquad \textit{i.e.}\ \Delta f \ge 0,$$

we call $f(\mathbf{x}_0)$ a minimum *at $\mathbf{x}_0 \in A$.*

4) Proper minimum. *If there is a $\delta > 0$, such that for all $\mathbf{x} \in A$ satisfying $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$,*

$$f(\mathbf{x}) > f(\mathbf{x}_0), \qquad \textit{i.e.}\ \Delta f > 0,$$

we call $f(\mathbf{x}_0)$ a proper minimum *at $\mathbf{x}_0 \in A$.*

When one considers a graph of a function, Definition 16.2 surely makes sense. It also follows immediately from this definition that we have the following theorem.

Theorem 16.4 *Let $\mathbf{x}_0 \in A$ be a point in the domain of the continuous function $f : A \to \mathbb{R}$. If for every $\delta > 0$ the increment Δf has both positive and negative values in any set of the form $A \cap B(\mathbf{x}_0, \delta)$, where $B(\mathbf{x}_0, \delta)$ denotes the open ball of centre $\mathbf{x}_0 \in A$ and radius $\delta > 0$, then f does not have a (local) extremum at $\mathbf{x}_0$.*

If we restrict the given function $f : A \to \mathbb{R}$ to the set $A \cap B(\mathbf{x}_0, \delta)$ for $\delta > 0$ sufficiently small, it follows from the results in Section 16.2 that we also have

Theorem 16.5 A necessary condition for local extremum. *If the continuous function $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^m$, has a local extremum at the point $\mathbf{x}_0 \in A^\circ$, then $\mathbf{x}_0$ is either a stationary point or an exceptional point.*

Note that Theorem 16.5 only gives a necessary, and not a sufficient condition. There may exist e.g. stationary points (saddle points) which are not extremum points. We shall later return to this problem.

Example 16.10 In this example we produce some functions in $\mathbb{R}^2$, which all have $(0, 0)$ as a stationary point and value zero.

We supply the investigation with sketches of the graphs and discussions of the sign of the function in the neighbourhood whenever this is necessary. Concerning the graphs the reader is also referred to Linear Algebra.

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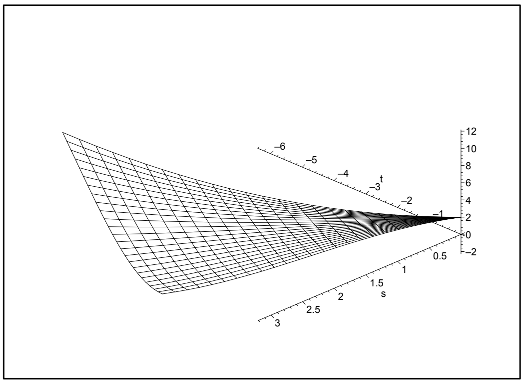

Figure 16.13: The graph of $z = x^2 + y^2$.

1) $z = f_1(x, y) = x^2 + y^2$.

   The graph of f_1 is a *paraboloid of revolution*.

   Since

   $$f_1(x, y) > 0 = f(0, 0) \qquad \text{for } (x, y) \neq (0, 0),$$

   the function f_1 has a *proper minimum* at $(0, 0)$. It is easily seen that this is also the *global minimum* of the function.

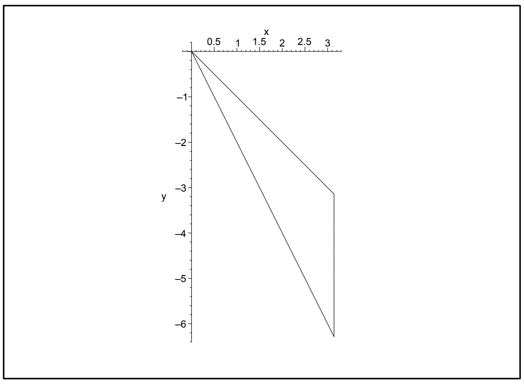

Figure 16.14: The graph of $z = (x - 2y)^2$.

2) $z = f_2(x, y) = x^2 - 4xy + 4y^2 = (x - 2y)^2$.

   The graph of f_2 is a *parabolic cylinder*. It follows immediately that

   $$f_2(x, y) \geq 0 = f(0, 0);$$

   but since

   $$f\left(x, \frac{x}{2}\right) = 0 = f(0, 0) \qquad \text{for all } x,$$

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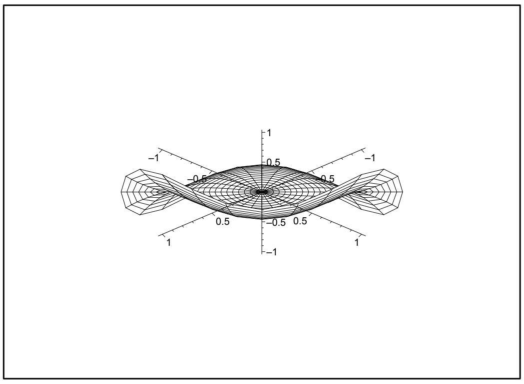

Figure 16.15: The graph of $z = x^3 + y^3$ sketched in MAPLE does not give the best picture.

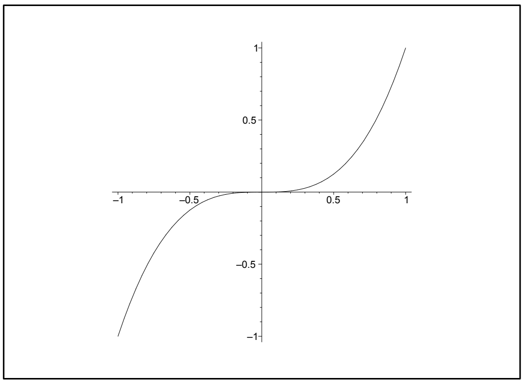

Figure 16.16: The restriction to the x-axis gives a better picture.

we see that $(0,0)$ is a *weak local minimum.* However, we also have in this case that 0 is a *global minimum.*

3) $z = f_3(x,y) = x^3 + y^3 = (x+y)(x^2 - xy + y^2)$.
Since $x^3 + y^3$ is of *odd degree* 3, we take e.g. the restriction of f_3 to the x-axis,

$$f_3(x,0) = x^3 \qquad \text{(is both } > \text{ and } < 0 \text{ in any open neighbourhood of } x = 0\text{)},$$

so f_3 has no extremum at $(0,0)$.

This can also be seen by analyzing the sign of the function. In fact, $x^2 - xy + y^2 \geq 0$ for all (x,y), thus $x^3 + y^3$ is everywhere of the same sign as $x + y$.

4) $z = f_4(x,y) = x^2 - y^2 = (x+y)(x-y)$.
The graph is a *hyperbolic paraboloid.* There is *no* extremum at $(0,0)$.

An *analysis of the sign* shows that $f_4(x,y)$ is 0 on the lines $x + y = 0$ and $x - y = 0$, and that $f_4(x,y)$ attains both positive and negative values in any neighbourhood of $(0,0)$.

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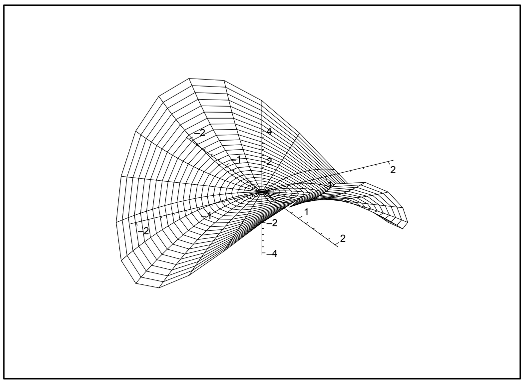

Figure 16.17: The graph of $z = x^2 - y^2$.

It is finally also possible to consider the restrictions

$$\begin{array}{lll} x-\text{axis:} & f_4(x,0) = x^2 > 0 & \text{for } x \neq 0, \\ y-\text{axis:} & f_5(0,y) = -y^2 < 0 & \text{for } y \neq 0, \end{array}$$

from which we obtain the same conclusion. $\Diamond$

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Example 16.11

A. *Examine whether the function*

$$f(x,y,z) = \exp\left(xy+z^2\right)$$

has a local extremum or not.

D. Here we shall not use the standard procedure but instead we suggest an *alternative* method. In fact, since exp is *strictly increasing*, the functions

$$\varphi(x,y,z) = xy+z^2 \quad \text{and} \quad f(x,y,z) = \exp(\varphi(x,y,z))$$

must have the same stationary points and extrema.

We therefore examine the simpler function $\varphi(x,y,z)$.

I. The equations for the stationary points for $\varphi(x,y,z)$ are

$$\frac{\partial\varphi}{\partial x} = y = 0, \qquad \frac{\partial\varphi}{\partial y} = x = 0, \qquad \frac{\partial\varphi}{\partial z} = 2z.$$

Hence it follows immediately that $(0,0,0)$ is the only *stationary point.* An *analysis of the sign* of the restriction

$$\varphi(x,y,0) = xy$$

to the plane $z=0$ shows that $\varphi(x,y,0)$ attains both positive and negative values in any neighbourhood of $(0,0,0)$, thus $(0,0,0)$ is *not* an extremum for φ, and therefore neither for f. $\Diamond$

16.3.2 Application of Taylor's formula

In most cases in the applications we may without loss of generality assume that $f : A \to \mathbb{R}$ is a C^2-function. If so, then it follows from *Taylor's formula*, cf. also Section 9.5, that

$$\Delta f = \mathrm{d}f + \frac{1}{2}\,\mathrm{d}^2 f + \varepsilon(\mathbf{h})\|\mathbf{h}\|^2,$$

with $\mathbf{x}_0$ as expansion point and $\mathbf{x} = \mathbf{x}_0 + \mathbf{h}$.

We shall in this section derive the so-called (r,s,t)*-method* in the 2-dimensional case, i.e. when $A \subseteq \mathbb{R}^2$.

If $\mathbf{x}_0$ is a *stationary point* of f, i.e. if $\bigtriangledown f(\mathbf{x}_0) = \mathbf{0}$, then clearly $df = 0$, so at a stationary point the increment becomes

$$\delta f = \frac{1}{2}\,\mathrm{d}^2 f + \varepsilon(\mathbf{h})\|\mathbf{h}\|^2,$$

where $\varepsilon(\mathbf{h}) \to 0$ for $\|\mathbf{h}\| \to 0$. It is *usually* not very difficult to calculate $\mathrm{d}^2 f$, so for small $\|\mathbf{h}\|$ we get an idea of the variation of the increment Δf.

Assume that the chosen point $\mathbf{x}_0 = (u,v) \in A \subseteq \mathbb{R}^2$ is a stationary point, and put $h := \|\mathbf{h}\|$ and

$$\mathbf{h} = (h_x, h_y) = (h\cos\varphi, h\sin\varphi).$$

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Then

$$\begin{aligned}\Delta f &= \frac{1}{2}\left\{f''_{xx}(u,v)h_x^2+2f''_{xy}(u,v)h_xh_y+f''_{yy}(u,v)h_y^2\right\}+\varepsilon(\mathbf{h})h^2\\ &= \frac{1}{2}\,h^2\left\{f''_{xx}(u,v)\cos^2\varphi+2f''_{xy}(u,v)\cos\varphi\sin\varphi+f''_{yy}(u,v)\sin^2\varphi\right\}+\varepsilon(\mathbf{h})h^2,\end{aligned}$$

where $\varepsilon(\mathbf{h})\to 0$ for $h=\|\mathbf{h}\|\to 0$.

For every fixed $h>0$ we let $\omega(h)$ denote the maximum of the continuous function

$$g_h(\varphi):=|\varepsilon(h\cos\varphi,h\sin\varphi)|,\qquad \text{for } \varphi\in[0,2\pi].$$

This maximum exists, because $[0,2\pi]$ is a closed and bounded interval. Then we get

$$|\varepsilon(\mathbf{h})|\le\omega(h)\qquad\text{for all }\varphi,\quad\text{and}\quad\omega(h)\to 0\text{ for }h\to 0+\,.$$

We then introduce the following *standard notation* for C^2-functions in $\mathbb{R}^2$ at the *stationary point* (u,v),

$$r:=f''_{xx}(u,v),\qquad s:=f''_{xy}(u,v),\qquad\text{and}\qquad t:=f''_{yy}(u,v).$$

Since

$$\cos^2\varphi=\frac{1+\cos 2\varphi}{2},\qquad 2\sin\varphi\cos\varphi=\sin 2\varphi,\qquad \sin^2\varphi=\frac{1-\cos 2\varphi}{2},$$

we get by insertion that

$$\begin{aligned}\Delta f &= \frac{1}{2}\,h^2\left\{r\,\frac{1+\cos 2\varphi}{2}+s\,\sin 2\varphi+t\,\frac{1-\cos 2\varphi}{2}\right\}+\varepsilon(\mathbf{h})h^2\\ &= \frac{1}{4}\,h^2\left\{r+t+(r-t)\cos 2\varphi+2s\,\sin 2\varphi\right\}+\varepsilon(\mathbf{h})h^2.\end{aligned}$$

The trick is then to analyze the vector $(r-t,2s)$. Its length is

$$K:=\sqrt{(r-t)^2+(2s)^2}=\sqrt{(r+t)^2-4\left(rt-s^2\right)},$$

and there exists an angle $\psi\in[0,2\pi]$, such that the vector is written

$$(r-t,2s)=K(\cos\psi,\sin\psi),$$

so writing the coordinates,

$$r-t=K\,\cos\psi\qquad\text{and}\qquad 2s=K\,\sin\psi.$$

Then by insertion,

$$\begin{aligned}\Delta f &= \frac{1}{4}\,h^2\{r+t+K(\cos\psi\cos 2\varphi+\sin\psi\sin 2\varphi)\}+\varepsilon(\mathbf{h})h^2\\ &= \frac{1}{4}\,h^2\{r+t+K\cos(2\varphi-\psi)\}+\varepsilon(\mathbf{h})h^2\\ &= \frac{1}{4}\,h^2\{F(\varphi)+4\varepsilon(\mathbf{h})\},\end{aligned}$$

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where

$$F(\varphi) = r + t + K\,\cos(2\varphi - \psi), \qquad \text{for } \varphi \in [0, 2\pi].$$

Since cosine has the range $[-1, 1]$, and $K > 0$, we conclude that F has the range

$$[r + t - K, r + t + K], \qquad \text{where } K = \sqrt{(r + t)^2 - 4\,(rt - s^2)}.$$

We put for short $\alpha := r + t - K$ and $\beta := r + t + K$, so the range of F is $[\alpha, \beta]$.

1) If $\alpha = r + t - K > 0$, then we can find $h_0 > 0$, such that $\omega(h) < \dfrac{1}{4}\,\alpha$ for $h < h_0$. This implies that $4|\varepsilon(\mathbf{h})| < \alpha$ for $h < h_0$, and hence

$$\Delta f \geq \frac{1}{4}\,h^2\{\alpha + 4\varepsilon(\mathbf{h})\} > 0 \qquad \text{for } 0 < h < h_0,$$

proving that f has a *proper minimum* at the stationary point (u, v).

2) If $\beta := r + t + K < 0$, we just copy the argument above and conclude that for some $h_0 > 0$,

$$\Delta f \leq \frac{1}{4}\,h^2\{\beta + 4\varepsilon(\mathbf{h})\} < 0 \qquad \text{for } 0 < h < h_0,$$

proving that f has a *proper maximum* at the stationary point (u, v).

3) If $\alpha < 0 < \beta$, then Δf is both positive and negative in any neighbourhood of $\mathbf{x}_0$, so $\mathbf{x}_0$ is not a point of extremum in this case.

4) Finally, if either $\alpha = 0$, or $\beta = 0$, then there exists a $\varphi_0 \in [0, 2\pi]$, such that $F(\varphi_0) = 0$, and the sign of Δf is then determined by the function $\varepsilon(\mathbf{h})$, which in this crude analysis is not under control. Hence, we cannot by this method alone decide, whether we have a local extremum at $\mathbf{x}_0$ or not.

We defined previously,

$$K = \sqrt{(r + t)^2 - 4\,(rt - s^2)}, \qquad \alpha = r = r + t - K \qquad \text{and} \qquad \beta = r + t + K.$$

We shall go through the possibilities above and translate the results to only involving r, s and t.

1) Since $\alpha > 0$, if and only if $r + t > K$, this inequality is equivalent to

$$rt > s^2 \qquad \text{and} \qquad r > 0 \text{ and } t > 0,$$

in which case we have a *proper minimum.*

2) Similarly, $\beta < 0$, if and only if $r + t < -K$, i.e. if and only if

$$rt > s^2 \qquad \text{and} \qquad r < 0 \text{ and } t < 0.$$

3) Then $\alpha < 0 < \beta$, if and only id $-K < r + t < K$, i.e. if and only if

$$rt < s^2.$$

4) Finally, $\alpha = 0$, or $\beta = 0$, if and only if $|r + t|K$, i.e. if and only if

$$rt = s^2.$$

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Summing up, we have proved

Theorem 16.6 The (r,s,t)-method. *Assume that (u,v) is a stationary point of the C^2-function $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^2$. Compute the numbers*

$$r = f''_{xx}(u,v), \qquad s = f''_{xy}(u,v), \qquad t = f''_{yy}(u,v).$$

1) We have a proper minimum at (u,v), if

$$rt > s^2 \qquad \text{and} \qquad r > 0 \text{ and } t > 0.$$

2) We have a proper maximum at (u,v), if

$$rt > s^2 \qquad \text{and} \qquad r < 0 \text{ and } t < 0.$$

3) There is no extremum at (u,v), if

$$rt < s^2.$$

4) Finally, if $rt = s^2$, it is an open question, whether there is a local extremum at $\mathbf{x}_0$ or not, and other methods should be applied.

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Theorem 16.6 above is the traditional way of tackling the question of local extrema for C^2-functions in two variables. It is not one of the author's favourites, because it is so easy to make errors, when one applies it. But since it is commonly used, it has for completeness been described here. One common pitfall is that one forgets to check, if the point (u, v) really is a stationary point. If it is not, then the first order terms become dominating over the second order terms, which actually are lying behind the (r, s, t)-method. Therefore, Theorem 16.6 can only be applied to points, which we know are also stationary points.

No matter the author does not like this method, there are of course given some examples below of its application.

Example 16.12

A. *In Example 16.10 it was shown that the function*

$$f(x,y) = x^4 + 4x^2y^2 + y^4 - 4x^3 - 4y^3, \qquad (x,y) \in \mathbb{R}^2,$$

has the stationary points

$$(0,0), \qquad (1,1), \qquad (3,0), \quad \textit{and} \quad (0,3).$$

Examine whether these points are also extrema.

D. This is an example where the (r, s, t)-method is of no use at the point $(0, 0)$. This illustrates that the mechanical (r, s, t)-method is *not* a universal method, which can handle all cases. We shall therefore here use the *alternative method* of the approximating polynomials. This is in general far better than the (r, s, t)-method, although this is not a universal method either.

Remark 16.4 Aside! None of the two methods above can solve the same problem of extrema for the function

$$g(x,y) = \exp\left(-\frac{1}{x^2+y^2}\right) \qquad \text{for } (x,y) \neq (0,0),$$

supplied by continuity with $g(0, 0) = 0$. It can be proved by using difference quotients that $g \in C^\infty(\mathbb{R}^2)$ and that

$$g^{(n)}_{x^k y^{n-k}}(0,0) := \frac{\partial^n g}{\partial x^k \partial y^{n-k}}(0,0) = 0 \qquad \text{for all } n \in \mathbb{N}_0 \text{ and } 0 \leq k \leq n,$$

so every approximating polynomial of degree at most n is degenerated to the zero polynomial. Nevertheless it follows immediately that because

$$g(x,y) > 0 = g(0,0) \qquad \text{for every } (x,y) \neq (0,0),$$

we must have a minimum at $(0, 0)$ for $g(x, y)$. $\Diamond$

I. 1) It follows for the point $(0, 0)$ that since the terms of third degree are dominating the terms of fourth degree, we must have in a neighbourhood of $(0, 0)$ that

$$f(x,y) \approx P_3(x,y) = -4x^3 - 4y^3 = -4(x+y)(x^2 - xy + y^2).$$

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Since

$$x^2 - xy + y^2 = \left(x - \frac{y}{2}\right)^2 + \frac{3}{4}\,y^2 > 0 \qquad \text{for } (x,y) \neq (0,0),$$

the polynomial $P_3(x,y)$ must have the same sign as $-(x+y)$, i.e. it is negative over the line $y=-x$ and positive under this line. We therefore conclude that $P_3(x,y)$, and hence also $f(x,y)$ attains both positive and negative values in any neighbourhood of $(0,0)$, where $f(0,0)=0$. Hence we cannot have an extremum at $(0,0)$.

2) For the other candidates the (r,s,t)-method is easier, because it is the essence of the determination of the approximative polynomial of *at most degree two*. One often forgets in the applications that this is the general idea behind the (r,s,t)-method. Note that we in 1) had to expand to the third degree, which is the reason why the (r,s,t)-method fails for $(0,0)$.

Since

$$r = 12x^2 + 8y^2 - 24x, \qquad s = 16xy, \qquad t = 8x^2 + 12y^2 - 24y,$$

we derive the following, using the (r,s,t)-method.

a) At $(1,1)$ we calculate

$$r=-4, \qquad s=16, \qquad t=-4 \quad \text{and} \quad rt-s^2=-240,$$

so according to the (r,s,t)-scheme there is no extremum at $(1,1)$.

b) At $(3,0)$ we calculate

$$r=36, \qquad s=0, \qquad t=72 \quad \text{and} \quad rt-s^2=2592,$$

so according to the (r,s,t)-scheme there is a proper minimum at $(3,0)$.

c) Since the function is symmetric in (x,y), the computations are obtained as in 2) with x and y interchanged. This changes nothing in the conclusion, so we have a proper minimum at $(0,3)$.

d) For completeness we note that we at $(0,0)$ get $r=s=t=0$, so nothing can be concluded by using the (r,s,t)-method alone.

There is, however, also an *alternative* method for the other points. This will here be illustrated at the point $(1,1)$.

a) First we *reset*, the problem, i.e. put $(x,y)=(1+h,1+k)$, so $(h,k)=(0,0)$ corresponds to the point $(x,y)=(1,1)$ under examination.

b) Insert this in the expression for $f(x,y)$ and write dots for terms of degree >2:

$$\begin{aligned}
f(x,y) &= (1+h)^4+4(1+h)^2(1+k)^2+(1+k)^4-4(1+h)^3-4(1+k)^3 \\
&= 1+4h+6h^2+\cdots+4(1+2h+h^2)(1+2k+k^2) \\
&\qquad +1+4k+6k^2+\cdots-4(1+3h+3h^2+\cdots) \\
&\qquad -4(1+3k+3k^2+\cdots) \\
&= (1+4h+6h^2)+4\left\{1+2h+h^2+2k+4hk+k^2+\cdots\right\} \\
&\qquad +(1+4k+6k^2)-4(1+3h+3h^2)-4(1+3k+3k^2)+\cdots \\
&= -2-2h^2+16hk-2k^2+\cdots\,,
\end{aligned}$$

i.e.

$$P_2(h,k) = -2-2(h^2-8hk+k^2) = -2-2\left\{(h-4k)^2-15k^2\right\}.$$

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Since $P_2(h,k)+2$ attains both negative values (for $k=0$ and $h \neq 0$) and positive values (for $h=4k$) in any neighbourhood of $(h,k)=(0,0)$, we conclude that $(x,y)=(1,1)$ is *not* an extremum. $\Diamond$

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Example 16.13

A. *Examine whether the function*

$$f(x,y) = 1 - 4x^2 - 4y^2 + x^2y^2, \qquad (x,y) \in \mathbb{R}^2.$$

has any extremum. Find the range $f(\mathbb{R}^2)$.

D. When we apply the standard procedure we are guided through the usual examination of the exceptional points (there are none) and of the stationary points. It is, however, here possible to make a shortcut by noticing that $f(x,y)$ only is a function in $u = x^2$ and $v = y^2$, i.e.

$$f(x,y) = 1 - 4x^2 - 4y^2 + x^2y^2 = g(u,v) = 1 - 4u - 4v + uv, \qquad u,\, v \geq 0.$$

I. The *stationary points* for $f(x,y)$, if any, must fulfil the equations

$$\frac{\partial f}{\partial x} = -2x(4-y^2) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = -2y(4-x^2) = 0.$$

This system is split into

$$x = 0 \quad \text{or} \quad y = 2 \quad \text{or} \quad y = -2,$$

and

$$y = 0 \quad \text{or} \quad x = 2 \quad \text{or} \quad x = -2.$$

Formally we get $3 \cdot 3$ possibilities, but four of them are not possible (e.g. x cannot at the same time be 0 and 2 or -2). We therefore get five *stationary points*,

$$(0,0), \qquad (2,2), \qquad (-2,-2), \qquad (-2,2), \qquad (2,-2).$$

We first apply the (r,s,t)-method. Here we get

$$r = 2y^2 - 8, \qquad s = 4xy, \qquad t = 2x^2 - 8,$$

from which

1) At $(0,0)$ we calculate

$$r = -8, \qquad s = 0, \qquad t = -8, \quad \text{and} \quad rt - s^2 = 64,$$

so it follows that we have a proper maximum at $(0,0)$.

2) Due to the symmetry we get the same calculations at the two points $(2,2)$ and $(-2,-2)$, namely

$$r = 0, \qquad s = 16, \qquad t = 0, \quad \text{and} \quad rt - s^2 = -256.$$

We conclude that there is no extremum at any of the two points $(2,2)$, $(-2,-2)$.

3) Due to the symmetry we get the same calculations at the two points $(2,-2)$ and $(-2,2)$, namely

$$r = 0, \qquad s = -16, \qquad t = 0, \quad \text{and} \quad rt - s^2 = -256.$$

Again, the conclusion is that there is no extremum at the points $(2,-2)$, $(-2,2)$,

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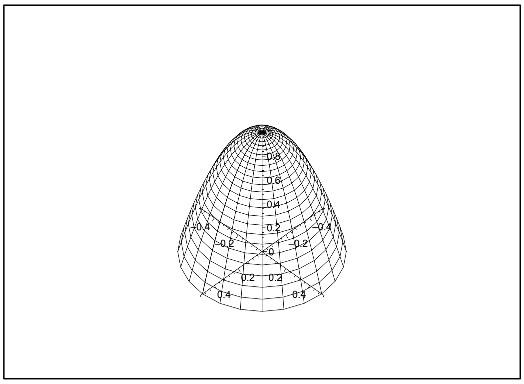

Figure 16.18: The graph of $z = 1 - 4x^2 - 4y^2$.

Then we turn to the alternative methods.

1) In $(0,0)$ we get

$$f(x,y) \approx P_2(x,y) = 1 - 4x^2 - 4y^2.$$

The graph of $z = P_2(x,y)$ is an *elliptic paraboloid of revolution.* Obviously we have a *proper local maximum* at $(0,0)$.

2) We take a shortcut by considering

$$g(u,v) = 1 - 4u - 4v + uv, \qquad u = x^2, \quad v = y^2,$$

instead. First, all four stationary points for f are seen to correspond to the only point $(u,v) = (4,4)$. It is therefore sufficient to examine $g(u,v)$ in the neighbourhood of $(4,4)$.

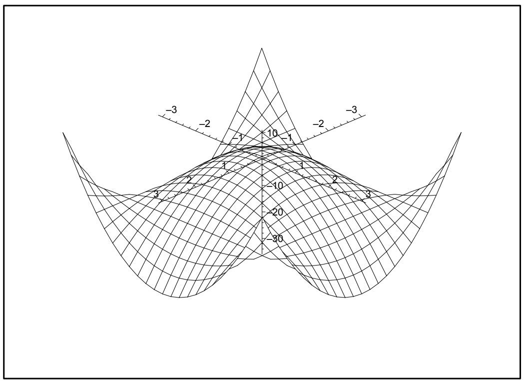

Figure 16.19: The graph of $f(x,y) = 1 - 4x^2 - 4y^2 + x^2 + y^2$.

The approximating polynomial for $g(u,v)$ expanded from $(4,4)$ of at most degree 2 is found by

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using:

$$
\begin{aligned}
&g(u,v) = 1 - 4u - 4v + uv, && g(4,4) = -15,\\
&\frac{\partial g}{\partial u} = -4 + v, && g'_u(4,4) = 0,\\
&\frac{\partial g}{\partial v} = -4 + u, && g'_v(4,4) = 0,\\
&\frac{\partial^2 g}{\partial u}\partial v = 1, && \\
&\frac{\partial^2 g}{\partial u^2} = \frac{\partial^2 g}{\partial v^2} = 0, && ,
\end{aligned}
$$

hence

$$P_3(u,v) = -15 + \frac{1}{2}\cdot(u-4)(v-4) = -15 + (u-4)(v-4).$$

It follows that $P_2(u,v)$ in the neighbourhood of $(4,4)$ attains values which are both > -15 and < -15. Thus $g(u,v)$ does *not* have an extremum at $(4,4)$. This implies that $f(x,y)$ does *not* have an extremum at $(\pm 2, \pm 2)$ (all four possible combinations of the sign).

3) The function $f(x,y)$ has only one local maximum,

$$f(0,0) = 1.$$

However, this value is *not* the global maximum. In fact, by rewriting

$$f(x,y) = 1 - 4x^2 - 4y^2 + x^2y^2 = (x^2-4)(y^2-4) - 15,$$

we see that the restriction to the line $y = x$ gives

$$f(x,x) = \left(x^2-4\right)^2 - 15 \to +\infty \qquad \text{for } x \to \pm\infty.$$

4) Note also that

$$f(x,0) = 1 - 4x^2 \to -\infty \qquad \text{for } x \to \pm\infty.$$

Thus, since f is *continuous* on the *connected* set $\mathbb{R}^2$, it follows from the *first main theorem for continuous functions* that the range is

$$f(\mathbb{R}^2) = \mathbb{R}. \qquad \Diamond$$

16.4 Extremum for continuous functions in three or more variables

In this section we show how to treat the extremum problem, when we consider a C^2-function $f : A \to \mathbb{R}$ first in three variables, i.e. $A \subseteq \mathbb{R}^3$, and then for general $\mathbb{R}^m$, $m \geq 2$.

Assume that $\mathbf{x}_0 \in A \subseteq \mathbb{R}^3$ is a stationary point, i.e. in the coordinates of the gradient,

$$\frac{\partial f}{\partial x_1}(\mathbf{x}_0) = 0, \qquad \frac{\partial f}{\partial x_2}(\mathbf{x}_0) = 0, \qquad \frac{\partial f}{\partial x_3}(\mathbf{x}_0) = 0.$$

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As previously we put $\Delta f = f(\mathbf{x}) - f(\mathbf{x}_0)$. Then we get by *Taylor's formula*,

$$\Delta f = \frac{1}{2}(\mathbf{h} \cdot \bigtriangledown)^2 f(\mathbf{x}_0) + \varepsilon(\mathbf{h})\|\mathbf{h}\|,$$

or, more elaborated,

$$\Delta f = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} h_i h_j \, \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0) + \varepsilon(\mathbf{h})\|\mathbf{h}\|^2.$$

To ease notation we put

$$M_{ij} := \frac{d^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0), \qquad \text{for } i,\, j = 1,\, 2,\, 3.$$

We note that since $f \in C^2$, the differentiations commute, so $M_{ij} = M_{ji}$. If we define the matrices

$$\mathbf{h}_- := (h_1, h_2, h_3), \qquad \mathbf{M} := \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}, \qquad \mathbf{h}_| := \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix},$$

then

$$\Delta f = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} h_i M_{ij} h_j + \varepsilon(\mathbf{h})\|\mathbf{h}\|^2 = \frac{1}{2}\,\mathbf{h}_- \mathbf{M} \mathbf{h}_| + \varepsilon(\mathbf{h})\|\mathbf{h}\|^2.$$

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Neglecting the "error term" $\varepsilon(\mathbf{h})\|\mathbf{h}\|^2$, which tends towards 0 faster than $\|\mathbf{h}\|^2$, the problem is reduced to a discussion of the bilinear form

$$F(\mathbf{h}) := \mathbf{h}_{-}\mathbf{M}\mathbf{h}_{|},$$

where as mentioned above the square matrix $\mathbf{M}$ is symmetric, $M_{ij} = M_{ji}$.

It is known from *Linear Algebra* that a real symmetric (3×3)-matrix has three real eigenvalues λ_1, λ_2, λ_3, counted by multiplicity., and that we can write

$$F(\mathbf{h}) = \mathbf{h}_{-}\mathbf{M}\mathbf{h}_{|} = \sum_{i=1}^{3} \lambda_i w_i^2,$$

where w_1, w_2, w_3 are some linear combinations of h_1, h_2, h_3, chosen such that $\|\mathbf{w}\| = \|\mathbf{h}\|$, where $\mathbf{w} = (w_1, w_2, w_3)$.

It is well-known from *Linear Algebra* that the eigenvalues are the roots in the equation

$$\det(\mathbf{M} - \lambda\mathbf{I}) = 0,$$

where $\mathbf{I}$ is the unit (3×3)-matrix.

The structure of the w_i's is not important here. Of importance for the analysis of possible extrema are only the eigenvalues λ_1, λ_2, λ_3, because we by using known results from *Linear Algebra* immediately get the following results.

1) The matrix $\mathbf{M}$ is *positive definite*, if and only if all its eigenvalues are *positive*. In this case f has a *proper minimum* at the stationary point $\mathbf{x}_0$.

2) The matrix $\mathbf{M}$ is *negative definite*, if and only if all its eigenvalues are *negative*. In this case f has a *proper maximum* at the stationary point $\mathbf{x}_0$.

3) The matrix $\mathbf{M}$ is *indefinite*, if and only if there are at least one positive and one negative eigenvalue. In this case f has *no extremum* at the stationary point $\mathbf{x}_0$,

4) The matrix $\mathbf{M}$ is *semidefinite*,, if and only if at least one of the eigenvalues is 0, and the nonzero eigenvalues have all the same sign. In this case nothing can be concluded based alone on the present analysis.

The analysis above was performed in $\mathbb{R}^3$, but it is trivial to extend it to any $\mathbb{R}^m$, $m \geq 2$. However, when $m \geq 4$, the computations soon become overwhelming, unless we have chosen very special cases.

Let us check the result for $m = 2$, already treated in Section 16.3.2. In this case,

$$\mathbf{M} = \begin{pmatrix} r & s \\ s & t \end{pmatrix},$$

and

$$\det(\mathbf{M} - \lambda\mathbf{I}) = \begin{vmatrix} r-\lambda & s \\ s & t-\lambda \end{vmatrix} = (r-\lambda)(t-\lambda) - s^2 = \lambda^2 - (r+t)\lambda + rt - s^2 = 0.$$

The eigenvalues are the roots of this equation,

$$\lambda = \frac{1}{2}\left\{r + t \pm \sqrt{(r+t)^2 - 4\left(rt - s^2\right)}\right\} = \frac{1}{2}\left(r + t \pm K\right),$$

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where K was introduced in Section 16.3.2.

When we go through the four bullets above, we again derive the (r, s, t)-scheme, so this is only a special case of this more general theory, in which we applied some matrix calculus.

Example 16.14 Consider the function

$$f(x,y,z) = x^3 + x^2 + 2xy - y^2 - z^2 \qquad \text{for } (x,y,z) \in \mathbb{R}^3,$$

in three real variables. Since $f \in C^\infty\left(\mathbb{R}^3\right)$, there are *no exception points*.

Then we turn to the *stationary points*. First we compute the coordinates of the gradient,

$$f'_x(x,y,z) = 3x^2 + 2x + 2y, \qquad f'_y(x,y,z) = 2x - 2y, \qquad f'_z(x,y,z) = -2z,$$

which we shall need to calculate the derivatives of second order.

The stationary points are the solutions of the system of equations,

$$3x^2 + 2x + 2y = 0, \qquad 2x - 2y = 0 \quad \text{and} \quad -2z = 0,$$

thus $z = 0$ (third equation), and $y = x$ (second equation). Eliminating y in the first equation we get

$$0 = 3x^2 + 4x = x(3x+4), \qquad \text{so } x = 0 \text{ or } x = -\frac{4}{3}.$$

We conclude that there are two *stationary points*,

$$(0,0,0) \qquad \text{and} \qquad \left(-\frac{4}{3}, -\frac{4}{3}, 0\right).$$

Then we calculate the matrix $\mathbf{M}$ consisting of the derivatives of second order,

$$\mathbf{M} = \begin{pmatrix} 6x+2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

1) First consider the stationary point $(0,0,0)$. The eigenvalues satisfy the equation

$$\begin{vmatrix} 2-\lambda & 2 & 0 \\ 2 & -2-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{vmatrix} = (-2-\lambda) \begin{vmatrix} 2-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix}$$

$$= -(\lambda+2)\left(\lambda^2 - 4 - 4\right) = -(\lambda+2)\left(\lambda^2 - 8\right) = 0,$$

so the eigenvalues are

$$\lambda_1 = -2\sqrt{2}, \qquad \lambda_2 = -2, \qquad \lambda_3 = +2\sqrt{2}.$$

Since λ_1 and λ_3 have opposite signs, we conclude that the matrix is indefinite at $(0,0,0)$, hence $(0,0,0)$ has not an extremum.

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2) Then at the stationary point $\left(-\frac{4}{3},-\frac{4}{3},0\right)$ we get instead the following equation for the eigenvalues,

$$\left|\begin{array}{ccc} -6-\lambda & 2 & 0 \\ 2 & -2-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{array}\right| = -(\lambda+2)((\lambda+6)(\lambda+2)-4) = -(\lambda+2)\left(\lambda^2+8\lambda+8\right) = 0,$$

so the eigenvalues are

$$\lambda_1 = -4-2\sqrt{2}, \qquad \lambda_2 = -2, \qquad \lambda_3 = -4+2\sqrt{2}.$$

They are all negative, so we conclude from the above that we have a *proper maximum* at

$$\left(-\frac{4}{3},-\frac{4}{3},0\right). \qquad \Diamond$$

Example 16.15 To illustrate the situation in higher dimensions we shall here consider the following very nice C^∞-function in four variables,

$$f(x,y,z,w) = x^3+y^3-3xz^2+3yw^2+6z+6w, \qquad \text{for } (x,y,z,w) \in \mathbb{R}^4.$$

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We first compute the coordinates of the gradient,

$$\frac{\partial f}{\partial x} = 3x^2 - 3z^2, \qquad \frac{\partial f}{\partial y} = 3y^2 - 3w^2,$$

$$\frac{\partial f}{\partial z} = -6xz + 6, \qquad \frac{\partial f}{\partial w} = -6yw + 6,$$

from which we for later use calculate the $\mathbf{M}$-matrix,

$$\mathbf{M} = 6 \begin{pmatrix} x & 0 & -z & 0 \\ 0 & y & 0 & -w \\ -z & 0 & -x & 0 \\ 0 & -w & 0 & -y \end{pmatrix}.$$

The stationary points are the solutions of the nonlinear system

$$3x^2 - 3z^2 = 0, \qquad 3y^2 - 3w^2 = 0,$$

$$-6xz + 6 = 0, \qquad -6yw + 6 = 0,$$

from which $z = \pm x$ and $xz = 1$, so $z = x = \pm 1$, and $w = \pm y$ and $yw = 1$, so $y = w = \pm 1$. By taking all possible combinations of the signs we find the following four stationary points,

$$(1,1,1,1), \qquad (1,-1,1,-1), \qquad (-1,1,-1,1), \quad \text{and} \quad (-1,-1,-1,-1).$$

Using brute force we get by expanding the determinant after the first columb,

$$\begin{aligned}
&\begin{vmatrix} x-\lambda & 0 & -z & 0 \\ 0 & y-\lambda & 0 & -w \\ -z & 0 & -x-\lambda & 0 \\ 0 & -w & 0 & -y-\lambda \end{vmatrix} \\
&\qquad = (x-\lambda) \begin{vmatrix} y-\lambda & 0 & -w \\ 0 & -x-\lambda & 0 \\ -w & 0 & -y-\lambda \end{vmatrix} - z \begin{vmatrix} 0 & -z & 0 \\ y-\lambda & 0 & -w \\ -w & 0 & -y-\lambda \end{vmatrix} \\
&\qquad = (x-\lambda)(-x-\lambda) \begin{vmatrix} y-\lambda & -w \\ -w & -y-\lambda \end{vmatrix} - z^2 \begin{vmatrix} y-\lambda & -w \\ -w & -y-\lambda \end{vmatrix} \\
&\qquad = \left(\lambda^2 - x^2 - z^2\right)\left(\lambda^2 - y^2 - w^2\right) = 0,
\end{aligned}$$

so the eigenvalues are $\lambda = \pm\sqrt{x^2+z^2}$ and $\lambda = \pm\sqrt{y^2+w^2}$.

Inserting the four stationary points we obtain in all cases the eigenvalues $\sqrt{2}$ and $-\sqrt{2}$, both of multiplicity 2. Since there are both positive and negative eigenvalues, the matrix is *indefinite*, and we do not have any extremum, $\Diamond$

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17 Examples of global and local extrema

17.1 MAPLE

It is possible in the examples to use some MAPLE commands. For completeness we mention the following:

with(VectorCalculus):
Gradient$(f(x,y),[x,y])$

or

Gradient$(f(x,y,z),[x,y,z])$

etc., where e.g. $f(x,y)$ is an explicit expression of a function, and $[x,y]$ is a list of coordinates. The results are given in the form

$$f'_x(x,y)e_x + f'_y(x,y)e_y,$$

and similarly.

It is also possible in this way to find the gradient in polar/spherical coordinates.

An alternative way in in $\mathbb{R}^3$:

with(Physics[Vectors]):
Gradient$(f(x,y,z),[x,y,z])$

which is the same as above, or, using

Nabla$(f(x,y,z),[x,y,z])$

instead.

One may also plot the *gradient field* of a given function f. The commands are:

with(plots):
gradplot$(f(x,y), x=a..b, y=c..d)$

or

gradplot3d$(f(x,y,z), x=a..b, y=c..d, z=h..k)$

in $\mathbb{R}^2$ and $\mathbb{R}^3$ resp..

However, most of the examples in this chapter are so simple that we shall only make little use of MAPLE.

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17.2 Examples of extremum for two variables

Example 17.1 *Find in each of the following cases first the stationary points of the given function $f : \mathbb{R}^2 \to \mathbb{R}$. Then check if f in any of these points has an extremum; whenever this is the case, decide whether it is a maximum or a minimum.*

1) $f(x, y) = x^2 + 2y^2 - 2x - 2y$.

2) $f(x, y) = x^2 + y^2 + 2xy$.

3) $f(x, y) = xy^2$.

4) $f(x, y) = 3x^3 + 4y^3 + 6xy^2 - 9x^2$.

5) $f(x, y) = (x^2 + y^2 - 2y)(x^2 + y^2 - 6y)$

6) $f(x, y) = x^4 + y^4 - 2x^2y^2$.

7) $f(x, y) = 3x^4 + 4y^4 - 4x^2y^2$.

8) $f(x, y) = (\sin x) \cos y$.

9) $f(x, y) = x^2 + y^2 + e^{xy}$.

10) $f(x, y) = xy + 2 \sinh(1 + x^2 + y^2)$.

11) $f(x, y) = x^3y - 2x^2y + xy^3$.

A Stationary points; extrema.

D Inspect the expression for a smart rearrangement. Find the stationary points. Check if these are extrema.

I 1) a) First variant. It is seen by inspection that

$$f(x, y) = x^2 + 2y^2 - 2x - 2y = (x - 1)^2 + 2\left(y - \frac{1}{2}\right)^2 - \frac{3}{2}.$$

We conclude that $\left(1, \frac{1}{2}\right)$ is the only stationary point and that it is a minimum.

b) Second variant. Traditionally the equations of the stationary points are

$$\frac{\partial f}{\partial x} = 2x - 2 = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 4y - 2 = 0,$$

from which follows that $\left(1, \frac{1}{2}\right)$ is the only stationary point.

i) First subvariant. The approximating polynomial of at most second degree is found by translating the coordinate system to the point $\left(1, \frac{1}{2}\right)$, so we introduce the new variables

$$x = x_1 + 1, \qquad y = y_1 + \frac{1}{2}.$$

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Then by insertion,

$$P_2(x_1,y_1)\,[=f(x,y)]\;=x_1^2+2y_1^2-\frac{3}{2}$$

[cf. the first variant], which clearly has a minimum for $(x_1,y_1)=(0,0)$, i.e. for $(x,y)=\left(1,\frac{1}{2}\right)$.

ii) Second subvariant. The (r,s,t)-method. It follows from

$$r=\frac{\partial^2 f}{\partial x^2}=2,\qquad s=\frac{\partial^2 f}{\partial x\partial y}=0,\qquad t=\frac{\partial^2 f}{\partial y^2}=4,$$

that r, $t>0$ and $s^2<rt$, so we conclude that we have a minimum.

2) a) Inspection. It is immediately seen that

$$f(x,y)=x^2+y^2+2xy=(x+y)^2,$$

which has a minimum $(=0)$ on the line $y=-x$. The points of this line are of course not proper minima.

b) The stationary points. These are the solutions of the equations

$$\frac{\partial f}{\partial x}=2x+2y=2(x+y)=0,\qquad \frac{\partial f}{\partial y}=2y+2x=2(x+y)=0,$$

thus every point on the line $y=-x$ is a stationary point.

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In this case we cannot conclude anything by the (r, s, t)-method. One should, however, be able to see that e.g.

$$\mathrm{d}f = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 2(x+y)(dx+dy) = d(x+y)^2,$$

so $f(x,y) = (x+y)^2$, and we are back to the first variant.

3) It follows from

$$\frac{\partial f}{\partial x} = y^2 = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 2xy = 0$$

that the stationary points are the points on the X axis $y = 0$. In a neighbourhood of any point $(x_0, 0)$, $x_0 \neq 0$ we see that xy^2 has the same sign as x_0, and it is 0 on the X axis. This means that we have a minimum (though not a proper minimum) for every $(x_0, 0)$, $x_0 > 0$, and a maximum (though not a proper maximum) for every $(x_0, 0)$, $x_0 < 0$. In the stationary point $(0, 0)$ we have neither a maximum nor a minimum, because the function in any neighbourhood of $(0, 0)$ takes on both positive and negative values.

4) When $f(x,y) = 3x^3 + 4y^3 + 6xy^2 - 9x^2$, the equations of the stationary points are

$$\begin{cases} \dfrac{\partial f}{\partial x} = 9x^2 + 6y^2 - 18x = 0, \\[2mm] \dfrac{\partial f}{\partial y} = 12y^2 + 12xy = 12y(y+x) = 0. \end{cases}$$

We get two possibilities from the latter condition:

$$y = 0 \qquad \text{or} \qquad y = -x.$$

a) If we put $y = 0$ into the first equation we get

$$9x^2 - 18x = 9x(x-2) = 0,$$

so we conclude that $(0, 0)$ and $(2, 0)$ are stationary points.

b) If we put $y = -x$ into the first equation we get

$$15x^2 - 18x = 15x\left(x - \frac{6}{5}\right) = 0,$$

so we get the stationary points $(0, 0)$ and $\left(\dfrac{6}{5}, -\dfrac{6}{5}\right)$.

Summing up we have three different stationary points

$$(0,0), \qquad (2,0) \quad \text{and} \quad \left(\frac{6}{5}, -\frac{6}{5}\right).$$

These are now considered one by one.

a) The point $(0, 0)$ is not an extremum, because e.g. $f(0, y) = 4y^3$ takes on both positive and negative values in any neighbourhood of $(0, 0)$.

In the other two cases we apply the (r, s, t)-method. We first calculate the general results

$$r = \frac{\partial^2 f}{\partial x^2} = 18x - 18, \quad s = \frac{\partial^2 f}{\partial x \partial y} = 12y, \quad t = \frac{\partial^2 f}{\partial y^2} = 24y + 12x.$$

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b) At $(2,0)$ we have $r = 18$, $s = 0$ and $t = 24$, so $rt > s^2$, $r > 0$, $t > 0$, corresponding to a proper minimum.

c) At $\left(\frac{6}{5}, -\frac{6}{5}\right)$ we have $r = \frac{18}{5}$, $s = -\frac{72}{5}$, $t = -\frac{72}{5}$, so $rt < s^2$ (e.g. s and t have different signs), and we have no extremum.

Summarizing we see that only $(2,0)$ is an extremum (a proper minimum).

5) It follows by the rearrangement

$$\begin{aligned} f(x,y) &= (x^2+y^2-2y)(x^2+y^2-6y) \\ &= \left(x^2+\{y-1\}^2-1\right)\left(x^2+\{y-3\}^2-3^2\right) \end{aligned}$$

that f is zero on the circles

$$x^2+(y-1)^2 = 1 \quad \text{and} \quad x^2+(y-3)^2 = 3^2.$$

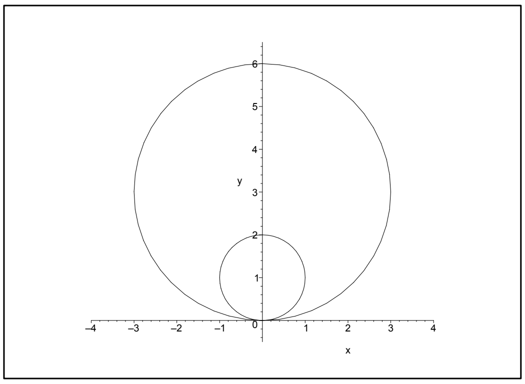

Figure 17.1: Zero curves for $f(x,y)$.

The function is positive inside both circles (i.e. inside the smaller circle), and outside both circles. It is negative at every point inside the larger circle and outside the smaller circle. The function is continuous and 0 on both circles, so it follows from the main theorem that we must have a local maximum inside the smaller circle, and a local minimum inside the larger disc (and outside the smaller disc). Finally, f is both positive and negative in any neighbourhood of $(0,0)$, so this point cannot be an extremum.

Remark. It follows from the above that one can apply the main theorem in a qualitative way to decide where we must have extrema. Such analyses of figures are very useful. $\Diamond$

The stationary points. We shall now start on the tough calculations of the example. It follows from

$$f(x,y) = (x^2+y^2-2y)(x^2+y^2-6y)$$

that

$$\frac{\partial f}{\partial x} = 2x(x^2+y^2-6y) + 2x(x^2+y^2-2y) = 4x(x^2+y^2-4y),$$

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and

$$\begin{aligned}\frac{\partial f}{\partial y} &= (2y-2)(x^2+y^2-6y)+(2y-6)(x^2+y^2-2y)\\ &= (2y-4)(x^2+y^2-6y)+2(x^2+y^2-6y)+(2y-4)(x^2+y^2-2y)-2(x^2+y^2-2y)\\ &= 4(y-2)(x^2+y^2-4y)-8y\\ &= 4\left\{(y-2)(x^2+y^2-4y)-2y\right\}.\end{aligned}$$

The two equations of the stationary points are therefore written more conveniently

$$(17.1)\quad \begin{cases} x(x^2+y^2-4y) &= 0,\\ (y-2)(x^2+y^2-4y) &= 2y.\end{cases}$$

It follows from the first equation that the stationary points (if any) either lies on the line $x=0$ or on the circle $x^2+(y-2)^2=2^2$.

a) If $x=0$, then we get from the latter equation (17.1) that

$$\begin{aligned}0 &= (y-2)(y^2-4y)-2y = y\{(y-2)(y-4)-2\}\\ &= y\left\{y^2-6y+6\right\} = y\left\{(y-3)^2-(\sqrt{3})^2\right\},\end{aligned}$$

so either $y=0$ or $y=3\pm\sqrt{3}$. Hence we get three stationary points,

$$(0,0),\qquad (0,3+\sqrt{3}),\qquad (0,3-\sqrt{3}).$$

b) If $x^2+y^2-4y=0$, then it follows from the latter equation of (17.1) that $y=0$, and thus $x=0$, so we find again $(0,0)$.

Summarizing we get the three stationary points

$$(0,0),\qquad (0,3+\sqrt{3}),\qquad (0,3-\sqrt{3}).$$

Alternatively the expression gives one the inspiration of using *polar coordinates*. We get

$$f(x,y)=(\varrho^2-2\varrho\sin\varphi)(\varrho^2-6\varrho\sin\varphi)=\varrho^4-8\varrho^3\sin\varrho+12\varrho^2\sin^2\varphi,$$

hence

$$\begin{aligned}\frac{\partial f}{\partial \varrho} &= 4\varrho^3-24\varrho^2\sin\varphi+24\varrho\sin^2\varphi = 4\varrho\left(\varrho^2-6\varrho\sin\varphi+6\sin^2\varphi\right),\\ \frac{\partial f}{\partial \varphi} &= -8\varrho^3\cos\varphi+24\varphi^2\sin\varphi\cos\varphi = 8\varrho^2\cos\varphi(-\varrho+3\sin\varphi).\end{aligned}$$

After a reduction the system of equations is written

$$(17.2)\quad \begin{cases} \varrho\left(\varrho^2-6\varrho\sin\varphi+6\sin^2\varphi\right)=0,\\ \varrho^2\cos\varphi(3\sin\varphi-\varrho)=0.\end{cases}$$

From the latter equation we get the three possibilities $\varrho=0$, $\cos\varphi=0$ and $3\sin\varphi=\varrho$.

a) If $\varrho=0$, then both equations are fulfilled so $(0,0)$ is a stationary point corresponding to $\varrho=0$, hence to $(0,0)$ in rectangular coordinates.

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b) If $\cos\varphi = 0$ (and $\varrho \geq 0$), then $\sin\varphi = \pm 1$.

i) When $\sin\varphi = -1$ it follows that

$$\varrho^2 - 6\varrho\sin\varphi + 6\sin^2\varphi = \varrho^2 + 6\varrho + 6 \geq 6 \quad \text{for } \varrho \geq 0,$$

so we are only left with the possibility $\varrho = 0$, which has already been treated above.

ii) When $\sin\varphi = 1$ (and $\varrho = y$, because $y = \varrho\sin\varphi$), then

$$\varrho^2 - 6\varrho\sin\varphi + 6\sin^2\varphi = \varrho^2 - 6\varrho + 6 = 0$$

has the two positive solutions $\varrho = 3 \pm \sqrt{3}$, corresponding to the stationary points

$$(0, 3+\sqrt{3}) \quad \text{and} \quad (0, 3-\sqrt{3})$$

in rectangular coordinates.

c) If $3\sin\varphi = \varrho$, then

$$0 = 4\varrho\left(\varrho^2 - 2\varrho\cdot 3\sin\varphi + \frac{2}{3}(3\sin\varphi)^2\right) = 4\varrho\left(\varrho^2 - 2\varrho^2 + \frac{2}{3}\,\varrho^2\right) = -\frac{8}{3}\,\varrho^2,$$

with the only solution $\varrho = 0$, which we have already treated above.

Summarizing we get in rectangular coordinates the following three stationary points

$$(0,0), \qquad (0, 3+\sqrt{3}), \qquad (0, 3-\sqrt{3}).$$

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Check of the type of the stationary points, inspection. It follows from the discussion of the sign in the beginning of the example that $(0,0)$ is *not* an extremum, because the function takes on both positive and negative values in any neighbourhood of $(0,0)$.

The point $(0,3-\sqrt{3})$ must be a *local maximum*. In fact, we have already by the main theorem concluded that there exists a local maximum in the smaller disc, and since $f \in C^{\infty}$, it can only be attained at a stationary point. Since $(0,3-\sqrt{3})$ is the only stationary point in the smaller disc the claim follows.

It follows analogously by the discussion of the sign that $(0,3+\sqrt{3})$ must be a *local minimum*.

Remark. Note that by the application of the main theorem we obtain a much simpler analysis than by the traditional *standard* methods in the following. We have almost done everything! $\Diamond$

Investigation of the type of the stationary points, standard procedure.

a) We first check $(0,0)$.

i) First variant, the (r,s,t)-method. This breaks totally down because

$$r=s=t=0,$$

and nothing can be concluded.

ii) Second variant. Approximating polynomials of at most second degree. This cannot be used either, because

$$P_2(x,y) \equiv 0,$$

and nothing can be concluded.

iii) Third variant. A dirty trick. If follows from

$$4ab=(a+b)^2-(a-b)^2, \quad \text{i.e. } ab=\frac{1}{4}\left\{(a+b)^2-(a-b)^2\right\},$$

that

$$a=x^2+y^2-2y \quad \text{and} \quad b=x^2+y^2-6y,$$

so

$$f(x,y)=\frac{1}{4}\left\{4(x^2+y^2-4y)^2-16y^2\right\}=(x^2+y^2-4y)^2-4y^2.$$

If we go towards $(0,0)$ along the line $y=0$, then $f(x,0)>0$, and if we go towards $(0,0)$ along the circle $x^2+y^2-4y=0$, then $f(x,y)<0$, so f takes on both positive and negative values in any neighbourhood of$(0,0)$. Therefore we cannot have an extremum at $(0,0)$.

Remark. The trick above is one of the very oldest in mathematics. The Egyptians did not have tables of multiplication, though tables of squared numbers. They used the trick above to calculate products. $\Diamond$

Erroneous variant. One might try to investigate the limit along all lines through $(0,0)$.

i) If $y=0$, then $f(x,0)=x^4>0$ for $x \neq 0$.

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ii) If $x = 0$, then

$$f(0, y) = (y^2 - 2y)(y^2 - 6^y) = y^2(2 - y)(6 - y) > 0 \quad \text{for } 0 < |y| < 2.$$

iii) If $y = ax$, $a \neq 0$, then

$$f(x, ax) = \cdots = x^2\{2a - (1 + a^2)x\}\{6a - (1 + a^2)x\},$$

where the product of the latter two factors tends towards $12a^2 > 0$ for $x \to 0$, and accordingly

$$f(x, ax) > 0 \quad \text{sufficiently close to } 0.$$

One might then **erroneously** conclude that $(0, 0)$ is a minimum, what it is **not**.

b) Let us return to the points $(0, 3 \pm \sqrt{3})$. These are checked by the (r, s, t)-method. First calculate

$$\begin{aligned} r &= \frac{\partial^2 f}{\partial x^2} = 4\{x^2 + y - 4y\} + 8x^2 = 12x^2 + 4y(y - 4), \\ s &= \frac{\partial^2 f}{\partial x \partial y} = 4x(2y - 4) = 8x(y - 2), \\ t &= \frac{\partial^2 f}{\partial y^2} = 4(x^2 + y^2 - 4y) + 4(y - 2)(2y - 4) - 8 = 4x^2 + 12y(y - 4). \end{aligned}$$

Since both stationary points satisfy $x = 0$ we can reduce in the following way

$$\begin{aligned} r_{|x=0} &= 4y(y - 4), \\ s_{|x=0} &= 0, \\ t_{|x=0} &= 12y(y - 4) = 3r_{|x=0}, \end{aligned}$$

thus

$$r_{|x=0}t_{|x=0} = 3\left(r_{|x=0}\right)^2 \geq 0 = s_{|x=0}^2.$$

We therefore have extremum when $r_{|x=0} \neq 0$.

i) We have at the point $(0, 3 + \sqrt{3})$ that $3 + \sqrt{3} > 4$, so $r > 0$, and $t > 0$, so we have a minimum.

ii) We have at the point $(0, 3 - \sqrt{3})$ that $3 - \sqrt{3} < 4$, so $r < 0$, and $t < 0$, and we have a maximum.

6) Here

$$f(x, y) = x^4 + y^4 - 2x^2y^2 = \left(x^2 - y^2\right)^2 = (x - y)^2(x + y)^2,$$

corresponding to a minimum on the lines $y = x$ and $y = -x$.

ALTERNATIVELY,

$$\begin{aligned} \frac{\partial f}{\partial x} &= 4x^3 - 4xy^2 = 4x(x^2 - y^2), \\ \frac{\partial f}{\partial y} &= 4y^3 - 4x^2y = 4y(y^2 - x^2). \end{aligned}$$

The stationary points are

$$(0, 0), \qquad (x, x) \text{ and } (x, -x), \quad x \in \mathbb{R},$$

corresponding to the fact that the stationary points form the set consisting of the two lines $y = x$ and $y = -x$. On these we have a minimum, though not a proper minimum.

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7) Here

$$f(x,y) = 3x^4 + 4y^4 - 4x^2y^2 = (4y^4 - 4x^2y^2 + x^4)02x^4 = \left(2y^2 - x^2\right)^2 + 2x^4,$$

so we have got a minimum for $y^2 = \dfrac{1}{2}\,x^2 = 0$, i.e. at $(0,0)$.

ALTERNATIVELY

$$\begin{aligned} \frac{\partial f}{\partial x} &= 12x^3 - 8xy^2 = 4x(3z^2 - 2y^2), \\ \frac{\partial f}{\partial y} &= 16y^3 - 8x^2y = 8y(2y^2 - x^2). \end{aligned}$$

It is almost obvious that $(0,0)$ is the only stationary point.

8) Here

$$\frac{\partial f}{\partial x} = \cos x \cdot \cos y \quad \text{and} \quad \frac{\partial f}{\partial y} = -\sin x \cdot \sin y.$$

These expressions are both zero, if and only if

$$(x,y) = \left(\frac{\pi}{2} + p\pi, q\pi\right), \qquad p,\, q \in \mathbb{Z},$$

hence

$$(x,y) = \left(p\pi, \frac{\pi}{2} + q\pi\right), \qquad p,\, q \in \mathbb{Z}.$$

These are the stationary points.

Since

$$f\left(\frac{\pi}{2} + p\pi, q\pi\right) = (-1)^p \cdot (-1)^q = (-1)^{p+q} \quad \text{og} \quad f\left(p\pi, \frac{\pi}{2} + q\pi\right) = 0,$$

and $|f(x,y)| \le 1$, it follows immediately that we have maxima at

$$\left(\frac{\pi}{2} + p\pi, q\pi\right) \qquad \text{for } p+q \text{ even},$$

and minima at

$$\left(\frac{\pi}{2} + p\pi, q\pi\right) \qquad \text{for } p+q \text{ odd}.$$

In the neighbourhood of any point of the form $\left(p\pi, \dfrac{\pi}{2} + q\pi\right)$ the function attains both positive and negative values, so these points are not extrema.

9) The equations of the stationary points for $f(x,y) = x^2 + y^2 + e^{xy}$ are

$$\frac{\partial f}{\partial x} = 2x + ye^{xy} = 0, \qquad \frac{\partial f}{\partial y} = 2y + xe^{xy} = 0.$$

These are clearly satisfied for $(x,y) = (0,0)$, so $(0,0)$ is a stationary point.

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Assume that $x \neq 0$. Then also $y \neq 0$, and thus $xy \neq 0$. Accordingly,

$$2x^2 = -xye^{xy} = 2y^2,$$

i.e. $x^2 = y^2$ and $xy \leq 0$, so $y = -x$. If this restriction is put into the first equation, we get

$$0 = 2x - xe^{-x^2} = x\left(2 - e^{-x^2}\right),$$

which has only the solution $x = 0$.

Remark. This is actually a "false solution". On the other hand, we have already checked that $(0,0)$ is a stationary point. $\Diamond$

Hence, $(0,0)$ is the only stationary point.

The problem of a possible extremum at $(0,0)$ can be solved in various ways.

a) It follows from

$$\begin{aligned} f(x,y) &= x^2 + y^2 + e^{xy} \\ &= x^2 + y^2 + 1 + xy + xy\varepsilon(xy) \qquad \text{(Taylor)} \\ &= 1 + \frac{1}{2}\left\{x^2 + y^2 + (x+y)^2\right\} + xy \cdot \varepsilon(x,y) \end{aligned}$$

that $(0,0)$ is a proper minimum

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b) We get from

$$\frac{\partial^2 f}{\partial x^2} = 2 + y^2 e^{xy}, \qquad r = 2,$$

$$\frac{\partial^2 f}{\partial x \partial y} = e^{xy} + xye^{xy}, \qquad s = 1,$$

$$\frac{\partial^2 f}{\partial y^2} = 2 + x^2 e^{xy}, \qquad t = 2,$$

that $rt > s^2$, and since $r > 0$ and $t > 0$, the stationary point $(0,0)$ must be a proper minimum.

10) The equations for the stationary points for $f(x,y) = xy + 2\sinh(1+x^2+y^2)$ are

$$\begin{aligned}\frac{\partial f}{\partial x} &= y + 4x\cosh(1+x^2+y^2) = 0,\\ \frac{\partial f}{\partial y} &= x + 4y\cosh(1+x^2+y^2) = 0.\end{aligned}$$

By inspection, $(0,0)$ is clearly a stationary point. If there were other stationary points they should fulfil $xy \neq 0$. Assume that $(x,y) \neq (0,0)$ is such a stationary point. Then

$$y^2 = -4xy\cosh(1+x^2+y^2) = x^2,$$

so $y^2 = x^2$ and $xy < 0$, thus $y = -x$. By insertion the condition is reduced to

$$0 = -x + 4x\cosh(1+2x^2) = x(4\cosh(1+2x^2) - 1).$$

Now $4\cosh(1+2x^2) - 1 > 0$, so $x = 0$ is the only solution (and strictly speaking we assumed that $x \neq 0$). On the other hand, he have already proved that $(0,0)$ is a stationary point. Accordingly, $(0,0)$ is the only stationary point.

Concerning extremum at $(0,0)$ we have again several possibilities.

a) It follows from

$$\frac{\partial^2 f}{\partial x^2} = 4\cosh(1+x^2+y^2) + x\{\cdots\}, \qquad r = 4\cosh 1,$$

$$\frac{\partial^2 f}{\partial x \partial y} = 1 + x\{\cdots\}, \qquad s = 1,$$

$$\frac{\partial^2 f}{\partial y^2} = 4\cosh(1+x^2+y^2) + y\{\cdots\}, \qquad t = 4\cosh 1,$$

that $rt > s^2$ and $r > 0$ and $t > 0$, corresponding to a minimum.

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b) It is this time more tricky just to inspect the function, though it is still possible:

$$\begin{aligned} f(x,y) &= xy + 2\sinh(1+x^2+y^2) \\ &= xy + 2\sinh 1 \cdot \cosh(x^2+y^2) + 2\cosh 1 \cdot \sinh(x^2+y^2) \\ &= xy + 2\sinh 1 \cdot \left\{1 + (x^2+y^2)\varepsilon(x^2+y^2)\right\} \\ &\qquad +2\cosh 1 \cdot \left\{x^2+y^2+(x^2+y^2)\varepsilon(x^2+y^2)\right\} \\ &= 2\sinh 1 + \frac{1}{2}(x+y)^2 + \left\{2\cosh 1 - \frac{1}{2}\right\}(x^2+y^2) \\ &\qquad +(x^2+y^2)\varepsilon(x^2+y^2). \end{aligned}$$

Since $\cosh 1 - \dfrac{1}{2} > 0$, it follows by this rearrangement that we have a minimum at $(0,0)$.

11) It follows from

$$\begin{aligned} f(x,y) &= x^3y - 2x^2y + xy^3 = xy\left\{x^2-2x+y^2\right\} \\ &= xy\left\{(x-1)^2+y^2-1\right\}, \end{aligned}$$

that the zero curves for $f(x,y)$ are the coordinate axes and the circle of centrum $(1,0)$ and radius 1.

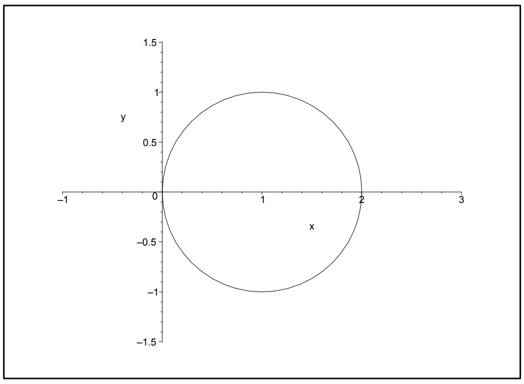

Figure 17.2: The zero curves for $f(x,y)$.

The plane is then divided into six subregions, in which the sign of the function is fixed. Every one of these open subregions has $(0,0)$ as a boundary point, so if one circles around $(0,0)$, then the sign of the function will be positive in every second subregion and negative in the others, because the sign is negative in the second quadrant and positive in the third quadrant.

The equations of the stationary points are

$$\frac{\partial f}{\partial x} = 3x^2y - 4xy + y^3 = y(3x^2-4x+y^2) = 0,$$

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and

$$\frac{\partial f}{\partial y} = x^3 - 2x^2 + 3xy^2 = x(x^2 - 2x + 3y^2) = 0.$$

If we put $y = 0$, then the first equation is fulfilled, and the second one is reduced to

$$0 = x(x^2 - 2x) = x^2(x-2).$$

Therefore, in this case we get the stationary points $(0,0)$ and $(2,0)$.

If instead $y^2 = 4x - 3x^2 = 3x\left(\frac{4}{3} - x\right) \geq 0$, the first equation is again fulfilled. Then note that this implies that $x \in \left[0, \frac{4}{3}\right]$. By insertion into the second equation we get

$$0 = x(x^2 - 2x + 12x - 9x^2) = x^2(10 - 8x) = 8x^2\left(\frac{5}{4} - x\right),$$

where the solutions are $x = 0$ and $x = \frac{5}{4} \in \left[0, \frac{4}{3}\right]$.

When $x = 0$ we get $y = 0$, so we find again the stationary point $(0,0)$.

When $x = \frac{5}{4}$ we get

$$y^2 = 4x - 3x^2 = 5 - \frac{73}{16} = \frac{5}{16}, \quad \text{i.e.} \quad y = \pm\frac{\sqrt{5}}{4},$$

corresponding to the stationary points

$$\left(\frac{5}{4}, \frac{\sqrt{5}}{4}\right) \quad \text{and} \quad \left(\frac{5}{4}, -\frac{\sqrt{5}}{4}\right).$$

Summarizing we have found the four stationary points

$$(0,0), \quad (2,0), \quad \left(\frac{5}{4}, \frac{\sqrt{5}}{4}\right), \quad \left(\frac{5}{4}, -\frac{\sqrt{5}}{4}\right).$$

It follows from the figure that $(0,0)$ and $(2,0)$ both lie on one of the zero curves, so

$$f(0,0) = f(2,0) = 0.$$

Then we conclude from the discussion of the sign that the function is both positive and negative in any neighbourhood of the points $(0,0)$ and $(2,0)$, so these cannot be extrema.

Since f is of class C^∞, and since the two closed half discs are bounded with the value 0 of the function on the boundaries, it follows from the second main theorem that we have a minimum somewhere in the interior of the upper half disc and a maximum somewhere in the interior of the lower half disc. These must necessarily be attained at stationary points. There is exactly one stationary point in each of these half discs, so we conclude that the local minimum is

$$f\left(\frac{5}{4}, \frac{\sqrt{5}}{4}\right) = \frac{5}{4} \cdot \frac{\sqrt{5}}{4}\left\{\frac{25}{16} - \frac{5}{2} + \frac{5}{16}\right\} = \frac{5\sqrt{5}}{16}\left(-\frac{10}{16}\right) = -\frac{25\sqrt{5}}{128},$$

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and the local maximum is

$$f\left(\frac{5}{4}, -\frac{\sqrt{5}}{4}\right) = \frac{25\sqrt{5}}{128}.$$

ALTERNATIVELY one may try the (r, s, t)-method. First calculate

$$\begin{aligned} r &= \frac{\partial^2 f}{\partial x^2} = 6xy - 4y = 2y(3x-2), \\ s &= \frac{\partial^2 f}{\partial x \partial y} = 3x^2 - 4x + 3y^2, \\ t &= \frac{\partial^2 f}{\partial y^2} = 6xy. \end{aligned}$$

For $(0,0)$ we cannot conclude anything, because $r = s = t = 0$.

For $(2,0)$ we get $r = 0$, $s = 12 - 8 = 4$ and $t = 0$, thus $rt < s^2$, and we have no extremum.

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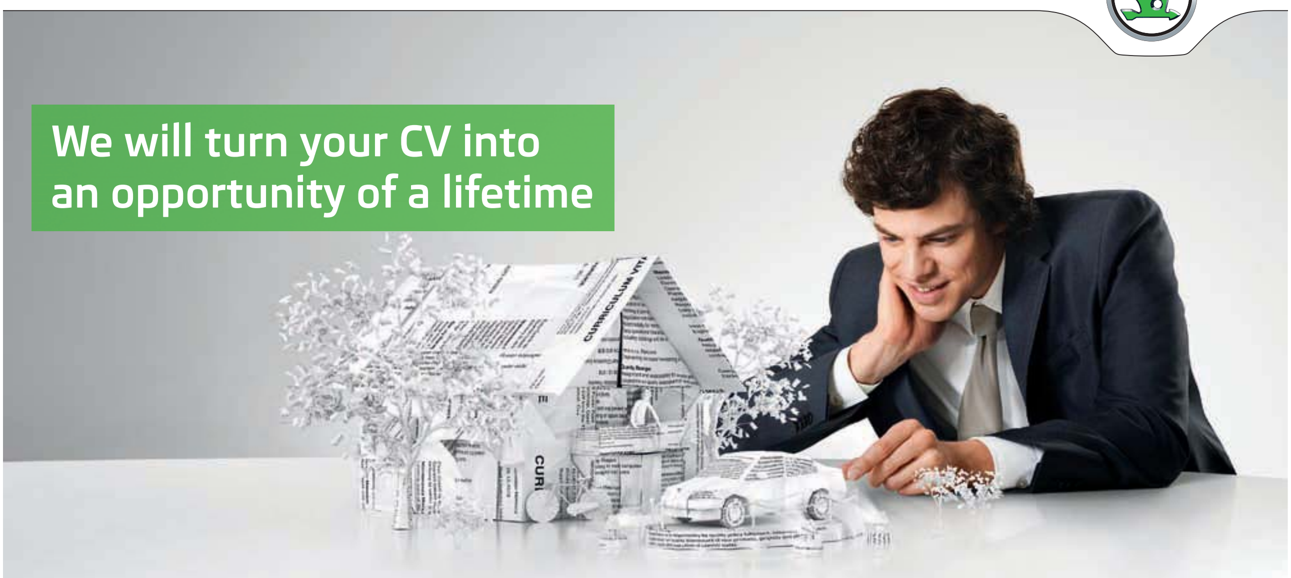

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For $\left(\dfrac{5}{4}, \dfrac{\sqrt{5}}{4}\right)$ we get

$$\begin{aligned} r &= \frac{\sqrt{5}}{4}\left(\frac{15}{4}-2\right) = \frac{7\sqrt{5}}{8}, \\ s &= \frac{75}{16} - 5 + \frac{15}{16} = \frac{5}{8}, \\ t &= 6 \cdot \frac{5}{4} \cdot \frac{\sqrt{5}}{4} = \frac{15\sqrt{5}}{8}. \end{aligned}$$

Since $r > 0$, $t > 0$ and $rt = \dfrac{525}{64} > \dfrac{25}{64} = s^2$, we conclude that we have a proper minimum at the point.

For $\left(\dfrac{5}{4}, -\dfrac{\sqrt{5}}{4}\right)$ we get [cf. the above]

$$r = -\frac{7\sqrt{5}}{8}, \quad s = \frac{5}{8}, \quad t = -\frac{15\sqrt{5}}{8},$$

so $r < 0$, $t < 0$ and $rt > s^2$, and we have a proper maximum at the point.

Finally, let us consider the point $(0,0)$. By using polar coordinates we get

$$\begin{aligned} f(x,y) &= \varrho^4 \cos^3\varphi \sin\varphi - 2\varrho^3 \cos^2\varphi \sin\varphi + \varrho^4 \cos\varphi \cdot \sin^3\varphi \\ &= \varrho^2\{-2\cos^2\varphi\,\sin\varphi + \varphi\cos\varphi\cdot\sin\varphi\}. \end{aligned}$$

When $\varrho \to 0+$, the first term dominates in most cases, and since the first term can take on both positive and negative values, we conclude that $(0,0)$ is not an extremum.

Example 17.2 *Let α be a constant. Check for each value of α if the function*

$$f(x,y) = x^2 + y^2 + \alpha xy + (x-y)^4$$

has an extremum at $(0,0)$. Whenever this is the case, check also if it is a proper extremum.

A Investigation of extrema.

D First method. Rewrite $f(x,y)$ and give a direct argument.

Second method. Apply the (r,s,t)-method.

I As $f(x,y)$ is a polynomial,$f(x,y)$ is of class C^∞. We note that $f(0,0) = 0$.

1) First method. This method is similar to (though not identical with) the procedure of finding the approximating polynomial of second degree. By rewriting the terms of second degree we get

$$\begin{aligned} f(x,y) &= x^2 + y^2 + \alpha xy + (x-y)^4 \\ &= \left\{x^2 + 2x\cdot\frac{\alpha}{2}\,y + \left(\frac{\alpha}{2}\,y\right)^2\right\} + \left\{1 - \frac{\alpha^2}{4}\right\}y^2 + (x-y)^4 \\ &= \left((x + \frac{\alpha}{2}\,y\right)^2 + \left\{1 - \left(\frac{\alpha}{2}\right)^2\right\}y^2 + (x-y)^4. \qquad (17.3) \end{aligned}$$

Then we split the investigation into various cases.

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a) If $|\alpha| < 2$, then all three terms of (17.3) are bigger than or equal to 0, and when $(x, y) \neq (0, 0)$, then at least one of them is bigger than zero. Thus we conclude that $(0, 0)$ is a proper minimum.
b) If $\alpha = 2$, then (17.3) reduces to

$$f(x, y) = (x + y)^2 + (x - y)^4 \quad [\geq 0].$$

If $(x, y) \neq (0, 0)$, then $x + y$ and $x - y$ cannot both be 0 and we conclude as above that we have a proper minimum at $(0, 0)$.
c) If $\alpha = -2$ (the difficult case) we write (17.3) in the form

$$f(x, y) = (x - y)^2 + (x - y)^4 = (x - y)^2\{1 + (x - y)^2\} \quad [\geq 0].$$

It follows that the minimum 0 is attained on the line $y = x$, so we conclude that we have a minimum, but not a proper minimum at $(0, 0)$.
d) If $|\alpha| > 2$, then $0 > 1 - \left(\frac{\alpha}{2}\right)^2 = -\beta^2$, where $\beta > 0$, and the function can be written

$$f(x, y) = \left(x + \frac{\alpha}{2}\, y\right)^2 - (\beta y)^2 + (x - y)^4,$$

and the approximating polynomial of second degree is

$$P_2(x, y) = \left(x + \frac{\alpha}{2}\, y\right)^2 - (\beta y)^2.$$

Since

$$P_2\left(x, -\frac{2}{\alpha}\, x\right) = 0 - \frac{4\beta^2}{\alpha^2}\, x^2 = -\left(\frac{2\beta x}{\alpha}\right)^2 < 0 \quad \text{for } x \neq 0,$$

and

$$P_2(x, 0) = x^2 > 0 \qquad \text{for } x \neq 0,$$

[in fact also $f(x, 0) > 0$ for $x \neq 0$], we conclude that $f(x, y)$ attains both positive and negative values in any neighbourhood of $(0, 0)$, and we do not have an extremum.

Summarizing we have

$\alpha < -2:$ $(0, 0)$ is not an extremum,

$\alpha = -2:$ $(0, 0)$ is a (non-proper) minimum,
$\alpha \in\,]-2, 2]:$ $(0, 0)$ is a proper minimum,

$\alpha > 2:$ $(0, 0)$ is not an extremum.

2) SECOND METHOD, the (r, s, t)-method. First note that the (r, s, t)-method can *only/* be applied if $f \in C^2$ and if $(0, 0)$ *indeed is* a stationary point. Therefore, we must first show that $(0, 0)$ is a stationary point.

It follows from

$$f(x, y) = x^2 + y^2 + \alpha xy + (x - y)^4, \qquad\qquad f(0, 0) = 0,$$

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that

$$\frac{\partial f}{\partial x} = 2x + \alpha y + 4(x-y)^3, \qquad \frac{\partial f}{\partial x}(0,0) = 0,$$

$$\frac{\partial f}{\partial y} = 2y + \alpha x - 4(x-y)^3, \qquad \frac{\partial f}{\partial y}(0,0) = 0,$$

so $(0,0)$ *is* a stationary point.

Furthermore,

$$\frac{\partial^2 f}{\partial x^2} = 2 + 12(x-y)^2, \qquad r = \frac{\partial^2 f}{\partial x^2}(0,0) = 2,$$

$$\frac{\partial^2 f}{\partial x \partial y} = \alpha - 12(x-y)^2, \qquad s = \frac{\partial^2 f}{\partial x \partial y}(0,0) = \alpha,$$

$$\frac{\partial^2 f}{\partial y^2} = 2 + 12(x-y)^2, \qquad t = \frac{\partial^2 f}{\partial y^2}(0,0) = 2.$$

A *sufficient* condition for extremum is that $rt > s^2$, i.e. $\alpha < 4$, which is fulfilled for $\alpha \in]-2,2[$. Since r, $t > 0$, the stationary point $(0,0)$ is a proper minimum, when $\alpha \in]-2,2[$.

If $|\alpha| > 2$, then $rt = 4 < \alpha^2 = s^2$, and we do *not* have an extremum in this case.

If $\alpha^2 = 4$, i.e. $\alpha = \pm 2$, then $s^2 = rt$, and we cannot conclude anything by the (r,s,t)-method.

Note that even if we get this negative result, there is no reason to stop. We have only demonstrated that one particular *method* does not work. Let us now investigate each of the cases $\alpha = 2$ and $\alpha = -2$ one by one.

a) If $\alpha = 2$, then $f(x,y)$ is written

$$f(x,y) = x^2 + y^2 + 2xy + (x-y)^4 = (x+y)^2 + (x-y)^4 \geq 0.$$

When $(x,y) \neq (0,0)$, at least one of the terms $(x+y)^2$ is $(x-y)^4$ positive, and we conclude that $(0,0)$ is a proper minimum.

b) Analogously, if $\alpha = -2$,

$$f(x,y) = x^2 + y^2 - 2xy + (x-y)^4 = (x-y)^2 + (x-y)^4 = (x-y)^2\{1 + (x-y)^2\}.$$

We conclude that $f(x,y) \geq 0$ and $f(x,x) = 0$ for every x, i.e. on the line. This shows that $(0,0)$ is a non-proper minimum. Summarizing we get

$|\alpha| > 2:$ $(0,0)$ is not an extremum.

$|\alpha| < 2:$ $(0,0)$ is a proper minimum,

$\alpha = 2:$ $(0,0)$ is a proper minimum,

$\alpha = -2:$ $(0,0)$ is a (non-proper) minimum.

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Example 17.3 *Let the function* $f : \mathbb{R}^2 \to \mathbb{R}$ *be given by*

$$f(x, y) = 3x^4 - 4x^2y + y^2.$$

Show that the restriction of f *to any straight line through* $(0, 0)$ *has a proper minimum at the point* $(0, 0)$. *Then prove that* f *does not have a minimum at this point.*
(Write $f(x, y)$ *as a product of two factors and check the sign of* f *in discs with centrum at the origo).*

A Extremum.

D Insert $x = 0$ and $y = \alpha x$ into $f(x, y)$, and prove that there is a minimum at $(0, 0)$ on each of these lines.

Consider possibly $f(x, y)$ as a polynomial in y of degree 2 for every x which will help to factorize $f(x, y)$. Use this factorization to discuss the sign of $f(x, y)$.

I 1) When $x = 0$ we get $f(0, y) = y^2$, which clearly has a minimum for $y = 0$, i.e. at $(0, 0)$.

2) When we restrict ourselves to the line $y = \alpha x$ we get

$$\begin{aligned} f(x, \alpha x) &= 3x^4 - 4\alpha x^3 + \alpha^2 x^2 = x^2(\alpha^2 - 4\alpha x + 3x^2) \\ &= x^2\{(\alpha - 2x)^2 - x^2\} = x^2(\alpha - x)(\alpha - 3x). \end{aligned}$$

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If $\alpha \neq 0$, then $(\alpha - x)(\alpha - 3x) > 0$ for $|x| < \left|\frac{\alpha}{3}\right|$, i.e.

$$f(x, \alpha x) > 0 \qquad \text{for } 0 < |x| < \frac{1}{3}|x|,$$

and $f(0,0) = 0$, proving that $(0,0)$ is a minimum.

If $\alpha = 0$, then $f(x,0) = 3x^4$, which clearly has a minimum for $x = 0$, i.e. at the point $(0,0)$.

Summarizing we get that the restriction of $f(x,y)$ to any straight line through $(0,0)$ has a minimum at $(0,0)$.

3) If we consider $f(x,y) = y^2 - 4x^2y + 3x^4$ as a polynomial of second degree in y for every fixed x, we get the roots

$$y = x^2 \qquad \text{and} \qquad y = 3x^2.$$

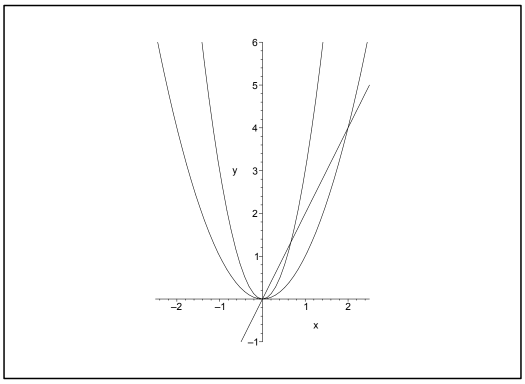

Figure 17.3: The function is positive inside both parabolas, and outside both parabolas, and negative between them.

Then we conclude that we have the factorization

$$f(x,y) = 3x^4 - 4x^2y + y^2 = (y - x^2)(y - 3x^2).$$

When we sketch the zero curves, the plane is divided as shown on the figure into four regions, where $f(x,y) > 0$ inside both parabolas or outside both parabolas, and $f(x,y) < 0$ between the two parabolas.

It follows from the figure that $f(x,y)$ is both positive and negative in any neighbourhood of the point $(0,0)$, where $f(0,0) = 0$.

Remark. When we consider the line on the figure we see that it will always lie totally in a "positive" region, when we are close to $(0,0)$. The trick of the example is that the zero curves are so "flat" in the neighbourhood of $(0,0)$ that they cannot be "caught" by any straight line. $\Diamond$

Alternatively we see that if we take the restriction to the curve given by the equation $y = 2x^2$, then

$$f(x, 2x^2) = 3x^4 - 8x^4 + 4x^4 = -x^4,$$

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which obviously has a (local) *maximum* for $x = 0$, i.e. at $(0,0)$, so $(0,0)$ cannot be an extremum. The construction of this alternative solution relies on that we can choose a curve in the "negative" region between the two zero curves approaching $(0,0)$.

Example 17.4 *Let the function* $f : \mathbb{R}^2 \to \mathbb{R}$ *be given by*

$$f(x,y) = -x^2 + 2xy^2 - y^4 + y^5.$$

Show that the restriction of f *to any straight line through* $(0,0)$ *has a proper maximum at this point. Then prove that* f *does not have an extremum at this point.*
(Find a restriction $f(g(y),y)$*, such that only the latter term on the right hand side remains).*

A Extremum.

D Insert $x = 0$, $y = 0$ and $y = \alpha x$ i $f(x,y)$. Then exploit that $-x^2 + 2xy^2 - y^4 = -(x-y^2)^2$.

I Clearly, $f(0,0) = 0$.

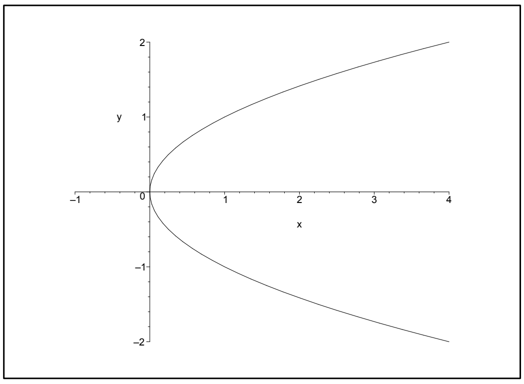

Figure 17.4: The parabola $x = y^2$.

If $x = 0$, then $f(0,y) = -y^4 + y^5 = -y^4(1-y)$, which clearly has a maximum for $y = 0$, i.e. at$(0,0)$.

If $y = 0$, then $f(x,0) = -x^2$, which clearly has a maximum for $x = 0$, i.e. at $(0,0)$.

If $y = \alpha x$, $\alpha \neq 0$, then

$$f(x,\alpha x) = -x^2 + 2\alpha^2 x^3 - \alpha^4 x^4 + \alpha^5 x^5 = -x^2 \left\{1 - 2\alpha^2 x + \alpha^4 x^2 - \alpha^5 x^3\right\}.$$

When $|x|$ is small, then the latter factor is positive, and it follows that $f(x,\alpha x)$ has a maximum for $x = 0$.

Finally, we can write $f(x,y)$ in the form

$$f(x,y) = -(x^2 - 2xy^2 + y^4) + y^5 = -(x-y^2)^2 + y^5.$$

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When we restrict ourselves to the parabola $x = y^2$, $y \in \mathbb{R}$, then

$$f(y^2, y) = y^2$$

has the same sign as y, so $f(x, y)$ attains both positive and negative values in any neighbourhood of $(0, 0)$. It follows that $(0, 0)$ is not an extremum.

Example 17.5 *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be given by*

$$f(x, y) = x^2y + xy^2 + x^2 + xy + 1.$$

check if f *has an extremum at* $(0, 0)$*.*

A Extremum.

D Here we have several possible methods:

1) First variant. The (r, s, t)-method.
2) Second variant. Factorize $x^2 + xy^2 + x^2 + xy$, and discuss the sign of it.
3) Third variant. Put $y = \alpha x$ into $f(x, y)$ for some choice of α.

I 1) First variant. This time (almost an exception) the (r, s, t)-method is the simplest one, so we shall start with it. From

$$\frac{\partial f}{\partial x} = 2xy + y^2 + 2x + y, \qquad \frac{\partial f}{\partial y} = x^2 + 2xy + x,$$

follows

$$\frac{\partial f}{\partial x}(0, 0) = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial y}(0, 0) = 0,$$

so $(0, 0)$ *is* a stationary point. (Note that this *must* be checked before we continue with the (r, s, t)-method).

Then it is quite easy to see that we have at $(0, 0)$

$$r = 2, \qquad s = 1 \quad \text{and} \quad t = 0,$$

so $rt = 0 < 1 = s^2$. We conclude that there is no extremum at $(0, 0)$.

2) Second variant. We get by a factorization,

$$f(x, y) = x^2y + xy^2 + x^2 + x + 1 = xy(x + y) + x(x + y) + 1.$$

The value of the function is $f(0, 0) = 1$, and the "disturbance"

$$(y + 1)x(x + y)$$

has the zero curves as shown on the figure. The sign of the "disturbance" changes whenever one passes a zero curve, so if one moves around $(0, 0)$, one will always pass through both positive and negative regions for $f(x, y) - f(0, 0)$. This means that $f(x, y)$ cannot have an extremum at $(0, 0)$, because $f(x, y)$ attains both values $> f(0, 0)$ and $< f(0, 0)$ in any neighbourhood of $(0, 0)$.

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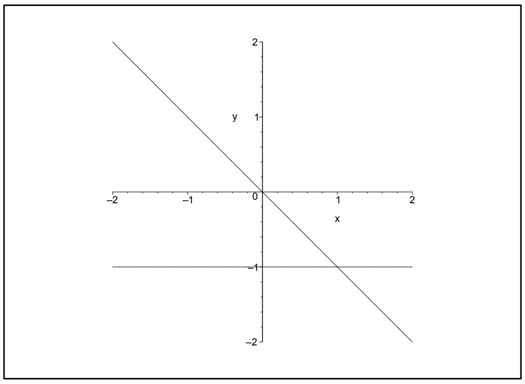


Figure 17.5: The zero curves are the lines $x+y=0$, $x=0$ and $y=-1$. In the subregions the product $(y+1)x(x+y)$ is positive and negative every second time.

3) Third variant Along the line $y = \alpha x$ we get

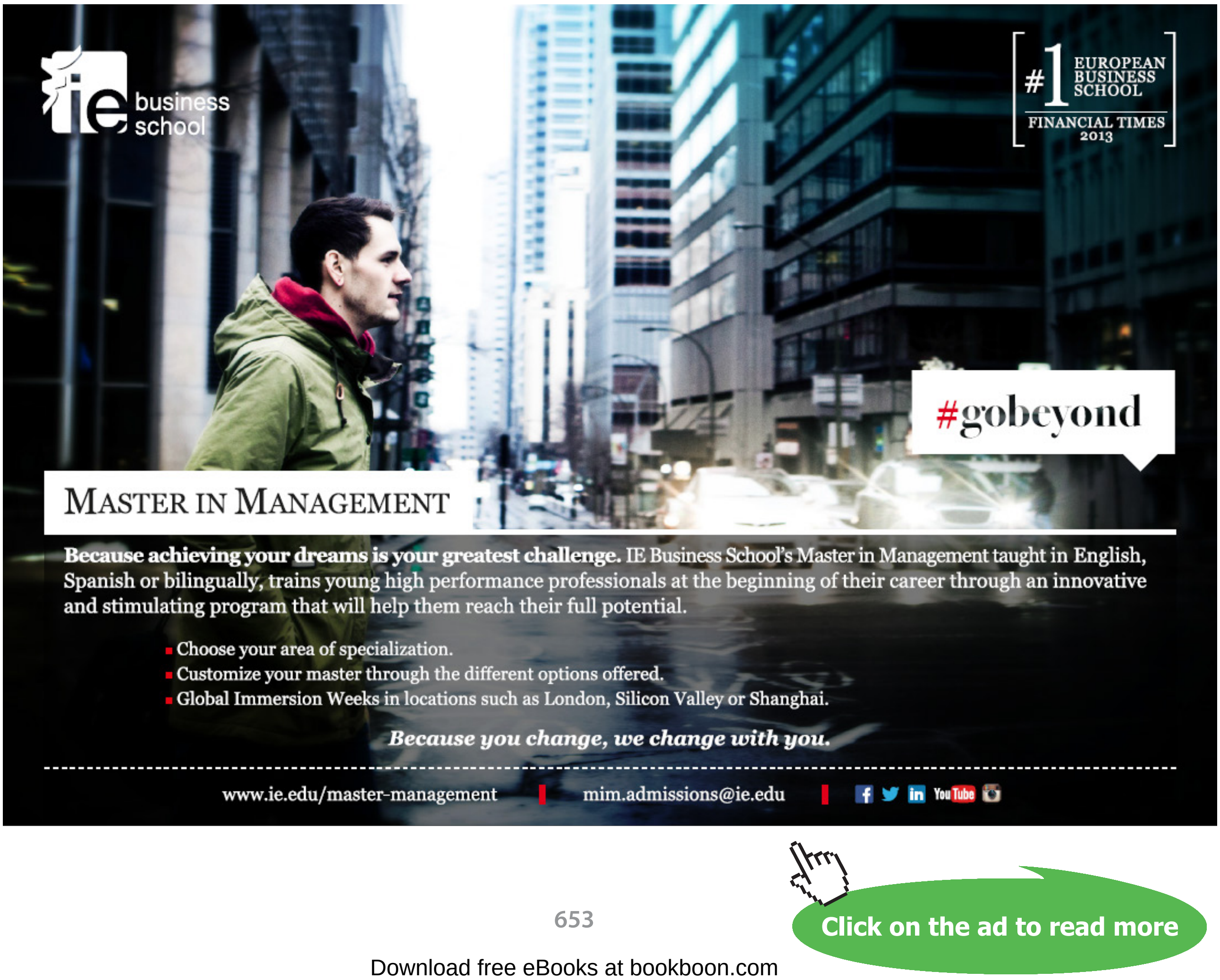
$$\varphi_\alpha(x) = f(x, \alpha x) = (\alpha + \alpha^2)x^3 + (1+\alpha)x^2 + 1 = 1 + (1+\alpha)x^2 + \cdots,$$

where the dots indicate terms of higher degree which are small in a neighbourhood of $(0,0)$ compared to x^2.

a) If $\alpha > -1$, then $\varphi_\alpha(x) \geq 1$ in a neighbourhood of $x = 0$.

b) If $\alpha < -1$, then $\varphi_\alpha(x) \leq 1$ in a neighbourhood of $x = 0$.

In both cases we have a strict inequality sign in a dotted neighbourhood, and we conclude that there is no extremum at $(0,0)$.



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Example 17.6 *Let $\alpha \neq 0$ be a constant. Consider the function*

$$f(x,y) = \alpha^3 xy + \frac{1}{x} + \frac{1}{y}, \qquad xy \neq 0.$$

Find the extremum of the function, and indicate for every value of α the type of the extremum.

A Extremum.

D Find the stationary points, if any, and check if they are extrema.

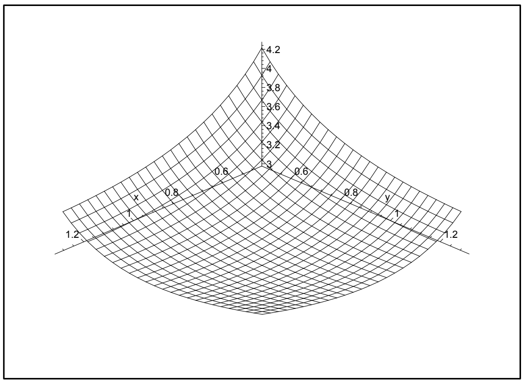

Figure 17.6: The graph of $f(x,y)$ for $\alpha = 1$. It is difficult to see on this figure that we have a minimum at $(1,1)$. The graph of $f(x,y)$ for $\alpha = -1$ gives even less information.

I The coordinates of the possible stationary points are the solutions of the equations

$$\frac{\partial f}{\partial x} = \alpha^3 y - \frac{1}{x^2} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = \alpha^3 x - \frac{1}{y^2} = 0,$$

accordingly,

$$\alpha^3 x^2 y = 1 \qquad \text{and} \qquad \alpha^3 x y^2 = 1 = \alpha^3 x^2 y,$$

so $y = x$.

We get by insertion $x = y = \dfrac{1}{\alpha}$, so there is just one stationary point, $\left(\dfrac{1}{\alpha}, \dfrac{1}{\alpha}\right)$, and the value of the function is here

$$f\left(\frac{1}{\alpha}, \frac{1}{\alpha}\right) = 3\alpha.$$

Furthermore,

$$\frac{\partial^2 f}{\partial x^2} = \frac{2}{x^3}, \quad \frac{\partial^2 f}{\partial x \partial y} = \alpha^3, \quad \frac{\partial^2 f}{\partial y^2} = \frac{2}{y^3},$$

hence

$$r = 2\alpha^3, \qquad s = \alpha^3, \qquad t = 2\alpha^3,$$

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and whence

$$rt - s^2 = 4\alpha^6 - \alpha^6 = 3\alpha^6 > 0,$$

showing that we have an extremum.

1) If $\alpha > 0$, then $r = 2\alpha^3 > 0$ and $t = 2\alpha^3 > 0$, and we have a proper minimum.
2) If instead $\alpha < 0$, we get analogously a proper maximum.

Example 17.7 *Check if the function*

$$f(x,y) = x^3 + xy^2 + 4xy - 3x - 4y, \qquad (x,y) \in \mathbb{R}^2,$$

has an extremum at $(1,0)$.

A Extremum.

D Here we have two variants:

First variant. Show that $(1,0)$ is a stationary point, and then apply the (r,s,t)-method.

Second variant. Translate the coordinate system to $(1,0)$ and argue directly.

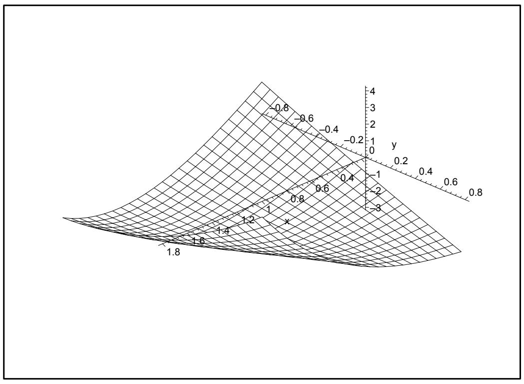

Figure 17.7: The surface in a neighbourhood of $(1,0)$. The figure does not give any hint of the type of the stationary point.

I First variant. It follows from

$$\frac{\partial f}{\partial x} = 3x^2 + y^2 + 4y - 3, \qquad \frac{\partial f}{\partial x}(1,0) = 0,$$
$$\frac{\partial f}{\partial y} = 2xy + 4x - 4, \qquad \frac{\partial f}{\partial y}(1,0) = 0,$$

that $(1,0)$ is a stationary point.

Now,

$$\frac{\partial^2 f}{\partial x^2} = 6x, \qquad \frac{\partial^2 f}{\partial x \partial y} = 2y + 4, \qquad \frac{\partial^2 f}{\partial y^2} = 2x,$$

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so we get at the point $(1,0)$ that

$$r = 6, \qquad s = 4, \qquad t = 2.$$

Since $rt = 12 < 16 = s^2$, there is no extremum at $(1,0)$.

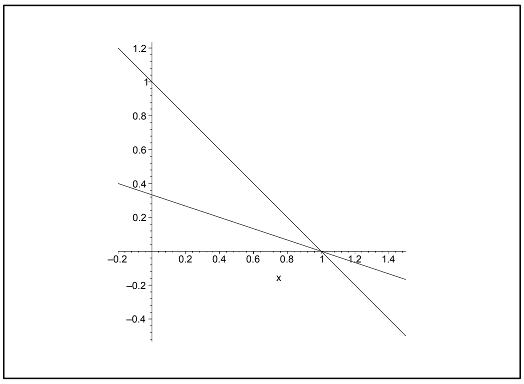

Figure 17.8: The zero curves of $(x+y-1)(x+3y-1)$.

Second variant. If we put $x = t+1$, then we shall check what happens in the neighbourhood of $(t,y) = (0,0)$. A simple computation gives

$$\begin{aligned} f(t+1,y) &= (t+1)^3 + (t+1)y^2 + 4(t+1)y - 3(t+1) - 4y \\ &= t^3 + 3t^2 + 3t + 1 + ty^2 + y^2 + 4ty + 4y - 3t - 3 - 4y \\ &= -2 + 3t^2 + 4ty + y^2 + t(t^2+y^2) \\ &= -2 + (t+y)(t+3y) + t(t^2+y^2). \end{aligned}$$

The term of second order

$$(t+y)(t+3y) = (x+y-1)(x+3y-1)$$

is both positive and negative in any neighbourhood of the point $(x,y) = (1,0)$, and since it in general dominates the term of third order $t(t^2+y^2) = (x-1)\{(x-1)^2+y^2\}$, there is no extremum at $(1,0)$.

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Example 17.8 *Given the function*

$$f(x,y) = e^{2y} + 4e^{y}\sin x, \qquad (x,y) \in \mathbb{R}^2.$$

1) Find the stationary points of f; check if f has a proper extremum.

2) Explain why f has both a maximum S and a minimum M on the rectangle $[0,2\pi]\times[0,1]$; Finally, find S and M.

A Extrema.

D Find the stationary points; check the values on the boundary.

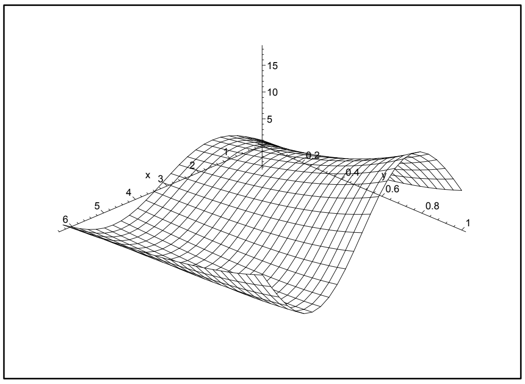

Figure 17.9: The graph of $f(x,t)$ for $(x,y) \in [0,2\pi]\times[0,1]$.

I 1) The equations of the stationary points are

$$\frac{\partial f}{\partial x} = 4e^{y}\cos x = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 2e^{2y} + 4e^{y}\sin x = 0.$$

It follows from the former equation that $\cos x = 0$, thus $x = \dfrac{\pi}{2} + p\pi$, $p \in \mathbb{Z}$.

If $x = \dfrac{\pi}{2} + 2p\pi$, $p \in \mathbb{Z}$, then

$$\frac{\partial f}{\partial y} = 2e^{2y} + 4e^{y} > 0,$$

and these x-values do not correspond to stationary points.

If $x = \dfrac{3\pi}{2} + 2p\pi$, $p \in \mathbb{Z}$, then

$$\frac{\partial f}{\partial y} = 2e^{2y} - 4e^{y} = 2e^{y}(e^{y}-2) = 0$$

for $y = \ln 2$. Thus the stationary points are

$$\left\{ \left(\frac{3\pi}{2} + 2p\pi, \ln 2 \right) \;\middle|\; p \in \mathbb{Z} \right\}.$$

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Furthermore,

$$\frac{\partial^2 f}{\partial x^2} = -4e^y \sin x, \qquad \frac{\partial^2 f}{\partial x \partial y} = 4e^y \cos x, \qquad \frac{\partial^2 f}{\partial y^2} = 4e^{2y} + 4e^y \sin x.$$

At the points $\left(\frac{3\pi}{2} + 2p\pi, \ln 2\right)$ we get $r = 8$, $s = 0$, $t = 16 - 8 = 8$, so $rt - s^2 = 64 > 0$, and we have an extremum. As $r > 0$ and $t > 0$, these are all minima.
The values of the function at these points are

$$f\left(\frac{3\pi}{2} + 2p\pi, \ln 2\right) = 4 - 8 = -4.$$

2) Since f is continuous (it is of class C^∞), and the rectangle

$$D = [0, 2\pi] \times [0, 1]$$

is closed and bounded (compact), it follows from the second main theorem for continuous functions that f has both a maximum and a minimum in D.

These can only be attained either at a stationary point

$$\left(\frac{3\pi}{2}, \ln 2\right) \in D,$$

or on the boundary.

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a) For the stationary point we get

$$f\left(\frac{3\pi}{2}, \ln 2\right) = -4.$$

b) Taking the restriction of f to the boundary curve $(x, y) = (x, 0)$, $x \in [0, 2\pi]$, we get:

$$\varphi_1(x) = 1 + 4\sin x, \qquad x \in [0, 2\pi],$$

with the maximum and minimum, respectively,

$$\varphi_1\left(\frac{\pi}{2}\right) = 1 + 4 = 5, \qquad \varphi_1\left(\frac{3\pi}{2}\right) = 1 - 4 = -3.$$

c) On the boundary curve $(x, y) = (0, y)$, $y \in [0, 1]$, or the boundary curve $(x, y) = (2\pi, y)$, $y \in [0, 1]$, we get the restriction of f:

$$\varphi_2(y) = e^{2y}, \qquad y \in [0, 1],$$

with the maximum and minimum, respectively,

$$\varphi_2(1) = e^2, \qquad \varphi_2(0) = 1.$$

d) On the boundary curve $(x, y) = (x, 1)$, $x \in [0, 2\pi]$, we get the restriction of f:

$$\varphi_3(x) = e^2 + 4e\,\sin x, \qquad x \in [0, 2\pi],$$

with the maximum and minimum, respectively,

$$\varphi_3\left(\frac{\pi}{2}\right) = e^2 + 4y = (e + 2)^2 - 4 > 10 > e^2$$

and

$$\varphi_3\left(\frac{3\pi}{2}\right) = e^2 - 4e = -e(4 - e) = (e - 2)^2 - 4 > -4.$$

Finally, by a numerical comparison of

a)

$$f\left(\frac{3\pi}{2}, \ln 2\right) = -4,$$

b)

$$f\left(\frac{\pi}{2}, 0\right) = 5, \qquad f\left(\frac{3\pi}{2}, 0\right) = -3,$$

c)

$$f(0, 1) = f(2\pi, 1) = e^2, \qquad f(0, 0) = f(2\pi, 0) = 1,$$

d)

$$f\left(\frac{\pi}{2}, 1\right) = e^2 + 4e > e^2, \qquad f\left(\frac{3\pi}{2}, 1\right) = e^2 - 4e > -4,$$

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we conclude that the maximum is

$$S = f\left(\frac{\pi}{2}, 1\right) = e^2 + 4e,$$

and the minimum is

$$M = f\left(\frac{3\pi}{2}, \ln 2\right) = -4.$$

Example 17.9 *Check if the function*

$$f(x,y) = 2\cosh(x+y) - e^{xy}, \qquad (x,y) \in \mathbb{R}^2,$$

has an extremum and $(0,0)$*, and indicate its type if there is an extremum.*

A Extremum.

D Either use known series expansions, or compute the Taylor coefficients.

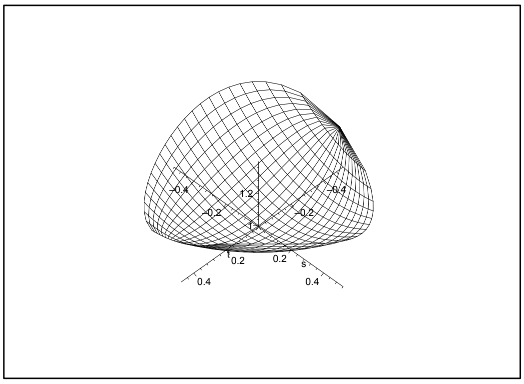

Figure 17.10: The graph in a neighbourhood of the point $(0,0)$.

I First method. We get by well-known series expansions from $(0,0)$,

$$\begin{aligned} f(x,y) &= 2\cosh(x+y) - e^{xy} \\ &= 2\left\{1 + \frac{1}{2}(x+y)^2 + \cdots\right\} - \{1 + xy + \cdots\} \\ &= 1 + (x+y)^2 - xy + \cdots, \end{aligned}$$

so the approximating polynomial of at most second degree is

$$P_2(x,y) = 10x^2 + xy + y^2 = 1 + \left(x + \frac{1}{2}\,y\right)^2 + \frac{3}{4}\,y^2.$$

It follows from the latter expression that $f(x,y)$ has a local minimum at $(0,0)$.

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Second method. As $f \in C^\infty$, we get by computation,

$$f(x,y) = 2\cosh(x+y) - e^{xy}, \qquad f(0,0) = 1,$$

$$f'_x(x,y) = 2\sinh(x+y) - y\,e^{xy}, \quad f'_x(0,0) = 0,$$

$$f'_y(x,y) = 2\sinh(x+y) - x\,e^{xy}, \quad f'_y(0,0) = 0.$$

It follows that $(0,0)$ is a stationary point. Furthermore,

$$f''_{xx}(x,y) = 2\cosh(x+y) - y^2 e^{xy}, \qquad f''_{xx}(0,0) = 2,$$

$$f''_{xy}(x,y) = 2\cosh(x+y) - xye^{xy} - e^{xy}, \quad f''_{xy}(0,0) = 1,$$

$$f''_{yy}(x,y) = 2\cosh(x+y) - x^2 e^{xy}, \qquad f''_{yy}(0,0) = 2.$$

Since $rt = 4 > 1 = s^2$, and $r, t > 0$, we conclude that we have a local minimum.

Remark. Because

$$f(x,x) = 2\cosh(2x) - \exp(x^2) \to -\infty \qquad \text{for } x \to +\infty,$$

the minimum above cannot be global.

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We have e.g.

$$f(x,0) = 2\cosh x - 1 \to +\infty \qquad \text{for } x \to +\infty,$$

and since f is continuous on the connected set $\mathbb{R}^2$, the range is $f(\mathbb{R}^2) = \mathbb{R}$.

Example 17.10 *Given the function*

$$f(x,y) = xy - y^2 - 2\ln x, \qquad (x,y) \in D,$$

where

$$D = \{(x,y) \in \mathbb{R}^2 \mid x > 0\}.$$

1) Find the approximating polynomial of at most second degree of the function f, where $(x_0,y_0) = (2,1)$ is used as point of expansion.

2) Check if the function f has an extremum at the point $(2,1)$, and indicate its type if it is an extremum.

A Approximating polynomials; extremum.

D Apply a series expansion from $(2,1)$, or alternatively the standard method by calculating the Taylor coefficients.

I First variant. Translate the coordinate system by

$$(x,y) = (x_1+2, y_1+1).$$

Then by insertion and a series expansion for $\ln(1+u)$,

$$\begin{aligned}
f(x,y) &= xy - y^2 - 2\ln x \\
\\
&= (x_1+2)(y_1+1) - (y_1+1)^2 - 2\ln(2+x_1) \\
&= 2 + x_1 + 2y_1 + x_1y_1 - y_1^2 - 2y_1 - 1 - 2\ln 2 - 2\ln\left(1 + \frac{x_1}{2}\right) \\
&= 1 - 2\ln 2 + x_1 + x_1y_1 - y_1^2 - 2\left\{\frac{x_1}{2} - \frac{1}{2}\,\frac{x_1 * 2}{4} + \cdots\right\} \\
&= 1 - 2\ln 2 + \left(\frac{x_1}{2}\right)^2 + 2\cdot\frac{x_1}{2}\cdot y_1 + y_1^2 - 2y_1^2 + \cdots \\
&= 1 - 2\ln 2 + \left\{\frac{1}{2}\,x_1 + y_1\right\}^2 - 2y_1^2 + \cdots .
\end{aligned}$$

The approximating polynomial of at most second degree is

$$\begin{aligned}
P_2(x,y) &= 1 - 2\ln 2 + \frac{1}{4}\,(x-2)^2 + (x-2)(y-1) - (y-1)^2 \\
&= 1 - 2\ln 2 + \frac{1}{4}\,\{(x-2) + 2(y-1)\}^2 - (y-1)^2.
\end{aligned}$$

As the first derivatives are zero, $(2,1)$ is a stationary point. It follows from the terms of second degree that $(2,1)$ is not an extremum.

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Second variant. By straightforward computations,

$$f(x,y) = xy - y^2 - 2\ln x, \quad f(2,1) = 1 - 2\ln 2,$$

$$f'_x(x,y) = y - \frac{2}{x}, \qquad f'_x(2,1) = 0,$$

$$f'_y(x,y) = x - 2y, \qquad f'_y(2,1) = 0,$$

$$f''_{xx}(x,y) = \frac{2}{x^2}, \qquad r = f''_{xx}(2,1) = \frac{1}{2},$$

$$f''_{xy}(x,y) = 1, \qquad s = f''_{xy}(2,1) = 1,$$

$$f''_{yy}(x,y) = -2, \qquad t = f''_{yy}(2,1) = -2.$$

The approximating polynomial of at most second degree is

$$P_2(x,y) = 1 - 2\ln 2 + \frac{1}{4}(x-2)^2 + (x-2)(y-1) - (y-1)^2.$$

As

$$f'_x(2,1) = f'_y(2,1) = 0,$$

we see that $(2,1)$ is a stationary point.

As $rt < 0$, there is no extremum at $(2,1)$.

Example 17.11 *Given the function*

$$f(x,y) = 2x^3 + 27x^2 - 60xy + 75y^2, \qquad (x,y) \in \mathbb{R}^2.$$

1) Find the stationary points of the function.

2) Check for each of the stationary points if f has an extremum; when this is the case one should indicate its type.

3) Find the range $f(\mathbb{R}^2)$ of the function.

A Stationary points; extremum; range.

D Solve the equations of the stationary points. Check the behaviour of the function in a neighbourhood of the stationary points. Finally, consider the restriction of f to the X axis.

I 1) The equations of the stationary points are

$$\begin{cases} \dfrac{\partial f}{\partial x} = 6x^2 + 54x - 60y = 0, & \text{i.e.} \quad x^2 + 9x - 10y = 0, \\[2mm] \dfrac{\partial f}{\partial y} = -60x + 150y = 0, & \text{i.e.} \quad -4x + 10y = 0. \end{cases}$$

We get by an addition that $x^2 + 5x = 0$, so either $x = 0$ or $x = -5$. By insertion into the latter equation we obtain the points $(0,0)$ and $(-5,-2)$. These are also satisfying the former equation, so the stationary points are $(0,0)$ and $(-5,-2)$.

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2) a) The value at $(0,0)$ is $f(0,0)=0$, and the approximating polynomial of at most second degree is

$$\begin{aligned} P_2(x,y) &= 27x^2-60xy+75y^2 \\ &= 2x^2+25x^2-60xy+36y^2+39y^2 \\ &= 2x^2+(5x-6y)^2+39y^2>0 \qquad \text{for } (x,y)\neq(0,0), \end{aligned}$$

hence $f(x,y)$ has a local minimum at $(0,0)$.

ALTERNATIVELY,

$$f''_{xx}(x,y)=12x+54, \quad r=f''_{xx}(0,0)=54,$$

$$f''_{xy}(x,y)=-60, \qquad s=f''_{xy}(0,0)=-60,$$

$$f''_{yy}(x,y)=150, \qquad t=f''_{yy}(0,0)=150.$$

We conclude from $rt=54\cdot 150>(-60)^2=s^2$, and $r,\ t>0$, that $f(x,y)$ has a proper minimum at $(0,0)$.

b) At $(-5,-2)$ we get

$$r=12\cdot(-5)+54=-6, \qquad s=-60, \qquad t=150.$$

It follows from $rt<0<s^2$ that there is no extremum at $(-5,-2)$.

ALTERNATIVELY we put $(x,y)=(h-5,k-2)$, from which

$$\begin{aligned} f(x,y) &= 2x^3+27x^2-60xy+75y^2 \\ &= 2(h-5)^3+27(h-5)^2-60(h-5)(k-2)+75(k-2)^2 \\ &= 2\left\{h^3-15h^2+75h-125\right\}+27\{h^2-10h+25\} \\ &\qquad -60\{hk-2h-5k+10\}+75\left\{k^2-4k+4\right\} \\ &= 2h^3+\left\{-3h^2-60hk+75k^2\right\}+125. \end{aligned}$$

The approximating polynomial of at most second degree in (h,k) is

$$P_2(h,k)=-3h^2-60hk+75k^2+125.$$

Since this expression attains values both $>$ and $<$ 125 in any neighbourhood of $(h,k)=(0,0)$, it follows that $(-5,-2)$ is not an extremum.

3) By taking the restriction of f to the X axis we get

$$\varphi(x)=f(x,0)=2x^3+27x^2=x^2\{2x+27\}, \qquad x\in\mathbb{R}.$$

Since already $\varphi(\mathbb{R})=\mathbb{R}$, we conclude that

$$f(\mathbb{R}^2)=\mathbb{R}.$$

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Example 17.12 *Given the point set*

$$A = \{(x,y) \in \mathbb{R}^2 \mid -4 \le x \le 4,\, 16 - 6y \le x^2 + y^2 \le 16\}.$$

1. *Sketch A, and show that the boundary ∂A consists of two circular arcs.*

We shall also consider the function

$$f(x,y) = \frac{y+8}{x^2+y^2+12}, \qquad (x,y) \in A.$$

2. *Show that the function f does not have a stationary point in the interior of A.*

3. *Find the range $f(A)$ of the function.*

A Range of a continuous function over a closed and bounded (i.e. compact) and connected set.

D Follow the guidelines and apply the main theorems.

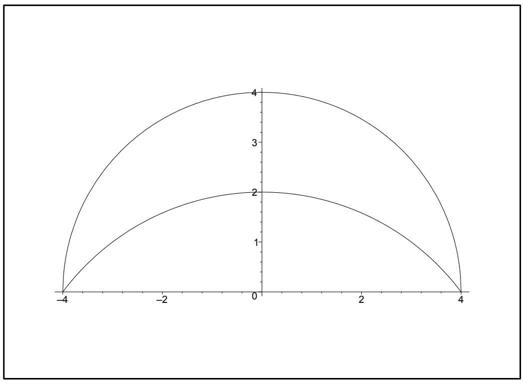

Figure 17.11: The domain A.

I 1) It follows immediately from $x^2 + y^2 \le 16 = 4^2$ that A lies inside the disc of centrum $(0,0)$ and radius 4.

We rearrange $16 - 6y \le x^2 + y^2$ as

$$25 = x^2 + y^2 + 6y + 9 = z^2 + (y+3)^2.$$

Then we can see that A lies outside the disc of centrum $(0,-3)$ and radius 5.

The domain is the half moon shaped region on the figure. It is closed and bounded and connected. The boundary clearly consists of two circular arcs which intersect at the points $(-4,0)$ and $(4,0)$.

2) The stationary points, if any, shall fulfil the equations

$$\frac{\partial f}{\partial x} = -\frac{2x(y+8)}{(x^2+y^2+12)^2} = 0, \quad \text{i.e. } x = 0 \text{ or } y = -8,$$

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$$\frac{\partial f}{\partial y} = \frac{(x^2+y^2+12)\cdot 1 - 2y(y+8)}{(x^2+y^2 12)^2} = \frac{x^2-y^2-16y+12}{(x^2+y^2+12)^2} = 0.$$

Here $y = -8$ is not possible in A, so we get $x = 0$ from the first equation. When this put into the second equation we get after a reduction that

$$y^2 + 16y - 12 = 0, \quad \text{from which} \quad y = -8 \pm \sqrt{76}.$$

The line $x = 0$ intersects A in the interval $[2, 4]$ on the Y axis. Since

$$-8 \pm \sqrt{76} \le -8 + \sqrt{76} < -8 + \sqrt{81} = -8 + 9 = 1 < 2,$$

neither of the two candidates $(0, -8 \pm \sqrt{76})$ (on the Y axis) lie in A, and f does not have a stationary point in A.

3) Since f is continuous on A, and A is closed and bounded, it follows from the second main theorem for continuous functions that the range $f(A)$ is closed and bounded.

Since A is also connected, cf. the figure, it follows from the first main theorem for continuous functions, that the range is connected.

Since $f(A) \subset \mathbb{R}$, it follows from the above that $f(A)$ is a closed interval, i.e.

$$f(A) = [M, S],$$

where M denotes the minimum and S denotes the maximum of f in A, because these exist according to the second main theorem.

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The maximum and the minimum must be found among the values at

a) the exceptional points (where f is not differentiable; there are none of them here),
b) the stationary points (none of the either),
c) the points on the boundary.

It follows that both the maximum and the minimum shall be found on the boundary.

Investigation of the boundary. The boundary is naturally split up into

a) $x^2 + y^2 = 16$, where $y \in [0, 4]$, (cf. the figure),
b) $x^2 + (y + 3)^2 = 25$, i.e. $x^2 = 25 - (y + 3)^2$, where $y \in [0, 2]$.

Now, x only occurs in the form x^2 in $f(x, y)$. It is therefore natural to eliminate x^2 and use y as a parameter.

a) The restriction of $f(x, y)$ to $x^2 + y^2 = 16$, $y \in [0, 4]$, is given by

$$g(y) = \frac{y + 8}{28}, \qquad y \in [0+, 4].$$

Clearly, $g(y)$ is increasing in this interval, so the candidates are

$$g(0) = f(\pm 4, 0) = \frac{8}{28} = \frac{2}{7}, \qquad g(4) = f(0, 4) = \frac{12}{28} = \frac{3}{7}.$$

b) The restriction of $f(x, y)$ to $x^2 + (y + 3)^2 = 25$, $y \in [0, 2]$, is

$$h(y) = \frac{y+8}{37+y^2-(y+3)^2} = \frac{y+8}{28-6y}, \qquad y \in [0, 2].$$

Both the numerator and the denominator are positive in the given interval. Furthermore, when y increases, then the numerator increases too, while the denominator decreases. Hence, $h(y)$ is increasing with the values at the end points

$$h(0) = f(\pm 4, 0) = \frac{2}{7}, \qquad h(2) = \frac{10}{16} = \frac{5}{8}.$$

Alternatively,

$$h'(y) = \frac{(28-y^2-6y) \cdot 1-(y+8)(-2y-6)}{(28-y^2-6y)^2} = \frac{y^2+28y+76}{(28-y^2-6y)^2} > 0 \qquad \text{for } y \in [0, 2],$$

hence $h(y)$ is increasing.

By a numerical comparison we get $M = \dfrac{2}{7}$ and $S = \dfrac{5}{8}$, so the range is

$$f(A) = \left[\frac{2}{7}, \frac{5}{8}\right].$$

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17.3 Examples of extremum for three variables

Example 17.13 *Find in each of the following cases the stationary points of the given function*

$$f : \mathbb{R}^3 \to \mathbb{R}.$$

Then check if f in these points has an extremum; whenever this is the case check if it is a maximum or a minimum.

1) $f(x,y,z) = x^2 + y^2 + z^2 + xyz$.

2) $f(x,y,z) = x^3 + y^3 + z^3 + xyz$.

3) $f(x,y,z) = x^4 + y^4 + z^4 - 4xyz$.

4) $f(x,y,z) = x\cos z + y^2$.

5) $f(x,y,z) = \exp(xy + yz + zx)$.

6) $f(x,y,z) = y^3 + \ln(1 + x^2 + z^2)$.

A Stationary points; extrema in three variables.

D Compute $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$, $\dfrac{\partial f}{\partial z}$; then find the stationary points; finally check if there are any extrema.

I 1) The equations of the stationary points are

$$\begin{cases} \dfrac{\partial f}{\partial x} = 2x + yz = 0, \\[2mm] \dfrac{\partial f}{\partial y} = 2y + xz = 0, \\[2mm] \dfrac{\partial f}{\partial z} = 2z + xy = 0, \end{cases}$$

i.e.

$$(17.4) \quad \begin{cases} x = -\dfrac{yz}{2}, \\[2mm] y = -\dfrac{xz}{2}, \\[2mm] z = -\dfrac{xy}{2}. \end{cases}$$

When we multiply the equations of (17.4) we get a *necessary condition* of stationary points,

$$xyz = -\frac{1}{8}(xyz)^2, \qquad \text{i.e.} \qquad xyz\{xyz + 8\} = 0.$$

Then either $xyz = 0$ or $xyz = -8 = (-2)^3$.

a) If $xyz = 0$, then one of the factors must be 0. Assume that $x = 0$. Then it follows from (17.4) that $y = z = 0$.
Analogously, if we assume that $y = 0$ or $z = 0$.

Summarizing we get in this case that $(0,0,0)$ is a stationary point.

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b) If $xyz \neq 0$, then all three factors are $\neq 0$. By insertion of the latter equation into (17.4), we get

$$y = -\frac{1}{2}\,xz = +\frac{1}{4}\,x^2 y,$$

hence $x^2 = 4$.

Analogously we get $y^2 = 4$ and $z^2 = 4$, so the candidates shall be found among $(\pm 2, \pm 2, \pm 2)$ with all possible combinations of the signs. By a simple test in (17.4) we see that we in this case get the stationary points

$$(2,2,-2), \quad (2,-2,2), \quad (-2,2,2), \quad (-2,-2,-2).$$

Summarizing we have the five stationary points

$$(0,0,0), \quad (2,2,-2), \quad (2,-2,2), \quad (-2,2,2), \quad (-2,-2,-2).$$

i) The point $(0,0,0)$ is a proper minimum, because

$$P_2(x,y,z) = x^2 + y^2 + z^2$$

is positive in any dotted neighbourhood of $(0,0,0)$.

Insertion. Note that

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial z^2} = 2,$$

$$\frac{\partial^2 f}{\partial x \partial y} = z, \qquad \frac{\partial^2 f}{\partial y \partial z} = x, \qquad \frac{\partial^2 f}{\partial z \partial x} = y,$$

so the approximating polynomial $P_2(x,y,z)$ from an expansion point (x_0,y_0,z_0) is

$$\begin{aligned} P_2(x,y,z) &= f(x_0,y_0,z_0) + (x-x_0)^2 + (y-y_0)^2 + (x-z_0)^2 \\ &\quad + z_0(x-x_0)(y-y_0) + x_0(y-y_0)(z-z_0) \\ &\quad + y_0(z-z_0)(x-x_0). \end{aligned}$$

When $|x_0| = |y_0| = |z_0| = 2$ and $x_0 y_0 z_0 = -8$, then either one or three of the factors are negative. $\Diamond$

ii) Assume that only one of the factors is negative. Due to the symmetry we can assume that $z_0 = -2$, hence $x_0 = y_0 = 2$. Then

$$\begin{aligned} P_2(x,y,z) &= f(x_0,y_0,z_0) + (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 \\ &\quad + 2(x-x_0)(y-y_0) + 2(y-y_0)(z-z_0) \\ &\quad + 2(x-x_0)(z-z_0) - 4(x-x_0)(y-y_0) \\ &= f(x_0,y_0,z_0) + \{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)\}^2 \\ &\qquad -4(x-x_0)(y-y_0). \end{aligned}$$

In the plane $z - z_0 = -(x-x_0) - (y-y_0)$ the term

$$-4(x-x_0)(y-y_0)$$

is both positive and negative in any neighbourhood of (x_0,y_0), so $(x_0,y_0,z_0) = (2,2,-2)$ is not an extremum.

It follows from the symmetry that neither $(2,-2,2)$ nor $(-2,2,2)$ are extrema.

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iii) If $(x_0, y_0, z_0) = (-2, -2, -2)$, then

$$\begin{aligned} P_2(x,y,z) &= f(-2,-2,-2) + (x+2)^2 + (y+2)^2 + (z+2)^2 \\ &\quad -2(x+2)(y+2) - 2(y+2)(z+2) - 2(z+2)(x+2) \\ &= f(-2,-2,-2) + (x+2)^2 + (y+2)^2 + (z+2)^2 \\ &\quad -2(x+2)(y+2) + 2(y+2)(z+2) - 2(z+2)(x+2) - 4(y+2)(z+2) \\ &= f(-2,-2,-2) + \{-(x+2) + (y+2) + (z+2)\}^2 - 4(y+2)(z+2). \end{aligned}$$

We see that in the plane $x+2 = (y+2)+(z+2)$ the term $-4(y+2)(z+2)$ is both positive and negative in any neighbourhood of $(y,z) = (-2,-2)$, so $(-2,-2,-2)$ is not an extremum.

The conclusion is that only $(0,0,0)$ is an extremum (a proper minimum).

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2) In this case,

$$\begin{cases} \dfrac{\partial f}{\partial x} = 3x^2 + yz = 0, \\[2mm] \dfrac{\partial f}{\partial y} = 3y^2 + xz = 0, \\[2mm] \dfrac{\partial f}{\partial z} = 3z^2 + xy = 0, \end{cases}$$

i.e.

$$(17.5) \quad \begin{cases} yz = -3x^2 \le 0, \\[2mm] xz = -3y^2 \le 0, \\[2mm] xy = -3z^2 \le 0. \end{cases}$$

From (17.5) we get the *necessary condition*

$$(yz) \cdot (zx) \cdot (xy) = (xyz)^2 = -27(xyz)^2$$

for a stationary point. The only possibility is $xyz = 0$. Since e.g. $x = 0$ implies that $y = z = 0$, and analogously for $y = 0$ and $z = 0$, it follows that $(0, 0, 0)$ is the only stationary point.

There is no extremum at$(0, 0, 0)$, because e.g. $f(x, 0, 0) = x^3$ is both positive and negative in any neighbourhood of $x_0 = 0$.

3) Here

$$\begin{cases} \dfrac{\partial f}{\partial x} = 4x^3 - 4yz = 0, \\[2mm] \dfrac{\partial f}{\partial y} = 4y^3 - 4xz = 0, \\[2mm] \dfrac{\partial f}{\partial z} = 4z^3 - 4xyz = 0, \end{cases}$$

so

$$(17.6) \quad \begin{cases} yz = x^3, \\ xz = y^3, \\ xy = z^3. \end{cases}$$

We get the following *necessary condition* for the stationary points

$$(yz) \cdot (xz) \cdot (xy) = (xyz)^2 = (xyz)^3,$$

i.e. either $xyz = 0$ or $xyz = 1$.

a) If $xyz = 0$, then e.g. $x = 0$, which immediately implies that $y = z = 0$. Analogously, if we assume $y = 0$ or $z = 0$. In this case we get the stationary point $(0, 0, 0)$.

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b) If $xyz = 1$, it follows from the first equation of (17.6) that

$$xyz = 1 = x^4,$$

i.e. $x = \pm 1$. Analogously we get $y = \pm 1$ and $z = \pm 1$. Therefore, the stationary points should be searched among $(\pm 1, \pm 1, \pm 1)$ with all possible eight combinations of the signs. By insertion into (17.6), i.e. testing these points, we find the stationary points

$$(1,1,1), \quad (1,-1,1), \quad (-1,1,-1), \qquad (-1,-1,1).$$

Summarizing we find that the function has five stationary points,

$$(0,0,0), \quad (1,1,1), \quad (1,-1,-1), \quad (-1,1,-1), \quad (-1,-1,1).$$

We shall in the following check each of them considering extremum.

a) The point $(0,0,0)$ is not an extremum, because $-4xyz$ is the dominating term in a dotted neighbourhood of $(0,0,0)$, and $-4xyz$ is both positive and negative in any neighbourhood of $(0,0,0)$.

Insertion. It follows from

$$\frac{\partial^2 f}{\partial x^2} = 12x^2, \qquad \frac{\partial^2 f}{\partial y^2} = 12y^2, \qquad \frac{\partial^2 f}{\partial z^2} = 12z^2,$$

$$\frac{\partial^2 f}{\partial x \partial y} = -4z, \qquad \frac{\partial^2 f}{\partial y \partial z} = -4x, \qquad \frac{\partial^2 f}{\partial x \partial z} = -4y,$$

that the approximating polynomial at a stationary point $(x_0, y_0, z_0) \neq (0,0,0)$ [of course also at $(0,0,0)$, but this is not relevant here] is given by

$$\begin{aligned} P_2(x,y,z) \;&=\; f(x_0,y_0,z_0) + 6\{x_0^2(x-x_0)^2 + y_0^2(y-y_0)^2 + z_0^2(z-z_0)^2\} \\ &\quad -4z_0(x-x_0)(y-y_0) - 4x_0(y-y_0)(z-z_0) - 4y_0(x-x_0)(z-z_0). \end{aligned}$$

When $|x_0| = |y_0| = |z_0| = 1$, this is written

$$\begin{aligned} P_2(x,y,z) \;&=\; f(x_0,y_0,z_0) + 2\{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\} \\ &\quad +2\left\{[z_0(x-x_0)]^2 - 2z_0(x-x_0)(y-y_0) + (y-y_0)^2\right\} \\ &\quad +2\left\{[x_0(y-y_0)]^2 - 2x_0(y-y_0)(z-z_0) + (z-z_0)^2\right\} \\ &\quad +2\left\{[y_0(x-x_0)]^2 - 2y_0(x-x_0)(z-z_0) + (x-x_0)^2\right\}, \end{aligned}$$

where we have used that. By a rearrangement,

$$\begin{aligned} &P_2(x,y,z) - f(x_0,y_0,z_0) \\ (17.7) = \;& 2\left\{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\right\} + 2\{z_0(x-x_0) - (y-y_0)\}^2 \\ &\quad +2\{x_0(y-y_0) - (z-z_0)\}^2 + 2\{y_0(x-x_0) - (z-z_0)\}^2. \end{aligned}$$

b) In the latter four stationary points the approximating polynomial is given by (17.7). It follows from this expression that they are all proper minima.

Summarizing we get that

$$(1,1,1), \quad (1,-1,-1), \quad (-1,1,-1), \quad (-1,-1,1)$$

are all proper minima, while $(0,0,0)$ is not an extremum.

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4) The equations of the stationary points are

$$\frac{\partial f}{\partial x} = \cos z = 0, \quad \frac{\partial f}{\partial y} = 2y = 0, \quad \frac{\partial f}{\partial z} = -x \sin z = 0,$$

so accordingly $z = \frac{\pi}{2} + p\pi$, $p \in \mathbb{Z}$, $y = 0$ and $x = 0$, and the stationary points are

$$\left(0, 0, \frac{\pi}{2} + p\pi\right), \qquad p \in \mathbb{Z}.$$

In all of these points, $f\left(0, 0, \frac{\pi}{2} + p\pi\right) = 0$. Since the restriction

$$f(x, 0, z) = x \cos z$$

is both positive and negative in any neighbourhood of any such point, none of them is an extremum.

5) If $f(x, y, z) = \exp(xy + yz + zx)$, then $f(x, y, z) > 0$, and the equations of the stationary points are

$$\begin{cases} \dfrac{\partial f}{\partial x} = (y + z) f(x, y, z) = 0, \\[2mm] \dfrac{\partial f}{\partial y} = (z + x) f(x, y, z) = 0, \\[2mm] \dfrac{\partial f}{\partial z} = (x + y) f(x, y, z) = 0, \end{cases}$$

i.e.

$$(17.8) \quad \begin{cases} y + z = 0, \\ z + x = 0, \\ x + y = 0. \end{cases}$$

The system (17.8) has only the solution $x = y = z = 0$, so $(0, 0, 0)$ is the only stationary point.

By a Taylor expansion,

$$\begin{aligned} f(x, y, z) &= \exp(xy + yz + zx) \\ &= 1 + xy + yz + xz + (x^2 + y^2 + z^2)\varepsilon(x, y, z), \end{aligned}$$

where $\varepsilon(x, y, z) \to 0$ for $(x, y, z) \to (0, 0, 0)$. Hence

$$P_2(x, y, z) = 1 + xy + yz + zx,$$

where e.g.

$$P_2(x, y, 0) - 1 = xy$$

attains both positive and negative values in any neighbourhood of $(x, y) = (0, 0)$. Thus there is no extremum at $(0, 0, 0)$.

6) If $f(x, y, z) = y^3 + \ln(1 + x^2 + z^2)$, the equations of the stationary points are

$$\frac{\partial f}{\partial x} = \frac{2x}{1 + x^2 + z^2} = 0, \quad \frac{\partial f}{\partial y} = 3y^2 = 0, \quad \frac{\partial f}{\partial z} = \frac{2z}{1 + x^2 + z^2} = 0.$$

It follows that $(0, 0, 0)$ is the only stationary point. The restriction $f(0, y, 0) = y^3$ is both positive and negative in any neighbourhood of $y_0 = 0$, so there is no extremum at $(0, 0, 0)$.

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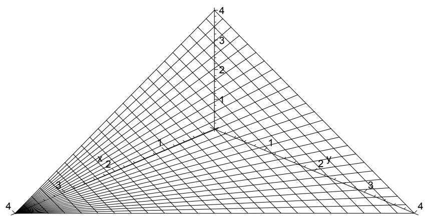
Example 17.14 *Examine in the same way as in* **Example 17.13** *the function*

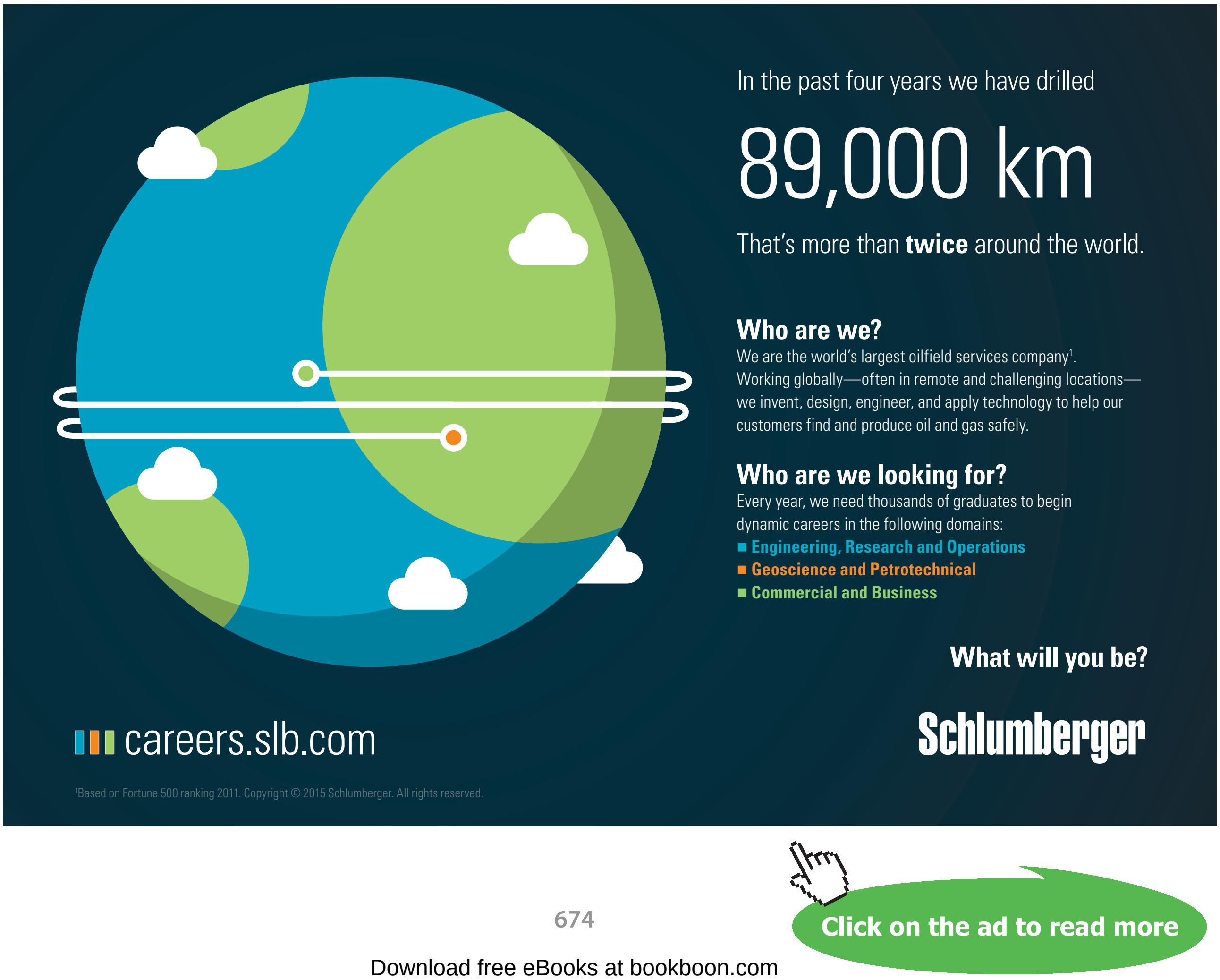
$$f(x,y,z) = xyz(4a - x - y - z), \qquad (x,y,z) \in \mathbb{R}_+^3.$$

A Stationary points; extrema.

D Find the possible stationary points; check if they are extrema.

I The function can by continuity be extended to its zero set. This is the surface of the tetrahedron on the figure, and it is obvious that $f(x,y,z) > 0$ in the open tetrahedron. The function must have a maximum in the tetrahedron, according to the second main theorem for continuous functions, and because $f(x,y,z)$ is of class C^∞, this maximum can only be attained at a stationary point in the interior of the tetrahedron.





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The equations of the stationary points are

$$\begin{cases} \dfrac{\partial f}{\partial x} = yz(4a-x-y-z) - xyz = yz(4a-2x-y-z) = 0, \\[2mm] \dfrac{\partial f}{\partial y} = xz(4a-x-y-z) - xyz = xz(4a-x-2y-z) = 0, \\[2mm] \dfrac{\partial f}{\partial z} = xy(4a-x-y-2z) = 0. \end{cases}$$

It follows from the assumptions $x > 0$, $y > 0$ and $z > 0$ that these equations are equivalent to

$$\begin{cases} x = 4a-x-y-z, \\ y = 4a-x-y-z, \\ z = 4a-x-y-z, \end{cases}$$

and it follows immediately that

$$x = 4a-x-y-z = y = z = a.$$

Hence, (a,a,a) is the only stationary point in the first octant.

It follows from the application of the second main theorem above that we have a maximum at (a,a,a), and the value of the function is here

$$f(a,a,a) = a^4.$$

Alternatively we compute

$$\frac{\partial^2 f}{\partial x^2} = -2yz, \qquad \frac{\partial^2 f}{\partial y^2} = -2xz, \qquad \frac{\partial^2 f}{\partial z^2} = -2xy,$$

$$\frac{\partial^2 f}{\partial x \partial y} = z(4a-2x-y-z) - yz = z(4a-2x-2y-z),$$

$$\frac{\partial^2 f}{\partial y \partial z} = x(4a-x-2y-2a), \qquad \frac{\partial^2 f}{\partial z \partial x} = y(4a-2x-y-2z),$$

hence

$$\frac{\partial^2 f}{\partial x^2}(a,a,a) = \frac{\partial^2 f}{\partial y^2}(a,a,a) = \frac{\partial^2 f}{\partial z^2}(a,a,a) = -2a^2,$$

$$\frac{\partial^2 f}{\partial y \partial z}(a,a,a) = \frac{\partial^2 f}{\partial x \partial z}(a,a,a) = \frac{\partial^2 f}{\partial x \partial z}(a,a,a) = -a^2.$$

Now $f(a,a,a) = a^4$, so

$$\begin{aligned} P_2(x,y,z) &= a^4 - a^2\left\{(x-a)^2+(y-a)^2+(z-a)^2\right. \\ &\qquad \left. +(x-a)(y-a)+(y-a)(z-a)+(z-a)(x-a)\right\} \\ &= a^4 - \frac{a^2}{2}\left\{(x-a)^2+(y-a)^2+(z-a)^2\right. \\ &\qquad \left. +[(x-a)+(y-a)+(z-a)]^2\right\}, \end{aligned}$$

and we see that (a,a,a) is a maximum.

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Example 17.15 *Let $f : \mathbb{R}^k \to \mathbb{R}$ be a polynomial of second degree. Prove that f has none or one or infinitely many stationary points.*
Then give for $k = 2$ examples of all three possibilities.

A Stationary points.

D Use some Linear Algebra on the system of equations of the stationary points.

I A general polynomial of second degree in $\mathbb{R}^k$ is of the form

$$f(\mathbf{x}) = \sum_{i,j=1}^{k} a_{ij}x_ix_j + \sum_{i=1}^{k} b_ix_i + c.$$

The equations of the stationary points are

$$\frac{\partial f}{\partial x_m} = \sum_{i=1}^{k}(a_{im} + a_{mi})x_i + b_m = 0, \qquad m = 1, \dots, k,$$

i.e. a system of k linear equations in k unknowns. It is known from Linear Algebra that such a system of equations has none or one or an infinity of solutions, and the claim is proved.

Let $k = 2$, and denote the variables by (x, y).

If $f(x, y) = x^2 + y^2$, we clearly have $(0, 0)$ as the only stationary point.

If $f(x, y) = y^2$, then we clearly have infinitely many stationary points, namely $(x, 0)$ for $x \in \mathbb{R}$.

Finally, let $f(x, y) = (x + y)^2 + x - y$. Then the equations of the de stationary points are

$$\frac{\partial f}{\partial x} = 2(x + y) + 1 = 0, \qquad \frac{\partial f}{\partial y} = 2(x + y) - 1 = 0.$$

This system of equations clearly has no solution.

A simpler system is

$$f(x, y) = x^2 + y,$$

where $\dfrac{\partial f}{\partial y} = 1 \neq 0$ for every (x, y). The structure is the same as in the example above. The difference is that both x^2 and y^2 occur in the former example, so the polynomial is of second degree in both x and y. This is not the case in the latter example.

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17.4 Examples of maxima and minima

Example 17.16 *Explain in each of the following cases why the indicated function has a maximum and a minimum, and find these values.*

1) $f(x, y) = 8\sqrt{x^2 + 3y^2} - 5x - y^2$ *for* $x^2 + 3y^2 \le 4$.

2) $f(x, y) = x^2 - 3y^2 - 3xy$ *for* $x^2 + y^2$.

3) $f(x, y) = xy + y^2 - 5y - 3\ln x$ *for* $x \ge 1$ *and* $0 \le y \le 5 - x$.

4) $f(x, y) = (x^2 + y^2 - 2y)(x^2 + y^2 - 6y)$ *for* $x^2 + y^2 \le 36$.

5) $f(x, y) = xy + \dfrac{64}{x} + \dfrac{64}{y}$ *for* $x \ge 1$, $y \ge 1$ *and* $xy \le 32$.

6) $f(x, y) = 3x^2 + 3y^2 - 2xy - 2x^2y^2$ *for* $x \ge 0$, $y \ge 0$ *and* $x^2 + y^2 \le 4$.

7) $f(x, y) = e^{-2y} + e^{-y} \sin x$ *for* $(x, y) \in [0, 2\pi] \times [0, 1]$.

8) $f(x, y) = x^4 + y^4 - x^2 + 2xy - y^2$ *for* $x^2 + y^2 \le 4$.

9) $f(x, y) = 8xy^2 - xy^3 - x^3y$ *for* $(x, y) \in [0, 4] \times [0, 8]$.

A Maximum and minimum for continuous functions on closed and bounded (i.e. compact) sets.

D Apply the second main theorem. Sketch the domain. Apply that the maximum and the minimum are either attained at an exception point or in a stationary point of on the boundary.

I All functions are continuous on a closed and bounded set,so by the second main theorem for continuous functions the function has both a maximum and a minimum on the set.

With 1) as the only exception, all the rest of the functions are of class C^∞ in the interior of their respective domains. We shall therefore in all these cases only find the stationary points and examine the boundary.

In 1) the function is continuous everywhere, and not differentiable at $(0, 0)$, so this is an exceptional point.

1) The domain is an ellipsoidal disc given by

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{\frac{2}{\sqrt{3}}}\right)^2 \le 1$$

of centrum $(0, 0)$ and half axes 2 and $\dfrac{2}{\sqrt{3}}$.

a) The exception point. The function is of class C^∞ everywhere inside the ellipse, except for $(0, 0)$, which is an *exception point*. The value of the function is here

$$f(0, 0) = 0.$$

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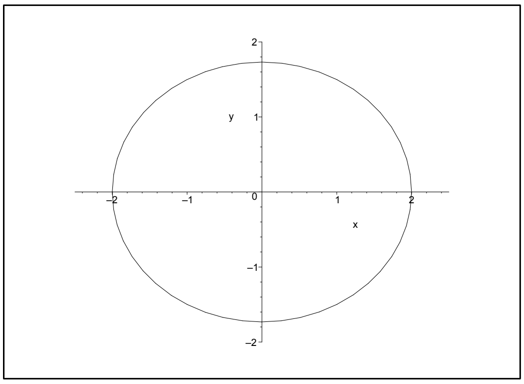

b) STATIONARY POINTS. In the domain given by $0 < x^2 + 3y^2 < 4$, the equations of the stationary points are

$$\frac{\partial f}{\partial x} = \frac{8x}{\sqrt{x^2+3y^2}} - 5 = 0, \qquad \frac{\partial f}{\partial y} = \frac{24y}{\sqrt{x^2+3y^2}} - 2y = 0.$$

The latter equation gives the following two possibilities

$$y = 0 \qquad \text{or} \qquad \sqrt{x^2+3y^2} = 12.$$

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The latter possibility is rejected, because $\sqrt{x^2+3y^2} \le 2$ in the given domain. We conclude that we necessarily have $y=0$, whenever we have a stationary point, if any.

When $y=0$ is put into the former equation of the stationary points we get

$$8\,\frac{x}{|x|} = 5,$$

which is never fulfilled. We conclude that there is no stationary point in the domain.

c) Examination of the boundary. On the boundary, $x^2+3y^2=4$, hence

$$y^2 = \frac{1}{3}(4-x^2), \qquad x \in [-2,2].$$

When $x \in [-2,2]$ is used as a parameter, we get along the boundary

$$\begin{aligned} f\left(x, \pm\sqrt{\frac{4-x^2}{3}}\right) &= 8 \cdot 2 - 5x - \frac{1}{3}(4-x^2) = \frac{1}{3}(x^2-15x+44) \\ &= \frac{1}{3}\left\{\left(x-\frac{15}{2}\right)^2 - \frac{49}{4}\right\} = \frac{1}{3}(x-y)(x-11). \end{aligned}$$

The maximum in the interval $[-2,2]$ is obtained for $x=-2$, corresponding to

$$f(-2,0) = 16+10 = 26,$$

and the minimum for $x=2$, corresponding to

$$f(2,0) = 16-10 = 6.$$

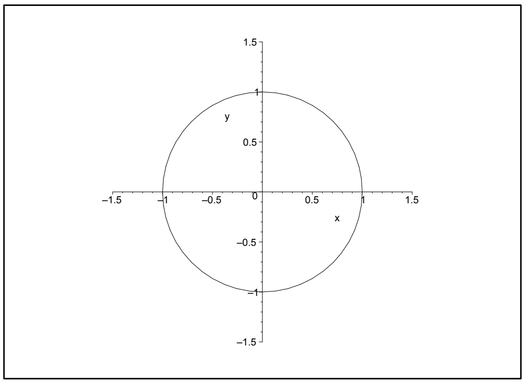
d) Numerical comparison. Summarizing the maximum and the minimum are included in the values

$$f(-2,0)=26, \qquad f(0,0)=0, \qquad f(x,0)=6.$$

It follows that

$$S = f(-2,0) = 26 \qquad \text{(maximum)} \qquad \text{and} \qquad M = f(0,0) = 0 \qquad \text{(minimum)}.$$

2) The domain is here the unit disc.



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a) Exception points. The function is everywhere of class C^∞, so there are no exception points.

b) Stationary points. The equations of the stationary points are

$$\frac{\partial f}{\partial x} = 2x - 3y = 0, \qquad \frac{\partial f}{\partial y} = -6y - 3x = 0,$$

and it follows that $(0,0)$ is the only stationary point in the domain. The value of the function is here

$$f(0,0) = 0.$$

c) Examination of the boundary. We shall use the following parametric description of the boundary,

$$x = \cos\varphi, \quad y = \sin\varphi, \qquad \varphi \in [0, 2\pi[,$$

hence by insertion,

$$\begin{aligned} g(\varphi) &= f(x,y) = x^2 - 3y^2 - 3xy = \cos^2\varphi - 3\sin^2\varphi - 3\cos\varphi\cdot\sin\varphi \\ &= \frac{1}{2}\{1+\cos 2\varphi - 3(1-\cos 2\varphi) - 3\sin 2\varphi\} = \frac{1}{2}\{4\cos 2\varphi - 3\sin 2\varphi - 2\} \\ &= \frac{5}{2}\left\{\frac{4}{5}\cos 2\varphi - \frac{3}{5}\sin 2\varphi\right\} - 1 = \frac{5}{2}\cos(2\varphi+\varphi_0) - 1, \end{aligned}$$

where

$$\cos\varphi_0 = \frac{4}{5} \qquad \text{and} \qquad \sin\varphi_0 = \frac{3}{5}.$$

Now $\cos(2\varphi+\varphi_0)$ goes twice through the interval $[-1,1]$, when φ goes through $[0,2\pi[$, and we find the maximum

$$\frac{5}{2} - 1 = \frac{3}{2},$$

and the minimum

$$-\frac{5}{2} - 1 = -\frac{7}{2}$$

on the boundary.

d) Numerical comparison. By comparison we see that the value $f(0,0) = 0$ from the stationary point lies between these two values on the boundary. We therefore conclude that

$$S = \frac{3}{2} \qquad \text{og} M = -\frac{7}{2}.$$

3) The domain is the closed triangle between the lines $x = 1$, $y = 0$ and $y = 5 - x$.

a) Stationary points. The stationary points are the solutions of the equations

$$\frac{\partial f}{\partial x} = y - \frac{3}{x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = x + 2y - 5 = 0,$$

from which $x + \dfrac{6}{x} - 5 = 0$. Since $x \geq 1$, we get

$$x^2 - 5x + 6 = (x-2)(x-3) = 0,$$

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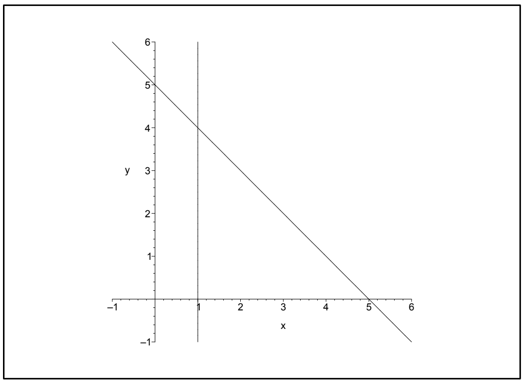

the roots of which are $x = 2$, corresponding to $y = \frac{3}{2}$, and $x = 3$, corresponding to $y = 1$. We get the two stationary points

$$\left(2, \frac{3}{2}\right) \qquad \text{og} \qquad (3,1).$$

The values of the function are here

$$f\left(2, \frac{3}{2}\right) = 2 \cdot \frac{3}{2} + \left(\frac{3}{2}\right)^2 - 5 \cdot \frac{3}{2} - 3\ln 2 = -\frac{9}{4} - 3\ln 2,$$

and

$$f(3,1) = 3 \cdot 1 + 1^2 - 5 \cdot 1 - 3\ln 3 = -1 - 3\ln 2.$$

b) Examination of the boundary.

i) If $y = 0$, then the restriction

$$f(x,0) = -3\ln x, \qquad x \in [1,5],$$

is monotonous with its maximum $f(1,0) = 0$ and its minimum $f(5,0) = -3\ln 5$.

ii) If $x = 1$, then the restriction

$$f(1,y) = y^2 - 4y, \qquad y \in [1,4]$$

has the maximum $f(1,0) = f(1,4) = 0$ and the minimum $f(1,2) = -4$.

iii) If $y = 5 - x$, then

$$\begin{aligned} f(x, 5-x) &= x(5-x) + (5-x)^2 - 5(5-x) - 3\ln x \\ &= -3\ln x \qquad\qquad \text{for } x \in [1,5], \end{aligned}$$

where the maximum is $f(1,4) = 0$ and the minimum is $f(5,0) = -3\ln 5$.

c) Numerical comparison. We shall compare

$$f\left(2, \frac{3}{2}\right) = -\frac{9}{4} - 3\ln 2, \qquad f(3,1) = -1 - 3\ln 3,$$

$$f(1,0) = 0, \qquad\qquad f(5,0) = -3\ln 5,$$

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$$f(1,0) = f(1,4) = 0, \qquad f(1,2) = -4,$$
$$f(1,4) = 0, \qquad f(5,0) = -3\ln 5.$$

The maximum is clearly

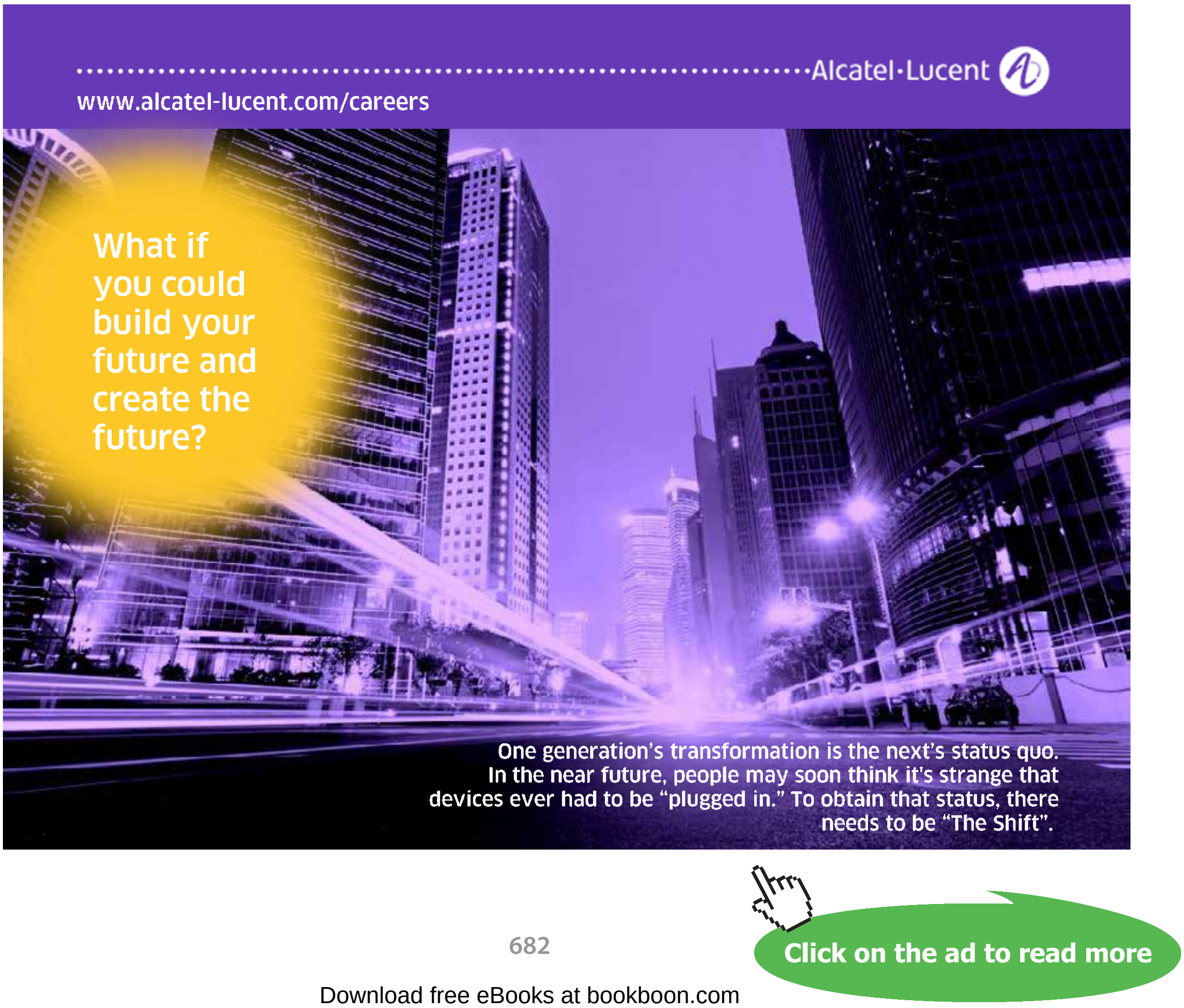
$$f(1,0) = f(1,4) = 0.$$

By using a pocket calculator we then get approximately,

$$-\frac{9}{4} - 3\ln 2 \approx -4.33, \quad -1 - 3\ln 3 \approx -4.30, \quad -3\ln 5 \approx -4.83.$$

We conclude that the minimum is

$$f(5,0) = -3\ln 5 \approx -4.83.$$



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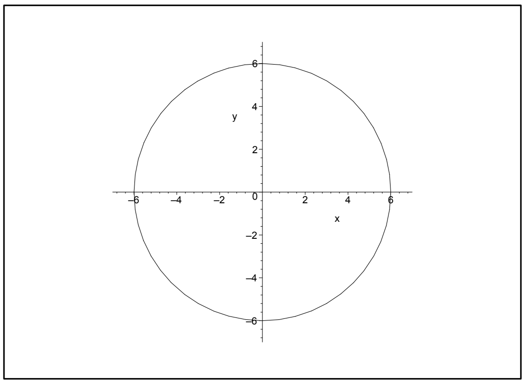
4) The domain is the closed disc of centrum $(0,0)$ and radius 6.



It is easy to find the three stationary points

$$(0,0), \qquad (0,3+\sqrt{3}) \quad \text{and} \quad (0,3-\sqrt{3}),$$

where $(0,0)$ is not an extremum, while $(0,3+\sqrt{3})$ is a minimum point, and $(0,3-\sqrt{3})$ is a maximum point. The function values are then

$$f(0,3+\sqrt{3}) = -36-24\sqrt{3}, \qquad f(0,3-\sqrt{3}) = -36+24\sqrt{3}.$$

Stationary points. We get by differentiation

$$\frac{\partial f}{\partial x} = 2x(x^2+y^2-6y)+2x(x^2+y^2-2y) = 4x(x^2+y^2-4y)$$

and

$$\begin{aligned}\frac{\partial f}{\partial y} &= (2y-2)(x^2+y^2-6y)+(2y-6)(x^2+y^2-2y)\\ &= (2y-4)(x^2+y^2-6y)+2(x^2+y^2-6y)\\ &\qquad +(2y-4)(x^2+y^2-2y)-2(x^2+y^2-2y)\\ &= 4(y-2)(x^2+y^2-4y)-8y\\ &= 4\left\{(y-2)(x^2+y^2-4y)-2y\right\}.\end{aligned}$$

The two equations for the stationary points are

$$(17.9)\quad \left\{\begin{array}{l} x(x^2+y^2-4y)=0,\\ (y-2)(x^2+y^2-4y)=2y.\end{array}\right.$$

It follows from the former equation that the stationary points either lie on the line $x=0$ or on the circle $x^2+(y-2)^2=2^2$

a) If $x=0$, it follows from the latter equation that

$$0=(y-2)(y^2-4y)-2y = y\{(y-2)(y-4)-2\} = y\{y^2-6y+6\},$$

which has the two solutions $y=0$ and $y=3\pm\sqrt{3}$. Thus we get in this case three stationary points

$$(0,0), \qquad (0,3+\sqrt{3}) \quad \text{and} \quad (0,3-\sqrt{3}).$$

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b) If $x^2 + y^2 - 4y = 0$, it follows from the latter equation of (17.9) that $y = 0$, and thus $x = 0$, so we find again $(0, 0)$.

Summarizing we get the three stationary points

$$(0,0), \quad (0, 3+\sqrt{3}) \quad \text{and} \quad (0, 3-\sqrt{3}).$$

Remark. Since we are searching the *maximum* and the *minimum* (and not the extrema), it is sufficient to calculate the value of the function in these points, and e.g. the (r, s, t)-investigation is totally superfluous (and a waste of time). $\Diamond$

By computation we get $f(0, 0) = 0$, and

$$\begin{aligned} f(0, 3+\sqrt{3}) &= \left\{(3+\sqrt{3})^2 - 2(3+\sqrt{3})\right\}\left\{(3+\sqrt{3})^2 - 6(3+\sqrt{3})\right\} \\ &= (12+6\sqrt{3}-6-2\sqrt{3})(12+6\sqrt{3}-18-6\sqrt{3}) \\ &= (6+4\sqrt{3})(-6) \\ &= -36-24\sqrt{3}, \end{aligned}$$

and analogously

$$f(0, 3-\sqrt{3}) = -36+24\sqrt{3}.$$

Examination of the boundary. We have on the boundary $x^2 + y^2 = 36$, so

$$\begin{aligned} f(x,y) &= (x^2+y^2-2y)(x^2+y^2-6y) \\ &= (36-2y)(36-6y) \\ &= 12(18-y)(6-y), \qquad \text{for } y \in [-6, 6], \end{aligned}$$

where the maximum is

$$f(0, -6) = 12 \cdot 24 \cdot 12 = 3456,$$

and the minimum is $f(0, 6) = 0$.

Numerical comparison. Since $f(0, 3+\sqrt{3}) < 0$, the maximum and the minimum of the function in the domain are respectively,

$$S = f(0, -6) = 3456 \quad \text{and} \quad M = f(0, 3+\sqrt{3}) = -36-24\sqrt{3}.$$

Alternative solution. The argument of the second main theorem concerning the existence of the maximum and the minimum is the same as above.

Write the function f in the following way:

$$\begin{aligned} f(x,y) &= (x^2+y^2-2y)(x^2+y^2-6y) \\ &= \left\{x^2+(y-1)^2-1\right\}\left\{x^2+(y-3)^2-3^2\right\}, \end{aligned}$$

and discuss the sign of f in the domain, i.e. sketch the zero sets (the circles of respectively centrum $(0, 1)$ and radius 1, and of centrum $(0, 3)$ and radius 3) inside the domain and find the signs in each of the thus defined subregions.

The function is positive inside the two sets of zero circles and also outside the same two sets of circles, while it is negative between the two zero circles.

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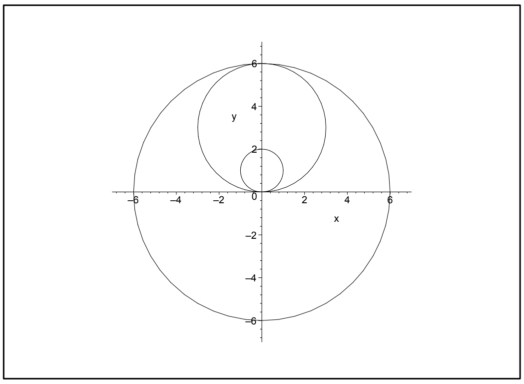

Figure 17.12: The zero curves inside the domain.

It follows immediately that the minimum must lie between the two sets of zero circles, i.e. in the set

$$K((0,3);3) \setminus \overline{K}((0,1);1) = A_1,$$

and also that the minimum point must be a stationary point, because A_1 is open. Because $3-\sqrt{3} < 2 = 1+1$, we see that $(0, 3+\sqrt{3})$ is the only stationary point in $A >_1$, so the *mininimum* is (originally only a local minimum)

$$f(0,3+\sqrt{3}) = -36 - 24\sqrt{3}.$$

It follows in the same way that $(0,3-\sqrt{3}) \in A_2 = K((0,1);1)$ must be a local maximum

$$f(0,3-\sqrt{3}) = -36 + 24\sqrt{3}.$$

Finally by using polar coordinates in the plane,

$$\begin{aligned} f(x,y) &= (r^2 - 2r\sin\varphi)(r^2 - 6r\sin\varphi) \\ &= r^2(r-2\sin\varphi)(r-6\sin\varphi), \qquad 0 \le r \le 6. \end{aligned}$$

In the remaining region A_3 both factors are positive, so

$$r - 2\sin\varphi > 0 \quad \text{og} \quad r - 6\sin\varphi > 0.$$

The product is largest when $\sin\varphi$ is smallest, i.e. when $\varphi = -\frac{\pi}{2}$, thus $\sin\varphi = -1$, corresponding to

$$f(x,y) = r^2(r+2)(r+6), \qquad 0 \le r \le 6.$$

This product is largest when r is largest, i.e. when $r = 6$ (the boundary), corresponding to $(x,y) = (0,-6)$, and

$$f(0,-6) = 3456.$$

When we compare with the other candidate above we conclude that the *maximum* in A is

$$f(0,-6) = 3456.$$

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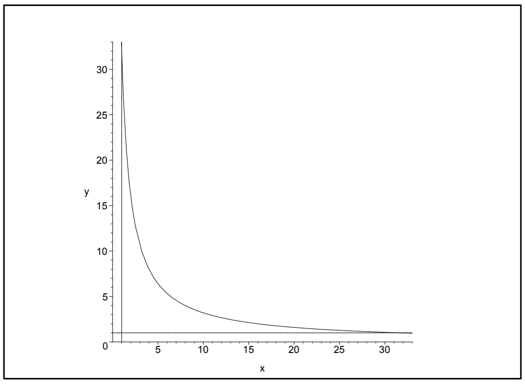

5) In this case it is difficult to sketch the domain because the hyperbola $xy = 32$ is very steep in the neighbourhood of $(1, 32)$, and very flat in the neighbourhood of $(32, 1)$. It is demonstrated on the figure what MAPLE does in this case.
The domain is bounded by the hyperbola $xy = 32$ and the lines $x = 1$ and $y = 1$.

a) STATIONARY POINTS. The stationary points are the solutions of the equations

$$\frac{\partial f}{\partial x} = y - \frac{64}{x^2} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = x - \frac{64}{y^2} = 0.$$

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From these we get

$$\frac{64}{x} = xy = \frac{64}{y},$$

so the stationary points lie on the line $y = x$. Then by insertion $x^3 = 64 = 4^3$, and thus $x = y = 4$. We conclude from

$$x = 4 \geq 1, \qquad y = 4 \geq 1 \quad \text{and} \quad xy = 16 \leq 32,$$

that we have the stationary point $(4, 4)$ in the domain. The value of the function is here

$$f(4,4) = 16 + 16 + 16 = 48.$$

NOTE that we shall not check if $(4, 4)$ is an extremum.

b) EXAMINATION OF THE BOUNDARY.

i) We get along the boundary curve $xy = 32$, $x \in [1, 32]$, that $y = \dfrac{32}{x}$, and the corresponding restriction is

$$f(x) = f\left(x, \frac{32}{x}\right) = 32 + \frac{64}{x} + 2x, \qquad \text{for } x \in [1, 32],$$

where

$$g'(x) = -\frac{64}{x^2} + 2 = 0 \qquad \text{for } x = \sqrt{32},$$

corresponding to a minimum. The y-value is $y = \dfrac{32}{\sqrt{32}} = \sqrt{32}$, and the value of the function at the point $(\sqrt{32}, \sqrt{32})$ is

$$f(\sqrt{32}, \sqrt{32}) = 32 + \frac{64}{\sqrt{32}} + \frac{64}{\sqrt{32}} = 32 + 4\sqrt{32} = 32 + 16\sqrt{2}.$$

At the end points of $xy = 32$, $x \in [1, 32]$, we get the values

$$f(1, 32) = f(32, 1) = 32 + 64 + \frac{64}{32} = 98.$$

ii) We get along the boundary curve $y = 1$, $x \in [1, 32]$, the following restriction

$$h(x) = f(x, 1) = x + \frac{64}{x} + 64,$$

where

$$h'(x) = 1 - \frac{64}{x^2} = 0 \qquad \text{for } x = 8,$$

corresponding to a minimum

$$f(8, 1) = 8 + \frac{64}{8} + 64 = 80.$$

At the end points we get

$$f(1, 1) = 1 + 64 + 64 = 129 \quad \text{and} \quad f(32, 1) = 32 + 2 + 64 = 98.$$

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iii) Due to the symmetry we get along the boundary curve $x = 1$ that the minimum is

$$f(1, 8) = 80,$$

and that the values at the end points are

$$f(1, 1) = 129 \qquad \text{and} \qquad f(1, 32) = 98.$$

iv) NUMERICAL COMPARISON. Summarizing the minimum is one of the values

$$f(4, 4) = 48, \quad f(\sqrt{32}, \sqrt{32}) = 32 + 16\sqrt{2}, \quad f(1, 8) = f(8, 1) = 80.$$

From $32 + 16\sqrt{2} > 32 + 16 = 48$ follows that the minimum is

$$M = f(4, 4) = 48.$$

Analogously the maximum is one of the numbers

$$f(1, 1) = 129 \qquad \text{and} \qquad f(1, 32) = f(32, 1) = 98,$$

hence the maximum is

$$S = f(1, 1) = 129.$$

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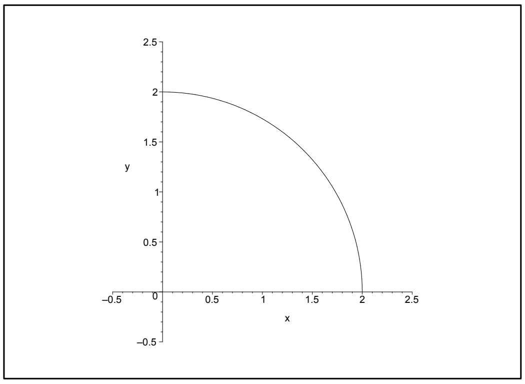
6) The domain is the quarter of a disc in the first quadrant of centrum $(0,0)$ and radius 2.



a) Stationary points. the stationary points are the solutions of the equations

$$(17.10)\quad \begin{cases} \dfrac{\partial f}{\partial x} = 6x - 2y - 4xy^2 = 0, \\ \dfrac{\partial f}{\partial y} = 6y - 2x - 4x^2y = 0. \end{cases}$$

When we add the two equations we get the following necessary condition

$$\begin{aligned} 0 &= 4x + 4y - 4xy^2 - 4x^2y \\ &= 4\{(x+y) - xy(x+y)\} \\ &= 4(x+y)(1-xy), \end{aligned}$$

so either $x + y = 0$ (not possible in this domain) or $xy = 1$,

Analogously it follows from (17.10) that

$$4x^2y^2 = 6x^2 - 2xy = 6y^2 - 2xy,$$

hence $x^2 = y^2$, which together with $xy = 1$ and $x > 0$ give $x = y = 1$. The only possibility is $(1,1)$, and by insertion into (17.10) we get that $(1,1)$ is a stationary point. Furthermore, we see that $(1,1)$ belongs to the domain. The value of the function at the point is

$$f(1,1) = 3 + 3 - 2 - 2 = 2.$$

b) Examination of the boundary. On the boundary curve $x^2 + y^2 = 4$ we shall use the parametric description

$$x = 2\cos\varphi, \qquad y = 2\sin\varphi, \qquad \varphi \in \left[0, \frac{\pi}{2}\right].$$

Then

$$\begin{aligned} g(\varphi) &= f(x,y) = 3x^2 + 3y^2 - 2xy(1+xy) \\ &= 3\cdot 4 - 2\cdot 4\cos\varphi\cdot\sin\varphi(1 + 4\cos\varphi\cdot\sin\varphi) \\ &= 12 - 4\sin 2\varphi\cdot(1 + 2\sin 2\varphi) \\ &= 12 - 4\sin 2\varphi - 8\sin^2 2\varphi \\ &= 12 - 8\left\{\sin^2 2\varphi + \frac{1}{2}\sin 2\varphi + \frac{1}{16}\right\} + \frac{8}{16} \\ &= \frac{25}{2} - 8\left(\sin 2\varphi + \frac{1}{4}\right)^2 \qquad \text{for } 2\varphi \in [0,\pi]. \end{aligned}$$

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Here $\sin 2\varphi \in [0,1]$ for $2\varphi \in [0,\pi]$, so the maximum of $g(\varphi)$ if obtained for $\sin 2\varphi = 0$, i.e. for either $\varphi = 0$ or $\varphi = \dfrac{\pi}{2}$, corresponding to

$$g(0) = g\left(\frac{\pi}{2}\right) = f(2,0) = f(0,2) = 12.$$

The minimum is obtained for $\sin 2\varphi = 1$, corresponding to $\varphi = \dfrac{\pi}{4}$, or $x = y = \sqrt{2}$, where

$$g\left(\frac{\pi}{4}\right) = f(\sqrt{2},\sqrt{2}) = 12 - 2\cdot 2\cdot(1+2) = 0.$$

ALTERNATIVELY we see that $z = xy$ runs through the interval $[0,2]$, when (x,y) runs through the arc of the quarter circle. This means that

$$g_1(z) = f(x,y) = 3x^2 + 3y^2 - 2xy(1+xy) = 12 - 2z(1+z),$$

which is largest in the interval $[0,2]$, when $z = 0$, which corresponds to $(x,y) = (2,0)$ or $(0,2)$, and smallest when $z = 2$, which corresponds to $(x,y) = (\sqrt{2},\sqrt{2})$.

For $x = 0$ we get the restriction $h(y) = f(0,y) = 3y^2$ with the minimum $f(0,0) = 0$ and the maximum

$$f(0,2) = f(2,0) = 12,$$

and where we exploit the symmetry of x and y.

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c) Numerical comparison. The minimum is one of the values

$$f(1,1) = 2, \quad f(\sqrt{2},\sqrt{2}) = 0, \quad f(0,0) = 0,$$

and the maximum is one of the values

$$f(1,1) = 2, \quad f(2,0) = f(0,2) = 12.$$

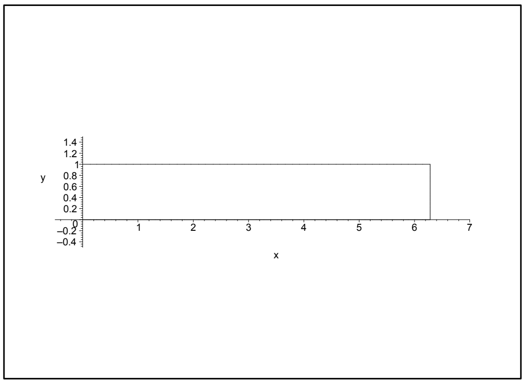
It follows that the minimum is

$$M = f(\sqrt{2},\sqrt{2}) = f(0,0)00,$$

and that the maximum is

$$S = f(2,0) = f(0,2) = 12.$$

7) The domain is here the rectangle $[0,2\pi] \times [0,1]$.



a) Stationary points. The stationary points are given by the equations

$$\begin{cases} \dfrac{\partial f}{\partial x} = e^{-y}\cos x = 0, \\[2mm] \dfrac{\partial f}{\partial y} = -2e^{-2y} - e^{-y}\sin x = 0. \end{cases}$$

It follows from the former of these that $\cos x = 0$, so either $x = \dfrac{\pi}{2}$ or $x = \dfrac{3\pi}{2}$. Then we get from the latter equation,

$$e^{-y}(2e^{-y} + \sin x) = 0.$$

Now $e^{-y} > 0$, hence $\sin x < 0$ and whence $x = \dfrac{3\pi}{2}$. Accordingly $2e^{-y} = 1$ and thus $y = \ln 2 \in]0,1[$.

The function has a stationary point in the domain, namely $\left(\dfrac{3\pi}{2}, \ln 2\right)$. The value of the function is here

$$f\left(\frac{3\pi}{2}, \ln 2\right) = e^{-2\ln 2} + e^{-\ln 2}\sin\left(\frac{3\pi}{2}\right) = -\frac{1}{4}.$$

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b) Examination of the boundary

i) The restriction along the boundary curve $y = 0$, $x \in [0, 2\pi]$ is

$$g(x) = f(x, 0) = 1 + \sin x.$$

It follows immediately that the maximum is $f\left(\frac{\pi}{2}, 0\right) = 2$ and the minimum is

$$f\left(\frac{3\pi}{2}, 0\right) = 0.$$

ii) The restriction along the boundary curve $y = 1$, $x \in [0, 2\pi]$, is

$$h(x) = f(x, 1) = \frac{1}{e^2} + \frac{1}{e}\,\sin x.$$

This gives the maximum

$$f\left(\frac{\pi}{2}, 1\right) = \frac{1}{e^2} + \frac{1}{2} \quad (< 2)$$

and the minimum

$$f\left(\frac{3\pi}{2}\right) = \frac{1}{e^2} - \frac{1}{e} = -\frac{e-1}{e^2} \quad (< 0).$$

iii) We get the same restriction $r(y) = e^{-2y}$, $y \in [0, 1]$, along the boundary curves $x = 0$ and $x = 2\pi$ which corresponds to the maximum $f(0, 0) = f(2\pi, 0) = 1$ and the minimum $f(0, 1) = f(2\pi, 1) = \dfrac{1}{e^2}$.

c) Numerical comparison. The maximum is one of the values

$$f\left(\frac{3\pi}{2}, \ln 2\right) = -\frac{1}{4}, \qquad f\left(\frac{\pi}{2}, 0\right) = 2,$$

$$f\left(\frac{\pi}{2}, 1\right) = \frac{1}{e^2} + \frac{1}{e}, \qquad f(0, 0) = f(2\pi, 0) = 1.$$

The minimum is one of the values

$$f\left(\frac{3\pi}{2}, \ln 2\right) = -\frac{1}{4}, \qquad f\left(\frac{3\pi}{2}, 0\right) = 0,$$

$$f\left(\frac{3\pi}{2}, 1\right) = -\frac{e-1}{e^2}, \qquad f(0, 1) = f(2\pi, 1) = \frac{1}{e^2}.$$

We conclude that the maximum is

$$S = f\left(\frac{\pi}{1}2, 0\right) = 2.$$

From $(e-2)^2 = e^2 - 4e + 4 > 0$ follows by a rearrangement that $e^2 > 4(e-1) > 0$, hence $0 < \dfrac{e-1}{e^2} < \dfrac{1}{4}$, and the minimum is

$$f\left(\frac{3\pi}{2}, \ln 2\right) = -\frac{1}{4}.$$

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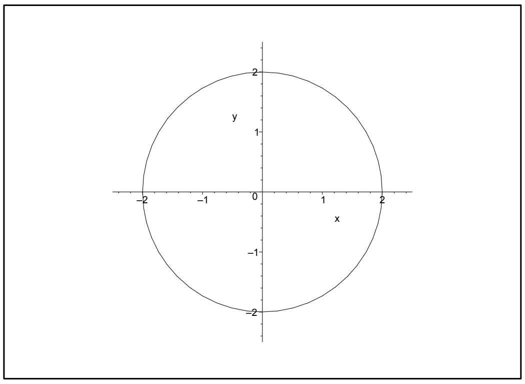

8) The domain is the disc of centrum $(0,0)$ and radius 2.

a) Stationary points. The stationary points are the solutions of the equations

$$\frac{\partial f}{\partial x} = 4x^3 - 2x + 2y = 0, \qquad \frac{\partial f}{\partial y} = 4y^3 + 2x - 2y = 0,$$

hence $4x^3 = 2x - 2y = -4y^3$, i.e. $y = -x$.
We get by insertion

$$0 = 4x^3 - 4x = 4x(x^2 - 1),$$

so we have the possibilities $x = 0, 1, -1$, corresponding to the candidates

$$(0,0), \qquad (1,-1), \quad \text{and} \quad (-1,1)$$

of the stationary points. It follows by insertion (i.e. by testing) that they are actually all stationary points. The values of the function are here

$$f(0,0) = 0 \quad \text{and} \quad f(1,-1) = f(-1,1) = 1 + 1 - 1 - 2 - 1 = -2.$$

b) Examination of the boundary. On the boundary $x^2 + y^2 = 4$,

$$\begin{aligned} f(x,y) &= x^4 + y^4 - x^2 + 2xy - y^2 \\ &= (x^2+y^2)^2 - 2x^2y^2 - (x^2+y^2) + 2xy \\ &= 12 + 2xy(1 - xy). \end{aligned}$$

i) First alternative. We get by the parametric description

$$x = 2\cos\varphi, \qquad y = 2\sin\varphi, \qquad \varphi \in [0, 2\pi[,$$

that

$$\begin{aligned} g_1(\varphi) &= f(x,y) = 12 + 2xy(1 - xy) \\ &= 12 + 2 \cdot 4\cos\varphi \cdot \sin\varphi(1 - 4\cos\varphi \cdot \sin\varphi) \\ &= 12 + 4\sin 2\varphi(1 - 2\sin 2\varphi) \\ &= 12 - 8\left(\sin^2 2\varphi - \frac{1}{2}\sin 2\varphi + \frac{1}{16}\right) + \frac{1}{2} \\ &= \frac{25}{2} - 8\left(\sin 2\varphi - \frac{1}{4}\right)^2. \end{aligned}$$

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The function $\left(\sin 2\varphi - \frac{1}{4}\right)^2$ is smallest when $\sin 2\varphi_0 = \frac{1}{4}$, corresponding to the maximum $g_1(\varphi) = \frac{25}{2}$.

The function $\left(\sin 2\varphi - \frac{1}{4}\right)^2$ is largest when $\sin 2\varphi_1 = -1$, i.e. when $(x, y) = (-\sqrt{2}, \sqrt{2})$ or $= (\sqrt{2}, -\sqrt{2})$, corresponding to the value of the function

$$f(-\sqrt{2}, \sqrt{2}) = f(\sqrt{2}, -\sqrt{2}) = 0.$$

ii) Second alternative. We see that $z = xy$ runs through $[-2, 2]$, when (x, y) runs through the circle of the equation $x^2 + y^2 = 4$. This means that it suffices to check

$$g_2(z) = 12 + 2xy(1 - xy) = 12 + 2z - 2z^2 = \frac{25}{2} - 2\left(z - \frac{1}{2}\right)^2$$

for $z \in [-2, 2]$. We get the maximum for $z = \frac{1}{2}$, corresponding to the value $\frac{25}{2} - 0 = \frac{25}{2}$.

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Remark. We mention that the corresponding (x,y)-values are

$$(17.11)\quad \begin{cases} \left(\dfrac{\sqrt{5}+\sqrt{3}}{2},\dfrac{\sqrt{5}-\sqrt{3}}{2}\right), & \left(\dfrac{\sqrt{5}-\sqrt{3}}{2},\dfrac{\sqrt{5}+\sqrt{3}}{2}\right), \\ \left(-\dfrac{\sqrt{5}+\sqrt{3}}{2},\dfrac{-\sqrt{5}+\sqrt{3}}{2}\right), & \left(\dfrac{-\sqrt{5}+\sqrt{3}}{2},-\dfrac{\sqrt{5}+\sqrt{3}}{2}\right), \end{cases}$$

but there is no need at all to find these point exactly to find the maximum and the minimum. ◊

The minimum is obtained for $z=-2$, corresponding to $g_2(-2)=0$ and $(x,y)=(-\sqrt{2},\sqrt{2})$ or $(\sqrt{2},-\sqrt{2})$.

c) Maximum and minimum. Summarizing we see that the maximum is

$$S=\max\left\{0,\frac{25}{2}\right\}=\frac{25}{2}.$$

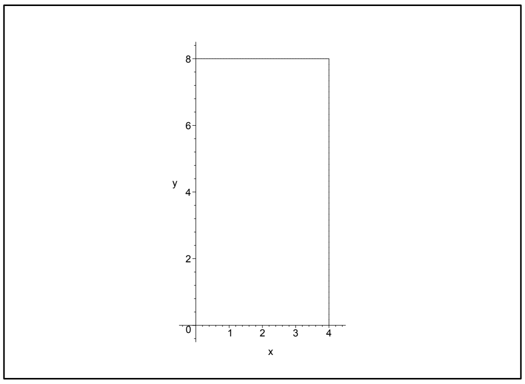
One is not asked of where the maximum is attained, so here is added that it is the value of the function at the points given by (17.11).

The minimum is

$$M=\min\{-2,0\}=-2.$$

It is the value of the functions in the points $(-1,1)$ and $(1,-1)$.

9) The domain is here the rectangle $[0,4]\times[0,8]$.



a) Stationary points. The stationary points are the solutions of the equations

$$(17.12)\quad \begin{cases} \dfrac{\partial f}{\partial x}=8y^2-y^3-3x^2y=y(8y-y^2-3x^2)=0, \\ \dfrac{\partial f}{\partial y}=16xy-3xy^2-x^3=x(16y-3y^2-x^2)=0. \end{cases}$$

We have $xy>0$ in the interior of the rectangle, so the equations are reduced to

$$y^2-8y+3x^2=0 \quad\text{and}\quad 3y^2-16y+x^2=0,$$

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hence

$$-3x^2 = y^2 - 8y = 9y^2 - 48y.$$

This is again reduced to

$$8y^2 - 40y = 8y(y-5) = 0.$$

Now $y > 0$, thus $y = 5 \in]0,8[$, hence $x^2 = 16y - 3y^2 = 80 - 75 = 5$, or (since $x > 0$), $x = \sqrt{5} \in]0,4[$. When we put $(x,y) = (\sqrt{5},5)$ into (17.12) we get

$$\frac{\partial f}{\partial x} = 5(40-25-15) = 0, \quad \frac{\partial f}{\partial y} = \sqrt{5}(80-75-5) = 0,$$

and we conclude that $(\sqrt{5},5)$ is a stationary point in the domain. The corresponding value of the function is

$$f(\sqrt{5},5) = 8\sqrt{5}\cdot 25 - \sqrt{5}\cdot 125 - 5\sqrt{5}\cdot 5 = \sqrt{5}\{200-125-25\} = 50\sqrt{5}.$$

b) Examination of the boundary. It follows from

$$f(x,y) = xy(8y - y^2 - x^2),$$

that $f(0,y) = f(x,0) = 0$.

The restriction on the boundary curve $x = 4$, $y \in [0,8]$, is

$$g(y) = -4y(16 - 8y + y^2) = -4y(y-4)^2$$

where

$$g'(y) = -4(y-4)^2 - 8y(y-4) = -4(y-4)\{y-4+2y\} = -12(y-4)\left(y - \frac{4}{3}\right).$$

Now $g'(y) = 0$ for $y = 4$ and for $y = \dfrac{4}{3}$, and the values of the function are

$$f\left(4,\frac{4}{3}\right) = -\frac{16}{3}\left(4 - \frac{4}{3}\right)^2 = -\frac{256}{27}\cdot 4 = -\frac{1024}{27}$$

and $f(4,4) = 0$, and at the end points

$$f(4,0) = 0, \qquad f(4,8) = -32\cdot 16 = -512.$$

The restriction to the boundary curve $y = 8$, $x \in [0,4]$, is

$$h(x) = 8x(64 - 64 - x^2) = -8x^3,$$

which clearly takes its maximum for $x = 0$ and minimum for $x = 4$, corresponding to the maximum $f(0,8) = 0$ and the minimum $f(4,8) = -512$.

c) Numerical comparison. The candidates of the minimum are

$$\begin{aligned} &f(\sqrt{5},5) = 50\sqrt{5}, && f(0,y) = f(x,0) = 0, \\ &f\left(4,\frac{4}{3}\right) = -\frac{1024}{27}, && f(4,8) = -256. \end{aligned}$$

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The candidates of the maximum are

$$f(\sqrt{5},5) = 50\sqrt{5}, \qquad f(0,y) = f(x,0) = f(4,4) = 0.$$

By comparison the get the minimum¿

$$M = f(4,8) = -512$$

and the maximum

$$S = f(\sqrt{5},5) = 50\sqrt{5}.$$

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Example 17.17 *We shall construct a cage for transportation of poultry from a board of the length 6 dm and the breadth b dm. The board is broken at two places and then a net of steel wire is stretched over it. Finally, two additional boards are added so that one gets a cage of a cross section of an equilateral trapeze. We want to break the given board in such a way that the volume of the cage* V *dm*^{3} *becomes as large as possible.*
First prove that

$$V(x,y) = (6-2x-x\cos y)bx\sin y.$$

Then explain why the function V *shall only be considered on the set* $[0,3]\times\left[\frac{\pi}{2},\pi\right]$. *Finally, find the maximum of the volume and the corresponding set* (x,y) *of coordinates.*

A Maximum.

D Analyze the text. Check the model and find the maximum.

By cutting the trapeze (chop off the two triangles, so one gets a rectangle) we get the height (i.e. the breadth of the trapeze)

$$h(x,y) = x\cdot\sin(\pi-y) = x\cdot\sin y.$$

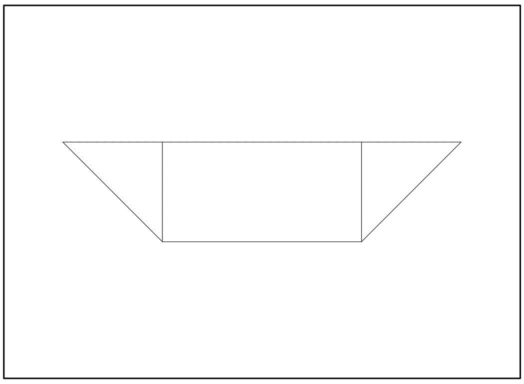

Figure 17.13: The skew lines are each of the length x, and the two obtuse angles are each of the size y.

Then compute the area,

$$\begin{aligned}A(x,y) &= (6-2x)(h(x,y)+2\cdot\frac{1}{2}\,h(x,y)\cdot x\cos(\pi-y) = (6-2x)x\sin y + x\sin y\{-x\cos y\}\\ &= (6x-2x^2)\sin y - x^2\sin y\cdot\cos y.\end{aligned}$$

Remark. It follows from the sign of the latter term that the areas of the corners are counted negatively, when $y\in\left]0,\frac{\pi}{2}\right[$, which is quite reasonable when one sketches the corresponding figure. $\Diamond$

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It follows from the above that the volume is

$$V(x,y) = b\,A(x,y) = \{(6x-2x^2)\sin y - x^2\sin y\,\cos y\}b.$$

Since $x \geq 0$ and $6-2x \geq 0$, we must have $x \in [0,3]$.

We can clearly choose y in the interval $[0,\pi]$; but since we shall find a maximum, we must have $-x^2\sin y\,\cos y \geq 0$, so $y \in \left[\frac{\pi}{3},\pi\right]$.

The task is then reduced to finding the maximum of the function

$$V(x,y) = \{(6x-2x^2)\sin y - x^2\sin y\,\cos y\}b$$

in the set $[0,3] \times \left[\frac{\pi}{2},\pi\right] = A$.

1) Stationary points. The stationary points in the interior of A are the solutions of the equations

$$\begin{cases} \dfrac{\partial V}{\partial x} = b\{(6-4x)\sin y - 2x\sin y\,\cos y\} = 0, \\[2ex] \dfrac{\partial V}{\partial y} = b\{(6x-2x^2)\cos y - x^2(\cos^2 y - \sin^2 y)\} = 0. \end{cases}$$

Since $b > 0$, $x > 0$ and $\sin y > 0$ in A°, these equations are reduced to

$$(17.13) \quad \begin{cases} (6-4x) - 2x\cos y = 0, \\[2ex] (6-2x)\cos y - x(2\cos^2 y - 1) = 0. \end{cases}$$

It follows from the former equation that

$$-1 < \cos y = \frac{3-2x}{x} = \frac{3}{x} - 2 < 0,$$

so $\frac{3}{2} < x < 3$ for possible stationary points. When the value above $\cos y$ is put into the latter equation of (17.13), then

$$\begin{aligned} 0 &= (6-2x)\cdot\frac{3-2x}{x} - x\left\{2\left(\frac{3-2x}{x}\right)^2 - 1\right\} \\ &= \frac{1}{x}\{(6-2x)(3-2x) - 2(3-2x)^2 + x^2\} \\ &= \frac{1}{x}\{(3-2x)[(6-2x)-(6-4x)] + x^2\} \\ &= \frac{1}{x}\{(3-2x)\cdot 2x + x^2\} = 2(3-2x) + x = 6-4x+x \\ &= 6-3x, \end{aligned}$$

hence $x = 2$, and thus $\cos y = \dfrac{3-4}{2} = -\dfrac{1}{2}$, corresponding to the candidate $(x,y) = \left(2,\dfrac{2\pi}{3}\right)$.

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TEST. That $\left(2, \frac{2\pi}{3}\right)$ really is a stationary point, follows from the computations

$$\begin{aligned}
\frac{\partial V}{\partial x} &= b\left\{(6-8)\sin\frac{2\pi}{3} - 2\cdot 2\,\sin\frac{2\pi}{3}\,\cos\frac{2\pi}{3}\right\} = b\,\sin\frac{2\pi}{3}\left(-2-2\cdot 2\cdot\left(-\frac{1}{2}\right)\right) = 0,\\
\frac{\partial V}{\partial y} &= b\left\{(12-8)\cos\frac{2\pi}{3} - 4\left(2\cos^2\frac{2\pi}{3}-1\right)\right\} = 4b\left\{-\frac{1}{2}-\left(\frac{1}{2}-1\right)\right\} = 0,
\end{aligned}$$

and we have tested our result. $\Diamond$

The value of the function at $\left(2, \frac{2\pi}{3}\right)$ is

$$\begin{aligned}
V\left(2, \frac{2\pi}{3}\right) &= b\left\{(12-8)\sin\frac{2\pi}{3} - 4\sin\frac{2\pi}{3}\,\cos\frac{2\pi}{3}\right\} = b\left\{4\cdot\frac{\sqrt{3}}{2} - 4\cdot\frac{\sqrt{3}}{2}\cdot\left(-\frac{1}{2}\right)\right\}\\
&= b\{2\sqrt{3}+\sqrt{3}\} = 3\sqrt{3}\,b.
\end{aligned}$$

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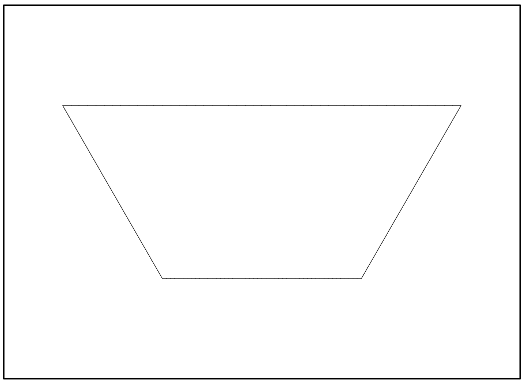

Figure 17.14: The form of the cage of maximum volume corresponding to the stationary point.

2) Examination of the boundary. We get $V(0,y) \equiv 0$ on the boundary curve $x = 0$, $y \in \left[\frac{\pi}{2}, \pi\right]$.

   On the boundary curve $x = 3$, $y \in \left[\frac{\pi}{2}, \pi\right]$, we have the restriction

$$V(3,y) = -9b \sin y \cdot \cos y = -\frac{9}{2}\, b \sin 2y,$$

   which has its minimum $V\left(3, \frac{\pi}{2}\right) = V(3,\pi) = 0$ and its maximum

$$V\left(3, \frac{3\pi}{4}\right) = \frac{9}{2}\, b.$$

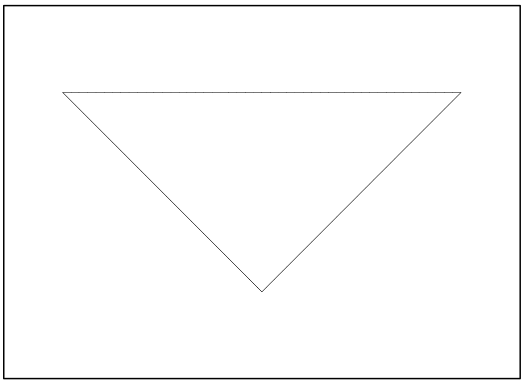

Figure 17.15: The triangle corresponding to $V\left(3, \frac{3\pi}{4}\right) = \frac{9}{2}\, b$.

Note that this case corresponds to a degenerated trapeze, i.e. to a rectangular triangle, cf. the figure.

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Figure 17.16: The rectangle corresponding to $V\left(\frac{3}{2},\frac{\pi}{2}\right) = \frac{9}{2}\,b$. The height is $\frac{3}{2}$, and each of the horizontal pieces have the length 3.

On the boundary curve $y = \frac{\pi}{2}$, $x \in [0,3]$, we get the restriction

$$V\left(x\frac{\pi}{2}\right) = (6x-2x^2)b = 2b\left\{\frac{9}{4} - \left(x-\frac{3}{2}\right)^2\right\},$$

with its minimum $V\left(3,\frac{\pi}{2}\right) = V(3,\pi) = 0$, and its maximum

$$V\left(\frac{3}{2},\frac{\pi}{2}\right) = \frac{9}{2}\,b.$$

Finally, $V(x,\pi) = 0$ on the boundary curve $y = \pi$, $x \in [0,3]$, which does not contribute to the candidates.

3) Numerical comparison. We conclude from $3\sqrt{3} > 3 \cdot \frac{3}{2} = \frac{9}{2}$ that the maximum is attained at the stationary point $(x,y) = \left(2,\frac{2\pi}{3}\right)$ with the value of the function

$$V\left(2,\frac{2\pi}{3}\right) = 3\sqrt{3}\,b.$$

The form of the corresponding cage is shown on the figure in connection with the stationary point.

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Example 17.18 *Check in each of the following cases if the given function has a maximum or a minimum or both or none of the kind. If it has a maximum or a minimum give the value of the function at these points.*

1) $f(x,y) = (x+y)\exp(-x^2-y^2)$ *for* $(x,y) \in \mathbb{R}^2$.

2) $f(x,y) = xy\exp(-x^2-y^2)$ *for* $(x,y) \in \mathbb{R}^2$.

3) $f(x,y) = \exp(x^2-y^2) - x^2 - y^2$ *for* $(x,y) \in \mathbb{R}^2$.

4) $f(x,y) = \exp(x^2+y^2) - 4xy$ *for* $(x,y) \in \mathbb{R}^2$. [cf. **Example 17.20**]

5) $f(x,y) = \dfrac{1+x^2}{x^2+y^2-2y-3}$ *for* $x^2+y^2 < 2y+3$.

6) $f(x,y) = x^3 + 2y^3$ *for* $(x,y) \in \mathbb{R}^2$.

7) $f(x,y) = 3xy + \ln(1-x^2-y^2)$ *for* $x^2+y^2 < 1$.

8) $f(x,y) = x^2 - 2x + y^2 + 3y + 5$ *for* $x^2+y^2 < 5$.

9) $f(x,y) = x + \tanh y$ *for* $x^2+y^2 < 2$.

10) $f(x,y) = x^2y - 2x^2 + 4y^2$ *for* $|x| < 2$ *og* $|y| < 1$.

11) $f(x,y) = (7x^2+4xy)\exp(-y^2)$ *for* $|x| < 1$.

A Extrema in open domain of C^∞-functions.

D Find the stationary points, if any. Check if they are extrema. Check also $f(x,y)$, when (x,y) tends towards the boundary or towards ∞ in the sense $x^2+y^2 \to +\infty$.

I 1) The domain is $\mathbb{R}^2$. It follows from the different magnitudes of the terms that

$$f(x,y) = (x+y)\exp(-x^2-y^2) \to 0 \quad \text{for } x^2+y^2 \to +\infty.$$

The stationary points are found by solving the equations

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \exp(-x^2-y^2) - 2x(x+y)\exp(-x^2-y^2) \\
&= \{1-2x(x+y)\}\exp(-x^2-y^2) = 0, \\
\frac{\partial f}{\partial y} &= \{1-2y(x+y)\}\exp(-x^2-y^2) = 0,
\end{aligned}$$

where we get the latter equation by a symmetric argument.

We conclude from the equations above that

$$2x(x+y) = 1 = 2y(x+y),$$

hence $x \neq 0$, $y \neq 0$, $x+y \neq 0$ and $y = x$, so $4x^2 = 1$. This implies that the stationary points are

$$\left(\frac{1}{2},\frac{1}{2}\right) \quad \text{and} \quad \left(-\frac{1}{2},-\frac{1}{2}\right).$$

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The values of the function are

$$f\left(\frac{1}{2},\frac{1}{2}\right)=\frac{1}{\sqrt{e}} \quad\text{and}\quad f\left(-\frac{1}{2},-\frac{1}{2}\right)=-\frac{1}{\sqrt{e}},$$

which compared with the examination of the boundary gives that the minimum is

$$M=f\left(-\frac{1}{2},-\frac{1}{2}\right)=-\frac{1}{\sqrt{e}},$$

and the maximum is

$$S=f\left(\frac{1}{2},\frac{1}{2}\right)=\frac{1}{\sqrt{e}}.$$

The function is of course 0 on the line $y=-x$.

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2) It follows from the rules of magnitudes that

$$f(x,y) = xy\exp(-x^2-y^2) \to 0 \quad \text{for } x^2+y^2 \to +\infty.$$

The stationary points are the solutions of the equations

$$\begin{aligned}\frac{\partial f}{\partial x} &= (y-2x^2y)\exp(-x^2-y^2)\\ &= y(1-2x^2)\exp(-x^2-y^2) = 0,\\ \frac{\partial f}{\partial y} &= x(1-2y^2)\exp(-x^2-y^2) = 0,\end{aligned}$$

where the latter equations follows by the symmetry.

These equations are reduced to the system

$$y(1-2x^2) = 0 \quad \text{og} \quad x(1-2y^2) = 0.$$

If $x = 0$, then $y = 0$, hence $(0,0)$ is a stationary point.
If $x = \pm\dfrac{1}{\sqrt{2}}$, then $y = \pm\dfrac{1}{\sqrt{2}}$. We find in total five stationary points,

$$(0,0), \quad \left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right), \quad \left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right), \quad \left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right), \quad \left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right).$$

The values of the function at these points are

$$f(0,0) = 0,$$

$$f\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right) = \frac{1}{2e},$$

$$f\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right) = f\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right) = -\frac{1}{2e}.$$

Summarizing the maximum is (by comparison)

$$S = f\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right) = \frac{1}{2e},$$

and the minimum is

$$M = f\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right) = f\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right) = -\frac{1}{2e}.$$

3) If $x = 0$, then

$$f(0,y) = \exp(-y^2) - y^2 \to -\infty \qquad \text{for } y \to +\infty.$$

If $y = 0$, then

$$f(x,0) = \exp(x^2) - x^2 \to +\infty \qquad \text{for } x \to +\infty.$$

We conclude that the function has neither a maximum nor a minimum in the domain $\mathbb{R}^2$.

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REMARK. Even though it is superfluous, we shall nevertheless for the exercise show how the possible stationary points are found. The corresponding equations are

$$\begin{aligned}\frac{\partial f}{\partial x} &= 2x\{\exp(x^2-y^2)-1\}=0,\\ \frac{\partial f}{\partial y} &= -2y\{\exp(x^2-y^2)+1\}=0.\end{aligned}$$

It follows from the latter equation that $y=0$, which put into the former one gives

$$2x\{\exp(x^2)-1\}=0.$$

This equation is only fulfilled for $x=0$, hence $(0,0)$ is the only stationary point. The value of the function is here $f(0,0)=0$, and it is obvious that $f(x,y)$ can be both positive and negative in any neighbourhood of $(0,0)$, so there exists no point in which a maximum or a minimum can be attained. $\Diamond$

4) It follows from the rules of magnitudes that

$$f(x,y)=\exp(x^2+y^2)-4xy\to+\infty \qquad \text{for } x^2+y^2\to+\infty,$$

and the function does not have a maximum.

The possible stationary points are the solutions of the equations

$$\begin{aligned}\frac{\partial f}{\partial x} &= 2x\exp(x^2+y^2)-4y=0,\\ \frac{\partial f}{\partial y} &= 2y\exp(x^2+y^2)-4x=0.\end{aligned}$$

These equations are reduced to

$$(17.14)\quad \left\{\begin{array}{l} x\exp(x^2+y^2)=2y,\\ y\exp(x^2+y^2)=2x.\end{array}\right.$$

We get by adding these equations,

$$(x+y)\exp(x^2+y^2)=2(x+y).$$

It follows from (17.14) that x and y must either be of the same sign or be 0. Therefore, if $x+y=0$, then $(x,y)=(0,0)$.

If $x+y\neq 0$, we get $x^2+y^2=\ln 2$. It follows in this case from (17.14) that

$$x^2\exp(x^2+y^2)=2xy=y^2\exp(x^2+y^2),$$

so $x^2=y^2$, or $y=x$. The stationary points $\neq(0,0)$ are then satisfying

$$x^2+y^2=2x^2=\ln 2,$$

i.e.

$$x=y=\pm\sqrt{\frac{\ln 2}{2}}.$$

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By insertion into (17.14) it follows that they are indeed stationary points, so we have three stationary points,

$$(0,0), \qquad \left(\sqrt{\frac{\ln 2}{2}}, \sqrt{\frac{\ln 2}{2}}\right), \qquad \left(-\sqrt{\frac{\ln 2}{2}}, -\sqrt{\frac{\ln 2}{2}}\right).$$

The values of the function here are $f(0,0) = 1$ and

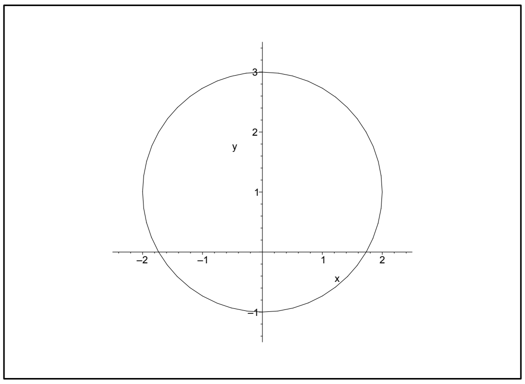
$$f\left(\sqrt{\frac{\ln 2}{2}}, \sqrt{\frac{\ln 2}{2}}\right) = f\left(-\sqrt{\frac{\ln 2}{2}}, -\sqrt{\frac{\ln 2}{2}}\right) = 2 - 2\ln 2 = 2(1-\ln 2).$$

From $\ln 2 > \frac{1}{2}$ follows that the minimum is

$$f\left(\sqrt{\frac{\ln 2}{2}}, \sqrt{\frac{\ln 2}{2}}\right) = f\left(-\sqrt{\frac{\ln 2}{2}}, -\sqrt{\frac{\ln 2}{2}}\right) = 2(1-\ln 2).$$

It was mentioned above that the function has no maximum.

5) The domain is the open disc of centrum $(0,1)$ and radius 2.



Clearly, the function

$$f(x,y) = \frac{1+x^2}{x^2+y^2-2y-3}, \qquad x^2+y^2 < 2y+3,$$

tends towards $-\infty$, when (x,y) tends to the circle $x^2+(y-1)^2 = 4$ from the inside. Therefore, the minimum does not exist.

The *stationary points* are the solutions of the equations

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{1}{(x^2+y^2-2y-3)^2}\left[2x\{x^2+y^2-2y-3\} - 2x(1+x^2)\right] = 0, \\ \frac{\partial f}{\partial y} &= \frac{1}{(x^2+y^2-2y-3)^2}\{-(1+x^2)\cdot 2(y-1)\} = 0. \end{aligned}$$

Inside the domain, these equations are reduced to

$$x\{(y-1)^2-5\} = 0 \quad \text{and} \quad y-1 = 0,$$

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hence $y = 1$ and $x = 0$. The only stationary point is the centre of the circle $(0, 1)$. This must necessarily be a maximum, because the maximum exists, and it can only be attained at a stationary point (because the function is of class C^∞). Hence the maximum is

$$S = f(0,1) = -\frac{1}{4}.$$

REMARK 1. One can also find the maximum in the following way without any calculation. Note that the numerator is positive, and the denominator is negative everywhere in the domain. Hence, we shall make the numerator as small as possible (for $x = 0$), and the denominator numerically as big as possible. The denominator can be written $(x-0)^2 + (y-1)^2 - 4$, so this situation occurs for $(x, y) = (0, 1)$. Since the optimum possibility for both the numerator and the denominator occurs for at least $x = 0$, we conclude that $(0, 1)$ is a maximum point and that the maximum is

$$S = f(0,1) = -\frac{1}{4}. \qquad \Diamond$$

REMARK 2. Another possible solution is the following: If x is kept fixed, then

$$f(x,y) = -\frac{1+x^2}{4-x^2-(y-1)^2}$$

is largest when $y = 1$, and $(x, 1)$ belongs to the domain, so the only condition on x is $|x| < 2$.

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Then the task is reduced to finding the maximum of

$$f(x,1)=\frac{x^2+1}{x^2-4}=1+\frac{5}{x^2-4}\qquad \text{for } x^2\in[0,4[.$$

This is obtained for $x=0$, so the maximum is

$$S=f(0,1)=1-\frac{5}{4}=-\frac{1}{4}.\qquad\Diamond$$

6) The function $f(x,y)=x^3+2y^3$ has neither a maximum nor a minimum in $\mathbb{R}^2$, because

$$f(x,0)=x^3\rightarrow\begin{cases}+\infty & \text{for } x\rightarrow+\infty,\\ -\infty & \text{for } x\rightarrow-\infty.\end{cases}$$

We mention – though it is not necessary – that $(0,0)$ is the only stationary point.

7) Clearly, the function

$$f(x,y)=3xy+\ln(1-x^2-y^2),\qquad x^2+y^2<1,$$

tends towards $-\infty$, when (x,y) is approaching the boundary $x^2+y^2=1$ of the unit disc (from the inside). Therefore, the function has no minimum.

Clearly, it has a maximum, because it is continuous on every closed subset of the open unit disc. We may choose this subset such that $f(x,y)=-C$ on the boundary of the subset, where $C>0$ is any c positive constant. According to the second main theorem, f has a maximum on the closed subset, and since $f(0,0)=0>-C$, we cannot have the maximum on the boundary. Since f is of class C^∞, the maximum must be attained at a stationary point.

The *stationary points* are the solutions of the equations

$$(17.15)\quad\begin{cases}\dfrac{\partial f}{\partial x}=3y-\dfrac{2x}{1-x^2-y^2}=0,\\[2mm] \dfrac{\partial f}{\partial y}=3x-\dfrac{2y}{1-x^2-y^2}=0.\end{cases}$$

Clearly, $(0,0)$ is a stationary point. It is almost obvious (due to the variation of $3xy$ in a neighbourhood of $(0,0)$) that the maximum is *not* attained at $(0,0)$.

We shall then find the stationary points $\neq(0,0)$, which must exist, cf. the discussion above. According to (17.15), such stationary points must satisfy

$$3y^2=\frac{2xy}{1-x^2-y^2}=3x^2,$$

thus $y^2=x^2$. It follows from $xy>0$ that x and y must have the same sign, so we conclude that $y=x$. By eliminating y we get

$$0=3y(1-x^2-y^2)-2x=x\{3-6x^2-2\}=x\{1-6x^2\}.$$

Since $x\neq 0$, we get $x=y=\pm\dfrac{1}{\sqrt{6}}$, and the stationary points are

$$(0,0),\qquad\left(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}}\right),\qquad\left(-\frac{1}{\sqrt{6}},-\frac{1}{\sqrt{6}}\right).$$

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The corresponding values are $f(0,0)=0$ (found previously, and we have already shown that this cannot be a maximum), and

$$f\left(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}}\right)=f\left(-\frac{1}{\sqrt{6}},-\frac{1}{\sqrt{6}}\right)=\frac{1}{2}-\ln\frac{3}{2},$$

which then necessarily must be the maximum. (This follows also from that $\ln\frac{3}{2}<\frac{1}{2}$).

8) Let

$$f(x,y)=x^2-2x+y^2-3y+5=(x-1)^2+\left(y-\frac{3}{2}\right)^2+\frac{7}{4}$$

for $x^2+y^2<(\sqrt{5})^2$.

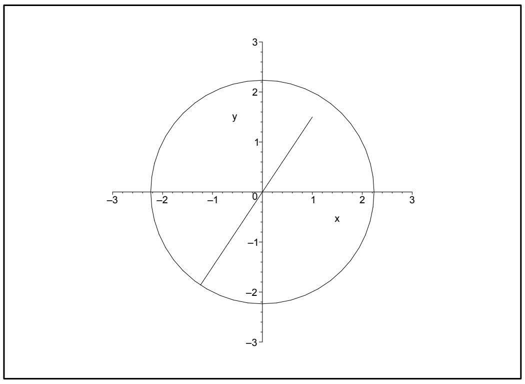

Figure 17.17: The domain with the longest possible line through $(1,\frac{3}{2})$ inside the domain.

Interpret $(x-1)^2+\left(y-\frac{3}{2}\right)^2$ as the square of the distance from (x,y) to the point $\left(1,\frac{3}{2}\right)$ in the domain. Clearly, $\left(1,\frac{3}{2}\right)$ is the only stationary point. It follows from the rearrangement that this is a minimum point corresponding to the minimum

$$M=f\left(1,\frac{3}{2}\right)=\frac{7}{4}.$$

The function $f(x,y)$ has a continuous extension to the closure of the domain given by $x^2+y^2\leq 5$. It follows from the above that the maximum exists (the second main theorem) and since there are no stationary points at hand, it must be attained at a boundary point. Then it follows from the geometric interpretation above that the maximum is obtained at the intersection of the line of the equation $y=\frac{3}{2}\,x$ and the boundary given by $x^2+y^2=5$ in the third quadrant. This implies that the function does not have a maximum in the given open domain.

Remark 1. For completeness it should be mentioned that the intersection point is

$$\left(-2\sqrt{\frac{5}{13}},-3\sqrt{\frac{5}{13}}\right).$$

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The distance from this point to $\left(1, \dfrac{3}{2}\right)$ is of geometrical reasons $\dfrac{1}{2}\sqrt{13}+\sqrt{5}$, so the maximum on the boundary (hence also in the *closed* domain is

$$f\left(-2\sqrt{\frac{5}{13}}, -3\sqrt{\frac{5}{13}}\right) = \left\{\frac{1}{2}\sqrt{13}+\sqrt{5}\right\}^2 + \frac{7}{4} = \frac{13}{4}+5+\sqrt{65}+\frac{7}{4} = 10+\sqrt{65}. \quad \Diamond$$

REMARK 2. It should be obvious that if one did not use the geometric interpretation above, then the task would give quite unpleasant computations. It is left to the reader to find the maximum on the boundary by inserting the parametric description

$$x = \sqrt{5}\cos t, \qquad y = \sqrt{5}\sin t, \qquad t \in [0, 2\pi[. \qquad \Diamond$$

9) Clearly, the function $f(x,y) = x + \tanh y$, $x^2+y^2 < 2$, has no stationary point (e.g. $\dfrac{\partial f}{\partial x} = 1 \neq 0$), and since the function is of class C^∞, and the domain is open (i.e. without boundary points) the function has neither a minimum nor a maximum in the domain.

10) Clearly, the function

$$f(x,y) = x^2y - 2x^2 + 4y^2, \qquad |x| < 2,\ |y| < 1,$$

is of class C^∞, so a possible maximum or minimum can only be attained at a stationary point. The stationary points are the solutions of the equations

$$\frac{\partial f}{\partial x} = 2xy - 4x = 2x(y-2x) = 0, \qquad \frac{\partial f}{\partial y} = x^2 + 8y = 0.$$

It follows from the former equation that either $x = 0$ or $y = 2x$.

a) If $x = 0$, then we get $y = 0$ from the latter equation and $(0,0)$ is a stationary point.

b) If $y = 2x$ is put into the latter equation, we get

$$0 = x^2 + 16x = x(x+16),$$

so either $x = 0$ (and $y = 0$ again as above) or $x = -16$. The latter is not possible inside the domain.

The only stationary point is $(0,0)$. The value of the function is here

$$f(0,0) = 0.$$

We conclude the task in the following way: The two restrictions

$$f(0,y) = 4y^2, \qquad |y| < 1,$$

and

$$f(x,0) = -2x^2, \qquad |x| < 2,$$

attain both positive and negative values at any point close to $(0,0)$, hence $f(0,0) = 0$ is neither a maximum nor a minimum. Since $(0,0)$ is the only possibility of extremum, there does not exist any.

ALTERNATIVELY (the standard procedure) the function is extended continuously to the boundary, and then we we continue by examining the values on the boundary.

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a) If $x = -2$ and $y \in [-1, 1]$ the restriction is

$$g_1(y) = f(-2, y) = 4y - 8 + 4y^2$$

where $g_1'(y) = 4 - 8y = 0$ for $y = \frac{1}{2}$. The value is

$$f\left(-2, \frac{1}{2}\right) = g_1\left(\frac{1}{2}\right) = 4 \cdot \frac{1}{2} - 8 + 4\left(\frac{1}{2}\right)^2 = 2 - 8 + 1 = -5.$$

At the end points of the interval we get the values of the function

$$f(-2, -1) = g_1(-1) = -4 - 8 + 4 = -8,$$

$$f(-2, 1) = g_1(1) = 4 - 8 + 4 = 0.$$

b) If $y = 1$ and $x \in [-2, 2]$ the restriction is

$$g_2(x) = f(x, 1) = x^2 - 2x^2 + 4 = 4 - x^2,$$

where $g_2(x) = -2x = 0$ for $x = 0$. The relevant values of the function are

$$f(-2, 1) = g_2(-2) = 4 - 8 + 4 = 0,$$

$$f(0, 1) = g_2(0) = 4,$$

$$f(2, 1) = g_2(2) = 4 - 8 + 4 = 0.$$

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Even if we have not examined the remaining two boundary curves, we can finish the task now, because $(0,0)$ is the only stationary point with the value of the function $f(0,0) = 0$. The boundary does *not* belong to the domain, and by continuous extensions we find the values at specially chosen boundary points

$$f(-2,-1) = -8 < f(0,0) = 0 < f(0,1) = 4.$$

Accordingly, f has neither a maximum nor a minimum in the open domain. In fact, due to the continuity we can inside the domain obtain values of the function as close to -8 as well to 4 as we wish.

11) The function

$$f(x,y) = (7x^2 + 4xy)\exp(-y^2), \qquad |x| < 1,$$

is of class C^∞, and it can be extended by the same definition to all of $\mathbb{R}^2$. This domain is open, so a possible maximum or minimum can only be attained at a stationary point.

Possible stationary points are the solutions of the equations

$$\frac{\partial f}{\partial x} = (14x + 4y)\exp(-y^2) = 0, \qquad \frac{\partial f}{\partial y} = \{4x - 2y(7x^2 + 4xy)\}\exp(-y^2).$$

Now $\exp(-y^2) \neq 0$, so these equations are equivalent to

$$7x = -2y \qquad \text{and} \qquad x\{2 - 7xy - 4y^2\} = 0.$$

If we only eliminate $7x$ in the term $7xy$ of the latter equation, we get

$$0 = x\{2 + 2y^2 - 4y^2\} = 2x\{1 - y^2\}.$$

Combining this with the equation $7x = -2y$ we get the possibilities

$$x = 0, \qquad \text{i.e. } y = 0, \qquad \text{hence } (x,y) = (0,0),$$

$$y = 1, \qquad \text{i.e. } x = -\frac{2}{7}, \quad \text{hence } (x,y) = \left(-\frac{2}{7}, 1\right),$$

$$y = -1, \quad \text{i.e. } x = \frac{2}{7}, \qquad \text{hence } (x,y) = \left(\frac{2}{7}, -1\right).$$

All three stationary points lie in the open domain, and the values of the function are

$$f(0,0) = 0,$$

and

$$f\left(-\frac{2}{7}, 1\right) = f(\left(\frac{2}{7}, -1\right) = \left(7 \cdot \frac{4}{49} - 4 \cdot \frac{2}{7} \cdot 1\right)\exp(-1) = -\frac{6}{7e}.$$

Examination of the boundary. As mentioned above the function can be extended to the boundary. As $f(-x,y) = f(x,y)$, it suffices to examine the restriction

$$g(y) = f(1,y) = (7 + 4y)\exp(-y^2), \qquad y \in \mathbb{R}.$$

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From

$$\begin{aligned} g'(y) &= \{4-2y(7+4y)\}\exp(-y^2) = (4-14y-8y^2)\exp(-y^2) \\ &= \left\{\left(y+\frac{7}{8}\right)^2 - \left(\frac{9}{8}\right)^2\right\}\exp(-y^2), \end{aligned}$$

follows $g'(y) = 0$ for

$$y = -\frac{7}{8} \pm \frac{9}{8} = \begin{cases} \dfrac{1}{4}, \\ -2, \end{cases}$$

corresponding to the boundary values

$$f\left(1,\frac{1}{4}\right) = g\left(\frac{1}{4}\right) = 8\exp\left(-\frac{1}{16}\right) > 0,$$

and

$$f(1,-2) = g(-2) = -\exp(-4).$$

Finally, it follows by the different magnitudes of the terms that

$$f(x,y) \to 0 \qquad \text{for } |y| \to +\infty.$$

Then by a numerical comparison

$$\begin{aligned} -\frac{6}{7e} &= f\left(-\frac{2}{7},1\right) = f\left(\frac{2}{7},-1\right) < -\frac{1}{e^4} = f(1,-2) < 0 = f(0,0) \\ &< 8\exp\left(-\frac{1}{16}\right) = f\left(1,\frac{1}{4}\right). \end{aligned}$$

Since $\left(1,\frac{1}{4}\right)$ is a boundary point which is not included in the domain, and $f\left(1,\frac{1}{4}\right)$ is the maximum in the closure, the function has no maximum in the domain given by $|x| < 1$.

On the other hand,

$$M = f\left(-\frac{2}{7},1\right) = f\left(\frac{2}{7},-1\right) = -\frac{6}{7e}$$

is a minimum in both the closed and open set, so the minimum exists.

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Example 17.19 *Find in each of the following cases the largest volume of a rectangular box for which the indicated condition is fulfilled.*

1) The sum of the 12 edges is given equal to $12a$.

2) The area of the surface of the box is given and equal to $6a^2$.

3) The length of the space diagonal of the box is given and equal to a.

A Maximum.

D Put the box into a rectangular coordinate system with one corner at $(0,0,0)$ and where the corresponding edges lie along the axes in the positive sense, i.e. the box can be described as the domain

$$[0,x]\times[0,y]\times[0,z].$$

Find the volume as a function of the edges. Exploit the condition in each sub-question to eliminate one of the variables. Indicate the domain, in which the remaining edges can vary. Finally, find the maximum.

I When the lengths of the edges are x, y, $z \geq 0$, then the volume is given by

$$V(x,y,z) = xyz.$$

1) The condition that the sum of the 12 edges is equal to $12a$ is written

$$4(x+y+z) = 12a, \qquad \text{i.e.} \qquad x+y+z = 3a.$$

Remark. For symmetrical reasons we may expect that the solution is given by $x = y = z = a$. The remaining part of the task is to prove that this hunch in this case is correct. $\Diamond$

First eliminate z,

$$z = 3a - x - y \geq 0, \qquad \text{i.e. } x \geq 0, \quad y \geq 0, \quad x+y \leq 3a.$$

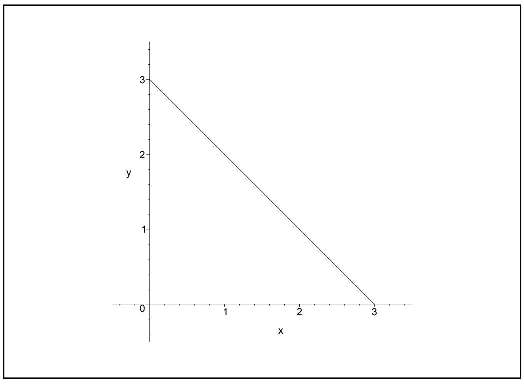

Figure 17.18: The triangle in which we shall find the maximum of $f(x,y) = xy(3a-x-y)$.

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By insertion we see that we shall find the maximum of the function

$$f(x, y) = xy(3a - x - y), \qquad x \geq 0,\ y \geq 0,\ x + y \leq 3a.$$

We have $f(x, y) > 0$ in the interior of the triangle, and on we boundary we have $f(x, y) = 0$. Now $f(x, y)$ is of class C^∞, so the maximum exists (second main theorem) and it can only be attained at an interior stationary point.

The equations of the stationary points are

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 3ay - 2xy - y^2 = y(3a - 2x - y) = 0, \\
\frac{\partial f}{\partial y} &= x(3a - x - 2y),
\end{aligned}$$

where we immediately get the latter expression by the symmetry.
Since $x > 0$ and $y > 0$ in the interior of the triangle A we get the reduced equations

$$2x + y = 3a \qquad \text{and} \qquad x + 2y = 3a,$$

and we find (a we guessed) $x = y = a$. The only stationary point is (a, a) corresponding to $z = 3a - a - a = a$, and as mentioned above this corresponds to a maximum,

$$S = V(a, a, a) = a^3.$$

2) In this case the condition is

$$2(xy + yz + xz) = 6a^2.$$

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REMARK. Due to the symmetry we may again expect the solution to be $x = y = z = a$. This satisfies at least the condition. The remaining part of the task is to prove that also this hunch is correct. $\Diamond$.

It follows from the condition that

$$z = \frac{3a^2 - xy}{x+y},$$

where $xy \le 3a^2$, $x \ge 0$, $y \ge 0$ and $(x,y) \ne (0,0)$, hence a troublesome expression.

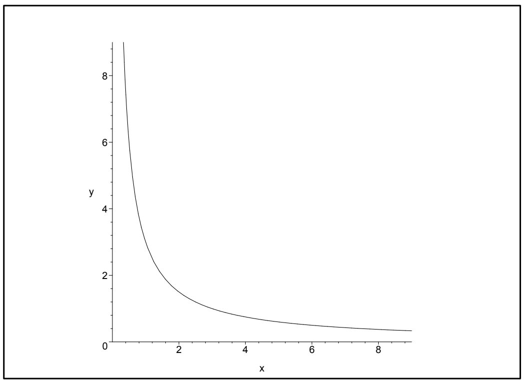

Figure 17.19: The infinite domain of the function in **Example 17.19.2**.

The volume function is given in this domain by

$$f(x,y) = xy \cdot \frac{3a^2 - xy}{x+y} = \frac{3a^2xy - x^2y^2}{x+y}.$$

We conclude from

$$0 \le f(x,y) \le \frac{y}{x+y} \cdot 3a^2 \cdot x \le x \cdot 3a^2,$$

that $f(x,y)$ can be extended by continuity from the first quadrant to the positive X and Y axes supplied by the point $(0,0)$ by putting the value of the function equal to 0. We have also the value 0 of the function of the branch of the hyperbola $xy = 3a^2$ which lies in the first quadrant. Finally, $f(x,y) \to 0$, when either $x \to +\infty$ or $y \to +\infty$ inside the open domain. Since $f(x,y) > 0$ is of class C^∞ in the open domain, it follows from the second main theorem that $f(x,y)$ has a maximum, which necessarily must be attained at a stationary point in the interior of the domain.

The equations of the stationary points are

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{1}{(x^2+y^2)} \left\{ (3a^2y - 2sy^2)(x+y) - (3a^2xy - x^2y^2) \right\} \\
&= \frac{y}{(x+y)^2} \left\{ 3a^2(x+y) - 2x^2y - 2xy^2 - 3a^2x + x^2y \right\} \\
&= \frac{y}{(x+y)^2} \left\{ 3a^2y - x^2y - 2xy^2 \right\} \\
&= \frac{y^2}{(x+y)^2} \left\{ 3a^2 - x^2 - 2xy \right\} = 0,
\end{aligned}$$

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and of symmetrical reasons,

$$\frac{\partial f}{\partial y} = \frac{x^2}{(x+y)^2}\left\{3a^2 - y^2 - 2xy\right\} = 0.$$

Since $x \neq 0$, $y \neq 0$ and $x + y \neq 0$ in the interior of the domain, these equations are reduced to

$$x^2 + 2xy = 3a^2, \qquad y^2 + 2xy = 3a^2,$$

accordingly $x^2 = y^2$, or $y = x$, because both x and y are positive. This implies that $3x^2 = 3a^2$, so $x = y = a$, and then finally,

$$z = \frac{3a^2 - a^2}{2a} = a.$$

Since (a, a) is the only stationary point, the maximum must be attained here. We find as expected that the maximum is

$$S = V(a, a, a) = a^3.$$

3) The condition is here that the length of the space diagonal of the box is equal to a, i.e.

$$\sqrt{x^2 + y^2 + z^2} = a.$$

Remark. As before the strong symmetry suggests that the solution must satisfy $x = y = z = \frac{a}{\sqrt{3}}$. We shall again prove this hunch. $\Diamond$.

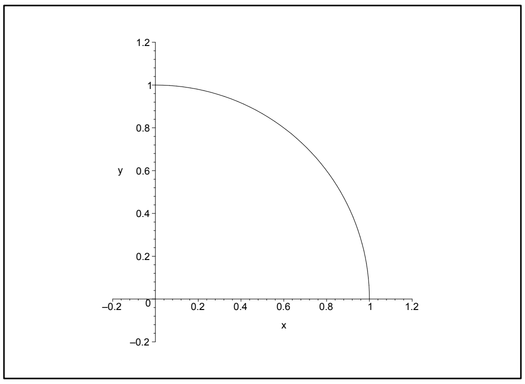

Figure 17.20: The domain of the function $xy\sqrt{a^2 - x^2 - y^2}$.

It follows that

$$z = \sqrt{a^2 - x^2 - y^2}, \qquad x \geq 0,\ y \geq 0,\ x^2 + y^2 \leq a^2.$$

By insertion we get that the task is reduced to finding the maximum of the function

$$f(x, y) = xy\sqrt{a^2 - x^2 - y^2}$$

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in the closed quarter disc

$$A = \{(x,y) \mid x \geq 0,\, y \geq 0,\, x^2 + y^2 \leq a^2\}.$$

Clearly, $f(x,y) = 0$ on the boundary ∂A of A. We have $f(x,y) > 0$ in the interior of A. Since f is of class C^∞ in the interior of A, and is continuous on A, it follows from the second main theorem that the maximum (which exists) must be attained at a stationary point in A°.

The stationary points satisfy the equations

$$\begin{aligned}
\frac{\partial f}{\partial x} &= y\sqrt{a^2 - x^2 - y^2} - \frac{x^2 y}{\sqrt{a^2 - x^2 - y^2}} = \frac{y}{\sqrt{a^2 - x^2 - y^2}}\{a^2 - 2x^2 - y^2\} = 0, \\
\frac{\partial f}{\partial y} &= \frac{x}{\sqrt{a^2 - x^2 - y^2}}\{a^2 - x^2 - 2y^2\} = 0,
\end{aligned}$$

where the latter equation follows by the symmetry.
We then derive the equations

$$y = 0 \qquad \text{or} \qquad 2x^2 + y^2 = a^2,$$

and

$$x = 0 \qquad \text{or} \qquad x^2 + 2y^2 = a^2.$$

Clearly, one of the coordinates must be 0, so

$$(0,0), \quad (a,0), \quad (-a,0), \quad (0,a), \quad (0,-a),$$

are all the possible stationary points. Unfortunately they all lie outside the interior of the domain, so none of the counts in the following. (Three of them lie on the boundary, because $f(x,y) = 0$, and the remaining two points do not lie at all in the closure of A).

Our only possibility is obtained when

$$2x^2 + y^2 = a^2 \qquad \text{and} \qquad x^2 + 2y^2 = a^2.$$

We get by a subtraction, $y^2 = x^2$, so $x^2 = y^2 = \dfrac{a^2}{3}$. Now $x^2 + y^2 = \dfrac{2}{3}\,a^2 < a^2$, so $\left(\dfrac{a}{\sqrt{3}}, \dfrac{a}{\sqrt{3}}\right)$ is the only stationary point in A°, corresponding to the maximum

$$S = f\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right) = \frac{a^3}{3\sqrt{3}} \qquad \left[= \left(\frac{a}{\sqrt{3}}\right)^3 \right],$$

and we see that $x = y = z = \dfrac{a}{\sqrt{3}}$ in agreement with our earlier hunch.

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Example 17.20 *Show by applying polar coordinates that the function*

$$f(x,y) = \exp(x^2+y^2) - 4xy, \qquad y \geq 0, \quad y \geq 0,$$

has a minimum and find this minimum. [cf. **Example 17.18.4**].

A Minimum by polar coordinates.

D Introduce the polar coordinates. Find the stationary points. Examine the boundary and what happens when $\varrho \to +\infty$.

I The domain is the first quadrant. This is described in polar coordinates by

$$x = \varrho\cos\varphi, \quad y = \varrho\sin\varphi, \qquad \varrho \in [0,+\infty[, \quad \varphi \in \left[0, \frac{\pi}{2}\right].$$

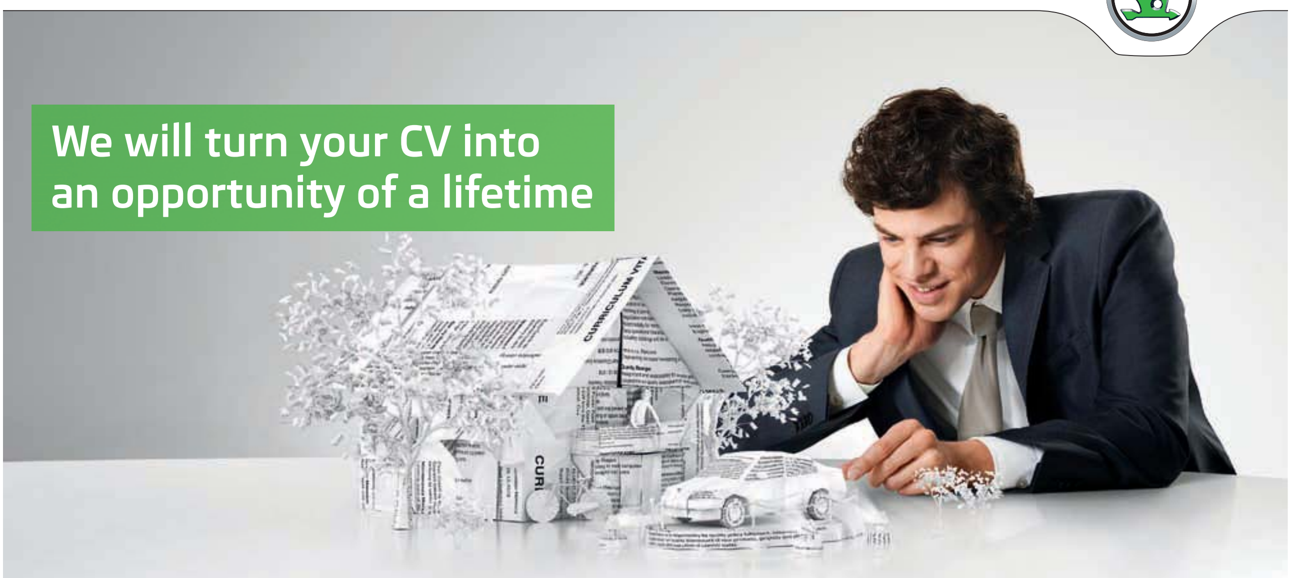
The function is

$$g(\varrho,\varphi) = f(x,y) = \exp(\varrho^2) - 4\varrho^2\cos\varphi\sin\varphi = \exp(\varrho^2) - 2\varrho^2\sin 2\varphi.$$

The equations of the stationary points become

$$(17.16) \quad \begin{cases} \dfrac{\partial g}{\partial \varrho} &= 2\varrho\exp(\varrho^2) - 4\varrho\sin 2\varphi = 2\varrho\{\exp(\varrho^2) - 2\sin 2\varphi\} = 0, \\[3mm] \dfrac{\partial g}{\partial \varphi} &= -4\varrho^2\cos 2\varrho = 0. \end{cases}$$

SIMPLY CLEVER
ŠKODA



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It follows from $\varrho > 0$ in the interior of the domain and the latter equation of (17.16) that the condition is

$$\cos 2\varphi = 0, \qquad \text{i.e.} \qquad \sin 2\varphi = \pm 1,$$

where the former equation of (17.16) shows that only $\sin 2\varphi = +1$ can be applied. Then

$$\varphi = \frac{\pi}{4} \in \left[0, \frac{\pi}{2}\right].$$

We get by insertion into for former equation of (17.16),

$$\exp(\varrho^2) = 2, \qquad \text{hence} \qquad \varrho = \sqrt{\ln 2}.$$

Thus, the only stationary point is

$$(\varrho, \varphi) = \left(\sqrt{\ln 2}, \frac{\pi}{4}\right),$$

corresponding in rectangular coordinates to

$$(x, y) = \left(\sqrt{\frac{\ln 2}{2}}, \sqrt{\frac{\ln 2}{2}}\right).$$

The value of the function is here

$$g\left(\sqrt{\ln 2}, \frac{\pi}{4}\right) = \exp(\ln 2) - 2\ln 2 = 2(1 - \ln 2).$$

If $\varrho \to +\infty$, then obviously $g(\varrho, \varphi) \to +\infty$.

We have on the boundary curves $\varphi = 0$ and $\varphi = \dfrac{\pi}{2}$, $\varrho \in [0, +\infty[$, that

$$g(\varrho, 0) = g\left(\varrho, \frac{\pi}{2}\right) = \exp(\varrho^2), \qquad \varrho \in [0, +\infty[,$$

so the minimum value is here $g(0,0) = 1 > 2(1 - \ln 2)$.

Summarizing, $f(x, y) = g(\varrho, \varphi)$ has a minimum in A for

$$(\varrho, \varphi) = \left(\sqrt{\ln 2}, \frac{\pi}{4}\right),$$

corresponding in rectangular coordinates to

$$(x, y) = \left(\sqrt{\frac{\ln 2}{2}}, \sqrt{\frac{\ln 2}{2}}\right).$$

The value of the function is here

$$g\left(\sqrt{\ln 2}, \frac{\pi}{4}\right) = f\left(\sqrt{\frac{\ln 2}{2}}, \sqrt{\frac{\ln 2}{2}}\right) = 2(1 - \ln 2),$$

in accordance with the result of **Example 17.18.4**.

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ALTERNATIVELY it is also possible only to use rectangular coordinates, and this is actually not that difficult. The equations of the stationary points

$$\frac{\partial f}{\partial x} = 2x\exp(x^2+y^2) - 4y = 0, \quad \frac{\partial f}{\partial y} = 2y\exp(x^2+y^2) - 4x = 0,$$

i.e.

$$x\exp(x^2+y^2) = 2y, \qquad y\exp(x^2+y^2) = 2x,$$

so

$$2y^2 = xy\exp(x^2+y^2) = 2x^2,$$

and hence $y = x$, because we only consider the open first quadrant. Then

$$x\exp(2x^2) = 2x, \quad \text{i.e.} \quad \exp(2x^2) = 2,$$

hence

$$y = x = +\sqrt{\frac{\ln 2}{2}}.$$

The only stationary point in the open first quadrant is

$$(x,y) = \left(\sqrt{\frac{\ln 2}{2}}, \sqrt{\frac{\ln 2}{2}}\right).$$

On the boundary either $f(x,0) = \exp(x^2)$, or $f(0,y) = \exp(y^2)$, with the minimum value $f(0,0) = 1 > 2(1 - \ln 2)$, so the minimum value is attained at the stationary point in the first quadrant.

Example 17.21 *Find the maximum and the minimum of the function*

$$f(x,y,z) = x^2+y^2+z^2, \quad x \geq 0, \quad y \geq 0, \quad z \geq 0, \quad x+y+z \leq 1.$$

A Maximum and minimum.

D The existence follows from the second main theorem. Then either argue geometrically, or use the standard method.

I The function $f(x,y,z) = x^2+y^2+z^2$ is continuous on the given closed and bounded domain. It follows from the second main theorem that f has both a maximum and a minimum.

1) THE ELEGANT GEOMETRICAL SOLUTION. The value

$$f(x,y,z) = x^2+y^2+z^2$$

can be interpreted as the square of the distance from $(0,0,0)$ to (x,y,z). This distance is smallest $(= 0)$ for $(x,y,z) = (0,0,0)$, and largest $(= 1)$ at the other corners $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ of the tetrahedron which describes A.

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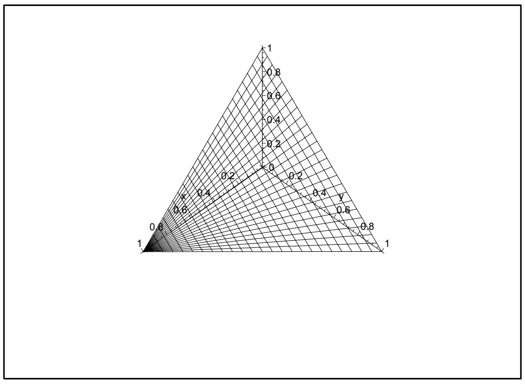

Figure 17.21: The domain.

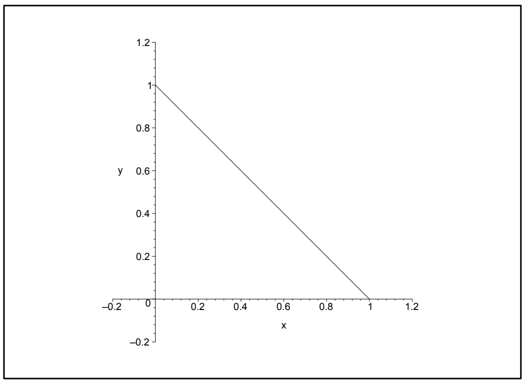

Figure 17.22: The projection onto the XY plane.

2) THE STANDARD PROCEDURE. It follows immediately that $(0,0,0)$ is the only possible stationary point, but since it lies on the boundary it is a matter of definition if it should be counted as a stationary point or not. It will always be treated as a boundary point, when we examine the boundary.

EXAMINATION OF THE BOUNDARY.

a) If $z = 0$, then $f(x,y,0) = x^2 + y^2$, which on the triangle in the XY plane is smallest at $(0,0)$ and largest at $(1,0)$ and $(0,1)$.
This is immediately seen, and it can also be obtained by another examination of the boundary in the XY plane on the triangle shown on the figure.

b) Due to the symmetry the same is true for the surfaces $x = 0$ and $y = 0$.

c) If $z = 1 - x - y$, then we get the same parametric domain as the domain above on the figure in the XY plane, and the restriction is given by

$$g(x,y) = f(x,y,1-x-y) = x^2 + y^2 + (1-x-y)^2.$$

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The equations of the possible stationary points are

$$\begin{aligned}\frac{\partial g}{\partial x} &= 2x - 2(1-x-y) = 2(2x+y-1) = 0,\\ \frac{\partial g}{\partial y} &= 2(x+2y-1) = 0,\end{aligned}$$

(the latter equation by symmetry). Accordingly, $y = x$ and $3(x+y) = 2$, i.e. $x = y = \dfrac{1}{3}$. Then by insertion,

$$g\left(\frac{1}{3},\frac{1}{3}\right) = 3 \cdot \frac{1}{3^3} = \frac{1}{3}.$$

i) We get on the boundary $x + y = 1$,

$$g(x, 1-x) = x^2 + (1-x)^2, \qquad x \in [0,1],$$

with its minimum

$$g\left(\frac{1}{2},\frac{1}{2}\right) = 2 \cdot \frac{1}{4} = \frac{1}{2} \qquad \text{for } x = \frac{1}{2},$$

and its maximum for $x = 0$ or $x = 1$, corresponding to

$$g(0,1) = g(1,0) = 1.$$

ii) We also get the restriction

$$\varphi(x) = x^2 + (1-x)^2, \qquad x \in [0,1],$$

on the boundary $y = 0$, and on the boundary $x = 0$ we get

$$\psi(y) = y^2 + (1-y)^2, \qquad y \in [0,1].$$

Anyone of these will lead to precisely the same investigation as above.

By comparison of these values we conclude that the minimum is

$$M = f(0,0,0) = 0,$$

and the maximum is

$$S = f(1,0,0) = f(0,1,0) = f(0,0,1) = 1.$$

The latter method is rather troublesome compared to the geometric interpretation.

Example 17.22 *Prove that the function*

$$f(x,y) = x^2 + 2y^2 - 2x, \qquad (x,y) \in \mathbb{R}^2,$$

has both a maximum and a minimum on the point set

$$A = \{(x,y) \mid x \geq 0,\, x^2 + y^2 \leq 2\},$$

and find these values.

A The second main theorem. Maximum and minimum.

D Apply the second main theorem for continuous functions. Then find the maximum and minimum.

I Since $f(x,y)$ is real and continuous on the closed and bounded domain A, it follows from the second main theorem for continuous functions that f has a maximum and a minimum on A. This proves the existence. We shall in the following give two methods for the explicit determination of these expressions.

1) Geometrical consideration. First rewrite f in the following way,

$$f(x,y) = x^2 + 2y^2 - 2x = (x-1)^2 + 2y^2 - 1 = 2\left\{\left(\frac{x-1}{\sqrt{2}}\right)^2 + y^2\right\} - 1.$$

It follows immediately from the first rearrangement that the minimum is attained at the point $(1,0) \in A$ of the value of the function

$$M = f(1,0) = -1.$$

From the latter rearrangement follows that

$$f(x,y) = 2\alpha^2 - 1$$

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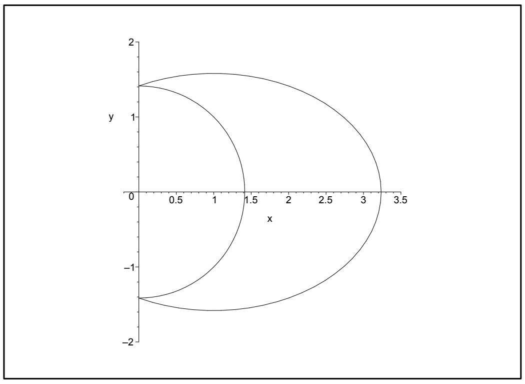

Figure 17.23: The largest possible ellipse, which intersects (touches) the domain.

is constant for every point on the ellipse of the equation

$$\left(\frac{x-1}{\sqrt{2}}\right)^2 + y^2 = \alpha^2.$$

Therefore, the maximum must be attained on the largest of these ellipses (characterized by $\alpha > 0$ being largest), which has points in common with A. It follows immediately from the figure that these maximum points must be $(x, y) = (0, \pm\sqrt{2})$ on the Y axis. Hence the maximum value is attained at these points,

$$S = f(0, \pm\sqrt{2}) = 4.$$

2) THE STANDARD PROCEDURE. The function is of class C^∞. Therefore we shall only find the stationary points in A° followed by an examination of the boundary and numerical comparisons.

a) STATIONARY POINTS. The possible stationary points are the solutions of the equations

$$\frac{\partial f}{\partial x} = 2(x-1) = 0, \qquad \frac{\partial f}{\partial y} = 4y = 0.$$

It follows that $(1, 0)$ is the only stationary point in A°. The value of the function is here

$$f(1, 0) = -1.$$

b) EXAMINATION OF THE BOUNDARY.

i) The restriction on the boundary $x = 0$, $y \in [-\sqrt{2}, \sqrt{2}]$, is

$$f(0, y) = 2y^2,$$

which has its minimum $f(0, 0) = 0$, and its maximum

$$f(0, \sqrt{2}) = f(0, -\sqrt{2}) = 4.$$

ii) The restriction on the boundary curve $x^2 + y^2 = 2$, $x \in [0, \sqrt{2}]$, is

$$\begin{aligned} g(x) &= f(x, y) = x^2 + 2y^2 - 2x = 2(x^2 + y^2) - x^2 - 2x \\ &= 4 + 1 - (x+1)^2 = 5 - (x+1)^2, \end{aligned}$$

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which has the minimum

$$g(\sqrt{2}) = 5 - (\sqrt{2}+1)^2 = 2 - 2\sqrt{2} = -2(\sqrt{2}-1) > -1,$$

and its maximum $g(0) = 4$, corresponding to $y = \pm\sqrt{2}$.

c) Numerical comparison. Summarizing, the minimum in A is

$$M = f(1,0) = -1,$$

and the maximum is

$$S = f(0,\sqrt{2}) = f(0,-\sqrt{2}) = 4.$$

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Example 17.23 *Given the function*

$$f(x,y) = x^2 + y^2 + e^{xy}, \qquad (x,y) \in \mathbb{R}^2.$$

First show that f does not have a maximum in $\mathbb{R}^2$. Then prove the following:

1) The function f has a minimum M on the disc $A = \overline{K}((0,0);10)$.

2) M is smaller than 100.

3) We have the estimate $f(x,y) \geq 100$ in the point set $\mathbb{R}^2 \setminus A$.

Finally, check if f has a minimum in $\mathbb{R}^2$.

A Extrema.

D Prove that $f(x,y) \to +\infty$ for $x^2+y^2 \to +\infty$. Then prove 1) – 3). Finally, argue for a minimum.

I Clearly,

$$(17.17)\quad f(x,y) = x^2 + y^2 + e^{xy} > x^2 + y^2,$$

thus $f(x,y) \to +\infty$ for $x^2+y^2 \to +\infty$. It follows that $f(x,y)$ has no maximum in $\mathbb{R}^2$.

1) Since $f(x,y)$ is continuous and $A = \overline{K}((0,0);10)$ is a closed and bounded set, $f(x,y)$ has according to the second main theorem for continuous functions a minimum value M on A.
2) Clearly, $M \leq f(0,0) = 1 < 100$.
3) If $(x,y) \in \mathbb{R}^2 \setminus A$, then it follows directly from (17.17) that

$$f(x,y) > 10^2 = 100.$$

Now $\mathbb{R}^2 = A \cup (\mathbb{R}^2 \setminus A)$, and $f(x,y) > 100$ on the entire $\mathbb{R}^2 \setminus A$, while there are points in A, for which $f(x,y) < 100$. Therefore, a possible minimum must lie in A, and since it exists according to 1) and is equal to M, we conclude that f has the minimum M in all of $\mathbb{R}^2$.

Additional remark. Since $f(x,y)$ is of class C^∞, the minimum M is attained at a stationary point. The equations of the stationary points are

$$(17.18)\quad \begin{cases} \dfrac{\partial f}{\partial x} = 2x + ye^{xy} = 0, \\[2ex] \dfrac{\partial f}{\partial y} = 2y + xe^{xy} = 0, \end{cases}$$

which has only $(0,0)$ as a stationary point, so

$$M = f(0,0) = 1.$$

That $(0,0)$ is the only stationary point follows from the following: Clearly, $xy \leq 0$. We have according to (17.18),

$$2x^2 = -xye^{xy} = 2y^2,$$

so $y = -x$. By insertion into the former equation of (17.18) we get

$$0 = 2x - x\exp(-x^2) = x(2 - \exp(-x^2)).$$

From $\exp(-x^2) < 2$ follows that $x = 0$, and thus $y = 0$, so $(0,0)$ is the only stationary point. $\Diamond$

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Example 17.24 *It is well-known that if a C^1-function $g : \mathbb{R} \to \mathbb{R}$ has precisely one stationary point x_0, which is a local minimum point x_0, then g has a global minimum at x_0.*
Show by considering

$$f(x,y) = x^2 + y^2(1+x)^3, \qquad (x,y) \in \mathbb{R}^2,$$

that no such result exists in general for functions $f : \mathbb{R}^2 \to \mathbb{R}$.

A Another illustration of the difference between one and several variables.

D Show that the given function f has precisely one stationary point in which there is a local minimum, and that this minimum is *not* a global minimum.

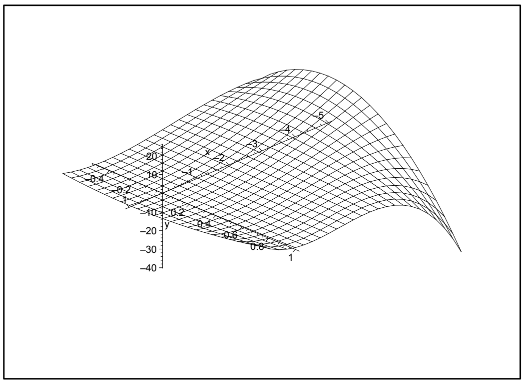

Figure 17.24: How the surface of the graph might look like for a counterexample.

I When

$$f(x,y) = x^2 + y^2(1+x)^3,$$

we get the following equations of the stationary points,

$$\frac{\partial f}{\partial x} = 2x + 3y^2(1+x)^2 = 0, \qquad \frac{\partial f}{\partial y} = 2y(1+x^3) = 0.$$

The latter equation shows that either $y = 0$ or $x = -1$.
If we put $y = 0$ into the former equation, we get $x = 0$, so $(0,0)$ is a stationary point.
If $x = -1$, we conclude from the former equation that $\dfrac{\partial f}{\partial x} = -2 \neq 0$.
This shows that $(0,0)$ is the only stationary point. The value of the function is here $f(0,0) = 0$.

Clearly, the approximating polynomial of at most second degree from $(0,0)$ is

$$P_2(x,y) = x^2 + y^2.$$

This structure shows that $(0,0)$ is a (local) minimum point.

We then get along the restriction $y = 1$ that

$$f(x,1) = x^2 + (x+1)^3 \to -\infty \qquad \text{for } x \to -\infty,$$

so $f(x,y)$ does not have a global minimum.

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Example 17.25 *Find the range of*

$$f(x,y) = -3y + 4y^2 + x^2y + y^3$$

on the open disc $K((0,0);1)$*, and on* $\mathbb{R}^2$*, resp.*

A Ranges.

D Find the possible stationary points; examine the boundary. Apply the main theorems for continuous functions.

I The function is of class C^∞ in $\mathbb{R}^2$. Restricted to the *closed* disc $\overline{K}((0,0);1)$ we have according to the second main theorem both a maximum and a minimum, and these are either attained at a stationary point or on the boundary.

Since the domain is connected, it follows from the first main theorem for continuous functions that the range is an interval.

Stationary points. The equations of the stationary points are

$$\frac{\partial f}{\partial x} = 2xy = 0, \qquad \frac{\partial f}{\partial y} = -3 + 8y + x^2 + 3y^2 = 0.$$

If $y = 0$, then $x = \pm\sqrt{3}$.

If $x = 0$, then $3y^2 + 8y - 3 = 0$, i.e. either $y = -3$ or $y = \dfrac{1}{3}$.

The stationary points are

$$(\sqrt{3}, 0), \quad (-\sqrt{3}, 0), \quad \left(0, \frac{1}{3}\right), \quad (0, -3).$$

Of these, only $\left(0, \dfrac{1}{3}\right)$ belongs to the open unit disc. The value of the function is here

$$f\left(0, \frac{1}{3}\right) = -3 \cdot \frac{1}{3} + 4 \cdot \frac{1}{9} + 0 + \frac{1}{27} = -\frac{14}{27}.$$

Examination of the boundary. We have on the boundary $x^2 + y^2 = 1$, which can also be written $x^2 = 1 - y^2$. Hence, we get the restriction

$$\begin{aligned} g(y) &= f(x,y)_{|x^2+y^2=1} = 4y^2 - 2y \\ &= \left(2y - \frac{1}{2}\right)^2 - \frac{1}{4} \\ &= 4\left(y - \frac{1}{4}\right)^2 - \frac{1}{4}, \qquad y \in [-1,1]. \end{aligned}$$

It follows from these rearrangements that the minimum on the boundary is $g\left(\dfrac{1}{4}\right) = -\dfrac{1}{4}$, and the maximum is

$$g(-1) = 4\left(-\frac{5}{4}\right)^2 - \frac{1}{4} = \frac{25}{4} - \frac{1}{4} = 6.$$

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Since the boundary is connected, the range of the boundary is the interval $\left[-\frac{1}{4}, 6\right]$.

The value of the function at the stationary point $\left(0, \frac{1}{3}\right)$ is smaller than the smallest value of the function on the boundary, because

$$-\frac{14}{27} < -\frac{1}{4}.$$

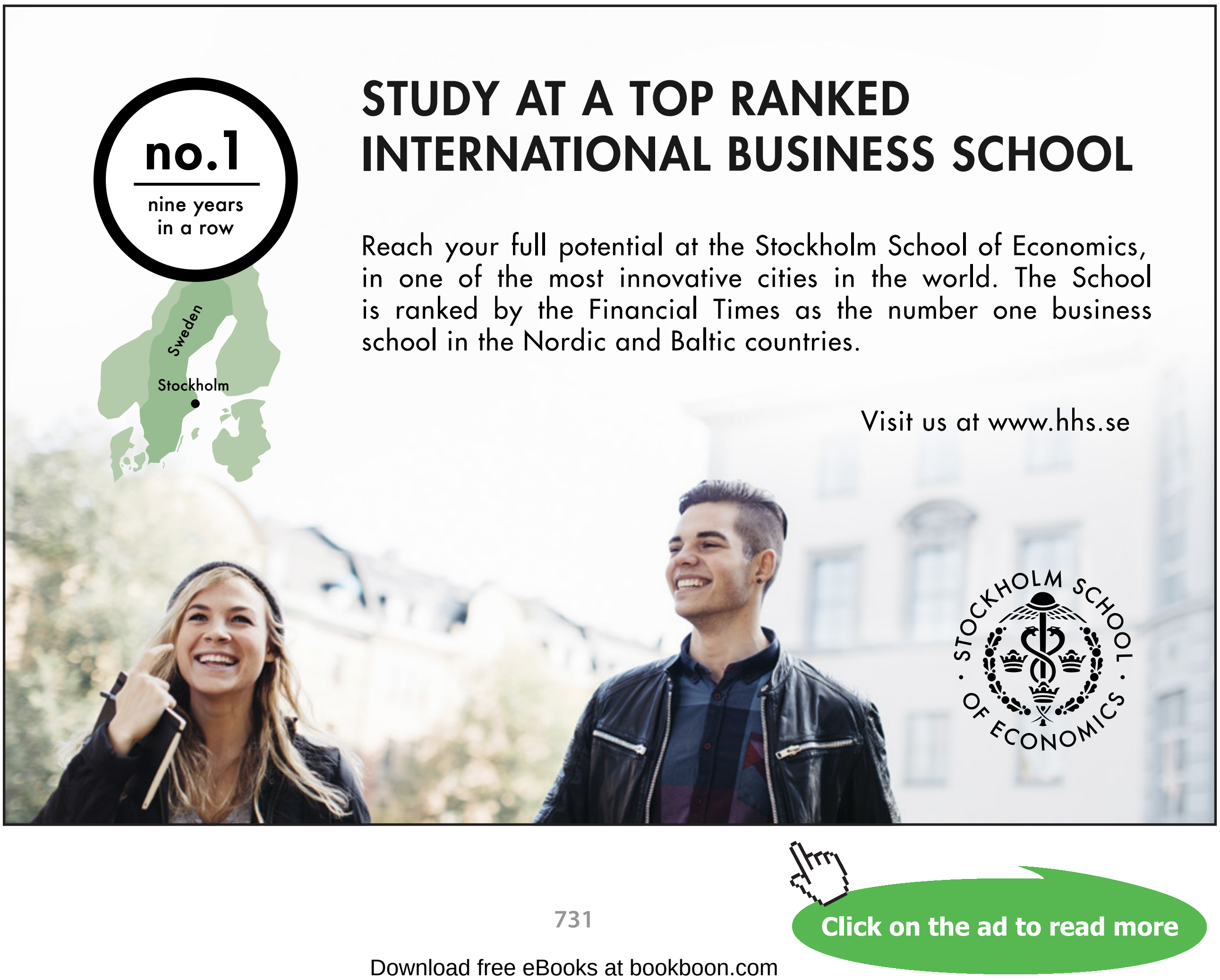
The range is connected, and the value 6 of the function on the boundary cannot be obtained in the interior, although we may come as close to this value as we wish. Hence

$$f(K((0,0);1)) = \left[-\frac{14}{27}, 6\right[.$$

The range of f over $\mathbb{R}^2$ is of course again $\mathbb{R}$. For instance, the restriction

$$f(0,y) = -3y + 4y^2 + y^3$$

tends towards $+\infty$ for $y \to +\infty$, and towards $-\infty$ for $y \to -\infty$, and by the first main theorem the range is an interval.



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Example 17.26 *When we approximate a function by an approximating polynomial based on Taylor's formula, the error is zero at the point of expansion, and it will usually increase with the distance form the point of expansion. We may get different results by using other polynomials for our approximation. One of the possibilities is to level out the error by demanding that* the integral of the square of the error *should be as small as possible. As an illustration we consider the function*

$$f(t) = \cos t, \qquad -\frac{\pi}{2} \le t \le \frac{\pi}{2},$$

and a polynomial

$$Q(t;x,y) = x - ty^2, \qquad -\frac{\pi}{2} \le t \le \frac{\pi}{2}.$$

Find x and y, such that the integral

$$I(x,y) = \int_{-\frac{pi}{2}}^{\frac{\pi}{2}} \{f(t) - Q(t;x,y)\}^2 dt$$

becomes as small as possible. Then compute the error of $t = \pm\frac{\pi}{2}$, partly by approximation by the found polynomial Q, and partly by using the Taylor polynomial of at most second degree P_2.

A Minimizing in L^2 norm.

D Compute $I(x,y)$ and minimize. Alternatively, compute $\frac{\partial I}{\partial x}$ and $\frac{\partial I}{\partial y}$ directly. Compare with the Taylor polynomial.

I Since $f(t) = \cos t$ is an even function, it is quite reasonable to approximate by a polynomial $Q(t;x,y)$ of even degree in t. Since $f(t) - Q(t;x,y)$ is even in t, we get

$$I(x,y) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{f(t) - Q(t;x,y)\}^2 dt = 2\int_0^{\frac{\pi}{2}} \{\cos t - x + yt^2\}^2 dt.$$

Clearly, $I(x,y) \ge 0$ is continuous, and $I(x,y) \to +\infty$ for $x^2 + y^2 \to +\infty$, so $I(x,y)$ has a minimum in $\mathbb{R}^2$. Now $I(x,y)$ is of class C^∞, so this minimum must be attained at a stationary point.

1) First variant. By differentiation under the sign of integration we get the equations of the possible stationary points

$$\begin{aligned}
\frac{\partial I}{\partial x} &= 2\int_0^{\frac{\pi}{2}} \frac{\partial}{\partial x}\left\{\cos t - x + yt^2\right\}^2 dt = -4\int_0^{\frac{\pi}{2}} (\cos t - x + yt^2)dt \\
&= -4\left[\sin t - xt + \frac{1}{3}\,yt^3\right]_0^{\frac{\pi}{3}} = -4\left(1 - \frac{\pi}{2}\,x + \frac{\pi^3}{24}\,y\right) = 0,
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial I}{\partial y} &= 2\int_0^{\frac{\pi}{2}} \frac{\partial}{\partial y}\left\{\cos t - x + yt^2\right\}^2 dt = 4\int_0^{\frac{\pi}{2}} t^2(\cos t - x + yt^2)dt \\
&= 4\int_0^{\frac{\pi}{2}} (t^2\cos t - xt^2 + yt^4)dt = \left[t^2\sin t + 2t\cos t - 2\sin t - \frac{x}{3}\,t^3 + \frac{y}{5}\,t^5\right]_0^{\frac{\pi}{2}} \\
&= 4\left(\frac{\pi^2}{4} - 2 - \frac{\pi^3}{24}\,x + \frac{\pi^5}{160}\,y\right) = 0.
\end{aligned}$$

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These are reduced to

$$x-\frac{\pi^2}{12}\,y=\frac{2}{\pi}\quad\text{and}\quad x-\frac{3\pi^2}{20}\,y=\frac{6}{\pi^3}(\pi^2-8),$$

hence

$$\left(\frac{3\pi^2}{20}-\frac{\pi^2}{12}\right)y=\frac{2}{\pi}-\frac{6}{\pi}+\frac{48}{\pi^3}=\frac{4}{\pi^3}(12-\pi^2),$$

and thus

$$y=\frac{15}{\pi^2}\cdot\frac{4}{\pi^3}(12-\pi^2)=\frac{60}{\pi^5}(12-\pi^2)$$

and accordingly

$$x=\frac{\pi^2}{12}\,y+\frac{2}{\pi}=\frac{\pi^2}{12}\cdot\frac{60}{\pi^5}(12-\pi^2)+\frac{2}{\pi}=\frac{1}{\pi^3}(60-5\pi^2)+\frac{2}{\pi}=\frac{3}{\pi^2}(20-\pi^2).$$

The only stationary point is

$$(x,y)=\left(\frac{3}{\pi^3}(20-\pi^2),\frac{60}{\pi^5}(12-\pi^2)\right),$$

and it must correspond to a minimum for $I(x,y)$. Note that $x>0$ and $y>0$.

2) Second variant. Alternatively we compute $I(x,y)$:

$$\begin{aligned}
I(x,y) &= 2\int_0^{\frac{\pi}{2}}\{\cos t-x+yt^2\}^2dt\\
&= 2\int_0^{\frac{\pi}{2}}\{\cos^2 t+x^2+y^2t^2-2x\cos t-2xyt^2+2yt^2\cos t\}dt\\
&= 2\left[\left\{\frac{1}{2}\,t+\frac{1}{2}\sin t\cos t\right\}+x^2t+\frac{y^2}{5}\,t^5-2x\sin t\right.\\
&\qquad\left.-\frac{2}{3}\,xyt^3+2y\{t^2\sin t+2t\cos t-2\sin t\}\right]_0^{\frac{\pi}{2}}\\
&= 2\left[\frac{\pi}{4}+\frac{\pi}{2}\,x^2+\frac{\pi^5}{160}\,y^2-2x-\frac{\pi^3}{12}\,xy+2y\left\{\frac{\pi^2}{4}-2\right\}\right]\\
&= \pi\,x^2+\frac{\pi^5}{80}\,y^2-\frac{\pi^3}{6}\,xy-4x+(\pi^2-8)y+\frac{\pi}{2}.
\end{aligned}$$

The equations of the stationary points,

$$\begin{aligned}
\frac{\partial I}{\partial x} &= 2\pi\,x-\frac{\pi^3}{6}\,y-4=0,\\
\frac{\partial I}{\partial y} &= \frac{\pi^5}{40}\,y-\frac{\pi^3}{6}\,x+\pi^2-8=0,
\end{aligned}$$

are identical with the equations of the first variant. The unique solution is

$$(x_0,y_0)=\left(\frac{3}{\pi^3}(20-\pi^2),\frac{60}{\pi^5}(12-\pi^2)\right)\approx(0.980\,162;0.417\,698),$$

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corresponding to a minimum for $I(x,y)$, given approximatively by

$$I(x_0,y_0) \approx 0.000\,936.$$

REMARK. This approximation is also called the *approximation in energy over the given interval*. It is seen that this is extremely good even for a polynomial of degree two (an error of less than 1 per thousand). This is the right concept of convergence in Communication Systems and other applications in the technical sciences. Unfortunately, most students are at this level still most familiar with the pointwise convergence, in spite of the fact that this concept in practice often is very awkward. It will below be demonstrated that the inaccuracy by the various pointwise approximations are bigger than the approximation in energy. ◊

We have found the approximation

$$Q(t) := Q(t;x_0,y_0) = \frac{3}{\pi^3}(20-\pi^2) - \frac{60}{\pi^5}(12-\pi^2)t^2 \approx 0.980\,162 - 0.417\,698t^2$$

of $\cos t$. The corresponding approximation by a *Taylor polynomial* is

$$P_2(t) = 1 - \frac{1}{2}\,t^2.$$

If $t = \pm\dfrac{\pi}{2}$, then $\cos t = 0$, and

$$P_2\left(\pm\frac{\pi}{2}\right) = 1 - \frac{\pi^2}{8} = \frac{8-\pi^2}{8} \approx -0.233\,701,$$

and

$$\begin{aligned} Q\left(\pm\frac{\pi}{2}\right) &= \frac{3}{\pi^3}(20-\pi^2) - \frac{60}{\pi^5}(12-\pi^2)\cdot\frac{\pi^2}{4} \\ &= \frac{1}{\pi^3}\{60-3\pi^2-180+15\pi^2\} = \frac{12\pi^2-120}{\pi^3} \\ &= \frac{12(\pi^2-10)}{\pi^3} \approx -0.050\,465. \end{aligned}$$

It is seen by comparison that the approximation in energy also gives a better pointwise result than the Taylor polynomial.

Example 17.27 *Consider the function*

$$f(x,y) = 3x^3 + 6xy^2 + 4y^3 - 9x^2, \qquad (x,y) \in A,$$

where A is given in the following way: We remove from the ellipsoidal disc given by $x^2+2y^2-3x \le 0$ those points which also satisfy $y < -\dfrac{1}{2}\,x$. Sketch A and then find the maximum and the minimum of the function.

A Maximum and minimum for a continuous function in a closed and bounded domain.

D Sketch the set A. Then refer to the second main theorem for continuous functions. Find the stationary points and examine the boundary points.

I Clearly, $f(x,y)$ is of class C^∞, even as a function in all of $\mathbb{R}^2$. Hence there are no exception points.

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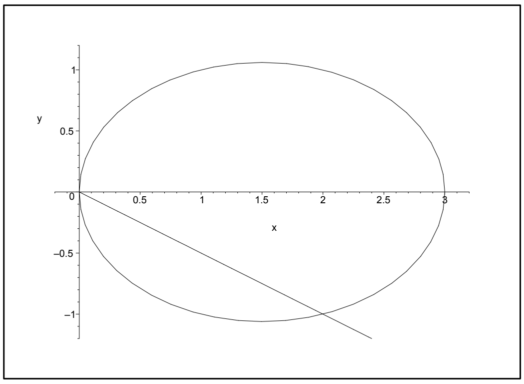

Figure 17.25: The closed set A lies above the line and inside the ellipse.

We first identify the ellipsoidal disc by the following rearrangement

$$0 \geq x^2+2y^2-3x = \left\{x^2 - 2\cdot\frac{3}{2}\,x + \left(\frac{3}{2}\right)^2\right\} + 2y^2 - \left(\frac{3}{2}\right)^2,$$

hence

$$\left(x-\frac{3}{2}\right)^2 + 2y^2 \leq \left(\frac{3}{2}\right)^2,$$

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or in the usual normed form

$$\left\{\frac{x-\frac{3}{2}}{\frac{3}{2}}\right\}^2+\left\{\frac{y}{\frac{3}{2\sqrt{2}}}\right\}^2\le 1.$$

It follows that the inequality describes a closed ellipsoidal disc of centrum $\left(\frac{3}{2},0\right)$ and of the half axes $a=\frac{3}{2}$ and $b=\frac{3}{2\sqrt{2}}$.

The domain A is the intersection of the closed ellipsoidal disc and the closed half plane $y\ge -\frac{1}{2}\,x$, i.e. that part of the ellipsoidal disc which lies above the line. It follows that A is closed and bounded.

According to the second main theorem for continuous functions, f has a maximum and a minimum in A. Since f is of class C^∞, these are among the values of the function at either the stationary points in the interior of A or at the points of the boundary.

First note that the line $y=-\frac{1}{2}\,x$ intersects the ellipse in two points given by

$$0=x^2+2y^2-3x=x^2+2\cdot\frac{x^2}{4}-3x=\frac{3}{2}\,x^2-3x=\frac{3}{2}\,x(x-2),$$

hence $x=0$ and $x=2$, corresponding to the points $(0,0)$ and $(2,-1)$.

1) Stationary points. The equations of the stationary points are

$$\begin{aligned}\frac{\partial f}{\partial x}&=9x^2+6y^2-18x=0,\\ \frac{\partial f}{\partial y}&=12xy+12y^2=12(x+y)y=0.\end{aligned}$$

The line $x+y=0$ does not intersect the interior of A, so we only get the possibility $y=0$. If we put this into the former equation we get

$$0=9x^2+0-18x=9x(x-2),$$

hence $x=0$ or $x=2$, corresponding to the stationary points $(0,0)$ and $(2,0)$. Only $(2,0)$ lies in A. Then compute the value of the function at this point,

$$f(2,0)=3\cdot 2^3+0+0-9\cdot 2^2=24-36=-12.$$

2) Examination of the boundary. The boundary is split into three obvious parts:

a) $y=-\frac{1}{2}\,x$, for $0\le x\le 2$ and $y\le 0$,

b) $y=-\sqrt{\frac{3x-x^2}{2}}$, for $2\le x\le 3$ and $y\le 0$,

c) $y=+\sqrt{\frac{3x-x^2}{2}}$, for $0\le x\le 3$ and $y\ge 0$.

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Notice that if (x,y) lies on the (boundary of the) ellipse, then

$$f(x,y) = 3x^3 + 6xy^2 + 4y^3 - 9x^2 = 3x\{x^2 + 2y^2 - 3x\} + 4y^3 = 4y^3,$$

which is a trick that will help us a lot in the following.

a) The restriction of $f(x,y)$ to $y = -\frac{1}{2}\,x$, $0 \le x \le 2$, is

$$\begin{aligned}\varphi(x) &= f\left(x, -\frac{1}{2}\,x\right) = 3x^3 + 6x \cdot \frac{x^2}{4} - 4 \cdot \frac{x^3}{8} - 9x^2 \\ &= 3x^3 + \frac{3}{2}\,x^3 - \frac{1}{2}\,x^3 - 9x^2 = 4x^3 - 9x^2, \qquad \text{for } 0 \le x \le 2,\end{aligned}$$

where

$$\varphi'(x) = 12x^2 - 18x = 12x\left(x - \frac{3}{2}\right), \qquad 0 < x < 2.$$

In the open interval we get $\varphi'(x) = 0$ for $x = \frac{3}{2}$, corresponding to $y = -\frac{3}{4}$, and

$$\varphi\left(\frac{3}{2}\right) = f\left(\frac{3}{2}, -\frac{3}{4}\right) = 4 \cdot \frac{27}{8} - 9 \cdot \frac{9}{4} = \frac{27}{2} - \frac{81}{4} = -\frac{27}{4}.$$

At the end points of the interval,

$$\varphi(0) = f(0,0) = 0 \text{ and } \varphi(2) = f(2,-1) = 4 \cdot 8 - 9 \cdot 4 = -4.$$

b) On the ellipsoidal boundary of A we reduce $f(x,y)$ to

$$f(x,y) = 4y^3.$$

It follows geometrically (consider the figure) that the maximum on this part of the boundary is

$$f\left(\frac{3}{2}, \frac{3}{2\sqrt{2}}\right) = 4 \cdot \left(\frac{3}{2\sqrt{2}}\right)^3 = \frac{27}{4\sqrt{2}},$$

and the minimum is

$$f(2,-1) = -4.$$

c) Numerical comparison. We shall find the maximum and the minimum among the values of the function,

$$f(2,0) = -12, \quad f\left(\frac{3}{2}, -\frac{3}{4}\right) = -\frac{27}{4}, \qquad f(0,0) = 0,$$

$$f(2,-1) = -4, \quad f\left(\frac{3}{2}, \frac{3}{2\sqrt{2}}\right) = \frac{27}{4\sqrt{2}},$$

thus the maximum is

$$S = f\left(\frac{3}{2}, \frac{3}{2\sqrt{2}}\right) = \frac{27}{4\sqrt{2}},$$

and the minimum is

$$f(2,0) = -12.$$

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Example 17.28 *Explain why the function given by*

$$f(x,y) = xy^2, \qquad (x,y) \in \overline{K}((0,0);1),$$

has both a maximum and a minimum, and find these values.

A Maximum and minimum.

D A continuous function on a closed, bounded set. Find the possible stationary points and examine the points of the boundary.

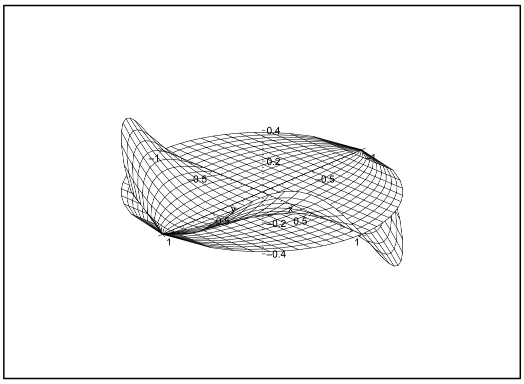

Figure 17.26: The graph of $f(x,y)$ over $\overline{K}((0,0);1)$.

I Since $f(x,y)$ is continuous (even of class C^∞), and $\overline{K}(\mathbf{0};1)$ is closed and bounded, it follows from the second main theorem for continuous functions that f has both a maximum and a minimum in $\overline{K}(\mathbf{0};1)$. These are either attained at a stationary point or at a boundary point.

Since

$$\frac{\partial f}{\partial x} = y^2 \qquad \text{and} \qquad \frac{\partial f}{\partial y} = 2xy$$

are 0 for $y = 0$, i.e. on the X axis, the set of stationary points is $[-1,1] \times \{0\}$. The value of $f(x,y)$ is here trivially $f(x,0) = 0$.

We use on the boundary the parametric description $(x,y) = (\cos t, \sin t)$, $t \in [0, 2\pi]$, thus the restriction to the boundary is given by

$$\varphi(t) = f(\cos t, \sin t) = \cos t \cdot \sin^2 t, \qquad t \in [0, 2\pi],$$

with the derivative

$$\varphi'(t) = -\sin^3 t + 2 \sin t \cdot \cos^3 t = \sin t \cdot (2\cos^2 t - \sin^2 t) = \sin t (3\cos^2 t - 1).$$

This expression is 0 for $t = 0$, π, or for $\cos t = \pm \dfrac{1}{\sqrt{3}}$, corresponding to $\sin^2 t = \dfrac{2}{3}$. When we put $t = 0$, π, we get the value 0 of the function.

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If we insert $\cos t = \pm \frac{1}{\sqrt{3}}$, $\sin^2 t = \frac{2}{3}$, we get

$$f(\cos t, \sin t) = \pm \frac{1}{\sqrt{3}} \cdot \frac{2}{3},$$

hence

$$f\left(\frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right) = \frac{2}{3\sqrt{3}} \qquad \text{is the maximum value,}$$

and

$$f\left(-\frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right) = -\frac{2}{3\sqrt{3}} \qquad \text{is the minimum value.}$$

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Example 17.29 *The function $f : \mathbb{R}^2 \to \mathbb{R}$ is given by*

$$f(x,y) = (x^2+y^2-1)(x^2+y^2-4).$$

1) Find the set of stationary points for f.

2) Show that f has a proper extremum at $(0,0)$, and indicate the type of this extremum.

3) Find the largest value which is attained by f in the disc

$$D = \{(x,y) \in \mathbb{R}^2 \mid (x-2)^2 + y^2 \le 4\}.$$

A Extremum.

D An alternative solution is to rewrite this problem as a 1-dimensional problem. Exploit this idea to solve as much of the problem as possible. Then compute the problem as indicated in the text.

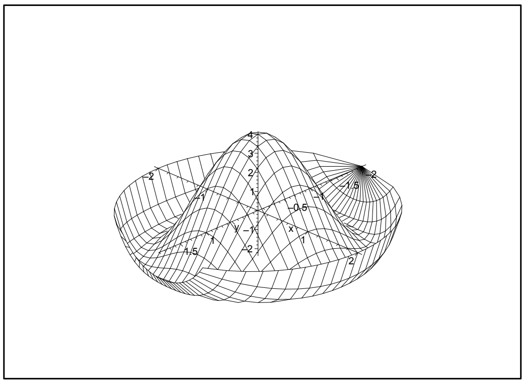

Figure 17.27: The surface $z = (x^2+y^2-1)(x^2+y^2-4)$ over D.

I The problem is actually 1-dimensional, because by switching to polar coordinates,

$$f(x,y) = f(\varrho\cos\varphi, \varrho\sin\varphi) = (\varrho^2-1)(\varrho^2-4) = \varrho^4 - 5\varrho^2 + 4 = g(\varrho), \qquad \varrho \ge 0,$$

and

$$g'(\varrho) = 4\varrho^3 - 10\varrho = 4\varrho\left(\varrho^2 - \frac{5}{2}\right)$$

is 0 for $\varrho = 0$ and for $\varrho = +\sqrt{\dfrac{5}{2}}$. Furthermore, $g(\varrho)$ is decreasing for $0 < \varrho < \sqrt{\dfrac{5}{2}}$ and increasing for $\varrho > \sqrt{\dfrac{5}{2}}$.

In particular, $(0,0)$ which corresponds to $\varrho = 0$ is a local maximum point of the value of the function

$$f(0,0) = 4.$$

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We get the minimum for $\varrho = \sqrt{\frac{5}{2}}$, corresponding to the value of the function

$$f\left(\sqrt{\frac{5}{2}}\cos\varphi, \sqrt{\frac{5}{2}}\sin\varphi\right) = g\left(\sqrt{\frac{5}{2}}\right) = \left(\frac{5}{2}-1\right)\left(\frac{5}{2}-4\right) = -\frac{9}{4}.$$

Since D contains both $(0,0)$ and $(4,0)$ on the X axis, cf. the figure, the maximum value is one of

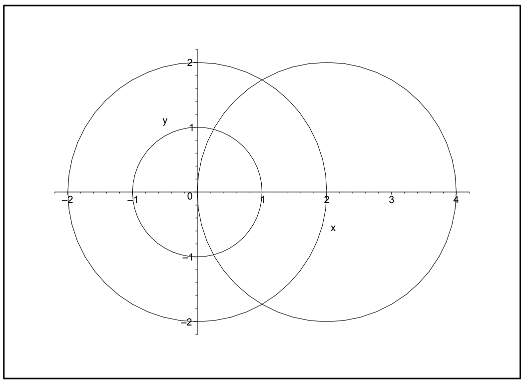

Figure 17.28: The domain D and the zero curves $x^2+y^2=1$ and $x^2+y^2=4$ for $f(x,y)$.

the values $g(0)$ and $g(4)$. We conclude from

$$f(4,0) = g(4) = (16-1)(16-4) = 15\cdot 12 = 180 > 4 = f(0,0),$$

that $f(4,0)=180$ is the maximum value in D.

Finally, the stationary points are necessarily $(0,0)$ and $\left\{(x,y) \;\middle|\; x^2+y^2=\frac{5}{2}\right\}$.

Then return to the very beginning, and handle the example according to the original intention.

1) The stationary points are the solutions of the equations

$$\frac{\partial f}{\partial x} = 2x(x^2+y^2-4)+2x(x^2+y^2-1) = 4x\left(x^2+y^2-\frac{5}{2}\right) = 0,$$

$$\frac{\partial f}{\partial y} = 2y(x^2+y^2-4)+2y(x^2+y^2-1) = 4y\left(x^2+y^2-\frac{5}{2}\right) = 0.$$

We conclude that the stationary points are

$$\left\{(x,y) \;\middle|\; x^2+y^2=\frac{5}{2}\right\} \cup \{(0,0)\}.$$

2) It follows from $f(x,y) = g(\varrho) = (1-\varrho^2)(4-\varrho^2)$ that $g(\varrho)$ is increasing when $\varrho \to 0$, hence $(0,0)$ is a maximum.

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ALTERNATIVELY,

$$\frac{\partial^2 f}{\partial x^2} = 4\left(x^2+y^2-\frac{5}{2}\right)+8x^2, \qquad r = 4\cdot\left(-\frac{5}{2}\right) = -10,$$

$$\frac{\partial^2 f}{\partial x \partial y} = 8xy, \qquad s = 0,$$

$$\frac{\partial^2 f}{\partial y^2} = 4\left(x^2+y^2-\frac{5}{2}\right)+8y^2, \qquad t = 4\cdot\left(-\frac{5}{2}\right) = -10,$$

so $rt - s^2 = 100 > 0$, and $r < 0$, $t < 0$, corresponding to that $(0,0)$ is a maximum point.

3) Since

$$f(x,y) = \left(\frac{5}{2}-1\right)\left(\frac{5}{2}-4\right) = -\frac{9}{4} \qquad \text{for } x^2+y^2 = \frac{5}{2},$$

only the examination of the boundary remains. A parametric description of the boundary curve of D is

$$(x,y) = (2+2\cos t, 2\sin t) = 2(1+\cos t, \sin t), \qquad t \in [0, 2\pi],$$

where

$$x^2+y^2 = 4\left(1+2\cos t+\cos^2 t+\sin^2 t\right) = 8(1+\cos t).$$

The restriction $h(t)$ to the boundary curve is given by

$$\begin{aligned} h(t) &= f(2(1+\cos t), 2\sin t) = \{8(1+\cos t)-1\}\{8(1+\cos t)-4\} \\ &= 4(7+8\cos t)(1+2\cos t) = 4(7+22\cos t+16\cos^2 t), \end{aligned}$$

where

$$h'(t) = -4\sin t(22+32\cos t),$$

which is only 0, when either $t=0$ or $t=\pi$ or $\cos t = -\dfrac{11}{16}$. We get by insertion

$$\begin{aligned} h(0) &= f(4,0) = 4(7+22+16) = 4\cdot 45 = 180, \\ h(\pi) &= f(0,0) = 4(7-22+16) = 4, \end{aligned}$$

and

$$\begin{aligned} h\left(\arccos\left(-\frac{11}{16}\right)\right) &= 4\left(7+22\cdot\left(-\frac{11}{16}\right)+16\cdot\left(-\frac{11}{16}\right)^2\right) \\ &= 4\left(7-2\cdot\frac{11^2}{16}+\frac{11^2}{16}\right) = 4\left(7-\frac{121}{16}\right) = 4\left(\frac{112-121}{16}\right) = -\frac{9}{4}. \end{aligned}$$

A numerical comparison gives that

$$f(4,0) = 180$$

is the maximum value in D.

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Example 17.30 *Given the function*

$$f(x,y) = xy \exp\left(y - x^2\right), \qquad (x,y) \in \mathbb{R}^2.$$

1) Find the stationary points of f.

2) Explain why f has both a maximum and a minimum in

$$A = \{(x,y) \mid x^2 - 3 \le y \le 0\},$$

and find the values of the function at these points.

3) Check if f has a global maximum and a global minimum in $\mathbb{R}^2$.

A Maximum and minimum.

D Find the stationary points and sketch A.

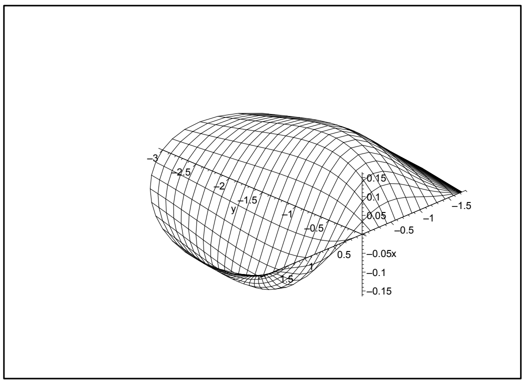

Figure 17.29: The graph of $f(x,y)$ over A.

I 1) The function is of class C^∞, and the partial derivatives are

$$\begin{aligned}\frac{\partial f}{\partial x} &= y \exp\left(y - x^2\right) - 2x^2 y \exp\left(y - x^2\right) = y\left(1 - 2x^2\right) \exp\left(y - x^2\right),\\ \frac{\partial f}{\partial y} &= x \exp\left(y - x^2\right) + xy \exp\left(y - x^2\right) = x(1+y) \exp\left(y - x^2\right).\end{aligned}$$

The exponential is never zero, so these two expressions are both equal to zero, if and only if

$$y(1 - 2x^2) = 0 \qquad \text{and} \qquad x(1+y) = 0.$$

If $x = 0$, the latter equation is fulfilled, and it follows from the former equation that $y = 0$.
If $y = -1$, the latter equation is again satisfied. It follows from the former equation that $x = \pm\dfrac{1}{\sqrt{2}}$.
Hence the possible stationary points are

$$(0,0), \qquad \left(\frac{1}{\sqrt{2}}, -1\right) \quad \text{and} \quad \left(-\frac{1}{\sqrt{2}}, -1\right).$$

Here $(0,0)$ lies on the boundary, so it will also enter the analysis of the boundary points.

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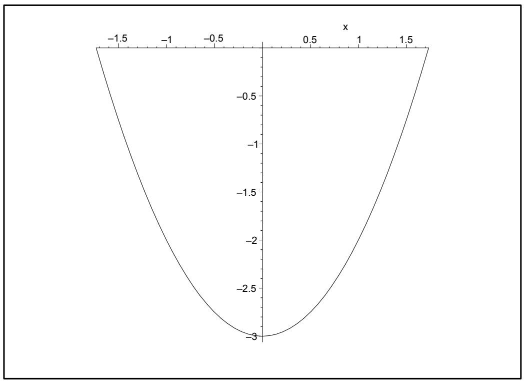

Figure 17.30: The domain A.

2) Since f is continuous and A is closed and bounded, it follows from the second main theorem for continuous functions that f has a maximum and a minimum in A. These points are either an inner stationary point or a boundary point. The inner stationary points are

$$\left(\frac{1}{\sqrt{2}},-1\right) \qquad \text{and} \qquad \left(-\frac{1}{\sqrt{2}},-1\right)$$

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with the values of the function

$$f\left(\frac{1}{\sqrt{2}},-1\right) = -\frac{1}{\sqrt{2}}\,\exp\left(-1-\frac{1}{2}\right) = -\frac{1}{\sqrt{2e^3}}$$

and

$$f\left(-\frac{1}{\sqrt{2}},-1\right) = \frac{1}{\sqrt{2e^3}}.$$

On the boundary curve $y=0$, $-\sqrt{3}\le x\le\sqrt{3}$,

$$f(x,0)=0.$$

On the boundary curve $y=x^2-3$, $-\sqrt{3}\le x\le\sqrt{3}$, we have the restriction

$$\varphi(x) = f\left(x,x^2-3\right) = x(x^2-3)\exp(x^2-3-x^2) = (x^3-3x)e^{-3}$$

with

$$\varphi'(x) = 3(x^2-1)e^{-3}.$$

We have already checked the end points $x=\pm\sqrt{3}$, so it only remains to compute

$$\varphi(1) = (1-3)e^{-3} = -2e^{-3} \quad\text{and}\quad \varphi(-1) = 2e^{-3}.$$

The maximum value and the minimum value are among

$$f\left(\frac{1}{\sqrt{2}},-1\right) = -\frac{1}{\sqrt{2e^3}},\qquad f\left(-\frac{1}{\sqrt{2}},-1\right) = \frac{1}{\sqrt{2e^3}},$$

$$f(1,-2) = -\frac{2}{e^3},\qquad f(-1,2) = \frac{2}{e^3},$$

hence

$$S = f\left(-\frac{1}{\sqrt{2}},-1\right) = \frac{1}{\sqrt{2e^3}} \quad\text{and}\quad M = f\left(\frac{1}{\sqrt{2}},-1\right) = -\frac{1}{\sqrt{2e^3}}.$$

3) We get along the curve $y=x^2$,

$$\psi(x) = f(x,x^2) = x^3,$$

which clearly has neither a maximum value nor a minimum value in $\mathbb{R}^2$, because the range is all of $\mathbb{R}$.

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Example 17.31 *1) Sketch the set*

$$B = \left\{(x,y) \in \mathbb{R}^2 \mid x \geq 0,\, 0 \leq y \leq 1-2x\right\},$$

and explain why the function $f(x,y) = xy$ has both a maximum S and a minimum value M on B.

2) Find S and M.

A Maximum value and minimum value.

D Solve the problem geometrically.

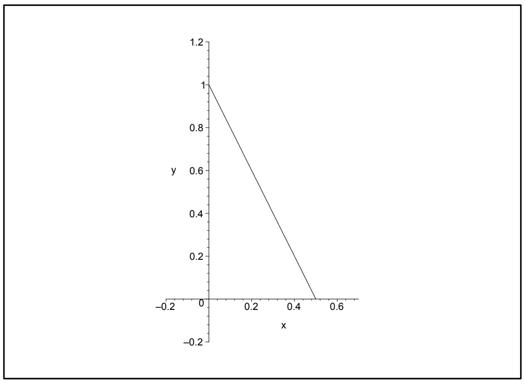

Figure 17.31: The domain B.

I 1) The set B is closed and bounded, and f is continuous. It follows from the second main theorem for continuous functions that f has a maximum and a minimum on B.

2) Clearly, $f(x,y) \geq 0$ on B, and the minimum value $M = 0$ must be attained on the axes.

Since $f(x,y) = C$ on the hyperbola $xy = C$, we conclude from considering the set of curves that the maximum value must be attained on the line $y = 1-2x$. Then consider the restriction

$$g(x) = f(x, 1-2x) = x - 2x^2, \qquad x \in \left[0, \frac{1}{2}\right],$$

of f to this boundary curve. First we vet $g'(x) = 1-4x$, which corresponds to a maximum for $x = \frac{1}{4}$, i.e.

$$S = g\left(\frac{1}{4}\right) = f\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{1}{8}.$$

Remark. An alternative way is of course to realize that $(0,0)$ is the only candidate of a stationary point. Since f is 0 on the axes, only the examination of the boundary line $y = 1-2x$ remains, and this was done above.

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Example 17.32 *Given the function*

$$f(x,y) = 32x^2y - 20x^3y - xy^3, \qquad (x,y) \in \mathbb{R}^2.$$

1) Find the stationary points of f.

2) Check for each of the points $(1,2)$, $(1,1)$ and $(0,0)$, if f has an extremum at the given point.

3) Find the range of the function f.

A Extremum.

D Follow the guidelines.

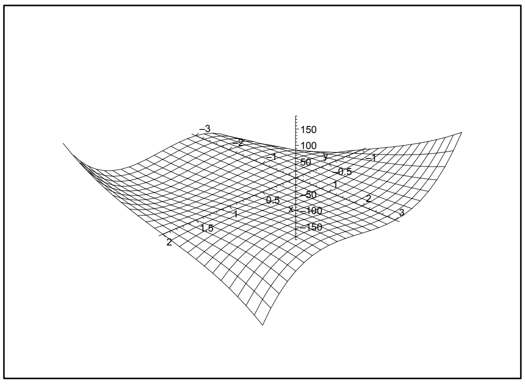

Figure 17.32: Part of the graph of f.

I 1) The stationary points are the solutions of the equations

$$\frac{\partial f}{\partial x} = 64xy - 60x^2y - y^3 = y(64x - 60x^2 - y^2) = 0,$$

$$\frac{\partial f}{\partial y} = 32x^2 - 20x^3 - 3xy^2 = x(32x - 20x^2 - 3y^2) = 0.$$

a) If $x = 0$, then the latter equation is fulfilled, and we obtain $y = 0$ from the first, so $(0,0)$ is a stationary point.

b) If $y = 0$, then the former equation is fulfilled, and the latter equation gives either $x = 0$ or $x = \frac{8}{5}$. Thus we get another stationary point $\left(\frac{8}{5}, 0\right)$.

c) If both $x \neq 0$ and $y \neq 0$, the equations are reduced to

$$y^2 = 64x - 60x^2 \quad \text{and} \quad 3y^2 = 32x - 20x^2.$$

By elimination of y,

$$32x - 20x^2 = 3y^2 = 192x - 180x^2,$$

from which we get the *necessary condition*

$$0 = 160x^2 - 160x = 160x(x-1).$$

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Since $x \neq 0$, we only get the *possibility* of $x = 1$.

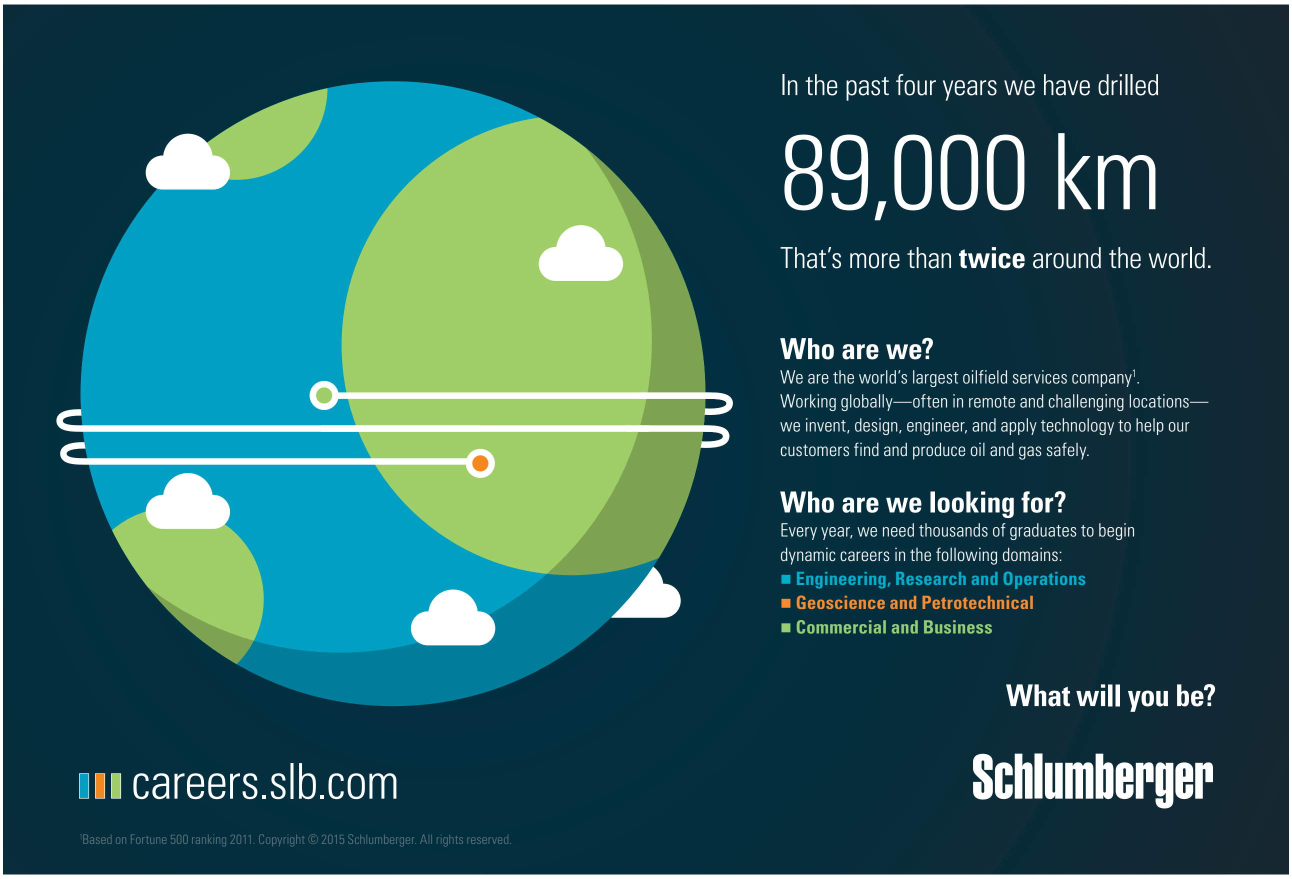
If we put $x = 1$ into the *original equations* (i.e. we test our possible solution), then

$$y(4 - y^2) = 0 \quad \text{and} \quad 12 - 3y^2 = 3(4 - y^2) = 0,$$

which are both satisfies for $y = \pm 2$.

Summarizing, the stationary points are

$$(0, 0), \qquad \left(\frac{8}{5}, 0\right), \qquad (1, 2), \qquad (1, -2).$$



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2) Examination of the extrema.

a) Since $(1,1)$ is not a stationary point, it cannot be an extremum.

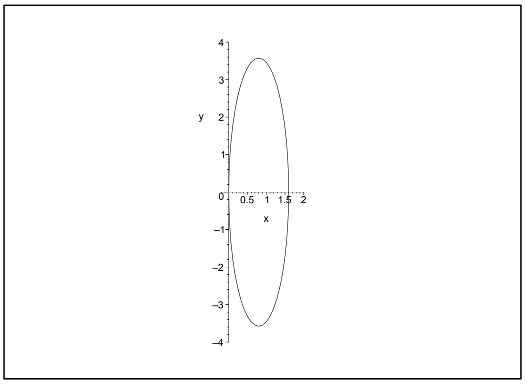

Figure 17.33: The function $f(x,y)$ changes its sign, whenever one crosses either one of the axes or the ellipse.

b) The point $(0,0)$ is not an extremum.

First variant. By an analysis of the sign of the function, cf. the figure,

$$\begin{aligned} f(x,y) &= 32x^2y-20x^3y-xy^3 = xy\left\{32x-20x^2-y^2\right\} = \cdots \\ &= \frac{64}{5}\,xy\left\{1-\frac{25}{16}\left(x-\frac{4}{5}\right)^2-\frac{5}{64}\,y^2\right\}, \end{aligned}$$

we see that f is both positive and negative in any neighbourhood of $(0,0)$, and we have no extremum.

Second variant. If we take the restriction of $f(x,y)$ to the line $y=x$, it follows, that

$$\varphi(x) = f(x,x) = 32x^3-21x^4 = x^3(32-21x)$$

is both positive and negative in any neighbourhood of $(0,0)$, and we cannot have an extremum.

c) When we check $(1,2)$, we have many methods at hand. We shall here restrict ourselves to two. First note that $(1,2)$ is in fact a stationary point, so it is possible that we have an extremum at the point.

First variant. Approximating polynomial of at most degree two from $(1,2)$.

If we put

$$x = x_1+1 \quad \text{and} \quad y = y_1+2,$$

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and neglect terms of higher order than 2 in (x_1, y_1) (symbolized by dots), we get

$$\begin{aligned}
f(x,y) &= 32x^2y - 20x^3y - xy^3 \\
&= 32(x_1+1)^2(y_1+2)^2 - 20(x_1+1)^3(y_1+2) - (x_1+1)(y_1+2)^3 \\
&= 32(x_1^2+2x_1+1)(y_1+2) - 20(1+3x_1+3x_1^2\cdots)(y_1+2) \\
&\qquad -(x_1+1)(8+12y_1+6y_1^2+\cdots) \\
&= 32(2+4x_1+2x_1^2+y_1+2x_1y_1+\cdots) - 20(2+6x_1+6x_1^2+y_1+3x_1y_1+\cdots) \\
&\qquad -(8+12y_1+6y_1^2+8x_1+12x_1y_1+\cdots) \\
&= (64-40-8)+(128-120-8)x_1+(32-20-12)y_1 \\
&\qquad +(64-120)x_1^2+(64-60-12)x_1y_1-6y_1^2+\cdots \\
&= 16 - 56x_1^2 - 8x_1y_1 - 6y_1^2 + \cdots \\
&= 16 - \frac{160}{3}\,x_1^2 - 6\left(\frac{2}{3}\,x_1 + y_1\right)^2 + \cdots\,,
\end{aligned}$$

proving that we have a local maximum for $(x_1, y_1) = (0, 0)$, i.e. for $(x, y) = (1, 2)$.

Second variant. The (r, s, t)-method.

First compute

$$\frac{\partial^2 f}{\partial x^2} = 64y - 120xy, \quad \frac{\partial^2 f}{\partial x \partial y} = 64x - 60x^2 - 3y^2, \quad \frac{\partial^2 f}{\partial y^2} = -6xy.$$

At the point $(1, 2)$,

$$r = 128 - 240 = -112, \qquad s = -8, \qquad t = 12,$$

hence

$$r < 0, \qquad t < 0 \quad \text{and} \quad rt > s^2.$$

It follows by the (r, s, t)-method that there is a proper maximum at the point $(1, 2)$.

3) The range is $\mathbb{R}$.

We have e.g.

$$f(x,x) = 32x^3 - 21x^4 \to -\infty \qquad \text{for } x \to +\infty,$$

and

$$f(x,-x) = -32x^3 + 21x^4 \to +\infty \qquad \text{for } x \to +\infty,$$

and the rest follows from that $f(x, y)$ is continuous.

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Example 17.33 *Given the function*

$$f(x,y) = (x+y^2)\exp(-2x^2), \qquad (x,y) \in \mathbb{R}^2.$$

1) Find the stationary points.

2) Explain why f has both a maximum S and a minimum M on the closed triangle with the vertices $(0,-1)$, $(0,1)$ and $(2,0)$; then find S and M.

3) Show that f does not have a (global) maximum in $\mathbb{R}^2$.

A Stationary points. Maximum and minimum.

D Follow the guidelines.

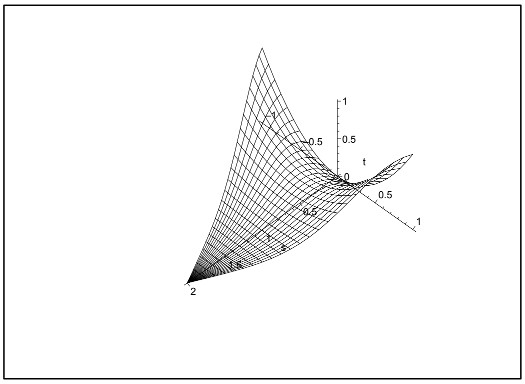

Figure 17.34: The graph of f over the triangle D.

I 1) The stationary points are the solutions of

$$\frac{\partial f}{\partial x} = \exp\left(-2x^2\right) 4x(x+y^2)\exp\left(-2x^2\right) = (1-4x^2-4xy^2)\exp(-2x^2) = 0,$$

and

$$\frac{\partial f}{\partial y} = 2y\exp\left(-2x^2\right) = 0.$$

We get from the latter equation that $y=0$, which by insertion into the former one gives $1-4x^2=0$, i.e. $x = \pm\frac{1}{2}$. The stationary points are

$$\left(\frac{1}{2},0\right) \quad \text{and} \quad \left(-\frac{1}{2},0\right).$$

From $f(x,y) \geq 0$ in D and $f(0,0)=0$ with $(0,0) \in D$ follows that

$$M = f(0,0) = 0.$$

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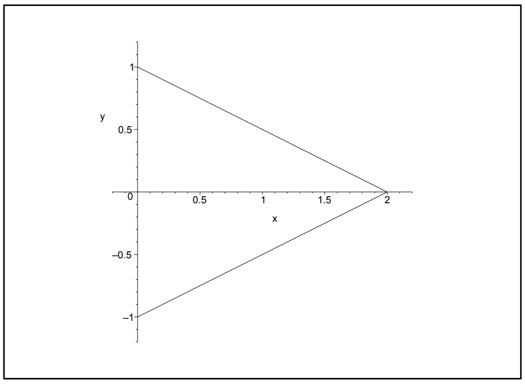

Figure 17.35: The domain D.

2) The triangle D is closed and bounded, and f is continuous on D. Hence, by the second main theorem for continuous functions, both S and M exist on D. These values are either attained at the stationary point $\left(\frac{1}{2}, 0\right) \in D$ or on the boundary ∂D.

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When x is fixed, we see that $f(x,y)$ is largest in D, when $y = \pm\left(1-\frac{x}{2}\right)$. This excludes $\left(\frac{1}{2},0\right)$, and the maximum must be attained on the line

$$y = \pm\left(1-\frac{x}{2}\right).$$

The restriction of f to this line is

$$\varphi(x) = f\left(x, \pm\left(1-\frac{x}{2}\right)\right) = \left(1+\frac{x^2}{4}\right)\exp\left(-2x^2\right), \quad x \in [0,2],$$

with the derivative

$$\begin{aligned}\varphi'(x) &= \frac{x}{2}\exp\left(-2x^2\right) - 4x\left(1+\frac{x^2}{4}\right)\exp\left(-2x^2\right)\\ &= \frac{x}{2}\exp\left(-2x^2\right)\cdot\{-7-2x^2\} \le 0, \quad x \in [0,2].\end{aligned}$$

Hence the maximum value is attained for $x = 0$, corresponding to

$$S = f(0,1) = f(0,-1) = 1.$$

3) Since $f(0,y) = y^2$, it follows that f does not have a global maximum in $\mathbb{R}^2$.

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Example 17.34 *Given the function*

$$f(x,y) = (x+y)^2 + 2\cos(2x+y), \qquad (x,y) \in \mathbb{R}^2.$$

1) Find the stationary points of f; check if f has proper extrema.

2) Let A be the point set

$$\{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le \pi,\ -2x \le y \le -x\}.$$

Prove that f has no stationary points in the interior of A.

Explain why f has both a maximum S and a minimum M on A, and find S and M.

3) Find the range $f(\mathbb{R}^2)$ of f.

A Extrema.

D Follow the guidelines

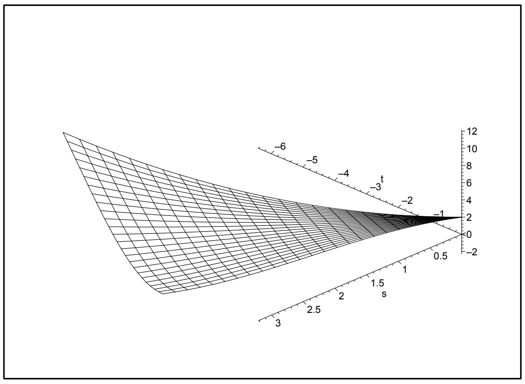

Figure 17.36: The graph of f the point set A.

I 1) The stationary points are the solutions of the equation $\nabla f = \mathbf{0}$, i.e.

$$\frac{\partial f}{\partial x} = 2(x+y) - 4\sin(2x+y) = 0,$$

$$\frac{\partial f}{\partial y} = 2(x+y) - 2\sin(2x+y) = 0.$$

It follows that $\sin(2x+y) = 0$ and $x+y = 0$, so

$$\sin(2x+y) = \sin(x+(x+y)) = \sin x = 0.$$

Accordingly, $x = p\pi$, $p \in \mathbb{Z}$, and $y = -x$. Thus the stationary points are

$$\{(p\pi, -p\pi) \mid p \in \mathbb{Z}\}.$$

The values of the function in these points are

$$f(p\pi, -p\pi) = 2\cos p\pi = 2 \cdot (-1)^p = \begin{cases} 2 & \text{for } p \text{ even}, \\ -2 & \text{for } p \text{ odd}. \end{cases}$$

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The extrema are now found by the (r,s,t)-method.

From

$$\frac{\partial^2 f}{\partial x^2} = 2-8\cos(2x+y) = 2-8\cos(x+(x+y)),$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2-4\cos(2x+y) = 2-4\cos(x+(x+y)),$$

$$\frac{\partial^2 f}{\partial y^2} = 2-2\cos(2x+y) = 2-2\cos(x+(x+y)),$$

follows that

$$r_p = \frac{\partial^2 f}{\partial x^2}(p\pi,-p\pi) = 2-8(-1)^p = \begin{cases} -6 & \text{for } p \text{ even,} \\ 10 & \text{for } p \text{ odd,} \end{cases}$$

$$s_p = \frac{\partial^2 f}{\partial x \partial y}(p\pi,-p\pi) = 2-4(-1)^p = \begin{cases} -2 & \text{for } p \text{ even,} \\ 6 & \text{for } p \text{ odd,} \end{cases}$$

$$t_p = \frac{\partial^2 f}{\partial y^2}(p\pi,-p\pi) = 2-2(-1)^p = \begin{cases} 0 & \text{for } p \text{ even,} \\ 4 & \text{for } p \text{ odd.} \end{cases}$$

If p is even, then

$$r_p = -6, \quad t_p = 0, \quad r_p \cdot t_p = 0 < (-2)^2 = s_p^2,$$

and we have no extremum.

If p is odd, then

$$r_p = 10, \quad t_p = 4, \quad r_p \cdot t_p = 40 > 36 = s_p^2,$$

and we have a proper minimum for p odd.

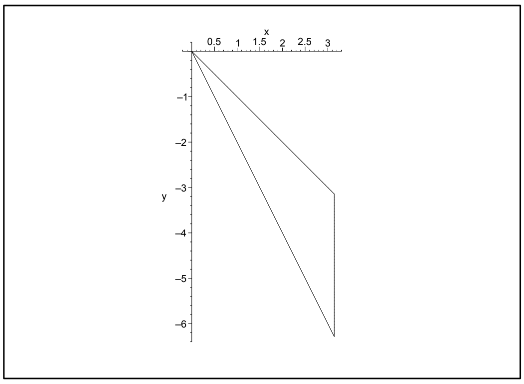

Figure 17.37: The domain A.

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2) It follows from the figure that $(0,0)$ and $(\pi,-\pi)$ are the only stationary points in A, and they both lie on the boundary of A.

Since $f(x,y)$ is continuous on the closed and bounded set A, it follows from the second main theorem for continuous functions that $f(x,y)$ has both a maximum and a minimum in A.

As proved above, the interior A° does not contain any stationary point, so the maximum and the minimum must be attained on the boundary of A.

Examination of the boundary.

a) If $y=-x$, $0\le x\le\pi$, then the restriction is

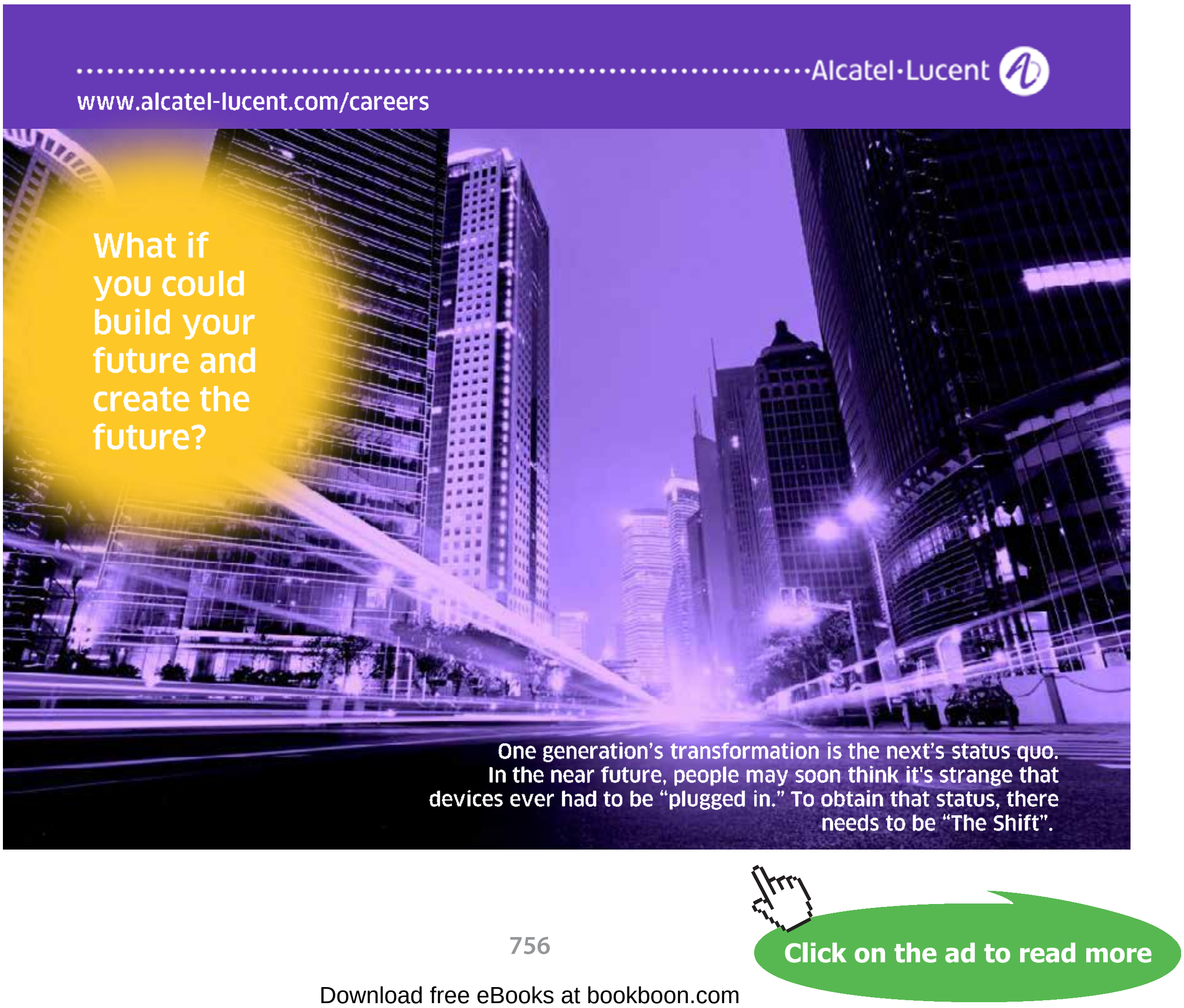
$$f(x,-x)=2\cos x.$$

This has its maximum $f(0,0)=2$ and its minimum $f(\pi,-\pi)=-2$.

b) If $y=-2x$, $0\le x\le\pi$, then the restriction is

$$f(x,-2x)=x^2+2\cos 0=x^2+2.$$

This has its maximum value $f(\pi,-2\pi)=\pi^2+2$ and its minimum value $f(0,0)=2$.



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c) If $x = \pi$, $-2\pi \le y \le -\pi$, then the restriction is

$$f(\pi, y) = (\pi + y)^2 + 2\cos(2\pi + y) = (\pi + y)^2 + 2\cos y$$

with

$$f'_y(\pi, y) = 2(\pi + y) - 2\sin y = 2\{\pi + y - \sin y\},$$

$$f''_{yy}(\pi, y) = 2\{1 - \cos y\} \ge 0,$$

so $f'_y(\pi, y)$ is increasing in y. From $f'_y(\pi, -\pi) = 0$ follows that $f'_y(\pi, y) < 0$ for $y \in [-2\pi, -\pi[$.

Since $f(\pi, -\pi) = 0 + 2\cos\pi = -2$ and $f(\pi, -2\pi) = \pi^2 + 2$, these two values are respectively the minimum and the maximum.

Summarizing,

$$S = f(\pi, -2\pi) = \pi^2 + 2, \qquad M = f(\pi, -\pi) = -2.$$

d) Clearly, $f(x, 0) \to +\infty$ for $x \to +\infty$, and $f(x, y) \ge -2$, so the range is contained in$[-2, +\infty[$.

Since

$$f((2n + 1)\pi, -(2n + 1)\pi) = -2, \qquad n \in \mathbb{Z},$$

and f is continuous on the connected set $\mathbb{R}^2$, it follows from the first main theorem of continuous functions that the range is connected, so

$$f(\mathbb{R}^2) = [-2, +\infty[.$$

Example 17.35 *Given the function*

$$f(x, y) = \frac{4}{3}\,x^3 - xy^2 + y, \qquad (x, y) \in \mathbb{R}^2.$$

1) Find the stationary points. Check for each of them if we have an extremum.

2) Find the maximum and the minimum of the function in the square $A = [0, 1] \times [0, 1]$.

A Extrema and maximum and minimum.

D Check the stationary points and the boundary points.

I 1) The stationary points are the solutions of the equations

$$\frac{\partial f}{\partial x} = 4x^2 - y^2 = (2x - y)(2x + y) = 0,$$

$$\frac{\partial f}{\partial y} = -2xy + 1 = 0, \quad \text{i.e.} \quad 2xy = 1.$$

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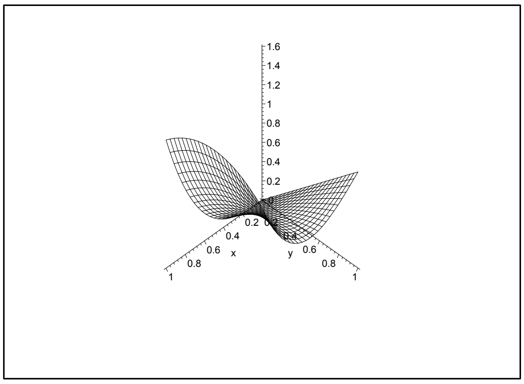

Figure 17.38: The graph of f over $A = [0,1] \times [0,1]$.

a) If $y = 2x$, then $1 = 2xy = 4x^2$, hence $x = \pm \dfrac{1}{2}$. In this case the stationary points are

$$\left(\frac{1}{2}, 1\right) \quad \text{and} \quad \left(-\frac{1}{2}, -1\right).$$

b) If $y = -2x$, then $1 = 2xy = -4x^2$, which is not fulfilled for any real x.

Thus the stationary points are $\left(\dfrac{1}{2}, 1\right)$ and $\left(-\dfrac{1}{2}, -1\right)$.

Search for extrema.

First method. The (r,s,t)-method. It follows from

$$\frac{\partial^2 f}{\partial x^2} = 8x, \qquad \frac{\partial^2 f}{\partial x \partial y} = -2y, \qquad \frac{\partial^2 f}{\partial y^2} = -2x,$$

that

$$\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} = -16x^2 < 0, \qquad \text{for } x \neq 0,$$

and we conclude that we have no extremum.

Second method. We shall find the approximating polynomial $P_2(x,y)$ of at most second degree in the neighbourhood of the stationary points.

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a) In the neighbourhood of $\left(\frac{1}{2},1\right)$ we put $x = \frac{1}{2}+\xi$ and $y = 1+\eta$. Then by insertion,

$$\begin{aligned}
f(x,y) &= \frac{4}{3}x^3 - xy^2 + y = \frac{4}{3}\left(\frac{1}{2}+\xi\right)^3 - \left(\frac{1}{2}+\xi\right)(1+\eta)^2 + 1 + \eta \\
&= \frac{4}{3}\left(\frac{1}{8}+\frac{3}{4}\xi+\frac{3}{2}\xi^2+\xi^3\right) - \left(\frac{1}{2}+\xi\right)(1+2\eta+\eta^2)+1+\eta \\
&= \frac{1}{6}+\xi+2\xi^2+\frac{4}{3}\xi^3-\frac{1}{2}-\eta-\frac{1}{2}\eta^2-\xi-2\xi\eta-\xi\eta^2+1+\eta \\
&= \left(1-\frac{1}{2}+\frac{1}{6}\right)+2\xi^2-2\xi\eta-\frac{1}{2}\eta^2+\cdots \\
&= \frac{2}{3}+2\left(\xi-\frac{1}{2}\eta\right)^2-\eta^2+\cdots,
\end{aligned}$$

and we see that $f(x,y)$ in any neighbourhood of $\left(\frac{1}{2},1\right)$ attains both values a value bigger than $f\left(\frac{1}{2},1\right)=\frac{2}{3}$ and values smaller than $f\left(\frac{1}{2},1\right)=\frac{2}{3}$, so there is no extremum.

b) In the neighbourhood of $\left(-\frac{1}{2},-1\right)$ we put instead $x = -\frac{1}{2}-\xi$ and $y=-1-\eta$. Then by similar calculations as above,

$$f(x,y) = -\frac{2}{3}-2\left(\xi-\frac{1}{2}\eta\right)^2+\eta^2+\cdots,$$

and we conclude that $\left(-\frac{1}{2},-1\right)$ is not an extremum.

2) Now f is continuous on the closed and bounded set A, hence it follows from the second main theorem for continuous functions that f has both a maximum and a minimum in A. Since f is of class C^∞ with no extremum at the stationary points, the maximum and the minimum must be attained at boundary points.

Examination of the boundary.

a) If $y=0$ and $0\le x\le 1$, then the restriction

$$f(x,0)=\frac{4}{3}x^3, \qquad x\in[0,1],$$

has its minimum $f(0,0)=0$ and its maximum $f(1,0)=\frac{4}{3}$.

b) If $x=0$ and $0\le y\le 1$, then the restriction

$$f(0,y)=y, \qquad y\in[0,1],$$

has its minimum $f(0,0)=0$ and its maximum $f(0,1)=1$.

c) If $x=1$ and $0\le y\le 1$, then the restriction

$$f(1,y)=\frac{4}{3}-y^2+y=\frac{4}{3}+\frac{1}{4}-\left(y-\frac{1}{2}\right)^2=\frac{19}{12}-\left(y-\frac{1}{2}\right)^2$$

has its minimum $f(1,0)=f(1,1)=\frac{4}{3}$, and its maximum $f\left(1,\frac{1}{2}\right)=\frac{19}{12}$.

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d) If $y = 1$ and $0 \le x \le 1$, then the restriction is

$$f(x,1) = \frac{4}{3}\,x^3 - x + 1,$$

where

$$f_x'(x,1) = 4x^2 - 1 = 0 \qquad \text{for } x = \frac{1}{2} \in [0,1].$$

Hence we have the possibilities

$$f(0,1) = 1, \qquad f\left(\frac{1}{2},1\right) = \frac{2}{3}, \qquad f(1,1) = \frac{4}{3},$$

and we see that this restriction has its minimum $f\left(\dfrac{1}{2},1\right) = \dfrac{2}{3}$, and its maximum

$$f(1,1) = \frac{4}{3}.$$

Summarizing, we get by a numerical comparison that $f(0,0) = 0$ is the minimum, and $f\left(1,\dfrac{1}{2}\right) = \dfrac{19}{12}$ is the maximum in A.

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Example 17.36 *Explain why the function*

$$f(x,y) = 2y\sqrt{x(1-x)} - y^2, \qquad (x,y) \in \left[\frac{1}{4},\frac{3}{4}\right] \times [0,3],$$

has both a maximum S and a minimum M.
Find these values as well as the range of the function.

A Maximum and minimum.

D Use the standard method.

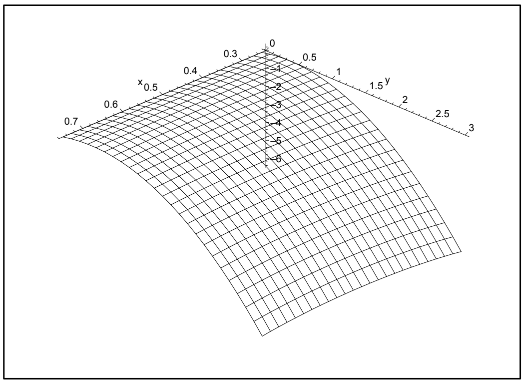

Figure 17.39: The graph of f. Notice the different scales on the axes.

I Since $f(x,y)$ is continuous and A is closed and bounded, it follows from the second main theorem for continuous functions that f has both a maximum and a minimum in A. Since f is of class C^∞ in the interior of the domain, these values are either obtained at a stationary point or at a boundary point.

The stationary points are the solutions of the equations

$$\frac{\partial f}{\partial x} = \frac{y(1-2x)}{\sqrt{x(1-x)}} = 0 \quad \text{og} \quad \frac{\partial f}{\partial y} = 2\sqrt{x(1-x)} - 2y = 0.$$

The former equation implies the possibilities of $y = 0$ or $x = \frac{1}{2}$. If $y = 0$, we consider only the boundary of A. Furthermore, $\frac{\partial f}{\partial y} \neq 0$ in A on this line.

If $x = \frac{1}{2}$, then

$$\frac{\partial f}{\partial y} = 2\sqrt{\frac{1}{2}\cdot\frac{1}{2}} - 2y = 1 - 2y = 0,$$

thus $y = \frac{1}{2}$, and the stationary point is $\left(\frac{1}{2},\frac{1}{2}\right) \in A$. The value of the function is here

$$f\left(\frac{1}{2},\frac{1}{2}\right) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

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THE BOUNDARY POINTS

1) If $y=0$ and $\frac{1}{4} \le x \le \frac{3}{4}$, then the restriction degenerates to $f(x,0)=0$.

2) If $y=3$ and $\frac{1}{4} \le x \le \frac{3}{4}$, then the restriction is given by

$$f(x,3) = 6\sqrt{x(1-x)} - 9 = 6\sqrt{\frac{1}{4} - \left(x-\frac{1}{2}\right)^2} - 9.$$

We get the maximum for $x = \frac{1}{2}$:

$$f\left(\frac{1}{2},3\right) = \frac{6}{2} - 9 = -6,.$$

and the minimum for $x = \frac{1}{4}$ and $x = \frac{3}{4}$, where

$$f\left(\frac{1}{4},3\right) = f\left(\frac{3}{4},3\right) = 6\,\frac{\sqrt{3}}{4} - 9 = \frac{3\sqrt{3}}{2} - 9.$$

3) If $x = \frac{1}{4}$ or $x = \frac{3}{4}$, and $0 \le y \le 3$, then the restriction is

$$f\left(\frac{1}{4},y\right) = f\left(\frac{3}{4},y\right) = 2y\sqrt{\frac{3}{16}}.y^2 = \frac{3}{16} - \left(y - \frac{\sqrt{3}}{4}\right)^2,$$

and we get the maximum for $y = \frac{\sqrt{3}}{4}$:

$$f\left(\frac{1}{4},\frac{\sqrt{3}}{4}\right) = f\left(\frac{3}{4},\frac{\sqrt{3}}{4}\right) = \frac{3}{16},$$

and the minimum for $y=3$,

$$f\left(\frac{1}{4},3\right) = f\left(\frac{3}{4},3\right) = 6\,\frac{\sqrt{3}}{4} - 9 = \frac{3\sqrt{3}}{2} - 9.$$

By a numerical comparison of the candidate above we conclude that the maximum is

$$f\left(\frac{1}{2},\frac{1}{2}\right) = \frac{1}{4},$$

and similarly the minimum,

$$f\left(\frac{1}{4},3\right) = f\left(\frac{3}{4},3\right) = \frac{3\sqrt{3}}{2} - 9.$$

Since f is continuous and the domain is connected we conclude from the first main theorem for continuous functions that the range is

$$M,S] = \left[\frac{3\sqrt{3}}{2} - 9, \frac{1}{4}\right].$$

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Example 17.37 *Given the function*

$$f(x,y) = (x+y)^2\left(8-(x^2+y^2)\right), \qquad (x,y) \in \mathbb{R}^2.$$

1) Find the set of stationary points for f.

2) Explain why f has both a maximum and a minimum in the set

$$A = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 9\},$$

and find those values.

3) Check if f has a maximum and a minimum on the set

$$B = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 16\},$$

and find the range $f(B)$.

A Stationary points; maximum and minimum.

D Use the standard method.

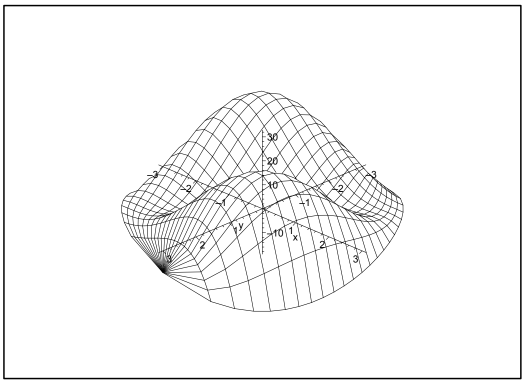

Figure 17.40: The graph of f over the set A.

I 1) The stationary points are the solutions of the equations

$$\begin{aligned}\frac{\partial f}{\partial x} &= 2(x+y)\left\{8-(x^2+y^2)\right\} - 2x(x+y)^2 \\ &= 2(x+y)\left\{8-x^2-(x^2+xy+y^2)\right\} = 0 \\ \frac{\partial f}{\partial y} &= 2(x+y)\left\{8-y^2-(x^2+xy+y^2)\right\} = 0.\end{aligned}$$

The obvious solution is $x+y=0$. The other possibility is

$$x^2+(x^2+xy^2) = 8 = y^2+(x^2+xy^2),$$

so $y = \pm x$. We have already found $y=-x$, so let $y=x$. Then $4x^2=8$, hence $x=y=\pm\sqrt{2}$.

Summarizing, the stationary points are

$$\{(x,-x) \mid x \in \mathbb{R}\} \cup \{(\sqrt{2},\sqrt{2})\} \cup \{(-\sqrt{2},-\sqrt{2})\}.$$

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2) The function f is continuous on the closed and bounded set A, so f has a maximum and a minimum according to the main theorem for continuous functions. These are either attained at singular points or at boundary points.

   a) If $y = -x$, then $f(x,-x) = 0$.

   b) If $(x,y) = \pm(\sqrt{2},\sqrt{2})$, then

   $$f(\pm\sqrt{2},\pm\sqrt{2}) = 8 \cdot (8-4) = 32$$

   (same sign at both places).

   c) If $x^2 + y^2 = 9$, then $x = 3\cos t$ and $y = 3\sin t$, $t \in [-\pi,\pi]$, so the restriction is

   $$\varphi(t) = f(x(t),y(t)) = 9(\cos t + \sin t)^2 \cdot (8-9) = -18\sin^2\left(t+\frac{\pi}{4}\right),$$

   and it follows that the maximum is 0 for $t = -\dfrac{\pi}{4}$ and $t = \dfrac{3\pi}{4}$, while the minimum is -18 for $t = \dfrac{\pi}{4}$ and $t = -\dfrac{3\pi}{4}$, i.e. for

   $$(x,y) = \pm\left(\frac{3\sqrt{2}}{2},\frac{3\sqrt{2}}{2}\right).$$

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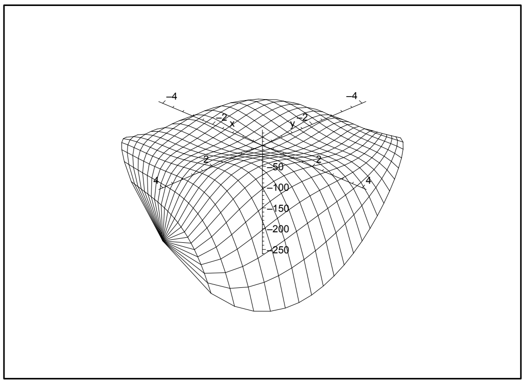

Figure 17.41: The graph of f over the set B.

Summarizing, the maximum is

$$f(\pm\sqrt{2},\pm\sqrt{2}) = 32 \qquad \text{(same sign at both places)},$$

and the minimum is

$$f\left(\pm\frac{3}{2}\sqrt{2},\pm\frac{3}{2}\sqrt{2}\right) = -18 \qquad \text{(same sign at both places)}.$$

3) The set B contains the same stationary points as A. The boundary is here $x^2+y^2=4^2=16$. If we put $x=4\cos t$ and $y=4\sin t$, we get the restriction

$$\varphi(t) = f(x(t),y(t)) = 16(\cos t+\sin t)^2\cdot(-8) = -256\sin^2\left(t+\frac{\pi}{4}\right).$$

The maximum on the boundary is 0, and the minimum is -256. The boundary is disjoint from B, so we shall never obtain the minimum, though the maximum in B is also

$$f(\pm\sqrt{2},\pm\sqrt{2}) = 32 \qquad \text{(same sign in both cases)}.$$

The set B is connected, and f is continuous on B. It follows from the first main theorem for continuous functions that the range is also connected. Finally, we conclude from the above that

$$f(B) =]-256,32].$$

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Example 17.38 *Let B denote the triangle of the vertices $\left(-\frac{1}{2}, 0\right)$, $(1,0)$ and $(0,1)$. Let the function $f : B \to \mathbb{R}$ be given by*

$$f(x,y) = (x+y)(2x+y) + x, \qquad (x,y) \in B.$$

1) Explain why f has a maximum S and a minimum M.

2) Find S and M and the points in which these values are attained.

A Maximum and minimum.

D Apply the second main theorem for continuous functions; find the stationary points and check the boundary points.

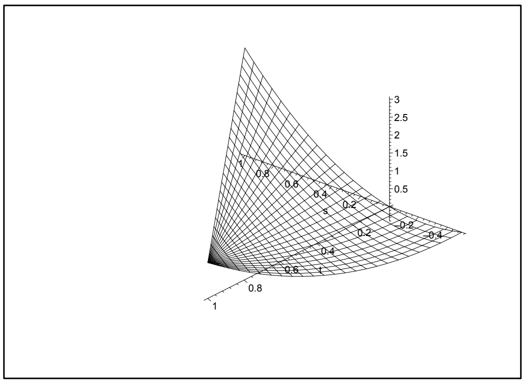

Figure 17.42: The graph of f over B.

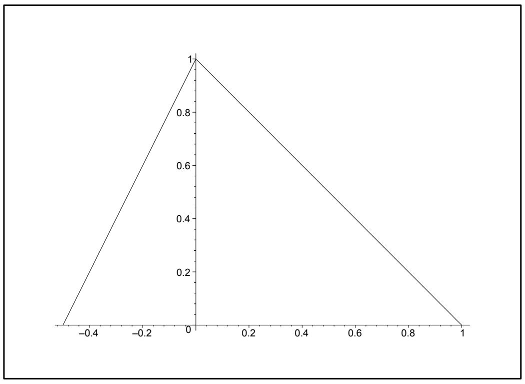

Figure 17.43: The domain B.

I 1) Since f is a real C^∞-function, and B is a closed and bounded domain, it follows from the second main theorem for continuous functions that f has both a maximum and a minimum in B.

Since f is of class C^∞, the maximum and the minimum can only be attained at either a stationary point or on the boundary of B.

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2) The possible *stationary points* are the solutions of the equations

$$\begin{aligned} f_x'(x,y) &= 2x+y+2(x+y)+1 = 4x+3y+1=0,\\ f_y'(x,y) &= 2x+y+x+y=3x+2y=0. \end{aligned}$$

Clearly, $(2,-3)$ is the only stationary point, and as it is in the fourth quadrant, it lies outside B.

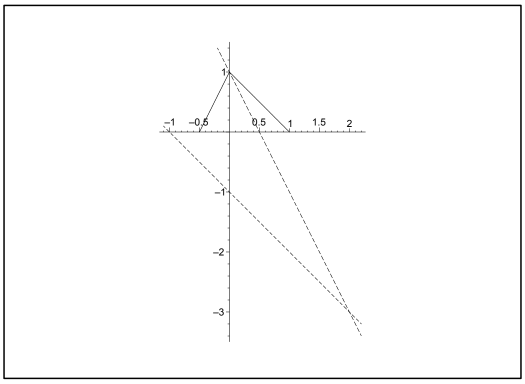

Figure 17.44: "Analysis of the sign" by means of the stationary point $(2,-3)$. The value of the function is <1 in the angular spaces over the acute angles, and the value of the function is >1 in the other two angular spaces.

THE BOUNDARY POINTS.

a) If $y=0$, $x \in \left[-\frac{1}{2}, 1\right]$, then we get the restriction

$$f(x,0) = x \cdot 2x + x = 2x^2 + x = 2\left(x+\frac{1}{4}\right)^2 - \frac{1}{8},$$

and it follows that the minimum and the maximum on this part of the boundary are

$$f\left(-\frac{1}{4}, 0\right) = -\frac{1}{8}, \quad \text{and} \quad f(1,0)=3, \qquad \text{respectively.}$$

b) If $x+y=1$, i.e. $y=1-x$, $x\in[0,1]$, then the restriction is

$$f(x,1-x) = (x+1)+x = 2x+1.$$

Clearly, the minimum and the maximum are on this part of the boundary

$$f(0,1)=1 \quad \text{og} \quad f(1,0)=3.$$

c) If $y=2x+1$, $x \in \left[-\frac{1}{2}, 0\right]$, then we get the restriction

$$f(x,2x+1) = 12x^2+8x+1 = 12\left(x+\frac{1}{3}\right)^2 - \frac{1}{3},$$

and the minimum and the maximum on this part of the boundary are

$$f\left(-\frac{1}{3}, +\frac{1}{3}\right) = -\frac{1}{3} \quad \text{and} \quad f(0,1)=1.$$

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Finally, by a numerical comparison,

$$M = f\left(-\frac{1}{3}, \frac{1}{3}\right) = -\frac{1}{3} \quad \text{and} \quad S = f(1,0) = 3.$$

REMARK. If we translate the coordinate system to the stationary point, i.e. if we put

$$x_1 = x - 2 \quad \text{og} \quad y_1 = y + 3,$$

then

$$f(x,y) = f_1(x_1, y_1) = (x_1 + y_1)(2x_1 + y_1) + 1.$$

A geometrical consideration shows that the minimum must be attained on the line $y = 2x + 1$, $x \in \left[-\frac{1}{2}, 0\right]$, and the maximum on the line $x + y = 1$, in accordance with the results above. $\Diamond$

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17.5 Examples of ranges of functions

Example 17.39 *Sketch the domain of the function*

$$f(x,y) = \sqrt{2x - x^2 - y^2} + \frac{1}{\sqrt{2y - x^2 - y^2}}.$$

Then find the range of the function.

A Domain and range.

D First find the domain. Then find the possible stationary points and check the values of the function on the boundary.

I The first term is defined (and ≥ 0) for

$$0 \leq 2x^2 - x^2 - y^2 = 1 - (x-1)^2 - y^2,$$

thus for

$$(x-1)^2 + y^2 \leq 1.$$

This inequality represents a closed disc of centrum $(1,0)$ and radius 1.

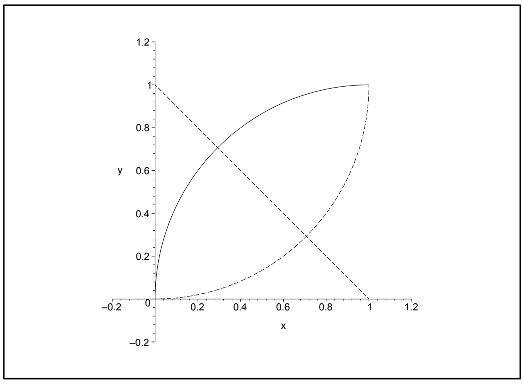

Figure 17.45: The domain D lies between the two circular arcs.

The second term is defined (and > 0) for

$$0 < 2y - x^2 - y^2 = 1 - x^2 - (y-1)^2, \text{ i.e. } x^2 + (y-1)^2 < 1.$$

This inequality describes an open disc of centrum $(0,1)$ and radius 1.

The domain D is the intersection of these two discs. It is neither open nor closed, because one of the boundary curves is contained in D, while the other is not.

Notice in particular that $0 < x < 1$ and $0 < y < 1$ for $(x,y) \in D$.

STATIONARY POINTS. The function is of class C^∞ in the interior D° of the domain. Therefore, there are no exception points. The equations of the possible stationary points are

$$\begin{aligned} 0 &= \frac{\partial f}{\partial x} = \frac{1-x}{\sqrt{2x - x^2 - y^2}} + \frac{x}{(\sqrt{2y - x^2 - y^2})^3}, \\ 0 &= \frac{\partial f}{\partial y} = -\frac{y}{\sqrt{2x - x^2 - y^2}} - \frac{1-y}{(\sqrt{2y - x^2 - y^2})^3}. \end{aligned}$$

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It was mentioned above that in particular $0 < x < 1$ and $0 < y < 1$ for $(x, y) \in D^\circ$, so $\dfrac{\partial f}{\partial x} > 0$ and $\dfrac{\partial f}{\partial y} < 0$ everywhere in D°. The equations of the stationary points are not fulfilled in D°, so there does not exist any of them.

Clearly, $f(x, y) \to +\infty$, when (x, y) tends to any point on that arc of ∂D, which does *not* belong to D. The set D is connected and f is continuous, so it follows from the first and the second main theorem for continuous functions that $f(D) = [a, +\infty[$, where the value a is attained at a point of the boundary which also lies in D, i.e. in $\partial D \cap D$. Clearly, a is a minimum value.

The restriction of the function to $\partial D \cap D$ is

$$f(x, y) = 0 + \frac{1}{\sqrt{1 - x^2 - (y-1)^2}}, \qquad (x-1)^2 + y^2 = 1,\ x > 0,\ y > 0.$$

This value of the function is smallest, when $1 - x^2 - (y-1)^2$ is largest. This means that $x^2 + (y-1)^2$, which can be interpreted as the square of the distance from $(0, 1)$ to (x, y) on the circular arc must be as small as possible. A geometric consideration shows that this point lies on both the line $x + y = 1$ and on the circle $(x-1)^2 + y^2 = 1$, as well as in the first quadrant, hence the searched point is

$$(x, y) = \left(1 - \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

The minimum value a is the value of the function at this point.

$$\begin{aligned}
a &= f\left(1 - \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{1 - \left(-\dfrac{1}{\sqrt{2}}\right)^2 - \left(\dfrac{1}{\sqrt{2}} - 1\right)^2}} \\
&= \frac{1}{\sqrt{1 - 2\left(1 - \dfrac{1}{\sqrt{2}}\right)^2}} = \frac{1}{\sqrt{1 - (\sqrt{2}-1)^2}} = \frac{1}{\sqrt{1 - 2 - 1 + 2\sqrt{2}}} \\
&= \frac{1}{\sqrt{2\sqrt{2} - 2}} \cdot \frac{\sqrt{2\sqrt{2}+2}}{\sqrt{2\sqrt{2}+2}} = \frac{\sqrt{2\sqrt{2}+2}}{2} = \sqrt{\frac{\sqrt{2}+1}{2}}.
\end{aligned}$$

We conclude that the range is

$$f(D) = \left[\frac{\sqrt{2\sqrt{2}+2}}{2}, +\infty\right[= \left[\sqrt{\frac{\sqrt{2}+1}{2}}, +\infty\right[.$$

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Example 17.40 *Let*

$$f(x,y) = -2xy^2 + 4x^2 + y^2 - 2x, \qquad (x,y) \in A.$$

Find the range of the function in the following cases:

1) The domain A is the closed half ellipsoidal disc given by

$$4x^2 + y^2 \le 1 \qquad \textit{and} \qquad x \ge 0.$$

2) The domain A is the open half ellipsoidal disc given by

$$4x^2 + y^2 < 1 \qquad \textit{and} \qquad x > 0.$$

3) The domain A is the whole plane.

A Maximum and minimum; range.

D From $f \in C^\infty$ follows that there are no exception point. The set A is connected in all three cases and the function is continuous, so the range is again connected, according to the first main theorem for continuous functions, hence an interval. We can in the three cases apply the following methods:

1) Since A is closed and bounded, we can apply the second main theorem, hence the maximum and minimum exist and they are attained at either a stationary point or at a boundary point.
2) Here A is bounded and open. The closure was treated in 1), so we can derive the range from 1).
3) In this case A is unbounded. We argue on the term of highest degree.

I The equations of the possible stationary points in the plane are

$$\begin{aligned}\frac{\partial f}{\partial x} &= -2y^2 + 8x - 2 = 0,\\ \frac{\partial f}{\partial y} &= -4xy + 2y = 2y(1-2x) = 0.\end{aligned}$$

From the latter equation we get the possibilities

a) $y = 0$ and b) $x = \frac{1}{2}$.

Thus:

a) If $y = 0$, then $\frac{\partial f}{\partial x} = 0$ for $x = \frac{1}{4}$, so $\left(\frac{1}{4}, 0\right)$ is a stationary point for f.

b) If $x = \frac{1}{2}$, then $\frac{\partial f}{\partial x} = -2y^2 + 4 - 2 = 0$ for $y = \pm 1$, so $\left(\frac{1}{2}, 1\right)$ and $\left(\frac{1}{2}, -1\right)$ are stationary points for f.

Summarizing, the stationary points for f in the plane are

$$\left(\frac{1}{4}, 0\right), \qquad \left(\frac{1}{2}, 1\right), \qquad \left(\frac{1}{2}, -1\right).$$

Of these, only the first one lies in the half ellipsoidal disc of the first two questions.

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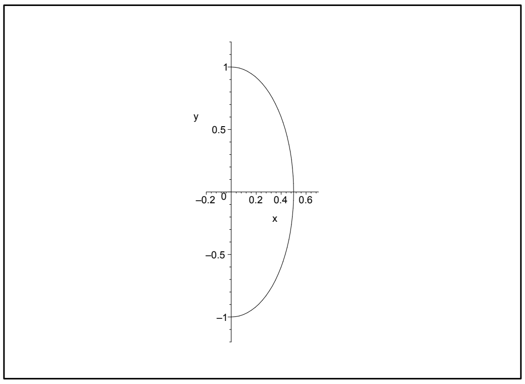

1) If $A = \{(x,y) \mid 4x^2 + y^2 \le 1 \text{ and } x \ge 0\}$, then in the stationary point of A,

$$f\left(\frac{1}{4},0\right) = 0 + 4 \cdot \frac{1}{16} + 0 - 2 \cdot \frac{1}{4} = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}.$$

The boundary falls naturally into two pieces:

a) If $x = 0$ and $y \in [-1,1]$, we get the restriction

$$\varphi(y) = f(0,y) = y^2, \qquad y \in [-1,1],$$

with the minimum value

$$\varphi(0) = f(0,0) = 0,$$

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and the maximum value

$$\varphi(1) = \varphi(-1) = f(0,1) = f(0,-1) = 1.$$

b) If $4x^2 + y^2 = 1$ and $x \ge 0$, then $y^2 = 1 - 4x^2$ and $x \in \left[0, \frac{1}{2}\right]$, so we get the restriction

$$\begin{aligned}\psi(x) &= f(x, \pm\sqrt{1-4x^2}) = -2x(y^2+1) + (4x^2+y^2) \\ &= -2x(1-4x^2+1)+1 = 8x^3 - 4x + 1, \qquad x \in \left[0, \frac{1}{2}\right],\end{aligned}$$

with the derivative

$$\psi'(x) = 24x^2 - 4 = 24\left(x^2 - \frac{1}{6}\right),$$

which in the interval $\left]0, \frac{1}{2}\right[$ is zero for $x = \frac{1}{\sqrt{6}}$. Hence

$$f\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{3}}\right) = f\left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{3}}\right) = -2\cdot\frac{1}{\sqrt{6}}\left(2 - \frac{4}{6}\right) + 1 = 1 - \frac{8}{3\sqrt{6}} < 0,$$

because $(3\sqrt{6})^2 = 9\cdot 6 = 54 < 8^2 = 64$. At the end points,

$$f(0,\pm 1) = 1 \quad \text{and} \quad f\left(\frac{1}{2}, 0\right) = \frac{8}{8} - \frac{4}{2} + 1 = 0.$$

The maximum and minimum values shall be found among the value at the stationary point

$$f\left(\frac{1}{4}, 0\right) = -\frac{1}{4},$$

and the values at the boundary points found above,

$$f(0,0) = 0, \quad f(0,\pm 1) = 1, \quad f\left(\frac{1}{2}, 0\right) = 0, \quad f\left(\frac{1}{\sqrt{6}}, \pm\frac{1}{\sqrt{3}}\right) = 1 - \frac{8}{3\sqrt{6}}.$$

Clearly, the *maximum value* is

$$S = f(0,1) = f(0,-1) = 1,$$

(both are boundary points). From

$$25\cdot 27 = 26^2 - 1 > 24^2 - 8^2 = 16\cdot 32,$$

follows that

$$\frac{25}{16} > \frac{32}{27} = \frac{64}{54},$$

so

$$\sqrt{\frac{25}{16}} = \frac{5}{4} > \sqrt{\frac{64}{54}} = \frac{8}{3\sqrt{6}},$$

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and hence

$$f\left(\frac{1}{\sqrt{6}}, \pm\frac{1}{\sqrt{3}}\right) = 1 - \frac{8}{3\sqrt{6}} > 1 - \frac{5}{4} = -\frac{1}{4} = f\left(\frac{1}{4}, 0\right).$$

This shows that the *minimum value* is attained at the stationary point and that the value is

$$M = f\left(\frac{1}{4}, 0\right) = -\frac{1}{4}.$$

The domain is connected, so it follows from the first main theorem that the range is also connected, i.e.

$$f(A) = \left[-\frac{1}{4}, 1\right].$$

2) The closure of A was treated in 1), where the minimum value was attained at a stationary point, in particular in an interior point, and where the maximum value is attained at a boundary point. We therefore conclude that the range is

$$f(A) = \left[-\frac{1}{4}, 1\right[.$$

3) By restriction to the line $y = x$,

$$g(x) = f(x, x) = -2x^3 + 5x^2 - 2x.$$

For large x the expression is dominated by $-2x^3$, and as $-2x^3 \to -\infty$ for $x \to +\infty$, and $-2x^3 \to +\infty$ for $x \to -\infty$, and as the range is an interval (the first main theorem again), we conclude that

$$f(A) = \mathbb{R}.$$

Example 17.41 *Let*

$$f(x,y) = 3x^3 + 4y^3 + 6xy^2 - 9x^2, \qquad (x,y) \in A.$$

Find the range of the function in the following cases.

1) *The domain A is the closed triangle of the vertices $(0,0)$, $(3,3)$ and $\left(3,-\frac{3}{2}\right)$. item The domain A is the interior of the point set of 1).*

2) *The domain A is the whole plane.*

A Maximum and minimum; range.

D From $f \in C^\infty\left(\mathbb{R}^2\right)$ follows that there are no exception points. First find the stationary points in the plane. Since A is connected in all three cases, it follows from the first main theorem that all the ranges are intervals.

1) Since A is closed and bounded, and f is continuous, it follows from the second main theorem that f has both a maximum and a minimum in A. These can only be attained at a stationary point or at a boundary point.
2) Because A is the interior of the set of 1), we can apply the results from 1).
3) Consider e.g. the restriction of f to the line $y = x$.

The possible stationary points in the whole plane must satisfy the equations

$$\frac{\partial f}{\partial x} = 9x^2 + 6y^2 - 18x = 0,$$
$$\frac{\partial f}{\partial y} = 12y^2 + 12xy = 12y(y+x) = 0.$$

It follows from the latter equation that either $y = 0$ or $y = -x$. We therefore get the following possibilities:

1) If $y = 0$, then either $x = 0$ or $x = 2$, and the stationary points in this case are

$$(0,0) \qquad \text{and} \qquad (2,0).$$

2) If $y = -x$, then

$$0 = 15x^2 - 18x = 15x\left(x - \frac{6}{5}\right),$$

which corresponds to the stationary points

$$(0,0) \qquad \text{and} \qquad \left(\frac{6}{5}, -\frac{6}{5}\right).$$

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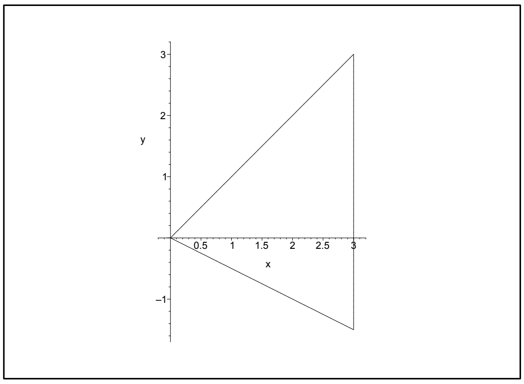

Summarizing, the stationary points in the plane are

$$(0,0), \qquad (2,0), \qquad \left(\frac{6}{5}, -\frac{6}{5}\right).$$

1) It follows from the figure that $(2,0)$ is the only *stationary point* in A. The value is here

$$f(2,0) = 3 \cdot 8 - 9 \cdot 4 = -12.$$

THE BOUNDARY.

a) The restriction to $y = x$ is

$$g_1(x) = f(x,x) = 13x^3 - 9x^2, \qquad x \in [0,3],$$

with the derivative

$$g_1'(x) = 39x^2 - 18 = 0 \qquad \text{for } x = +\sqrt{\frac{6}{13}} \in [0,3].$$

The candidates are

$$f\left(\sqrt{\frac{6}{13}}, \sqrt{\frac{6}{13}}\right) = 13\sqrt{\frac{6}{13}} \cdot \frac{6}{13} - 9 \cdot \frac{6}{13} = \frac{6}{13}(\sqrt{6 \cdot 13} - 9) < 0,$$

because $\sqrt{6 \cdot 13} - 9 = \sqrt{78} - 9 < \sqrt{81} - 9 = 0$, and

$$f(0,0) = 0, \qquad f(3,3) = 13 \cdot 27 - 81 = 270.$$

b) The restriction to $x = 3$ is

$$g_2(y) = 81 + 41y^3 + 18y^2 - 81 = 4y^3 + 18y^2$$

for $y \in \left[-\frac{3}{2}, 3\right]$, with the derivative

$$g_2'(y) = 12y^2 + 36y = 12y(y+3) = 0 \qquad \text{for } y = 0 \in \left[-\frac{3}{2}, 3\right].$$

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The candidates are

$$f\left(3,-\frac{3}{2}\right)=4\left(-\frac{3}{2}\right)^3+18\left(\frac{3}{2}\right)^2=-4\cdot\frac{27}{8}+18\cdot\frac{9}{4}=\frac{54}{2}=27,$$

$$f(3,0)=0 \qquad \text{and} \qquad f(3,3)=270.$$

c) The restriction to $y=-\frac{1}{2}\,x$ is

$$\begin{aligned} g_3(x) &= 3x^3+4\left(-\frac{1}{2}\,x\right)^3+6x\left(-\frac{1}{2}\,x\right)^2-9x^2 \\ &= 3x^3-\frac{1}{2}\,x^3+\frac{3}{2}\,x^3-9x^2 \\ &= 4x^3-9x^2 \qquad \text{for } x\in[0,3], \end{aligned}$$

with the derivative

$$g_3'(x)=12x^2-18x=12x\left(x-\frac{3}{2}\right)=0$$

for ($x=0$ and) $x=\frac{3}{2}\in[0,3]$. The candidates are

$$f\left(\frac{3}{2},-\frac{3}{4}\right)=4\left(\frac{3}{2}\right)^3-9\left(\frac{3}{2}\right)^2=\frac{4\cdot 27}{8}-\frac{81}{4}=-\frac{27}{4},$$

$$f(0,0)=0 \qquad \text{and} \qquad f\left(3,-\frac{3}{2}\right)=27.$$

Summarizing, all the candidates are

$$f(2,0)=-12,$$

(the stationary point), and

$$f\left(\sqrt{\frac{6}{13}},\sqrt{\frac{6}{13}}\right)=6\sqrt{\frac{6}{13}}-\frac{54}{13}, \quad f(3,0)=0, \quad f\left(\frac{3}{2},-\frac{3}{4}\right)=-\frac{27}{4},$$

$$f(0,0)=0, \qquad f(3,3)=270, \qquad f\left(3,-\frac{3}{2}\right)=27,$$

(at boundary points).
By a numerical comparison we find the *minimum value* at a stationary point

$$M=f(2,0)=-12,$$

and the *maximum value* at a boundary point,

$$S=f(3,3)=270.$$

It follows from the first main theorem that the range is the interval

$$f(A)=[-12,270].$$

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2) When we remove the boundary of 1), we also remove the maximum value from the range. However, due to the continuity we can in A obtain values as close to 270 as we wish, so the range becomes

$$f(A) = [-12, 270[.$$

3) The restriction to the line $y = x$ is

$$g(x) = 13x^3 - 9x^2 = x^2(13x - 9), \qquad x \in \mathbb{R}.$$

The range is clearly

$$f(A) = \mathbb{R}.$$

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Example 17.42 *Let*

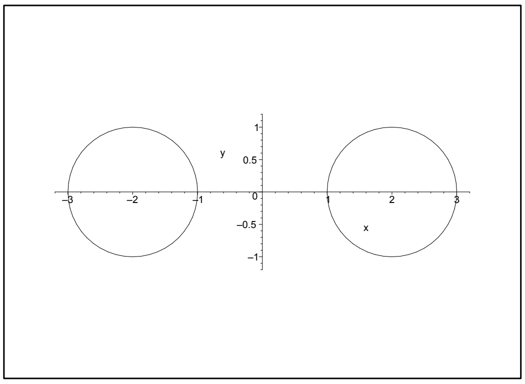
$$f(x,y) = x + y + \sqrt{|4x| - x^2 - y^2 - 3}, \qquad (x,y) \in A.$$

1) Find the domain of the function.

2) Find the range $f(A)$.

A Domain and range.

D Sketch the set A. Consider the two main theorems of continuous functions.



I 1) It follows from the rearrangement

$$f(x,y) = x + y + \sqrt{|4x| - x^2 - y^2 - 3} = x + y + \sqrt{1 - (|x| - 2)^2 - y^2}$$

that the domain of f is the union of two closed discs,

$$A = \overline{K}((2,0);1) \cup \overline{K}((-2,0);1).$$

This set is closed and bounded, but not connected, so we can only apply the second main theorem. We conclude that f has a maximum and a minimum of both of its connected components.

2) The investigation is now split into the two cases of $x > 0$ and $x < 0$.

a) If $x > 0$, then

$$f(x,y) = x + y + \sqrt{1 - (x-2)^2 - y^2}, \qquad (x-2)^2 + y^2 \le 1.$$

The equations of the stationary points are

$$\frac{\partial f}{\partial x} = 1 - \frac{x-2}{\sqrt{1-(x-2)^2-y^2}} = 0, \quad \frac{\partial f}{\partial y} = 1 - \frac{y}{\sqrt{1-(x-2)^2-y^2}} = 0,$$

hence

$$y = x - 2 = \sqrt{1-(x-2)^2-y^2},$$

so $y \ge 0$ (and $x \ge 2$), and $y = \sqrt{1-2y^2}$, thus

$$1 - 2y^2 = y^2, \quad \text{or} \quad y = +\frac{1}{\sqrt{3}} > 0,$$

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and $x = 2 + \frac{1}{\sqrt{3}}$.

In this connected component, the only stationary point is $\left(2 + \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, corresponding to the value

$$f\left(2 + \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = 2 + \frac{2}{\sqrt{3}} + \sqrt{1 - \frac{2}{3}} = 2 + \frac{3}{\sqrt{3}} = 2 + \sqrt{3}.$$

The square root is 0 on the *boundary*, so we shall only find the maximum and the minimum of $x+y$ on the boundary. A geometric analysis shows that the points must lie on the line $y = x - 2$. The radius of the circle is 1, so we get the points

$$\left(2 - \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \quad \text{and} \quad \left(2 + \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),$$

corresponding to the values

$$f\left(2 - \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = 2 - \frac{2}{\sqrt{2}} = 2 - \sqrt{2} \text{ and } f\left(2 + \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 2 + \sqrt{2}.$$

It follows by a numerical comparison of

$$f\left(2 + \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = 2 + \sqrt{3}, \qquad f\left(2 - \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = 2 - \sqrt{2},$$

$$f\left(2 + \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 2 + \sqrt{2},$$

that the range is

$$[2 - \sqrt{2}, 2 + \sqrt{3}].$$

ALTERNATIVELY the boundary is described by

$$x = 2 + \cos\theta, \quad y = \sin\theta, \qquad \theta \in [0, 2\pi],$$

so the restriction becomes

$$g_1(\theta) = x + y + 0 = 2 + \cos\theta + \sin\theta$$

with the derivative

$$g_1'(\theta) = -\sin\theta + \cos\theta = -\sqrt{2}\sin\left(\theta - \frac{\pi}{4}\right),$$

which is 0 for $\theta = \frac{\pi}{4}$ or for $\theta = \frac{5\pi}{4}$. This gives us the candidates

$$f\left(2 + \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 2 + \sqrt{2} \qquad \text{for } \theta = \frac{\pi}{4},$$

$$f\left(2 - \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = 2 - \sqrt{2} \qquad \text{for } \theta = \frac{5\pi}{4},$$

and formally (though not in reality)

$$g_1(0) = g_1(2\pi) = f(3,0) = 3 \quad (< 2 + \sqrt{2}) \quad \text{for } \theta = 0 \text{ and } \theta = 2\pi.$$

The range of the connected subregion is

$$[2 - \sqrt{2}, 2 + \sqrt{3}].$$

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3) If $x < 0$, then

$$f(x,y) = x + y + \sqrt{1-(x+2)^2-y^2}, \qquad (x+2)^2+y^2 \le 1,$$

and the equations of the stationary points are

$$\begin{aligned}\frac{\partial f}{\partial x} &= 1 - \frac{x+2}{\sqrt{1-(x+2)^2-y^2}} = 0,\\ \frac{\partial f}{\partial y} &= 1 - \frac{y}{\sqrt{1-(x+2)^2-y^2}} = 0.\end{aligned}$$

The possible stationary points satisfy

$$y = x+2 = \sqrt{1-(x+2)^2-y^2} \ge 0,$$

thus as before

$$y = +\frac{1}{\sqrt{3}} \qquad \text{and} \qquad x = -2+\frac{1}{\sqrt{3}},$$

corresponding to the value

$$f\left(-2+\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = -2+\frac{2}{\sqrt{3}}+\sqrt{1-\frac{2}{3}} = -2+\sqrt{3}.$$

The investigation of the boundary is similar the the previous one (we have again two variants), so we find the candidates

$$\left(-2-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \qquad \text{and} \qquad \left(-2+\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}\right)$$

of the corresponding values of the function

$$f\left(-2-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -2-\sqrt{2} \quad \text{and} \quad f\left(-2+\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -2+\sqrt{2}.$$

We conclude by a numerical comparison that the maximum and the minimum values on this connected component are

$$S = f\left(-2+\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = -2+\sqrt{3}$$

and

$$M = f\left(-2-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -2-\sqrt{2},$$

corresponding to the subinterval

$$[-2-\sqrt{2}, -2+\sqrt{3}].$$

Summarizing, the range becomes

$$f(A) = [-2-\sqrt{2}, -2+\sqrt{3}] \cup [2-\sqrt{2}, 2+\sqrt{3}],$$

which is the union of two closed intervals with no points in common.

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Example 17.43 *Find the range of the function*

$$f(x,y,z) = xy + z^2, \qquad x^2+y^2+z^2 \le a^2.$$

A the range of a polynomial in three variables over a closed ball, i.e. a closed and bounded and connected set.

D According to the second main theorem, f has a maximum value S and a minimum value M on this set. Then the first main theorem implies that the range is connected, hence the interval $[M,S]$. The maximum and the minimum values are either attained at a stationary point or at a boundary point, because a polynomial does not have exceptional points. Therefore, find the possible stationary points, and the examine the behaviour of the function on the boundary. Finally, make a numerical comparison.

I The equations of the possible stationary points are

$$\frac{\partial f}{\partial x} = y = 0, \quad \frac{\partial f}{\partial y} = x = 0, \quad \frac{\partial f}{\partial z} = 2z = 0.$$

Clearly, $(0,0,0)$ is the only stationary point, and the value is here $f(0,0,0)=0$.

We shall use spherical coordinates by the EXAMINATION OF THE BOUNDARY,

$$\begin{cases} x = a\sin\theta\cos\varphi, \\ y = a\sin\theta\sin\varphi, \\ z = a\cos\theta, \end{cases} \qquad \text{where } \theta \in [0,\pi] \text{ and } \varphi \in [0,2\pi].$$

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By insertion we get the following restriction to the boundary

$$\begin{aligned} f(x,y,z) &= xy+z^2 = a^2\sin^2\theta\cos\varphi\sin\varphi + a\cos^2\theta \\ &= \frac{a^2}{2}\left\{\sin^2\theta\sin 2\varphi + 2\cos^2\theta\right\} \\ &= \frac{a^2}{2}\left\{2+\sin^2\theta(\sin 2\varphi - 2)\right\}. \end{aligned}$$

These rearrangements show that $f(x,y,z)$ is largest on the boundary (the sphere), when $\sin^2\theta = 0$, and the corresponding maximum value is

$$S = \frac{a^2}{2}\{2+0\} = a^2 \qquad (>0).$$

Furthermore, $f(x,y,z)$ is smallest on the boundary (the sphere), when $\sin^2\theta = 1$ and $\sin 2\varphi = -1$, which corresponds to the minimum value

$$M = \frac{a^2}{2}\{2+1\cdot(-1-2)\} = -\frac{a^2}{2}.$$

Since the ball is connected and the function is continuous, it follows from the first main theorem that the range is

$$f(A) = \left[-\frac{a^2}{2}, a^2\right].$$

REMARK. Since we shall not explicitly indicate where the maximum and the minimum values are attained, we shall only argue instead of doing some heavy computations.. $\Diamond$

Example 17.44 *Given the function*

$$f(x,y,z) = \sqrt{x}+\sqrt{y}+\sqrt{z}+\sqrt{2-x-y-z}, \qquad (x,y,z)\in A.$$

1) Find the domain A and sketch it.

2) Explain why the function has both a maximum value S and a minimum value M.

3) Find S and M and the points in which they are attained.

4) Find the range $f(A)$ of the function.

A Domain, maximum and minimum and range.

D Use the standard methods and some symmetry arguments.

I 1) The function is defined (and continuous) for $x\geq 0$, $y\geq 0$, $z\geq 0$ and $x+y+z\leq 2$, i.e. in the closed tetrahedron

$$A = \{(x,y,z)\mid 0\leq x\leq 2,\, 0\leq y\leq 2-x,\, 0\leq z\leq 2-x-y\}$$

shown on the figure.

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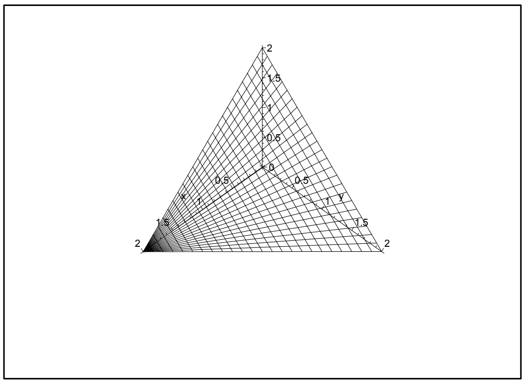

Figure 17.46: The domain A.

2) Since A is closed and bounded, and f is continuous on A, it follows from the second main theorem for continuous functions that f has both a maximum and a minimum on A.

   Since f is of class C^∞ in the interior of A, there are no exception points, so the maximum and the minimum lie either in the interior stationary points or on the boundary.

   Since A is connected, also $f(A)$ is connected, due to the first main theorem for continuous functions, and the range is the interval

   $$f(A) = [M, S],$$

   cf. 4).

3) The equations of the possible stationary points in the interior or A are

   $$\frac{\partial f}{\partial x} = \frac{1}{2}\,\frac{1}{\sqrt{x}} - \frac{1}{2}\,\frac{1}{\sqrt{2-x-y-z}} = 0,$$

   $$\frac{\partial f}{\partial y} = \frac{1}{2}\,\frac{1}{\sqrt{y}} - \frac{1}{2}\,\frac{1}{\sqrt{2-x-y-z}} = 0.$$

   $$\frac{\partial f}{\partial z} = \frac{1}{2}\,\frac{1}{\sqrt{z}} - \frac{1}{2}\,\frac{1}{\sqrt{2-x-y-z}} = 0.$$

   Since x, y, $z > 0$ and $x + y + z < 2$ in the interior of A we get

   $$x = y = z = 2 - x - y - z \quad (-2 - 3x),$$

   thus $x = y = z = \dfrac{1}{2} > 0$. The only stationary point in the interior of A is $\left(\dfrac{1}{2}, \dfrac{1}{2}, \dfrac{1}{2}\right)$, which we intuitively could expect from the symmetry. The value of the function at this stationary point is

   $$f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = 4\sqrt{\frac{1}{2}} = 2\sqrt{2}.$$

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THE BOUNDARY.

a) First consider the boundary surface B, which also lies in the (X,Y)-plane, i.e. where $z=0$. The restriction is given by

$$\varphi(x,y) = f(x,y,0) = \sqrt{x} + \sqrt{y} + \sqrt{2-x-y}, \qquad (x,y) \in B,$$

where

$$B = \{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le 2-x\}.$$

This restriction φ is of class C^∞ in the interior of B, so it has a maximum and a minimum

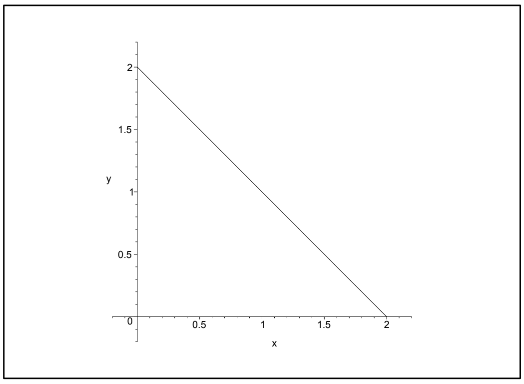

Figure 17.47: The part of the boundary B in the (X,Y)-plane.

in B according to the second main theorem. It follows from the conditions of a (restricted) stationary point

$$\frac{\partial \varphi}{\partial x} = \frac{1}{2}\,\frac{1}{\sqrt{x}} - \frac{1}{2}\,\frac{1}{\sqrt{2-x-y}} = 0,$$

$$\frac{\partial \varphi}{\partial y} = \frac{1}{2}\,\frac{1}{\sqrt{y}} - \frac{1}{2}\,\frac{1}{\sqrt{2-x-y}} = 0,$$

that

$$0 < x = y = 2-x-y \qquad (= 2-2x),$$

hence $x = y = \dfrac{2}{3}$, which corresponds to

$$\varphi\left(\frac{2}{3}, \frac{2}{3}\right) = f\left(\frac{2}{3}, \frac{2}{3}, 0\right) = 3\sqrt{\frac{2}{3}} = \sqrt{6}.$$

This is the "stationary point" on the part of the boundary in the (X,Y)-plane.

There is a similar "examination of the boundary" connected with this part of the boundary. If e.g. $y=0$, we get the restriction

$$\psi(x) = \varphi(x,0) = f(x,0,0) = \sqrt{x} + \sqrt{2-x}, \qquad x \in [0,2].$$

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Using the symmetry we get $\psi'(x) = 0$ for $x = 1$, which corresponds to the candidate

$$\psi(1) = \varphi(1,0) = f(1,0,0) = 2.$$

If instead $x = 0$, then similarly,

$$f(0,1,0) = 2.$$

The remaining boundary curve of the surface in the (X,Y)-plane lies on the line $x+y=2$, so the restriction becomes

$$\eta(x) = \varphi(x, 2-x) = \sqrt{x} + \sqrt{2-x}, \qquad x \in [0,2],$$

which is identical with $\psi(x)$ from above. Therefore we get the candidate

$$\eta(1) = \varphi(1,1) = f(1,1,0) = 2.$$

b) It follows by the symmetry that we have analogous results in those parts of the boundary surface which lie in the planes $y = 0$ and $x = 0$, respectively. We see that we have at the same time examined the *boundary curves* of the remaining oblique part of the boundary surface, so we are only missing the investigation of the "interior" of this remaining part of the boundary surface.

c) The restriction to the "interior" of the oblique boundary surface lying in the plane $z = 2 - x - y$ is given by

$$\Theta(x,y) = f(x,y,2-x-y) = \sqrt{x} + \sqrt{y} + \sqrt{2-x-y},$$

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where the domain of the parameter is the domain B in the (X,Y)-plane. Since we formally have $\Theta(x,y)=\varphi(x,y)$ over the same domain, we can reuse the results from φ, hence the interesting point is $\left(\frac{2}{3}\,,\,\frac{2}{3}\right)$. The value of the function is here

$$\Theta\left(\frac{2}{3}\;\frac{2}{3}\right)=f\left(\frac{2}{3}\,,\,\frac{2}{3}\,,\,\frac{2}{3}\right)=\varphi\left(\frac{2}{3}\,,\,\frac{2}{3}\right)=\sqrt{6}.$$

The CONCLUSION is that we have the following candidates of S and M:

Stationary point:

$$f\left(\frac{1}{2}\,,\,\frac{1}{2}\,,\,\frac{1}{2}\right)=2\sqrt{2}\quad(=\sqrt{8}).$$

Interior point on the boundary surfaces in $x=0$, $y=0$, $z=0$, respectively:

$$f\left(0,\;\frac{2}{3}\,,\,\frac{2}{3}\right)=f\left(\frac{2}{3}\,,\,0\,,\,\frac{2}{3}\right)=f\left(\frac{2}{3}\,,\,\frac{2}{3}\,,\,0\right)=\sqrt{6}.$$

Interior point of the oblique part of the boundary surface lying in $x+y+z=2$:

$$f\left(\frac{2}{3}\,,\,\frac{2}{3}\,,\,\frac{2}{3}\right)=\sqrt{6}.$$

Interior points on the edges lying on one of the axes:

$$f(1,0,0)=f(0,1,0)=f(0,0,1)=2\qquad(=\sqrt{4}).$$

Interior points lying on one of the oblique edges:

$$f(1,1,0)=f(1,0,1)=f(0,1,1)=2\qquad(=\sqrt{4}).$$

The corners:

$$f(0,0,0)=f(2,0,0)=f(0,2,0)=f(0,0,2)=\sqrt{2}.$$

It follows by a numerical comparison that the maximum value is

$$S=f\left(\frac{1}{2}\,,\,\frac{1}{2}\,,\,\frac{1}{2}\right)=2\sqrt{2},$$

and the minimum value is

$$M=f(0,0,0)=f(2,0,0)=f(0,2,0)=f(0,0,2)=\sqrt{2}.$$

REMARK. Note that the examination of the boundary which is here given in all details may be quite large. In some cases one could have made a shortcut, but for pedagogical reasons we have here only skipped trivial arguments of symmetry. ◊

4) Since A is connected and f is continuous, it follows from the first main theorem for continuous functions that the range is the interval

$$f(A)=[\sqrt{2},2\sqrt{2}].$$

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Example 17.45 *Given the function*

$$f(x,y) = \exp\left(-x^4 - y^2\right), \qquad (x,y) \in \mathbb{R}^2.$$

1) Show without differentiating that f has a proper maximum at $(0,0)$.

2) Find the range of the function.

3) Show that f has both a maximum and a minimum on the set

$$A = \left\{(x,y) \in \mathbb{R}^2 \mid 2x^2 + y^2 \leq 6\right\},$$

and find these values.

A Maximum and minimum without differentiating.

D Analyze $f(x,y)$.

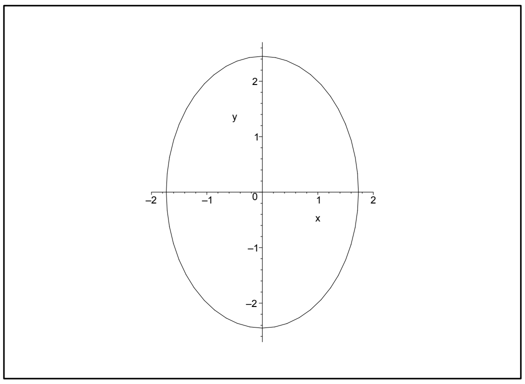

Figure 17.48: The domain A.

I 1) Clearly, $x^4 + y^2 \geq 0$, and we have only equality at $(0,0)$. Since exp is strictly increasing, f must have a proper maximum at $(0,0)$ where $f(0,0) = 1$.

2) Since $x^4 + y^2 \to +\infty$ for $x^2 + y^2 \to +\infty$, and since exp is strictly increasing and continuous where

$$\exp\left(-\left(x^4 + y^2\right)\right) \to 0 \qquad \text{for } x^2 + y^2 \to +\infty,$$

it follows that the range is $]0,1]$.

3) Since A is a closed ellipsoidal disc (in particular a closed and bounded set), and since f is continuous, it follows from the second main theorem for continuous functions that f has both a maximum value and a minimum value on A. From $(0,0) \in A$ and 1) follow that the maximum value is

$$S = f(0,0) = 1.$$

Clearly, f is constant on the curves

$$x^4 + y^2 = C \geq 0$$

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of the value $f(x,y) = e^{-C}$. We shall therefore only find the largest C, for which we can find an (x,y) on the curve, also lying in A. Clearly, such a point (x,y) must lie in ∂A, so $y^2 = 6 - 2x^2$. Hence, the constant C is the maximum value of the function

$$\varphi(x) = x^4 + y^2 = x^4 - 2x^2 + 6 = \left(x^2 - 1\right)^2 + 5 \qquad \text{for } x \in [-\sqrt{3}, \sqrt{3}].$$

If we put $t = x^2 \in [0,3]$, it follows that we shall find the maximum value of

$$\psi(t) = (t-1)^2 + 5, \qquad t \in [0,3].$$

The only possibilities are $t = 0$, $t = 1$ and $t = 3$. We get by insertion

$$\psi(0) = 5, \qquad \psi(1) = 5,$$

and

$$\psi(3) = \varphi(\pm\sqrt{3}) = 4 + 5 = 9 = C,$$

and we conclude that the minimum value of f is

$$f(\pm\sqrt{3}, 0) = e^{-9}.$$

Example 17.46 *1) Find the domain A of the function*

$$f(x,y) = \sqrt{x} + \sqrt{y} + \sqrt{2 - x - y},$$

and sketch A.

2) Explain why f has both a maximum value and a minimum value on A.

3) Then find the maximum value and the minimum value of f as well as the points in which they are attained.

4) Finally, find the range $f(A)$.

A Domain; maximum and minimum; range.

D Standard example.

I 1) The function is defined when

$$x \geq 0, \qquad y \geq 0 \quad \text{and} \quad x + y \leq 2,$$

i.e. in the closed triangle A with the corners $(0,0)$, $(2,0)$ and $(0,2)$.

2) Since A is closed and bounded, and f is continuous in A, it follows from the second main theorem for continuous functions that f has both a maximum and a minimum in A.

3) Since $f \in C^1$ in the interior of A, the maximum and the minimum can only be attained at either a stationary point in A° or in a point on the boundary ∂A.

The stationary points shall satisfy the equations

$$\frac{\partial f}{\partial x} = \frac{1}{2}\,\frac{1}{\sqrt{x}} - \frac{1}{2}\,\frac{1}{\sqrt{2-x-y}} = 0, \quad \frac{\partial f}{\partial y} = \frac{1}{2}\,\frac{1}{\sqrt{y}} - \frac{1}{2}\,\frac{1}{\sqrt{2-x-y}} = 0,$$

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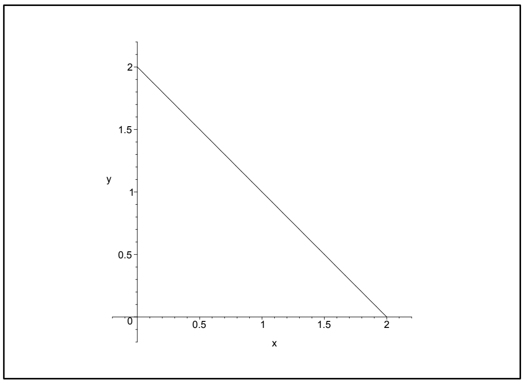

Figure 17.49: The domain A.

hence $\sqrt{x} = \sqrt{y} = \sqrt{2-x-y}$, i.e. $x = y = 2-2x$, and the only stationary point is

$$(x,y) = \left(\frac{2}{3}, \frac{2}{3}\right).$$

The value of the function is here

$$f\left(\frac{2}{3}, \frac{2}{3}\right) = 3\sqrt{\frac{2}{3}} = \sqrt{6}.$$

REMARK. Since we shall only find the maximum and the minimum on a closed and bounded set, a numerical comparison is sufficient, and we do not have to go through an elaborated examination of extrema. ◊

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THE BOUNDARY:

a) We get on the boundary curve $y = 0$, $x \in [0, 2]$, the restriction

$$\varphi(x) = f(x, 0) = \sqrt{x} + \sqrt{2-x},$$

which for symmetric reasons has its maximum for $x = 1$ and its minimum for $x = 0$ and $x = 2$.

ALTERNATIVELY,

$$\varphi'(x) = \frac{1}{2}\frac{1}{\sqrt{x}} - \frac{1}{2}\frac{1}{\sqrt{2-x}} = 0 \qquad \text{for } x = 1.$$

the interesting values are

$$f(0, 0) = f(2, 0) = \sqrt{2} \quad \text{and} \quad f(1, 0) = 2.$$

b) By interchanging the letters $(x, y) \to (y, x)$ we get on the boundary curve $x = 0$, $y \in [0, 2]$ the same function as above. Hence the candidates are

$$f(0, 0) = f(0, 2) = \sqrt{2} \quad \text{and} \quad f(0, 1) = 2.$$

c) We get on the boundary curve $y = 2 - x$, $x \in [0, 2]$ the restriction

$$\psi(x) = f(x, 2-x) = \sqrt{x} + \sqrt{2-x} = \varphi(x),$$

cf. above, so the candidates are

$$f(0, 2) = f(2, 0) = \sqrt{2} \quad \text{and} \quad f(1, 1) = 2.$$

All things considered, we have found the candidates

$$f\left(\frac{2}{3}, \frac{2}{3}\right) = \sqrt{6}, \qquad \text{(stationary point)},$$

$$f(1, 0) = f(0, 1) = f(1, 1) = 2,$$

$$f(0, 0) = f(2, 0) = f(0, 2) = \sqrt{2}.$$

Thence by comparison, the minimum value is

$$M = f(0, 0) = f(2, 0) = f(0, 2)\sqrt{2},$$

and the maximum value is

$$S = f\left(\frac{2}{3}, \frac{2}{3}\right) = \sqrt{6}.$$

4) Since f is continuous and the triangle A is connected, it follows from the first main theorem for continuous functions that the range $f(A)$ is also connected, hence an interval. Then it follows from 3) that

$$f(A) = [M, S] = [\sqrt{2}, \sqrt{6}].$$

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Example 17.47 *The function $f : A \to \mathbb{R}$ is given by*

$$f(x,y) = \sqrt{2-x-y}, \quad A = \{(x,y) \in \mathbb{R}^2 \mid -2 \le x \le 1,\, x^2 \le y \le 2-x\}.$$

1) Explain why f has both a maximum value S and a minimum value M.

2) Find S and M and the range $f(A)$ of the function.

A Maximum and minimum and range.

D Apply the standard methods.

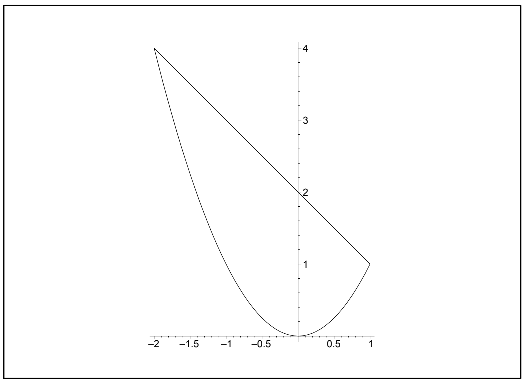

Figure 17.50: The domain A.

I 1) The domain A is indicated on the figure. Clearly, A is closed and bounded.

It follows that f is defined and continuous for $y \le 2 - x$, thus in particular in A. It follows from the second main theorem for continuous functions that f has both a maximum value S and a minimum value M on A. Since f is of class C^∞ in the interior of A, these values are either attained at an inner stationary point or on the boundary.

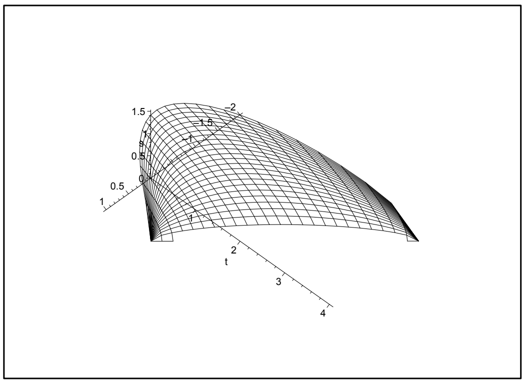

Figure 17.51: The graph of f over A.

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2) We get in the interior of A,

$$\frac{\partial f}{\partial x} = -\frac{1}{2}\,\frac{1}{\sqrt{2-x-y}}, \qquad \frac{\partial f}{\partial y} = -\frac{1}{2}\,\frac{1}{\sqrt{2-x-y}}.$$

These equations are never 0 in the interior of A, so f has no stationary points. We conclude that S and M are attained at boundary points.

a) When $y = 2-x$, the restriction is $f(x, 2-x) = 0$. As $f(x, y) \geq 0$, this must be the minimum value $M = 0$.

b) When $y = x^2$, the restriction is

$$\begin{aligned} f(x, x^2) &= \sqrt{2-x-x^2} = \sqrt{2+\frac{1}{4}-\left(x^2+x+\frac{1}{4}\right)} \\ &= \sqrt{\left(\frac{3}{2}\right)^2-\left(x+\frac{1}{2}\right)^2}, \qquad x \in [-2, 1]. \end{aligned}$$

It follows immediately that the maximum value is obtained for $x = -\dfrac{1}{2}$, i.e.

$$S = f\left(-\frac{1}{2}, \frac{1}{4}\right) = \frac{3}{2} \quad \text{and} \quad M = f(x, 2-x) = 0.$$

Since A is connected, it follows from the first main theorem for continuous functions that the range is

$$f(A) = [M, S] = \left[0, \frac{3}{2}\right].$$

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Example 17.48 *1) Sketch the set*

$$A = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 2 \text{ and } y \le 0\};$$

Indicate in particular the boundary ∂A on the figure. Explain why A is bounded and closed.

2) Explain why

$$f(x,y) = (x+y)\sqrt{2-x^2-y^2}, \qquad (x,y) \in A,$$

has a minimum value M and a maximum value S. Find these values as well as the points where they are attained.

3) Finally, explain why the range $f(A)$ of the function is an interval, and find $f(A)$.

A Maximum and minimum. The first and second main theorems for continuous functions.

D Use the standard methods.

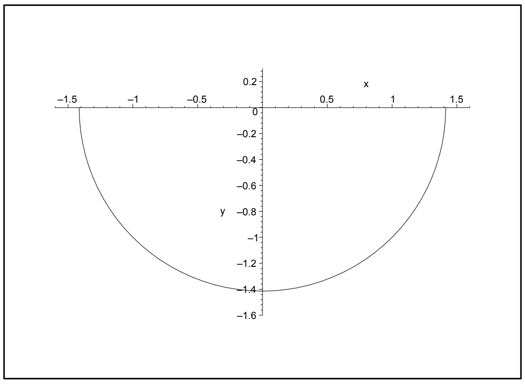

Figure 17.52: The point set A is the closed half disc in the lower half plane of centrum $(0,0)$ and radius $\sqrt{2}$.

I 1) The point set A is the intersection of the closed disc of centrum $(0,0)$ and radius $\sqrt{2}$, and the closed lower half plain. An intersection of two closed sets is also closed, so A is closed. Since A is contained in a disc of finite radius, A is bounded.

2) The function

$$f(x,y) = (x+y)\sqrt{2-x^2-y^2}$$

is defined and continuous on A.

Since A is closed and bounded and f is continuous, it follows from the second main theorem for continuous functions that f has a minimum value M and a maximum value S on A.

Since f is of class C^∞ in the interior of A, the values S and M can only be attained at an interior stationary point in A° or on the boundary ∂A.

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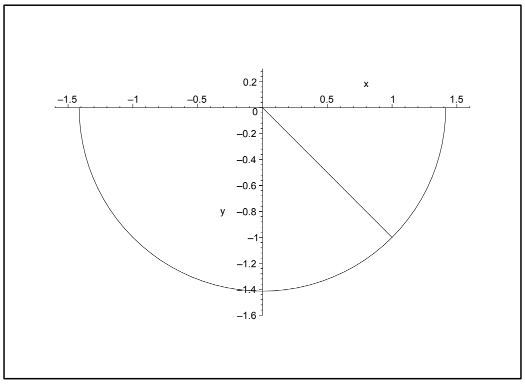

Figure 17.53: The zero lines of the function $f(x,y)$ in A. There are the union of the half circle and the oblique line $y=-x$. The function is negative to the left of this oblique line, and positive to the right of it.

When we examine the sign of f we see that f can be both positive and negative. When we specify S and M, we can exclude the zero curves, i.e. that part of the boundary which lies on the circular arc, as well as the points of A, which lie on the line $y=-x$.

In particular, the examination of the boundary is reduced to the segment $y=0$, $x \in [-\sqrt{2}, \sqrt{2}]$, on the X-axis, where we also can exclude the end points because the value is here 0. The restriction of f to this part of the boundary is

$$\varphi(x) = f(x,0) = x\sqrt{2-x^2}, \qquad x \in [-\sqrt{2}, \sqrt{2}],$$

with the derivative

$$\varphi'(x) = \sqrt{2-x^2} - \frac{x^2}{\sqrt{2-x^2}} = \frac{2(1-x^2)}{\sqrt{2-x^2}}, \qquad \text{for } x \in]-\sqrt{2}, \sqrt{2}[.$$

It follows that $\varphi'(x)=0$ for $x=\pm 1$. Hence, on the boundary we get the following candidates of S and M (because we have already excluded the end points and the circular arc),

$$f(1,0) = 1 \qquad \text{og} \qquad f(-1,0) = -1.$$

The possible candidates of the *stationary points* in the interior of A (i.e. where $2-x^2-y^2>0$ and $y<0$) are the solutions of the equations

$$\begin{cases} \dfrac{\partial f}{\partial x} = \sqrt{2-x^2-y^2} - \dfrac{x(x+y)}{\sqrt{2-x^2-y^2}} = \dfrac{2-2x^2-xy-y^2}{\sqrt{2-x^2-y^2}} = 0, \\[2ex] \dfrac{\partial f}{\partial y} = \dfrac{1-x^2-xy-2y^2}{\sqrt{s-x^2-y^2}} = 0, \end{cases}$$

which are reduced in the interior of A to

$$(17.19) \quad \begin{cases} 2x^2+xy+y^2 = 2, \\[1ex] x^2+xy+2y^2 = 2. \end{cases}$$

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Thus $y^2 = x^2$, i.e. either $y = x$ or $y = -x$. By the analysis of the sign of f, neither of the stationary points on $y = -x$ can be a maximum or a minimum (the value of the function is here zero). In the chase of the candidates the investigation is now reduced to the line segment $y = x$, $x \in]-1, 0[$, (because $y \le 0$). Then we get by insertion into either of the equations of (17.19) that

$$2 = 2x^2 + xy + y^2 = 4x^2,$$

hence $x = y = -\dfrac{1}{\sqrt{2}}$. We note again that f may have stationary points on $y = -x$, but these are of no importance because the only *relevant* stationary point is $\left(-\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}\right)$, corresponding to the value of the function

$$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\sqrt{2}.$$

We have now found three candidates of M and S:

$$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\sqrt{2}, \quad f(-1,0) = -1, \quad f(1,0) = 1.$$

It follows by a numerical comparison that

$$M = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\sqrt{2} \quad \text{and} \quad S = f(1,0) = 1.$$

3) Since A is convex, A is in particular connected. Since f is continuous in A, it follows from the first main theorem for continuous functions that the range (here a subset of $\mathbb{R}$) is connected, hence an interval. Now f has according to 2) a minimum and a maximum, so we finally get

$$f(A) = [M, S] = [-\sqrt{2}, 1].$$

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Example 17.49 *Let*

$$A = \{(x,y) \in \mathbb{R}^2 \mid x^2 - 1 \le y \le 3\},$$

and consider the function $f : A \to \mathbb{R}$ *given by*

$$f(x,y) = 1 - x^2 - y + 2x^2 y, \qquad (x,y) \in A.$$

1) Sketch A*, and explain why the function* f *has both a maximum value* S *and a minimum value* M*.*

2) Find the stationary points of the function f*.*

3) Find S *and* M*.*

4) Find the range $f(A)$ *of the function.*

A Maximum and minimum and range of a function.

D Standard example.

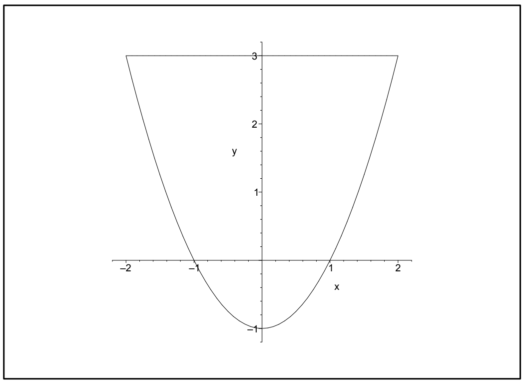

Figure 17.54: The domain A.

I 1) The set A is closed and bounded, and the polynomial f is continuous on A. It follows from the second main theorem for continuous functions that f has both a maximum value S and a minimum value M on A. Since f is of class C^∞ in the interior of A, the values S and M are either attained at a stationary point or at a boundary point.

2) The equations of the stationary points are

$$\begin{cases} \dfrac{\partial f}{\partial x} = -2x + 4xy = 0, \quad \text{i.e.} \quad 4x\left(y - \dfrac{1}{2}\right) = 0, \\[2ex] \dfrac{\partial f}{\partial y} = -1 + 2x^2 = 0, \quad \text{i.e.} \quad x^2 = \dfrac{1}{2}. \end{cases}$$

We get from the latter equation that $x = \pm\dfrac{1}{\sqrt{2}}$, which put into the first equation gives $y = \dfrac{1}{2}$. Hence, the stationary points are

$$\left(-\frac{1}{\sqrt{2}}, \frac{1}{2}\right) \quad \text{and} \quad \left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right),$$

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and we note that they both lie in the interior of A:

$$x^2 - 1 = -\frac{1}{2} < y = \frac{1}{2} < 3.$$

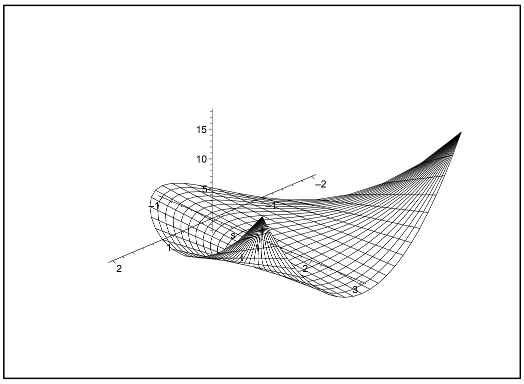

Figure 17.55: The graph of f over A.

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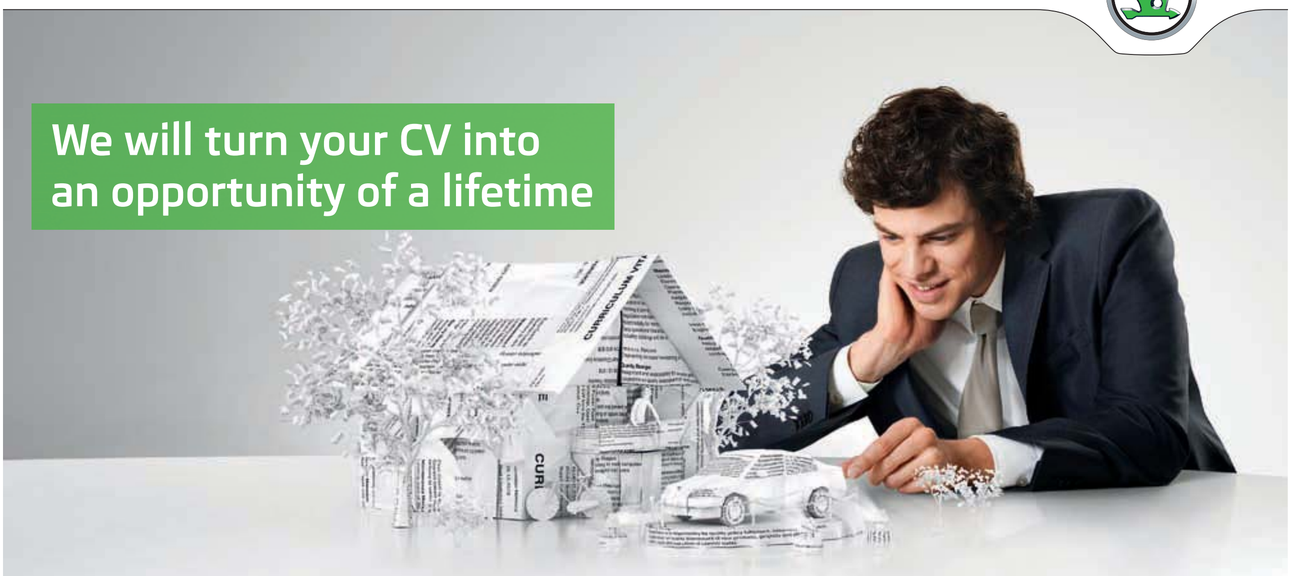

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3) The values at the stationary points are

$$f\left(\pm\frac{1}{\sqrt{2}},\frac{1}{2}\right)=1-\frac{1}{2}-\frac{1}{2}+2\cdot\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{2}.$$

THE BOUNDARY.

a) By restriction to the line segment $y=3$, $x\in[-2,2]$,

$$\varphi(x)=f(x,3)=1-x^2-3+6x^2=5x^2-2,$$

which clearly is smallest when $x=0$, and largest when $x=\pm2$. We get the candidates

$$f(0,3)=-2 \quad\text{and}\quad f(\pm2,3)=18.$$

b) Considering the restriction to the parabolic arc we have two alternatives:

i) By the restriction to $y=x^2-1$, $x\in[-2,2]$, we get

$$\psi(x)=f(x,x^2-1)=2(1-x^2)+2x^2(x^2-1)=2(x^2-1)^2,$$

which is smallest when $x^2=1$, and largest when $x^2=(\pm2)^2$.The candidates are

$$f(\pm1,0)=1-1=0 \quad\text{and}\quad f(\pm2,3)=18.$$

ii) ALTERNATIVELY, $x^2=y+1$, $y\in[-1,3]$, so

$$\eta(y)=-2y+2(y+1)y=2y^2,\qquad y\in[-1,3],$$

which is smallest for $y=0$, and largest for $y=3$, corresponding to

$$f(\pm1,0)=0 \qquad\text{and}\qquad f(\pm2,3)=18.$$

Then by a numerical comparison of the candidates,

$$\text{SP:}\quad f\left(\pm\frac{1}{\sqrt{2}},\frac{1}{2}\right)=\frac{1}{2},$$

$$\text{a)}\qquad f(0,3)=-2 \quad\text{and}\quad f(\pm2,3)=18,$$

$$\text{b)}\qquad f(\pm1,0)=0 \quad\text{and}\quad f(\pm2,3)=18,$$

we finally get

$$S=f(\pm2,3)=18 \quad\text{and}\quad M=f(0,3)=-2.$$

c) Since A is connected, and f is continuous, it follows from the first main theorem for continuous functions, that $f(A)\subseteq\mathbb{R}$ is connected, hence an interval.
We have shown in 3) that $M=-2$ and $S=18$, so the range is

$$f(A)=[-2,18].$$

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Example 17.50 *Find the interval of range of the function*

$$f(x,y) = xy + 64\left(\frac{1}{x}+\frac{1}{y}\right), \qquad (x,y) \in A,$$

where

$$A = \{(x,y) \in \mathbb{R}^2 \mid x \geq 1,\, y \geq 1,\, xy \leq 32\}.$$

A Maximum, minimum, interval of range.

D Sketch A and conclude that A is closed and bounded. Apply the first and second main theorems for continuous functions. Find the possible stationary points. Check the boundary. Conclude by a numerical comparison.

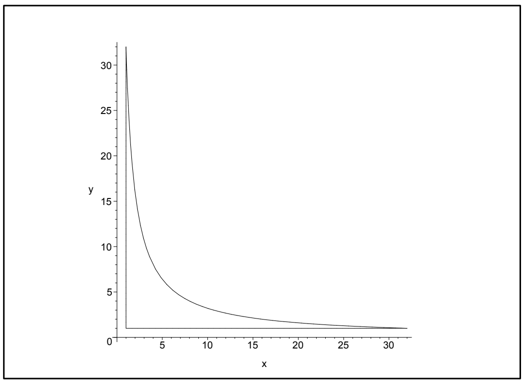

Figure 17.56: The closed and bounded domain A.

I It follows from the figure and the definition of A that A is connected and closed and bounded. Since $f(x,y)$ is *continuous* on the *closed and bounded* set A, it follows from the second main theorem for continuous functions that $f(x,y)$ has a maximum and a minimum on A. Since f is of class C^∞, the maximum and minimum values are to be found among the values at the possible inner stationary points and at the boundary points.

Since A is also *connected*, and f is *continuous*, it follows from the first main theorem for continuous functions that the range is connected, hence a closed interval, which must necessarily be

$$f(A) = [f_{\text{min}}, f_{\text{max}}]\,.$$

The equations of the possible stationary points are

$$\frac{\partial f}{\partial x} = y - \frac{64}{x^2} = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial y} = x - \frac{64}{t^2} = 0,$$

thus

$$x^2y = 64 \qquad \text{and} \qquad xy^2 = 64.$$

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Now $x, y > 0$ in A, so it follows by a division that $\dfrac{x}{y} = 1$, i.e. $y = x$. Then by insertion $x^3 = 64$, so $x = y = 4$. Since $xy = 16 < 32$ and $4 > 1$ we conclude that $(4,4)$ lies in the interior of A. Hence $(4,4)$ is a stationary point, and the value of the function is here

$$f(4,4) = 16 + 64\left(\frac{1}{4} + \frac{1}{4}\right) = 16 + 32 = 48.$$

THE BOUNDARY.

1) For $y = 1$, $1 \le x \le 32$, we get the restriction

$$\varphi(x) = f(x,1) = x + \frac{64}{x} + 64$$

where

$$\varphi'(x) = 1 - \frac{64}{x^2} = 0 \qquad \text{for } x = 8.$$

We notice the values

$$f(1,1) = \varphi(1) = 1 + 64 + 64 = 129,$$

$$f(32,1) = \varphi(32) = 32 + 2 + 64 = 98,$$

$$f(8,1) = \varphi(8) = 8 + 8 + 64 = 80.$$

2) For $x = 1$, $1 \le y \le 32$, it follows from the symmetry,

$$f(1,1) = 129, \qquad f(1,32) = 98, \qquad f(1,8) = 80.$$

3) If $xy = 32$, i.e. $y = \dfrac{32}{x}$, $1 \le x \le 32$, we get the restriction

$$\psi(x) = xy + 64 \cdot \frac{x+y}{xy} = 32 + 2\left(x + \frac{32}{x}\right) = 2x + \frac{64}{x} + 32$$

where

$$\psi'(x) = 2 - \frac{64}{x^2} = 0 \qquad \text{for } x = \sqrt{32} = 4\sqrt{2} = y.$$

This is the only additional value, because we have already checked the end points of the interval above,

$$f(\sqrt{32}, \sqrt{32}) = \psi(\sqrt{32}) = 32 + (\sqrt{32} + \sqrt{32}) = 32 + 8\sqrt{2}.$$

Then compare the values of the candidates,

$$f(4,4) = 48, \qquad f(1,1) = 129, \qquad f(8,1) = f(1,8) = 80,$$

$$f(32,1) = f(1,32) = 98, \qquad f(\sqrt{32}, \sqrt{32}) = 32 + 8\sqrt{2} < 48.$$

It follows that the minimum value is

$$f(\sqrt{32}, \sqrt{32}) = 32 + 8\sqrt{2},$$

and the maximum value is

$$f(1,1) = 129.$$

The range is connected, so the interval of the range is

$$f(A) = [32 + 8\sqrt{2}, 129].$$

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Example 17.51 *Given the function*

$$f(x,y) = 2\ln\left(1+x^2+y^2\right) + x\sqrt{2} + y, \qquad x^2+y^2 \le 4.$$

1) Explain why the function has a maximum value S and a minimum value M.

2) Show that the stationary points of the function are $(-\sqrt{2},-1)$ and $\left(-\frac{1}{3}\sqrt{2},-\frac{1}{3}\right)$.

3) Find S and M.

4) Find the range of the function.

A Maximum and minimum, range.

D Sketch a figure. Follow the guidelines.

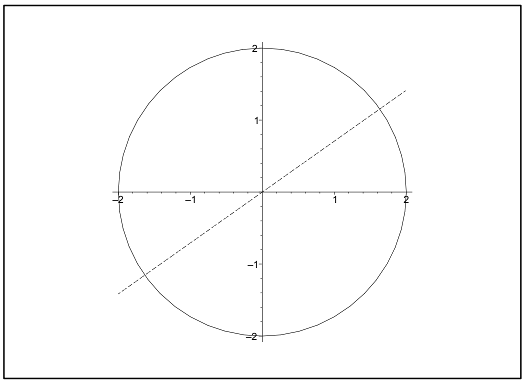

Figure 17.57: The domain A and the line $y = \dfrac{1}{\sqrt{2}}\,x$.

I 1) The domain A is a closed disc of centrum$(0,0)$ and radius 2, thus A is closed and bounded and connected.

Clearly, f is continuous on A and of class C^∞ in the interior of A. According to the second main theorem for continuous functions, f has a maximum value S and a minimum value M on A. These values are either attained at a stationary point or on the boundary, because there are no exceptional points.

Note also that A is connected, so the range $f(A) = [M,S]$ is connected according to the the first main theorem for continuous functions. This will be used in 4).

2) The equations of the stationary points are

$$\frac{\partial f}{\partial x} = \frac{4x}{1+x^2+y^2} + \sqrt{2} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{4y}{1+x^2+y^2} + 1 = 0.$$

It follows that at a stationary point we must have $x \neq 0$ and $y \neq 0$, so the equations are equivalent to

$$\frac{4xy}{1+x^2+y^2} = -\sqrt{2}\,y \quad \text{and} \quad \frac{4xy}{1+x^2+y^2} = -x, \quad x \neq 0, \quad y \neq 0.$$

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Accordingly, the possible stationary points must lie on the line $x = \sqrt{2}\,y$.

When we eliminate x in e.g. the latter equation of the stationary points, then

$$-1 = \frac{4y}{1+x^2+y^2} = \frac{4y}{1+3y^2},$$

hence

$$3y^2 + 4y + 1 = 0.$$

The solutions are $y = -1$ and $y = -\dfrac{1}{3}$. From $x = \sqrt{2}\,y$ follows that the only *possible* stationary points are

$$(-\sqrt{2}, -1) \qquad \text{and} \qquad \left(-\frac{1}{3}\sqrt{2}, -\frac{1}{3}\right).$$

It remains to be proved that they are both stationary points.

a) They satisfy the equations:

i) For $(-\sqrt{2}, -1)$ we get

$$\frac{-4\sqrt{2}}{1+2+1} + \sqrt{2} = -\sqrt{2} + \sqrt{2} = 0 \quad \text{and} \quad \frac{-4}{1+2+1} + 1 = -1 + 1 = 0.$$

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ii) We get for $\left(-\frac{1}{3}\sqrt{2}, -\frac{1}{3}\right)$,

$$\frac{-\frac{4}{3}\sqrt{2}}{1+\frac{2}{9}+\frac{1}{9}} + \sqrt{2} = \frac{-\frac{4}{3}\sqrt{2}}{\frac{4}{3}} + \sqrt{2} = 0 \text{ and } \frac{-\frac{4}{3}}{1+\frac{2}{9}+\frac{1}{9}} + 1 = \frac{-\frac{4}{3}}{\frac{4}{3}} + 1 = 0.$$

b) They both lie in A:

i) For $(-\sqrt{2}, -1)$ we get

$$x^2 + y^2 = 2 + 1 = 3 < 4.$$

ii) For $\left(-\frac{1}{3}\sqrt{2}, -\frac{1}{3}\right) = \frac{1}{3}(-\sqrt{2}, 1)$ we get

$$x^2 + y^2 = \frac{1}{9} \cdot 3 = \frac{1}{3} < 4.$$

We have now proved that the stationary points are

$$(-\sqrt{2}, -1) \qquad \text{and} \qquad \left(-\frac{1}{3}\sqrt{2}, -\frac{1}{3}\right).$$

3) The values of the function at the stationary points are

$$f(-\sqrt{2}, -1) = 2\ln(1+2+1) - \sqrt{2}\cdot\sqrt{2} - 1 = 2\ln 4 - 3,$$

$$f\left(-\frac{1}{3}\sqrt{2}, -\frac{1}{3}\right) = 2\ln\left(1 + \frac{2}{9} + \frac{1}{9}\right) - \frac{1}{3}\sqrt{2}\cdot\sqrt{2} - \frac{1}{3} = 2\ln\frac{4}{3} - 1.$$

THE BOUNDARY.

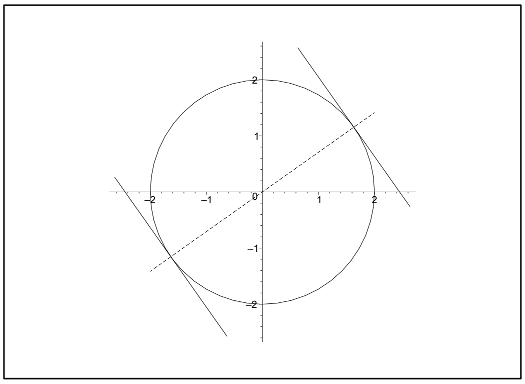

Figure 17.58: The geometrical analysis of the maximum and the minimum on the boundary.

On the boundary $2\ln(1+x^2+y^2) = 2\ln 5$ is constant. The maximum and minimum values on the boundary are attained at those points in which $x\sqrt{2}+y$ attains its maximum and minimum value on the circle. Geometrically these are given by the two tangents of the circle, which are parallel to the line $x\sqrt{2}+y=0$, i.e. at the intersection points of the circle $x^2+y^2=4$ and

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perpendicular to the line $x = \sqrt{2}\,y$, on which the stationary points lie. This gives the new candidates

$$\left(2\sqrt{\frac{2}{3}}, \frac{2}{\sqrt{3}}\right) \qquad \text{and} \qquad \left(-2\sqrt{\frac{2}{3}}, -\frac{2}{\sqrt{3}}\right).$$

Since all candidates of S and M lie on the line $x = \sqrt{2}\,y$, we get the following one-dimensional variant of the identification of S and M:

The restriction to the line $x = \sqrt{2}\,y$ is

$$g(y) = 2\left(1+3y^2\right) + 3y, \qquad y \in \left[-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right],$$

where

$$g'(y) = \frac{12y}{1+3y^2} + 3, \qquad y \in \left]-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right[.$$

As before we get $g'(y) = 0$ for $y = -1$ and $y = -\frac{1}{3}$. The variation of $g'(y)$ is:

y	$-\frac{2}{\sqrt{3}} \le y < -1$	-1	$-1 < y < -\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3} < y \le \frac{2}{\sqrt{3}}$
$g'(y)$	$+$	0	$-$	0	$+$

Hence we have a local maximum for $y = -1$ and a local minimum for $y = -\frac{1}{3}$.

The candidates of S are

$$g(-1) = 2\ln 4 - 3 \qquad \text{and} \qquad g\left(\frac{2}{\sqrt{3}}\right) = 2\ln 5 + 2\sqrt{3}.$$

It follows that

$$S = g\left(\frac{2}{\sqrt{3}}\right) = f\left(2\sqrt{\frac{2}{3}}, \frac{2}{\sqrt{3}}\right) = 2 < ln5 + 2\sqrt{3} \quad [\approx 6,6830].$$

Analogously, the candidates of M are

$$\left(-\frac{2}{\sqrt{3}}\right) = 2\ln 5 - 2\sqrt{3} \quad \text{and} \quad g\left(-\frac{1}{3}\right) = 2\ln\frac{4}{3} - 1.$$

By a numerical comparison on a pocket calculator we get

$$M = g\left(-\frac{1}{3}\right) = f\left(-\frac{1}{3}\sqrt{2}, -\frac{1}{3}\right) = 2\ln\frac{4}{3} - 1 \quad [\approx -0,4246].$$

Alternatively a parametric description of the boundary is

$$(x,y) = (2\cos\varphi, 2\sin\varphi), \qquad \varphi \in [0, 2\pi].$$

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Hence the restriction to the boundary becomes

$$h(\varphi) = f(2\,\cos\varphi, 2\,\sin\varphi) = 2\ln 5 + 2\sqrt{2}\,\cos\varphi + 2\sin\varphi$$

where

$$h'(\varphi) = -2\sqrt{2}\,\sin\varphi + 2\cos\varphi = 0$$

for $\tan\varphi = \dfrac{1}{\sqrt{2}}$, corresponding to

$$\cos\varphi = \pm\frac{1}{\sqrt{1+\tan^2\varphi}} = \pm\frac{1}{\sqrt{1+\frac{1}{2}}} = \pm\sqrt{\frac{2}{3}}$$

and

$$\sin\varphi = \cos\varphi\tan\varphi = \pm\sqrt{\frac{2}{3}}\cdot\sqrt{\frac{1}{2}} = \pm,\frac{1}{\sqrt{3}},$$

where the signs are belonging together.

We get the candidates

$$f\left(2\sqrt{\frac{2}{3}},\frac{2}{\sqrt{3}}\right) \qquad \text{and} \qquad \left(-2\sqrt{\frac{2}{3}},-\frac{2}{\sqrt{3}}\right)$$

with the values of the function

$$\left(2\sqrt{\frac{2}{3}},\frac{2}{\sqrt{3}}\right) = 2\ln\left(1+\frac{8}{3}+\frac{4}{3}\right) + 2\sqrt{\frac{2}{3}}\cdot\sqrt{2} + \frac{2}{\sqrt{3}} = 2\ln 5 + 2\sqrt{3},$$

and

$$f\left(-2\sqrt{\frac{2}{3}},-\frac{2}{\sqrt{3}}\right) = 2\ln 5 - 2\sqrt{3}.$$

Numerical comparison.

We shall find the maximum and the minimum value among

$$f(-\sqrt{2},-1) = 2\ln 4 - 3, \qquad f\left(-\frac{1}{3}\sqrt{2},-\frac{1}{3}\right) = 2\ln\frac{4}{3} - 1,$$

$$f\left(2\sqrt{\frac{2}{3}},\frac{2}{\sqrt{3}}\right) = 2\ln 5 + 2\sqrt{3}, \qquad f\left(-2\sqrt{\frac{2}{3}},-\frac{2}{\sqrt{3}}\right) = 2 < ln5 - \sqrt{3}.$$

Clearly,

$$S = f\left(2\sqrt{\frac{2}{3}},\frac{2}{\sqrt{3}}\right) = 2\ln 5 + 2\sqrt{3} \approx 6,6830.$$

Then by a numerical comparison on a pocket calculator,

$$M = f\left(-\frac{1}{3}\sqrt{2},-\frac{1}{3}\right) = 2\ln\frac{4}{3} - 1 \approx -0.4246.$$

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4) Since A is connected, and f is continuous on A, the range $f(A)$ is connected according to the first main theorem for continuous functions. We have already found S and M in 3), so the range is the interval

$$f(A) = [M, S] = \left[2 \ln \frac{4}{3} - 1, 2 \ln 5 + 2\sqrt{3}\right].$$

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Example 17.52 *Given the function*

$$f(x,y) = 2y\sin x - x, \qquad (x,y) \in A,$$

where the domain A is given by the inequalities

$$-\frac{\pi}{2} \le x \le \frac{\pi}{2}, \qquad 0 \le y \le \cos x.$$

1) Explain why the function has a maximum value S and a minimum value M.

2) Find S and M.

3) Find the range of the function.

A Maximum and minimum values; range.

D Sketch A, and apply the second and the first main theorem for continuous functions.

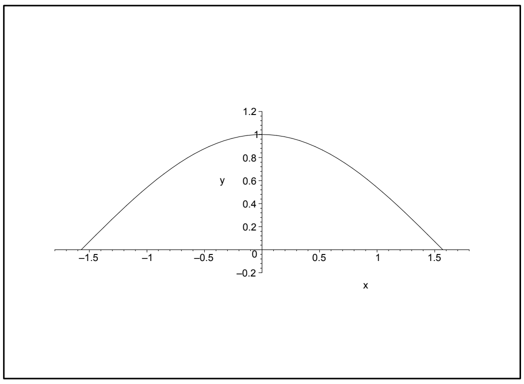

Figure 17.59: The set A.

I 1) The set A is closed and bounded, and f is continuous. According to the second main theorem for continuous functions, f has both a maximum value S and a minimum value M on A.

2) Since f is of class C^∞, the values S and M are either attained at an inner stationary point or at a boundary point.

STATIONARY POINTS.
The equations of the stationary points are

$$\frac{\partial f}{\partial x} = 2y\cos x - 1 = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 2\sin x = 0.$$

It follows from the latter equation that $x = p\pi$, $p \in \mathbb{Z}$, i.e. $x = 0$, if the point shall also lie in A. When this is put into the former equation we get $2y - 1 = 0$, so $y = \frac{1}{2}$. Thus the only stationary point in A is $\left(0, \frac{1}{2}\right)$. The value of the function at this point is

$$f\left(0,\frac{1}{2}\right) = 2 \cdot \frac{1}{2} \cdot \sin 0 - 0 = 0.$$

THE BOUNDARY.

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a) On the boundary curve $y = 0$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we have the restriction

$$\varphi(x) = f(x,0) = -x,$$

with the maximum value

$$\varphi\left(-\frac{\pi}{2}\right) = f\left(-\frac{\pi}{2}.0\right) = \frac{\pi}{2}$$

and the minimum value

$$\varphi\left(\frac{\pi}{2}\right) = f\left(\frac{\pi}{2},0\right) = -\frac{\pi}{2}.$$

b) On the boundary curve $y = \cos x$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we get the restriction

$$\psi(x) = f(x,\cos x) = 2\cos x \cdot \sin x - x = \sin 2x - x,$$

where

$$\psi'(x) = 2\cos 2x - 1 = 0, \qquad \cos 2x = \frac{1}{2},$$

so

$$x = \left(\pm\frac{\pi}{3} + 2p\pi\right)\cdot\frac{1}{2} = \pm\frac{\pi}{6} + p\pi, \qquad p \in \mathbb{Z}.$$

Here $p = 0$, because we shall stay inside A, hence

$$\psi\left(\frac{\pi}{6}\right) = f\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right) = \sin\frac{\pi}{3} - \frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{\pi}{6} = \frac{3\sqrt{3}-\pi}{6},$$

and

$$\psi\left(-\frac{\pi}{6}\right) = f\left(-\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right) = -\sin\frac{\pi}{3} + \frac{\pi}{6} = -\frac{3\sqrt{3}-\pi}{6}.$$

Then by a numerical comparison,

$$S = f\left(-\frac{\pi}{2},0\right) = \frac{\pi}{2} \quad\text{and}\quad M = f\left(\frac{\pi}{2},0\right) = -\frac{\pi}{2}.$$

c) Since A is connected, and f is continuous, it follows from the first main theorem for continuous functions that $f(A)$ is connected, i.e. an interval. When we apply the results from 2), we get

$$f(A) = [M,S] = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

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18 Formulæ

Some of the following formulæ can be assumed to be known from high school. It is highly recommended that one *learns most of these formulæ in this appendix by heart.*

18.1 Squares etc.

The following simple formulæ occur very frequently in the most different situations.

$$\begin{array}{ll}
(a+b)^2 = a^2+b^2+2ab, & a^2+b^2+2ab = (a+b)^2, \\
(a-b)^2 = a^2+b^2-2ab, & a^2+b^2-2ab = (a-b)^2, \\
(a+b)(a-b) = a^2-b^2, & a^2-b^2 = (a+b)(a-b), \\
(a+b)^2 = (a-b)^2+4ab, & (a-b)^2 = (a+b)^2-4ab.
\end{array}$$

18.2 Powers etc.

Logarithm:

$$\begin{array}{rcl}
\ln|xy| = & \ln|x|+\ln|y|, & x,y \neq 0, \\
\ln\left|\dfrac{x}{y}\right| = & \ln|x|-\ln|y|, & x,y \neq 0, \\
\ln|x^r| = & r\ln|x|, & x \neq 0.
\end{array}$$

Power function, fixed exponent:

$$(xy)^r = x^r \cdot y^r, x,y>0 \quad \text{(extensions for some } r\text{)},$$

$$\left(\frac{x}{y}\right)^r = \frac{x^r}{y^r}, x,y>0 \qquad \text{(extensions for some } r\text{)}.$$

Exponential, fixed base:

$$a^x \cdot a^y = a^{x+y}, \quad a>0 \quad \text{(extensions for some } x,\, y\text{)},$$
$$(a^x)^y = a^{xy}, a>0 \qquad \text{(extensions for some } x,\, y\text{)},$$

$$a^{-x} = \frac{1}{a^x}, a>0, \qquad \text{(extensions for some } x\text{)},$$

$$\sqrt[n]{a} = a^{1/n}, a \geq 0, \qquad n \in \mathbb{N}.$$

Square root:

$$\sqrt{x^2} = |x|, \qquad x \in \mathbb{R}.$$

Remark 18.1 It happens quite frequently that students make errors when they try to apply these rules. They must be mastered! In particular, as one of my friends once put it: "If you can master the square root, you can master everything in mathematics!" Notice that this innocent looking square root is one of the most difficult operations in Calculus. Do not forget the *absolute value!* $\diamond$

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18.3 Differentiation

Here are given the well-known rules of differentiation together with some rearrangements which sometimes may be easier to use:

$$\{f(x) \pm g(x)\}' = f'(x) \pm g'(x),$$

$$\{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x) = f(x)g(x)\left\{\frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}\right\},$$

where the latter rearrangement presupposes that $f(x) \neq 0$ and $g(x) \neq 0$.
If $g(x) \neq 0$, we get the usual formula known from high school

$$\left\{\frac{f(x)}{g(x)}\right\}' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

It is often more convenient to compute this expression in the following way:

$$\left\{\frac{f(x)}{g(x)}\right\} = \frac{d}{dx}\left\{f(x) \cdot \frac{1}{g(x)}\right\} = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2} = \frac{f(x)}{g(x)}\left\{\frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}\right\},$$

where the former expression often is *much easier* to use in practice than the usual formula from high school, and where the latter expression again presupposes that $f(x) \neq 0$ and $g(x) \neq 0$. Under these assumptions we see that the formulæ above can be written

$$\frac{\{f(x)g(x)\}'}{f(x)g(x)} = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)},$$

$$\frac{\{f(x)/g(x)\}'}{f(x)/g(x)} = \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}.$$

Since

$$\frac{d}{dx}\ln|f(x)| = \frac{f'(x)}{f(x)}, \qquad f(x) \neq 0,$$

we also name these the *logarithmic derivatives*.

Finally, we mention the rule of **differentiation of a composite function**

$$\{f(\varphi(x))\}' = f'(\varphi(x)) \cdot \varphi'(x).$$

We first differentiate the function itself; then the insides. This rule is a 1-dimensional version of the so-called *Chain rule*.

18.4 Special derivatives.

Power like:

$$\frac{d}{dx}(x^\alpha) = \alpha \cdot x^{\alpha-1}, \qquad \text{for } x > 0, \quad \text{(extensions for some } \alpha\text{)}.$$

$$\frac{d}{dx}\ln|x| = \frac{1}{x}, \qquad \text{for } x \neq 0.$$

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Exponential like:

$$\frac{d}{dx}\exp x = \exp x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\left(a^x\right) = \ln a \cdot a^x, \qquad \text{for } x \in \mathbb{R} \text{ and } a > 0.$$

Trigonometric:

$$\frac{d}{dx}\sin x = \cos x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\cos x = -\sin x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\tan x = 1 + \tan^2 x = \frac{1}{\cos^2 x}, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, p \in \mathbb{Z},$$

$$\frac{d}{dx}\cot x = -\left(1 + \cot^2 x\right) = -\frac{1}{\sin^2 x}, \qquad \text{for } x \neq p\pi, p \in \mathbb{Z}.$$

Hyperbolic:

$$\frac{d}{dx}\sinh x = \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\cosh x = \sinh x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\tanh x = 1 - \tanh^2 x = \frac{1}{\cosh^2 x}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\coth x = 1 - \coth^2 x = -\frac{1}{\sinh^2 x}, \qquad \text{for } x \neq 0.$$

Inverse trigonometric:

$$\frac{d}{dx}\text{ Arcsin } x = \frac{1}{\sqrt{1-x^2}}, \qquad \text{for } x \in \left]-1, 1\right[,$$

$$\frac{d}{dx}\text{ Arccos } x = -\frac{1}{\sqrt{1-x^2}}, \qquad \text{for } x \in \left]-1, 1\right[,$$

$$\frac{d}{dx}\text{ Arctan } x = \frac{1}{1+x^2}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\text{ Arccot } x = \frac{1}{1+x^2}, \qquad \text{for } x \in \mathbb{R}.$$

Inverse hyperbolic:

$$\frac{d}{dx}\text{ Arsinh } x = \frac{1}{\sqrt{x^2+1}}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\text{ Arcosh } x = \frac{1}{\sqrt{x^2-1}}, \qquad \text{for } x \in \left]1, +\infty\right[,$$

$$\frac{d}{dx}\text{ Artanh } x = \frac{1}{1-x^2}, \qquad \text{for } |x| < 1,$$

$$\frac{d}{dx}\text{ Arcoth } x = \frac{1}{1-x^2}, \qquad \text{for } |x| > 1.$$

Remark 18.2 The derivative of the trigonometric and the hyperbolic functions are to some extent exponential like. The derivatives of the inverse trigonometric and inverse hyperbolic functions are power like, because we include the logarithm in this class. $\Diamond$

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18.5 Integration

The most obvious rules are dealing with linearity

$$\int \{f(x)+\lambda g(x)\}\,dx = \int f(x)\,dx + \lambda \int g(x)\,dx, \qquad \text{where } \lambda \in \mathbb{R} \text{ is a constant,}$$

and with the fact that differentiation and integration are "inverses to each other", i.e. modulo some arbitrary constant $c \in \mathbb{R}$, which often tacitly is missing,

$$\int f'(x)\,dx = f(x).$$

If we in the latter formula replace $f(x)$ by the product $f(x)g(x)$, we get by reading from the right to the left and then differentiating the product,

$$f(x)g(x) = \int \{f(x)g(x)\}'\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx.$$

Hence, by a rearrangement

The rule of partial integration:

$$\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx.$$

The differentiation is moved from one factor of the integrand to the other one by changing the sign and adding the term $f(x)g(x)$.

Remark 18.3 This technique was earlier used a lot, but is almost forgotten these days. It must be revived, because MAPLE and pocket calculators apparently do not know it. It is possible to construct examples where these devices cannot give the exact solution, unless you first perform a partial integration yourself. $\Diamond$

Remark 18.4 This method can also be used when we estimate integrals which cannot be directly calculated, because the antiderivative is not contained in e.g. the catalogue of MAPLE. The idea is by a succession of partial integrations to make the new integrand smaller. $\Diamond$

Integration by substitution:

If the integrand has the special structure $f(\varphi(x))\cdot\varphi'(x)$, then one can change the variable to $y=\varphi(x)$:

$$\int f(\varphi(x))\cdot\varphi'(x)\,dx = \text{“}\int f(\varphi(x))\,d\varphi(x)\text{''} = \int_{y=\varphi(x)} f(y)\,dy.$$

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Integration by a monotonous substitution:

If $\varphi(y)$ is a *monotonous* function, which maps the y-interval *one-to-one* onto the x-interval, then

$$\int f(x)\,dx = \int_{y=\varphi^{-1}(x)} f(\varphi(y))\varphi'(y)\,dy.$$

Remark 18.5 This rule is usually used when we have some "ugly" term in the integrand $f(x)$. The idea is to put this ugly term equal to $y = \varphi^{-1}(x)$. When e.g. x occurs in $f(x)$ in the form $\sqrt{x}$, we put $y = \varphi^{-1}(x) = \sqrt{x}$, hence $x = \varphi(y) = y^2$ and $\varphi'(y) = 2y$. $\Diamond$

18.6 Special antiderivatives

Power like:

$$\int \frac{1}{x}\,dx = \ln|x|, \qquad \text{for } x \neq 0. \text{ (Do not forget the numerical value!)}$$

$$\int x^\alpha\,dx = \frac{1}{\alpha+1}x^{\alpha+1}, \qquad \text{for } \alpha \neq -1,$$

$$\int \frac{1}{1+x^2}\,dx = \text{ Arctan } x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{1-x^2}\,dx = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|, \qquad \text{for } x \neq \pm 1,$$

$$\int \frac{1}{1-x^2}\,dx = \text{ Artanh } x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{1-x^2}\,dx = \text{ Arcoth } x, \qquad \text{for } |x| > 1,$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \text{ Arcsin } x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = -\text{ Arccos } x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \text{ Arsinh } x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \ln(x + \sqrt{x^2+1}), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{x}{\sqrt{x^2-1}}\,dx = \sqrt{x^2-1}, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \text{ Arcosh } x, \qquad \text{for } x > 1,$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \ln|x + \sqrt{x^2-1}|, \qquad \text{for } x > 1 \text{ eller } x < -1.$$

There is an error in the programs of the pocket calculators TI-92 and TI-89. The *numerical signs* are missing. It is obvious that $\sqrt{x^2-1} < |x|$ so if $x < -1$, then $x + \sqrt{x^2-1} < 0$. Since you cannot take the logarithm of a negative number, these pocket calculators will give an error message.

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Exponential like:

$$\int \exp x \, dx = \exp x, \qquad \text{for } x \in \mathbb{R},$$

$$\int a^x \, dx = \frac{1}{\ln a} \cdot a^x, \qquad \text{for } x \in \mathbb{R}, \text{ and } a > 0, a \neq 1.$$

Trigonometric:

$$\int \sin x \, dx = -\cos x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \cos x \, dx = \sin x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \tan x \, dx = -\ln|\cos x|, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \cot x \, dx = \ln|\sin x|, \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos x} \, dx = \frac{1}{2} \ln\left(\frac{1+\sin x}{1-\sin x}\right), \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin x} \, dx = \frac{1}{2} \ln\left(\frac{1-\cos x}{1+\cos x}\right), \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos^2 x} \, dx = \tan x, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin^2 x} \, dx = -\cot x, \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z}.$$

Hyperbolic:

$$\int \sinh x \, dx = \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \cosh x \, dx = \sinh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \tanh x \, dx = \ln \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \coth x \, dx = \ln|\sinh x|, \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh x} \, dx = \operatorname{Arctan}(\sinh x), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\cosh x} \, dx = 2 \operatorname{Arctan}(e^x), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh x} \, dx = \frac{1}{2} \ln\left(\frac{\cosh x - 1}{\cosh x + 1}\right), \qquad \text{for } x \neq 0,$$

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$$\int \frac{1}{\sinh x}\,dx = \ln\left|\frac{e^x-1}{e^x+1}\right|, \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh^2 x}\,dx = \tanh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh^2 x}\,dx = -\coth x, \qquad \text{for } x \neq 0.$$

18.7 Trigonometric formulæ

The trigonometric formulæ are closely connected with circular movements. Thus $(\cos u, \sin u)$ are the coordinates of a point P on the unit circle corresponding to the angle u, cf. figure A.1. This geometrical interpretation is used from time to time.

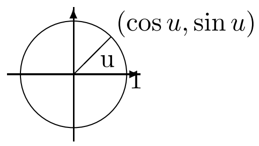

Figure 18.1: The unit circle and the trigonometric functions.

The fundamental trigonometric relation:

$$\cos^2 u + \sin^2 u = 1, \qquad \text{for } u \in \mathbb{R}.$$

Using the previous geometric interpretation this means according to *Pythagoras's theorem*, that the point P with the coordinates $(\cos u, \sin u)$ always has distance 1 from the origo $(0,0)$, i.e. it is lying on the boundary of the circle of centre $(0,0)$ and radius $\sqrt{1} = 1$.

Connection to the complex exponential function:

The *complex exponential* is for imaginary arguments defined by

$$\exp(\mathrm{i}\, u) := \cos u + \mathrm{i}\ \sin u.$$

It can be checked that the usual functional equation for exp is still valid for complex arguments. In other word: The definition above is extremely conveniently chosen.

By using the definition for $\exp(\mathrm{i}\, u)$ and $\exp(-\mathrm{i}\, u)$ it is easily seen that

$$\cos u = \frac{1}{2}(\exp(\mathrm{i}\, u) + \exp(-\mathrm{i}\, u)),$$

$$\sin u = \frac{1}{2i}(\exp(\mathrm{i}\, u) - \exp(-\mathrm{i}\, u)),$$

.

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Moivre's formula: We get by expressing $\exp(\mathrm{i}nu)$ in two different ways:

$$\exp(\mathrm{i}nu) = \cos nu + \mathrm{i}\,\sin nu = (\cos u + \mathrm{i}\,\sin u)^n.$$

Example 18.1 If we e.g. put $n = 3$ into Moivre's formula, we obtain the following typical application,

$$\begin{aligned}\cos(3u) + \mathrm{i}\,\sin(3u) &= (\cos u + \mathrm{i}\,\sin u)^3\\ &= \cos^3 u + 3\mathrm{i}\cos^2 u\cdot\sin u + 3\mathrm{i}^2\cos u\cdot\sin^2 u + \mathrm{i}^3\sin^3 u\\ &= \{\cos^3 u - 3\cos u\cdot\sin^2 u\} + \mathrm{i}\{3\cos^2 u\cdot\sin u - \sin^3 u\}\\ &= \{4\cos^3 u - 3\cos u\} + \mathrm{i}\{3\sin u - 4\sin^3 u\}\end{aligned}$$

When this is split into the real- and imaginary parts we obtain

$$\cos 3u = 4\cos^3 u - 3\cos u, \qquad \sin 3u = 3\sin u - 4\sin^3 u. \quad \diamondsuit$$

Addition formulæ:

$$\sin(u+v) = \sin u\,\cos v + \cos u\,\sin v,$$
$$\sin(u-v) = \sin u\,\cos v - \cos u\,\sin v,$$
$$\cos(u+v) = \cos u\,\cos v - \sin u\,\sin v,$$
$$\cos(u-v) = \cos u\,\cos v + \sin u\,\sin v.$$

Products of trigonometric functions to a sum:

$$\sin u\,\cos v = \frac{1}{2}\sin(u+v) + \frac{1}{2}\sin(u-v),$$
$$\cos u\,\sin v = \frac{1}{2}\sin(u+v) - \frac{1}{2}\sin(u-v),$$
$$\sin u\,\sin v = \frac{1}{2}\cos(u-v) - \frac{1}{2}\cos(u+v),$$
$$\cos u\,\cos v = \frac{1}{2}\cos(u-v) + \frac{1}{2}\cos(u+v).$$

Sums of trigonometric functions to a product:

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right),$$
$$\sin u - \sin v = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right),$$
$$\cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right),$$
$$\cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right).$$

Formulæ of halving and doubling the angle:

$$\sin 2u = 2\sin u\,\cos u,$$
$$\cos 2u = \cos^2 u - \sin^2 u = 2\,\cos^2 u - 1 = 1 - 2\,\sin^2 u,$$
$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}} \qquad \text{followed by a discussion of the sign,}$$
$$\cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}} \qquad \text{followed by a discussion of the sign,}$$

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18.8 Hyperbolic formulæ

These are very much like the trigonometric formulæ, and if one knows a little of Complex Function Theory it is realized that they are actually identical. The structure of this section is therefore the same as for the trigonometric formulæ. The reader should compare the two sections concerning similarities and differences.

The fundamental relation:

$$\cosh^2 x - \sinh^2 x = 1.$$

Definitions:

$$\cosh x = \frac{1}{2}\left(\exp(x) + \exp(-x)\right), \qquad \sinh x = \frac{1}{2}\left(\exp(x) - \exp(-x)\right).$$

"Moivre's formula":

$$\exp(x) = \cosh x + \sinh x.$$

This is trivial and only rarely used. It has been included to show the analogy.

Addition formulæ:

$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y),$$

$$\sinh(x-y) = \sinh(x)\cosh(y) - \cosh(x)\sinh(y),$$

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y),$$

$$\cosh(x-y) = \cosh(x)\cosh(y) - \sinh(x)\sinh(y).$$

Formulæ of halving and doubling the argument:

$$\sinh(2x) = 2\sinh(x)\cosh(x),$$

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x) = 2\cosh^2(x) - 1 = 2\sinh^2(x) + 1,$$

$$\sinh\left(\frac{x}{2}\right) = \pm\sqrt{\frac{\cosh(x)-1}{2}} \qquad \text{followed by a discussion of the sign,}$$

$$\cosh\left(\frac{x}{2}\right) = \sqrt{\frac{\cosh(x)+1}{2}}.$$

Inverse hyperbolic functions:

$$\operatorname{Arsinh}(x) = \ln\left(x + \sqrt{x^2+1}\right), \qquad x \in \mathbb{R},$$

$$\operatorname{Arcosh}(x) = \ln\left(x + \sqrt{x^2-1}\right), \qquad x \geq 1,$$

$$\operatorname{Artanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \qquad |x| < 1,$$

$$\operatorname{Arcoth}(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right), \qquad |x| > 1.$$

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18.9 Complex transformation formulæ

$$\cos(\mathrm{i}x) = \cosh(x), \qquad \cosh(\mathrm{i}x) = \cos(x),$$

$$\sin(\mathrm{i}x) = \mathrm{i}\sinh(x), \qquad \sinh(\mathrm{i}x) = \mathrm{i}\sin x.$$

18.10 Taylor expansions

The generalized binomial coefficients are defined by

$$\binom{\alpha}{n} := \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{1\cdot 2\cdots n},$$

with n factors in the numerator and the denominator, supplied with

$$\binom{\alpha}{0} := 1.$$

The Taylor expansions for *standard functions* are divided into *power like* (the radius of convergency is finite, i.e. $=1$ for the standard series) and*exponential like* (the radius of convergency is infinite).
Power like:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad |x|<1,$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x|<1,$$

$$(1+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j, \qquad n\in\mathbb{N}, x\in\mathbb{R},$$

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \qquad \alpha\in\mathbb{R}\setminus\mathbb{N}, |x|<1,$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n}, \qquad |x|<1,$$

$$\operatorname{Arctan}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \qquad |x|<1.$$

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Exponential like:

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \qquad x \in \mathbb{R}$$

$$\exp(-x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} x^n, \qquad x \in \mathbb{R}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}, \qquad x \in \mathbb{R},$$

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}, \qquad x \in \mathbb{R},$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}, \qquad x \in \mathbb{R},$$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}, \qquad x \in \mathbb{R}.$$

18.11 Magnitudes of functions

We often have to compare functions for $x \to 0+$, or for $x \to \infty$. The simplest type of functions are therefore arranged in an hierarchy:

1) logarithms,

2) power functions,

3) exponential functions,

4) faculty functions.

When $x \to \infty$, a function from a higher class will always dominate a function form a lower class. More precisely:

A) A *power function* dominates a *logarithm* for $x \to \infty$:

$$\frac{(\ln x)^\beta}{x^\alpha} \to 0 \qquad \text{for } x \to \infty, \quad \alpha,\, \beta > 0.$$

B) An *exponential* dominates a *power function* for $x \to \infty$:

$$\frac{x^\alpha}{a^x} \to 0 \qquad \text{for } x \to \infty, \quad \alpha,\, a > 1.$$

C) The *faculty function* dominates an *exponential* for $n \to \infty$:

$$\frac{a^n}{n!} \to 0, \qquad n \to \infty, \quad n \in \mathbb{N}, \quad a > 0.$$

D) When $x \to 0+$ we also have that a *power function* dominates the *logarithm:*

$$x^\alpha \ln x \to 0-, \qquad \text{for } x \to 0+, \quad \alpha > 0.$$

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Index

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