Real Functions in One Variable - Complex...
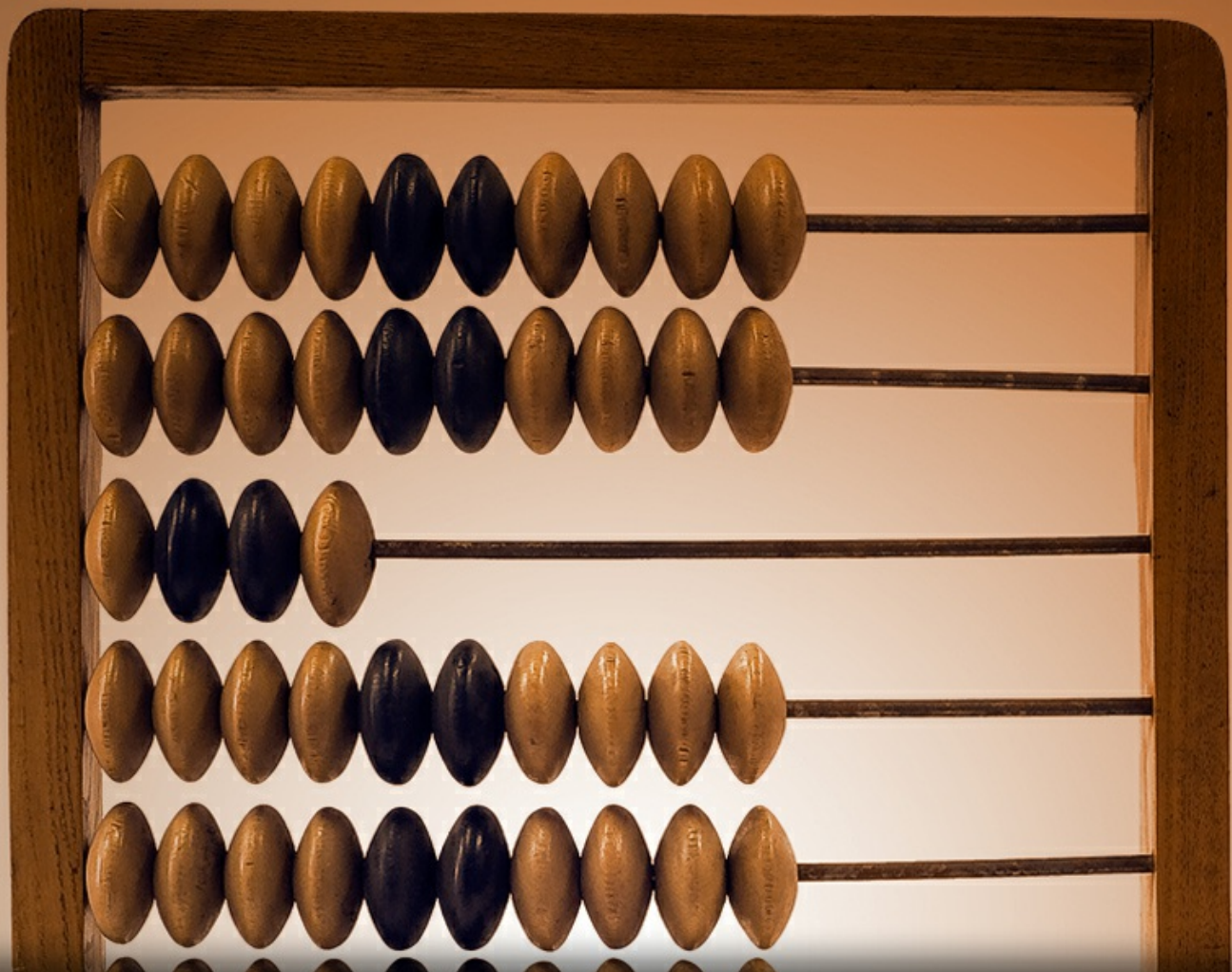
Leif Mejlbro

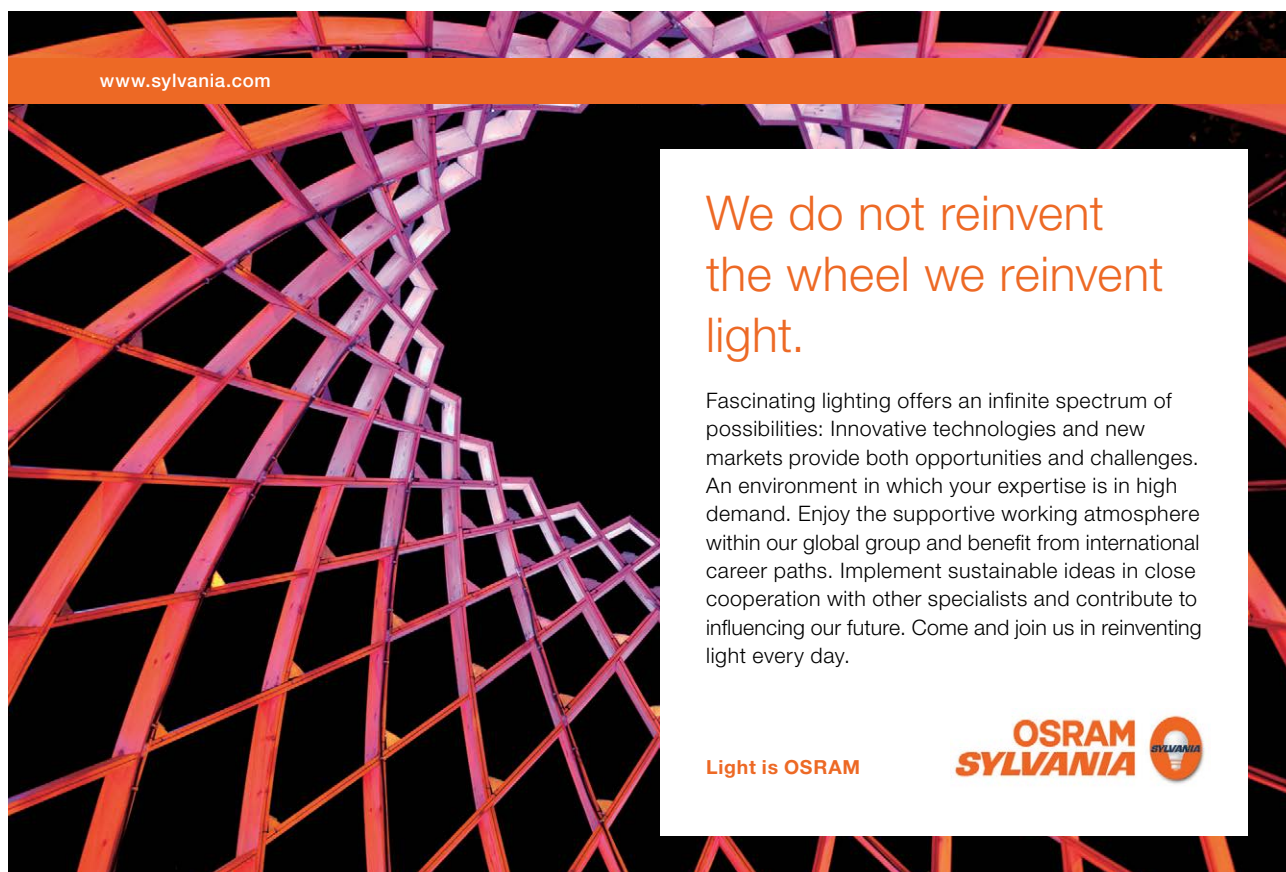Leif Mejlbro

Real Functions in One Variable

Complex Numbers Examples

Calculus 1c-4

Real Functions in One Variables
Complex Numbers Examples – Calculus Analyse 1c-4
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Contents

Preface

In this volume I present some examples of *Complex Numbers*, cf. also *Calculus 1a, Functions of One Variable*. Since my aim also has been to demonstrate some solution strategy I have as far as possible structured the examples according to the following form

A *Awareness*, i.e. a short description of what is the problem.

D *Decision*, i.e. a reflection over what should be done with the problem.

I *Implementation*, i.e. where all the calculations are made.

C *Control*, i.e. a test of the result.

This is an ideal form of a general procedure of solution. It can be used in any situation and it is not linked to Mathematics alone. I learned it many years ago in the Theory of Telecommunication in a situation which did not contain Mathematics at all. The student is recommended to use it also in other disciplines.

One is used to from high school immediately to proceed to **I**. *Implementation*. However, examples and problems at university level are often so complicated that it in general will be a good investment also to spend some time on the first two points above in order to be absolutely certain of what to do in a particular case. Note that the first three points, **ADI**, can always be performed.

This is unfortunately not the case with **C** *Control*, because it from now on may be difficult, if possible, to check one's solution. It is only an extra securing whenever it is possible, but we cannot include it always in our solution form above.

I shall on purpose not use the logical signs. These should in general be avoided in Calculus as a shorthand, because they are often (too often, I would say) misused. Instead of $\wedge$ I shall either write "and", or a comma, and instead of $\vee$ I shall write "or". The arrows $\Rightarrow$ and $\Leftrightarrow$ are in particular misunderstood by the students, so they should be totally avoided. Instead, write in a plain language what you mean or want to do.

It is my hope that these examples, of which many are treated in more ways to show that the solutions procedures are not unique, may be of some inspiration for the students who have just started their studies at the universities.

Finally, even if I have tried to write as careful as possible, I doubt that all errors have been removed. I hope that the reader will forgive me the unavoidable errors.

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Leif Mejlbro
27th July 2007

</div>

1 Calculation of complex numbers

Example 1.1 *Write the following complex numbers in the form $a + ib$. It is latently given that a and b are real numbers.*

1) $(5 + i)(1 + 9i)$,

2) $i + 1^2 + i^3 + i^4$,

3) $\dfrac{1}{1 + 3i} + \dfrac{1}{(1 + 3i)^2}$,

4) $\dfrac{1}{(1 + i)^4}$.

A. Simple exercises in complex numbers.

D. Calculate the numbers; in the case of the fractions in (3) we multiply the numerator and the denominator by the complex conjugated of the denominator.

I. 1) $(5 + i)(1 + 9i) = 5 - 9 + 45i + i = -4 + 46i$.

    2) $i + i^2 + i^3 + i^4 = i - 1 - i + 1 = 0$.

    3) From $\dfrac{1}{1 + 3i} = \dfrac{1 - 3i}{10}$ we get

$$
\begin{aligned}
\frac{1}{1 + 3i} + \frac{1}{(1 + 3i)^2} &= \frac{1 - 3i}{10} + \left(\frac{1 - 3i}{10}\right)^2 \\
&= \frac{1}{100}\{10 - 30i + 1 - 9 - 6i\} = \frac{1}{50} - \frac{9}{25}\, i.
\end{aligned}
$$

    4) From $(1 + i)^4 = (2i)^2 = -4$ we get $\dfrac{1}{(1 + i)^4} = -\dfrac{1}{4}$.

Example 1.2 *Write* $\left(1 + \dfrac{3}{1 + i}\right)^2$ *in the form $a + ib$.*

A. Simple exercise in complex numbers.

D. There are several possible solutions. The safest method is first to get rid of the denominator $1 + i$.

I. By a straight forward calculation,

$$
\begin{aligned}
\left(1 + \frac{3}{1 + i}\right)^2 &= \left(1 + \frac{3(1 - i)}{(1 + i)(1 - i)}\right)^2 = \left(1 + \frac{3}{2}\{1 - i\}\right)^2 \\
&= \frac{1}{4}\{2 + 3 - 3i\}^2 = \frac{1}{4}(5 - 3i)^2 = \frac{1}{4}(25 - 9 - 30i) \\
&= \frac{16 - 30i}{4} = 4 - \frac{15}{2}\, i.
\end{aligned}
$$

Example 1.3 *Write the following complex numbers in the form $a + ib$:*

1) $(3 + 2i)(1 + i)^2 + (5 + i)(1 + 9i)$;

2) $\dfrac{1}{1 + 3i} + \dfrac{1}{(1 + 3i)^2} - \dfrac{9}{25i}$.

A. Simple calculations of complex numbers. Some of them have already been calculated in Example 1.1.

D. Calculate (without using Example 1.1).

I. 1)

$$
\begin{aligned}
(3 + 2i)&(1 + i)^2 + (5 + i)(1 + 9i) \\
&= (3 + 2i) \cdot 2i + (5 - 9 + 5i + 45i) \\
&= -4 + 6i - 4 + 50i \\
&= -8 + 56i.
\end{aligned}
$$

    2)

$$
\begin{aligned}
\frac{1}{1 + 3i} &+ \frac{1}{(1 + 3i)^2} - \frac{9}{25i} \\
&= \frac{1 - 3i}{10} + \frac{(1 - 3i)^2}{100} + \frac{9i}{25} \\
&= \frac{1}{100} \left\{ 10 - 30i + 1 - 9 - 6i + 36u \right\} \\
&= \frac{2}{100} \\
&= \frac{1}{50}.
\end{aligned}
$$

Example 1.4 *Write the following complex numbers in the form $a + ib$:*

1) $\dfrac{i}{(1 + i)^4} + \dfrac{1}{2} \left(\dfrac{1}{\sqrt{3} - i} - \dfrac{1}{\sqrt{3} + i} \right)$,

2) $1 + i + i^2 + i^3$.

A. Simple manipulations with complex Numbers. Notice that (2) is the same as Example 1.1 (2).

D. Just calculate.

I. 1) Da $(1 + i)^4 = \{(1 + i)\}^2 = (2i)^2 = -4$, fås

$$
\frac{1}{(1 + i)^4} + \frac{1}{2} \left(\frac{1}{\sqrt{3} - i} - \frac{1}{\sqrt{3} + i} \right) = -\frac{1}{4} + \frac{1}{2} \cdot \frac{(\sqrt{3} + i) - (\sqrt{3} - i)}{3 + 1} = -\frac{1}{4} + \frac{1}{4} i.
$$

    2) $i + i^2 + i^2 + i^4 = i - 1 - i + 1 = 0.$

Example 1.5 *Write each of the following complex numbers in the form $a + ib$:*

1) $(1 + 2i)^2 + \dfrac{1}{2}(1 + i)(1 + i^8)$,

2) $\dfrac{2 + 5i}{(3 - 7i)^2} + (2 + 3i)(3 - 4i)$.

A. Simple manipulations with complex numbers.

D. Just calculate.

I. 1)

$$(1 + 2i)^2 + \frac{1}{2}(1 + i)(1 + i^8)$$

$$= 1 - 4 + 4i + \frac{1}{2}(1 + i) \cdot 2 = -2 + 5i.$$

2)

$$\frac{2+5i}{(3-7i)^2} + (2+3i)(3-4i)$$

$$= \frac{2+5i}{(9+49)^2}(0-49+42i) + 6 + 12 - 8i + 9i$$

$$= \frac{1}{58^2}(2+5i)(-40+42i) + 18 + i$$

$$= \frac{1}{58^2}(-80-210+84i-200i) + 18 + i$$

$$= \frac{1}{58^2}(-290-116i) + 18 + i$$

$$= -\frac{5}{58} - \frac{1}{29}i + 18 + i$$

$$= \frac{1039}{58} + \frac{28}{29}i.$$

Example 1.6 *Write the following complex numbers in the form $a + ib$:*

1) $(4)_{-\pi}$,

2) $(2)_{\frac{4\pi}{3}}$,

3) $(6)_{\frac{21\pi}{4}}$.

A. Simple manipulations with complex numbers.

D. Just calculate.

I. 1) $(4)_{-\pi} = 4\{\cos(-\pi) + i \sin(-\pi)\} = -4.$

2) $(2)_{\frac{4\pi}{3}} = 2\left\{\cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right)\right\} = 2\left\{-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right\} = -1 - i\sqrt{3}.$

3)

$$(6)_{\frac{21\pi}{4}} = (6)_{\frac{5\pi}{4}} = 6\left\{\cos\left(\frac{5\pi}{4}\right) + i \sin\left(\frac{5\pi}{4}\right)\right\}$$

$$= 6\left\{-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right\} = -3\sqrt{2} - 3\sqrt{2}\,i.$$

Example 1.7 *Find modulus and the principal argument of the following complex numbers:*

1) $2 + 2i$,

2) $\sqrt{3} - i$,

3) $-\dfrac{1}{6} + \dfrac{i}{2\sqrt{3}}$.

A. Simple manipulations with complex numbers.

D. Find the modulus and then the corresponding angle in the interval $]-\pi, \pi]$.

I. 1) It follows immediately from

$$2 + 2i = 2\sqrt{2} \cdot \left(\frac{1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}} \right) = 2\sqrt{2} \left\{ \cos\frac{\pi}{4} + i\,\sin\frac{\pi}{4} \right\},$$

that

$$r = 2\sqrt{2} \qquad \text{og} \qquad \Theta = \frac{\pi}{4}.$$

2) It follows immediately from

$$\sqrt{3} - i = 2 \left(\frac{\sqrt{3}}{2} - i\,\frac{1}{2} \right) = 2 \left\{ \cos\left(-\frac{\pi}{6}\right) + i\,\sin\left(-\frac{\pi}{6}\right) \right\},$$

that

$$r = 2 \qquad \text{og} \qquad \Theta = -\frac{\pi}{6}.$$

3) Here,

$$r = \left| -\frac{1}{6} + i\,\frac{1}{2\sqrt{3}} \right| = \sqrt{\frac{1}{36} + \frac{1}{12}} = \frac{1}{3},$$

thus

$$-\frac{1}{6} + i\,\frac{1}{2\sqrt{3}} = \frac{1}{3} \left\{ -\frac{1}{2} + i\,\frac{\sqrt{3}}{2} \right\} = \frac{1}{3} \left\{ \cos\left(\frac{2\pi}{3}\right) + i\,\sin\left(\frac{2\pi}{3}\right) \right\}.$$

Hence,

$$r = \frac{1}{3} \qquad \text{og} \qquad \Theta = \frac{2\pi}{3}.$$

Example 1.8 *Write the following complex numbers in the form r_v, where v denotes the principal argument.*

1) $-4 - 4i$,

2) $2\sqrt{3} - 6i$,

3) $\dfrac{4}{1+i}$.

A. Simple manipulations with complex numbers.

D. First find the modulus, then the argument.

I. 1) Mental arithmetic:

$$-4 - 4i = 4\sqrt{2}\left\{ -\frac{1}{\sqrt{2}} - i\,\frac{1}{\sqrt{2}} \right\} = (4\sqrt{2})_{-\frac{3\pi}{4}}.$$

2) Here,

$$2\sqrt{3} - 6i = 4\sqrt{3}\left\{ \frac{1}{2} - i\,\frac{\sqrt{3}}{2} \right\} = (4\sqrt{3})_{-\frac{\pi}{3}}.$$

3) From $1 + i = (\sqrt{2})_{\frac{\pi}{4}}$ follows that

$$\frac{4}{1+i} = (2\sqrt{2})_{-\frac{\pi}{4}}.$$

Example 1.9 *Write the following complex numbers in the form $a + ib$:*

1) $6_{\frac{3\pi}{2}}$,

2) $14_{15\pi}$,

3) $\left(\dfrac{1}{4}\right)_{-\frac{13\pi}{6}}$,

4) $2_{-\frac{\pi}{8}}$.

A. Simple manipulations with complex numbers.

D. Just calculate.

I. 1) It follows from the definition that

$$6_{\frac{3\pi}{2}} = 6\left\{ \cos\left(\frac{3\pi}{3}\right) + i\,\sin\left(\frac{3\pi}{2}\right) \right\} = -6i.$$

2) Similarly, we get $14_{15\pi} = 14_\pi = -14$.

3) Here,

$$
\begin{aligned}
\left(\frac{1}{4}\right)_{-\frac{13\pi}{6}} &= \left(\frac{1}{4}\right)_{-\frac{\pi}{6}} = \frac{1}{4}\left\{\cos\left(-\frac{\pi}{6}\right) + i\,\sin\left(-\frac{\pi}{6}\right)\right\} \\
&= \frac{1}{4}\left(\frac{\sqrt{3}}{2} - i\,\frac{1}{2}\right) = \frac{\sqrt{3}}{8} - i\,\frac{1}{8}.
\end{aligned}
$$

4) Applying

$$
\cos\frac{u}{2} = +\sqrt{\frac{1+\cos u}{2}}, \qquad \sin\frac{u}{2} = +\sqrt{\frac{1-\cos u}{2}}, \qquad u \in [0,\pi],
$$

where $u = \dfrac{\pi}{4}$, we get

$$
\begin{aligned}
2_{-\frac{\pi}{8}} &= 2\left\{\cos\left(-\frac{\pi}{8}\right) + i\,\sin\left(-\frac{\pi}{8}\right)\right\} = 2\left\{\cos\frac{\pi}{8} - i\,\sin\frac{\pi}{8}\right\} \\[2ex]
&= 2\left\{\sqrt{\frac{1 + \cos\dfrac{\pi}{4}}{2}} - i\sqrt{\frac{1 - \cos\dfrac{\pi}{4}}{2}}\right\} \\[3ex]
&= 2\left\{\sqrt{\frac{1 + \dfrac{\sqrt{2}}{2}}{2}} - i\sqrt{\frac{1 - \dfrac{\sqrt{2}}{2}}{2}}\right\} \\[3ex]
&= \sqrt{2 + \sqrt{2}} - i\sqrt{2 - \sqrt{2}}.
\end{aligned}
$$

Example 1.10 *Calculate $(1+i)^8$. (Consider the various possibilities of calculation).*

A. Calculation of a complex number, where it is indicated that there way of calculation is not unique.

D. Perform the following variants:

    1) Apply the modulus and the argument.
    2) Apply that $(1+i)^8 = \{(1+i)^2\}^4$, etc.
    3) Apply the binomial formula.
    4) Try some other methods.

I. 1) From $1 + i = \sqrt{2}\left\{\cos\dfrac{\pi}{4} + i\,\sin\dfrac{\pi}{4}\right\} = (\sqrt{2})_{\frac{\pi}{4}}$, we get

$$
(1+i)^8 = \left((\sqrt{2})^8\right)_{8\cdot\frac{\pi}{4}} = (2^4)_{2\pi} = 16.
$$

    2) By a small manipulation we get

$$
(1+i)^8 = \left\{(1+i)^2\right\}^4 = (2i)^4 = 2^4 i^4 = 16.
$$

    3) Using the binomial formula we get the following rather ridiculous variant,

$$
\begin{aligned}
(1+i)^8 &= \sum_{n=0}^{8} \binom{8}{n} i^n \\
&= 1 + 8i + 28i^2 + 56i^3 + 70i^4 + 56i^5 + 28i^6 + 8i^7 + i^8 \\
&= 1 + 8i - 28 - 56i + 70 + 56i - 28 - 8i + 1 \\
&= 1 - 28 + 70 - 28 + 1 = 72 - 56 = 16.
\end{aligned}
$$

    4) Finally, one may of course successively calculate

$$
(1+i)^2 = 2i, \qquad (1+i)^3 = -2 + 2i, \quad \dots, \quad (1+i)^8 = 16,
$$

which is even worse than the application of the binomial formula, though one finally gets the right result.

Example 1.11 *Reduce the expression*

$$\frac{(\sqrt{3}+i)^6}{(2-2i\sqrt{3})^3}$$

by using the modulus and the argument.

A. Simple manipulations with complex numbers.

D. Find modulus and argument for $\sqrt{3}+i$ and $2-i\,2\sqrt{3}$, and then calculate the quotient.

I. We get

$$\sqrt{3}+i = 2\left\{\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\right\} = 2\left\{\cos\frac{\pi}{6} + i\,\sin\frac{\pi}{6}\right\} = (2)_{\frac{\pi}{6}},$$

and

$$2-i\,2\sqrt{3} = 4\left\{\frac{1}{2} - i\,\frac{\sqrt{3}}{2}\right\} = 4\left\{\cos\left(-\frac{\pi}{3}\right) + i\,\sin\left(-\frac{\pi}{3}\right)\right\} = (4)_{-\frac{\pi}{3}}.$$

Hence

$$(\sqrt{3}+i)^6 = \left(2^6\right)_{6\cdot\frac{\pi}{6}} = -64, \qquad (2-i\,2\sqrt{3})^3 = \left(4^3\right)_{-\frac{\pi}{3}\cdot 3} = -64,$$

from which we conclude that

$$\frac{(\sqrt{3}+i)^6}{(2-i\,2\sqrt{3})^3} = 1.$$

Example 1.12 *Find the modulus of the complex number*

$$\frac{(2-3i)^2}{(8+6i)^2}.$$

A. Simple manipulations with complex numbers.

D. Find the absolute value of numerator and denominator and then just calculate.

I. We get immediately

$$\left|\frac{(2-3i)^2}{(8+6i)^2}\right| = \frac{|2-3i|^2}{|8+6i|^2} = \frac{4+9}{64+36} = \frac{13}{100}.$$

Example 1.13 *In former days (i.e. before the 1950s) one used tables for addition of complex numbers. Suppose e.g. that $a = r_v$ and $b = s_w$ are two given complex numbers, written in the form of modulus-/argument. We shall also express the complex number $a+b$ in the form of modulus-/argument. Assume that we have a table of $1 + \varrho_\varphi$ for given modulus r and principal argument v, when ϱ and φ are given. Describe how one can apply such a table to find the modulus and argument of the sum $r_v + s_w$.*

A. Addition of complex numbers given in the form of modulus-/argument, assuming that the function $(R)_\alpha = 1 + (\varrho)_\varphi$ is given in a table.

Adverse element. The description is not possible in a simple way, because one shall encounter a similar problem without being able to solve the problem.

It is historically not correct to claim that one before the 1950s only added complex numbers in this way. I do not remember where I found this claim, but I must say that I feel that the usage belittles without justification all mathematicians before 1950.

D. Multiply $1 + (r)_v$ and $1 + (s)_w$ and analyze.

I. The most obvious would be to multiply

$$(R)_\alpha = 1 + (r)_v \quad \text{and} \quad (S)_\beta = 1 + (s)_w.$$

This gives

$$
\begin{aligned}
(RS)_{\alpha+\beta} &= \{1 + (r)_v\}\{1 + (s)_w\} \\
&= 1 + (rs)_{v+w} + \{(r)_v + (s)_w\} \\
&= (T)_\gamma + \{(r)_v + (s)_w\},
\end{aligned}
$$

where $(T)_\gamma = 1 + (rs)_{v+w}$ is known in its form of modulus-/argument by the assumption. This gives apparently the result

$$(r)_v + (s)_w = (RS)_{\alpha+\beta} - (T)_\gamma.$$

The problem here is that we on the right hand side shall *subtract* two complex numbers written in the form of modulus-/argument, thus in principle the same task as our starting poin, so we have ended in a vicious circle.

Example 1.14 *Write each of the following complex numbers in the form $a + ib$, where a and b are real numbers.*

1) $1 + i^2 + 2i^4$,

2) $\dfrac{1 + 3i}{2 - i} + (2 + 3i)(3 - 4i)$,

3) $\dfrac{2 + 5i}{(3 - 7i)^2}$,

4) $\dfrac{6 + 7i}{7i - 3} - \dfrac{2 + 5i}{3 + 7i}$.

A. Simple manipulations with complex numbers.

D. Reduce; in fractions we multiply the numerator and the denominator by the complex conjugated of the denominator.

I. 1)

$$1 + i^2 + 2i^4 = 1 - 1 + 2 = 2.$$

2)

$$
\begin{aligned}
\frac{1 + 3i}{2 - i} + (2 + 3i)(3 - 4i) &= \frac{1 + 3i}{2 - i} \cdot \frac{2 + i}{2 + i} + 6 + 12 - 8i + 9 \\
&= \frac{1}{5}(2 - 3 + 6i + i) + 18 + i \\
&= \frac{89}{5} + \frac{12}{5}i.
\end{aligned}
$$

3)

$$\begin{aligned}
\frac{2+5i}{(3-7i)^2} &= \frac{2+5i}{(3-7i)^2} \cdot \frac{(3+7i)^2}{(3+7i)^2} = \frac{(2+5i)(9-49+42i)}{(9+49)^2} \\
&= \frac{1}{58^2}\,(2+5i)(-40+42i) \\
&= \frac{1}{58^2}\,(-80-210+84i-200i) \\
&= \frac{1}{58^2}\,(-290-116i) = -\frac{5}{58}-\frac{1}{29}\,i.
\end{aligned}$$

4)

$$\begin{aligned}
\frac{6+7i}{7i-3}-\frac{2+5i}{3+7i} &= \frac{6+7i}{7i-3}\cdot\frac{-3-7i}{-3-7i}-\frac{2+5i}{3+7i}\cdot\frac{3-7i}{3-7i} \\
&= \frac{1}{49+9}\,\{-18+49-42i-21i\}-\frac{1}{49+9}\,\{6+35-14i+15i\} \\
&= \frac{1}{58}\,(31-63i-42-i) = -\frac{5}{29}-\frac{32}{29}\,i.
\end{aligned}$$

2   Geometry of complex numbers

Example 2.1 *Show that if*

$$\frac{x - iy}{x + iy} = a + ib,$$

then $a^2 + b^2 = 1$.

A. Show that the absolute value of the fraction is 1. We must of course assume that $(x, y) \neq (0, 0)$.

D. Find the moduli of numerator and denominator separately.

I. The claim follows from the calculation

$$a^2 + b^2 = |a + ib|^2 = \left|\frac{x - iy}{x + iy}\right|^2 = \frac{|x - iy|^2}{|x + iy|^2} = \frac{x^2 + y^2}{x^2 + y^2} = 1.$$

Example 2.2 *Let $z_0 \neq 0$ be a given complex number. Indicate the complex number which corresponds to the mirror image of z_0 in*

1) the origo,

2) the real axis,

3) the imaginary axis,

4) the line $y = x$ in the first and third quadrant.

Let $z_0 = a + ib$. Express all the mirror images mentioned above by a, b and i, as well by z_0, $\overline{z}_0$ and i. Sketch a figure.

A. Mirror images in the complex plane.

D. The sketch of the figure is left out, due to some failure in the programme. Analysis.

I. 1) Mirror image in the origo.

$$z_0 = a + ib \qquad \text{is transferred into} \qquad - z_0 = -a - ib.$$

    2) Mirror image in the real axis:

$$z_0 = a + ib \qquad \text{is transferred into} \qquad \overline{z}_0 = a - ib.$$

    3) Mirror image in the imaginary axis:

$$z_0 = a + ib \qquad \text{is transferred into} - \overline{z}_0 = -a + ib.$$

    4) Mirror image in the line $y = x$:

$$z_0 = a + ib \qquad \text{is transferred into} \qquad b + ia = i\overline{z}_0.$$

Example 2.3 *Let z and w be two complex numbers, of which at least one of them has modulus 1. Assume that $\overline{z}w \neq 1$. Prove that*

$$\left| \frac{z - w}{1 - \overline{z}w} \right| = 1.$$

A. Manipulations with modulus.

D. Consider the cases $|z| = 1$ and $|w| = 1$ separately. Calculate.

I. 1) If $|z| = 1$, then $z\overline{z} = |z|^2 = 1$, and

$$\left| \frac{z - w}{1 - \overline{z}w} \right| = \frac{1}{|z|} \cdot \frac{|z - w|}{|1 - \overline{z}w|} = \frac{|z - w|}{|z - z\overline{z}w|} = \frac{|z - w|}{|z - w|} = 1.$$

   2) If $|w| = 1$, then also $|\overline{w}| = 1$ and $w\overline{w} = |w|^2 = 1$. In general we have $|\overline{\zeta}| = |\zeta|$ for every $\zeta \in \mathbb{C}$. Thus

$$\left| \frac{z - w}{1 - \overline{z}w} \right| = \frac{1}{|\overline{w}|} \cdot \frac{|z - w|}{|1 - \overline{z}w|} = \frac{|z - w|}{|\overline{w} - \overline{z}w\overline{w}|} = \frac{|z - w|}{|\overline{z} - w|} = \frac{|z - w|}{|z - w|} = 1.$$

Example 2.4 *Prove that if $|z| = 1$, then*

$$\left| \frac{az + b}{\overline{b}z + \overline{a}} \right| = 1$$

for all complex numbers a, b, for which $az + b \neq 0$.

A. Manipulations with complex numbers.

D. Note that $z\overline{z} = 1$.

I. A division by $|z| = 1$ gives

$$\left| \frac{az + b}{\overline{b}z + \overline{a}} \right| = \frac{|az + b|}{|b\overline{z} + a|} \cdot \frac{1}{|z|} = \frac{|az + b|}{|bz\overline{z} + az|} = \frac{|az + b|}{|az + b|} = 1.$$

Example 2.5 *Prove that if $|z| = r > 0$, then*

$$\mathrm{Re}(z) = \frac{1}{2}\left(z + \frac{r^2}{z} \right), \qquad \mathrm{Im}(z) = \frac{1}{2i}\left(z - \frac{r^2}{z} \right).$$

A. Manipulation with modulus.

D. Use that

$$r^2 = |z|^2 = z\overline{z} > 0,$$

and

$$\mathrm{Re}\, z = \frac{1}{2}(z + \overline{z}), \qquad \mathrm{Im}\, z = \frac{1}{2i}(z - \overline{z}).$$

I. We find immediately that

$$\frac{1}{2}\left(z + \frac{r^2}{z}\right) = \frac{1}{2}\left(z + \frac{z\overline{z}}{z}\right) = \frac{1}{2}(z + \overline{z}) = \text{Re } z,$$

and

$$\frac{1}{2i}\left(z - \frac{r^2}{z}\right) = \frac{1}{2i}(z - \overline{z}) = \text{Im } z.$$

Example 2.6 *Find all complex numbers $z \neq 1$, for which*

$$\text{Re}\left(\frac{z+1}{z-1}\right) = 0.$$

Find all complex numbers $z \neq 1$, for which

$$\text{Im}\left(\frac{z+1}{z-1}\right) = 0.$$

A. Calculations with the real and imaginary part.

D. Multiply both the numerator and the denominator by $\overline{z} - 1 \neq 0$ and then split into real and imaginary part.

I. By using the trick we get

$$\begin{aligned}
\frac{z+1}{z-1} &= \frac{z+1}{z-1} \cdot \frac{\overline{z}-1}{\overline{z}-1} = \frac{1}{|z-1|^2}\left\{z\overline{z} + \overline{z} - z - 1\right\} \\
&= \frac{1}{|z-1|^2}\left\{|z|^2 - 1 - 2i\,\text{Im } z\right\}.
\end{aligned}$$

If $z \neq 1$, then

$$\text{Re}\left(\frac{z+1}{z-1}\right) = 0, \quad \text{if and only if } |z| = 1,\ z \neq 1,$$

i.e. the unit circle without one point.

If $z \neq 1$, then

$$\text{Im}\left(\frac{z+1}{z-1}\right) = 0, \quad \text{if and only if } \text{Im } z = 0,\ \text{i.e. } z \in \mathbb{R} \setminus \{1\}.$$

Example 2.7 *Let $z = x + iy$. Prove that $|x| + |y| \leq \sqrt{2}|z|$.*

A. An estimate of modulus.

D. Square and use an inequality for real numbers.

I. If $a, b \in \mathbb{R}$, it is well-known that

$$0 \leq (a - b)^2 = a^2 + b^2 - 2ab,$$

thus

$$2ab \leq a^2 + b^2.$$

Hence,

$$(|x| + |y|)^2 = x^2 + y^2 + 2|xy| \leq 2(x^2 + y^2) = 2|z|^2,$$

from which we get

$$|x| + |y| \leq \sqrt{2}|z|.$$

Example 2.8 *Apply the interpretation of a complex number as a point in the complex plane to sketch each of the following point sets on a figure:*

1) $M_1 = \{z \in \mathbb{C} \mid |z - i| < 2\}$.

2) $M_2 = \{z \in \mathbb{C} \mid |z - i| = |z + 1|\}$.

3) $M_3 = \{z \in \mathbb{C} \mid |z - 3| + |z + 4i| = 5\}$.

It is here useful to consider $|a - b|$ as the distance between the two points in the complex plane, corresponding to the numbers a and b.

A. Point sets in the complex plane.

D. Interpret $|a - b|$ as the distance between a and b, and sketch the point sets.
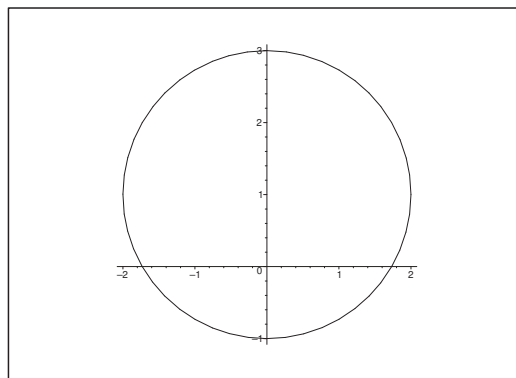

Figure 1: The point set in (1) is the open disc.

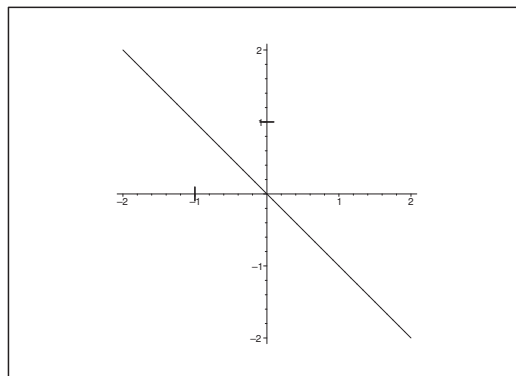**I.** 1) The set M_1 is the open disc of centre $z = i$ and radius 2.

Figure 2: The point set in (2) consists of the points on the line $y = -x$.

2) The set M_2 consists of the points z, which lie at the same distance from the points i and -1, i.e. the set M_2 consists of the points on the line $y = -x$.
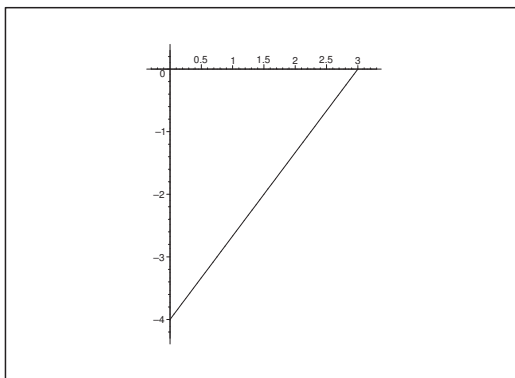
Figure 3: The point set in (3) consists of the line segment from $-4i$ to 3.

3) The set M_3 consists of the points z, for which the sum of the distances from z to either 3 or $-4i$ is 5. This is either an ellipse, a straight line segment of the empty set. Since the distance between 3 and $-4i$ is 5, the set M_3 is the line segment between 3 and $-4i$.

Example 2.9 *Find in each of the following cases the set of complex numbers z, which satisfy the given condition. Sketch figures, which show these sets.*

(1) $|z - 1| = 3$,   (2) $|z - 1| = |x + i|$,   (3) $|2z - 1| = |z - i|$.

A. Point sets in the complex plane.
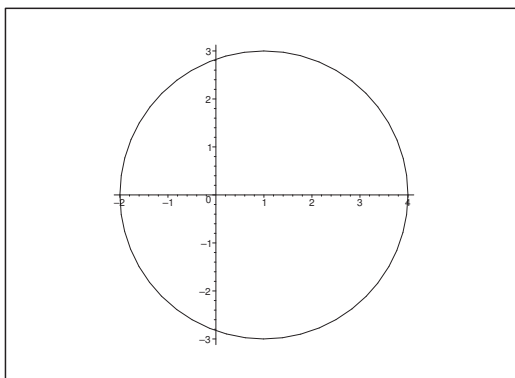
D. Give the equations a geometric interpretation.

Figure 4: The point set, for which $|x - 3| = 3$.

I. 1) The points z, which satisfy $|z - 1| = 3$, have all the distance 3 from 1, i.e. they are the points of the circle of centre 1 and radius 3.

    2) The points z, which satisfy $|z - 1| = |z + i|$, have the same distance from the points 1 and $-i$. They lie on the line described by $y = -x$.
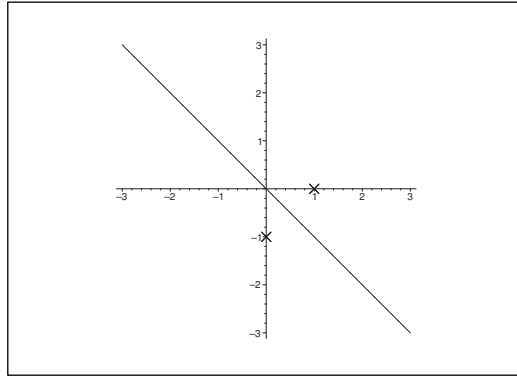
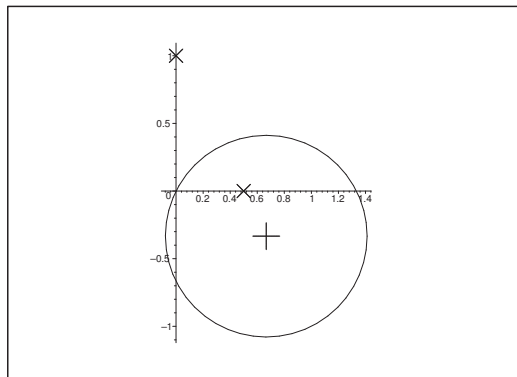Figure 5: the set of points z, which has the same distance to 1 and $-i$.

Figure 6: The set of points z, which satisfy $|2z - 1| = |z - i|$.

3) The points z, which satisfyr $|2z - 1| = |z - i|$, describe a circle. This is easily seen by the calculation

$$|2z - 1|^2 = |2x - 1 + 2iy|^2 = |z - i|^2 = |x + i(y - 1)|^2,$$

i.e.

$$(2x - 1)^2 + 4y^2 = x^2 + (y - 1)^2,$$

hence

$$
\begin{aligned}
0 &= (2x - 1)^2 - x^2 + 4y^2 - (y - 1)^2 \\
&= 3x^2 - 4x + 1 + 3y^2 + 2y - 1 \\
&= 3\left\{ x^2 - 2 \cdot \frac{2}{3}x + \frac{4}{9} + y^2 + 2 \cdot \frac{1}{3}y + \frac{1}{9} - \frac{5}{9} \right\}
\end{aligned}
$$

which can be written as the equation of a circle,

$$\left(x - \frac{2}{3} \right)^2 + \left(y + \frac{1}{3} \right)^2 = \frac{5}{9},$$

where the centre is $\left(\dfrac{2}{3}, -\dfrac{1}{3}\right)$, and the radius is $\dfrac{\sqrt{5}}{3}$.

REMARK. It follows from the rearrangement

$$2\left|z - \frac{1}{2}\right| = |z - i|$$

that the point set can be described as the points z, the distances os which to $z_1 = i$ is the double of the distance to $z_2 = \dfrac{1}{2}$. $\lozenge$

Example 2.10 *A complex number $A = a_1 + i\,a_2$ may be considered as a vector, which we here for clarity denote by $\mathbf{A}$. Let $\cdot$ denote the scalar product of vectors, and let $\hat{\mathbf{A}}$ denote the vector which is obtained by rotating $\mathbf{A}$ the angle of $\dfrac{\pi}{2}$.*

1) Prove that if $B = i\,A$, then $\mathbf{B} = \hat{\mathbf{A}}$.

2) Prove that for complex numbers A and B we have

$$\overline{A}\,B = \mathbf{A} \cdot \mathbf{B} - i(\mathbf{A} \cdot \hat{\mathbf{B}}).$$

3) What is the geometric interpretation of

$$|\mathrm{Im}(\overline{A}B)|?$$

4) Show that

$$\mathrm{Re}(\overline{A}B - \overline{B}A) = 0.$$

5) Show that

$$\mathrm{Re}\{(\overline{A} + \overline{B})(A - B)\} = 0 \qquad \textit{if and only if} \qquad |A|^2 = |B|^2.$$

6) The results in (5) and (2) give a geometric theorem of parallelograms. Which theorem?

A. Correspondence between the complex plane and vectors in the Euclidean plane.

D. Calculate; if necessary, sketch a figure; analyze.

I. 1) Since we obtain the vector $\hat{\mathbf{A}}$ by rotating $\mathbf{A}$ the angle $+\dfrac{\pi}{2}$, we get

$$\mathbf{B} = \hat{\mathbf{A}} \sim A \cdot (1)_{\frac{\pi}{2}} = iA = B.$$

2) Straight forward calculations give

$$\overline{A}B = (a_1 - i\,a_2) \cdot (b_1 + i\,b_2) = a_1 b_1 + a_2 b_2 - i\{a_1(-b_2) + a_2 b_1\},$$

and

$$\begin{aligned}
\mathbf{A} \cdot \mathbf{B} - i(\mathbf{A} \cdot \hat{\mathbf{B}}) &= (a_1, a_2) \cdot (b_1, b_2) - i(a_1, a_2) \cdot (-b_2, b_1) \\
&= a_1 b_1 + a_2 b_2 - i\{a_1(-b_2) + a_2 b_1\} = \overline{A}B.
\end{aligned}$$

3) From (2) follows

$$\begin{aligned}
|\mathrm{Im}(\overline{A}B)| &= |\mathbf{A} \cdot \hat{\mathbf{B}}| = |A| \cdot |\mathbf{B}| \cdot \left|\cos\left(\frac{\pi}{2} + \theta\right)\right| \\
&= |A| \cdot |B| \cdot |\sin\theta|,
\end{aligned}$$

where θ is the angle between $\mathbf{A}$ and $\mathbf{B}$.

This number is equal to the area of the parallelogram, which is defined by $\mathbf{A}$ and $\mathbf{B}$.

4) From (2) follows

$$\mathrm{Re}\{\overline{A}B - \overline{B}A\} = \mathbf{A} \cdot \mathbf{B} - \mathbf{B} \cdot \mathbf{A} = 0.$$
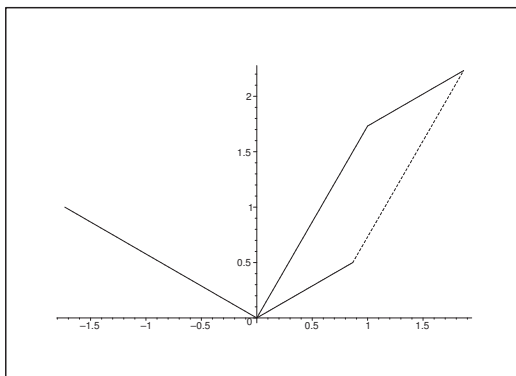
Figure 7: The parallelogram defined by $\mathbf{A}$ and $\mathbf{B}$, and the corresponding vector $\hat{\mathbf{B}}$.

5) By a calculation,

$$
\begin{aligned}
(\overline{A+B})(A-B) &= (\overline{A}+\overline{B})(A-B) \\
&= |A|^2 - |B|^2 - (\overline{A}B - \overline{B}A),
\end{aligned}
$$

thus, according to (4),

$$
\mathrm{Re}\{(\overline{A}+\overline{B})(A-B)\} = |A|^2 - |B|^2.
$$

It is immediately seen that

$$
\mathrm{Re}\left\{(\overline{A}+\overline{B})(A-B)\right\} = 0 \qquad \text{if and only if} \qquad |A| = |B|.
$$

6) According to (2),

$$
(\overline{A+b})(A-B) = (\mathbf{A}+\mathbf{B}) \cdot (\mathbf{A}-\mathbf{B}) - i((\mathbf{A}+\mathbf{B}) \cdot (\mathbf{A} \,\hat{-}\, \mathbf{B})),
$$

hence,

$$
\mathrm{Re}\left\{(\overline{A}+\overline{B})(A-B)\right\} = (\mathbf{A}+\mathbf{B}) \cdot (\mathbf{A}-\mathbf{B}).
$$

According to (5) this expression is equal to 0, if and only if $|A| = |B|$.

It follows from the above that

$$
(\mathbf{A}+\mathbf{B}) \cdot (\mathbf{A}-\mathbf{B}) = 0,
$$

if and only if $|A| = |B|$, i.e. if and only if the parallelogram defined by $\mathbf{A}$ and $\mathbf{B}$ is a en rhombus.

The condition is also equivalent to that the parallelogram defined by $\mathbf{A} - \mathbf{B}$ and $\mathbf{A} + \mathbf{B}$ is a rectangle, and thus equivalent to the theorem that the diagonals, $2|A|$ and $2|B|$, have the same length.

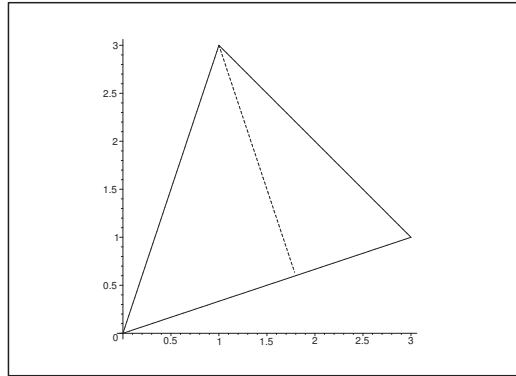Figure 8: The triangle T with its corners 0, a and b, and with the height h from b onto the line $0a$.

Example 2.11 *In the complex plane we choose two complex numbers a and b, which together with 0 form the three corners of a triangle T. The angle from a to b measured in the positive direction is called θ.*

1) Show that

$$\overline{a}b = |a||b|\cos\theta + i|a||b|\sin\theta.$$

2) Apply (1) to prove the cosine relation

$$|a - b|^2 = |a|^2 + |b|^2 - 2|a||b|\cos\theta.$$

   Hint: Start e.g. by proving that

$$\overline{a}b + a\overline{b} = 2\,Re(\overline{a}b).$$

3) Apply (1) to prove that the area $A(T)$ of the triangle T can be expressed by

$$A(T) = \left| \frac{1}{2}\,Im(\overline{a}b) \right|.$$

4) Let z_1, z_2, z_3 be three complex numbers in the complex plane, and let T' denote the triangle of corners z_1, z_2 and z_3. Prove that the area of T' is given by

$$A(T') = \left| \frac{1}{2}\,Im\left\{\overline{z}_1 z_2 + \overline{z}_2 z_3 + \overline{z}_3 z_1\right\} \right|.$$

A. Triangle in the complex plane. Area of this triangle.

D. Sketch a figure. Use polar coordinates. Follow the guidelines.

I. 1) If $a = r\,e^{i\varphi}$ and $b = R\,e^{i(\varphi+\theta)}$, then

$$\begin{aligned}
\overline{a}b &= r\,e^{-i\varphi} \cdot R\,e^{i(\varphi+\theta)} = |a| \cdot |b| \cdot e^{i\theta} \\
&= |a| \cdot |b| \cdot \cos\theta + i\,|a| \cdot |b| \cdot \sin\theta.
\end{aligned}$$

2) It is immediately seen that

$$\overline{a}b + a\overline{b} = \overline{a}b + \overline{\overline{a}b} = 2\,\mathrm{Re}(\overline{a}b).$$

Combining this result with (1) we get

$$
\begin{aligned}
|a - b|^2 &= (\overline{a} - \overline{b})(a - b) = |a|^2 + |b|^2 - \overline{a}b - a\overline{b} \\
&= |a|^2 + |b|^2 - 2|a||b|\cos\theta.
\end{aligned}
$$

3) The area of T is given by $\dfrac{1}{2}\,|a|\cdot h$, where a geometric consideration shows that $h = |b||\sin\theta|$ is the length of the height from b onto the side $0a$. When we apply the imaginary part from (1), we get

$$A(T) = \frac{1}{2}\,|a|\cdot|b|\cdot|\sin\theta| = \left|\frac{1}{2}\,\mathrm{Im}(\overline{a}b)\right|.$$
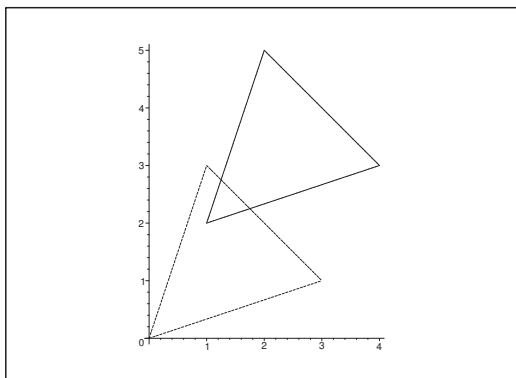
Figure 9: The triangle T' and the translated triangle T with one of its corners in 0.

4) By a parallel transform of the amount $-z_1$ we see that

$$A(T') = A(T),$$

where T is the triangle given by 0, $z_1 - z_1$ and $z_3 - z_1$. Then it follows from (3) that

$$
\begin{aligned}
A(T') &= A(T) = \left| \frac{1}{2} \operatorname{Im}\{(\overline{z}_2 - \overline{z}_1)(z_3 - z_1)\} \right| \\
&= \left| \frac{1}{2} \operatorname{Im}\left\{ \overline{z}_2 z_3 - \overline{z}_2 z_1 - \overline{z}_1 z_3 + \overline{z}_1 z_1 \right\} \right| \\
&= \left| \frac{1}{2} \operatorname{Im}\left\{ -z_1 \overline{z}_2 + \overline{z}_2 z_3 - \overline{z}_1 z_3 \right\} \right|,
\end{aligned}
$$

because $\overline{z}_1 z_1 = |z_1|^2$ is real.

Then note that $\operatorname{Im}\overline{a} = -\operatorname{Im}a$, thus

$$\operatorname{Im}(-z_1 \overline{z}_2) = \operatorname{Im}(\overline{z}_1 z_2), \qquad \operatorname{Im}(-\overline{z}_1 z_3) = \operatorname{Im}(\overline{z}_3 z_1),$$

hence by insertion,

$$A(T') = A(T) = \left| \frac{1}{2} \operatorname{Im}\left\{ \overline{z}_1 z_2 + \overline{z}_2 z_3 + \overline{z}_3 z_1 \right\} \right|.$$

Example 2.12 *Show that for every pair of complex numbers z_1 and z_2,*

$$||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|.$$

A. Estimates of the absolute value of a sum.

D. Calculate $|z_1 + z_2|^2$ and estimate.

I. First calculate

$$
\begin{aligned}
|z_1 + z_2|^2 &= (\overline{z}_1 + \overline{z}_2)(z_1 + z_2) \\
&= |z_1|^2 + |z_2|^2 + \overline{z}_1 z_2 + \overline{z}_2 z_1 \\
&= |z_1|^2 + |z_2|^2 + 2\operatorname{Re}(\overline{z}_1 z_2).
\end{aligned}
$$

From

$$-2\,|z_1||z_2| \le 2\operatorname{Re}(\overline{z}_1 z_2) \le 2\,|z_1||z_2|,$$

we get

$$
\begin{aligned}
\left(|z_1| - |z_2|\right)^2 &= |z_1|^2 + |z_2|^2 - 2\,|z_1||z_2| \\
&= |z_1|^2 + |z_2|^2 + 2\operatorname{Re}(\overline{z}_1 z_2) = |z_1 + z_2|^2 \\
&\le |z_1|^2 + |z_2|^2 + 2\,|z_1||z_2| = \left(|z_1| + |z_2|\right)^2,
\end{aligned}
$$

i.e.

$$\left(|z_1| - |z_2|\right)^2 \le |z_1 + z_2|^2 \le \left(|z_1| + |z_2|\right)^2.$$

Finally, by taking the square root,

$$\big||z_1| - |z_2|\big| \le |z_1 + z_2| \le |z_1| + |z_2|.$$

Example 2.13 *Let $a \neq b$ be complex numbers, and let k be a positive real number, $k \neq 1$. Prove that*

$$\{z \in \mathbb{C} \mid |z - a| = k \cdot |z - b|\}$$

describes a circle in the complex plane. Specify the centre and the radius of the circle.

A. Point set in the complex plane.

D. Square the condition and then reduce.

I. The condition is equivalent to

$$|z - a|^2 = k^2 |z - b|^2.$$

In general,

$$
\begin{aligned}
(1)\ |z - c|^2 &= (\overline{z} - \overline{c})(z - c) \\
&= |z|^2 - \overline{z}\,c - z\,\overline{c} + |c|^2 \\
&= |z|^2 - 2\,\mathrm{Re}(\overline{z}\,c) + |c|^2.
\end{aligned}
$$

Putting $c = a$, resp. $c = b$, we get

$$|z|^2 - 2\,\mathrm{Re}(\overline{z}\,a) + |a|^2 = k^2 \left\{ |z|^2 - 2\,\mathrm{Re}(\overline{z}\,b) + |b|^2 \right\},$$

which can be written

$$(1 - k^2)|z|^2 - 2\,\mathrm{Re}\left\{ \overline{z}\left(a - k^2 b\right) \right\} + |a|^2 - k^2 |b|^2 = 0.$$

Division by $1 - k^2 \neq 0$, followed by a rearrangement, gives

$$|z|^2 - 2\,\mathrm{Re}\left\{ \overline{z} \cdot \frac{a - k^2 b}{1 - k^2} \right\} + \left| \frac{a - k^2 b}{1 - k^2} \right|^2 = \left| \frac{a - k^2 b}{1 - k^2} \right|^2 - \frac{|a|^2 - k^2 |b|^2}{1 - k^2},$$

hence by (1),

$$
\begin{aligned}
\left| z - \frac{a - k^2 b}{1 - k^2} \right|^2 \\
&= \frac{1}{(1 - k^2)^2} \left\{ |a - k^2 b|^2 - (1 - k^2)|a|^2 + k^2(1 - k^2)|b|^2 \right\} \\
&= \frac{1}{(1 - k^2)^2} \left\{ |a|^2 + k^4 |b|^2 - 2\,\mathrm{Re}\left\{ k^2\,\overline{a}\,b \right\} - (1 - k^2)|a|^2 + (k^2 - k^4)|b|^2 \right\} \\
&= \frac{1}{(1 - k^2)^2} \left\{ k^2 |a|^2 + k^2 |b|^2 - k^2 \cdot 2\,\mathrm{Re}(\overline{a}b) \right\} \\
&= \frac{k^2}{(1 - k^2)^2} |a - b|^2.
\end{aligned}
$$

This shows that

$$
\begin{aligned}
\{x \in \mathbb{C} \mid |z - a| &= k|z - b|\} \\
&= \left\{ z \in \mathbb{C} \;\middle|\; \left| z - \frac{a - k^2 b}{1 - k^2} \right| = \frac{k|a - b|}{|1 - k^2|} \right\},
\end{aligned}
$$

and the set is a circle with the complex centre

$$\frac{a - k^2 b}{1 - k^2}$$

and of radius

$$r = \frac{k}{|1 - k^2|}\, |a - b|.$$

3 The equation of second degree

Example 3.1 *1) Find the complex numbers z, which satisfy the equation $\overline{z} = z^2$.*

2) For which complex numbers z is $z^2 = |z|^2$?

A. Solution of non-typical equations.

D. Compare the moduli and the arguments.

I. 1) Obviously, $z = 0$ is a solution.

When $z \neq 0$, the moduli must still be identical, thus $|z| = |z|^2$, i.e. $|z| = 1$. Then by a multiplication by z,

$$z^3 = z\overline{z} = |z|^2 = 1 = (1)_{2p\pi}, \qquad z = (1)_{\frac{2p\pi}{3}},$$

and we see that the solutions are

$$z = 0, \quad z = 1, \quad z = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad z = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

Notice that we obtain *four* solutions, which is *not* a contradiction to the fundamental theorem of algebra, because $z^2 - \overline{z}$ is *not* a polynomial in Z.

2) By putting $z = x + iy$ we immediately get

$$x^2 - y^2 + 2ixy = x^2 + y^2,$$

and then by a splitting into real and imaginary part,

$$2y^2 = 0 \qquad \text{and} \qquad 2xy = 0.$$

Hence, $y = 0$ and $x \in \mathbb{R}$, i.e. $z \in \mathbb{R}$.

C. We see in both cases that the found solutions are in fact correct.

Example 3.2 *Find in the form $z = a + ib$, a, $b \in \mathbb{R}$, the solutions of the equation*

$$z^2 + 2z - 2 - 4i = 0.$$

A. Equation of second degree.

D. Solve the equation. This can be done in more than one way, although the methods in principle rely on the same idea.

I. First variant. From $(z + 1)^2 = z^2 + 2z + 1$, we get the inspiration of performing the following rearrangement,

$$
\begin{aligned}
0 &= z^2 + 2z - 2 - 4i = (z^2 + 2z + 1) - (3 + 4i) \\
&= (z + 1)^2 - (4 + i^2 + 4i) \\
&= (z + 1)^2 - (2 + i)^2 = (z + 3 + i)(z - 1 - i).
\end{aligned}
$$

It is immediately seen that the roots are $-3 - i$ and $1 + i$.

Second variant. An unconscious application of the solution formula from high school gives the awkward solutions

$$z = -1 \pm \sqrt{1 + 2 + 4i} = -1 \pm \sqrt{3 + 4i}.$$

We assume that $\sqrt{3 + 4i} = a + ib$ for some a, $b \in \mathbb{R}$. Then by squaring,

$$a^2 - b^2 + i\, 2ab = 3 + 4i, \quad \text{i.e.} \quad a^2 - b^2 = 3, \quad 2ab = 4,$$

thus

$$(a^2 + b^2)^2 = (a^2 - b^2)^2 + (2ab)^2 = 9 + 16 = 25,$$

from which $a^2 + b^2 = 5$. Together with $a^2 - b^2 = 3$ this gives $a^2 = 4$ and $b^2 = 1$. Now, $ab > 0$, so a and b have the same sign, i.e. $\sqrt{3 + 4i} = \pm(2 + i)$. Then by insertion

$$z = -1 \pm (2 + i) = \begin{cases} -3 - i, \\ 1 + i \end{cases}$$

C. The solutions of *normed* equations of second degree are checked by using that the sum of the roots is the coefficient of z with opposite sign, and the product of the roots is equal to the constant term. In the present case we get

$$(-3 - i) + (1 + i) = -2,$$

$$(-3 - i)(1 + i) = -3 + 1 + i(-3 - 1) = -2 - 4i.$$

Q.E.D.

Example 3.3 *Find in the form $z = a + ib$, a, $b \in \mathbb{R}$, the solutions of the equation*

$$z^2 - (5 + 5i)z + 13i = 0.$$

A. Complex equation of second degree.

D. Apply the solution formula from high school in its complex form.

I. We get by the solution formula,

$$
\begin{aligned}
z &= \frac{1}{2}\left\{5 + 5i \pm \sqrt{5^2(1 + i)^2 - 4 \cdot 13i}\right\} \\
&= \frac{1}{2}\left\{5 + 5i \pm \sqrt{5^2 \cdot 2i - 52i}\right\} \\
&= \frac{1}{2}\left\{5 + 5i \pm \sqrt{-2i}\right\} = \frac{1}{2}\left\{5 + 5i \pm \sqrt{(1 - i)^2}\right\} \\
&= \begin{cases} \dfrac{1}{2}(5 + 5i + 1 - i) = 3 + 2i, \\[2mm] \dfrac{1}{2}(5 + 5i - 1 + i) = 2 + 3i. \end{cases}
\end{aligned}
$$

C. The sum of the roots is

$$(3 + 2i) + (2 + 3i) = 5 + 5i,$$

i.e. the coefficient of z of the opposite sign,

The product of the roots is

$$(3 + 2i) \cdot (2 + 3i) = 6 - 5 + 4i + 9i = 13i,$$

i.e. equal to the constant term. Q.E.D.

Example 3.4 *Find in the form $z = a + ib$, a, $b \in \mathbb{R}$, the solutions of the equation*

$$iz^2 - (2 + 3i)z + 1 + 5i = 0.$$

A. A non-normed complex equation of second degree.

D. First multiply by $-i$; then solve the equation.

I. When we multiply by $-i$ we obtain the equivalent *normed* equation

(2) $z^2 - (3 - 2i)z + 5 - i = 0.$

Then by the usual solution formula known from high school,

$$
\begin{aligned}
z &= \frac{1}{2}\left\{3 - 2i \pm \sqrt{(3 - 2i)^2 - 4(5 - i)}\right\} \\
&= \frac{1}{2}\left\{3 - 2i \pm \sqrt{9 - 4 - 12i - 20 + 4i}\right\} \\
&= \frac{1}{2}\left\{3 - 2i \pm \sqrt{-15 - 8i}\right\} \\
&= \frac{1}{2}\left\{3 - 2i \pm \sqrt{(4i)^2 + 1 - 2 \cdot 4i \cdot 1}\right\} \\
&= \frac{1}{2}\left\{3 - 2i \pm \sqrt{(1 - 4i)^2}\right\} = \frac{1}{2}\left\{3 - 2i \pm (1 - 4i)\right\} \\
&= \begin{cases} 2 - 3i, \\ 1 + i. \end{cases}
\end{aligned}
$$

C. The sum of the roots is

$$(2 - 3i) + (1 + i) = 3 - 2i,$$

which is the coefficient of z of the opposite sign in the *normed* equation (2).

The product of the roots is

$$(2 - 3i) \cdot (1 + i) = 2 + 3 - 3i + 2i = 5 - i,$$

which is equal to the constant term in (2).

Thus, the solutions are correct for (2). Finally, when (2) is multiplied by i, we obtain the equivalent original equation.

Example 3.5 *Find the solutions of the equations*

1) $(z+1)^2 = 3 + 4i$.

2) $(z+1)^4 = 3 + 4i$.

A. Two disguised binomial equations which can be solved by taking a square root.

D. Solve the equations, where the second equations can be derived from the first one.

I. 1) From

$$(z+1)^2 = 3 + 4i = 4 - 1 + 2 \cdot 2 \cdot i = (2+i)^2,$$

we get $z + 1 = \pm(2 + i)$, hence

$$z = \begin{cases} 1+i, \\ -3-i. \end{cases}$$

2) As above, $(z+1)^4 = (2+i)^2$, hence by (1),

$$(z+1)^2 = \begin{cases} 2+i, \\ -2-i. \end{cases}$$

The four roots are

$$z = -1 \pm \sqrt{2+i}, \quad \text{og} \quad z = -1 \pm \sqrt{-2-i} = -1 \pm i\sqrt{2+i}.$$

REMARK. Let $\sqrt{2+i} = a + ib$, where a and $b \in \mathbb{R}$. Then we get by a squaring,

$$a^2 - b^2 = 2 \qquad \text{and} \qquad 2ab = 1,$$

from which

$$(a^2 + b^2)^2 = (a^2 - b^2)^2 + (2ab)^2 = 5,$$

i.e.

$$a^2 + b^2 = \sqrt{5} \qquad \text{and from previously} \qquad a^2 - b^2 = 2.$$

This gives

$$a = \pm\sqrt{\frac{\sqrt{5}+2}{2}}, \qquad b = \pm\sqrt{\frac{\sqrt{5}-2}{2}}, \qquad ab > 0,$$

hence,

$$\sqrt{2+i} = \pm\left\{ \sqrt{\frac{\sqrt{5}+2}{2}} + i\sqrt{\frac{\sqrt{5}-2}{2}} \right\}.$$

We can therefore write the four roots,

$$-1 + \sqrt{\frac{\sqrt{5}+2}{2}} + i\sqrt{\frac{\sqrt{5}-2}{2}}, \qquad -1 - \sqrt{\frac{\sqrt{5}+2}{2}} - i\sqrt{\frac{\sqrt{5}-2}{2}},$$

$$-1 + \sqrt{\frac{\sqrt{5}-2}{2}} - i\sqrt{\frac{\sqrt{5}+2}{2}}, \qquad -1 - \sqrt{\frac{\sqrt{5}-2}{2}} + i\sqrt{\frac{\sqrt{5}+2}{2}}. \qquad \Diamond$$

Example 3.6 *Let $\pm\sqrt{w}$ denote the two solutions of the equation $z^2 = w$. Find the values of the following expressions written in the form $a + ib$:*

1) $\pm\sqrt{1+i}$,

2) $\pm\sqrt{1 \pm \sqrt{i}}$,

3) $\pm\sqrt{\pm\sqrt{i}}$.

A. Splitting into real and imaginary part by a square root. A very difficult example!

D. Let $z^2 = w$ for given w. Insert $z = a + ib$ and find two equations for a and b. Then find in each case the corresponding equation without the symbols $\pm\sqrt{\cdot}$.

I. First some general theory.

Let $z^2 = w = u + iv$, where u and v are given real numbers. If we put $z = a + ib$, we get by insertion and separation into real and imaginary parts that

$$a^2 - b^2 = u \qquad \text{og} \qquad 2ab = v.$$

Then

$$(a^2 + b^2)^2 = (a^2 - b^2)^2 + \{2ab\}^2 = u^2 + v^2,$$

and thus

$$a^2 + b^2 = \sqrt{u^2 + v^2} = |w|.$$

Since we already have that $a^2 - b^2 = u$, we get

$$a^2 = \frac{|w| + u}{2}, \qquad b^2 = \frac{|w| - u}{2}, \qquad 2ab = v,$$

and we can find $z = a + ib$. The latter equation is used to check whether a and b have the same or opposite sign.

1) $z = \pm\sqrt{1+i}$.

Here, $z^2 = (a + ib)^2 = 1 + i$, thus $u = 1$ and $|w| = \sqrt{2}$, and $ab > 0$. It follows from the above that

$$a = \pm\sqrt{\frac{\sqrt{2}+1}{2}}, \qquad b = \pm\sqrt{\frac{\sqrt{2}-1}{2}}, \qquad ab > 0,$$

hence

$$\pm\sqrt{1+i} = \pm\left\{\sqrt{\frac{\sqrt{2}+1}{2}} + i\sqrt{\frac{\sqrt{2}-1}{2}}\right\}.$$

C. TEST. We get by a squaring

$$\left\{\sqrt{\frac{\sqrt{2}+1}{2}} + i\sqrt{\frac{\sqrt{2}-1}{2}}\right\}^2$$

$$= \frac{\sqrt{2}+1}{2} - \frac{\sqrt{2}-1}{2} + 2i\sqrt{\frac{\sqrt{2}+1}{2} \cdot \frac{\sqrt{2}-1}{2}} = 1 + i. \qquad \text{Q.E.D.}$$

2) $z = \pm\sqrt{1 \pm \sqrt{i}}$.

Let us first find $\pm\sqrt{i}$. Since $i = (1)_{\frac{\pi}{2}}$, it should now be well-known that

$$\pm\sqrt{i} = \pm\left(\frac{1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}\right) = \pm\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\,i\right).$$

For the first one of the two possibilities of $1 \pm \sqrt{i}$ we get

a)

$$w = 1 + \frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}, \qquad |w| = \sqrt{1 + \frac{1}{2} + \sqrt{2} + \frac{1}{2}} = \sqrt{2 + \sqrt{2}},$$

$$u = 1 + \frac{\sqrt{2}}{2} = \frac{2 + \sqrt{2}}{2}, \qquad 2ab = \frac{\sqrt{2}}{2} > 0,$$

$$a^2 = \frac{1}{2}\left\{\sqrt{2 + \sqrt{2}} + \frac{1}{2}\,(2 + \sqrt{2})\right\} = \frac{1}{4}\left\{2\sqrt{2 + \sqrt{2}} + (2 + \sqrt{2})\right\},$$

$$b^2 = \frac{1}{4}\left\{2\sqrt{2 + \sqrt{2}} - (2 + \sqrt{2})\right\},$$

thus

$$z = \pm\left\{\frac{1}{2}\sqrt{2\sqrt{2 + \sqrt{2}} + (2 + \sqrt{2})} + \frac{i}{2}\sqrt{2\sqrt{2 + \sqrt{2}} - (2 + \sqrt{2})}\right\}.$$

b) For the second one we have instead

$$w = 1 - \frac{\sqrt{2}}{2} - i\,\frac{\sqrt{2}}{2}, \qquad |w| = \sqrt{1 + \frac{1}{2} - \sqrt{2} + \frac{1}{2}} = \sqrt{2 - \sqrt{2}},$$

$$u = 1 - \frac{\sqrt{2}}{2} = \frac{2 - \sqrt{2}}{2}, \qquad 2ab = -\frac{\sqrt{2}}{2} < 0,$$

$$a^2 = \frac{1}{2}\left\{\sqrt{2 - \sqrt{2}} + \frac{1}{2}\,(2 - \sqrt{2})\right\} = \frac{1}{4}\left\{2\sqrt{2 - \sqrt{2}} + (2 - \sqrt{2})\right\},$$

$$b^2 = \frac{1}{4}\left\{2\sqrt{2 - \sqrt{2}} - (2 - \sqrt{2})\right\},$$

thus

$$z = \pm\left\{\frac{1}{2}\sqrt{2\sqrt{2 - \sqrt{2}} + (2 - \sqrt{2})} - \frac{i}{2}\sqrt{2\sqrt{2 - \sqrt{2}} - (2 - \sqrt{2})}\right\}.$$

Hence the four roots are

$$\frac{1}{2}\sqrt{2\sqrt{2+\sqrt{2}}+(2+\sqrt{2})}+\frac{i}{2}\sqrt{2\sqrt{2+\sqrt{2}}-(2+\sqrt{2})},$$

$$-\frac{1}{2}\sqrt{2\sqrt{2+\sqrt{2}}+(2+\sqrt{2})}-\frac{i}{2}\sqrt{2\sqrt{2+\sqrt{2}}(2+\sqrt{2})},$$

$$\frac{1}{2}\sqrt{2\sqrt{2-\sqrt{2}}+(2-\sqrt{2})}-\frac{i}{2}\sqrt{2\sqrt{2-\sqrt{2}}-(2-\sqrt{2})},$$

$$-\frac{1}{2}\sqrt{2\sqrt{2-\sqrt{2}}+(2-\sqrt{2})}+\frac{i}{2}\sqrt{2\sqrt{2-\sqrt{2}}-(2-\sqrt{2})}.$$

C. It is possible to perform the test by noting that $z = \pm\sqrt{1 \pm \sqrt{i}}$ are the four roots of the equation $(z^2 - 1)^2 = i$.

a) If

$$z = \pm\frac{1}{2}\left\{\sqrt{2\sqrt{2+\sqrt{2}} + (2+\sqrt{2})} + i\sqrt{2\sqrt{2+\sqrt{2}} - (2+\sqrt{2})}\right\},$$

then

$$\begin{aligned}
z^2 &= \frac{1}{4}\left\{2(2+\sqrt{2}) + 2i\sqrt{4(2+\sqrt{2}=)} - (2+\sqrt{2})^2\right\} \\
&= 1 + \frac{1}{2}\left\{\sqrt{2} + i\sqrt{8 + 4\sqrt{2} - 4 - 2 - 4\sqrt{2}}\right\} \\
&= 1 + \frac{1}{\sqrt{2}}(1+i),
\end{aligned}$$

hence

$$(z^2 - 1)^2 = \frac{1}{2}(1+i)^2 = \frac{1}{2} \cdot 2i = i. \qquad \text{Q.E.D.}$$

b) If

$$z = \pm, \frac{1}{2}\left\{\sqrt{2\sqrt{2-\sqrt{2}} + (2-\sqrt{2})} - i\sqrt{2\sqrt{2-\sqrt{2}} - (2-\sqrt{2})}\right\},$$

then

$$\begin{aligned}
z^2 &= \frac{1}{4}\left\{2(2-\sqrt{2}) - 2i\sqrt{4(2-\sqrt{2})} - (2-\sqrt{2})^2\right\} \\
&= 1 + \frac{1}{2}\left\{-\sqrt{2} - i\sqrt{8 - 4\sqrt{2} - 4 - 2 + 4\sqrt{2}}\right\} \\
&= 1 - \frac{1}{\sqrt{2}}(1+i),
\end{aligned}$$

hence

$$(z^2 - 1)^2 = \frac{1}{2}(1+i)^2 = i. \qquad \text{Q.E.D.}$$

3) $z = \pm\sqrt{\pm\sqrt{i}}$.

In this case, $z^2 = \pm\sqrt{i}$, i.e.

$$z^4 = i = (1)_{\frac{\pi}{2}+2p\pi}, \qquad p \in \mathbb{Z},$$

and

$$z = (1)_{\frac{\pi}{8}+p\cdot\frac{pi}{2}}, \qquad p = 0,\ 1,\ 2,\ 3,$$

where we have solve the equation as a binomial equation. Here,

$$
\begin{aligned}
(1)_{\frac{\pi}{8}} &= \cos\frac{\pi}{8} + i\,\sin\frac{\pi}{8} \\
&= \sqrt{\frac{1+\cos\frac{\pi}{4}}{2}} + i\sqrt{\frac{1-\cos\frac{\pi}{4}}{2}} \\
&= \frac{1}{2}\sqrt{2+\sqrt{2}} + \frac{i}{2}\sqrt{2-\sqrt{2}}, \\
(1)_{\frac{\pi}{8}+\frac{\pi}{2}} &= (1)_{\frac{\pi}{8}}\cdot i = -\frac{1}{2}\sqrt{2-\sqrt{2}} + \frac{i}{2}\sqrt{2+\sqrt{2}}, \\
(1)_{\frac{\pi}{8}+\pi} &= -(1)_{\frac{\pi}{8}} = -\frac{1}{2}\sqrt{1+\sqrt{2}} - \frac{i}{2}\sqrt{2-\sqrt{2}}, \\
(1)_{\frac{\pi}{8}+\frac{3\pi}{2}} &= (1)_{\frac{\pi}{8}}\cdot(-i) = \frac{1}{2}\sqrt{2-\sqrt{2}} - \frac{i}{2}\sqrt{2+\sqrt{2}}.
\end{aligned}
$$

All things put together we see that for $\pm\sqrt{\pm\sqrt{i}}$ we get

$$
\pm\frac{1}{2}\left\{\sqrt{2+\sqrt{2}} + i\sqrt{2-\sqrt{2}}\right\}, \qquad \pm\frac{1}{2}\left\{\sqrt{2-\sqrt{2}} - i\sqrt{2+\sqrt{2}}\right\}.
$$

Example 3.7 *Solve the equation of second degree, $z^2 - 4iz - 1 + 4i = 0$. Then find the roots of the polynomial*

$$
P(z) = z^4 - 4iz^2 - 1 + 4i.
$$

A. An equation of second degree and another one in disguise.

D. Solve the equations, either by means of a formula or by inspection.

I. 1) We get by inspection,

$$
z^2 - 4iz - 1 + 4i = (z^2 - 1) - 4i(z - 1) = (z - 1)(z + 1 - 4i),
$$

so the first polynomial has the roots $-1 + 4i$ and 1.

2) When we here replace z^2 by z, we obtain the equation in (1). We therefore conclude that $P(z) = 0$, if and only if either

$$
(i)\ \ z^2 = 1, \quad \text{or} \quad (ii)\ \ z^2 = -1 + 4i.
$$

a) When $z^2 = 1$, the roots are ± 1.

b) Then we solve $z^2 = -1 + 4i$. If we put $z = a + ib$, we get

$$
a^2 - b^2 = -1, \qquad 2ab = 4, \qquad a^2 + b^2 = \sqrt{17},
$$

hence

$$
a = \pm\sqrt{\frac{\sqrt{17}-1}{2}} = \pm\frac{1}{2}\sqrt{2\sqrt{17}-2},
$$

$$
b = \pm\sqrt{\frac{\sqrt{17}+1}{2}} = \pm\frac{1}{2}\sqrt{2\sqrt{17}+2}.
$$

Since $a \cdot b > 0$, the roots are

$$\pm \frac{1}{2} \left\{ \sqrt{2\sqrt{17} - 2} + i\sqrt{2\sqrt{17} + 2} \right\}.$$

C. TEST.

1) If $z = \pm 1$, then $P(\pm 1) = 1 - 4i - 1 + 4i = 0$. Q.E.D.

2) If

$$z = \pm \frac{1}{2} \left\{ \sqrt{2\sqrt{17} - 2} + i\sqrt{2\sqrt{17} + 2} \right\},$$

then

$$
\begin{aligned}
z^2 &= \frac{1}{4} \left\{ 2 \cdot (-2) + 2i\sqrt{4 \cdot 17 - 4} \right\} \\
&= \frac{1}{4} \left\{ -4 + 2i \cdot 4 \cdot 2 \right\} = -1 + 4i,
\end{aligned}
$$

hence by insertions,

$$
\begin{aligned}
P(z) &= (-1 + 4i)^2 - 4i(-1 + 4i) - 1 + 4i \\
&= (-1 + 4i) \cdot \{-1 + 4i + 1 - 4i\} = 0. \qquad \text{Q.E.D.}
\end{aligned}
$$

Example 3.8 *Find the solutions z of the equation*

$$z^2 - z + 1 - i = 0.$$

Then find all complex solutions y of the equation

$$e^{2y} - e^y + 1 - i = 0.$$

A. Complex roots.

D. One may either use the solution formula, or find a root by inspection.

I. 1) By a small inspection we see that

$$0 = z^2 - z + 1 - i = (z^2 + 1) - (z + i) = (z + i)(z - 1 - i),$$

thus the roots are $1 + i$ and $-i$.

2) If we instead apply the solution formula, we get

$$
\begin{aligned}
z &= \frac{1}{2} \pm \sqrt{\frac{1}{4} - 1 + i} = \frac{1}{2} \pm \frac{1}{2} \sqrt{-3 + 4i} \\
&= \frac{1}{2} \pm \frac{1}{2} \sqrt{-4 + 1 + 2 \cdot 2i \cdot 1} = \frac{1}{2} \pm \frac{1}{2}\sqrt{(1 + 2i)^2} \\
&= \frac{1}{2} \{1 \pm (1 + 2i)\} = \left\{ \begin{array}{l} 1 + i, \\ -i. \end{array} \right.
\end{aligned}
$$

3) If we put $e^y = z$, we get the same equation as before, so either

$$e^y = 1 + i = \exp\left(\ln 2 + i\left\{\frac{\pi}{4} + 2p\pi\right\}\right),$$

or

$$e^y = -i = \exp\left(i\left\{-\frac{\pi}{2} + 2p\pi\right\}\right).$$

From this we immediately get the solutions, either

$$y = \ln 2 + i\left\{\frac{\pi}{4} + 2p\pi\right\}, \qquad p \in \mathbb{Z},$$

or

$$y = i\left\{-\frac{\pi}{2} + 2p\pi\right\}, \qquad p \in \mathbb{Z}.$$

Example 3.9 *1) Solve the complex equation*

$$z^2 - (1 + 2i)z + 1 - 5i = 0.$$

2) Is it possible to write some of the solutions in the form

$$z = \sqrt{2}\, e^{it}, \qquad t \in \mathbb{R}?$$

A. A complex equation of second degree.

D. Use the solution formula, known from high school. Find modulus and argument of the solutions and compare.

I. 1) First we find the discriminant

$$
\begin{aligned}
B^2 - 4AC &= \{-(1+2i)\}^2 - 4(1-5i) \\
&= 1 - 4 + 4i - 4 + 20i = -7 + 24i \\
&= 3^2 - 4^2 + 2 \cdot 3 \cdot 4i = (3 + 4i)^2.
\end{aligned}
$$

Then we get the solutions

$$z = \frac{1}{2}\{1 + 2i \pm (3 + 4i)\} = \begin{cases} 2 + 3i, \\ -1 - i. \end{cases}$$

2) Since $|2 + 3i| = \sqrt{4 + 9} = \sqrt{13} \neq \sqrt{2}$, this solution is not possible.

Concerning the second solution we get

$$-1 - i = \sqrt{2} \cdot \exp\left(-i\frac{3\pi}{4} + 2ip\pi\right), \qquad p \in \mathbb{Z},$$

thus

$$z = -1 - i = \sqrt{2} \cdot e^{it}$$

for

$$t = -\frac{3\pi}{4} + 2p\pi, \qquad p \in \mathbb{Z}.$$

4 The binomial equation

Example 4.1 *Let w be a solution of the equation $w^n = 1$. Show that when $w \neq 1$, then*

$$1 + w + w^2 + \cdots + w^{n-1} = 0.$$

A. Prove an identity.

D. The trick is to multiply by $w - 1 \neq 0$.

I. By the multiplication by $w - 1 \neq 0$, followed by a calculation and a reduction we get by the assumption that

$$
\begin{aligned}
(w - 1)&(w^{n-1} + w^{n-2} + \cdots + w + 1) \\
&= (w^n - w^{n-1}) + (w^{n-1} - w^{n-2}) + \cdots + (w^2 - w) + (w - 1) \\
&= w^n - 1 = 0.
\end{aligned}
$$

Then one of the factors must be 0. Now, $w - 1 \neq 0$, so we conclude that

$$1 + w + w^2 + \cdots + w^{n-1} = 0.$$

Example 4.2 *Solve the equations*

$$(1) \quad z^3 = 1, \qquad (2) \quad z^3 = i.$$

Sketch figures!

A. Two binomial equations.

D. Solve by using modulus and argument. Alternatively it is possible to find a root by inspection.
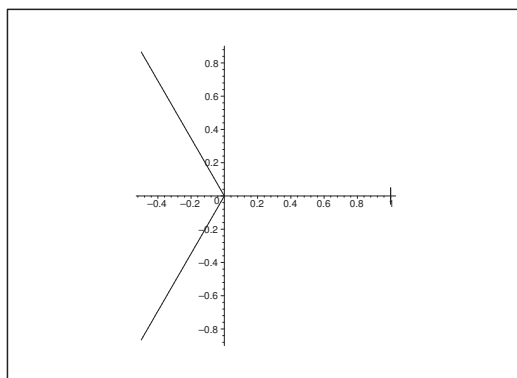

Figure 10: The solutions of $z^3 = 1$.

I. 1) **First variant.** From $z^3 = 1 = (1)_{2p\pi}$ we get

$$z = (1)_{\frac{2p\pi}{3}}, \qquad p = 0, 1, 2,$$

thus

$$z = 1 \qquad \text{og} \qquad z = -\frac{1}{2} \pm i\,\frac{\sqrt{3}}{2}.$$

Second variant. It is immediately seen that $z = 1$ is a solution, and since

$$0 = z^3 - 1 = (z-1)(z^2 + z + 1),$$

the other solutions must be roots in $z^2 + z + 1$, thus

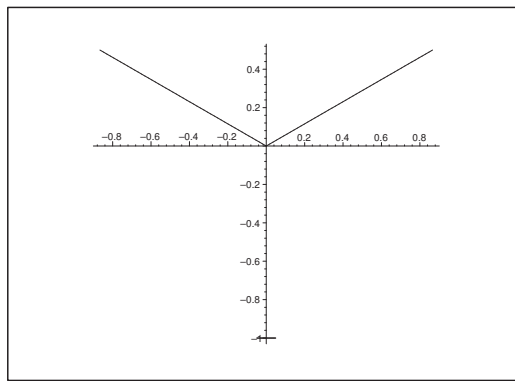$$z = 1 \qquad \text{and} \qquad z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}.$$

Figure 11: The solutions of $z^3 = i$.

2) **First variant.** From $z^3 = i = (1)_{\frac{\pi}{2} + 2p\pi}$, we get

$$z = (1)_{\frac{\pi}{6} + \frac{2p\pi}{3}}, \qquad p = 0, 1, 2,$$

i.e. we get the three solutions

$$(1)_{\frac{\pi}{6}} = \frac{\sqrt{3}}{2} + i\,\frac{1}{2},$$

$$(1)_{\frac{\pi}{6} + \frac{2\pi}{3}} = (1)_{\frac{5\pi}{6}} = -\frac{\sqrt{3}}{2} + i\,\frac{1}{2},$$

$$(1)_{\frac{\pi}{6} + \frac{4\pi}{3}} = (1)_{\frac{3\pi}{2}} = -i.$$

Second variant. It is immediately seen that $z = -i$ is a solution. Since

$$0 = z^3 - i = (z+i)(z^2 - iz - 1),$$

the remaining two solutions are roots in $z^2 - iz - 1$, thus

$$z = \frac{1}{2}\left\{i \pm \sqrt{-1+4}\right\} = \pm\frac{\sqrt{3}}{2} + i\,\frac{1}{2}.$$

Hence the three solutions are

$$-i, \qquad \text{and} \qquad \pm \frac{\sqrt{3}}{2} + i \frac{1}{2}.$$

Example 4.3 *Solve the equations*

(1) $z^4 = -4,$ (2) $z^3 = 1 + i.$

A. Two binomial equations.

D. Solve by means of modulus and argument. In (1) it is alternatively possible instead to perform two successive square roots, or, one can split the polynomial into its factors. Hence, we have three variants.

I. 1) **First variant.** From $z^4 = -4 = (4)_{\pi + 2p\pi}$ we get

$$z = (\sqrt{2})_{\frac{\pi}{4} + p \cdot \frac{\pi}{2}} = (1 + i) \cdot i^p, \qquad p = 0,\, 1,\, 2,\, 3.$$

The solutions are

$$1 + i, \qquad -1 + i, \qquad , -1 - i, \qquad 1 - i.$$

Second variant. From $z^4 = -4$ we get

$$z^2 = \pm 2i = \pm (1 + i)^2.$$

a) The equation $z^2 = (1 + i)^2$ has the solutions $z = \pm(1 + i)$.

b) The equation $z^2 = -(1 + i)^2$ has the solutions $z = \mp i(1 + i) = \pm(1 - i)$.

All things considered we get the solutions $\pm 1 \pm i$ with all four possible combinations of the signs.

Third variant. Write the equation in the form $z^4 + 4 = 0$. Then by a rearrangement,

$$\begin{aligned}
0 &= z^4 + 4 = (z^4 + 4z^2 + 4) - 4z^2 = (z^2 + 2)^2 - (2z)^2 \\
&= (z^2 + 2z + 2)(z^2 - 2z + 2) = \{(z + 1)^2 + 1\}\{(z - 1)^2 + 1\},
\end{aligned}$$

from which we get the solutions

$$-1 \pm i \qquad \text{og} \qquad 1 \pm i.$$

2) From $z^3 = 1 + i = (\sqrt{2})_{\frac{\pi}{4} + 2p\pi}$ we get

$$z = (\sqrt[6]{2})_{\frac{\pi}{12} + \frac{2p\pi}{3}}, \qquad p = 0,\, 1,\, 2.$$

Here,

$$\begin{aligned}
(\sqrt[6]{2})_{\frac{\pi}{12}} &= \sqrt[6]{2} \left\{ \cos \frac{\pi}{12} + i \sin \frac{\pi}{12} \right\} \\
&= \sqrt[6]{2} \left\{ \sqrt{\frac{1 + \cos \frac{\pi}{6}}{2}} + i \sqrt{\frac{1 - \cos \frac{\pi}{6}}{2}} \right\} \\
&= \sqrt[6]{2} \left\{ \sqrt{\frac{1}{2} \left(1 + \frac{\sqrt{3}}{2} \right)} + i \sqrt{\frac{1}{2} \left(1 - \frac{\sqrt{3}}{2} \right)} \right\} \\
&= \sqrt[6]{2} \left\{ \sqrt{\frac{1}{8} (4 + 2\sqrt{3})} + i \sqrt{\frac{1}{8} (4 - 2\sqrt{3})} \right\} \\
&= \frac{\sqrt[6]{3}}{2\sqrt{2}} \left\{ \sqrt{(\sqrt{3} + 1)^2} + i \sqrt{(\sqrt{3} - 1)^2} \right\} \\
&= \frac{1}{2 \sqrt[3]{2}} \left\{ \sqrt{3} + 1 + i(\sqrt{3} - 1) \right\}.
\end{aligned}$$

Furthermore, $\dfrac{\pi}{12} + \dfrac{2\pi}{3} = \dfrac{9\pi}{12} = \dfrac{3\pi}{4}$, thus

$$(\sqrt[6]{2})_{\frac{3\pi}{4}} = \sqrt[6]{2} \left\{ \frac{1}{\sqrt{2}} (-1 + i) \right\} = \frac{1}{\sqrt[3]{2}} (-1 + i).$$

Finally,

$$
\begin{aligned}
\left(\sqrt[6]{2}\right)_{\frac{\pi}{12}+\frac{4\pi}{3}} &= \left(\sqrt[6]{2}\right)_{\frac{3\pi}{4}} \cdot (1)_{\frac{2\pi}{3}} = \frac{1}{\sqrt[3]{2}}\left(-1+i\right) \cdot \frac{1}{2}\left(-\sqrt{3}+i\right) \\
&= \frac{1}{2\sqrt[3]{2}}\{\sqrt{3}-1-i(\sqrt{3}+1)\}.
\end{aligned}
$$

Example 4.4 *1) Solve the equation $x^4 + 16 = 0$.*

2) Write the polynomial $P(x) = x^4 + 16$ as a product of polynomials of second degree of real coefficients.

A. Binomial equation and factorization of a polynomial.

D. Find the complex roots, and use the solutions it the factorization of $P(x)$. Alternatively, add a term and subtract it again.

I. 1) We write the binomial equation in the form $z^2 = -16 = (2^4)_{\pi+2p\pi}$, from which

$$
z = (2)_{\frac{\pi}{4}+p\frac{\pi}{2}}, \qquad p = 0,\ 1,\ 2,\ 3.
$$

Since $(2)_{\frac{\pi}{4}} = \sqrt{2} + i\sqrt{2}$ and $(1)_{\frac{\pi}{2}} = i$, the solutions are given by

$$
\sqrt{2}+i\sqrt{2}, \qquad -\sqrt{2}+i\sqrt{2}, \qquad -\sqrt{2}-i\sqrt{2}, \qquad \sqrt{2}-i\sqrt{2}.
$$

2) **First variant.** It follows from (1) that

$$
\begin{aligned}
P(x) &= x^4 + 16 \\
&= (x-\sqrt{2}-i\sqrt{2})(x-\sqrt{2}+i\sqrt{2})(x+\sqrt{2}-i\sqrt{2})(x+\sqrt{2}+i\sqrt{2}) \\
&= \left\{(x-\sqrt{2})^2 + 2\right\}\left\{(x+\sqrt{2})^2 + 2\right\} \\
&= (x^2 - 2\sqrt{2}\,x + 4)(x^2 + 2\sqrt{2}\,x + 4).
\end{aligned}
$$

Second variant. Alternatively,

$$
\begin{aligned}
P(x) &= x^4 + 16 = \left(x^2\right)^2 + 4^2 + 2 \cdot 4 \cdot x^2 - 8x^2 \\
&= \left(x^2 + 4\right)^2 - \left(2\sqrt{2}\,x\right)^2 \\
&= \left(x^2 - 2\sqrt{2}\,x + 4\right)\left(x^2 + 2\sqrt{2}\,x + 4\right)
\end{aligned}
$$

by the rule $a^2 - b^2 = (a+b)(a-b)$.

Example 4.5 *1) Solve the equation $x^8 + 1 = 0$.*

2) Write the polynomial $P(x) = x^8 + 1$ as a product of polynomials of second degree of real coefficients.

A. A binomial equation and a factorization of a real polynomial into real factors of second degree.

The second item is very difficult.

D. 1) We solve this as a binomial equation.

2) a) Complex conjugated roots are paired.

b) Alternatively add a term and subtract it again, and apply the rule $a^2 - b^2 = (a+b)(a-b)$.

I. 1) From $z^8 = -1 = (1)_{\pi+2p\pi}$ we get the solutions

$$z = (1)_{\frac{\pi}{8}+p\frac{\pi}{4}}.$$

First calculate

$$(1)_{\frac{\pi}{8}} = \cos\frac{\pi}{8} + i\sin\frac{\pi}{8} = \sqrt{\frac{1+\cos\frac{\pi}{4}}{2}} + i\sqrt{\frac{1-\cos\frac{\pi}{4}}{2}}$$

$$= \sqrt{\frac{1+\frac{1}{\sqrt{2}}}{2}} + i\sqrt{\frac{1-\frac{1}{\sqrt{2}}}{2}} = \frac{1}{2}\sqrt{2+\sqrt{2}} + i\frac{1}{2}\sqrt{2-\sqrt{2}},$$

and

$$(1)_{\frac{3\pi}{8}} = \cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8} = \sqrt{\frac{1+\cos\dfrac{3\pi}{4}}{2}} + i\sqrt{\frac{1-\cos\dfrac{3\pi}{4}}{2}}$$

$$= \frac{1}{2}\sqrt{2-\sqrt{2}} + i\frac{1}{2}\sqrt{2+\sqrt{2}}.$$

Since $(1)_{\frac{\pi}{2}} = i$, one easily derive the eight solutions by successively multiplying by i:

$$\frac{1}{2}\sqrt{2+\sqrt{2}} + i\frac{1}{2}\sqrt{2-\sqrt{2}}, \qquad -\frac{1}{2}\sqrt{2+\sqrt{2}} - i\frac{1}{2}\sqrt{2-\sqrt{2}},$$
$$\frac{1}{2}\sqrt{2-\sqrt{2}} - i\frac{1}{2}\sqrt{2+\sqrt{2}}, \qquad -\frac{1}{2}\sqrt{2-\sqrt{2}} + i\frac{1}{2}\sqrt{2+\sqrt{2}},$$
$$\frac{1}{2}\sqrt{2+\sqrt{2}} - i\frac{1}{2}\sqrt{2-\sqrt{2}}, \qquad -\frac{1}{2}\sqrt{2+\sqrt{2}} + i\frac{1}{2}\sqrt{2-\sqrt{2}},$$
$$\frac{1}{2}\sqrt{2-\sqrt{2}} + i\frac{1}{2}\sqrt{2+\sqrt{2}}, \qquad -\frac{1}{2}\sqrt{2-\sqrt{2}} - i\frac{1}{2}\sqrt{2+\sqrt{2}}.$$

First variant. When complex conjugated roots are paired we get the factors of second degree

$$\left(x - \frac{1}{2}\sqrt{2+\sqrt{2}} - i\frac{1}{2}\sqrt{2-\sqrt{2}}\right)\left(x - \frac{1}{2}\sqrt{2+\sqrt{2}} + i\frac{1}{2}\sqrt{2-\sqrt{2}}\right)$$

$$= \left(x - \frac{1}{2}\sqrt{2+\sqrt{2}}\right)^2 + \frac{1}{4}(2-\sqrt{2}) = x^2 - \sqrt{2+\sqrt{2}}\,x + 1,$$

$$\left(x + \frac{1}{2}\sqrt{2+\sqrt{2}} - i\frac{1}{2}\sqrt{2-\sqrt{2}}\right)\left(x + \frac{1}{2}\sqrt{2+\sqrt{2}} + i\frac{1}{2}\sqrt{2-\sqrt{2}}\right)$$

$$= x^2 + \sqrt{2+\sqrt{2}}\,x + 1,$$

$$\left(x - \frac{1}{2}\sqrt{2-\sqrt{2}} + i\frac{1}{2}\sqrt{2+\sqrt{2}}\right)\left(x - \frac{1}{2}\sqrt{2-\sqrt{2}} - i\frac{1}{2}\sqrt{2+\sqrt{2}}\right)$$

$$= x^2 - \sqrt{2-\sqrt{2}}\,x + 1,$$

$$\left(x + \frac{1}{2}\sqrt{2-\sqrt{2}} - i\frac{1}{2}\sqrt{2+\sqrt{2}}\right)\left(x + \frac{1}{2}\sqrt{2-\sqrt{2}} + i\frac{1}{2}\sqrt{2+\sqrt{2}}\right)$$

$$= x^2 + \sqrt{2-\sqrt{2}}\,x + 1.$$

Finally,

$$x^8 + 1 = \left(x^2 - \sqrt{2+\sqrt{2}}\,x + 1\right)\left(x^2 + \sqrt{2+\sqrt{2}}\,x + 1\right) \cdot$$
$$\cdot \left(x^2 - \sqrt{2-\sqrt{2}}\,x + 1\right)\left(x^2 - \sqrt{2-\sqrt{2}}\,x + 1\right).$$

Second variant. ALTERNATIVELY we introduce the missing products,

$$
\begin{aligned}
P(x) &= x^8 + 1 = \left(x^4\right)^2 + 2x^4 + 1 - 2x^4 = \left(x^4 + 1\right)^2 - \left(\sqrt{2}\,x^2\right)^2 \\
&= \left(x^4 + \sqrt{2}\,x^2 + 1\right)\left(x^4 - \sqrt{2}\,x^2 + 1\right) \\
&= \left\{x^4 + 2x^2 + 1 - (2 - \sqrt{2})x^2\right\}\left\{x^4 + 2x^2 + 1 - (2 + \sqrt{2})x^2\right\} \\
&= \left\{(x^2+1)^2 - \left(\sqrt{2-\sqrt{2}}\,x\right)^2\right\}\left\{(x^2+1)^2 - \left(\sqrt{2+\sqrt{2}}\,x\right)^2\right\} \\
&= \left(x^2 + \sqrt{2 - \sqrt{2}}\,x + 1\right)\left(x^2 - \sqrt{2 - \sqrt{2}}\,x + 1\right)\cdot \\
&\quad \cdot \left(x^2 + \sqrt{2 + \sqrt{2}}\,x + 1\right)\left(x^2 - \sqrt{2 + \sqrt{2}}\,x + 1\right).
\end{aligned}
$$

C. TEST. A tedious calculation gives

$$
\begin{aligned}
&\left(x^2 - \sqrt{2 + \sqrt{2}}\,x + 1\right)\left(x^2 + \sqrt{2 + \sqrt{2}}\,x + 1\right)\cdot \\
&\quad \cdot \left(x^2 - \sqrt{2 - \sqrt{2}}\,x + 1\right)\left(x^2 + \sqrt{2 - \sqrt{2}}\,x + 1\right) \\
&= \left\{(x^2 + 1)^2 - (2 + \sqrt{2})x^2\right\}\left\{(x^2 + 1)^2 - (2 - \sqrt{2})x^2\right\} \\
&= \left(x^2 + 1\right)^4 - \{2 + \sqrt{2} + 2 - \sqrt{2}\}x^2\left(x^2 + 1\right)^2 \\
&\qquad\qquad + (2 + \sqrt{2})(2 - \sqrt{2})x^4 \\
&= \left(x^2 + 1\right)^4 - 4x^2\left(x^2 + 1\right)^2 + 2x^4 \\
&= \left(x^8 + 4x^6 + 6x^4 + 4x^2 + 1\right) - \left(4x^6 + 8x^4 + 4x^2\right) + 2x^4 \\
&= x^8 + 1. \qquad \text{Q.E.D.}
\end{aligned}
$$

Example 4.6 *1) Solve the equation $x^6 - 1 = 0$.*

2) Factorize the polynomial $P(x) = x^6 - 1$ into a product of polynomials of first and second degree, all of real coefficients.

A. 1) Binomial equation.

2) Real factorization of a polynomial.

D. 1) Use the modulus and the argument.

2) There are at least two possible solutions:

    a) A trivial factor is $x^3 - 1$.

    b) Complex conjugated factors are multiplied.

I. 1) From $z^6 = 1 = (1)_{2p\pi}$ we obtain

$$
z = (1)_{p\frac{\pi}{3}}, \qquad p = 0,\ 1,\ 2,\ 3,\ 4,\ 5,
$$

thus the solutions are

$$(1)_0 = 1, \qquad (1)_{\frac{\pi}{3}} = \frac{1}{2} + i\frac{\sqrt{3}}{2}, \qquad (1)_{\frac{2\pi}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2},$$

$$(1)_\pi, \qquad (1)_{\frac{4\pi}{3}} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}, \qquad (1)_{\frac{5\pi}{3}} = \frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

2) **First variant.** Since $x^3 - 1$ is a trivial divisor, we get

$$\begin{aligned} P(x) &= x^6 - 1 = \left(x^3\right)^2 - 1 = \left(x^3 - 1\right)\left(x^3 + 1\right) \\ &= (x-1)(x^2 + x + 1) \cdot (x+1)(x^2 - x + 1). \end{aligned}$$

Second variant. If we instead apply the result from (1), we get

$$\begin{aligned} P(x) &= x^6 - 1 \\ &= (x-1)(x+1)\left(x - \frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\left(x - \frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \cdot \\ &\qquad \cdot \left(x + \frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\left(x + \frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\ &= (x-1)(x+1)\left\{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}\right\}\left\{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}\right\} \\ &= (x-1)(x+1)(x^2 - x + 1)(x^2 + x + 1). \end{aligned}$$

Example 4.7 *Solve the following equation with respect to $z \in \mathbb{C}$.*

(1) $z^3 = 1$,   (2) $z^3 = -8$,   (3) $z^6 + 7z^3 - 8 = 0$.

A. Two binomial equations and a composition of an equation of second order and a binomial equation.

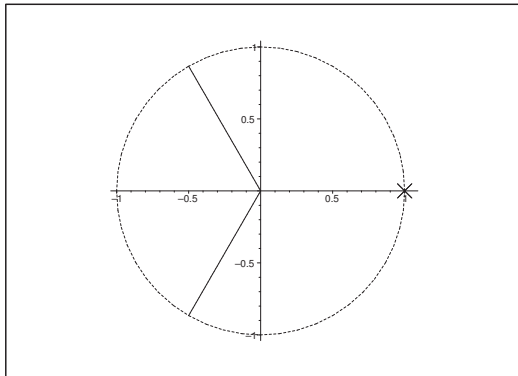**D.** First solve (1). Apply (1) in the solution of (2), and then use both to solve (3).

Figure 12: Graphical solution of the binomial equation $z^3 = 1$.

I. 1) From $z^3 = (1)_{2p\pi}$ we get $z = (1)_{\frac{2p\pi}{3}}$, $p = 0$, 1, 2, thus the solutions are

$$1, \qquad -\frac{1}{2} + i\,\frac{\sqrt{3}}{2}, \qquad -\frac{1}{2} - i\,\frac{\sqrt{3}}{2}.$$
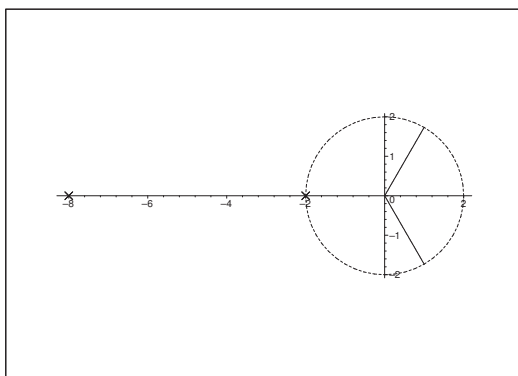

Figure 13: Graphical solution of the binomial equation $z^3 = -8$.

2) From $z^3 = (-2)^3 \cdot (1)_{2p\pi}$ we get

$$z = -2 \cdot (1)_{\frac{2p\pi}{3}}, \qquad p = 0,\ 1,\ 2.$$

Thus, by (1) the solutions are

$$-2, \qquad 1 - i\sqrt{3}, \qquad 1 + i\sqrt{3}.$$

3) By putting $w = z^3$ the equation is reduced to

$$0 = w^2 + 7w - 8 = (w - 1)(w + 8) = (z^3 - 1)(z^3 + 8).$$

We have already solved the equation $z^3 = 1$ in (1), and the equation $z^3 = -8$ is solved in (2). Hence, all solutions are given by

$$1, \qquad -\frac{1}{2} \pm i\,\frac{\sqrt{3}}{2}, \qquad -2, \qquad 1 \pm i\sqrt{3}.$$

Example 4.8 *In this example we shall derive exact expression for* $\cos\dfrac{\pi}{5}$ *and* $\cos\dfrac{3\pi}{5}$.

1) *Indicate the roots of the binomial equation* $z^5 = -1$ *in polar coordinates. Sketch the roots in the complex plane.*

2) *The real polynomial* $x^5 + 1$ *can be written as a product of a polynomial of first degree and two polynomials of second degree, all of which have real coefficients.*

Show that the two polynomials of second degree are

$$x^2 - ax + 1 \qquad and \qquad x^2 - bx + 1,$$

where

$$a = 2 \cos\frac{\pi}{5} \qquad og \qquad b = 2 \cos\frac{3\pi}{5}.$$

3) *Apply the factorization of* $x^5 + 1$ *to find* $a + b$ *and* ab. *Then set up an equation of second degree, having* a *and* b *as its roots.*

4) *Find the exact values of* $\cos\dfrac{\pi}{5}$ *and* $\cos\dfrac{3\pi}{5}$.

A. Calculation of $\cos\dfrac{\pi}{5}$ and $\cos\dfrac{3\pi}{5}$ by means of complex numbers.

D. Follow the guidelines. We have here several variants.

I. 1) From

$$z^5 = -1 = e^{i(\pi + 2p\pi)}, \qquad p \in \mathbb{Z},$$

we get immediately that the roots in polar coordinates are

$$z_p = \exp\left(i \cdot \frac{1 + 2p}{5}\,\pi\right), \qquad p = 0, 1, 2, 3, 4.$$

When $p = 2$ we get $z_2 = -1$, and since $z^5 + 1$ has real coefficients, the other roots must necessarily be pairwise complex conjugated,

$$z_0 = \cos\frac{\pi}{5} + i \cdot \sin\frac{\pi}{5} = \overline{z}_4,$$

$$z_1 = \cos\frac{3\pi}{5} + i \cdot \sin\frac{3\pi}{5} = \overline{z}_3,$$
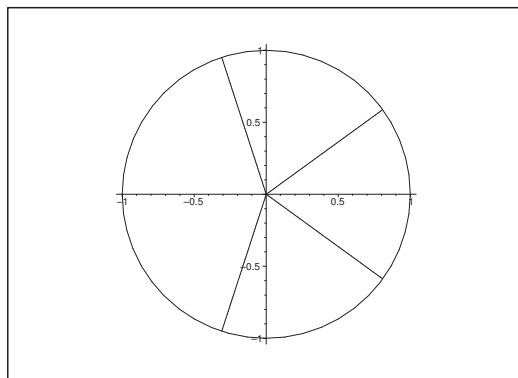
Figure 14: The roots of $z^5 = -1$.

thus

$$z_3 = \cos\frac{3\pi}{5} - i \cdot \sin\frac{3\pi}{5} \quad \text{and} \quad z_4 = \cos\frac{\pi}{5} - i \cdot \sin\frac{\pi}{5},$$

and $z_2 = -1$.

2) It follows from the result of (1) that the two factors of second degree are

$$\left\{ x - \cos\frac{\pi}{5} - i\,\sin\frac{\pi}{5} \right\} \left\{ x - \cos\frac{\pi}{5} + i\,\sin\frac{\pi}{5} \right\}$$
$$= \left(x - \cos\frac{\pi}{5} \right)^2 + \sin^2\frac{\pi}{5}$$
$$= x^2 - 2\left\{ \cos\frac{\pi}{5} \right\} x + \cos^2\frac{\pi}{5} + \sin^2\frac{\pi}{5}$$
$$= x^2 - ax + 1,$$

and

$$\left\{ x - \cos\frac{3\pi}{5} - i\,\sin\frac{3\pi}{5} \right\} \left\{ x - \cos\frac{3\pi}{5} + i\,\sin\frac{3\pi}{5} \right\}$$
$$= \left(x - \cos\frac{3\pi}{5} \right)^2 + \sin^2\frac{3\pi}{5}$$
$$= x^2 - 2\left\{ \cos\frac{3\pi}{5} \right\} x \cos^2\frac{3\pi}{5} + \sin^2\frac{3\pi}{5}$$
$$= x^2 - bx + 1.$$

ALTERNATIVELY we can calculate these in the following way:

$$\begin{aligned} \{x - z_0\}\{x - z_4\} &= \left\{ x - e^{i\frac{\pi}{5}} \right\}\left\{ x - e^{-i\frac{\pi}{5}} \right\} \\ &= x^2 - \left\{ e^{i\frac{\pi}{5}} + e^{-i\frac{\pi}{5}} \right\} x + 1 \\ &= x^2 - 2\left\{ \cos\frac{\pi}{5} \right\} x + 1 \\ &= x^2 - ax + 1, \end{aligned}$$

and analogously

$$\begin{aligned}
\{x - z_1\}\{x - z_3\} &= \left\{x - e^{i\frac{3\pi}{5}}\right\}\left\{x - e^{-i\frac{3\pi}{5}}\right\} \\
&= x^2 - \left\{e^{i\frac{3\pi}{5}} + e^{-i\frac{3\pi}{5}}\right\}x + 1 \\
&= x^2 - 2\left\{\cos\frac{3\pi}{5}\right\}x + 1 \\
&= x^2 - bx + 1,
\end{aligned}$$

where we have used Euler's formulæ.

For later use we notice that

$$a = 2\cos\frac{\pi}{5} = \operatorname{Re} z_0 > 0, \qquad b = 2\cos\frac{3\pi}{5} = \operatorname{Re} z_1 < 0.$$

This is e.g. seen by a consideration of a figure, or by using that $\frac{\pi}{5} \in \left]0, \frac{\pi}{2}\right[$, hence $\cos\frac{\pi}{5} > 0$ and $\frac{3\pi}{5} \in \left]\frac{\pi}{2}, \pi\right[$, thus $\cos\frac{3\pi}{5} < 0$.

3) **First variant.** Since

$$x^5 + 1 = (x+1)(x^2 - ax + 1)(x^2 - bx + 1),$$

we get for $x \neq -1$ that

$$(x^2 - ax + 1)(x^2 - bx + 1) = \frac{x^5 + 1}{x + 1} = x^4 - x^3 + x^2 - x + 1.$$

On the other hand,
$$(x^2 - ax + 1)(x^2 - bx + 1)$$
$$= x^4 - (a+b)x^3 + (2+ab)x^2 - (a+b)x + 1.$$
When we identify the coefficients we get

$$a + b = 1, \quad \text{and} \quad 2 + ab = 1, \quad \text{i.e.} \quad ab = -1.$$

This means that a and b are the roots of the λ polynomial

$$(\lambda - a)(\lambda - b) = \lambda^2 - (a+b)\lambda + ab = \lambda^2 - \lambda - 1.$$

Second variant. By a straight forward calculation we get
$$\begin{aligned}
x^5 + 1 &= (x+1)(x^2 - ax + 1)(x^2 - bx + 1) \\
&= (x+1)(x^4 - (a+b)x^3 + (2+ab)x^2 - (a+b)x + 1) \\
&= x^5 + (1-a-b)x^4 + (2+ab-a-b)x^3 \\
&\quad + (2+ab-a-b)x^2 + (1-a-b)x + 1.
\end{aligned}$$
By identification of the coefficients we get

$$\begin{cases} 1-a-b=0, \\ 2+ab-a-b=0, \end{cases} \text{i.e.} \begin{cases} a+b=1, \\ ab=a+b-2=1-2=-1, \end{cases}$$

hence $a + b = 1$ and $ab = -1$.
Then continue as in the first variant,

$$(\lambda - a)(\lambda - b) = \lambda^2 - (a+b)\lambda + ab = \lambda^2 - \lambda - 1,$$

thus $a = 2 \cos \dfrac{\pi}{5}$ and $b = 2 \cos \dfrac{3\pi}{5}$ are the roots of the equation

$$\lambda^2 - \lambda - 1 = 0.$$

4) Since $a = 2 \cos \dfrac{\pi}{5} > 0$ and $b = 2 \cos \dfrac{3\pi}{5} < 0$ are the roots of the equation

$$\lambda^2 - \lambda - 1 = 0,$$

we get by the usual solution formula,

$$\lambda = \frac{1}{2} \left\{ 1 \pm \sqrt{1 + 4 \cdot 1} \right\},$$

thus

$$a = \frac{1 + \sqrt{5}}{2} > 0, \quad \text{og} \quad b = \frac{1 - \sqrt{5}}{2} < 0,$$

så

$$\cos \frac{\pi}{5} = \frac{a}{2} = \frac{1 + \sqrt{5}}{4}, \qquad \cos \frac{3\pi}{5} = \frac{b}{2} = \frac{1 - \sqrt{5}}{4}.$$

Example 4.9 *Solve the equation $z^6 + 64 = 0$. All solutions should be given in the form $z = a + i\,b$, where a and b are real numbers. (Find the exact values of a and b). Sketch the solutions in the complex plane.*

A. Binomial equation.

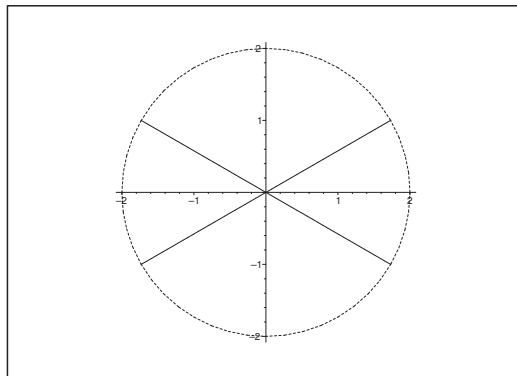**D.** Write -64 in polar coordinates and solve the equation.

Figure 15: The solutions of the equation $z^6 + 64 = 0$.

I. We get by a rearrangement,

$$z^6 = -64 = 2^6 \cdot \exp(i(2p+1)\pi), \qquad p \in \mathbb{Z},$$

thus

$$z = 2 \exp\left(i\left\{p \cdot \frac{\pi}{3} + \frac{\pi}{6}\right\}\right), \qquad p \in \{0, 1, 2, 3, 4, 5\}.$$

Hence

$$
\begin{aligned}
p = 0: &\quad z_0 = 2\left\{\cos\left(\frac{\pi}{6}\right) + i\,\sin\left(\frac{\pi}{6}\right)\right\} = \sqrt{3} + i, \\
p = 1: &\quad z_1 = 2\left\{\cos\left(\frac{\pi}{2}\right) + i\,\sin\left(\frac{\pi}{2}\right)\right\} = 2i, \\
p = 2: &\quad z_2 = 2\left\{\cos\left(\frac{5\pi}{6}\right) + i\,\sin\left(\frac{5\pi}{6}\right)\right\} = -\sqrt{3} + i, \\
p = 3: &\quad z_3 = 2\left\{\cos\left(\frac{7\pi}{6}\right) + i\,\sin\left(\frac{7\pi}{6}\right)\right\} = -\sqrt{3} - i, \\
p = 4: &\quad z_4 = 2\left\{\cos\left(\frac{3\pi}{2}\right) + i\,\sin\left(\frac{3\pi}{2}\right)\right\} = -2i, \\
p = 5: &\quad z_5 = 2\left\{\cos\left(\frac{11\pi}{6}\right) + i\,\sin\left(\frac{11\pi}{6}\right)\right\} = \sqrt{3} - i.
\end{aligned}
$$

Example 4.10 *Solve the equation $z^3 = i$. All solutions should be given in the form $a + i b$.*

A. Binomial equation.

D. Write the right hand side in polar coordinates, and then calculate.
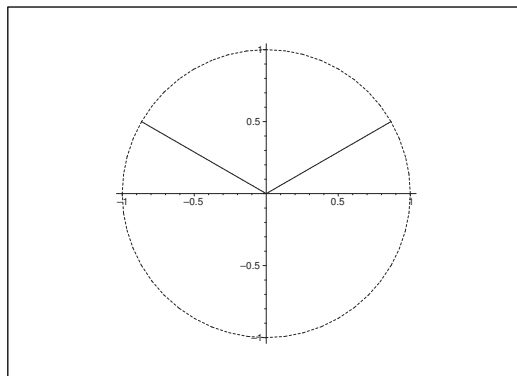
Figure 16: The three roots lie on the unit circle separated by the angle $\dfrac{2\pi}{3}$.

I. From

$$z^3 = i = \exp\left(i\left\{\frac{\pi}{2} + 2p\pi\right\}\right), \qquad p \in \mathbb{Z},$$

we get

$$z = \exp\left(i\left\{\frac{\pi}{6} + p \cdot \frac{2\pi}{3}\right\}\right), \qquad p = 0,\, 1,\, 2.$$

We get more explicitly,

$$
\begin{aligned}
z_0 &= \cos\frac{\pi}{6} + i\,\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + i\,\frac{1}{2}, \\
z_1 &= \cos\frac{5\pi}{6} + i\,\sin\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + i\,\frac{1}{2}, \\
z_2 &= \cos\frac{9\pi}{6} + i\,\sin\frac{9\pi}{6} = -i.
\end{aligned}
$$

5 The complex exponential

Example 5.1 *Write the following complex numbers in the form $a + ib$:*

(1) $e^{i\frac{\pi}{2}}$;   (2) $3e^{1+\pi i}$;   (3) $\dfrac{1 - e^{i\frac{\pi}{2}}}{1 + e^{i\frac{\pi}{2}}} + e^{i\frac{\pi}{4}} - e^{-i\frac{\pi}{4}}$.

A. Complex exponential.

D. Just calculate; apply, if necessary the complex definitions of the trigonometric functions.

I. 1) $e^{i\frac{\pi}{2}} = \cos\dfrac{\pi}{2} + i\,\sin\dfrac{\pi}{2} = i$.

2) $3e^{1+i\pi} = 3e\{\cos\pi + i\,\sin\pi\} = -3e$.

3) From $e^{i\frac{\pi}{2}} = i$ we get

$$\frac{1 - e^{i\frac{\pi}{2}}}{1 + e^{i\frac{\pi}{2}}} + e^{i\frac{\pi}{4}} - e^{-i\frac{\pi}{4}} = \frac{1 - i}{1 + i} + 2i\,\sin\frac{\pi}{4}$$

$$= \frac{-i(1 + i)}{1 + i} + i\sqrt{2} = i(\sqrt{2} - 1).$$

Example 5.2 *Prove in each of the following cases that one can find complex numbers A and B, such that*

$$q(t) = \ Re\left\{A\,e^{Bt}\right\},$$

and find such numbers.

1) $q(t) = \cos 3t + \sin 3t,$

2) $q(t) = e^{-t}\cos 2t - e^{-t}\sin 2t,$

3) $q(t) = e^{t}\cos t - 3e^{t}\sin t,$

4) $q(t) = k_1 e^{\alpha t}\cos\beta t + k_2 e^{\alpha t}\sin\beta t.$

In (4) the constants k_1, k_2, α and β are real.

A. Reduction of mixed exponential/trigonometric expressions to pure exponentials.

D. The easiest way is to start from (4), because the other three cases can be derived from (4). However, for pedagogical reasons we solve the example in the given succession.

I. First notice that since

$$e^{i\beta t} = \cos\beta t + i\,\sin\beta t \qquad \text{for } \beta \in \mathbb{R},$$

we have

$$\cos\beta t = \ Re\left\{e^{i\beta t}\right\}, \qquad \sin\beta t = \ Re\left\{-i\,e^{i\beta t}\right\}.$$

1) In this case we get

$$\begin{aligned}
q(t) &= \cos 3t + \sin 3t = \ Re\left\{e^{3it} - i\,e^{3it}\right\}\\
&= \ Re\left\{(1-i)e^{3it}\right\}, \qquad A = 1-i, \quad B = 3i.
\end{aligned}$$

2) Here,

$$\begin{aligned}
q(t) &= e^{-t}\cos 2t - e^{-t}\sin 2t = e^{-t}\{\cos 2t - \sin 2t\}\\
&= e^{-t}\,Re\left\{e^{2it} + i\,e^{2it}\right\}\\
&= \ Re\left\{(1+i)e^{(-1+2i)t}\right\}, \qquad A = 1+i, \quad B = -1+2i.
\end{aligned}$$

3) Similarly,

$$\begin{aligned}
q(t) &= e^{t}\cos t - 3e^{t}\sin t = e^{t}\,Re\left\{e^{it} + 3ie^{it}\right\}\\
&= \ Re\left\{(1+3i)e^{(1+i)t}\right\}, \qquad A = 1+3i, \quad B = 1+i.
\end{aligned}$$

4) In general,

$$\begin{aligned}
q(t) &= e^{\alpha t}\left\{k_1\cos\beta t + k_2\sin\beta\right\}\\
&= e^{\alpha t}\,Re\left\{k_1 e^{i\beta t} - ik_2 e^{i\beta t}\right\}\\
&= \ Re\left\{(k_1 - ik_2)\,e^{(\alpha + i\beta)t}\right\}, \qquad A = k_1 - ik_2, \quad B = \alpha + i\beta.
\end{aligned}$$

Example 5.3 *Prove that $e^z \neq 0$ for every $z \in \mathbb{C}$.*

A. Exponential.

D. Prove that $|e^z| > 0$ for every $z \in \mathbb{C}$.

I. If $z = x + iy$, x, $y \in \mathbb{R}$, then

$$|e^z| = |e^{x+iy}| = |e^x| \cdot |e^{iy}| = e^x > 0,$$

thus $e^z \neq 0$ for every $z \in \mathbb{C}$.

Example 5.4 *Apply Moivre's formula to express $\cos 3v$ and $\sin 3v$ by $\cos v$ and $\sin v$. Then use the addition formulæ to express $\sin 4v$ by $\cos v$ and $\sin v$.*

A. Trigonometric relations.

D. Apply Moivre's formula.

I. From

$$\cos 3v + i \sin 3c = (\cos v + i \sin v)^3$$
$$= \cos^3 v + 3i \cos^2 v \sin v - 3 \cos v \sin^2 v - i \sin^3 v,$$

we get by a splitting into real and imaginary part,

$$\cos 3v = \cos^3 v - 3 \cos v \sin^2 v = 4 \cos^3 v - 3 \cos v,$$

$$\sin 3v = 3 \cos^2 v \sin v - \sin^3 v = 3 \sin v - 4 \sin^3 v.$$

Then by the addition formulæ,

$$
\begin{aligned}
\sin 4v &= \sin(3v + v) \\
&= \sin 3v \cos v + \cos 3v \sin v \\
&= 3 \cos^3 v \sin v - \sin^3 v \cos v + \cos^3 v \sin v - 3 \cos v \sin^3 v \\
&= 4 \cos^3 v \sin v - 4 \sin^3 v \cos v.
\end{aligned}
$$

REMARK. Notice that we have ALTERNATIVELY,

$$
\begin{aligned}
\sin 4v &= 2 \sin 2v \cos 2v = 4 \sin v \cos v (\cos^2 v - \sin^2 v) \\
&= 4 \sin v \cos^3 v - 4 \sin^3 v \cos v.
\end{aligned}
$$

Example 5.5 *Show by means of Euler's formulæ,*

$$\sin 3x \cos 5x = \frac{1}{2} (\sin 8x - \sin 2x).$$

A. Trigonometric formula.

D. Apply Euler's formulæ on the left hand side and then just calculate.

I. By Euler's formulæ we get

$$
\begin{aligned}
\sin 3x \cos 5x &= \frac{1}{2i}\left\{e^{3ix} - e^{-3ix}\right\} \cdot \frac{1}{2}\left\{e^{5ix} + e^{-5ix}\right\} \\
&= \frac{1}{2} \cdot \frac{1}{2i}\left\{e^{8ix} - e^{2ix} + e^{-2ix} - e^{-8ix}\right\} \\
&= \frac{1}{2}\left\{\frac{1}{2i}\left(e^{8ix} - e^{-8ix}\right) - \frac{1}{2i}\left(e^{2ix} - e^{-2ix}\right)\right\} \\
&= \frac{1}{2}\left(\sin 8x - \sin 2x\right).
\end{aligned}
$$

Example 5.6 *Apply Euler's formulæ in order to write* $\cos 2x \sin^2 3x$ *as a simple sum.*
Then calculate

$$
\int_0^{\frac{\pi}{4}} \cos 2x \sin^2 3x \, dx.
$$

A. Trigonometric formula and an integration.

D. Apply Euler's formulæ and reduce. Then integrate.

I. We get by means of Euler's formulæ,

$$
\begin{aligned}
\cos 2x \sin^2 3x &= \frac{1}{2}\left\{e^{2ix} + e^{-2ix}\right\}\left\{\frac{1}{2i}\left(e^{3ix} - e^{-3ix}\right)\right\}^2 \\
&= \frac{1}{2}\left\{e^{2ix} + e^{-2ix}\right\} \cdot \left(-\frac{1}{4}\right)\left\{e^{6ix} - 2 + e^{-6ix}\right\} \\
&= \frac{1}{2}\left\{e^{2ix} + e^{-2ix}\right\} \cdot \left(-\frac{1}{4}\right)(-2) \\
&\quad - \frac{1}{4} \cdot \frac{1}{2}\left\{e^{2ix} + e^{-2ix}\right\}\left\{e^{6ix} + e^{-6ix}\right\} \\
&= \frac{1}{2}\cos 2x - \frac{1}{4} \cdot \frac{1}{2}\left\{e^{8ix} + e^{4ix} + e^{-4ix} + e^{-8ix}\right\} \\
&= \frac{1}{2}\cos 2x - \frac{1}{4}\cos 8x - \frac{1}{4}\cos 4x.
\end{aligned}
$$

Then by insertion into the integrand,

$$
\begin{aligned}
\int_0^{\frac{\pi}{4}} \cos 2x \sin^2 3x \, dx &= \int_0^{\frac{\pi}{4}}\left\{\frac{1}{2}\cos 2x - \frac{1}{4}\cos 8x - \frac{1}{4}\cos 4x\right\} dx \\
&= \left[\frac{1}{4}\sin 2x - \frac{1}{32}\sin 8x - \frac{1}{16}\sin 4x\right]_0^{\frac{\pi}{4}} \\
&= \frac{1}{4} - 0 - 0 = \frac{1}{4}.
\end{aligned}
$$

Example 5.7 *Express* $\cos 5x$ *and* $\sin 5x$ *by* $\cos x$ *and* $\sin x$.

A. Trigonometric formulæ.

D. Apply Moivre's formula and then split into the real and the imaginary part.

I. According to Moivre's formula,

$$\cos 5x + i \sin 5x = (\cos x + i \sin x)^5$$
$$\cos^5 x + 5i \cos^4 x \sin x - 10 \cos^3 x \sin^2 x$$
$$-10i \cos^2 x \sin^3 x + 5 \cos x \sin^4 x + i \sin^5 x.$$

Then by splitting into real and imaginary part,

$$
\begin{aligned}
\cos 5x &= \cos^5 x - 10 \cos^3 x \sin^2 x + 5 \cos x \sin^2 x \\
&= \cos^5 x - 10 \cos^3 x (1 - \cos^2 x) + 5 \cos x (1 - \cos^2 x)^2 \\
&= \cos^5 x + 10 \cos^5 x - 10 \cos^3 x + 5 \cos x (1 - 2 \cos^2 x + \cos^4) \\
&= 16 \cos^5 x - 20 \cos^3 x + 5 \cos x,
\end{aligned}
$$

and

$$
\begin{aligned}
\sin 5x &= 5\cos^4 x \sin x - 10\cos^2 x \sin^3 x + \sin^5 x \\
&= 5(1 - \sin^2 x)^2 \sin x - 10(1 - \sin^2 x) \sin^3 x + \sin^5 x \\
&= 5(1 - 2\sin^2 x + \sin^4 x) \sin x - 10\sin^3 x + 10\sin^5 x + \sin^5 x \\
&= 5(1 - 2\sin^2 x + \sin^4 x) \sin x - 10\sin^3 x + 11\sin^5 x \\
&= 16\sin^5 x - 20\sin^3 x + 5\sin x.
\end{aligned}
$$

Example 5.8 *Given the complex number*

$$
f(x) = \frac{1}{1 + \cos 2x - i \sin 2x}, \qquad x \in \left] -\frac{\pi}{2}, \frac{\pi}{2} \right[.
$$

1) Find $M(x) = |f(x)|$. Reduce the expression. Sketch the graph of $y = M(x)$.

2) Find $A(x) = \operatorname{Arg} f(x)$. Reduce the expression. Sketch the graph of $y = A(x)$.

3) Indicate on a figure, where $f(x)$ lies in the complex plane.

A. Complex function in one real variable. Modulus, argument and range.

D. Whenever necessary, we multiply the numerator and the denominator by the complex conjugated of the denominator.

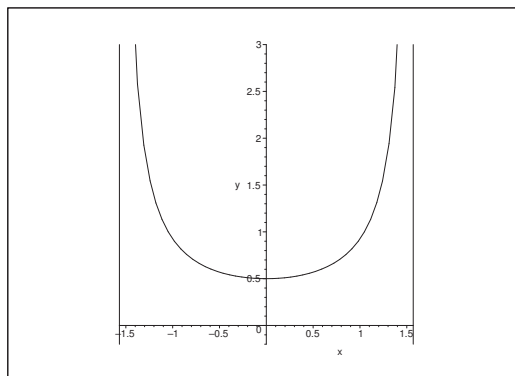Figure 17: The graph of $y = M(x) = \dfrac{1}{2\cos x}$, $x \in \left] -\dfrac{\pi}{2}, \dfrac{\pi}{2} \right[$.

I. 1) From

$$
\begin{aligned}
|1 + \cos 2x + i \sin 2x| &= \sqrt{(1 + \cos 2x)^2 + \sin^2 2x} = \sqrt{2 + 2\cos 2x} \\
&= \sqrt{2\cos^2 x + 2\sin^2 x + 2\cos^2 x - 2\sin^2 x} \\
&= \sqrt{4\cos^2 x} = |2\cos x| \\
&= 2\cos x \qquad \text{for } x \in \left] -\frac{\pi}{2} \right[,
\end{aligned}
$$

we conclude that

$$M(x) = \frac{1}{2\cos x} \qquad \text{for } x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[.$$
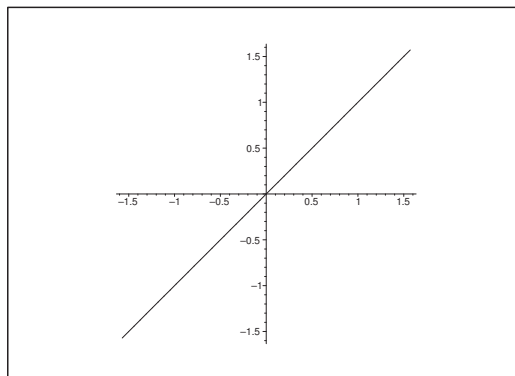
Figure 18: The graph of $y = A(x) = x$, $x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.

2) From

$$\begin{aligned}
f(x) &= \frac{1}{1 + \cos 2x - i \sin 2x} \cdot \frac{1 + \cos 2x + i \sin 2x}{1 + \cos 2x + i \sin 2x} \\
&= \frac{1}{4\cos^2 x}\left\{\cos^2 x + \sin^2 x + \cos^2 x - \sin^2 x + 2i \sin x \cos x\right\} \\
&= \frac{1}{2\cos x}\left\{\cos x + i \sin x\right\} = \frac{1}{2\cos x} e^{ix},
\end{aligned}$$

we get

$$\text{Arg } f(x) = x \qquad \text{for } x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[.$$

3) Since we also have

$$f(x) = \frac{1}{2\cos x}\left\{\cos x + i \sin x\right\} = \frac{1}{2} + i\,\frac{\tan x}{2}, \qquad x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[,$$

the image curve must be the straight line $u = \dfrac{1}{2}$ in the UV-plane.
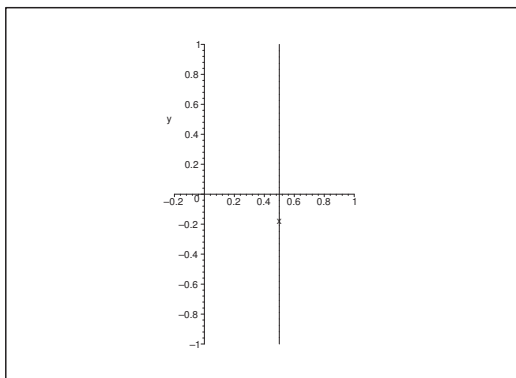
Figure 19: The image curve $u = \dfrac{1}{2}$.

Example 5.9 *1) Find the modulus and the principal argument of the complex number*

$$1 + e^{ix}, \; -\pi < x < \pi,$$

first by a calculation, and then by a geometric consideration.

2) Find modulus and principal argument of the complex number $e^{e^{ix}}$.

A. Complex numbers.

D. Follow the given guidelines.

I. 1) a) By a straight forward calculation we get

$$
\begin{aligned}
1 + e^{ix} &= (1 + \cos x) + i \sin x = 2 \cos^2 \frac{x}{2} + 2i \cos \frac{x}{2} \cdot \sin \frac{x}{2} \\
&= 2 \cos \frac{x}{2} \cdot \left\{ \cos \frac{x}{2} + i \sin \frac{x}{2} \right\} = 2 \cos \frac{x}{2} \cdot \exp\left(i \frac{x}{2} \right),
\end{aligned}
$$

hence the modulus is $2 \cos \dfrac{x}{2} > 0$, $-\pi < x < \pi$, and the principal argument is $\dfrac{x}{2} \in \left] -\dfrac{\pi}{2}, \dfrac{\pi}{2} \right[$.

b) A variant of (a) is

$$
\begin{aligned}
1 + e^{ix} &= \exp\left(i \frac{x}{2} \right) \cdot \left\{ \exp\left(i \frac{x}{2} \right) + \exp\left(-i \frac{x}{2} \right) \right\} \\
&= 2 \cos \frac{x}{2} \cdot \exp\left(i \frac{x}{2} \right).
\end{aligned}
$$

c) A geometric analysis shows that $1 + e^{ix}$ is described by a circle of centre $(1,0)$ and radius 1.

The centre angle x from $(1,0)$ corresponds to the periphery angle $\dfrac{x}{2}$ from $(0,0)$. Thus, the principal argument is

$$\frac{x}{2} \in \left] -\frac{\pi}{2}, \frac{\pi}{2} \right[.$$

Since the periphery angle is $\dfrac{x}{2}$, the modulus is equal to the closest of the shorter sides in an rectangular triangle with the hypothenuse of length 2, hence the modulus is $2 \cos \dfrac{x}{2}$.
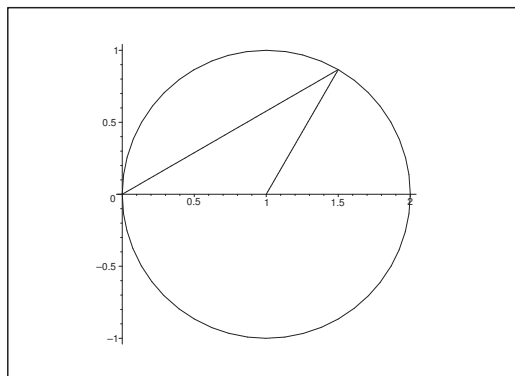
Figure 20: A centre angle is always twice the corresponding periphery angle.
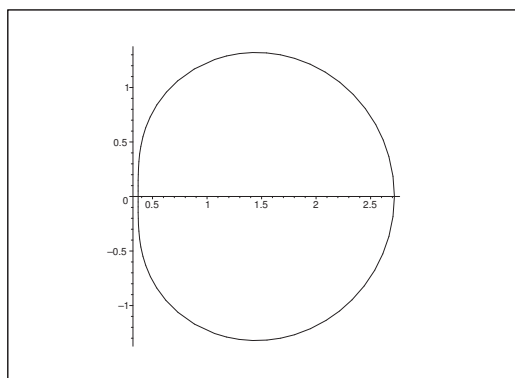
Figure 21: The graph of $e^{e^{ix}}$, $x \in \mathbb{R}$.

2) Now,

$$e^{e^{ix}} = \exp(\cos x + i \sin x) = e^{\cos x} \cdot e^{i \sin x},$$

so the modulus is $e^{\cos x}$.

Since $\sin x \in [-1, 1] \subset \,]-\pi, \pi[$, the principal argument is $\sin x$.

Example 5.10 *Find modulus and argument of the complex number*

$$A = \frac{(3 + i\sqrt{3})^4}{(1 + i\sqrt{3})^3}.$$

A. Modulus and argument of a complex number.

D. First find the modulus and the argument of $3 + i\sqrt{3}$ and $1 + i\sqrt{3}$.

Alternatively we can calculate A directly.

I. First variant. We have

$$|3 + i\sqrt{3}| = \sqrt{3^2 + 3} = \sqrt{12} = 2\sqrt{3},$$

and thus

$$3 + i\sqrt{3} = 2\sqrt{3}\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = (2\sqrt{3})_{\frac{\pi}{6}}.$$

Analogously,

$$|1 + i\sqrt{3}| = \sqrt{1+3} = 2,$$

and thus

$$1 + i\sqrt{3} = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 2_{\frac{\pi}{3}}.$$

Hence,

$$
\begin{aligned}
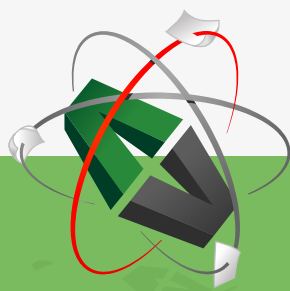A &= \frac{(3+i\sqrt{3})^4}{(1+i\sqrt{3})^3} = \left\{(2\sqrt{3})_{\frac{\pi}{6}}\right\}^4 \cdot \left\{2_{\frac{\pi}{3}}\right\}^{-3} \\
&= \left\{(2\sqrt{3})^4 \cdot 2^{-3}\right\}_{4\cdot\frac{\pi}{6}-3\cdot\frac{\pi}{3}} = \left\{\frac{144}{8}\right\}_{\frac{2\pi}{3}-2\pi} \\
&= 18_{-\frac{\pi}{3}}.
\end{aligned}
$$

It follows that the modulus is $|A| = 18$ and that the principal argument is $v = -\dfrac{\pi}{3}$.

Second variant. ALTERNATIVELY just calculate,

$$
\begin{aligned}
A &= \frac{(3+i\sqrt{3})^4}{(1+i\sqrt{3})^3} = \frac{(9-3+6i\sqrt{3})^2}{(1+i\sqrt{3})^3} = \frac{(6+6i\sqrt{3})^2}{(1+i\sqrt{3})^3} \\
&= 36 \cdot \frac{(1+i\sqrt{3})^2}{(1+i\sqrt{3})^3} = \frac{36}{1+i\sqrt{3}} = \frac{36}{4}\cdot(1-i\sqrt{3}) \\
&= 18\left(\frac{1}{2} - i\,\frac{\sqrt{3}}{2}\right) = 18_{-\frac{\pi}{3}}.
\end{aligned}
$$

Example 5.11 *Let $F(t)$ and $G(t)$ be complex functions in the real variable t. Put*

$$F_1(t) = \mathrm{Re}(F(t)), \qquad F_2(t) = \mathrm{Im}(F(t)),$$

$$G_1(t) = \mathrm{Re}(G(t)), \qquad G_2(t) = \mathrm{Im}(G(t)).$$

It is well-known that

$$F'(t) = G(t) \qquad \text{if and only if} \qquad F_1'(t) = G_1(t) \text{ and } F_2'(t) = G_2(t).$$

1) Let $F(t) = \dfrac{1}{1+2i}\, e^{(1+2i)t}$, $t \in \mathbb{R}$.

Find $F'(t)$, and then calculate the integrals

$$\int e^t \cos 2t\, dt, \qquad \int e^t \sin 2t\, dt.$$

2) Let α and β be real numbers fulfilling $\alpha + i\,\beta \neq 0$. Find the integrals

$$\int e^{\alpha t} \cos \beta t\, dt, \qquad \int e^{\alpha t} \sin \beta t\, dt.$$

A. Integral of a mixed trigonometric and exponential expression.

D. Follow the guidelines given in (1), and then split into real and imaginary part. Then copy this method in (2).

I. 1) When $F(t) = \dfrac{1}{1+2i}\, e^{(1+2i)t}$, we get

$$F'(t) = e^{(1+2i)t} = e^t \cos 2t + i\, e^t \sin 2t.$$

On the other hand,

$$
\begin{aligned}
F(t) &= \frac{1 - 2i}{|1 + 2i|^2}\, e^{(1+2i)t} \\
&= \frac{1}{5}\,(1 - 2i)e^t(\cos 2t + i\,\sin 2t) \\
&= \frac{1}{5}\,e^t\{\cos 2t + 2\,\sin 2t\} \\
&\qquad\qquad + i\,\frac{1}{5}\,e^t\{\sin 2t - 2\,\cos 2t\}.
\end{aligned}
$$

Thus by taking the real and imaginary parts,

$$
\begin{aligned}
\int F_1'(t)\,dt &= \int e^t\cos 2t\,dt = F_1(t) \\
&= \frac{1}{5}\,e^t\{\cos 2t + 2\,\sin 2t\}, \\
\int F_2'(t)\,dt &= \int e^t\sin 2t\,dt = F_2(t) \\
&= \frac{1}{5}\,e^t\{\sin 2t - 2\,\cos 2t\}.
\end{aligned}
$$

2) In general we put

$$
\begin{aligned}
F(t) &= \frac{1}{\alpha + i\beta}\, e^{(\alpha + i\beta)t} \\
&= \frac{\alpha - i\beta}{\alpha^2 + \beta^2}\, e^{\alpha t}\{\cos\beta t + i\,\sin\beta t\} \\
&= \frac{1}{\alpha^2 + \beta^2}\, e^{\alpha t}\{\alpha\,\cos\beta t + \beta\,\sin\beta t\} \\
&\qquad\qquad + i\cdot\frac{1}{\alpha^2 + \beta^2}\, e^{\alpha t}\{\alpha\,\sin\beta t - \beta\,\cos\beta t\}.
\end{aligned}
$$

By splitting

$$
F'(t) = e^{(\alpha + i\beta)t} = e^{\alpha t}\cos\beta t + i\,e^{\alpha t}\sin\beta t,
$$

into real and imaginary part we get

$$
\begin{aligned}
\int F_1'(t)\,dt &= \int e^{\alpha t}\cos\beta t\,dt = F_1(t) \\
&= \frac{1}{\alpha^2 + \beta^2}\, e^{\alpha t}\{\alpha\,\cos\beta t + \beta\,\sin\beta t\}, \\
\int F_2'(t)\,dt &= \int e^{\alpha t}\sin\beta t\,dt = F_2(t) \\
&= \frac{1}{\alpha^2 + \beta^2}\, e^{\alpha t}\{\alpha\,\sin\beta t - \beta\,\cos\beta t\}.
\end{aligned}
$$

Example 5.12 *Consider a linear differential equation of first order*

$$\frac{dx}{dt} + a\,x = q(t), \qquad t \in I,$$

where a is a constant. It is well-known that we have a standard solution formula for such a differential equation. We shall in this example show how one in special cases can find a solution by applying the complex exponential.

1) *First consider the differential equation*

   (3) $\dfrac{dx}{dt} + 2\,x = \cos \omega t, \qquad t \in \mathbb{R},$

   where $\omega > 0$ is a constant. Consider the complex equation

   $$\frac{dx}{dt} + 2\,x = e^{i\omega t}, \qquad t \in \mathbb{R}.$$

   Find a solution of the form $x = c \cdot e^{i\omega t}$ of this differential equation and then use this solution to find the complete solution of (3).

2) *Similarly, find the complete solution of*

   $$\frac{dx}{dt} + 2\,x = \sin \omega t, \qquad t \in \mathbb{R}.$$

3) *As an example from the applications we consider a network containing the resistance R, a coil of induction L, and a voltage generator of the voltage $E(t)$ at time t.*
   We shall put $E(t) = E_0 \cos \omega t$.

   It is possible to show that the current intensity $I(t)$ satisfies the differential equation

   (4) $L\dfrac{dI}{dt} + R\,I(t) = E_0 \cos \omega t.$

   Find by means of the complex exponential a solution $I(t) = I_0(t)$ of this differential equation, and then the complete solution.

   In practice one is only interested in $I_0(t)$ and not in the other solutions of (4). Explain why.

A. Linear differential equations of first order of constant coefficients. There are given some guidelines.

D. Follow the guidelines.

I. 1) By insertion of $x = c \cdot e^{i\omega t}$ into the left hand side of (3) we get

   $$\frac{dx}{dt} + 2\,x = c\,i\,\omega\,e^{i\omega t} + 2\,c\,e^{i\omega t} = (2 + i\omega)c\,e^{i\omega t},$$

   which is equal to $e^{i\omega t}$ for $c = \dfrac{1}{2 + i\omega} = \dfrac{2 - i\omega}{4 + \omega^2}$. The coefficients of (3) being real, we conclude that

   $$\mathrm{Re}\left\{\frac{2 - i\omega}{4 + \omega^2}\,e^{i\omega t}\right\}$$

is a solution of (3), hence

$$
\begin{aligned}
x &= \operatorname{Re}\left\{\frac{2 - i\omega}{4 + \omega^2}\, e^{i\omega t}\right\} + c\, e^{-2t} \\
&= \frac{1}{\omega^2 + 4}\left\{2\cos\omega t + \omega\,\sin\omega t\right\} + c\, e^{-2t}, \qquad t \in \mathbb{R}, \quad c \in \mathbb{R},
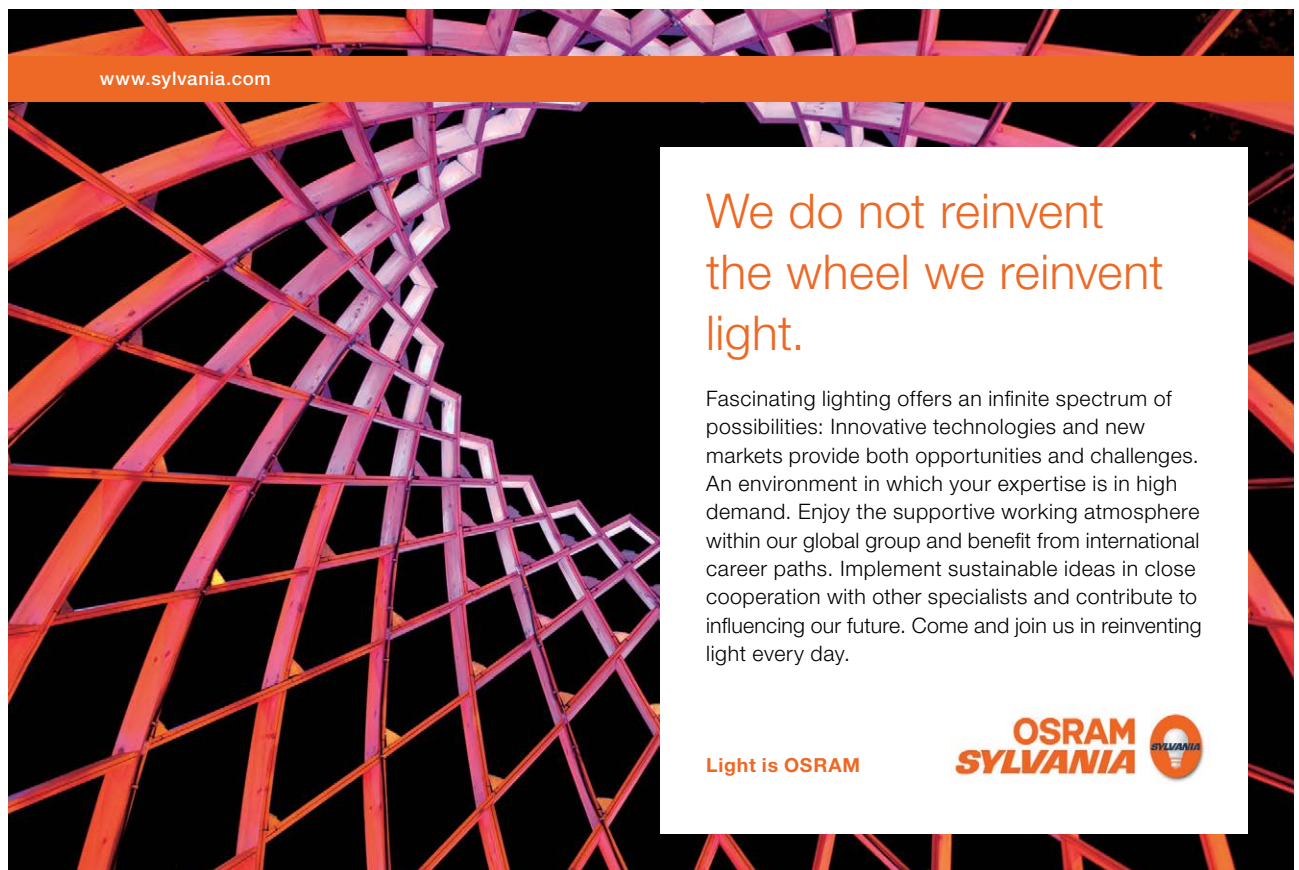\end{aligned}
$$

is the complete solution of (3), as the corresponding homogeneous equation has the complete solution $c \cdot e^{-2t}$, $c \in \mathbb{R}$.

2) From $\operatorname{Im}\left\{e^{i\omega t}\right\} = \sin\omega t$, follows analogously that the complete solution of

$$
\frac{dx}{dt} + 2\,x = \sin\omega t, \qquad t \in \mathbb{R},
$$

is given by

$$
\begin{aligned}
x &= \operatorname{Im}\left\{\frac{2 - i\omega}{\omega^2 + 4}\, e^{i\omega t}\right\} + c\, e^{-2y} \\
&= \frac{1}{\omega^2 + 4}\left\{2\sin\omega t - \omega\,\cos\omega t\right\} + c\, e^{-2t}, \qquad t \in \mathbb{R}, \quad c \in \mathbb{R}.
\end{aligned}
$$

3) Then consider (4), i.e.

$$L\frac{dI}{dt} + R\,I(t) = E_0 \cos \omega t.$$

The corresponding homogeneous equation has the complete solution $c \cdot \exp\left(-\dfrac{R}{L}\,t\right)$, $c \in \mathbb{R}$.

Considering the corresponding complex equation

$$L\frac{dI}{dt} + R\,I(t) = E_0 e^{i\omega t},$$

we get by insertion of $I(t) = c\,e^{i\omega t}$ that

$$Lc \cdot i\omega\,e^{i\omega t} + Rc\,e^{i\omega t} = (R + i\omega L)c\,e^{i\omega t} = E_0 e^{i\omega t},$$

which is fulfilled for

$$c = \frac{E_0}{R + i\omega L} = E_0 \cdot \frac{R - i\omega L}{R^2 + \omega^2 L^2}.$$

Hence, a particular solution of (4) is e.g.

$$
\begin{aligned}
x &= \frac{E_0}{R^2 + \omega^2 L^2}\,\mathrm{Re}\left\{(R - i\omega L)e^{i\omega t}\right\} \\
&= \frac{E_0}{R^2 + \omega^2 L^2}\,\{R\cos\omega t + \omega L \sin\omega t\},
\end{aligned}
$$

and the complete solution of (4) is then

$$x = \frac{E_0}{R^2 + \omega^2 L^2}\,\{R\cos\omega t + \omega L\,\sin\omega t\} + c \cdot \exp\left(-\frac{R}{L}\,t\right), \qquad t \in \mathbb{R},$$

where $c \in \mathbb{R}$ is an arbitrary constant.

Now, $\dfrac{R}{L} > 0$, so the term $c \cdot \exp\left(-\dfrac{R}{L}\,t\right) \to 0$ for $t \to +\infty$. It remains the pure oscillation given by

$$I_0(t) = \frac{E_0}{R^2 + \omega^2 L^2}\,\{R\cos\omega t + \omega L\,\sin\omega t\}, \qquad t \in \mathbb{R}.$$

Example 5.13 *In this example we show that if a complex function $h(z) \neq 0$ satisfies the two properties:*

K 1. $h(z_1 + z_2) = h(z_1) \cdot h(z_2)$ *for every z_1, $z_2 \in \mathbb{C}$,*

K 2. $\dfrac{d}{dt}\{h(R \cdot t)\} = R \cdot h(R \cdot t)$ *for every $R \in \mathbb{C}$ and $t \in \mathbb{R}$,*

then er $h(z)$ is equal to the complex exponential. We therefore assume that $h(z)$ satisfies **K 1** *and* **K 2**.

1) Apply **K 1** *to show that $h(0) = 1$.*

2) Apply **K 2** *and (1) to show that*

$$h(t) = e^t \qquad for\ t \in \mathbb{R}.$$

3) Put $h(i \cdot t) = \alpha(t) + i\,\beta(t)$. Apply **K 2** *to prove that*

$$\alpha'(t) = -\beta(t) \quad and \quad \beta'(t) = \alpha(t) \qquad for\ t \in \mathbb{R}.$$

4) Apply (1) and (3) to prove that

$$h(i\,t) = \cos t + i\,\sin t \qquad for\ t \in \mathbb{R}.$$

5) Finally, apply **K 1**, *(2) and (4) to prove that*

$$h(x + i\,y) = e^x\{\cos y + i\,\sin y\} \qquad for\ all\ x,\ y \in \mathbb{R}.$$

A. A theoretical derivation of the exponential. There are given some guidelines.

D. Follow the guidelines.

I. 1) If we choose $z_1 = z_2 = 0$, then it follows from **K 1** that

$$h(0) = h(0 + 0) = h(0) \cdot h(0) = h(0)^2,$$

so we have either $h(0) = 0$ or $h(0) = 1$.

If $h(0) = 0$, then it follows from **K 1** that

$$h(z) = h(z + 0) = h(z) \cdot h(0) = 0,$$

hence $h \equiv 0$, which was excluded by the assumptions. Therefore, $h(0) = 1$.

2) Then let $R = 1$. It follows from **K 2** that $h(t)$, $t \in \mathbb{R}$, satisfies the differential equation

$$\frac{dh}{dt} = h(t),$$

with the complete solution $h(t) = c \cdot e^t$. Then it follows from (1) that $h(0) = 1 = c \cdot e^0 = c$, hence $c = 1$, or

$$h(t) = e^t, \qquad t \in \mathbb{R}.$$

3) According to **K 2**,

$$\frac{d}{dt}\, h(i \cdot t) = i\, h(i\, t),$$

which can also be written

$$\alpha'(t) + i\,\beta'(t) = i\{\alpha(t) + i\,\beta(t)\} = -\beta(t) + i\,\alpha(t).$$

By a separation into real and imaginary parts we get

$$\alpha'(t) = -\beta(t), \quad \beta'(t) = \alpha(t), \qquad \text{for } t \in \mathbb{R}.$$

4) It follows from (3) that $\alpha(t)$ and $\beta(t)$ both satisfy the differential equation

(5) $\dfrac{d^2 x}{dt^2} + x = 0.$

They must therefore be among the solutions of (5). The characteristic polynomial $R^2 + 1$ has the roots $\pm i$, so the complete solution is

$$c_1 \cos t + c_2 \sin t.$$

From (1) we get $h(0) = 1 = \alpha(0) + i\,\beta(0)$, hence

$$\alpha(0) = 1 = c_1 \cdot 1 + c_2 \cdot 0,$$

i.e. $\alpha(t) = \cos t + c_2 \sin t$. Then it follows from this and from (3) that

$$\beta(t =) = -\alpha'(t) = -(-\sin t) - c_2 \cdot \cos t = \sin t - c_2 \cos t.$$

From $\beta(0) = 0 = 0 - c_2$ we get $c_2 = 0$, so

$$\alpha(t) = \cos t \qquad \text{and} \qquad \beta(t) = \sin t.$$

All things put together we have

$$h(i\, t) = \cos t + i\,\sin t, \qquad t \in \mathbb{R}.$$

5) Finally,

$$h(x + i\, y) = h(x) \cdot h(i\, y) = e^x (\cos y + i\,\sin y),$$

where we in the first equality have used **K 1**, and in the second one, (2) and (4).

Example 5.14 *Find all solutions of the equation*

$$e^{iz} - 8i = 0.$$

A. Complex equation.

D. The equation is rearranged, and then the result follows by taking the complex logarithm.

I. From

$$e^{iz} = 8i = \exp\left(\ln 8 + i\left\{\frac{\pi}{2} + 2p\pi\right\}\right), \qquad p \in \mathbb{Z},$$

we get

$$
\begin{aligned}
z &= \frac{1}{i}\left(\ln 8 + i\left\{\frac{\pi}{2} + 2p\pi\right\}\right) \\
  &= \frac{\pi}{2} + 2p\pi - i\ln 8, \qquad p \in \mathbb{Z}.
\end{aligned}
$$

Example 5.15 *Find modulus and principal argument of the complex number*

$$A = -\frac{(1+i)^7}{(\sqrt{3}-i)^4}.$$

A. Modulus and principal argument.

D. Consider the numerator and the denominator separately, and then write them in polar coordinates.

I. If the sign in front of the fraction is incorporated in the numerator, this can be written as

$$
\begin{aligned}
-(1+i)^7 &= e^{i\pi} \cdot \left\{ \sqrt{2} \cdot \exp\left(i\frac{\pi}{4} \right) \right\}^7 = (\sqrt{2})^7 \cdot \exp\left(i\pi + i\frac{7\pi}{4} \right) \\
&= 8\sqrt{2} \cdot \exp\left(i\frac{11\pi}{4} \right) = 8\sqrt{2} \cdot \exp\left(i\frac{3\pi}{4} \right).
\end{aligned}
$$

The denominator is written

$$(\sqrt{3}-i)^4 = \left\{ 2 \cdot \exp\left(-i\frac{\pi}{6} \right) \right\}^4 = 2^4 \cdot \exp\left(-i\frac{2\pi}{3} \right),$$

thus

$$
\begin{aligned}
A &= \frac{8\sqrt{2}\,\exp\left(i\dfrac{3\pi}{4} \right)}{16 \cdot \exp\left(-i\dfrac{2\pi}{3} \right)} = \frac{1}{\sqrt{2}} \cdot \exp\left(i\left\{ \frac{3\pi}{4} + \frac{2\pi}{3} \right\} \right) \\
&= \frac{1}{\sqrt{2}} \cdot \exp\left(i\frac{17\pi}{12} \right).
\end{aligned}
$$

It follows that the modulus is

$$|A| = \frac{1}{\sqrt{2}},$$

and that the principal argument is

$$\frac{17\pi}{12} - 2\pi = -\frac{7\pi}{12} \in \,]-\pi, \pi[.$$

Example 5.16 *Find modulus and principal argument of the complex number*

$$A = \frac{(1 - i\sqrt{3})^8}{(\sqrt{3}-i)^{10}}.$$

A. Modulus and principal argument of a fraction.

D. Consider numerator and denominator separately, and write them in polar coordinates.

I. The numerator is

$$(1 - i\sqrt{3})^8 = \left\{ 2 \exp\left(-i\frac{\pi}{3}\right) \right\}^8 = 2^8 \exp\left(-i\frac{8\pi}{3}\right) = 2^8 \exp\left(-i\frac{2\pi}{3}\right).$$

The denominator is

$$(\sqrt{3} - i)^{10} = \left\{ 2 \exp\left(-i\frac{\pi}{6}\right) \right\}^{10} = 2^{10} \exp\left(-i\frac{5\pi}{3}\right) = 2^{10} \exp\left(i\frac{\pi}{3}\right).$$

Then by insertion,

$$A = \frac{(1 - i\sqrt{3})^8}{(\sqrt{3} - i)^{10}} = \frac{2^8 \exp\left(-\dfrac{2\pi}{3}\right)}{2^{10} \exp\left(i\dfrac{\pi}{3}\right)} = \frac{1}{4} \exp(-i\pi) = -\frac{1}{4} = \frac{1}{4} \exp(i\pi).$$

Thus, the modulus is $|A| = \dfrac{1}{4}$, and the principal argument is Arg $A = \pi$.

Example 5.17 *For which real t is the numbers e^{-it} and $1 + e^{2it}$ complex conjugated?*

A. Conjugated numbers.

D. Since $\overline{e^{-it}} = e^{it}$, when $t \in \mathbb{R}$ is real, it is easy to set up the equation.

I. As mentioned above, the conjugated of e^{-it} is e^{it}. Thus, we get the condition

$$e^{it} = 1 + e^{2it}, \qquad \text{i.e.} \qquad \left(e^{it}\right)^2 - e^{it} + 1 = 0.$$

This is an equation of second degree in e^{it} with the roots

$$e^{it} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - 1} = \frac{1}{2}\left(1 \pm i\sqrt{3}\right) = \exp\left(i\left\{\pm\frac{\pi}{3} + 2p\pi\right\}\right),$$

hence

$$t = \pm\frac{\pi}{3} + 2p\pi, \qquad p \in \mathbb{Z}.$$

Example 5.18 *1) Find the solutions $t \in \mathbb{R}$ of each of the two equations*

$$(a) \quad e^{2it} - 1 = 2i \cdot e^{it}, \qquad (b) \quad e^{2it} - 1 = 2e^{it}.$$

2) Sketch the curves

$$2e^{it}, \quad 2ie^{it}, \quad e^{2it} - 1, \qquad t \in [0, 2\pi[,$$

in the complex plane, and compare with the results from (1).

A. Complex equations and curves.

D. Solve the two equations of second degree in e^{it}.

I. 1) a) By a rearrangement,

$$0 = e^{2it} - 2ie^{it} - 1 = (e^{it})^2 - 2 \cdot i \cdot e^{it} + i^2 = \left(e^{it} - i \right)^2,$$

hence the solutions are identical with the solutions of

$$e^{it} = i = \exp\left(i \left\{ \frac{\pi}{2} + 2p\pi \right\} \right), \qquad p \in \mathbb{Z}.$$
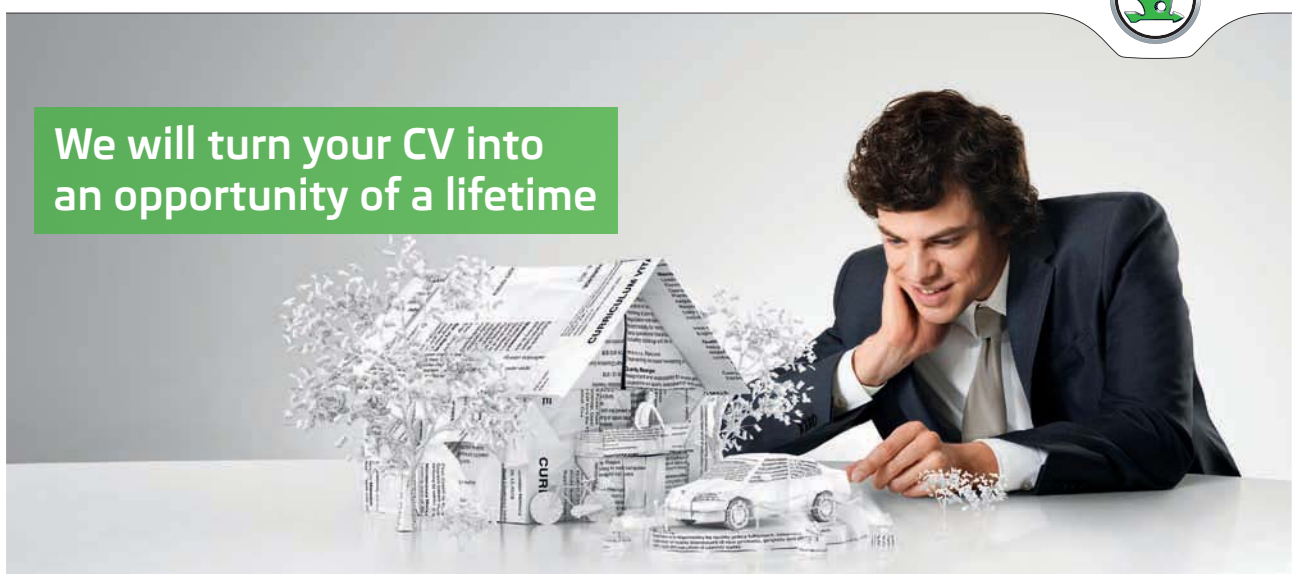
This gives

$$t = \frac{\pi}{2} + 2p\pi, \qquad p \in \mathbb{Z}.$$

b) Here we get by a rearrangement

$$0 = e^{2it} - 2\,e^{it} - 1 = (e^{it})^2 - 2\,e^{it} + 1 - 2) = \left(e^{it} - 1\right)^2 - 2,$$

hence

$$e^{it} = 1 \pm \sqrt{2} = \begin{cases} 1 + \sqrt{2}, \\ (\sqrt{2} - 1)\,e^{i\pi} = \dfrac{1}{\sqrt{2} + 1}\,e^{i\pi}. \end{cases}$$

If we take the complex logarithm and then divide by i, we either get

$$t = 2p\pi - i\,\ln(1 + \sqrt{2}), \qquad p \in \mathbb{Z},$$

or

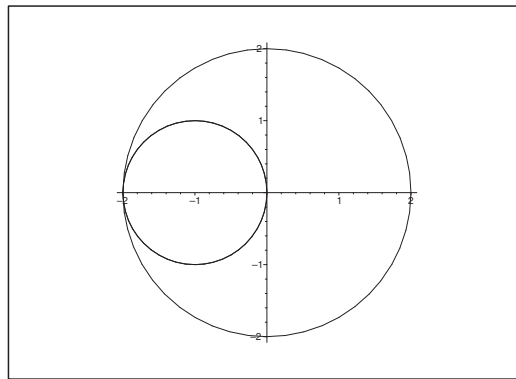$$t = \pi + 2p\pi + i\,\ln(1 + \sqrt{2}), \qquad p \in \mathbb{Z}.$$

Figure 22: The circle of centre $(0,0)$ is the image for both $2e^{it}$ and $2ie^{it}$. The smaller circle is the image of $e^{2it} - 1$. The only point in common is $(-2, 0)$.

c) The first two curves are as *point sets* (but not as curves) identical with the circle of centre $(0,0)$ and radius 2. The smaller circle represents the curve $e^{2it} - 1$. We note that the only common point of the two circle is $(-2, 0)$.
   If $t = \dfrac{\pi}{2} + 2p\pi$, $p \in \mathbb{Z}$, then $e^{it} = i$, hence

$$e^{2it} - 1 = i^2 - 1 = -2, \quad \text{og} \quad 2i\,e^{it} = 2i \cdot i = -2.$$

If

$$t = 2p\pi - i\,\ln(1 + \sqrt{2}), \qquad p \in \mathbb{Z},$$

then

$$2e^{it} = 2(1 + \sqrt{2}) \quad \text{og} \quad e^{2it} - 1 = (1 + \sqrt{2})^2 - 1 = 3 + 2\sqrt{2} - 1 = 2(1 + \sqrt{2}).$$

If

$$t = \pi + 2p\pi + i \ln(1 + \sqrt{2}), \qquad p \in \mathbb{Z},$$

then

$$2e^{it} = 2\,\frac{-1}{\sqrt{2}+1} = 2\{1 - \sqrt{2}\},$$

and

$$e^{2it} - 1 = \{1 - \sqrt{2}\}^2 - 1 = 3 - 2\sqrt{2} - 1 = 2\{1 - \sqrt{2}\}.$$

Example 5.19 .

1) Find the position in the complex plane of the complex numbers A, for which $Re(A^2) = 0$.

2) For which complex numbers z do we have

$$e^{iz} = 2 + 2i\sqrt{3}?$$

A. Complex numbers and equations.

D. In (1) we write $A = x + iy$. In (2) we use polar coordinates.

I. 1) Put $A = x + iy$. Then $A^2 = x^2 - y^2 + 2ixy$, thus

$$0 = \ \mathrm{Re}\left(A^2\right) = x^2 - y^2,$$

and the set of solutions is the union of the two straight lines $y = x$ and $y = -x$.
REMARK. It is also possible to find the solutions by using polar coordinates, $z = r\exp(i\varphi)$, because $z^2 = r^2 \exp(2\varphi\, i)$ is purely imaginary. This gives us the conditions

$$2\varphi = \pm\frac{\pi}{2} + 2p\pi,$$

from which

$$\varphi = \pm\frac{\pi}{4} + p\pi, \qquad p \in \mathbb{Z},$$

etc., where we still have to discuss every possibility. I judge here that the polar version is more sophisticated than the rectangular one. ◊

2) We have in polar coordinates

$$e^{2iz} = 2 + 2i\sqrt{3} = 4\left\{\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right\} = 4 \cdot \exp\left(i\,\frac{\pi}{3}\right),$$

hence

$$2iz = 2\ln 2 + i\left\{\frac{\pi}{3} + 2p\pi\right\}, \qquad p \in \mathbb{Z},$$

and the set of solutions is

$$z = \frac{\pi}{6} + p\pi - i\ln 2, \qquad p \in \mathbb{Z}.$$

6   Roots in a polynomial

Example 6.1 *Let*

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

be any complex polynomial.

1) Show that there exist two polynomials $Q(z)$ and $R(z)$ of real coefficients, such that

$$P(z) = Q(z) + i\,R(z), \qquad z \in \mathbb{C}.$$

2) Show that if x_0 is a real p-tuple root in both $Q(z)$ and $R(z)$, then x_0 is also a p-tuple root in $P(z)$. Do we also have a converse result?

3) Prove that if $P(z)$ has real roots, then these are common roots of the two real polynomials $Q(z)$ and $R(z)$.

4) Solve the equation

$$z^5 - 2z^4 + z^3 - iz^2 + 2iz - i = 0.$$

A. Real roots in complex polynomials.

D. There is a guideline, and finally the theory is applied in (4).

I. 1) This is a trivial statement,

$$\begin{aligned}
P(z) &= a_n z^n + \cdots + a_1 z + a_0 \\
&= (\{\operatorname{Re} a_n\} z^n + \cdots + \{\operatorname{Re} a_1\} z + \operatorname{Re} a_0) \\
&\quad + i(\{\operatorname{Im} a_n\} z^n + \cdots + \{\operatorname{Im} a_1\} z + \operatorname{Im} a_0) \\
&= Q(z) + i\,R(z).
\end{aligned}$$

2) According to the assumption,

$$Q(z) = (z - x_0)^p Q_1(z) \quad \text{og} \quad R(z) = (z - x_0)^p R_1(z),$$

så

$$P(z) = (z - x_0)^p \{Q_1(z) + i\,R_1(z)\}.$$

It follows immediately that x_0 is a p-tuple root in $P(z)$.

Assume conversely that

$$P(z) = (z - x_0)^p P_1(z) = Q(z) + i\,R(z).$$

If $z = x$ is real, then we immediately get

$$(x - x_0)^p \operatorname{Re} P_1(x) = Q(x) \quad \text{and} \quad (x - x_0)^p \operatorname{Im} P_1(x) = R(x),$$

and x_0 is a p-tuple root in the two *real* polynomials $Q(x)$ and $R(x)$. Since the coefficients are not changed, when x is replaced by z, this also holds for $Q(z)$ and $R(z)$.

REMARK. It is here *important* that x_0 is real. If $z_0 \in \mathbb{C} \setminus \mathbb{R}$ is a p-tuple root, the result does not hold in general.
EXAMPLE. If $P(z) = z - i$, then $Q(z) = z$ and $R(z) = -1$. It is obvious that $z = i$ is a root of $P(z)$, and of course neither a root in $Q(z)$ nor in $R(z)$. $\Diamond$

3) This follows immediately from (2).

4) If we split the polynomial as above, we get

$$\begin{aligned}
P(z) &= z^5 - 2z^4 + z^3 - iz^2 + 2iz - i \\
&= \left(z^5 - 2z^4 + z^3\right) - i\left(z^2 - 2z + 1\right) \\
&= \left(z^3 - i\right)\left(z^2 - 2z + 1\right) = (z - i)\left(z^2 - iz - i\right)(z - 1)^2.
\end{aligned}$$

Here, $z^2 - iz - i$ has formally the roots $z = \dfrac{1}{2}\left(i \pm \sqrt{-1 + 4i}\right)$. If we put $\sqrt{-1 + 4i} = a + ib$, we get

$$a^2 - b^2 = -1 \qquad \text{og} \qquad 2ab = 4,$$

thus

$$\left(a^2 + b^2\right)^2 = \left(a^2 - b^2\right)^2 + (2ab)^2 = 1 + 16 = 17,$$

and hence

$$a^2 + b^2 = \sqrt{17}, \qquad a^2 - b^2 = -1, \quad \text{and} \quad ab > 0,$$

i.e.

$$a = \pm\sqrt{\frac{\sqrt{17}-1}{2}}, \qquad b = \pm\sqrt{\frac{\sqrt{17}+1}{2}}, \qquad ab > 0.$$

Then

$$z = \frac{1}{2}\left\{ i \pm \left(\sqrt{\frac{\sqrt{17}-1}{2}} + i\sqrt{\frac{\sqrt{17}+1}{2}} \right) \right\}.$$

Thus the five solutions of $P(z) = 0$ are

$$z = 1 \quad \text{(double root)}, \qquad\qquad z = i,$$

and

$$z = \frac{1}{2}\sqrt{\frac{\sqrt{17}-1}{2}} + i\,\frac{1}{2}\left\{ 1 + \sqrt{\frac{\sqrt{17}+1}{2}} \right\},$$

$$z = -\frac{1}{2}\sqrt{\frac{\sqrt{17}-1}{2}} - i\,\frac{1}{2}\left\{ \sqrt{\frac{\sqrt{17}+1}{2}} - 1 \right\}.$$

Example 6.2 *Let the polynomial $P(x)$ be given by*

$$P(x) = x^4 - 2x^3 - x^2 + 2x + 10.$$

It is given that $P(x)$ has a root of the form $\lambda(1+i)$, $\lambda \in \mathbb{R}$.

1) Find $\lambda \in \mathbb{R}$, such that $P(\lambda(1+i)) = 0$.

2) Find all the complex roots of $P(x)$.

A. Complex roots of a polynomial.

D. Put $x = \lambda(1+i)$ and find $\lambda \in \mathbb{R}$. Then exploit that $\lambda(1-i)$ must also be a root.

I. 1) When $x = \lambda(1+i)$, $\lambda \in \mathbb{R}$, we get

$$\begin{aligned}
P(\lambda(1+i)) &= \lambda^4(1+i)^4 - 2\lambda^3(1+i)^3 - \lambda^2(1+i)^2 + 2\lambda(1+i) + 10 \\
&= \lambda^4(2i)^2 - 2\lambda^3 \cdot 2i(1+i) - \lambda^2 \cdot 2i + 2\lambda + 2i\lambda + 10 \\
&= -4\lambda^4 + 4\lambda^3 + 2\lambda + 10 + i\left\{ -4\lambda^3 - 2\lambda^2 + 2\lambda \right\}.
\end{aligned}$$

This expression is zero, if and only if both the real part and the imaginary part are zero, i.e. if and only if

$$2\lambda^4 - 2\lambda^3 - \lambda - 5 = 0 \text{ and } 2\lambda^3 + \lambda^2 - \lambda = \lambda(2\lambda^2 + \lambda - 1) = 0.$$

From the latter equation we get the possibilities

$$\lambda = 0, \qquad \lambda = -1 \quad \text{and} \quad \lambda = \frac{1}{2}.$$

Of these only $\lambda = -1$ also fulfils the first equation. We can only conclude that $z = -1 - i$ is a root.

2) Since the polynomial has real coefficients, and $z = -1 - i$ is a root, we must have that $z = -1 + i$ is also a root, hence

$$(x + 1)^2 + 1 = x^2 + 2x + 2$$

must be a divisor in $P(x)$. By division we get

$$P(x) = x^4 - 2x^3 - x^2 + 2x + 10 = (x^2 + 2x + 2)(x^2 - 4x + 5).$$

Finally we conclude that the roots are

$$x = -1 \pm i \qquad \text{og} \qquad x = 2 \pm i.$$

Example 6.3 *1) Sketch the set S of complex numbers z, which satisfy the inequality*

$$|z - (2 + i)| \leq 3.$$

Consider the polynomial

$$P(z) = z^4 + 2z^3 - z^2 + 2isz + (11 + s^2),$$

where s is a complex constant.

2) Determine the values of s, for which $z = 1 + i$ is a root of $P(z)$, and mark them on the figure in (1).

3) Find for $s \in S$ all complex roots in $P(z)$.

A. Complex numbers, and a polynomial containing a complex parameter.

D. Apply the geometric interpretation; insert the root and solve the equation in s.

I. 1) The set S of complex numbers z, satisfying

$$|z - (2 + i)| \leq 3,$$

is the closed disc of centre $z_0 = 2 + i$ and radius 3.

2) Insertion of the root $z = 1 + i$ in $P(z)$ gives the equation of second degree in s:

$$
\begin{aligned}
0 &= (1 + i)^4 + 2(1 + i)^3 - (1 + i)^2 + 2i(1 + i)s + 11 + s^2 \\
&= (2i)^2 + 2(1 + i) \cdot 2i - 2i + 2(-1 + i)s + 11 + s^2 \\
&= -4 - 4 + 2i - 2(1 - i)s + 11 + s^2 \\
&= s^2 - 2(1 - i)s + (1 - i)^2 - (1 - i)^2 + 3 + 2i \\
&= \{s - (1 - i)\}^2 + 3 + 4i \\
&= \{s - (1 - i)\}^2 + (2 + i)^2 \\
&= \{s - (1 - i)\}^2 - (1 - 2i)^2.
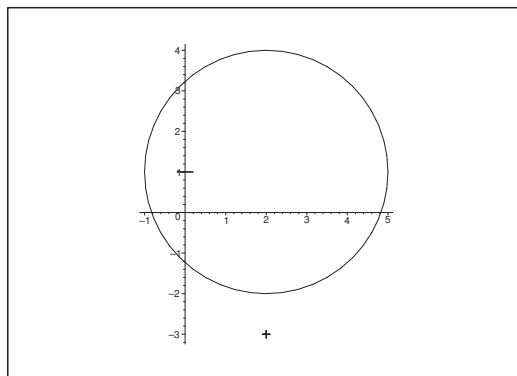\end{aligned}
$$

Figure 23: The domain S and the two points $s = i$ and $s = 2 - 3i$.

Hence,

$$s = 1 - i \pm (1 - 2i) = \begin{cases} 2 - 3i, \\ \\ i. \end{cases}$$

It follows from $|i - (2 + i)| = 2 < 3$ and $|2 - 3i - (2 + i)| = 4 > 3$ that only $s = i$ belongs to the set S.

3) When $s = i \ (\in S)$, then

$$P(z) = z^4 + 2z^3 - z^2 - 2z + 10$$

is a polynomial of real coefficients. We know already that $z = 1 + i$ is a root, thus $\overline{z} = 1 - i$ must also be a root, corresponding to the fact that

$$(z - 1 - i)(z - 1 + i) = z^2 - 2z + 2$$

is a divisor in $P(z)$. By division we get

$$\begin{aligned} P(z) &= z^4 + 2z^3 - z^2 - 2z + 10 \\ &= (z^2 - 2z + 2)(z^2 + 4z + 5), \end{aligned}$$

and the roots are

$$z = 1 \pm i \qquad \text{and} \qquad z = -1 \pm i.$$

Example 6.4 *1) Find all complex roots of the polynomial*

$$P(z) = z^7 + z^6 + 8z + 8.$$

2) Let w_1 and w_2 denote the two non-real roots of numerically smallest principal argument. Write w_1 and w_2 in the form $a + ib$ (a, $b \in \mathbb{R}$), and calculate $w_1 + w_2$ and $w_1 w_2$.

3) Write the polynomial $P(z)$ as a product of four real polynomials, where one is of first degree, and the others are of second degree.

A. Polynomial with complex roots.

D. Find the roots by going through the succession (3), (1), (2) of the questions above.

I. 3. We get immediately the factorization

$$
\begin{aligned}
P(z) &= z^7 + z^6 + 8z + 8 \\
&= (z+1)(z^6 + 8) \\
&= (z+1)(z^2 + 2)(z^4 - 2z^2 + 4) \\
&= (z+1)(z^2 + 2)(z^4 + 4z^2 + 4 - 6z^2) \\
&= (z+1)(z^2)\{(z^2 + 2)^2 - (\sqrt{6}\, z)^2\} \\
&= (z+1)(z^2 + 2)(z^2 + \sqrt{6}\, z + 2)(z^2 - \sqrt{6}\, z + 2).
\end{aligned}
$$

1. It follows from the factorization that the roots are

$$z = -1, \quad z = \pm i\sqrt{2}, \quad z = \frac{1}{2}\left(-\sqrt{6} \pm i\sqrt{2}\right), \quad z = \frac{1}{2}\left(\sqrt{6} \pm i\sqrt{2}\right).$$

2. The meaning of the example is here that we shall solve the binomial equation

$$z^6 = -8 = 2^3 e^{i\pi}$$

in polar coordinates, where $z = -1$ has the principal argument π. Thus we get

$$z = \sqrt{2} \cdot \exp\left(i\,\frac{\pi}{6} + ip\,\frac{\pi}{3}\right), \qquad p = 0,\ 1,\ 2,\ 3,\ 4,\ 5.$$

The two non-real roots of numerically smallest principal values are

$$w_1 = \sqrt{2} \cdot \left\{\cos\frac{\pi}{6} + i\,\sin\frac{\pi}{6}\right\} = \frac{\sqrt{6}}{2} + i\,\frac{\sqrt{2}}{2},$$

$$w_2 = \sqrt{2} \cdot \left\{\cos\frac{\pi}{6} - i\,\sin\frac{\pi}{6}\right\} = \frac{\sqrt{6}}{2} - i\,\frac{\sqrt{2}}{2}.$$

We immediately get from this that

$$w_1 + w_2 = \sqrt{6} \quad \text{and} \quad w_1 \cdot w_2 = \sqrt{2} \cdot e^{i\frac{\pi}{6}} \cdot \sqrt{2}\, e^{-i\frac{\pi}{6}} = 2,$$

so w_1 and w_2 are the roots of the polynomial $z^2 - \sqrt{6}\, z + 2$, which also can be seen from the factorization in (3).

3. Just in case, we calculate (3) once more, because the original task was the following: According to the above, $z^2 - \sqrt{6}\, z + 2$ is a divisor, and it is easy to see that $z^2 + 2$ is also a divisor. Then by division by polynomials,

$$P(z) = (z+1)(z^2 + 2)(z^2 + \sqrt{6}\, z + 2)(z^2 - \sqrt{6}\, z + 2).$$

Example 6.5 *1) Let $A = \sqrt{6} + \sqrt{2} + i(\sqrt{6} - \sqrt{2})$, and let $B = A^2$. Write B in the form $a + ib$, a, $b \in \mathbb{R}$, and find modulus and argument of B.*

2) Find all complex solutions of the equation

$$z^2 = 8(\sqrt{3} + i).$$

3) Show that

$$\cos\frac{\pi}{12} = \frac{\sqrt{6} + \sqrt{2}}{4}.$$

A. Complex numbers.

D. Follow the guidelines given above.

I. 1) When $A = (\sqrt{6} + \sqrt{2}) + i(\sqrt{6} - \sqrt{2})$, then

$$\begin{aligned} B &= A^2 = (\sqrt{6} + \sqrt{2})^2 - (\sqrt{6} - \sqrt{2})^2 + 2i(\sqrt{6} + \sqrt{2})(\sqrt{6} - \sqrt{2}) \\ &= 4\sqrt{12} + 2i(6 - 2) = 8\sqrt{3} + 8i, \end{aligned}$$

which we also can write in the following way

$$B = 8(\sqrt{3} + i) = 16\left(\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\right) = 16\left\{\cos\frac{\pi}{6} + i\,\sin\frac{\pi}{6}\right\} = 16 \cdot \exp\left(i\,\frac{\pi}{6}\right).$$

Thus, B has modulus 16 and argument $\dfrac{\pi}{6}$.

2) It follows from (1) that the solutions os

$$z^2 = 8(\sqrt{3} + i) = B = A^2,$$

are

$$z = \pm A = \begin{cases} \sqrt{6} + \sqrt{2} + i(\sqrt{6} - \sqrt{2}) &= 4\exp\left(i\,\dfrac{\pi}{12}\right), \\ -\sqrt{6} - \sqrt{2} - i(\sqrt{6} - \sqrt{2}) &= -4\exp\left(i\,\dfrac{\pi}{12}\right) \\ &= 4\exp\left(i\,\dfrac{13\pi}{12}\right). \end{cases}$$

3) Now,

$$A = (\sqrt{6} + \sqrt{2}) + i(\sqrt{6} - \sqrt{2}) = 4\exp\left(i\,\frac{\pi}{12}\right) = 4\left\{\cos\frac{\pi}{12} + i\,\sin\frac{\pi}{12}\right\},$$

so by a splitting into the real part and the imaginary part, followed by a division by 4, we get

$$\cos\frac{\pi}{12} = \frac{\sqrt{6} + \sqrt{2}}{4} \qquad \text{and} \qquad \sin\frac{\pi}{12} = \frac{\sqrt{6} - \sqrt{2}}{4}.$$

REMARK. *Alternatively* one can here use the well-used old fashioned trigonometry. From

$$\cos 2v = 2\cos^2 v - 1 \qquad \text{and} \qquad \cos\frac{\pi}{12} > 0$$

follows that

$$
\begin{aligned}
\cos\frac{\pi}{12} &= +\sqrt{\frac{1}{2}\left(1+\cos\frac{\pi}{6}\right)} = \sqrt{\frac{1}{2}\left(1+\frac{\sqrt{3}}{2}\right)} \\
&= \frac{1}{2}\sqrt{2+\sqrt{3}} = \frac{1}{4}\sqrt{8+4\sqrt{3}} \\
&= \frac{1}{4}\sqrt{6+2+2\sqrt{12}} \\
&= \frac{1}{4}\sqrt{(\sqrt{6})^2+(\sqrt{2})^2+2\cdot\sqrt{6}\cdot\sqrt{2}} \\
&= \frac{\sqrt{6}+\sqrt{2}}{4}.
\end{aligned}
$$

Example 6.6 *Let*

$$P(z) = z^4 + 2z^3 + 3z^2 + 2z + 2.$$

We are given the information that $z = i$ is a root of $P(z)$.

Prove that $P(z)$ can be written as a product of two polynomials of second degree, and then find all roots of $P(z)$.

A. Factorization of a polynomial of fourth degree.

D. Solve either by inspection or by using the information that $z = i$ is a root.

I. First variant. By inspection we get that

$$
\begin{aligned}
P(z) &= z^4 + 2z^3 + 3z^2 + 2z + 2 \\
&= (z^4 + 2z^3 + 2z^2) + (z^2 + 2z + 2) \\
&= (z^2 + 1)(z^2 + 2z + 2),
\end{aligned}
$$

and we see that the four roots are

$$i, \qquad -i, \qquad -1 + i, \qquad -1 - i.$$

Second variant. Since $z = i$ is a root, and $P(z)$ has real coefficients, we also have that $z = -i$ is a root, so

$$(z - i)(z + i) = z^2 + 1$$

is a divisor in $P(z)$. By division we get

$$P(z) = (z^2 + 1)(z^2 + 2z + 2),$$

and it follows again that the four simple roots are

$$i, \qquad -i, \qquad -1 + i, \qquad -1 - i.$$

Example 6.7 *Find all complex solutions of the equation*

$$e^{2z} - 2e^z + 2 = 0.$$

A. An equation of second order in e^z in disguise.

D. Solve the equation of second order in e^z, and then find z.

I. When $w = e^z$, the equation is written

$$w^2 - 2w + 2 = (w - 1)^2 + 1 = 0,$$

thus

$$w = e^z = 1 \pm i = \sqrt{2} \exp\left(i\left\{\pm\frac{\pi}{4} + 2p\pi\right\}\right), \qquad p \in \mathbb{Z}.$$

Hence

$$z = \frac{1}{2}\ln 2 + i\left\{\pm\frac{\pi}{4} + 2p\pi\right\}, \qquad p \in \mathbb{Z}.$$

Example 6.8 *Find all complex solutions of the equation*

$$e^{2z} - 2e^z + 4 = 0.$$

A. An equation of second degree in $w = e^z$ in disguise.

D. Solve the equation of second degree in w, and then find z.

I. When $w = e^z$, the equation is written

$$w^2 - 2w + 4 = (w - 1)^2 + 3 = 0.$$

The roots are

$$w = e^z = 1 \pm i\sqrt{3} = 2\left\{\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right\} = 2\,\exp\left(i\left\{\pm\frac{\pi}{3} + 2p\pi\right\}\right), \quad p \in \mathbb{Z},$$

hence

$$z = \ln 2 + i\left\{\pm\frac{\pi}{3} + 2p\pi\right\}, \qquad p \in \mathbb{Z}.$$

Example 6.9 *Find all $t \in \mathbb{R}$, for which $1 + e^{2it} = e^{it}$.*

A. Complex solutions of a non-linear equation.

D. Consider the equation as an equation of second degree in e^{it}.

I. When $w = e^{it}$, the equation is written

$$w^2 - w + 1 = 0,$$

with the solutions

$$
\begin{aligned}
e^{it} &= w = \frac{1}{2} \pm \sqrt{\frac{1}{4} - 1} = \frac{1 \pm i\sqrt{3}}{2} = \exp\left(\pm i\,\frac{\pi}{3}\right) \\
&= \exp\left(i\left\{\pm\frac{\pi}{3} + 2p\pi\right\}\right),
\end{aligned}
$$

thus

$$t = \pm\frac{\pi}{3} + 2p\pi, \qquad p \in \mathbb{Z}.$$

Example 6.10 *Given the equation $z^3 + z - 10 = 0$. Obviously, this equation has a real rod. Find this real root, and then find all roots.*

A. Equation of third degree with a real root.

D. From the factorization $10 = 1 \cdot 2 \cdot 5$ follows that the only possible rational roots are ± 1, ± 2, ± 5 and ± 10. We see immediately that $z = 2$ is a root. Then

$$z^3 + z - 10 = (z - 2)(z^2 + 2z + 5) = (z - 2)\left\{(z + 1)^2 + 4\right\},$$

shows that all roots are

$$-2, \qquad -1 + 2i, \qquad -1 - 2i.$$

Example 6.11 *Find all zeros of the polynomial*

$$P(z) = z^4 - (3 - 2i)z^2 + 8 + 6i, \qquad z \in \mathbb{C}.$$

A. Roots in a complex equation of fourth degree.

D. The equation is considered as an equation of second order in z^2.

I. By the usual solution formula we get

$$
\begin{aligned}
z^2 &= \frac{1}{2}(3-2i) \pm \frac{1}{2}\sqrt{(3-2i)^2 - 4(8+6i)} \\
&= \frac{1}{2}(3-2i) \pm \sqrt{9-4-12i-32-24i} \\
&= \frac{1}{2}(3-2i) \pm \frac{1}{2}\sqrt{-27-36i} = \frac{1}{2}(3-2i) \pm \frac{3}{2}\sqrt{-3-4i} \\
&= \frac{1}{2}(3-2i) \pm \frac{3}{2}\sqrt{1-4-4i} = \frac{1}{2}(3-2i) \pm \frac{3}{2}\sqrt{(1-2i)^2} \\
&= \frac{1}{2}(3-2i) \pm \frac{1}{2}(3-6i) \\
&= \begin{cases} \frac{1}{2}(3-2i+3-6i) = 3-4i = (2-i)^2, \\[2mm] \frac{1}{2}(3-2i-3+6i) = 2i = (1+i)^2. \end{cases}
\end{aligned}
$$

It follows from the last rearrangements that the roots are

$$1+i, \qquad -1-i, \qquad 2-i, \qquad -2+i.$$