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# Exercises for A youtube Calculus Workbook Part II

## a flipped classroom model

Frédéric Mynard

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# Contents




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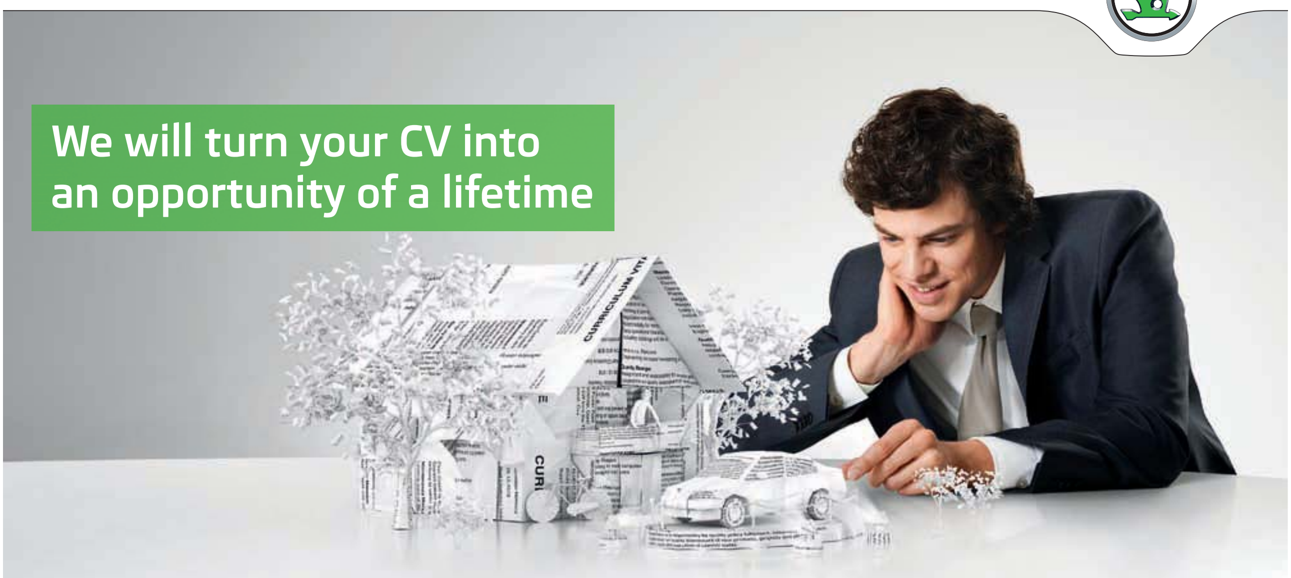


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# Preface

This book of exercises is a companion to A youtube Calculus Workbook (Part II), which is itself a companion to a play-list of 109 youtube video lectures covering material consistent with a second semester long college level Calculus course.

Part I was similarly a companion to a play-list of 94 youtube instructional videos consistent with a semester long first Calculus course at the college level. This book incorporated exercises and sample tests. In contrast, the second part is split into *the workbook* which is a set of extensive notes on the material covered in the videos, to simplify study from the playlist, and the present book of exercises.

The latter includes, for 31 topics, a worksheet of exercises without solutions, which are typically meant to be either worked out in class with the help of the teacher or assigned, a homework set consisting of exercises similar to those of the worksheet, and the complete solutions of the homework sets. It is organized to mirror the structure of the *workbook,* which itself references the parts of this *exercises book* that can be worked on at various points in your study.

Four Mock Tests with full solutions, as well as a Sample Final with full solutions are included to test yourself on a regular basis. This is meant for self study, or use in a flipped classroom setting as outlined below.

I hope that only few errors are left, but some are bound to remain. I welcome feedback and comments at calculusvideos@gmail.com.



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# To the instructor: A brief description of a flipped classroom model

I have used the material in *A youtube Calculus Workbook (Part II)* and the exercises in the present book for a one semester 4 credit-hours college level second semester Calculus course, using a flipped classroom model. The basic pattern is that students are assigned, for each class period, a set of videos to watch, and a homework set on the previous lesson. Class time is devoted to questions and to the worksheets. The homework exercises are very similar to those of the previous worksheet.

During class time, after grading the homework and after possible questions on the videos, I would typically give a very quick recap of what they have learned from the videos by questioning students. After that, students work on the worksheet while I would walk around the classroom to help them individually whenever needed. On regular intervals and when I see that many have been through a couple of exercises, I would go to the board to go over the solutions of the first exercises. Students then return to working on the next exercises, and this process is iterated until the end of the class period. Worksheets often contain more exercises than can be covered in one class period, and exercises of various levels of difficulties. This way, the teacher may chose a subset of exercises adapted to his/her class and/or his/her choice of homework exercises.



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## Self-graded homework

The homework is self-graded: students take a solution set when entering the class and have 5 to 10 minutes to read the solutions and grade themselves. This is why homework solutions in this book contain indications of the number of points to attribute to various parts of a question.

When done, they return their work to the teacher with the solutions on which they have entered their grade. The teacher can then keep track of the grades, and verify the work of the students as needed. My experience is that the first couple of homework sets may necessitate feedback for the students to adjust to the expectations for self-grading, but after that, students are typically grading themselves very fairly and even tend to be harsher on themselves than expected. At any rate, I find this method very fruitful, for it devotes a specific time every class period to *actually read* solutions to exercises. This way, students develop a better understanding of the way to present their work and justify their answers, and of the expectations in that respect. Questions on whether they should get credit or not for something that seems different from the solution are not rare but can lead to fruitful discussions.



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## Mock Tests

Before each test, the students take a mock test in class, in the conditions of the test (no notes or books). I encourage them to get prepared for the test for that day, so they can benefit from the mock test the most. At the end of the class period, they keep their work and pick up the full solutions of the mock test on their way out. This way they can see what mistakes they've made, what they knew and what they need to review further for the actual test.

The actual test is modeled on the review test. The Mock Test thus also serves the purpose of making expectations for the test (and for the extent to which answers should be justified) very clear.



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## Schedule

Expectations and due dates should be clearly explained and spelled out in the syllabus.

Here is a sample schedule for a 15 weeks course organized in 3 class periods per week:

| date | videos to watch before; workbook sections | homework due | worksheet |
|---|---|---|---|
| | week 1 | | |
| | introduction going over the organization of the class. Video M1-1 in class if time | | |
| | M1 videos 1-4 ($\approx 50\,min$) Sections 1.1-1.3 | | M1A |
| | M1 video 5-7 ($\approx 40\,min$) Sections 1.4-1.6 | HW M1A | M1B |
| | week 2 | | |
| | M1 videos 8-10 ($\approx 30\,min$) Sections 1.7-1.9 | HW M1B | M1C |
| | M2 videos 1-4 ($\approx 35\,min$) Sections 2.1-2.2 | HW M1C | M2A |
| | M2 videos 5-7 ($\approx 45\,min$) Sections 2.3-2.5 | HW M2A | M2B |
| | week 3 | | |
| | M2 videos 8-9 ($\approx 35\,min$) Sections 2.6-2.7 | HW M2B | M2C |
| | M3 videos 1-4 ($\approx 45\,min$) Sections 3.1-3.4 | HW M2C | M3 |
| | M4 videos 1-2 ($\approx 25\,min$) Sections 4.1-4.2 | HW M3 | M4A |
| | week 4 | | |
| | M4 videos 3-6 ($\approx 35\,min$) Sections 4.3-4.5 | HW M4A | M4B |
| | | HW M4B | MOCK TEST 1 |
| | TEST 1 | | |



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| date | videos to watch before; workbook sections | homework due | worksheet |
|---|---|---|---|
| | week 5 | | |
| | M5 videos 1-3 ($\approx 35\,min$)<br>Sections 5.1-5.3 | | M5A |
| | M5 videos 4-6 ($\approx 40\,min$)<br>Section 5.4 | HW M5A | M5B |
| | M6 videos 1-3 ($\approx 35\,min$)<br>Sections 6.1-6.2 | HW M5B | M6A |
| | week 6 | | |
| | M6 videos 4-5 ($\approx 30\,min$)<br>Sections 6.3-6.4 | HW M6A | M6B |
| | M6 videos 6-7 ($\approx 25\,min$)<br>Section 6.5 | HW M6B | M6C |
| | M7 videos 1-2 ($\approx 25\,min$)<br>Section 7.1 | HW M6C | M7A |
| | week 7 | | |
| | M7 videos 3-6 ($\approx 40\,min$)<br>Section 7.2-7.3 | HW M7A | M7B |
| | | HW M7B | MOCK TEST 2 |
| | TEST 2 | | |
| | week 8 | | |
| | M8 videos 1-5 ($\approx 45\,min$)<br>Sections 8.1-8.3 | | M8A |
| | M8 videos 6-9 ($\approx 50\,min$)<br>Sections 8.4-8.6 | HW M8A | M8B |
| | M9 videos 1-3 ($\approx 35\,min$)<br>Sections 9.1-9.2 | HW M8B | M9A |
| | week 9 | | |
| | M9 videos 4-7 ($\approx 50\,min$)<br>Sections 9.3-9.5 | HW M9A | M9B |
| | | HW M9B | MOCK TEST 3 |
| | TEST 3 | | |



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| date | videos to watch before; workbook sections | homework due | worksheet |
|---|---|---|---|
| | week 10 | | |
| | M10 videos 1-3 ($\approx 40\,min$)<br>Sections 10.1-10.3 | | M10A |
| | M10 videos 4-8 ($\approx 60\,min$)<br>Sections 10.4-10.5 | HW M10A | M10B |
| | M10 videos 9-10 ($\approx 40\,min$)<br>Sections 10.6-10.9 | HW M10B | M10C |
| | week 11 | | |
| | M11 videos 1-3 ($\approx 45\,min$)<br>Sections 11.1-11.3 | HW M10C | M11A |
| | M11 videos 4-6 ($\approx 40\,min$)<br>Sections 11.4-11.6 | HW M11A | M11B |
| | M12 videos 1-3 ($\approx 35\,min$)<br>Sections 12.1-12.3 | HW M11B | M12 |
| | week 12 | | |
| | M13 videos 1-5 ($\approx 55\,min$)<br>Sections 13.1-13.3 | HW M12 | M13 |
| | M14 video 1<br>Section 13.4 | HW M13 | M14 |
| | | HW M14 | MOCK TEST 4 |
| | TEST 4 | | |
| | week 13 | | |
| | M15 videos 1-4 ($\approx 45\,min$)<br>Sections 14.1-14.2 | | M15A |
| | M15 videos 5-8 ($\approx 55\,min$)<br>Sections 14.3-14.6 | HW M15A | M15B |
| | M15 videos 9-12 ($\approx 60\,min$)<br>Sections 14.7-14.10 | HW M15B | M15C |
| | week 14 | | |
| | M16 videos 1-3 ($\approx 30\,min$)<br>Sections 15.1-15.3 | HW M15C | catch up M15/M16 |
| | M16 videos 4-5 ($\approx 45\,min$)<br>Sections 15.4-15.5 | | M16 |
| | | HW M16 | review |
| | week 15 | | |
| | catch up/review | | |
| | Sample Final | | |
| | FINAL | | |



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## Your flipped classroom

Should you decide to use *A youtube Calculus Workbook (Part II)* for a flipped classroom course, the present volume of exercises and what I described above can only be seen as an example. The homework sets and solutions give you an idea of how they are structured, but you should probably make your own, if anything because the solutions to those in this book are freely available. You may find that you need more extensive comments on how to assign credit to your homework sets, for instance using margin notes in solutions. The present examples are in the format given to my students. Instructions to grade are minimal as I was present to clarify whenever needed.

You may want to spend more time on part of the material and omit other parts. It is my hope that the material presented here can easily be adapted to your needs.

Beyond the material, there are many parameters in the outline for the course that you may adapt to your needs. In particular, depending on the number of students in your class, you may want to have students work in groups, have them present solutions, etc.

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# 1 M1: Natural Logarithm and Exponential

The Worksheet and Homework set M1A should be worked on after studying the material from sections 1.1, 1.2, and 1.3 of the youtube workbook.

## 1.1 M1A Worksheet: Natural Logarithm

1. Express the following logarithms in terms of ln 5 and ln 7:
   (a) $\ln\left(\frac{1}{125}\right)$;
   (b) $\ln 1225$;
   (c) $\ln 7\sqrt{7}$;
   (d) $\frac{\ln 35+\ln(1/7)}{\ln 25}$;
   (e) $\ln(9.8)$.

2. Use the properties of logarithms to simplify the expressions:
   (a) $\ln\sec\theta + \ln\cos\theta$ ;
   (b) $\ln(8x+4) - 2\ln 2$;
   (c) $3\ln\sqrt[3]{t^2-1} - \ln(t-1)$;

3. Differentiate:
   (a) $y = \ln\left(x^3\right)$;
   (b) $y = \frac{\ln t}{t}$;
   (c) $y = \ln\left(\frac{10}{x}\right)$;
   (d) $y = \ln(\sin x)$;
   (e) $y = \ln(\ln(\ln x))$;
   (f) $y = \frac{x\ln x}{1+\ln x}$;
   (g) $y = \ln\left(\frac{(x^2+1)^5}{\sqrt{1-x}}\right)$.

4. Use logarithmic differentiation to find the derivative of
   (a) $y = \dfrac{1}{t(t+1)(t+2)}$;
   (b) $y = \sqrt[3]{\dfrac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}$



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5. Integrate:
   (a) $\int_{-1}^{0} \frac{3}{3x-2}\,dx$
   (b) $\int \frac{8r}{4r^2-5}\,dr$
   (c) $\int_{2}^{4} \frac{dx}{x\ln x}$
   (d) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot t\,dt$
   (e) $\int \frac{3\sec^2 t}{6+3\tan t}\,dt$

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## 1.2 M1A Homework set: Natural Logarithm

1. Write
$$2\ln x + 5\ln(x+4) - \frac{1}{2}\ln(x^2+1)$$
as a single logarithm.

2. Write
$$\ln\left(\frac{x^2\sqrt{x+3}}{(x+1)^3}\right)$$
as a linear combination of logarithms.

3. Differentiate:

   (a) $f(x) = x\ln x$;

   (b) $f(x) = \sqrt{\ln x}$;

   (c) $y = \ln(x^2+1)$;

   (d) $f(x) = \frac{x^2(x+1)}{\sqrt{x^2+1}(x^3+2)}$ using logarithmic differentiation;

4. Integrate

   (a) $\int \frac{2x+1}{x^2+x+1}\,dx$

   (b) $\int \tan(2x)\,dx$

   (c) $\int \frac{\sin x}{1-5\cos x}\,dx$



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## 1.3 M1A Homework set: Solutions

NAME:

GRADE: /13

1. [1] Write
$$2\ln x + 5\ln(x+4) - \frac{1}{2}\ln(x^2+1)$$
as a single logarithm.

*Solution.*
$$\begin{aligned} 2\ln x + 5\ln(x+4) - \frac{1}{2}\ln(x^2+1) &= \ln\left(x^2\right) + \ln\left((x+4)^5\right) - \ln\left((x^2+1)^{\frac{1}{2}}\right) \\ &= \ln\left(\frac{x^2(x+4)^5}{\sqrt{x^2+1}}\right). \end{aligned}$$

2. [1] Write
$$\ln\left(\frac{x^2\sqrt{x+3}}{(x+1)^3}\right)$$
as a linear combination of logarithms.

*Solution.*
$$\ln\left(\frac{x^2\sqrt{x+3}}{(x+1)^3}\right) = 2\ln x + \frac{1}{2}\ln(x+3) - 3\ln(x+1).$$

3. Differentiate:

(a) [1] $f(x) = x\ln x$;
*Solution.* Using the product rule:
$$f'(x) = \ln x + x\cdot\frac{1}{x} = \ln x + 1.$$

(b) [1] $f(x) = \sqrt{\ln x}$;
*Solution.* Using the Chain rule:
$$f'(x) = \frac{1}{2\sqrt{\ln x}}\cdot\frac{1}{x}.$$

(c) [1] $y = \ln(x^2+1)$;
*Solution.* Using the Chain rule:
$$\frac{dy}{dx} = \frac{2x}{x^2+1}.$$



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(d) [2] $f(x) = \frac{x^2(x+1)}{\sqrt{x^2+1}(x^3+2)}$ using logarithmic differentiation;

*Solution.* Letting $y = \frac{x^2(x+1)}{\sqrt{x^2+1}(x^3+2)}$, we have

$$\ln y = 2\ln x + \ln(x+1) - \frac{1}{2}\ln(x^2+1) - \ln(x^3+2),$$

so that

$$\frac{y'}{y} = \frac{2}{x} + \frac{1}{x+1} - \frac{x}{x^2+1} - \frac{3x^2}{x^3+2},$$

and

$$y' = \frac{x^2(x+1)}{\sqrt{x^2+1}(x^3+2)} \cdot \left(\frac{2}{x} + \frac{1}{x+1} - \frac{x}{x^2+1} - \frac{3x^2}{x^3+2}\right).$$

4. Integrate

(a) [2] $\int \frac{2x+1}{x^2+x+1}\,dx$

*Solution.* Let $u = x^2+x+1$. Then $du = (2x+1)\,dx$. Thus

$$\int \frac{2x+1}{x^2+x+1}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|x^2+x+1| + C.$$

(b) [2] $\int \tan(2x)\,dx$

*Solution.* Let $u = \cos 2x$. Then $du = -2\sin 2x\,dx$ and $\sin 2x dx = -\frac{1}{2}du$. Thus

$$\int \tan(2x)\,dx = \int \frac{\sin 2x}{\cos 2x}\,dx = -\frac{1}{2}\int \frac{du}{u} = -\frac{1}{2}\ln|\cos 2x| + C.$$

(c) [2] $\int \frac{\sin x}{1-5\cos x}\,dx$

*Solution.* Let $u = 1-5\cos x$. Then $du = 5\sin x\,dx$ and $\sin x\,dx = \frac{1}{5}du$. Thus

$$\int \frac{\sin x}{1-5\cos x}\,dx = \frac{1}{5}\int \frac{du}{u} = \frac{1}{5}\ln|1-5\cos x| + C.$$



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The Worksheet and Homework set M1B should be worked on after studying the material from sections 1.4, 1.5, and 1.6 of the youtube workbook.

## 1.4 M1B Worksheet: inverse functions

1. Are the following functions one-to-one?

   (a) $f(x) = 2x^3 - 1$;

   (b) $f(x) = x^3 - 2x - 1$;

   (c) $f(x) = \ln x$;

   (d) $f(x) = \ln(x^2)$;

   (e) $f(x) = 2\cos x$.

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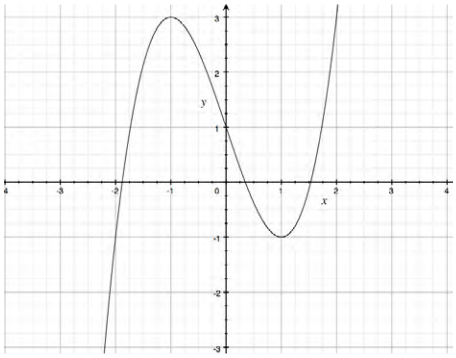
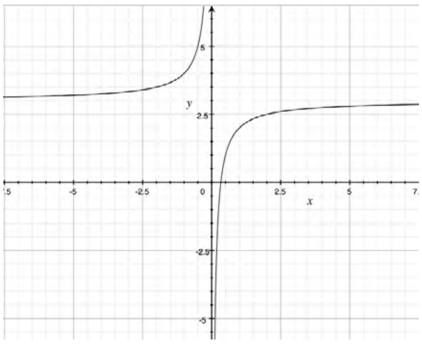
2. Are the following graphs of one-to-one functions? If yes, draw the graph of the inverse function.

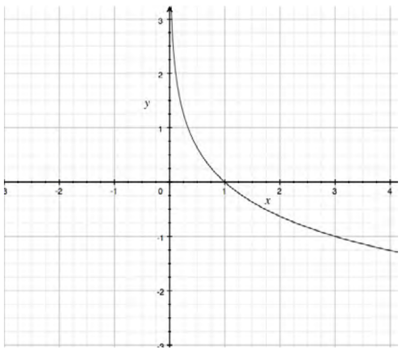
(a)



(b)



(c)



3. In the following questions, the function $f$ has an inverse. Without solving for the inverse find $\left(f^{-1}\right)'(a)$:

(a) $f(x) = x^3 + 3x + 2;\ a = -2;$

(b) $f(x) = \sqrt{x^5 + 2x^3 + x + 2};\ a = \sqrt{2};$

(c) $f(x) = 3 + \ln x\,;\ a = 3.$

4. Are the following functions one-to-one? Find the inverse when it exists.

(a) $f(x) = x^3 - 4;$

(b) $f(x) = \frac{5x+2}{2x+3};$

(c) $f(x) = \sqrt{x^2 + 4};$

(d) $f(x) = \sqrt{x + 4}.$



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## 1.5 M1B Homework set: Inverse functions

1. Let $f(x) = \frac{2x}{x-1}$.
   (a) Find a formula for $f^{-1}$;
   (b) Calculate $(f^{-1})'$ directly from (a), and then verify with the formula for the derivative of an inverse function.

2. Let $f(x) = 2x^2 + 1$ for $x \geq 0$.
   (a) Find a formula for $f^{-1}$;
   (b) Calculate $(f^{-1})'$ directly from (a), and then verify with the formula for the derivative of an inverse function.

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## 1.6 M1B Homework set: Solutions

NAME:

GRADE: /10

1. Let $f(x) = \frac{2x}{x-1}$.

(a) [2] Find a formula for $f^{-1}$;

*Solution.* Let $y = \frac{2x}{x-1}$ and solve for *x*:

$$(x-1)y = 2x \iff x(y-2) = y \iff x = \frac{y}{y-2}.$$

Thus $f^{-1}(y) = \frac{y}{y-2}$ with $y \neq 2$, or, if you prefer, $f^{-1}(x) = \frac{x}{x-2}$, with $x \neq 2$.

(b) [3] Calculate $(f^{-1})'$ directly from (a), and then verify with the formula for the derivative of an inverse function.

*Solution.* If $f^{-1}(x) = \frac{x}{x-2}$ then

$$(f^{-1})'(x) = \frac{1 \cdot (x-2) - 1 \cdot x}{(x-2)^2} = -\frac{2}{(x-2)^2}.$$

On the other hand, we have the formula

$$\left(f^{-1}\right)'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Here $f(x) = \frac{2x}{x-1}$ so that

$$f'(x) = \frac{2(x-1) - 2x}{(x-1)^2} = -\frac{2}{(x-1)^2}$$

and $\frac{1}{f'(x)} = -\frac{(x-1)^2}{2}$. Thus

$$\begin{aligned}
\left(f^{-1}\right)'(x) &= \frac{1}{f'(f^{-1}(x))} \\
&= -\frac{(\frac{x}{x-2} - 1)^2}{2} \\
&= -\frac{(\frac{x-(x-2)}{(x-2)})^2}{2} \\
&= -\frac{\left(\frac{2}{(x-2)}\right)^2}{2} \\
&= -\frac{2}{(x-2)^2}.
\end{aligned}$$



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2. Let $f(x) = 2x^2 + 1$ for $x \geq 0$.

(a) [2] Find a formula for $f^{-1}$;

*Solution.* Let $y = 2x^2 + 1$ with $x \geq 0$. Then $x^2 = \frac{y-1}{2}$, which has as only non-negative solution

$$x = \sqrt{\frac{y-1}{2}} = f^{-1}(y),$$

so that

$$f^{-1}(x) = \sqrt{\frac{x-1}{2}};\ x \geq 1.$$

(b) [3] Calculate $(f^{-1})'$ directly from (a), and then verify with the formula for the derivative of an inverse function.

*Solution.* If $f^{-1}(x) = \sqrt{\frac{x-1}{2}}$ then

$$\left(f^{-1}\right)'(x) = \frac{1}{2} \cdot \frac{1}{2\sqrt{\frac{x-1}{2}}} = \frac{\sqrt{2}}{4\sqrt{x-1}}.$$

On the other hand, we have the formula

$$\left(f^{-1}\right)'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Here $f(x) = 2x^2 + 1$ so that $f'(x) = 4x$ and $\frac{1}{f'(x)} = \frac{1}{4x}$. Thus

$$\left(f^{-1}\right)'(x) = \frac{1}{4 \cdot \sqrt{\frac{x-1}{2}}} = \frac{\sqrt{2}}{4\sqrt{x-1}}.$$



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The Worksheet and Homework set M1C should be worked on after studying the material from sections 1.7, 1.8, and 1.9 of the youtube workbook.

## 1.7 M1C Worksheet: natural exponential

1. Solve the following equations for $t$:
   (a) $e^{-0.3t} = 27$;
   (b) $e^{kt} = \frac{1}{5}$;
   (c) $e^{x^2} e^{2x+1} = e^t$.

2. Solve for $x$:
   (a) $e^{2x+5} = 3$;
   (b) $e^{e^x} - 5 = 0$;
   (c) $e^{2x} + e^x - 6 = 0$;
   (d) $\ln(3x+2) = -1$;
   (e) $e^{3x+1} < 5$.

3. Differentiate:
   (a) $f(x) = e^{-7x}$;
   (b) $f(x) = e^{3-5x}$;
   (c) $f(x) = xe^{2x} - e^x$;
   (d) $y = (9x^2 - 6x + 2)e^{3x}$;
   (e) $f(x) = \frac{e^x}{1+e^x}$;
   (f) $f(x) = \cos e^x + e^{\cos x}$;
   (g) $y = e^{\sin t}\left(\ln t^2 + 1\right)$.

4. Find $\frac{dy}{dx}$ along the curve
$$\ln xy = e^{x+y}.$$

5. Integrate:
   (a) $\displaystyle\int e^{5x} - 3e^{-x}\,dx$
   (b) $\displaystyle\int_{\ln 2}^{\ln 3} e^x\,dx$
   (c) $\displaystyle\int \frac{e^{\sqrt{t}}}{\sqrt{t}}\,dt$
   (d) $\displaystyle\int e^{2x}\cos e^{2x}\,dx$
   (e) $\displaystyle\int \frac{e^{\frac{1}{x}}}{x^2}\,dx$
   (f) $\displaystyle\int \frac{e^t}{1+e^t}\,dt$
   (g) $\displaystyle\int \frac{dt}{1+e^t}$



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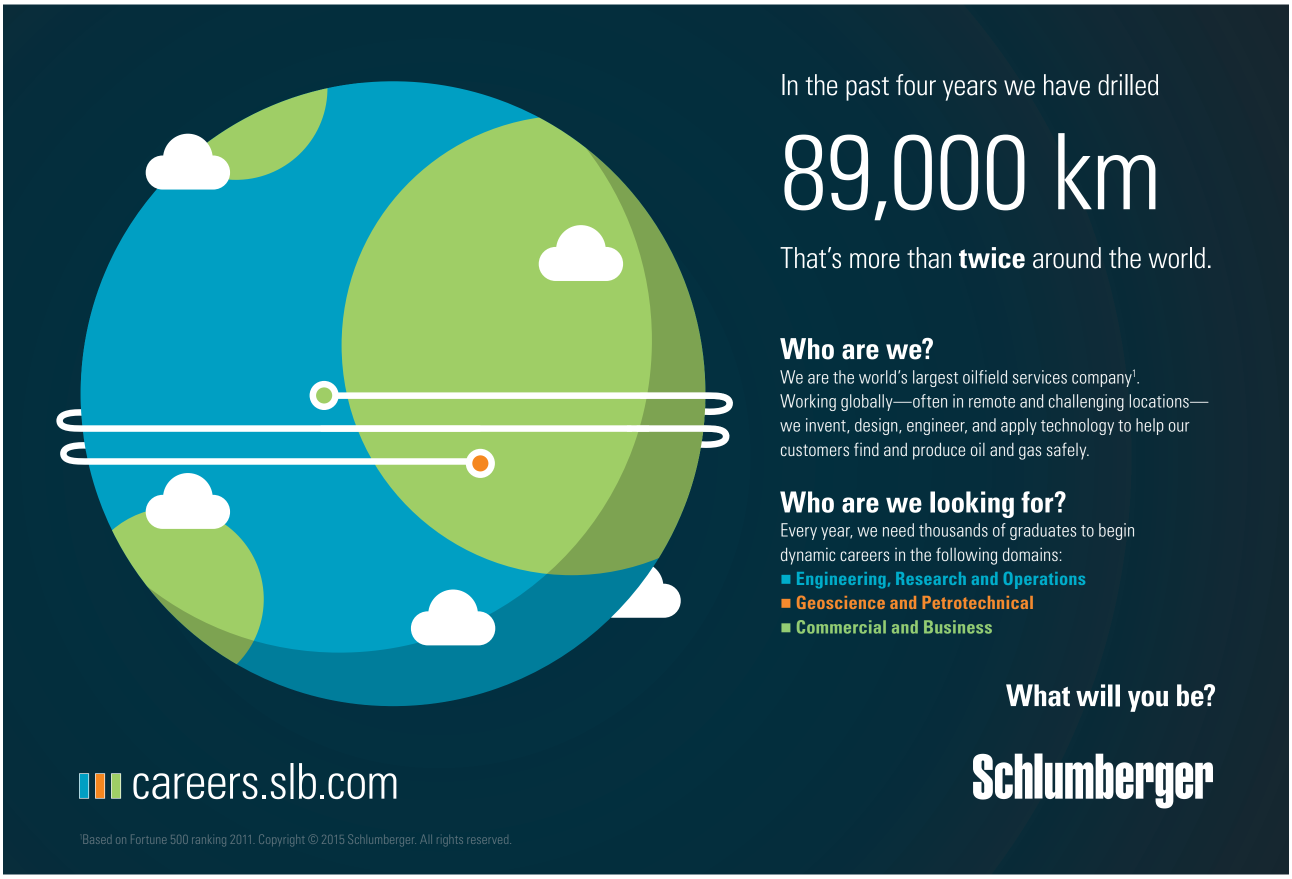
## 1.8 M1C Homework set: natural exponential

1. Solve the following equations for $x$:
   (a) $e^{x^2} e^{4x+4} = 1.$
   (b) $e^{5x+1} = 2;$
   (c) $\ln(1-2x) = 2;$

2. Differentiate:
   (a) $f(x) = e^{\pi x};$
   (b) $f(x) = x^2 e^x - e^{x^2};$
   (c) $f(x) = \frac{xe^x}{x+e^x};$
   (d) $f(x) = \sin e^x + e^{\sin x};$

3. Integrate:
   (a) $\int e^{2x} + 3e^{3x}\, dx$
   (b) $\int_{\ln 2}^{\ln 3} e^{2x}\, dx$
   (c) $\int e^x \sin e^x\, dx$
   (d) $\int \frac{e^{\frac{1}{x^2}}}{x^3}\, dx$



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## 1.9 M1C Homework set: Solutions

NAME:

GRADE: / 11

1. Solve the following equations for $x$:

(a) [1] $e^{x^2}e^{4x+4} = 1$.

*Solution.* $e^{x^2}e^{4x+4} = e^{x^2+4x+4}$ and $1 = e^0$, so that the equation rewrites as

$$e^{x^2+4x+4} = e^0,$$

which is equivalent to

$$x^2 + 4x + 4 = 0 = (x+2)^2,$$

because the natural exponential is one-to-one. Thus $x = -2$ is the unique solution.

(b) [1] $e^{5x+1} = 2$;

*Solution.*

$$e^{5x+1} = 2 \iff 5x + 1 = \ln 2 \iff x = \frac{\ln 2 - 1}{5}.$$

(c) [1] $\ln(1-2x) = 2$;

*Solution.*

$$\ln(1-2x) = 2 \iff 1 - 2x = e^2 \iff x = \frac{1-e^2}{2}.$$

2. Differentiate:

(a) [1] $f(x) = e^{\pi x}$;

*Solution.* Using the Chain Rule

$$f'(x) = \pi e^{\pi x}.$$

(b) [1] $f(x) = x^2e^x - e^{x^2}$;

*Solution.* Using the Chain Rule and Product Rule

$$f'(x) = 2xe^x + x^2e^x - 2xe^{x^2}.$$



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(c) [1] $f(x) = \frac{xe^x}{x+e^x}$;

*Solution.* Using the Quotient Rule and Product Rule

you do not need to simplify to get credit

$$\begin{aligned} f'(x) &= \frac{(x+1)e^x(x+e^x) - (1+e^x)(xe^x)}{(x+e^x)^2} \\ &= \frac{e^x\left(x^2+x+xe^x+e^x-x-xe^x\right)}{(x+e^x)^2} \\ &= \frac{e^x\left(x^2+e^x\right)}{(x+e^x)^2}. \end{aligned}$$

(d) [1] $f(x) = \sin e^x + e^{\sin x}$;

*Solution.* Using the Chain Rule

$$f'(x) = e^x\cos(e^x) + \cos x e^{\sin x}.$$

3. Integrate:

(a) [1] $\int e^{2x} + 3e^{3x}\,dx$

*Solution.* Using $\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$ (established by substitution), we have

$$\int e^{2x} + 3e^{3x}\,dx = \frac{e^{2x}}{2} + e^{3x} + C.$$

(b) [1] $\int_{\ln 2}^{\ln 3} e^{2x}\,dx$

*Solution.* Using $\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$ and the Fundamental Theorem of Calculus, we have

$$\int_{\ln 2}^{\ln 3} e^{2x}\,dx = \left[\frac{e^{2x}}{2}\right]_{\ln 2}^{\ln 3} = \frac{1}{2}\left(e^{2\ln 3} - e^{2\ln 2}\right) = \frac{e^{\ln 9} - e^{\ln 4}}{2} = \frac{5}{2}.$$

(c) [1] $\int e^x \sin e^x\,dx$

*Solution.* By the substitution with $u = e^x$ and $du = e^x\,dx$, we have

$$\int e^x \sin e^x\,dx = \int \sin u\,du = -\cos u + C = -\cos(e^x) + C.$$

(d) [1] $\int \frac{e^{\frac{1}{x^2}}}{x^3}\,dx$

*Solution.* By the substitution with $u = \frac{1}{x^2}$ and $du = -\frac{1}{2}\cdot\frac{dx}{x^3}$, we have

$$\int \frac{e^{\frac{1}{x^2}}}{x^3}\,dx = -2\int e^u\,du = -2e^u + C = -2e^{\frac{1}{x^2}} + C.$$



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# 2 M2: More transcendental functions

The Worksheet and Homework set M2A should be worked on after studying the material from sections 2.1 and 2.2 of the youtube workbook.

## 2.1 M2A Worksheet: general exponential and logarithm

1. Write as a power of *E*:
   (a) $5^{\sqrt{13}}$;
   (b) $x^{\sin x}$;
   (c) $(\sin x)^x$;
   (d) $(x^2)^x$;
   (e) $3^{x^2}$;
   (f) $x^{x^2}$.

2. Evaluate:
   (a) $\log_{10} 10000$;
   (b) $\log_{10} 0.1$;
   (c) $\log_2 \sqrt{2}$;
   (d) $\log_2 \frac{1}{32}$;
   (e) $\log_3 3^{\sqrt{7}}$;
   (f) $10^{\log_{10} 5+\log_{10} 13}$.

3. Evaluate the limits:
   (a) $\lim_{t\to\infty} 3^{-t^2}$;
   (b) $\lim_{t\to-\infty} 2^{-t^3}$;
   (c) $\lim_{t\to 2^+} \log_2 \left(x^2+x-6\right)$.

4. Differentiate:
   (a) $f(x) = 5^x - x^5$;
   (b) $f(x) = 4^{\frac{1}{x}}$;
   (c) $g(t) = 3^t t^3$;
   (d) $y = 3^{2^{x^2}}$;
   (e) $f(x) = x^x$;
   (f) $h(t) = (\sin t)^t$;
   (g) $y = x^{\cos x}$;
   (h) $y = (\ln x)^x$;
   (i) $y = (\ln x)^{\sin x}$.



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5. Find an equation of the tangent line to $y = 7^x$ at $(1, 7)$.
6. Find $\frac{dy}{dx}$ along the curve $x^y = y^x$.
7. Integrate:
   (a) $\displaystyle\int_0^1 2^t - 2t^2 \, dt$

   (b) $\displaystyle\int \frac{\log_3 x}{x} \, dx$

   (c) $\displaystyle\int 7^{\sin\theta} \cos\theta \, d\theta$

   (d) $\displaystyle\int_0^1 3^{-2t} \, dt$

   (e) $\displaystyle\int \frac{3^x}{3^x + 2} \, dx$




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## 2.2 M2A Homework set: general exponential and logarithm

1. Write as a power of *e*:
   (a) $3^{\sqrt{7}}$;
   (b) $x^{\tan x}$;
   (c) $(x^2)^x$;
   (d) $x^{\sqrt{x}}$.

2. Evaluate:
   (a) $\log_5 125$;
   (b) $\log_3 \frac{1}{27}$;
   (c) $\log_2 0.5$;
   (d) $5^{\log_5 3+2\log_5 7}$.

3. Differentiate:
   (a) $f(x) = 3^x - x^3$;
   (b) $f(x) = 2^{x^2}$;
   (c) $f(x) = (\cos x)^x$;
   (d) $y = (\cos x)^{\ln x}$.

4. Find an equation of the tangent line to $y = 2^{3x}$ at (1, 8).

5. Integrate:
   (a) $\int_0^1 3^t + t^3\, dt$
   (b) $\int \frac{1}{x\log_2 x}\, dx$
   (c) $\int 3^{\cos\theta} \sin\theta\, d\theta$
   (d) $\int \frac{2^x}{(2^x+2)^2}\, dx$



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## 2.3 M2A Homework set: Solutions

NAME:

GRADE: /24

1. Write as a power of $e$:
   (a) [1] $3^{\sqrt{7}} = e^{\sqrt{7}\ln 3}$.
   (b) [1] $x^{\tan x} = e^{\tan x \ln x}$.
   (c) [1] $(x^2)^x = e^{x \ln x^2} = e^{2x \ln x}$.
   (d) [1] $x^{\sqrt{x}} = e^{\sqrt{x}\ln x}$.

2. Evaluate:
   (a) [1] $\log_5 125 = \log_5 5^3 = 3$.
   (b) [1] $\log_3 \dfrac{1}{27} = \log_3(3^{-3}) = -3$.
   (c) [1] $\log_2 0.5 = \log_2(2^{-1}) = -1$.
   (d) [1] $5^{\log_5 3 + 2\log_5 7} = 5^{\log_5 3} \cdot 5^{\log_5 7^2} = 3 \cdot 7^2 = 147$.

3. Differentiate:
   (a) [1] $f(x) = 3^x - x^3$;

$$f'(x) = 3^x \ln 3 - 3x^2.$$

   (b) [1] $f(x) = 2^{x^2}$;

$$f'(x) = 2^{x^2} \cdot 2x \ln 2.$$

   (c) [2] $f(x) = (\cos x)^x = e^{x\ln(\cos x)}$. Thus

$$\begin{aligned} f'(x) &= e^{x \ln \cos x} \cdot (x \ln \cos x)' \\ &= (\cos x)^x \cdot \left( \ln \cos x - \frac{x \sin x}{\cos x} \right). \end{aligned}$$

   (d) [2] $y = (\cos x)^{\ln x} = e^{\ln x \ln(\cos x)}$. Thus

$$\begin{aligned} f'(x) &= e^{\ln x \ln(\cos x)} \cdot (\ln x \ln(\cos x))' \\ &= (\cos x)^{\ln x} \cdot \left( \frac{\ln(\cos x)}{x} - \frac{\ln x \sin x}{\cos} \right). \end{aligned}$$

4. [2] Find an equation of the tangent line to $y = 2^{3x}$ at $(1, 8)$.
   *Solution.* The tangent line is the line through $(1, 8)$ of slope $\frac{dy}{dx}_{|x=1}$. Moreover,

$$\frac{dy}{dx} = 2^{3x} \cdot 3 \ln 2,$$



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so that the line has slope 24ln2 and thus equation

$$y - 8 = 24\ln 2(x - 1).$$

5. Integrate:

(a) [2]

$$\int_0^1 3^t + t^3\,dt = \left[\frac{3^t}{\ln 3} + \frac{t^4}{4}\right]_0^1 = \frac{2}{\ln 3} + \frac{1}{4}.$$

(b) [2] If $u = \log_2 x$ then $du = \frac{dx}{x \ln 2}$, so that

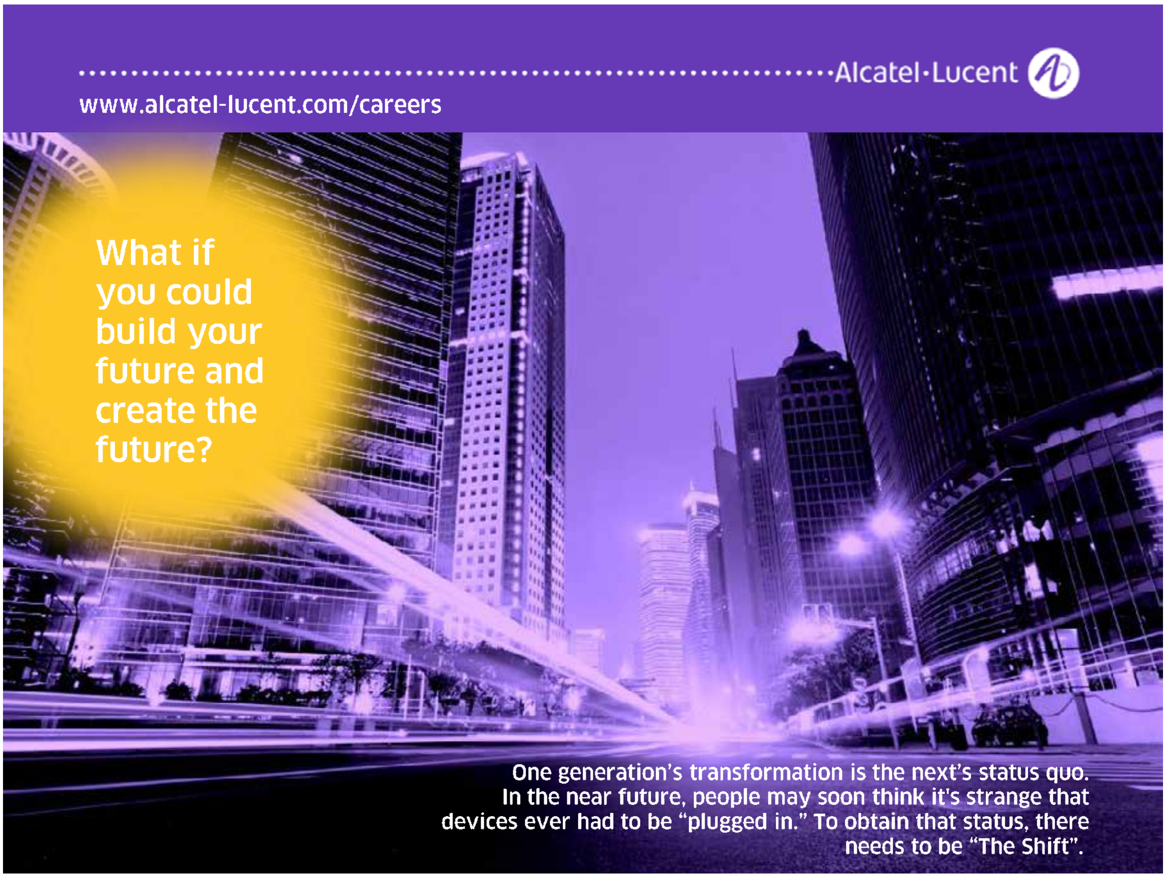
$$\int \frac{1}{x \log_2 x}\,dx = \ln 2 \int \frac{du}{u} = \ln 2 \cdot \ln|\log_2 x| + C.$$

(c) [2] If $u = \cos\theta$ then $du = -\sin\theta\,d\theta$ so that

$$\int 3^{\cos\theta}\sin\theta\,d\theta = -\int 3^u\,du = -\frac{3^u}{\ln 3} + C = -\frac{3^{\cos\theta}}{\ln 3} + C.$$

(d) [2]If $u = 2^x + 2$ then $du = 2^x \ln 2\,dx$ so that

$$\int \frac{2^x}{(2^x+2)^2}\,dx = \frac{1}{\ln 2}\int \frac{du}{u^2} = -\frac{1}{u\ln 2} + C = -\frac{1}{\ln 2\,(2^x+2)} + C.$$



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The Worksheet and Homework set M2B should be worked on after studying the material from sections 2.3, 2.4, and 2.5 of the youtube workbook.

## 2.4 M2B Worksheet: inverse trig functions

1. Find the exact value of:
   (a) $\arcsin(\sin \frac{\pi}{6})$
   (b) $\arcsin(\sin \frac{2\pi}{3})$
   (c) $\arctan(-1)$
   (d) $\arccos(\cos 2\pi)$
   (e) $\arctan(\tan \frac{3\pi}{4})$
   (f) $\cos(\arcsin \frac{1}{2})$
   (g) $\tan(\arcsin \frac{2}{3})$

2. Find the following limits:
   (a) $\lim_{x\to\infty} \arctan(e^x)$
   (b) $\lim_{x\to 0^+} \arctan(\ln x)$
   (c) $\lim_{x\to\infty} \arccos\left(\frac{1+x^3}{3+2x^3}\right)$

3. Differentiate:
   (a) $f(x) = \arctan\sqrt{x}$ ;
   (b) $f(x) = \sqrt{\arctan x}$ ;
   (c) $f(x) = \arcsin(5x+2)$;
   (d) $f(x) = \arccos(e^{5x})$;
   (e) $f(x) = x\arcsin x + \tan(\arccos x)$;
   (f) $f(x) = x\ln(\arctan x)$;
   (g) $y = e^{\arcsin x} + \arccos(\ln x)$.

4. Find $\frac{dy}{dx}$ along the curve
$$\arctan(xy) = 1 + x^2 y.$$



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5. Find an equation of the tangent line to $y = 2\arccos(\frac{x}{2})$ at $x = 1$.
6. Integrate:

(a) $$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{6}{\sqrt{1-t^2}}\, dt$$

(b) $$\int_0^1 \frac{4}{t^2+1}\, dt$$

(c) $$\int \frac{dx}{1+9x^2}$$

(d) $$\int \frac{dt}{\sqrt{1-4t^2}}$$

(e) $$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\, dx$$

(f) $$\int \frac{\arctan x}{1+x^2}\, dx$$

(g) $$\int \frac{x+9}{x^2+9}\, dx$$

(h) $$\int \frac{t^2}{\sqrt{1-t^6}}\, dt$$

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## 2.5 M2B Homework set: inverse trig functions

1. Find the exact value of:
   (a) $\arccos(\cos \frac{\pi}{6})$
   (b) $\arccos(\cos \frac{4\pi}{3})$
   (c) $\cos(\arcsin \frac{2}{3})$

2. Differentiate:
   (a) $f(x) = \sqrt{\arccos x}$ ;
   (b) $f(x) = \arctan(5x^2 + 2x + 1)$;
   (c) $f(x) = \arcsin(\ln x)$;
   (d) $f(x) = x^2 \arctan x + 2e^{\arccos x}$;

3. Find an equation of the tangent line to $y = \arcsin(\frac{x}{2})$ at $x = 1$.

4. Integrate:
   (a) $\int \frac{t+1}{\sqrt{1-t^2}}\, dt$
   (b) $\int_0^1 \frac{4}{4t^2+1}\, dt$
   (c) $\int \frac{dt}{\sqrt{1-16t^2}}$
   (d) $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x}\, dx$
   (e) $\int \frac{dx}{(1+x^2)\arctan x}$



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## 2.6 M2B Homework set: Solutions

NAME:

GRADE: /19

1. Find the exact value of:

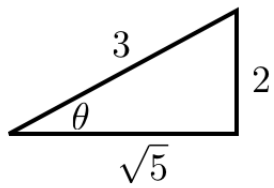
(a) [1]

$$\arccos(\cos\frac{\pi}{6}) = \frac{\pi}{6}.$$

(b) [1]

$$\arccos(\cos\frac{4\pi}{3}) = \arccos(\cos\frac{2\pi}{3}) = \frac{2\pi}{3}.$$

(c) [1] $\cos(\arcsin\frac{2}{3})$. Let $\theta = \arcsin\frac{2}{3}$, that is, $\sin\theta = \frac{2}{3}$ and $\theta \in [0, \frac{\pi}{2}]$:




where we use the Pythagorean to find the missing side in this triangle. Thus

$$\cos\theta = \frac{\sqrt{5}}{3}.$$

2. Differentiate:

(a) [1] $f(x) = \sqrt{\arccos x}$;

$$f'(x) = -\frac{1}{2\sqrt{\arccos x}} \cdot \frac{1}{\sqrt{1-x^2}}.$$

(b) [1] $f(x) = \arctan(5x^2 + 2x + 1)$;

$$f'(x) = \frac{10 + 2}{1 + (5x^2 + 2x + 1)^2}.$$

(c) [1] $f(x) = \arcsin(\ln x)$;

$$f'(x) = \frac{1}{x\sqrt{1 - (\ln x)^2}}.$$



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(d) [1] $f(x) = x^2 \arctan x + 2e^{\arccos x}$;

$$f'(x) = 2x \arctan x + \frac{x^2}{1+x^2} - \frac{2e^{\arccos x}}{\sqrt{1-x^2}}.$$

3. [2] Find an equation of the tangent line to $y = \arcsin(\frac{x}{2})$ at $x = 1$.

*Solution.* The point of tangency is

$$\left(1, \arcsin\frac{1}{2}\right) = \left(1, \frac{\pi}{6}\right).$$

Moreover,

$$\frac{dy}{dx} = \frac{1}{2\sqrt{1-(\frac{x}{2})^2}}$$

takes the value $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ at $x = 1$. Thus, the tangent line has equation

$$y - \frac{\pi}{6} = \frac{\sqrt{3}}{3}(x-1).$$

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4. Integrate:

(a) [2]

$$\begin{aligned}\int \frac{t+1}{\sqrt{1-t^2}}\,dt &= \int \frac{t}{\sqrt{1-t^2}}\,dt + \int \frac{dt}{\sqrt{1-t^2}} \\ &= -\frac{1}{2}\int \frac{du}{\sqrt{u}} + \arcsin t + C,\end{aligned}$$

using $u = 1 - t^2$, so that $du = -2t\,dt$ . Thus

$$\int \frac{t+1}{\sqrt{1-t^2}}\,dt = -\sqrt{1-t^2} + \arcsin t + C$$

(b) [2]

$$\begin{aligned}\int_0^1 \frac{4}{4t^2+1}\,dt &= 2\int_0^1 \frac{2dt}{(2t)^2+1} \\ &= 2\int_0^2 \frac{du}{u^2+1} = 2\left[\arctan u\right]_0^2 = 2\arctan 2.\end{aligned}$$

(c) [2]

$$\begin{aligned}\int \frac{dt}{\sqrt{1-16t^2}} &= \int \frac{dt}{\sqrt{1-(4t)^2}} \\ &= \frac{1}{4}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{4}\arcsin u + C \\ &= \frac{1}{4}\arcsin(4t) + C.\end{aligned}$$

(d) [2] $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x}\,dx$. With $u = \sin x$ and $du = \cos x\,dx$, we have

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x}\,dx = \int_0^1 \frac{du}{1+u^2} = \left[\arctan u\right]_0^1 = \frac{\pi}{4}.$$

(e) [2] $\int \frac{dx}{(1+x^2)\arctan x}$. With $u = \arctan x$ and $du = \frac{dx}{x^2+1}$, we have

$$\int \frac{dx}{(1+x^2)\arctan x} = \int \frac{du}{u} = \ln|u| + C = \ln|\arctan x| + C.$$



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The Worksheet and Homework set M2C should be worked on after studying the material from sections 2.6 and 2.7 of the youtube workbook.

## 2.7 M2C Worksheet: hyperbolic functions

1. Differentiate the following functions:
   (a) $f(x) = 3\cosh(2x) + 2\sinh(3x)$
   (b) $f(x) = x\tanh x + \operatorname{sech}(x^2)$
   (c) $f(x) = \operatorname{arccosh}(2x) - 3x \operatorname{arcsinh} x$
   (d) $f(x) = \frac{\sinh x}{\cosh(x^3)}$

2. Evaluate the following integrals:
   (a) $\displaystyle\int \frac{\sinh x}{5+\cosh x}\,dx$
   (b) $\displaystyle\int x^2\tanh(x^3)\,dx$
   (c) $\displaystyle\int \frac{dx}{\sqrt{4x^2-9}}$
   (d) $\displaystyle\int \frac{e^x\,dx}{\sqrt{e^{2x}-9}}$
   (e) $\displaystyle\int \frac{dx}{\sqrt{4x^2+9}}$
   (f) $\displaystyle\int \frac{e^{2x}\,dx}{\sqrt{e^{4x}-16}}$



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## 2.8 M2C Homework set: hyperbolic functions

1. Differentiate the following functions:

   (a) $f(x) = 5\cosh(3x) - 7\sinh(5x)$

   (b) $f(x) = \tanh(x^2) + 2\operatorname{csch} x$

   (c) $f(x) = \frac{\operatorname{arccosh}(2x)}{\operatorname{arcsinh}(3x)}$

2. Evaluate the following integrals:

   (a) $\displaystyle\int \frac{\cosh x}{\sinh x - 5}\,dx$

   (b) $\displaystyle\int x\tanh(x^2)\,dx$

   (c) $\displaystyle\int \frac{dx}{\sqrt{16x^2+25}}$

   (d) $\displaystyle\int \frac{e^{3x}\,dx}{\sqrt{e^{6x}-4}}$

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## 2.9 M2C Homework set: Solutions

NAME:

GRADE: /16

1. [5] Differentiate the following functions:

(a) [1] $f(x) = 5\cosh(3x) - 7\sinh(5x)$

*Solution.*

$$f'(x) = 15\sinh 3x - 35\cosh(5x).$$

(b) [2] $f(x) = \tanh(x^2) + 2\operatorname{csch} x$

*Solution.*

$$f'(x) = 2x\operatorname{sech}^2(x^2) - 2\cosh x \operatorname{csch} x.$$

(c) [2] $f(x) = \frac{\operatorname{arccosh}(2x)}{\operatorname{arcsinh}(3x)}$

*Solution.*

$$\begin{aligned} f'(x) &= \frac{\frac{2\operatorname{arcsinh}(3x)}{\sqrt{4x^2-1}} - \frac{3\operatorname{arccosh}(2x)}{\sqrt{9x^2+1}}}{(\operatorname{arcsinh}(3x))^2} \\ &= \frac{2\sqrt{9x^2+1}\operatorname{arcsinh}(3x) - 3\sqrt{4x^2-1}\operatorname{arccosh}(2x)}{\sqrt{(4x^2-1)(9x^2+1)}\,(\operatorname{arcsinh}(3x))^2}. \end{aligned}$$

2. [11] Evaluate the following integrals:

(a) [2]

$$\int \frac{\cosh x}{\sinh x - 5}\,dx$$

*Solution.* Let $u = \sinh x - 5$. Then $du = \cosh x\,dx$, so that

$$\int \frac{\cosh x}{\sinh x - 5}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|\sinh x - 5| + C.$$

(b) [3]

$$\int x\tanh(x^2)\,dx$$

*Solution.* Let $u = x^2$. Then $du = 2x\,dx$ and

$$\begin{aligned} \int x\tanh(x^2)\,dx &= \frac{1}{2}\int \tanh u\,du \\ &= \frac{1}{2}\int \frac{\sinh u}{\cosh u}\,du. \end{aligned}$$



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Let $v = \cosh u$ so that $dv = \sinh u \, du$ and

$$\begin{aligned} \int x \tanh(x^2)\,dx &= \frac{1}{2}\int \frac{dv}{v} = \frac{1}{2}\ln|v| + C \\ &= \frac{1}{2}\ln|\cosh(x^2)| + C \\ &= \frac{1}{2}\ln\left(\cosh(x^2)\right) + C. \end{aligned}$$

(c) [3]

$$\int \frac{dx}{\sqrt{16x^2+25}}$$

*Solution.* Let $u = \frac{4x}{5}$ and thus $du = \frac{4}{5}\,dx$ in

$$\begin{aligned} \int \frac{dx}{\sqrt{16x^2+25}} &= \int \frac{dx}{\sqrt{25\left(\left(\frac{4x}{5}\right)^2+1\right)}} \\ &= \frac{1}{5}\cdot\frac{5}{4}\int\frac{du}{\sqrt{u^2+1}} \\ &= \frac{1}{4}\operatorname{arcsinh} u + C \\ &= \frac{1}{4}\operatorname{arcsinh}\left(\frac{4x}{5}\right) + C. \end{aligned}$$

(d) [3]

$$\int \frac{e^{3x}dx}{\sqrt{e^{6x}-4}}$$

*Solution.* Let $u = \frac{1}{2}e^{3x}$. Then $u^2 = \frac{e^{6x}}{4}$ and $du = \frac{3}{2}e^{3x}\,dx$. Thus

$$\begin{aligned} \int \frac{e^{3x}dx}{\sqrt{e^{6x}-4}} &= \int \frac{e^{3x}\,dx}{\sqrt{4\left(\left(\frac{e^{3x}}{2}\right)^2-1\right)}} \\ &= \frac{1}{2}\cdot\frac{2}{3}\int\frac{du}{\sqrt{u^2-1}} \\ &= \frac{1}{3}\operatorname{arccosh} u + C \\ &= \frac{1}{3}\operatorname{arccosh}\left(\frac{e^{3x}}{2}\right) + C. \end{aligned}$$



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# 3 Rule of De l'Hospital

The Worksheet and Homework set M3 should be worked on after studying the material from sections 3.1, 3.2, 3.3, and 3.4 of the youtube workbook.

## 3.1 M3 Worksheet: Rule of De l'Hospital

Evaluate the following limits:

1. $\lim\limits_{x\to\infty} \dfrac{x}{e^x}$
2. $\lim\limits_{x\to\infty} \dfrac{\ln x}{\sqrt{x}}$
3. $\lim\limits_{x\to 0} \dfrac{\tan x}{x}$
4. $\lim\limits_{x\to\pi^-} \dfrac{\sin x}{1-\cos x}$
5. $\lim\limits_{x\to 0} \dfrac{x+\sin(2x)}{x-\sin(2x)}$
6. $\lim\limits_{x\to 0} \dfrac{e^x+e^{-x}-x^2-2}{\sin^2 x - x^2}$
7. $\lim\limits_{x\to 0^+} x^2 \ln x$
8. $\lim\limits_{x\to \frac{\pi}{4}} (1-\tan x)\sec(2x)$
9. $\lim\limits_{x\to 0} \csc x - \dfrac{1}{x}$
10. $\lim\limits_{x\to 0^+} x^{\sin x}$
11. $\lim\limits_{x\to -\infty} x^2 e^x$
12. $\lim\limits_{x\to 0^+} |\ln x|^x$
13. $\lim\limits_{x\to 1} x^{\frac{1}{x-1}}$
14. $\lim\limits_{x\to 0^+} \dfrac{\ln(\sin x)}{\ln(\tan x)}$



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## 3.2 M3 Homework set: Rule of De l'Hospital

Evaluate the following limits:

1. $\displaystyle\lim_{x\to\infty} \frac{x^2}{e^{2x}}$
2. $\displaystyle\lim_{x\to\infty} \frac{x^2}{\ln x}$
3. $\displaystyle\lim_{x\to\pi^-} \frac{\sin x}{1+\cos x}$
4. $\displaystyle\lim_{x\to 0} \frac{\cos 2x - \cos x}{\sin^2 x}$
5. $\displaystyle\lim_{x\to 1} \csc(\pi x)\ln x$
6. $\displaystyle\lim_{x\to 0} \csc x - \cot x$
7. $\displaystyle\lim_{x\to 0^+} (\sin x)^{\tan x}$
8. $\displaystyle\lim_{x\to -\infty} x^2 e^x$

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## 3.3 M3 Homework set: Solutions

NAME:

GRADE: /13

Evaluate the following limits:

1. [1]

$$\lim_{x\to\infty}\frac{x^2}{e^{2x}} \overset{H}{=} \lim_{x\to\infty}\frac{2x}{2e^{2x}} \overset{H}{=} \lim_{x\to\infty}\frac{2}{4e^{2x}} = 0.$$

2. [1]

$$\lim_{x\to\infty}\frac{x^2}{\ln x} \overset{H}{=} \lim_{x\to\infty}\frac{2x}{\frac{1}{x}} = \lim_{x\to\infty} 2x^2 = \infty.$$

3. [1]

$$\lim_{x\to\pi^-}\frac{\sin x}{1+\cos x} \overset{H}{=} \lim_{x\to\pi^-}\frac{\cos x}{-\sin x} = +\infty,$$

because $\cos x \approx -1$ for $x \approx \pi$ and $-\sin x \to 0^-$ as $x \to \pi^-$.

4. [2]

$$\lim_{x\to 0}\frac{\cos 2x - \cos x}{\sin^2 x} \overset{H}{=} \lim_{x\to 0}\frac{-2\sin 2x + \sin x}{2\sin x\cos x} \overset{H}{=} \lim_{x\to 0}\frac{-4\cos 2x + \cos x}{-2\sin^2 x + 2\cos^2 x} = -\frac{3}{2}.$$

5. [2]

$$\lim_{x\to 1}\csc(\pi x)\ln x = \lim_{x\to 1}\frac{\ln x}{\sin(\pi x)} \overset{H}{=} \lim_{x\to 1}\frac{\frac{1}{x}}{\pi\cos(\pi x)} = -\frac{1}{\pi}.$$

6. [2]

$$\begin{aligned}\lim_{x\to 0}\csc x - \cot x &= \lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right)\\ &= \lim_{x\to 0}\frac{1-\cos x}{\sin x} \overset{H}{=} \lim_{x\to 0}\frac{\sin x}{\cos x} = 0.\end{aligned}$$

7. [2] $\lim_{x\to 0^+}(\sin x)^{\tan x}$. It is an indeterminate form of the type $0^0$. Since $(\sin x)^{\tan x} = e^{\tan x\cdot\ln(\sin x)}$,we need to find

$$\begin{aligned}\lim_{x\to 0^+}\tan x\cdot\ln(\sin x) &= \lim_{x\to 0^+}\frac{\ln(\sin x)}{\cot x}\\ &\overset{H}{=} \lim_{x\to 0^+}\frac{\frac{\cos x}{\sin x}}{-\frac{1}{\sin^2 x}} = \lim_{x\to 0^+} -\sin x\cos x = 0,\end{aligned}$$

so that

$$\lim_{x\to 0^+}(\sin x)^{\tan x} = e^0 = 1$$

8. [2]

$$\lim_{x\to -\infty} x^2 e^x = \lim_{x\to -\infty}\frac{x^2}{e^{-x}} \overset{H}{=} \lim_{x\to -\infty}\frac{2x}{-e^{-x}} \overset{H}{=} \lim_{x\to -\infty}\frac{2}{e^{-x}} = 0.$$



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# 4 M4: Integration review and Integration by parts

The Worksheet and Homework set M4A should be worked on after studying the material from sections 4.1 and 4.2 of the youtube workbook.

## 4.1 M4A Worksheet: Review of integration

Evaluate:

1. $\displaystyle\int \frac{2x+1}{\sqrt{2x^2+2x-5}}\,dx$
2. $\displaystyle\int \frac{u+2}{u+1}\,du$
3. $\displaystyle\int \frac{x-3}{x^2+4}\,dx$
4. $\displaystyle\int \frac{dx}{x+2\sqrt{x}}$
5. $\displaystyle\int_0^{\frac{3\pi}{2}} |\sin 2x|\,dx$
6. $\displaystyle\int \cot 3x\,dx$
7. $\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{2-2\cos 2x}\,dx$
8. $\displaystyle\int \frac{e^{2t}}{9+e^{4t}}\,dt$
9. $\displaystyle\int \frac{1}{t\sqrt{1-(\ln t)^2}}\,dt$
10. $\displaystyle\int \frac{2x+3}{\sqrt{1-4x^2}}\,dx$
11. $\displaystyle\int \frac{2+3\ln t}{t\sqrt{1-(\ln t)^2}}\,dt$
12. $\displaystyle\int \frac{4}{x^2+4x+8}\,dx$
13. $\displaystyle\int \frac{dx}{\sqrt{-9+10x-x^2}}$



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## 4.2 M4A Homework set: Review of Integration

Evaluate the following integrals:

1. $\displaystyle\int \frac{dx}{\sqrt{x}(1+\sqrt{x})^3}$
2. $\displaystyle\int \frac{4}{x^2-8x+17}\,dx$
3. $\displaystyle\int_0^{\frac{\pi}{6}} \sqrt{1+\cos 6x}\,dx$
4. $\displaystyle\int \frac{10x^2+3x+1}{5x-1}\,dx$
5. $\displaystyle\int \frac{5x+3}{\sqrt{1-x^2}}\,dx$




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## 4.3 M4A Homework set: Solutions

NAME:

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1. [2] Letting $u = 1 + \sqrt{x}$ , we have $du = \frac{dx}{2\sqrt{x}}$, so that

$$\int \frac{dx}{\sqrt{x}(1+\sqrt{x})^3} = 2\int \frac{du}{u^3} = 2 \cdot \frac{u^{-2}}{-2} + C = -\frac{1}{(1+\sqrt{x})^2} + C.$$

2. [2] By completing the square, we see that

$$x^2 - 8x + 17 = (x-4)^2 + 1$$

so that, letting $u = x - 4$, we have:

$$\int \frac{4}{x^2 - 8x + 17}\,dx = \int \frac{4}{u^2+1}\,du = 4\arctan(x-4) + C.$$

3. [2] Using the double angle formula, we have $\sqrt{1+\cos 6x} = \sqrt{1 + 2\cos^2 3x - 1} = \sqrt{2}|\cos 3x|$ . Moreover $\cos 3x \geq 0$ if $x \in [0, \frac{\pi}{6}]$. Thus

$$\int_0^{\frac{\pi}{6}} \sqrt{1+\cos 6x}\,dx = \sqrt{2}\int_0^{\frac{\pi}{6}} \cos 3x\,dx = \sqrt{2}\left[\frac{1}{3}\sin 3x\right]_0^{\frac{\pi}{6}} = \frac{\sqrt{2}}{3}.$$

4. [3] By long division, we have:

$$\begin{array}{r|l} & \quad 2x + 1 \\ \cline{2-2} 5x - 1 & 10x^2 + 3x + 1 \\ & 10x^2 - 2x \\ \cline{2-2} & \qquad 5x + 1 \\ & \qquad 5x - 1 \\ \cline{2-2} & \qquad\qquad 2 \end{array}$$

so that

$$\frac{10x^2+3x+1}{5x-1} = 2x + 1 + \frac{2}{5x-1}$$

and

$$\int \frac{10x^2+3x+1}{5x-1}\,dx = \int 2x+1\,dx + 2\int \frac{dx}{5x-1} = x^2 + x + \frac{2}{5}\ln|5x-1| + C.$$

5. [2]

$$\begin{aligned} \int \frac{5x+3}{\sqrt{1-x^2}}\,dx &= 5\int \frac{x}{\sqrt{1-x^2}}\,dx + 3\int \frac{dx}{\sqrt{1-x^2}} \\ &= -\frac{5}{2}\int \frac{du}{\sqrt{u}} + 3\arcsin x + C \text{ using } u = 1 - x^2 \text{ and } du = 2x\,dx \\ &= -5\sqrt{1-x^2} + 3\arcsin x + C \end{aligned}$$



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The Worksheet and Homework set M4B should be worked on after studying the material from sections 4.3, 4.4 and 4.5 of the youtube workbook.

## 4.4 M4B Worksheet: Integration by parts

Evaluate

1. $\int x \ln x \, dx$
2. $\int (x+1)e^{2x} \, dx$
3. $\int x^2 e^{3x} \, dx$
4. $\int \ln(x^2+1) \, dx$
5. $\int x^2 \sin(2x) \, dx$
6. $\int x \arctan x \, dx$
7. $\int x \sin(x^2) \, dx$
8. $\int_1^e \frac{\ln x}{x^2} \, dx$
9. Let $n$ be a positive integer. Evaluate
$$\int_0^{2\pi} x \sin(nx) \, dx.$$
10. Let $R$ be the region of the plane bounded by the curve $y = \ln x$, the $x$-axis, and $x = e$. Find the area of this region.



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## 4.5 M4B Homework set: Integration by parts

Evaluate

1. $\displaystyle\int_0^1 xe^{5x}\,dx$

2. $\displaystyle\int x^2\cos x\,dx$

3. $\displaystyle\int e^{2x}\sin(3x)\,dx$

4. $\displaystyle\int \ln(x+2)\,dx$

5. $\displaystyle\int \arcsin(2x)\,dx$

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## 4.6 M4B Homework set: Solutions

NAME:

GRADE: /16

1. [2]

$$\int_0^1 xe^{5x}\,dx$$

Integrating by parts with $u = x$, $dv = e^{5x}\,dx$, we have $du = dx$ and $v = \frac{1}{5}e^{5x}$; thus

$$\int_0^1 xe^{5x}\,dx = \left[\frac{1}{5}xe^{5x}\right]_0^1 - \frac{1}{5}\int e^{5x}\,dx = \left[\frac{xe^{5x}}{5} - \frac{e^{5x}}{25}\right]_0^1 = \frac{4}{25}e^5 + \frac{1}{25}.$$

2. [4: 2 for each integration by parts]

$$\int x^2\cos x\,dx$$

Integrating by parts with $u = x^2$ and $dv = \cos x\,dx$, we have $du = 2x\,dx$ and $v = \sin x$; thus

$$\int x^2\cos x\,dx = x^2\sin x - 2\int x\sin x\,dx.$$

Integrating by parts again with $u = x$ and $dv = \sin x\,dx$, we have $du = dx$ and $v = -\cos x$, so that

$$\begin{aligned}\int x^2\cos x\,dx &= x^2\sin x - 2\left(-x\cos x + \int \cos x\,dx\right)\\ &= x^2\sin x + 2x\cos x - 2\sin x + C\end{aligned}$$

3. [5: 2 for each integration by parts; 1 for solving for $I$]

$$\int e^{2x}\sin(3x)\,dx$$

Integrating by parts with $u = e^{2x}$ and $dv = \sin 3x\,dx$, we have $du = 2e^{2x}\,dx$ and $v = -\frac{1}{3}\cos 3x$, so that

$$\int e^{2x}\sin(3x)\,dx = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\int e^{2x}\cos 3x\,dx.$$

Integrating again by parts with $u = e^{2x}$ and $dv = \cos 3x\,dx$, we have $du = 2e^{2x}$ and $v = \frac{1}{3}\sin 3x$, so that

$$\begin{aligned}\int e^{2x}\sin(3x)\,dx &= -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\left(\frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\int e^{2x}\sin 3x\,dx\right)\\ I &= -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{9}e^{2x}\sin 3x - \frac{4}{9}I,\end{aligned}$$



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where $I = \int e^{2x} \sin 3x \, dx$ is the integral we want to calculate. Solving for $I$, we obtain:

$$I(1 + \frac{4}{9}) = -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{9} e^{2x} \sin 3x,$$

that is,

$$I = -\frac{9}{39} e^{2x} \cos 3x + \frac{2}{13} e^{2x} \sin 3x.$$

4. [2]

$$\int \ln(x+2) \, dx$$

Integrating by parts with $u = \ln(x+2)$ and $dv = dx$, we have $du = \frac{dx}{x+2}$ and $v = x + 2$ (we can, of course, choose a constant of integration that fits our situation), so that

$$\int \ln(x+2) \, dx = (x+2) \ln(x+2) - \int (x+2) \frac{dx}{x+2} = (x+2) \ln(x+2) - x + C.$$

5. [3: 2 by parts; 1 for last integral]

$$\int \arcsin(2x) \, dx$$

Integrating by parts with $u = \arcsin(2x)$ and $dv = dx$, we have $du = \frac{2dx}{\sqrt{1-4x^2}}$ and $v = x$, so that

$$\int \arcsin(2x) \, dx = x \arcsin(2x) - \int \frac{2x}{\sqrt{1-4x^2}} \, dx.$$

Using the substitution $u = 1 - 4x^2$ in the remaining integral, we have $du = -8x \, dx$, so that

$$\int \arcsin x \, dx = x \arcsin x + \frac{1}{4} \int \frac{du}{\sqrt{u}} = x \arcsin(2x) + \frac{1}{2} \sqrt{1-4x^2} + C.$$



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# Mock Test 1

*You should give yourself two hours to do the following test on your own, then, and only then, move to the solutions to evaluate your work. Show all your work to get credit. You should not need a calculator.*

1. Evaluate
   (a) $\log_5 \sqrt{5}$
   (b) $\arccos(\cos \frac{5\pi}{3})$

2. Differentiate
   (a) $f(x) = \sin(e^x)$
   (b) $f(x) = \ln(x^3) + e^{\frac{1}{2}\ln(x)}$
   (c) $f(x) = 2^{\cos x}$
   (d) $f(x) = x^{x^2}$
   (e) $f(x) = \arcsin(x) + \log_3(x^2+1)$
   (f) $f(x) = \arctan(\ln x)$
   (g) $f(x) = x^3 \text{arcsinh}\, 3x$
   (h) $f(x) = \cosh(\sin x) - 2\sinh(\cos x)$

3. Evaluate the following limits:
   (a) $\displaystyle\lim_{x\to 1} \frac{\ln x}{x-1}$
   (b) $\displaystyle\lim_{x\to +\infty} xe^{-x}$
   (c) $\displaystyle\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x}}$
   (d) $\displaystyle\lim_{x\to 0} \frac{\cos x - 1}{x^2}$

4. Evaluate the following integrals:
   (a) $\displaystyle\int_0^1 \left(3^t + 4t^3\right)\, dt$
   (b) $\displaystyle\int \frac{e^x}{4+e^{2x}}\, dx$
   (c) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x}\, dx$
   (d) $\displaystyle\int \frac{x}{\sqrt{4+x^2}}\, dx$
   (e) $\displaystyle\int \frac{3e^{3x}}{\sqrt{1-e^{6x}}}\, dx$



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(f) $\displaystyle\int_0^1 \frac{1}{(1+x^2)(1+(\arctan x)^2)}\,dx$

(g) $\displaystyle\int x\sin(3x)\,dx$

(h) $\displaystyle\int_0^{\frac{1}{2}} \arctan(2x)\,dx$

(i) $\displaystyle\int_1^e \frac{dx}{x\sqrt{1-(\ln x)^2}}$

(j) $\displaystyle\int \frac{dt}{\sqrt{t^2+4}}$

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# Mock Test 1 Solutions

1. Evaluate

   (a) $\log_5 \sqrt{5} = \log_5(5^{\frac{1}{2}}) = \frac{1}{2}$

   (b) $\arccos(\cos \frac{5\pi}{3}) = \arccos(\cos \frac{\pi}{3}) = \frac{\pi}{3}$ because $\cos \frac{5\pi}{3} = \cos \frac{\pi}{3}$ and $\frac{\pi}{3} \in [0, \pi]$.

2. Differentiate

   (a) $f(x) = \sin(e^x)$

$$f'(x) = \cos(e^x)e^x.$$

   (b) $f(x) = \ln(x^3) + e^{\frac{1}{2}\ln(x)} = 3\ln x + x^{\frac{1}{2}}$, so that

$$f'(x) = \frac{3}{x} + \frac{1}{2\sqrt{x}}.$$

   (c) $f(x) = 2^{\cos x}$

$$f'(x) = -2^{\cos x} \ln 2 \sin x.$$

   (d) $f(x) = x^{x^2} = e^{x^2 \ln x}$, so that

$$\begin{aligned} f'(x) &= e^{x^2 \ln x} \left(x^2 \ln x\right)' \\ &= x^{x^2} \left(2x \ln x + x\right). \end{aligned}$$

   (e) $f(x) = \arcsin(x) + \log_3(x^2 + 1)$

$$f'(x) = \frac{1}{\sqrt{1 - x^2}} + \frac{2x}{(x^2 + 1)\ln 3}.$$

   (f) $f(x) = \arctan(\ln x)$

$$f'(x) = \frac{1}{(1 + (\ln x)^2)x}.$$

   (g) $f(x) = x^3 \text{arcsinh}\, 3x$

$$f'(x) = 3x^2 \text{arcsinh}\, 3x + 3x^3 \frac{1}{\sqrt{9x^2 + 1}}.$$

   (h) $f(x) = \cosh(\sin x) - 2\sinh(\cos x)$

$$f'(x) = \sinh(\sin x)\cos x + 2\cosh(\cos x)\sin x.$$



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3. Evaluate the following limits:

(a) $\displaystyle \lim_{x\to 1}\frac{\ln x}{x-1} \overset{H}{=} \lim_{x\to 1}\frac{\frac{1}{x}}{1} = 1.$

(b) $\displaystyle \lim_{x\to +\infty} xe^{-x} = \lim_{x\to\infty}\frac{x}{e^x} \overset{H}{=} \lim_{x\to\infty}\frac{1}{e^x} = 0.$

(c) $\displaystyle \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x}} = \lim_{x\to 0} e^{\frac{1}{x}\ln\left(\frac{\sin x}{x}\right)}$

and, in view of

$$\lim_{x\to 0}\frac{\sin x}{x} \overset{H}{=} \lim_{x\to 0}\frac{\cos x}{1} = 1,$$

we conclude that

$$\begin{aligned}
\lim_{x\to 0}\frac{\ln\left(\frac{\sin x}{x}\right)}{x} \overset{H}{=} \lim_{x\to 0}\frac{\frac{x}{\sin x}\cdot\frac{x\cos x-\sin x}{x^2}}{1} &= \lim_{x\to 0}\frac{x\cos x-\sin x}{x\sin x}\\
&\overset{H}{=} \lim_{x\to 0}\frac{\cos x - x\sin x - \cos x}{\sin x + x\cos x}\\
&\overset{H}{=} \lim_{x\to 0}\frac{-\sin x - x\cos x}{2\cos x - x\sin x} = \frac{0}{2} = 0.
\end{aligned}$$

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Thus,

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x}} = \lim_{x\to 0} e^{\frac{1}{x}\ln\left(\frac{\sin x}{x}\right)} = e^0 = 1.$$

(d) $\displaystyle \lim_{x\to 0}\frac{\cos x - 1}{x^2} \overset{H}{=} \lim_{x\to 0}\frac{-\sin x}{2x} \overset{H}{=} \lim_{x\to 0}\frac{-\cos x}{2} = -\frac{1}{2}.$

4. Evaluate the following integrals:

(a) $\displaystyle \int_0^1 \left(3^t + 4t^3\right)\, dt = \left[\frac{3^t}{\ln 3} + t^4\right]_0^1 = \frac{3}{\ln 3} + 1 - \frac{1}{\ln 3} = \frac{2}{\ln 3} + 1.$

(b) Letting $u = e^x$, we have $du = e^x\,dx$ and $u^2 = e^{2x}$, so that

$$\int \frac{e^x}{4+e^{2x}}\,dx = \int \frac{du}{4+u^2} = \frac{1}{2}\arctan\left(\frac{u}{2}\right) + C = \frac{1}{2}\arctan\left(\frac{e^x}{2}\right) + C$$

(c) Letting $u = \sin x$, we have $du = \cos x\,dx$, so that, taking into account sin 0 =0 and $\sin\frac{\pi}{2} = 1$:

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x}\,dx = \int_0^1 \frac{du}{1+u^2} = [\arctan u]_0^1 = \arctan 1 - \arctan 0 = \frac{\pi}{4}.$$

(d) Letting $u = 4 + x^2$, we have $du = 2x\,dx$ so that

$$\int \frac{x}{\sqrt{4+x^2}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \sqrt{u} + C = \sqrt{4+x^2} + C.$$

(e) Letting $u = e^{3x}$, we have $du = 3e^{3x}\,dx$ and $u^2 = (e^{3x})^2 = e^{6x}$, so that

$$\int \frac{3e^{3x}}{\sqrt{1-e^{6x}}}\,dx = \int \frac{du}{\sqrt{1-u^2}} = \arcsin u + C = \arcsin(e^{3x}) + C.$$

(f) Letting $u = \arctan x$, we have $du = \frac{dx}{1+x^2}$, so that, taking into account that $\arctan 0 = 0$ and $\arctan 1 = \frac{\pi}{4}$, we have:

$$\int_0^1 \frac{1}{(1+x^2)(1+(\arctan x)^2)}\,dx = \int_0^{\frac{\pi}{4}} \frac{du}{1+u^2} = [\arctan u]_0^{\frac{\pi}{4}} = \arctan(\frac{\pi}{4}).$$

(g) We proceed by integration by parts with $u = x$ and $dv = \sin(3x)\,dx$, so that $du = dx$ and $v = -\frac{1}{3}\cos(3x)$ and

$$\int x\sin(3x)\,dx = -\frac{1}{3}x\cos(3x) + \frac{1}{3}\int \cos(3x)\,dx = -\frac{1}{3}x\cos(3x) + \frac{1}{9}\sin(3x) + C.$$



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(h) We proceed by integration by parts with $u = \arctan(2x)$ and $dv = dx$, so that $du = \frac{2}{1+4x^2}\,dx$ and $v = x$ and

$$\int_0^{\frac{1}{2}} \arctan(2x)\,dx = [x\arctan(2x)]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{2x}{1+4x^2}\,dx.$$

For the remaining integral, we proceed by substitution with $u = 1+4x^2$ so that $du = 8x\,dx$ and $2x\,dx = \frac{1}{4}\,du$. Thus

$$\begin{aligned}\int_0^{\frac{1}{2}} \arctan(2x)\,dx &= [x\arctan(2x)]_0^{\frac{1}{2}} - \frac{1}{4}\int_1^2 \frac{du}{u} \\ &= \frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{4}[\ln u]_1^2 = \frac{\pi}{8} - \frac{1}{4}\ln 2.\end{aligned}$$

(i) Letting $u = \ln x$, we have $du = \frac{dx}{x}$, so that, taking into account that $\ln 1 = 0$ and $\ln e = 1$, we have:

$$\int_1^e \frac{dx}{x\sqrt{1-(\ln x)^2}} = \int_0^1 \frac{du}{\sqrt{1-u^2}} = [\arcsin u]_0^1 = \arcsin 1 - \arcsin 0 = \frac{\pi}{2}.$$

(j) $$\int \frac{dt}{\sqrt{t^2+4}} = \int \frac{dt}{\sqrt{4\left(\left(\frac{t}{2}\right)^2+1\right)}} = \frac{1}{2}\int \frac{dt}{\sqrt{\left(\frac{t}{2}\right)^2+1}}$$

so that, letting $u = \frac{t}{2}$ and $du = \frac{dt}{2}$, we have

$$\int \frac{dt}{\sqrt{t^2+4}} = \int \frac{du}{\sqrt{u^2+1}} = \operatorname{arcsinh} u + C = \operatorname{arcsinh}\frac{t}{2} + C.$$



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# 5 M5: Trigonometric integrals and trigonometric substitutions

The Worksheet and Homework set M5A should be worked on after studying the material from sections 5.1, 5.2 and 5.3 of the youtube workbook.

## 5.1 M5A Worksheet: Trig integrals

Evaluate the following integrals:

1. $\displaystyle\int 5\cos(3x) - \sin(2x) + \cot(5x)\,dx$
2. $\displaystyle\int \cos^2(3x)\,dx$
3. $\displaystyle\int \cos^2 x\ \sin x\,dx$
4. $\displaystyle\int \cos^4(2x)\ \sin^3(2x)\,dx$
5. $\displaystyle\int \sin^6 x\,dx$
6. $\displaystyle\int \sin(2x)\ \cos(2x)\,dx$
7. $\displaystyle\int \cos(\pi x)\ \cos(4\pi x)\,dx$
8. Given two *different* integers $k$ and $n$, calculate
$$\int_0^\pi \sin(nx)\sin(kx)\,dx$$
9. What if $n = k$ in the previous question?
10. $\displaystyle\int \tan^2(3x)\,dx.$



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## 5.2 M5A Homework set: Trig integrals

Evaluate the following integrals:

1. $\displaystyle\int 2\cos(2x) - 3\sin(4x) + \tan x\, dx$

2. $\displaystyle\int \cos^3 x \sin^5 x\, dx$

3. $\displaystyle\int \sin^4(2x)\, dx$

4. $\displaystyle\int \cos(2x)\sin(5x) + 2\sin x \sin(8x)\, dx$


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## 5.3 M5A Homework solutions

NAME:

GRADE: /12

1. [3 (1 for each piece)]

$$\begin{aligned}\int 2\cos(2x) - 3\sin(4x) + \tan x\,dx &= \sin 2x + \frac{3}{4}\cos 4x + \int \frac{\sin x}{\cos x}\,dx \\ &= \sin 2x + \frac{3}{4}\cos 4x - \int \frac{du}{u} \text{ for } u = \cos x \\ &= \sin 2x + \frac{3}{4}\cos 4x - \ln|\cos x| + C\end{aligned}$$

2. [3: 1 for *u*; 1 for form, 1 for integral] Letting $u = \sin x$ (and thus $du = \cos x\,dx$ ) and observing that

$$\cos^3 x = \cos^2 x \cdot \cos x = (1 - \sin^2 x) \cdot \cos x,$$

we have

$$\begin{aligned}\int \cos^3 x \sin^5 x\,dx &= \int (1 - \sin^2 x)\sin^5 x\ \cos x\,dx \\ &= \int (1 - u^2)u^5\,du = \int u^5 - u^7\,du \\ &= \frac{\sin^6 x}{6} - \frac{\sin^8 x}{8} + C.\end{aligned}$$

3. [4: 2 for each iteration] When powers are all even, we use the double angle formulas, namely here,

$$\sin^2(2x) = \frac{1 - \cos 4x}{2},$$

so that

$$\begin{aligned}\int \sin^4(2x)\,dx &= \int \left(\frac{1 - \cos 4x}{2}\right)^2\,dx \\ &= \frac{1}{4}\int \cos^2 4x - 2\cos 4x + 1\,dx.\end{aligned}$$

Since

$$\cos^2 4x = \frac{1 + \cos 8x}{2},$$



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we obtain

$$\begin{aligned}
\int \sin^4(2x)\,dx &= \frac{1}{4}\int \frac{1}{2}\cos 8x - 2\cos 4x + \frac{3}{2}\,dx \\
&= \frac{1}{64}\sin 8x - \frac{1}{8}\sin 4x + \frac{3}{8}x + C.
\end{aligned}$$

4. [2: 1 for each piece] Using the formulas

$$\begin{aligned}
\cos a \sin b &= \frac{1}{2}\left(\sin(b-a) + \sin(a+b)\right) \\
\sin a \sin b &= \frac{1}{2}\left(\cos(a-b) - \cos(a+b)\right),
\end{aligned}$$

we have

$$\begin{aligned}
\int \cos(2x)\sin(5x) + 2\sin x \sin(8x)\,dx &= \int \frac{1}{2}\sin 3x + \frac{1}{2}\sin 7x + \cos 7x - \cos 9x\,dx \\
&= -\frac{1}{6}\cos 3x - \frac{1}{14}\cos 7x + \frac{1}{7}\sin 7x - \frac{1}{9}\sin 9x + C.
\end{aligned}$$


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The Worksheet and Homework set M5B should be worked on after studying the material from section 5.4 of the youtube workbook.

## 5.4 M5B Worksheet: Trig substitution

Evaluate

1. $\displaystyle\int \frac{\sqrt{25-4x^2}}{x}\,dx$

2. $\displaystyle\int \frac{dx}{x\sqrt{16+9x^2}}$

3. $\displaystyle\int \frac{x^2}{\sqrt{2x-x^2}}\,dx$

4. $\displaystyle\int \frac{x^2}{(9-x^2)^{\frac{3}{2}}}\,dx$

5. $\displaystyle\int \frac{x}{\sqrt{x^2+1}}\,dx$

6. $\displaystyle\int \frac{dx}{(x^2+9)^2}$

7. $\displaystyle\int \frac{dx}{(x^2+x+1)^2}$



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## 5.5 M5B homework set: trig substitution

Evaluate

1. $\displaystyle\int \frac{x}{\sqrt{x^2+9}}\,dx$
2. $\displaystyle\int \frac{\sqrt{x^2-4}}{x}\,dx$
3. $\displaystyle\int \frac{\sqrt{x^2+4}}{x}\,dx$
4. $\displaystyle\int \frac{dx}{(x^2+9)^3}$

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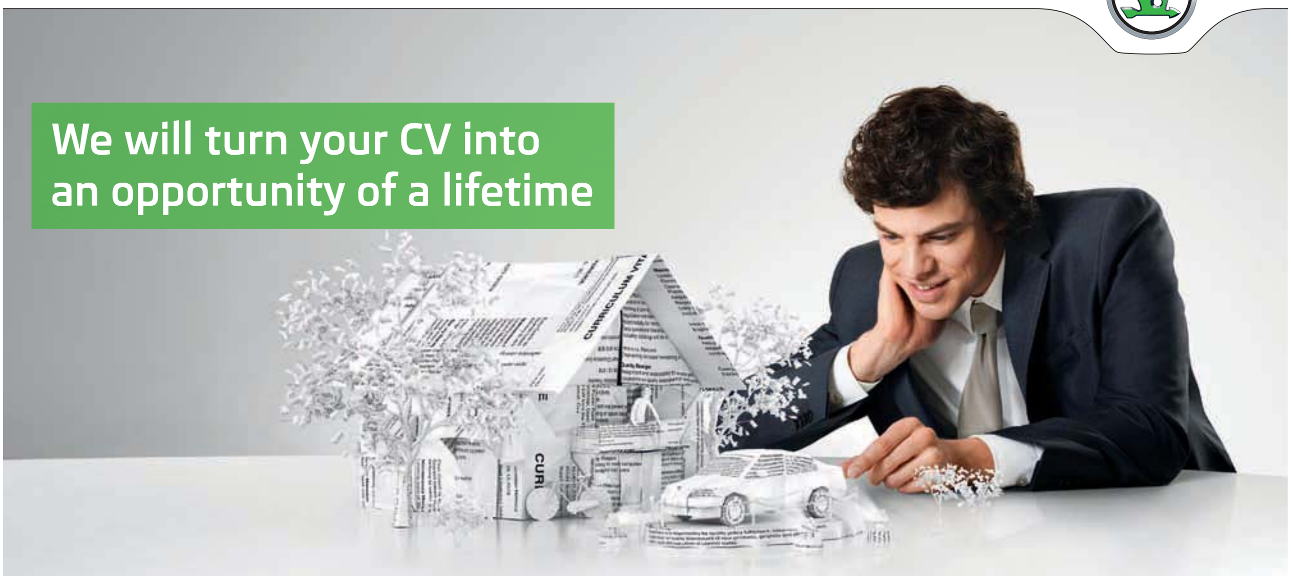


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## 5.6 M5B homework set solutions

NAME:

GRADE: /18

1. [2] There is no need for trig substitution, for we can use the regular substitution $u = x^2 + 9$, with $du = 2x\,dx$, to the effect that

$$\int \frac{x}{\sqrt{x^2+9}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \sqrt{u} + C = \sqrt{x^2+9} + C.$$

2. [4:2 till $\tan\theta - \theta$ ; 2 to go back to $x$] We use the trig substitution $x = 2\sec\theta$ on the interval $[2, \infty)$ for $x$, that is, for $0 \le \theta < \frac{\pi}{2}$. Then $dx = 2\sin\theta\sec^2\theta\,d\theta$ and

$$\begin{aligned}\int \frac{\sqrt{x^2-4}}{x}\,dx &= \int \frac{\sqrt{4(\sec^2\theta - 1)}}{2\sec\theta}\,2\sin\theta\sec^2\theta\,d\theta \text{ and } \sec^2\theta - 1 = \tan^2\theta \\ &= \int \frac{2|\tan\theta|}{2\sec\theta}\,2\sin\theta\sec^2\theta\,d\theta \text{ and } 0 \le \theta < \tfrac{\pi}{2} \\ &= 2\int \tan^2\theta\,d\theta = 2\int \sec^2\theta - 1\,d\theta \\ &= 2(\tan\theta - \theta) + C\end{aligned}$$

and $\cos\theta = \frac{2}{x} = \frac{\text{adjacent}}{\text{hypothenuse}}$ so that the opposite side is $\sqrt{x^2-4}$ by the Pythagorean Theorem, and $\tan\theta = \frac{\sqrt{x^2-4}}{2}$. Moreover, $\theta = \arccos(\frac{2}{x})$. Thus

$$\int \frac{\sqrt{x^2-4}}{x}dx = \sqrt{x^2-4} - 2\arccos(\frac{2}{x}) + C$$

on $[2, \infty)$.

3. [6: 2 till $\int \sec^2\theta\csc\theta d\theta$ , 2 by parts, 2 going back to $x$] We use the trig substitution $x = 2\tan\theta$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, so that $dx = 2\sec^2\theta\,d\theta$ and

$$\begin{aligned}\int \frac{\sqrt{x^2+4}}{x}\,dx &= \int \frac{\sqrt{4(\tan^2\theta + 1)}}{2\tan\theta}\,2\sec^2\theta\,d\theta \text{ and } \tan^2\theta + 1 = \sec^2\theta \\ &= \int \frac{2|\sec\theta|}{2\tan\theta}\cdot 2\sec^2\theta\,d\theta \text{ and } \theta \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \text{ so } \sec\theta > 0 \\ &= 2\int \sec^2\theta\csc\theta\,d\theta.\end{aligned}$$



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Integrating by part with $u = \csc\theta$ and $dv = \sec^2\theta\, d\theta$, we have $du = -\cos\theta\csc^2\theta\, d\theta$ and $v = \tan\theta$, so that

$$\begin{aligned}\int \frac{\sqrt{x^2+4}}{x}\,dx &= 2\csc\theta\tan\theta + 2\int \tan\theta\cos\theta\csc^2\theta\, d\theta\\ &= 2\sec\theta + 2\int \csc\theta\, d\theta\\ &= 2\sec\theta + 2\ln|\csc\theta - \cot\theta| + C\end{aligned}$$

using

$$\int \csc x\, dx = \ln|\csc x - \cot x| + C.$$

Since $\tan\theta = \frac{x}{2} = \frac{\text{opposite}}{\text{adjacent}}$, we conclude that the hypotenuse has length $\sqrt{x^2+4}$ and $\sec\theta = \frac{\sqrt{x^2+4}}{2}$, $\csc\theta = \frac{\sqrt{x^2+4}}{x}$ and $\cot\theta = \frac{2}{x}$, so that

$$\int \frac{\sqrt{x^2+4}}{x}\,dx = \sqrt{x^2+4} + 2\ln\left|\frac{\sqrt{x^2+4}-2}{x}\right| + C.$$

4. [6: 2 till $\int \cos^4\theta d\theta$, 2 for function of $\theta$, 2 to go back to $x$] We use trig substitution with $x = 3\tan\theta$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, so that $dx = 3\sec^2\theta\, d\theta$ and

$$\begin{aligned}\int \frac{dx}{(x^2+9)^3} &= 3\int \frac{\sec^2\theta}{\left(9\tan^2\theta+9\right)^3}\,d\theta\\ &= \frac{3}{9^3}\int \frac{\sec^2\theta}{\left(\sec^2\theta\right)^3}\,d\theta\\ &= \frac{1}{243}\int \cos^4\theta\, d\theta\\ &= \frac{1}{243}\int \left(\frac{1+\cos(2\theta)}{2}\right)^2 d\theta\\ &= \frac{1}{972}\int 1 + 2\cos(2\theta) + \cos^2(2\theta)\, d\theta\\ &= \frac{1}{972}\left(\theta + \sin(2\theta) + \int \frac{1+\cos(4\theta)}{2}\,d\theta\right)\\ &= \frac{1}{972}\left(\frac{3}{2}\theta + \sin(2\theta) + \frac{1}{8}\sin(4\theta)\right) + C.\\ &= \frac{\theta}{648} + \frac{1}{486}\sin\theta\,\cos\theta + \frac{1}{4}\sin(2\theta)\,\cos(2\theta) + C\\ &= \frac{\theta}{648} + \frac{1}{486}\sin\theta\,\cos\theta + \frac{1}{2}\sin\theta\,\cos\theta(2\cos^2\theta - 1) + C\end{aligned}$$

Moreover, $\tan\theta = \frac{x}{3} = \frac{\text{opposite}}{\text{adjacent}}$ so that $\sin\theta = \frac{x}{\sqrt{x^2+9}}$ and $\cos\theta = \frac{3}{\sqrt{x^2+9}}$. Thus

$$\int \frac{dx}{(x^2+9)^3} = \frac{1}{648}\arctan\left(\frac{x}{3}\right) + \frac{3x}{x^2+9}\left(\frac{1}{486} + \frac{9}{x^2+9} - \frac{1}{2}\right) + C.$$



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# 6 M6: Partial Fractions

The Worksheet and Homework set M6A should be worked on after studying the material from sections 6.1 and 6.2 of the youtube workbook.

## 6.1 M6A Worksheet: partial fractions; non-repeated linear factors

Evaluate

1. $\displaystyle\int \frac{6x^2+11x-1}{2x+1}\,dx$
2. $\displaystyle\int \frac{7x+4}{x^2+5x-14}\,dx$
3. $\displaystyle\int \frac{-5x^3-17x^2+x+27}{x^2+4x+3}\,dx$
4. $\displaystyle\int \frac{2x^4-2x^3-2x^2+6x-2}{x^3-x^2-2x}\,dx$
5. $\displaystyle\int \frac{-5x-16}{5x^2+5x-10}\,dx$
6. $\displaystyle\int \frac{19x^2+2x-11}{(x^2-1)(2x-1)(3x+2)}\,dx$



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## 6.2 M6A Homework set: partial fractions; non-repeated linear factors

Evaluate

1. $\displaystyle\int \frac{15x^2+x-6}{3x-1}\,dx$

2. $\displaystyle\int \frac{x+23}{x^2-3x-10}\,dx$

3. $\displaystyle\int \frac{4x^3-5x^2-15x-3}{x^2-x-6}\,dx$




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## 6.3 M6A Homework set: Solutions

NAME:

GRADE: /16

1. [4: 2 for long division, 2 for integral]

$$\int \frac{15x^2+x-6}{3x-1}\,dx$$

*Solution.* By long division

$$\begin{array}{r@{}l} & \quad\;\; 5x+2 \\ 3x-1\big) & \overline{\;15x^2 \;\; +x-6} \\ & -15x^2+5x \\ \hline & \qquad\quad 6x-6 \\ & \qquad -6x+2 \\ \hline & \qquad\qquad -4 \end{array}$$

we have

$$\frac{15x^2+x-6}{3x-1} = 5x+2-\frac{4}{3x-1},$$

so that

$$\begin{aligned} \int \frac{15x^2+x-6}{3x-1}\,dx &= \frac{5}{2}x^2+2x-4\int\frac{dx}{3x-1} \\ &= \frac{5}{2}x^2+2x-\frac{4}{3}\int\frac{du}{u} \text{ where } u=3x-1,\ du=3dx \\ &= \frac{5}{2}x^2+2x-\frac{4}{3}\ln|3x-1|+C. \end{aligned}$$

2. [5: 1 for form of decomposition, 1 for each coefficient, 2 for integral]

$$\int \frac{x+23}{x^2-3x-10}\,dx$$

*Solution.* Since

$$\frac{x+23}{x^2-3x-10} = \frac{x+23}{(x+2)(x-5)} = \frac{A}{x+2}+\frac{B}{x-5},$$

and $A=-3$, $B=4$ are easily obtained by the handcover method, we conclude that

$$\int \frac{x+23}{x^2-3x-10}\,dx = \int \frac{-3}{x+2}+\frac{4}{x-5}\,dx = -3\ln|x+2|+4\ln|x-5|+C.$$



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3. [7: 2 for long division, 1 for form of decomposition, 2 for coefficients, 2 for integral]

$$\int \frac{4x^3 - 5x^2 - 15x - 3}{x^2 - x - 6}\, dx$$

*Solution*. By long division

$$\begin{array}{r}
4x - 1 \\
x^2 - x - 6 \overline{\smash{\big)}\; 4x^3 - 5x^2 - 15x - 3} \\
\underline{-\,4x^3 + 4x^2 + 24x \phantom{{}-3}} \\
-\,x^2 \;\; + 9x - 3 \\
\underline{x^2 \;\;\; - x - 6} \\
8x - 9
\end{array}$$

we have

$$\frac{4x^3 - 5x^2 - 15x - 3}{x^2 - x - 6} = 4x - 1 + \frac{8x - 9}{x^2 - x - 6}.$$

Moreover

$$\frac{8x - 9}{x^2 - x - 6} = \frac{8x - 9}{(x + 2)(x - 3)} = \frac{A}{x + 2} + \frac{B}{x - 3},$$

and $A = 5$, $B = 3$ are easily obtained by handcover method. Thus

$$\begin{aligned}
\int \frac{4x^3 - 5x^2 - 15x - 3}{x^2 - x - 6}\, dx &= \int 4x - 1 + \frac{5}{x + 2} + \frac{3}{x - 3}\, dx \\
&= 2x^2 - x + 5 \ln|x + 2| + 3 \ln|x - 3| + C.
\end{aligned}$$



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The Worksheet and Homework set M6B should be worked on after studying the material from sections 6.3 and 6.4 of the youtube workbook.

## 6.4 M6B Worksheet: partial fractions (repeated linear factors; non-repeated irreducible quadratic factors)

Evaluate

1. $\displaystyle\int \frac{5x^2+5x+27}{x^3+9x}\,dx$
2. $\displaystyle\int \frac{x+4}{x^2-4x+9}\,dx$
3. $\displaystyle\int \frac{4x+1}{2x^2+x+3}\,dx$
4. $\displaystyle\int \frac{4x^4+3x^3+5x^2+5x+1}{x^3+x^2+x}\,dx$
5. $\displaystyle\int \frac{dx}{(x^2-1)(x^3+x^2+x+1)}$



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## 6.5 M6B Homework set: partial fractions (repeated linear factors; non-repeated irreducible quadratic factors)

Evaluate

1. $\displaystyle\int \frac{3x+1}{x^2+4x+5}\,dx$

2. $\displaystyle\int \frac{4x+8}{x^2+4x+5}\,dx$

3. $\displaystyle\int \frac{64x^5+64}{x^4+16x^2}\,dx$

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## 6.6 M6B Homework set: Solutions

NAME:

GRADE: /12

Evaluate

1. [4pts: 1 for completing the square, 1 for splitting, 1 for each integral]

$$\int \frac{3x+1}{x^2+4x+5}\,dx$$

*Solution*. Note that $x^2+4x+5$ has discriminant $-4<0$, so that it is irreducible. Thus we complete the square:

$$x^2+4x+5=(x+2)^2-4+5=(x+2)^2+1,$$

and

$$\begin{aligned}
\int \frac{3x+1}{x^2+4x+5}\,dx &= \int \frac{3x+1}{(x+2)^2+1}\,dx \\
&= \int \frac{3(u-2)+1}{u^2+1}\,du \text{ for } u=x+2 \\
&= 3\int \frac{u}{u^2+1}\,du - 5\int \frac{du}{u^2+1} \\
&= \frac{3}{2}\ln(u^2+1) - 5\arctan u + C \\
&= \frac{3}{2}\ln\left(x^2+4x+5\right) - 5\arctan(x+2) + C.
\end{aligned}$$

2. [2 pts]

$$\int \frac{4x+8}{x^2+4x+5}\,dx$$

*Solution*. We can proceed by substitution with $u=x^2+4x+5$, so that $du=2x+4\,dx$ and

$$\int \frac{4x+8}{x^2+4x+5}\,dx = \int \frac{2du}{u} = 2\ln u + C = 2\ln|x^2+4x+5| + C.$$

3. [6pts:1 for long division, 1 for form of decomposition, 2 for coefficients, 2 for integrals]

$$\int \frac{64x^5+64}{x^4+16x^2}\,dx$$



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*Solution*. By long division

$$\begin{array}{r|l} & \quad 64x \\ \hline x^4+16x^2 & \ 64x^5 \qquad\qquad\quad + 64 \\ & -64x^5 - 1024x^3 \\ \hline & \qquad -1024x^3 \end{array}$$

we see that

$$\frac{64x^5+64}{x^4+16x^2} = 64x + \frac{64-1024x^3}{x^4+16x^2}.$$

Moreover

$$\begin{aligned} \frac{64-1024x^3}{x^4+16x^2} &= \frac{64-1024x^3}{x^2(x^2+16)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+16} \\ &= \frac{Ax(x^2+16)+B(x^2+16)+Cx^3+Dx^2}{x^2(x^2+16)}, \end{aligned}$$

so that

$$64-1024x^3 = (A+C)x^3 + (B+D)x^2 + 16Ax + 16B,$$

and, identifying the coefficients of same degree, we have

$$\begin{cases} A+C=-1024 \\ B+D=0 \\ 16A=0 \\ 16B=64 \end{cases} \iff \begin{cases} A=0 \\ B=4 \\ C=-1024 \\ D=-4 \end{cases}.$$

Thus,

$$\begin{aligned} \int \frac{x^5}{x^4+16x^2}\,dx &= \int 64x + \frac{4}{x^2} - \frac{1024x+4}{x^2+16}\,dx \\ &= 32x^2 - \frac{4}{x} - 512\int \frac{du}{u} - 4\int \frac{dx}{x^2+16} \text{ for } u = x^2+16 \\ &= 32x^2 - \frac{4}{x} - 512\ln(x^2+16) - \arctan\left(\frac{x}{4}\right) + C. \end{aligned}$$



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The Worksheet and Homework set M6C should be worked on after studying the material from section 6.5 of the youtube workbook.

## 6.7 M6C Worksheet: partial fractions (repeated irreducible quadratic factors)

1. What is the form of the decomposition into partial fractions for

$$\frac{2x+1}{(x^2+x-6)(x^2-x-2)(x^2+x+1)^2}\ ?$$

Evaluate:

2. $\displaystyle\int \frac{3x}{(x^2+9)^2}\,dx$
3. $\displaystyle\int \frac{4x+8}{\left(x^2+4x+5\right)^2}\,dx$
4. $\displaystyle\int \frac{x+3}{(x^2+4x+5)^2}\,dx$
5. $\displaystyle\int \frac{dx}{\left(4x^2+9\right)^3}$
6. $\displaystyle\int \frac{7x^5-2x^4+11x^3-3x^2+2x+1}{\left(x^4-1\right)\left(x^2+1\right)}\,dx$

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## 6.8 M6C Homework set: partial fractions (repeated irreducible quadratic factors)

1. What is the form of the decomposition into partial fractions for

$$\frac{2x+1}{(x^2+x-6)(x^2-x-2)(x^2+x+1)^2}\ ?$$

Evaluate:

2. $\displaystyle\int \frac{6x-6}{(x^2-2x+5)^4}\,dx$

3. $\displaystyle\int \frac{1}{(4x^2+16)^2}\,dx$

4. $\displaystyle\int \frac{3x^4+29x^3+139x^2+309x+324}{x\left(x^2+6x+18\right)^2}\,dx$



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## 6.9 M6C Homework set: Solutions

NAME:

GRADE: /19

1. [2pt] What is the form of the decomposition into partial fractions for

$$\frac{2x+1}{(x^2+x-6)(x^2-x-2)(x^2+x+1)^2}\,?$$

*Solution.*

$$\begin{aligned}\frac{2x+1}{(x^2+x-6)(x^2-x-2)(x^2+x+1)^2} &= \frac{2x+1}{(x-2)(x+3)(x-2)(x+1)(x^2+x+1)^2}\\ &= \frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{C}{x+3}+\frac{D}{x+1}+\frac{Ex+F}{x^2+x+1}\\ &\quad +\frac{Gx+H}{(x^2+x+1)^2}.\end{aligned}$$

Evaluate:

2. [2pt]

$$\int \frac{6x-6}{(x^2-2x+5)^4}\,dx$$

*Solution.* Let $u = x^2-2x+5$. Then $du = 2x-2$ so that $6x-6\,dx = 3\,du$ and

$$\int \frac{6x-6}{(x^2-2x+5)^4}\,dx = 3\int \frac{du}{u^4} = -\frac{3}{3u^3} = -\frac{1}{u^3}+C = -\frac{1}{(x^2-2x+5)^3}+C.$$

3. [6pts: 1 for right substitution; 1 till $\int \cos^2\theta\,d\theta$, 2 for explicit function of $\theta$, 2 for back to $x$]

$$\int \frac{1}{(4x^2+16)^2}\,dx$$

*Solution.* Let $x = 2\tan\theta$ for $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$. Then $dx = 2\sec^2\theta\,d\theta$ and

$$\begin{aligned}\int \frac{1}{(4x^2+16)^2}\,dx &= \frac{1}{16}\int \frac{dx}{(x^2+4)^2}\\ &= \frac{1}{16}\int \frac{2\sec^2\theta}{\left(4(1+\tan^2\theta)\right)^2}\,d\theta\\ &= \frac{1}{128}\int \frac{\sec^2\theta}{(\sec^2\theta)^2}\,d\theta\\ &= \frac{1}{128}\int \cos^2\theta\,d\theta\\ &= \frac{1}{256}\int 1+\cos(2\theta)\,d\theta\\ &= \frac{1}{256}\left(\theta+\frac{1}{2}\sin 2\theta\right)+C = \frac{1}{256}\left(\theta+\sin\theta\cos\theta\right)+C.\end{aligned}$$



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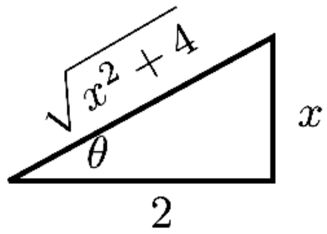
To express this as a function of $x$, note that $\tan\theta = \frac{x}{2}$ and $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, so that $\theta = \arctan\left(\frac{x}{2}\right)$, and moreover, we can represent the situation in the triangle:




Thus,

$$\cos\theta\sin\theta = \frac{2x}{x^2+4}$$

and

$$\int \frac{1}{(4x^2+16)^2}\,dx = \frac{1}{256}\left(\arctan\left(\frac{x}{2}\right) + \frac{2x}{x^2+4}\right) + C.$$

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4. [9pts: 1 for form of decomposition; 2 for coefficients; 1 for completing the square; 3 for $\int \frac{2x+5}{x^2+6x+18}\,dx$ (1 splitting, 1 for each integral); 2 for $\int \frac{x+3}{(x^2+6x+18)^2}\,dx$ ]

$$\int \frac{3x^4+29x^3+139x^2+309x+324}{x\left(x^2+6x+18\right)^2}\,dx$$

*Solution.* The form of the decomposition into partial fractions is

$$\begin{aligned}\frac{3x^4+29x^3+139x^2+309x+324}{x\left(x^2+6x+18\right)^2} &= \frac{A}{x}+\frac{Bx+C}{x^2+6x+18}+\frac{Dx+E}{(x^2+6x+18)^2}\\ &= \frac{A(x^2+6x+18)^2+(Bx+C)x(x^2+6x+18)+Dx^2+Ex}{x\left(x^2+6x+18\right)^2},\end{aligned}$$

so that, identifying the numerators and reordering terms on the righthand side, we have

$$\begin{aligned}3x^4+29x^3+139x^2+309x+324 &= x^4(A+B)+x^3(12A+6B+C)\\ &\quad +x^2(72A+18B+6C+D)+x(216A+18C+E)+324A,\end{aligned}$$

so that, identifying the coefficients of same degree, we have

$$\begin{cases}A+B=3\\ 12A+6B+C=29\\ 72A+18B+6C+D=139\\ 216A+18C+E=309\\ 324A=324\end{cases} \iff \begin{cases}A=1\\ B=2\\ C=5\\ D=1\\ E=3\end{cases}.$$

Thus

$$\int \frac{3x^4+29x^3+139x^2+309x+324}{x\left(x^2+6x+18\right)^2}\,dx = \int \frac{1}{x}+\frac{2x+5}{x^2+6x+18}+\frac{x+3}{(x^2+6x+18)^2}\,dx.$$

Completing the square, we have

$$x^2+6x+18=(x+3)^2-9+18=(x+3)^2+9$$

and, letting $u=x+3$, we have:

$$\begin{aligned}\int \frac{3x^4+29x^3+139x^2+309x+324}{x\left(x^2+6x+18\right)^2}\,dx &= \int \frac{dx}{x}+\int \frac{2(x+3)-1}{(x+3)^2+9}\,dx+\int \frac{x+3}{\left((x+3)^2+9\right)^2}\,dx\\ &= \ln|x|+\int \frac{2u}{u^2+9}\,du-\int \frac{du}{u^2+9}+\int \frac{u}{(u^2+9)^2}\,du\\ &= \ln|x|+\ln(u^2+9)-\frac{1}{3}\arctan\left(\frac{u}{3}\right)+\frac{1}{2}\int \frac{dv}{v^2},\\ &\quad \text{where } v=u^2+9 \text{ and } dv=2u\,du\\ &= \ln|x|+\ln(x^2+6x+18)-\frac{1}{3}\arctan\left(\frac{x+3}{3}\right)\\ &\quad -\frac{1}{2(x^2+6x+18)}+C.\end{aligned}$$



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# 7 M7: Improper Integrals

The Worksheet and Homework set M7A should be worked on after studying the material from section 7.1 of the youtube workbook.

## 7.1 M7A Worksheet: improper integrals, type I

Are the following convergent or divergent? If convergent, find the value.

1. $\displaystyle\int_0^{\infty} xe^{-x^2}\,dx$
2. $\displaystyle\int_0^{\infty} xe^{-2x}\,dx$
3. $\displaystyle\int_1^{\infty} \frac{x^2}{x^3+5}\,dx$
4. $\displaystyle\int_{-\infty}^{0} \frac{dx}{x^2+5}$
5. $\displaystyle\int_1^{\infty} \frac{\ln x}{x}\,dx$
6. $\displaystyle\int_{-\infty}^{\infty} x^2e^{-x^3}\,dx$
7. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{9+x^6}\,dx$
8. $\displaystyle\int_0^{\infty} \frac{dt}{t^2+3t+2}$



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## 7.2 M7A Homework set: Improper Integrals, type I

1. $\displaystyle\int_0^\infty xe^{-3x}\,dx$

2. $\displaystyle\int_{-\infty}^\infty xe^{-x^2}\,dx$

3. $\displaystyle\int_1^\infty \frac{x^3}{x^4+10}\,dx$

4. $\displaystyle\int_1^\infty \frac{\ln x}{x^3}\,dx$

5. $\displaystyle\int_0^\infty \frac{x}{16+x^4}\,dx$

6. $\displaystyle\int_0^\infty \frac{dx}{x^2+4x+4}$

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## 7.3 M7A Homework set: Solutions

NAME:

GRADE: /20

1. [3: 2 for integral, 1 for limit] Using integration by parts with $u = x$ and $dv = e^{-3x}\,dx$ so that $du = dx$ and $v = -\frac{1}{3}e^{-3x}$, we obtain:

$$\begin{aligned}\int_0^\infty xe^{-3x}\,dx &= \lim_{t\to\infty}\int_0^t xe^{-3x}\,dx = \lim_{t\to\infty}\left[-\frac{1}{3}xe^{-3x}\right]_0^t + \frac{1}{3}\int_0^t e^{-3x}\,dx \\ &= \lim_{t\to\infty}\left[-\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x}\right]_0^t \\ &= \lim_{t\to\infty} -\frac{1}{3}te^{-3t} - \frac{1}{9}e^{-3t} + \frac{1}{9} = \frac{1}{9}.\end{aligned}$$

2. [4: 2 for each part] Letting $u = -x^2$ and $du = -2x\,dx$, we have

$$\begin{aligned}\int_0^\infty xe^{-x^2}\,dx &= \lim_{t\to\infty}\int_0^t xe^{-x^2}\,dx = \lim_{t\to\infty} -\frac{1}{2}\int_0^{-t^2} e^u\,du \\ &= \lim_{t\to\infty} -\frac{1}{2}\left(e^{-t^2}-1\right) = \frac{1}{2}.\end{aligned}$$

Similarly,

$$\int_{-\infty}^0 xe^{-x^2}\,dx = \lim_{t\to-\infty} -\frac{1}{2}\int_{-t^2}^0 e^u\,du = \lim_{t\to-\infty}\frac{1}{2}\left(e^{-t^2}-1\right) = -\frac{1}{2}$$

so that

$$\int_{-\infty}^\infty xe^{-x^2}\,dx = \int_{-\infty}^0 xe^{-x^2}\,dx + \int_0^\infty xe^{-x^2}\,dx = -\frac{1}{2} + \frac{1}{2} = 0.$$

3. [3: 2 integral, 1 limit] Using $u = x^4 + 10$ and $du = 4x^3\,dx$, we have

$$\begin{aligned}\int_1^\infty \frac{x^3}{x^4+10}\,dx &= \lim_{t\to\infty}\int_1^t \frac{x^3}{x^4+10}\,dx \\ &= \lim_{t\to\infty}\frac{1}{4}\int_{11}^{t^4+10}\frac{du}{u} \\ &= \lim_{t\to\infty}\frac{1}{4}\left[\ln|u|\right]_{11}^{t^4+10} \\ &= \lim_{t\to\infty}\frac{1}{4}\left(\ln\left|t^4+10\right| - \ln 11\right) = \infty,\end{aligned}$$

and the integral is divergent.



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4. [4: 2 for integral, 2 for limit] We proceed by parts with $u = \ln x$ and $dv = \frac{dx}{x^3}$ so that $du = \frac{dx}{x}$ and $v = -\frac{1}{2x^2}$:

$$\begin{aligned}\int_1^\infty \frac{\ln x}{x^3}\,dx &= \lim_{t\to\infty}\int_1^t \frac{\ln x}{x^3}\,dx = \lim_{t\to\infty}\left[-\frac{\ln x}{2x^2}\right]_1^t + \frac{1}{2}\int_1^t \frac{dx}{x^3} \\ &= \lim_{t\to\infty}\left[-\frac{\ln x}{2x^2} - \frac{1}{4x^2}\right]_1^t \\ &= \lim_{t\to\infty} -\frac{\ln t}{2t^2} - \frac{1}{4t^2} + \frac{1}{4}\end{aligned}$$

and

$$\lim_{t\to\infty}\frac{\ln t}{t^2} \overset{H}{=} \lim_{t\to\infty}\frac{\frac{1}{t}}{2t} = 0$$

so that

$$\int_1^\infty \frac{\ln x}{x^3}\,dx = \lim_{t\to\infty} -\frac{\ln t}{2t^2} - \frac{1}{4t^2} + \frac{1}{4} = \frac{1}{4}.$$

5. [3: 2 integral, 1 limit] Let $u = x^2$. Then $du = 2x\,dx$ and

$$\begin{aligned}\int_0^\infty \frac{x}{16+x^4}\,dx &= \lim_{t\to\infty}\frac{1}{2}\int_0^{t^2}\frac{du}{16+u^2} = \lim_{t\to\infty}\frac{1}{8}\left[\arctan\left(\frac{u}{4}\right)\right]_0^{t^2} \\ &= \lim_{t\to\infty}\frac{1}{8}\arctan\left(\frac{t^2}{4}\right) = \frac{1}{8}\cdot\frac{\pi}{2} = \frac{\pi}{16}.\end{aligned}$$

6. [3: 2 integral, 1 limit]

$$\begin{aligned}\int_0^\infty \frac{dx}{x^2+4x+4} &= \int_0^\infty \frac{dx}{(x+2)^2} \overset{u=x+2}{=} \int_2^\infty \frac{du}{u^2} \\ &= \lim_{t\to\infty}\left[-\frac{1}{u}\right]_2^t = \lim_{t\to\infty}\frac{1}{2} - \frac{1}{t} = \frac{1}{2}.\end{aligned}$$



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The Worksheet and Homework set M7B should be worked on after studying the material from section 7.2 and 7.3 of the youtube workbook.

## 7.4 M7B Worksheet: Improper integral type II and comparison

Are the following integrals convergent or divergent? If convergent, find the value.

1. $\displaystyle\int_{-1}^{0} \frac{dx}{x^2}$

2. $\displaystyle\int_{0}^{\pi} \sec x \, dx$

3. $\displaystyle\int_{0}^{1} \frac{e^x}{e^x - 1} \, dx$

4. $\displaystyle\int_{0}^{2} \frac{dx}{\sqrt{x}}$

5. $\displaystyle\int_{0}^{1} \frac{dx}{\sqrt{1 - x^2}}$

6. $\displaystyle\int_{0}^{1} x^2 \ln x \, dx$

7. $\displaystyle\int_{0}^{\infty} \frac{1}{\sqrt{x}(1 + x)} \, dx$

   (Hint: use $u = \sqrt{x}$ )

   For what value of $p$ are the following integrals convergent?

8. $\displaystyle\int_{0}^{1} \frac{dx}{x^p}$

9. $\displaystyle\int_{0}^{1} x^p \ln x \, dx.$

   Are the following convergent or divergent? You do not need to find the value when convergent.

10. $\displaystyle\int_{1}^{\infty} \frac{\cos^2 x}{1 + x^2} \, dx$

11. $\displaystyle\int_{0}^{\frac{\pi}{2}} \frac{dx}{x \sin x}$

    (Hint: show that $\sin x \leq x$ for $x \geq 0$)

12. $\displaystyle\int_{0}^{1} \frac{|\sin x|}{\sqrt{x}} \, dx$



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## 7.5 M7B Homework set: Improper integral type II and comparison

Are the following integrals convergent or divergent? If convergent, find the value.

1. $\displaystyle\int_{-3}^{0} \frac{dx}{x}$
2. $\displaystyle\int_{0}^{3} \frac{dx}{x^2 + x - 2}$
3. $\displaystyle\int_{-1}^{2} \frac{dx}{x^2 - 2x + 1}$
4. $\displaystyle\int_{0}^{2} \frac{dx}{\sqrt{x}}$
5. $\displaystyle\int_{0}^{\ln \frac{1}{2}} \frac{e^x\,dx}{\sqrt{1 - e^{2x}}}$
6. $\displaystyle\int_{0}^{\infty} \frac{dx}{x^2}$

Are the following convergent or divergent? You do not need to find the value when convergent.

7. $\displaystyle\int_{0}^{\infty} e^{-x^4}\,dx$
8. $\displaystyle\int_{0}^{\frac{\pi}{2}} \frac{dx}{x \cos x}$

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## 7.6 M7B Homework set Solutions

NAME:

GRADE: /23

Are the following integrals convergent or divergent? If convergent, find the value.

1. [2pt]
$$\int_{-3}^{0} \frac{dx}{x}$$

*Solution*. The only discontinuity on $[-3, 0]$ is 0. Thus

$$\int_{-3}^{0} \frac{dx}{x} = \lim_{t\to 0^-} \int_{-3}^{t} \frac{dx}{x} = \lim_{t\to 0^-} [\ln|x|]_{-3}^{t} = \lim_{t\to 0^-} \ln|t| - \ln 3 = -\infty,$$

and the integral is divergent.

2. [3pt: 1 for discontinuity, 2 for integral]
$$\int_{0}^{3} \frac{dx}{x^2+x-2}$$

*Solution*. Since $x^2+x-2 = (x-1)(x+2)$, the only discontinuity on $[0, 3]$ is 1. Moreover

$$\begin{aligned}\frac{1}{(x-1)(x+2)} &= \frac{A}{x-1} + \frac{B}{x+2} \\ &= \frac{1}{3}\left(\frac{1}{x-1} - \frac{1}{x+2}\right)\end{aligned}$$

using handcover method. Thus

$$\begin{aligned}\int_{0}^{1} \frac{dx}{x^2+x-2} &= \lim_{t\to 1^-} \frac{1}{3}\left[\ln|x-1| - \ln|x+2|\right]_0^1 \\ &= \lim_{t\to 1^-} \frac{1}{3}\left(\ln|t-1| - \ln 3 + \ln 2\right) = -\infty,\end{aligned}$$

so that $\int_0^1 \frac{dx}{x^2+x-2}$, and thus also $\int_0^3 \frac{dx}{x^2+x-2}$, is divergent.

3. [3pt: 1 for discontinuity, 1 for integral]
$$\int_{-1}^{2} \frac{dx}{x^2-2x+1}$$

*Solution.* Since $x^2-2x+1 = (x-1)^2$, the only discontinuity on $[-1, 2]$ is 1. Moreover

$$\int_{-1}^{1} \frac{dx}{(x-1)^2} = \lim_{t\to 1^-} \left[-\frac{1}{x-1}\right]_{-1}^{t} = \lim_{t\to 1^-} -\frac{1}{t-1} - \frac{1}{2} = \infty$$



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is divergent and thus, so is $\int_{-1}^{2} \frac{dx}{x^2-2x+1}$.

4. [2pts]

$$\int_0^2 \frac{dx}{\sqrt{x}}$$

*Solution*. The only discontinuity is at 0. Thus

$$\int_0^2 \frac{dx}{\sqrt{x}} = \lim_{t\to 0^+} \int_t^2 \frac{dx}{\sqrt{x}} = \lim_{t\to 0^+} \left[2\sqrt{x}\right]_t^2 = \lim_{t\to 0^+} 2\sqrt{2} - 2\sqrt{t} = 2\sqrt{2}.$$

5. [3pts:2 for integrals, 1 for value]

$$\int_0^{\ln\frac{1}{2}} \frac{e^x dx}{\sqrt{1-e^{2x}}}$$

*Solution*. The only discontinuity is at 0 and

$$\begin{aligned}
\int_0^{\ln\frac{1}{2}} \frac{e^x dx}{\sqrt{1-e^{2x}}} &= \lim_{t\to 0^+} \int_t^{\ln\frac{1}{2}} \frac{e^x}{\sqrt{1-(e^x)^2}}\,dx \\
&= \lim_{t\to 0^+} \int_{e^t}^{\frac{1}{2}} \frac{du}{\sqrt{1-u^2}} \text{ for } u = e^x \\
&= \lim_{t\to 0^+} \left[\arcsin u\right]_{e^t}^{\frac{1}{2}} \\
&= \lim_{t\to 0^+} \arcsin\frac{1}{2} - \arcsin e^t \\
&= \arcsin\frac{1}{2} - \arcsin 1 = \frac{\pi}{6} - \frac{\pi}{2} = -\frac{\pi}{3}.
\end{aligned}$$

6. [3pt: 1 to split, 2 for integral]

$$\int_0^\infty \frac{dx}{x^2}$$

*Solution*. This integral is improper for two reasons: the function $\frac{1}{x^2}$ is discontinuous at 0, and the interval is not bounded.

$$\int_0^\infty \frac{dx}{x^2} = \int_0^1 \frac{dx}{x^2} + \int_1^\infty \frac{dx}{x^2}.$$

While $\int_1^\infty \frac{dx}{x^2}$ is convergent ($p > 1$),

$$\int_0^1 \frac{dx}{x^2} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{x^2} = \lim_{t\to 0^+} \left[-\frac{1}{x}\right]_t^1 = \lim_{t\to 0^+} \frac{1}{t} - 1 = \infty$$

is divergent, and thus, $\int_0^\infty \frac{dx}{x^2}$ is divergent.



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Are the following convergent or divergent? You do not need to find the value when convergent.

7. [3pt: 1 to split, 1 to compare, 1 for integral]

$$\int_0^\infty e^{-x^4}\,dx$$

*Solution*. For $x \geq 1$, we have $x4 \geq x$ and thus

$$e^{-x} \geq e^{-x^4} \geq 0.$$

Since

$$\int_1^\infty e^{-x}\,dx = \lim_{t\to\infty}\left[-e^{-x}\right]_1^t = \lim_{t\to\infty}\frac{1}{e} - \frac{1}{e^t} = \frac{1}{e},$$

we conclude by comparison that $\int_1^\infty e^{-x^4}\,dx$ is also convergent. Since $\int_0^1 e^{-x^4}\,dx$ is finite, the integral

$$\int_0^\infty e^{-x^4}\,dx = \int_0^1 e^{-x^4}\,dx + \int_1^\infty e^{-x^4}\,dx$$

is convergent.

8. [4pt: 1 to split, 2 to compare, 1 for integral]

$$\int_0^{\frac{\pi}{2}} \frac{dx}{x\cos x}$$

*Solution.* Theе function $\frac{1}{x\cos x}$ is discontinuous both at 0 and $\frac{\pi}{2}$. Let us consider $\int_0^{\frac{\pi}{3}} \frac{dx}{x\cos x}$. For $0 \leq x \leq \frac{\pi}{3}$, $\frac{1}{2} \leq \cos x \leq 1$ so that $1 \leq \frac{1}{\cos x} \leq 2$ and

$$\frac{1}{x} \leq \frac{1}{x\cos x} \leq \frac{2}{x}.$$

Thus $\int_0^{\frac{\pi}{3}} \frac{dx}{x\cos x}$ is convergent if and only if $\int_0^{\frac{\pi}{3}} \frac{dx}{x}$ is. Moreover,

$$\int_0^{\frac{\pi}{3}} \frac{dx}{x} = \lim_{t\to0^+}\left[\ln|x|\right]_t^{\frac{\pi}{3}} = \lim_{t\to0^+}\ln\frac{\pi}{3} - \ln t = \infty$$

is divergent. Thus $\int_0^{\frac{\pi}{3}} \frac{dx}{x\cos x}$, and hence $\int_0^{\frac{\pi}{2}} \frac{dx}{x\cos x}$, is divergent.



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# Mock Test 2

*You should give yourself two hours to do the following test on your own, then, and only then, move to the solutions to evaluate your work. Show all your work to get credit. You should not need a calculator.*

Evaluate the integrals in questions 1 through 7:

1. [5]
$$\int \cos^3 x \sin^2 x \, dx$$

2. [10]
$$\int_0^{\pi/4} \cos^4 x \, dx$$

3. [10]
$$\int \frac{\sqrt{4-x^2}}{x^2} \, dx$$

4. [10]
$$\int \frac{x^3}{x^2+x-6} \, dx$$

5. [10]
$$\int_2^3 \frac{x^2+1}{(x-1)^3} \, dx$$

6. [10]
$$\int \frac{3x^2-x+9}{x^3+9x} \, dx$$

7. [10]
$$\int_{-3}^{-2} \frac{2\, dx}{x^2+6x+10}$$

8. [5] Give the form of the decomposition into partial fractions (you do NOT need to find the coefficients) of
$$\frac{2x^4+3}{x^2(x-1)(x^2+4)^2(x^2+x-2)}$$



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9. [5] Is the following integral convergent or divergent? If convergent, find its value

$$\int_1^\infty xe^{-x^2}\,dx$$

10. [5] Is the following integral convergent or divergent? If convergent, find its value

$$\int_1^\infty \frac{dx}{3x+5}$$

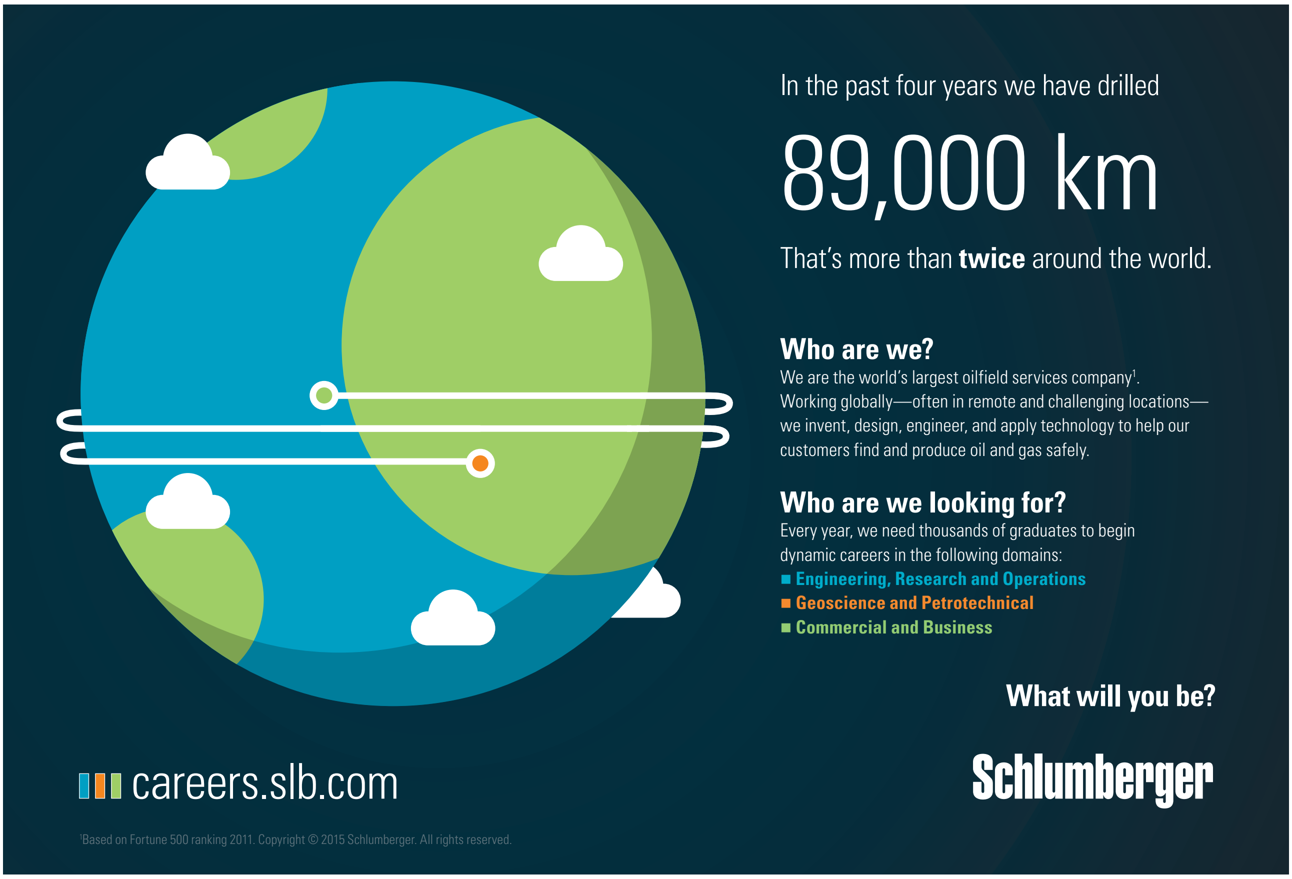
11. [10] Is the following integral convergent or divergent? If convergent, find its value

$$\int_{-1}^{2} \frac{2x}{x^3+2x^2}\,dx$$

12. [10] Are the following improper integrals convergent or divergent. Justify your answer

(a) $\displaystyle\int_1^\infty \frac{x^2}{\sqrt{3+x^8}}\,dx$

(b) $\displaystyle\int_1^\infty \frac{1+|\sin x|}{x}\,dx$



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# Mock Test 2 Solutions

1.

$$\int \cos^3 x \sin^2 x\, dx$$

*Solution.* Let $u = \sin x$ then $du = \cos x\, dx$ and

$$\begin{aligned}
\int \cos^3 x \sin^2 x\, dx &= \int \cos^2 x \sin^2 x\ \cos x\, dx \\
&= \int (1 - \sin^2 x) \sin^2 x\ \cos x\, dx \\
&= \int (1 - u^2) u^2\, du \\
&= \int u^2 - u^4\, du = \frac{u^3}{3} - \frac{u^5}{5} + C \\
&= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.
\end{aligned}$$

2.

$$\int_0^{\pi/4} \cos^4 x\, dx$$

*Solution.* We use the double angle formula $\cos^2 x = \frac{\cos(2x)+1}{2}$ :

3.

$$\int \frac{\sqrt{4 - x^2}}{x^2}\, dx$$

*Solution.* Let $x = 2\sin\theta$ where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = 2\cos\theta\, d\theta$ and

$$\begin{aligned}
\int \frac{\sqrt{4 - x^2}}{x^2}\, dx &= \int \frac{\sqrt{4 - 4\sin^2\theta}}{4\sin^2\theta}\, 2\cos\theta\, d\theta \\
&= \int 4\cos\theta \frac{\sqrt{1 - \sin^2\theta}}{4\sin^2\theta}\, d\theta \\
&= \int \frac{\cos\theta\sqrt{\cos^2\theta}}{\sin^2\theta}\, d\theta \\
&= \int \frac{\cos^2\theta}{\sin^2\theta}\, d\theta = \int \frac{1 - \sin^2\theta}{\sin^2\theta}\, d\theta \\
&= \int \csc^2\theta - 1\, d\theta = -\cot\theta - \theta + C.
\end{aligned}$$



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Moreover, $\sin\theta = \frac{x}{2}$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, so that $\theta = \arcsin(\frac{x}{2})$ and, using a right triangle, $\cot\theta = \frac{\sqrt{4-x^2}}{x}$. Thus

$$\int \frac{\sqrt{4-x^2}}{x^2}\,dx = -\frac{\sqrt{4-x^2}}{x} - \arcsin(\frac{x}{2}) + C.$$

4.

$$\int \frac{x^3}{x^2+x-6}\,dx$$

*Solution.* The degree of the numerator is not less than that of the denominator, so that we need to start with long division

$$\begin{array}{r|l} & \quad x-1 \\ \hline x^2+x-6 & \ x^3 \\ & -x^3-x^2+6x \\ \hline & \quad -x^2+6x \\ & \quad\ x^2+x-6 \\ \hline & \qquad 7x-6 \end{array}$$

Thus

$$\int \frac{x^3}{x^2+x-6}\,dx = \int x-1+\frac{7x-6}{x^2+x-6}\,dx$$

and

$$\frac{7x-6}{x^2+x-6} = \frac{7x-6}{(x-2)(x+3)} = \frac{A}{x-2}+\frac{B}{x+3}.$$

Using for instance the hand-cover method, we find $A = \frac{8}{5}$ and $B = \frac{-27}{-5}$. Thus

$$\begin{aligned} \int \frac{x^3}{x^2+x-6}\,dx &= \int x-1+\frac{8}{5}\cdot\frac{1}{x-2}+\frac{27}{5}\cdot\frac{1}{x+3}\,dx \\ &= \frac{x^2}{2}-x+\frac{8}{5}\ln|x-2|+\frac{27}{5}\ln|x+3|+C \end{aligned}$$

5.

$$\int_2^3 \frac{x^2+1}{(x-1)^3}\,dx$$

*Solution.* The decomposition into partial fractions is

$$\frac{x^2+1}{(x-1)^3} = \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}.$$



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To find $A$, $B$ and $C$, we identify the numerators after rewriting the left-hand side with denominator $(x-1)^3$:

$$x^2+1 = A(x-1)^2 + B(x-1) + C,$$

and we set $x = 1$ to the effect that $2 = C$. Differentiating, we obtain

$$2x = 2A(x-1) + B,$$

and we set $x = 1$ to the effect that $2 = B$. differentiating, we obtain

$$2 = 2A$$

to the effect that $A = 1$. Thus

$$\begin{aligned}
\int_2^3 \frac{x^2+1}{(x-1)^3}\,dx &= \int_2^3 \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{2}{(x-1)^3}\,dx \\
&= \left[\ln|x-1| - \frac{2}{x-1} - \frac{1}{(x-1)^2}\right]_2^3 \\
&= \left(\ln 2 - 1 - \frac{1}{4}\right) - (-2-1) = \ln 2 + \frac{7}{4}.
\end{aligned}$$

6.

$$\int \frac{3x^2-x+9}{x^3+9x}\,dx$$

*Solution.* The decomposition into partial fractions is

$$\frac{3x^2-x+9}{x^3+9x} = \frac{3x^2-x+9}{x(x^2+9)} = \frac{A}{x} + \frac{Bx+C}{x^2+9}.$$

Identifying the numerators of fractions with denominator $x(x^2+9)$ we have

$$\begin{aligned}
3x^2-x+9 &= A(x^2+9) + Bx^2 + Cx \\
&= x^2(A+B) + Cx + 9A
\end{aligned}$$



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so that the equality holds if $A + B = 3$, $C = -1$ and $9A = 9$. Thus $A = 1$, $B = 3 - A = 2$ and $C = -1$. Therefore

$$\begin{aligned}\int \frac{3x^2 - x + 9}{x^3 + 9x}\,dx &= \int \frac{1}{x} + \frac{2x-1}{x^2+9}\,dx \\ &= \ln|x| + \int \frac{2x}{x^2+9}\,dx - \int \frac{dx}{x^2+9} \\ &= \ln|x| + \ln(x^2+9) - \frac{1}{3}\arctan(\frac{x}{3}) + C.\end{aligned}$$

7.

$$\int_{-3}^{-2} \frac{2\,dx}{x^2 + 6x + 10}$$

We complete the square at the denominator:

$$x^2 + 6x + 10 = (x+3)^2 - 9 + 10 = (x+3)^2 + 1,$$

so that

$$\int_{-3}^{-2} \frac{2\,dx}{x^2 + 6x + 10} = 2\int_{-3}^{-2} \frac{dx}{(x+3)^2 + 1}.$$

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Letting $u = x + 3$, we have

$$\int_{-3}^{-2} \frac{2\,dx}{x^2 + 6x + 10} = 2\int_0^1 \frac{du}{u^2+1} = 2\left[\arctan u\right]_0^1 = 2 \cdot \frac{\pi}{4} = \frac{\pi}{2}.$$

8. *Solution:* Give the form of the decomposition into partial fractions (you do NOT need to find the coefficients) of

$$\begin{aligned}\frac{2x^4+3}{x^2(x-1)(x^2+4)^2(x^2+x-2)} &= \frac{2x^4+3}{x^2(x-1)(x^2+4)^2(x-1)(x+2)} \\ &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{x+2} + \frac{Fx+G}{x^2+4} + \frac{Hx+I}{(x^2+4)^2}\end{aligned}$$

9. Is the following integral convergent or divergent? If convergent, find its value

$$\int_1^\infty xe^{-x^2}\,dx$$

*Solution.* By definition

$$\int_1^\infty xe^{-x^2}\,dx = \lim_{t\to\infty}\int_1^t xe^{-x^2}\,dx.$$

In this definite integral, we use the substitution $u = -x^2$ so that $du = -2x\,dx$ and we have

$$\begin{aligned}\int_1^\infty xe^{-x^2}\,dx &= \lim_{t\to\infty}\int_1^t xe^{-x^2}\,dx \\ &= \lim_{t\to\infty} -\frac{1}{2}\int_{-1}^{-t^2} e^u\,du \\ &= \lim_{t\to\infty} -\frac{1}{2}\left[e^u\right]_{-1}^{-t^2} = \lim_{t\to\infty}\frac{1}{2e} - \frac{1}{2e^{t^2}} = \frac{1}{2e}.\end{aligned}$$

10. Is the following integral convergent or divergent? If convergent, find its value

$$\int_1^\infty \frac{dx}{3x+5}$$

*Solution.* This integral is divergent because

$$\begin{aligned}\int_1^\infty \frac{dx}{3x+5} &= \lim_{t\to\infty}\int_1^t \frac{dx}{3x+5} \\ &\overset{u=3x+5}{=} \lim_{t\to\infty}\frac{1}{3}\int_8^{3t+5}\frac{du}{u} \\ &= \lim_{t\to\infty}\frac{1}{3}\left(\ln|3t+5| - \ln 8\right) = \infty.\end{aligned}$$



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11. Is the following integral convergent or divergent? If convergent, find its value

$$\int_{-1}^{2} \frac{2x}{x^3+2x^2}dx$$

*Solution.* This is an improper integral because $x^3+2x^2 = x^2(x+2)$ so that $\frac{2x}{x^3+2x^2}$ is discontinuous at 0. Moreover

$$\int_{-1}^{2} \frac{2x}{x^3+2x^2}dx = \int_{-1}^{0} \frac{2x}{x^3+2x^2}dx + \int_{0}^{2} \frac{2x}{x^3+2x^2}dx$$

is convergent if and only if both integrals are convergent. Since

$$\frac{2x}{x^3+2x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2},$$

we see that

$$2x = Ax(x+2) + B(x+2) + Cx^2.$$

Setting $x=-2$ gives $-4=4C$ and $C=-1$. Setting $x=0$ gives $0=2B$ and $B=0$ ([1]). Differentiating, we have

$$2 = A(x+2) + Ax + 2Cx,$$

where setting $x=0$ yields $2=2A$ and $A=1$. That is

$$\begin{aligned} \int_0^2 \frac{2x}{x^3+2x^2}\,dx &= \lim_{t\to 0^+} \int_t^2 \frac{1}{x} - \frac{1}{x+2}\,dx \\ &= \lim_{t\to 0^+} \left[\ln|x| - \ln|x+2|\right]_t^2 \\ &= \infty, \end{aligned}$$

because $\lim_{t\to 0^+} \ln|t| = -\infty$. Thus the integral is divergent.

12. Are the following improper integrals convergent or divergent. Justify your answer

(a) $$\int_1^\infty \frac{x^2}{\sqrt{3+x^8}}\,dx$$

*Solution.* We use comparison: $3+x^8 \geq x^8$, so that $\sqrt{3+x^8} \geq x^4$ and

$$0 \leq \frac{x^2}{\sqrt{3+x^8}} \leq \frac{x^2}{x^4} = \frac{1}{x^2}.$$

Since $\int_1^\infty \frac{dx}{x^2}$ is convergent, we conclude by comparison that $\int_1^\infty \frac{x^2}{\sqrt{3+x^8}}\,dx$ is convergent.



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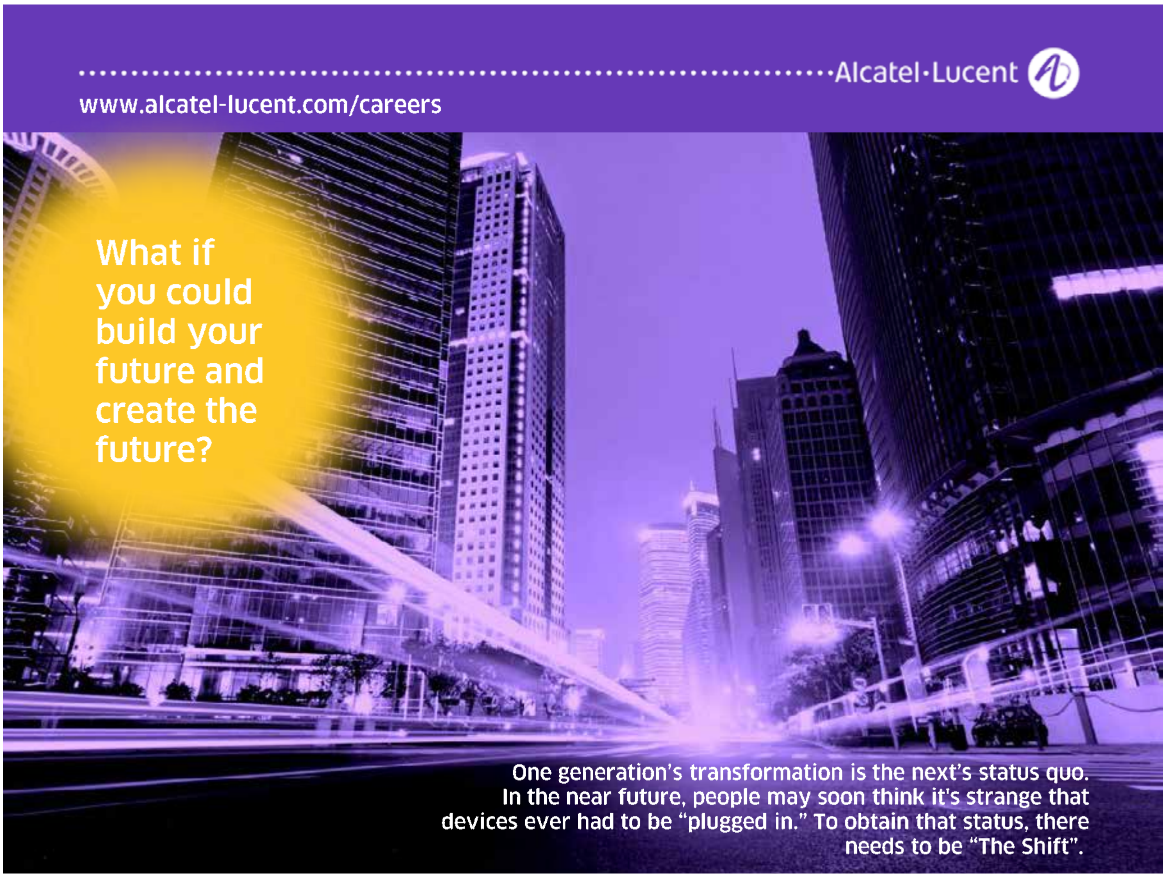
(b) $\displaystyle\int_1^\infty \frac{1+|\sin x|}{x}\,dx$

*Solution.* Since $1+|\sin x| \geq 1$ we conclude that

$$0 \leq \frac{1}{x} \leq \frac{1+|\sin x|}{x}$$

for all $x \geq 1$, so that, by comparison $\int_1^\infty \frac{1+|\sin x|}{x}\,dx$ is divergent, because $\int_1^\infty \frac{dx}{x}$ is divergent.

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# 8 M8: Parametric Curves

The Worksheet and Homework set M8A should be worked on after studying the material from sections 8.1, 8.2, and 8.3 of the youtube workbook.

## 8.1 M8A Worksheet: parametric curves

1. Find an equation of the tangent line to $\begin{cases} x = t^4 + 1 \\ y = t^3 + t \end{cases}$ at $t = -2$.

2. Find an equation of the tangent line to $\begin{cases} x = \cos\theta + \sin 2\theta \\ y = \sin\theta + \cos 2\theta \end{cases}$ at $\theta = \frac{\pi}{4}$.

3. Verify that $(0, 0)$ is a point of self-intersection of $\begin{cases} x = \sin t \\ y = \sin(t + \sin t) \end{cases}$. Find an equation of the tangent lines at $(0, 0)$.

4. Find the points of the curve where the tangent is horizontal or vertical, then sketch the curve
$$\begin{cases} x = \cos 3\theta \\ y = 2\sin\theta \end{cases}$$

5. At what point does the curve $\begin{cases} x = 1 - 2\cos^2 t \\ y = (\tan t)\left(1 - 2\cos^2 t\right) \end{cases}$ cross itself? Find the equations of the tangents at that point.

6. For which value of *t* is the curve $\begin{cases} x = \cos t \\ y = \cos 2t \end{cases}$ $, 0 < t < \pi$ concave up?



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## 8.2 M8A Homework set: parametric curves

1. Find an equation of the tangent line to $\begin{cases} x = t^3 - 2t \\ y = t^2 - 3 \end{cases}$ for $t = \sqrt{2}$ and for $t = -\sqrt{2}$, then sketch the graph and the tangent(s).

2. Find horizontal and vertical tangent lines to $\begin{cases} x = 1 + 2\cos t \\ y = -2 + 3\sin t \end{cases}$ then sketch the curve.

3. At what point(s) does the curve $\begin{cases} x = \sin(2t) \\ y = \cos t \end{cases}$ cross itself? Find the tangent lines at this point, then sketch the curve.

4. For which value of $t$ is the curve $\begin{cases} x = 2\sin t \\ y = \cos t \end{cases}$ $, 0 < t < \pi$ concave up?

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## 8.3 M8A Homework set: Solutions

NAME:

GRADE: /29

1. [8pts: 1 for formula for $\frac{dy}{dx}$, 2 for the equations of tangent lines, 2 for variations, 2 for special points, 1 for sketch] Find an equation of the tangent line to $\begin{cases} x = t^3 - 2t \\ y = t^2 - 3 \end{cases}$ for $t = \sqrt{2}$ and for $t = -\sqrt{2}$, then sketch the graph and the tangent(s).

*Solution*. The points of parameters $\sqrt{2}$ and $-\sqrt{2}$ are the same:

$$\left(x(\sqrt{2}), y(\sqrt{2})\right) = (0, -1) = \left(x(-\sqrt{2}), y(-\sqrt{2})\right).$$

Since

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{2t}{3t^2 - 2},$$

the slope of the tangent line at $(0, -1)$ for $t = \sqrt{2}$ is $\frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2}$ while the slope of the tangent at $(0, -1)$ for $t = -\sqrt{2}$ is $-\frac{\sqrt{2}}{2}$. The two corresponding tangent lines have equations

$$y + 1 = \frac{\sqrt{2}}{2}x \text{ and } y + 1 = -\tfrac{\sqrt{2}}{2}x.$$

To sketch the curve, we first study the variations of $x(t)$ and $y(t)$:

$$\begin{aligned} x'(t) &= 3t^2 - 2 = 3\left(t - \sqrt{\frac{2}{3}}\right)\left(t + \sqrt{\frac{2}{3}}\right) \\ y'(t) &= 2t \end{aligned}$$

so that the variations can be represented in the chart:

| $t$ | $(-\infty, -\sqrt{\frac{2}{3}})$ | $(-\sqrt{\frac{2}{3}}, 0)$ | $(0, \sqrt{\frac{2}{3}})$ | $(\sqrt{\frac{2}{3}}, \infty)$ |
|---|---|---|---|---|
| $x'(t)$ | $+$ | $-$ | | $+$ |
| $x(t)$ | $\nearrow$ | $\searrow$ | | $\nearrow$ |
| $y'(t)$ | $-$ | | $+$ | |
| $y(t)$ | $\searrow$ | | $\nearrow$ | |

where

$$x\left(-\sqrt{\frac{2}{3}}\right) = \frac{4}{3}\sqrt{\frac{2}{3}}$$



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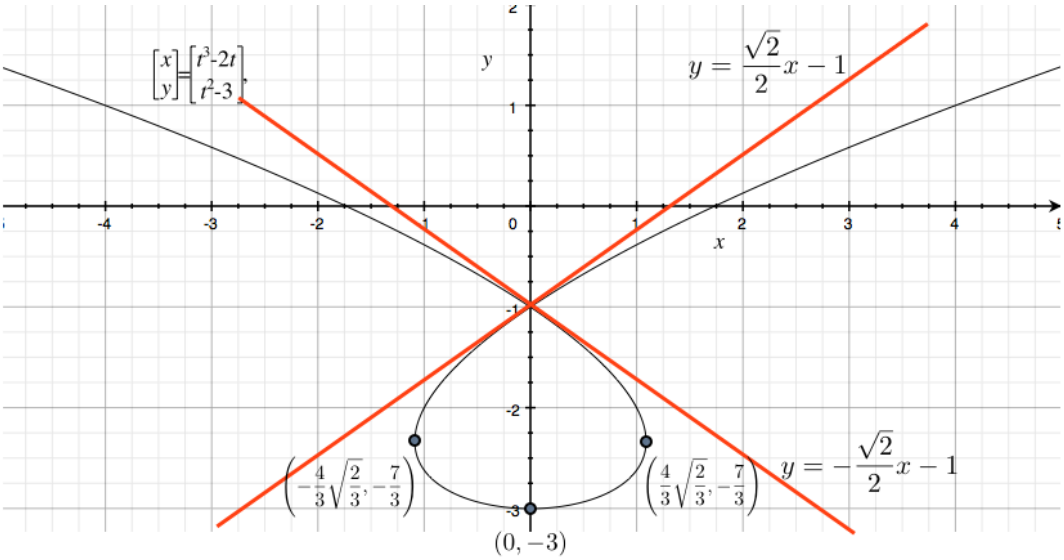
is a local maximum for *x*,

$$x\left(\sqrt{\frac{2}{3}}\right) = -\frac{4}{3}\sqrt{\frac{2}{3}}$$

is a local minimum for *x*, and $y(0) = -3$ is the absolute minimum for *y*. Hence, the important following points are important features of the curve:

$$\begin{array}{ll} (0,-1) & \text{(self-intersection)} \\ (0,-3) & (y \text{ is minimal}) \\ \left(-\frac{4}{3}\sqrt{\frac{2}{3}}, -\frac{7}{3}\right) & (\text{local min for } x) \\ \left(\frac{4}{3}\sqrt{\frac{2}{3}}, -\frac{7}{3}\right) & (\text{local max for } x). \end{array}$$

Following the variations of *x* and *y*, we obtain the following sketch:




2. [8 pts: 1 for $\frac{dy}{dx}$, 2 for horizontal tangents, 2 for vertical tangents, 2 for variations, 1 for sketch] Find horizontal and vertical tangent lines to $\begin{cases} x = 1 + 2\cos t \\ y = -2 + 3\sin t \end{cases}$ then sketch the curve.

*Solution.* We only need to consider $t \in [0, 2\pi]$ as *x* and *y* are both $2\pi$-periodic. The slope of the tangent line at the point of parameter *t* is

$$\frac{y'(t)}{x'(t)} = \frac{3\cos t}{-2\sin t},$$



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so that the curve has an horizontal tangent when $\cos t = 0$, that is, for $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$, and a vertical tangent when $\sin t = 0$, that is, when $t = 0$ or $t = \pi$. The horizontal tangent lines go through

$$\left(x\left(\frac{\pi}{2}\right), y\left(\frac{\pi}{2}\right)\right) = (1,1) \text{ and } \left(x\left(\frac{3\pi}{2}\right), y\left(\frac{3\pi}{2}\right)\right) = (1,-5)$$

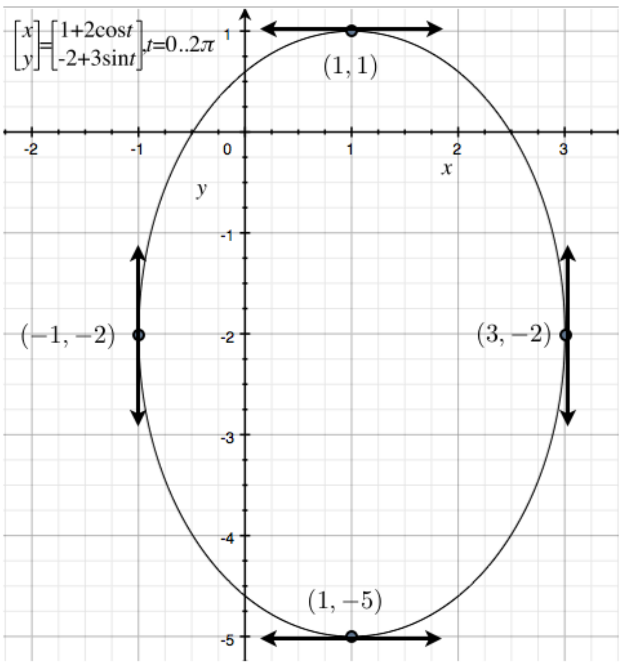
and have therefore equations $y = 1$ and $y = -5$. The vertical tangent lines go through

$$\left(x(0), y(0)\right) = (3,-2) \text{ and } \left(x(\pi), y(\pi)\right) = (-1,-2)$$

and have therefore equations $x = 3$ and $x = -1$. To sketch the graph, we may either recognize an equation of ellipse centered at $(1,-2)$, or study the variation of $x(t)$ and $y(t)$:

| $t$ | $(0, \frac{\pi}{2})$ | $(\frac{\pi}{2}, \pi)$ | $(\pi, \frac{3\pi}{2})$ | $(\frac{3\pi}{2}, 2\pi)$ |
|---|---|---|---|---|
| $x'$ | $-$ | | $+$ | |
| $x$ | $\searrow$ | | $\nearrow$ | |
| $y'$ | $+$ | $-$ | | $+$ |
| $y$ | $\nearrow$ | $\searrow$ | | $\nearrow$ |

Thus *x* has an absolute minimum of –1 when $t = \pi$ and *y* has a maximum of 1 when $t = \frac{\pi}{2}$ and a minimum of –5 when $t = \frac{3\pi}{2}$, and we obtain



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3. [9pts: 2 for self intersection point; 2 for tangents; 2 for variations, 2 for special points, 1 for sketch] At what point(s) does the curve $\begin{cases} x = \sin(2t) \\ y = \cos t \end{cases}$ cross itself? Find the tangent lines at this point, then sketch the curve.

*Solution.* We can restrict ourselves to $t \in [0, 2\pi]$. The curve crosses itself if

$$\begin{cases} \sin(2t_1) = \sin(2t_2) \\ \cos(t_1) = \cos(t_2) \end{cases} \iff \begin{cases} \sin(t_1)\cos(t_1) = \sin(t_2)\cos(t_2) \\ \cos(t_1) = \cos(t_2) \end{cases} \iff \begin{cases} \cos(t_1)\left(\sin t_1 - \sin t_2\right) = 0 \\ \cos(t_1) = \cos(t_2) \end{cases},$$

that is, if $\cos(t_1) = \cos(t_2) = 0$ or if both $\sin t_1 = \sin t_2$ and $\cos t_1 = \cos t_2$ which means $t_1 = t_2$ if we restrict ourselves to $t \in [0, 2\pi]$. Thus the only self intersection correspond to $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$ both of which give the point

$$\left(x\left(\frac{\pi}{2}\right), y\left(\frac{\pi}{2}\right)\right) = (0, 0).$$

The corresponding tangent lines have slope

$$\frac{y'}{x'}\left(\frac{\pi}{2}\right) = \frac{-\sin\left(\frac{\pi}{2}\right)}{2\cos\pi} = \frac{1}{2} \text{ and } \frac{y'}{x'}\left(\frac{\pi}{2}\right) = \frac{-\sin\left(\frac{3\pi}{2}\right)}{2\cos 3\pi} = -\frac{1}{2}$$




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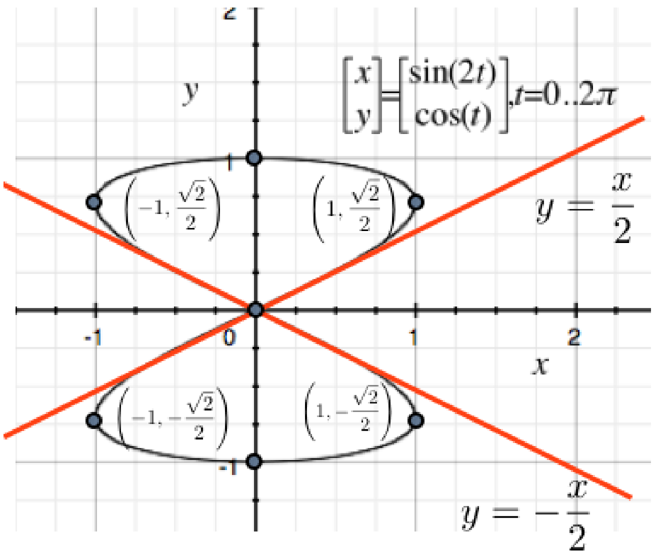
and have therefore equations $y = \frac{x}{2}$ and $y = -\frac{x}{2}$. To sketch the graph, we study the variations of $x(t)$ and $y(t)$:

| $t$ | $(0, \frac{\pi}{4})$ | $(\frac{\pi}{4}, \frac{3\pi}{4})$ | $(\frac{3\pi}{4}, \pi)$ | $(\pi, \frac{5\pi}{4})$ | $(\frac{5\pi}{4}, \frac{7\pi}{4})$ | $(\frac{7\pi}{4}, 2\pi)$ |
|---|---|---|---|---|---|---|
| $x'$ | $+$ | $-$ | $+$ | | $-$ | $+$ |
| $x$ | $\nearrow$ | $\searrow$ | $\nearrow$ | | $\searrow$ | $\nearrow$ |
| $y'$ | | $-$ | | | $+$ | |
| $y$ | | $\searrow$ | | | $\nearrow$ | |

Thus the curve has the following important points:

$$
\begin{array}{rl}
(0,1) & \text{where } y \text{ is maximal} \\
\left(1, \dfrac{\sqrt{2}}{2}\right) & \text{where } x \text{ is maximal} \\
(0,0) & \text{where the curve self-intersects for } t = \frac{\pi}{2}; \frac{3\pi}{2} \\
\left(-1, -\dfrac{\sqrt{2}}{2}\right) & \text{where } x \text{ is minimal} \\
(0,-1) & \text{where } y \text{ is minimal} \\
\left(1, -\dfrac{\sqrt{2}}{2}\right) & \text{where } x \text{ is minimal}
\end{array}
$$

We obtain this way the following sketch:



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4. [4pts: 2 for $\frac{d^2y}{dx^2}$, 2 sign and for interpretation in terms of concavity] For which value of $t$ is the curve $\begin{cases} x = 2\sin t \\ y = \cos t \end{cases}$ $, 0 < t < \pi$ concave up?

*Solution*. Since

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{1}{x'(t)} \cdot \frac{d}{dt}\left(\frac{y'(t)}{x'(t)}\right) \\
&= \frac{1}{2\cos t} \cdot \frac{d}{dt}\left(\frac{-\sin t}{2\cos t}\right) \\
&= -\frac{1}{4}\sec^3 t,
\end{aligned}$$

we conclude that

$$\begin{aligned}
\frac{d^2y}{dx^2} &< 0 \text{ on } \left(0, \tfrac{\pi}{2}\right) \\
\frac{d^2y}{dx^2} &> 0 \text{ on } \left(\tfrac{\pi}{2}, \pi\right)
\end{aligned}$$

so that the curve is concave down on $\left(0, \frac{\pi}{2}\right)$ and concave up on $\left(\frac{\pi}{2}, \pi\right)$.

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The Worksheet and Homework set M8B should be worked on after studying the material from sections 8.4, 8.5, and 8.6 of the youtube workbook.

## 8.4 M8B Worksheet: length and surface areas

1. Find the area bounded by the curve

$$\begin{cases} x = t - \frac{1}{t} \\ y = t + \frac{1}{t} \end{cases}$$

   and the line $y = \frac{5}{2}$.

2. Find the arc length of

$$\begin{cases} x = \frac{1}{2}\ln(1+t^2) \\ y = \arctan t \end{cases}, \ t \in [0,1].$$

3. Find the arc length of

$$\begin{cases} x = 2\cos\theta + \cos 2\theta + 1 \\ y = 2\sin\theta + \sin 2\theta \end{cases}, \ \theta \in [0, 2\pi].$$

4. Find the length of one arch of the cycloid

$$\begin{cases} x = a\,(\theta - \sin\theta) \\ y = a(1 - \cos\theta) \end{cases}, \ \theta \in [0, 2\pi].$$

5. Find the surface area of a sphere of radius *r*.

6. Find the surface area generated by rotating the curve $y = x^3$, $0 \le x \le 1$ about the *x*-axis.

7. Find the surface area generated by rotating

$$\begin{cases} x = 3\cos^3\theta \\ y = 3\sin^3\theta \end{cases}, \theta \in [0, \frac{\pi}{2}]$$

   about the *x*-axis.



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## 8.5 M8B Homework set: length and surface areas

1. Find the area bounded by the curve

$$\begin{cases} x = \sin t \\ y = e^t \end{cases}, t \in \left[0, \frac{\pi}{2}\right]$$

and the lines $x = 1$ and $y = 0$.

2. Sketch the curve and find the length of the curve

$$\begin{cases} x(t) = & e^t \cos t \\ y(t) = & e^t \sin t \end{cases} t \in [0, \pi].$$

3. Find the surface area of the surface of revolution obtained by revolving the curve

$$\begin{cases} x(t) = & 3t - t^3 \\ y(t) = & 3t^2 \end{cases} t \in [0, 1]$$

about the $x$-axis.

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## 8.6 M8B Homework set: Solutions

NAME:

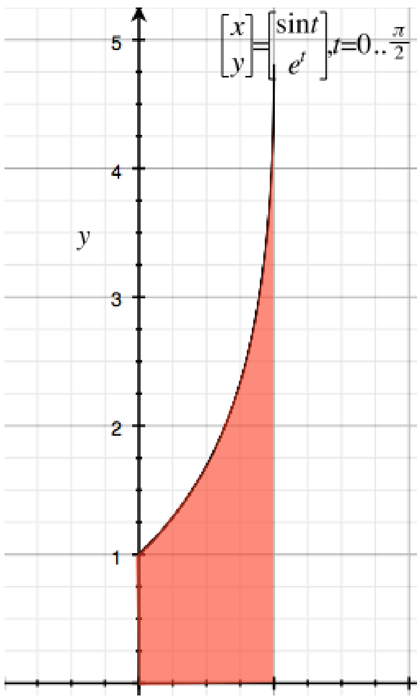
GRADE: /17

1. [5pts: 2 for setting up integral, 3 to calculating it] Find the area bounded by the curve

$$\begin{cases} x = \sin t \\ y = e^t \end{cases}, t \in \left[0, \frac{\pi}{2}\right]$$

and the lines $x = 1$ and $y = 0$.

*Solution.* Sketching the region, we obtain:




Thus, we see that the curve

$$\begin{cases} x = \sin t \\ y = e^t \end{cases}, t \in \left[0, \frac{\pi}{2}\right]$$

is traversed once with $(0, 1)$ corresponding to $t = 0$ and $(1, e^{\frac{\pi}{2}})$ for $t = \frac{\pi}{2}$. The area under the curve is thus

$$\int_0^{\frac{\pi}{2}} y(t)\, x'(t)\, dt = \int_0^{\frac{\pi}{2}} e^t \cos t\, dt,$$



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which we evaluate by integrating by parts twice:

$$\begin{aligned}\int_0^{\frac{\pi}{2}} e^t \cos t\, dt &= \left[e^t \sin t\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^t \sin t\, dt \\ &= \left[e^t \sin t - e^t \cos t\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^t \cos t\, dt,\end{aligned}$$

so that, solving for the desired integral:

$$\int_0^{\frac{\pi}{2}} e^t \cos t\, dt = \frac{1}{2}\left[e^t \sin t - e^t \cos t\right]_0^{\frac{\pi}{2}} = \frac{e^{\frac{\pi}{2}}+1}{2}.$$

2. [8 pts: 5 to sketch the curve; 1 to setup the length, 2 to calculate it] Sketch the curve and find the length of the curve

$$\begin{cases} x(t) = & e^t \cos t \\ y(t) = & e^t \sin t \end{cases} \quad t \in [0, \pi].$$

*Solution.* To sketch the curve, we need to find the intervals of increase and decrease of both $x(t)$ and $y(t)$. Note that

$$\begin{aligned} x'(t) &= e^t(\cos t - \sin t) \\ y'(t) &= e^t(\cos t + \sin t) \end{aligned}$$

so that $x'(t)$ is of the sign of $\cos t - \sin t$, and $y'(t)$ is of the sign of $\cos t + \sin t$. As easily seen on the trig circle, if $t \in [0, \pi]$ then

$$\cos t > \sin t \iff 0 \le t < \frac{\pi}{4},$$

and

$$\sin t < -\cos t \iff \frac{3\pi}{4} < t \le \pi.$$

Thus we have

| $t$ | $[0, \frac{\pi}{4}]$ | $(\frac{\pi}{4}, \frac{3\pi}{4})$ | $[\frac{3\pi}{4}, \pi]$ |
|---|---|---|---|
| $x'(t)$ | $+$ | $-$ | $-$ |
| $x(t)$ | $\nearrow$ | $\searrow$ | $\searrow$ |
| $y'(t)$ | $+$ | $+$ | $-$ |
| $y(t)$ | $\nearrow$ | $\nearrow$ | $\searrow$ |



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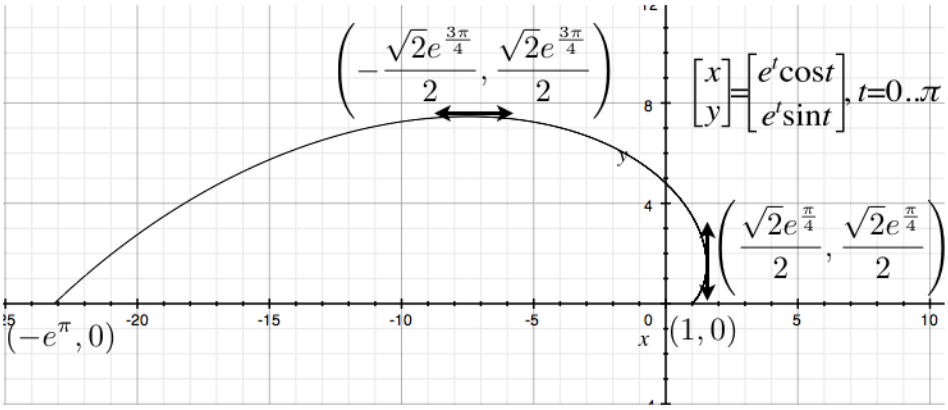
with an horizontal tangent for $t = \frac{3\pi}{4}$ because $y'(\frac{3\pi}{4}) = 0$ and $x'(\frac{3\pi}{4}) \neq 0$ and a vertical tangent for $t = \frac{\pi}{4}$ where $x' = 0$ and $y' \neq 0$. Note that

$$\begin{aligned}
(x(0), y(0)) &= (1, 0) \\
\left(x(\frac{\pi}{4}), y(\frac{\pi}{4})\right) &= \left(\frac{\sqrt{2}e^{\frac{\pi}{4}}}{2}, \frac{\sqrt{2}e^{\frac{\pi}{4}}}{2}\right) \\
\left(x(\frac{3\pi}{4}), y(\frac{3\pi}{4})\right) &= \left(-\frac{\sqrt{2}e^{\frac{3\pi}{4}}}{2}, \frac{\sqrt{2}e^{\frac{3\pi}{4}}}{2}\right) \\
(x(\pi), y(\pi)) &= (-e^{\pi}, 0)\,.
\end{aligned}$$

Following this information, we obtain the following sketch:







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To find the length of the curve, we apply the formula:

$$\begin{aligned}
L &= \int_0^{\pi} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt \\
&= \int_0^{\pi} \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\cos t + \sin t)^2}\, dt \\
&= \int_0^{\pi} \sqrt{e^{2t}\left(2\cos^2 t + 2\sin^2 t - 2\sin t\cos t + 2\sin t\cos t\right)}\, dt \\
&= \int_0^{\pi} \sqrt{2}e^t\, dt \\
&= \sqrt{2}\left[e^t\right]_0^{\pi} = \sqrt{2}(e^{\pi} - 1).
\end{aligned}$$

3. [4pts: 2 to setup integral, 2 to calculate] Find the surface area of the surface of revolution obtained by revolving the curve

$$\begin{cases} x(t) = & 3t - t^3 \\ y(t) = & 3t^2 \end{cases} \quad t \in [0, 1]$$

about the *x*-axis.

*Solution.* We apply the formula

$$\begin{aligned}
A &= 2\pi \int_0^1 y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt \\
&= 2\pi \int_0^1 3t^2\sqrt{(3 - 3t^2)^2 + (6t)^2}\, dt \\
&= 6\pi \int_0^1 t^2\sqrt{9 + 9t^4 - 18t^2 + 36t^2}\, dt \\
&= 6\pi \int_0^1 t^2\sqrt{(3 + 3t^2)^2}\, dt \\
&= 6\pi \int_0^1 t^2(3 + 3t^2)\, dt \\
&= 6\pi \int_0^1 3t^2 + 3t^4\, dt \\
&= 6\pi \left[t^3 + \frac{3}{5}t^5\right]_0^1 = 6\pi \cdot \frac{8}{5} = \frac{48\pi}{5}.
\end{aligned}$$



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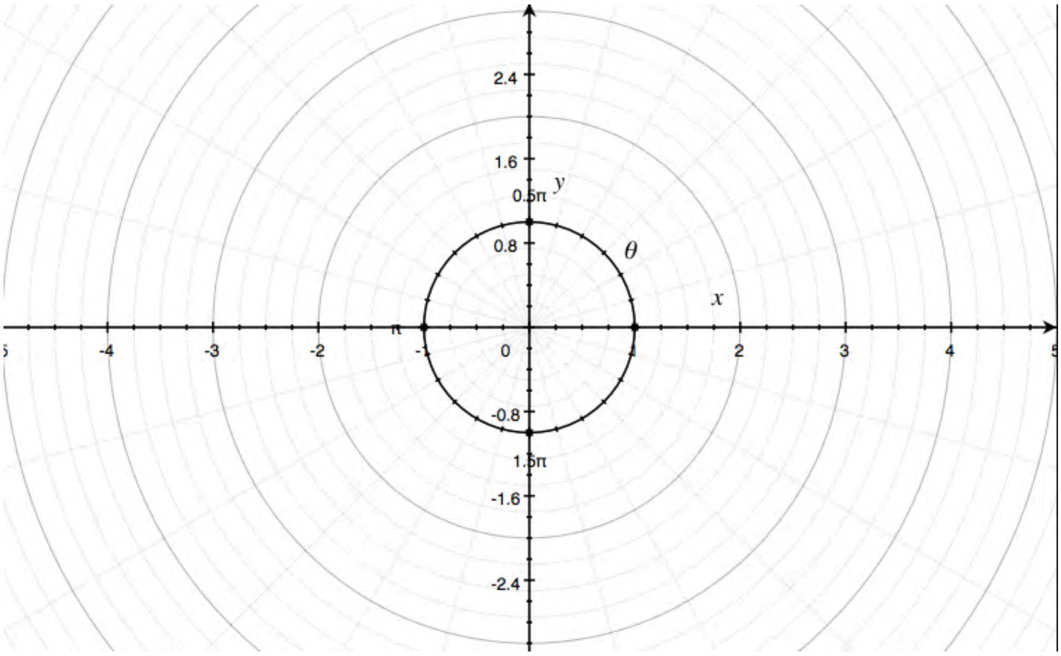
# 9 Polar curves

The Worksheet and Homework set M9A should be worked on after studying the material from sections 9.1 and 9.2 of the youtube workbook.

## 9.1 M9A Worksheet: polar coordinates

1. Plot below the following points given by their polar coordinates: a) $(2, 0)$; b) $(1, \pi)$; c) $(-2, \frac{3\pi}{2})$; d) $(1, \frac{\pi}{4})$; e) $(-1, \frac{\pi}{3})$.




2. Find 3 different pairs of polar coordinates for the point of Cartesian coordinates $(-\sqrt{3}, 1)$.
3. Find Cartesian coordinates for the points of polar coordinates $(2, -\frac{\pi}{3})$, $(0, \frac{\pi}{6})$, $(2, \frac{\pi}{4})$ and $(-2, -\frac{\pi}{2})$.
4. Find Cartesian equations for the following polar curve and sketch the curves:
   (a) $r = 3$
   (b) $\theta = -\frac{\pi}{3}$
   (c) $r = 2\cos\theta$
   (d) $r = 4\sin\theta$
5. Find polar equations for the following Cartesian curves:
   (a) $x^2 + y^2 = 16$
   (b) $x^2 + y^2 = x$
   (c) $x = 2$
   (d) $y = 4$



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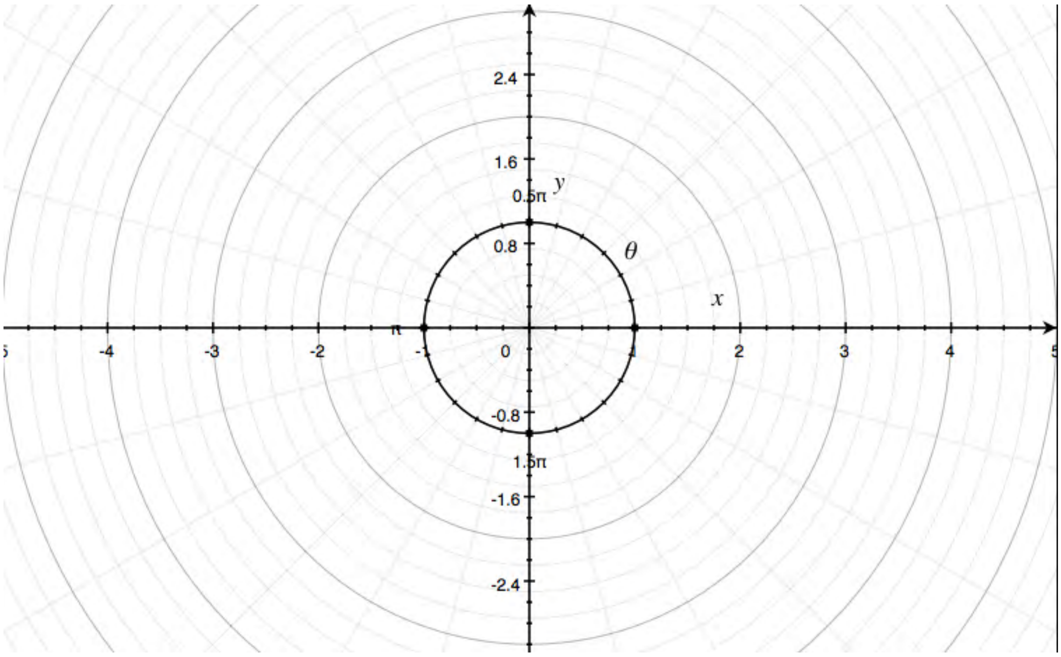
6. Sketch the polar regions on the polar grid below
   (a) $0 \leq r \leq 3$
   (b) $0 \leq \theta \leq \frac{\pi}{4}$ and $0 \leq r \leq 3$







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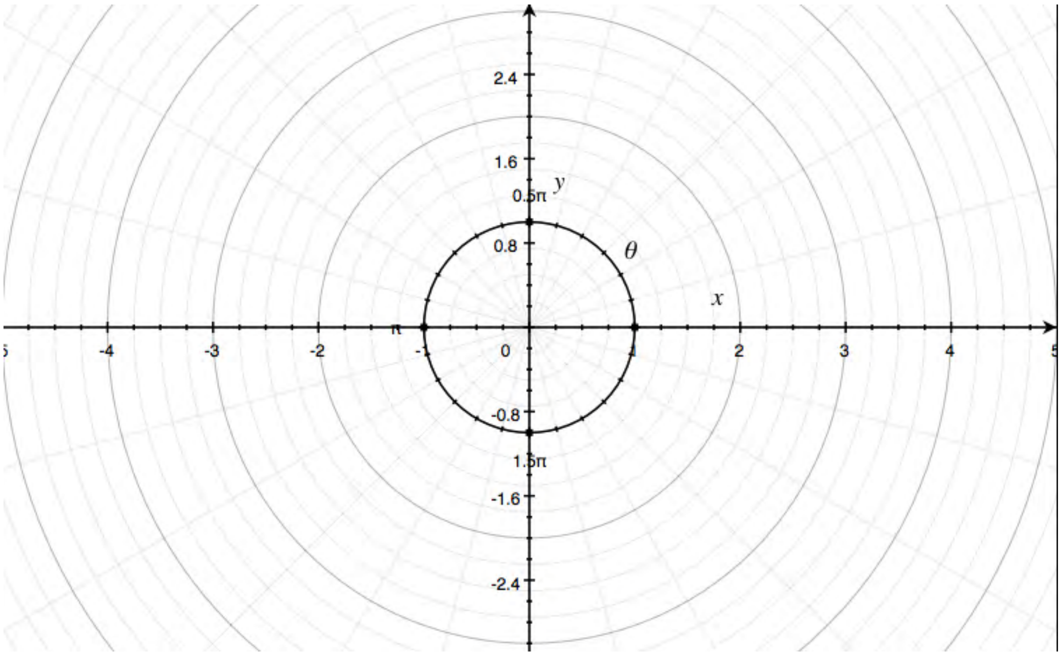
## 9.2 M9A Homework set: polar coordinates

1. Plot below the following points given by their polar coordinates: a) $(1,0)$; b) $(0,1)$; c) $(-1,\frac{\pi}{2})$; d) $(1,-\frac{\pi}{3})$; e) $(-1,\frac{2\pi}{3})$.




2. Find 2 different pairs of polar coordinates for the point of Cartesian coordinates $(-1,1)$.
3. Find Cartesian coordinates for the points of polar coordinates $(2,\frac{\pi}{6})$, $(0,\frac{\pi}{12})$, $(2,\frac{\pi}{4})$ and $(1,-\frac{3\pi}{2})$.
4. Find Cartesian equations for the following polar curve and sketch the curves:
   (a) $\theta = \frac{\pi}{6}$
   (b) $r = 3\cos\theta$
5. Find polar equations for the following Cartesian curves:
   (a) $x^2 + y^2 = y$
   (b) $y = -1$



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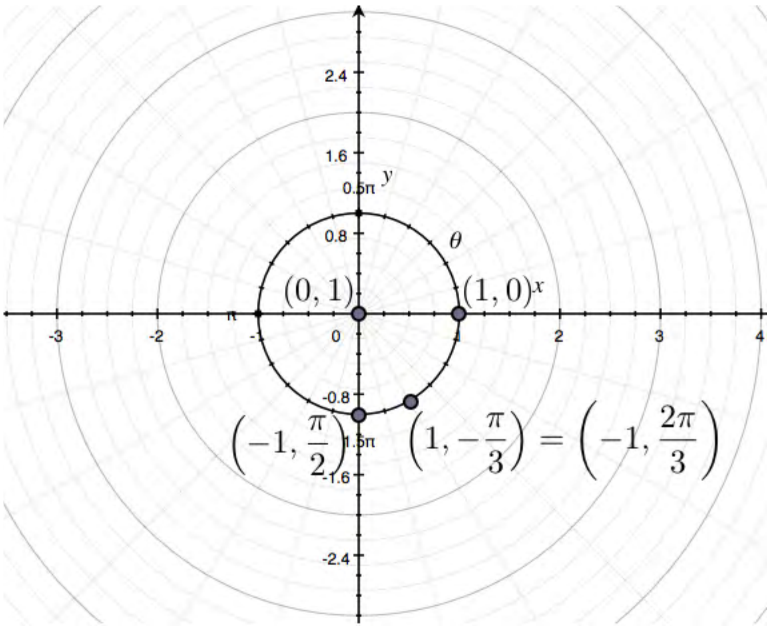
## 9.3 M9A Homework set: Solutions

NAME:

GRADE: /20

1. [5pts: 1 for each point] Plot below the following points given by their polar coordinates:
   a) $(1,0)$; b) $(0,1)$; c) $(-1,\frac{\pi}{2})$; d) $(1,-\frac{\pi}{3})$; e) $(-1,\frac{2\pi}{3})$.




2. [2pts] Find 2 different pairs of polar coordinates for the point of Cartesian coordinates $(-1,1)$.
   *Solution.* As $\tan\theta=\frac{y}{x}=-1$, the point is located on the line $\theta=\frac{3\pi}{4}$ and $r=\sqrt{x^2+y^2}=\sqrt{2}$. Thus, polar coordinates for this point are

$$\left(\sqrt{2},\frac{3\pi}{4}\right) \text{ or } (-\sqrt{2},-\tfrac{\pi}{4}).$$



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3. [4pts: 1 for each] Find Cartesian coordinates for the points of polar coordinates $(2, \frac{\pi}{6})$, $(0, \frac{\pi}{12})$, $(2, \frac{\pi}{4})$ and $(1, -\frac{3\pi}{2})$.

   *Solution.* Using

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

   we obtain

   - for the Cartesian coordinates of the point of polar coordinates $(2, \frac{\pi}{6})$:

$$\begin{cases} x = 2\cos\frac{\pi}{6} = \sqrt{3} \\ y = 2\sin\frac{\pi}{6} = 1 \end{cases}$$

   - any point with $r = 0$, in particular $(0, \frac{\pi}{12})$ is the origin (0, 0) of the Cartesian system.
   - for the Cartesian coordinates of the point of polar coordinates $(2, \frac{\pi}{4})$:

$$\begin{cases} x = 2\cos\frac{\pi}{4} = \sqrt{2} \\ y = 2\sin\frac{\pi}{4} = \sqrt{2} \end{cases}$$

   - We note that the half ray $\theta = -\frac{3\pi}{2}$, $r \geq 0$ is the positive $y$-axis. Thus the Cartesian coordinates of the point of polar coordinates $(1, -\frac{3\pi}{2})$ are $(0, 1)$.

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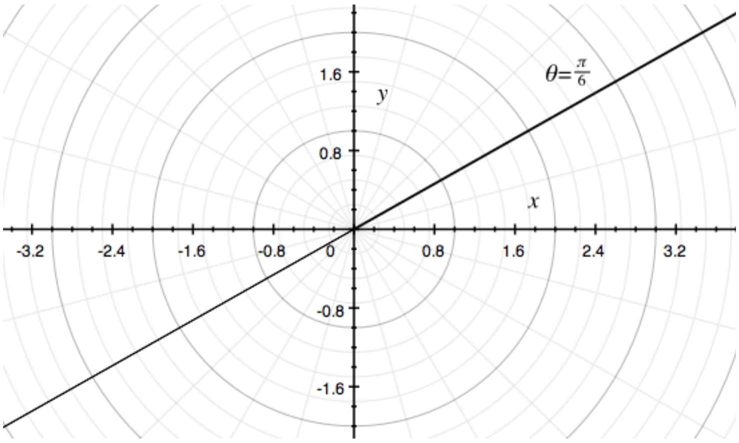
4. Find Cartesian equations for the following polar curve and sketch the curves:

(a) [2pt] $\theta = \frac{\pi}{6}$

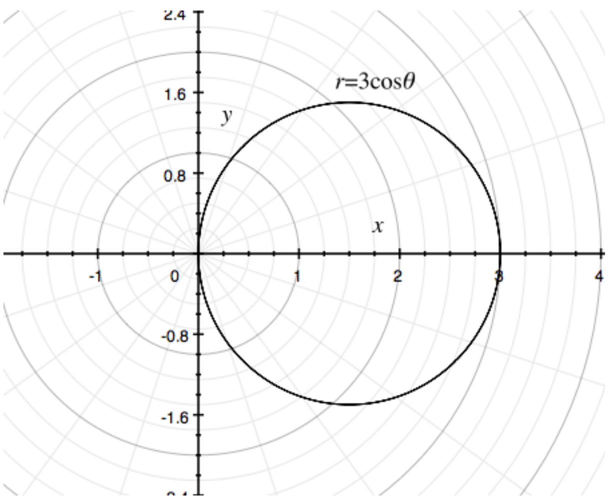
*Solution*. This is the line through the origin of slope $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$, that is, $y = \frac{\sqrt{3}}{3}x$.




(b) [3pts] $r = 3\cos\theta$

*Solution*. Note that the origin, or pole, is on the curve for $\cos\frac{\pi}{2} = 0$. On the other hand, for $r \neq 0$, the equation $r^2 = 3r\cos\theta$ is equivalent, and rewrites interms of *x* and *y* as

$$
\begin{aligned}
x^2 + y^2 = 3x \quad &\iff \quad (x^2 - 3x) + y^2 = 0 \\
&\iff \quad \left(x - \frac{3}{2}\right)^2 - \frac{9}{4} + y^2 = 0 \text{ by completing the square} \\
&\iff \quad \left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4}.
\end{aligned}
$$



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5. Find polar equations for the following Cartesian curves:

(a) [2pts] $x^2 + y^2 = y$

*Solution.* $x^2 + y^2 = r^2$ and $y = r\sin\theta$ so that

$$r^2 = r\sin\theta \iff r(r - \sin\theta) = 0 \iff r = \sin\theta$$

because the pole $r = 0$ is on the curve $r = \sin\theta$ .

(b) [2pts] $y = -1$

*Solution.* Since $y = r\sin\theta$ , a polar equation is

$$r\sin\theta = -1,$$

which we may rewrite

$$r = -\frac{1}{\sin\theta} = -\csc\theta$$

(thus obtaining a curve of the form $r = f(\theta)$) because values of $\theta$ making $\sin\theta = 0$ do not correspond to points on the curve ($0 \neq -1$!).

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The Worksheet and Homework set M9B should be worked on after studying the material from sections 9.3, 9.3, and 9.5of the youtube workbook.

## 9.4 M9B Worksheet: polar curves

1. Consider the polar curve $r = \sin 3\theta$ .
   (a) Find the slope of the tangent at $\theta = 0$ and at $\theta = \frac{\pi}{4}$.
   (b) Find all the points of the curve where $|r|$ is maximal, and show that at these points the tangent line is perpendicular to the radius joining the pole to the point.

2. Find the slope of the tangent line
   (a) to $r = \cos 2\theta$ at $\theta = \frac{\pi}{4}$
   (b) to $r = \sin 4\theta$ at $\theta = \frac{\pi}{16}$
   (c) to $r = 4\sin\theta$ at $\theta = 0$ and at $\theta = \frac{\pi}{2}$.

3. Find the arc length of
   (a) $r = \cos 3\theta$
   (b) $r = \theta$ , $0 \leq \theta \leq \pi$ .

4. Find the area of
   (a) the region bounded by $r = 2\cos\theta$
   (b) one leaf of $r = \cos 3\theta$
   (c) the inner loop of $r = 1 - 2\cos\theta$
   (d) the region inside of $r = 2$ and outside $r = 2 - 2\sin\theta$ .



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## 9.5 M9B Homework set: polar curves

1. Consider the polar curve $r = 2 + 4\sin 2\theta$ .
   (a) Find the slope of the tangent at $\theta = 0$ and at $\theta = \frac{\pi}{6}$.
   (b) Find all points where $|r|$ is a (local) maximum and show that the tangent line at such points is perpendicular to the radius connecting the point to the origin.

2. Find the slope of the tangent line to $r = \cos 3\theta$ at $\theta = \frac{\pi}{6}$ and at $\theta = \frac{\pi}{4}$ .
3. Find the arc length of
   (a) $r = \frac{e^\theta}{\sqrt{2}}$ for $0 \le \theta \le \pi$.
   (b) $r = \sqrt{1 + \sin 2\theta}$ for $0 \le \theta \le \sqrt{2}\pi$

4. Find the area of the region
   (a) inside the cardioid $r = a(1 + \cos\theta)$, $a > 0$
   (b) shared by the cardioids $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$ .



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## 9.6 M9B Homework set: Solutions

NAME:

GRADE: /25

1. Consider the polar curve $r = 2 + 4\sin 2\theta$ .

(a) [4pts: 2 for each] Find the slope of the tangent at $\theta = 0$ and at $\theta = \frac{\pi}{6}$.
*Solution*. The slope at the point of parameter $\theta$ is

$$\frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{8\cos 2\theta\sin\theta + 2\cos\theta + 4\sin 2\theta\cos\theta}{8\cos 2\theta\cos\theta - 2\sin\theta - 4\sin 2\theta\sin\theta},$$

so that

$$\frac{dy}{dx}_{|\theta=0} = \frac{2}{8} = \frac{1}{4}$$

and

$$\frac{dy}{dx}_{|\theta=\frac{\pi}{6}} = \frac{8\cos\frac{\pi}{3}\sin\frac{\pi}{6} + 2\cos\frac{\pi}{6} + 4\sin\frac{\pi}{3}\cos\frac{\pi}{6}}{8\cos\frac{\pi}{3}\cos\frac{\pi}{6} - 2\sin\frac{\pi}{6} - 4\sin\frac{\pi}{3}\sin\frac{\pi}{6}} = \frac{2+\sqrt{3}+3}{2\sqrt{3}-1-\sqrt{3}} = \frac{5+\sqrt{3}}{\sqrt{3}-1}.$$

(b) [4: 2 for finding points, 2 for slope and perpendicular] Find all points where $|r|$ is a (local) maximum and show that the tangent line at such points is perpendicular to the radius connecting the point to the origin.

*Solution*. $|r|$ is a local maximal when $\sin 2\theta = \pm 1$, that is, when $2\theta = \frac{\pi}{2} + k\pi$ , equivalently, $\theta = \frac{\pi}{4} + k\frac{\pi}{2}$, that is $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$. When $\theta = \frac{\pi}{4}$ or $\theta = \frac{5\pi}{4}$, the radius is carried by the line $y = x$ of slope $1$, and the slope of the tangent is

$$\frac{dy}{dx}_{|\theta=\frac{\pi}{4}} = \frac{3\sqrt{2}}{-3\sqrt{2}} = -1 \text{ and } \frac{dy}{dx}_{|\theta=\frac{\pi}{4}} = \frac{-3\sqrt{2}}{3\sqrt{2}} = -1$$

so that the tangent line is perpendicular to the radius (as the product of the slopes is –1). Similarly, when $\theta = \frac{3\pi}{4}$ or $\theta = \frac{7\pi}{4}$, the radius is carried by the line $y = -x$ of slope –1, and the slope of the tangent is

$$\frac{dy}{dx}_{|\theta=\frac{3\pi}{4}} = \frac{-3\sqrt{2}}{-3\sqrt{2}} = 1 \text{ and } \frac{dy}{dx}_{|\theta=\frac{7\pi}{4}} = \frac{3\sqrt{2}}{3\sqrt{2}} = 1$$

of that the tangent line is perpendicular to the radius (as the product of the slopes is –1).



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2. [4pts: 2 for each] Find the slope of the tangent line to $r = \cos 3\theta$ at $\theta = \frac{\pi}{6}$ and at $\theta = \frac{\pi}{4}$ .

*Solution*. The slope at the point of parameter $\theta$ is

$$\frac{dy}{dx} = \frac{r' \sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{-3\sin 3\theta \sin\theta + \cos 3\theta\cos\theta}{-3\sin 3\theta\cos\theta - \cos 3\theta \sin\theta},$$

so that

$$\frac{dy}{dx}_{|\theta=\frac{\pi}{4}} = \frac{-\frac{3}{2} - \frac{1}{2}}{-\frac{3}{2} + \frac{1}{2}} = 2.$$

On the other hand, for $\theta = \frac{\pi}{6}$ we have $r = \cos\frac{\pi}{2} = 0$, so that the line $\theta = \frac{\pi}{6}$ of slope $\tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$ is the tangent.

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3. Find the arc length of

(a) [2pts] $r = \frac{e^\theta}{\sqrt{2}}$ for $0 \le \theta \le \pi$.

*Solution.*

$$\begin{aligned} L &= \int_0^\pi \sqrt{r^2 + (r')^2}\, d\theta \\ &= \int_0^\pi \sqrt{\frac{e^{2\theta}}{2} + \frac{e^{2\theta}}{2}}\, d\theta \\ &= \int_0^\pi e^\theta\, d\theta = \left[e^\theta\right]_0^\pi = e^\pi - 1. \end{aligned}$$

(b) [3pts] $r = \sqrt{1 + \sin 2\theta}$ for $0 \le \theta \le \sqrt{2}\pi$

*Solution.*

$$\begin{aligned} L &= \int_0^{\sqrt{2}\pi} \sqrt{r^2 + (r')^2}\, d\theta \\ &= \int_0^{\sqrt{2}\pi} \sqrt{1 + \sin 2\theta + \left(\frac{2\cos 2\theta}{2\sqrt{1+\sin 2\theta}}\right)^2}\, d\theta \\ &= \int_0^{\sqrt{2}\pi} \sqrt{1 + \sin 2\theta + \frac{\cos^2(2\theta)}{1+\sin 2\theta}}\, d\theta \\ &= \int_0^{\sqrt{2}\pi} \sqrt{\frac{1 + 2\sin 2\theta + \sin^2(2\theta) + \cos^2(2\theta)}{1+\sin 2\theta}}\, d\theta \\ &= \int_0^{\sqrt{2}\pi} \sqrt{\frac{2\,(1+\sin 2\theta)}{1+\sin 2\theta}}\, d\theta = \sqrt{2}\int_0^{\sqrt{2}\pi} d\theta = 2\pi. \end{aligned}$$

4. Find the area of the region

(a) [3pts] inside the cardioid $r = a(1 + \cos\theta)$, $a > 0$

*Solution.* Since $r(0) = r(2\pi)$ the curve is a closed curve and it encloses a region of area

$$\begin{aligned} A &= \frac{1}{2}\int_0^{2\pi} r^2\, d\theta = \frac{1}{2}\int_0^{2\pi} a^2 + a^2\cos^2\theta + 2a\cos\theta\, d\theta \\ &= \frac{a^2}{4}\int_0^{2\pi} 3 + \cos(2\theta)\, d\theta + a\int_0^{2\pi} \cos\theta\, d\theta \end{aligned}$$

because $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$. Thus

$$A = \frac{a^2}{4}\left[3\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi} + a\left[\sin\theta\right]_0^{2\pi} = \frac{3\pi a^2}{2}.$$



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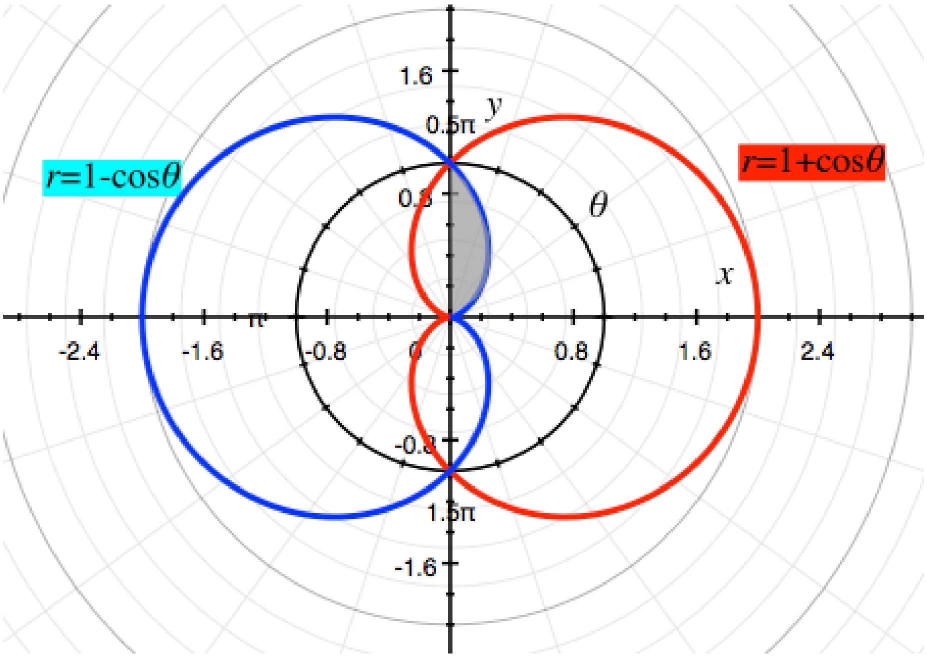
(b) [5pts: 2 for setup, 3 for calculation] shared by the cardioids $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$ .

*Solution*. Sketching the curves, we obtain:




Because

$$1 - \cos(\pi - \theta) = 1 + \cos\theta,$$

the two curves are reflection of one another about the $y$-axis, and they intersect on the $y$-axis (at the pole, for $\theta = \frac{\pi}{2}$ and for $\theta = \frac{3\pi}{2}$). Thus, by symmetry, we see that the desired area is 4 times the shaded area, which we can interpret as the area of the region bounded by the angular sector $0 \le \theta \le \frac{\pi}{2}$ and the (blue) polar curve $\theta = 1 - \cos\theta$ . Thus

$$\begin{aligned} A &= 4 \cdot \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 - \cos\theta)^2 \, d\theta \\ &= 2 \int_0^{\frac{\pi}{2}} 1 - 2\cos\theta + \cos^2\theta \, d\theta \\ &= 2 \int_0^{\frac{\pi}{2}} 1 - 2\cos\theta + \frac{1 + \cos(2\theta)}{2} \, d\theta \\ &= \left[ 3\theta - 4\sin\theta + \frac{1}{2}\sin(2\theta) \right]_0^{\frac{\pi}{2}} = \frac{3\pi}{2} - 4. \end{aligned}$$



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# Mock Test 3

*You should give yourself two hours to do the following test on your own, then, and only then, move to the solutions to evaluate your work. Show all your work to get credit.*

1. [30] Consider the curve

$$\begin{cases} x(t) = & \cos 2t \\ y(t) = & 2\sin 3t \end{cases} \quad t \in [0, 2\pi].$$

   (a) [10] Find the points on the curve where the tangent is horizontal, and those where the tangent is vertical.

   (b) [10] What are the intersection points of the curve and the $x$-axis? Show that one of them is a point of self-intersection of the curve and find the tangent lines at this point.

   (c) [10] Sketch the curve.

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2. [20] Consider the parametric curve

$$\begin{cases} x(t) = & 3(t - \sin t) \\ y(t) = & 3(1 - \cos t) \end{cases} \quad t \in [0, 2\pi].$$

(a) [10] What is the length of the curve?

(b) [10] What is the area enclosed between the curve and the $x$-axis?

3. [10] Find the surface area of the surface of revolution obtained by revolving the curve

$$\begin{cases} x(t) = & 2\sin^3 t \\ y(t) = & 2\cos^3 t \end{cases} \quad t \in \left[0, \frac{\pi}{2}\right]$$

about the $x$-axis.

4. [20] Consider the curve of polar equation

$$r = \frac{1}{4}\theta^2.$$

(a) Describe the tangent lines at the points of parameter

$$\theta = 0, \quad \frac{\pi}{2}, \quad \pi.$$

(b) Sketch the curve.

(c) Find the area enclosed by the curve and the angular sector ranging from $\theta = 0$ to $\theta = \frac{3\pi}{2}$.

(d) Find the length of the part of the curve corresponding to the same angular sector.

5. [20] Consider the curve of polar equation

$$r = \sin(3\theta).$$

(a) Describe the tangent lines at the tip of each "petal" of this curve.

(b) Sketch the curve.

(c) Find the area enclosed by one petal.



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# Mock Test 3 Solutions

1. [30] Consider the curve

$$\begin{cases} x(t) = & \cos 2t \\ y(t) = & 2\sin 3t \end{cases} \quad t \in [0, 2\pi].$$

(a) [10] Find the points on the curve where the tangent is horizontal, and those where the tangent is vertical.

*Solution*. The slope of the tangent line to the curve at the point of parameter $t$ is

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{6\cos 3t}{-2\sin 2t}.$$

The tangent is horizontal whenever $\frac{dy}{dx} = 0$, that is, whenever $\cos 3t = 0$ and $\sin 2t \neq 0$. Note that

$$\cos 3t = 0 \iff 3t = \frac{\pi}{2}+k\pi \iff t = \frac{\pi}{6}+k\frac{\pi}{3} \iff t \in \left\{\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}\right\},$$

and

$$\sin 2t = 0 \iff 2t = k\pi \iff t = k\frac{\pi}{2} \iff t \in \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\right\}.$$

At $\theta = \frac{\pi}{2}$:

$$\lim_{t\to\frac{\pi}{2}} \frac{dy}{dx} = \lim_{t\to\frac{\pi}{2}} \frac{6\cos 3t}{-2\sin 2t} \overset{H}{=} \lim_{t\to\frac{\pi}{2}} \frac{-18\sin 3t}{-4\cos 2t} = \frac{18}{4} = \frac{9}{2},$$

and at $\theta = \frac{3\pi}{2}$,

$$\lim_{t\to\frac{3\pi}{2}} \frac{dy}{dx} = \lim_{t\to\frac{3\pi}{2}} \frac{6\cos 3t}{-2\sin 2t} \overset{H}{=} \lim_{t\to\frac{3\pi}{2}} \frac{-18\sin 3t}{-4\cos 2t} = \frac{-18}{4} = -\frac{9}{2}.$$

Thus, the tangent is horizontal for $t \in \left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}$ that is at the points

$$\left(\frac{1}{2}, 2\right), \left(\frac{1}{2}, -2\right).$$

On the other hand, the tangent is vertical when $\sin 2t = 0$ and $\cos 3t \neq 0$ that is for $t = 0, \pi, 2\pi$, that is, at the point $(1, 0)$.



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(b) [10] What are the intersection points of the curve and the $x$-axis? Show that one of them is a point of self-intersection of the curve and find the tangent lines at this point.

*Solution*. The curve intersects the $x$-axis if $y = 2\sin 3t = 0$, that is,

$$3t = k\pi \iff t = k\frac{\pi}{3} \iff t \in \left\{0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi\right\}.$$

For $t = 0, \pi, 2\pi$, we obtain the same point $(1, 0)$ but it is not a point of self-intersection for the tangent line is always the vertical line $x = 1$, as seen in the previous question.

For $t = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$, we obtain the same point $(-\frac{1}{2}, 0)$. The latter is a point of self-intersection, for the slope of the tangent line is different for $t = \frac{\pi}{3}$ and $t = \frac{4\pi}{3}$:

$$\frac{dy}{dx}_{|t=\frac{\pi}{3}} = \frac{-6}{-\sqrt{3}} = 2\sqrt{3} \text{ and } \frac{dy}{dx}_{|t=\frac{4\pi}{3}} = \frac{6}{-\sqrt{3}} = -2\sqrt{3}.$$

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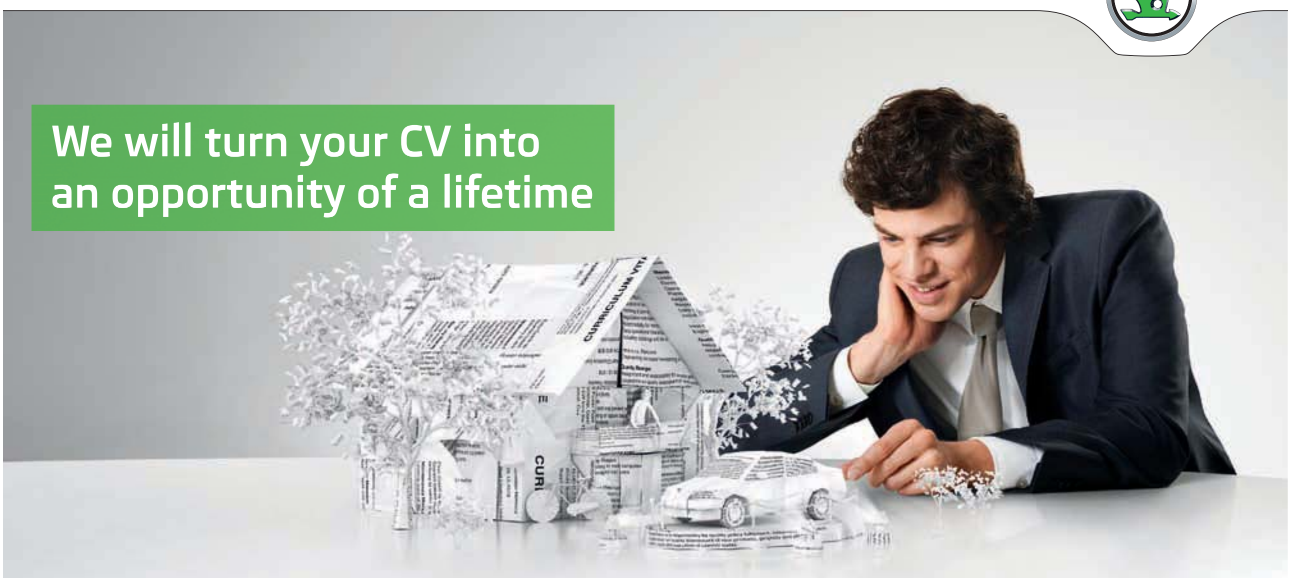


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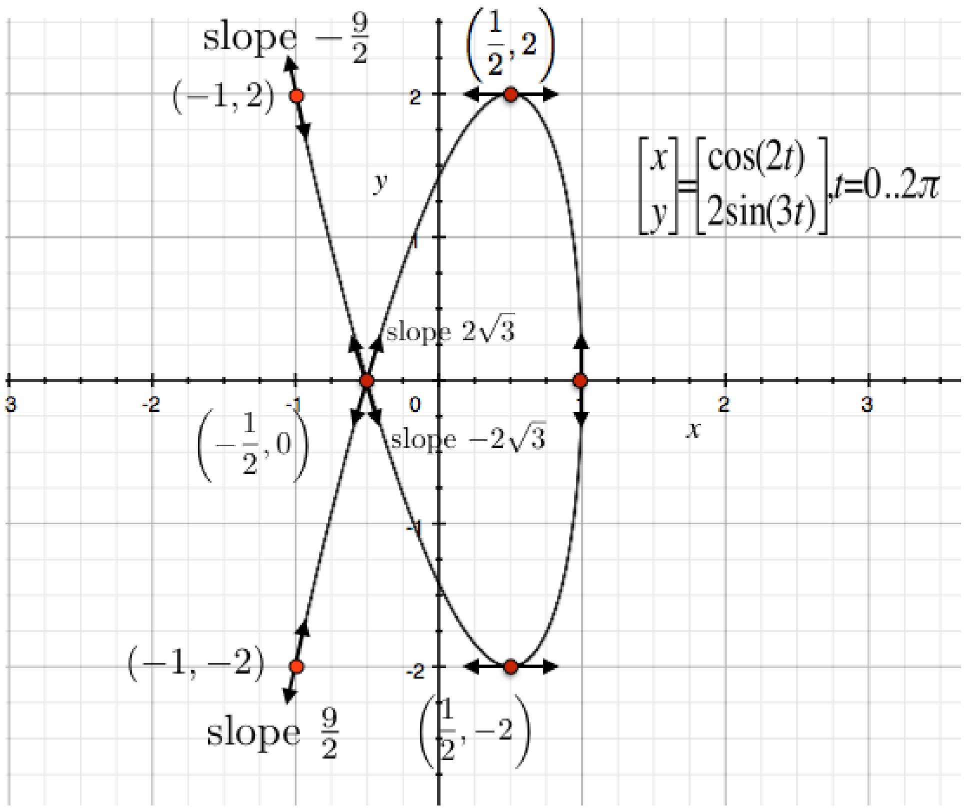
(c) [10] Sketch the curve.

*Solution.* We study the variations of $x(t)$ and $y(t)$:

| $t$ | $(0,\frac{\pi}{6})$ | $(\frac{\pi}{6},\frac{\pi}{2})$ | $(\frac{\pi}{2},\frac{5\pi}{6})$ | $(\frac{5\pi}{6},\pi)$ | $(\pi,\frac{7\pi}{6})$ | $(\frac{7\pi}{6},\frac{3\pi}{2})$ | $(\frac{3\pi}{2},\frac{11\pi}{6})$ | $(\frac{11\pi}{6},2\pi)$ |
|---|---|---|---|---|---|---|---|---|
| $x'(t)$ | $-$ | | $+$ | | $-$ | | $+$ | |
| $x(t)$ | $\searrow$ | | $\nearrow$ | | $\searrow$ | | $\nearrow$ | |
| $y'(t)$ | $+$ | $-$ | $+$ | $-$ | | $+$ | $-$ | $+$ |
| $y(t)$ | $\nearrow$ | $\searrow$ | $\nearrow$ | $\searrow$ | | $\nearrow$ | $\searrow$ | $\nearrow$ |

and thus we obtain the following sketch:




2. [20] Consider the parametric curve

$$\begin{cases} x(t) = & 3(t-\sin t) \\ y(t) = & 3(1-\cos t) \end{cases} \quad t \in [0, 2\pi].$$



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(a) [10] What is the length of the curve?

*Solution*. The length is given by

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt \\ &= \int_0^{2\pi} \sqrt{(3-3\cos t)^2 + 9\sin^2 t}\, dt \\ &= \int_0^{2\pi} \sqrt{9 + 9\cos^2 t + 9\sin^2 t - 18\cos t}\, dt \\ &= \int_0^{2\pi} \sqrt{18(1-\cos t)}\, dt. \end{aligned}$$

Using the double angle formula $\cos t = 1 - 2\sin^2 \frac{t}{2}$ so that

$$L = 3\sqrt{2}\int_0^{2\pi} \sqrt{2\sin^2\left(\frac{t}{2}\right)}\, dt = 3\sqrt{2}\cdot\sqrt{2}\int_0^{2\pi} \left|\sin\frac{t}{2}\right|\, dt.$$

Moreover, if $t \in [0, 2\pi]$ then $\frac{t}{2} \in [0, \pi]$ and $\sin\left(\frac{t}{2}\right) \geq 0$. Thus,

$$L = 6\int_0^{2\pi} \sin\frac{t}{2}\, dt = 12\left[-\cos\frac{t}{2}\right]_0^{2\pi} = 24.$$

(b) [10] What is the area enclosed between the curve and the *x*-axis?

*Solution*. Since $y(t) \geq 0$ for all *t*, and the curve is traversed once as *t* increases from 0 to $2\pi$, we have

$$\begin{aligned} A &= \int_0^{2\pi} y(t)\cdot x'(t)\, dt = \int_0^{2\pi} 3(1-\cos t)\cdot 3\,(1-\cos t)\; dt \\ &= 9\int_0^{2\pi} 1 - 2\cos t + \cos^2 t\, dt \\ &= 9\int_0^{2\pi} 1 - 2\cos t + \frac{1+\cos 2t}{2}\, dt \\ &= 9\left[\frac{3}{2}t - 2\sin t + \frac{1}{4}\sin 2t\right]_0^{2\pi} = 27\pi. \end{aligned}$$

3. [10] Find the surface area of the surface of revolution obtained by revolving the curve

$$\begin{cases} x(t) = & 2\sin^3 t \\ y(t) = & 2\cos^3 t \end{cases} \quad t \in \left[0, \frac{\pi}{2}\right]$$

about the *x*-axis.



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*Solution.* In this context,

$$
\begin{aligned}
A &= 2\pi \int_0^{\frac{\pi}{2}} y\sqrt{x'^2 + y'^2}\, dt \\
&= 2\pi \int_0^{\frac{\pi}{2}} 2\cos^3 t\sqrt{36\sin^4 t\cos^2 t + 36\cos^4 t\sin^2 t}\, dt \\
&= 4\pi \int_0^{\frac{\pi}{2}} \cos^3 t\sqrt{36\sin^2 t\cos^2 t\left(\sin^2 t + \cos^2 t\right)}\, dt \\
&= 24\pi \int_0^{\frac{\pi}{2}} \cos^3 t \cdot \sin t\cos t\, dt \\
&= 24\pi \int_0^{\frac{\pi}{2}} \cos^4 t\, \sin t\, dt.
\end{aligned}
$$

Let $u = \cos t$. Then $du = -\sin t\, dt$ and

$$A = -24\pi \int_1^0 u^4\, du = 24\pi \left[\frac{u^5}{5}\right]_0^1 = \frac{24\pi}{5}.$$

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4. [20] Consider the curve of polar equation

$$r = \frac{1}{4}\theta^2.$$

(a) Describe the tangent lines at the points of parameter

$$\theta = 0, \quad \frac{\pi}{2}, \quad \pi.$$

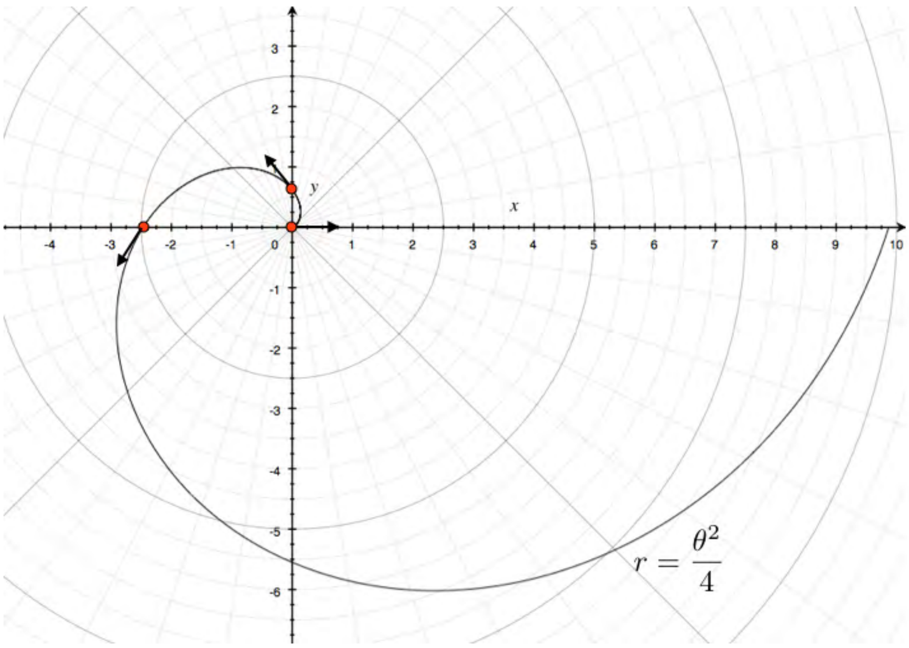
*Solution*. For $\theta = 0$, $r = 0$ so that the line $\theta = 0$ is also the tangent line to the curve. On the other hand, the slope of the tangent line at $\theta$ is given by

$$\frac{dy}{dx} = \frac{r' \sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{\frac{\theta}{2}\sin\theta + \frac{\theta^2}{4}\cos\theta}{\frac{\theta}{2}\cos\theta - \frac{\theta^2}{4}\sin\theta},$$

so that for $\theta = \frac{\pi}{2}$, $\frac{dy}{dx} = -\frac{2}{\theta} = -\frac{4}{\pi}$ and for $\theta = \pi$, $\frac{dy}{dx} = \frac{\theta}{2} = \frac{\pi}{2}$.

(b) Sketch the curve.

*Solution*. $r$ is an increasing function of $\theta$ for $\theta \in [0, 2\pi]$, so that the curve is a spiral. Including the points for $\theta - 0, \frac{\pi}{2}, \pi$ for which we have the slope of the tangents, we obtain:




(c) Find the area enclosed by the curve and the angular sector ranging from $\theta = 0$ to $\theta = \frac{3\pi}{2}$.

*Solution*. The area is

$$A = \frac{1}{2}\int_0^{\frac{3\pi}{2}} r^2\, d\theta = \frac{1}{2}\int_0^{\frac{3\pi}{2}} \frac{\theta^4}{16}\, d\theta = \frac{1}{160}\left[\theta^5\right]_0^{\frac{3\pi}{2}} = \frac{243\pi^5}{5120}.$$



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(d) Find the length of the part of the curve corresponding to the same angular sector.

*Solution*. The length is given by

$$\begin{aligned} L &= \int_0^{\frac{3\pi}{2}} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \\ &= \int_0^{\frac{3\pi}{2}} \sqrt{\frac{\theta^4}{16} + \frac{\theta^2}{4}}\, d\theta \\ &= \int_0^{\frac{3\pi}{2}} \sqrt{\frac{\theta^2}{4}\left(\frac{\theta^2}{4} + 1\right)}\, d\theta \\ &= \int_0^{\frac{3\pi}{2}} \frac{\theta}{2}\sqrt{\frac{\theta^2}{4} + 1}\, d\theta. \end{aligned}$$

Let $u = \frac{\theta^2}{4} + 1$. Then $du = \frac{\theta}{2}\, d\theta$ and

$$L = \int_1^{\frac{9\pi^2}{16}+1} \sqrt{u}\, du = \left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^{\frac{9\pi^2}{16}+1} = \frac{2}{3}\left(\left(\frac{9\pi^2}{16} + 1\right)^{\frac{3}{2}} - 1\right).$$

5. [20] Consider the curve of polar equation

$$r = \sin(3\theta).$$

(a) Describe the tangent lines at the tip of each "petal" of this curve.

*Solution*. Note that $|r| \leq 1$ so that $r$ is maximal when $\sin(3\theta) = \pm 1$, that is, if

$$3\theta = \frac{\pi}{2} + k\pi \iff \theta = \frac{\pi}{6} + k\frac{\pi}{3} \overset{\theta\in[0,2\pi)}{\iff} \theta \in \left\{\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}\right\}.$$

Moreover, that corresponds to only 3 "tips of petals", for $r(\frac{\pi}{6}) = 1$ and $r(\frac{7\pi}{6}) = -1$ represent the same point, $r(\frac{\pi}{2}) = -1$ and $r(\frac{3\pi}{2}) = 1$ represent the same point, and $r(\frac{5\pi}{6}) = 1$ and $r(\frac{11\pi}{6}) = -1$ represent the same point. At any of these points, the tangent line is perpendicular to the radius joining the pole to the point, for this radius has slope $\tan\theta$ while the tangent has slope

$$\frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{3\cos(3\theta)\sin\theta + \sin(3\theta)\cos\theta}{3\cos(3\theta)\cos\theta - \sin(3\theta)\sin\theta} = -\cot\theta$$

whenever $\cos(3\theta) = 0$, which happens at each one of these points. Thus, for $\theta \in \{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}\}$ the product of the slopes is –1 and the two lines are perpendicular. At $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, $\tan\theta$ is not defined, but the radius is vertical and the tangent horizontal.



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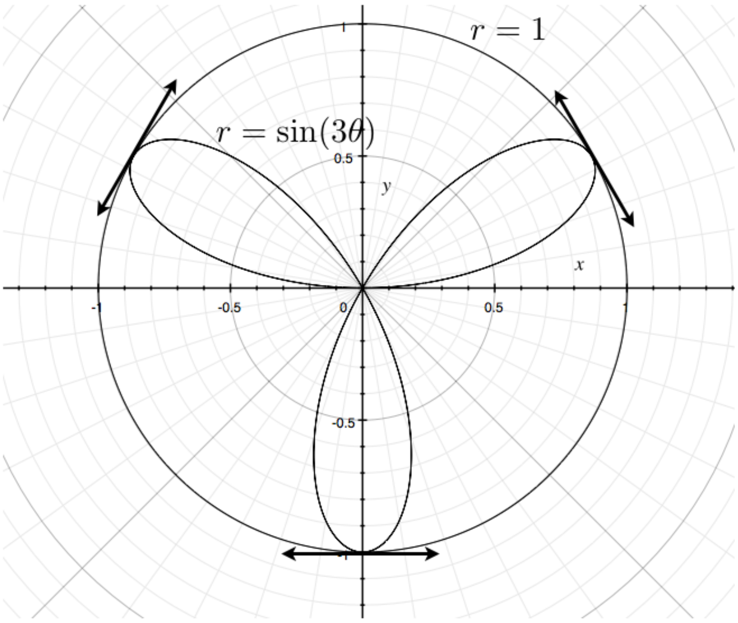
(b) Sketch the curve.

*Solution*. The variations of $r$ as a function of $\theta$ are as follows:

| $\theta$ | $(0, \frac{\pi}{6})$ | $(\frac{\pi}{6}, \frac{\pi}{2})$ | $(\frac{\pi}{2}, \frac{5\pi}{6})$ | $(\frac{5\pi}{6}, \frac{7\pi}{6})$ | $(\frac{7\pi}{6}, \frac{3\pi}{2})$ | $(\frac{3\pi}{2}, \frac{11\pi}{6})$ | $(\frac{11\pi}{6}, 2\pi)$ |
|---|---|---|---|---|---|---|---|
| $r'(\theta)$ | $+$ | $-$ | $+$ | $-$ | $+$ | $-$ | $+$ |
| $r(\theta)$ | $\nearrow$ | $\searrow$ | $\nearrow$ | $\searrow$ | $\nearrow$ | $\searrow$ | $\nearrow$ |

In particular, $r = 0$ for $\theta \in \{0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi\}$, which gives us the direction of the tangent lines at the pole.

Taking all this into account, we obtain:




(c) Find the area enclosed by one petal.

*Solution*. We see that all 3 petals are identical. The area enclosed by the first petal corresponds to

$$\begin{aligned} A &= \frac{1}{2}\int_0^{\frac{\pi}{3}} \sin^2(3\theta)\, d\theta \\ &= \frac{1}{2}\int_0^{\frac{\pi}{3}} \frac{1-\cos(6\theta)}{2}\, d\theta \\ &= \frac{1}{4}\left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\frac{\pi}{3}} = \frac{\pi}{12}. \end{aligned}$$



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# 10 M10: Sequences and Series

The Worksheet and Homework set M10A should be worked on after studying the material from sections 10.1, 10.2, and 10.3 of the youtube workbook.

## 10.1 M10A Worksheet: Sequences

1. Write the first 5 terms of the sequence $\{a_n\}_{n=1}^{\infty}$ if
   (a) $a_n = \frac{2n-1}{n^2+1}$
   (b) $a_n = (-1)^n \frac{n}{n+1}$
   (c) $\begin{cases} a_1 = 2 \\ a_{n+1} = \sqrt{2+\sqrt{a_n}} \end{cases}$

2. Find the limit of the sequence $\{a_n\}_{n=1}^{\infty}$ if
   (a) $a_n = \frac{3}{n^3}$
   (b) $a_n = \frac{2n+1}{1-3n}$
   (c) $a_n = \frac{2n^2+2}{n+5}$
   (d) $a_n = \frac{(\ln n)^2}{n}$
   (e) $a_n = \sin\left(\frac{(\ln n)^2}{n}\right)$
   (f) $a_n = \frac{\cos n}{e^n}$
   (g) $a_n = (-1)^n \cdot \frac{5}{n+2}$
   (h) $a_n = 5 \cdot \left(\frac{2}{3}\right)^n$
   (i) $a_n = 5 \cdot \left(\frac{3}{2}\right)^n$
   (j) $a_n = \frac{3^n}{e^n+n}$
   (k) $a_n = \frac{5^n n!}{(2n)!}$

3. For each sequence $\{a_n\}_{n=1}^{\infty}$ below, say if its bounded above? below? (eventually) increasing? (eventually) decreasing? Justify your answers.
   (a) $a_n = \frac{n+3}{n+2}$
   (b) $a_n = \frac{e^n}{n}$
   (c) $a_n = \frac{\cos(n\pi)}{n^2}$
   (d) $a_n = e^{\frac{1}{n}}$



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## 10.2 M10A Homework set: Sequences

1. Write the first 5 terms of the sequence $\{a_n\}_{n=1}^{\infty}$ if
   (a) $a_n = (-1)^{n+1} \cdot \frac{2n+4}{3n-1}$
   (b) $\begin{cases} a_1 = 3 \\ a_{n+1} = \frac{3}{2+a_n} \end{cases}$

2. Find the limit of the sequence $\{a_n\}_{n=1}^{\infty}$ if
   (a) $a_n = \frac{3n^5+2}{n^3+1}$
   (b) $a_n = \frac{5n^3+1}{2n^3-3}$
   (c) $a_n = \sin\left(\arctan\left(\frac{e^n}{n^2}\right)\right)$
   (d) $a_n = (-1)^n \cdot \frac{\cos\left(\frac{1}{n}\right)-1}{\sin\left(\frac{1}{n}\right)}$
   (e) $a_n = \frac{2^n}{e^n+3n}$
   (f) $a_n = \frac{10^n}{n!}$

3. For each sequence $\{a_n\}_{n=1}^{\infty}$ below, say if its bounded above? below? (eventually) increasing? (eventually) decreasing? Justify your answers.
   (a) $a_n = \frac{2n+3}{3n-2}$
   (b) $a_n = \frac{(-1)^n}{\ln(n+1)}$

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## 10.3 M10A Homework set: Solutions

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1. [5] Write the first 5 terms of the sequence $\{a_n\}_{n=1}^{\infty}$ if

(a) $[2.5 = 0.5 \times 5\,] a_n = (-1)^{n+1} \cdot \frac{2n+4}{3n-1}$

*Solution.*

$$a_1 = 3,\ a_2 = -\frac{8}{5},\ a_3 = \frac{5}{4},\ a_4 = -\frac{12}{11},\ a_5 = 1.$$

(b) $[2.5 = 0.5 \times 5\,] \begin{cases} a_1 = 3 \\ a_{n+1} = \frac{3}{2+a_n} \end{cases}$

*Solution.*

$$a_1 = 3,\ a_2 = \frac{3}{2+3} = \frac{3}{5},\ a_3 = \frac{3}{2+\frac{3}{5}} = \frac{15}{13},\ a_4 = \frac{3}{2+\frac{15}{13}} = \frac{39}{41},\ a_5 = \frac{3}{2+\frac{39}{41}} = \frac{123}{121}$$

2. [10] Find the limit of the sequence $\{a_n\}_{n=1}^{\infty}$ if

(a) [1] $a_n = \frac{3n^5+2}{n^3+1}$

*Solution.* $\left\{\frac{3n^5+2}{n^3+1}\right\}_{n=1}^{\infty}$ is divergent because

$$\lim_{n\to\infty} \frac{3n^5+2}{n^3+1} = \lim_{x\to\infty} \frac{3x^5+2}{x^3+1} = \infty.$$

(b) [1] $a_n = \frac{5n^3+1}{2n^3-3}$

*Solution.*

$$\lim_{n\to\infty} \frac{5n^3+1}{2n^3-3} = \lim_{x\to\infty} \frac{5x^3+1}{2x^3-3} = \frac{5}{2}.$$

(c) $[2] a_n = \sin\left(\arctan\left(\frac{e^n}{n^2}\right)\right)$

*Solution.* Since

$$\lim_{n\to\infty} \frac{e^n}{n^2} = \lim_{x\to\infty} \frac{e^x}{x^2} \overset{H}{=} \lim_{x\to\infty} \frac{e^x}{2x} \overset{H}{=} \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

and $\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$, we conclude that

$$\lim_{n\to\infty} \sin\left(\arctan\left(\frac{e^n}{n^2}\right)\right) = \sin\frac{\pi}{2} = 1.$$



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(d) [2] $a_n = (-1)^n \cdot \frac{\cos\left(\frac{1}{n}\right)-1}{\sin\left(\frac{1}{n}\right)}$

*Solution.* Since

$$\lim_{n\to\infty} \frac{\cos\left(\frac{1}{n}\right)-1}{\sin\left(\frac{1}{n}\right)} = \lim_{x\to\infty} \frac{\cos\left(\frac{1}{x}\right)-1}{\sin\left(\frac{1}{x}\right)} \underset{t=\frac{1}{x}}{=} \lim_{t\to 0} \frac{\cos t - 1}{\sin t} \overset{H}{=} \lim_{t\to 0} \frac{-\sin t}{\cos t} = \frac{0}{1} = 0,$$

we conclude that $\lim_{n\to\infty} |a_n| = 0$ and thus $\lim_{n\to\infty} a_n = 0$.

(e) [2] $a_n = \frac{2^n}{e^n+3n}$

*Solution.*

$$\begin{aligned} \lim_{n\to\infty} \frac{2^n}{e^n+3n} &= \lim_{x\to\infty} \frac{2^x}{e^x+3x} \overset{H}{=} \lim_{x\to\infty} \frac{2^x \ln 2}{e^x+3} \overset{H}{=} \lim_{x\to\infty} \frac{2^x (\ln 2)^2}{e^x} \\ &= \lim_{x\to\infty} (\ln 2)^2 \cdot \left(\frac{2}{e}\right)^x = 0 \end{aligned}$$

because $0 < \frac{2}{e} < 1$ so that $\lim_{x\to\infty} \left(\frac{2}{e}\right)^x = 0$.

(f) [2] $a_n = \frac{10^n}{n!}$

*Solution.* For $n > 10$, we have

$$\frac{10^n}{n!} = \frac{10 \cdot 10 \ldots \cdot 10 \ldots \ldots 10 \cdot 10}{1 \cdot 2 \cdot \ldots 10 \ldots (n-1) \cdot n} = \frac{10^{10}}{10!} \cdot \frac{10}{11} \cdot \frac{10}{12} \cdot \ldots \cdot \frac{10}{n-1} \cdot \frac{10}{n} \leq \frac{10^{10}}{10!} \cdot \frac{10}{n}$$

and $\lim_{n\to\infty} \frac{10}{n} = 0$, so that by the Squeeze Theorem, $\lim_{n\to\infty} a_n = 0$.

3. [6] For each sequence $\{a_n\}_{n=1}^{\infty}$ below, say if its bounded above? below? (eventually) increasing? (eventually) decreasing? Justify your answers.

(a) [3: 1 decreasing, 1 bounded above, 1 bounded below] $a_n = \frac{2n+3}{3n-2}$

*Solution.* Since $a_n = f(n)$ for $f(x) = \frac{2x+3}{3x-2}$ and

$$f'(x) = \frac{2(3x-2) - 3(2x+3)}{(3x-2)^2} = -\frac{13}{(3x-2)^2} < 0$$

we conclude that $f(x)$ is decreasing, and therefore $\{a_n\}_{n=1}^{\infty}$ is also decreasing. Thus, $\{a_n\}_{n=1}^{\infty}$ is bounded above by $a_1 = 5$ and below by 0, because the sequence has only positive terms. Thus, $\{a_n\}_{n=1}^{\infty}$ is bounded.



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(b) [3: 1 neither increasing nor decreasing, 2 bounded] $a_n = \frac{(-1)^n}{\ln(n+1)}$

*Solution*. The sequence $\{a_n\}_{n=1}^{\infty}$ is neither (eventually) increasing nor decreasing, for it takes alternatively positive and negative values. On the other hand, it is bounded, for $|a_n| = \frac{1}{\ln(n+1)}$ is decreasing so that

$$0 \leq |a_n| \leq \frac{1}{\ln 2} \Longrightarrow -\frac{1}{\ln 2} \leq a_n \leq \frac{1}{\ln 2}.$$

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The Worksheet and Homework set M10B should be worked on after studying the material from sections 10.4 and 10.5 of the youtube workbook.

## 10.4 M10B Worksheet: Sequences defined inductively

1. Is the sequence $\{a_n\}_{n=1}^{\infty}$ defined by

$$\begin{cases} a_1 = 2 \\ a_{n+1} = \frac{1}{3-a_n} \quad \text{for all } n \end{cases}$$

convergent? If yes, find its limit.

2. Same question if

$$\begin{cases} a_1 = -1 \\ a_{n+1} = \frac{1}{3-a_n} \quad \text{for all } n \end{cases}$$

3. Is the sequence $\{a_n\}_{n=1}^{\infty}$ defined by

$$\begin{cases} a_1 = 1 \\ a_{n+1} = 3 - \frac{1}{a_n} \quad \text{for all } n \end{cases}$$

convergent? If yes, find its limit.

4. To show the *continued fraction* equality

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2+\frac{1}{2+\dots}}}$$

consider the sequence $\{a_n\}_{n=1}^{\infty}$ defined by

$$\begin{cases} a_1 = 1 \\ a_{n+1} = 1 + \frac{1}{1+a_n} \quad \text{for all } n \end{cases}.$$

(a) Find the first 8 terms (with a calculator).

(b) Show that $\{a_{2n}\}_{n=1}^{\infty}$ is convergent.

(c) Show that $\{a_{2n+1}\}_{n=1}^{\infty}$ is convergent.

(d) Show that $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n+1}$ and conclude about the continued fraction equality.



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## 10.5 M10B Homework set: Sequences defined inductively

1. Consider a sequence $\{a_n\}_{n=1}^{\infty}$ defined by

$$\begin{cases} a_1 \\ a_{n+1} = (a_n - 1)^3 + 1 \quad \text{for all } n \end{cases}.$$

   (a) What is the limit of $\{a_n\}_{n=1}^{\infty}$ if $a_1 = \frac{1}{2}$? Justify your answer.

   (b) What is the limit of $\{a_n\}_{n=1}^{\infty}$ if $a_1 = \frac{3}{2}$? Justify your answer.

   (c) What is the limit of $\{a_n\}_{n=1}^{\infty}$ if $a_1 = 3$? Justify your answer.

2. Consider a sequence $\{a_n\}_{n=1}^{\infty}$ defined by

$$\begin{cases} a_1 = 1 \\ a_{n+1} = 1 + x - \frac{x^2}{3} \quad \text{for all } n \end{cases}.$$

   (a) Find the first 8 terms (with a calculator).

   (b) Show that

$$\frac{3}{2} \leq a_{2n} \leq a_{2n+1} \leq 2$$

   for all $n$.

   (c) Show that $\{a_{2n}\}_{n=1}^{\infty}$ is convergent.

   (d) Show that $\{a_{2n+1}\}_{n=1}^{\infty}$ is convergent.

   (e) Admitting that $\lim_{n\to\infty} a_{2n+1} - a_{2n} = 0$, deduce $\lim_{n\to\infty} a_n$.



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## 10.6 M10B Homework set: Solutions

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1. [10] Consider a sequence $\{a_n\}_{n=1}^{\infty}$ defined by

$$\begin{cases} a_1 \\ a_{n+1} = (a_n - 1)^3 + 1 \quad \text{for all } n \end{cases}.$$

(a) [4: either 3 for proving direct formula for $a_n$ and 1 for the limit, or 1 for non-decreasing, 1 for bounded above, 2 for limit] What is the limit of $\{a_n\}_{n=1}^{\infty}$ if $a_1 = \frac{1}{2}$? Justify your answer.

*Solution*. The first few terms are

$$a_1 = 1 - \frac{1}{2},\ a_2 = 1 - \frac{1}{8},\ a_3 = 1 - \frac{1}{8^3}, \ldots$$

so that it seems that

$$a_n = 1 - \frac{1}{2^{3^{n-1}}},$$

for all *n*, which we set out to prove by induction. This is true for $n = 1$.
Assume it is true for some *n*. Then

$$\begin{aligned} a_{n+1} &= (a_n - 1)^3 + 1 = \left(1 - \frac{1}{2^{3^{n-1}}} - 1\right)^3 + 1 \\ &= \left(-\frac{1}{2^{3^{n-1}}}\right)^3 + 1 \\ &= 1 - \frac{1}{2^{3^n}}. \end{aligned}$$

Thus

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1 - \frac{1}{2^{3^{n-1}}} = 1.$$

**Alternatively**, we see from the first few terms that the sequence seems non-decreasing and bounded above by 1.



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We prove it by induction, that is, we prove that $a_n \le a_{n+1}$ and $a_n \le 1$ for all $n$. We have already verified it for $n = 1$ for $a_1 = \frac{1}{2} \le a_2 = \frac{7}{8}$ and $a_1 \le 1$. Assume now that for some $n$, $a_n \le a_{n+1}$ and $a_n \le 1$. Note that $f(x) = (x-1)^3 + 1$ is a non-decreasing function for $f'(x) = 3(x-1)^2 \ge 0$. Thus

$$\begin{aligned} a_n \le a_{n+1} &\implies a_{n+1} = f(a_n) \le f(a_{n+1}) = a_{n+2} \\ a_n \le 1 &\implies a_{n+1} = f(a_n) \le f(1) = 1. \end{aligned}$$

The sequence $\{a_n\}_{n=1}^{\infty}$ is non-decreasing and bounded above, therefore convergent. Let $L := \lim_{n\to\infty} a_n$. Then $f(L) = L$, so that

$$\begin{aligned} (L-1)^3 + 1 = L &\iff (L-1)^3 - (L-1) = 0 \\ &\iff (L-1)\left((L-1)^2 - 1\right) = 0 \\ &\iff L = 1 \text{ or } (L-1)^2 = 1 \\ &\iff L = 1 \text{ or } L - 1 = \pm 1 \\ &\iff L = 1 \text{ or } L = 0 \text{ or } L = 2. \end{aligned}$$

Since $\{a_n\}_{n=1}^{\infty}$ is non-decreasing with $a_1 = \frac{1}{2}$, $L \neq 0$, and since $\{a_n\}_{n=1}^{\infty}$ is bounded above by 1, $L \neq 2$. Thus $\lim_{n\to\infty} a_n = 1$.

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(b) [4 similarly] What is the limit of $\{a_n\}_{n=1}^{\infty}$ if $a_1 = \frac{3}{2}$? Justify your answer.

*Solution*. The first few terms are

$$a_1 = 1 + \frac{1}{2},\ a_2 = 1 + \frac{1}{8},\ a_3 = 1 + \frac{1}{8^3}, \ldots$$

so that it seems that

$$a_n = 1 + \frac{1}{2^{3^{n-1}}},$$

for all *n*, which we set out to prove by induction. This is true for $n = 1$.
Assume it is true for some *n*. Then

$$\begin{aligned} a_{n+1} &= (a_n - 1)^3 + 1 = \left(1 + \frac{1}{2^{3^{n-1}}} - 1\right)^3 + 1 \\ &= \left(\frac{1}{2^{3^{n-1}}}\right)^3 + 1 \\ &= 1 + \frac{1}{2^{3^n}}. \end{aligned}$$

Thus

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1 + \frac{1}{2^{3^{n-1}}} = 1.$$

**Alternatively**, we see from the first few terms that the sequence seems non-increasing and bounded below by 1. We prove it by induction, that is, we prove that $a_n \geq a_{n+1}$ and $a_n \geq 1$ for all *n*. We have already verified it for $n = 1$ for $a_1 = \frac{3}{2} \geq a_2 = \frac{9}{8}$ and $a_1 \geq 1$. Assume now that for some *n*, $a_n \geq a_{n+1}$ and $a_n \geq 1$. Since $f(x) = (x-1)^3 + 1$ is a non-decreasing function

$$\begin{aligned} a_n \geq a_{n+1} &\implies a_{n+1} = f(a_n) \geq f(a_{n+1}) = a_{n+2} \\ a_n \geq 1 &\implies a_{n+1} = f(a_n) \geq f(1) = 1. \end{aligned}$$

The sequence $\{a_n\}_{n=1}^{\infty}$ is non-increasing and bounded below, therefore convergent. We already have seen that the possible limits are 0, 1, and 2, and the we conclude that $\lim_{n\to\infty} a_n = 1$ for 2 is not possible because $\{a_n\}_{n=1}^{\infty}$ is non-increasing with $a_1 = \frac{3}{2}$ and 0 is not possible because $\{a_n\}_{n=1}^{\infty}$ is bounded below by 1.



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(c) [2pts] What is the limit of $\{a_n\}_{n=1}^{\infty}$ if $a_1 = 3$? Justify your answer.
*Solution*. The first few terms are

$$a_1 = 3,\ a_2 = 9,\ a_3 = 513,\ a_4 = 134217729, \ldots$$

so that it seems that the sequence grows without bounds. Indeed, we prove by induction that

$$a_n = 2^{3^{n-1}} + 1$$

for all *n*. It is true for *n* = 1. Assume it is true for some *n*. Then

$$a_{n+1} = (a_n - 1)^3 + 1 = \left(2^{3^{n-1}}\right)^3 + 1 = 2^{3n} + 1,$$

and the claim is true by induction. Thus $\lim_{n\to\infty} a_n = \infty$.

2. [11] Consider a sequence $\{a_n\}_{n=1}^{\infty}$ defined by

$$\begin{cases} a_1 = 1 \\ a_{n+1} = 1 + x - \frac{x^2}{3} \quad \text{for all } n \end{cases}.$$

(a) [2] Find the first 8 terms (with a calculator).
*Solution*.

$$\begin{array}{lcl} a_1 & = & 1 \\ a_2 & = & \dfrac{5}{3} \approx 1.66 \\ a_3 & = & \dfrac{47}{27} \approx 1.741 \\ a_4 & \approx & 1.731 \\ a_5 & \approx & 1.73226204616 \\ a_6 & \approx & 1.73201811397 \\ a_7 & \approx & 1.73205586493 \\ a_8 & \approx & 1.73205002518 \end{array}$$

(b) [2] Show that

$$\frac{3}{2} \le a_{2n} \le a_{2n+1} \le 2$$

for all *n*.



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*Solution*. We proceed by induction. In (a), we have already verified the case $n = 1$.

Assume that it is true for some $n$. Note that $f(x) = 1 + x - \frac{x^2}{3}$ is decreasing for $x > \frac{3}{2}$ for $f'(x) = 1 - \frac{2}{3}x$. Thus

$$\begin{aligned} \frac{3}{2} \leq a_{2n} \leq a_{2n+1} \leq 2 \quad &\Longrightarrow \quad f(2) \leq f(a_{2n+1}) \leq f(a_{2n}) \leq f\left(\frac{3}{2}\right) \\ &\Longrightarrow \quad \frac{3}{2} \leq f(2) = \frac{5}{3} \leq a_{2n+2} \leq a_{2n+1} \leq 1.75 \leq 2, \end{aligned}$$

which proves the property by induction.

(c) [2] Show that $\{a_{2n}\}_{n=1}^{\infty}$ is convergent.

*Solution*. By the previous question $\{a_{2n}\}_{n=1}^{\infty}$ is bounded. We show by induction that it is non-decreasing, that is, that $a_{2n} \leq a_{2n+2}$ for all $n$. We have already verified it for $n = 1$ because $a_2 < a_4$. Assume that it is true for some $n$. Then, since all terms of the sequence are at least $\frac{3}{2}$, we can use the fact that $f$ is non-decreasing, and thus, that $f \circ f$ is increasing to the effect that

$$a_{2n} \leq a_{2n+2} \Longrightarrow a_{2n+2} = f \circ f(a_{2n}) \leq f \circ f(a_{2n+2}) = a_{2n+4},$$

which completes the proof that $\{a_{2n}\}_{n=1}^{\infty}$ is non-decreasing. Since it is also bounded, it is convergent.

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(d) [2] Show that $\{a_{2n+1}\}_{n=1}^{\infty}$ is convergent.

*Solution*. By (b), $\{a_{2n+1}\}_{n=1}^{\infty}$ is bounded. We show by induction that it is non-increasing, that is, that $a_{2n+1} \geq a_{2n+3}$ for all *n*. We have already verified it for $n = 1$ because $a_3 > a_5$. Assume that it is true for some *n*. Then, since all terms of the sequence are at least $\frac{3}{2}$, we can use the fact that *f* is non-decreasing, and thus, that $f \circ f$ is increasing to the effect that

$$a_{2n+1} \geq a_{2n+3} \Longrightarrow a_{2n+3} = f \circ f(a_{2n+1}) \leq f \circ f(a_{2n+3}) = a_{2n+5},$$

which completes the proof that $\{a_{2n+1}\}_{n=1}^{\infty}$ is non-increasing. Since it is also bounded, it is convergent.

(e) [3] Admitting that $\lim_{n\to\infty} a_{2n+1} - a_{2n} = 0$, deduce $\lim_{n\to\infty} a_n$.

*Solution*. If $\lim_{n\to\infty} a_{2n+1} - a_{2n} = 0$ and $\{a_{2n}\}_{n=1}^{\infty}$ and $\{a_{2n+1}\}_{n=1}^{\infty}$ are both convergent, then

$$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} a_n$$

and $\{a_n\}_{n=1}^{\infty}$ converges. Let $L := \lim_{n\to\infty} a_n$. Then $f(L) = L$ because *f* is continuous. Thus

$$1 + L - \frac{L^2}{3} = L \iff L^2 = 3 \iff L = \pm\sqrt{3}.$$

Moreover, only $\sqrt{3}$ is between $\frac{3}{2}$ and $2$ as terms of $a_n$ for $n > 1$. Thus

$$\lim_{n\to\infty} a_n = \sqrt{3}.$$



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The Worksheet and Homework set M10C should be worked on after studying the material from sections 10.6, 10.7, 10.8, and 10.9 of the youtube workbook.

## 10.7 M10C Worksheet: Series

1. Are the following series convergent or divergent? When convergent, find the sum.

   (a) $\sum_{n=1}^{\infty} \frac{3^{n+2}}{2^{n-1}}$

   (b) $\sum_{n=1}^{\infty} \frac{e^n}{n^3}$

   (c) $\sum_{n=1}^{\infty} \frac{3^n}{2^{2n}}$

   (d) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+4}$

   (e) $\sum_{n=1}^{\infty} (-1)^n \frac{2^{n+2}}{3^{n-1}}$

   (f) $\sum_{n=1}^{\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}}$

   (g) $\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$

   (h) $\frac{5}{2} + \frac{7}{3} + \frac{9}{4} + \frac{11}{5} \dots$

   (i) $3 + \sqrt{3} + \sqrt[3]{3} + \sqrt[4]{3} + \dots$

   (j) $\sum_{n=1}^{\infty} \frac{e^n}{2^{2n}} + \frac{3}{4n^2-1}$

2. Justify the equality

$$0.\overline{99} = 1.$$

3. Represent as a fraction of two integers the decimal number

$$2.\overline{345}$$

4. A rubber ball falls an initial height of 10 meters. Whenever it hits the ground, it bounces up two-thirds of the previous height. What is the total distance covered by the ball before it comes to rest?



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## 10.8 M10C Homework set: Series

1. Are the following series convergent or divergent? When convergent, find the sum.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{2n}}{5^{n-1}}$

   (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{e^n}{5^{n-1}}$

   (c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3n^2+1}{2n^2+3}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{16n^2-8n-3}$

   (f) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{2^{2n}} + \frac{4}{n^2+n}$

2. Represent as a fraction of two integers the decimal number

$$3.\overline{12}$$

3. (Bonus) Achilles and a Tortoise have a race. The tortoise gets a 1000 feet head start, but Achilles runs 10 ft/s while the tortoise only runs 0.01 ft/s. Zeno of Elea (ca. 490–430 BC) proposed the following paradox, known as *Zeno's paradox*, to show that motion is nothing but an illusion: When Achilles reaches the tortoise's starting point, she has moved ahead, although by a short distance. By the time Achilles reaches that new point, the tortoise has again moved a short distance ahead, and so on. Thus, it seems that Achilles can never catch up. Using series, explain why it is not the case, and calculate the time it takes Achilles to pass the Tortoise.



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## 10.9 M10C Homework set: Solutions

NAME:

GRADE: /13

BONUS: /5

1. Are the following series convergent or divergent? When convergent, find the sum.

(a) [1] *Solution.*

$$\sum_{n=1}^{\infty} \frac{2^{2n}}{5^{n-1}} = \sum_{n=1}^{\infty} 4\left(\frac{4}{5}\right)^{n-1} = \frac{4}{1-\frac{4}{5}} = 20$$

because the series is geometric with first term 4 and common ratio $\frac{4}{5}$.

(b) [1] *Solution.*

$$\sum_{n=1}^{\infty} (-1)^n \frac{e^n}{5^{n-1}} = \sum_{n=1}^{\infty} -e \cdot \left(-\frac{e}{5}\right)^{n-1} = \frac{-e}{1+\frac{e}{5}} = -\frac{5e}{5+e}$$

because the series is geometric with first term $-e$ and common ratio $-\frac{e}{5}$, and $|-\frac{e}{5}| < 1$.

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(c) [1] *Solution*. The series

$$\sum_{n=1}^{\infty}(-1)^n \frac{3n^2+1}{2n^2+3} = \sum_{n=1}^{\infty} a_n$$

is divergent by the $n^{th}$ term Test, because

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{3n^2+1}{2n^2+3} = \frac{3}{2} \neq 0$$

so that $\lim_{n\to\infty} a_n \neq 0$.

(d) [2]

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$$

*Solution*. Since

$$\frac{n}{(n+1)!} = \frac{(n+1)-1}{(n+1)!} = \frac{n+1}{(n+1)!} - \frac{1}{(n+1)!} = \frac{1}{n!} - \frac{1}{(n+1)!},$$

the partial sum is

$$s_n = \sum_{i=1}^{n} \frac{i}{(i+1)!} = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{1}{n!} - \frac{1}{(n+1)!} = 1 - \frac{1}{(n+1)!}$$

so that

$$\lim_{n\to\infty} s_n = 1 = \sum_{n=1}^{\infty} \frac{n}{(n+1)!}.$$

(e) [2]

$$\sum_{n=1}^{\infty} \frac{1}{16n^2-8n-3}$$

*Solution*. Since

$$\frac{1}{16n^2-8n-3} = \frac{1}{(4n-3)(4n+1)} = \frac{A}{4n-3} + \frac{B}{4n+1}$$

and $A$ and $B$ are easily found to be $\frac{1}{4}$ and $-\frac{1}{4}$ respectively, the partial sum is

$$\begin{aligned} s_n &= \sum_{i=1}^{n} \frac{1}{16n^2-8n-3} = \frac{1}{4}\left(1 - \frac{1}{5} + \frac{1}{5} - \frac{1}{9} + \frac{1}{9} - \frac{1}{13} + \ldots + \frac{1}{4n-3} - \frac{1}{4n+1}\right) \\ &= \frac{1}{4}\left(1 - \frac{1}{4n+1}\right) \end{aligned}$$



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so that

$$\lim_{n\to\infty} s_n = \frac{1}{4} = \sum_{n=1}^{\infty} \frac{1}{16n^2 - 8n - 3}.$$

(f) [4: 2 for telescoping, 1 for geometric, 1 for sum]

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{2^{2n}} + \frac{4}{n^2+n}$$

*Solution*. The series

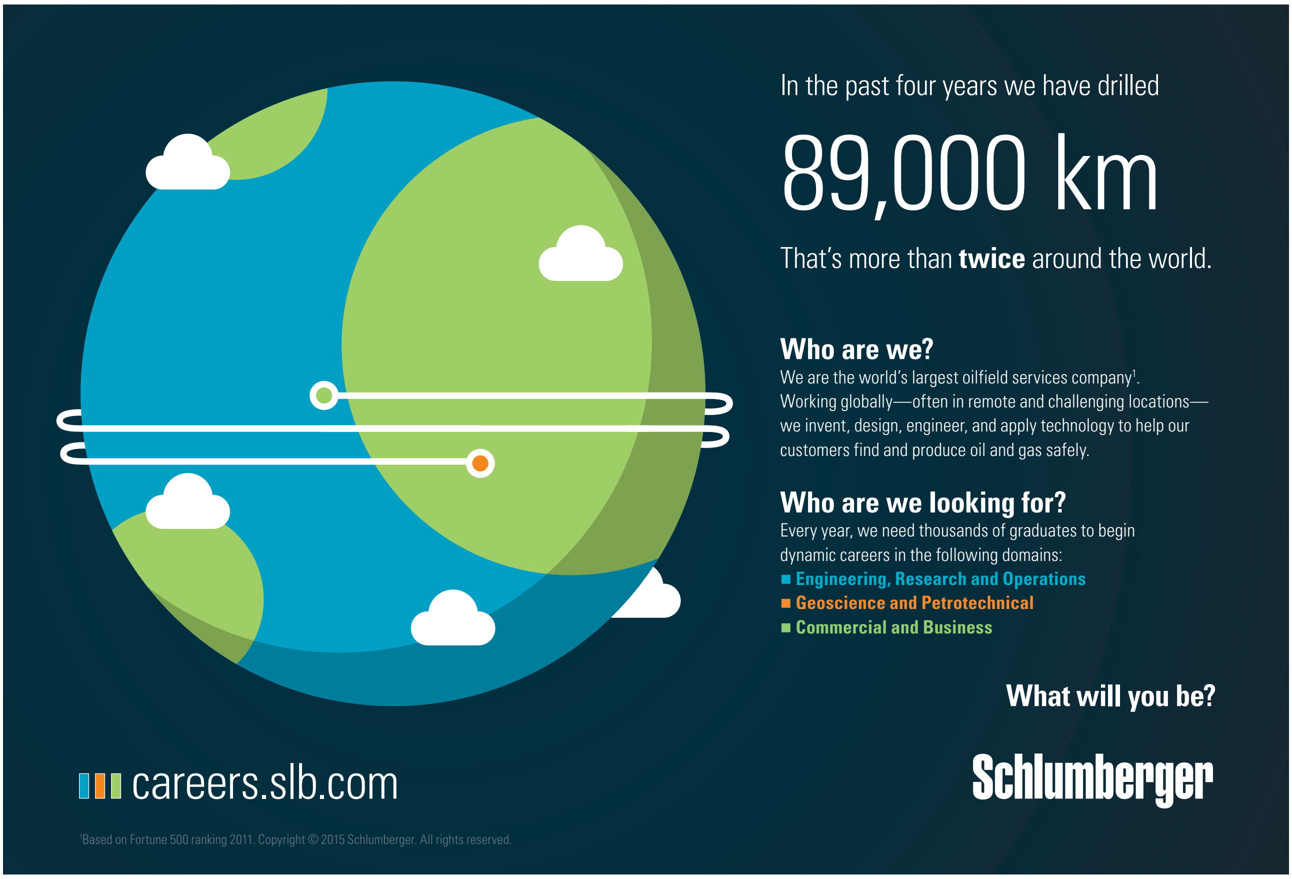
$$\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{2^{2n}} = \sum_{n=1}^{\infty} \left(-\frac{3}{4}\right)^n = \frac{-\frac{3}{4}}{1+\frac{3}{4}} = -\frac{3}{7}$$

because the series is geometric of common ratio $-\frac{3}{4}$ and first term $-\frac{3}{4}$. On the other hand

$$\frac{1}{n^2+n} = \frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} = \frac{1}{n} - \frac{1}{n+1}$$

so that the partial sum for $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ is

$$s_n = \sum_{i=1}^{n} \frac{1}{i^2+i} = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \ldots + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}$$



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so that

$$\lim_{n\to\infty} s_n = 1 = \sum_{n=1}^{\infty} \frac{1}{n^2+n}.$$

Thus

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{2^{2n}} + \frac{4}{n^2+n} = \sum_{n=1}^{\infty}\left(-\frac{3}{4}\right)^n + 4\sum_{n=1}^{\infty}\frac{1}{n^2+n} = -\frac{3}{7} + 4 = \frac{25}{7}.$$

2. [2] Represent as a fraction of two integers the decimal number

$$3.\overline{12}$$

*Solution*.

$$\begin{aligned} 3.\overline{12} &= 3 + \frac{12}{100} + \frac{12}{(100)^2} + \ldots + \frac{12}{(100)^n} + \ldots \\ &= 3 + \sum_{n=1}^{\infty} 12\left(\frac{1}{100}\right)^n = 3 + \frac{\frac{12}{100}}{1-\frac{1}{100}} \\ &= 3 + \frac{12}{99} = 3 + \frac{4}{33} = \frac{103}{33}. \end{aligned}$$

3. [Bonus:+5] Achilles and a Tortoise have a race. The tortoise gets a 1000 feet head start, but Achilles runs 10 ft/s while the tortoise only runs 0.01 ft/s. Zeno of Elea (ca. 490–430 BC) proposed the following paradox, known as *Zeno's paradox*, to show that motion is nothing but an illusion: When Achilles reaches the tortoise's starting point, she has moved ahead, although by a short distance. By the time Achilles reaches that new point, the tortoise has again moved a short distance ahead, and so on. Thus, it seems that Achilles can never catch up. Using series, explain why it is not the case, and calculate the time it takes Achilles to pass the Tortoise.

*Solution*. When Achilles reaches the tortoise starting point, he has covered 1000 feet at 10ft/s, hence 100s have passed. Meanwhile, the tortoise has moved ahead by $0.01 \times 100 = 1\,ft$. Achilles covers this feet in $0.1$ second, and the tortoise has then moved ahead by $0.01 \times 0.1 = 0.001\,ft$. It takes Achilles $0.0001$ second to cover this distance, and meanwhile the tortoise moved ahead by $0.01 \times 0.0001 = 0.000001\,ft$. As we iterate this process, the limit of the distance between Achilles and the tortoise is 0 and the time it takes Achilles to reach the tortoise is the sum of the geometric series

$$\begin{aligned} 100 + 0.1 + 0.0001 + 0.0000001 + \ldots &= \sum_{n=1}^{\infty} 100\left(\frac{1}{1000}\right)^{n-1} = \frac{100}{1-\frac{1}{1000}} \\ &= \frac{10^5}{999} = 100.\overline{1001}\,sec. \end{aligned}$$



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# 11 M11: Integral Test and Comparison Test

The Worksheet and Homework set M11A should be worked on after studying the material from sections 11.1, 11.2, and 11.3 of the youtube workbook.

## 11.1 M11A Worksheet: Integral Test

1. Are the following series convergent or divergent?

   (a) $\sum_{n=1}^{\infty} \frac{4}{\sqrt[7]{n^4}}$

   (b) $\sum_{n=1}^{\infty} \frac{2}{\sqrt[5]{n^8}}$

   (c) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

   (d) $\sum_{n=1}^{\infty} \frac{e^n}{n}$

   (e) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

   (f) $\sum_{n=1}^{\infty} \frac{1}{2^n}$

   (g) $\sum_{n=1}^{\infty} \frac{n+1}{(n^2+2n+4)^2}$

   (h) $\sum_{n=1}^{\infty} \frac{3}{n^2+2n+2}$

   (i) $\sum_{n=1}^{\infty} \frac{n}{e^n}$

2. Justify the convergence of the following series and find the number of terms to be added to obtain an estimate of the sum with an error less than $10^{-3}$:

   (a) $\sum_{n=1}^{\infty} \frac{2}{n^4}$

   (b) $\sum_{n=1}^{\infty} \frac{n^2}{e^{n^3}}$

   (c) $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^2}$



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## 11.2 M11A Homework set: Integral Test

1. Are the following series convergent or divergent?

   (a) $\sum_{n=1}^{\infty} \frac{2n+3}{n^2+3n+2}$

   (b) $\sum_{n=1}^{\infty} \frac{4}{\sqrt[6]{n^7}}$

   (c) $\sum_{n=1}^{\infty} \frac{n+1}{2n+3}$

   (d) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

   (e) $\sum_{n=1}^{\infty} \frac{3^{n-1}}{2^n}$

   (f) $\sum_{n=1}^{\infty} \frac{1}{n^2+4n+5}$

2. Justify the convergence of

$$\sum_{n=1}^{\infty} \frac{n^2}{e^n}$$

   and find the number of terms to be added to obtain an estimate of the sum with an error less than $10^{-3}$.

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## 11.3 M11A Homework set: Solutions

NAME:

GRADE: /22

1. Are the following series convergent or divergent?

(a) [4pts:1 for checking assumptions, 2 for integrals, 1 for conclusion] $\sum_{n=1}^{\infty} \frac{2n+3}{n^2+3n+2}$

*Solution*. This series is of the form $\sum_{n=1}^{\infty} f(n)$ where $f(x) = \frac{2x+3}{x^2+3x+2}$ is non-negative continuous and decreasing on $[1,\infty)$ for

$$f'(x) = \frac{2(x^2+3x+2)-(2x+3)^2}{(x^2+3x+2)^2} = \frac{-2x^2-6x-5}{(x^2+3x+2)^2} < 0 \text{ for } x > 0.$$

Thus, by the Integral Test, $\sum_{n=1}^{\infty} \frac{2n+3}{n^2+3n+2}$ is convergent if and only if $\int_1^{\infty} f(x)\,dx$ is. Moreover,

$$\begin{aligned}\int_1^{\infty} \frac{2x+3}{x^2+3x+2}\,dx &= \lim_{t\to\infty} \int_1^t \frac{2x+3}{x^2+3x+2}\,dx \\ &= \lim_{t\to\infty} \int_6^{t^2+3t+2} \frac{du}{u} \text{ for } u = x^2+3x+2 \\ &= \lim_{t\to\infty} [\ln|u|]_6^{t^2+3t+2} = \infty,\end{aligned}$$

so that $\sum_{n=1}^{\infty} \frac{2n+3}{n^2+3n+2}$ is divergent.

(b) [1pt] $\sum_{n=1}^{\infty} \frac{4}{\sqrt[6]{n^7}}$

*Solution*.This is a *p*-series for $p = \frac{7}{6} > 1$, and therefore $\sum_{n=1}^{\infty} \frac{4}{\sqrt[6]{n^7}}$ is convergent.

(c) [1pt] $\sum_{n=1}^{\infty} \frac{n+1}{2n+3}$

*Solution*. The general term $a_n = \frac{n+1}{2n+3}$ does not converge to 0 ($\lim_{n\to\infty} a_n = \frac{1}{2} \neq 0$), and thus, by the $n^{th}$ Term Test, $\sum_{n=1}^{\infty} \frac{n+1}{2n+3}$ is divergent.

(d) [4pts:1 for checking assumptions, 2 for integrals, 1 for conclusion] $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

*Solution*. This series is of the form $\sum_{n=1}^{\infty} f(n)$ where $f(x) = \frac{1}{x(\ln x)^3}$ is non-negative continuous and decreasing on $[2,\infty)$ for

$$f'(x) = -\frac{(\ln x)^3 + 3(\ln x)^2}{x^2(\ln x)^6} < 0 \text{ for } x > 1.$$



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By the Integral Test, $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$ is convergent if and only if $\int_2^\infty f(x)\,dx$ is. Moreover

$$\begin{aligned}\int_2^\infty \frac{dx}{x(\ln x)^3} &= \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{du}{u^3}, \text{ using } u = \ln x \\ &= \lim_{t\to\infty} \left[-\frac{1}{2u^2}\right]_{\ln 2}^{\ln t} = \frac{1}{2(\ln 2)^2},\end{aligned}$$

so that $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$ is convergent.

(e) [1pt]$\sum_{n=1}^{\infty} \frac{3^{n-1}}{2^n}$

*Solution*. The series

$$\sum_{n=1}^{\infty} \frac{3^{n-1}}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{3}{2}\right)^{n-1}$$

is geometric of common ratio $\frac{3}{2} \geq 1$. Thus $\sum_{n=1}^{\infty} \frac{3^{n-1}}{2^n}$ is divergent.

(f) [4pts: 1 for checking assumptions, 2 for integrals, 1 for conclusion]$\sum_{n=1}^{\infty} \frac{1}{n^2+4n+5}$

*Solution*. This series is of the form $\sum_{n=1}^{\infty} f(n)$ where $f(x) = \frac{1}{x^2+4x+5}$ is non-negative continuous and decreasing on $[1,\infty)$ for

$$f'(x) = \frac{-(2x+4)}{(x^2+4x+5)^2} < 0 \text{ for } x \geq 0.$$

By the Integral Test, $\sum_{n=1}^{\infty} \frac{1}{n^2+4n+5}$ is convergent if and only if $\int_1^\infty f(x)\,dx$ is. Moreover

$$\begin{aligned}\int_1^\infty \frac{1}{x^2+4x+5}\,dx &= \lim_{t\to\infty} \int_1^t \frac{dx}{(x+2)^2+1} \\ &= \lim_{t\to\infty} \int_3^{t+2} \frac{du}{u^2+1} \text{ for } u = x+2 \\ &= \lim_{t\to\infty} [\arctan u]_3^{t+2} = \frac{\pi}{2} - \arctan 3,\end{aligned}$$

so that $\sum_{n=1}^{\infty} \frac{1}{n^2+4n+5}$ is convergent.

2. [7pts: 1 for assumptions, 3 for integral, 1 for conclusion of CV, 1 for formula for $R_n$, 1 for conclusion] Justify the convergence of

$$\sum_{n=1}^{\infty} \frac{n^2}{e^n}$$

and find the number of terms to be added to obtain an estimate of the sum with an error less than $10^{-3}$.



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*Solution*. This series is of the form $\sum_{n=1}^{\infty} f(n)$ where $f(x) = x^2e^{-x}$ is non-negative continuous and decreasing on $[2, \infty)$ for

$$f'(x) = 2xe^{-x} - x^2e^{-x} = xe^{-x}(2 - x) < 0 \text{ for } x > 2.$$

By the Integral Test $\sum_{n=1}^{\infty} \frac{n^2}{e^n}$ is convergent if and only if $\int_2^{\infty} f(x)\,dx$ is. Moreover,

$$\int_2^{\infty} x^2e^{-x}\,dx = \lim_{t\to\infty} \int_2^t x^2e^{-x}\,dx$$

and we integrate $\int_2^t x^2e^{-x}\,dx$ using integration by parts twice with $dv = e^{-x}\,dx$ in both case:

$$\begin{aligned}
\int_2^t x^2e^{-x}\,dx &= \left[-x^2e^{-x}\right]_2^t + 2\int_2^t xe^{-x}\,dx \\
&= \left[-x^2e^{-x}\right]_2^t + 2\left(\left[-xe^{-x}\right]_2^t + \int_2^t e^{-x}\,dx\right) \\
&= \left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_2^t \\
&= 10e^{-2} - e^{-t}\left(t^2 + 2t + 2\right).
\end{aligned}$$

Since

$$\lim_{t\to\infty} e^{-t}\left(t^2 + 2t + 2\right) = \lim_{t\to\infty} \frac{t^2 + 2t + 2}{e^t} \overset{H}{=} \lim_{t\to\infty} \frac{2t + 2}{e^t} \overset{H}{=} \lim_{t\to\infty} \frac{2}{e^t} = 0,$$

we conclude that

$$\int_2^{\infty} x^2e^{-x}\,dx = 10e^{-2}$$

and therefore $\sum_{n=1}^{\infty} \frac{n^2}{e^n}$ is convergent. Moreover the error $R_n$ committed in approximating $\sum_{n=1}^{\infty} \frac{n^2}{e^n}$ by the $n^{th}$ partial sum $s_n$ satisfies

$$R_n \leq \int_n^{\infty} x^2e^{-x}\,dx = e^{-n}\left(n^2 + 2n + 2\right).$$

Thus, $R_n \leq 10^{-2}$ if $e^{-n}\left(n^2 + 2n + 2\right) \leq 10^{-2}$, and plugging in integer values in $e^{-n}\left(n^2 + 2n + 2\right)$ shows that $R_9 > 10^{-2}$ and $R_{10} < 10^{-2}$, so that we need to use $s_{10}$ to guarantee an error less than $10^{-2}$.



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The Worksheet and Homework set M11B should be worked on after studying the material from sections 11.4, 11.5, and 11.6 of the youtube workbook.

## 11.4 M11B Worksheet: Comparison for series

1. Are the following series convergent or divergent? Justify your answers.

   (a) $\sum_{n=1}^{\infty} \frac{4}{\sqrt{n^5+3}}$

   (b) $\sum_{n=1}^{\infty} \frac{2}{\sqrt{2n-1}}$

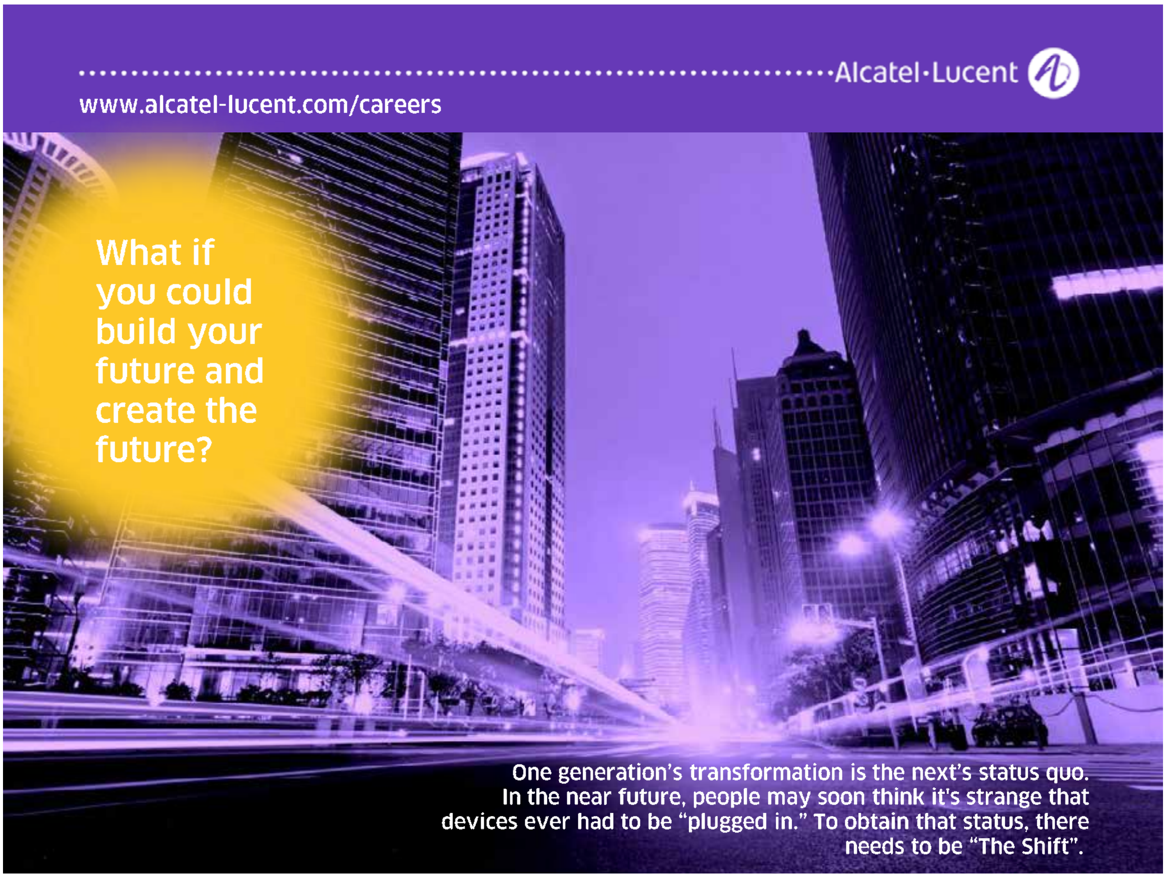
   (c) $\sum_{n=1}^{\infty} \frac{2}{4^n+3}$

   (d) $\sum_{n=1}^{\infty} \frac{3^n}{e^{n+1}-1}$

   (e) $\sum_{n=1}^{\infty} \frac{2n+3}{3n^3+4n+1}$

   (f) $\sum_{n=1}^{\infty} \frac{4n^2+n-1}{\sqrt{3n^5+n^3+3}}$

   (g) $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2}$



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(h) $\sum_{n=1}^{\infty} \frac{1+\sin n}{n}$

(i) $\sum_{n=1}^{\infty} \frac{1}{n!}$

(j) $\sum_{n=1}^{\infty} \frac{\ln n}{n^2+2}$

(k) $\sum_{n=1}^{\infty} \frac{n+1}{n3^n}$

(l) $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}}$

2. Justify that the following series are convergent and find $n$ such that the $n^{th}$ partial sum gives an estimate of the sum with an error of at most $10^{-3}$.

(a) $\sum_{n=1}^{\infty} \frac{2}{4^n+3}$

(b) $\sum_{n=1}^{\infty} \frac{n}{n^3+4}$

(c) $\sum_{n=1}^{\infty} \frac{n}{(n^2+2)(\ln n)^2}$



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## 11.5 M11B Homework set: Comparison for Series

1. Are the following series convergent or divergent? Justify your answers.

   (a) $\sum_{n=1}^{\infty} \frac{2}{\sqrt[3]{n^4+1}}$

   (b) $\sum_{n=1}^{\infty} \frac{1}{2\sqrt[3]{n^2}-\sqrt{n}}$

   (c) $\sum_{n=1}^{\infty} \frac{5^n}{e^{2n}+4}$

   (d) $\sum_{n=1}^{\infty} \frac{2n^2+1}{n^3+n+1}$

   (e) $\sum_{n=1}^{\infty} \frac{2n+1}{\sqrt{5n^6+n^2+3}}$

   (f) $\sum_{n=1}^{\infty} \frac{1+\cos n}{n^2}$

   (g) $\sum_{n=1}^{\infty} \frac{2n+1}{n^2 2^n}$

   (h) $\sum_{n=1}^{\infty} \frac{1}{1+\ln n}$

2. Justify that the following series are convergent and find $n$ such that the $n^{th}$ partial sum gives an estimate of the sum with an error of at most $10^{-3}$.

   (a) $\sum_{n=1}^{\infty} \frac{n}{(n+1)5^n}$

   (b) $\sum_{n=1}^{\infty} \frac{2n}{\sqrt{n^5+n+1}}$



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## 11.6 M11B Homework set: Solutions

NAME:

GRADE: /24

1. [16] Are the following series convergent or divergent? Justify your answers.

(a) [2]$\sum_{n=1}^{\infty} \frac{2}{\sqrt[3]{n^4+1}}$

*Solution.*

$$0 \le \frac{2}{\sqrt[3]{n^4+1}} \le \frac{2}{n^{\frac{4}{3}}}$$

and $\sum_{n=1}^{\infty} \frac{2}{n^{\frac{4}{3}}}$ is a convergent $p$-series for $p = \frac{4}{3} > 1$. By Comparison Test, $\sum_{n=1}^{\infty} \frac{2}{\sqrt[3]{n^4+1}}$ is convergent.

(b) [2]$\sum_{n=1}^{\infty} \frac{1}{2\sqrt[3]{n^2}-\sqrt{n}}$

*Solution.*

$$2\sqrt[3]{n^2} - \sqrt{n} \le 2n^{\frac{2}{3}} \Longrightarrow 0 \le \frac{1}{2n^{\frac{2}{3}}} \le \frac{1}{2\sqrt[3]{n^2} - \sqrt{n}}$$

and $\sum_{n=1}^{\infty} \frac{1}{2n^{\frac{2}{3}}}$ is a divergent $p$-series ($p = \frac{2}{3} \le 1$). By Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{2\sqrt[3]{n^2}-\sqrt{n}}$ is also divergent.

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(c) [2]$\sum_{n=1}^{\infty} \frac{5^n}{e^{2n}+4}$

*Solution.*

$$\frac{5^n}{e^{2n}+4} \leq \frac{5^n}{e^{2n}} = \left(\frac{5}{e^2}\right)^n$$

and $\sum_{n=1}^{\infty} \left(\frac{5}{e^2}\right)^n$ is a convergent geometric series for the common ratio is $\frac{5}{e^2} < 1$. By Comparison Test, $\sum_{n=1}^{\infty} \frac{5^n}{e^{2n}+4}$ is convergent.

(d) [2]$\sum_{n=1}^{\infty} \frac{2n^2+1}{n^3+n+1}$

*Solution.* Let $a_n = \frac{2n^2+1}{n^3+n+1}$ and $b_n = \frac{1}{n}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^3+n}{n^3+n+1} = 2 > 0.$$

Moreover, $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent *p*-series $(p = 1 \leq 1)$. By the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{2n^2+1}{n^3+n+1}$ is also divergent.

(e) [2]$\sum_{n=1}^{\infty} \frac{2n+1}{\sqrt{5n^6+n^2+3}}$

*Solution.* Let $a_n = \frac{2n+1}{\sqrt{5n^6+n^2+3}}$ and let $b_n = \frac{n}{\sqrt{n^6}} = \frac{1}{n^2}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3\left(2+\frac{1}{n}\right)}{\sqrt{n^6\left(5+\frac{1}{n^4}+\frac{3}{n^6}\right)}} = \lim_{n\to\infty} \frac{2+\frac{1}{n}}{\sqrt{5+\frac{1}{n^4}+\frac{3}{n^6}}} = \frac{2}{\sqrt{5}} > 0.$$

Moreover $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent *p*-series $(p = 2 > 1)$. By the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{2n+1}{\sqrt{5n^6+n^2+3}}$ is convergent.

(f) [2]$\sum_{n=1}^{\infty} \frac{1+\cos n}{n^2}$

*Solution.* Since $-1 \leq \cos n \leq 1$,

$$0 \leq \frac{1+\cos n}{n^2} \leq \frac{2}{n^2}.$$

Moreover, $\sum_{n=1}^{\infty} \frac{2}{n^2}$ is a convergent *p*-series $(p = 2 > 1)$. By Comparison Test, $\sum_{n=1}^{\infty} \frac{1+\cos n}{n^2}$ is also convergent.



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(g) [2]$\sum_{n=1}^{\infty} \frac{2n+1}{n^2 2^n}$

*Solution*. Let $a_n = \frac{2n+1}{n^2 2^n}$ and $b_n = \frac{1}{n2^n}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n+1}{n} = 2 > 0.$$

Moreover

$$0 \leq b_n = \frac{1}{n2^n} \leq \frac{1}{2^n}$$

and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series of common ratio $\frac{1}{2}$ and is therefore convergent. By Comparison $\sum_{n=1}^{\infty} b_n$ is convergent. By the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{2n+1}{n^2 2^n}$ is convergent.

(h) [2]$\sum_{n=1}^{\infty} \frac{1}{1+\ln n}$

*Solution*. Note that

$$1 + \ln n \leq n + 1$$

because $f(x) = \ln x - x$ satisfies

$$f'(x) = \frac{1}{x} - 1 < 0 \text{ for all } x > 1$$

and $f(1) = -1 < 0$ and thus $f(x) < 0$ for $x \geq 1$. Thus

$$0 \leq \frac{1}{n+1} \leq \frac{1}{1+\ln n}$$

and $\sum_{n=1}^{\infty} \frac{1}{n+1}$ is divergent by Limit Comparison, for

$$\lim_{n\to\infty} \frac{n}{n+1} = 1 > 0$$

and $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent $p$-series $(p = 1 \leq 1)$. Therefore, by Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{1+\ln n}$ is divergent.



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2. [8] Justify that the following series are convergent and find $n$ such that the $n^{th}$ partial sum gives an estimate of the sum with an error of at most $10^{-3}$.

(a) [4: 2 convergence, 2 for finding $n$ ] $\sum_{n=1}^{\infty} \frac{n}{(n+1)5^n}$

*Solution.* Since

$$\frac{n}{(n+1)5^n} \leq \frac{1}{5^n}$$

and $\sum_{n=1}^{\infty} \frac{1}{5^n}$ is a geometric series of common ratio $\frac{1}{5} < 1$, hence convergent, we conclude by the Comparison Test that $\sum_{n=1}^{\infty} \frac{n}{(n+1)5^n}$ is also convergent. Moreover

$$R_n = \sum_{i=n+1}^{\infty} \frac{i}{(i+1)5^i} \leq T_n = \sum_{i=n+1}^{\infty} \frac{1}{5^i} = \frac{\frac{1}{5^{n+1}}}{1-\frac{1}{5}} = \frac{1}{4 \cdot 5^n}.$$

Thus $R_n \leq 10^{-3}$ whenever

$$\frac{1}{4 \cdot 5^n} \leq 10^{-3} \iff 5^n \geq \frac{10^3}{4} \iff n \geq \log_5{(250)} \approx 3.4.$$

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Therefore,

$$s_4 = \frac{1}{10} + \frac{2}{75} + \frac{3}{500} + \frac{4}{3125}$$

approximates $\sum_{n=1}^{\infty} \frac{n}{(n+1)5^n}$ with an error less than $10^{-3}$.

(b) [4:2 convergence, 2 for finding $n$ ]$\sum_{n=1}^{\infty} \frac{2n}{\sqrt{n^5+n+1}}$

*Solution*. Since

$$0 \le \frac{2n}{\sqrt{n^5+n+1}} \le \frac{2n}{n^{\frac{5}{2}}} = \frac{2}{n^{\frac{3}{2}}}$$

and $\sum_{n=1}^{\infty} \frac{2}{n^{\frac{3}{2}}}$ is a convergent $p$-series ($p = \frac{3}{2} > 1$), we conclude by the Comparison Test that $\sum_{n=1}^{\infty} \frac{2n}{\sqrt{n^5+n+1}}$ is convergent. Moreover

$$R_n = \sum_{i=n+1}^{\infty} \frac{2i}{\sqrt{i^5+i+1}} \le T_n = \sum_{i=n+1}^{\infty} \frac{2}{i^{\frac{3}{2}}} \le 2\int_n^{\infty} \frac{dx}{x^{\frac{3}{2}}}$$

and

$$\int_n^{\infty} \frac{dx}{x^{\frac{3}{2}}} = \lim_{t\to\infty} \left[-\frac{2}{\sqrt{x}}\right]_n^t = \frac{2}{\sqrt{n}}$$

so that $R_n \le \frac{4}{\sqrt{n}}$. Thus, $R_n \le 10^{-3}$ whenever

$$\frac{4}{\sqrt{n}} \le 10^{-3} \iff 4000 \le \sqrt{n} \iff 16 \cdot 10^6 \le n$$

and $s_{16\cdot 10^6}$ approximates $\sum_{n=1}^{\infty} \frac{2n}{\sqrt{n^5+n+1}}$ with an error of at most $10^{-3}$.



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# 12 M12: Alternating Series Test

The Worksheet and Homework set M12 should be worked on after studying the material from sections 12.1, 12.2, and 12.3 of the youtube workbook.

## 12.1 M12 Worksheet: Alternating Series

1. Are the following series absolutely convergent, conditionally convergent or divergent?

   (a) $\frac{1}{2} - \frac{1}{5} + \frac{1}{10} - \frac{1}{17} + \ldots$

   (b) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{e^n}$

   (c) $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{2} + \ldots$

   (d) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \ldots$

   (e) $\sum_{n=1}^{\infty} (-1)^n \frac{n^2-1}{2n^2+1}$

   (f) $\sum_{n=1}^{\infty} (-1)^n \frac{n+1}{n^2+2n+3}$

   (g) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$

   (h) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\ln n}{3n+1}$

   (i) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[n]{3}}$

   (j) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$

   (k) $\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n+1} - \sqrt{n}\right)$

2. How many terms of $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$ will suffice to get an approximation within $10^{-4}$ of the sum $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$?

3. How many terms of $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ will suffice to get an approximation within 0.01 of the sum $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$?



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## 12.2 M12 Homework set: Alternating Series

1. Are the following series absolutely convergent, conditionally convergent or divergent?

   (a) $\frac{1}{2} - \frac{1}{8} + \frac{1}{18} - \frac{1}{32} + \ldots$

   (b) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^n}{e^{n-1}}$

   (c) $2 - \frac{4}{5} + \frac{8}{25} - \frac{16}{125} + \ldots$

   (d) $\sum_{n=1}^{\infty} (-1)^n \frac{n^3+n+1}{\sqrt{2n^7+n^5+4}}$

   (e) $\sum_{n=1}^{\infty} (-1)^n \frac{2n+1}{n^5+3}$

   (f) $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{1}{n}\right)$

   (g) $\sum_{n=1}^{\infty} (-1)^n \frac{e^{\frac{1}{n}}}{n}$

2. How many terms of $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ will suffice to get an approximation within $10^{-4}$ of the sum $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$?

3. How many terms of $\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n}$ will suffice to get an approximation within 0.002 of the sum $\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n}$?

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## 12.3 M12 Homework set: Solutions

NAME:

GRADE: /26

1. Are the following series absolutely convergent, conditionally convergent or divergent?

(a) [2]$\frac{1}{2} - \frac{1}{8} + \frac{1}{18} - \frac{1}{32} + \dots$

*Solution*. This is the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2n^2},$$

which is alternating. Moreover

$$\left| (-1)^{n+1} \frac{1}{2n^2} \right| = \frac{1}{2n^2}$$

and $\sum_{n=1}^{\infty} \frac{1}{2n^2}$ is a convergent *p*-series (for $p = 2 > 1$). Thus $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2n^2}$ is absolutely convergent.

(b) [2]$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^n}{e^{n-1}}$

*Solution*.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^n}{e^{n-1}} = \sum_{n=1}^{\infty} 3 \left( -\frac{3}{e} \right)^{n-1}$$

is a geometric series of common ratio $-\frac{3}{e}$ and $|-\frac{3}{e}| > 1$. Thus $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^n}{e^{n-1}}$ is divergent.

(c) [2]$2 - \frac{4}{5} + \frac{8}{25} - \frac{16}{125} + \dots$

*Solution*. This is the geometric series

$$\sum_{n=1}^{\infty} 2 \left( -\frac{2}{5} \right)^{n-1}$$

of common ratio $-\frac{2}{5}$ and $|-\frac{2}{5}| < 1$, so that

$$2 - \frac{4}{5} + \frac{8}{25} - \frac{16}{125} + \dots = \sum_{n=1}^{\infty} 2 \left( -\frac{2}{5} \right)^{n-1}$$



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is absolutely convergent for

$$\left|2\left(-\frac{2}{5}\right)^{n-1}\right| = 2\left(\frac{2}{5}\right)^{n}$$

is the general term of a convergent geometric series.

(d) [4: 2 convergent, 2 non-absolutely conv.] $\sum_{n=1}^{\infty}(-1)^n\frac{n^3+n+1}{\sqrt{2n^7+n^5+4}}$

*Solution.* Let $a_n = \frac{n^3+n+1}{\sqrt{2n^7+n^5+4}}$ and $b_n = \frac{n^3}{\sqrt{n^7}} = \frac{1}{\sqrt{n}}$. Since

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\sqrt{n}(n^3+n+1)}{\sqrt{n^7\left(2+\frac{1}{n^2}+\frac{4}{n^7}\right)}} = \lim_{n\to\infty}\frac{n^{\frac{7}{2}}\left(1+\frac{1}{n^2}+\frac{1}{n^3}\right)}{n^{\frac{7}{2}}\sqrt{2+\frac{1}{n^2}+\frac{4}{n^7}}} = \frac{1}{\sqrt{2}} > 0$$

and $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is a divergent $p$-series (for $p=\frac{1}{2}\leq 1$), we conclude by Limit Comparison Test that $\sum_{n=1}^{\infty}\frac{n^3+n+1}{\sqrt{2n^7+n^5+4}}$ is divergent, and therefore $\sum_{n=1}^{\infty}(-1)^n\frac{n^3+n+1}{\sqrt{2n^7+n^5+4}}$ is **not** absolutely convergent.

On the other hand, if $f(x) = \frac{x^3+x+1}{\sqrt{2x^7+x^5+4}}$ then

$$\begin{aligned}
f'(x) &= \frac{(3x^2+1)\sqrt{2x^7+x^5+4} - \frac{(14x^6+5x^4)(x^3+x+1)}{2\sqrt{2x^7+x^5+4}}}{2x^7+x^5+4} \\
&= \frac{(6x^2+2)(2x^7+x^5+4) - (14x^9+14x^7+14x^6+5x^7+5x^5+5x^4)}{2\left(2x^7+x^5+4\right)^{\frac{3}{2}}} \\
&= \frac{-2x^9-8x^7-3x^5-5x^4+24x^2+8}{2\left(2x^7+x^5+4\right)^{\frac{3}{2}}} < 0 \text{ for } x > 2
\end{aligned}$$

so that $\left\{\frac{n^3+n+1}{\sqrt{2n^7+n^5+4}}\right\}_{n=1}^{\infty}$ is eventually decreasing with limit 0. By the Alternating Series Test, $\sum_{n=1}^{\infty}(-1)^n\frac{n^3+n+1}{\sqrt{2n^7+n^5+4}}$ is convergent, thus conditionally convergent.

(e) [2] $\sum_{n=1}^{\infty}(-1)^n\frac{2n+1}{n^5+3}$

*Solution.* Let $a_n = \frac{2n+1}{n^5+3}$ and $b_n = \frac{n}{n^5} = \frac{1}{n^4}$. Then

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{2n^5+n^4}{n^5+3} = 2 > 0,$$

and $\sum_{n=1}^{\infty}b_n = \sum_{n=1}^{\infty}\frac{1}{n^4}$ is a convergent $p$-series (for $p=4>1$). By the Limit Comparison Test, $\sum_{n=1}^{\infty}\frac{2n+1}{n^5+3}$ is convergent, so that $\sum_{n=1}^{\infty}(-1)^n\frac{2n+1}{n^5+3}$ is absolutely convergent.



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(f) $[2]\sum_{n=1}^{\infty}(-1)^n \cos\left(\frac{1}{n}\right)$

*Solution.* $\lim_{n\to\infty}\cos\left(\frac{1}{n}\right) = \cos 0 = 1$ so that $\left\{(-1)^n \cos\left(\frac{1}{n}\right)\right\}_{n=1}^{\infty}$ does not converge to 0. By the $n^{th}$ term Test, $\sum_{n=1}^{\infty}(-1)^n \cos\left(\frac{1}{n}\right)$ is divergent.

(g) [4: 2 convergent, 2 non-absolutely conv.] $\sum_{n=1}^{\infty}(-1)^n \frac{e^{\frac{1}{n}}}{n}$

*Solution.* Since

$$\frac{1}{n} \leq \frac{e^{\frac{1}{n}}}{n} \leq \frac{e}{n} \tag{12.3.1}$$

and $\sum_{n=1}^{\infty}\frac{1}{n}$ is a divergent $p$-series $(p = 1 \leq 1)$, we conclude by the Comparison Test that $\sum_{n=1}^{\infty}\frac{e^{\frac{1}{n}}}{n}$ is divergent, so that $\sum_{n=1}^{\infty}(-1)^n\frac{e^{\frac{1}{n}}}{n}$ is **not** absolutely convergent.

On the other hand, (12.3.1) gives that $\lim_{n\to\infty}\frac{e^{\frac{1}{n}}}{n} = 0$ because $\lim_{n\to\infty}\frac{1}{n} = 0$, and $\left\{\frac{e^{\frac{1}{n}}}{n}\right\}_{n=1}^{\infty}$ is decreasing because $\frac{1}{n+1} \leq \frac{1}{n}$ and, since $e^x$ is increasing, $e^{\frac{1}{n+1}} \leq e^{\frac{1}{n}}$, so that $\frac{e^{\frac{1}{n+1}}}{n+1} \leq \frac{e^{\frac{1}{n}}}{n}$. By the Alternating Series Test, $\sum_{n=1}^{\infty}(-1)^n\frac{e^{\frac{1}{n}}}{n}$ is convergent, hence conditionally convergent.

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2. [4: 2 convergent, 2 to find *n*] How many terms of $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ will suffice to get an approximation within $10^{-4}$ of the sum $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$?

*Solution*. This series is convergent by the Alternating Series Test for $\left\{\frac{1}{n^2}\right\}_{n=1}^{\infty}$ is decreasing with limit 0. Thus the remainder $R_n=\sum_{i=n+1}^{\infty}\frac{(-1)^i}{i^2}$ satisfies

$$|R_n|\leq\frac{1}{(n+1)^2}.$$

Hence the error made in the approximation $s_n\approx\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ is less than $10^{-3}$ whenever

$$\frac{1}{(n+1)^2}\leq 10^{-4} \iff n+1\geq 10^2 \iff n\geq 99.$$

Thus 99 terms would be necessary to guarantee this accuracy.

3. [4: 2 convergent, 2 to find *n*] How many terms of $\sum_{n=1}^{\infty}\frac{(-1)^n n}{4^n}$ will suffice to get an approximation within 0.002 of the sum $\sum_{n=1}^{\infty}\frac{(-1)^n n}{4^n}$?

*Solution*. This series is convergent by the Alternating Series Test for $\left\{\frac{n}{4^n}\right\}_{n=1}^{\infty}$ is decreasing with limit 0 for

$$\left(\frac{x}{4^x}\right)'=\frac{4^x-x4^x\ln 4}{4^{2x}}=\frac{1-x\ln 4}{4^x}<0 \text{ for } x>\tfrac{1}{\ln 4}$$

and

$$\lim_{n\to\infty}\frac{n}{4^n}=\lim_{x\to\infty}\frac{x}{4^x}\overset{H}{=}\lim_{x\to\infty}\frac{1}{4^x\ln 4}=0.$$

Thus, the remainder $R_n=\sum_{i=n+1}^{\infty}\frac{(-1)^i i}{4^i}$ satisfies

$$|R_n|\leq\frac{n+1}{4^{n+1}}$$

Hence the error made in the approximation $s_n\approx\sum_{n=1}^{\infty}\frac{(-1)^n n}{4^n}$ is less than 0.002 whenever

$$\frac{n+1}{4^{n+1}}\leq 0.002$$

which is true for $n\geq 6$. Thus 6 terms would be necessary to guarantee this accuracy.



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# 13 M13: Ratio and Root Tests

The Worksheet and Homework set M13 should be worked on after studying the material from sections 13.1, 13.2, and 13.3 of the youtube workbook.

## 13.1 M13 Worksheet: Ratio and Root Tests

1. Are the following series convergent or divergent?
   (a) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^4}{4^n}$

   (b) $\sum_{n=1}^{\infty}\frac{(-10)^n}{n!}$

   (c) $\sum_{n=1}^{\infty}\frac{2\cdot 4\cdot 6 \ldots (2n)}{n!}$

   (d) $\sum_{n=1}^{\infty}\left(\frac{3n+2}{1+7n}\right)^n$

   (e) $\sum_{n=1}^{\infty}\frac{4^n n^2}{n!}$

   (f) $\sum_{n=1}^{\infty}\frac{3^n n!}{(n+3)!}$

   (g) $\sum_{n=1}^{\infty}\frac{n!}{e^{n^3}}$

   (h) $\sum_{n=1}^{\infty}\frac{(2n)^n}{n^{2n}}$

   (i) $\sum_{n=1}^{\infty}\frac{n!}{n^{2n}}$

   (j) $\sum_{n=1}^{\infty}(-1)^n\sqrt[n]{2}$

   (k) $\sum_{n=1}^{\infty}\frac{(-2)^{2n}}{n^n}$

   (l) $\sum_{n=1}^{\infty}\frac{1}{(\ln n)^n}$

2. For what positive integers $k$ is the series

$$\sum_{n=1}^{\infty}\frac{(n!)^2}{(kn)!}$$

   convergent?



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## 13.2 M13 Homework set: Ratio and Root Test

Are the following series convergent or divergent?

1. $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$

2. $\sum_{n=1}^{\infty} \frac{(-5)^n}{n!3^n}$

3. $\sum_{n=1}^{\infty} \left(\frac{3n+2}{1+7n}\right)^n$

4. $\sum_{n=1}^{\infty} \frac{3^n n^3 n!}{(2n)!}$

5. $\sum_{n=1}^{\infty} \frac{(3n)^{2n}}{(2n)^{3n}}$

6. $\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^n$

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### 13.3 M13 Homework set: Solutions

NAME:

GRADE: /12

Are the following series convergent or divergent?

1. [2] $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$

   *Solution*. Since
   $$\left|\frac{a_{n+1}}{a_n}\right| = \frac{3^{n+1}}{(n+1)^3} \cdot \frac{n^3}{3^n} = 3\left(\frac{n}{n+1}\right)^3$$
   and
   $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 3 > 1,$$
   we conclude by the Ratio Test that $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$ is divergent.

2. [2] $\sum_{n=1}^{\infty} \frac{(-5)^n}{n!3^n}$

   *Solution*. Since
   $$\left|\frac{a_{n+1}}{a_n}\right| = \frac{5^{n+1}}{(n+1)!3^{n+1}} \cdot \frac{n!3^n}{5^n} = \frac{5}{3} \cdot \frac{1}{n+1},$$
   and
   $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 < 1,$$
   we conclude by the Ratio Test that $\sum_{n=1}^{\infty} \frac{(-5)^n}{n!3^n}$ is absolutely convergent, hence convergent.

3. [2] $\sum_{n=1}^{\infty} \left(\frac{3n+2}{1+7n}\right)^n$

   *Solution*. Since
   $$\sqrt[n]{|a_n|} = \frac{3n+2}{1+7n}$$
   we have
   $$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \frac{3}{7} < 1.$$
   Thus, by the Root Test, $\sum_{n=1}^{\infty} \left(\frac{3n+2}{1+7n}\right)^n$ is convergent.



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4. [2] $\sum_{n=1}^{\infty} \frac{3^n n^3 n!}{(2n)!}$

*Solution*. Since

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \frac{3^{n+1}(n+1)^3(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{3^n n^3 n!} = 3 \cdot \left(\frac{n+1}{n}\right)^3 \cdot \frac{(n+1)!}{n!} \cdot \frac{(2n)!}{(2n+2)!} \\ &= 3 \cdot \left(\frac{n+1}{n}\right)^3 \cdot \frac{n+1}{(2n+1)(2n+2)}, \end{aligned}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 3 \cdot 1 \cdot 0 = 0$$

and, by the Ratio Test, $\sum_{n=1}^{\infty} \frac{3^n n^3 n!}{(2n)!}$ is convergent.

5. [2] $\sum_{n=1}^{\infty} \frac{(3n)^{2n}}{(2n)^{3n}}$

*Solution*. Since

$$\sqrt[n]{|a_n|} = \frac{(3n)^2}{(2n)^3} = \frac{9n^2}{8n^3} = \frac{9}{8n},$$

we have

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = 0$$

so that, by the Root Test, $\sum_{n=1}^{\infty} \frac{(3n)^{2n}}{(2n)^{3n}}$ is convergent.

6. [2] $\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^n$

*Solution*. Since

$$\sqrt[n]{|a_n|} = \frac{\ln n}{n},$$

we have

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} \overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0,$$

so that, by the Root Test, $\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^n$ is convergent.



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# 14 M14: Strategies to test series for convergence

The Worksheet and Homework set M14 should be worked on after studying the material from section 13.4 of the youtube workbook.

## 14.1 M14 Worksheet: review of numerical series

1. Are the following series convergent or divergent. *Fully justify your answer.*

   (a) $\sum_{n=1}^{\infty} \frac{3n^2+n+1}{n^2+2n+1}$

   (b) $\sum_{n=1}^{\infty} \frac{2}{n^2+2n}$

   (c) $\sum_{n=1}^{\infty} \frac{2n^2+5}{n^3+n}$

   (d) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n^2+5}{n^3+n}$

   (e) $\sum_{n=1}^{\infty} 3n^{-2.3}$

   (f) $\sum_{n=1}^{\infty} \frac{n^2}{e^n}$

   (g) $\sum_{n=1}^{\infty} \left(\frac{7n^2}{5+5n^2}\right)^n$

   (h) $\sum_{n=1}^{\infty} \left(\frac{2n+1}{7n+5}\right)^n$

   (i) $\sum_{n=1}^{\infty} \frac{\sqrt{2n^3+n+1}}{5n^3+2n+5}$

   (j) $\sum_{n=1}^{\infty} \frac{2n^2+1}{\sqrt{n^5+2n+1}}$

   (k) $\sum_{n=1}^{\infty} \frac{\sin n}{2n^4+3n}$

   (l) $\sum_{n=1}^{\infty} \frac{5^n n!}{(2n)!}$

   (m) $\sum_{n=1}^{\infty} \left(\sqrt[n]{5}-1\right)^{2n}$

   (n) $\sum_{n=1}^{\infty} \frac{4^n}{(2n)!}$

   (o) $\sum_{n=1}^{\infty} \frac{\left(5n^2\right)^n}{n^{5n}}$

   (p) $\sum_{n=1}^{\infty} (-1)^n 2^{\frac{1}{n}}$



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2. Justify that the series below is convergent. How large does $n$ have to be for the $n^{th}$-partial sum $s_n$ of the series

$$\sum_{n=1}^{\infty} \frac{n}{e^n}$$

to be an approximation of the sum with an error less than $10^{-3}$?

3. Justify that the series below is convergent. How large does $n$ have to be for the $n^{th}$-partial sum $s_n$ of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$$

to be an approximation of the sum with an error less than $10^{-3}$?

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## 14.2 M14 Homework set: Review of numerical series

1. Are the following series convergent or divergent. *Fully justify your answer.*

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+1}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^2+n}$

(d) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}\frac{n-1}{n^2+n}$

(e) $\displaystyle\sum_{n=1}^{\infty} n^{-1.7}$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{e^{n^3}}$

(g) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3n}{1+8n}\right)^n$

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{n^3+2n^2+5}$

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(\frac{n}{2})}{n^2+4n}$

(j) $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{n!}$

(k) $\displaystyle\sum_{n=1}^{\infty} \left(\sqrt[n]{2}-1\right)^n$

(l) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{(2n+1)!}$

(m) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)^n}{n^{2n}}$

(n) $\displaystyle\sum_{n=1}^{\infty} (-1)^n 2^{\frac{1}{n}}$



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2. Justify that the series below is convergent. How large does $n$ have to be for the $n^{th}$-partial sum $s_n$ of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+4}$$

to be an approximation of the sum with 3 exact decimal places?

3. Justify that the series below is convergent. How large does $n$ have to be for the $n^{th}$-partial sum $s_n$ of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+4}$$

to be an approximation of the sum with 3 exact decimal places?

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## 11.3 M14 Homework set: solutions

NAME:

GRADE: /37

1. Are the following series convergent or divergent. *Fully justify your answer.*

(a) [2]

$$\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+1}$$

*Solution.* Since $\lim_{n\to\infty} \frac{n^2-1}{n^2+1} = 1 \neq 0$, $\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+1}$ is divergent by the $n^{th}$ term Test.

(b) [2]

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n}$$

*Solution.* Since

$$0 \leq \frac{1}{n^2+n} \leq \frac{1}{n^2}$$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent *p*-series ($p = 2 > 1$), we conclude by Comparison that $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ is convergent.

(c) [2]

$$\sum_{n=1}^{\infty} \frac{n-1}{n^2+n}$$

*Solution.* Let $b_n := \frac{1}{n}$ and $a_n := \frac{n-1}{n^2+n}$ . Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n-1}{n^2+n} \cdot n = \lim_{n\to\infty} \frac{n^2-n}{n^2+n} = 1 > 0$$

Thus, by Limit Comparison, $\sum_{n=1}^{\infty} \frac{n-1}{n^2+n}$ is divergent, because $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent *p*-series ($p = 1 \leq 1$).



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(d) [2]

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n-1}{n^2+n}$$

*Solution.* (c) shows that this series is not absolutely convergent. However, $\lim_{n\to\infty}\frac{n-1}{n^2+n}=0$ and $\left\{\frac{n-1}{n^2+n}\right\}_{n=1}^{\infty}$ is eventually decreasing. Indeed, if $f(x):=\frac{x-1}{x^2+x}$ then

$$f'(x)=\frac{x^2+x-(2x+1)(x-1)}{(x^2+x)^2}=\frac{-x^2+2x+1}{(x^2+x)^2}$$

is negative outside of the roots of $-x^2+2x+1$, which are, $\frac{-2\pm\sqrt{8}}{-2}=1\pm\sqrt{2}$. In particular, $f'(x)<0$ if $x>1+\sqrt{2}$. By the Alternating Series Test, we conclude that $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n-1}{n^2+n}$ converges.

(e) [2]

$$\sum_{n=1}^{\infty}n^{-1.7}$$

*Solution.*

$$\sum_{n=1}^{\infty}n^{-1.7}=\sum_{n=1}^{\infty}\frac{1}{n^{1.7}}$$

is a *p*-series for $p=1.7>1$, and is therefore convergent.

(f) [2]

$$\sum_{n=1}^{\infty}\frac{n^2}{e^{n^3}}$$

*Solution.* Let $a_n:=\frac{n^2}{e^{n^3}}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right|=\frac{(n+1)^2}{e^{(n+1)^3}}\cdot\frac{e^{n^3}}{n^2}=\left(\frac{n+1}{n}\right)^2\cdot e^{n^3-(n^3+3n^2+3n+1)},$$

so that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2\cdot e^{-(3n^2+3n+1)}=1\cdot 0=0<1.$$

Thus, by the Ratio Test, $\sum_{n=1}^{\infty}\frac{n^2}{e^{n^3}}$ is convergent.



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Alternatively, you could use the Integral Test, by showing that the continuous and positive function

$$f(x) = x^2 e^{-x^3}$$

is also eventually decreasing, and by calculating the integral

$$\int_1^\infty x^2 e^{-x^3}\,dx \overset{u=x^3}{=} \lim_{t\to\infty} \frac{1}{3}\int_1^{t^3} e^{-u}\,du = \lim_{t\to\infty}\left[-\frac{e^{-u}}{3}\right]_1^{t^3} = \frac{1}{3e}.$$

Since this integral is convergent, by the Integral Test, so is $\sum_{n=1}^\infty f(n) = \sum_{n=1}^\infty \frac{n^2}{e^{n^3}}$.

(g) [2]

$$\sum_{n=1}^\infty \left(\frac{3n}{1+8n}\right)^n$$

*Solution.* If $a_n := \left(\frac{3n}{1+8n}\right)^n$ is the general tern, then $\sqrt[n]{|a_n|} = \frac{3n}{1+8n}$ so that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \frac{3}{8} < 1$. By the Root Test, $\sum_{n=1}^\infty \left(\frac{3n}{1+8n}\right)^n$ is convergent.

(h) [2]

$$\sum_{n=1}^\infty \frac{\sqrt{n^2-1}}{n^3+2n^2+5}$$

*Solution.* Let $a_n := \frac{\sqrt{n^2-1}}{n^3+2n^2+5}$ and $b_n := \frac{\sqrt{n^2}}{n^3} = \frac{1}{n^2}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n^2-1}}{n^3+2n^2+5}\cdot n^2 = \lim_{n\to\infty} \frac{n^3\sqrt{1-\frac{1}{n^2}}}{n^3\left(1+\frac{2}{n}+\frac{5}{n^3}\right)} = 1 > 0.$$

By the Limit Comparison Test, we conclude that $\sum_{n=1}^\infty \frac{\sqrt{n^2-1}}{n^3+2n^2+5}$ is convergent because $\sum_{n=1}^\infty \frac{1}{n^2}$ is a convergent *p*-series ($p = 2 > 1$).

(i) [2]

$$\sum_{n=1}^\infty \frac{\cos(\frac{n}{2})}{n^2+4n}$$

*Solution.* Since $|\cos(\frac{n}{2})| \le 1$, we have

$$\left|\frac{\cos(\frac{n}{2})}{n^2+4n}\right| \le \frac{1}{n^2+4n} \le \frac{1}{n^2}$$

and $\sum_{n=1}^\infty \frac{1}{n^2}$ is a convergent *p*-series ($p = 2 > 1$), so that, by Comparison, $\sum_{n=1}^\infty \frac{\cos(\frac{n}{2})}{n^2+4n}$ is (absolutely) convergent.



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(j) [2]

$$\sum_{n=1}^{\infty} \frac{10^n}{n!}$$

*Solution.* Let $a_n := \frac{10^n}{n!}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{10^{n+1}}{(n+1)!} \cdot \frac{n!}{10^n} = \frac{10}{n+1},$$

so that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$. By the Ratio Test, $\sum_{n=1}^{\infty} \frac{10^n}{n!}$ is convergent.

(k) [2]

$$\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)^n$$

*Solution.* Let $a_n := \left(\sqrt[n]{2} - 1\right)^n$. Then $\sqrt[n]{|a_n|} = 2^{\frac{1}{n}} - 1$ and

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = 2^0 - 1 = 0 < 1,$$

so that, by the Root Test, $\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)^n$ is convergent.

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(l) [2]

$$\sum_{n=1}^{\infty} \frac{2^n}{(2n+1)!}$$

*Solution*. Let $a_n = \frac{2^n}{(2n+1)!}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}}{(2n+3)!} \cdot \frac{(2n+1)!}{2^n} = \frac{2}{(2n+2)(2n+3)}$$

and $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$. Thus, by the Ratio Test, $\sum_{n=1}^{\infty} \frac{2^n}{(2n+1)!}$ is convergent.

(m) [2]

$$\sum_{n=1}^{\infty} \frac{(2n)^n}{n^{2n}}$$

*Solution*. Let

$$a_n := \frac{(2n)^n}{n^{2n}} = \left(\frac{2n}{n^2}\right)^n = \left(\frac{2}{n}\right)^n.$$

Then

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{2}{n} = 0 < 1,$$

so that, by the Root Test, $\sum_{n=1}^{\infty} \frac{(2n)^n}{n^{2n}}$ is convergent.

(n) [2]

$$\sum_{n=1}^{\infty} (-1)^n 2^{\frac{1}{n}}$$

*Solution*. Note that

$$\lim_{n\to\infty} \left|(-1)^n 2^{\frac{1}{n}}\right| = \lim_{n\to\infty} 2^{\frac{1}{n}} = 2^0 = 1$$

so that $\left\{(-1)^n 2^{\frac{1}{n}}\right\}_{n=1}^{\infty}$ is divergent. By the *n*<sup></sup>$^{th}$ term Test, $\sum_{n=1}^{\infty} (-1)^n 2^{\frac{1}{n}}$ is divergent.

2. [5: 2 to justify convergence, 3 for remainder] Justify that the series below is convergent. How large does *n* have to be for the *n*$^{th}$-partial sum $s_n$ of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+4}$$

to be an approximation of the sum with 3 exact decimal places?



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*Solution.* To justify convergence, we could note that

$$0 \leq \frac{1}{n^2+4} \leq \frac{1}{n^2}$$

and that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent $p$-series ($p = 2 > 1$), so that, by Comparison, $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ is convergent.

Moreover the error made in the approximation

$$s_n \approx \sum_{n=1}^{\infty} \frac{1}{n^2+4} \tag{14.3.1}$$

satisfies

$$R_n \leq \int_n^{\infty} \frac{dx}{x^2+4} \leq \int_n^{\infty} \frac{dx}{x^2} = \frac{1}{n},$$

so that we need $n = 10000$ to guarantee an error less than $10^{-4}$.

**Alternatively**, we can also use the Integral Test because $f(x) = \frac{1}{x^2+4}$ is a continuous, positive, decreasing function on $[1, \infty)$, with antiderivative $F(x) = \frac{1}{2}\arctan(\frac{x}{2})$. Thus $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ converges, because the integral

$$\int_1^{\infty} \frac{dx}{x^2+4} = \lim_{t\to\infty} \left[\frac{1}{2}\arctan(\frac{x}{2})\right]_1^t = \frac{\pi}{4} - \frac{1}{2}\arctan\frac{1}{2}$$

is convergent. The error made in the approximation

$$s_n \approx \sum_{n=1}^{\infty} \frac{1}{n^2+4} \tag{14.3.2}$$

satisfies

$$R_n \leq \int_n^{\infty} \frac{dx}{x^2+4} = \frac{\pi}{4} - \frac{1}{2}\arctan(\frac{n}{2}).$$

Thus, to have 3 exact decimal places in (14.3.2), it is enough that

$$\begin{aligned}
\frac{\pi}{4} - \frac{1}{2}\arctan(\frac{n}{2}) \leq 10^{-4} \quad &\iff \quad \frac{\pi}{2} - 2\cdot 10^{-4} \leq \arctan(\frac{n}{2}) \\
&\iff \quad \tan\left(\frac{\pi}{2} - 2\cdot 10^{-4}\right) \leq \frac{n}{2} \\
&\iff \quad 2\tan\left(\frac{\pi}{2} - 2\cdot 10^{-4}\right) \approx 9999.9 \leq n
\end{aligned}$$



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3. [4:1 for convergence, 3 for remainder] Justify that the series below is convergent. How large does $n$ have to be for the $n^{th}$-partial sum $s_n$ of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+4}$$

to be an approximation of the sum with 3 exact decimal places?

*Solution*. By the previous question, the series is absolutely convergent, hence convergent. Moreover, $\left\{\frac{1}{n^2+4}\right\}_{n=1}^{\infty}$ is decreasing with limit 0, so that by the Alternating Series Test, the error made in the approximation

$$s_n \approx \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+4}$$

satisfies

$$|R_n| \leq \frac{1}{(n+1)^2+4}.$$

Thus to have 3 exact decimal places, it is enough that

$$\frac{1}{(n+1)^2+4} \leq 10^{-4} \iff 10^4 - 4 \leq (n+1)^2 \iff \sqrt{10^4-4} - 1 \approx 98.9 \leq n,$$

so that $s_{99}$ is an approximation with 3 exact decimal places.

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# Mock Test 4

*You should give yourself two hours to do the following test on your own, then, and only then, move to the solutions to evaluate your work.*

*Show all your work to get credit. No calculator should be needed.*

1. Find the limit of the following sequences. Justify your answers.

   (a) $\left\{\frac{\cos^2 n}{2^n}\right\}_{n=1}^{\infty}$

   (b) $\left\{n\sin\left(\frac{1}{n}\right)\right\}_{n=1}^{\infty}$

2. Justify that the series $\sum_{n=1}^{\infty} 3\frac{2^{n+1}}{5^n}$ is convergent and give the value of its sum.
3. Justify that the series $\sum_{n=1}^{\infty} \frac{4}{n^2+2n}$ is convergent and give the value of its sum.
4. For each of the following series, say if it is convergent. Fully justify your answer.

   (a) $\sum_{n=2}^{\infty} \frac{\sqrt{n^3+2}}{\sqrt{n^4-1}}$

   (b) $\sum_{n=1}^{\infty} \frac{n^3-3}{n^6+1}$

   (c) $\sum_{n=1}^{\infty}(-1)^n \arctan(n)$

   (d) $\sum_{n=3}^{\infty} \frac{(-1)^{n+1}n}{(n+1)(n-2)}$

   (e) $\sum_{n=1}^{\infty} n^4 e^{-n^5}$

   (f) $\sum_{n=1}^{\infty}(\frac{3n}{2n+1})^n$

   (g) $\sum_{n=1}^{\infty} \frac{\cos(3n)}{n^3+2n}$

   (h) $\sum_{n=1}^{\infty} \frac{(2n+1)!}{2^{n+1}}$

   (i) $\sum_{n=1}^{\infty} \frac{5^n}{3^n-n^2}$



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5. Consider the series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt[3]{n^2}}.$$

(a) Is the series absolutely convergent, conditionally convergent or divergent?

(b) If it is convergent, find $n$ such that the $n^{th}$ remainder of this series is less than $10^{-2}$.

6. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}.$$

(a) Is the series absolutely convergent, conditionally convergent or divergent?

(b) If it is convergent, find $n$ such that the $n^{th}$ remainder of this series is less than $10^{-2}$.

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# Mock Test 4 Solutions

1. Find the limit of the following sequences. Justify your answers

(a) [5] $\left\{\frac{\cos^2 n}{2^n}\right\}_{n=1}^{\infty}$

*Solution.* Since $0 \leq \cos^2 n \leq 1$, we have

$$0 \leq \frac{\cos^2 n}{2^n} \leq \frac{1}{2^n}$$

for all *n*, and $\lim_{n\to\infty} \frac{1}{2^n} = 0$. Thus, by the Squeeze Theorem,

$$\lim_{n\to\infty} \frac{\cos^2 n}{2^n} = 0.$$

(b) [5] $\left\{n \sin\left(\frac{1}{n}\right)\right\}_{n=1}^{\infty}$

*Solution.*

$$\begin{aligned}\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) &= \lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)\\ &\underset{t=\frac{1}{x}}{=} \lim_{t\to 0} \frac{\sin t}{t}\\ &\overset{H}{=} \lim_{t\to 0} \frac{\cos t}{1} = 1.\end{aligned}$$

2. [10] Justify that the series $\sum_{n=1}^{\infty} 3\frac{2^{n+1}}{5^n}$ is convergent and give the value of its sum.

*Solution.* The series

$$\sum_{n=1}^{\infty} 3\frac{2^{n+1}}{5^n} = \sum_{n=1}^{\infty} 6\left(\frac{2}{5}\right)^n$$

is a geometric series of common ratio $\frac{2}{5}$ and first term $\frac{12}{5}$. Since $|\frac{2}{5}| < 1$, the series is convergent and

$$\sum_{n=1}^{\infty} 3\frac{2^{n+1}}{5^n} = \frac{\frac{12}{5}}{1-\frac{2}{5}} = \frac{12}{3} = 4.$$



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3. [10] Justify that the series $\sum_{n=1}^{\infty} \frac{4}{n^2+2n}$ is convergent and give the value of its sum.

*Solution*. Since

$$\frac{4}{n^2+2n} = \frac{4}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2} = \frac{2}{n} - \frac{2}{n+2},$$

the partial sum $s_n$ can be written

$$\begin{aligned} s_n : &= \sum_{i=1}^{n} \frac{4}{i^2+2i} = 2\sum_{i=1}^{n}\left(\frac{1}{i} - \frac{1}{i+2}\right) \\ &= 2\left(1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \ldots + \frac{1}{n-1} - \frac{1}{n+1} + \frac{1}{n} - \frac{1}{n+2}\right) \\ &= 2\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) \end{aligned}$$

so that

$$\lim_{n\to\infty} s_n = 2 \cdot \frac{3}{2} = 3 = \sum_{n=1}^{\infty} \frac{4}{n^2+2n}.$$

4. [45] For each of the following series, say if it is convergent. Fully justify your answer.

(a) [5] $\sum_{n=2}^{\infty} \frac{\sqrt{n^3+2}}{\sqrt{n^4-1}}$

*Solution*. Since $\sqrt{n^3+2} \geq \sqrt{n^3}$ and $\frac{1}{\sqrt{n^4-1}} \geq \frac{1}{\sqrt{n^4}}$ for all $n \geq 2$, we conclude that

$$\frac{\sqrt{n^3+2}}{\sqrt{n^4-1}} \geq \sqrt{\frac{n^3}{n^4}} = \frac{1}{n^{\frac{1}{2}}}$$

for all $n \geq 2$. Moreover, $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ is a divergent *p*-series ($p = \frac{1}{2} \leq 1$). By Comparison, $\sum_{n=2}^{\infty} \frac{\sqrt{n^3+2}}{\sqrt{n^4-1}}$ is also divergent.

(b) [5] $\sum_{n=1}^{\infty} \frac{n^3-3}{n^6+1}$

*Solution*. Since $n^3 - 3 \leq n^3$ and $\frac{1}{n^6+1} \leq \frac{1}{n^6}$ for all $n \geq 1$, we conclude that

$$\frac{n^3-3}{n^6+1} \leq \frac{n^3}{n^6} = \frac{1}{n^3}$$

for all $n \geq 1$. Moreover, $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent *p*-series ($p = 3 > 1$). By Comparison, $\sum_{n=1}^{\infty} \frac{n^3-3}{n^6+1}$ is also convergent.



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(c) [5] $\sum_{n=1}^{\infty}(-1)^n \arctan(n)$

*Solution*. Since

$$\lim_{n\to\infty} \arctan(n) = \lim_{x\to\infty} \arctan x = \frac{\pi}{2},$$

we conclude that $\lim_{n\to\infty}(-1)^n \arctan(n) \neq 0$, so that, by the $n^{th}$ term Test, the series $\sum_{n=1}^{\infty}(-1)^n \arctan(n)$ is divergent.

(d) [5] $\sum_{n=3}^{\infty} \frac{(-1)^{n+1}n}{(n+1)(n-2)}$

*Solution*. This is an alternating series and

$$b_n = \left|\frac{(-1)^{n+1}n}{(n+1)(n-2)}\right| = \frac{n}{(n+1)(n-2)}$$

satisfies $\lim_{n\to\infty} b_n = 0$ and $\{b_n\}_{n=3}^{\infty}$ is decreasing for

$$\left(\frac{x}{(x+1)(x-2)}\right)' = \frac{(x+1)(x-2) - x(x-2+x+1)}{(x+1)^2(x-2)^2} = \frac{-2-x^2}{(x+1)^2(x-2)^2} < 0.$$

Thus, $\sum_{n=3}^{\infty} \frac{(-1)^{n+1}n}{(n+1)(n-2)}$ is convergent by the Alternating Series Test.

(e) [5] $\sum_{n=1}^{\infty} n^4 e^{-n^5}$

*Solution*. Let $a_n := n^4 e^{-n^5}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^4}{e^{(n+1)^5}} \cdot \frac{e^{n^5}}{n^4} = \left(\frac{n+1}{n}\right)^4 \cdot e^{n^5-(n+1)^5},$$

so that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$$

for

$$\lim_{n\to\infty} e^{n^5-(n+1)^5} = \lim_{n\to\infty} e^{-\left(5n^4+10n^3+10n^2+5n+1\right)} = \lim_{x\to\infty} e^{-x} = 0.$$

By the Ratio Test, $\sum_{n=1}^{\infty} n^4 e^{-n^5}$ is convergent.



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***Remark:*** We could alternatively have used the Integral Test, noting that $f(x) = x^4 e^{-x^5}$ is positive continuous and eventually decreasing, and that $\int_1^\infty f(x)\,dx$ is convergent, all of which would need explicit justifications.

(f) [5]$\sum_{n=1}^\infty (\frac{3n}{2n+1})^n$

*Solution*. Let $a_n := \left(\frac{3n}{2n+1}\right)^n$. Then $\sqrt[n]{|a_n|} = \frac{3n}{2n+1}$ so that

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \frac{3}{2} > 1.$$

By the Root Test, the series $\sum_{n=1}^\infty (\frac{3n}{2n+1})^n$ is divergent.

(g) [5]$\sum_{n=1}^\infty \frac{\cos(3n)}{n^3+2n}$

*Solution*. Since

$$0 \le \left|\frac{\cos(3n)}{n^3+2n}\right| \le \frac{1}{n^3+2n} \le \frac{1}{n^3}$$

and $\sum_{n=1}^\infty \frac{1}{n^3}$ is a convergent *p*-series ($p = 3 > 1$), we conclude by the Comparison Test that $\sum_{n=1}^\infty \frac{\cos(3n)}{n^3+2n}$ is absolutely convergent, hence convergent.

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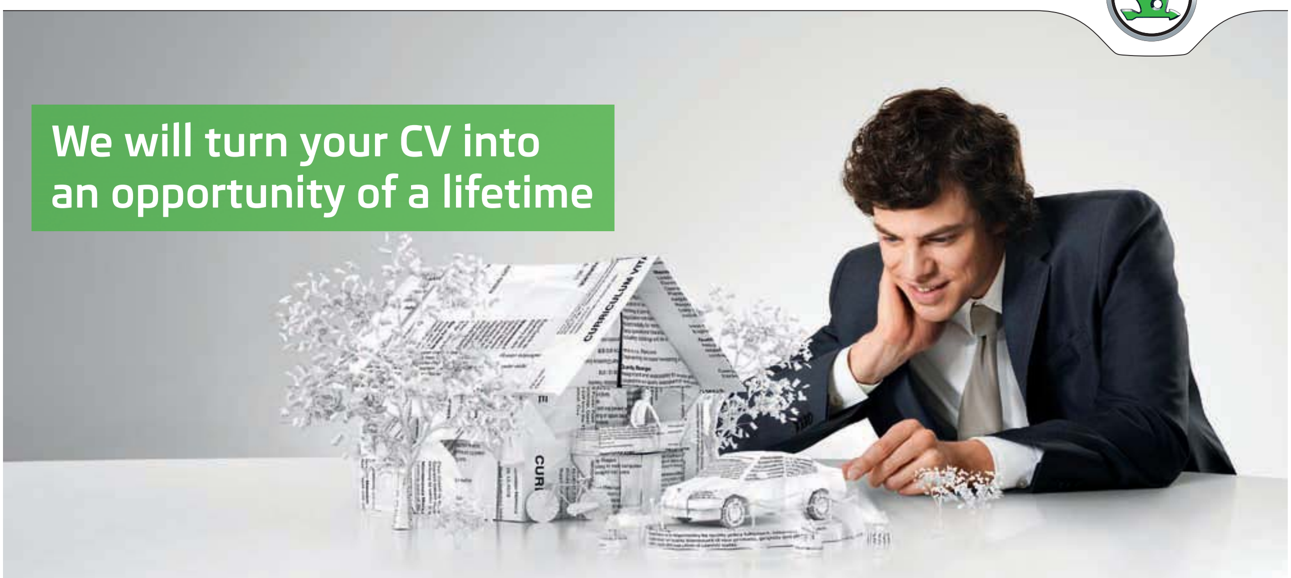


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(h) [5]$\sum_{n=1}^{\infty} \frac{(2n+1)!}{2^{n+1}}$

*Solution.* Let $a_n := \frac{(2n+1)!}{2^{n+1}}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(2n+3)!}{2^{n+2}} \cdot \frac{2^{n+1}}{(2n+1)!} = \frac{(2n+3)(2n+2)}{2}$$

so that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$. By the Root Test, $\sum_{n=1}^{\infty} \frac{(2n+1)!}{2^{n+1}}$

(i) [5]$\sum_{n=1}^{\infty} \frac{5^n}{3^n - n^2}$

*Solution.* Since

$$\frac{5^n}{3^n - n^2} \geq \left(\frac{5}{3}\right)^n$$

for all *n*, and $\lim \left(\frac{5}{3}\right)^n = \infty$, we conclude that $\lim \frac{5^n}{3^n - n^2} \neq 0$. By the *n*^th^ term Test, $\sum_{n=1}^{\infty} \frac{5^n}{3^n - n^2}$ is divergent.

5. Consider the series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt[3]{n^2}}.$$

(a) [10] Is the series absolutely convergent, conditionally convergent or divergent?

*Solution.* The series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt[3]{n^2}}$ is not absolutely convergent for

$$\left|\frac{(-1)^n}{\sqrt[3]{n^2}}\right| = \frac{1}{n^{\frac{2}{3}}}$$

and $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{3}}}$ is a divergent *p*-series ($p = \frac{2}{3} \leq 1$). On the other hand, $\left\{\frac{1}{n^{\frac{2}{3}}}\right\}_{n=1}^{\infty}$ is a decreasing sequence with limit 0, and $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt[3]{n^2}}$ is alternating. Thus, $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt[3]{n^2}}$ is convergent by the Alternating Series Test. Thus $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt[3]{n^2}}$ is conditionally convergent.



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(b) [5] If it is convergent, find $n$ such that the $n^{th}$ remainder of this series is less than $10^{-2}$.

*Solution*. Since the series is convergent by the Alternating Series Test, the $n^{th}$ remainder $R_n$ satisfies

$$|R_n| \leq \frac{1}{(n+1)^{\frac{2}{3}}}.$$

Thus $|R_n| \leq 10^{-2}$ whenever

$$100 \leq (n+1)^{\frac{2}{3}} \iff (100)^{\frac{3}{2}} \leq n+1 \iff 999 \leq n.$$

6. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}.$$

(a) [5] Is the series absolutely convergent, conditionally convergent or divergent?

*Solution*.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$$

is a convergent *p*-series ($p = \frac{3}{2} > 1$) and is thus absolutely convergent.

(b) [5] If it is convergent, find $n$ such that the $n^{th}$ remainder of this series is less than $10^{-2}$.

*Solution*. As a *p*-series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$ is convergent by the Integral Test, so that the $n^{th}$ remainder $R_n$ satisfies

$$R_n \leq \int_n^{\infty} \frac{dx}{x^{\frac{3}{2}}} = \lim_{t\to\infty} \left[\frac{-2}{\sqrt{x}}\right]_n^t = \frac{2}{\sqrt{n}}.$$

Thus $R_n \leq 10^{-2}$ whenever

$$100 \leq \frac{\sqrt{n}}{2} \iff 40000 \leq n.$$



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# 15 M15: Power Series and Taylor Series

The Worksheet and Homework set M15A should be worked on after studying the material from sections 14.1 and 14.2 of the youtube workbook.

## 15.1 M15A Worksheet: Intervals of convergence

Find the interval of convergence for the following power series:

1. $\sum_{n=0}^{\infty} \frac{2^{n+1}}{3^n}(x-1)^n$
2. $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}(x+2)^n$
3. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{3^n n}$
4. $\sum_{n=1}^{\infty} \frac{(x-5)^n}{(3n)^n}$
5. $\sum_{n=0}^{\infty} \frac{n!}{10^n}(x+1)^n$
6. $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$
7. $\sum_{n=0}^{\infty}(-1)^{n+1} \frac{x^{2n}}{3^n}$
8. $\sum_{n=0}^{\infty} \left(\frac{n+1}{4n+2}\right)^n x^{2n}$
9. $\sum_{n=1}^{\infty} \frac{2^n}{n}(x+1)^n$
10. $\sum_{n=1}^{\infty}(-1)^n \frac{4^n}{n}(x-3)^n$



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## 15.2 M15A Homework set: Intervals of convergence

Find the interval of convergence for the following power series:

1. $\sum_{n=0}^{\infty} \frac{2^n}{5^{n+1}}(x+3)^n$

2. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2 x^n}{5^n \sqrt{n^5}}$

3. $\sum_{n=2}^{\infty} \frac{(x-2)^n}{(\sqrt{\ln n})^n}$

4. $\sum_{n=0}^{\infty} \left(\frac{n+3}{8n+7}\right)^n x^{3n}$

5. $\sum_{n=1}^{\infty} \frac{n!}{2^n}(x+1)^n$




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## 15.3 M15A Homework set: Solutions

NAME:

GRADE: /21

Find the interval of convergence for the following power series:

1. [3] $\sum_{n=0}^{\infty} \frac{2^n}{5^{n+1}}(x+3)^n$

   *Solution*. This is a geometric series of common ratio

   $$\frac{2(x+3)}{5}$$

   for

   $$\sum_{n=0}^{\infty} \frac{2^n}{5^{n+1}}(x+3)^n = \sum_{n=0}^{\infty} \frac{1}{5} \cdot \left(\frac{2(x+3)}{5}\right)^n,$$

   so that it is convergent exactly when

   $$\frac{2}{5}|x+3| < 1 \iff |x+3| < \frac{5}{2} \iff -\frac{11}{2} < x < -\frac{1}{2},$$

   and the interval of convergence is

   $$I = \left(-\frac{11}{2}, -\frac{1}{2}\right).$$

   *Remark:* Under this condition on *x*,

   $$\sum_{n=0}^{\infty} \frac{2^n}{5^{n+1}}(x+3)^n = \frac{\frac{1}{5}}{1-\frac{2(x+3)}{5}} = -\frac{1}{2x+1}.$$

2. [5: 2 for interval, 1 for testing each endpoint, 1 for interval] $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n^2x^n}{5^n\sqrt{n^5}}$

   *Solution*. Let

   $$a_n := \frac{(-1)^{n+1}n^2x^n}{5^n\sqrt{n^5}} = \frac{(-1)^{n+1}x^n}{5^n n^{\frac{1}{2}}}.$$

   Then

   $$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{n+1}}{5^{n+1}\cdot(n+1)^{\frac{1}{2}}} \cdot \frac{5^n \cdot n^{\frac{1}{2}}}{|x|^n} = \frac{1}{5} \cdot \left(\frac{n}{n+1}\right)^{\frac{1}{2}} \cdot |x|,$$



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so that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|}{5}$ is less than 1 if $|x| < 5$. The interval of convergence is centered at 0 of radius 5. Thus, the end points to be tested are $\pm 5$.

At $x = 5$,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2 x^n}{5^n \sqrt{n^5}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{5^n n^{\frac{1}{2}}} \cdot 5^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{\frac{1}{2}}},$$

which is convergent by the Alternating Series Test, for $\left\{\frac{1}{n^{\frac{1}{2}}}\right\}_{n=1}^{\infty}$ is a decreasing sequence with limit 0.

At $x = -5$,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2 x^n}{5^n \sqrt{n^5}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{5^n n^{\frac{1}{2}}} \cdot (-5)^n = \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n^{\frac{1}{2}}} = \sum_{n=1}^{\infty} \frac{-1}{n^{\frac{1}{2}}},$$

which is a divergent *p*-series ($p = \frac{1}{2} \leq 1$). Thus, the interval of convergence is

$$I = (-5, 5].$$

3. [3] $\sum_{n=2}^{\infty} \frac{(x-2)^n}{\left(\sqrt{\ln n}\right)^n}$

*Solution.* Let

$$a_n := \frac{(x-2)^n}{\left(\sqrt{\ln n}\right)^n} = \left(\frac{x-2}{\sqrt{\ln n}}\right)^n.$$

Then

$$\sqrt[n]{|a_n|} = \frac{|x-2|}{\sqrt{\ln n}}$$

and $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 0$ for all *x*. Thus, by the Root Test, the interval of convergence is

$$I = (-\infty, \infty).$$

4. [7: 2 for interval, 2 for testing each endpoint, 1 for interval] $\sum_{n=0}^{\infty} \left(\frac{n+3}{8n+7}\right)^n x^{3n}$

*Solution.* Let $a_n := \left(\frac{n+3}{8n+7}\right)^n x^{3n}$. Then

$$\sqrt[n]{|a_n|} = \frac{n+3}{8n+7}|x|^3$$



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so that $\lim_{n\to\infty}\sqrt[n]{|a_n|}=\frac{|x|^3}{8}$ is less than 1 whenever $|x|^3<8$, that is, whenever $|x|<2$. Thus, the interval of convergence is centered at 0 and of radius 2.

At $x=2$,

$$\sum_{n=0}^{\infty}\left(\frac{n+3}{8n+7}\right)^n 2^{3n}=\sum_{n=0}^{\infty}\left(\frac{8(n+3)}{8n+7}\right)^n$$

is divergent by the *n*^th^ term Test, for $\left(\frac{8n+24}{8n+7}\right)^n=e^{n\ln\left(\frac{8n+24}{8n+7}\right)}$ and

$$\begin{aligned}
\lim_{n\to\infty} n\ln\left(\frac{8n+24}{8n+7}\right) &= \lim_{x\to\infty}\frac{\ln(8x+24)-\ln(8x+7)}{\frac{1}{x}}\\
&\overset{H}{=} \lim_{x\to\infty}\frac{\frac{8}{8x+24}-\frac{8}{8x+7}}{-\frac{1}{x^2}}\\
&= \lim_{x\to\infty}-x^2\cdot\frac{64x+56-64x-192}{(8x+24)(8x+7)}\\
&= \lim_{x\to\infty}\frac{136x^2}{64x^2+248x+168}=\frac{136}{64},
\end{aligned}$$

so that $\lim_{n\to\infty}\left(\frac{8n+24}{8n+7}\right)^n=e^{\frac{136}{64}}\neq 0$.

Similarly, at $x=-2$,

$$\sum_{n=0}^{\infty}\left(\frac{n+3}{8n+7}\right)^n(-2)^{3n}=\sum_{n=0}^{\infty}(-1)^n\left(\frac{8(n+3)}{8n+7}\right)^n$$

by the *n*^th^ term Test, because we have seen that $\lim_{n\to\infty}\left(\frac{8n+24}{8n+7}\right)^n\neq 0$ and thus $\lim_{n\to\infty}(-1)^n\left(\frac{8n+24}{8n+7}\right)^n\neq 0$. Thus the interval of convergence is

$$I=(-2,2).$$

5. [3] $\sum_{n=1}^{\infty}\frac{n!}{2^n}(x+1)^n$

*Solution.* Let $a_n:=\frac{n!}{2^n}(x+1)^n$. Then

$$\left|\frac{a_{n+1}}{a_n}\right|=\frac{(n+1)!}{2^{n+1}}|x+1|^{n+1}\cdot\frac{2^n}{n!|x+1|^n}=\frac{n+1}{2}\cdot|x+1|$$

so that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\infty$ for all $x\neq-1$ and $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=0$ if $x=-1$. Thus the interval of convergence is

$$I=\{-1\}.$$



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The Worksheet and Homework set M15B should be worked on after studying the material from sections 14.3, 14.4, 14.5, and 14.6 of the youtube workbook.

## 15.4 M15B Worksheet: power series representation

1. Find power series representations, and specify the interval of validity, for the following functions:

   (a) $f(x) = \frac{2x}{3+x}$

   (b) $f(x) = \frac{1}{x^2+x-6}$

   (c) $f(x) = \ln(1+x)$

   (d) $f(x) = \arctan x$

   (e) $f(x) = \frac{x}{(x+2)^2}$

2. Estimate

$$\int_0^1 \frac{dx}{16+x^4}$$

   with an error at most $10^{-5}$.

3. Find

   (a) a power series decomposition for

$$f(x) = \ln(4-x)$$

   (b) Deduce the exact value of $\sum_{n=1}^{\infty} \frac{3^{n+1}}{(n+1)4^{n+1}}$



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## 15.5 M15B Homework set: power series representation

1. Find power series representations, and specify the interval of validity, for the following functions:

   (a) $f(x) = \frac{x+1}{4+2x}$

   (b) $f(x) = \frac{x+2}{x^2+4x+17}$

   (c) $f(x) = \arctan(x^2)$

2. Estimate

$$\int_0^{\frac{1}{2}} \arctan(x^2)\,dx$$

   with an error at most $10^{-5}$.

3. Find

   (a) a power series decomposition for

$$f(x) = \ln(5+x)$$

   (b) Deduce the exact value of $\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)5^n}$

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## 15.6 M15B Homework set: Solutions

NAME:

GRADE: /22

1. Find power series representations, and specify the interval of validity, for the following functions:

(a) [3] $f(x) = \frac{x+1}{4+2x}$

*Solution*. Note that

$$\begin{aligned}
f(x) &= \frac{x+1}{4+2x} = \frac{1}{2} - \frac{1}{2x+4} \text{ by long division} \\
&= \frac{1}{2} - \frac{1}{4} \cdot \frac{1}{1+\frac{x}{2}} = \frac{1}{2} - \frac{1}{4} \cdot \frac{1}{1-(-\frac{x}{2})} \\
&= \frac{1}{2} - \frac{1}{4} \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n \text{ for } |\tfrac{x}{2}| < 1 \\
&= \frac{1}{2} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^{n+2}} x^n \text{ for } |x| < 2.
\end{aligned}$$

(b) [3] $f(x) = \frac{x+2}{x^2+4x+17}$

*Solution*. Since

$$f(x) = \frac{x+2}{x^2+4x+17} = \frac{x+2}{(x+2)^2+1} = (x+2) \cdot \frac{1}{1 - \left(-(x+2)^2\right)},$$

we conclude that for $|(x+2)^2| < 1$, that is, for $|x+2| < 1$, we have

$$\begin{aligned}
f(x) &= (x+2) \cdot \sum_{n=0}^{\infty} \left(-(x+2)^2\right)^n \\
&= \sum_{n=0}^{\infty} (-1)^n (x+2)^{2n+1},
\end{aligned}$$

for $|x+2| < 1$, that is, on the interval $(-3, -1)$.

(c) [4:2 for rep. of derivative, 1 for integral, 1 for constant] $f(x) = \arctan(x^2)$

*Solution*. Since

$$f'(x) = \frac{2x}{1+(x^2)^2} = \frac{2x}{1-(-x^4)},$$



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we conclude that for $|-x^4| < 1$, that is, for $|x| < 1$, we have

$$f'(x) = 2x \cdot \sum_{n=0}^{\infty} \left(-x^4\right)^n = 2\sum_{n=0}^{\infty}(-1)^n x^{4n+1}.$$

By term-by-term integration, a power series representation of $f$ on $(-1,1)$ is of the form

$$f(x) = \int f(x)\,dx = C + 2\sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+2}}{4n+2}. \tag{15.6.1}$$

Moreover, $f(0) = \arctan 0 = 0$, and, from (15.6.1), $f(0) = C$, so that $C = 0$. Thus

$$\arctan(x^2) = 2\sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+2}}{4n+2} \text{ for } -1 < x < 1. \tag{15.6.2}$$

2. [5: 2 for rep. of antiderivative, 1 for numerical series rep. of definite integral, 1 for remainder estimate, 1 for partial sum] Estimate

$$\int_0^{\frac{1}{2}} \arctan(x^2)\,dx$$

with an error at most $10^{-5}$.

*Solution*. Using (15.6.2), we have

$$\int \arctan(x^2)\,dx = 2\sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+3}}{(4n+2)(4n+3)} + C \text{ on } (-1,1).$$

In particular

$$\int_0^{\frac{1}{2}} \arctan(x^2)\,dx = \left[2\sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+3}}{(4n+2)(4n+3)}\right]_0^{\frac{1}{2}} = 2\sum_{n=0}^{\infty}(-1)^n \frac{1}{2^{4n+3}(4n+2)(4n+3)}.$$

The series on the right-hand side is converging by the Alternating Series Test, so that,

$$|R_n| \le \frac{1}{2^{4n+7}(4n+6)(4n+7)}.$$

Thus, for $n=1$, $|R_n| \le \frac{1}{2^{11}\times 10 \times 11} \approx 4.4 \cdot 10^{-6}$ and

$$s_1 = 2\left(\frac{1}{2^3 \times 2 \times 3} - \frac{1}{2^7 \times 6 \times 7}\right) = \frac{37}{896}$$

is an estimate of $\int_0^{\frac{1}{2}} \arctan(x^2)\,dx$ with an error less than $10^{-5}$.



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3. Find

(a) [4: 2 for rep. of derivative, 1 for integral, 1 for constant] a power series decomposition for

$$f(x) = \ln(5+x)$$

*Solution.* Since

$$\begin{aligned} f'(x) &= \frac{1}{5+x} = \frac{1}{5} \cdot \frac{1}{1-(-\frac{x}{5})} \\ &= \frac{1}{5} \sum_{n=0}^{\infty} \left(-\frac{x}{5}\right)^n \text{ for } |\tfrac{x}{5}| < 1 \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{5^{n+1}} x^n \text{ for } |x| < 5, \end{aligned}$$

we have by term-by-term integration that

$$f(x) = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{5^{n+1}(n+1)} x^{n+1} \text{ on } (-5,5). \tag{15.6.3}$$


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Moreover, $f(0) = \ln 5$ and from (15.6.3), $f(0) = C$. Thus $C = \ln 5$ and

$$f(x) = \ln 5 + \sum_{n=0}^{\infty} \frac{(-1)^n}{5^{n+1}(n+1)} x^{n+1} \text{ on } (-5, 5).$$

(b) [3] Deduce the exact value of $\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)5^n}$

*Solution*. According to the previous question ([2]),

$$f(1) = \ln 5 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)5^{n+1}} = \ln 5 + \frac{1}{5} + \frac{1}{5} \cdot \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)5^n}$$

because $1 \in (-5, 5)$. But $f(1) = \ln(5+1) = \ln 6$. Thus

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)5^n} = 5 \left( \ln 6 - \ln 5 - \frac{1}{5} \right).$$



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The Worksheet and Homework set M15C should be worked on after studying the material from sections 14.7, 14.8, 14.9, and 14.10 of the youtube workbook.

## 15.7 M15C Worksheet: Taylor series

1. Find the Taylor series of $f(x) = \sin x$ at $\frac{\pi}{2}$
2. Show that the series obtained above represents sin *x* for all *x*.
3. Find the Taylor series of $f(x) = \ln x$ at 2
4. Find the Taylor series of $f(x) = \frac{1}{\sqrt{x}}$ at 9
5. Find the MacLaurin series of $f(x) = \cos(x^2)$
6. Find the MacLaurin series of $f(x) = (1+x)^{-3}$
7. Find the MacLaurin series of $f(x) = xe^{-2x}$
8. Find the MacLaurin series of $f(x) = \sin(x^3)$
9. Find the MacLaurin series of $f(x) = \sin^2 x$
10. Find the MacLaurin series of $f(x) = x\cos(3x)$
11. If $f(x) = \sum_{n=0}^{\infty} 2^n x^n$ find $f^{(21)}(0)$.
12. Estimate the error when $\sqrt{e} = e^{\frac{1}{2}}$ is approximated by the first four terms of the MacLaurin series of $e^x$.



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## 15.8 M15C Homework set: Taylor Series

1. Find the Taylor series of $f(x) = \cos x$ at $\frac{\pi}{2}$
2. Show that the series obtained above represents $\cos x$ for all $x$.
3. Find the Taylor series of $f(x) = \frac{1}{x^2}$ at 1
4. Find the MacLaurin series of $f(x) = x^2 \cos(x^3)$
5. Find the MacLaurin series of $f(x) = x \arctan(x^2)$
6. If $f(x) = \sum_{n=0}^{\infty} \frac{n}{2^n}(x-2)^n$ find $f^{(37)}(2)$.

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## 15.9 M15C Homework set: Solutions

NAME:

GRADE: /19

1. [4] Find the Taylor series of $f(x) = \cos x$ at $\frac{\pi}{2}$

   *Solution*. Since

$$\begin{array}{rcl} f^{(0)}(x) & = & \cos x \\ f^{(1)}(x) & = & -\sin x \\ f^{(2)}(x) & = & -\cos x \\ f^{(3)}(x) & = & \sin x \\ f^{(4)}(x) & = & \cos x = f^{(0)}(x), \end{array}$$

   the successive derivatives then repeat this pattern, in which derivatives of even order are of the form $\pm\cos x$ and derivatives of odd order are of the form $\pm\sin x$. More explicitly,

$$f^{(2n)}(x) = (-1)^n \cos x \text{ and } f^{(2n+1)}(x) = (-1)^{n+1} \sin x.$$

   Thus,

$$f^{(2n)}\left(\frac{\pi}{2}\right) = 0 \text{ and } f^{(2n+1)}\left(\frac{\pi}{2}\right) = (-1)^{n+1}$$

   and the Taylor series is therefore

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} \left(x - \frac{\pi}{2}\right)^{2n+1}.$$

2. [3] Show that the series obtained above represents $\cos x$ for all *x*.

   *Solution*. Let $R_n = \cos x - \sum_{i=0}^{n} \frac{(-1)^{i+1}}{(2i+1)!} \left(x - \frac{\pi}{2}\right)^{2i+1}$. The successive derivatives of *f* are of the form $\pm\cos x$ or $\pm\sin x$ . At any rate,

$$|f^{(n+1)}(x)| \leq 1 \text{ for all } x,$$

   so that, in view of Taylor's Inequality,

$$|R_n(x)| \leq \frac{|x - \frac{\pi}{2}|^{n+1}}{(n+1)!}.$$



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Since $\lim_{n\to\infty} \frac{|x-\frac{\pi}{2}|^{n+1}}{(n+1)!} = 0$ for all $x$, we conclude that $\lim_{n\to\infty} |R_n(x)| = 0$ for all $x$, that is

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} \left(x - \frac{\pi}{2}\right)^{2n+1} \text{ for all } x.$$

3. [4] Find the Taylor series of $f(x) = \frac{1}{x^2}$ at 1

*Solution.* Note that

$$\begin{aligned} f^{(0)}(x) &= x^{-2} \\ f^{(1)}(x) &= -2x^{-3} \\ f^{(2)}(x) &= (-2)(-3)x^{-4} \\ f^{(3)}(x) &= (-2)(-3)(-4)x^{-5} \\ &\vdots \\ f^{(n)}(x) &= (-1)^n (n+1)! x^{-(n+2)}, \end{aligned}$$

so that

$$\frac{f^{(n)}(1)}{n!} = \frac{(-1)^n (n+1)!}{n!} = (-1)^n (n+1)$$

and the Taylor series of $f$ at 1 is

$$\sum_{n=0}^{\infty} (-1)^n (n+1)(x-1)^n.$$

4. [3] Find the MacLaurin series of $f(x) = x^2 \cos(x^3)$

*Solution.* Since

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \text{ for all } x,$$

we have

$$\begin{aligned} x^2 \cos\left(x^3\right) &= x^2 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(x^3\right)^{2n} \\ &= x^2 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{6n} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{6n+2}. \end{aligned}$$



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5. [3] Find the MacLaurin series of $f(x) = x\arctan(x^2)$

*Solution*. If $x \in [-1,1]$ then $x^2 \in [-1,1]$. For such an $x$, in view of

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \text{ for } -1 \leq x \leq 1,$$

we conclude that

$$\begin{aligned} x\arctan\left(x^2\right) &= x \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(x^2\right)^{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{4n+3} \text{ for } -1 \leq x \leq 1. \end{aligned}$$

6. [2] If $f(x) = \sum_{n=0}^{\infty} \frac{n}{2^n}(x-2)^n$ find $f^{(37)}(2)$.

*Solution*. Since the coefficient of $(x-2)^n$ in the power series representation centered at 2 of $f$ is necessarily $\frac{f^{(n)}(2)}{n!}$, we have in particular

$$\frac{f^{(37)}(2)}{37!} = \frac{37}{2^{37}} \Longrightarrow f^{(37)}(2) = \frac{37! \times 37}{2^{37}}.$$

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# 16 M16: Applications of power series

The Worksheet and Homework set M16 should be worked on after studying the material from sections 15.1, 15.2, 15.3, 15.4, and 15.5 of the youtube workbook.

## 16.1 M16 Worksheet: applications of power series

1. Find the exact values of the following numerical series:

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{2^{4n+2}(2n)!}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n3^n}$

   (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^n}{n!2^n}$

   (d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n-1} \pi^{2n+1}}{2^{4n+2}(2n+1)}$

2. Evaluate the following limits:

   (a) $\displaystyle\lim_{x\to 0} \frac{\arctan x - x}{\sin x - x}$

   (b) $\displaystyle\lim_{x\to 0} \frac{\cos x - 1}{e^x - 1 - x}$

   (c) $\displaystyle\lim_{x\to 0} \frac{e^x \cos x - 1 - x}{\sin x - x}$

3. Estimate the following integrals with 2 exact decimal places:

   (a) $\displaystyle\int_0^1 \cos(x^2)\,dx$

   (b) $\displaystyle\int_0^{\frac{1}{2}} \arctan(x^3)\,dx$

4. Find a numerical series representation for

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin x}{x}\,dx$$



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5. Find

(a) a power series representation for

$$f(x) = \left(4 + x^2\right)^{-\frac{3}{2}}$$

and write out the first 4 terms.

(b) a power series representation for

$$\int \frac{dx}{\sqrt{\left(4 + x^2\right)^3}}$$

and write out the first 4 terms.

(c) a numerical series representation of

$$\int_0^1 \frac{dx}{\sqrt{\left(4 + x^2\right)^3}}$$

(d) how many terms of this numerical series need to be added in order to get an estimate of $\int_0^1 \frac{dx}{\sqrt{(4+x^2)^3}}$ with an error less than $10^{-3}$.



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## 16.2 M16 Homework set: Applications of power Series

1. Find the exact values of the following numerical series:

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^{n+1}}{n!3^n}$

   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{6^{2n+1}(2n+1)!}$

2. Evaluate the following limits:

   (a) $\displaystyle\lim_{x\to 0} \frac{\arctan x - x}{\cos x - 1}$

   (b) $\displaystyle\lim_{x\to 0} \frac{\cos x - 1}{e^x - 1 - x}$

   (c) $\displaystyle\lim_{x\to 0} \frac{e^x \sin x - x - x^2}{\cos x - 1 - \frac{x^2}{2}}$

3. Estimate the following integral

$$\int_0^1 e^{-x^3}\, dx$$

   with 3 exact decimal places.

4. Use power series to find an estimate of

$$\int_0^1 \frac{dx}{\sqrt{1+x^4}}$$

   with an error of at most $10^{-1}$.



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## 16.3 M16 Homework set: Solutions

NAME:

GRADE: /24

1. [4] Find the exact values of the following numerical series:

(a) [2]

$$\sum_{n=0}^{\infty} \frac{2^{n+1}}{n!3^n} = 2\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{2}{3}\right)^n = 2e^{\frac{2}{3}}.$$

(b) [2]

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{6^{2n+1}(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \left(\frac{\pi}{6}\right)^{2n+1} = \sin\frac{\pi}{6} = \frac{1}{2}.$$

2. [8] Evaluate the following limits:

(a) [2]

$$\lim_{x\to 0} \frac{\arctan x - x}{\cos x - 1} = \lim_{x\to 0} \frac{\left(-\frac{x^3}{3} + \frac{x^5}{5} - \dots\right)}{\left(-\frac{x^2}{2} + \frac{x^4}{4!} - \dots\right)} = \lim_{x\to 0} x \cdot \frac{\left(-\frac{1}{3} + \frac{x^2}{5} - \dots\right)}{\left(-\frac{1}{2} + \frac{x^2}{4!} - \dots\right)} = 0 \times \frac{2}{3} = 0.$$

(b) [2]

$$\lim_{x\to 0} \frac{\cos x - 1}{e^x - 1 - x} = \lim_{x\to 0} \frac{\left(-\frac{x^2}{2} + \frac{x^4}{4!} - \dots\right)}{\left(\frac{x^2}{2} + \frac{x^3}{3!} + \dots\right)} = \frac{-\frac{1}{2}}{\frac{1}{2}} = -1.$$

(c) [4:2 for $e^x \sin x$, 2 for rest]

$$\lim_{x\to 0} \frac{e^x \sin x - x - x^2}{\cos x - 1 - \frac{x^2}{2}}$$

*Solution*. Note that

$$\begin{aligned} e^x \sin x &= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!}\dots\right)\cdot\left(x - \frac{x^3}{6} + \frac{x^5}{5!} - \dots\right) \\ &= x + x^2 + x^3\left(-\frac{1}{6} + \frac{1}{2}\right) + x^4\left(\frac{1}{3!} - \frac{1}{3!}\right) + x^5\left(\frac{1}{4!} - \frac{1}{12} + \frac{1}{5!}\right) + \dots \\ &= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \dots \end{aligned}$$

so that

$$\lim_{x\to 0} \frac{e^x \sin x - x - x^2}{\cos x - 1 - \frac{x^2}{2}} = \lim_{x\to 0} \frac{\frac{x^3}{3} - \frac{x^5}{30} + \dots}{\frac{x^4}{4!} - \frac{x^6}{6!} + \dots} = \lim_{x\to 0} \frac{1}{x} \cdot \frac{\frac{1}{3} - \frac{x^2}{30} + \dots}{\frac{1}{4!} - \frac{x^2}{6!} + \dots} = \infty.$$



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3. [6: 2 for series $\int e^{-x^3}\,dx$ , 1 for numerical series, 1 for upper bound of $|R_n|$ , 1 for finding $n$, 1 for $s_5$] Estimate the following integral

$$\int_0^1 e^{-x^3}\,dx$$

with 3 exact decimal places.

*Solution.* Using the power series representation of $e^x$ and term-by-term integration, we have

$$\begin{aligned}\int e^{-x^3}\,dx &= \int \sum_{n=0}^{\infty} \frac{1}{n!}\left(-x^3\right)^n\,dx \\ &= \int \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{3n}\,dx \\ &= C + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(3n+1)} x^{3n+1},\end{aligned}$$

so that

$$\int_0^1 e^{-x^3}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(3n+1)}.$$

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This series is convergent by the Alternating Series Test, so that the error made in the approximation

$$s_n \approx \int_0^1 e^{-x^3}\,dx$$

satisfies

$$|R_n| \leq \frac{1}{(n+1)!(3n+4)}.$$

To ensure 3 exact decimal places, we require $|R_n| \leq 10^{-4}$, which in turn is ensured by $(n+1)!(3n+4) \geq 10^4$, which amounts to $n \geq 5$. Thus

$$s_5 = 1 - \frac{1}{4} + \frac{1}{14} - \frac{1}{60} + \frac{1}{312} - \frac{1}{2280} \approx 0.80752\ldots$$

is an approximation of the integral with at least 3 exact decimal places.

4. [6: 2 for series for $\frac{1}{\sqrt{1+x^4}}$, 1 for term-by-term integration, 1 for numerical series, 1 for upper bound of $|R_n|$, 1 for find *n*] Use power series to find an estimate of

$$\int_0^1 \frac{dx}{\sqrt{1+x^4}}$$

with an error of at most $10^{-1}$.

*Solution.* Since

$$\frac{1}{\sqrt{1+x^4}} = \left(1+x^4\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \left(x^4\right)^n$$

for $|x^4| < 1$, that is, for $|x| < 1$, we conclude by term-by-term integration that

$$\int \frac{dx}{\sqrt{1+x^4}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \frac{x^{4n+1}}{4n+1},$$

so that

$$\begin{aligned}\int_0^1 \frac{dx}{\sqrt{1+x^4}} &= \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \frac{1}{4n+1} \\ &= 1 - \frac{1}{10} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2 \times 9} - \ldots\end{aligned}$$



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Since this series is alternating and converging by the alternating series test, the error made in the approximation

$$s_n \approx \int_0^1 \frac{dx}{\sqrt{1+x^4}}$$

satisfies

$$|R_n| \leq \binom{-\frac{1}{2}}{n+1} \frac{1}{4n+5},$$

so that the error is less than $10^{-1}$ whenever

$$\left| \binom{-\frac{1}{2}}{n+1} \frac{1}{4n+5} \right| \leq 10^{-1},$$

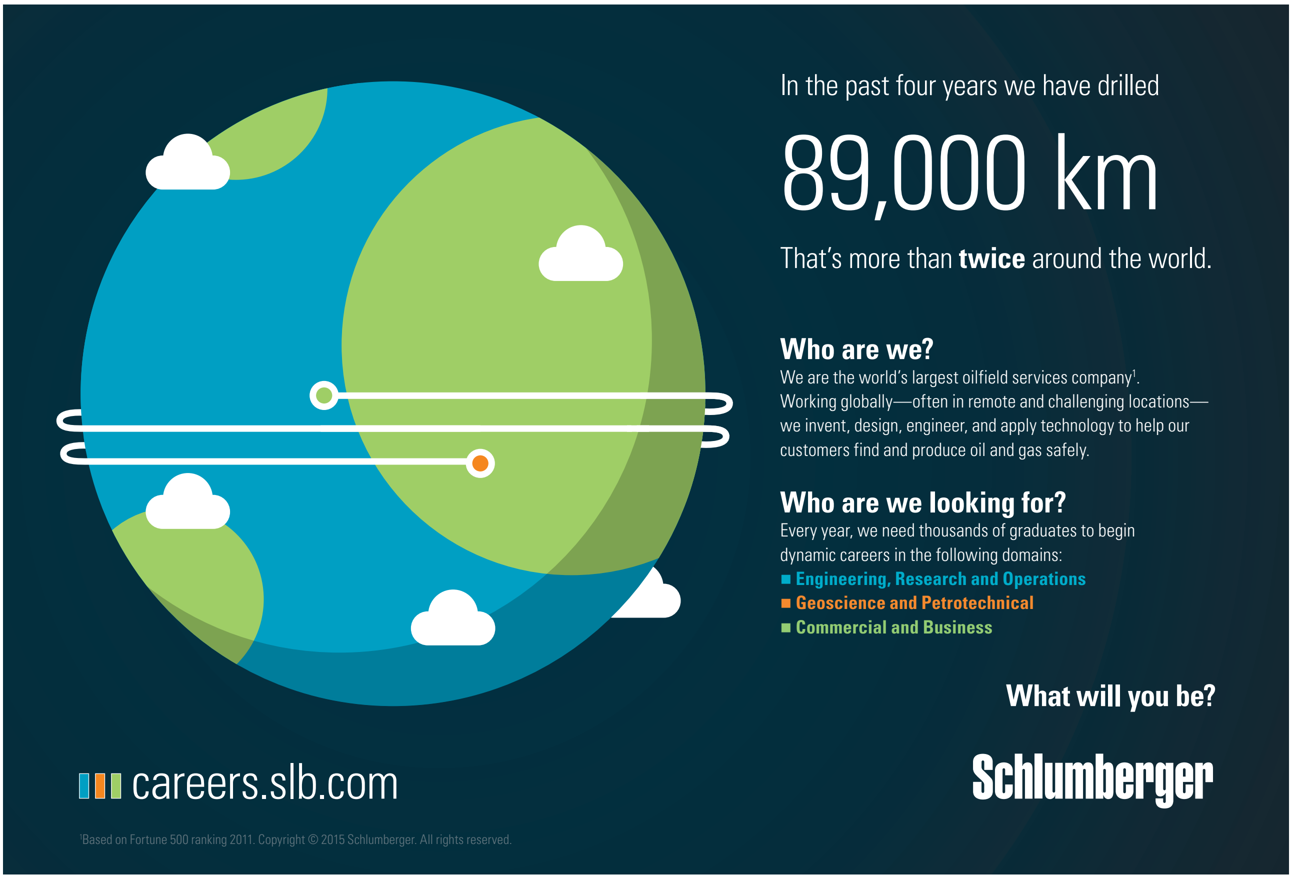
which occurs for $n = 1$ for

$$\binom{-\frac{1}{2}}{2} \cdot \frac{1}{9} = \frac{1}{24} < 10^{-1}.$$

Thus

$$s_1 = 1 - \frac{1}{10} = 0.9 \approx \int_0^1 \frac{dx}{\sqrt{1+x^4}}$$

with an error less than $10^{-1}$.



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# Sample Final Exam

*You should give yourself two hours to do the following test on your own, then, and only then, move to the solutions to evaluate your work.*

*Show all your work and justify all your answers to get credit.*

1. [15] Differentiate

   (a) $f(x) = x^{\arcsin x}$

   (b) $f(x) = \log_3 (\arctan x)$

   (c) $f(x) = e^{3x} \ln(x)$

   Evaluate the following integrals:

2. [5]
$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\, dx$$

3. [5]
$$\int_0^1 \left(4^t + t^4\right)\, dt$$

4. [5]
$$\int \frac{5-x}{\sqrt{1-x^2}}\, dx$$

5. [10]
$$\int x^2 e^{3x}\, dx$$

6. [5]
$$\int x \cos(x^2)\, dx$$

7. [10]
$$\int \sin^3 x \cos^4 x\, dx$$



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8. [10]

$$\int_0^1 \frac{2x^3+3x^2-16x+2}{x^2+x-12}\,dx$$

9. [10] Is the following integral convergent or divergent? If convergent, find its value

$$\int_0^\infty \frac{x}{(x^2+3)^3}\,dx$$

10. [5] Is the improper integral $\int_1^\infty \frac{1+|\cos x|}{x}\,dx$ convergent or divergent. Justify your answer.

11. [10] Find the area of the surface generated by revolving the curve $y=\sqrt{x},\ 0\le x\le 1$ about the $x$-axis.

12. [25] Consider the polar curve $r=\sin(2\theta)$

   (a) Find the slope of the tangent line to the curve at the points corresponding to $\theta=0,\ \frac{\pi}{4}$ and $\frac{\pi}{2}$.
   (b) Sketch the curve.
   (c) Setup an integral to calculate the length of the portion of the curve corresponding to $0\le\theta\le\frac{\pi}{2}$. **Do not** evaluate the integral.
   (d) Find the area enclosed by the same portion of the curve and the $x$-axis.

13. [35] For each of the following series, say if it is convergent. Fully justify your answer.

   (a) $\sum_{n=1}^\infty \frac{n}{n+1}$.
   (b) $\sum_{n=1}^\infty \frac{(-1)^n n}{n^2+n+2}$
   (c) $\sum_{n=2}^\infty \frac{n^3+2}{n^4-2}$.
   (d) $\sum_{n=1}^\infty \left(\frac{n^2}{3n^2+4}\right)^n$.
   (e) $\sum_{n=1}^\infty \frac{\sqrt{n^4+n^2+1}}{n^4+1}$
   (f) $\sum_{n=1}^\infty n^2e^{-n^3}$.
   (g) $\sum_{n=1}^\infty \frac{2^n n}{(2n+1)!3^n}$.

14. [10] Find the interval of convergence of $\sum_{n=1}^\infty \frac{(-1)^n}{(n+2)3^n}(x+1)^n$.



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15. [5] Find the sum of

$$\sum_{n=0}^{\infty} \frac{(-1)^n \, \pi^{2n}}{2n! \quad 2^{4n}}$$

16. [0] Find the Taylor series of $e^{2x}$ at 1. What is its radius of convergence?

17. (a) [10] Establish a representation as power series (together with the corresponding radius of convergence) for $f(x) = \ln(1+x) + 1$.

(b) [5] Deduce the sum of $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)2^{n+1}}$.

18. [10] Find an estimate, with three exact decimal places, of

$$\int_0^{\frac{1}{2}} \arctan(x^2)\, dx$$

19. [5] Find

$$\lim_{x \to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5}$$

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# Sample Final Solutions

1. [15] Differentiate

(a) $f(x) = x^{\arcsin x}$

*Solution.* Since

$$f(x) = e^{\arcsin x \ln x},$$

we conclude with the Chain Rule that

$$\begin{aligned} f'(x) &= e^{\arcsin x \ln x} (\arcsin x \ln x)' \\ &= x^{\arcsin x} \left( \frac{\ln x}{\sqrt{1-x^2}} + \frac{\arcsin x}{x} \right). \end{aligned}$$

(b) $f(x) = \log_3 (\arctan x)$

*Solution.*

$$f'(x) = \frac{1}{\ln 3 \cdot \arctan x \cdot (1+x^2)}.$$

(c) $f(x) = e^{3x} \ln(x)$

*Solution.*

$$f'(x) = 3e^{3x} \ln x + \frac{e^{3x}}{x}.$$

Evaluate the following integrals:

2. [5]

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\, dx$$

*Solution.* Let $u = \cos x$. Then $du = -\sin x\, dx$ so that

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\, dx = -\int_1^0 \frac{du}{1+u^2} = [\arctan u]_0^1 = \arctan 1 = \frac{\pi}{4}.$$



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3. [5]

$$\int_0^1 \left(4^t + t^4\right)\, dt$$

*Solution.*

$$\int_0^1 \left(4^t + t^4\right)\, dt = \left[\frac{4^t}{\ln 4} + \frac{t^5}{5}\right]_0^1 = \frac{4}{\ln 4} + \frac{1}{5} - \frac{1}{\ln 4} = \frac{3}{\ln 4} + \frac{1}{5}.$$

4. [5]

$$\int \frac{5-x}{\sqrt{1-x^2}}\, dx$$

*Solution.*

$$\begin{aligned}
\int \frac{5-x}{\sqrt{1-x^2}}\, dx &= 5\int \frac{dx}{\sqrt{1-x^2}} - \int \frac{x}{\sqrt{1-x^2}}\, dx \\
&= 5 \arcsin x + \frac{1}{2}\int \frac{du}{\sqrt{u}} \text{ for } u = 1 - x^2 \\
&= 5 \arcsin x + \sqrt{1-x^2} + C.
\end{aligned}$$

5. [10]

$$\int x^2 e^{3x}\, dx$$

*Solution.* We proceed by parts with $u = x^2$ and $dv = e^{3x}\, dx$ so that $du = 2x\, dx$ and $v = \frac{e^{3x}}{3}$:

$$\int x^2 e^{3x}\, dx = \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\int x e^{3x}\, dx.$$

To calculate the remaining integral, we proceed again by parts with $u = x$ and $dv = e^{3x}\, dx$ so that $du = dx$ and $v = \frac{e^{3x}}{3}$:

$$\begin{aligned}
\int x^2 e^{3x}\, dx &= \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\int x e^{3x}\, dx \\
&= \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\left(\frac{1}{3}x e^{3x} - \frac{1}{3}\int e^{3x}\, dx\right) \\
&= e^{3x}\left(\frac{x^2}{3} - \frac{2x}{9} + \frac{2}{27}\right) + C.
\end{aligned}$$



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6. [5]

$$\int x\cos(x^2)\,dx$$

*Solution.* Let $u = x^2$. Then $du = 2x\,dx$ and

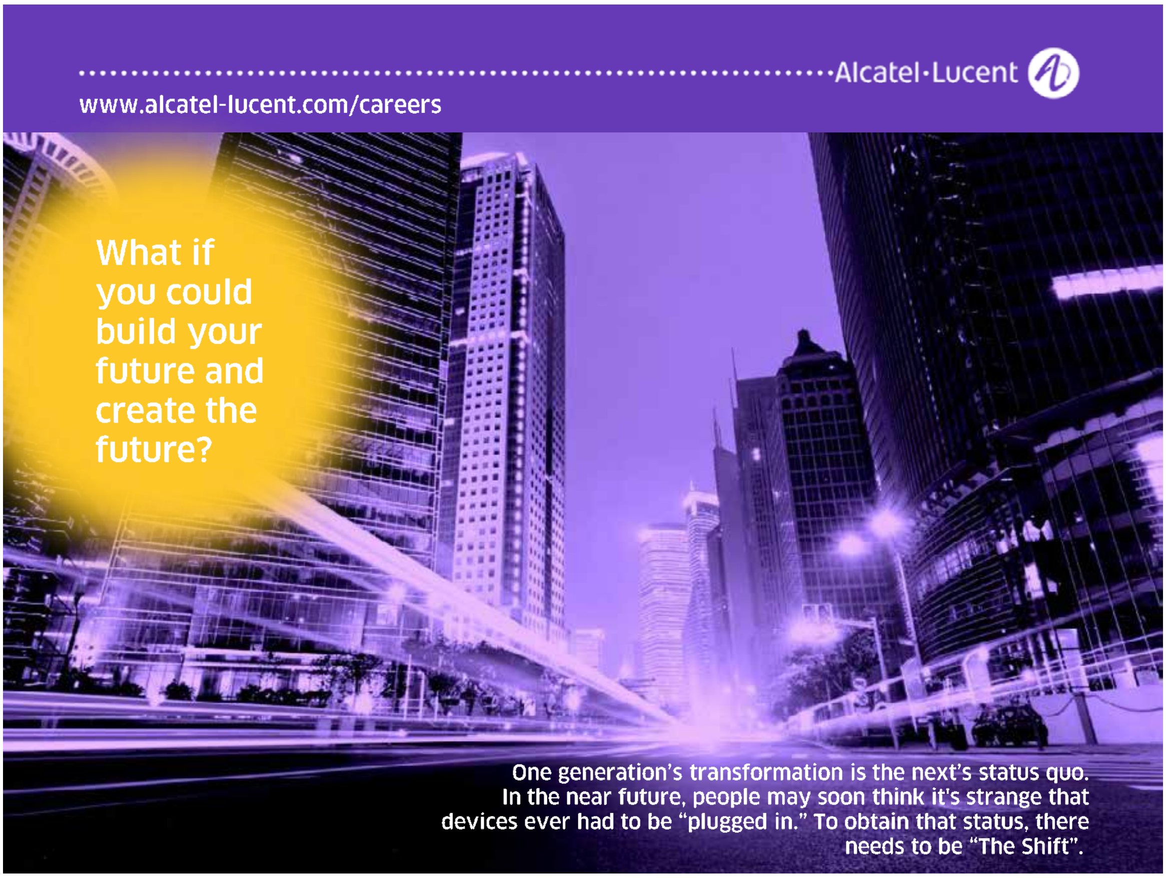
$$\int x\cos(x^2)\,dx = \frac{1}{2}\int \cos u\,du = \frac{1}{2}\sin(x^2) + C.$$

7. [10]

$$\int \sin^3 x\cos^4 x\,dx$$

*Solution.* Let $u = \cos x$ and $du = -\sin x\,dx$. Then

$$\begin{aligned}
\int \sin^3 x\cos^4 x\,dx &= -\int \sin^2 x\cos^4 x\,(-\sin x\,dx) \\
&= -\int (1-u^2)\,u^4\,du \\
&= \int u^6 - u^4\,du \\
&= \frac{u^7}{7} - \frac{u^5}{5} + C \\
&= \frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + C.
\end{aligned}$$



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8. [10]

$$\int_0^1 \frac{2x^3+3x^2-16x+2}{x^2+x-12}\,dx$$

*Solution.* By long division

$$\begin{array}{r|l} & \quad 2x \quad +1 \\ \hline x^2+x-12 & 2x^3+3x^2-16x+2 \\ & -2x^3-2x^2+24x \\ \hline & \qquad x^2+8x+2 \\ & \qquad -x^2-x+12 \\ \hline & \qquad\qquad 7x+14 \end{array}$$

we see that

$$\int_0^1 \frac{2x^3+3x^2-16x+2}{x^2+x-12}\,dx = \int_0^1 2x+1+\frac{7x+14}{x^2+x-12}\,dx.$$

Moreover,

$$\frac{7x+14}{x^2+x-12} = \frac{7x+14}{(x-3)(x+4)} = \frac{A}{x-3}+\frac{B}{x+4}$$

and using the handcover method, we easily obtain $A=5$ and $B=2$. Thus

$$\begin{aligned}\int_0^1 \frac{2x^3+3x^2-16x+2}{x^2+x-12}\,dx &= \int_0^1 2x+1+\frac{5}{x-3}+\frac{2}{x+4}\,dx. \\ &= \left[x^2+x+5\ln|x-3|+2\ln|x+4|\right]_0^1 \\ &= 2+5\ln 2+2\ln 5-5\ln 3-2\ln 4.\end{aligned}$$

9. [10] Is the following integral convergent or divergent? If convergent, find its value

$$\int_0^\infty \frac{x}{(x^2+3)^3}\,dx$$

*Solution.* By definition

$$\begin{aligned}\int_0^\infty \frac{x}{(x^2+3)^3}\,dx &= \lim_{t\to\infty}\int_0^t \frac{x}{(x^2+3)^3}\,dx \\ &= \lim_{t\to\infty}\frac{1}{2}\int_3^{t^2+3}\frac{du}{u^3} \text{ for } u=x^2+3 \\ &= \lim_{t\to\infty}\left[-\frac{1}{4u^2}\right]_3^{t^2+3} \\ &= \lim_{t\to\infty}\frac{1}{36}-\frac{1}{4(t^2+3)^2}=\frac{1}{36}.\end{aligned}$$



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10. [5] Is the improper integral $\int_1^\infty \frac{1+|\cos x|}{x}\,dx$ convergent or divergent. Justify your answer.

*Solution.* Since

$$0 \le \frac{1}{x} \le \frac{1+|\cos x|}{x}$$

and $\int_1^\infty \frac{dx}{x}$ is divergent, we conclude by comparison that $\int_1^\infty \frac{1+|\cos x|}{x}\,dx$ is divergent.

11. [10] Find the area of the surface generated by revolving the curve $y=\sqrt{x},\ 0\le x\le 1$ about the $x$-axis.

*Solution.*

$$\begin{aligned} A &= 2\pi\int_0^1 y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\ &= 2\pi\int_0^1 \sqrt{x}\sqrt{1+\left(\frac{1}{2\sqrt{x}}\right)^2}\,dx \\ &= 2\pi\int_0^1 \sqrt{x}\sqrt{\frac{4x+1}{4x}}\,dx \\ &= 2\pi\cdot\frac{1}{2}\int_0^1 \sqrt{4x+1}\,dx \\ &= \frac{\pi}{4}\int_1^5 \sqrt{u}\,du \text{ for } u=4x+1 \\ &= \frac{\pi}{4}\cdot\frac{2}{3}\left[u^{\frac{3}{2}}\right]_1^5 = \frac{\pi}{6}\left(5^{\frac{3}{2}}-1\right). \end{aligned}$$

12. [25] Consider the polar curve $r=\sin(2\theta)$

(a) Find the slope of the tangent line to the curve at the points corresponding to $\theta=0,\ \frac{\pi}{4}$ and $\frac{\pi}{2}$.

*Solution.* For $\theta=0$ and $\theta=\frac{\pi}{2}$, we have $r=0$, so that the value of $\theta$ determines the direction of the tangent line. Specifically, the tangent is horizontal when $\theta=0$, and vertical when $\theta=\frac{\pi}{2}$. On the other hand, for $\theta=\frac{\pi}{4}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{r'\sin\theta+r\cos\theta}{r'\cos\theta-r\sin\theta} = \frac{2\cos(2\theta)\sin\theta+\sin(2\theta)\cos\theta}{2\cos(2\theta)\cos\theta-\sin(2\theta)\sin\theta} \\ &\overset{\theta=\frac{\pi}{4}}{=} \frac{\cos\frac{\pi}{4}}{-\sin\frac{\pi}{4}} = -1 \end{aligned}$$

is the slope of the tangent line.



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(b) Sketch the curve.

*Solution.* Since $r'(\theta) = 2\cos(2\theta)$, $r' > 0$ if

$$2k\pi < 2\theta < \frac{\pi}{2} + 2k\pi \iff k\pi < \theta < \frac{\pi}{4} + k\pi$$

or if

$$2k\pi + \frac{3\pi}{2} < 2\theta < 2(k+1)\pi \iff k\pi + \frac{3\pi}{4} < \theta < (k+1)\pi.$$

Thus $r' > 0$ if $\theta \in \left(0, \frac{\pi}{4}\right)$ or $\theta \in \left(\pi, \frac{5\pi}{4}\right)$ or $\theta \in \left(\frac{3\pi}{4}, \pi\right)$ or $\theta \in \left(\frac{7\pi}{4}, 2\pi\right)$, which leads to the following variations of $r$ as a function of $\theta$:

<table>
<tr><td>$\theta$</td><td>$(0, \frac{\pi}{4})$</td><td>$(\frac{\pi}{4}, \frac{3\pi}{4})$</td><td>$(\frac{3\pi}{4}, \pi)$</td><td>$(\pi, \frac{5\pi}{4})$</td><td>$(\frac{5\pi}{4}, \frac{7\pi}{4})$</td><td>$(\frac{7\pi}{4}, 2\pi)$</td></tr>
<tr><td>$r'(\theta)$</td><td>$+$</td><td>$-$</td><td>$+$</td><td>$+$</td><td>$-$</td><td>$+$</td></tr>
<tr><td>$r(\theta)$</td><td>$\nearrow$</td><td>$\searrow$</td><td colspan="2">$\nearrow$</td><td>$\searrow$</td><td>$\nearrow$</td></tr>
</table>

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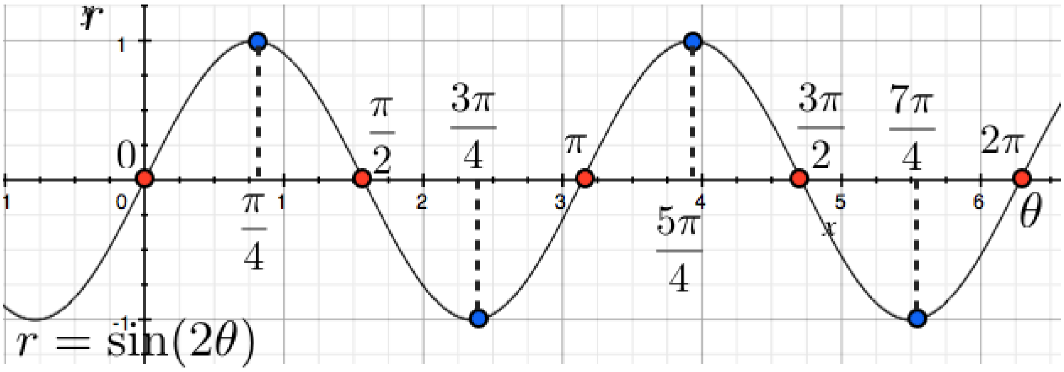
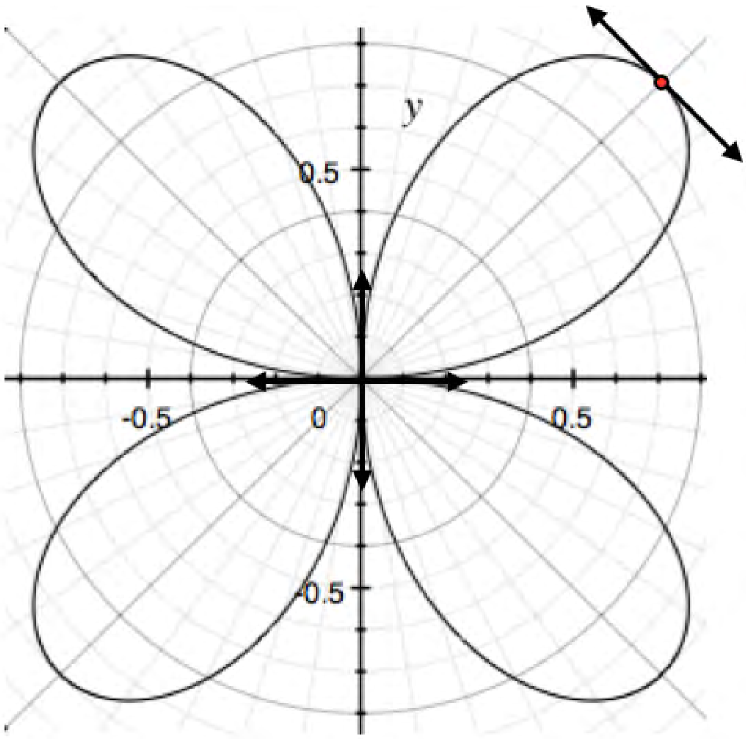
We deduce the following sketch for $r(\theta)$




and thus, the following sketch for the polar curve $r = \sin 2\theta$ :




(c) Setup an integral to calculate the length of the portion of the curve corresponding to $0 \le \theta \le \frac{\pi}{2}$. **Do not** evaluate the integral.

*Solution.* The length is given by

$$\begin{aligned} L &= \int_0^{\frac{\pi}{2}} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \\ &= \int_0^{\frac{\pi}{2}} \sqrt{\sin^2(2\theta) + 4\cos^2(2\theta)}\, d\theta \end{aligned}$$



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(d) Find the area enclosed by the same portion of the curve and the $x$-axis.

*Solution.*

$$\begin{aligned} A &= \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^2(2\theta)\, d\theta \\ &= \frac{1}{4}\int_0^{\frac{\pi}{2}} 1-\cos(4\theta)\, d\theta \\ &= \frac{1}{4}\left[\theta - \frac{1}{4}\sin(4\theta)\right]_0^{\frac{\pi}{2}} = \frac{\pi}{8}. \end{aligned}$$

13. [35] For each of the following series, say if it is convergent. Fully justify your answer.

(a) $\sum_{n=1}^{\infty} \frac{n}{n+1}$.

*Solution.* Since

$$\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0,$$

the series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ is divergent by the $n^{th}$ Term Test.

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+n+2}$

*Solution.* The sequence $\left\{\frac{n}{n^2+n+2}\right\}_{n=1}^{\infty}$ is eventually decreasing for

$$\left(\frac{x}{x^2+x+2}\right)' = \frac{x^2+x+2-2x^2-x}{(x^2+x+2)^2} = \frac{2-x^2}{(x^2+x+2)^2} < 0$$

for all $x \geq 2$, and

$$\lim_{n\to\infty} \frac{n}{n^2+n+2} = 0,$$

so that the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+n+2}$ is convergent by the Alternating Series Test.

(c) $\sum_{n=2}^{\infty} \frac{n^3+2}{n^4-2}$.

*Solution.* Since $n^3+2 \geq n^3$ and $\frac{1}{n^4-2} \geq \frac{1}{n^4}$, we have

$$\frac{n^3+2}{n^4-2} \geq \frac{n^3}{n^4} = \frac{1}{n} \geq 0$$

for all $n \geq 2$. Moreover, $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent $p$-series for $p = 1 \leq 1$. By Comparison, $\sum_{n=2}^{\infty} \frac{n^3+2}{n^4-2}$ is also divergent.



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(d) $\sum_{n=1}^{\infty} \left(\frac{n^2}{3n^2+4}\right)^n$.

*Solution.* Let $a_n := \left(\frac{n^2}{3n^2+4}\right)^n$. Then

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{n^2}{3n^2+4} = \frac{1}{3} < 1.$$

By the Root Test, the series $\sum_{n=1}^{\infty} \left(\frac{n^2}{3n^2+4}\right)^n$ is convergent.

(e) $\sum_{n=1}^{\infty} \frac{\sqrt{n^4+n^2+1}}{n^4+1}$

*Solution.* Let $a_n := \frac{\sqrt{n^4+n^2+1}}{n^4+1}$ and $b_n := \frac{\sqrt{n^4}}{n^4} = \frac{1}{n^2}$. Then

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{n^2\sqrt{n^4+n^2+1}}{n^4+1} \\ &= \lim_{n\to\infty} \frac{n^4\sqrt{1+\frac{1}{n^2}+\frac{1}{n^4}}}{n^4\left(1+\frac{1}{n^4}\right)} = 1 > 0.\end{aligned}$$

Moreover, $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent $p$-series for $p = 2 > 1$. Thus, by the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{\sqrt{n^4+n^2+1}}{n^4+1}$ is also convergent.

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(f) $\sum_{n=1}^{\infty} n^2 e^{-n^3}$.

*Solution.* Let $a_n := n^2 e^{-n^3}$. Then

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{(n+1)^2}{e^{(n+1)^3}} \cdot \frac{e^{n^3}}{n^2} = \frac{(n+1)^2}{n^2} \cdot e^{n^3-(n+1)^3} \\ &= \frac{(n+1)^2}{n^2} e^{-3n^2-3n-1},\end{aligned}$$

so that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$. By the Ratio Test, $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ is convergent.

(g) $\sum_{n=1}^{\infty} \frac{2^n n}{(2n+1)!3^n}$.

*Solution.* Let $a_n := \frac{2^n n}{(2n+1)!3^n}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}(n+1)}{(2n+3)!3^{n+1}} \cdot \frac{(2n+1)!3^n}{2^n n} = \frac{2}{3} \cdot \frac{n+1}{n} \cdot \frac{1}{(2n+2)(2n+3)},$$

so that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$. By the Ratio Test, $\sum_{n=1}^{\infty} \frac{2^n n}{(2n+1)!3^n}$ is convergent.

14. [10] Find the interval of convergence of $\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+2)3^n}(x+1)^n$.

*Solution.* Let $a_n := \frac{(-1)^n}{(n+2)3^n}(x+1)^n$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x+1|^{n+1}}{(n+3)3^{n+1}} \cdot \frac{(n+2)3^n}{|x+1|^n} = \frac{1}{3} \cdot \frac{n+2}{n+3} \cdot |x+1|$$

so that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x+1|}{3}.$$

By the Ratio Test, the interval of convergence is centered at –1 and has radius 3. Moreover, when $x = -4$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+2)3^n}(-3)^n = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n+2} = \sum_{n=1}^{\infty} \frac{1}{n+2} = \sum_{n=3}^{\infty} \frac{1}{n},$$

which is a divergent *p*-series ($p = 1 \leq 1$).

When $x = 2$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+2)3^n} 3^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+2},$$



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which is convergent by the Alternating Series Test, for $\left\{\frac{1}{n+2}\right\}_{n=1}^{\infty}$ is decreasing with limit 0. Thus, the interval of convergence is

$$I = (-4, 2].$$

15. [5] Find the sum of

$$\sum_{n=0}^{\infty} \frac{(-1)^n \, \pi^{2n}}{2n! \quad 2^{4n}}$$

*Solution.* Since

$$\sum_{n=0}^{\infty} \frac{(-1)^n \, \pi^{2n}}{2n! \quad 2^{4n}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} \left(\frac{\pi}{4}\right)^{2n}$$

and for all *x*,

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} x^{2n} = \cos x,$$

we conclude that

$$\sum_{n=0}^{\infty} \frac{(-1)^n \, \pi^{2n}}{2n! \quad 2^{4n}} = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}.$$

16. [10] Find the Taylor series of $e^{2x}$ at 1. What is its radius of convergence?

*Solution.* If $f(x) = e^{2x}$ then

$$\begin{array}{rcl} f'(x) & = & 2e^{2x} \\ f''(x) & = & 2^2 e^{2x} \\ & \vdots & \\ f^{(n)}(x) & = & 2^n e^{2x}, \end{array}$$

so that

$$\frac{f^{(n)}(1)}{n!} = \frac{2^n e^2}{n!}$$

and the desired Taylor series is

$$\sum_{n=0}^{\infty} \frac{2^n e^2}{n!} (x-1)^n.$$



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To determine its radius of convergence, we use the Ratio Test with $a_n := \frac{2^n e^2}{n!}(x-1)^n$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}e^2|x-1|^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n e^2 |x-1|^n} = \frac{2}{n+1}|x-1|$$

so that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$ for all $x$, and the interval of convergence of this series is $(-\infty, \infty)$, that is, the radius of convergence is infinite.

17. (a) [10] Establish a representation as power series (together with the corresponding radius of convergence) for $f(x) = \ln(1+x) + 1$.

*Solution.* Since

$$f'(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$$

for $|x| < 1$, we deduce by term-by-term integration that

$$f(x) = C + \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1}$$

for $|x| < 1$. Since $f(0) = 1 + \ln 1 = 1$ and $f(0) = C$, we deduce that

$$f(x) = 1 + \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1} \text{ for } x \in (-1,1). \tag{16.3.1}$$

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(b) [5] Deduce the sum of $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)2^{n+1}}$.

*Solution*. Note that according to (16.3.1),

$$f\left(\frac{1}{2}\right) = 1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)} \left(\frac{1}{2}\right)^{n+1},$$

so that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)2^{n+1}} = f\left(\frac{1}{2}\right) - 1 = \ln\left(\frac{3}{2}\right) - 1.$$

18. [10] Find an estimate, with three exact decimal place, of

$$\int_0^{\frac{1}{2}} \arctan(x^2)\,dx$$

*Solution*. Since

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \text{ for all } -1 \le x \le 1,$$

we have

$$\arctan(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{4n+2} \text{ on } [-1,1].$$

Integrating term-by-term, we have

$$\int \arctan(x^2)\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(4n+3)} x^{4n+3} \text{ for } |x| < 1.$$

Thus

$$\int_0^{\frac{1}{2}} \arctan(x^2)\,dx = \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(4n+3)} x^{4n+3}\right]_0^{\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(4n+3)2^{4n+3}}.$$

This series is convergent by the Alternating Series Test. Thus the error made in the approximation

$$s_n := \sum_{i=0}^{n} \frac{(-1)^i}{(2i+1)(4i+3)2^{4i+3}} \approx \int_0^{\frac{1}{2}} \arctan(x^2)\,dx$$

satisfies

$$|R_n| \le \frac{1}{(2n+3)(4n+7)2^{4n+7}}.$$



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We can guarantee at least 3 exact decimal places by requiring $|R_n| \leq 10^{-4}$, which is true whenever

$$(2n+3)(4n+7)2^{4n+7} \geq 10^4.$$

Since $5 \times 11 \times 2^{11} = 112640 \geq 10^5$, we have

$$s_1 = \frac{1}{24} - \frac{1}{2688} \approx \int_0^{\frac{1}{2}} \arctan(x^2)\, dx$$

with at least 4 exact decimal places.

19. [5] Find

$$\lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5}$$

*Solution*. Since

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \dots$$

for all *x*, we conclude that

$$\frac{\sin x - x + \frac{x^3}{6}}{x^5} = \frac{\frac{x^5}{120} - \frac{x^7}{5040} + \dots}{x^5} = \frac{1}{120} - \frac{x^2}{5040} + \frac{x^4}{9!} - \dots$$

so that

$$\lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5} = \frac{1}{120}.$$



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# Endnotes

1. we could also have simplified the function by $x$ to $\frac{2}{x^2+2x} = \frac{2}{x(x+2)}$
2. note the change of starting index between the 2 series



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