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Examples of General Elementary Series

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Calculus 3c-2

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Contents

Introduction

Here follows a collection of examples of *general, elementary series*. The reader is also referred to *Calculus 3b*. The main subject is *Power series*; but first we must consider series in general. We shall in *Calculus 3c-3* return to the power series.

Finally, even if I have tried to write as careful as possible, I doubt that all errors have been removed. I hope that the reader will forgive me the unavoidable errors.

Leif Mejlbro
14th May 2008

1 Partial sums and telescopic series

Example 1.1 *Prove that the series*

$$\sum_{n=1}^{\infty}\left(\frac{1}{n+4}-\frac{1}{n+6}\right)$$

is convergent and find its sum.

We shall in this chapter only use the definition of the convergence as the limit of the partial sums of the series. In this particular case we have

$$\begin{aligned} s_N &= \sum_{n=1}^{N}\left(\frac{1}{n+4}-\frac{1}{n+6}\right) \\ &= \left\{\frac{1}{5}-\frac{1}{7}\right\}+\left\{\frac{1}{6}-\frac{1}{8}\right\}+\left\{\frac{1}{7}-\frac{1}{9}\right\}+\cdots+\left\{\frac{1}{N+2}-\frac{1}{N+4}\right\} \\ &\quad +\left\{\frac{1}{N+3}-\frac{1}{N+5}\right\}+\left\{\frac{1}{N+4}-\frac{1}{N+6}\right\}. \end{aligned}$$

The sum is finite, and we see that all except four terms disappear, so

$$s_N=\frac{1}{5}+\frac{1}{6}-\frac{1}{N+5}-\frac{1}{N+6}\to\frac{1}{5}+\frac{1}{6}-0-0=\frac{11}{30}\quad\text{for } N\to\infty.$$

It follows by the definition that the series is convergent and its sum is

$$\sum_{n=1}^{\infty}\left(\frac{1}{n+4}-\frac{1}{n+6}\right)=\lim_{N\to\infty}s_N=\frac{11}{30}.$$

Remark 1.1 Since

$$\sum_{n=1}^{N}\frac{1}{n+4}=\sum_{n=5}^{N+4}\frac{1}{n}\quad\text{and}\quad\sum_{n=1}^{N}\frac{1}{n+6}=\sum_{n=7}^{N+6}\frac{1}{n}$$

(finite sums with the same terms; check!), we get more well-arranged (the sum can be split, because it is *finite*)

$$\begin{aligned} s_N &= \sum_{n=1}^{N}\frac{1}{n+4}-\sum_{n=1}^{N}\frac{1}{n+6}=\sum_{n=5}^{N+4}\frac{1}{n}-\sum_{n=7}^{N+6}\frac{1}{n} \\ &= \left\{\frac{1}{5}+\frac{1}{6}+\sum_{n=7}^{N+4}\frac{1}{n}\right\}-\left\{\sum_{n=7}^{N+4}\frac{1}{n}+\frac{1}{N+5}+\frac{1}{N+6}\right\} \\ &= \frac{1}{5}+\frac{1}{6}-\frac{1}{N+5}-\frac{1}{N+6}\to\frac{1}{5}+\frac{1}{6}=\frac{11}{30}\quad\text{for } N\to\infty, \end{aligned}$$

etc.

Example 1.2 *Prove that the given series is convergent and find its sum*

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}.$$

Since we have a rational function in e.g. $x = 2n$, we start by decomposing the term

$$\frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\,\frac{1}{2n-1} - \frac{1}{2}\,\frac{1}{2n+1}.$$

Then calculate the N-th partial sum

$$\begin{aligned} s_N &= \sum_{n=1}^{N} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\sum_{n=1}^{N} \frac{1}{2n-1} - \frac{1}{2}\sum_{n=1}^{N} \frac{1}{2n+1} \\ &= \frac{1}{2}\sum_{n=0}^{N-1} \frac{1}{2n+1} - \frac{1}{2}\sum_{n=1}^{N} \frac{1}{2n+1} = \frac{1}{2} - \frac{1}{2}\cdot\frac{1}{2N+1}. \end{aligned}$$

Since the sequence of partial sums is convergent,

$$s_N = \frac{1}{2} - \frac{1}{2}\cdot\frac{1}{2N+1} \to \frac{1}{2} \qquad \text{for } N \to \infty,$$

the *series* is convergent and its sum is

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \lim_{N\to\infty} s_N = \frac{1}{2}.$$

Example 1.3 *Prove that the given series is convergent and find its sum,*

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}.$$

We get by a decomposition,

$$\frac{1}{n^2-1} = \frac{1}{(n-1)(n+1)} = \frac{1}{2}\cdot\frac{1}{n-1} - \frac{1}{2}\cdot\frac{1}{n+1}.$$

Se sequence of partial sums is then

$$\begin{aligned}
s_N &= \sum_{n=2}^{N}\frac{1}{n^2-1} = \frac{1}{2}\sum_{n=2}^{N}\frac{1}{n-1} - \frac{1}{2}\sum_{n=2}^{N}\frac{1}{n+1} \\
&= \frac{1}{2}\sum_{n=1}^{N-1}\frac{1}{n} - \frac{1}{2}\sum_{n=3}^{N+1}\frac{1}{n} \qquad \text{(the same insides, check the first and the last terms)} \\
&= \left\{\frac{1}{2}+\frac{1}{4}+\frac{1}{2}\sum_{n=3}^{N-1}\frac{1}{n}\right\} - \frac{1}{2}\left\{\sum_{n=3}^{N-1}\frac{1}{n}+\frac{1}{N}+\frac{1}{N+1}\right\} \qquad \text{(remove some terms)} \\
&= \frac{3}{4} - \frac{1}{2}\cdot\frac{1}{N} - \frac{1}{2}\cdot\frac{1}{N+1} \qquad \text{(cancel the two identical sums)} \\
&\to \frac{3}{4} \qquad \text{for } N\to\infty.
\end{aligned}$$

It follows by the definition that the series is convergent and its sum is

$$\sum_{n=2}^{\infty}\frac{1}{n^2-1} = \lim_{N\to\infty} s_N = \frac{3}{4}.$$

Example 1.4 *Prove that the given series is convergent and find its sum*

$$\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}}.$$

This is a nontypical case, though one may still copy the method of decomposition. since

$$\sqrt{n^2+n} = \sqrt{n+1}\cdot\sqrt{n},$$

it follows by a division that

$$\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}} = \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}\cdot\sqrt{n}} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}.$$

Then calculate the *sequence* of partial sums,

$$\begin{aligned}
s_N &= \sum_{n=1}^{N}\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}} = \sum_{n=1}^{N}\frac{1}{\sqrt{n}} - \sum_{n=1}^{N}\frac{1}{\sqrt{n+1}} \\
&= \sum_{n=1}^{N}\frac{1}{\sqrt{n}} - \sum_{n=2}^{N+1}\frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{N+1}} = 1 - \frac{1}{\sqrt{N+1}}.
\end{aligned}$$

Since the *sequence* of partial sums is convergent

$$s_N = 1 - \frac{1}{\sqrt{N+1}} \to 1 \qquad \text{for } N \to \infty,$$

the *series* is also convergent and its sum is

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}} = \lim_{N\to\infty} s_N = 1.$$

Remark 1.2 We see from the expression of the sequence of partial sums that the convergence is very slow, so it is not a good idea here to use a pocket calculator.

Example 1.5 . *Prove that the given series is convergent and find its sum,*

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}.$$

We first decompose,

$$\frac{2n+1}{n^2(n+1)^2} = \frac{(n^2+2n+1)-n^2}{n^2(n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2}.$$

This gives us the sequence of partial sums

$$\begin{aligned} s_N &= \sum_{n=1}^{N} \frac{2n+1}{n^2(n+1)^2} = \sum_{n=1}^{N} \frac{1}{n^2} - \sum_{n=1}^{N} \frac{1}{(n+1)^2} = \sum_{n=1}^{N} \frac{1}{n^2} - \sum_{n=2}^{N+1} \frac{1}{n^2} \\ &= \left\{1 + \sum_{n=2}^{N} \frac{1}{n^2}\right\} - \left\{\sum_{n=2}^{N} \frac{1}{n^2} + \frac{1}{(N+1)^2}\right\} \\ &= 1 - \frac{1}{(N+1)^2} \to 1 \qquad \text{for } N \to \infty. \end{aligned}$$

It follows by the definition that the series is convergent and its sum is

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = \lim_{N\to\infty} s_N = 1.$$

Example 1.6 *Prove that the given series is convergent and find its sum,*

$$\sum_{n=1}^{\infty} \frac{3n+4}{n(n+1)(n+2)}.$$

We get by a decomposition that

$$\frac{3n+4}{n(n+1)(n+2)} = \frac{2}{n} - \frac{1}{n+1} - \frac{1}{n+2}.$$

The sequence og partial sums becomes

$$
\begin{aligned}
s_N &= \sum_{n=1}^{N} \frac{3n+4}{n(n+1)(n+2)} = 2\sum_{n=1}^{N}\frac{1}{n} - \sum_{n=1}^{N}\frac{1}{n+1} - \sum_{n=1}^{N}\frac{1}{n+2} \\
&= 2\sum_{n=1}^{N}\frac{1}{n} - \sum_{n=2}^{N+1}\frac{1}{n} - \sum_{n=3}^{N+2}\frac{1}{n} \qquad\qquad \text{(same insides)} \\
&= \left\{\sum_{n=1}^{N}\frac{1}{n} - \sum_{n=2}^{N+1}\frac{1}{n}\right\} + \left\{\sum_{n=1}^{N}\frac{1}{n} - \sum_{n=3}^{N+2}\frac{1}{n}\right\} \qquad \left(\text{by writing } 2\sum_{n=1}^{N}\frac{1}{n} = \sum_{n=1}^{N}\frac{1}{n} + \sum_{n=1}^{N}\frac{1}{n}\right) \\
&= \left\{1 + \sum_{n=2}^{N}\frac{1}{n} - \sum_{n=2}^{N}\frac{1}{n} - \frac{1}{N+1}\right\} + \left\{1 + \frac{1}{2} + \sum_{n=3}^{N}\frac{1}{n} - \sum_{n=3}^{N}\frac{1}{n} - \frac{1}{N+1} - \frac{1}{N+2}\right\} \\
&= \left\{1 - \frac{1}{N+1}\right\} + \left\{\frac{3}{2} - \frac{1}{N+1} - \frac{1}{N+2}\right\} \\
&= \frac{5}{2} - \frac{2}{N+1} - \frac{1}{N+2} \to \frac{5}{2} \qquad \text{for } N \to \infty.
\end{aligned}
$$

It follows by the definition that the series converges towards the sum

$$
\sum_{n=1}^{\infty} \frac{3n+4}{n(n+1)(n+2)} = \lim_{N\to\infty} s_N = \frac{5}{2}.
$$

Example 1.7 *Prove that the given series is convergent and find its sum,*

$$\sum_{n=1}^{\infty} \frac{2^n + n^2 + n}{2^{n+1}n(n+1)}.$$

By using a decomposition like method we get

$$\frac{2^n + n^2 + n}{2^{n+1}n(n+1)} = \frac{1}{2} \cdot \frac{1}{n(n+1)} + \frac{1}{2^{n+1}} = \frac{1}{2}\,\frac{1}{n} - \frac{1}{2}\,\frac{1}{n+1} + \frac{1}{2^{n+1}}.$$

The sequence of partial sums is

$$\begin{aligned}
s_N &= \sum_{n=1}^{N} \frac{2^n + n^2 + n}{2^{n+1}n(n+1)} = \frac{1}{2}\sum_{n=1}^{N}\frac{1}{n} - \frac{1}{2}\sum_{n=1}^{N}\frac{1}{n+1} + \sum_{n=1}^{N}\frac{1}{2^{n+1}} \\
&= \frac{1}{2}\sum_{n=1}^{N}\frac{1}{n} - \frac{1}{2}\sum_{n=2}^{N+1}\frac{1}{n} + \frac{1}{2}\left(1 - \frac{1}{2^N}\right) \\
&= \left\{\frac{1}{2} + \frac{1}{2}\sum_{n=2}^{N}\frac{1}{n}\right\} - \left\{\frac{1}{2}\sum_{n=2}^{N}\frac{1}{n} + \frac{1}{2}\,\frac{1}{N+1}\right\} + \frac{1}{2} - \frac{1}{2^{N+1}} \\
&= \frac{1}{2} - \frac{1}{2}\,\frac{1}{N+1} + \frac{1}{2} - \frac{1}{2^{N+1}} = 1 - \frac{1}{2}\,\frac{1}{N+1} - \frac{1}{2^{N+1}} \to 1 \quad \text{for } N \to \infty.
\end{aligned}$$

By the definition, the series is convergent and its sum is

$$\sum_{n=1}^{\infty} \frac{2^n + n^2 + n}{2^{n+1}n(n+1)} = \lim_{N\to\infty} s_N = 1.$$

Example 1.8 *Check if the given series is convergent or divergent,*

$$\sum_{n=1}^{\infty} \frac{2+(-1)^n}{2^n}.$$

First estimate each term,

$$0 < a_n = \frac{2+(-1)^n}{2^n} \leq \frac{3}{2^n} = b_n.$$

Then the *larger* series

$$\sum_{n=1}^{\infty} b_n = 3\sum_{n=1}^{\infty}\frac{1}{2^n} = 3, \qquad \text{(a quotien series)},$$

is convergent, so it follows from the *criterion of comparison* that the smaller series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{2+(-1)n}{2^n}$$

is also convergent.

Remark 1.3 We can in this case even find the sum. which will give us an alternative proof. The sequence of partial sums is

$$
\begin{aligned}
s_N &= \sum_{n=1}^{N} \frac{2+(-1)^n}{2^n} = 2\sum_{n=1}^{N}\left(\frac{1}{2}\right)^n + \sum_{n=1}^{N}\left(-\frac{1}{2}\right)^n \\
&= 2\left\{1-\frac{1}{2^N}\right\} + \frac{\left(-\frac{1}{2}\right)-\left(-\frac{1}{2}\right)^{N+1}}{1-\left(-\frac{1}{2}\right)} \qquad \text{(quotient series)} \\
&= 2-\frac{2}{2^N}-\frac{1}{3}-\frac{2}{3}\cdot\left(-\frac{1}{2}\right)^{N+1} \to \frac{5}{3} \quad \text{for } N\to\infty.
\end{aligned}
$$

It follows that the series converges towards the sum

$$
\sum_{n=1}^{\infty} \frac{2+(-1)^n}{2^n} = \lim_{N\to\infty} s_N = \frac{5}{3}.
$$

2 Simple convergence criteria for series

Example 2.1 *Check if the given series is convergent or divergent,*

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n(n-1)}}.$$

Criterion of comparison. Since

$$\sqrt{n(n-1)} < \sqrt{n \cdot n} = n, \quad \text{we have} \quad \frac{1}{\sqrt{n(n-1)}} > \frac{1}{n}.$$

Therefore, if we put $a_n = 1/\sqrt{n(n-1)}$ and $b_n = 1/n$, we get

$$\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{1}{n(n-1)} \geq \sum_{n=2}^{\infty} \frac{1}{n} = \sum_{n=2}^{\infty} b_n.$$

Since the smaller series is divergent (the *harmonic series* is divergent), the larger series is also, thus

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n(n-1)}} \qquad \text{is divergent.}$$

Criterion of equivalence. Putting as above

$$a_n = \frac{1}{\sqrt{n(n-1)}} \quad \text{and} \quad b_n = \frac{1}{n}, \qquad n \geq 2,$$

we see that both $a_n > 0$ and $b_n > 0$. Since

$$\frac{b_n}{a_n} = \frac{\sqrt{n(n-1)}}{n} = \sqrt{\frac{n-1}{n}} = \sqrt{1 - \frac{1}{n}} \to 1 \quad \text{for } n \to \infty,$$

the series $\sum a_n$ and $\sum b_n$ are equivalent. Since the harmonic series is divergent, it follows that

$$\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n} \qquad \text{is divergent.}$$

By the criterion of equivalence,

$$\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n(n-1)}} \qquad \text{is divergent.}$$

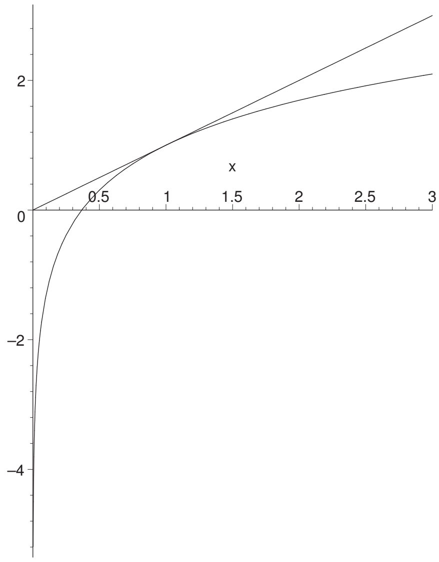
Example 2.2 *Check if the given series is convergent or divergent,*

$$\sum_{n=1}^{\infty} \frac{1}{1+\ln n}.$$

It follows either by the magnitudes or by a graphical consideration that

$$0 < 1 + \ln n \le n \qquad \text{for every } n \in \mathbb{N}.$$

Hence,



$$a_n = \frac{1}{1+\ln n} \ge \frac{1}{n} = b_n,$$

and thus

$$\sum_{n=1}^{\infty} a_n \ge \sum_{n=1}^{\infty} b_n.$$

Since the smaller series $\sum b_n$ is divergent (the harmonic series), the larger series $\sum a_n$ is by the **criterion of comparison** also divergent.

We have proved that

$$\sum_{n=1}^{\infty} \frac{1}{1+\ln n} \qquad \text{is divergent.}$$

Example 2.3 *Check if the given series is convergent or divergent,*

$$\sum_{n=1}^{\infty} ne^{-n^2}.$$

Criterion of comparison. By putting $a_n = ne^{-n^2} > 0$, and e.g. $b_n = 1/n^2$, it is seen that

$$0 < a_n = n \cdot e^{-n^2} = \frac{1}{n^2} \cdot n^3 e^{-n^2} < \frac{1}{n^2} = b_n \quad \text{for } n \geq N \quad (\text{in fact for } n \in \mathbb{N}),$$

since the magnitudes assure that at $n^3 \cdot e^{-n^2} \to 0$ for $n \to \infty$.

Since the larger series $\sum b_n = \sum n^{-2}$ is convergent, it follows by the **criterion of comparison** that

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} ne^{-n^2} \qquad \text{is konvergent.}$$

Remark 2.1 Another choice of b_n could be $b_n = e^{-n}$ or $b_n = \exp(-n^2/2)$. In both cases we also prove the convergence.

Criterion of quotients. If we put $a_n = n\exp(-n^2) > 0$, it follows that

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)\exp(-(n+1)^2)}{n\exp(-n^2)} = \left(1 + \frac{1}{n}\right) e^{-2n-1} \to 0 < 1 \quad \text{for } n \to \infty,$$

and the convergence follows by the criterion of quotients.

Criterion of roots. If we put $a_n = n\exp(-n^2) > 0$, it follows that

$$\sqrt[n]{a_n} = \sqrt[n]{n} \cdot e^{-n} \to 1 \cdot 0 = 0 < 1 \qquad \text{for } n \to \infty,$$

and the convergence follows by the criterion of roots.

Example 2.4 *Check if the given series is convergent or divergent,*

$$\sum_{n=1}^{\infty} \frac{1}{2n^2 - \sqrt{n}}.$$

Criterion of equivalence. If we put

$$a_n = \frac{1}{2n^2 - \sqrt{n}} > 0 \quad \text{og} \quad b_n = \frac{1}{2n^2}, \qquad \text{for } n \in \mathbb{N},$$

it follows that

$$\frac{b_n}{a_n} = \frac{2n^2 - \sqrt{n}}{2n^2} = 1 - \frac{1}{2}\,\frac{1}{n\sqrt{n}} \to 1 \qquad \text{for } n \to \infty,$$

thus (a_n) and (b_n) are equivalent sequences. Since

$$\sum_{n=1}^{\infty} b_n = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \text{is convergent,}$$

the criterion of equivalence shows that

$$\sum_{n=1}^{\infty} \frac{1}{2n^2-\sqrt{n}} \qquad \text{is konvergent.}$$

Criterion of comparison. Since $\sqrt{n} \le n \le n^2$ for $n \in \mathbb{N}$, we have

$$2n^2 - \sqrt{n} \ge 2n^2 - n^2 = n^2,$$

thus

$$0 < a_n = \frac{1}{2n^2-\sqrt{n}} \le \frac{1}{n^2} = b_n, \qquad n \in \mathbb{N}.$$

The larger series $\sum b_n = \sum 1/n^2$ is convergent, hence the smaller series

$$\sum_{n=1}^{\infty} \frac{1}{2n^2-\sqrt{n}} \qquad \text{is also convergent.}$$

Example 2.5 *Check if the given series is convergent or divergent,*

$$\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}.$$

Criterion of equivalence. Put

$$a_n = \frac{1}{n+\sqrt{n}} > 0 \quad \text{and} \quad b_n = \frac{1}{n} > 0.$$

Then

$$\frac{b_n}{a_n} = \frac{n+\sqrt{n}}{n} = 1 + \frac{1}{\sqrt{n}} \to 1 \qquad \text{for } n \to \infty,$$

so (a_n) and (b_n) are equivalent. The harmonic series

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{is divergent,}$$

so we conclude by the criterion of equivalence that

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}} \quad \text{is also divergent.}$$

Criterion of comparison. Since $\sqrt{n} \le n$ for $n \in \mathbb{N}$, we have $n + \sqrt{n} \le 2n$, so

$$a_n = \frac{1}{n+\sqrt{n}} \ge \frac{1}{2n} = b_n.$$

Since the harmonic series is divergent, the smaller series

$$\sum_{n=1}^{\infty} b_n = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \qquad \text{is divergent.}$$

By the criterion of comparison, the larger series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}} \qquad \text{is divergent.}$$

Example 2.6 *Check if the series*

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{2n^2+1}}$$

is convergent or divergent.

Since

$$\sqrt{2n^2+1} = n\sqrt{2+\frac{1}{n^2}} > n,$$

it follows by the **criterion of comparison** that

$$0 < \sum_{n=1}^{\infty} \frac{1}{n\sqrt{2n^2+1}} < \sum_{n=1}^{\infty} \frac{1}{n \cdot n} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

so the series is convergent.

We can also apply the **criterion of equivalence**, but it will only be a variant of the above.

Remark 2.2 Since $a_n = 1/(n\sqrt{2n^2+1})$ approximately behaves like a fractional rational function, we cannot use the criteria of quotients or roots:

$$\sqrt[n]{a_n} \to 1, \quad \text{and} \quad \frac{a_{n+1}}{a_n} \to 1, \qquad \text{for } n \to \infty.$$

Example 2.7 *Check if the given series is convergent or divergent,*

$$\sum_{n=1}^{\infty} \frac{n+2}{(n+1)\sqrt{n+3}}.$$

Criterion of equivalence. Put

$$a_n = \frac{n+2}{(n+1)\sqrt{n+3}} > 0.$$

By counting the degrees we see that it would be reasonable to compare with $b_n = 1/\sqrt{n}$. Since

$$\frac{b_n}{a_n} = \frac{(n+1)\sqrt{n+3}}{(n+2)\sqrt{n}} = \frac{n+1}{n+2}\sqrt{\frac{n+3}{n}} = \left(1-\frac{1}{n+2}\right)\sqrt{1+\frac{3}{n}} \to 1 \quad \text{for } n \to \infty,$$

it follows that (a_n) and (b_n) are equivalent. Then compare $b_n = 1/\sqrt{n}$ and $c_n = 1/n$. We see that

$$b_n = \frac{1}{\sqrt{n}} \geq \frac{1}{n} = c_n.$$

The harmonic series is divergent, so $\sum c_n = \sum \frac{1}{n}$ is divergent. The larger series $\sum b_n$ is also divergent, so according to the criterion of equivalence

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n+2}{(n+1)\sqrt{n+3}} \qquad \text{is divergent.}$$

Remark 2.3 The above is rather difficult. It will later be proved that $\sum n^{-\alpha}$ is divergent, when $\alpha \leq 1$. This is true here, where $\alpha = \frac{1}{2}$.

Criterion of comparison. Since

$$a_n = \frac{n+2}{(n+1)\sqrt{n+3}} \geq \frac{1}{\sqrt{n+3}} > \frac{1}{n} = b_n \qquad \text{for } n \geq 3,$$

and $\sum b_n = \sum 1/n$ is divergent (the harmonic series again), the larger series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n+2}{(n+1)\sqrt{n+3}} \qquad \text{divergent.}$$

Example 2.8 *Check if the given series is convergent or divergent,*

$$\sum_{n=1}^{\infty} \frac{4n^2+5n-2}{n(n^2+1)^{3/2}}.$$

Criterion of equivalence. Put

$$a_n = \frac{4n^2+5n-2}{n(n^2+1)^{3/2}} > 0.$$

By counting the degrees we are led to choose $b_n = 4/n^2$. Then

$$\frac{b_n}{a_n} = \frac{n(n^2+1)^{3/2}}{4n^2+5n-2} \cdot \frac{4}{n^2} = \frac{n \cdot n^3 \left(1+\dfrac{1}{n^2}\right)^{3/2}}{n^2\left(4+\dfrac{5}{n}-\dfrac{2}{n^2}\right)} \cdot \frac{4}{n^2} \to 1 \quad \text{for } n \to \infty,$$

so (a_n) and (b_n) are equivalent. Since

$$\sum_{n=1}^{\infty} b_n = 4\sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \text{is convergent,}$$

it follows that also

$$\sum_{n=1}^{\infty} \frac{4n^2+5n-2}{n(n^2+1)^{3/2}} \qquad \text{is convergent.}$$

Criterion of comparison. Since

$$0 < a_n = \frac{4n^2+5n-2}{n(n^2+1)^{3/2}} = \frac{n^2}{n^4} \cdot \frac{4+\dfrac{5}{n}-\dfrac{2}{n^2}}{\left(1+\dfrac{1}{n^2}\right)^{3/2}} \leq \frac{4+5-0}{(1+0)^{3/2}} \cdot \frac{1}{n^2} = \frac{9}{n^2} = b_n,$$

where the larger series

$$\sum_{n=1}^{\infty} b_n = 9\sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \text{is convergent,}$$

the smaller series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{4n^2+5n-2}{n(n^2+1)^{3/2}} \qquad \text{is also convergent.}$$

Example 2.9 *Check if the given series is convergent or divergent,*

$$\sum_{n=1}^{\infty} \sqrt{\frac{n-\ln n}{n^2+10n^3}}.$$

The series looks horrible, but if we use the principle of taking the dominating factors outside the expression, then the task becomes fairly easy:

$$(1)\quad a_n = \sqrt{\frac{n-\ln n}{n^2+10n^3}} = \sqrt{\frac{n\left(1-\frac{1}{n}\ln n\right)}{n^3\left(10+\frac{1}{n}\right)}} = \frac{1}{n}\sqrt{\frac{1-\frac{1}{n}\ln n}{10+\frac{1}{n}}}.$$

It follows by the magnitude that the latter factor converges towards $1/\sqrt{10}$ for $n \to \infty$. We have now two variants.

Criterion of equivalence. If we put

$$b_n = \frac{1}{\sqrt{10}\cdot n},$$

it follows from the above that

$$\frac{b_n}{a_n} = \sqrt{\frac{10-\frac{10}{n}\ln n}{10+\frac{1}{n}}} \to 1 \qquad \text{for } n \to \infty,$$

thus (a_n) and (b_n) are equivalent. Since $\sum b_n = \dfrac{1}{\sqrt{10}}\sum \dfrac{1}{n}$ is divergent, we have that

$$\sum_{n=1}^{\infty} \sqrt{\frac{n-\ln n}{n^2+10n^3}} \qquad \text{is also divergent.}$$

Criterion of comparison. Since $1/4 < 1/\sqrt{10}$, it follows from (1) that there is an $N \in \mathbb{N}$, such that

$$a_n \ge \frac{1}{4}\cdot\frac{1}{n} = b_b \qquad \text{for every } n \ge N.$$

Since the smaller series

$$\sum_{n=N}^{\infty} b_n = \frac{1}{4}\sum_{n=N}^{\infty}\frac{1}{n} \qquad \text{is divergent,}$$

then the larger series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\sqrt{\frac{n-\ln n}{n^2+10n^3}} \qquad \text{is also divergent.}$$

Example 2.10 *Check if the given series is convergent or divergent,*

$$\sum_{n=1}^{\infty}(\sqrt{1+n^2}-n).$$

The insides $\sqrt{1+n^2}-n$ is of the type "$\infty-\infty$", so we must first make a rearrangement

$$a_n=\sqrt{1+n^2}-n=\frac{(\sqrt{1+n^2})^2-n^2}{\sqrt{1+n^2}+n}=\frac{1}{n+\sqrt{1+n^2}}=\frac{1}{n}\cdot\frac{1}{1+\sqrt{1+\frac{1}{n^2}}}.$$

It follows immediately from this that

$$a_n\geq\frac{1}{2n}=b_n>0.$$

Since the smaller series $\sum b_n=\frac{1}{2}\sum\frac{1}{n}$ is divergent, the larger series

$$\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}(\sqrt{1+n^2}-n)\qquad\text{is also divergent}$$

according to the **criterion of comparison**.

We can **alternatively** apply the **criterion of equivalence** with $b_n=\dfrac{1}{2n}$, because

$$\frac{b_n}{a_n}=\frac{1}{2}\left\{1+\sqrt{1+\frac{1}{n^2}}\right\}\to\frac{1}{2}\{1+1\}=1\qquad\text{for } n\to\infty,$$

thus (a_n) and (b_n) are equivalent. Since

$$\sum_{n=1}^{\infty}b_n=\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\qquad\text{is divergent,}$$

we also have that

$$\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}(\sqrt{1+n^2}-n)\qquad\text{is divergent.}$$

Example 2.11 *Check if the given series is convergent or divergent,*

$$\sum_{n=1}^{\infty} \frac{\exp(-\sqrt{n})}{\sqrt{n}}.$$

Whenever the exponential function appears together with a term which is almost polynomial, one should immediately think of the different magnitudes and try to make a comparison with a known convergent series.
Here we choose the standard series

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}, \qquad \text{which is convergent.}$$

We shall isolate the factor $1/n^2$. This gives the following estimate

$$0 < a_n = \frac{\exp(-\sqrt{n})}{\sqrt{n}} = \frac{1}{n^2} \cdot \frac{n^2 \exp(-\sqrt{n})}{\sqrt{n}} = \frac{1}{n^2} \cdot \left\{ (\sqrt{n})^3 e^{-\sqrt{n}} \right\}.$$

Due to the law of magnitudes, $x^3 e^{-x} \to 0$ for $n \to \infty$.

Then put $x = \sqrt{n} \to \infty$ for $n \to \infty$. We see that there exists an $N \in \mathbb{N}$, such that

$$0 < a_n < \frac{1}{n^2} = b_n \qquad \text{for } n \geq N.$$

Since the larger series

$$\sum_{n=N}^{\infty} b_n = \sum_{n=N}^{\infty} \frac{1}{n^2} \qquad \text{is convergent,}$$

then the smaller series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\exp(-\sqrt{n})}{\sqrt{n}} \qquad \text{is also ckonvergent}$$

since we only add a finite number of terms $\sum_{n=1}^{N-1} a_n$.

Example 2.12 *Check if the given series is convergent or divergent,*

$$\sum_{n=1}^{\infty} \frac{1}{n \ln\left(1 + \frac{1}{n}\right)}.$$

Criterion of comparison. We get from

$$0 < n \ln\left(1 + \frac{1}{n}\right) \leq \ln 2 \cdot n,$$

that

$$a_n = \frac{1}{n \ln\left(1 + \frac{1}{n}\right)} \geq \frac{1}{\ln 2} \cdot \frac{1}{n} = b_n > 0.$$

Since the smaller series

$$\sum_{n=1}^{\infty} b_n = \frac{1}{\ln 2} \sum_{n=1}^{\infty} \frac{1}{n} \qquad \text{is divergent,}$$

the larger series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n \ln\left(1+\frac{1}{n}\right)} \qquad \text{is also divergent.}$$

Alternatively we prove that the **necessary condition** of convergence is not fulfilled. In fact, we get by Taylor's formula

$$\ln(1+x) = x + x\varepsilon(x).$$

If we put $x = 1/n \to 0$ for $n \to \infty$, then

$$n \ln\left(1+\frac{1}{n}\right) = n\left\{\frac{1}{n} + \frac{1}{n}\varepsilon\left(\frac{1}{n}\right)\right\} = 1 + \varepsilon\left(\frac{1}{n}\right) \to 1 \quad \text{for } n \to \infty,$$

hence

$$a_n = \frac{1}{n \ln\left(1+\frac{1}{n}\right)} \to \frac{1}{1} = 1 \neq 0 \qquad \text{for } n \to \infty.$$

The necessary condition of convergence is *not* fulfilled, hence the series is divergent.

Example 2.13 *Check if the given series is convergent or divergent,*

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{Arccot } n.$$

We get by Taylor's formula that

$$\text{Arctan } x = x + x\varepsilon(x).$$

Put $x = 1/n$. Then

$$0 < a_n = \frac{1}{n} \text{ Arccot } n = \frac{1}{n} \text{ Arctan } \frac{1}{n} = \frac{1}{n}\left\{\frac{1}{n} + \frac{1}{n}\varepsilon\left(\frac{1}{n}\right)\right\} = \frac{1}{n^2}\left\{1 + \varepsilon\left(\frac{1}{n}\right)\right\},$$

hence

$$0 < a_n = \frac{1}{n^2}\left\{1 + \varepsilon\left(\frac{1}{n}\right)\right\} \le \frac{2}{n^2} = b_n \qquad \text{for } n \ge N.$$

Since the larger series

$$\sum_{n=N}^{\infty} b_n = 2\sum_{n=N}^{\infty} \frac{1}{n^2} \qquad \text{is convergent,}$$

it follows by the **criteria of comparison** that the smaller series

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ Arccot } n \qquad \text{is also convergent.}$$

Example 2.14 *Check if the given series is convergent or divergent,*

$$\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n^2}\right).$$

We get by Taylor's formula that

$$\ln(1+x) = x + x\varepsilon(x).$$

We even get by a graphical consideration that

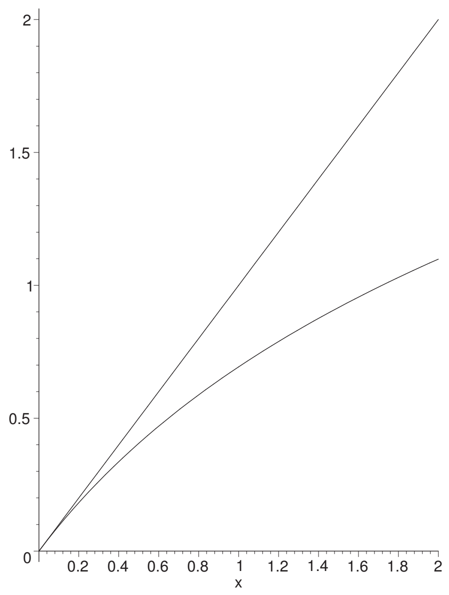
$$0 < \ln(1+x) < x \qquad \text{for } x > 0.$$

If we put $x = 1/n^2$, it follows that

$$0 < a_n = \ln\left(1 + \frac{1}{n^2}\right) < \frac{1}{n^2} = b_n, \qquad \text{for } n \in \mathbb{N}.$$

The larger series

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \text{is convergent,}$$



hence the smaller series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n^2}\right) \qquad \text{is also convergent}$$

by the **criterion of comparison**.

Example 2.15 *Check if the given series is convergent or divergent,*

$$\sum_{n=1}^{\infty} \frac{(3n)!3^n}{n^{3n}2^{2n}}.$$

This example either assumes Stirling's formula or the criterion of quotients combined with a pocket calculator.

It follows from **Stirling's formula**

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left\{1+\varepsilon\left(\frac{1}{n}\right)\right\}$$

when n is replaces by $3n$ that

$$(3n)! = \sqrt{6\pi n}\left(\frac{3n}{e}\right)^{3n} \left\{1+\varepsilon\left(\frac{1}{n}\right)\right\},$$

hence

$$a_n = \frac{(3n)!3^n}{n^{3n}2^{2n}} = \frac{\sqrt{6\pi n}\cdot 3^{3n}\cdot n^{3n}\cdot 3^n}{e^{3n}\cdot n^{3n}\cdot 4^n}\left\{1+\varepsilon\left(\frac{1}{n}\right)\right\} = \sqrt{6\pi n}\cdot\left(\frac{3^4}{4e^3}\right)^n \left\{1+\varepsilon\left(\frac{1}{n}\right)\right\}.$$

We get by a calculation on a pocket calculator that

$$a = \frac{3^4}{4e^3} \approx 1,008 > 1,$$

hence

$$a_n = \sqrt{6\pi n} \cdot a^n \left\{ 1 + \varepsilon\left(\frac{1}{n}\right)\right\} \to \infty,$$

and the **necessary condition of convergence** is *not* fulfilled, and the series is divergent.

Alternatively we apply the **criterion of quotients**. Since

$$a_n = \frac{(3n)!3^n}{n^{3n} \cdot 2^{2n}} > 0,$$

we get

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(3n+3)!3^{n+1}}{(n+1)^{3n+3} \cdot 2^{2n+2}} \cdot \frac{n^{3n} \cdot 2^{2n}}{(3n)!3^n} = \frac{(3n+3)(3n+2)(3n+1)3}{\left(1+\dfrac{1}{n}\right)^{3n} \cdot (n+1)^3 \cdot 2^2} \\ &\to \frac{3^4}{e^3 \cdot 4} \approx 1,008 > 1 \qquad \text{for } n \to \infty,\end{aligned}$$

where we again have used our pocket calculator.
Since the limit value is > 1, it follows from the **criterion of quotients** that the series is divergent.

Example 2.16 *Check if the series*

$$\sum_{n=2}^{\infty} \left(\sqrt[n]{n} - 1\right)^n$$

is convergent.

The structure invites an application of the criterion of roots. The criterion of comparison may also be applied. Anyway, an application of the criterion of quotients will be rather messy, although it is also possible to succeed in this case. See below.

Initial investigation. Since $\sqrt[n]{n} > 1$, we have $\left(\sqrt[n]{n} - 1\right)^n > 0$ for $n \geq 2$. Since even $\sqrt[n]{n} \to 1$ for $n \to \infty$, there exists an N, such that

$$0 \leq \sqrt[n]{n} - 1 \leq \frac{1}{2} \quad \text{for every } n \geq N \quad (\text{one may here even choose } N = 2).$$

1) **Criterion of roots**. Since

$$\sqrt[n]{|a_n|} = \sqrt[n]{n} - 1 \to 0 < 1 \qquad \text{for } n \to \infty,$$

it follows from the criterion of roots that the series is convergent.

2) **Criterion of comparison**. Since

$$0 < \sqrt[n]{n} - 1 \le \frac{1}{2} \qquad \text{for every } n \ge 2,$$

we get the estimate

$$0 < \sum_{n=2}^{\infty} \left(\sqrt[n]{n} - 1 \right)^n < \sum_{n=2}^{\infty} \left(\frac{1}{2} \right)^n = \frac{1}{2},$$

and the series is convergent.

3) The **criterion of quotients** becomes very messy:

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{\left(\sqrt[n+1]{n+1} - 1 \right)^{n+1}}{\left(\sqrt[n]{n} - 1 \right)^n} = \left\{ \frac{\sqrt[n+1]{n+1} - 1}{\sqrt[n]{n} - 1} \right\}^n \cdot \left(\sqrt[n+1]{n+1} - 1 \right).$$

This does not look nice. We can, however manage it by noting that the latter factor $\to 0$ for $n \to \infty$, and by convincing oneself that the former factor can be estimated upwards by 1, by proving that $\sqrt[n]{n} - 1$ is decreasing in n for $n \ge 3$, which means that the numerator is smaller than the denominator.

Example 2.17 *Check if the series*

$$\sum_{n=1}^{\infty} \frac{2^n}{n^7}$$

is convergent.

First variant. According to the rules of magnitudes,

$$a_n = \frac{2^n}{n^7} \to \infty \qquad \text{for } n \to \infty,$$

hence the **necessary condition of convergence** is not fulfilled, and the series is divergent.

Second variant. Since $a_n = 2^n/n^7 > 0$, we get by the **criterion of roots** that

$$0 < \sqrt[n]{a_n} = \sqrt[n]{\frac{2^n}{n^7}} = \frac{2}{\left(\sqrt[n]{n}\right)^7} \to 2 > 1 \quad \text{for } n \to \infty,$$

showing that the series is divergent.

Third variant. Choosing the same a_n we get by the **criterion of quotients** that

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)^7} \cdot \frac{n^7}{2^n} = \frac{2}{\left(1+\frac{1}{n}\right)^7} \to 2 > 1 \qquad \text{for } n \to \infty,$$

and we conclude that the series is divergent.

Example 2.18 *Check if the series*

$$\sum_{n=2}^{\infty} \left(\frac{\ln n}{n}\right)^n$$

is convergent.

First variant. The structure invites to an application of the **criterion of roots**. Put

$$a_n = \left(\frac{\ln n}{n}\right)^n > 0 \qquad \text{for } n \geq 2.$$

Then

$$\sqrt[n]{a_n} = \frac{1}{n} \ln n \to 0 < 1 \qquad \text{for } n \to \infty,$$

by the laws of magnitudes. Then the series is convergent by the **criterion of roots**.

Second variant. Put

$$f(x) = \frac{\ln x}{x}, \qquad \text{med} \qquad f'(x) = \frac{1 - \ln x}{x^2}.$$

Then we have a global maximum for $x = e$, thus

$$0 < \frac{\ln n}{n} = f(n) \le f(e) = \frac{\ln e}{e} = \frac{1}{e} \qquad \text{for } n \ge 3.$$

Then

$$0 < a_n = \left(\frac{\ln n}{n}\right)^n < \frac{1}{e^n} = b_n \qquad \text{for } n \ge 3.$$

The larger series is convergent, because it is a quotient series of quotient $1/e \in]0,1[$,

$$\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \left(\frac{1}{e}\right)^n$$

and thus convergent. Then it follows by the **criterion of comparison** that the smaller series

$$\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \left(\frac{\ln n}{n}\right)^n$$

is also convergent.

Third variant. It is possible to apply also the criterion of quotients, but this will give a terrible mess, so the details are ere left out.

Example 2.19 *Check if the series*

$$\sum_{n=1}^{\infty} \frac{2^n + n^2}{3^n}$$

is convergent.

First variant. Criterion of roots. Since $a_n > 0$ and

$$0 < \sqrt[n]{a_n} = \sqrt[n]{\frac{2^n + n^2}{3^n}} = \frac{2}{3}\sqrt[n]{1 + \frac{n^2}{2^n}} \to \frac{2}{3} < 1 \qquad \text{for } n \to \infty,$$

it follows from the criterion of roots that the series is convergent.

Second variant. Criterion of quotients. Since $a_n > 0$ and

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1} + (n+1)^2}{3^{n+1}} \cdot \frac{3^n}{2^n + n^2} = \frac{2}{3} \cdot \frac{1 + \dfrac{(n+1)^2}{2^{n+1}}}{1 + \dfrac{n^2}{2^n}} \to \frac{2}{3} < 1 \quad \text{for } n \to \infty,$$

it follows from the criterion of quotients that the series is convergent.

Third variant. Criterion of comparison. Since

$$0 < \frac{2^n + n^2}{3^n} \le \frac{2^n + 2^n}{3^n} = 2 \cdot \left(\frac{2}{3}\right)^n,$$

and the larger quotient series $\sum 2 \cdot \left(\frac{2}{3}\right)^n$ is convergent, it follows from the criterion of comparison that the given series is convergent.

Addition, fourth variant. By using the theory of power series it is possible explicitly to find its sum. We have for $|x| < 1$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,$$

and we are allowed to differentiate each term separately

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} \quad \text{og} \quad \frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2}, \quad |x| < 1.$$

Then

$$\sum_{n=1} n^2 x^n = x^2 \sum_{\substack{n=2 \\ (n=1)}}^{\infty} n(n-1)x^{n-2} + x \sum_{n=1}^{\infty} n x^{n-1} = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2}.$$

Choosing $x = \frac{1}{3}$, which immediately gives the convergence, we get

$$\sum_{n=1}^{\infty} \frac{2^n + n^2}{3^n} = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n + \sum_{n=1}^{\infty} n^2 \left(\frac{1}{3}\right)^n = \frac{\frac{2}{3}}{1-\frac{2}{3}} + \frac{2 \cdot \frac{1}{9}}{\frac{8}{27}} + \frac{\frac{1}{3}}{\frac{4}{9}} = 2 + \frac{3}{4} + \frac{3}{4} = \frac{7}{2}.$$

Example 2.20 *Check if the series*

$$\sum_{n=1}^{\infty} \frac{2^n + 3^n}{6^n}$$

is convergent.

First variant. Find the sum directly. Every term is positive, so we may split the series. Then it is reduced to two convergent quotient series,

$$\sum_{n=1}^{\infty} \frac{2^n + 3^n}{6^n} = \sum_{n=1}^{\infty} \frac{2^n}{6^n} + \sum_{n=1}^{\infty} \frac{3^n}{6^n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{\frac{1}{3}}{1-\frac{1}{3}} + \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{3}{2}.$$

Second variant. Criterion of comparison . Since

$$0 < a_n = \frac{2^n + 3^n}{6^n} < \frac{3^n + 3^n}{6^n} = 2 \cdot \frac{1}{2^n} = b_n,$$

and the larger quotient series $\sum b_n$ (quotient $1/2 < 1$) is convergent, the smaller series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{2^n + 3^n}{6^n} \quad \text{is also convergent.}$$

Third variant. Criterion of roots. Since $a_n > 0$, and

$$\sqrt[n]{a_n} = \sqrt[n]{\frac{2^n + 3^n}{6^n}} = \frac{3}{6}\sqrt[n]{1 + \left(\frac{2}{3}\right)^n} \to \frac{3}{6} \cdot 1 = \frac{1}{2} < 1 \quad \text{for } n \to \infty,$$

it follows from the criterion of roots that the series is convergent.

Fourth variant. Criterion of quotients. Since $a_n > 0$, and

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1} + 3^{n+1}}{6^{n+1}} \cdot \frac{6^n}{2^n + 3^n} = \frac{6^n}{6^{n+1}} \cdot \frac{3^{n+1}}{3^n} \cdot \frac{1 + \left(\frac{2}{3}\right)^{n+1}}{1 + \left(\frac{2}{3}\right)^n} \to \frac{1}{2} < 1 \quad \text{for } n \to \infty,$$

it follows from the criterion of quotients that the series is convergent.

Example 2.21 *Check if the series*

$$\sum_{n=1}^{\infty} \frac{5^{n-1}}{n^2+n}$$

is convergent.

1) Prove directly that the series is *crudely divergent.* It follows from the laws of magnitudes that

$$a_n = \frac{5^{n-1}}{n^2+n} \to \infty \neq 0 \qquad \text{for } n \to \infty,$$

hence the necessary condition of convergence is *not* fulfilled.

2) Alternatively we get by the *criterion of roots* that

$$\sqrt[n]{|a_n|} = \sqrt[n]{\frac{5^{n-1}}{n^2+n}} = \frac{5}{\sqrt[n]{5} \cdot \sqrt[n]{n} \cdot \sqrt[n]{n+1}} \to 5 > 1 \quad \text{for } n \to \infty,$$

hence the series is divergent.

3) Alternatively every $a_n > 0$, so by using the criterion of quotients,

$$0 < \frac{a_{n+1}}{a_n} = \frac{5^n}{(n+1)(n+2)} \cdot \frac{n(n+1)}{5^{n-1}} = 5 \cdot \frac{n}{n+2} \to 5 > 1 \quad \text{for } n \to \infty,$$

and the series is divergent.

Example 2.22 *Check if the series*

$$\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}$$

is convergent.

Since $\dfrac{n^2+1}{n^3+1}$ is a quotient between two polynomials, the criteria of roots and of quotients will both give the limit value 1, so nothing can be concluded by applying these two criteria.

It follows instead by the equivalence

$$a_n = \frac{n^2+1}{n^3+1} = \frac{n^2}{n^3} \cdot \frac{1+\dfrac{1}{n^2}}{1+\dfrac{1}{n^3}} = \frac{1}{n} \cdot \frac{1+\dfrac{1}{n^2}}{1+\dfrac{1}{n^3}} \sim \frac{1}{n} = b_n,$$

that the series behaves approximately like the divergent harmonic series. Then by the *criterion of equvivalence* se series

$$\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1} \qquad \text{is divergent.}$$

Example 2.23 *Check if the series*

$$\sum_{n=0}^{\infty} \frac{(n+1)^{2n}}{(2n)!}$$

is convergent.

Whenever the faculty function occurs, apply only the *criterion of quotients* and avoid the criterion of roots.

Criterion of quotients. First check the assumption

$$a_n = \frac{(n+1)^{2n}}{(2n)!} > 0, \qquad \text{OK.}$$

Then we get [be aware of the calculation of a_{n+1}]

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{(n+2)^{2(n+1)}}{(2n+2)!} \cdot \frac{(2n)!}{(n+1)^{2n}} = \left(\frac{n+2}{n+1}\right)^{2(n+1)} \cdot \frac{(n+1)^2}{(2n+2)(2n+1)} \\ &= \left\{\left(1+\frac{1}{n+1}\right)^{n+1}\right\}^2 \cdot \frac{1}{4} \cdot \frac{1+\frac{1}{n}}{1+\frac{1}{2n}}. \end{aligned}$$

When we apply the *standard sequence* $\left(1+\frac{1}{n}\right)^n \to e$ for $n \to \infty$, we get

$$\frac{a_{n+1}}{a_n} \to e^2 \cdot \frac{1}{4} \cdot \frac{1+0}{1+0} = \left(\frac{e}{2}\right)^2 > 1 \qquad \text{for } n \to \infty.$$

It follows from the *criterion of quotients* that the series is divergent.

Remark 2.4 It is possible also to use the *Criterion of roots* (excluded here) if we apply Stirling's formula.

Remark 2.5 If we instead use the rather sophisticated estimate

$$(2n-j+1)j \le (n+1)n \qquad \text{for } j = 1, \dots, n,$$

(prove this), one may directly prove the *coarse divergence*,

$$a_n = \frac{(n+1)^{2n}}{(2n)!} = \frac{(n+1)^{2n}}{\prod_{j=1}^{n}(2n-j+1)j} \ge \frac{(n+1)^{2n}}{(n+1)^n n^n} = \left(1+\frac{1}{n}\right)^n > 1,$$

thus a_n does not converge towards 0 for $n \to \infty$, proving that we have coarse divergence.

Example 2.24 *Check if the series*

$$\sum_{n=1}^{\infty} \frac{n^{2n+1}}{9^n (n!)^2}$$

is convergent.

The structure indicates that we should apply the *criterion of quotients*. Since $a_n > 0$ and

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(n+1)^{2n+3}}{9^{n+1}((n+1)!)^2} \cdot \frac{9^n (n!)^2}{n^{2n+1}} = \frac{1}{9} \cdot \frac{(n+1)^{2n+3}}{(n+1)^2} \cdot \frac{1}{n^{2n+1}} = \frac{1}{9}\left(\frac{n+1}{n}\right)^{2n+1} \\ &= \frac{1}{9}\left(1+\frac{1}{n}\right) \cdot \left\{\left(1+\frac{1}{n}\right)^n\right\}^2 \to \frac{1}{9} \cdot 1 \cdot e^2 = \left(\frac{e}{3}\right)^2 < 1 \qquad \text{for } n \to \infty,\end{aligned}$$

it follows from the **criterion of quotients** that the series is convergent.

Example 2.25 *Check if the series*

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \tan \frac{1}{\sqrt{n}}$$

is convergent.

Since $\tan x > x$ for $0 < x \le 1$, we get

$$a_n = \frac{1}{\sqrt{n}} \tan \frac{1}{\sqrt{n}} \ge \frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{n}} = \frac{1}{n} = b_n.$$

The smaller series $\sum b_n$ is the divergent harmonic series, hence it follows from the *criterion of comparison* that the larger series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \tan \frac{1}{\sqrt{n}}$$

is divergent.

Example 2.26 *Check if the series*

$$\sum_{n=1}^{\infty} \frac{n!}{2^{(n^2)}}$$

is convergent.

Since the faculty function occurs, one should use the *criterion of quotients* and avoid the criterion of roots.

Criterion of quotients. First check the assumption

$$a_n = \frac{n!}{2^{(n^2)}} > 0, \qquad \text{OK.}$$

Da $(n+1)^2 = n^2 + 2n + 1$, fås dernæst, at

$$a_{n+1} = \frac{(n+1)!}{2^{n^2+2n+1}}.$$

It follows from the laws of magnitudes that

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{2^{n^2+2n+1}} \cdot \frac{2^{(n^2)}}{n!} = \frac{n+1}{2^{2n+1}} \to 0 < 1 \quad \text{for } n \to \infty.$$

(The exponential function dominates any polynomial).
It follows from the *criterion of quotients* that the series is convergent.

Example 2.27 *Check if the series*

$$\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}}$$

is convergent.

The faculty function occurs we use the *Criterion of quotients*.

First check the assumption,

$$a_n = \frac{(2n)!}{n^{2n}} > 0, \qquad \text{OK.}$$

Then calculate a_{n+1} separately (in order to avoid errors),

$$a_{n+1} = \frac{(2\{n+1\})!}{(n+1)^{2(n+1)}} = \frac{(2n+2)!}{(n+1)^{2n+2}}.$$

Finally, check the quotient,

$$\begin{aligned} 0 \quad < \quad \frac{a_{n+1}}{a_n} &= \frac{(2n+2)!}{(n+1)^{2n+2}} \cdot \frac{n^{2n}}{(2n)!} = \frac{(2n+2)(2n+1)\cdot(2n)!}{(2n)!} \cdot \left(\frac{n}{n+1}\right)^{2n} \cdot \frac{1}{(n+1)^2} \\ &= \frac{(2n+2)(2n+1)}{(n+1)^2} \cdot \frac{1}{\left\{\left(1+\dfrac{1}{n}\right)^n\right\}^2} \to \frac{4}{e^2} = \left(\frac{2}{e}\right)^2 < 1, \end{aligned}$$

because $\left(1+\dfrac{1}{n}\right)^n \to e$ for $n \to \infty$ (a standard sequence).
Then the *criterion of quotients* shows that the series is convergent.

Example 2.28 *Find all values of the constant $a \in \mathbb{R}_+$, for which the series*

$$\sum_{n=1}^{\infty} \frac{a^n n!}{n^n}$$

is convergent.

From $a > 0$ follows that $b_n = \dfrac{a^n n!}{n^n} > 0$, hence we get for the quotient

$$\frac{b_{n+1}}{b_n} = \frac{a^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{a^n n!} = \frac{a}{\left(1+\dfrac{1}{n}\right)^n} \to \frac{a}{e} \quad \text{for } n \to \infty.$$

We conclude by the *criterion of quotients* that the series is convergent for $0 < a < e$, and divergent for $a > e$.

Investigation of the possible convergence when $a = e$. We cannot conclude anything from the criterion of quotients itself, but since $\left(1+\dfrac{1}{n}\right)^n$ is *increasing*, it follows that $\left(1+\dfrac{1}{n}\right)^n < e$ for every n, thus

$$\frac{b_{n+1}}{b_n} = \frac{e}{\left(1+\dfrac{1}{n}\right)^n} > \frac{e}{e} = 1 \qquad \text{for every } n \in \mathbb{N},$$

which shows that (b_n) is *increasing*. Since every $b_n > 0$, we conclude that the *necessary condition* of convergence is *not* fulfilled. Hence the series is divergent for $a = e$.

Alternatively we apply *Stirling's formula*

$$n! = \sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n\left\{1+\varepsilon\left(\frac{1}{n}\right)\right\}.$$

If $a = e$, then

$$b_n = \frac{e^n n!}{n^n} = \left(\frac{e}{n}\right)^n\cdot\sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n\left\{1+\varepsilon\left(\frac{1}{n}\right)\right\} = \sqrt{2\pi n}\left\{1+\varepsilon\left(\frac{1}{n}\right)\right\},$$

hence $b_n \to \infty \neq 0$ for $n \to \infty$, and the *necessary condition* of convergence is *not* satisfied.

We conclude that we have convergence for $0 < a < e$ and divergence for $a \geq e$.

Addition. If we put

$$c_n = \frac{e^n n!}{n^{n+1/2}}, \qquad n \in \mathbb{N},$$

then

$$\frac{c_{n+1}}{c_n} = \frac{e}{\left(1+\dfrac{1}{n}\right)^{n+1/2}} = \frac{1}{\sqrt{1+\dfrac{1}{n}}}\cdot\frac{e}{\left(1+\dfrac{1}{n}\right)^n} \to 1 \quad \text{for } n \to \infty.$$

Then by Taylor's formula (with three terms),

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + x^3\varepsilon(x).$$

Writing $x = 1/n$, we get

$$\ln\left\{\left(1+\frac{1}{n}\right)^{n+1/2}\right\} = \left(n+\frac{1}{2}\right)\left\{\frac{1}{n} - \frac{1}{2}\frac{1}{n^2} + \frac{1}{3}\frac{1}{n^3} + \frac{1}{n^3}\varepsilon\left(\frac{1}{n}\right)\right\} = 1 + \frac{1}{12}\frac{1}{n^2} + \frac{1}{n^2}\varepsilon\left(\frac{1}{n}\right),$$

hence

$$\frac{c_{n+1}}{c_n} = \frac{e}{\left(1+\dfrac{1}{n}\right)^{n+1/2}} = \frac{e}{\exp\left(1+\dfrac{1}{12}\dfrac{1}{n^2}+\dfrac{1}{n^2}\varepsilon\left(\dfrac{1}{n}\right)\right)} = \frac{1}{\exp\left(\dfrac{1}{12}\dfrac{1}{n^2}+\dfrac{1}{n^2}\varepsilon\left(\dfrac{1}{n}\right)\right)} < 1$$

for $n \geq N$. This shows that (c_n) is *decreasing* eventually, so (c_n) is bounded

$$0 < c_n = \frac{e^n n!}{n^{n+1/2}} < k_1,$$

thus

$$n! < k_1\cdot e^{-n}\, n^{n+1/2} = k_1\sqrt{n}\cdot\left(\frac{n}{e}\right)^n.$$

On the other hand, (b_n) is increasing, so

$$b_n = \frac{e^n n!}{n^n} > k_2 > 0,$$

and thus

$$n! > k_2 \cdot e^{-n}\, n^n = k_2 \left(\frac{n}{e}\right)^n .$$

Therefore, there exist positive constants k_1, k_2, such that

$$k_2 \cdot \left(\frac{n}{e}\right)^n < n! < k_1\sqrt{n} \cdot \left(\frac{n}{e}\right)^n \qquad \text{for } n \in \mathbb{N},$$

and we are pretty close of a *proof* of Stirling's formula.

Example 2.29 *Prove that the series*

$$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^p}$$

is divergent for every $p \in \mathbb{R}$.

According to the laws of magnitudes, to every $p \in \mathbb{R}$ there exists an $N_p \in \mathbb{N} \setminus \{1\}$, such that

$$(\ln n)^p < n \qquad \text{for } n \geq N_p.$$

Then

$$a_n = \frac{1}{(\ln n)^p} > \frac{1}{n} = b_n \qquad \text{for alle } n \geq N_p.$$

The smaller series is the divergent harmonic series, so it follows from the *criterion of comparison* that the larger series is also divergent. Since $p \in \mathbb{R}$ was any number, the claim is proved.

Example 2.30 *Check in each of the following cases if the given series is conditionally convergent, absolutely convergent or divergent.*

(1) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{n+1}{n} + \frac{n}{n+1} - 2\right).$

(2) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{n+1}{n} + \frac{n}{n+1} - 1\right).$

(3) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \ln\left(\frac{n+1}{n} + \frac{n}{n+1} - 1\right).$

1) It follows from the rearrangement

$$\frac{n+1}{n} + \frac{n}{n+1} - 2 = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} \sim \frac{1}{n^2}$$

and the *criterion of equivalence* that the series is *absolutely convergent.*

2) It follows from the rearrangement [cf. (1)]

$$\frac{n+1}{n}+\frac{n}{n+1}-1=1+\frac{1}{n(n+1)}\to 1\neq 0 \qquad \text{for } n\to\infty$$

that the *necessary condition* of convergence is *not* satisfied, so the series is (coarsely) divergent.

3) It follows from (2) and a *Taylor expansion* that

$$\begin{aligned}\ln\left(\frac{n+1}{n}+\frac{n}{n+1}-1\right)&=\ln\left(1+\frac{1}{n(n+1)}\right)\\&=\frac{1}{n(n+1)}+\frac{1}{n(n+1)}\,\varepsilon\left(\frac{1}{n(n+1)}\right)\sim\frac{1}{n^2}.\end{aligned}$$

Then it follows from the *criterion of equivalence* that the series is absolutely convergent.

3 The integral criterion

Example 3.1 *Check if the given series is convergent or divergent,*

$$\sum_{n=1}^{\infty} \frac{\sqrt{2n-1}\,\ln(4n+1)}{n(n+1)}.$$

It follows from the *integral criterion* that $\sum n^{-\alpha}$ is convergent for $\alpha > 1$. We shall need this result below.

Let alone the logarithmic term we see by counting the degrees of the other terms that we should compare with

$$\sum_{n=1}^{\infty} \frac{\sqrt{2}}{n^{\alpha}}, \qquad \text{hvor } \alpha = \frac{3}{2} > 1.$$

Since $\frac{3}{2} = \frac{5}{4} + \frac{1}{4}$, where we still have $\frac{5}{4} > 1$, we can dominate the logarithmic term by $n^{1/4}$, because the laws of magnitudes give

$$\ln(4n+1) \le \frac{1}{\sqrt{2}} \cdot \sqrt[4]{n} \qquad \text{for } n \ge N.$$

Then

$$0 < a_n = \frac{\sqrt{2n-1}\,\ln(4n+1)}{n(n+1)} \le \frac{\sqrt{2n} \cdot \frac{1}{\sqrt{2}}\sqrt[4]{n}}{n \cdot n} = \frac{1}{n^{5/4}} \qquad \text{for } n \ge N.$$

Since $\alpha = \frac{5}{4} > 1$, the larger series

$$\sum \frac{1}{n^{5/4}} \qquad \text{is convergent.}$$

By the *criterion of comparison* the smaller series

$$\sum_{n=1}^{\infty} \frac{\sqrt{2n-1}\,\ln(4n+1)}{n(n+1)} \qquad \text{is also convergent.}$$

Example 3.2 *Check if the given series is convergent or divergent,*

$$\sum_{n=2}^{\infty} \frac{\operatorname{Arctan} n}{n \ln n}.$$

In this case we compare with a series which according to the *integral criterion* is divergent.

Now, $\operatorname{Arctan} n \ge \operatorname{Arctan} 1 = \frac{\pi}{4}$, thus

$$a_n = \frac{\operatorname{Arctan} n}{n \ln n} \ge \frac{\pi}{4} \cdot \frac{1}{n \ln n} = b_n \qquad \text{for } n \ge 2.$$

Since $f(x) = x \ln x$ tends increasingly towards ∞, it follows that $\dfrac{1}{n \ln n}$ tends decreasingly towards 0. Since

$$\int_2^t \frac{dx}{x \ln x} = [\ln(\ln x)]_2^t = \ln(\ln t) - \ln(\ln 2) \to \infty \quad \text{for } t \to \infty,$$

it follows by the *integral criterion* that the series

$$\sum_{n=2}^{\infty} b_n = \frac{\pi}{4} \sum_{n=2}^{\infty} \frac{1}{n \ln n} \qquad \text{is divergent.}$$

By the *criterion of comparison* the larger series

$$\sum_{n=2}^{\infty} \frac{\text{Arctan } n}{n \ln n} \qquad \text{is also divergent.}$$

Example 3.3 *Prove the inequalities*

$$\frac{\pi}{4} < \sum_{n=1}^{\infty} \frac{1}{n^2+1} < \frac{1}{2} + \frac{\pi}{4}.$$

We shall use the **integral criterion**.

1) **Identification of the function**. Clearly, we shall choose

$$f(x) = \frac{1}{x^2+1} \qquad \text{for } x \in [1, \infty[.$$

2) **Assumptions**. Obviously, $f(x) = \dfrac{1}{x^2+1}$ tends *decreasingly towards* 0 for $x \to \infty$ in $[1, \infty[$.

3) By the *integral criterion*, $\sum_{n=1}^{\infty} f(n)$ and $\int_1^{\infty} f(x)\,dx$ are both convergent (or divergent) at the same time. We get in case of convergence

$$\int_1^{\infty} f(x)\,dx < \sum_{n=1}^{\infty} \frac{1}{n^2+1} < f(1) + \int_1^{\infty} f(x)\,dx.$$

4) When we *calculate the integral* we get

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{dx}{1+x^2} = [\text{Arctan } x]_1^{\infty} = \frac{\pi}{4}.$$

Therefore we have convergence.

5) Since $f(1) = \dfrac{1}{1^2+1} = \dfrac{1}{2}$, we get by insertion into the estimates of (3) that

$$\frac{\pi}{4} < \sum_{n=1}^{\infty} \frac{1}{n^2+1} < \frac{1}{2} + \frac{\pi}{4}.$$

Example 3.4 . *Prove the inequalities*

$$\frac{1}{8} < \sum_{n=2}^{\infty} \frac{1}{n^3} < \frac{1}{4}.$$

As a rule of thumb we shall only go through harder estimates by either *Leibniz's criterion* or by the *integral criterion*. This series is *not* alternating, so n Leibniz's criterion cannot be used.

Instead we shall try the *integral criterion*.

1) **Identification of the function**. Obviously, we shall choose

$$f(x) = \frac{1}{x^3} \qquad \text{for } x \in [2, \infty[.$$

2) **Assumptions**. Clearly, $f(x) = \dfrac{1}{x^3}$ is a) decreasing on $[2, \infty[$, and b) tends towards 0 for $x \to \infty$.

3) Then by the *integral criterion*,

$$\int_2^{\infty} f(x)\,dx \leq \sum_{n=2}^{\infty} \frac{1}{n^3} \leq f(2) + \int_2^{\infty} f(x)\,dx.$$

4) **Calculation of the integral** gives

$$\int_2^{\infty} f(x)\,dx = \int_2^{\infty} \frac{1}{x^3}\,dx = \left[-\frac{1}{2}\cdot\frac{1}{x^2}\right]_2^{\infty} = \frac{1}{8}.$$

5) Since $f(2) = \frac{1}{2^3} = \frac{1}{8}$, we get by insertion into the estimates of (3) that

$$\frac{1}{8} < \sum_{N=2}^{\infty} \frac{1}{N^3} < \frac{1}{8} + \frac{1}{8} = \frac{1}{4}.$$

Example 3.5 *Prove that the series*

$$\sum_{n=1}^{\infty} \ln\left(\frac{\sqrt{1+n^2}}{n}\right)$$

is convergent, and that its sum is smaller that $\frac{\pi}{4}$.

It follows immediately by the rearrangement

$$f(x) = \ln\left(\frac{\sqrt{1+x^2}}{x}\right) = \frac{1}{2}\ln\left(\frac{1+x^2}{x^2}\right) = \frac{1}{2}\ln\left(1+\frac{1}{x^2}\right), \quad x \in [1,\infty[,$$

that $f(x)$ is decreasing and that $f(x) \to 0$ for $x \to \infty$.

Then we get by partial integration,

$$\begin{aligned}
\int f(x)\,dx &= \frac{1}{2}\int 1\cdot\ln(1+x^2) - \int 1\cdot \ln x\,dx \\
&= \frac{1}{2}\left\{x\cdot\ln(1+x^2) - \int \frac{2x^2}{1+x^2}\,dx\right\} - \left\{x\cdot\ln x - \int \frac{x}{x}\,dx\right\} \\
&= \frac{1}{2}x\left\{\ln(1+x^2) - 2\ln x\right\} - \int\left(1 - \frac{1}{1+x^2}\right)dx + x \\
&= \frac{1}{2}x\ln\left(1+\frac{1}{x^2}\right) + \text{ Arctan } x.
\end{aligned}$$

The estimate $\ln(1+y) \le y$ for $y > -1$ follows easily from the graph. Then by putting $y = 1/x^2$,

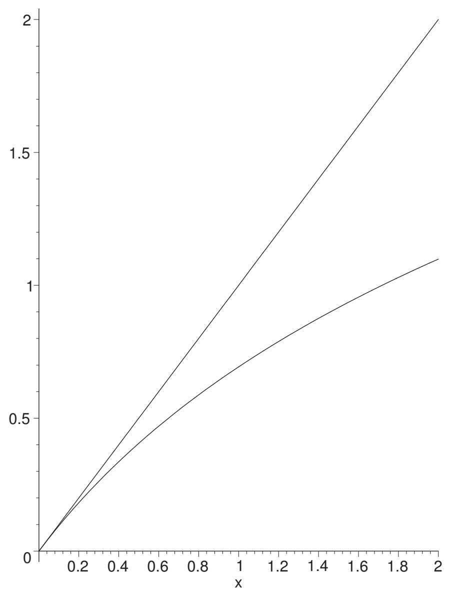
$$0 < x\ln\left(1+\frac{1}{x^2}\right) \le x\cdot\frac{1}{x^2} = \frac{1}{x} \to 0 \quad \text{for } x \to \infty.$$

This implies that the improper integral is convergent

$$\begin{aligned}
\int_1^{\infty} f(x)\,dx &= \lim_{x\to\infty}\left\{\frac{1}{2}x\ln\left(1+\frac{1}{x^2}\right) + \text{ Arctan } x\right\} - \frac{1}{2}\cdot 1\cdot\ln(1+1) - \text{ Arctan } 1 \\
&= \frac{\pi}{2} - \frac{1}{2}\ln 2 - \frac{\pi}{4} = \frac{\pi}{4} - \frac{1}{2}\ln 2.
\end{aligned}$$

From the *integral criterion* follows that the series

$$\sum_{n=1}^{\infty} \ln\left(\frac{\sqrt{1+n^2}}{n}\right) = \sum_{n=1}^{\infty} f(n)$$



is convergent and that we have the estimate

$$\sum_{n=1}^{\infty} \ln\left(\frac{\sqrt{1+n^2}}{n}\right) < f(1) + \int_1^{\infty} f(x)\,dx = \frac{1}{2}\ln 2 + \frac{\pi}{4} - \frac{1}{2}\ln 2 = \frac{\pi}{4}.$$

Example 3.6 *Check for each of the following series if it is convergent or divergent,*

$$(1)\quad \sum_{n=1}^{\infty}(n-\sqrt{n^2-1}), \qquad (2)\quad \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}(n-\sqrt{n^2-1}).$$

Notice that we have the well-known result $(n-\sqrt{n^2-1})(n+\sqrt{n^2-1}) = 1$.

It is obvious that

$$(n-\sqrt{n^2-1})(n+\sqrt{n^2-1}) = n^2-(n^2-1) = 1.$$

Since $0 \le \sqrt{n^2-1} < n$, it follows in particular that

$$(2)\quad 0 < n-\sqrt{n^2-1} = \frac{1}{n+\sqrt{n^2-1}} \begin{cases} \le \dfrac{1}{n}, \\ > \dfrac{1}{2n}, \end{cases} \qquad \text{for every } n \in \mathbb{N}.$$

1) It follows from (2) that $n-\sqrt{n^2-1} > \dfrac{1}{2n}$. Since $\sum_{n=1}^{\infty}\dfrac{1}{2n}$ is divergent, we conclude from the *criterion of comparison* that $\sum_{n=1}^{\infty}(n-\sqrt{n^2-1})$ is divergent.

2) It follows from (2) that

$$0 < \frac{1}{\sqrt{n}}(n - \sqrt{n^2-1}) \le \frac{1}{n^{3/2}}.$$

Since $\sum_{n=1}^{\infty} \dfrac{1}{n^{3/2}}$ is convergent by the *integral criterion*, because $\left(\alpha = \dfrac{3}{2} > 1\right)$, it follows from the *criterion of comparison* that the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}(n - \sqrt{n^2-1})$$

is convergent.

Example 3.7 *Let α be a real number bigger than 1. Prove that*

$$\sum_{n=k}^{\infty} \frac{1}{n^{\alpha}} < \frac{\alpha}{\alpha-1} \cdot \frac{1}{k^{\alpha-1}} \qquad \text{for } k \in \mathbb{N}.$$

Hint: First use the integral criterion to prove that

$$\sum_{n=k}^{\infty} \frac{1}{n^{\alpha}} < \frac{1}{k^{\alpha}} + \frac{1}{(\alpha-1)k^{\alpha-1}} \qquad \text{for } k \in \mathbb{N}.$$

It is obvious that when $\alpha > 1$, then $f(t) = \dfrac{1}{t^{\alpha}}$ is decreasing for $t > 0$ with the limit value. By considering an area we get

$$\begin{aligned}
\sum_{n=k}^{\infty} \frac{1}{n^{\alpha}} &< \frac{1}{k^{\alpha}} + \int_k^{\infty} \frac{1}{t^{\alpha}}\,dt = \frac{1}{k^{\alpha}} + \left[\frac{t^{1-\alpha}}{1-\alpha}\right]_k^{\infty} \\
&= \frac{1}{k^{\alpha}} + \frac{1}{(\alpha-1)k^{\alpha-1}} = \frac{\alpha-1+k}{(\alpha-1)k^{\alpha}} \\
&= \frac{k\alpha-(k-1)(\alpha-1)}{(\alpha-1)k^{\alpha}} \le \frac{\alpha}{\alpha-1} \cdot \frac{1}{k^{\alpha-1}},
\end{aligned}$$

because $k \ge 1$ and $\alpha > 1$.

4 Small theoretical examples

Example 4.1 *Let* $\sum_{n=1}^{\infty} a_n$ *be an infinite series of positive terms. Prove or disprove the following claim:*

- *If* $\sum_{n=1}^{\infty} a_n$ *is divergent, then* $\sum_{n=1}^{\infty} a_n^2$ *is also divergent.*

This claim is wrong. In fact, if we choose $a_n = \dfrac{1}{n}$, then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n} \qquad \text{is divergent,}$$

and

$$\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \text{is convergent.}$$

We know in general that if $a_n = \dfrac{1}{n^\alpha}$, then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^\alpha} \quad \text{is divergent for } \alpha \le 1,$$

and

$$\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{\infty} \frac{1}{n^{2\alpha}} \quad \text{is convergent for } 2\alpha > 1.$$

We see that we get counterexamples of the claim, if only

$$\frac{1}{2} < \alpha \le 1.$$

Example 4.2 *Let $\sum_{n=1}^{\infty} a_n$ be an infinite series of positive terms. Prove or disprove the following claim:*

- *If $\sum_{n=1}^{\infty} a_n^2$ is convergent, then $\sum_{n=1}^{\infty} \dfrac{a_n}{n}$ is also convergent.*

The claim is true, which is proved by a small trick. It follows from

$$0 \le \left(a_n - \frac{1}{n}\right)^2 = a_n^2 + \frac{1}{n^2} - 2\frac{a_n}{n}$$

by a rearrangement that

$$0 < \frac{a_n}{n} \le \frac{1}{2}a_n^2 + \frac{1}{2}\frac{1}{n^2} = b_n.$$

By the assumption, the larger series is convergent,

$$\sum_{n=1}^{\infty} b_n = \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2}$$

as a sum of two convergent series of positive terms. Using the *criterion of comparison* the smaller series

$$\sum_{n=1}^{\infty} \frac{a_n}{n} \qquad \text{is convergent.}$$

Remark 4.1 By a modification of the proof above it follows that

- If $\sum_{n=1}^{\infty} a_n^2$ is convergent, then $\sum_{n=1}^{\infty} \dfrac{a_n}{n^\alpha}$ is convergent for every $\alpha > \dfrac{1}{2}$.

5 Conditional convergence and Leibniz's criterion

Example 5.1 *Prove that the following series is conditionally convergent*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln \cosh n}.$$

Since $\ln \cosh n > 0$, and $\ln \cosh n$ tends increasingly towards $+\infty$, we see that

$$\frac{1}{\ln \cosh n} \to 0 \qquad \textit{decreasingly} \text{ for } n \to \infty.$$

In particular, the *necessary condition* for convergence is fulfilled.

Absolute convergence? Using the definition of cosh, we get the following estimate for every $n \in \mathbb{N}$,

$$0 < \ln \cosh n = \ln\left(\frac{e^n + e^{-n}}{2}\right) < \ln\left(\frac{e^n + e^{+n}}{2}\right) = n,$$

hence

$$b_n = \frac{1}{\ln \cosh n} > \frac{1}{n} = a_n.$$

The smaller series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (the harmonic series), hence the larger series

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{\ln \cosh n}$$

is also divergent according to the *criterion of comparison.* This proves that the series is *not* er absolutely convergent.

Conditional convergence? The series is *alternating,* $\sum_{n=1}^{\infty} (-1)^n b_n$, where $b_n = \dfrac{1}{\ln \cosh n} \to 0$ is *decreasing* for $n \to \infty$. It follows from **Leibniz's criterion** that the series is convergent.

Since the series is convergent, though not absolutely convergent, it must be conditionally convergent.

Example 5.2 *Prove that the following series is conditionally convergent,*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln \sinh n}.$$

Since $\ln \sinh n > 0$ for $n \in \mathbb{N}$, and $\ln \sinh n$ increases to $+\infty$, it follows that

$$\frac{1}{\ln \sinh n} \to 0 \qquad \textit{is decreasing} \text{ for } n \to \infty.$$

In particular, the **necessary condition** for convergence is fulfilled.

Absolute convergence? By using the definition of sinh we obtain the following estimate for every $n \in \mathbb{N}$

$$0 < \ln \sinh n = \ln \left(\frac{e^n - e^{-n}}{2} \right) < \ln \left(\frac{e^n + e^n}{2} \right) = n,$$

hence

$$b_n = \frac{1}{\ln \sinh n} > \frac{1}{n} = a_n.$$

The smaller series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (the harmonic series), so the larger series

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{\ln \sinh n}$$

is divergent according to the **criterion of comparison**. This shows that the original series cannot be absolutely convergent.

Conditional convergence? The series is *alternating*, $\sum_{n=1}^{\infty} (-1)^n b_n$, where $b_n = \frac{1}{\ln \sinh n} \to 0$ *decreases* for $n \to \infty$. Then it follows from **Leibniz's criterion** that the series is convergent.

As the series is convergent, though not absolutely convergent, it is conditionally convergent.

Example 5.3 *Check if the given series is absolutely convergent, conditionally convergent or divergent,*

$$\sum_{n=1}^{\infty}(-1)^n \cos\left(\frac{2\pi}{n}\right).$$

The necessary condition for convergence? From

$$|a_n| = \left|\cos\left(\frac{2\pi}{n}\right)\right| \to \cos 0 = 1 \neq 0 \qquad \text{for } n \to \infty,$$

follows that the necessary condition of convergence is *not* fulfilled, hence the series is (coarsely) divergent.

Example 5.4 *Check if the given series is absolutely convergent, conditionally convergent or divergent,*

$$\sum_{n=1}^{\infty}(-1)^n \sin\left(\frac{2\pi}{n}\right).$$

The necessary condition for convergence? As $\sin\left(\dfrac{2\pi}{n}\right) \to 0$ for $n \to \infty$, this condition is satisfied.

Absolute convergence? We see that

$$|a_n| = \left|\sin\left(\frac{2\pi}{n}\right)\right| = \sin\left(\frac{2\pi}{n}\right) \qquad \text{for } n \geq 4$$

where $\dfrac{2\pi}{n} \in \left]0, \dfrac{\pi}{0}\right]$ for $n \geq 4$. Hence, by a consideration of a graph we get for $n \geq 4$ that

$$|a_n| = \sin\left(\frac{2\pi}{n}\right) \geq \frac{2}{\pi} \cdot \frac{2\pi}{n} = \frac{4}{n}.$$

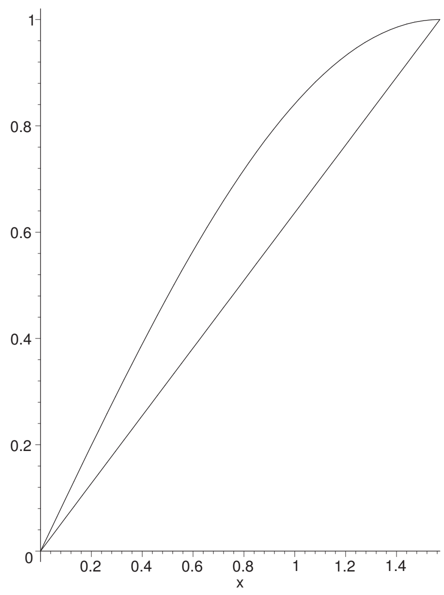
The smaller series $\sum_{n=4}^{\infty} \dfrac{4}{n} = 4\sum_{n=4}^{\infty} \dfrac{1}{n}$ is divergent, hence the larger series $\sum_{n=4}^{\infty} |a_n|$ is also divergent according to the **criterion of comparison**, and the series is *not* absolutely convergent.

Conditional convergence? As $|a_n| = \sin\left(\dfrac{2\pi}{n}\right)$ for $n \geq 4$ *decreases* towards 0, and the series is *alternating* for $n \geq 4$,

$$\sum_{n=4}^{\infty}(-1)^n \sin\left(\frac{2\pi}{n}\right) = \sum_{n=4}^{\infty}(-1)^n |a_n|,$$

it follows from **Leibniz's criterion** that the series is convergent.

As the series is convergent, though not absolutely convergent, it is conditionally convergent.



Example 5.5 *Check if the given series is absolutely convergent, conditionally convergent or divergent,*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sinh n}.$$

Necessary condition for convergence? From $\sinh n \to \infty$ for $n \to \infty$, follows that

$$\frac{(-1)^n}{\sinh n} \to 0 \qquad \text{for } n \to \infty,$$

and the necessary condition is fulfilled.

Absolute convergence? We have e.g. that

$$\sinh n = \frac{e^n - e^{-n}}{2} > \frac{e^n}{4}, \qquad \text{implies} \qquad 0 < \frac{1}{\sinh n} < \frac{2}{e^n}.$$

The larger series

$$\sum_{n=1}^{\infty} \frac{4}{e^n} = 4 \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$$

is a quotient series of quotient $1/e \in]0,1[$, hence convergent.

According to the **criterion of comparison** the smaller series

$$\sum_{n=1}^{\infty} \frac{1}{\sinh n} \quad \text{is convergent.}$$

We conclude that the original series is *absolutely convergent*.

Example 5.6 *Check if the following series is absolutely convergent, conditionally convergent or divergent,*

$$\sum_{n=1}^{\infty}(-1)^n \cosh\left(\frac{1}{n}\right).$$

Necessary condition for convergence? It follows from

$$|a_n| = \cosh\left(\frac{1}{n}\right) \to \cosh 0 = 1 \neq 0 \qquad \text{for } n \to \infty,$$

that the necessary condition for convergence is *not* fulfilled, so the series is (coarsely) divergent.

Example 5.7 *Check if the following series is absolutely convergent, conditionally convergent or divergent,*

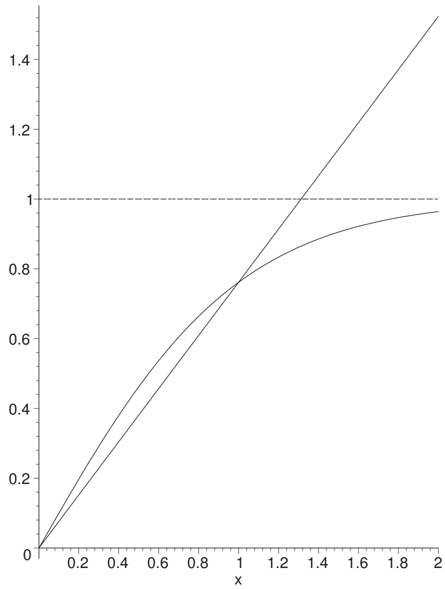
$$\sum_{n=1}^{\infty}(-1)^n \tanh\left(\frac{1}{n}\right).$$

Necessary condition for convergence? As $\tanh\left(\frac{1}{n}\right) \to 0$ for $n \to \infty$, this condition is satisfied.

Absolute convergence? The graph of $\tanh x$ is concave (cf. the figure), hence

$$0 < \tanh(1)\cdot x < \tanh(x) \qquad \text{for } x \in]0,1[.$$

When $x = 1/n$, we get the estimate



$$\tanh\left(\frac{1}{n}\right) > \tanh(1)\cdot\frac{1}{n} > 0.$$

Since the smaller series

$$\sum_{n=1}^{\infty} \tanh(1) \cdot \frac{1}{n} = \tanh(1) \sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent (the harmonic series), the larger series $\sum_{n=1}^{\infty} \tanh\left(\frac{1}{n}\right)$ is also divergent by the **criterion of comparison**. This shows that the original series is *not* absolutely convergent.

Conditional convergence? As $\tanh(x)$ is increasing, we get

1) $\tanh\left(\frac{1}{n}\right)$ decreasing for $n \to \infty$.

2) $\tanh\left(\frac{1}{n}\right) \to 0$ for $n \to \infty$ is proved above.

3) $\sum_{n=1}^{\infty}(-1)^n \tanh\left(\frac{1}{n}\right)$ is due to the factor $(-1)^n$ *alternating*.

The series is convergent according to **Leibniz's criterion**.

Since the series is convergent, though not absolutely convergent, it is *conditionally convergent*.

Example 5.8 *Check if the following series is absolutely convergent, conditionally convergent or divergent,*

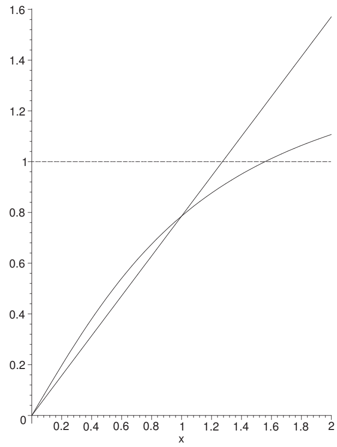
$$\sum_{n=1}^{\infty}(-1)^n \text{ Arctan}\left(\frac{1}{n}\right).$$

Necessary condition for convergence? Because of $\text{Arctan}\left(\frac{1}{n}\right) \to 0$ for $n \to \infty$, this condition is fulfilled.

Absolute convergence? The graph of Arctan x is concave (cf. the figure), hence

$$0 < \frac{\pi}{4}\,x \leq \text{ Arctan } x \qquad \text{for } x \in \,]0,1].$$

Putting $x = 1/n$ this gives the estimate



$$\text{Arctan}\left(\frac{1}{n}\right) \geq \frac{\pi}{4}\cdot\frac{1}{n} \qquad \text{for } n \in \mathbb{N}.$$

The smaller series

$$\sum_{n=1}^{\infty}\frac{\pi}{4}\cdot\frac{1}{n} = \frac{\pi}{4}\sum_{n=1}^{\infty}\frac{1}{n}$$

is divergent (the harmonic series), hence the larger series $\sum_{n=1}^{\infty}$ Arctan $\left(\frac{1}{n}\right)$ is also divergent according to the **criterion of comparison**, and the series is *not* absolutely convergent.

Conditional convergence? Since $\text{Arctan}\left(\frac{1}{n}\right) \to 0$ is *decreasing* for $n \to \infty$, and the series $\sum_{n=1}^{\infty}(-1)^n$ Arctan $\left(\frac{1}{n}\right)$ is *alternating*, it follows form **Leibniz's criterion** that the series is convergent.

As the series is convergent, though not absolutely convergent, it is *conditionally convergent*.

Example 5.9 *Check if the following series is absolutely convergent, conditionally convergent or divergent,*

$$\sum_{n=1}^{\infty}(-1)^n \text{ Arccot}\left(\frac{1}{n}\right).$$

Necessary condition for convergence? Since

$$|a_n| = \text{ Arccot}\left(\frac{1}{n}\right) \to \frac{\pi}{2} \neq 0 \qquad \text{for } n \to \infty,$$

the **necessary condition** for convergence is *not* fulfilled, and the series is (coarsely) divergent.

Example 5.10 *Check if the series*

$$\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{\sqrt{n}}{1+n}$$

is absolutely convergent, conditionally convergent or divergent.

Use the flow diagram in *Calculus 3b*.

1) **Is the series coarsely divergent?** It follows from

$$|a_n| = \frac{\sqrt{n}}{1+n} = \frac{1}{\sqrt{n}+\dfrac{1}{\sqrt{n}}} \to 0 \quad \text{for } n \to \infty,$$

that the series is *not* coarsely divergent.

2) **Is the series absolutely convergent?** Since

$$|a_n| = \frac{\sqrt{n}}{n+1} \geq \frac{1}{n+1},$$

we get (the harmonic series)

$$\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{\sqrt{n}}{n+1} \geq \sum_{n=1}^{\infty}\frac{1}{n+1} = \sum_{n=2}^{\infty}\frac{1}{n} = \infty.$$

Then it follows form the **criterion of comparison** that the series is *not* absolutely convergent.

3) **Is the series conditionally convergent?** Obviously, the series is *alternating*. The auxiliary function is

$$f(x) = \frac{\sqrt{x}}{x+1}, \quad x \in [1,\infty[, \quad \text{where } f(x) \to 0 \quad \text{for } x \to \infty.$$

Since

$$f'(x) = \frac{1}{2\sqrt{x}}\cdot\frac{1}{x+1} - \frac{\sqrt{x}}{(x+1)^2} = \frac{x+1-2x}{2\sqrt{x}\cdot(x+1)^2} = -\frac{x-1}{2\sqrt{x}\cdot(x+1)^2} < 0$$

for $x > 1$, we see that $f(x)$ *decreases* for $x > 1$.

According to **Leibniz's criterion** the series is convergent.

As the series is convergent, though not absolutely convergent, it is *conditionally convergent*.

Example 5.11 *Check if the series*

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{2n - \sqrt{n} \cdot \ln n}$$

is absolutely convergent, conditionally convergent or divergent.

Necessary condition of convergence? It follows from $2n - \sqrt{n} \cdot \ln n \neq 0$ and

$$2n - \sqrt{n} \cdot \ln n = \sqrt{n}\{2\sqrt{n} - \ln n\} = n\left\{2 - \frac{\ln n}{\sqrt{n}}\right\} \to \infty \quad \text{for } n \to \infty,$$

that

$$\frac{(-1)^n}{2n - \sqrt{n} \cdot \ln n} \to \infty \qquad \text{for } n \to \infty,$$

hence the necessary condition of convergence is fulfilled.

Absolute convergence? As $\ln n/\sqrt{n} \to 0$ for $n \to \infty$, we have

$$0 < 2n - \sqrt{n} \cdot \ln n = n\left\{2 - \frac{\ln n}{\sqrt{n}}\right\} < 2n \qquad \text{for } n \geq 2.$$

Hence

$$\frac{1}{2n - \sqrt{n} \cdot \ln n} > \frac{1}{2} \cdot \frac{1}{n}.$$

The smaller series $\sum_{n=2}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n}$ (the harmonic series) is divergent, so the larger series

$$\sum_{n=2}^{\infty} \frac{1}{2n - \sqrt{n} \cdot \ln n}$$

is also divergent according to the **criterion of comparison**. Thus the series is *not* absolutely convergent.

Conditional convergence? Two of the conditions of Leibniz's criterion have already been proved, namely that the series is alternating and $|a_n| \to 0$ for $n \to \infty$. We shall show that the sequence is decreasing eventually. We introduce the auxiliary function $(t \sim \sqrt{n})$

$$\varphi(t) = 2t^2 - t \cdot \ln(t^2) = 2t^2 - 2t \ln t, \qquad t \geq \sqrt{2}.$$

Now

$$\varphi'(t) = 4t - 2\ln t - 2 = 2(2t - \ln t - 1) > 0 \qquad \text{for } t \geq \sqrt{2},$$

so $\varphi(t)$ is increasing, and

$$|a_n| = \frac{1}{\varphi(\sqrt{n})} = \frac{1}{2n - \sqrt{n} \cdot \ln n} \quad \text{is decreasing for } n \to \infty.$$

Then it follows from **Leibniz's criterion** that the series is convergent.

As the series is convergent, though not absolutely convergent, it is *conditionally convergent*.

Example 5.12 *Check if the series*

$$\sum_{n=1}^{\infty}(-1)^n \frac{\ln n}{n+1}$$

is absolutely convergent, conditionally convergent or divergent.

Apply the flow diagram from *Calculus 3b*.

1) **Is the series coarsely divergent?** We have

$$|a_n| = \frac{\ln n}{n+1} \to 0 \qquad \text{for } n \to \infty$$

by the law of magnitudes, so the series is *not* coarsely divergent.

2) **Is the series absolutely convergent?** Since $\ln n > 1$ for $n \geq 3$, we get the following estimate of the numerical series.

$$\sum_{n=1}^{\infty} \frac{\ln n}{n+1} \geq \sum_{n=3}^{\infty} \frac{1}{n+1} = \sum_{n=4}^{\infty} \frac{1}{n} = \infty \quad \text{(the harmonic series).}$$

It follows from the **criterion of comparison** that the series is *not* absolutely convergent.

3) **Is the series conditionally convergent?** There are several criteria in the literature for conditional convergence, but at this stage one may assume that *Leibniz's criterion* is the only known one to most readers.

   a) As $\ln n/(n+1) \geq 0$, the factor $(-1)^n$ shows that the series is *alternating*.

   b) If we put $f(x) = \dfrac{\ln x}{x+1}$, it follows from 1. that $f(x) \to 0$ for $x \to \infty$.

   c) It follows from

$$f'(x) = \frac{1}{x(x+1)} - \frac{\ln x}{(x+1)^2} = \frac{x+1-x\ln x}{x(x+1)^2}$$

   that $f'(x) < 0$, at least for $x \geq 4$, thus $f(x)$ is decreasing eventually.

   Then it follows from **Leibniz's criterion** that the series is convergent.

   As the series is convergent, though not absolutely convergent, it is *conditionally convergent.*

Example 5.13 *Check if the series*

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{3^{n-1}}$$

is convergent or divergent. In case of convergence, check if the series is conditionally convergent or absolutely convergent.

We have from the magnitudes,

$$0 < |a_n| = 3 \cdot \frac{n}{3^n} \leq \frac{1}{2^n} \qquad \text{for } n \geq n_0.$$

The larger series $\sum 2^{-n}$ is convergent, hence the smaller series is also convergent by the **criterion of comparison**, and the series is *absolutely convergent.*

Alternatively the convergence is obtained by the **criterion of roots**,

$$\sqrt[n]{|a_n|} = \sqrt[n]{3} \cdot \sqrt[n]{n} \cdot \frac{1}{3} \to \frac{1}{3} < 1 \qquad \text{for } n \to \infty.$$

Alternatively the convergence is obtained by the **criterion of quotients**,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{3(n+1)}{3^{n+1}} \cdot \frac{3^n}{3n} = \frac{n+1}{n} \cdot \frac{1}{3} \to \frac{1}{3} < 1 \qquad \text{for } n \to \infty.$$

Remark 5.1 It is possible directly to find the sum. In fact,

$$\varphi(x) := \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad \text{for } x \in]-1,1[,$$

hence by differentiation each term (which is legal for power series in their open interval of convergence),

$$\varphi'(x) = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} \qquad \text{for } x \in]-1,1[.$$

Choosing $x = -\frac{1}{3} \in]-1,1[$, we get

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3^{n-1}} = \sum_{n=1}^{\infty} n\left(-\frac{1}{3}\right)^{n-1} = \varphi'\left(-\frac{1}{3}\right) = \frac{1}{\left(1+\frac{1}{3}\right)^2} = \frac{9}{16}.$$

Example 5.14 *Check if the series*

$$\sum_{n=10}^{\infty} \frac{(-1)^n}{n-3\sqrt{n}}$$

is convergent or divergent. In case of convergence, check if the series is conditionally convergent or absolutely convergent.

Necessary condition for convergence? Since $n - 3\sqrt{n} > 0$ for $n \geq 10$, and

$$\frac{1}{n-3\sqrt{n}} = \frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{n}-3} \to 0 \qquad \text{for } n \to \infty,$$

this condition is fulfilled.

Absolute convergence? Since

$$\frac{1}{n-3\sqrt{n}} > \frac{1}{n},$$

and $\sum_{n=10}^{\infty} \dfrac{1}{n}$ is divergent, it follows from the **criterion of comparison** that the numerical series

$$\sum_{n=10}^{\infty} \frac{1}{n-3\sqrt{n}} \qquad \text{is divergent,}$$

and the series is *not* absolutely convergent.

Conditional convergence? The series is alternating, and $a_n \to 0$ for $n \to \infty$. Thus we shall only prove that

$$(3) \quad \frac{1}{n-3\sqrt{n}}$$

is *decreasing* in order to apply Leibniz's criterion. However, the denominator

$$n - 3\sqrt{n} = \sqrt{n} \cdot (\sqrt{n} - 3)$$

is clearly increasing, so (3) is decreasing. Hence it follows from **Leibniz's criterium** that the series is convergent.

As the series is convergent, though not absolutely convergent, it is *conditional convergent.*

Example 5.15 *Check if the series*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

is convergent or divergent. In case of convergence, check if the series is conditionally convergent or absolutely convergent.

Necessary condition for convergence Since $1/\sqrt{n} \to 0$ for $n \to \infty$, this condition is fulfilled.

Absolute convergence? Since $\dfrac{1}{\sqrt{n}} \geq \dfrac{1}{n}$ for every $n \in \mathbb{N}$, and since the harmonic series $\sum_{n=1}^{\infty} \dfrac{1}{n}$ is divergent, it follows from the **criterion of comparison** that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

is also divergent. This proves that the series is *not* absolutely convergent.

Conditional convergence? The series is alternating, and $\dfrac{1}{\sqrt{n}} \to 0$ is *decreasing* for $n \to \infty$. Hence, it follows from **Leibniz's criterion** that the series is convergent.

As the series is convergent, though not absolutely convergent, it is *conditionally convergent.*

Example 5.16 *Check if the series*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(e^n + e^{-n})}$$

is convergent or divergent. In case of convergence, check if the series is conditionally convergent or absolutely convergent.

Remark 5.2 Since $\ln(e^n + e^{-n}) = \ln(2\cosh n)$, this example is almost identical with Example 5.1.

Necessary condition for convergence? Since

$$0 < \frac{1}{\ln(e^n + e^{-n})} < \frac{1}{\ln(e^n + 0)} = \frac{1}{n} \to 0 \qquad \text{for } n \to \infty,$$

this condition is satisfied.

Absolute convergence? It follows from

$$|a_n| = \frac{1}{\ln(e^n + e^{-n})} \geq \frac{1}{\ln(e^n + e^n)} = \frac{1}{n + \ln 2} \geq \frac{1}{2n},$$

where $\sum \frac{1}{2n}$ is divergent, and the **criterion of comparison** that $\sum |a_n|$ is divergent. Hence the series is *not* absolutely convergent.

Conditional convergence? The series is alternating and $|a_n| \to 0$ for $n \to \infty$. Since

$$\ln(e^n + e^{-n}) = \ln(2\cosh n) \to \infty \qquad \text{is increasing for } n \to \infty,$$

because both ln and cosh are increasing, we have

$$\frac{1}{\ln(e^n + e^{-n})} \to 0 \quad \text{decreasingly}.$$

It follows from **Leibniz's criterion** that the series is convergent.

As the series is convergent, though not absolutely convergent, it is *conditionally convergent.*

Example 5.17 *Check if the series*

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$$

is convergent or divergent. In case of convergence, check if the series is conditionally convergent or absolutely convergent.

Note that $\ln n > 0$ for every $n \geq 2$.

Necessary condition for convergence? This is fulfilled, because

$$\frac{1}{\ln n} \to 0 \qquad \text{for } n \to \infty.$$

Absolute convergence? We see that

$$\frac{1}{\ln n} > \frac{1}{n},$$

and since $\sum_{n=2}^{\infty} \frac{1}{n}$ is divergent, it follows from the **criterion of comparison** that $\sum_{n=2}^{\infty} \dfrac{1}{\ln n}$ is also divergent. This proves that the series is *not* absolutely convergent.

Conditional convergence? The series is *alternating*, and

$$\frac{1}{\ln n} \to 0 \qquad \textit{aftagende} \text{ for } n \to \infty.$$

Hence the series is convergent according to **Leibniz's criterion**.

As the series is convergent, though not absolutely convergent, it is *conditionally convergent.*

Example 5.18 *Check if the series*

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$$

is convergent or divergent. In case of convergence, check if the series is conditionally convergent or absolutely convergent.

Apply the flow diagram.

1) **Is the series coarsely divergent?** Since $n(\ln n)^2 \to \infty$, it is obvious that

$$a_n = \frac{(-1)^n}{n(\ln n)^2} \to 0 \qquad \text{for } n \to \infty,$$

and the series is *not* coarsely divergent.

2) **Is the series absolutely convergent?** Concerning the numerical series we get the auxiliary function

$$f(x) = \frac{1}{x(\ln x)^2}, \qquad \text{for } x \in [2, \infty[.$$

The denominator tends increasingly towards ∞ (look at the derivative) for $x \to \infty$ in $[2, \infty[$, hence $f(x)$ tends *decreasingly towards* 0 for $x \to \infty$ in the same interval.

According to the **integral criterion**

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \qquad \text{and} \qquad \int_2^{\infty} f(x)\, dx$$

have the same property of convergence. Since the integral

$$\int_2^{\infty} f(x)\, dx = \int_2^{\infty} \frac{dx}{x(\ln x)^2} = \left[-\frac{1}{\ln x}\right]_2^{\infty} = \frac{1}{\ln 2}$$

is convergent, the series is *absolutely convergent*, where we apply the **integral criterion**.

Remark 5.3 One should here not be misled by the changing of sign $(-1)^n$ and immediately start with Leibniz's criterion. *This is a waste of time!* Leibniz's criterion will only show the convergence, so anyway one shall afterwards check the absolute convergens.

Example 5.19 *Check if the series*

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{1+n^2}$$

is convergent or divergent. In case of convergence, check if it is conditionally or absolutely convergent.

Necessary condition for convergence? We see that

$$|a_n| = \frac{n^2}{1+n^2} = 1 - \frac{1}{1+n^2} \to 1 \neq 0 \qquad \text{for } n \to \infty,$$

so the **necessary condition** for convergence is *not* fulfilled, and the series is (coarsely) divergent.

Example 5.20 *Check if the series*

$$\sum_{n=1}^{\infty} (-1)^n \left\{1 - \cos\left(\frac{1}{n}\right)\right\}$$

is convergent or divergent. In case of convergence, check if the series is conditionally or absolutely convergent.

Necessary condition for convergence? As

$$1 - \cos\left(\frac{1}{n}\right) \to 1 - \cos 0 = 1 - 1 = 0 \qquad \text{for } n \to \infty,$$

this condition is fulfilled.

Absolute convergence? Since $|a_n| = 1 - \cos\frac{1}{n}$, we get by a Taylor expansion

$$0 < 1 - \cos\frac{1}{n} = 1 - \left\{1 - \frac{1}{2}\frac{1}{n^2} + \frac{1}{n^2}\,\varepsilon\left(\frac{1}{n}\right)\right\} = \frac{1}{2} \cdot \frac{1}{n^2} + \frac{1}{n^2}\,\varepsilon\left(\frac{1}{n}\right) \sim \frac{1}{2} \cdot \frac{1}{n^2}.$$

Now

$$\sum_{n=1}^{\infty} \frac{1}{2} \cdot \frac{1}{n^2}$$

is convergent, so by the **criterion of equivalence**,

$$\sum_{n=1}^{\infty} \left\{1 - \cos \frac{1}{n}\right\} \qquad \text{is convergent,}$$

and we have proved the *absolute convergence*.

Alternatively,

$$0 < 1 - \cos \frac{1}{n} = 2 \sin^2 \frac{1}{2n} < \frac{1}{4n^2},$$

As $\sum 1/n^2$ is convergent, it follows by the **criterion of comparison** that $\sum\{1 - \cos \frac{1}{n}\}$ is convergent, from which follows that we have *absolute convergence*.

Example 5.21 *Prove that the series below is conditionally convergent and find its sum:*

$$\sum_{n=}^{\infty} \frac{(-1)^n(n+1)}{n(2n+1)}.$$

Necessary condition for convergence. Clearly,

$$|a_n| = \frac{n+1}{n(2n+1)} = \frac{1+\dfrac{1}{n}}{2n+1} \to 0 \qquad \text{for } n \to \infty,$$

so the condition is fulfilled.

Absolute convergence? As

$$|a_n| = \frac{n+1}{n(2n+1)} \geq \frac{n+1}{n(2n+2)} = \frac{1}{2n},$$

and $\sum \frac{1}{2n}$ is divergent, the larger series $\sum |a_n|$ is also divergent, and the series is not absolutely convergent.

Conditional convergence. Since

$$\frac{1}{|a_n|} = \frac{n(2n+1)}{n+1} = \frac{n(2n+2)-n-1+1}{n+1} = 2n-1+\frac{1}{n+1} \to \infty \quad \textit{is increasing},$$

we must have that $|a_n| \to 0$ is *decreasing.* The series is alternating, so it follows from **Leibniz's criterion** that the series is convergent.

As the series is convergent, though not absolutely convergent, it is *conditionally convergent.*

Sum. We get by a decomposition,

$$\frac{n+1}{n(2n+1)} = \frac{1}{n} - \frac{1}{2n+1},$$

hence the sectional sequence becomes

$$s_N = \sum_{n=1}^{N} \frac{(-1)^n(n+1)}{n(2n+1)} = \sum_{n=1}^{N} \frac{(-1)^n}{n} - \sum_{n=1}^{N} \frac{(-1)^n}{2n+1}.$$

Now

$$\sum_{n=1}^{N} \frac{(-1)^n}{n} \to \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\ln(1+1) = -\ln 2,$$

and

$$-\sum_{n=1}^{N} \frac{(-1)^n}{2n+1} \to 1 - \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \text{ Arctan } 1 = 1 - \frac{\pi}{4},$$

where we in both cases apply Abel's theorem, i.e. both series are convergent according to **Leibniz's criterion**, and

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n, \quad \text{Arctan } x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}, \quad |x| < 1.$$

Then it follows by taking the limit that

$$\sum_{n=1}^{\infty} \frac{(-1)^n(n+1)}{n(2n+1)} = \lim_{N\to\infty} s_N = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} - \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{\pi}{4} - \ln 2.$$

Example 5.22 *Prove that the series below is conditionally convergent and find its sum*

$$\sum_{n=1}^{\infty} \frac{(-1)^n(n+1)}{n(n+2)},$$

Necessary condition for convergence? Clearly,

$$0 < |a_n| = \frac{n+1}{n(n+2)} \le \frac{1}{n} \to 0 \qquad \text{for } n \to \infty,$$

so $|a_n| \to 0$ for $n \to \infty$, and the condition is satisfied.

Absolute convergence? We see that

$$|a_n| = \frac{n+1}{n(n+2)} \ge \frac{1}{n+2},$$

and the smaller series $\sum_{n=2}^{\infty} \frac{1}{n+2} = \sum_{n=3}^{\infty} \frac{1}{n}$ is divergent. Hence also $\sum_{n=1}^{\infty} |a_n|$ is divergent, and the series is not absolutely convergent.

Conditional convergence. We see that

$$\frac{1}{|a_n|} = \frac{n(n+2)}{n+1} = n+1 - \frac{1}{n+1} \to \infty \qquad \textit{is increasing},$$

so $|a_n| \to 0$ *decreasingly*. The series is alternating, so it follows from **Leibniz's criterion** that the series is convergent.

As the series is convergent, though not absolutely convergent, it is conditionally convergent.

Sum. We get by a decomposition,

$$\frac{n+1}{n(n+2)} = \frac{1}{2}\,\frac{1}{n} + \frac{1}{2}\,\frac{1}{n+2},$$

hence the sectional sequence becomes

$$\begin{aligned} s_N &= \sum_{n=1}^{N} \frac{(-1)^n(n+1)}{n(n+2)} = \frac{1}{2}\sum_{n=1}^{N} \frac{(-1)^n}{n} + \frac{1}{2}\sum_{n=1}^{N} \frac{(-1)^n}{n+2} = \frac{1}{2}\sum_{n=1}^{N} \frac{(-1)^n}{n} + \frac{1}{2}\sum_{n=3}^{N+2} \frac{(-1)^{n-2}}{n} \\ &= \sum_{n=1}^{N} \frac{(-1)^n}{n} - \frac{1}{2}\left\{-\frac{1}{1} + \frac{1}{2}\right\} + \frac{1}{2}\left\{\frac{(-1)^{N+1}}{N+1} + \frac{(-1)^N}{N+2}\right\},. \end{aligned}$$

By taking the limit and applying a known sum we finally get

$$\sum_{n=1}^{\infty} \frac{(-1)^n(n+1)}{n(n+2)} = \lim_{N\to\infty} s_N = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} + \frac{1}{4} + 0 = \frac{1}{4} - \ln 2.$$

Example 5.23 *Prove that the series*

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} = 1-\frac{1}{2}+\frac{1}{3}+\cdots$$

is convergent. Find an approximation s^* of the sum of the series, such that

$$|s-s^*| < 10^{-1}.$$

It is easily proved by **Leibniz's criterion** that the series is convergent.

Since $|s-s_n| \le \dfrac{1}{n+1}$ (the absolute value of the first neglected term), we can choose $n=9$, thus

$$s^* = s_9 = 1-\frac{1}{2}+\frac{1}{3}-\cdots+\frac{1}{9} \approx 0,745\,635.$$

Remark 5.4 It is easy to prove that

$$\ln 2 \approx 0,693\,147.$$

This shows that the error here is $< 10^{-1}$.

Example 5.24 *Prove that the series*

$$\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{n}{4n^2-1}$$

is convergent and find its sum. Check if the series is absolutely convergent.

Necessary condition for convergence. This follows from $\dfrac{n}{4n^2-1} \to 0$ for $n\to\infty$.

Absolutely convergence? Since $\dfrac{n}{4n^2-1} \ge \dfrac{n}{4n^2} = \dfrac{1}{4}\,\dfrac{1}{n}$, and $\sum_{n=1}^{\infty}\dfrac{1}{4}\cdot\dfrac{1}{n}$ is divergent, it follows that $\sum|a_n|$ is divergent, and the series is not absolutely convergent.

Conditional convergence. This can be proved by using **Leibniz's criterium**, but this not necessary here. In fact, we get by a decomposition that

$$\frac{n}{4n^2-1} = \frac{n}{(2n-1)(2n+1)} = \frac{1}{4}\cdot\frac{1}{2n-1}+\frac{1}{4}\cdot\frac{1}{2n+1},$$

and the sectional sequence becomes

$$\begin{aligned}
s_N &= \sum_{n=1}^{N}(-1)^{n-1}\frac{n}{4n^2-1} = \frac{1}{4}\sum_{n=1}^{N}\frac{(-1)^{n-1}}{2n-1}+\frac{1}{4}\sum_{n=1}^{N}\frac{(-1)^{n-1}}{2n+1}\\
&= \frac{1}{4}\sum_{n=1}^{N}\frac{(-1)^{n-1}}{2n-1}+\frac{1}{4}\sum_{n=2}^{N+1}\frac{(-1)^{n}}{2n-1} \qquad \text{(change of index)}\\
&= \frac{1}{4}-\frac{1}{4}\sum_{n=2}^{N}\frac{(-1)^{n}}{2n-1}+\frac{1}{4}\sum_{n=2}^{N}\frac{(-1)^{n}}{2n-1}+\frac{1}{4}\,\frac{(-1)^{N+1}}{2N+1}\\
&= \frac{1}{4}+\frac{1}{4}\,\frac{(-1)^{N+1}}{2N+1} \to \frac{1}{4} \qquad \text{for } N\to\infty.
\end{aligned}$$

We conclude that the series is convergent.

As the series is convergent, though not absolutely convergent, it is conditionally convergent, and its sum is according to the above,

$$\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{n}{4n^2-1}=\lim_{N\to\infty}s_N=\frac{1}{4}.$$

Example 5.25 *Check the values of $\alpha \in \mathbb{R}$ for which the series*

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n^\alpha \ln n}$$

is absolutely convergent, conditionally convergent or divergent, respectively.

Necessary condition for convergence?. It follows from

$$|a_n| = \frac{1}{n^\alpha \ln n} \to \infty \quad \text{for } n \to \infty, \qquad \text{when } \alpha < 0,$$

that the series is coarsely divergent for $\alpha < 0$.

Absolute convergence. Since

$$|a_n| = \frac{1}{n^\alpha \ln n} \le \frac{1}{\ln 2} \cdot \frac{1}{n^\alpha} \qquad \text{for } n \ge 2,$$

and $\sum_{n=2}^{\infty} n^{-\alpha}$ is convergent for $\alpha > 1$, the series is absolutely convergent for $\alpha > 1$.

Conditional convergence? If $0 \le \alpha \le 1$, then

$$\frac{1}{n^\alpha \ln n} \ge \frac{1}{n \ln n}.$$

Now, $\dfrac{1}{n \ln n} \to 0$ decreasingly, so by the **integral criterion**

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} \sim \int_2^{\infty} \frac{dx}{x \ln x} = [\ln(\ln x)]_2^{\infty} = \infty$$

and the series is not absolutely convergent for $0 \le \alpha \le 1$.

On the other hand, $\dfrac{1}{n^\alpha \ln n} \to 0$ is *decreasing* for $n \to \infty$ and $0 \le \alpha \le 1$, and since the series is alternating, it is convergent according to **Leibniz's criterion**.

As the series is convergent, though not absolutely convergent for $0 \le \alpha \le 1$, it is conditionally convergent.

Conclusion. The series is

1) Absolutely convergent for $\alpha > 1$.

2) Conditionally convergent for $0 \le \alpha \le 1$.

3) Coarsely divergent for $\alpha < 0$.

Example 5.26 *Check the values of α for which the series*

$$\sum_{n=1}^{\infty} \frac{n^2(-1)^n}{(n^2+1)^\alpha}$$

os

1) absolutely convergent,

2) convergent,

3) conditional convergent.

1) It follows by the **criterion of equivalence** and

$$|a_n| = \frac{n^2}{(n^2+1)^\alpha} \sim \frac{n}{n^{2\alpha}} = \frac{1}{n^{2\alpha-1}},$$

that the series is absolutely convergent, if and only if $2\alpha - 1 > 1$, hence $\alpha > 1$.

2) If we put $f(x) = \dfrac{x}{(x^2+1)^\alpha}$, it follows that $f(x) \to 0$ for $x \to \infty$, if and only if $\alpha > \dfrac{1}{2}$, which we shall assume in the following. Now,

$$f'(x) = -\frac{(2\alpha-1)x^2-1}{(x^2+1)^{\alpha+1}} < 0 \qquad \text{for } x > \frac{1}{\sqrt{2\alpha-1}},$$

so $|a_n| = f(n) \to 0$ *decreasingly* for $n > 1/\sqrt{2\alpha-1}$, $n \to \infty$. The series is alternating, thus it follows from **Leibniz's criterion** that the series is convergent, if and only if $\alpha > \dfrac{1}{2}$.

3) The series is conditionally convergent, when it is convergent and not absolutely convergent. According to 1. and 2. this happens when $\dfrac{1}{2} < \alpha \leq 1$.

As a conclusion we get that the series is

1) absolutely convergent for $\alpha > 1$,

2) convergent for $\alpha > \dfrac{1}{2}$,

3) conditionally convergent for $\dfrac{1}{2} < \alpha \leq 1$,

4) coarsely divergent for $\alpha \leq \dfrac{1}{2}$.

Example 5.27 *Prove that the series*

$$(4) \quad \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n+1}{n}$$

is divergent. Then prove that if one introduces parentheses into (4) in the following way

$$(a_1 + a_2) + (a_3 + a_4) + (a_5 + a_6) + \cdots,$$

then we get a convergent series.

Necessary condition for convergence?. Since

$$|a_n| = \frac{2n+1}{n} = 2 + \frac{1}{n} \to 2 \neq 0 \qquad \text{for } n \to \infty,$$

this cannot be fulfilled, so (4) is coarsely divergent.

Convergence by introducing parentheses. First calculate

$$\begin{aligned} a_{2n-1} + a_{2n} &= (-1)^{2n-2}\left(2 + \frac{1}{2n-1}\right) + (-1)^{2n-1}\left(2 + \frac{1}{2n}\right) \\ &= 2 + \frac{1}{2n-1} - 2 - \frac{1}{2n} = \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{2n(2n-1)} \sim \frac{1}{4n^2}. \end{aligned}$$

Here $\sum_{n=1}^{\infty} \frac{1}{4n^2}$ is convergent, so it follows from the **criterion of equivalence** that

$$\sum_{n=1}^{\infty} (a_{2n-1} + a_{2n}) = \sum_{n=1}^{\infty} \frac{1}{2n(2n-1)}$$

is convergent.

Example 5.28 *Check in each of the cases below if the given series is conditionally convergent, absolutely convergent or divergent.*

$$\sum_{n=1}^{\infty} (-1)^n \left(1 + \frac{1}{n^2}\right), \qquad \sum_{n=1}^{\infty} \frac{\cos n\pi}{\sqrt{n}}, \qquad \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}.$$

1) **Necessary condition for convergence**. It follows from

$$\left|(-1)^n \left(1 + \frac{1}{n^2}\right)\right| = 1 + \frac{1}{n^2} \to 1 \neq 0 \quad \text{for } n \to \infty,$$

that the necessary condition for convergence is *not* fulfilled, so $\sum_{n=1}^{\infty} (-1)^n \left(1 + \frac{1}{n^2}\right)$ is (coarsely) divergent.

2) Now $\cos n\pi = (-1)^n$, so the series can more conveniently also be written $\sum_{n=1}^{\infty} (-1)^n / \sqrt{n}$.

a) **Necessary condition for convergence**? This follows from

$$\left|\frac{(-1)^n}{\sqrt{n}}\right| = \frac{1}{\sqrt{n}} \to 0 \qquad \text{for } n \to \infty.$$

b) **Absolute convergence**? Since $\sum_{n=1}^{\infty} \dfrac{1}{\sqrt{n}}$ is divergent, the series is not absolutely convergent.

c) **Conditional convergence**? The series is alternating and $\dfrac{1}{\sqrt{n}} \to 0$ is *decreasing* for $n \to \infty$. It therefore follows from **Leibniz's criterion** that the series is convergent. As it is not absolutely convergent, it is conditionally convergent.

3) Clearly, $a_n = \dfrac{(n!)^2}{(2n)!} > 0$, so $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} |a_n|$, and the series is either absolutely convergent or divergent.

We get from the **criterion of quotients** that

$$\frac{a_{n+1}}{a_n} = \frac{((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2} = \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4} \cdot \frac{n+1}{n+\frac{1}{2}} \to \frac{1}{4} < 1$$

for $n \to \infty$. It therefore follows that the series is convergent, thus absolutely convergent.

Example 5.29 *Check if each of the series*

$$(1)\quad \sum_{n=1}^{\infty} \sin\left(n\,\frac{\pi}{6}\right)\,\frac{n^2+2}{n^2+3n+2}, \qquad og \qquad (2)\quad \sum_{n=1}^{\infty}(-1)^{n+1}\ln\left(1+\frac{2}{n^2}\right)$$

is divergent, conditionally convergent or absolutely convergent.

We shall use the flow diagram from *Calculus 3b*, i.e. first check if $a_n \to 0$. If "yes", then continue with "absolute convergence". Note that (2) invites to an early application of Leibniz's criterion, which is here a waste of time.

1) Since $\sin\left(n\,\frac{\pi}{6}\right) = 1$ for $n = 3+12p$, $p \in \mathbb{N}_0$, we have $\sin\left(n\,\frac{\pi}{6}\right) = 1$. From

$$\frac{n^2+2}{n^2+3n+2} = \frac{1+\frac{2}{n^2}}{1+\frac{3}{n}+\frac{2}{n^2}} \to 1 \neq 0 \qquad \text{for } n \to \infty,$$

follows that

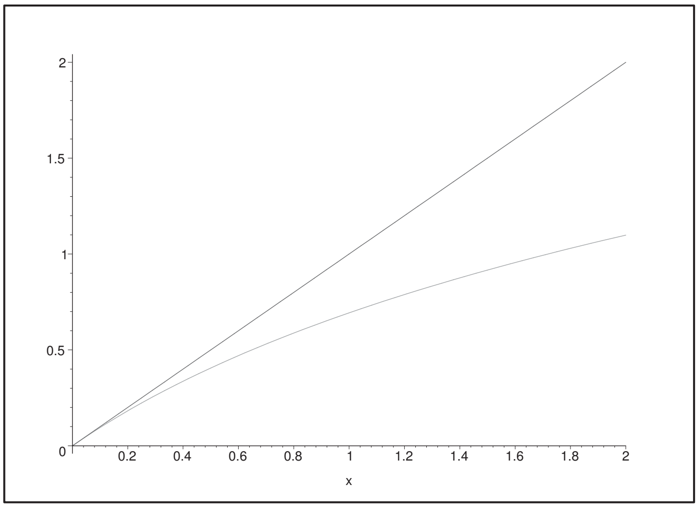
$$a_{3+12p} \to 1 \qquad \text{for } p \to \infty.$$

The necessary condition for convergence is *not* fulfilled, hence the series is *coarsely divergent.*

2) Using a well-known graph (cf. the figure) we have the estimate

$$0 < \ln(1+x) < x \qquad \text{for every } x > 0.$$

[**Alternatively**, $\ln(1+x) = x + x\varepsilon(x)$, etc.]



Then for $x = \frac{2}{n^2}$,

$$0 < |a_n| = \left|(-1)^{n+1}\ln\left(1+\frac{2}{n^2}\right)\right| = \ln\left(1+\frac{2}{n^2}\right) < \frac{2}{n^2} \to 0 \quad \text{for } n \to \infty.$$

Thus, (a) the necessary condition for convergence is fulfilled, and

$$(b) \qquad \sum_{n=1}^{\infty} |a_n| < \sum_{n=1}^{\infty} \frac{2}{n^2} = 2 \cdot \frac{\pi^2}{6} = \frac{\pi^2}{3}$$

is convergent, hence the series is *absolutely convergent*.

Remark 5.5 Leibniz's criterion it not at all mentioned in this proof. $\Diamond$

Example 5.30 *Check in each of the following cases if the series is conditionally convergent, absolutely convergent or divergent.*

$$(1) \quad \sum_{n=1}^{\infty} \frac{2^n n!}{(2n)!}, \qquad (2) \sum_{n=8}^{\infty} (-1)e^{-p}.$$

1) The faculty function occurs, hence the **criterion of quotients** is the most natural criterion to apply. All terms are positive, the the series is either absolutely convergent or divergent. It follows by the **criterion of quotients** from

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{2^n n!} = \frac{2(n+1)}{(2n+2)(2n+1)} = \frac{1}{2n+1} \to 0 < 1$$

for $n \to \infty$ that the series is absolutely convergent.

2) From $0 < \dfrac{1}{e} < 1$ follows that

$$\sum_{p=8}^{\infty} |(-1)^p e^{-p}| = \sum_{p=8}^{\infty} \left(\frac{1}{e}\right)^p = \frac{1}{e^8} \cdot \frac{1}{1 - \frac{1}{e}} < \infty,$$

so the series is absolutely convergent.

Remark 5.6 The series is a quotient series, so we can find its exact sum,

$$\sum_{p=8}^{\infty} (-1)^p e^{-p} = \sum_{p=8}^{\infty} \left(-\frac{1}{e}\right)^p = \left(-\frac{1}{e}\right)^8 \cdot \frac{1}{1 - \left(-\frac{1}{e}\right)} = e^{-8} \cdot \frac{e}{e+1} = \frac{e^{-7}}{e+1}.$$

Example 5.31 *Check in each of the following cases if the series is absolutely convergent, conditionally convergent or divergent,*

$$\sum_{n=1}^{\infty}(-1)^n\,\frac{e^n}{n^2}, \qquad \sum_{n=1}^{\infty}(-1)^n\,\frac{(n!)^3}{(3n)!}.$$

1) We have by the laws of magnitudes

$$\frac{e^n}{n^2}\to\infty \qquad \text{for } n\to\infty,$$

hence the necessary condition for convergence is not satisfied, and $\sum_{n=1}^{\infty}(-1)^n\,e^n/n^2$ is (coarsely) divergent.

2) If we put $a_n=\dfrac{(n!)^3}{(3n)!}>0$, then

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{\{(n+1)!\}^3}{\{3(n+1)\}!}\cdot\frac{(3n)!}{(n!)^3}=\left\{\frac{(n+1)!}{n!}\right\}^3\cdot\frac{(3n)!}{(3n+3)!}\\ &= \frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)}=\frac{1}{3\left(3-\dfrac{1}{n+1}\right)\left(3-\dfrac{2}{n+1}\right)}\\ &\to \frac{1}{3^3}=\frac{1}{27}<1 \qquad \text{for } n\to\infty.\end{aligned}$$

Then by the **criterion of quotients**,

$$\sum_{n=1}^{\infty}(-1)^n\,\frac{(n!)^3}{(3n)!}$$

is absolutely convergent.

Example 5.32 *Check in each of the following cases, if the series is absolutely convergent, conditionally convergent or divergent.*

$$(1)\ \ \sum_{n=1}^{\infty}(-1)^n\,\frac{\ln n}{n}, \qquad (2)\ \ \sum_{n=1}^{\infty}\frac{(-4)^n(2n)!}{(3n)!}.$$

1) The series $\sum_{n=1}^{\infty}(-1)^n\,\dfrac{\ln n}{n}$ is not absolutely convergent. It is conditionally convergent.

a) Applying e.g. the **criterion of comparison** we get

$$\sum_{n=1}^{\infty}\left|(-1)^n\,\frac{\ln n}{n}\right|\ge \ln 2\sum_{n=2}^{\infty}\frac{1}{n}=\infty,$$

which shows that we do not have absolute convergence.

b) Due to the different magnitudes, the function $\dfrac{\ln x}{x}$ tends towards mod 0 for $x \to \infty$, and

$$\frac{d}{dx}\left(\frac{\ln x}{x}\right) = \frac{1}{x^2} - \frac{\ln x}{x^2} = \frac{1-\ln x}{x^2} < 0 \qquad \text{for } x > e,$$

thus for $n \geq 3$ we see that $\dfrac{\ln n}{n}$ tends decreasingly towards 0. Furthermore, the series is alternating, so it follows from **Leibniz's criterion** that the series is convergent, hence conditionally convergent.

2) This series is absolutely convergent. In fact, if we put

$$a_n = \left|\frac{(-4)^n (2n)!}{(3n)!}\right| = \frac{4^n \cdot (2n)!}{(3n)!} \neq 0,$$

then

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{4^{n+1}(2n+2)!}{(3n+3)!} \cdot \frac{(3n)!}{4^n \cdot (2n)!} = \frac{4(2n+2)(2n+1)}{(3n+3)(3n+2)(3n+1)} \\
&= \frac{8}{3} \cdot \frac{2n+1}{(3n+2)(3n+1)} \to 0 \qquad \text{for } n \to \infty.
\end{aligned}$$

We conclude from the **criterion of quotients** that the series is absolutely convergent.

Example 5.33 *1) Prove that*

$$\sqrt{n+1}-\sqrt{n}\to 0 \qquad \textit{for } n\to\infty.$$

Hint: Apply the formula

$$a-b=\frac{a^2-b^2}{a+b}.$$

2) Check if the series

$$\sum_{n=1}^{\infty}(-1)^{n-1}(\sqrt{n+1}-\sqrt{n})=(\sqrt{2}-1)-(\sqrt{3}-\sqrt{2})+\cdots+(-1)^{n-1}(\sqrt{n+1}-\sqrt{n})+\cdots$$

is absolutely convergent, conditionally convergent or divergent.

3) When we remove all the parentheses of the series in (2), we get the series

$$\sqrt{2}-1-\sqrt{3}+\sqrt{2}+\sqrt{4}-\sqrt{3}-\cdots.$$

Is this series convergent or divergent?

1) It follows from

$$\sqrt{n+1}-\sqrt{n}=\frac{(\sqrt{n+1})^2-(\sqrt{n})^2}{\sqrt{n+1}+\sqrt{n}}=\frac{1}{\sqrt{n+1}+\sqrt{n}},$$

that $\sqrt{n+1}-\sqrt{n}\to 0$ *decreasingly* for $n\to\infty$.

Alternatively,

$$\begin{aligned}\sqrt{n+1}-\sqrt{n} &= \sqrt{n}\left\{\sqrt{1+\frac{1}{n}}-1\right\}=\sqrt{n}\left\{1+\frac{1}{2}\,\frac{1}{n}+\frac{1}{n}\,\varepsilon\left(\frac{1}{n}\right)-1\right\}\\ &= \frac{1}{2}\,\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n}}\,\varepsilon\left(\frac{1}{n}\right)\to 0 \qquad \text{for } n\to\infty.\end{aligned}$$

2) If we put $a_n=\sqrt{n+1}-\sqrt{n}=\dfrac{1}{\sqrt{n+1}+\sqrt{n}}$, then

a) $a_n\to 0$ for $n\to\infty$, thus the series is not coarsely divergent.

b) $a_n\sim\dfrac{1}{2\sqrt{n}}$, and $\sum_{n=1}^{\infty}\dfrac{1}{2\sqrt{n}}$ is divergent, hence the series is not absolutely convergent.

c) We see that $\sum_{n=1}^{\infty}(-1)^{n-1}(\sqrt{n+1}-\sqrt{n})=\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ is *alternating*, and that $a_n\to 0$ is *decreasing*, so it follows from **Leibniz's criterion** that the series is convergent. Since it is not absolutely convergent, it must be conditionally convergent.

d) When we remove all the parentheses we see that $|b_n|\geq 1$ for every n, and the necessary condition for convergence is not satisfied. The series is coarsely divergent.

Example 5.34 *1) Prove that the series*

$$\sum_{n=1}^{\infty}\frac{1}{1+2+\cdots+n}=1+\frac{1}{1+2}+\cdots+\frac{1}{1+2+\cdots+n}+\cdots,$$

is convergent with the sum 2.

(Hint: One may without proof use that $1+2+\cdots+n=\dfrac{1}{2}n(n+1)$, $n\in\mathbb{N}$*).*

2) Prove that the series

$$\sum_{n=1}^{\infty}\frac{1}{1-2+\cdots+(-1)^{n-1}n}=1+\frac{1}{1-2}+\cdots+\frac{1}{1-2+\cdots+(-1)^{n-1}n}+\cdots$$

is conditionally convergent with sum 0.

Hint: One may use without proof that

$$1-2+\cdots+(-1)^{n-1}n=\frac{1}{2}(-1)^{n-1}\left\{n+\frac{1}{2}\left(1+(-1)^{n-1}\right)\right\},\quad n\in\mathbb{N}.$$

1) When we consider the sectional sequence s_N, we get by a decomposition,

$$\begin{aligned}s_N &= \sum_{n=1}^{N}\frac{1}{1+2+\cdots+n}=\sum_{n_1}^{N}\frac{2}{n(n+1)}=2\sum_{n=1}^{N}\left(\frac{1}{n}-\frac{1}{n+1}\right)\\ &= 2\sum_{n=1}^{N}\frac{1}{n}-2\sum_{n=2}^{N+1}\frac{1}{n}=2-\frac{2}{N+1}\to 2 \qquad \text{for } N\to\infty,\end{aligned}$$

and the series is according to the definition (absolutely) convergent with the sum

$$\sum_{n=1}^{\infty}\frac{1}{1+2+\cdots+n}=2.$$

2) We first apply the hint,

$$\sum_{n=1}^{\infty}\frac{1}{1-2+\cdots+(-1)^{n-1}n}=\sum_{n=1}^{\infty}\frac{2(-1)^{n-1}}{n+\frac{1}{2}\{1+(-1)^{n-1}\}}.$$

a) Now, $\left|\dfrac{2(-1)^{n-1}}{n+\frac{1}{2}\{1+(-1)^{n-1}\}}\right|\sim\dfrac{2}{n}$, and $\sum\dfrac{2}{n}$ is divergent. Thus it follows from the **criterion of equivalence** that the series is not absolutely convergent.

b) The series is alternating, and $\dfrac{2}{n+\frac{1}{2}\{1+(-1)^{n-1}\}}$ is *weakly decreasing* towards 0. Hence the series is convergent by **Leibniz's criterium**, and therefore conditionally convergent according to (a).

Sum. We see

$$a_n = \left| \frac{1}{1-2+\cdots+(-1)^{n-1}n} \right| = \frac{2}{n+\dfrac{1}{2}\{1+(-1)^{n-1}\}} \to 0$$

weakly decreasing for $n \to \infty$.

Then calculate the sectional subsequence s_{2N},

$$\begin{aligned} s_{2N} &= \sum_{n=1}^{\infty}(-1)^{n-1}a_n = \sum_{n=1}^{2N} \frac{2(-1)^{n-1}}{n+\frac{1}{2}\{1+(-1)^{n-1}\}} \\ &= \sum_{p=1}^{N} \left\{ \frac{2(-1)^{2p-1-1}}{(2p-1)+\frac{1}{2}\{1+(-1)^{2p-1-1}\}} + \frac{2(-1)^{2p-1}}{2p+\frac{1}{2}\{1+(-1)^{2p-1}\}} \right\} \\ &= \sum_{p=1}^{N} \left(\frac{2}{2p} - \frac{2}{2p} \right) = 0 \to 0 \qquad \text{for } N \to \infty. \end{aligned}$$

We can now continue in different ways:

a) The series is convergent, thus $s_N \to s$ for $N \to \infty$. The subsequence (s_{2N}) converges both towards s and towards 0, and its limit is unique. Hence $s = 0$, and the sum is 0.

b) Since $s_{2N+1} = a_{2n+1} + s_{2N} = a_{2N+1} = \dfrac{1}{N+1}$, we have

$$s_n = \begin{cases} 0 & \text{for } n \text{ lige,} \\ \dfrac{2}{n+1} & \text{for } n \text{ ulige,} \end{cases}$$

and it follows that $s_n \to 0$ for $n \to \infty$.

As a conclusion we finally get

$$\sum_{n=1}^{\infty} \frac{1}{1-2+\cdots+(-1)^{n-1}n} = 0,$$

where the convergence is conditional.

6 Series of functions; uniform convergence

Example 6.1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1-function, for which $f(0) = 0$. Prove that if the series $\sum_{n=1}^{\infty} a_n$ (real terms) is absolutely convergent, then the series $\sum_{n=1}^{\infty} f(a_n)$ is also absolutely convergent.*

We shall apply Taylor's formula from e.g. *Calculus 1b*.

1) We assume that the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. Then especially, $|a_n| \to 0$ for $n \to \infty$.

2) Since $f \in C^1(\mathbb{R})$ we get by Taylor's formula that

$$f(x) = f(0) + \frac{f'(0)}{1!}\, x + x\varepsilon(x) = f'(0) \cdot x + x\varepsilon(x).$$

3) Since (a_n) is bounded, we can find constants C and N, such that

$$|\varepsilon(a_n)| \le C \qquad \text{for } n \ge N.$$

Without loss of generality we may assume that $N = 1$.

4) Since

$$f(a_n) = f'(0) \cdot a_n + a_n \varepsilon(a_n)$$

where

$$|f(a_n)| \le |f'(0)| \cdot |a_n| + |a_n| \cdot C = (|f'(0)| + C) \cdot |a_n|,$$

and since $\sum_{n=1}^{\infty} |a_n|$ is convergent, and $|f'(0)| + C < \infty$, we get by applying the **criterion of comparison** that $\sum_{n=1}^{\infty} |f(a_n)|$ is convergent, so $\sum_{n=1}^{\infty} f(a_n)$ is absolutely convergent.

Remark 6.1 It is here essential that $f(0) = 0$ and that $f'(0)$ is finite. Consider for instance the function $f(x) = \sqrt{|x|}$ which is not differentiable at 0 (vertical half tangent), while $\sum_{n=1}^{\infty} \dfrac{1}{n^2}$ is convergent. In this case we see that

$$\sum_{n=1}^{\infty} f\left(\frac{1}{n^2}\right) = \sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent.

Example 6.2 *Find all values $x \in \mathbb{R}$, for which the series*

$$\sum_{n=0}^{\infty} (1-x)x^n$$

is convergent, and find for each of these values the sum function of the series.

Then check if the series is uniformly convergent in

(1) $]-1,1[$, (2) $\left]-\dfrac{1}{2}, \dfrac{1}{2}\right[$.

1) When $|x| < 1$, the series is the usual quotient series of quotient x, hence the series is here convergent with the sum function

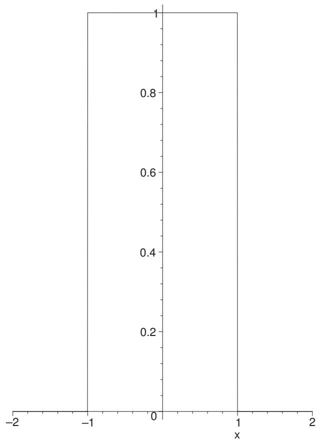
$$f(x) = \sum_{n=0}^{\infty} (1-x)x^n = \frac{1-x}{1-x} = 1 \qquad \text{for } x \in]-1,1[.$$

2) When r $x = 1$, all terms of the series are 0, thus $f(1) = 0$.

3) When $x = -1$ or $|x| > 1$, we see that $|(1-x)x^n|$ does not converge towards 0 for $n \to \infty$, and the series is coarsely divergent.

Conclusion. The series is convergent for $x \in]-1,1]$ with the sum function

$$f(x) = \begin{cases} 1 & \text{for } x \in]-1,1[, \\ 0 & \text{for } x = 1. \end{cases}$$



1) Every term $(1-x)x^n$ is continuous in $]-1,1]$, while the sum function $f(x)$ is *not* continuous at 1. Hence, the convergence cannot be uniform in this interval.

2) When $x \in \left]-\frac{1}{2},\frac{1}{2}\right[$, we get the estimate

$$|s_N(x)-1| = \left|\sum_{n=0}^{N}(1-x)x^n - 1\right| = \left|\sum_{n=0}^{N}x^n - \sum_{n=0}^{N}x^{n+1} - 1\right| = |x|^{N+1} \le \frac{1}{2^{N+1}},$$

which tends towards 0 for $N \to \infty$ *independently* of $x \in \left]-\frac{1}{2},\frac{1}{2}\right[$, hence the convergence is uniform in $\left]-\frac{1}{2},\frac{1}{2}\right[$.

Alternatively we have the convergent majoring series

$$\sum_{n=0}^{\infty}|1-x|\cdot|x|^n \le \frac{3}{2}\sum_{n=0}^{\infty}\frac{1}{2^n} = 3, \quad \text{for } x \in \left]-\frac{1}{2},\frac{1}{2}\right[,.$$

We conclude again that the series is uniformly convergent i $\left]-\frac{1}{2},\frac{1}{2}\right[$.

Example 6.3 *Check if the series*

$$\sum_{n=0}^{\infty}(1-x)x^n$$

is uniformly convergent in the interval $I = [0,1[$.

This is a tricky example, because the sum function $f(x) = 1$ is continuous in $[0,1[$, so one is misled to think that the convergence is uniform. This is not true!

When $x \in [0,1[$, we get as in Example 6.2 that

$$|s_N(x) - 1| = |x|^{N+1}.$$

By choosing

$$x_N = \frac{1}{\sqrt[N+1]{2}} \in [0,1[,$$

it follows that

$$|s_N(x_N) - 1| = \frac{1}{2} \qquad \text{for every } N \in \mathbb{N},$$

and the convergence is not uniform.

Example 6.4 *1) Prove that the series*

$$(5)\quad \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$$

is convergent for every $x \in \mathbb{R}$, and find its sum function.

2) Then prove that (5) is not uniformly convergent in the interval $\mathbb{R}$.

3) Prove also that (5) is uniformly convergent in any interval of the form $[a,b]$, where $0 < a < b$.

4) Find

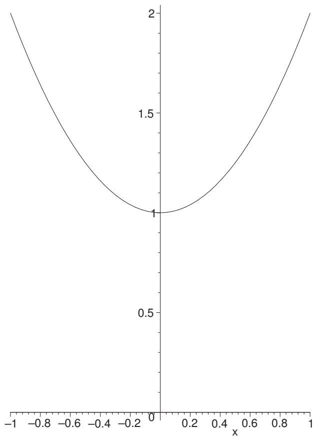
$$\sum_{n=0}^{\infty} \int_1^2 \frac{x^2}{(1+x^2)^n}\, dx.$$

1) If $x = 0$, then $f(0) = 0$.
If $x \neq 0$, then the series is a quotient series of quotient $1/(1+x^2) \in]0,1[$ (i.e. convergent), and its sum is

$$\sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n} = \frac{x^2}{1-\dfrac{1}{1+x^2}} = \frac{x^2(1+x^2)}{x^2} = 1+x^2.$$

The sum function is

$$f(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n} = \begin{cases} 1+x^2 & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$



2) Every term $x^2/(1+x^2)^n$ is continuous, while the sum function is *not* continuous at $x = 0$. It follows that the convergence *cannot* be uniform in any interval, which contains 0, and in particular not in $\mathbb{R}$.

3) Finally, we can prove the uniform convergence in $[a,b]$, where $0 < a < b$, in two ways:

   a) directly by the definition (estimate such that x disappears),

   b) find a convergent majoring series.

   a) We shall prove that

$$|s_N(x) - (1+x^2)| \le a_N \to 0 \qquad \text{for } N \to \infty, \quad x \in [a,b],$$

   where a_N does not depend on x. We get

$$\begin{aligned}
|s_N(x) - (1+x^2)| &= \left|\sum_{n=0}^{N} \frac{x^2}{(1+x^2)^n} - (1+x^2)\right| = \left|\sum_{n=0}^{N} \frac{1+x^2-1}{(1+x^2)^n} - (1+x^2)\right| \\
&= \left|\sum_{n=0}^{N} \frac{1}{(1+x^2)^{n-1}} - \sum_{n=0}^{N} \frac{1}{(1+x^2)^n} - (1+x^2)\right| \\
&= \left|(1+x^2) + \sum_{n=0}^{N-1} \frac{1}{(1+x^2)^n} - \sum_{n=0}^{N} \frac{1}{(1+x^2)^n} - (1+x^2)\right| \\
&= \frac{1}{(1+x^2)^N} \le \frac{1}{(1+a^2)^N} \to 0 \quad \text{for } N \to \infty \quad \text{independent of } x \ge a,
\end{aligned}$$

   and we have proved that the series is *uniformly convergence*, even in the half infinite intervals $[a, \infty[$, where $a > 0$.

   b) **Alternatively** we get for $0 < a \le x \le b < \infty$ the following estimate

$$\sum_{n=0}^{\infty} \left|\frac{x^2}{(1+x^2)^n}\right| \le \sum_{n=0}^{\infty} \frac{b^2}{(1+a^2)^n} = b^2 \sum_{n=0}^{\infty} \left(\frac{1}{1+a^2}\right)^n = \frac{b^2(1+a^2)}{a^2}.$$

   Notice that the numerator is estimated *from above*, while the denominator is estimated *from below* by some smaller positive number, and also that the quotient $1/(1+a^2) \in]0,1[$, hence the quotient series is convergent with the given sum. Since we have obtained a convergent majoring series we conclude that the original series is uniformly convergent.

4) According to 3) the convergence is *uniform* in the closed and bounded (i.e. compact) interval $[1,2]$. Then we can interchange summation and integration, so we get by 1)

$$\begin{aligned}
\sum_{n=0}^{\infty} \int_1^2 \frac{x^2}{(1+x^2)^n}\,dx &= \int_1^2 \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}\,dx = \int_1^2 (1+x^2)\,dx \\
&= \left[x + \frac{x^3}{3}\right]_1^2 = 1 + \frac{7}{3} = \frac{10}{3}.
\end{aligned}$$

   It is possible, though very difficult directly to calculate the sum of the integrals. We shall leave out these tedious details.

Example 6.5 *Prove that the series*

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^4} \qquad \textit{and} \qquad \sum_{n=1}^{\infty} \frac{1}{n^2+x^2}$$

are uniformly convergent in the interval $\mathbb{R}$.

We use in both cases the **criterion of majoring series**.

1) Since $|\cos nx| \le 1$, each term of the series can be estimated *from above* by

$$\left|\frac{\cos nx}{n^4}\right| \le \frac{1}{n^4}, \qquad \text{for alle } x \in \mathbb{R}.$$

Then er $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is a *convergent majoring series* (x does not occur), so $\sum_{n=1}^{\infty} \frac{\cos nx}{n^4}$ is *uniformly convergent* in $\mathbb{R}$.

Remark 6.2 Here $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$ is one of the standard series. $\Diamond$

2) Each term of the series is estimated *from above* by *decreasing* the denominator,

$$\left|\frac{1}{n^2+x^2}\right| = \frac{1}{n^2+x^2} \le \frac{1}{n^2}, \qquad \text{da } x^2 \ge 0.$$

Now, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a *convergent majoring seris* (x does not occur), hence $\sum_{n=1}^{\infty} \frac{1}{n^2+x^2}$ is *uniformly convergent* i $\mathbb{R}$.

Remark 6.3 Here $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ is one of the standard series. $\Diamond$

Example 6.6 *Prove that the series*

$$\sum_{n=1}^{\infty} \frac{x^n}{n^{3/2}}$$

is uniformly convergent for $|x| \le 1$, *and divergent for* $|x| > 1$.

1) Since

$$\left|\frac{x^n}{n^{3/2}}\right| \le \frac{1}{n^{3/2}} \qquad \text{for } |x| \le 1,$$

and $\frac{3}{2} > 1$, the series $\sum_{n=1}^{\infty} 1/n^{3/2}$ is a convergent majoring series, thus $\sum_{n=1}^{\infty} x^n/n^{3/2}$ is uniformly convergent for $|x| \le 1$.

2) If $|x| > 1$, then it follows by the magnitudes that

$$\left|\frac{x^n}{n^{3/2}}\right| = \frac{|x|^n}{n^{3/2}} \to \infty \qquad \text{for } n \to \infty.$$

The necessary condition for convergence is not fulfilled, so the series is coarsely divergent for $|x| > 1$.

Example 6.7 *Prove that the series*

$$\sum_{n=1}^{\infty} \frac{2\cos nx + 3\sin n^2 x}{n\sqrt{n}}$$

is uniformly convergent in the interval $\mathbb{R}$.

Each term of the series is estimated by

$$\left|\frac{2\cos nx + 3\sin n^2 x}{n\sqrt{n}}\right| \le \frac{2+3}{n\sqrt{n}} = \frac{5}{n^{3/2}}, \qquad \text{for alle } x \in \mathbb{R},$$

so a majoring series (in which x does not occur) is

$$\sum_{n=1}^{\infty} \frac{5}{n^{3/2}}.$$

Now, $\alpha = \dfrac{3}{2} > 1$, so the majoring series is convergent.

We conclude that the series is uniformly convergent in $\mathbb{R}$.

Example 6.8 *Prove that the series*

$$\sum_{n=1}^{\infty} \frac{2^{nx}}{n!}$$

is uniformly convergent in the interval $]-\infty, k]$, *where* k *is any number in* $\mathbb{R}_+$.

We shall find a convergent majoring series. It follows from

$$2^{nx} \le 2^{nk} = (2^k)^n,$$

that

$$0 < \sum_{n=1}^{\infty} \frac{2^{nx}}{n!} \le \sum_{n=1}^{\infty} \frac{1}{n!}\left(2^k\right)^n = \exp\left(2^k\right) - 1 \qquad \text{for } x \le k.$$

Thus we have constructed a convergent majoring series, and the claim is proved.

Remark 6.4 We see that we have *pointwisely* everywhere,

$$\sum_{n=1}^{\infty} \frac{2^{nx}}{n!} = \exp\left(2^x\right) - 1, \qquad x \in \mathbb{R},$$

and this convergence is even *uniform* over each interval of the type $x \in]-\infty, k]$.

Example 6.9 *Prove that the series*

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^x}$$

is uniformly convergent in the interval $[k, \infty[$*, where* k *is any real number bigger than 1. Then prove that the sum function tends towards* ∞ *for* x *tending towards 1 from the right.*

When $n \geq 3$, then $\ln n > 1$, and thus $n(\ln n)^x \geq n(\ln n)^k$ for $x \geq k$ and $n \geq 3$. This gives us the estimate

$$0 < \sum_{n=3}^{\infty} \frac{1}{n(\ln n)^x} \leq \sum_{n=3}^{\infty} \frac{1}{n(\ln n)^k} \qquad \text{for } x \geq k.$$

Now, $\dfrac{1}{n(\ln n)^k}$ is decreasing in n for $n \geq 3$ and $k > 1$. Since

$$\int_3^{\infty} \frac{dt}{t(\ln t)^k} = \left[-\frac{1}{k-1}\cdot\frac{1}{(\ln t)^{k-1}}\right]_3^{\infty} = \frac{1}{k-1}\cdot\frac{1}{(\ln 3)^{k-1}},$$

we get by the **integral criterion** that $\sum_{n=3}^{\infty} \dfrac{1}{n(\ln n)^k}$ is a convergent majoring series, so the series is uniformly convergent, because nothing is changed by adding the function $\dfrac{1}{2(\ln 2)^x}$, even if it tends towards ∞ for $x \to \infty$.

Assume that $x > 1$. Since $\dfrac{1}{n(\ln n)^x}$ is decreasing in n for $n \geq 3$, we get by the **integral criterion** the following estimate from below,

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^x} &= \frac{1}{2(\ln 2)^x} + \sum_{n=3}^{\infty} \frac{1}{n(\ln n)^x} \geq \frac{1}{2(\ln 2)^x} + \int_3^{\infty} \frac{dt}{t(\ln t)^x} \\ &= \frac{1}{2(\ln 2)^x} + \left[\frac{(\ln t)^{1-x}}{1-x}\right]_3^{\infty} = \frac{1}{2(\ln 2)^x} + \frac{1}{x-1}\cdot\frac{1}{(\ln 3)^{x-1}}.\end{aligned}$$

Then the claim follows from the fact that the lower estimate clearly tends towards ∞ for $x \to 1+$.

Example 6.10 *Prove that the series*

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$$

is uniformly convergent for every $x \in \mathbb{R}$.

Also prove that

$$\int_0^{\pi} \left(\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}\right) dx = 2\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3}.$$

Each term in the series is estimated *from above* by

$$\left|\frac{\sin nx}{n^2}\right| \leq \frac{1}{n^2} \qquad \text{for every } x \in \mathbb{R}.$$

Then $\sum_{n=1}^{\infty} \dfrac{1}{n^2} = \dfrac{\pi^2}{6}$ is a convergent majoring series (a standard series), so the series itself is uniformly convergent in $\mathbb{R}$.

Since the convergence is *uniform*, and $[0, \pi]$ is a *bounded* interval, we may interchange summation and integration,

$$\int_0^{\pi} \left(\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}\right) dx = \sum_{n=1}^{\infty} \frac{1}{n^2}\int_0^{\pi} \sin nx\, dx = \sum_{n=1}^{\infty} \frac{1}{n^2}\left[-\frac{1}{n}\cos nx\right]_0^{\pi} = \sum_{n=1}^{\infty} \frac{1}{n^3}\{1-(-1)^n\}.$$

As

$$1-(-1)^n = \begin{cases} 2 & \text{for } n \text{ odd}, \\ 0 & \text{for } n \text{ even}, \end{cases}$$

we shall only sum over *odd* indices, thus

$$\int_0^{\pi}\left(\sum_{n=1}^{\infty}\frac{\sin nx}{n^2}\right)dx=\sum_{n=1}^{\infty}\frac{1}{n^3}\{1-(-1)^n\}=2\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}.$$

Remark 6.5 The exact value of $\sum_{n=1}^{\infty}\dfrac{1}{(2n-1)^3}$ is yet not known; but one may of course find approximating values.

Example 6.11 *1) Prove that the series*

$$(6) \quad \sum_{n=0}^{\infty} \sin x \cdot (\cos x)^{2n}$$

is pointwise convergent for every $x \in [0,\pi]$, *and find its sum for every* $x \in [0,\pi]$.

2) Check if the series (6) is uniformly convergent in $[0,\pi]$.

3) Prove that the series (6) is uniformly convergent in $\left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$.

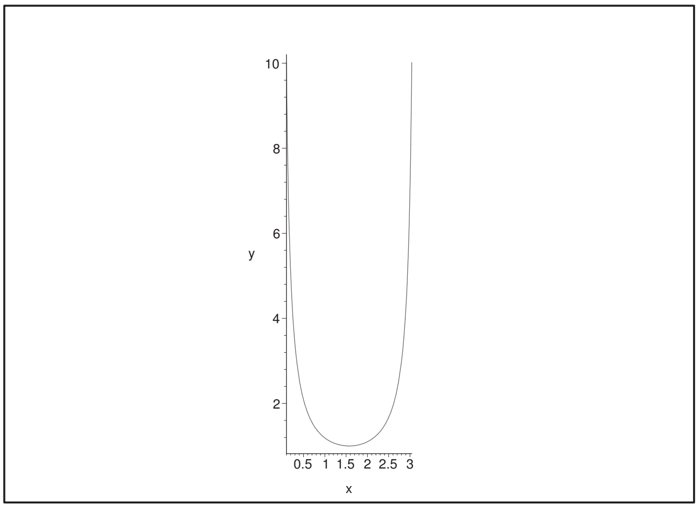
1) When $x \in]0,\pi[$, the quotient fulfils $0 \leq \cos^2 x < 1$, so the quotient series is pointwise convergent i $]0,\pi[$ and its sum function is

$$\sum_{n=0}^{\infty} \sin x \cdot (\cos x)^{2n} = \sin x \sum_{n=0}^{\infty} \{\cos^2 x\}^n = \frac{\sin x}{1-\cos^2 x} = \frac{1}{\sin x}.$$

When either $x = 0$ or $x = \pi$, every term is 0, and the sum is 0.

As a conclusion we get pointwise convergence in $[0,\pi]$ and the sum function is here given by

$$f(x) = \begin{cases} 1/\sin x & \text{for } x \in]0,\pi[, \\ 0 & \text{for } x = 0 \text{ or } x = \pi. \end{cases}$$



2) Since every term of the series is a continuous function, and the sum function is *not* continuous at the end points, we conclude that the series *cannot* be uniformly convergent in the interval $[0,\pi]$.

3) If $x \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$, then $|\cos x| \le \frac{1}{2}$. Hence we get the estimate

$$\left|\sum_{n=0}^{\infty} \sin x \cdot (\cos x)^{2n}\right| \le \sum_{n=0}^{\infty} 1 \cdot \left(\frac{1}{2}\right)^{2n} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^{n} = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}.$$

The series has the convergent majoring series $\sum 4^{-n}$ in the interval $\left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$, so it must be uniformly convergent in this interval.

Example 6.12 *Prove that the series*

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{(n+x)^2}$$

is uniformly convergent for $x \ge 0$, and then calculate

$$\int_0^1 f(x)\,dx = \int_0^1 \left\{\sum_{n=1}^{\infty} \frac{1}{(n+x)^2}\right\} dx.$$

We assume that $x \ge 0$. Every term is estimated *from above* by *decreasing* the denominator,

$$0 < \frac{1}{(n+x)^2} \le \frac{1}{(n+0)^2} = \frac{1}{n^2} \qquad \text{for every } x \ge 0.$$

Now, $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ is a convergent majoring series (a standard series). Hence, it follows that the series is uniformly convergent for $x \ge 0$.

Since the series is uniformly convergent in the bounded interval $[0,1]$, we may interchange summation and integration. Hereby we get

$$\begin{aligned}
\int_0^1 f(x)\,dx &= \int_0^1 \left\{\sum_{n=1}^{\infty} \frac{1}{(n+x)^2}\right\} dx = \sum_{n=1}^{\infty} \int_0^1 \frac{1}{(n+x)^2}\,dx \\
&= \sum_{n=1}^{\infty} \left[-\frac{1}{n+x}\right]_{x=0}^{1} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1}\right) \qquad \text{(telescoping series)} \\
&= \lim_{N\to\infty} \left\{1 - \frac{1}{N+1}\right\} = 1.
\end{aligned}$$

Example 6.13 *1) Prove that the series*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2+x^2}$$

is convergent for every $x \in \mathbb{R}$.

2) Explain why we have the following inequalities for every $x \in \mathbb{R}$ and every $n \in \mathbb{N}$,

$$2|x| \le x^2+1 \le x^2+n.$$

3) Define a function f by

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2+x^2}, \qquad x \in \mathbb{R}.$$

Prove that f is differentiable and that

$$f'(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^n x}{(n^2+x^2)^2}, \qquad x \in \mathbb{R}.$$

(Hint: Apply the result of (2)).

1) We get by a crude estimate that

$$\left|\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2+x^2}\right| \le \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

and we see that the series has a convergent majoring series, so it is even uniformly convergent, and its sum function $f(x)$ is continuous.

2) Now, $x^2-2|x|+1 = (|x|-1)^2 \ge 0$, so we obtain the left hand inequality by a rearrangement. The right hand inequality is trivial.

3) *Formally* we get by termwise differentiation,

$$\text{“}f'(x)\text{”} = 2\sum_{n=1}^{\infty} \frac{(-1)^n x}{(n^2+x^2)^2}.$$

However, the following estimate

$$\left|2\sum_{n=1}^{\infty} \frac{(-1)^n x}{(n^2+x^2)^2}\right| \le \sum_{n=1}^{\infty} \frac{2|x|}{n^2+x^2} \cdot \frac{1}{n^2+x^2} \le \sum_{n=1}^{\infty} 1 \cdot \frac{1}{n^2} < \infty$$

shows that the formally differentiated series has a convergent majoring series, so it is uniformly convergent. Hence, f is differentiable with the derivative

$$f'(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^n x}{(n^2+x^2)^2}.$$

Example 6.14 *1) Prove that the series*

$$\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{x+n} = \frac{1}{1+x} - \frac{1}{x+2} + \cdots + \frac{(-1)^{n-1}}{x+n} + \cdots$$

is convergent for every $x \in [0, \infty[$

2) Let f *be the sum function of the series of (1), thus*

$$(7)\quad f(x) = \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{x+n}, \qquad x \in [0, \infty[.$$

Prove that f *is differentiable and that*

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x), \qquad x \in [0, \infty[.$$

1) If $x \leq 0$, then $|f_n(x)| = \dfrac{1}{x+n} \to 0$ decreasingly. Now, $f_n(x) = (-1)^{n-1}|f_n(x)|$, so the series is alternating, and therefore convergent according to **Leibniz's criterion**. Hence, the series of $f(x)$ is pointwise convergent for every $x \in [0, \infty[$.

2) The formally termwise differentiated series is

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(x+n)^2}, \qquad x \in [0, \infty[.$$

Clearly, this series has the convergent majoring series $\sum \dfrac{1}{n^2}$, so it is uniformly convergent. The series of $f(x)$ is pointwise convergent, and the termwise differentiated series is uniformly convergent, so it follows that $f(x)$ is differentiable with the derivative

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(x+n)^2}, \qquad x \in [0, \infty[.$$