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Examples of Differential Equations of Second...

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Calculus 4c-4

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Contents

Introduction

Here follows the continuation of a collection of examples from *Calculus 4c-1, Systems of differential systems*. The reader is also referred to *Calculus 4b* and to *Complex Functions*.

We focus in particular on the linear differential equations of second order of variable coefficients, although the amount of examples is far from exhausting.

It should no longer be necessary rigourously to use the ADIC-model, described in *Calculus 1c* and *Calculus 2c*, because we now assume that the reader can do this himself.

Even if I have tried to be careful about this text, it is impossible to avoid errors, in particular in the first edition. It is my hope that the reader will show some understanding of my situation.

Leif Mejlbro
23rd May 2008

1 Linear differential equations of second order with variable coefficients

Example 1.1 *Solve the differential equation*

$$t(t-1)\frac{d^2x}{dt^2}+t\frac{dx}{dt}-x=t$$

in the interval $]1,\infty[$*, given that the corresponding homogeneous equation has the solution* $x=t$*.*

There are here several variants of solutions. We shall here produce three of them.

1) **Norm the equation and apply a solution formula.**
Since $t(t-1)>0$ in $]1,\infty[$, we get by **norming**, i.e. division by the coefficients of the term of second order,

$$\frac{d^2x}{dt^2}+\frac{1}{t-1}\frac{dx}{dt}-\frac{1}{t(t-1)}x=\frac{1}{t-1}, \qquad \text{thus } u(t)=\frac{1}{t-1}.$$

Hence

$$\Omega(t)=\exp\left(\int\frac{dt}{t-1}\right)=t-1.$$

Since $\varphi_1(t) = t$, it follows from the solution formula that the complete solution is

$$\begin{aligned}
\varphi(t) &= \varphi_1(t)\left\{\int \frac{1}{\varphi_1(t)^2\Omega(t)}\left[c_2+\int \varphi_1(t)\Omega(t)u(t)\,dt\right]dt+c_1\right\}\\
&= t\left\{\int \frac{1}{t^2(t-1)}\left[c_2+\int t(t-1)\cdot\frac{1}{t-1}\,dt\right]dt+c_1\right\}\\
&= t\int \frac{1}{t^2(t-1)}\left\{c_2+\frac{1}{2}t^2\right\}dt+c_1 t \qquad \text{(decompose)}\\
&= \frac{1}{2}\,t\int \frac{1}{t-1}\,dt+c_1 t+c_2 t\int\left(\frac{1}{t-1}-\frac{1}{t}-\frac{1}{t^2}\right)dt\\
&= \frac{1}{2}\,t\ln(t-1)+c_1 t+c_2\left\{t\ln\left(\frac{t-1}{t}\right)+1\right\}.
\end{aligned}$$

2) **Inspection**. (Elegant, though a difficult method). When $t > 1$ it follows by some small rearrangements that

$$\begin{aligned}
t &= t(t-1)\frac{d^2x}{dtr}+t\frac{dx}{dt}-x=\left\{(t^2-t)\frac{d^2x}{dt^2}+(2t-1)\frac{dx}{dt}\right\}+\left\{-(2t-1)\frac{dx}{dt}+t\frac{dx}{dt}-x\right\}\\
&= \frac{d}{dt}\left\{t(t-1)\frac{dx}{dt}\right\}-\left\{(t-1)\frac{dx}{dt}+1\cdot x\right\}=\frac{d}{dt}\left\{t(t-1)\frac{dx}{dt}-(t-1)x\right\}\\
&= \frac{d}{dt}\left\{t^2(t-1)\left[\frac{1}{t}\frac{dx}{dt}-\frac{1}{t^2}x\right]\right\}=\frac{d}{dt}\left\{t^2(t-1)\frac{d}{dt}\left(\frac{x}{t}\right)\right\}.
\end{aligned}$$

By integration of the equation

$$\frac{d}{dt}\left\{t^2(t-1)\frac{d}{dt}\left(\frac{x}{t}\right)\right\}=t$$

we get

$$t^2(t-1)\frac{d}{dt}\left(\frac{x}{t}\right)=\frac{1}{2}t^2+c_2, \qquad \left[\text{smarter } =\frac{1}{2}(t^2-1)+\tilde{c}_2\right],$$

hence by a decomposition,

$$\frac{d}{dt}\left(\frac{x}{t}\right)=\frac{1}{2}\cdot\frac{1}{t-1}+c_2\cdot\frac{1}{t^2(t-1)}=\frac{1}{2}\cdot\frac{1}{t-1}+c_2\left\{\frac{1}{t-1}-\frac{1}{t}-\frac{1}{t^2}\right\}.$$

Since $t > 1$, it follows by another integration,

$$\frac{x}{t}=\frac{1}{2}\ln(t-1)+c_1+c_2\left\{\ln\left(\frac{t-1}{t}\right)+\frac{1}{t}\right\},$$

hence

$$x=\frac{1}{2}\,t\ln(t-1)+c_1 t+c_2\left\{t\ln\left(\frac{t-1}{t}\right)+1\right\} \qquad \text{for } t>1.$$

3) **The standard method**. We first norm the equation

$$\frac{d^2x}{dt^2}+\frac{1}{t-1}\,\frac{dx}{dt}-\frac{1}{t(t-1)}x=\frac{1}{t-1}\qquad\text{for } t>1.$$

Given that $y_1(t)=t$ is a solution of the homogeneous equation, where $y_1(t)\neq 0$ for $t>1$, a linearly independent solution of the homogeneous equation can be computed by means of a formula,

$$\begin{aligned}y_2(t) &= y_1(t)\int\frac{1}{(y_1(t))^2}\,\exp\left(-\int f_1(t)\,dt\right)dt=t\int\frac{1}{t^2}\,\exp\left(-\int\frac{dt}{t-1}\right)dt\\ &= t\int\frac{1}{t^2}\cdot\frac{1}{t-1}\,dt=t\int\left\{\frac{1}{t-1}-\frac{1}{t}-\frac{1}{t^2}\right\}dt=t\ln\left(\frac{t-1}{t}\right)+1.\end{aligned}$$

Then a particular solution is also given by a standard formula,

$$(1)\quad y_0(t)=y_1(t)\int\frac{W_1(t)}{W(t)}\,dt+y_2(t)\int\frac{W_2(t)}{W(t)}\,dt,$$

where

$$W(t)=\begin{vmatrix}y_1(t) & y_2(t)\\ y_1'(t) & y_2'(t)\end{vmatrix}=\begin{vmatrix}t & t\ln\left(\dfrac{t-1}{t}\right)+1\\ 1 & \ln\left(\dfrac{t-1}{t}\right)+\dfrac{1}{t-1}\end{vmatrix}=\frac{t}{t-1}-1=\frac{1}{t-1},$$

$$W_1(t)=\begin{vmatrix}0 & t\ln(\dfrac{t-1}{t})+1\\ \dfrac{1}{t-1} & \ln\left(\dfrac{t-1}{t}\right)+\dfrac{1}{t-1}\end{vmatrix}=-\frac{t}{t-1}\,\ln\left(\frac{t-1}{t}\right)-\frac{1}{t-1}$$

and

$$W_2(t)=\begin{vmatrix}t & 0\\ 1 & \dfrac{1}{t-1}\end{vmatrix}=\frac{t}{t-1}.$$

Then insert into (1) and we obtain the particular solution

$$\begin{aligned}y_0(t) &= y_1(t)\int\frac{W_1(t)}{W(t)}\,dt+y_2(t)\int\frac{W_2(t)}{W(t)}\,dt\\ &= -t\int(t\ln(t-1)-t\ln t+t)dt+\left\{t\ln\left(\frac{t-1}{t}\right)+1\right\}\int t\,dt\\ &= -t\left[\frac{t^2}{2}\ln\left(\frac{t-1}{t}\right)+t-\int\frac{t^2}{2}\left(\frac{1}{t-1}-\frac{1}{t}\right)dt\right]-\frac{t^2}{2}\left\{t\ln\left(\frac{t-1}{t}\right)+1\right\}\\ &= -\frac{1}{2}t^2+\frac{t}{2}\int\left(\frac{t^2-1+1}{t-1}-t\right)dt=-\frac{1}{2}t^2+\frac{t}{2}\int\left(t+1+\frac{1}{t-1}-t\right)dt\\ &= -\frac{1}{2}t^2+\frac{t}{2}\{t+\ln(t-1)\}=\frac{1}{2}t\ln(t-1).\end{aligned}$$

Summing up we get the complete solution

$$x = \frac{1}{2}\, t\ln(t-1) + c_1 t + c_2\left\{ t\ln\left(\frac{t-1}{t}\right)+1\right\}, \qquad t>1,$$

where c_1 and c_2 are arbitrary constants.

Example 1.2 *Find the complete solution of the following differential equation, given that $\varphi_1(t) = \cosh t$ is a solution of the corresponding homogeneous differential equation,*

$$\frac{d^2x}{dt^2} - \tanh t \cdot \frac{dx}{dt} - (1-\tanh^2 t)x = e^t, \qquad t \in \mathbb{R}.$$

Here again there are several variants.

1) The equation is already normed, so we can immediately apply the solution formula,

$$x(t) = \varphi_1(t)\left\{\int \frac{1}{\varphi_1(t)^2\Omega(t)}\left(c_1 + \int \varphi_1(t)\Omega(t)u(t)\,dt\right)dt + c_2\right\}$$

where $\varphi_1(t) = \cosh t$, $u(t) = e^t$ and

$$\Omega(t) = \exp(-\int \tanh t\,dt) = \frac{1}{\cosh t}.$$

This gives us

$$\begin{aligned} x(t) &= \cosh t\left\{\int \frac{\cosh t}{\cosh^2 t}\left(c_1 + \int \frac{\cosh t}{\cosh t}\, e^t\,dt\right)dt + c_2\right\} \\ &= \cosh t\left\{\int \frac{1}{\cosh t}(c_1+e^t)dt\right\} + c_2\cosh t \\ &= c_2\cosh t + c_1\cosh t\int \frac{2e^t}{e^{2t}+1}\,dt + \cosh t\int \frac{2e^{2t}}{e^{2t}+1}\,dt \\ &= c_2\cosh t + 2c_1\cosh t\cdot \operatorname{Arctan}(e^t) + \cosh t\cdot \ln(1+e^{2t}). \end{aligned}$$

2) If we put $x = \cosh t \cdot y$, then

$$\frac{dx}{dt} = \cosh t\,\frac{dy}{dt} + \sinh t\cdot y,$$

$$\frac{d^2x}{dt^2} = \cosh t\,\frac{d^2y}{dt^2} + 2\sinh t\cdot\frac{dy}{dt} + \cosh t\cdot y,$$

hence by insertion into the differential equation,

$$\begin{aligned} e^t &= \frac{d^2x}{dt^2} - \tanh t\,\frac{dx}{dt} - \frac{1}{\cosh^2 t}\,x \\ &= \cosh t\,\frac{d^2y}{dt^2} + 2\sinh t\,\frac{dy}{dt} + \cosh t\cdot y - \sinh t\,\frac{dy}{dt} - \frac{\sinh^2 t}{\cosh t}\,y - \frac{y}{\cosh t} \\ &= \cosh t\cdot\frac{d^2y}{dt^2} + \sinh t\cdot\frac{dy}{dt} = \frac{d}{dt}\left\{\cosh t\cdot\frac{dy}{dt}\right\}. \end{aligned}$$

Then by an integration,

$$\cosh t \cdot \frac{dy}{dt} = e^t + c_2,$$

thus

$$\frac{dy}{dt} = \frac{e^t}{\cosh t} + \frac{c_2}{\cosh t} = \frac{2e^t}{e^t + e^{-t}} + 2c_2 \cdot \frac{e^t}{e^{2t}+1} = \frac{2e^{2t}}{e^{2t}+1} + 2c_2 \cdot \frac{e^t}{e^{2t}+1}.$$

By another integration,

$$y = \frac{x}{\cosh t} = \ln(1+e^{2t}) + 2c_2 \operatorname{Arctan}(e^t) + c_1,$$

hence

$$x = \cosh t \cdot \ln(1+e^{2t}) + c_1 \cosh t + c_2 \cosh t \cdot \operatorname{Arctan}(e^t).$$

3) We rearrange the equation by **inspection** in the following way,

$$\begin{aligned} e^t &= \frac{d^2x}{dt^2} - \left\{\tanh t \cdot \frac{dx}{dt} + \frac{d}{dt}(\tanh t)\cdot x\right\} \\ &= \frac{d^2x}{dt^2} - \frac{d}{dt}\{\tanh t \cdot x\} = \frac{d}{dt}\left\{\frac{dx}{dt} - \frac{\sinh t}{\cosh t}\,x\right\} \\ &= \frac{d}{dt}\left\{\cosh t\left[\frac{1}{\cosh t}\,\frac{dx}{dt} + \frac{d}{dt}\left(\frac{1}{\cosh t}\right)\cdot y\right]\right\} \\ &= \frac{d}{dt}\left\{\cosh t \cdot \frac{d}{dt}\left(\frac{x}{\cosh t}\right)\right\}. \end{aligned}$$

Then by an integration,

$$\cosh t \cdot \frac{d}{dt}\left(\frac{x}{\cosh t}\right) = e^t + c_2,$$

thus

$$\frac{d}{dt}\left(\frac{x}{\cosh t}\right) = \frac{2e^{2t}}{1+e^{2t}} + \frac{2c_2e^t}{1+e^{2t}}.$$

Finally, by another integration,

$$x = \cosh t \cdot \ln(1+e^{2t}) + 2c_2\cosh t \cdot \operatorname{Arctan}(e^t) + c_1\cosh t.$$

4) By the standard solution formula,

$$x_0(t) = \varphi_1(t)\int \frac{W_1(t)}{W(t)}\,dt + \varphi_2(t)\int \frac{W_2(t)}{W(t)}\,dt,$$

where

$$\begin{aligned} \varphi_2(t) &= \varphi_1(t)\int \frac{1}{\varphi_1(t)^2}\,\exp\left(-\int f_1(t)\,dt\right)dt = \cosh t\int \frac{\cosh t}{\cosh^2 t}\,dt = \cosh t\int \frac{2e^t}{e^{2t}+1}\,dt \\ &= 2\cosh t\cdot \operatorname{Arctan}(e^t), \end{aligned}$$

$$\begin{aligned} W(t) &= \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix} = \begin{vmatrix} \cosh t & 2\cosh t\cdot\operatorname{Arctan}(e^t) \\ \sinh t & 2\sinh t\cdot\operatorname{Arctan}(e^t) + 2\cosh t\cdot\dfrac{e^t}{e^{2t}+1} \end{vmatrix} \\ &= \begin{vmatrix} \cosh t & 0 \\ \sinh t & 1 \end{vmatrix} = \cosh t, \end{aligned}$$

$$W_1(t) = \begin{vmatrix} 0 & \varphi_2 \\ u(t) & \varphi_2' \end{vmatrix} = -2\cosh t\cdot\operatorname{Arctan}(e^t)\cdot e^t,$$

$$W_2(t) = \begin{vmatrix} \varphi_1 & 0 \\ \varphi_1' & u(t) \end{vmatrix} = \cosh t\cdot e^t.$$

By insertion,

$$\begin{aligned}
x_0(t) &= \cosh t \int \frac{-2\cosh t\cdot \mathrm{Arctan}(e^t)}{\cosh t}\, e^t\,dt + 2\cosh t\cdot \mathrm{Arctan}(e^t)\int \frac{\cosh t}{\cosh t}\, e^t\,dt \\
&= -2\cosh t \int_{u=e^t} \mathrm{Arctan}\ u\,du + 2\cosh t\cdot e^t\mathrm{Arctan}(e^t) \\
&= -2\cosh t\left[e^t\mathrm{Arctan}(e^t) - \int \frac{e^t\cdot e^t}{1+e^{2t}}\,dt\right] + 2\cosh t\cdot e^t\cdot \mathrm{Arctan}(e^t) \\
&= 2\cosh t \int \frac{e^2 t}{1+e^{2t}}\,dt = \cosh t\cdot \ln(1+e^{2t}).
\end{aligned}$$

Finally, we add the complete solution of the homogeneous equation to get

$$x = \cosh t\cdot \ln(1+e^{2t}) + c_1\cosh t + c_2\cosh t\cdot \mathrm{Arctan}(e^t).$$

Example 1.3 *Find the complete solution of the following differential equation, given that $\varphi_1(t) = \sin t$ is a solution of the corresponding homogeneous differential equation,*

$$\frac{d^2x}{dt^2} - 2\tan t\cdot \frac{dx}{dt} + 3x = 3\tan t, \qquad t\in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[.$$

Again there are several possible methods of solutions:

1) The equation is already normed, so we can immediately apply the solution formula,

$$x(t) = \varphi_1(t)\left\{\int \frac{1}{\varphi_1(t)^2\Omega(t)}\left(c_1 + \int \varphi_1(t)\Omega(t)u(t)\,dt\right)dt + c_2\right\},$$

where $\varphi_1(t) = \sin t$, $u(t) = 3\tan t$ and

$$\Omega(t) = \exp\left(-2\int \frac{\sin t}{\cos t}\,dt\right) = \cos^2 t.$$

Then we get for $t \neq 0$,

$$\begin{aligned}
x &= \sin t\left\{\int \frac{1}{\sin^2 t\cos^2 t}\left(c_1 + \int \sin t\cdot \cos^2 t\cdot 3\tan t\,dt\right)dt + c_2\right\} \\
&= \sin t\left\{c_1\int \frac{\sin^2 t+\cos^2 t}{\sin^2 t\cos^2 t}\,dt + \int \frac{1}{\sin^2 t\cos^2}\left(\int 3\sin^2 t\cdot \cos t\,dt\right)dt\right\} + c_2\sin t \\
&= c_1\sin t\int\left(\frac{1}{\cos^2 t} + \frac{1}{\sin^2 t}\right)dt + \sin t\int \frac{\sin t}{\cos^2 t}\,dt + c_2\sin t \\
&= c_1\sin t(\tan t - \cot t) + \tan t + c_2\sin t \\
&= \tan t + c_2\sin t + c_1\cdot \frac{\sin^2 t - \cos^2 t}{\cos t} \\
&= \tan t + c_2\sin t - c_1\,\frac{\cos 2t}{\cos t}.
\end{aligned}$$

By a continuous extension (or just checking) it follows that the final result holds in all of $\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$.

2) If we put $x = \varphi_1(t) \cdot y = \sin t \cdot y$, we get

$$\frac{dx}{dt} = \sin t \cdot \frac{dy}{dt} + \cos t \cdot y,$$

$$\frac{d^2x}{dt^2} = \sin t \cdot \frac{d^2y}{dt^2} + 2\cos t \cdot \frac{dy}{dt} - \sin t \cdot y,$$

which we put into the equation,

$$\begin{aligned} 3\tan t &= \frac{d^2x}{dt^2} - 2\tan t \cdot \frac{dx}{dt} + 3x \\ &= \sin t \cdot \frac{d^2y}{dt^2} + 2\cos t \cdot \frac{dy}{dt} - \sin t \cdot y - 2\frac{\sin^2 t}{\cos t}\,\frac{dy}{dt} - 2\sin t \cdot y + 3\sin t \cdot y \\ &= \sin t \cdot \frac{d^2y}{dt^2} + 2\,\frac{\cos^2 t - \sin^2 t}{\cos t} \cdot \frac{dy}{dt}. \end{aligned}$$

This is a differential equation of first order in $\dfrac{dy}{dt}$. When $t \neq 0$ we get by division by $\sin t$ that

$$\frac{d}{dt}\left(\frac{dy}{dt}\right) + 4\,\frac{\cos 2t}{\sin 2t}\,\frac{dy}{dt} = \frac{3}{\cos t}, \qquad t \neq 0.$$

This equation is most elegantly solved by multiplying by $\sin^2 2t$, because we then can write the resulting left hand side in the form $\dfrac{d}{dt}\left\{\sin^2 2t\,\dfrac{dy}{dt}\right\}$. We shall here only use the well-known solution formula. The homogeneous equation has the solution

$$\frac{4}{\sin^2 2t} = \frac{1}{\cos^2 t \sin^2 t}.$$

Then we get by the formula,

$$\frac{dy}{dt} = \frac{1}{\cos^2 t \sin^2 t}\left\{c_2' + 3\int \frac{\cos^2 t \cdot \sin^2 t}{\cos t}\,dt\right\} = \frac{4c_2'}{\sinh^2 2t} + \frac{\sin^3 t}{\cos^2 t \cdot \sin^2 t} = \frac{4c_2'}{\sin^2 2t} + \frac{\sin t}{\cos^2 t}.$$

By another integration,

$$y = -2c_2 \cot 2t + \frac{1}{\cos t} + c_1, \qquad t \neq 0,$$

hence

$$x = \tan t + c_1 \sin t + 2c_2' \,\frac{\cos^2 t - \sin^2 t}{2\sin t \cdot \cos t} \cdot \sin t = \tan t + c_1 \sin t + c_2 \cdot \frac{\sin^2 t - \cos^2 t}{\cos t}.$$

3) We compute for $t \neq 0$

$$\begin{aligned} \varphi_2(t) &= \varphi_1 \int \frac{1}{\varphi_1(t)^2}\,\exp(-\int f_1(t)\,dt)dt = \sin \int \frac{1}{\sin^2 t}\,\exp\left(+2\int \frac{\sin t}{\cos t}\,dt\right)dt \\ &= \sin t \int \frac{1}{\sin^2 t \cdot \cos^2 t}\,dt = \sin t \int \frac{\cos^2 t + \sin^2 t}{\sin^2 t \cdot \cos^2 t}\,dt \\ &= \sin t \int \left\{\frac{1}{\cos^2 t} + \frac{1}{\sin^2 t}\right\} dt = \sin t\{\tan t - \cot t\} \\ &= \sin t \cdot \frac{\sin^2 t - \cos^2 t}{\cos t \cdot \sin t} = \frac{\sin^2 t - \cos^2 t}{\cos t}. \end{aligned}$$

Then

$$
\begin{aligned}
W(t) &= \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix} = \begin{vmatrix} \sin t & \dfrac{\sin^2 t - \cos^2 t}{\cos t} \\ \cos t & 3\sin t + \dfrac{\sin^3 t}{\cos^2 t} \end{vmatrix} = 3\sin^2 t + \frac{\sin^4 t}{\cos^2 t} - \sin^2 t + \cos^2 t \\
&= 2\sin^2 t + \cos^2 t + \frac{\sin^4 t}{\cos^2 t} = \frac{\sin^4 t + 2\sin^2 t\cos^2 t + \cos^4 t}{\cos^2 t} = \frac{(\sin^2 t + \cos^2 t)^2}{\cos^2 t} \\
&= \frac{1}{\cos^2 t}.
\end{aligned}
$$

Thus a particular solution is given by

$$\begin{aligned}
x_0(t) &= \varphi_2(t)\int \frac{\varphi_1(t)u(t)}{W(t)}\,dt - \varphi_1(t)\int \frac{\varphi_2(t)u(t)}{W(t)}\,dt \\
&= \frac{\sin^2 t - \cos^2 t}{\cos t}\int \sin t\cdot\cos^3 t\cdot 3\tan t\,dt - \sin t\int \frac{\sin^2 t-\cos^2 t}{\cos t}\cdot\cos^2 t\cdot 3\tan t\,dt \\
&= \frac{\sin^2 t - \cos^2 t}{\cos t}\int 3\sin^2 t\cos t\,dt - \sin t\int 3(1-2\cos^2 t)\sin t\,dt \\
&= \frac{\sin^2 t - \cos^2 t}{\cos t}\cdot\sin^3 t + \sin t\{3\cos t - 2\cos^3 t\} \\
&= \frac{\sin t}{\cos t}\{\sin^4 t - \cos^2\sin^2 t + 3\cos^2 t - 2\cos^4 t\} \\
&= \tan t\{(1-\cos^2 t)^2 - \cos^2 t(1-\cos^2 t) + 3\cos^2 t - 2\cos^4 t\} \\
&= \tan t\{1 - 2\cos^2 t + \cos^4 t - \cos^2 t + \cos^4 t + 3\cos^2 t - 2\cos^4 t\} \\
&= \tan t.
\end{aligned}$$

Finally, the complete solution is

$$x = \tan t + c_1\sin t + c_2\cdot\frac{\cos^2 - \sin^2 t}{\cos t}.$$

Example 1.4 *Find the complete solution of the following differential equation, given that* $\varphi_1(t) = \dfrac{1}{\sqrt{t}}\cos t$ *is a solution of the corresponding homogeneous differential equation,*

$$t^2\frac{d^2x}{dt^2} + t\frac{dx}{dt} + \left(t^2 - \frac{1}{4}\right)x = t\sqrt{t}.$$

Here we shall give four different solution methods.

1) Since we have the factor $\sqrt{t}$ in the denominator, it is quite reasonable to put $x = \dfrac{y}{\sqrt{t}}$ and then derive a differential equation instead in y. We find

$$\frac{dx}{dt} = \frac{1}{\sqrt{t}}\frac{dy}{dt} - \frac{1}{2t\sqrt{t}}\,y,$$

$$\frac{d^2x}{dt^2} = \frac{1}{\sqrt{t}}\frac{d^2y}{dt^2} - \frac{1}{t\sqrt{t}}\frac{dy}{dt} + \frac{3}{4}\frac{1}{t^2\sqrt{t}}\,y.$$

Then by putting these expressions into the differential equation,

$$\begin{aligned}
t\sqrt{t} &= t^2\frac{d^2x}{dt^2} + t\frac{dx}{dt} + \left(t^2 - \frac{1}{4}\right)x \\
&= t\sqrt{t}\frac{d^2y}{dt^2} - \sqrt{t}\frac{dy}{dt} + \frac{3}{4}\frac{1}{\sqrt{t}}\,y + \sqrt{t}\frac{dy}{dt} - \frac{1}{2\sqrt{t}}\,y + t\sqrt{t}\,y - \frac{1}{4\sqrt{t}}\,y \\
&= t\sqrt{t}\frac{d^2y}{dt^2} + t\sqrt{t}\,y,
\end{aligned}$$

which by a division by $t\sqrt{t}$ is reduced to

$$\frac{d^2y}{dt^2}+y=1.$$

The complete solution is

$$y=1+c_1\cos t+c_2\sin t,$$

hence

$$x=\frac{1}{\sqrt{t}}+c_1\,\frac{\cos t}{\sqrt{t}}+c_2\,\frac{\sin t}{\sqrt{t}}.$$

2) When we norm the equation, we get

$$\frac{d^2x}{dt^2}+\frac{1}{t}\,\frac{dx}{dt}+\left(1-\frac{1}{4t^2}\right)x=\frac{1}{\sqrt{t}},\qquad t>0.$$

Now,

$$\Omega(t)=\exp\left(\int\frac{1}{t}\,dt\right)=t,$$

and $\varphi_1(t)=\dfrac{1}{\sqrt{t}}\cos t$ and $u(t)=\dfrac{1}{\sqrt{t}}$. We therefore formally get for $\cos t\neq 0$ that

$$\begin{aligned}
x(t) &= \varphi_1(t)\left(\int\frac{1}{\varphi_1(t)^2\Omega(t)}\left\{\varphi_1(t)\Omega(t)u(t)\,dt+c_2\right\}+c_1\right)\\
&= \frac{1}{\sqrt{t}}\cos t\left(\int\frac{t}{\cos^2 t}\cdot\frac{1}{t}\left\{c_2+\int\frac{\cos t}{\sqrt{t}}\cdot t\cdot\frac{1}{\sqrt{t}}\,dt\right\}dt+c_1\right)\\
&= \frac{1}{\sqrt{t}}\,\cos t\left(\int\frac{1}{\cos^2 t}\{c_2+\sin t\}dt\right)+c_1\cdot\frac{\cos t}{\sqrt{t}}\\
&= c_2\cdot\frac{1}{\sqrt{t}}\cdot\cos t\cdot\tan t+c_1\cdot\frac{cost}{\sqrt{t}}+\frac{1}{\sqrt{t}}\cdot\cos t\cdot\frac{1}{\cos t}\\
&= \frac{1}{\sqrt{t}}+c_1\,\frac{\cos t}{\sqrt{t}}+c_2\,\frac{\sin t}{\sqrt{t}}.
\end{aligned}$$

3) We get by norming the equation,

$$\frac{d^2x}{dt^2}+\frac{1}{t}\,\frac{dx}{dt}+\left(1-\frac{1}{4t^2}\right)x=\frac{1}{\sqrt{t}},$$

hence

$$\begin{aligned}
\varphi_2(t) &= \varphi_1(t)\int\frac{1}{\varphi_1(t)^2}\exp(-\int f_1(t)\,dt)dt=\frac{\cos t}{\sqrt{t}}\int\frac{t}{\cos^2 t}\,\exp\left(-\int\frac{dt}{t}\right)dt\\
&= \frac{\cos t}{\sqrt{t}}\int\frac{dt}{\cos^2 t}=\frac{\cos t}{\sqrt{t}}\cdot\tan t=\frac{\sin t}{\sqrt{t}}.
\end{aligned}$$

The Wroński determinant is

$$W(t)=\begin{vmatrix}\varphi_1 & \varphi_2\\ \varphi_1' & \varphi_2'\end{vmatrix}=\begin{vmatrix}\dfrac{\cos t}{\sqrt{t}} & \dfrac{\sin t}{\sqrt{t}}\\ -\dfrac{\sin t}{\sqrt{t}}-\dfrac{1}{2}\,\dfrac{\cos t}{t\sqrt{t}} & \dfrac{\cos t}{\sqrt{t}}-\dfrac{1}{2}\,\dfrac{\sin t}{t\sqrt{t}}\end{vmatrix}=\begin{vmatrix}\dfrac{\cos t}{\sqrt{t}} & \dfrac{\sin t}{\sqrt{t}}\\ -\dfrac{\sin t}{\sqrt{t}} & \dfrac{\cos t}{\sqrt{t}}\end{vmatrix}=\frac{1}{t}.$$

Then a particular solution is

$$\begin{aligned}x_0(t) &= \frac{\sin t}{\sqrt{t}}\int\frac{\cos t}{\sqrt{t}}\cdot t\cdot\frac{1}{\sqrt{t}}\,dt-\frac{\cos t}{\sqrt{t}}\int\frac{\sin t}{\sqrt{t}}\cdot t\cdot\frac{1}{\sqrt{t}}\,dt\\ &= \frac{\sin t}{\sqrt{t}}\int\cos t\,dt-\frac{\cos t}{\sqrt{t}}\int\sin t\,dt=\frac{\sin^2 t}{\sqrt{t}}+\frac{\cos^2 t}{\sqrt{t}}=\frac{1}{\sqrt{t}}.\end{aligned}$$

Summing up the complete solution is

$$x=\frac{1}{\sqrt{t}}+c_1\,\frac{\cos t}{\sqrt{t}}+c_2\,\frac{\sin t}{\sqrt{t}}.$$

4) If we put $x=y\cdot\varphi_1(t)=y\cdot\dfrac{1}{\sqrt{t}}\cos t$, we get

$$\frac{dx}{dt}=\frac{\cos t}{\sqrt{t}}\,\frac{dy}{dt}-\left(\frac{\cos t}{2t\sqrt{t}}+\frac{\sin t}{\sqrt{t}}\right)y,$$

$$\frac{d^2x}{dt^2} = \frac{\cos t}{\sqrt{t}}\,\frac{d^2y}{dt^2}-\left(\frac{\cos t}{t\sqrt{t}}+\frac{2\sin t}{\sqrt{t}}\right)\frac{dy}{dt}+\left(\frac{\sin t}{2t\sqrt{t}}+\frac{3\cos t}{4t^2\sqrt{t}}+\frac{\sin t}{2t\sqrt{t}}-\frac{\cos t}{\sqrt{t}}\right)y,$$

hence by insertion,

$$\begin{aligned}t\sqrt{t} &= t\sqrt{t}\cdot\cos t\cdot\frac{d^2y}{dt^2}-(\sqrt{t}\cos t+2t\sqrt{t}\sin t)\frac{dy}{dt}\\ &\quad+\left(\sqrt{t}\sin t+\frac{3}{4}\,\frac{\cos t}{\sqrt{t}}-t\sqrt{t}\cos t\right)y+\sqrt{t}\cos t\,\frac{dy}{dt}\\ &\quad-\left(\frac{\cos t}{2\sqrt{t}}+\sqrt{t}\sin t\right)y+\left(t\sqrt{t}\cos t-\frac{\cos t}{4\sqrt{t}}\right)y\\ &= t\sqrt{t}\cdot\cos t\cdot\frac{d^2y}{dt^2}-2t\sqrt{t}\cdot\sin t\cdot\frac{dy}{dt}.\end{aligned}$$

A multiplication by $\dfrac{\cos t}{t\sqrt{t}}$ reduces this equation to

$$\cos t=\cos^2 t\cdot\frac{d}{dt}\left(\frac{dy}{dt}\right)-2\sin t\cdot\cos t\cdot\frac{dy}{dt}=\frac{d}{dt}\left\{\cos^2 t\,\frac{dy}{dt}\right\},$$

hence by an integration,

$$\cos^2 t\,\frac{dy}{dt}=\sin t+c_2.$$

Then [for $\cos t \neq 0$],

$$\frac{dy}{dt} = \frac{\sin t}{\cos^2 t} + \frac{c_2}{\cos^t} = \frac{d}{dt}\left\{\frac{1}{\cos t} + c_2 \tan t\right\},$$

thus

$$y = \frac{1}{\cos t} + c_1 + c_2 \tan t.$$

Finally, we get

$$x = \frac{\cos t}{\sqrt{t}} \cdot y = \frac{1}{\sqrt{t}} + c_1 \cdot \frac{\cos t}{\sqrt{t}} + c_2 \cdot \frac{\sin t}{\sqrt{t}}.$$

Example 1.5 *Find the complete solution of the following differential equation, given that* $\varphi_1(t) = t \cdot \cosh t$ *is a solution of the corresponding homogeneous differential equation,*

$$t^2 \frac{d^2x}{dt^2} - 2t \frac{dx}{dt} - (t^2-2)x = t^3, \qquad t \in \mathbb{R}_+.$$

Here we have at least four different solution methods.

1) **Intuition**. The structure of $\varphi_1(t)$ invites one to put $x = t \cdot y$. Then

$$\frac{dx}{dt} = t\frac{dy}{dt} + y \qquad \text{og} \qquad \frac{d^2x}{dt^2} = t\frac{d^2y}{dtr} + 2\frac{dy}{dt}.$$

If these results are put into the differential equation, we get

$$\begin{aligned} t^3 &= t^2\frac{d^2x}{dt^2} - 2t\frac{dx}{dt} - (t^2-2)x = t^3\frac{d^2x}{dt^2} + 2t^2\frac{dy}{dt} - 2t^2\frac{dy}{dt} - 2ty - t^3y + 2ty \\ &= t^3\frac{d^2y}{dt^2} - t^3y. \end{aligned}$$

If we divide by $t^3 > 0$, then

$$\frac{d^2y}{dt^2} - y = 1,$$

which is immediately solved,

$$y = -1 + c_1\cosh t + c_2\sinh t.$$

Then

$$x = t \cdot y = -t + c_1 t \cdot \cosh t + c_2 t \cdot \sinh t.$$

2) **Solution formula**. We first norm the equation,

$$\frac{d^2x}{dt^2} - \frac{2}{t}\frac{dx}{dt} - \left(1 - \frac{2}{t^2}\right)x = t, \qquad t > 0.$$

Then

$$\Omega(t) = \exp\left(-\int \frac{2}{t}\,dt\right) = \frac{1}{t^2}.$$

Since $\varphi_1(t) = t \cdot \cosh t$ and $u(t) = t$, it follows that

$$\begin{aligned} x &= \varphi_1(t)\left\{\int \frac{1}{\varphi_1(t)^2\Omega(t)}\left(\varphi_1(t)\Omega(t)u(t)\,dt + c_2\right) + c_1\right\} \\ &= t\cdot\cosh t\left\{\frac{t^2}{t^2\cosh^2 t}\left(c_2 + \int t\cdot\cosh t\cdot\frac{1}{t^2}\cdot t\,dt\right)dt\right\} + c_1 t\cdot\cosh t \\ &= t\cdot\cosh t\left\{\int\frac{1}{\cosh^2 t}(c_2+\sinh t)dt\right\} + c_1\cdot t\cdot\cosh t \\ &= c_1 t\cosh t + c_2 t\cosh t\cdot\tanh t - t\cosh t\cdot\frac{1}{\cosh t} \\ &= -t + c_1 t\cosh t + c_2 t\sinh t. \end{aligned}$$

3) First norm the equation,

$$\frac{d^2x}{dt^2} - \frac{2}{t}\,\frac{dx}{dt} - \left(1 - \frac{2}{t^2}\right) x = t, \qquad t > 0.$$

Using that $\varphi_1(t) = t \cdot \cosh t$ we get

$$\begin{aligned}\varphi_2(t) &= \varphi_1(t)\int \frac{1}{\varphi_1(t)^2}\exp(-\int f_1(t)\,dt)dt = t\cdot\cosh t\int \frac{1}{t^2\cosh^2 t}\exp\left(+\int \frac{2}{t}\,dt\right)dt \\ &= t\cdot\cosh t\int \frac{dt}{\cosh^2 t} = t\cdot\cosh t\cdot\tanh t = t\cdot\sinh t.\end{aligned}$$

The Wroński determinant is

$$\begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix} = \begin{vmatrix} t\cosh t & t\sinh t \\ t\sinh t + \cosh t & t\cosh t + \sinh t \end{vmatrix} = t^2.$$

A particular solution is

$$x_0(t) = \varphi_2(t)\int \frac{\varphi_1 u}{W}\,dt - \varphi_1(t)\int \frac{\varphi_1 u}{W}\,dt = t\cdot\sinh t\int \frac{t\cosh t\cdot t}{t^2}\,dt - t\cdot\cosh^2 t = -t.$$

The complete solution is

$$x = -t + c_2 t\cosh t + c_2 t\sinh t.$$

4) If we put $x = \varphi_1(t)\cdot y = t\cdot\cosh t\cdot y$, then

$$\frac{dx}{dt} = t\cdot\cosh t\cdot\frac{dy}{dt} + (\cosh t + t\cdot\sinh t)y,$$

$$\frac{d^2x}{dt^2} = t\cdot\cosh t\cdot\frac{d^2y}{dt^2} + 2(\cosh t + t\cdot\sinh t)\frac{dy}{dt} + (2\sinh t + t\cdot\cosh t)y.$$

Then by insertion into the equation,

$$\begin{aligned}t^3 &= t^2\,\frac{d^2x}{dt^2} - 2t\,\frac{dx}{dt} - (t^2-2)x \\ &= t^3\cosh t\cdot\frac{d^2y}{dt^2} + 2(t^2\cosh t + t^3\sinh t)\frac{dy}{dt} \\ &\qquad +(2t^2\sinh t + t^3\cosh t)y - 2t^2\cosh t\cdot\frac{dy}{dt} \\ &\qquad -(2t\cosh t + 2t^2\sinh t)y - (t^3\cosh t - 2t\cosh t)y \\ &= t^3\cosh t\cdot\frac{d^2y}{dt^2} + 2t^3\sinh t\cdot\frac{dy}{dt},\end{aligned}$$

which is reduced to

$$\cosh t\cdot\frac{d}{dt}\left(\frac{dy}{dt}\right) + 2\sinh t\cdot\frac{dy}{dt} = 1.$$

If we multiply by $\cosh t$, we get

$$\cosh t = \cosh^2 t\cdot\frac{d}{dt}\left(\frac{dy}{dt}\right) + 2\sinh t\cdot\cosh t\cdot\frac{dy}{dt} = \frac{d}{dt}\left(\cosh^2 t\,\frac{dy}{dt}\right).$$

Then by an integration,

$$\cosh^2 t\,\frac{dy}{dt} = \sinh t + c_2,$$

thus

$$\frac{dy}{dt} = \frac{\sinh t}{\cosh^2 t} + \frac{c_2}{\cosh^2 t}.$$

By another integration,

$$y = -\frac{1}{\cosh t} + c_1 + c_2 \tanh t,$$

and hence

$$x = t\cdot \cosh t \cdot y = -t + c_1 t \cosh t + c_2 t \cdot \sinh t.$$

Example 1.6 *Find the complete solution of the following differential equation, given that* $\varphi_1(t) = t$ *is a solution of the corresponding homogeneous differential equation,*

$$(1-t^2)\frac{d^2x}{dt^2} - t\,\frac{dx}{dt} + x = t, \qquad t \in]-1,1[.$$

We have at least four different variants of solutions:

1) **Intuition**. If we add $-t\,\dfrac{dx}{dt} + t\,\dfrac{dx}{dt} = 0$, and then perform some small rearrangements, we get

$$\begin{aligned} t &= (1-t^2)\frac{d^2x}{dt^2} - t\,\frac{dx}{dt} + x \\ &= \left\{(1-t^2)\frac{d^2x}{dt^2} - 2t\,\frac{dx}{dt}\right\} + \left\{t\,\frac{dx}{dt} + x\right\} \\ &= \frac{d}{dx}\left\{(1-t^2)\frac{dx}{dt}\right\} + \frac{d}{dt}\{tx\} = \frac{d}{dt}\left\{(1-t^2)\frac{dx}{dt} + tx\right\}. \end{aligned}$$

We get by an integration,

$$(1-t^2)\frac{dx}{dt} + tx = \frac{1}{2}(t^2-1) + c \qquad \left[= \frac{1}{2}t^2 + \left(c_2 - \frac{1}{2}\right)\right].$$

Then divide by $(\sqrt{1-t^2})^3$,

$$-\frac{1}{2}\,\frac{1}{\sqrt{1-t^2}} + c_2\,\frac{1}{(\sqrt{1-t^2})^3} = \frac{1}{\sqrt{1-t^2}}\,\frac{dx}{dt} + \frac{t}{(\sqrt{1-t^2})^3}\,x = \frac{d}{dt}\left(\frac{x}{\sqrt{1-t^2}}\right).$$

This equation can immediately be integrated,

$$\begin{aligned} \frac{x}{\sqrt{1-t^2}} &= -\frac{1}{2}\int \frac{dt}{\sqrt{1-t^2}} + c_1 + c_2\int \frac{1}{(\sqrt{1-t^2})^3}\,dt \\ &= -\frac{1}{2}\,\text{Arcsin}\ t + c_1 + c_2\int_{t=\sin u} \frac{\cos u}{\cos^3 u}\,du \\ &= -\frac{1}{2}\,\text{Arcsin}\ t + c_1 + c_2\tan(\text{Arcsin}\ t) \\ &= -\frac{1}{2}\,\text{Arcsin}\ t + c_1 + c_2\,\frac{t}{\sqrt{1-t^2}}, \end{aligned}$$

and we end up with the complete solution

$$x = -\frac{1}{2}\sqrt{1-t^2}\cdot \operatorname{Arcsin} t + c_1\sqrt{1-t^2} + c_2 t.$$

2) If we put $x = \varphi_1(t)\cdot y = t\cdot y$, then

$$\frac{dx}{dt} = t\,\frac{dy}{dt} + y \qquad \text{and} \qquad \frac{d^2x}{dt^2} = t\,\frac{d^2y}{dt^2} + 2\,\frac{dy}{dt}.$$

Then by insertion into the differential equation,

$$\begin{aligned} t &= (1-t^2)\frac{d^2x}{dt^2} - t\,\frac{dx}{dt} + x = (1-t^2)t\,\frac{d^2y}{dt^2} + 2(1-t^2)\frac{dy}{dt} - t^2\,\frac{dy}{dt} - ty + ty \\ &= \left\{(t-t^3)\frac{d^2y}{dt^2} + (1-3t^2)\frac{dy}{dt}\right\} + \frac{dy}{dt} = \frac{d}{dt}\left\{t(1-t^2)\frac{dy}{dt} + y\right\}. \end{aligned}$$

Thus by an integration

$$t(1-t^2)\frac{dy}{dt} + y = \frac{1}{2}(t^2-1) + c_2 \qquad \left[= \frac{1}{2}\,t^2 + \left(c_2 - \frac{1}{2}\right)\right].$$

We norm the equation,

$$\frac{dy}{dt} + \frac{t}{t^2(1-t^2)}\, y = -\frac{1}{2}\cdot\frac{1}{t} + c_2 \cdot \frac{1}{t(t-1)}.$$

Now,

$$\begin{aligned}\exp\left(-\int \frac{t}{t^2(1-t^2)}\,dt\right) &= \exp\left(-\frac{1}{2}\int\left\{\frac{1}{t^2}+\frac{1}{1-t^2}\right\}2t\,dt\right)\\ &= \exp\left(-\frac{1}{2}\ln(t^2)+\frac{1}{2}\ln(1-t^2)\right) = \frac{\sqrt{1-t^2}}{|t|}, \quad 0<|t|<1,\end{aligned}$$

so we get by the usual solution formula that

$$\begin{aligned} y &= \frac{\sqrt{1-t^2}}{t}\int \frac{t}{\sqrt{1-t^2}}\left(-\frac{1}{2}\cdot\frac{1}{t}+c_1\cdot\frac{1}{t(1-t^2)}\right)dt + c_2\cdot\frac{\sqrt{1-t^2}}{t}\\ &= -\frac{1}{2}\cdot\frac{\sqrt{1-t^2}}{t}\cdot \text{Arcsin}\, t + c_2\cdot\frac{\sqrt{1-t^2}}{t} + c_1\cdot\frac{\sqrt{1-t^2}}{t}\int\frac{1}{(\sqrt{1-t^2})^3}\,dt.\end{aligned}$$

Since

$$\int\frac{1}{(\sqrt{1-t^2})^3}\,dt = \int_{t=\sin u}\frac{\cos u}{\cos^3 u}\,du = \tan(\text{Arcsin}\, t) = \frac{t}{\sqrt{1-t^2}},$$

we finally get

$$y = -\frac{1}{2}\cdot\frac{\sqrt{1-t^2}}{t}\cdot\text{Arcsin}\, t + c_1 + c_2\cdot\frac{\sqrt{1-t^2}}{t},$$

and the complete solution of the original equation is

$$x = t\cdot y = -\frac{1}{2}\sqrt{1-t^2}\cdot\text{Arcsin}\, t + c_1 t + c_2\sqrt{1-t^2}.$$

3) First norm the equation,

$$\frac{d^2x}{dt^2} - \frac{t}{1-t^2}\,\frac{dx}{dt} + \frac{1}{1-t^2}\,x = \frac{t}{1-t^2}.$$

Here $\varphi_1(t) = t$ and $u(t) = \dfrac{t}{1-t^2}$ and

$$\Omega(t) = \exp\left(-\int\frac{t}{1-t^2}\,dt\right) = \sqrt{1-t^2},$$

so

$$\begin{aligned} x &= t\left\{\int\frac{1}{t^2\sqrt{1-t^2}}\left(c_1+\int t\cdot\sqrt{1-t^2}\cdot\frac{t}{1-t^2}\,dt\right)dt + c_2\right\}\\ &= c_2 t + c_1 t\int\frac{dt}{t^2\sqrt{1-t^2}} + t\left\{\int\frac{1}{t^2\sqrt{1-t^2}}\left(\int\frac{t^2}{\sqrt{1-t^2}}\,dt\right)dt\right\}.\end{aligned}$$

A couple of computations give

$$\int \frac{dt}{t^2\sqrt{1-t^2}} = \int_{t=\sin u} \frac{\cos u}{\sin^2 u \cdot \cos u}\, du = -\cot(\text{Arcsin } t) = -\frac{\sqrt{1-t^2}}{t},$$

and

$$\begin{aligned}\int \frac{t^2}{\sqrt{1-t^2}}\, dt &= \int_{t=\sin u}^{2} u\, du = \frac{1}{2}\int_{t=\sin u}(1-\cos 2u)du \\ &= \frac{1}{2}\left[u - \frac{1}{2}\sin 2u\right]_{t=\sin u} = \frac{1}{2}\text{Arcsin } t - \frac{1}{2}t\sqrt{1-t^2},\end{aligned}$$

hence

$$\int \frac{1}{t^2\sqrt{1-t^2}}\left(\int \frac{t^2}{\sqrt{1-t^2}}\, dt\right) dt = \frac{1}{2}\int \frac{\text{Arcsin } t}{t^2\sqrt{1-t^2}}\, dt - \frac{1}{2}\int \frac{t\sqrt{1-t^2}}{t^2\sqrt{1-t^2}}\, dt,$$

where we for $t \neq 0$ see that

$$\begin{aligned}\frac{1}{2}\int \frac{\text{Arcsin } t}{t^2\sqrt{1-t^2}}\, dt &= \frac{1}{2}\int_{t=\sin u} \frac{u\cos u}{\sin^2 u \cos u}\, du = \frac{1}{2}\int_{t=\sin u} \frac{u}{\sin^2 u}\, du \\ &= \frac{1}{2}[-u\cos u]_{t=\sin u} + \frac{1}{2}\int_{t=\sin u} \frac{\cos u}{\sin u}\, du \\ &= -\frac{1}{2}\text{Arcsin } t \cdot \frac{\sqrt{1-t^2}}{t} + \frac{1}{2}\ln|t|\end{aligned}$$

and

$$-\frac{1}{2}\int \frac{t\sqrt{1-t^2}}{t^2\sqrt{1-t^2}}\, dt = -\frac{1}{2}\ln|t|, \qquad t \neq 0,$$

thus

$$\int \frac{1}{t^2\sqrt{1-t^2}}\left(\int \frac{t^2}{\sqrt{1-t^2}}\, dt\right) dt = -\frac{1}{2}\text{Arcsin } t \cdot \frac{\sqrt{1-t^2}}{t},$$

because the additional terms $\frac{1}{2}\ln|t| - \frac{1}{2}\ln|t| = 0$ cancel. Finally, by insertion,

$$x = -\frac{1}{2}\text{Arcsin } t \cdot \sqrt{1-t^2} - c_1\sqrt{1-t^2}t + c_2 t.$$

4) The normed equation is

$$\frac{d^2x}{dt^2} - \frac{t}{1-t^2}\frac{dx}{dt} + \frac{1}{1-t^2}\, x = \frac{t}{1-t^2},$$

so

$$\begin{aligned}\varphi_2(t) &= \varphi_1(t)\int \frac{1}{\varphi_1(t)^2}\, \exp(-\int f_1(t)\, dt)dt = t\int \frac{dt}{t^2\sqrt{1-t^2}} = t\int_{t=\sin u} \frac{\cos u}{\sin^2 u \cdot \cos u}\, du \\ &= -t\cot(\text{Arcsin } t) = -\sqrt{1-t^2}.\end{aligned}$$

The Wroński determinant is

$$W(t) = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix} = \begin{vmatrix} t & -\sqrt{1-t^2} \\ 1 & \dfrac{t}{\sqrt{1-t^2}} \end{vmatrix} = \frac{t^2}{\sqrt{1-t^2}} + \sqrt{1-t^2} = \frac{1}{\sqrt{1-t^2}}.$$

Then a particular solution is given by

$$\begin{aligned}
x_0(t) &= \varphi_2(t)\int \frac{\varphi_1 u}{W}\,dt - \varphi_1(t)\int \frac{\varphi_2 u}{W}\,dt \\
&= -\sqrt{1-t^2}\int t\cdot\sqrt{1-t^2}\cdot\frac{t}{1-t^2}\,dt + t\int \sqrt{1-t^2}\cdot\sqrt{1-t^2}\cdot\frac{t}{1-t^2}\,dt \\
&= -\sqrt{1-t^2}\int \frac{t^2}{\sqrt{1-t^2}}\,dt + t\int t\,dt = \sqrt{1-t^2}\int \frac{1-t^2-1}{\sqrt{1-t^2}}\,dt + \frac{1}{2}\,t^3 \\
&= \sqrt{1-t^2}\int \left\{\sqrt{1-t^2} - \frac{1}{\sqrt{1-t^2}}\right\}dt + \frac{1}{2}\,t^3 = \sqrt{1-t^2}\int_{t=\sin u}(\cos^2 u - 1)du + \frac{1}{2}\,t^3 \\
&= \sqrt{1-t^2}\int_{t=\sin u}\frac{1}{2}(\cos 2u - 1)du + \frac{1}{2}\,t^3 \\
&= \sqrt{1-t^2}\left[\frac{1}{4}\sin(2\,\mathrm{Arcsin}\,t\right] - \frac{1}{2}\sqrt{1-t^2}\cdot\mathrm{Arcsin}\,t + \frac{1}{2}\,t^3 \\
&= -\frac{1}{2}\sqrt{1-t^2}\cdot\mathrm{Arcsin}\,t + \frac{1}{2}\,t^3 + \frac{1}{2}(t-t^3) = -\frac{1}{2}\sqrt{1-t^2}\cdot\mathrm{Arcsin}\,t + \frac{1}{2}\cdot t.
\end{aligned}$$

Since $\frac{1}{2}\,t$ is a solution of the homogeneous equation, the complete solution is

$$x = -\frac{1}{2}\sqrt{1-t^2}\,\mathrm{Arcsin}\,t + c_1 t + c_2\sqrt{1-t^2}.$$

Example 1.7 *Find the complete solution of the differential equation*

$$(2)\;\; t(t+1)\frac{d^2x}{dt^2} + (2-t^2)\frac{dx}{dt} - (2+t)x = (t+1)^2, \qquad t\in\mathbb{R}_+,$$

by first guessing some power function solution of the corresponding homogeneous differential equation. Then find the solution of (2) through the line element $\left(1, e-\frac{5}{2}, e+\frac{1}{2}\right)$*, and prove that this is an increasing function.*

If $x = t^n$, then $\dfrac{dx}{dt} = nt^{n-1}$ and $\dfrac{d^2x}{dt^2} = n(n-1)t^{n-2}$. By putting these expressions into the left hand side of (2) we get

$$\begin{aligned}
&t(t+1)\frac{d^2x}{dt^2} + (2-t^2)\frac{dx}{dt} - (2+t)x \\
&\quad = (t^2+t)n(n-1)t^{n-2} + (2-t^2)nt^{n-1} - 2t^n - t^{n+1} \\
&\quad = n(n-1)t^n + n(n-1)t^{n-1} + 2nt^{n-1} - nt^{n+1} - 2t^n - t^{n+1} \\
&\quad = (n^2-n-2)t^n + n(n+1)t^{n-1} - (n+1)t^{n+1} \\
&\quad = (n+1)\left\{(n-2)t^n + nt^{n-1} - t^{n+1}\right\}.
\end{aligned}$$

This expression is identically 0 for $t > 0$, if and only if $n = -1$. We conclude that

$$y_1(t) = \frac{1}{t}, \qquad t > 0,$$

is a solution of the homogeneous equation.

In order to proceed we then must norm the equation. Now, $t(t+1) > 0$ for $t > 0$, so this is possible in $\mathbb{R}_+$, and

$$\frac{d^2x}{dt^2} + \frac{2-t^2}{t(t+1)}\,\frac{dx}{dt} - \frac{t+2}{t(t+1)}\,x = \frac{t+1}{t}.$$

Then we apply a solution formula. We first compute

$$\Omega(t) = \exp\left(\int \frac{2-t^2}{t(t+1)}\,dt\right).$$

By a decomposition,

$$\frac{t^2-2}{t(t+1)} = \frac{t^2+t-t-2}{t^2+t} = 1-\frac{t+2}{t(t+1)} = 1-\frac{2}{t}+\frac{1}{1+t},$$

thus

$$\int \frac{t^2-2}{t(t+1)}\,dt = \int\left\{1-\frac{2}{t}+\frac{1}{1+t}\right\}dt = t-2\ln t+\ln(1+t)$$

for $t>0$, and hence

$$\Omega(t) = \exp\left(-\int \frac{t^2-2}{t(t+1)}\,dt\right) = \frac{t^2}{t+1}\cdot e^{-t}$$

(notice the change of sign).

A linearly independent solution of the homogeneous equation is obtained by putting $c_1=1$ and $c_2=0$ and $u=0$ in some formula, by which

$$y_2(t) = y_1(t)\int \frac{dt}{y_1(t)^2\Omega(t)} = \frac{1}{t}\int t^2\cdot\frac{t+1}{t^2}\cdot e^t\,dt = \frac{1}{t}\int (t+1)e^t\,dt = \frac{1}{t}\cdot t\,e^t = e^t.$$

A particular solution of the inhomogeneous equation is obtained by putting $c_1=c_2=0$ and $u(t)=\dfrac{t+1}{t}$ into some formula, thus

$$\begin{aligned}
y_0(t) &= \int \frac{1}{y_1(t)^2\Omega(t)}\left\{\int y_1(t)\Omega(t)u(t)\,dt\right\}dt \\
&= \frac{1}{t}\int t^2\cdot\frac{t+1}{t^2}\,e^t\left\{\int\frac{1}{t}\cdot\frac{t^2}{t+1}\cdot e^{-t}\cdot\frac{t+1}{t}\,dt\right\}dt \\
&= \frac{1}{t}\int (t+1)e^t\left\{\int e^{-t}dt\right\}dt = -\frac{1}{t}\int (t+1)e^t\cdot e^{-t}dt \\
&= -\frac{1}{t}\int (t+1)dt = -\frac{1}{t}\left\{\frac{1}{2}t^2+t\right\} = -1-\frac{1}{2}\,t.
\end{aligned}$$

The complete solution is

$$y(t) = -1-\frac{1}{2}\cdot t+c_1\cdot\frac{1}{t}+c_2\cdot e^t.$$

Of course, there are other **variants** of the computations above. We shall not give them here.

If

$$y(t) = -1-\frac{t}{2}+c_1\cdot\frac{1}{t}+c_2\cdot e^t,$$

then

$$y'(t) = -\frac{1}{2}-c_1\cdot\frac{1}{t^2}+c_2\cdot e^t.$$

Then by the line element

$$(1,\tilde{y}(1),\tilde{y}'(1)) = \left(1, e-\frac{5}{2}, e+\frac{1}{2}\right)$$

we get

$$e - \frac{5}{2} = \tilde{y}(1) = -1 - \frac{1}{2} + c_1 + c_2 \cdot e, \qquad \text{thus } c_1 + ec_2 = e - 1,$$

and

$$e + \frac{1}{2} = \tilde{y}'(t) = -\frac{1}{2} - c_1 + c_2 \cdot e, \qquad \text{thus } -c_1 + ec_2 = e + 1.$$

It follows immediately that $c_1 = -1$ and $c_2 = 1$, and the wanted solution is

$$\tilde{y}(t) = -1 - \frac{t}{2} - \frac{1}{t} + e^t.$$

Since

$$\tilde{y}'(t) = -\frac{1}{2} + \frac{1}{t^2} + e^t \geq -\frac{1}{2} + 0 + e^0 = -\frac{1}{2} + 1 = \frac{1}{2} > 0,$$

it follows that $\tilde{y}(t)$ is increasing.

Remark 1.1 Here we have used that $\dfrac{1}{t^2} > 0$ and that $e^t > e^0 = 1$ for $t > 0$.

Example 1.8 *Solve the differential equation*

$$(1+t^2)\frac{d^2x}{dt^2} + 2t\,\frac{dx}{dt} - 2x = 4t^2 + 2, \qquad t \in \mathbb{R}_+,$$

given that is has some polynomial of second degree as a solution.

We demonstrate three variants of solutions:

1) It is easily seen that the solution of the homogeneous equation is

$$x = c_1 t + c_2(1 + t \cdot \text{Arctan } t).$$

According to the hint we may guess a particular solution of the form $x = at^2 + b$, where we can neglect the term $c \cdot t$, because it is a solution of the homogeneous equation. By insertion into the left hand side of the equation we get

$$2a(1+t^2) + 2 \cdot 2at^2 - 2at^2 - 2b = 4at^2 + (2a - 2b).$$

This expression is equal to $4t^2 + 2$, if $a = 1$ and $b = 0$. Hence the complete solution is

$$x = t^2 + c_1 t + c_2(1 + t \cdot \text{Arctan } t).$$

2) We apply again that the solution of the homogeneous equation is

$$x = c_1 t + c_2(1 + t \cdot \text{Arctan } t).$$

Choose $\varphi_1(t) = 1$. When we norm the equation, we get

$$\frac{d^2x}{dt^2} + \frac{2t}{1+t^2}\,\frac{dx}{dt} - \frac{2}{1+t^2}\,x = \frac{4t^2+2}{t^2+1},$$

so

$$\Omega(t) = \exp\left(\int \frac{2t}{1+t^2}\,dt\right) = t^2+1.$$

Then by a solution formula,

$$\begin{aligned}
x(t) &= t\left(\int \frac{1}{t^2(t^2+1)}\left\{c_1+\int (t^2+1)t\cdot\frac{4t^2+2}{t^2+1}\,dt\right\}dt+c_2\right)\\
&= c_2t + c_1t\int\left(\frac{1}{t^2}-\frac{1}{t^2+1}\right)dt + t\int\frac{1}{t^2(t^2+1)}\{\int(4t^3+2t)dt\}dt\\
&= t\int\frac{t^4+t^2}{t^2(t^2+1)}\,dt + c_2 - c_1t\left(\frac{1}{t}+\text{ Arctan } t\right)\\
&= t^2+c_2t-c_1(1+t\cdot\text{Arctan } t).
\end{aligned}$$

3) If we multiply the equation by t, we obtain by some deftness that

$$\begin{aligned}
&(t+t^3)\frac{d^2x}{dt^2}+2t^2\frac{dx}{dt}-2tx = \left\{(t+t^3)\frac{d^2x}{dt^2}+(1+3t^2)\frac{dx}{dt}\right\}-\left\{(1+t^2)\frac{dx}{dt}+2tx\right\}\\
&\quad= \frac{d}{dt}\left\{t(1+t^2)\frac{dx}{dt}-(1+t^2)c\right\} = \frac{d}{dt}\left\{t^2(t+t^2)\left[\frac{1}{t}\cdot\frac{dx}{dt}-\frac{1}{t^2}\cdot x\right]\right\}\\
&\quad= \frac{d}{dt}\left\{t^2(t^2+1)\frac{d}{dt}\left(\frac{x}{t}\right)\right\} = 4t^3+2t.
\end{aligned}$$

Then by an integration for $t \neq 0$,

$$t^2(t^2+1)\frac{d}{dt}\left(\frac{x}{t}\right) = t^4+t^2-c_2,$$

so if we divide by $t^2(t^2+1)$ (for $t \neq 0$), we get

$$\frac{d}{dt}\left(\frac{x}{t}\right) = 1 - c_2\left(\frac{1}{t^2}-\frac{1}{t^2+1}\right).$$

Another integration gives

$$\frac{x}{t} = t + c_1 + c_2\left(\frac{1}{t}+\text{ Arctan } t\right),$$

and thus

$$x = t^2+c_1t+c_2(1+t\cdot\text{Arctan } t).$$

Example 1.9 *Find the complete solution of the differential equation*

$$(3)\quad t^2\frac{d^2x}{dt^2} - 2t\,\frac{dx}{dt} + (t^2+2)x = t^3$$

in each of the intervals $]-\infty,0[$ *and* $]0,\infty[$*, given that the homogeneous differential equation corresponding to (3) has the solution* $\varphi_1(t) = t\cdot\sin t$*.*
Explain why anyone of the found solutions can be extended in precisely one way by one of the other solutions to a single solution of (3) all over $\mathbb{R}$*.*
Then find the complete solution of (3) in $\mathbb{R}$*.*
Find those line elements $(0,x_0,p)$*, which allow at least one solution of (3).*

We give here four variants of solutions.

1) The factor t in $\varphi_1(t) = t\cdot\sin t$ invites one to put $x = t\cdot y$ in order to obtain a simpler equation in y. We get by a couple of computations,

$$\frac{dx}{dt} = t\,\frac{dy}{dt} + y \qquad \text{and} \qquad \frac{d^2x}{dt^2} = t\,\frac{d^2y}{dt^2} + 2\,\frac{dy}{dt},$$

hence by insertion into (3),

$$t^3 = t^2\frac{d^2x}{dt^2} - 2t\,\frac{dx}{dt} + (t^2+2)x = t^3\frac{d^2y}{dt^2} + 2t^2\,\frac{dy}{dt} - 2t^2\frac{dy}{dt} - 2ty + t^3y + 2ty = t^3\frac{d^2y}{dt^2} + t^3y.$$

Thus, if $t \neq 0$, then

$$\frac{d^2y}{dt^2} + y = 1,$$

the complete solution of which is

$$y = 1 + c_1 \sin t + c_2 \cos t.$$

Then we obtain the complete solution of the original equation by a multiplication by t,

$$x = t + c_1 t \sin t + c_2 t \cos t, \qquad t \neq 0.$$

2) If we instead put $x = \varphi_1(t)y = t \sin t \cdot y$, then

$$\frac{dx}{dt} = t \sin t \, \frac{dy}{dt} + (t \cos t + \sin t)y,$$

$$\frac{d^2x}{dt^2} = t \sin t \, \frac{d^2y}{dt^2} + 2(t \cos t + \sin t)\frac{dy}{dt} + (-t \sin t + 2 \cos t)y,$$

hence by insertion into (3),

$$\begin{aligned} t^3 &= t^2 \frac{d^2x}{dt^2} - 2t \, \frac{dx}{dt} + (t^2+2)x \\ &= t^3 \sin t \, \frac{d^2y}{dt^2} + 2t^2(t \cos t + \sin t)\frac{dy}{dt} \\ &\qquad + t^2(-t \sin t + 2 \cos t)y - 2t^2 \sin t \, \frac{dy}{dx} \\ &\qquad - 2t(t \cos t + \sin t)y + (t^2+2)t \sin t \cdot y \\ &= t^3 \sin t \, \frac{d^2y}{dt^2} + 2t^3 \cos t \, \frac{dy}{dt}. \end{aligned}$$

When $t \neq p\pi$, $p \in \mathbb{Z}$, this equation is reduced to

$$\frac{d^2y}{dt^2} + 2 \, \frac{\cos t}{\sin t} \, \frac{dy}{dt} = \frac{1}{\sin t}.$$

Since

$$\exp\left(-2 \int \frac{\cos t}{\sin t} \, dt\right) = \frac{1}{\sin^2 t},$$

it follows that

$$\frac{dy}{dt} = \frac{1}{\sin^2 t} \int \frac{\sin^2 t}{\sin t} \, dt - \frac{c_2}{\sin^2 t} = -\frac{\cos t}{\sin^2 t} - \frac{c_2}{\sin^2 t} = \frac{d}{dt}\left\{\frac{1}{\sin t} + c_2 \cot t\right\}.$$

Another integration gives

$$y = \frac{1}{\sin t} + c_1 + c_2 \cdot \frac{\cos t}{\sin t},$$

and we find the complete solution

$$x = t \sin t \cdot y = t + c_1 t \sin t + c_2 t \cos t, \qquad t \neq p\pi, \; p \in \mathbb{Z}.$$

3) In the following variant we start by norming the equation for $t \neq 0$,

$$\frac{d^2x}{dt^2} - \frac{2}{t}\,\frac{dx}{dt} + \left(1 + \frac{2}{t^2}\right) x = t.$$

Since $\varphi_1(t) = t\sin t$, it follows for $t \neq p\pi$, $p \in \mathbb{Z}$ that

$$\varphi_2(t) = t\sin t \int \frac{t^2}{t^2\sin^2 t}\,dt = -t\sin t\cdot\cot t = -t\cos t.$$

The Wroński determinant is

$$W(t) = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix} = \begin{vmatrix} t\sin t & -t\cos t \\ t\cos t + \sin t & t\sin t - \cos t \end{vmatrix} = \begin{vmatrix} t\sin t & -t\cos t \\ t\cos t & t\sin t \end{vmatrix} = t^2.$$

Then a particular solution is given by

$$\begin{aligned} x_0(t) &= \varphi_2(t)\int \frac{\varphi_1 u}{W}\,dt - \varphi_1(t)\int\frac{\varphi_2 u}{W}\,dt = -t\cos t\int\frac{t\sin t\cdot t}{t^2}\,dt + t\sin t\int\frac{t\cos t\cdot t}{t^2}\,dt \\ &= t\cos^2 t + t\sin^2 t = t. \end{aligned}$$

Summing up the complete solution is

$$x = t + c_1 t\sin t + c_2 t\cos t, \qquad t \neq p\pi,\ p\in\mathbb{Z}.$$

4) We norm again (3),

$$\frac{d^2x}{dt^2} - \frac{2}{t}\,\frac{dx}{dt} + \left(1 + \frac{2}{t^2}\right) x = t, \qquad t \neq 0.$$

Since $\varphi_1(t) = t\sin t$ and $u(t) = t$, and

$$\Omega(t) = \exp\left(-\int \frac{2}{t}\,dt\right) = \frac{1}{t^2},$$

the complete solution is for $t \neq p\pi$, $p \in \mathbb{Z}$,

$$\begin{aligned} x &= t\sin t\left\{\int \frac{t^2}{t^2\sin^2 t}\left[-c_2 + \int\frac{t\sin t}{t^2}\cdot t\,dt\right]dt + c_1\right\} \\ &= t\sin t\int\frac{1}{\sin^2 t}\{-c_2 + \int \sin t\,dt\}dt + c_1 t\sin t \\ &= c_2 t\sin t\cot t + c_1 t\sin t - t\sin\int\frac{\cos t}{\sin^2 t}\,dt \\ &= t + c_1 t\sin t + c_2 t\cos t, \qquad t\neq p\pi,\ p\in\mathbb{Z}. \end{aligned}$$

Remark 1.2 It is only sufficient with the restriction $t \neq 0$ in the first variant. All the methods of the other three variants require that $t \neq p\pi$, $p \in \mathbb{Z}$, so we have strictly speaking some extension problems in these variants at all these points.

Investigation of the possible extensions. Depending on the choice of method we shall either check the continuation to all points $t = p\pi$, $p \in \mathbb{Z}$ (in the second, third and fourth variant) or just to $t = 0$ (in the first variant). We shall here only check the possible extension to $t = 0$, as if we had applied the first variant. The extensions to the other points are quite similar.

Hence we assume that

$$x = t + c_1 t \sin t + c_2 t \cos t, \qquad \text{for } t \neq 0.$$

Then

$$\frac{dx}{dt} = 1 + c_1(t\cos t + \sin t) + c_2(-t\sin t + \cos t),$$

$$\frac{d^2x}{dt^2} = c_1(-t\sin t + 2\cos t) + c_2(-t\cos t - 2\sin t).$$

Clearly, this solution holds in all of $\mathbb{R}$. The problem is if there are other possibilities of extensions. We note that

$$(4)\quad \lim_{t\to 0} x(t) = 0, \quad \lim_{t\to 0} x'(t) = 1 + c_2, \quad \lim_{t\to 0} x''(t) = 2c_1,$$

no matter it the limit is taken from the left hand side or from the right hand side. Thus, any C^2 solution must have uniquely determined constants c_1 and c_2 by (4). We therefore conclude that the natural extension

$$x = t + c_1 t \sin t + c_2 t \cos t, \qquad t \in \mathbb{R},$$

is the complete solution of (3) all over $\mathbb{R}$.

Finally, it follows from (4) that the line elements $(0, x_0, p)$, through which we have a solution of (3), are given by $(0, 0, 1 + c_2)$, hence the line element must necessarily have the form $(0, 0, p)$, so $x_0 = 0$ and $c_2 = p - 1$. Since there is no restriction on c_1, there are *infinitely many* solutions through every line element of the form $(0, 0, p)$,

$$x = t + c_1 t \sin t + (p-1) t \cos t, \qquad t \in \mathbb{R}; \quad c_1 \in \mathbb{R}.$$

The explanation of this phenomenon is that $t \sin t$ has a zero of second order at 0, so we cannot "catch" this function by means of the line element, which is only of the first order.

On the other hand, the extensions to $p\pi$, $p \in \mathbb{Z} \setminus \{0\}$, are all unique, because here $t \sin t$ has only zeros of first order, so the line element can "catch" $t \sin t$.

Example 1.10 *Find the complete solution of the differential equation*

$$4t^3\frac{d^2x}{dt^2}+3t\,\frac{dx}{dt}-3x=0, t\in\mathbb{R}_+,$$

by first finding the solutions of the form t^α, where α is a real number.

1) If we put $x=t^\alpha$, $t>0$, then

$$\begin{aligned}0 &= 4t^3\frac{d^2x}{dt^2}+3t\,\frac{dx}{dt}-3x=4t^3\cdot\alpha(\alpha-1)t^{\alpha-2}+3t\alpha\cdot t^{\alpha-1}-3t^\alpha\\ &= 4\alpha(\alpha-1)t^{\alpha+1}+3\alpha t^\alpha-3t^\alpha=(\alpha-1)t^\alpha\{4\alpha t+3\}.\end{aligned}$$

This equation is fulfilled for every $t>0$, if and only if $\alpha=1$, thus $x=\varphi_1(t)=t$ is a solution of the homogeneous equation.

2) We norm the equation,

$$\frac{d^2x}{dt^2}+\frac{3}{4}\cdot\frac{1}{t^2}\,\frac{dx}{dt}-\frac{3}{4t^3}\,x=0, \qquad t>0.$$

Then a solution which is linearly independent of φ_1 is given by

$$\begin{aligned}\varphi_2(t) &= \varphi_1(t)\int \frac{1}{\varphi_1(t)^2}\,\exp\left(-\frac{3}{4}\int\frac{1}{t^2}\,dt\right)dt = t\int\frac{1}{t^2}\,\exp\left(\frac{3}{4}\cdot\frac{1}{t}\right)\\ &= -t\int\exp\left(\frac{3}{4}\cdot\frac{1}{t}\right)d\left(\frac{1}{t}\right) == -\frac{4}{3}\,t\,\exp\left(\frac{3}{4}\cdot\frac{1}{t}\right).\end{aligned}$$

Here $-\dfrac{4}{3}$ can be included in the arbitrary constant, so the complete solution is

$$x(t) = c_1 t + c_2 t\exp\left(\frac{3}{4t}\right),\qquad t>0,$$

where c_1 and c_2 are arbitrary constants.

Example 1.11 *Find the complete solution of the differential equation*

$$\frac{d^2x}{dt^2} - \cosh t\,\frac{dx}{dt} - (1-\coth^2 t)x = e^t$$

in the interval $]0,\infty[$, *given that* $x = \sinh t$ *is a solution of the corresponding homogeneous equation.*

We give here four variants.

1) It is seen by **inspection** that the equation can be written in the following way,

$$\begin{aligned}e^t &= \frac{d^2x}{dt^2} - \left\{\coth t\cdot\frac{dx}{dt} + \frac{d}{dt}\{\coth t\}\cdot x\right\} = \frac{d^2x}{dt^2} - \frac{d}{dt}(\coth t\cdot x) = \frac{d}{dt}\left\{\frac{dx}{dt} - \frac{\cosh t}{\sinh t}\,x\right\}\\ &= \frac{d}{dt}\left\{\sinh t\left[\frac{1}{\sinh t}\cdot\frac{dx}{dt} - \frac{\cosh t}{\sinh^2 t}\cdot x\right]\right\} = \frac{d}{dt}\left\{\sinh t\,\frac{d}{dt}\left\{\frac{x}{\sinh t}\right\}\right\}.\end{aligned}$$

Then by an integration,

$$\sinh t\cdot\frac{d}{dt}\left(\frac{x}{\sinh t}\right) = e^t + c_2,$$

hence by a rearrangement,

$$\frac{d}{dt}\left(\frac{x}{\sinh t}\right) = \frac{e^t}{\sinh t} + \frac{c_2}{\sinh t} = \frac{2e^{2t}}{e^{2t}-1} + \frac{2c_2e^t}{e^{2t}-1} = \frac{d}{dt}\ln(e^{2t}-1) + c_2\left(\frac{1}{e^t-1} - \frac{1}{e^t+1}\right)e^t.$$

Another integration gives

$$\frac{x}{\sinh t} = \ln(e^{2t}-1) + c_1 + c_2\ln\left(\frac{e^t-1}{e^t+1}\right),$$

so finally we obtain (with various equivalent variants)

$$\begin{aligned}x &= \sinh t\cdot\ln(e^{2t}-1) + c_1\sinh t + c_2\sinh t\cdot\ln\left(\frac{e^t-1}{e^t+1}\right)\\ &= \sinh t\cdot\ln(2e^t\sinh t) + c_1\sinh t + c_2\sinh t\cdot\ln\left(\tanh\frac{t}{2}\right)\\ &= \sinh t\cdot\{t+\ln\sinh t\} + c_1'\sinh t + c_2\sinh t\cdot\ln\left(\tanh\frac{t}{2}\right).\end{aligned}$$

2) It we put $x = \varphi_1(t) \cdot y = \sinh t \cdot y$, then

$$\frac{dx}{dt} = \sinh t \cdot \frac{dy}{dt} + \cosh t \cdot y,$$

$$\frac{d^2x}{dt^2} = \sinh t \cdot \frac{d^2y}{dt^2} + 2\cosh t \cdot \frac{dy}{dt} + \sinh t \cdot y,$$

hence by an insertion into the differential equation

$$\begin{aligned} e^t &= \frac{d^2x}{dt^2} - \coth t \cdot \frac{dx}{dt} + \frac{1}{\sinh^2 t}\, x \\ &= \sinh t \cdot \frac{d^2y}{dt^2} + 2\cosh t \cdot \frac{dy}{dt} + \sinh t \cdot y - \cosh t \frac{dy}{dt} - \frac{\cosh^2 t}{\sinh t}\, y + \frac{1}{\sinh t}\, y \\ &= \sinh t \cdot \frac{d^2y}{dt^2} + \cosh t \cdot \frac{dy}{dt} = \frac{d}{dt}\left(\sinh t \cdot \frac{dy}{dt}\right). \end{aligned}$$

Then by an integration,

$$\sinh t \cdot \frac{dy}{dt} = e^t + c_2,$$

from which we e.g. get

$$\frac{dy}{dt} = \frac{2e^{2t}}{e^{2t}-1} + \frac{c_2}{2\sinh t \frac{t}{2} \cosh \frac{t}{2}} = \frac{d}{dt}\ln(e^{2t}-1) + c_2 \cdot \frac{1}{\tanh \frac{t}{2}} \cdot \frac{1}{2\cosh^2 \dfrac{t}{2}}.$$

Another integration gives

$$y = \ln(e^{2t}-1) + c_1 + c_2 \ln\left(\tanh \frac{t}{2}\right),$$

so finally,

$$x = \sinh t \cdot y = \sinh t \cdot \ln(e^{2t}-1) + c_1 \sinh t + \sinh t \cdot \ln \tanh \frac{t}{2}.$$

3) The equation is already normed, so $f_1(t) = -\coth t$, thus

$$\begin{aligned} \varphi_2(t) &= \varphi_1(t) \int \frac{1}{\varphi_1(t)^2}\, \exp(-\int f_1(t)\, dt) dt \\ &= \sinh t \int \frac{1}{\sinh^2 t}\, \exp\left(\int \frac{\cosh t}{\sinh t}\, dt\right) dt \\ &= \sinh t \int \frac{1}{\sinh^2 t}\, \exp(\ln \sinh t) dt = \sinh t \int \frac{\sinh t}{\cosh^2 t - 1}\, dt \\ &= \sinh t \cdot \frac{1}{2} \int \left(\frac{1}{\cosh t - 1} - \frac{1}{\cosh t + 1}\right) d\cosh t \\ &= \sinh t \cdot \frac{1}{2} \ln\left(\frac{\cosh t - 1}{\cosh t + 1}\right) == \sinh t \cdot \ln \tanh \frac{t}{2}. \end{aligned}$$

The Wroński determinant is

$$W(t) = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix} = \begin{vmatrix} \sinh t & \sinh t \cdot \ln\tanh\dfrac{t}{2} \\ \cosh t & \cosh t \cdot \ln\tanh\dfrac{t}{2} + \sinh t \cdot \dfrac{1}{2}\left(\dfrac{\sinh t}{\cosh t - 1} - \dfrac{\sinh t}{\cosh + 1}\right) \end{vmatrix}$$

$$= \begin{vmatrix} \sinh t & 0 \\ 0 & \sinh t \cdot \dfrac{\sinh t}{\cosh^2 t - 1} \end{vmatrix} = \begin{vmatrix} \sinh t & 0 \\ 0 & 1 \end{vmatrix} = \sinh t.$$

When we apply a solution formula we get the particular solution

$$\begin{aligned} x_0(t) &= \varphi_2(t)\int \frac{\varphi_1 u}{W}\,dt - \varphi_1(t)\int\frac{\varphi_2 u}{W}\,dt \\ &= \sinh t \cdot \ln\tanh\frac{t}{2}\int\frac{\sinh t \cdot e^t}{\sinh t}\,dt - \sinh t\int\frac{\sinh t \cdot \ln(\tanh\frac{t}{2})e^t}{\sinh t}\,dt \\ &= e^t \sinh t \cdot \ln\tanh\frac{t}{2} - \sinh t\int \ln\left(\tanh\frac{t}{2}\right)e^t\,dt \\ &= e^t\sinh t\cdot\ln\tanh\frac{t}{2} - \sinh t\cdot\ln(\tanh\frac{t}{2})\cdot e^t + \sinh t\int\frac{e^t}{\sinh t}\,dt \\ &= 0 + \sinh t \cdot \int\frac{2e^{2t}}{e^{2t}-1}\,dt = \sinh t\cdot\ln(e^{2t}-1). \end{aligned}$$

Summing up the complete solution is

$$x = \sinh t \cdot \ln(e^{2t} - 1) + c_1 \sinh t + c_2 \sinh t \cdot \ln \tanh \frac{t}{2}.$$

4) Since $\varphi_1(t) = \sinh t$ and $u(t) = e^t$, and

$$\Omega(t) = \exp(-\int \coth t \, dt) = \frac{1}{\sinh t},$$

we get that

$$\begin{aligned}
x(t) &= \sinh t \left(\int \frac{\sinh t}{\sinh^2 t} \left\{ c_1 + \int \frac{\sinh t}{\sinh t} \cdot e^t \, dt \right\} dt + c_2 \right) \\
&= \sinh t \left(\int \frac{1}{\sinh t} (c_1 + e^t) dt \right) + c_2 \sinh t \\
&= \sinh t \int \frac{2e^{2t}}{e^{2t} - 1} \, dt + c_2 \sinh t + c_1 \sinh t \int \frac{d \cosh t}{\cosh^2 t - 1} \\
&= \sinh t \cdot \ln(e^{2t} - 1) + c_2 \sinh t + \frac{1}{2} c_1 \sinh t \cdot \ln \left(\frac{\cosh t - 1}{\cosh t + 1} \right).
\end{aligned}$$

Example 1.12 *Find the complete solution of the differential equation*

$$t(t^2 + 1) \frac{d^2 y}{dt^2} + (4t^2 + 2) \frac{dy}{dt} + 2ty = 2, \qquad t \in \mathbb{R}_+,$$

given that the corresponding homogeneous equation has the solution

$$y = \frac{1}{t}, \qquad t \in \mathbb{R}_+.$$

Hint:

$$\int \frac{4t^2 + 2}{t^3 + t} \, dt = \ln(t^4 + t^2), \qquad t \in \mathbb{R}_+.$$

First method. We get by some deftness that

$$\begin{aligned}
2 &= t(t^2 + 1) \frac{d^2 y}{dt^2} + (4t^2 + 2) \frac{dy}{dt} + 2ty \\
&= (t^2 + 1) \left\{ t \frac{d^2 y}{dt^2} + 1 \cdot \frac{dy}{dt} \right\} + (3t^2 + 1) \frac{dy}{dt} + 2ty \\
&= (t^2 + 1) \frac{d}{dt} \left\{ t \frac{dy}{dt} + 1 \cdot y \right\} - (t^2 + 1) \frac{dy}{dt} + (3t^2 + 1) \frac{dy}{dt} + 2ty \\
&= (t^2 + 1) \frac{d}{dt} \left\{ \frac{d}{dt}(ty) \right\} + 2t^2 \frac{dy}{dt} + 2ty \\
&= (t^2 + 1) \frac{d}{dt} \left\{ \frac{d}{dt}(ty) \right\} + 2t \left\{ t \frac{dy}{dt} + 1 \cdot y \right\} \\
&= (t^2 + 1) \frac{d}{dt} \left\{ \frac{d}{dt}(ty) \right\} + \frac{d}{dt}(t^2 + 1) \cdot \frac{d}{dt}(ty) \\
&= \frac{d}{dt} \left\{ (t^2 + 1) \frac{d}{dt}(ty) \right\},
\end{aligned}$$

thus

$$\frac{d}{dt}\left\{(t^2+1)\frac{d}{dt}(ty)\right\} = 2.$$

Then by an integration,

$$(t^2+1)\frac{d}{dt}(ty) = 2t + c_2, \quad \text{dvs.} \quad \frac{d}{dt}(ty) = \frac{2t}{t^2+1} + \frac{c_2}{t^2+1}.$$

And by another integration,

$$ty = \ln(1+t^2) + c_2 \operatorname{Arctan} t + c_1,$$

and the complete solution is

$$y = \frac{\ln(1+t^2)}{t} + c_1 \cdot \frac{1}{t} + c_2 \cdot \frac{\operatorname{Arctan} t}{t}, \qquad t \in \mathbb{R}_+,$$

where c_1 and c_2 are arbitrary constants.

Second method. The standard method. First *norm* the equation,

$$\frac{d^2y}{dt^2} + \frac{4t^2+2}{t(t^2+1)}\,\frac{dy}{dt} + \frac{2}{t^2+1}\,y = \frac{2}{t(t^2+1)}.$$

Since $\varphi_1(t) = \dfrac{1}{t}$, we have

$$\varphi_2(t) = \frac{1}{t}\int t^2 \exp\left(-\int \frac{4t^2+2}{t^3+t}\,dt\right)dt = \frac{1}{t}\int t^2 \cdot \frac{1}{t^4+t^2}\,dt = \frac{1}{t}\int \frac{dt}{t^2+1} = \frac{\operatorname{Arctan} t}{t}.$$

Then we compute the Wrońskian,

$$W(t) = \begin{vmatrix} \dfrac{1}{t} & \dfrac{\operatorname{Arctan} t}{t} \\ -\dfrac{1}{t} & -\dfrac{\operatorname{Arctan} t}{t^2} + \dfrac{1}{t(t^2+1)} \end{vmatrix} = \frac{1}{t^2(t^2+1)}.$$

A particular solution is

$$\begin{aligned}
\varphi(t) &= \varphi_2(t)\int \frac{\varphi_1(t)u(t)}{W(t)}\,dt - \varphi_1(t)\int \frac{\varphi_2(t)u(t)}{W(t)}\,dt \\
&= \frac{1}{t}\operatorname{Arctan} t \int \frac{1}{t}\cdot t^2(1+t^2)\cdot\frac{2}{t(t^2+1)}\,dt - \frac{1}{t}\int \frac{\operatorname{Arctan} t}{t}\cdot t^2(1+t^2)\cdot \frac{2}{t(t^2+1)}\,dt \\
&= \frac{1}{t}\operatorname{Arctan} t\cdot 2t - \frac{1}{t}\int 2\operatorname{Arctan} t\,dt = 2\operatorname{Arctan} t - \frac{1}{t}\left\{2t\operatorname{Arctan} t - \int \frac{2t}{t^2+1}\,dt\right\} \\
&= \frac{1}{t}\,\ln(1+t^2).
\end{aligned}$$

The complete solution is

$$y = \frac{1}{t}\,\ln(1+t^2) + \frac{c_1}{t} + c_2\,\frac{\operatorname{Arctan} t}{t}, \qquad t \in \mathbb{R}_+,$$

where c_1 and c_2 are arbitrary constants.

Example 1.13 *Consider the differential equation*

$$(5)\quad t(1+t)\frac{d^2y}{dt^2}+(2+3t)\frac{dy}{dt}+y=0, \qquad t\in\mathbb{R}_+.$$

1) Prove that $y=t^{-1}$, $t\in\mathbb{R}_+$ is a solution of (5), and then find the complete solution.

Hint: Exploit that $\dfrac{2+3t}{t(1+t)}=\dfrac{2}{t}+\dfrac{1}{1+t}$.

2) Find the complete solution of the differential equation

$$t(1+t)\frac{d^2y}{dt^2}+(2+3t)\frac{dy}{dt}+y=\frac{1}{1+t}, \qquad t\in\mathbb{R}_+.$$

First variant. This problem is immediately solved by some reformulations of the linear, inhomogeneous equation of (2),

$$\begin{aligned}\frac{1}{1+t} &= t(1+t)\frac{d^2y}{dt^2}+(2+3t)\frac{dy}{dt}+y\\ &= \left\{(t+t^2)\frac{d^2y}{dt^2}+(1+2t)\frac{dy}{dt}\right\}+\left\{(1+t)\frac{dy}{dt}+1\cdot y\right\}=\frac{d}{dt}\left\{(t+t^2)\frac{dy}{dt}\right\}+\frac{d}{dt}\{(1+t)y\}\\ &= \frac{d}{dt}\left\{(1+t)\left[t\frac{dy}{dt}+1\cdot y\right]\right\}=\frac{d}{dt}\left\{(1+t)\frac{d}{dt}(t\cdot y)\right\}.\end{aligned}$$

If this is integrated, we obtain with some arbitrary constant c_2 that

$$(1+t)\frac{d}{dt}(t\cdot y)=c_2+\ln(1+t), \qquad \text{for } t>0,$$

hence by a rearrangement

$$\frac{d}{dt}(t\cdot y)=\frac{\ln(1+t)}{1+t}+\frac{c_2}{1+t}, \qquad \text{for } t>0.$$

By another integration we obtain with another arbitrary constant c_1,

$$t\cdot y=\frac{1}{2}\{\ln(1+t)\}^2+c_1+c_2\ln(1+t), \qquad \text{for } t>0,$$

and the complete solution is er

$$y=\frac{1}{2}\cdot\frac{\{\ln(1+t)\}^2}{t}+\frac{c_1}{t}+c_2\cdot\frac{\ln(1+t)}{t} \qquad \text{for } t>0,$$

where c_1 and c_2 are arbitrary constants.

In particular, $y=t^{-1}$ is a solution of the homogeneous solution, and

$$y=\frac{c_1}{t}+c_2\cdot\frac{\ln(1+t)}{t}$$

is the complete solution of the homogeneous equation.

Second variant. The standard method.

1) A check for $t > 0$ shows that

$$t(1+t)\frac{d^2}{dt^2}\left(\frac{1}{t}\right)+(2+3t)\frac{d}{dt}\left(\frac{1}{t}\right)+\frac{1}{t}=t(1+t)\frac{d}{dt}\left(-\frac{1}{t^2}\right)-(2+3t)\frac{1}{t^2}+\frac{1}{t}$$
$$=\frac{2t(1+t)}{t^3}-\frac{2+3t}{t^2}+\frac{1}{t}=\frac{1}{t^2}\{2+2t-2-3t+t\}=0.$$

It follows that $\varphi_1(t)=\dfrac{1}{t}$ is a solution of (1.13).

Then norm the equation,

$$\frac{d^2y}{dt^2}+\frac{2+3t}{t(1+t)}\,\frac{dy}{dt}+\frac{y}{t(t+1)}=0,$$

where the right hand side in the inhomogeneous case is $\dfrac{1}{t(1+t)^2}$.

It follows from the equation above that

$$f_1(t) = \frac{2+3t}{t(1+t)} = \frac{2}{t} + \frac{1}{1+t},$$

so a linearly independent solution is given by

$$\begin{aligned}\varphi_2(t) &= \varphi_1(t)\int \exp\left(-\int f_1(t)\,dt\right)\frac{1}{\varphi_1(t)^2}\,dt = \frac{1}{t}\int \exp\left(-\int\left\{\frac{2}{t}+\frac{1}{1+t}\right\}dt\right)t^2\,dt \\ &= \frac{1}{t}\int \exp(-\ln|t^2(1+t)|)\cdot t^2\,dt = \frac{1}{t}\int \frac{t^2}{t^2(1+t)}\,dt = \frac{1}{t}\int\frac{dt}{1+t} = \frac{\ln(1+t)}{t}.\end{aligned}$$

The complete solution of the homogeneous equation (5) is then

$$y = c_1\cdot\frac{1}{t} + c_2\cdot\frac{\ln(1+t)}{t} \qquad \text{for } t>0.$$

2) The normed, inhomogeneous equation is now

$$\frac{d^2y}{dt^2} + \left(\frac{2}{t}+\frac{1}{1+t}\right)\frac{dy}{dt} + \frac{1}{t(t+1)}\,y = \frac{1}{t(1+t)^2},$$

thus

$$u(t) = \frac{1}{t(1+t)^2}.$$

The Wrońskian can be computed in various ways:

$$(a)\;\; W(t) = \exp\left(-\int\left(\frac{2}{t}+\frac{1}{1+t}\right)dt\right) = \frac{1}{t^2(1+t)}.$$

$$(b)\;\; W(t) = \begin{vmatrix}\varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2'\end{vmatrix} = \begin{vmatrix}\dfrac{1}{t} & \dfrac{\ln(1+t)}{t} \\ -\dfrac{1}{t^2} & -\dfrac{\ln(1+t)}{t^2}+\dfrac{1}{t(1+t)}\end{vmatrix} = \frac{1}{t^2(1+t)}.$$

We get (NB, only one of the many possible variants, $\Omega(t) = W(t)^{-1}$)

$$\begin{aligned}\varphi_0(t) &= \varphi_2(t)\int\frac{\varphi_1(t)u(t)}{W(t)}\,dt - \varphi_1(t)\int\frac{\varphi_2(t)u(t)}{W(t)}\,dt \\ &= \frac{\ln(1+t)}{t}\int\frac{1}{t}\cdot\frac{1}{t(1+t)^2}\cdot t^2(1+t)\,dt - \frac{1}{t}\int\frac{\ln(1+t)}{t}\cdot\frac{1}{t(1+t)^2}\cdot t^2(1+t)\,dt \\ &= \frac{\ln(1+t)}{t}\int\frac{dt}{1+t} - \frac{1}{t}\int\frac{\ln(1+t)}{1+t}\,dt = \frac{\{\ln(1+t)\}^2}{t} - \frac{1}{2}\cdot\frac{\{\ln(1+t)\}^2}{t} \\ &= \frac{1}{2}\cdot\frac{\{\ln(1+t)\}^2}{t}.\end{aligned}$$

Due to the linearity the complete solution is

$$y = \frac{1}{2}\cdot\frac{\{\ln(1+t)\}^2}{t} + c_1\cdot\frac{1}{t} + c_2\cdot\frac{\ln(1+t)}{t}, \qquad t>0,$$

where c_1 and c_2 are arbitrary constants.

Example 1.14 *Consider the differential equation*

(6) $t^2\dfrac{d^2y}{dt^2}+t\dfrac{dy}{dt}-y=0, \qquad t\in\mathbb{R}_+.$

1) Prove that (6) has the solution $y=t$, $t\in\mathbb{R}_+$, and then find the complete solution.

2) Find the complete solution of the differential equation

$$t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}-y=t, \qquad t\in\mathbb{R}_+.$$

This example can be solved in many ways.

First variant.

1) If we put $y=t$ into the equation, we get

$$t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}-y=t^2\cdot 0+t\cdot 1-t=0,$$

proving that $y=t$ is a solution of the homogeneous equation, so $\varphi_1(t)=t$.
By norming, i.e. division by t^2, we get for $t>0$ that

$$\frac{d^2y}{dt^2}+\frac{1}{t}\,\frac{dy}{dt}-\frac{1}{t^2}\,y=0, \qquad \left[\text{and } =\frac{1}{t}\text{ i spørgsmål (2)}\right].$$

Since

$$\Omega(t)=\exp\left(\int f_1(t)\,dt\right)=\exp\left(\int\frac{1}{t}\,dt\right)=\exp(\ln t)=t,$$

we get

$$\varphi_2(t)=\varphi_1(t)\int\frac{1}{\varphi_1(t)^2}\cdot\frac{1}{\Omega(t)}\,dt=t\int\frac{dt}{t^2\cdot t}=t\int t^{-3}\,dt=-\frac{1}{2}\,t\cdot t^{-2}=-\frac{1}{2}\cdot\frac{1}{t}.$$

The complete solution of the homogeneous equation is

$$y(t)=c_1t+c_2\cdot\frac{1}{t}, \qquad t>0,$$

where c_1 and c_2 are arbitrary constants.

2) Since $u(t)=\dfrac{1}{t}$ by the norming, we get the particular solution

$$\begin{aligned}y_0(t) &= \varphi_1(t)\int\frac{1}{\varphi_1(t)^2\Omega(t)}\left\{\varphi_1(t)\Omega(t)u(t)\,dt\right\}dt=t\int\frac{1}{t^2\cdot t}\left\{\int t\cdot t\cdot\frac{1}{t}\,dt\right\}dt\\ &= t\int\frac{1}{t^3}\cdot\frac{1}{2}\cdot t^2\,dt=\frac{1}{2}\,t\,\ln t.\end{aligned}$$

Then the complete solution is by the linearity,

$$y(t)=\frac{1}{2}\,t\,\ln t+c_1t+c_2\cdot\frac{1}{t}, \qquad t>0,$$

where c_1 and c_2 are arbitrary constants.

Second variant.

1) If we put $y = t^a$ into the equation, then

$$t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} - y = a(a-1)t^a + at^a - t^a = (a^2-1)t^a.$$

This expression is identical 0 for $t > 0$, if $a = \pm 1$, so the complete solution is

$$y(t) = c_1 t + c_2 \cdot \frac{1}{t}, \qquad t > 0,$$

where c_1 and c_2 are arbitrary constants.

2) Then note that the right hand side of the (non-normed) equation

$$t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} - y = t,$$

is a solution of the homogeneous equation. We therefore *guess* a particular solution of the form

$$y_0(t) = a \cdot t \ln t,$$

where

$$\frac{dy_0}{dt} = a(1 + \ln t) \quad \text{and} \quad \frac{d^2y_0}{dt^2} = \frac{a}{t}.$$

Then by insertion,

$$t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dy} - y = a\{t + t + t\ln t - t\ln t\} = 2at,$$

which is equal to t for $a = \frac{1}{2}$.

The complete solution is

$$y(t) = \frac{1}{2}\, t\, \ln t + c_1 t + c_2 \cdot \frac{1}{t}, \qquad t > 0,$$

where c_1 and c_2 are arbitrary constants.

Third variant.

1) By the monotonous substitution $t = e^u$, $u = \ln t$, and the chain rule we get,

$$\frac{dy}{dt} = \frac{du}{dt} \cdot \frac{dy}{du} = \frac{1}{t}\,\frac{dy}{du}, \qquad \text{dvs. } t\,\frac{dy}{dt} = \frac{dy}{du},$$

and

$$t^2 \frac{d^2y}{dt^2} = t^2 \frac{d}{dt}\left\{\frac{1}{t}\,\frac{dy}{du}\right\} = t^2 \cdot \left(-\frac{1}{t^2}\right)\frac{dy}{du} + t^2 \cdot \frac{1}{t^2}\,\frac{d^2y}{du^2} = \frac{d^2y}{du^2} - \frac{dy}{du},$$

hence

$$t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} - y = \frac{d^2y}{du^2} - \frac{dy}{du} + \frac{dy}{du} - y = \frac{d^2y}{du^2} - y = t = e^u,$$

and thus

$$\frac{d^2y}{du^2} - y = e^u, \qquad u = \ln t, \quad t > 0.$$

The characteristic equation $R^2 - 1 = 0$ has the roots $R = \pm 1$, so the homogeneous equation has the complete solution

$$y = c_1 e^u + c_2 e^{-u} = c_1 t + c_2 \frac{1}{t}, \qquad t > 0,$$

where c_1 and c_2 are arbitrary constants.

2) Since the right hand side is of the form $e^u = \varphi_1(u)$, we *guess* a solution of the form $y = aue^u$ where

$$\frac{dy}{du} = a(u+1)e^u \quad \text{og} \quad \frac{d^2y}{du^2} = a(u+2)e^u.$$

Then by insertion,

$$\frac{d^2y}{du^2} - y = a(u+2)e^u - aue^u = 2ae^u,$$

which is equal to e^u for $a = \frac{1}{2}$, and the complete solution is

$$y = \frac{1}{2}\, ue^u + c_1 e^u + c_2 e^{-u} = \frac{1}{2}\, t \ln t + c_1 t + c_2 \frac{1}{t}, \quad t > 0,$$

where c_1 and c_2 are arbitrary constants.

Fourth variant We get by some clever manipulation on the inhomogeneous equation that

$$\begin{aligned} t &= t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} - y = \left\{ t^2 \frac{d^2y}{dt^2} + 2t\frac{dy}{dt} \right\} - \left\{ t\frac{dy}{dt} + 1 \cdot y \right\} = \frac{d}{dt}\left\{ t^2 \frac{dy}{dt} - ty \right\} \\ &= \frac{d}{dt}\left\{ t^3 \left(\frac{1}{t}\frac{dy}{dt} - \frac{1}{t^2}\, y \right) \right\} = \frac{d}{dt}\left\{ t^3 \frac{d}{dt}\left(\frac{y}{t}\right) \right\}, \end{aligned}$$

thus

$$\frac{d}{dt}\left\{ t^3 \frac{d}{dt}\left(\frac{y}{t}\right) \right\} = t.$$

Then by an integration,

$$t^3 \frac{d}{dt}\left(\frac{y}{t}\right) = -2c_2 + \frac{t^2}{2}, \qquad \text{dvs.} \quad \frac{d}{dt}\left(\frac{y}{t}\right) = -\frac{2c_2}{t^3} + \frac{1}{2t},$$

and by another integration,

$$\frac{y}{t} = c_1 + \frac{c_2}{t^2} + \frac{1}{2}\ln t,$$

and the complete solution of the inhomogeneous equation is

$$y = \frac{1}{2}\, t \ln t + c_1 t + c_2 \frac{1}{t}, \qquad t > 0,$$

where c_1 and c_2 are arbitrary constants.

2 Euler's differential equation

Example 2.1 *Consider the differential equation*

$$(7)\quad t^2\frac{d^2y}{dt^2}+3t\frac{dy}{dt}+y=0, \qquad t\in\mathbb{R}_+.$$

1) Prove that (7) has a solution of the form $y=t^\alpha$*, and then find the complete solution.*

2) Find the complete solution of the differential equation

$$t^2\frac{d^2y}{dt^2}+3t\frac{dy}{dt}+y=-2, \qquad t\in\mathbb{R}_+.$$

1) If we put $y=t^\alpha$, then

$$0=\alpha(\alpha-1)t^\alpha+3\alpha t^\alpha+t^\alpha=(\alpha^2+2\alpha+1)t^\alpha,$$

which is satisfied for $t>0$, when $\alpha=-1$, and $y=\dfrac{1}{t}$ is a solution.

The complete solution can now be found in several ways.

a) If one knows the theory of **Euler's differential equation**, then since

$$\alpha^2 + 2\alpha + 1 = (\alpha+1)^2$$

has $\alpha = -1$ as a root of multiplicity two, one concludes that the complete solution is

$$y = c_1 \cdot \frac{1}{t} + c_2 \cdot \frac{\ln t}{t}, \quad t \in \mathbb{R}_+, \qquad c_1,\, c_2 \text{ arbitrary.}$$

b) The equation is normed,

$$\frac{d^2y}{dt^2} + \frac{3}{t}\,\frac{dy}{dt} + \frac{1}{t^2}\,y = 0, \qquad t \in \mathbb{R}_+.$$

Then a linearly independent solution is given by

$$y_2(t) = y_1(t) \int \exp\left(-\int \frac{3}{t}\,dt\right) \frac{1}{y_1(t)^2}\,dt = \frac{1}{t}\int \frac{1}{t^3}\,t^4 dt = \frac{\ln t}{t}, \quad t > 0.$$

The complete solution is

$$y = c_1 \cdot \frac{1}{t} + c_2 \cdot \frac{\ln t}{t}, \quad t \in \mathbb{R}_+; \quad c_1,\, c_2 \text{ arbitrære.}$$

2) Here we also have several variants.

a) **Guessing**. It follows immediately that $y = -2$ is a particular solution, hence the complete solution is

$$y = -2 + c_1 \cdot \frac{1}{t} + c_2 \cdot \frac{\ln t}{t}, \quad t \in \mathbb{R}_+; \quad c_1,\, c_2 \text{ arbitrære.}$$

b) **Simple manipulation**. Rewrite the equation in the following way,

$$\begin{aligned} -2 &= t^2\frac{d^2y}{dt^2} + 3t\frac{dy}{dt} + y = \left\{t^2\frac{d^2y}{dt^2} + 2t\frac{dy}{dt}\right\} + \left\{t\frac{dy}{dt} + y\right\} \\ &= \frac{d}{dt}\left\{t^2\frac{dy}{dt} + ty\right\} = \frac{d}{dt}\left\{t\left(t\frac{dy}{dt} + y\right)\right\} = \frac{d}{dt}\left\{t\frac{d}{dt}(ty)\right\}. \end{aligned}$$

Then by integration,

$$t\frac{d}{dt}(ty) = -2t + c_2, \quad \text{thus } \frac{d}{dt}(ty) = -2 + \frac{1}{t}\,c_2.$$

And by another integration,

$$ty = -2t + c_2 \ln t + c_1,$$

and we finally get

$$y = -2 + c_1 \cdot \frac{1}{t} + c_2 \cdot \frac{\ln t}{t}, \quad t \in \mathbb{R}_+; \quad c_1,\, c_2 \text{ arbitrary.}$$

c) **An alternative solution**. When the equation is normed we get

$$\frac{d^2y}{dt^2} + \frac{3}{t}\,\frac{dy}{dt} + \frac{1}{t^2}\,y = -\frac{2}{t^2}, \qquad t > 0,$$

so $u(t) = -\dfrac{2}{t^2}$ and $y_1(t) = \dfrac{1}{t}$, and

$$\Omega(t) = \exp\left(\int \frac{3}{t}\,dt\right) = t^3.$$

Then a particular solution is

$$\begin{aligned} y(t) &= y_1(t)\int \frac{1}{y_1(t)^2\Omega(t)}\left\{\int \Omega(t)y_1(t)u(t)\,dt\right\}dt = \frac{1}{t}\int \frac{1}{t^3}\cdot t^3\left\{\int \frac{1}{t}\cdot t^3\left(-\frac{2}{t^2}\right)dt\right\}dt \\ &= -2\cdot\frac{1}{t}\int\frac{1}{t}\left\{\int dt\right\}dt = -2\cdot\frac{1}{t}\int dt = -2. \end{aligned}$$

Summing up the complete solution becomes

$$y = -2 + c_1\cdot\frac{1}{t} + c_2\cdot\frac{\ln t}{t}, \quad t\in\mathbb{R}_+, \quad c_1,\,c_2 \text{ arbitrære.}$$

d) **An alternative solution**. The normed equation is

$$\frac{d^2y}{dt^2} + \frac{3}{t}\,\frac{dy}{dt} + \frac{1}{t^2}\,y = -\frac{2}{t^2}, \qquad t > 0.$$

The Wrońskian is

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} \dfrac{1}{t} & \dfrac{\ln t}{t} \\ -\dfrac{1}{t^2} & \dfrac{1-\ln t}{t^2} \end{vmatrix} = \frac{1}{t^3}\begin{vmatrix} 1 & \ln t \\ -1 & 1-\ln t \end{vmatrix} = \frac{1}{t^3}.$$

Then a particular solution is

$$\begin{aligned} y(t) &= y_2(t)\int\frac{y_1(t)u(t)}{W(t)}\,dt - y_1(t)\int\frac{y_2(t)u(t)}{W(t)}\,dt \\ &= \frac{\ln t}{t}\int t^3\cdot\frac{1}{t}\left(-\frac{2}{t^2}\right)dt - \frac{1}{t}\int t^3\cdot\frac{\ln t}{t}\left(-\frac{2}{t^2}\right)dt \\ &= -2\frac{\ln t}{t}\int dt + \frac{2}{t}\int \ln t\,dt = -2\ln t + \frac{2}{t}\{t\ln t - t\} = -2. \end{aligned}$$

The complete solution is

$$y = -2 + c_1\cdot\frac{1}{t} + c_2\cdot\frac{\ln t}{t}, \quad t > 0; \quad c_1,\,c_2 \text{ arbitrary.}$$

e) Since $y_1(t) = \dfrac{1}{t}$ is a solution of the homogeneous equation, we obtain a simpler equation in z, if we put

$$y = y_1(t)\cdot z = \frac{1}{t}\cdot z.$$

It follows from

$$\frac{dy}{dt} = \frac{1}{t}\,\frac{dz}{dt} - \frac{1}{t^2}\,z, \qquad \frac{d^2y}{dt^2} = \frac{1}{t}\,\frac{d^2z}{dt^2} - \frac{2}{t^2}\,\frac{dz}{dt} + \frac{2}{t^3}\,z,$$

that the differential equation is written

$$\begin{aligned} -2 &= t^2\frac{d^2y}{dt^2} + 3t\frac{dy}{dt} + y = \left\{t\frac{d^2z}{dt^2} - 2\frac{dz}{dt} + \frac{2}{t}\,z\right\} + \left\{3\frac{dz}{dt} - \frac{3}{t}\,z\right\} + \frac{1}{t}\,z \\ &= t\frac{d^2z}{dt^2} + \frac{dz}{dt} = t\frac{d}{dt}\left(\frac{dz}{dt}\right) + \frac{dt}{dt}\cdot\frac{dz}{dt} = \frac{d}{dt}\left(t\frac{dz}{dt}\right). \end{aligned}$$

Thus be have found the equation

$$\frac{d}{dt}\left(t\frac{dz}{dt}\right) = -2.$$

When this is integrated, we get

$$t\frac{dz}{dt} = -2t + c_2, \quad \text{thus } \frac{dz}{dt} = -2 + c_2\cdot\frac{1}{t}.$$

Then by another integration,

$$z = \int\left(-2 + c_2\cdot\frac{1}{t}\right)dt = -2t + c_1 + c_2\ln t.$$

Finally, we get the complete solution

$$y = \frac{z}{t} = -2 + c_1\cdot\frac{1}{t} + c_2\cdot\frac{\ln t}{t}, \quad t > 0; \quad c_1, c_2 \text{ arbitrære}.$$

Example 2.2 *Consider the differential equation*

$$(8)\quad t^2\frac{d^2y}{dt^2} - 3t\,\frac{dy}{dt} + 4y = 0, \qquad t \in \mathbb{R}_+.$$

1) Prove that (8) has a solution of the form $y = t^\alpha$, $t \in \mathbb{R}_+$, and then find the complete solution.

2) Find the complete solution of the differential equation

$$t^2\frac{d^2y}{dt^2} - 3t\,\frac{dy}{dt} + 4y = t^4, \qquad t \in \mathbb{R}_+.$$

1) If we put $y = t^\alpha$ into (8), then

(9) $\alpha(\alpha-1)t^\alpha - 3\alpha t^\alpha + 4t^\alpha = (\alpha^2 - 4\alpha + 4)t^\alpha = (\alpha-2)^2 t^\alpha.$

This expression is identical 0, if $\alpha = 2$, hence a solution is $y_1 = t^2$, $t \in \mathbb{R}_+$.

Remark 2.1 Since the equation is an Euler equation and $\alpha = 2$ is a root of multiplicity 2, a linearly independent solution is given by $t^2 \ln t$. However, this can no longer be assumed to be known. Therefore we continue with the following more difficult way.

We norm the inhomogeneous equation,

$$\frac{d^2y}{dt^2} - \frac{3}{t}\frac{dy}{dt} + \frac{4}{t^2}\,y = t^2, \qquad t \in \mathbb{R}_+,$$

so $f_1(t) = -\dfrac{3}{t}$ and $u(t) = t^2$. Then by using the solution formula we get a linearly independent solution of the homogeneous equation,

$$\begin{aligned} y_2(t) &= y_1(t)\int \frac{1}{y_1(t)^2}\exp\left(-\int f_1(t)\,dt\right)dt = t^2\int\frac{1}{t^4}\exp\left(\frac{3}{t}\,dt\right)dt \\ &= t^2\int\frac{1}{t^4}\exp(3\ln t)dt = t^2\int\frac{1}{t^4}\cdot t^3\,dt = t^2\int\frac{1}{t}\,dt = t^2\ln t. \end{aligned}$$

The complete solution of the homogeneous equation is

$$y = c_1 t^2 + c_2 t^2 \ln t, \qquad t \in \mathbb{R}_+.$$

2) a) We guess the solution $y = ct^4$ of the inhomogeneous equation, thus $\alpha = 4$. When we put this into (9), we get

$$t^2\frac{d^2y}{dt^2} - 3t\,\frac{dy}{dt} + 4y = (4-2)^2ct^4 = 4ct^4 = t^4,$$

which is satisfied for $c = \frac{1}{4}$, and the complete solution is

$$y = \frac{1}{4}\,t^4 + c_1t^2 + c_2t^2\ln t, \qquad t \in \mathbb{R}_+.$$

b) **Alternatively** we apply the horrible standard solution formula, where we have

$$W(t) = \left|\begin{array}{cc} t^2 & t^2\ln t \\ 2t & 2t\ln t + t \end{array}\right| = t^3,$$

$$W_1(t) = \left|\begin{array}{cc} 0 & t^2\ln t \\ t^2 & 2t\ln t + t \end{array}\right| = -t^4\ln t,$$

$$W_2(t) = \left|\begin{array}{cc} t^2 & 0 \\ 2t & t^2 \end{array}\right| = t^4.$$

Then

$$\begin{aligned} y_0(t) &= y_(t)\int\frac{W_1(t)}{W(t)}\,dt + y_2(t)\int\frac{W_2(t)}{W(t)}\,dt = t^2\int\frac{-t^4\ln t}{t^3}\,dt + t^2\ln t\int\frac{t^4}{t^3}\,dt \\ &= -t^2\int t\ln t\,dt + t^2\ln t\int t\,dt = -t^2\left\{\frac{t^2}{2}\,\ln t - \frac{1}{2}\int\frac{t^2}{t}\,dt\right\} + t^2\ln t\cdot\frac{t^2}{2} \\ &= \frac{1}{4}\,t^2\cdot t^2 = \frac{1}{4}\,t^4, \end{aligned}$$

and the complete solution is

$$y = \frac{1}{4}\,t^4 + c_1t^2 + c_2t^2\ln t, \qquad t \in \mathbb{R}_+.$$

c) **Alternatively** we insert into another solution formula, in which we use

$$\Omega(t) = \exp\left(\int f_1(t)\,dt\right) = \exp\left(-\int\frac{3}{t}\,dt\right) = \frac{1}{t^3}.$$

Then the complete solution is

$$\begin{aligned} y(t) &= y_1(t)\left\{\int\frac{1}{y_1(t)^2\Omega(t)}\left[c_1+\int y_1(t)\Omega(t)u(t)\,dt\right]dt+c_2\right\} \\ &= t^2\left\{\int\frac{t^3}{t^4}\left[c_1+\int t^2\cdot\frac{1}{t^3}\cdot t^2\,dt\right]dt+c_2\right\} = t^2\left\{\int\frac{1}{t}\left[c_1+\int t\,dt\right]dt + c_2\right\} \\ &= t^2\left\{\int\left(\frac{c_1}{t}+\frac{1}{t}\cdot\frac{t^2}{2}\right)dt + c_2\right\} = c_2t^2 + c_1t^2\ln t + t^2\int\frac{t}{2}\,dt \\ &= \frac{t^4}{4} + c_2t^2 + c_1t^2\ln t, \qquad t > 0. \end{aligned}$$

Example 2.3 *Find the complete solution of the differential equation*

$$t^3\frac{d^2x}{dt^2}+3t^2\frac{dx}{dt}+tx=2, \qquad t\in\mathbb{R}_+,$$

given that the corresponding homogeneous differential equation has a solution of the form $x=t^\alpha$ for some $\alpha\in\mathbb{R}$.

If we put $x=t^\alpha$, we get for the homogeneous equation that

$$t^3\frac{d^2x}{dt^2}+3t^2\frac{dx}{dt}+tx=\{\alpha(\alpha-1)+3\alpha+1\}t^{\alpha+1}=(\alpha+1)^2t^{\alpha+1}=0.$$

It follows that $x=\dfrac{1}{t}$, $t>0$ is a solution of the homogeneous equation.

The rest of he example can now be solved in many ways.

1) By norming we get the equation

$$\frac{d^2x}{dt^2}+\frac{3}{t}\frac{dx}{dt}+\frac{1}{t^2}x=\frac{2}{t^3}, \qquad t>0.$$

Now,

$$\Omega(t)=\exp(\int f_1(t)\,dt)=\exp\left(\int\frac{3}{t}\,dt\right)=t^3,$$

and $u(t)=2/t^3$, hence since $\varphi_1(t)=1/t$ we get the complete solution

$$\begin{aligned}x &= \frac{1}{t}\left\{\int\frac{t^2}{t^3}\left[c_2+\int\frac{1}{t}\cdot t^3\cdot\frac{2}{t^3}\,dt\right]dt+c_1\right\}=c_1\frac{1}{t}+c_2\frac{1}{t}\int\frac{dt}{t}+\frac{1}{t}\int\frac{1}{t}\left\{\int\frac{2}{t}\,dt\right\}dt\\ &= c_1\cdot\frac{1}{t}+c_2\cdot\frac{\ln t}{t}+\frac{1}{t}\int 2\frac{\ln t}{t}\,dt\\ &= \frac{(\ln t)^2}{t}+c_1\cdot\frac{1}{t}+c_2\cdot\frac{\ln t}{t}, \qquad t\in\mathbb{R}_+;\quad c_1,c_2\in\mathbb{R}.\end{aligned}$$

2) First we norm the equation

$$\frac{d^2x}{dt^2}+\frac{3}{t}\frac{dx}{dt}+\frac{1}{t^2}x=\frac{2}{t^3}, \qquad t>0.$$

From $\varphi_1(t)=1/t$ follows that a linearly independent solution of the homogeneous equation is given by

$$\varphi_2(t)=\frac{1}{t}\int t^2\exp\left(-\int\frac{3}{t}\,dt\right)dt=\frac{1}{t}\int t^2\exp(-3\ln t)dt=\frac{1}{t}\int t^2\cdot\frac{1}{t^3}\,dt=\frac{\ln t}{t}.$$

The corresponding Wrońskian is

$$W(t)=\begin{vmatrix}\dfrac{1}{t} & \dfrac{\ln t}{t}\\ -\dfrac{1}{t^2} & \dfrac{1-\ln t}{t^2}\end{vmatrix}=\frac{1}{t^3}\begin{vmatrix}1 & \ln t\\ -1 & 1-\ln t\end{vmatrix}=\frac{1}{t^3}\begin{vmatrix}1 & \ln t\\ 0 & 1\end{vmatrix}=\frac{1}{t^3}.$$

Then a particular solution is given by

$$\begin{aligned}\varphi_0(t) &= \frac{\ln t}{t}\int \frac{1}{t}\cdot t^3\cdot\frac{2}{t^3}\,dt - \frac{1}{t}\int \frac{\ln t}{t}\cdot t^3\cdot \frac{2}{t^3}\,dt = 2\frac{\ln t}{t}\int\frac{dt}{t} - \frac{1}{t}\int 2\frac{\ln t}{t}\,dt \\ &= 2\frac{(\ln t)^2}{t} - \frac{(\ln t)^2}{t} = \frac{(\ln t)^2}{t}.\end{aligned}$$

Summing up the complete solution is

$$x = \frac{(\ln t)^2}{t} + c_1\cdot\frac{1}{t} + c_2\cdot\frac{\ln t}{t}, \qquad t\in\mathbb{R}_+, \quad c_1, c_2 \text{ arbitrary.}$$

3) The equation is an *Euler equation in disguise*. In fact a division by $t>0$ shows that

$$(10)\quad t^2\frac{d^2x}{dt^2} + 3t\frac{dx}{dt} + x = \frac{2}{t}, \qquad t>0,$$

with the characteristic polynomial for the Euler equation $(R+1)^2$. Since $R=-1$ is a double root, we guess that the complete solution of the homogeneous equation is

$$x = c_1\cdot\frac{1}{t} + c_2\cdot\frac{\ln t}{t}, \qquad t>0.$$

Check. Clearly, $1/t$ and $(\ln t)/t$ are linearly independent. We have already proved that $1/t$ is a solutions. Then put $x = (\ln t)/t$. We have

$$\frac{dx}{dt} = \frac{1-\ln t}{t^2} \quad\text{and}\quad \frac{d^2x}{dt^2} = \frac{2\ln t - 3}{t^3},$$

hence by insertion into the differential equation,

$$t^3\frac{d^2x}{dt^2} + 3t^2\frac{dx}{dt} + tx = (2\ln t - 3) + 3(1-\ln t) + \ln t = 0,$$

and $(\ln t)/t$ also fulfils the homogeneous differential equation, and the claim is proved.

Then we *guess* a particular solution of (10). The apparently obvious choice c/t does not apply, because it is already a solution of the homogeneous equation. The same is true for $c(\ln t)/t$. Therefore, we try instead with $x = c\cdot x_0$, where $x_0 = (\ln t)^2/t$.

Check. If $x_0 = (\ln t)^2/t$, then

$$\frac{dx_0}{dt} = -\frac{1}{t^2}(\ln t)^2 + \frac{2\ln t}{t^2} = \frac{-(\ln t)^2 + 2\ln t}{t^2}$$

and

$$\frac{d^2x_0}{dt^2} = \frac{2}{t^3}\{(\ln t)^2 - 2\ln t\} + \frac{1}{t^3}\{-2\ln t + 2\} = \frac{1}{t^3}\{2(\ln t)^2 - 6\ln t + 2\}.$$

These expressions are put into the left hand side of the original equation,

$$t^3\frac{d^2x_0}{dt^2} + 3t^2\frac{dx_0}{dt} + tx_0 = 2(\ln t)^2 - 6\ln t + 2 - 3(\ln t)^2 + 6\ln t + (\ln t)^2 = 2,$$

which is precisely the right hand side of the equation, so $c = 1$, and x_0 is a particular solution.

Summing up the complete solution is

$$x = \frac{(\ln t)^2}{t} + c_1 \cdot \frac{1}{t} + c_2 \cdot \frac{\ln t}{t}, \qquad t \in \mathbb{R}_+; \quad c_1, c_2 \quad \text{arbitrary}.$$

4) The Euler differential equation can be solved by using the **monotonous substitution**

$$u = \ln t, \qquad t = e^u; \qquad t \in \mathbb{R}_+, \qquad u \in \mathbb{R}.$$

Then by the chain rule,

$$\frac{dx}{dt} = \frac{du}{dt} \cdot \frac{dx}{du} = \frac{1}{t} \cdot \frac{dx}{du},$$

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{1}{t} \cdot \frac{dx}{du}\right) = -\frac{1}{t^2} \cdot \frac{dx}{du} + \frac{1}{t^2} \cdot \frac{d^2x}{du^2} = \frac{1}{t^2}\left\{\frac{d^2x}{du^2} - \frac{dx}{du}\right\}.$$

By this change of variable the equation is uniquely transferred into

$$(11)\quad 2 = t^3\frac{d^2x}{dt^2}+3t^2\frac{dx}{dt}+tx = t\left\{\frac{d^2x}{du^2}-\frac{dx}{du}\right\}+3t\cdot\frac{dx}{du}+tx = e^u\left\{\frac{d^2x}{du^2}+2\frac{dx}{du}+x\right\},$$

hence by a reduction,

$$\frac{d^2x}{du^2}+2\frac{dx}{du}+x = 2e^{-u}.$$

This equation has constant coefficients. The characteristic polynomial $R^2+2R+1=(R+1)^2$ has $R=-1$ as a double root. The complete solution of the corresponding homogeneous equation is

$$x = c_1e^{-u}+c_2ue^{-u},\qquad u\in\mathbb{R};\quad c_1,\, c_2 \text{ arbitrary.}$$

We can find a particular solution in various ways:

a) **Guessing**. Suppose that $x = u^2e^{-u}$. Then

$$\frac{dx}{du} = (-u^2+2u)e^{-u}\quad\text{and}\quad\frac{d^2x}{du^2} = (u^2-4u+2)e^{-u}.$$

Then by insertion,

$$\frac{d^2x}{du^2}+2\frac{dx}{du}+x = (u^2-4u+2)e^{-u}+2(-u^2+2u)e^{-u}+u^2e^{-u} = 2e^{-u},$$

proving that $x=u^2e^{-u}$ is a particular solution.

b) **Alternative solution**. Since $\Omega(u)=\exp(\int 2\,du)=e^{2u}$, a particular solution is given by

$$x = e^{-u}\int\frac{1}{(e^{-u})^2e^{2u}}(\int e^{-u}e^{2u}\cdot 2\cdot e^{-u}du)du = e^{-u}\int\{\int 2\,du\}du = e^{-u}\int 2u\,du = u^2e^{-u}.$$

c) **Alternative solution**. If we put $x_1=e^{-u}$ and $x_2=ue^{-u}$, then the corresponding Wrońskian is

$$W(u) = \begin{vmatrix} e^{-u} & ue^{-u} \\ -e^{-u} & (1-u)e^{-u}\end{vmatrix} = e^{-2u}\begin{vmatrix} 1 & u \\ -1 & 1-u\end{vmatrix} = e^{-2u}\begin{vmatrix} 1 & u \\ 0 & 1\end{vmatrix} = e^{-2u}.$$

Then a particular solution is given by the formula

$$\begin{aligned} x &= ue^{-u}\int e^{2u}\cdot e^{-u}\cdot 2e^{-u}du - e^{-u}\int e^{2u}\cdot ue^{-u}\cdot 2e^{-u}du \\ &= ue^{-u}\int 2\,du - e^{-u}\int 2u\,du = 2u^2e^{-u}-u^2e^{-u} = u^2e^{-u}.\end{aligned}$$

d) The original transformed equation (11) is now rewritten as

$$\begin{aligned} 2 &= e^u\frac{d^2x}{du^2}+2e^u\frac{dx}{du}+e^ux = \left\{e^u\frac{d}{du}\left(\frac{dx}{du}\right)+e^u\frac{dx}{du}\right\}+\left\{e^u\frac{dx}{du}+e^ux\right\} \\ &= \frac{d}{du}\left\{e^u\frac{dx}{du}\right\}+\frac{d}{du}\{e^ux\} = \frac{d^2}{du^2}\{e^ux\}.\end{aligned}$$

Then by two integrations of the equation

$$\frac{d^2}{du^2}\{e^x\} = 2$$

we get $e^ux=u^2$, hence a particular integral is $x=u^2e^{-u}$.

Summing up the complete solution is [where $u = \ln t$]

$$x = u^2e^{-u} + c_1e^{-u} + c_2ue^{-u} = \frac{(\ln t)^2}{t} + c_1 \cdot \frac{1}{t} + c_2 \cdot \frac{\ln t}{t}, \qquad t \in \mathbb{R}_+; \quad c_1, c_2 \text{ arbitrary.}$$

5) **Intuition**. By some deftness we see that

$$\begin{aligned} 2 &= t^3\frac{d^2x}{dt^2} + 3t^2\frac{dx}{dt} + tx = t^2\left\{t\frac{d}{dt}\left(\frac{dx}{dt}\right) + 1 \cdot \frac{dx}{dt} + \frac{dx}{dt}\right\} + t\left\{t\frac{dx}{dt} + 1 \cdot x\right\} \\ &= t^2\left\{\frac{d}{dt}\left(t\frac{dx}{dt}\right) + \frac{dx}{dt}\right\} + t\frac{d}{dt}(tx) = t^2\frac{d}{dt}\left\{t \cdot \frac{dx}{dt} + 1 \cdot x\right\} + t\frac{d}{dt}(tx) \\ &= t^2\frac{d}{dt}\left\{\frac{d}{dt}(tx)\right\} + t\frac{d}{dt}(tx) = t\left\{t\frac{d}{dt}\left[\frac{d}{dt}(tx)\right] + 1 \cdot \frac{d}{dt}(tx)\right\} = t\frac{d}{dt}\left\{t\frac{d}{dt}(tx)\right\}. \end{aligned}$$

The equation is therefore equivalent to

$$\frac{d}{dt}\left\{t\frac{d}{dt}(tx)\right\} = \frac{2}{t}, \qquad t > 0.$$

By integration of this equation we get

$$t\frac{d}{dt}(tx) = c_2 + \int \frac{2}{t}\,dt = c_2 + 2\ln t,$$

hence by a rearrangement,

$$\frac{d}{dt}(tx) = \frac{c_2}{t} + 2\,\frac{\ln t}{t}.$$

Then by another integration,

$$tx = c_1 + c_2\int\frac{dt}{t} + \int\frac{2\ln t}{t}\,dt = c_1 + c_2\ln t + (\ln t)^2.$$

and we finally obtain the complete solution

$$x = \frac{(\ln t)^2}{t} + c_1 \cdot \frac{1}{t} + c_2 \cdot \frac{\ln t}{t}, \qquad t \in \mathbb{R}_+; \quad c_1, c_2 \text{ arbitrary.}$$

Example 2.4 *Find the complete solution of the differential equation*

$$t^2\frac{d^2x}{dt^2} - 2t\frac{dx}{dt} + 2x = 2t^2, \qquad t \in \mathbb{R}_+.$$

Type: An Euler differential equation. The equation is solved below in three different variants.

1) **The standard method**. If we change variable $u = \ln t$, $t = e^u$, then it follows that the equation is equivalent to the inhomogeneous differential equation

$$\frac{d^2x}{du^2} - 3\frac{dx}{du} + 2x = 2e^{2u}$$

of second order and of constant coefficients. The corresponding characteristic equation,

$$\lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2) = 0,$$

has the solutions $\lambda_1 = 1$ and $\lambda_2 = 2$, so we conclude that the homogeneous equation has the complete solution

$$x = c_1 e^u + c_2 e^{2u} = c_1 t + c_2 t^2, \quad c_1,\, c_2 \in \mathbb{R}; \quad t \in \mathbb{R}_+.$$

Then we can either find a particular solution by guessing or by using the Wrońskian method.

a) **Guessing**. The right hand side $2e^{2u}$ is a solution of the homogeneous equation. We therefore guess on $x = c \cdot ue^{2u}$ instead. Then

$$x = cue^{2u}, \quad \frac{dx}{du} = 2cue^{2u} + ce^{2u}, \quad \frac{d^2x}{dt^2} = 4cue^{2u} + 4ce^{2u},$$

which give by insertion

$$\frac{d^2x}{du^2} - 3\frac{dx}{du} + 2x = 4cue^{2u} + 4ce^{2u} - 6cue^{2u} - 3ce^{2u} + 2cue^{2u} = ce^{2u}.$$

This is equal to $2e^{2u}$, if $c = 2$. Hence a particular solution is e.g. $x = 2ue^{2u}$, and then we get by the linearity the complete solution (where $u = \ln t$),

$$x = 2t^2 \ln t + c_1 t + c_2 t^2, \quad c_1,\, c_2 \in \mathbb{R}; \quad t \in \mathbb{R}_+.$$

b) **The Wrońskian method**. If we put

$$\varphi_1(u) = e^u \qquad \text{and} \qquad \varphi_2(u) = e^{2u},$$

then the Wrońskian is

$$W(u) = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix} = \begin{vmatrix} e^u & e^{2u} \\ e^u & 2e^{2u} \end{vmatrix} = e^{3u}.$$

Then a particular solution is given by

$$\begin{aligned}
\varphi_0(u) &= \varphi_2(u) \int \frac{\varphi_1(u)q(u)}{W(u)}\, du - \varphi_1(u) \int \frac{\varphi_2(u)q(u)}{W(u)}\, du \\
&= e^{2u} \int \frac{e^u \cdot 2e^{2u}}{e^{3u}}\, du - e^u \int \frac{e^{2u} \cdot 2e^{2u}}{e^{3u}}\, du \\
&= e^{2u} \int 2\, du - e^u \int 2e^u\, du = 2ue^{2u} - 2e^{2n} \\
&= 2t^2 \ln t - 2t^2.
\end{aligned}$$

Since $-2t^2$ already is a solution of the homogeneous equation, the complete solution is given by

$$x = 2t^2 \ln t + c_1 t + c_2 t^2, \quad c_1,\, c_2 \in \mathbb{R}; \quad t \in \mathbb{R}_+.$$

2) **Guessing**. If we put $x = t^n$ into the Euler differential equation, then

$$t^2 \frac{d^2x}{dt^2} - 2t\frac{dx}{dt} + 2x = \{n(n-1)-2n+2\}t^n = (n-1)(n-2)t^n.$$

This expression is equal to 0 when $n = 1$ and $n = 2$. Since $t^1 = t$ and t^2 are linearly independent, it follows from the existence and uniqueness theorem for linear differential equations that the complete solution of the homogeneous equation is

$$x = c_1 t + c_2 t^2, \qquad c_1,\, c_2 \in \mathbb{R}; \quad t \in \mathbb{R}_+.$$

Here the right hand side is a solution of the homogeneous equation, so we guess instead a particular solution of the form $x = c \cdot t^2 \ln t$, $t \in \mathbb{R}_+$. Then

$$x = ct^2 \ln t, \quad \frac{dx}{dt} = 2ct \ln t, \quad \frac{d^2x}{dt^2} = 2c \ln t + 3c,$$

which is put into the Euler differential equation, giving

$$t^2 \frac{d^2x}{dt^2} - 2t\frac{dx}{dt} + 2x = 2ct^2 \ln t + 3ct^2 - 4ct^2 \ln t - 2ct^2 + 2xt^2 \ln t = ct^2.$$

This expression is equal to $2t^2$, when $c = 2$. Then it follows from the linearity that the complete solution is

$$x = 2t^2 \ln t + c_1 t + c_2 t^2, \quad c_1,\, c_2 \in \mathbb{R}; \quad t \in \mathbb{R}_+.$$

3) By a **"divine inspiration"** we divide the equation by $t^3 > 0$. Hereby we obtain the equivalent equation

$$\begin{aligned}\frac{2}{t} &= \frac{1}{t}\,\frac{d^2x}{dt^2} - \frac{2}{t^2}\,\frac{dx}{dt} + \frac{2}{t^3}\,x = \left\{\frac{1}{t}\,\frac{d}{dt}\left(\frac{dx}{dt}\right) + \frac{d}{dt}\left(\frac{1}{t}\right)\frac{dx}{dt}\right\} - \left\{\frac{1}{t^2}\,\frac{dx}{dt} + \frac{d}{dt}\left(\frac{1}{t^2}\right)x\right\}\\ &= \frac{d}{dt}\left\{\frac{1}{t}\,\frac{dx}{dt} - \frac{1}{t^2}\,x\right\} = \frac{d}{dt}\left\{\frac{1}{t}\,\frac{dx}{dt} + \frac{d}{dt}\left(\frac{1}{t}\right)\cdot x\right\} = \frac{d^2}{dt^2}\left(\frac{x}{t}\right),\end{aligned}$$

thus

$$\frac{d^2}{dt^2}\left(\frac{x}{t}\right) = \frac{2}{t}.$$

When this is integrated we get

$$\frac{d}{dt}\left(\frac{x}{t}\right) = 2\ln t + c_2', \qquad c_1' \in \mathbb{R}, \quad t \in \mathbb{R},$$

hence by another integration,

$$\begin{aligned}\frac{x}{t} &= c_2't + c_1 + 2\int \ln t \cdot 1\,dt = c_2't + c_1 + 2t\cdot\ln t - 2t\\ &= 2t\cdot\ln t + c_1 t + c_2 t^2, \qquad c_1,\, c_2 \in \mathbb{R}; \quad t \in \mathbb{R}_?.\end{aligned}$$

The complete solution is

$$x = 2t^2\ln t + c_1 t + c_2 t^2, \qquad c_1,\, c_2 \in \mathbb{R}; \quad t \in \mathbb{R}_+.$$

Example 2.5 *Consider the differential equation*

$$2t\frac{d^2y}{dt^2} + (6+t)\frac{dy}{dt} + 2y = 0, \qquad t \in \mathbb{R}_+.$$

Prove that $y = t^{-2}$, $t \in \mathbb{R}_+$, is a solution, and then find the complete solution.

If we put $y = t^{-2}$ into the differential equation, we get

$$2t\cdot(-2)\cdot(-3)t^{-4} + (6+t)\cdot(-2)t^{-3} + 2t^{-2} = \{12-12\}t^{-3} + \{-2+2\}t^{-2} = 0,$$

proving that $y_1 = t^{-2}$ is a solution.

Then we norm the equation,

$$\frac{d^2y}{dt^2} + \left(\frac{1}{2} + \frac{3}{t}\right)\frac{dy}{dt} + \frac{1}{t}\,y = 0.$$

A linearly independent solution of the homogeneous equation is then

$$\begin{aligned}y_2 &= \frac{1}{t^2}\int \exp\left(-\int\left(\frac{1}{2} + \frac{3}{t}\right)dt\right)t^4\,dt = \frac{1}{t^2}\int \exp\left(-\frac{t}{2}\right)t\,dt\\ &= \frac{1}{t^2}\left\{-2\exp\left(-\frac{t}{2}\right)t + 2\int \exp\left(-\frac{t}{2}\right)dt\right\}\\ &= \frac{1}{t^2}\left\{-2t\exp\left(-\frac{t}{2}\right) - 4\exp\left(-\frac{t}{2}\right)\right\}.\end{aligned}$$

The complete solution is

$$y = c_1 \cdot \frac{1}{t^2} + c_2 \cdot \frac{t+2}{t^2} \exp\left(-\frac{t}{2}\right), \quad c_1,\, c_2 \text{ arbitrary}, \quad t \in \mathbb{R}_+.$$

Alternatively we put $z = t^2 y$, and then the equation can be rewritten in the following way:

$$\begin{aligned}
0 &= 2t\frac{d^2y}{dt^2} + (6+t)\frac{dy}{dt} + 2y = 2t\frac{d^2}{dt^2}\left\{\frac{z}{t^2}\right\} + (6+t)\frac{d}{dt}\left\{\frac{z}{t^2}\right\} + \frac{2}{t^2}\,z \\
&= 2t\frac{d}{dt}\left(-\frac{2}{t^3}\,z + \frac{1}{t^2}\,\frac{dz}{dt}\right) + (6+t)\left(-\frac{2}{t^3}\,z + \frac{1}{t^2}\,\frac{dz}{dt}\right) + \frac{2}{t^2}\,z \\
&= 2t\left\{\frac{1}{t^2}\,\frac{d^2z}{dt^2} - \frac{4}{t^3}\,\frac{dz}{dt} + \frac{6}{t^4}\,z\right\} - \frac{12}{t^3}\,z + \frac{6}{t^2}\,\frac{dz}{dt} - \frac{2}{t^2}\,z + \frac{1}{t}\,\frac{dz}{dt} + \frac{2}{t^2}\,z \\
&= \frac{2}{t}\,\frac{d^2z}{dt^2} + \left(-\frac{2}{t^2} + \frac{1}{t}\right)\frac{dz}{dt} = 2\exp\left(-\frac{t}{2}\right)\left\{\frac{\exp(t/2)}{t}\,\frac{d^2z}{dt^2} + \left(\frac{1}{2} - \frac{1}{t}\right)\frac{\exp(t/2)}{t}\,\frac{dz}{dt}\right\} \\
&= 2\exp\left(-\frac{t}{2}\right)\frac{d}{dt}\left\{\frac{\exp(t/2)}{t}\,\frac{d}{dt}(t^2y)\right\}.
\end{aligned}$$

If $t \in \mathbb{R}_+$, then the equation is equivalent to

$$\frac{d}{dt}\left\{\frac{\exp(t/2)}{t}\,\frac{d}{dt}(t^2y)\right\} = 0,$$

hence by an integration and a rearrangement,

$$\frac{d}{dt}(t^2y) = \tilde{c}_2 t \exp\left(-\frac{t}{2}\right).$$

Then we get by another integration,

$$t^2y = c_1 + \tilde{c}_2 \int t\exp\left(-\frac{t}{2}\right) dt = c_1 - 2\tilde{c}_2(t+2)\exp\left(-\frac{t}{2}\right),$$

hence with $c_2 = -2\tilde{c}_2$,

$$y = c_1 \cdot \frac{1}{t^2} + c_2 \cdot \frac{t+2}{t^2}\exp\left(-\frac{t}{2}\right).$$

Example 2.6 *Consider the differential equation*

$$(12)\quad t\frac{d^2y}{dt^2}-(t+1)\frac{dy}{dt}+y=0,\qquad t\in\mathbb{R}_+.$$

1) Prove that (12) has the solution $y=e^t$, $t\in\mathbb{R}_+$, and find then the complete solution.

2) Find the complete solution of the differential equation

$$t\frac{d^2y}{dt^2}-(t+1)\frac{dy}{dt}+y=t^2,\qquad t\in\mathbb{R}_+.$$

1) a) If we put $y=e^t$ into (12), then

$$te^t-(t+1)e^t+e^t=0,$$

and we see that $y=e^t$ satisfies the differential equation so $y_1(t)=e^t$.

b) Then we norm the equation (note that we have assumed that $t>0$),

$$\frac{d^2y}{dt^2}-\left(1+\frac{1}{t}\right)\frac{dy}{dt}+\frac{1}{t}\,y=0,\qquad t\in\mathbb{R}_+.$$

Then

$$f_1(t)=-\left(1+\frac{1}{t}\right),\qquad t>0.$$

A linearly independent solution of (12) is then

$$\begin{aligned}y_2(t)&=y_1(t)\int\frac{1}{y_1(t)^2}\,\exp\left(-\int f_1(t)\,dt\right)dt\\&=e^t\int e^{-2t}\,\exp\left(+\int\left(1+\frac{1}{t}\right)dt\right)dt\\&=e^t\int e^{-2t}\,\exp(t+\ln t)\,dt=e^t\int e^{-t}t\,dt\\&=e^{-t}\left\{-e^{-t}(t+1)\right\}=-(t+1).\end{aligned}$$

The complete solution of the homogeneous equation is

$$y=c_1e^t+c_2(t+1),\qquad t\in\mathbb{R}_+,\quad c_1,\,c_2\text{ arbitrary}.$$

Remark 2.2 The check of the solution is straightforward.

Remark 2.3 Warning. If one forgets to norm, then one will obtain the following **wrong** variant,

$$e^t\int e^{-2t}\exp\left(\int(t+1)dt\right)dt=e^t\int e^{-2t}\exp\left(\frac{1}{2}t^2+t\right)dt=e^t\int\exp\left(\frac{1}{2}\,t^2-t\right)dt,$$

which **cannot** be expressed by elementary functions, and which furthermore is **not** a solution of the homogeneous equation. A check gives e.g. that

$$t\frac{d^2y}{dt^2}-(t+1)\frac{dy}{dt}+y=(t^2-1)\exp\left(\frac{t^2}{2}\right)\neq 0.$$

2) a) **Guessing**. When we count the degrees, we see that if y is a polynomial of degree n, then the left hand side of (12) is again a polynomial of degree n. Hence we guess on

$$y = at^2 + bt + c, \qquad \text{where } \frac{dy}{dt} = 2at + b \text{ and } \frac{d^2y}{dt^2} = 2a.$$

Then by insertion into the left hand side of the equation,

$$\begin{aligned} t\frac{d^2y}{dt^2} - (t+1)\frac{dy}{dt} + y &= 2at - (t+1)(2at+b) + at^2 + bt + c = -at^2 + (2a-b-2a+b)t + c - b \\ &= -at^2 + c - b. \end{aligned}$$

This expression is equal to t^2, when

$$a = -1 \qquad \text{and} \qquad b = c.$$

Now $b = c$ corresponds to

$$bt + c = b(t+1),$$

which is a solution of the homogeneous equation. Hence the complete solution of the inhomogeneous equation is

$$y = -t^2 + c_1 e^t + c_2(t+1), \qquad t > 0, \quad c_1,\ c_2 \text{ arbitrære},$$

because we can include $b = c$ into the constant c_2.

b) **Alternative solution**. First **norm** the equation

$$\frac{d^2y}{dt^2}-\left(1+\frac{1}{t}\right)\frac{dy}{dt}+\frac{1}{t}\,y=t, \qquad t>0,$$

thus

$$f_1(t)=-\left(1+\frac{1}{t}\right) \qquad \text{and} \qquad u(t)=t.$$

Hence we get the auxiliary function

$$\Omega(t)=\exp\left(\int f_1(t)\,dt\right)=\exp\left(-\int\left(1+\frac{1}{t}\right)dt\right)=\frac{1}{te^t}.$$

Since $y_1(t)=e^t$, a particular solution is e.g.

$$\begin{aligned}
y_0(t) &= y_1(t)\int\frac{1}{y_1(t)^2\Omega(t)}\left(\int y_1(t)\Omega(t)u(t)\,dt\right)dt=e^t\int e^{-2t}\cdot te^t\left(\int e^t\cdot\frac{1}{te^t}\cdot t\,dt\right)dt \\
&= e^t\int te^{-t}\left(\int dt\right)dt=e^t\int t^2e^{-t}dt=e^t(-t^2-2t-2)e^{-t}=-t^2-2(t+1).
\end{aligned}$$

The complete solution is

$$\begin{aligned}
y(t) &= -t^2-2(t+1)+c_1e^t+\tilde{c}_2(t+1) \\
&= -t^2+c_1e^t+c_2(t+1), \qquad t>0, \quad c_1,,c_2 \text{ arbitrære},
\end{aligned}$$

where $c_2=\tilde{c}_2-2$.

Remark 2.4 Warning. If we forget to norm the equation, we get the following **erroneous variant**

$$\tilde{\Omega}(t)=\exp\left(-\int(t+1)dt\right)=\exp\left(-\frac{(t+1)^2}{2}\right)$$

with the **wrong** "solution"

$$\tilde{y}_0(t)=e^t\int e^{-2t}\exp\left(\frac{(t+1)^2}{2}\right)\left\{\int e^t\exp\left(-\frac{(t+1)^2}{2}\right)t^2dt\right\}dt.$$

This function cannot be expressed by elementary functions.

c) **The Wrońskian method**. When we **norm** the equation, we get as before,

$$\frac{d^2y}{dt^2}-\left(1+\frac{1}{t}\right)\frac{dy}{dt}+\frac{1}{t}\,y=t, \qquad \text{dvs. } u(t)=t.$$

Put $y_1(t)=e^t$ and $y_2(t)=t+1$ [taken from (1)]. Then

$$W(t)=\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t)\end{vmatrix}=\begin{vmatrix} e^t & t+1 \\ e^t & 1\end{vmatrix}=-te^t.$$

Now, compute

$$W_1(t)=\begin{vmatrix} 0 & y_2(t) \\ u(t) & y_2'(t)\end{vmatrix}=\begin{vmatrix} 0 & t+1 \\ t & 1\end{vmatrix}=-t^2-t,$$

$$W_2(t) = \begin{vmatrix} y_1(0) & 0 \\ y_1'(t) & u(t) \end{vmatrix} = \begin{vmatrix} e^t & 0 \\ e^t & t \end{vmatrix} = te^t.$$

Then a particular solution is given by

$$\begin{aligned} y_0(t) &= y_1(t)\int \frac{W_1(t)}{W(t)}\,dt + y_2(t)\int \frac{W_1(t)}{W(t)}\,dt \\ &= e^t \int \frac{-t^2-t}{-te^t}\,dt + (t+1)\int \frac{te^t}{-te^t}\,dt \\ &= e^t \int (t+1)e^{-t}dt - (t+) \int dt \qquad \text{(partial integration)} \\ &= e^t\left\{-(t+1)e^{-t} + \int e^{-t}dt\right\} - t(t+1) \\ &= -t-1-1-t^2-t = -t^2-2(t+1). \end{aligned}$$

Since $-2(t+1) = -2y_2(t)$, The complete solution is

$$y(t) = -t^2 + c_1e^t + c_2(t+1), \qquad t>0, \quad c_1,\, c_2 \text{ arbitrære.}$$

Remark 2.5 **Warning**. If we forget to norm the equation, then we get the wrong solution

$$\tilde{y}_0(t) = \cdots = -\frac{t^2}{2}(t+1) + (-t^2-3t-3).$$

A check in the equation shows that this is not a solution.

Remark 2.6 The equation can *in principle* be solved by means of the power series method, and the result looks *apparently* nice. We get for instance for the **homogeneous equation** (12) after some computations,

$$\sum_{n=0}^{\infty}(n-1)\{(n+1)a_{n+1}-a_n\}t^n = 0,$$

hence by the **identity theorem**

$$(n-1)\{(n+1)a_{n+1}-a_n\} = 0 \qquad \text{for } n \in \mathbb{N}_0,$$

(the summation domain). There is, however, a trap here because $n-1=0$ for $n=1$, thus

$$0\cdot\{2a_2-a_1\} = 0 \qquad \text{for } n=1.$$

This equation is fulfilled for all choices of the constants a_1 and a_2, so we conclude that a_1 are a_2 the **arbitrary** constants. The trap is, that we cannot get this result if we first divide by $n-1$ to obtain

$$(n+1)a_{n+1} = a_n.$$

The error is of course that we divide by 0, when $n=1$.
The correct variant is

$$(n+1)a_{n+1} = a_n \qquad \text{for } n \in \mathbb{N}_0 \setminus \{1\},$$

where a_1 and a_2 are arbitrary constants. It **is** possible to solve this equation, but it requires some deftness. The trick is to multiply by $n!$ and then define $b_n := n!a_n$).

Example 2.7 *Consider the differential equation*

$$(13)\quad t\frac{d^2y}{dt^2}-\frac{dy}{dt}-(t+1)y=0, \qquad t\in\mathbb{R}_+.$$

1) Prove that (13) has the solution $y=e^{-t}$, $t\in\mathbb{R}_+$, and then find the complete solution.

2) Find the complete solution of the differential equation

$$t\frac{d^2y}{dt^2}-\frac{dy}{dt}-(t+1)y=t^2, \qquad t\in\mathbb{R}_+.$$

The equation can be solved in different ways. We first demonstrate the **traditional one**:

1) If we put $y=e^{-t}$, then

$$te^{-t}+e^{-t}-(t+1)e^{-t}=0,$$

and $y=e^{-t}$ is a solution of the homogeneous equation.

2) By norming the inhomogeneous equation we get

$$\frac{d^2y}{dt^2}-\frac{1}{t}\,\frac{dy}{dt}-\left(1+\frac{1}{t}\right)y=t.$$

Hence a linearly independent solution of the homogeneous equation is given by

$$\begin{aligned}\varphi_2(t) &= e^{-t}\int e^{2t}\exp\left(\int\frac{1}{t}\,dt\right)dt=e^{-t}\int te^{2t}\,dt=e^{-t}\left\{\frac{1}{2}\,te^{2t}-\frac{1}{2}\int e^{2t}\,dt\right\}\\ &= \frac{1}{4}\,e^{-t}(2te^{2t}-e^{2t})=\frac{1}{4}(2t-1)e^t.\end{aligned}$$

The complete solution of the homogeneous equation is

$$y=c_1e^{-t}+c_2(2t-1)e^t, \qquad c_1,\, c_2 \text{ arbitrary.}$$

There are several ways to find a particular solution of the inhomogeneous equation.

a) **Guessing**. If we put $y=at+b$, then

$$t\frac{d^2y}{dt^2}-\frac{dy}{dt}-(t+1)=-a-(t+1)(at+b)=-at^2-(a+b)t-(a+b).$$

This expression is equal to t^2, if $a=-1$ and $b=1$. The complete solution is

$$y(t)=1-t+c_1e^{-t}+c_2te^t(2t-1).$$

b) **Computation by a solution formula**. We first identify $u(t)=t$, cf. the normed equation, and we have already shown in (1) that $y_1(t)=e^{-t}$. Then we get

$$\Omega(t)=\exp\left(\int f_1(t)\,dt\right)=\exp\left(-\int\frac{1}{t}\,dt\right)=\frac{1}{t}.$$

By insertion into the solution formula we obtain the complete solution

$$
\begin{aligned}
y(t) &= y_1(t)\left\{\int \frac{1}{y_1(t)^2\Omega(t)}\left[\tilde{c}_2+\int y_1(t)\Omega(t)u(t)\,dt\right]dt+c_1\right\}\\
&= e^{-t}\left\{\int te^{2t}\left[\tilde{c}_2+\int e^{-t}\cdot\frac{1}{t}\cdot t\,dt\right]dt+c_1\right\}\\
&= c_1e^{-t}+\tilde{c}_2e^{-t}\int te^{2t}\,dt+e^{-t}\int te^{2t}\left(\int e^{-t}\,dt\right)dt\\
&= c_1e^{-t}+\tilde{c}_2e^{-t}\left\{\frac{1}{2}\,te^{2t}-\frac{1}{4}\,e^{2t}\right\}-e^{-t}\int te^{t}\,dt\\
&= c_1e^{-t}+\frac{1}{4}\,\tilde{c}_2e^{t}(2t-1)-e^{-t}\{te^{t}-e^{t}\}\\
&= 1-t+c_1e^{-t}+c_2(2t-1)e^{t},
\end{aligned}
$$

where we have put $c_2=\frac{1}{4}\tilde{c}_2$.

c) **Computation by the Wrońskian method**. When we put $y_1(t) = e^{-t}$ and $y_2(t) = (2t-1)e^t$ and $u(t) = t$ we get

$$W(t) = \begin{vmatrix} e^{-t} & (2t-1)e^t \\ -e^{-t} & (2t+1)e^t \end{vmatrix} = \begin{vmatrix} e^{-t} & (2t-1)e^t \\ 0 & 4^t e^t \end{vmatrix} = 4t,$$

$$W_1(t) = \begin{vmatrix} 0 & (2t-1)e^t \\ t & (2t+1)e^t \end{vmatrix} = -t(2t-1)e^t,$$

$$W_2(t) = \begin{vmatrix} e^{-t} & 0 \\ -e^{-t} & t \end{vmatrix} = te^{-t}.$$

Then by insertion,

$$\begin{aligned} y_0(t) &= y_1(t)\int \frac{W_1(t)}{W(t)}\,dt + y_2(t)\int \frac{W_2(t)}{W(t)}\,dt = e^{-t}\int \frac{-t(2t-1)e^t}{4t}\,dt + (2t-1)e^t\int \frac{te^{-t}}{4t}\,dt \\ &= -\frac{1}{4}\,e^{-t}\int (2t-1)e^t\,dt + \frac{1}{4}(2t-1)e^t\int e^{-t}\,dt \\ &= -\frac{1}{4}\,e^{-t}\left\{(2t-1)e^t - 2\int e^t\,dt\right\} - \frac{1}{4}(2t-1) = -\frac{1}{4}(2t-1) + \frac{1}{2} - \frac{1}{4}(2t-1) \\ &= \frac{1}{4}\{1-2t+2-2t+1\} = 1-t, \end{aligned}$$

and the complete solution is

$$y = 1 - t + c_1 e^{-t} + c_2(2t-1)e^t, \qquad t \in \mathbb{R},$$

where c_1 and c_2 are arbitrary constants.

Finally, it is possible also to solve the equation directly by some clever tricks.

Since $y = e^{-t}$ is a solution of the simpler equation $\dfrac{dy}{dt} + y = 0$, the idea is to rewrite the equation (13) as a differential equation of first order in $z = \dfrac{dy}{dt} + y$. When we add $t\dfrac{dy}{dt} - t\dfrac{dy}{dt} = 0$ to the left hand side of (13), we get

$$\begin{aligned} t\frac{d^2y}{dt^2} - \frac{dy}{dt} - (t+1)y &= t\frac{d^2y}{dt^2} + t\frac{dy}{dt} - (t+1)\left\{\frac{dy}{dt} + y\right\} \\ &= t\frac{d}{dt}\left\{\frac{dy}{dt} + y\right\} - (t+1)\left\{\frac{dy}{dt} + y\right\} \qquad \left[= t\,\frac{dz}{dt} - (t+1)z\right] \\ &= t^2e^t\left[\frac{1}{te^t}\,\frac{d}{dt}\left\{\frac{dy}{dt} + y\right\} - \frac{1}{te^t}\left(1 + \frac{1}{t}\right)\cdot\left\{\frac{dy}{dt} + y\right\}\right] \\ &= t^2e^t\left[\frac{1}{te^t}\,\frac{d}{dt}\left\{\frac{dy}{dt} + y\right\} + \frac{d}{dt}\left(\frac{1}{te^t}\right)\cdot\left\{\frac{dy}{dt} + y\right\}\right] \\ &= t^2e^t\,\frac{d}{dt}\left\{\frac{1}{te^t}\left[\frac{dy}{dt} + y\right]\right\} = t^2e^t\,\frac{d}{dt}\left\{\frac{1}{te^{2t}}\,\frac{d}{dt}(e^t y)\right\}. \end{aligned}$$

Then we can solve (2) [and intrinsically also (1)] by a couple of simple integrations, because

$$t^2e^t\,\frac{d}{dt}\left\{\frac{1}{te^{2t}}\,\frac{d}{dt}(e^t y)\right\} = t^2$$

can be written as

$$\frac{d}{dt}\left\{\frac{1}{te^{2t}}\,\frac{d}{dt}\left(e^t y\right)\right\} = e^{-t}.$$

When this equation is integrated, we get with some arbitrary constant $\tilde{c}_2$ that

$$\frac{1}{te^{2t}}\,\frac{d}{dt}\left(e^t y\right) = -e^{-t} + \tilde{c}_2,$$

which we reformulate as

$$\frac{d}{dt}\left(e^t y\right) = \tilde{c}_2 te^{2t} - te^t.$$

By another integration and another arbitrary constant c_1 we get

$$e^t y = c_1 + \tilde{c}_2 \int te^{2t}\,dt - \int te^t\,dt,$$

hence

$$y = c_1 e^{-t} + \tilde{c}_2 e^{-t}\left\{\frac{1}{2}\,te^{2t} - \frac{1}{4}\,e^{2t}\right\} - (t-1) = 1 - t + c_1 e^{-t} + c_2(2t-1)e^t,$$

where $c_2 = \frac{1}{4}\,\tilde{c}_2$.

3 The exponential matrix

Example 3.1 *Find* $\exp(t\mathbf{A})$ *in each of the following cases.*

$$(1)\quad \mathbf{A} = \begin{pmatrix} -4 & 0 \\ 0 & 5 \end{pmatrix}, \qquad (2)\quad \mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

$$(3)\quad \mathbf{A} = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix}, \qquad (4)\quad \mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

1) Since $\mathbf{A}$ is a diagonal matrix, we immediately get

$$\exp(t\mathbf{A}) = \begin{pmatrix} e^{-4t} & 0 \\ 0 & e^{5t} \end{pmatrix}.$$

2) We here find for the same reason,

$$\exp(t\mathbf{A}) = \begin{pmatrix} e^{t} & 0 & 0 \\ 0 & e^{3t} & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix}.$$

3) The **eigenvalues** are the roots of the polynomial

$$\begin{vmatrix} 1-\lambda & -5 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 - 1 + 5 = \lambda^2 + 4,$$

hence $\lambda = \pm 2i$. Since the roots are imaginary and complex conjugated, we first compute

$$\mathbf{A}^2 = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix} = -2^2\mathbf{I}.$$

Then it follows by the definition of the exponential series that

$$\begin{aligned}
\exp(\mathbf{A}t) &= \sum_{n=0}^{\infty} \frac{1}{n!}\,\mathbf{A}^n\, t^n \quad \text{(split into even and odd indices)} \\
&= \sum_{n=0}^{\infty} \frac{1}{(2n)!}\,\mathbf{A}^{2n}\, t^{2n} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\,\mathbf{A}^{2n+1}\, t^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, 2^{2n}\, t^{2n}\,\mathbf{I} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, 2^{2n}\, t^{2n+1}\,\mathbf{A} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\,(2t)^{2n}\,\mathbf{I} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\,(2t)^{2n+1}\cdot\frac{1}{2}\,\mathbf{A} \\
&= \cos(2t)\,\mathbf{I} + \sin(2t)\cdot\frac{1}{2}\,\mathbf{A} = \begin{pmatrix} \cos 2t + \frac{1}{2}\sin 2t & -\frac{5}{2}\sin 2t \\ \frac{1}{2}\sin 2t & \cos 2t - \frac{1}{2}\sin 2t \end{pmatrix}.
\end{aligned}$$

4) It is seen by inspection that the eigenvalues are $\lambda = 0$ and $\lambda = 1$. Since

$$\mathbf{A}^2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \mathbf{A},$$

it follows by induction that $\mathbf{A}^n = \mathbf{A}$ for every $n \in \mathbb{N}$. Then the defining series becomes

$$\begin{aligned}\exp(\mathbf{A}t) &= \mathbf{I} + \sum_{n=1}^{\infty} \frac{1}{n!}\,\mathbf{A}^n\, t^n = \mathbf{I} + \sum_{n=1}^{\infty} \frac{t^n}{n!}\,\mathbf{A} = \mathbf{I} + (e^t - 1)\mathbf{A} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} e^t - 1 & e^t - 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e^t & e^t - 1 \\ 0 & 1 \end{pmatrix}.\end{aligned}$$

Example 3.2 *The matrix* $\begin{pmatrix} e^t & -te^t \\ 0 & e^t \end{pmatrix}$ *is of the form* $\exp(t\mathbf{A})$ *Find the matrix* $\mathbf{A}$ *by exploiting the properties of the columns in a matrix of the form* $\exp(t\mathbf{A})$.

It follows from

$$\begin{vmatrix} e^t & -te^t \\ 0 & e^t \end{vmatrix} = e^{2t} > 0, \qquad \text{and} \qquad \begin{pmatrix} e^0 & -0\cdot e^0 \\ 0 & e^0 \end{pmatrix} = \mathbf{I} \quad \text{for } t = 0,$$

that the matrix can be written in the form $\exp(t\mathbf{A})$.

Then we can find the matrix $\mathbf{A}$ in various ways:

1) We find by a differentiation,

$$\frac{d}{dt}\exp(t\mathbf{A}) = \mathbf{A}\exp(t\mathbf{A}) = \begin{pmatrix} e^t & -e^t - te^t \\ 0 & e^t \end{pmatrix}.$$

By putting $t = 0$, we get

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

2) ("The hint"). The equation $\dfrac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ has the complete solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1e^t - c_2te^t \\ c_2e^t \end{pmatrix}.$$

Hence

$$\begin{aligned}\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} (c_1-c_2)e^t - c_2te^t \\ c_2e^t \end{pmatrix} = \begin{pmatrix} c_1e^t - c_2te^t \\ c_2e^t \end{pmatrix} + \begin{pmatrix} -c_2e^t \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\end{aligned}$$

and we conclude that

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Example 3.3 *Find by means of the eigenvalue method the complete solution of*

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix} \mathbf{x}(t).$$

Then find $\exp(t\mathbf{A})$.

Clearly, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -2$. The corresponding eigenvectors are cross vectors of the first row $(1-\lambda, 1)$, thus a possibility is

$$\lambda_1 = 1, \qquad \mathbf{v}_1 = (1, \lambda_1 - 1) = (1, 0),$$

$$\lambda_2 = -2, \qquad \mathbf{v}_2 = (1, \lambda_2 - 1) = (1, -3).$$

The complete solution is

$$\mathbf{x} = c_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} e^t & e^{-2t} \\ 0 & -3e^{-2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

There are now a couple of ways to find the exponential matrix $\exp(t\mathbf{A})$.

1) First note that $\exp(\mathbf{A}t) = \mathbf{\Phi}(t)[\mathbf{\Phi}(0)]^{-1}$. Since

$$\mathbf{\Phi}(t) = \begin{pmatrix} e^t & e^{-2t} \\ 0 & -3e^{-2t} \end{pmatrix}, \qquad \text{we have} \qquad \mathbf{\Phi}(0) = \begin{pmatrix} 1 & 1 \\ 0 & -3 \end{pmatrix},$$

where $\det \mathbf{\Phi}(0) = -3$, hence

$$\mathbf{\Phi}(0)^{-1} = \frac{1}{-3} \begin{pmatrix} -3 & -1 \\ 0 & 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 3 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{3} \\ 0 & -\frac{1}{3} \end{pmatrix},$$

and whence

$$\exp(\mathbf{A}t) = \begin{pmatrix} e^t & e^{-2t} \\ 0 & -3e^{-2t} \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{3} \\ 0 & -\frac{1}{3} \end{pmatrix} = \begin{pmatrix} e^t & \frac{1}{3}e^t - \frac{1}{3}e^{-2t} \\ 0 & e^{-2t} \end{pmatrix}.$$

2) **Definition by a series**. Since $\mathbf{A}$ and $\mathbf{I}$ commute, we obtain by subtracting $\lambda_1 \mathbf{I} = \mathbf{I}$ and then adding it again,

$$\exp(\mathbf{A}t) = \exp((\mathbf{A} - \mathbf{I})t + \mathbf{I}t) = e^t \exp(\mathbf{B}t),$$

where

$$\mathbf{B} = \mathbf{A} - \mathbf{I} = \begin{pmatrix} 0 & 1 \\ 0 & -3 \end{pmatrix},$$

and

$$\mathbf{B}^2 = \begin{pmatrix} 0 & -3 \\ 0 & 9 \end{pmatrix} = -3 \begin{pmatrix} 0 & 1 \\ 0 & -3 \end{pmatrix} = -3\mathbf{B}.$$

Then by induction,

$$\mathbf{B}^n = (-3)^{n-1}\mathbf{B}, \qquad n \in \mathbb{N},$$

which gives

$$
\begin{aligned}
\exp(\mathbf{A}t) &= e^t \exp(\mathbf{B}t) = e^t \left\{ \mathbf{I} + \sum_{n=1}^{\infty} \frac{1}{n!} \mathbf{B}^n\, t^n \right\} = e^t \left\{ \mathbf{I} + \sum_{n=1}^{\infty} \frac{1}{n!} (-3)^{n-1}\, t^n \cdot \mathbf{B} \right\} \\
&= e^t \left\{ \mathbf{I} - \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n!} (-3t)^n\, \mathbf{B} \right\} = e^t \left\{ \mathbf{I} - \frac{1}{3}(e^{-3t} - 1)\mathbf{B} \right\} \\
&= \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix} - \left(e^{-2t} - e^t\right) \begin{pmatrix} 0 & \frac{1}{3} \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{3}e^t - \frac{1}{3}e^{-2t} \\ 0 & e^{-2t} - e^t \end{pmatrix} \\
&= \begin{pmatrix} e^t & \frac{1}{3}e^t - \frac{1}{3}e^{-2t} \\ 0 & e^{-2t} \end{pmatrix}.
\end{aligned}
$$

Example 3.4 *Find the complete solution of the given system below*

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & 4 \\ -2 & -3 \end{pmatrix} \mathbf{x}(t).$$

The **eigenvalues** are the roots of the polynomial

$$\begin{vmatrix} 1-\lambda & 4 \\ -2 & -3-\lambda \end{vmatrix} = (\lambda-1)(\lambda+3)+8 = \lambda^2+2\lambda+5 = (\lambda+1)^2+4,$$

thus $\lambda = -1 \pm 2i$. since we have complex conjugated eigenvalues, we have at least four reasonable methods of solution.

1) The **eigenvalue method**. If $\lambda = -1+2i$, then the corresponding eigenvector is a cross vector of $(2-2i, 4)$, e.g.

$$\mathbf{v} = \begin{pmatrix} 2 \\ -1+i \end{pmatrix} = \alpha + i\beta = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then a fundamental matrix is given by

$$\begin{aligned} \mathbf{\Phi}(t) &= e^{-t}\{\cos 2t(\alpha \ \ \beta) + \sin 2t(-\beta \ \ \alpha)\} = e^{-t}\left\{\cos 2t \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} + \sin 2t \begin{pmatrix} 0 & 2 \\ -1 & -1 \end{pmatrix}\right\} \\ &= e^{-t} \begin{pmatrix} 2\cos 2t & 2\sin 2t \\ -\cos 2t - \sin 2t & \cos 2t - \sin 2t \end{pmatrix}. \end{aligned}$$

The complete solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} 2\cos 2t \\ -\cos 2t - \sin 2t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 2\sin 2t \\ \cos 2t - \sin 2t \end{pmatrix}.$$

2) The **exponential matrix**. If we choose $a + i\omega = -1+2i$, then

$$\begin{aligned} \exp(\mathbf{A}t) &= e^{-t}\left\{\cos 2t + \frac{1}{2}\sin 2t\right\} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2} e^{-t} \sin 2t \begin{pmatrix} 1 & 4 \\ -2 & -3 \end{pmatrix} \\ &= e^{-t} \begin{pmatrix} \cos 2t + \sin 2t & 2\sin 2t \\ -\sin 2t & \cos 2t - \sin 2t \end{pmatrix}. \end{aligned}$$

The complete solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} \cos 2t + \sin 2t \\ -\sin 2t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 2\sin 2t \\ \cos 2t - \sin 2t \end{pmatrix}.$$

3) It follows from $\lambda = -1 \pm 2i$, that the **real structure of solution** necessarily must be of the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 e^{-t}\cos 2t + a_2 e^{-t}\sin 2t \\ b_1 e^{-t}\cos 2t + b_2 e^{-t}\sin 2t \end{pmatrix}.$$

Since

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (-a_1+2a_2)e^{-t}\cos 2t + (-2a_1-a_2)e^{-t}\sin 2t \\ (-b_1+2b_2)e^{-t}\cos 2t + (-2b_1-b_2)e^{-t}\sin 2t \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 4 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} (a_1+4b_1)e^{-t}\cos 2t+(a_2+4b_2)e^{-t}\sin 2t \\ (-2a_1-3b_1)e^{-t}\cos 2t+(-2b_1-b_2)e^{-t}\sin 2t \end{pmatrix},$$

we obtain by an identification of the coefficients that

$$\begin{cases} -a_1+2a_2=a_1+4b_1, & \text{dvs. } b_1=-\frac{1}{2}a_1+\frac{1}{2}a_2, \\ -b_1+2b_2=-2a_1-3b_1, & \text{dvs. } b_1+b_2=-a_1. \end{cases}$$

We see that

$$b_2=-\frac{1}{2}a_1-\frac{1}{2}a_2 \qquad \text{and} \qquad b_1=-\frac{1}{2}a_1+\frac{1}{2}a_2.$$

If we put $a_1=2c_1$ and $a_2=2c_2$, then the solution is given by

$$\begin{aligned}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 2c_1e^{-t}\cos 2t+2c_2e^{-t}\sin 2t \\ (-c_1+c_2)e^{-t}\cos 2t+(-c_1-c_2)e^{-t}\sin 2t \end{pmatrix} \\ &= c_1e^{-t}\begin{pmatrix} 2\cos 2t \\ -\cos 2t-\sin 2t \end{pmatrix}+c_2e^{-t}\begin{pmatrix} 2\sin 2t \\ \cos 2t-\sin 2t \end{pmatrix}.\end{aligned}$$

4) The **"fumbling method"**. It follows from

$$\frac{dx_1}{dt}=x_1+4x_2, \qquad \text{thus } x_2=\frac{1}{4}\left\{\frac{dx_1}{dt}-x_1\right\},$$

and

$$\frac{dx_2}{dt}=-2x_1-3x_2,$$

that

$$4\frac{dx_2}{dt}=\frac{d^2x_1}{dt^2}-\frac{dx_1}{dt}=-8x_1-12x_2=-8x_1-3\frac{dx_1}{dt}+3x_1,$$

hence by a rearrangement,

$$\frac{d^2x_1}{dt^2}+2\frac{dx_1}{dt}+5x_1=0, \qquad \text{where } R^2+2R+5=(R+1)^2+2^2.$$

The complete solution is

$$x_1=c_1e^{-t}\cos 2t+c_2e^{-t}\sin 2t,$$

where

$$\frac{dx_1}{dt}=c_1\left\{-e^{-t}\cos 2t-2e^{-t}\sin 2t\right\}+c_2\left\{-e^{-t}\sin 2t+2e^{-t}\cos 2t\right\}.$$

Hence

$$\begin{aligned} x_2 &= \frac{1}{4}\left\{\frac{dx_1}{dt}-x_1\right\}=\frac{1}{4}c_1\left\{-2e^{-t}\cos 2t-2e^{-t}\sin 2t\right\}+\frac{1}{4}c_2\left\{2e^{-t}\cos 2t-2e^{-t}\sin 2t\right\} \\ &= c_1\frac{1}{2}e^{-t}(-\cos 2t-\sin 2t)+\frac{1}{2}c_2e^{-t}(\cos 2t-\sin 2t),\end{aligned}$$

and summing up,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=c_1e^{-t}\begin{pmatrix} \cos 2t \\ -\frac{1}{2}\cos 2t-\frac{1}{2}\sin 2t \end{pmatrix}+c_2e^{-t}\begin{pmatrix} \sin 2t \\ \frac{1}{2}\cos 2t-\frac{1}{2}\sin 2t \end{pmatrix}.$$

Example 3.5 *Given the linear system*

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t \\ 1 \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Compute $\exp(\mathbf{A}t)$
Then compute by means of the general solution formula the particular solution (x, y) *of the system, for which*

$$(x(0), y(0)) = (v_1, v_2).$$

1) **Direct determination of** $\exp(\mathbf{A}t)$. First we note by induction that

$$\mathbf{A}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \qquad n \in \mathbb{N}.$$

Then by the exponential series,

$$\exp(t\mathbf{A}) = \mathbf{I} + \sum_{n=1}^{\infty} \frac{1}{n!} t^n \, \mathbf{A}^n = \mathbf{I} + \sum_{n=1}^{\infty} \frac{1}{n!}\, t^n \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

It follows from

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \cdot n = \sum_{n=1}^{\infty} \frac{t^n}{(n-1)!} = \sum_{n=0}^{\infty} \frac{t^{n+1}}{n!} = te^t,$$

that

$$\exp(t\mathbf{A}) = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}.$$

2) **Variant of the eigenvalue method**. Clearly, $\lambda = 1$ has algebraic multiplicity 2 and geometrical multiplicity 1. Therefore the complete solution of the corresponding homogeneous equation must have the structure

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 e^t + a_2 te^t \\ b_1 e^t + b_2 te^t \end{pmatrix}$$

where

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (a_1 + a_2)e^t + a_2 te^t \\ (b_1 + b_2)e^t + b_2 te^t \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (a_1 + b_1)e^t + (a_2 + b_2)te^t \\ b_1 e^t + b_2 te^t \end{pmatrix}.$$

When we identify the coefficients, we get

$$\begin{cases} a_1 + a_2 = a_1 + b_1, & \text{hence } b_1 = a_2, \\ b_1 + b_2 = b_1, & \text{hence } b_2 = 0. \end{cases}$$

Choose a_1 and a_2 a the arbitrary constants. Then the solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 e^t + a_2 t e^t \\ a_2 e^t \end{pmatrix} = a_1 \begin{pmatrix} e^t \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} te^t \\ e^t \end{pmatrix} = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

The fundamental matrix becomes

$$\mathbf{\Phi}(t) = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix} \qquad \text{where } \mathbf{\Phi}(0) = \mathbf{I} = \mathbf{\Phi}(0)^{-1}.$$

Hence

$$\exp(\mathbf{A}t) = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1} = \mathbf{\Phi}(t) = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}.$$

We shall now apply the complicated **solution formula**. We find

$$\begin{aligned}
\begin{pmatrix} x \\ y \end{pmatrix} &= e^{\mathbf{A}t}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}t}\begin{pmatrix} \tau \\ 1 \end{pmatrix} d\tau \\
&= \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}\int_0^t \begin{pmatrix} e^{-\tau} & -\tau e^{-\tau} \\ 0 & e^{-\tau} \end{pmatrix}\begin{pmatrix} \tau \\ 1 \end{pmatrix} d\tau \\
&= \begin{pmatrix} v_1 e^t + v_2 te^t \\ v_2 e^t \end{pmatrix} + \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}\int_0^t \begin{pmatrix} \tau e^{-\tau} - \tau e^{-\tau} \\ e^{-\tau} \end{pmatrix} d\tau \\
&= \begin{pmatrix} v_1 e^t + v_2 te^t \\ v_2 e^t \end{pmatrix} + \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}\begin{pmatrix} 0 \\ 1-e^{-t} \end{pmatrix} \qquad \left[\text{because } \int_0^t \begin{pmatrix} 0 \\ e^{-\tau} \end{pmatrix} = \begin{pmatrix} 0 \\ 1-e^{-t} \end{pmatrix}\right] \\
&= \begin{pmatrix} v_1 e^t + v_2 te^t \\ v_2 e^t \end{pmatrix} + \begin{pmatrix} te^t - t \\ e^t - 1 \end{pmatrix} = \begin{pmatrix} v_1 e^t + (v_2+1)te^t - t \\ (v_2+1)e^t - 1 \end{pmatrix}.
\end{aligned}$$

Even if these computations are unexpectedly easy to perform, it follows that it also here would be a better procedure just to guess a particular solution of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 t + a_2 \\ b_1 t + b_2 \end{pmatrix} \qquad \left[= \begin{pmatrix} -t \\ -1 \end{pmatrix}\right].$$

The details are left to the reader.

Example 3.6 *Consider the system*

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}, \qquad t \in \mathbb{R}.$$

1) Find $\exp(t\mathbf{A})$ *for* $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

2) Apply the general solution formula to find that particular solution $\mathbf{x}(t)$*, for which* $\mathbf{x}(0) = (1,0)^T$.

1) As usual we we have several solution possibilities.

a) Since $\mathbf{A}^2 = \mathbf{I}$, it follows immediately by the exponential series that

$$\begin{aligned}
\exp(t\mathbf{A}) &= \sum_{n=0}^{\infty} \frac{1}{n!} t^n \mathbf{A}^n = \mathbf{I}\sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} + \mathbf{A}\sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} \\
&= \begin{pmatrix} \cosh t & 0 \\ 0 & \cosh t \end{pmatrix} + \begin{pmatrix} 0 & \sinh t \\ \sinh t & 0 \end{pmatrix} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}.
\end{aligned}$$

b) The eigenvalues are the roots of the polynomial

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = (\lambda-1)(\lambda+1),$$

thus $\lambda = \pm 1$.

If $\lambda = 1$, then one eigenvector is $(1,1)$.

If $\lambda = -1$, then one eigenvector is $(1,-1)$.

The complete solution of the homogeneous system is

$$\begin{pmatrix} x \\ y \end{pmatrix} = a\begin{pmatrix} e^t \\ e^t \end{pmatrix} + b\begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} = \begin{pmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}.$$

Then we get the fundamental matrix,

$$\mathbf{\Phi}(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix}$$

where

$$\mathbf{\Phi}(0) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{og} \quad \mathbf{\Phi}(0)^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

The exponential matrix is

$$\exp(t\mathbf{A}) = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1} = \frac{1}{2}\begin{pmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}.$$

c) We shall now apply the solution formula:

$$\begin{aligned}
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \exp(t\mathbf{A})\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \exp(t\mathbf{A})\int_0^t \exp(-\tau\mathbf{A})\begin{pmatrix} e^{-\tau} \\ e^{-\tau} \end{pmatrix} d\tau \\
&= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}\int_0^t \begin{pmatrix} \cosh\tau & -\sinh\tau \\ -\sinh\tau & \cosh\tau \end{pmatrix}\begin{pmatrix} e^{-\tau} \\ e^{-\tau} \end{pmatrix} d\tau \\
&= \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix} + \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}\int_0^t \begin{pmatrix} e^{-2\tau} \\ e^{-2\tau} \end{pmatrix} d\tau \\
&= \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix} + \frac{1}{2}(1 - e^{-2t})\begin{pmatrix} \cosh t + \sinh t \\ \sinh t + \cosh t \end{pmatrix} \\
&= \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix} + \frac{1}{2}(1 - e^{-2t})\begin{pmatrix} e^t \\ e^t \end{pmatrix} = \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix} + \begin{pmatrix} \sinh t \\ \sinh t \end{pmatrix} \\
&= \begin{pmatrix} e^t \\ 2\sinh t \end{pmatrix} \qquad \left[= \begin{pmatrix} e^t \\ e^t - e^{-t} \end{pmatrix}\right].
\end{aligned}$$

Example 3.7 *Consider the homogeneous linear system*

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\,\mathbf{x}, \qquad \textit{where } \mathbf{A} = \begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix}.$$

1) Find the complete solution.

2) Find the exponential matrix $\exp(\mathbf{A}t)$.

Here we give five variants.

Overriding we first find the characteristic polynomial, which is applied in all the following variants,

$$\begin{vmatrix} 1-\lambda & 2 \\ 4 & -1-\lambda \end{vmatrix} = (\lambda-1)(\lambda+1)-8 = \lambda^2-9 = (\lambda-3)(\lambda+3).$$

We conclude that the eigenvalues are $\lambda = \pm 3$.

First variant. The eigenvalue method.

1) Consider the (2×2) matrix $\mathbf{A}$. An eigenvector corresponding to λ is a cross vector of e.g. $(1-\lambda, 2)$, where we can choose $\mathbf{v} = \left(1, \dfrac{\lambda-1}{2}\right)$ by adding the factor $\dfrac{1}{2}$.

If $\lambda = 3$, then an eigenvector is $\mathbf{v}_1 = (1,1)$.

If $\lambda = -3$, then an eigenvector is $\mathbf{v}_2 = (1,-2)$.

The complete solution is

$$(14)\quad \mathbf{x} = c_1 e^{3t}\begin{pmatrix}1\\1\end{pmatrix} + c_2 e^{-3t}\begin{pmatrix}1\\-2\end{pmatrix} = \begin{pmatrix}e^{3t} & e^{-3t}\\ e^{3t} & -2e^{-3t}\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix},$$

where c_1 and c_2 are arbitrary constants, and $t \in \mathbb{R}$.

2) By (14) a fundamental matrix is given by

$$\mathbf{\Phi}(t) = \begin{pmatrix}e^{3t} & e^{-3t}\\ e^{3t} & -2e^{-3t}\end{pmatrix}, \quad \text{where } \mathbf{\Phi}(0) = \begin{pmatrix}1 & 1\\ 1 & -2\end{pmatrix}, \quad \det \mathbf{\Phi}(0) = -3,$$

hence

$$\mathbf{\Phi}(0)^{-1} = -\frac{1}{3}\begin{pmatrix}-2 & -1\\ -1 & 1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}2 & 1\\ 1 & -1\end{pmatrix},$$

and

$$\begin{aligned}\exp(\mathbf{A}t) &= \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1} = \frac{1}{3}\begin{pmatrix}e^{3t} & e^{-3t}\\ e^{3t} & -2e^{-3t}\end{pmatrix}\begin{pmatrix}2 & 1\\ 1 & -1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}2e^{3t}+e^{-3t} & e^{3t}-e^{-3t}\\ 2e^{3t}-2e^{-3t} & e^{3t}+2e^{-3t}\end{pmatrix}\\ &= \frac{1}{3}\begin{pmatrix}3\cosh(3t)+\sinh(3t) & 2\sinh(3t)\\ 4\sinh(3t) & 3\cosh(3t)-\sinh(3t)\end{pmatrix},\end{aligned}$$

where we have given the result in two equivalent versions.

Second variant. Direct determination of the exponential matrix. In this case it will be convenient to interchange the two questions of the example.

1) It follows from

$$\mathbf{A}^2 = \begin{pmatrix}1 & 2\\ 4 & -1\end{pmatrix}\begin{pmatrix}1 & 2\\ 4 & -1\end{pmatrix} = \begin{pmatrix}9 & 0\\ 0 & 9\end{pmatrix} = 9\mathbf{I} = 3^2\mathbf{I},$$

that

$$\mathbf{A}^{2n} = 3^{2n}\mathbf{I} \quad \text{and} \quad \mathbf{A}^{2n+1} = \mathbf{A}^{2n}\mathbf{A} = 3^{2n}\mathbf{A}.$$

Then we get by the exponential series (again with two equivalent versions)

$$\begin{aligned}\exp(\mathbf{A}t) &= \sum_{n=0}^{\infty}\frac{t^n}{n!}\mathbf{A}^n = \sum_{n=0}^{\infty}\frac{t^{2n}}{(2n)!}\mathbf{A}^{2n} + \sum_{n=0}^{\infty}\frac{t^{2n+1}}{(2n+1)!}\mathbf{A}^{2n+1} = \sum_{n=0}^{\infty}\frac{(3t)^{2n}}{(2n)!}\mathbf{I} + \frac{1}{3}\sum_{n=0}^{\infty}\frac{(3t)^{2n+1}}{(2n+1)!}\mathbf{A}\\ &= \cosh(3t)\mathbf{I} + \frac{1}{3}\sinh(3t)\mathbf{A} = \frac{1}{3}\begin{pmatrix}3\cosh(3t)+\sinh(3t) & 2\sinh(3t)\\ 4\sinh(3t) & 3\cosh(3t)-\sinh(3t)\end{pmatrix}\\ &= \frac{1}{3}\begin{pmatrix}2e^{3t}+e^{-3t} & e^{3t}-e^{-3t}\\ 2e^{3t}-2e^{-3t} & e^{3t}+2e^{-3t}\end{pmatrix}.\end{aligned}$$

2) The complete solution is obtained by taking all the possible linear combinations of the columns of $\exp(\mathbf{A}t)$. The common factor $\frac{1}{3}$ can be included in the arbitrary constants, so

$$\begin{aligned}\mathbf{x}(t) &= c_1\begin{pmatrix}2e^{3t}+e^{-3t}\\ 2e^{3t}-2e^{-3t}\end{pmatrix}+c_2\begin{pmatrix}e^{3t}-e^{-3t}\\ e^{3t}+2e^{-3t}\end{pmatrix}=(2c_1+c_2)e^{3t}\begin{pmatrix}1\\1\end{pmatrix}+(c_1-c_2)e^{-3t}\begin{pmatrix}1\\-2\end{pmatrix}\\ &= \tilde{c}_1e^{3t}\begin{pmatrix}1\\1\end{pmatrix}+\tilde{c}_2e^{-3t}\begin{pmatrix}1\\-2\end{pmatrix},\end{aligned}$$

where $t \in \mathbb{R}$, and where c_1, c_2, $\tilde{c}_1$ and $\tilde{c}_2$ are arbitrary constants.

Third variant. The exponential matrix computed by a formula. Here we only show (2).

Since $n = 2$ and the eigenvalues are $\lambda = 3$ and $\mu = -3$, thus $\lambda \neq \mu$, we get

$$\begin{aligned}\exp(\mathbf{A}t) &= \frac{e^{\lambda t}-e^{\mu t}}{\lambda-\mu}\mathbf{A}+\frac{\lambda e^{\mu t}-\mu e^{\lambda t}}{\lambda-\mu}\mathbf{I}=\frac{1}{6}(e^{3t}-e^{-3t})\mathbf{A}+\frac{1}{6}(3e^{-3t}+3e^{3t})\mathbf{I}\\ &= \frac{1}{3}\sinh(3t)\mathbf{A}+\cosh(3t)\mathbf{I},\end{aligned}$$

and the example is then concluded as in one of the other variants.

Fourth variant. The fumbling method. We first expand the system of equations,

$$\frac{dx_1}{dt} = x_1 + 2x_2, \quad \text{thus } 2x_2 = \frac{dx_1}{dt} - x_1,$$

$$\frac{dx_2}{dt} = 4x_1 - x_2, \quad \text{thus } \frac{d}{dt}(2x_2) + 2x_2 = 8x_1.$$

By eliminating x_2 we get

$$\frac{d}{dt}(2x_2) + 2x_2 = \left(\frac{d^2x_1}{dt^2} - \frac{dx_1}{dt}\right) + \left(\frac{dx_1}{dt} - x_1\right) = \frac{d^2x_1}{dt^2} - x_1 = 8x_1,$$

hence

$$\frac{d^2x_1}{dt^2} - 9x_1 = 0 \quad \text{med } R^2 - 9 = (R+3)(R-3) = 0.$$

The complete solution is

$$x_1 = c_1e^{3t} + c_2e^{-3t}, \qquad c_1, c_2 \text{ arbitrary},$$

so by insertion,

$$2x_2 = \frac{dx_1}{dt} - x_1 = (3c_1e^{3t} - 3c_2e^{-3t}) - (c_1e^{3t} + c_2e^{-3t}) = 2c_1e^{3t} - 4c_2e^{-3t},$$

thus

$$x_2 = c_1e^{3t} - 2c_2e^{-3t},$$

where c_1 and c_2 are the same arbitrary constants as for x_1.

Summing up we have

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1e^{3t} + c_2e^{-3t} \\ c_1e^{3t} - 2c_2e^{-3t} \end{pmatrix} = c_1e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2e^{-3t}\begin{pmatrix} 1 \\ -2 \end{pmatrix},$$

where c_1 and c_2 are arbitrary constants.

Then we continue as in the first variant.

Fifth variant. Caley-Hamilton's theorem. Recall that the characteristic polynomial is $\lambda^2 - 9$. Then we get by Caley-Hamilton's theorem that

$$\mathbf{A}^2 - 9\mathbf{I} = \mathbf{0}, \qquad \text{thus } \mathbf{A}^2 = 9\mathbf{I}.$$

This implies that there exist functions $\varphi(t)$ and $\psi(t)$, such that

$$\exp(\mathbf{A}t) = \varphi(t)\mathbf{I} + \psi(t)\mathbf{A}, \quad \varphi(0) = 1 \text{ og } \psi(0) = 0.$$

We then conclude from

$$\frac{d}{dt}\exp(\mathbf{A}t) = \mathbf{A}\,\exp(\mathbf{A}t)$$

and

$$\frac{d}{dt}\exp(\mathbf{A}t) = \varphi'(t)\mathbf{I} + \psi'(t)\mathbf{A},$$

and

$$\mathbf{A}\,\exp(\mathbf{A}t) = \varphi(t)\mathbf{A} + \psi(t)\mathbf{A}^2 = 9\psi(t)\mathbf{I} + \varphi(t)\mathbf{A},$$

by identifying the coefficients of $\mathbf{I}$ and $\mathbf{A}$ that

(15) $\varphi'(t) = 9\psi(t)$ and $\psi'(t) = \varphi(t)$,

where

$$\begin{cases} \varphi(0) = 1, & \varphi'(0) = 9\psi(0) = 0, \\ \psi(0) = 0, & \psi'(t) = \varphi(0) = 1. \end{cases}$$

If we put the latter equation of (15) into the first one, i.e. eliminating $\varphi(t)$, then

$$\psi''(t) = 9\psi(t), \qquad \psi(0) = 0, \quad \psi'(0) = 1,$$

The complete solution is

$$\psi(t) = c_1 \sinh(3t) + c_2 \cosh(3t).$$

We conclude from $\psi(0) = 0$ that $c_2 = 0$, and from $\psi'(0) = 1$ that $c_1 = \dfrac{1}{3}$, so

$$\psi(t) = \frac{1}{3}\sinh(3t).$$

Finally,

$$\varphi(t) = \psi'(t) = \cosh(3t),$$

so summing up,

$$\exp(\mathbf{A}t) = \varphi(t)\mathbf{I} + \psi(t)\mathbf{A} = \cosh(3t)\mathbf{I} + \frac{1}{3}\sinh(3t)\mathbf{A},$$

and we continue as in anyone of the previous variants.

Example 3.8 *Let* $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 2 & -3 \end{pmatrix}$.

1) Find the complete solution of the homogeneous system of differential equations,

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\,\mathbf{x}, \qquad t \in \mathbb{R}.$$

2) Prove that

$$\exp(\mathbf{A}t) = \begin{pmatrix} 2e^{-t} - e^{-2t} & -e^{-t} + e^{-2t} \\ 2e^{-t} - 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{pmatrix}, \qquad t \in \mathbb{R}.$$

3) Find the solution of the inhomogeneous system of differential equations

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\,\mathbf{x} + \begin{pmatrix} 3e^{t} \\ 0 \end{pmatrix}, \qquad t \in \mathbb{R},$$

for which $\mathbf{x}(0) = (2,1)^T$.

Again there are several variants.

1) **The brazen** (though legal) **variant**.

   a) Start with (2). Then $\exp(\mathbf{A}t)$ is the unique solution of

   $$\frac{d}{dt}\mathbf{B}(t) = \mathbf{A}\,\mathbf{B}(t), \qquad \mathbf{B}(0) = \mathbf{I},$$

   so we shall only check that these equations are fulfilled for

   $$\mathbf{B}(t) = \begin{pmatrix} 2e^{-t} - e^{-2t} & -e^{-t} + e^{-2t} \\ 2e^{-t} - 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{pmatrix} = e^{-t}\begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} + e^{-2t}\begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}.$$

Clearly, $\mathbf{B}(0) = \mathbf{I}$, and

$$\frac{d}{dt}\mathbf{B}(t) = e^{-t}\begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix} + e^{-2t}\begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix}$$

and

$$\begin{aligned} \mathbf{A}\,\mathbf{B}(t) &= e^{-t}\begin{pmatrix} 0 & -1 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} + e^{-2t}\begin{pmatrix} 0 & -1 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix} \\ &= e^{-t}\begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix} + e^{-2t}\begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix} = \frac{d}{dt}\mathbf{B}(t) \end{aligned}$$

and we conclude that $\mathbf{B}(t)$ also fulfils the system of differential equations, and we have proved (2).

b) (Question (1)). It follows from (2) that the complete solution of (1) is

$$\begin{aligned} \mathbf{x}(t) &= c_1\begin{pmatrix} 2e^{-t} - e^{-2t} \\ 2e^{-t} - 2e^{-2t} \end{pmatrix} + c_2\begin{pmatrix} -e^{-t} + e^{-2t} \\ -e^{-t} + 2e^{-2t} \end{pmatrix} \\ &= (2c_1 - c_2)e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-c_1 + c_2)e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}. \end{aligned}$$

c) We guess a solution of the form $\mathbf{x}(t) = e^t(a,b)^T$. Then by a rearrangement,

$$e^t\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \frac{d\mathbf{x}}{dt} - \mathbf{A}\,\mathbf{x} = e^t\begin{pmatrix} a \\ b \end{pmatrix} - e^t\begin{pmatrix} 0 & -1 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = e^t\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix},$$

which corresponds to the system of equations

$$a + b = 3 \qquad \text{and} \qquad -2a + 4b = 0.$$

It follows immediately that it has the solution $a = 2$ and $b = 1$.
The complete solution is

$$\mathbf{x}(t) = e^t\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \tilde{c}_1 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \tilde{c}_2 e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

The initial condition gives $\tilde{c}_1 = \tilde{c}_2 = 0$, hence the solution is

$$\mathbf{x}(t) = e^t\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad t \in \mathbb{R}.$$

2) The **standard method**

a) The **eigenvalue method**. The characteristic polynomial is

$$\begin{vmatrix} 0-\lambda & -1 \\ 2 & -3-\lambda \end{vmatrix} = \lambda(\lambda+3)+2 = \lambda^2+3\lambda+2 = (\lambda+1)(\lambda+2),$$

so the eigenvalues are $\lambda = -1$ and $\lambda = -2$. A corresponding eigenvector is a cross vector of $(-\lambda, -1)$, thus e.g. $(1, -\lambda)$.

If $\lambda = -1$, then an eigenvector is $\mathbf{v}_1 = (1, 1)$.

If $\lambda = -2$, then an eigenvector is $\mathbf{v}_2 = (1, 2)$.

The complete solution is

$$\mathbf{x}(t) = c_1 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} e^{-t} & e^{-2t} \\ e^{-t} & 2e^{-2t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

where c_1 and c_2 are arbitrary constants.

b) It follows from (a) that a fundamental matrix is

$$\mathbf{\Phi}(t) = \begin{pmatrix} e^{-t} & e^{-2t} \\ e^{-t} & 2e^{-2t} \end{pmatrix} \quad \text{where} \quad \mathbf{\Phi}(0) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

It follows from $\det \mathbf{\Phi}(0) = 1$ that

$$\mathbf{\Phi}(0)^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix},$$

hence

$$\exp(\mathbf{A}t) = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1} = \begin{pmatrix} e^{-t} & e^{-2t} \\ e^{-t} & 2e^{-2t} \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -2e^{-t} - e^{-2t} & -e^{-t} + e^{-2t} \\ 2e^{-t} - 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{pmatrix}.$$

c) It is of course possible to apply the same method as in the first variant. Here we shall, however, demonstrate the horrible solution formula,

$$\mathbf{x}(t) \exp(\mathbf{A}t)\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \exp(\mathbf{A}t)\int_0^t \exp(-\mathbf{A}\tau)\begin{pmatrix} 3e^{\tau} \\ 0 \end{pmatrix} d\tau,$$

just to see how big the computations become.

We always first compute the integrand separately,

$$\exp(-\mathbf{A}\tau)\begin{pmatrix} 3e^{\tau} \\ 0 \end{pmatrix} = \begin{pmatrix} -e^{2\tau} + 2e^{\tau} & e^{2\tau} - e^{\tau} \\ -2e^{2\tau} + 2e^{\tau} & 2e^{2\tau} - e^{\tau} \end{pmatrix}\begin{pmatrix} 3e^{\tau} \\ 0 \end{pmatrix} = \begin{pmatrix} -3e^{3\tau} + 6e^{2\tau} \\ -6e^{3\tau} + 6e^{2\tau} \end{pmatrix}.$$

Then we compute the integral,

$$\int_0^t \exp(-\mathbf{A}\tau)\begin{pmatrix} 3e^{\tau} \\ 0 \end{pmatrix} d\tau = \int_0^t \begin{pmatrix} -3e^{3\tau} + 6e^{2\tau} \\ -6e^{3\tau} + 6e^{2\tau} \end{pmatrix} d\tau = \begin{pmatrix} -e^{3t} + 3e^{2t} \\ -2e^{3t} + 3e^{2t} \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Finally, we put the above into the solution formula,

$$\begin{aligned}
\mathbf{x}(t) &= \exp(\mathbf{A}t)\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \exp(\mathbf{A}t)\begin{pmatrix} -e^{3t}+3e^{2t} \\ -2e^{3t}+3e^{2t} \end{pmatrix} - \exp(\mathbf{A}t)\begin{pmatrix} 2 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 2e^{-t}-e^{-2t} & -e^{-t}+e^{-2t} \\ 2e^{-t}-2e^{-2t} & -e^{-t}+2e^{-2t} \end{pmatrix}\begin{pmatrix} -e^{3t}+3e^{2t} \\ -2e^{3t}+3e^{2t} \end{pmatrix} = \begin{pmatrix} 2e^{t}-1 & -e^{t}+1 \\ 2e^{t}-2 & -e^{t}+2 \end{pmatrix}\begin{pmatrix} -e^{t}+3 \\ -2e^{t}+3 \end{pmatrix} \\
&= \begin{pmatrix} (2e^{t}-1)(-e^{t}+3)+(e^{-t}+1)(-2e^{t}+3) \\ (2e^{t}-2)(e^{-t}+3)+(e^{-t}+2)(-2e^{t}+3) \end{pmatrix} = \begin{pmatrix} -2e^{2t}+7e^{-t}-3+2e^{2t}-5e^{t}+3 \\ -2e^{2t}+8e^{t}-6+2e^{2t}-7e^{t}+6 \end{pmatrix} \\
&= \begin{pmatrix} 2e^{t} \\ e^{t} \end{pmatrix}.
\end{aligned}$$

4 Cayley-Hamilton's theorem

Example 4.1 *Let*

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \det \mathbf{A} = ad - bc \neq 0.$$

Find $\mathbf{A}^{-1}$. *Then formulate a simple mnemonic rule for the inversion of regular* (2×2) *matrices.*

If we define

$$\mathbf{B} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

where we have interchanged the diagonal elements and changed the signs of the remaining elements, then

$$\mathbf{AB} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = \det \mathbf{A} \cdot \mathbf{I},$$

and

$$\mathbf{BA} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = \det \mathbf{A} \cdot \mathbf{I},$$

and it follows that

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\,\mathbf{B} = \frac{1}{ab-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Mnemonic rule. If $\mathbf{A}$ is a regular (2×2) matrix, then the inverse matrix $\mathbf{A}^{-1}$ is obtained by interchanging the diagonal elements, change the signs of the remaining two elements, and finally dividing by $\det \mathbf{A}$.

Example 4.2 *1) Find a fundamental matrix of the homogeneous system*

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 0 & 2 \\ 5 & -3 \end{pmatrix}\mathbf{x}, \qquad t \in \mathbb{R}.$$

2) Find the complete solution of the system

$$(16) \quad \frac{d\mathbf{x}}{dt} = \begin{pmatrix} 0 & 2 \\ 5 & -3 \end{pmatrix}\mathbf{x} - e^t\begin{pmatrix} 6 \\ 0 \end{pmatrix}, \qquad t \in \mathbb{R}.$$

Hint: Guess a solution of (16).

1) The characteristic polynomial

$$P(\lambda) = \begin{vmatrix} -\lambda & 2 \\ 5 & -3-\lambda \end{vmatrix} = \lambda^2 + 3\lambda - 10 = \left(\lambda + \frac{3}{2}\right)^2 - \left(\frac{7}{2}\right)^2 = (\lambda-2)(\lambda+5)$$

has the roots $\lambda = 2$ and $\lambda = -5$. We may now proceed in various ways:

a) The **eigenvalue method.**

If $\lambda = 2$, then $\mathbf{A} - 2\mathbf{I} = \begin{pmatrix} -2 & 2 \\ 5 & -5 \end{pmatrix}$. By taking the cross vector we see that we can select the eigenvector $\mathbf{v}_1 = (1, 1)$.

If $\lambda = -5$, then $\mathbf{A} + 5\mathbf{I} = \begin{pmatrix} 5 & 2 \\ 5 & 2 \end{pmatrix}$. By taking the cross vector we see that we can select the eigenvector $\mathbf{v}_2 = (-2, 5)$.

The complete solution of the homogeneous system is

$$\mathbf{x} = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-5t} \begin{pmatrix} -2 \\ 5 \end{pmatrix} = \begin{pmatrix} e^{2t} & -2e^{-5t} \\ e^{2t} & 5e^{-5t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

It follows that a fundamental matrix is given by

$$\begin{pmatrix} e^{2t} & -2e^{-5t} \\ e^{2t} & 5e^{-5t} \end{pmatrix}.$$

b) The **exponential matrix found by means of Caley-Hamilton's theorem**.

Since $P(\lambda) = \left(\lambda + \frac{3}{2}\right)^2 - \left(\frac{7}{2}\right)^2$ it follows by Cayley-Hamilton's theorem that

$$\left(\mathbf{A} + \frac{3}{2}\mathbf{I}\right)^2 - \left(\frac{7}{2}\right)^2 \mathbf{I} = \mathbf{0}, \quad \text{thus} \quad \mathbf{B}^2 = \left(\mathbf{A} + \frac{3}{2}\mathbf{I}\right)^2 = \left(\frac{7}{2}\right)^2 \mathbf{I},$$

where we have put $\mathbf{B} = \mathbf{A} + \frac{3}{2}\mathbf{I}$.

Now, $\mathbf{A}$ and $\mathbf{I}$ commute, so we get by the exponential series that

$$\begin{aligned}
\exp(\mathbf{A}t) &= \exp\left(\left(-\frac{3}{2}\mathbf{I} + \left(\mathbf{A} + \frac{3}{2}\mathbf{I}\right)\right)t\right) = \exp\left(-\frac{3}{2}\,t\right)\exp(\mathbf{B}t) \\
&= \exp\left(-\frac{3}{2}\,t\right)\left\{\sum_{n=0}^{\infty}\frac{1}{(2n)!}\,\mathbf{B}^{2n} + \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\,\mathbf{B}^{2n+1}\right\} \\
&= \exp\left(-\frac{3}{2}\,t\right)\left\{\sum_{n=0}^{\infty}\frac{1}{(2n)!}\left(\frac{7}{2}\right)^{2n}\mathbf{I} + \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(\frac{7}{2}\right)^{2n}\mathbf{B}\right\} \\
&= \exp\left(-\frac{3}{2}\,t\right)\left\{\cosh\left(\frac{7}{2}\,t\right)\mathbf{I} + \frac{2}{7}\sinh\left(\frac{7}{2}\,t\right)\mathbf{B}\right\} \\
&= \begin{pmatrix} \frac{1}{2}e^{2t} + \frac{1}{2}e^{-5t} & 0 \\ 0 & \frac{1}{2}e^{2t} + \frac{1}{2}e^{-5t} \end{pmatrix} + \frac{1}{7}\left(e^{2t} - e^{-5t}\right)\begin{pmatrix} \frac{3}{2} & 2 \\ 5 & -\frac{3}{2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{5}{7}e^{2t} + \frac{2}{7}e^{-5t} & \frac{2}{7}e^{2t} - \frac{2}{7}e^{-5t} \\ \frac{5}{7}e^{2t} - \frac{5}{7}e^{-5t} & \frac{2}{7}e^{2t} + \frac{5}{7}e^{-5t} \end{pmatrix},
\end{aligned}$$

which is the exponential matrix corresponding to the problem.

2) Since e^t are not of the same type as e^{2t} or e^{-5t}, a reasonable guess is a solution of the form

$$\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix} \qquad \text{where } \frac{d\mathbf{x}}{dt} = \begin{pmatrix} a \\ b \end{pmatrix} e^t.$$

Then by insertion into the matrix equation, followed by a multiplication by e^{-t},

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 5 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} 6 \\ 0 \end{pmatrix}, \text{ dvs. } \begin{pmatrix} -1 & 2 \\ 5 & -4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}.$$

By Cramer's formulæ or just some fumbling we find

$$a = 4 \qquad \text{and} \qquad b = 5,$$

and the complete solution is

$$\mathbf{x}(t) = \begin{pmatrix} 4 \\ 5 \end{pmatrix} e^t + c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2\begin{pmatrix} -2 \\ 5 \end{pmatrix} e^{-5t}, \qquad t \in \mathbb{R},$$

where c_1 and c_2 are arbitrary constants.

Alternatively one may apply the unfortunate solution formula

$$\mathbf{x}_0(t) = \mathbf{\Phi}(t)\int \mathbf{\Phi}(t)^{-1}\mathbf{u}(t)\,dt$$

in order to obtain a particular solution.

a) It follows from

$$\mathbf{\Phi}(t) = \begin{pmatrix} e^{2t} & -2e^{-5t} \\ e^{2t} & 5e^{-5t} \end{pmatrix} \qquad \text{where } \det\mathbf{\Phi}(t) = 7e^{-3t}$$

that

$$\mathbf{\Phi}(t)^{-1} = \frac{e^{3t}}{7}\begin{pmatrix} 5e^{-5t} & 2e^{-5t} \\ -e^{2t} & e^{2t} \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 5e^{-2t} & 2e^{-2t} \\ -e^{5t} & e^{5t} \end{pmatrix}.$$

b) The integrand is

$$\mathbf{\Phi}(t)^{-1}\mathbf{u}(t) = \frac{1}{7}\begin{pmatrix} 5e^{-2t} & 2e^{-2t} \\ -e^{5t} & e^{5t} \end{pmatrix}\begin{pmatrix} -6e^{t} \\ 0 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} -30e^{-t} \\ 6e^{6t} \end{pmatrix}.$$

c) Then by integrating each coordinate separately,

$$\int \mathbf{\Phi}(t)^{-1}\mathbf{u}(t)\,dt = \frac{1}{7}\begin{pmatrix} 30e^{-t} \\ e^{6t} \end{pmatrix}.$$

d) A particular solution is

$$\mathbf{\Phi}(t)\int \mathbf{\Phi}(t)^{-1}\mathbf{u}(t)\,dt = \frac{1}{7}\begin{pmatrix} e^{2t} & -2e^{-5t} \\ e^{2t} & 5e^{-5t} \end{pmatrix}\begin{pmatrix} 30e^{-t} \\ e^{6t} \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 28e^{t} \\ 35e^{t} \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} e^{t}.$$

e) The complete solution is

$$\mathbf{x}(t) = \begin{pmatrix} 4 \\ 5 \end{pmatrix} e^{t} + \begin{pmatrix} e^{2t} & -2e^{-5t} \\ e^{2t} & 5e^{-5t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \qquad t \in \mathbb{R},$$

where c_1 and $c_2 \in \mathbb{R}$ are arbitrary constants.

Example 4.3 *Let*

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be any (2×2) *matrix, and let*

$$p(\lambda) = \det(\mathbf{A}-\lambda\mathbf{I}) = \lambda^2-(a+d)\lambda+(ad-bc) = (\lambda-\lambda_1)(\lambda-\lambda_2)$$

be the corresponding characteristic polynomial. Prove that

$$p(\mathbf{A}) := (\mathbf{A}-\lambda_1\mathbf{I})(\mathbf{A}-\lambda_2\mathbf{I}) = \mathbf{0}.$$

When we identify the coefficients of the characteristic polynomial we get

$$\lambda_1+\lambda_2 = a+d \qquad \text{and} \qquad \lambda_1\lambda_2 = ad-bc.$$

A computation of $p(\mathbf{A})$ gives

$$\begin{aligned} p(\mathbf{A}) &= (\mathbf{A}-\lambda_1\mathbf{I})(\mathbf{A}-\lambda_2\mathbf{I}) = \mathbf{A}^2-(\lambda_1+\lambda_2)\mathbf{A}+\lambda_1\lambda_2\mathbf{I} \\ &= \mathbf{A}^2-(a+d)\mathbf{A}+(ad-bc)\mathbf{I} = \mathbf{A}\{\mathbf{A}-(a+d)\mathbf{I}\}+(ad-bc)\mathbf{I} \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}-\begin{pmatrix} a+d & 0 \\ 0 & a+d \end{pmatrix}\right\}+\det\mathbf{A}\cdot\mathbf{I} \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -d & b \\ c & -a \end{pmatrix}+\det\mathbf{A}\cdot\mathbf{I} = \mathbf{A}\{-\det\mathbf{A}\cdot\mathbf{A}^{-1}\}+\det\mathbf{A}\cdot\mathbf{I} = \mathbf{0}, \end{aligned}$$

where we have applied the result of Example 4.1.

Example 4.4 *Let* $\mathbf{A}$ *be any* (2×2) *matrix.*

1) Prove that

$$\mathbf{A}^2 = (\lambda+\mu)\mathbf{A}-\lambda\mu\mathbf{I},$$

where λ *and* μ *are the two roots of the characteristic polynomial for* $\mathbf{A}$*.*
Hint: Apply the result of Example 4.3.

2) Conclude from (1) that there exist functions $\varphi(t)$ *and* $\psi(t)$*, such that*

$$\exp(\mathbf{A}t) = \varphi(t)\mathbf{I}+\psi(t)\mathbf{A},$$

where $\varphi(0)=1$ *and* $\psi(0)=0$*.*

1) It follows from Example 4.3 that

$$p(\mathbf{A}) = (\mathbf{A}-\lambda\mathbf{I})(\mathbf{A}-\mu\mathbf{I}) = \mathbf{A}^2-(\lambda+\mu)\mathbf{A}+\lambda\mu\mathbf{I} = \mathbf{0}.$$

Then by a rearrangement,

$$\mathbf{A}^2 = (\lambda+\mu)\mathbf{A}-\lambda\mu\mathbf{I}.$$

2) The exponential matrix is defined by the matrix series

$$\exp(\mathbf{A}t) = \mathbf{I} + \sum_{n=1}^{\infty} \frac{1}{n!}\mathbf{A}^n t^n = \mathbf{I} + \mathbf{A}t + \sum_{n=2}^{\infty} \frac{1}{n!}\mathbf{A}^n t^n,$$

where the radius of convergence is ∞.

According to (1) every $\mathbf{A}^n$, $n \geq 2$, can be written as a linear combination of $\mathbf{I}$ and $\mathbf{A}$, by using the recursion formula

$$\mathbf{A}^n = \mathbf{A}^2\,\mathbf{A}^{n-2} = (\lambda+\mu)\mathbf{A}^{n-1} - \lambda\mu\mathbf{A}^{n-2}, \qquad n \geq 2.$$

The coefficients only increase as a polynomial, while the denominator $n!$ is of factorial size. Therefore, this process cannot change the radius of convergence, so we conclude that $\exp(\mathbf{A}t)$ can also be written as a linear combination of $\mathbf{I}$ and $\mathbf{A}$, where the coefficients $\varphi(t)$ and $\psi(t)$ are given by power series in t of radius of convergence ∞,

$$\exp(\mathbf{A}t) = \varphi(t)\mathbf{I} + \psi(t)\mathbf{A}.$$

If we here put $t = 0$, then

$$\exp(\mathbf{A}0) = \mathbf{I} = \varphi(0)\mathbf{I} + \psi(0)\mathbf{A},$$

and it follows that $\varphi(0) = 1$ and $\psi(0) = 0$, because we may assume that $\mathbf{I}$ and $\mathbf{A}$ are linearly independent. (There is nothing to prove if this is not the case).

Example 4.5 *Let* $\mathbf{A}$ *be a* (2×2) *matrix. We proved in Example 4.4 that*

$$\mathbf{A}^2 = (\lambda + \mu)\mathbf{A} - \lambda\mu\mathbf{I},$$

and

$$\exp(\mathbf{A}t) = \varphi(t)\mathbf{I} + \psi(t)\mathbf{A}, \qquad \varphi(0) = 1 \text{ and } \psi(0) = 0.$$

Prove that φ *and* ψ *are the unique solutions of the system of equations*

$$\begin{cases} \varphi'(t) = -\lambda\mu\psi(t), \\ \varphi''(t) = (\lambda+\mu)\varphi'(t) - \lambda\mu\varphi(t) = 0, \end{cases} \qquad \varphi(0) = 1,\ \psi(0) = 0.$$

Then find $\exp(\mathbf{A}t)$ *explicitly.*
Hint: Split the investigation into the to cases (1) $\lambda \neq \mu$*, and (2)* $\lambda = \mu$*.*

We shall only check the differential equation for $\exp(\mathbf{A}t)$,

$$\frac{d}{dt}\exp(\mathbf{A}t) = \frac{d}{dt}\left\{\mathbf{I} + \sum_{n=1}^{\infty} \frac{1}{n!}\,\mathbf{A}^n t^n\right\} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!}\,\mathbf{A}^n t^{n-1} = \mathbf{A}\,\exp(\mathbf{A}t).$$

Since

$$\frac{d}{dt}\exp(\mathbf{A}t) = \varphi'(t)\mathbf{I} + \psi'(t)\mathbf{A}$$

and

$$\mathbf{A}\,\exp(\mathbf{A}t) = \varphi(t)\mathbf{A} + \psi(t)\mathbf{A}^2 = (\lambda+\mu)\psi(t)\mathbf{A} - \psi(t)\lambda\mu\mathbf{I} + \varphi(t)\mathbf{A},$$

it follows by identification of the coefficients that

$$\varphi'(t) = -\lambda\mu\psi(t), \qquad \varphi'(0) = -\lambda\mu\psi(0) = 0,$$

$$\psi'(t) = (\lambda+\mu)\psi(t) + \varphi(t), \qquad \psi'(0) = 0 + \varphi(0) = 1.$$

When the latter equation is differentiated, we get

$$\psi''(t) = (\lambda+\mu)\psi'(t) + \varphi'(t) = (\lambda+\mu)\psi'(t) - \lambda\mu\psi(t),$$

hence by a rearrangement,

$$\frac{d^2\psi}{dt^2} - (\lambda+\mu)\frac{d\psi}{dt} + \lambda\mu\psi(t) = 0, \qquad \psi(0) = 0,\ \psi'(0) = 1,$$

and of course

$$\varphi'(t) = -\lambda\mu\psi(t), \qquad \varphi(0) = 1.$$

The characteristic polynomial for the differential equation of ψ is

$$R^2 - (\lambda+\mu)R + \lambda\mu = (R-\lambda)(R-\mu),$$

with the roots λ and μ.

1) If $\lambda \neq \mu$, then

$$\psi(t) = c_1 e^{\lambda t} + c_2 e^{\mu t}$$

where

$$\psi(0) = c_1 + c_2 = 0 \qquad \text{and} \qquad \psi'(0) = \lambda c_1 + \mu c_2 = 1,$$

hence $c_1 = \dfrac{1}{\lambda - \mu}$ and $c_2 = -\dfrac{1}{\lambda - \mu}$, thus

$$\psi(t) = \frac{1}{\lambda - \mu}(e^{\lambda t} - e^{\mu t}).$$

We have found that

$$\varphi'(t) = -\lambda\mu\psi(t) = \frac{-\mu}{\lambda - \mu} \cdot \lambda e^{\lambda t} + \frac{\lambda}{\lambda - \mu} \cdot \mu e^{\mu t}, \quad \varphi(0) = 1,$$

so we get by an integration,

$$\begin{aligned} \varphi(t) &= 1 + \int_0^t \left\{ -\frac{\mu}{\lambda - \mu}\, \lambda e^{\lambda\tau} + \frac{\lambda}{\lambda - \mu}\, \mu e^{\mu\tau} \right\} d\tau \\ &= 1 + \frac{\lambda}{\lambda - \mu}\, e^{\mu t} - \frac{\mu}{\lambda - \mu}\, e^{\lambda t} - \left\{ \frac{\lambda}{\lambda - \mu} - \frac{\mu}{\lambda - \mu} \right\} \\ &= \frac{\lambda}{\lambda - \mu}\, e^{\mu t} - \frac{\mu}{\lambda - \mu}\, e^{\lambda t}. \end{aligned}$$

Summing up we have

$$\exp(\mathbf{A}t) = \varphi(t)\mathbf{I} + \psi(t)\mathbf{A} = \left\{ \frac{\lambda}{\lambda - \mu}\, e^{\mu t} - \frac{\mu}{\lambda - \mu}\, e^{\lambda t} \right\} \mathbf{I} + \frac{1}{\lambda - \mu}\, (e^{\lambda t} - e^{\mu t})\mathbf{A}.$$

2) If $\mu = \lambda$, then

$$\psi(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}, \quad \text{where } \psi(0) = c_1 = 0,$$

and

$$\psi'(t) = c_2 e^{\lambda t} + c_2 \lambda t e^{\lambda t}, \quad \text{where } \psi'(0) = c_2 = 1.$$

Then

$$\psi(t) = t e^{\lambda t}.$$

Now, $\varphi'(t) = -\lambda^2 \psi(t)$ and $\psi(0) = 1$, so

$$\varphi(t) = 1 - \lambda^2 \int_0^t \tau e^{\lambda\tau} d\tau = 1 - \lambda \left[\tau e^{\lambda\tau}\right]_0^t + \left[e^{\lambda\tau}\right]_0^t = 1 - \lambda t e^{\lambda t} + e^{\lambda t} - 1 = (1 - \lambda t) e^{\lambda t}.$$

Summing up we get

$$\exp(\mathbf{A}t) = \varphi(t)\mathbf{I} + \psi(t)\mathbf{A} = (1 - \lambda t) e^{\lambda t}\mathbf{I} + t e^{\lambda t}\mathbf{A}.$$

Example 4.6 *Let* $\mathbf{A}$ *be a* (2×2) *matrix, the characteristic polynomial of which has the root* λ *of multiplicity 2. Compute* $\exp(\mathbf{A}t)$ *by using that*

$$\exp(\mathbf{A}t) = \exp(\lambda t\mathbf{I}+(\mathbf{A}-\lambda\mathbf{I})t) = e^{\lambda t}\exp((\mathbf{A}-\lambda\mathbf{I})t).$$

When λ is a double root, then the characteristic polynomial is $(R-\lambda)^2$, and it follows from either Cayley-Hamilton's theorem or Example 4.3 that

$$(\mathbf{A}-\lambda\mathbf{I})^2 = \mathbf{0}.$$

Then

$$\exp((\mathbf{A}-\lambda\mathbf{I})t) = \mathbf{I}+(\mathbf{A}-\lambda I)t+\sum_{n=2}^{\infty}\frac{1}{n!}(\mathbf{A}-\lambda\mathbf{I})^n t^n = \mathbf{I}+(\mathbf{A}-\lambda\mathbf{I})t+\mathbf{0} = (1-\lambda t)\mathbf{I}+t\mathbf{A},$$

and hence

$$\exp(\mathbf{A}t) = e^{\lambda t}\exp((\mathbf{A}-\lambda\mathbf{I})t) = (1-\lambda t)e^{\lambda t}\mathbf{I}+te^{\lambda t}\mathbf{A},$$

cf. the result of Example 4.5 (2).

Example 4.7 *Let* $\mathbf{A}$ *be a* (2×2) *matrix, the characteristic polynomial of which has the root* λ *of multiplicity 2. Prove that*

$$\mathbf{\Phi}(t) = e^{\lambda t}\mathbf{I}+te^{\lambda t}(\mathbf{A}-\lambda\mathbf{I})$$

is a solution of

$$\frac{d\mathbf{\Phi}}{dt} = \mathbf{A}\,\mathbf{\Phi}(t), \qquad \mathbf{\Phi}(0) = \mathbf{I},$$

and conclude that

$$\exp(\mathbf{A}t) = e^{\lambda t}\mathbf{I}+te^{\lambda t}(\mathbf{A}-\lambda\mathbf{I}).$$

Since

$$\frac{d\mathbf{\Phi}}{dt} = \lambda e^{\lambda t}\mathbf{I}+(1+\lambda t)e^{\lambda t}(\mathbf{A}-\lambda\mathbf{I}) = -\lambda^2 te^{\lambda t}\mathbf{I}+(1+\lambda t)e^{\lambda t}\mathbf{A}$$

and

$$\begin{aligned}\mathbf{A}\,\mathbf{\Phi}(t) &= e^{\lambda t}\mathbf{A}+te^{\lambda t}(\mathbf{A}-\lambda\mathbf{I})(\mathbf{A}-\lambda\mathbf{I}+\lambda\mathbf{I}) = e^{\lambda t}\mathbf{A}+\lambda te^{\lambda t}(\mathbf{A}-\lambda\mathbf{I})\\ &= -\lambda^2 te^{\lambda t}\mathbf{I}+(1+\lambda t)e^{\lambda t}\mathbf{A},\end{aligned}$$

it follows that

$$\frac{d\mathbf{\Phi}}{dt} = \mathbf{A}\,\mathbf{\Phi}(t).$$

Since

$$\mathbf{\Phi}(0) = \mathbf{I}+0\cdot e^0(\mathbf{A}-\lambda\mathbf{I}) = \mathbf{I},$$

the initial conditions are also fulfilled. Due to the uniqueness the only solution is $\exp(\mathbf{A}t)$, so

$$\exp(\mathbf{A}t) = e^{\lambda t}\mathbf{I} + te^{\lambda t}(\mathbf{A} - \lambda\mathbf{I}).$$

Example 4.8 *Let* $\mathbf{A}$ *be a* (3×3) *matrix, the characteristic polynomial of which* $p(x) = (x - \lambda)^3$ *has the root* λ *of multiplicity 3.*

1) Prove that $p(\mathbf{A}) := (\mathbf{A} - \lambda\mathbf{I})^3 = \mathbf{0}$.

2) Let

$$\mathbf{\Phi}(t) = e^{\lambda t}\mathbf{I} + te^{\lambda t}(\mathbf{A} - \lambda\mathbf{I}) + \frac{1}{2}\,t^2e^{\lambda t}(\mathbf{A} - \lambda\mathbf{I})^2.$$

Prove that $\mathbf{\Phi}(t) = \exp(\mathbf{A}t)$.

1) This follows immediately form Cayley-Hamilton's theorem.

2) Clearly, $\mathbf{\Phi}(0) = \mathbf{I}$. Then we just check,

$$\frac{d\mathbf{\Phi}}{dt} = \lambda e^{\lambda t}\mathbf{I}+(1+\lambda t)e^{\lambda t}(\mathbf{A}-\lambda\mathbf{I})+\left(t+\frac{\lambda}{2}\,t^2\right)(\mathbf{A}-\lambda\mathbf{I})^2$$

and

$$\begin{aligned}\mathbf{A}\,\mathbf{\Phi}(t) &= e^{\lambda t}(\mathbf{A}-\lambda\mathbf{I}+\lambda\mathbf{I})+te^{\lambda t}(\mathbf{A}-\lambda\mathbf{I})(\mathbf{A}-\lambda\mathbf{A}+\lambda\mathbf{I})+\frac{1}{2}\,t^2e^{\lambda t}(\mathbf{A}-\lambda\mathbf{I})^2(\mathbf{A}-\lambda\mathbf{I}+\lambda\mathbf{I})\\ &= \lambda e^{\lambda t}\mathbf{I}+e^{\lambda t}(\mathbf{A}-\lambda\mathbf{I})+\lambda te^{\lambda t}(\mathbf{A}-\lambda\mathbf{I})\\ &\quad +te^{\lambda t}(\mathbf{A}-\lambda\mathbf{I})^2+\frac{\lambda}{2}\,t^2e^{\lambda t}(\mathbf{A}-\lambda\mathbf{I})^2+\frac{1}{2}\,t^2e^{\lambda t}(\mathbf{A}-\lambda\mathbf{I})^3\\ &= \lambda e^{\lambda t}\mathbf{I}+(1+\lambda t)e^{\lambda t})(\mathbf{A}-\lambda\mathbf{I})+\left(t+\frac{\lambda}{2}\,t^2\right)(\mathbf{A}-\lambda\mathbf{I})^2+\mathbf{0},\end{aligned}$$

hence $\mathbf{\Phi}(t)$ also satisfies the matrix differential equation

$$\frac{d\mathbf{\Phi}}{dt} = \mathbf{A}\,\mathbf{\Phi}(t), \qquad \mathbf{\Phi}(0)=\mathbf{I}.$$

Now, the exponential matrix is the unique solution of this matrix differential equation, so we conclude that

$$\exp(\mathbf{A}t) = \mathbf{\Phi}(t) = e^{\lambda t}\mathbf{I}+te^{\lambda t}(\mathbf{A}-\lambda\mathbf{I})+\frac{1}{2}\,t^2e^{\lambda t}(\mathbf{A}-\lambda\mathbf{I})^2.$$

Example 4.9 *Let $\mathbf{A}$ be a (3×3) matrix, the characteristic polynomial of which $p(x)=(x-\lambda)^2(x-\mu)$ has a double root λ and a simple real root $\mu(\neq\lambda)$.*

1) Prove that

$$p(\mathbf{A}) := (\mathbf{A}-\lambda\mathbf{I})^2(\mathbf{A}-\mu\mathbf{I}) = \mathbf{0}.$$

2) Prove that

$$\mathbf{\Phi}(t) := \frac{-e^{\lambda t}}{(\lambda-\mu)^2}(\mathbf{A}-\lambda\mathbf{I})(\mathbf{A}-\mu\mathbf{I})+\frac{e^{\lambda t}}{\lambda-\mu}(\mathbf{A}-\mu\mathbf{I})+\frac{te^{\lambda t}}{\lambda-\mu}(\mathbf{A}-\mu\mathbf{I})(\mathbf{A}-\mu\mathbf{I})+\frac{e^{\mu t}}{(\lambda-\mu)^2}(\mathbf{A}-\lambda\mathbf{I})^2$$

is a solution of

$$\frac{d\mathbf{\Phi}}{dt} = \mathbf{A}\,\mathbf{\Phi}(t), \qquad \mathbf{\Phi}(0)=\mathbf{I},$$

and then conclude that $\mathbf{\Phi}(t) = \exp(\mathbf{A}t)$.

1) This follows immediately from Cayley-Hamilton's theorem.

2) We get by differentiation,

$$\begin{aligned}\frac{d\mathbf{\Phi}}{dt} &= \left\{\frac{-\lambda}{(\lambda-\mu)^2}+\frac{1}{\lambda-\mu}\right\}e^{\lambda t}(\mathbf{A}-\lambda\mathbf{I})(\mathbf{A}-\mu\mathbf{I})+\frac{\lambda}{\lambda-\mu}e^{\lambda t}(\mathbf{A}-\mu\mathbf{I})\\ &\quad +\frac{\lambda}{\lambda-\mu}te^{\lambda t}(\mathbf{A}-\lambda\mathbf{I})(\mathbf{A}-\mu\mathbf{I})+\frac{\mu}{(\lambda-\mu)^2}e^{\mu t}(\mathbf{A}-\lambda\mathbf{I})^2,\end{aligned}$$

which should be compared with

$$\begin{aligned}\mathbf{A}\,\mathbf{\Phi}(t) &= -\frac{e^{\lambda t}}{(\lambda-\mu)^2}(\mathbf{A}-\lambda\mathbf{I})(\mathbf{A}-\mu)(\mathbf{A}-\lambda\mathbf{I}+\lambda\mathbf{I})+\frac{e^{\lambda t}}{\lambda-\mu}(\mathbf{A}-\mu\mathbf{I})(\mathbf{A}-\lambda\mathbf{I}+\lambda\mathbf{I})\\ &\quad +\frac{te^{\lambda t}}{\lambda-\mu}(\mathbf{A}-\lambda\mathbf{I})(\mathbf{A}-\mu\mathbf{I})(\mathbf{A}-\lambda\mathbf{I}+\lambda\mathbf{I})+\frac{e^{\mu t}}{(\lambda-\mu)^2}(\mathbf{A}-\lambda\mathbf{I})^2(\mathbf{A}-\mu\mathbf{I}+\mu\mathbf{I}),\end{aligned}$$

where we have used that all matrices commute. Now,

$$(\mathbf{A}-\lambda\mathbf{I})^2(\mathbf{A}-\mu\mathbf{I})=\mathbf{0},$$

so this expression is reduced to

$$\begin{aligned}\mathbf{A}\,\mathbf{\Phi}(t) &= \frac{-\lambda}{(\lambda-\mu)^2}e^{\lambda t}(\mathbf{A}-\lambda\mathbf{I})(\mathbf{A}-\mu\mathbf{I})+\frac{1}{\lambda-\mu}e^{\lambda t}(\mathbf{A}-\lambda\mathbf{I})(\mathbf{A}-\mu\mathbf{I})\\ &\quad +\frac{\lambda}{\lambda-\mu}e^{\lambda t}(\mathbf{A}-\mu\mathbf{I})-\frac{\lambda}{\lambda-\mu}te^{\lambda t}(\mathbf{A}-\mu\mathbf{I})(\mathbf{A}-\lambda\mathbf{I})+\frac{\mu}{(\lambda-\mu)^2}e^{\mu t}(\mathbf{A}-\lambda\mathbf{I})^2.\end{aligned}$$

It follows by the comparison that $\mathbf{\Phi}(t)$ satisfies the matrix differential equation

$$\frac{d\mathbf{\Phi}}{dt} = \mathbf{A}\,\mathbf{\Phi}(t).$$

Since

$$\begin{aligned}
\mathbf{\Phi}(0) &= \frac{-1}{(\lambda-\mu)^2}(\mathbf{A}-\lambda\mathbf{I})(\mathbf{A}-\mu\mathbf{I}) + \frac{1}{\lambda-\mu}(\mathbf{A}-\mu\mathbf{I}) + \frac{1}{(\lambda-\mu)^2}(\mathbf{A}-\lambda\mathbf{I})^2 \\
&= \frac{1}{(\lambda-\mu)^2}(\mathbf{A}-\lambda\mathbf{I})(\mu-\lambda)\mathbf{I} + \frac{1}{\lambda-\mu}(\mathbf{A}-\mu\mathbf{I}) \\
&= -\frac{1}{\lambda-\mu}(\mathbf{A}-\lambda\mathbf{I}) + \frac{1}{\lambda-\mu}(\mathbf{A}-\mu\mathbf{I}) = \frac{\lambda-\mu}{\lambda-\mu}\mathbf{I} = \mathbf{I},
\end{aligned}$$

it follows that $\mathbf{\Phi}(t)$ fulfils the same initial conditions as $\exp(\mathbf{A}t)$. Since this solution is unique, we conclude that

$$\mathbf{\Phi}(t) = \exp(\mathbf{A}t).$$

Example 4.10 *Let $\mathbf{A}$ be an $(n \times n)$ matrix, the characteristic polynomial $p(x)$ has λ as an n-tuple root.*

1) Prove that $p(\mathbf{A}) := (\mathbf{A} - \lambda\mathbf{I})^n = \mathbf{0}$.

Hint: Apply a coordinate transformation $\mathbf{S}$, such that $\mathbf{A} = \mathbf{S\Lambda S}^{-1}$, where $\mathbf{\Lambda}$ is an upper triangular matrix with only λs in the diagonal.

2) Prove that

$$\exp(\mathbf{A}t) = e^{\lambda t}\exp((\mathbf{A}-\lambda\mathbf{I})t) = e^{\lambda t}\mathbf{I} + \sum_{j=1}^{n-1}\frac{1}{j!}\,t^j e^{\lambda t}(\mathbf{A}-\lambda\mathbf{I})^j.$$

1) By using the hint we get

$$(\mathbf{A}-\lambda\mathbf{I})^n = (\mathbf{S\Lambda S}^{-1} - \lambda^n\mathbf{SIS}^{-1})^n = \mathbf{S}^n(\mathbf{\Lambda}-\lambda\mathbf{I})^n\mathbf{S}^{-n}.$$

Now, $\mathbf{\Lambda} - \lambda\mathbf{I}$ is an upper triangular matrix with only zeros in the diagonal. Hence, $(\mathbf{\Lambda} - \lambda\mathbf{I})^n = \mathbf{0}$, and the claim is proved.

2) Since $\mathbf{A}$ and $\mathbf{I}$ commute, and

$$(\mathbf{A}-\lambda\mathbf{I})^j = (\mathbf{A}-\lambda\mathbf{I})^n(\mathbf{A}-\lambda\mathbf{I})^{j-n} = \mathbf{0} \qquad \text{for } j \ge n,$$

it follows that

$$\begin{aligned}
\exp(\mathbf{A}t) &= \exp(\lambda\mathbf{I}t + (\mathbf{A}-\lambda\mathbf{I})t) = e^{\lambda t}\exp((\mathbf{A}-\lambda\mathbf{I})t) \\
&= e^{\lambda t}\sum_{j=0}^{\infty}\frac{1}{j!}\,t^j(\mathbf{A}-\lambda\mathbf{I})^j = e^{\lambda t}\sum_{j=0}^{n-1}\frac{1}{j!}\,t^j(\mathbf{A}-\lambda\mathbf{I})^j.
\end{aligned}$$

Example 4.11 *Let $\mathbf{A}$ be an $(n \times n)$ matrix, the characteristic polynomial of which $p(x)$ has the n mutually different eigenvalues $\lambda_1, \dots, \lambda_n$, everyone of multiplicity 1.*

1) Prove that $p(\mathbf{A}) := \prod_{j=1}^{n} (\mathbf{A} - \lambda_j \mathbf{I}) = \mathbf{0}$.

2) Let $q(x)$ be the polynomial of degree grad $n-1$ defined by

$$q(x) = \sum_{j=1}^{n} \left\{ \prod_{\substack{k=1\\k\neq j}}^{n} \frac{x - \lambda_k}{\lambda_j - \lambda_k} \right\}.$$

Prove that $q(\lambda_j) = 1$ for $j = 1, \dots, n$, and conclude that $q(x) = 1$ identically.

3) Conclude from (2) that

$$q(\mathbf{A}) := \sum_{j=1}^{n} \left\{ \prod_{\substack{k=1\\k\neq j}}^{n} \frac{1}{\lambda_j - \lambda_k} (\mathbf{A} - \lambda_k \mathbf{I}) \right\} = \mathbf{I}.$$

4) Prove that

$$\mathbf{\Phi}(t) := \sum_{j=1}^{n} \left\{ \prod_{\substack{k=1\\k\neq j}}^{n} \frac{1}{\lambda_j - \lambda_k} (\mathbf{A} - \lambda_k \mathbf{I}) \right\} e^{\lambda_j t}$$

is a solution of

$$\frac{d\mathbf{\Phi}}{dt} = \mathbf{A}\,\mathbf{\Phi}(t), \qquad \mathbf{\Phi}(0) = \mathbf{I},$$

and conclude that

$$\exp(\mathbf{A}t) = \sum_{j=1}^{n} \left\{ \prod_{\substack{k=1\\k\neq j}}^{n} \frac{1}{\lambda_j - \lambda_k} (\mathbf{A} - \lambda_k \mathbf{I}) \right\} e^{\lambda_j t}.$$

1) There exists a coordinate transformation $\mathbf{S}$, such that

$$\mathbf{A} = \mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1},$$

where $\mathbf{\Lambda}$ is a diagonal matrix with the diagonal elements λ_j and zeros outside the diagonal. Then

$$p(\mathbf{A}) = \mathbf{S}^n \left\{ \prod_{j=1}^{n} (\mathbf{\Lambda} - \lambda_j \mathbf{I}) \right\} \mathbf{S}^{-n} = \mathbf{0},$$

because $\mathbf{\Lambda} - \lambda_j \mathbf{I}$ only contains elements $\neq 0$ in the diagonal, and because the j-th diagonal element is also 0. Since j goes through $\{1, \dots, n\}$, it follows that the product matrix is $\mathbf{0}$.

2) When we put $x = \lambda_{j_0}$, then

$$q(\lambda_{j_0}) = \sum_{j=1}^{n} \left\{ \prod_{\substack{k=1 \\ k\neq j}} \}^n \frac{\lambda_{j_0} - \lambda_k}{\lambda_j - \lambda_k} \right\} = \prod_{\substack{k=1 \\ k\neq j_0}}^{n} \frac{\lambda_{j_0} - \lambda_k}{\lambda_{j_0} - \lambda_k} = 1.$$

Clearly, the polynomial $q(x) - 1$ has at most degree $n-1$, and since it has n zeros, $\lambda_1, \dots, \lambda_n$, we must have $q(x) - 1 = 0$ identically, so $q(x) = 1$ identically.

3) It follows immediately from $x \mapsto \mathbf{A}$ and $1 \mapsto \mathbf{I}$ that

$$q(\mathbf{A}) = \sum_{j=1}^{n} \left\{ \prod_{\substack{k=1 \\ k\neq j}}^{n} \frac{1}{\lambda_j - \lambda_k} (\mathbf{A} - \lambda_k \mathbf{I}) \right\} = \mathbf{I}.$$

4) Then by (3),

$$\mathbf{\Phi}(0) = \sum_{j=1}^{n} \left\{ \prod_{\substack{k=1 \\ k\neq j}}^{n} \frac{1}{\lambda_j - \lambda_k} (\mathbf{A} - \lambda_k \mathbf{I}) \right\} = q(\mathbf{A}) = \mathbf{I}.$$

Furthermore,

$$\frac{d\mathbf{\Phi}}{dt} = \sum_{j=1}^{n} \left\{ \prod_{\substack{k=1 \\ k\neq j}}^{n} \frac{\lambda_j}{\lambda_j - \lambda_k} (\mathbf{A} - \lambda_k \mathbf{I}) \right\} e^{\lambda_j t}$$

and

$$\begin{aligned} \mathbf{A}\,\mathbf{\Phi}(t) &= \sum_{j=1}^{n} \left\{ \prod_{\substack{k=1 \\ k\neq j}}^{n} \frac{1}{\lambda_j - \lambda_k} (\mathbf{A} - \lambda_j \mathbf{I}) \right\} (\mathbf{A} - \lambda_j \mathbf{I} + \lambda_j \mathbf{I}) e^{\lambda_j t} \\ &= \mathbf{0} + \sum_{j=1}^{n} \left\{ \prod_{\substack{k=1 \\ k\neq j}}^{n} \frac{\lambda_j}{\lambda_j - \lambda_k} (\mathbf{A} - \lambda_k \mathbf{I}) \right\} e^{\lambda_j t} = \frac{d\mathbf{\Phi}}{dt}. \end{aligned}$$

Since $\mathbf{\Phi}(t)$ and $\exp(\mathbf{A}t)$ fulfil the same differential equation and same initial conditions, and since the solution is unique, we conclude that

$$\exp(\mathbf{A}t) = \mathbf{\Phi}(t) = \sum_{j=1}^{n} \left\{ \prod_{\substack{k=1 \\ k\neq j}}^{n} \frac{1}{\lambda_j - \lambda_k} (\mathbf{A} - \lambda_k \mathbf{I}) \right\} e^{\lambda_j t}.$$

Example 4.12 *Compute* $\exp(\mathbf{A}t)$ *for*

$$(1)\ \mathbf{A}=\begin{pmatrix}1&1\\0&1\end{pmatrix},\qquad (2)\ \mathbf{A}=\begin{pmatrix}0&1\\-2&3\end{pmatrix},\qquad (3)\ \mathbf{A}=\begin{pmatrix}0&1\\-1&2\end{pmatrix}.$$

1) The characteristic polynomial is $(\lambda-1)^2$, where $\lambda=1$ is a double root. Then it follows from Example 4.7 that

$$\exp(\mathbf{A}t)=e^t\mathbf{I}+te^t(\mathbf{A}-\mathbf{I})=\begin{pmatrix}e^t&0\\0&e^t\end{pmatrix}+\begin{pmatrix}0&te^t\\0&0\end{pmatrix}=\begin{pmatrix}e^t&te^t\\0&e^t\end{pmatrix}.$$

2) The characteristic polynomial is

$$\begin{vmatrix}-\lambda&1\\-2&3-\lambda\end{vmatrix}=\lambda(\lambda-3)+2=\lambda^2-3\lambda+2=(\lambda-1)(\lambda-2).$$

where $\lambda=1$ and $\lambda=2$ are simple roots. We get by using the formula of Example 4.11 that

$$\begin{aligned}\exp(\mathbf{A}t)&=\frac{1}{1-2}(\mathbf{A}-2\mathbf{I})e^t+\frac{1}{2-1}(\mathbf{A}-\mathbf{I})e^{2t}=\begin{pmatrix}2&-1\\2&-1\end{pmatrix}e^t+\begin{pmatrix}-1&1\\-2&2\end{pmatrix}e^{2t}\\&=\begin{pmatrix}2e^t-e^{2t}&e^{2t}-e^t\\2e^t-2e^{2t}&2e^{2t}-e^t\end{pmatrix}.\end{aligned}$$

3) The characteristic polynomial is

$$\begin{vmatrix} -\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = \lambda(\lambda-2)+1 = \lambda^2-2\lambda+1 = (\lambda-1)^2,$$

where $\lambda = 1$ is a double root. Then by Example 4.7,

$$\exp(\mathbf{A}t) = e^t\mathbf{I} + te^t(\mathbf{A}-\mathbf{I}) = e^t\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + e^t\begin{pmatrix} -t & t \\ -t & t \end{pmatrix} = \begin{pmatrix} (1-t)e^t & te^t \\ -te^t & (1+t)e^t \end{pmatrix}.$$

Example 4.13 *Compute* $\exp(\mathbf{A}t)$ *for*

$$(1)\ \ \mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (2)\ \ \mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

1) Clearly, $\lambda = 1$ is a root of multiplicity 3, hence by Example 4.8,

$$\begin{aligned}
\exp(\mathbf{A}t) &= e^t\mathbf{I} + te^t(\mathbf{A}-\mathbf{I}) + \frac{1}{2}\,t^2e^t(\mathbf{A}-\mathbf{I})^2 \\
&= e^t\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + te^t\begin{pmatrix} 0 & t & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\,t^2e^t\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{pmatrix}.
\end{aligned}$$

2) Clearly, $\lambda = 1$ is a double root, and $\mu = 2$ is a simple root. It then follows from Example 4.9 that

$$\begin{aligned}
\exp(\mathbf{A}t) &= -\frac{e^t}{(1-2)^2}(\mathbf{A}-\mathbf{I})(\mathbf{A}-2\mathbf{I}) + \frac{e^t}{1-2}(\mathbf{A}-2\mathbf{I}) \\
&\qquad + \frac{te^t}{1-2}(\mathbf{A}-\mathbf{I})(\mathbf{A}-2\mathbf{I}) + \frac{e^{2t}}{(1-2)^2}(\mathbf{A}-\mathbf{I})^2 \\
&= -(t+1)e^t\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} - e^t\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&\qquad + e^{2t}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{2t} \end{pmatrix}.
\end{aligned}$$

Example 4.14 *Compute* $\exp(\mathbf{A}t)$ *for*

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ -4 & 7 & -2 \\ -5 & 7 & -1 \end{pmatrix}.$$

First we find the characteristic polynomial

$$\begin{aligned} \begin{vmatrix} -\lambda & 1 & 0 \\ -4 & 7-\lambda & -2 \\ -5 & 7 & -1-\lambda \end{vmatrix} &= -\lambda(\lambda-7)(\lambda+1)+10-4(\lambda+1)-14\lambda \\ &= -\lambda^3+6\lambda^2+7\lambda+10-4\lambda-4-14\lambda = -\lambda^3+6\lambda^2-11\lambda+6 \\ &= -(\lambda-1)(\lambda-2)(\lambda-3). \end{aligned}$$

The eigenvalues are $\lambda = 1$, 2, 3. They are all simple.

The equations of the eigenvectors are

$$\begin{cases} -\lambda x_1 + x_2 & = 0, \\ -4x_1 + (7-\lambda)x_2 - 2x_3 & = 0, \\ -5x_1 + 7x_2 - (\lambda+1)x_3 & = 0, \end{cases} \qquad \text{dvs.} \quad \begin{cases} x_2 = \lambda x_1, \\ x_3 = \dfrac{7\lambda-5}{\lambda+1}\, x_1. \end{cases}$$

If $\lambda = 1$, then we can choose the eigenvector $(1,1,1)$.
If $\lambda = 2$, then we can choose the eigenvector $(1,2,3)$.
If $\lambda = 3$, then we can choose the eigenvector $(1,3,4)$.
The complete solution of the corresponding system of differential equations is

$$\mathbf{x} = c_1 e^t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} e^t & e^{2t} & e^{3t} \\ e^t & 2e^{2t} & 3e^{3t} \\ e^t & 3e^{2t} & 4e^{3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

Hence a fundamental matrix is

$$\mathbf{\Phi}(t) = \begin{pmatrix} e^t & e^{2t} & e^{3t} \\ e^t & 2e^{2t} & 3e^{3t} \\ e^t & 3e^{2t} & 4e^{3t} \end{pmatrix}, \quad \text{med } \mathbf{\Phi}(0) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{pmatrix}.$$

Here,

$$\det \mathbf{\Phi}(0) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = -1.$$

Then by an inversion of the matrix,

$$\mathbf{\Phi}(0)^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -3 & 2 \\ -1 & 2 & -1 \end{pmatrix}.$$

Finally, we get

$$\begin{aligned}\exp(\mathbf{A}t) &= \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1} = \begin{pmatrix} e^t & e^{2t} & e^{3t} \\ e^t & e^{2t} & 3e^{3t} \\ e^t & 3e^{2t} & 4e^{3t} \end{pmatrix}\begin{pmatrix} 1 & 1 & -1 \\ 1 & -3 & 2 \\ -1 & 2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} e^t+e^{2t}-e^{3t} & e^t-3e^{2t}+2e^{3t} & -e^t+2e^{2t}-e^{3t} \\ e^t+2e^{2t}-3e^{3t} & e^t-6e^{2t}+6e^{3t} & -e^t+4e^{2t}-3e^{3t} \\ e^t+3e^{2t}-4e^{3t} & e^t-9e^{2t}+8e^{3t} & -e^t+6e^{2t}-4e^{3t} \end{pmatrix}.\end{aligned}$$

Example 4.15 *Compute* $\exp(\mathbf{A}t)$ *for*

$$\mathbf{A} = \begin{pmatrix} 2 & -3 & 2 \\ 2 & -5 & 4 \\ 3 & -9 & 7 \end{pmatrix}.$$

The characteristic polynomial is

$$\begin{aligned}\begin{vmatrix} 1-\lambda & -3 & 2 \\ 2 & -5-\lambda & 4 \\ 3 & -9 & 7-\lambda \end{vmatrix} &= \begin{vmatrix} 2-\lambda & -3 & 2 \\ 2\lambda-2 & 1-\lambda & 0 \\ 1 & -4+\lambda & 3-\lambda \end{vmatrix} = (\lambda-1)\begin{vmatrix} 2-\lambda & -3 & 2 \\ 2 & -1 & 0 \\ 1 & -4+\lambda & 3-\lambda \end{vmatrix} \\ &= (\lambda-1)\{(\lambda-2)(3-\lambda)+4\lambda-16+2-6\lambda+18\} \\ &= (\lambda-1)\{(\lambda-2)(3-\lambda)-2\lambda+4\} \\ &= (\lambda-1)(\lambda-2)(1-\lambda) = -(\lambda-1)^2(\lambda-2).\end{aligned}$$

Clearly, $\lambda = 1$ is a double root, and $\mu = 2$ is a simple root. Then by Example 4.9,

$$\begin{aligned}\exp(\mathbf{A}t) &= -\frac{e^{\lambda t}}{(\lambda-\mu)^2}(\mathbf{A}-\lambda\mathbf{I})(\mathbf{A}-\mu\mathbf{I}) + \frac{e^{\lambda t}}{\lambda-\mu}(\mathbf{A}-\mu\mathbf{I}) \\ &\qquad + \frac{te^{\lambda t}}{\lambda-\mu}(\mathbf{A}-\lambda\mathbf{I})(\mathbf{A}-\mu\mathbf{I}) + \frac{e^{\mu t}}{(\lambda-\mu)^2}(\mathbf{A}-\lambda\mathbf{I})^2 \\ &= -(1+t)e^t(\mathbf{A}-\mathbf{I})(\mathbf{A}-2\mathbf{I}) - e^t(\mathbf{A}-2\mathbf{I}) + e^{2t}(\mathbf{A}-\mathbf{I})^2 \\ &= -e^t\mathbf{A}(\mathbf{A}-2\mathbf{I}) - te^t(\mathbf{A}-\mathbf{I})(\mathbf{A}-2\mathbf{I}) + e^{2t}(\mathbf{A}-\mathbf{I})^2 \\ &= -e^t\{(\mathbf{A}-\mathbf{I})^2-\mathbf{I}\} - te^t\{(\mathbf{A}-\mathbf{I})^2-(\mathbf{A}-\mathbf{I})\} + e^{2t}(\mathbf{A}-\mathbf{I})^2.\end{aligned}$$

A small computation gives

$$(\mathbf{A}-\mathbf{I})^2 = \begin{pmatrix} 1 & -3 & 2 \\ 2 & -6 & 4 \\ 3 & -9 & 6 \end{pmatrix}\begin{pmatrix} 1 & -3 & 2 \\ 2 & -6 & 4 \\ 3 & -9 & 6 \end{pmatrix} = \begin{pmatrix} 1 & -3 & 2 \\ 2 & -6 & 4 \\ 3 & -9 & 6 \end{pmatrix} = \mathbf{A}-\mathbf{I}.$$

Then by insertion,

$$\begin{aligned}\exp(\mathbf{A}t) &= -e^t(\mathbf{A}-2\mathbf{I}) - te^t\mathbf{0} + e^{2t}(\mathbf{A}-\mathbf{I}) = -e^t\begin{pmatrix}0 & -3 & 2\\ 2 & -7 & 4\\ 3 & -9 & 5\end{pmatrix} + e^{2t}\begin{pmatrix}1 & -3 & 2\\ 2 & -6 & 4\\ 3 & -9 & 6\end{pmatrix}\\ &= \begin{pmatrix} e^{2t} & 3e^t-3e^{2t} & 2e^{2t}-2e^t\\ 2e^{2t}-2e^t & 7e^t-6e^{2t} & 4e^{2t}-4e^t\\ 3e^{2t}-3e^t & 9e^t-9e^{2t} & 6e^{2t}-5e^t\end{pmatrix}.\end{aligned}$$

Example 4.16 *Compute* $\exp(\mathbf{A}t)$ *for*

$$\mathbf{A} = \begin{pmatrix}1 & -1 & 1\\ -2 & 4 & -1\\ -3 & 5 & -1\end{pmatrix}.$$

The characteristic polynomial is

$$\begin{aligned}\begin{vmatrix}1-\lambda & -1 & 1\\ -2 & 4-\lambda & -1\\ -3 & 5 & -1-\lambda\end{vmatrix} &= -\begin{vmatrix}\lambda-1 & 1 & -1\\ 2 & \lambda-4 & 1\\ 3 & -5 & \lambda+1\end{vmatrix}\\ &= -\{(\lambda-1)(\lambda-4)(\lambda+1)+3+10+3(\lambda-4)-2(\lambda+1)+5(\lambda-1)\}\\ &\qquad -\{(\lambda-1)(\lambda-4)(\lambda+1)+6\lambda-6\}\\ &= -(\lambda-1)\{\lambda^2-3\lambda+2\} = -(\lambda-1)^2(\lambda-2).\end{aligned}$$

We see that $\lambda = 1$ is a double root and that $\mu = 2$ is a simple root. Then by Example 4.9 using the reductions from Example 4.15 (the same polynomial),

$$\exp(\mathbf{A}t) = -e^t\{(\mathbf{A}-\mathbf{I})^2-\mathbf{I}\} - te^t\{(\mathbf{A}-\mathbf{I})^2-(\mathbf{A}-\mathbf{I})\} + e^{2t}(\mathbf{A}-\mathbf{I})^2.$$

It therefore follows from

$$(\mathbf{A}-\mathbf{I})^2 = \begin{pmatrix}0 & -1 & 1\\ -2 & 3 & -1\\ -3 & 5 & -2\end{pmatrix}\begin{pmatrix}0 & -1 & 1\\ -2 & 3 & -1\\ -3 & 5 & -2\end{pmatrix} = \begin{pmatrix}-1 & 2 & -1\\ -3 & 6 & -3\\ -4 & 8 & -4\end{pmatrix}$$

that

$$\begin{aligned}\exp(\mathbf{A}t) &= -e^t\begin{pmatrix}-2 & 2 & -1\\ -3 & 5 & -3\\ -4 & 8 & -5\end{pmatrix} - te^t\begin{pmatrix}-1 & 3 & -2\\ -1 & 3 & -2\\ -1 & 3 & -2\end{pmatrix} + e^{2t}\begin{pmatrix}-1 & 2 & -1\\ -3 & 6 & -3\\ -4 & 8 & -4\end{pmatrix}\\ &= \begin{pmatrix}(2+t)e^t-e^{2t} & -(2+3t)e^t+2e^{2t} & (1+2t)e^t-e^{2t}\\ (3+t)e^t-3e^{2t} & -(5+3t)e^t+6e^{2t} & (3+2t)e^t-3e^{2t}\\ (4+t)e^t-4e^{2t} & -(8+3t)e^t+8e^{2t} & (5+2t)e^t-4e^{2t}\end{pmatrix}.\end{aligned}$$