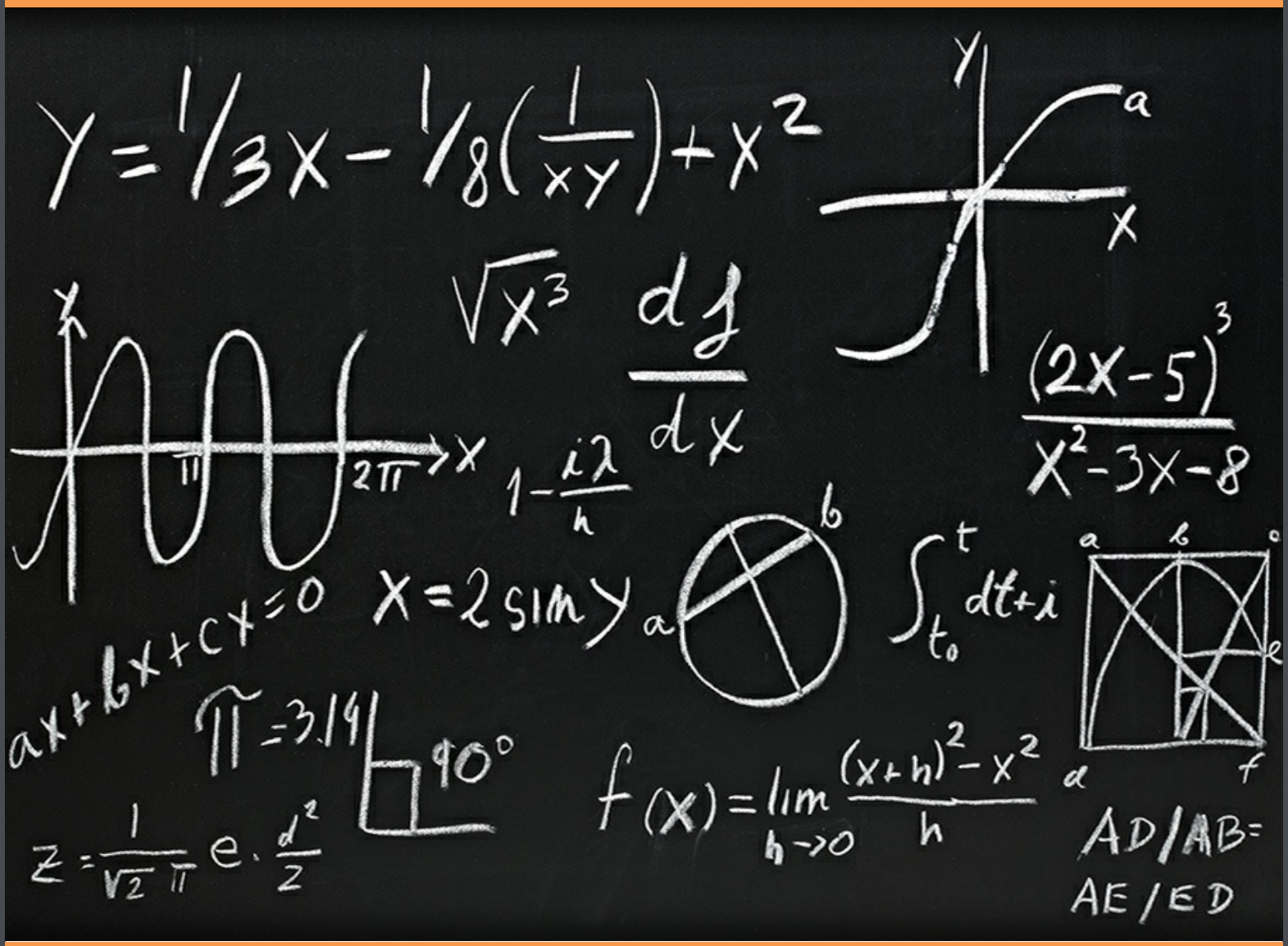
A youtube Calculus Workbook (Part II)

Frédéric Mynard



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A youtube Calculus Workbook (Part II)
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Preface

With the explosion of resources available on the internet, virtually anything can be learned on your own, using free online resources. Or can it, really? If you are looking for instructional videos to learn Calculus, you will probably have to sort through thousands of hits, navigate through videos of inconsistent quality and format, jump from one instructor to another, all this without written guidance.

This free e-book is a guide through a play-list of Calculus instructional videos. The play-list and the book are divided into 16 thematic learning modules. The format, level of details and rigor, and progression of topics are consistent with a semester long college level Calculus II course, the first volume covering the equivalent of a Calculus I course. The continuity of style should help you learn the material more consistently than jumping around the many options available on the internet. The book further provides simple summary of videos, written definitions and statements, worked out examples – even though fully step by step solutions are to be found in the videos – and an index.

The present book is a guide to instructional videos, and as such can be used for self study, or as a textbook for a Calculus course following the flipped classroom model.

An **essential** companion to this book is the exercise manual Exercises for A youtube Calculus Workbook Part II: a flipped classroom model, which also outlines and discusses the structure for a flipped classroom course based on this material.

For future reference, the play list of all the videos is available at:

https://www.youtube.com/playlist? list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL.

If you need to review any part of Calculus I, please refer to the first youtube workbook, whose associated play-list is available at:

http://www.youtube.com/playlist? list=PL265CB737C01F8961.

In particular, undefined notions or Theorems we may refer to that are not stated in the present book can be found in the first volume.

I hope that only few errors are left in this book, but some are bound to remain. I welcome feedback and comments at calculusvideos@gmail.com.

1 M1: Natural Logarithm and Exponential

1.1 Natural Logarithm: definition and logarithm laws

Watch the video at

https://www.youtube.com/watch?v=HETqWLBLsYc&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=1

Abstract

The natural logarithm function is defined as the antiderivative of $\frac{1}{x}$ on $(0,\infty)$ that takes the value 0 at 1. Its basic properties are introduced and proved.

Definition 1.1.1. The *natural logarithm* function, denoted ln, is defined on by

$$\ln x := \int_1^x \frac{dt}{t}.$$

Properties of the natural logarithm function: Let x and y be two positive numbers and let $r \in (-\infty,\infty)$.

$$\ln(1) = 0 \tag{1.1.1}$$

$$\ln(xy) = \ln x + \ln y \tag{1.1.2}$$

$$\ln\left(\frac{x}{y}\right) = \ln x - \ln y \tag{1.1.3}$$

$$\ln(x^r) = r\ln x \tag{1.1.4}$$

$$(\ln x)' = \frac{1}{x}, \tag{1.1.5}$$

so that ln is an increasing function on $(0,\infty)$.

Example 1.1.2. *Expand* $\ln\left(\frac{3x^2}{\sqrt{x+2}}\right)$.

$$\ln\left(\frac{3x^2}{\sqrt{x+2}}\right) = \ln 3 + 2\ln x - \frac{1}{2}\ln(x+2).$$

Example 1.1.3. Express as a single logarithm $\ln 3 + \frac{1}{3}\ln 8$.

$$\ln 3 + \frac{1}{3}\ln 8 = \ln\left(3\cdot 8^{\frac{1}{3}}\right) = \ln 6.$$

1.2 Calculus of Logarithms

Watch the videos at

https://www.youtube.com/watch?v=JjwQvDwhUrQ&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=2

and at

https://www.youtube.com/watch?v=z1oupT5GyaU&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=3

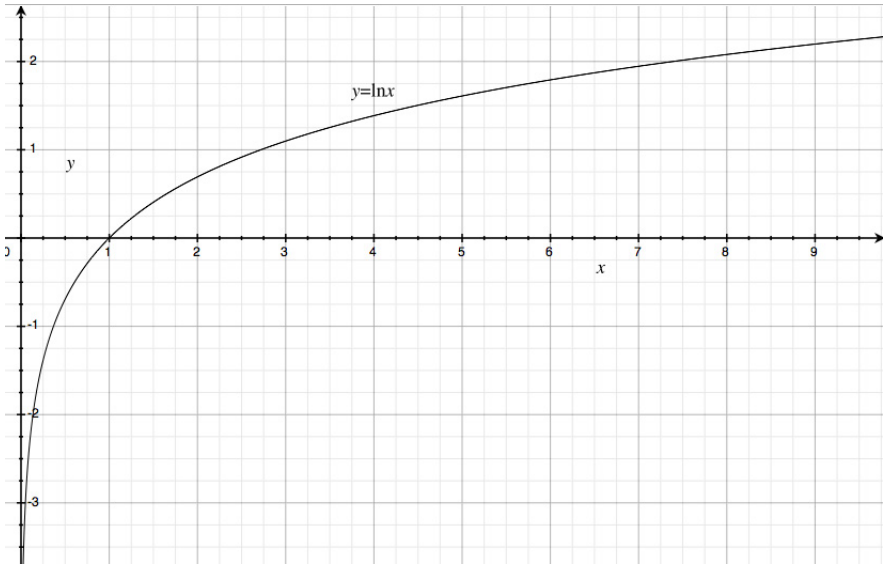
Abstract

These two videos go over more properties of the natural logarithm function, in particular its graph, and over derivatives and integrals involving the natural logarithm function.

Since, $(\ln x)' = \frac{1}{x} > 0$ and $(\ln x)'' = -\frac{1}{x^2} < 0$, the function ln is increasing and concave down on its domain $(0, \infty)$. Moreover

$$\begin{aligned} \lim_{x\to\infty} \ln x &= +\infty \\ \lim_{x\to 0^+} \ln x &= -\infty, \end{aligned}$$

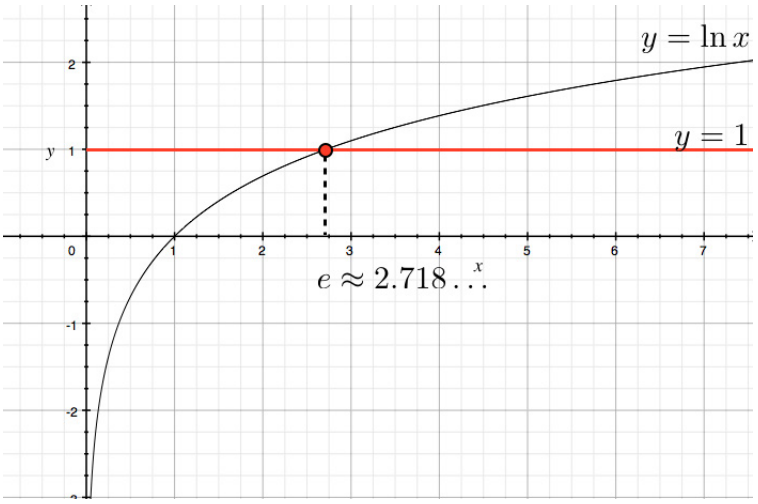
and the graph of the function is



Definition 1.2.1. The *Euler number e* is by definition the unique number such that

$$\ln e = 1,$$

as shown below:



Calculus of logarithm:

$$\begin{aligned}
(\ln x)' &= \frac{1}{x} \\
(\ln |x|)' &= \frac{1}{x} \\
\int \frac{dx}{x} &= \ln|x| + C \text{ on } (0,\infty) \text{ or on } (-\infty, 0). \\
(\ln(f(x)))' &= \frac{f'(x)}{f(x)}.
\end{aligned}$$

EXAMPLE 1.2.2. Differentiate:

1) $f(x) = \ln(x^2 + 10)$;

Solution.

$$f'(x) = \frac{2x}{x^2 + 10}.$$

2) $f(x) = \cos(\ln x)$;

Solution.

$$f'(x) = -\frac{\sin(\ln x)}{x}.$$

3) $f(x) = \ln(3\sqrt[5]{x})$.

Solution.

$$f(x) = \ln 3 + \frac{1}{5}\ln x \Longrightarrow f'(x) = \frac{1}{5x}.$$

EXAMPLE 1.2.3. Find an equation of the tangent line to $y = \ln\left(x^3 - 7\right)$ at $x = 2$.

Solution. The point of tangency is

$$\left(2, \ln(2^3 - 7)\right) = (2, \ln 1) = (2, 0)\,,$$

and the slope is $\frac{dy}{dx}_{|x=2}$. Since,

$$\frac{dy}{dx} = \frac{3x^2}{x^3 - 7},$$

the slope is $\frac{12}{1} = 12$ and the tangent line has equation

$$y = 12(x - 2).$$

EXAMPLE 1.2.4. Find $\frac{dy}{dx}$ along the curve

$$\ln(xy) = y \sin x.$$

Solution. We differentiate implicitly

$$\frac{y + x\frac{dy}{dx}}{xy} = \frac{dy}{dx}\sin x + y\cos x,$$

and solve for $\frac{dy}{dx}$:

$$y + x\frac{dy}{dx} = xy\sin x\frac{dy}{dx} + xy^2\cos x \Longrightarrow \frac{dy}{dx} = \frac{xy^2\cos x - y}{x - xy\sin x}.$$

EXAMPLE 1.2.5. Evaluate the following integrals:

1) $\int_1^2 \frac{4+u^2}{u^3}\,du$;

Solution.

$$\int_1^2 \frac{4+u^2}{u^3}\,du = \int_1^2 \frac{4}{u^3} + \frac{1}{u}\,du = \left[\frac{-2}{u^2} + \ln|u|\right]_1^2 = (-\frac{1}{2} + \ln 2) - (-2+0) = \frac{3}{2} + \ln 2.$$

2) $\int_e^6 \frac{dx}{x\ln x}$;

Solution. Let $u = \ln x$. Then $du = \frac{dx}{x}$ and

$$\int_e^6 \frac{dx}{x\ln x} = \int_1^{\ln 6} \frac{du}{u} = [\ln|u|]_1^{\ln 6} = \ln(\ln 6).$$

3) $\int \frac{\cos x}{2+\sin x}\,dx$;

Solution. Let $u = 2 + \sin x$, then $du = \cos x\,dx$ and

$$\int \frac{\cos x}{2+\sin x}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln(2+\sin x) + C.$$

4) $\int \tan x\,dx$.

Solution. Since $\tan x = \frac{\sin x}{\cos x}$, we can let $u = \cos x$ and $du = -\sin x\,dx$ to the effect that

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{du}{u} = -\ln|u| + C = \ln|\cos x|^{-1} + C = \ln|\sec x| + C.$$

1.3 Logarithmic Differentiation

Watch the video at

https://www.youtube.com/watch?v=6sjx6oAkWTE&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=4

Abstract

This video illustrates on examples the technique of logarithmic differentiation.

The basic idea behind *logarithmic differentiation* is that to differentiate a function $f(x)$ that involves products, quotients and/or powers, we can consider $\ln f(x)$ and take advantage of the Laws of Logarithm to turn it into a simpler expression to differentiate, and then note that

$$(\ln f(x))' = \frac{f'(x)}{f(x)} \Longrightarrow f'(x) = f(x) \cdot (\ln f(x))'.$$

Example 1.3.1. differentiate

$$f(x) = \frac{(x^2+1)^3 x \ln x}{x^3\sqrt{x^4+3}}.$$

Solution.

$$\ln f(x) = 3\ln(x^2+1) + \ln x + \ln(\ln x) - 3\ln x - \frac{1}{2}\ln(x^4+3),$$

so that

$$(\ln f(x))' = \frac{6x}{x^2+1} - \frac{2}{x} + \frac{1}{x\ln x} - \frac{1}{2}\frac{4x^3}{x^4+3},$$

that is

$$\begin{aligned} \frac{f'(x)}{f(x)} &= \left(\frac{6x}{x^2+1} - \frac{2}{x} + \frac{1}{x\ln x} - \frac{2x^3}{x^4+3}\right) \Longrightarrow \\ f'(x) &= f(x)\left(\frac{6x}{x^2+1} - \frac{2}{x} + \frac{1}{x\ln x} - \frac{2x^3}{x^4+3}\right) \\ &= \left(\frac{(x^2+1)^3 x\ln x}{x^3\sqrt{x^4+3}}\right)\left(\frac{6x}{x^2+1} - \frac{2}{x} + \frac{1}{x\ln x} - \frac{2x^3}{x^4+3}\right). \end{aligned}$$

Exercises

you are now prepared to work on the Practice Problems, and Homework set M1A in the manual of exercises.

1.4 One-to-one functions and inverse functions

Watch the video at

https://www.youtube.com/watch?v=a0oKcvT2ECA&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL

Abstract

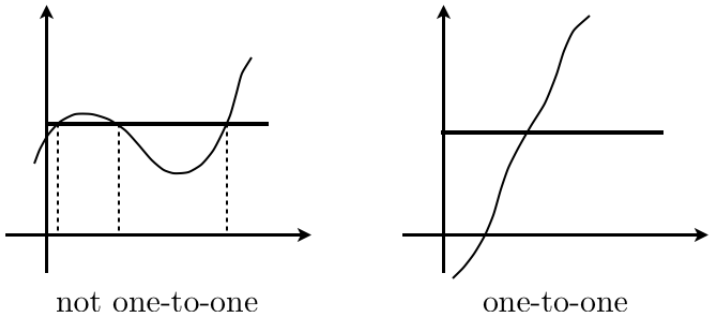
This video defines one-to-one functions and considers criteria, and defines the inverse function of a one-to-one function.

Definition 1.4.1. A function f is *one-to-one* on an interval I if for every x_1 and x_2 in I

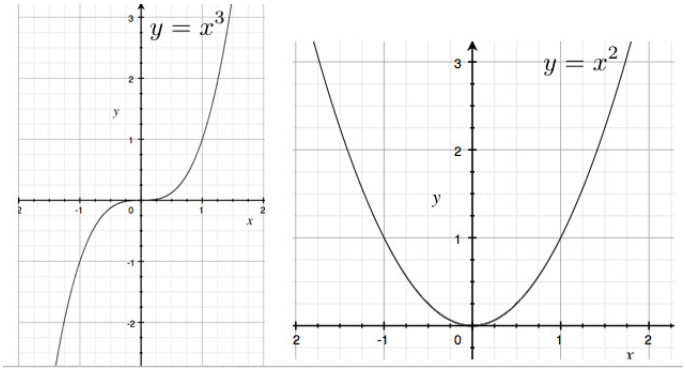
$$x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2).$$

In other words, f is one-to-one if it never takes on the same value twice.

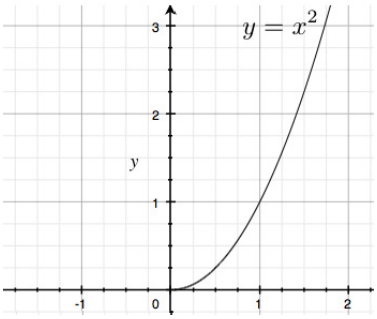
Proposition 1.4.2 (Horizontal line Test). *A function is one-to-one if and only if horizontal lines intersect the graph in at most one point.*



Example 1.4.3. For instance $f(x) = x^3$ is one-to-one on its domain $(-\infty, \infty)$, while $g(x) = x^2$ is not:



while the restriction of x^2 to the non-negative reals, $h(x) = x^2$ for $x \geq 0$, is one-to-one:



Recall

Definition 1.4.4. A function *f* is *increasing* on an interval *I* if for every x_1 and x_2 in *I*

$$x_1 < x_2 \Longrightarrow f(x_1) < f(x_2),$$

and *decreasing* on *I* if for every x_1 and x_2 in *I*

$$x_1 < x_2 \Longrightarrow f(x_1) > f(x_2).$$

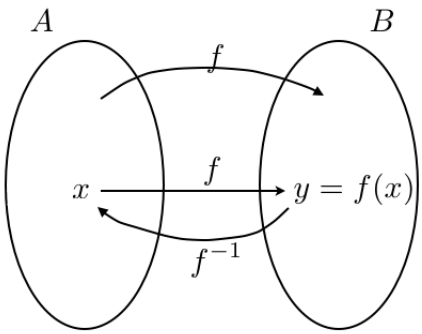
A function is *monotone* on *I* if it is either increasing or decreasing on *I*.

Proposition 1.4.5. *If f is monotone on an interval, then it is one-to-one on this interval.*

Definition 1.4.6. Let *f* be a function with domain *A*. The *range of f* is the set of values assumed by *f*, that is, the set $\{f(x) : x \in A\}$.

Definition 1.4.7. If *f* is a one-to-one function with domain *A* and range *B*, then it admits an *inverse function,* denoted f^{-1}, that has domain *B* and range *A* and is defined by

$$y = f(x) \iff f^{-1}(y) = x.$$



1.5 Finding inverse functions

Watch the video at

https://www.youtube.com/watch?v=ddc_t0n0xhQ&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=6

Abstract

In this video, examples are provided to check that a function is one-to-one, to evaluate the inverse function at a particular point, and to find a formula for the inverse of a given function. Additionally, it is shown that the graph of the inverse function of a one-to-one function is obtained from the graph of the function by reflection about $y = x$.

Example 1.5.1. If f is one-to-one and $f(2) = 9$, what is $f^{-1}(9)$?

Solution.

$$f(2) = 9 \underset{f \text{ one-to-one}}{\iff} 2 = f^{-1}(9).$$

EXAMPLE 1.5.2. Let $f(x) = x + \cos x$. Is f one-to-one? What is $f^{-1}(1)$?

Solution. Since

$$f'(x) = 1 - \sin x \geq 0,$$

the function is increasing on its domain, hence one-to-one by Proposition 1.4.5. Thus it has an inverse function and $f^{-1}(1) = 0$ because

$$f(0) = 0 + \cos 0 = 1.$$

To find an algebraic expression for f^{-1} from an algebraic expression defining a one-to-one function f, remember that by definition, in this case:

$$y = f(x) \iff f^{-1}(y) = x.$$

Thus, start from $y = f(x)$, and solve for x!

EXAMPLE 1.5.3. Is $f(x) = 3 - 2x$ one-to-one? If yes, find a formula for f^{-1}.

Solution. Since f is decreasing on $(-\infty, \infty)$, it is one-to-one by Proposition 1.4.5. Moreover

$$y = 3 - 2x \iff x = \frac{3 - y}{2} = f^{-1}(y),$$

so that $f^{-1}(x) = \frac{3-x}{2}$.

EXAMPLE 1.5.4. Find the inverse function of

$$f(x) = \frac{4x - 1}{2x + 3}.$$

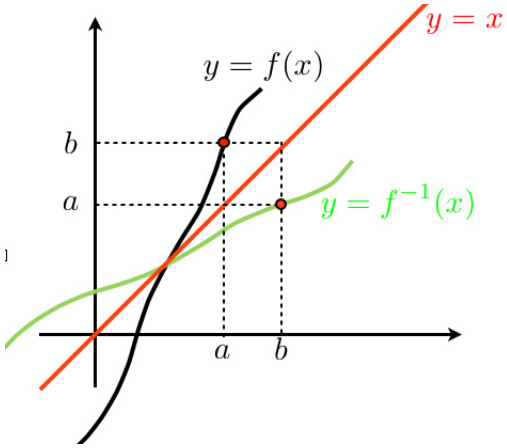
Solution. Note that f has domain $(-\infty, -\frac{3}{2}) \cup (-\frac{3}{2}, \infty)$. If we were asked to find the range of f, one way to do that would be to find the domain of f^{-1}. Let $y = \frac{4x-1}{2x+3}$. Then

$$(2x + 3)y = 4x - 1 \iff x(2y - 4) = -(1 + 3y) \iff x = -\frac{1 + 3y}{2y - 4} = f^{-1}(y),$$

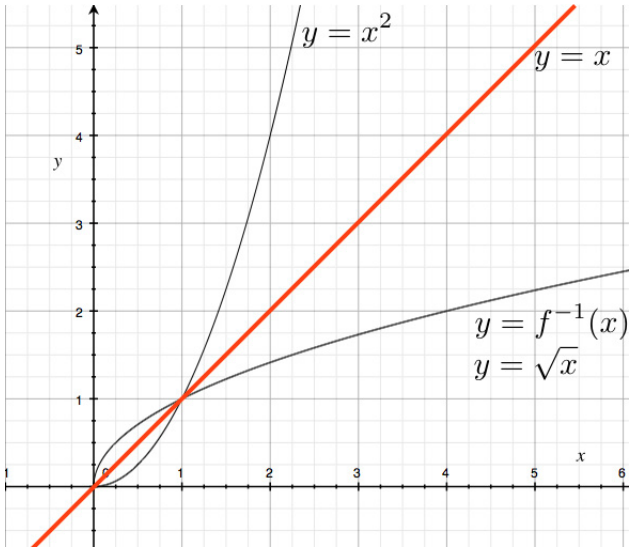
so that $f^{-1}(x) = -\frac{1+3x}{2x-4}$, whose domain is $(-\infty, 2) \cup (2, \infty)$, which is also the range of f.

Graph of inverse functions:

PROPOSITION 1.5.5. *If f is a one-to-one function, the graph of f^{-1} is obtained from the graph of f by reflection about the line $y = x$:*



EXAMPLE 1.5.6. The function $f(x) = x^2$ for $x \geq 0$ is one-to-one and its inverse function is $f^{-1}(x) = \sqrt{x}$:



1.6 Calculus of inverse functions

Watch the video at

https://www.youtube.com/watch?v=eN_4_naF-4U&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=7

Abstract

This video examines how differentiability and continuity transfer from a one-to-one function to its inverse function, and establishes a formula for the derivative of the inverse function.

Proposition 1.6.1. *If f is one-to-one and continuous on an interval I, then f^{-1} is also continuous on $f(I)$.*

Theorem 1.6.2. *If f is one-to-one and differentiable at $b = f^{-1}(a)$, and $f'(b) \neq 0$, then f^{-1} is differentiable at a and*

$$\left(f^{-1}\right)'(a) = \frac{1}{f'(f^{-1}(a))}.$$

In Leibniz notations, with

$$y = f^{-1}(x) \iff f(y) = x$$

this translates as

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

EXAMPLE 1.6.3. Let $f(x) = x^5 - x^3 + 2x$. Is it one-to-one? If yes, find $\left(f^{-1}\right)'(2)$.

Solution. Note that

$$f'(x) = 5x^4 - 3x^2 + 2 = 5u^2 - 3u + 2 \text{ for } u = x^2$$

and $5u^2 - 3u + 2 = 0$ has no solution because its discriminant ([1]) is $-31 < 0$, so that $f'(x) > 0$ because $f'(0) > 0$ and f' does not change sign. Thus f is an increasing function, and is thus one-to-one (Proposition 1.4.5). According to Theorem 1.6.2, we have

$$\left(f^{-1}\right)'(2) = \frac{1}{f'\left(f^{-1}(2)\right)}.$$

Moreover $f^{-1}(2) = 1$ because $f(1) = 2$. Thus $f'(f^{-1}(2)) = f'(1) = 4$ and

$$\left(f^{-1}\right)'(2) = \frac{1}{4}.$$

EXAMPLE 1.6.4. Consider $f(x) = \sqrt{x-2}$.

1) Show that f is one-to-one;
   Solution. Since $f'(x) = \frac{1}{2\sqrt{x-2}} > 0$ on $(2, \infty)$, f is increasing, hence one-to-one, on its domain.

2) Use Theorem 1.6.2 to calculate $\left(f^{-1}\right)'(2)$;
   Solution.

$$\left(f^{-1}\right)'(2) = \frac{1}{f'\left(f^{-1}(2)\right)}$$

   and

$$f(x) = 2 \iff \sqrt{x-2} = 2 \iff x = 6,$$

   so that $f^{-1}(2) = 6$. Thus

$$\left(f^{-1}\right)'(2) = \frac{1}{\frac{1}{2\sqrt{6-2}}} = 4.$$

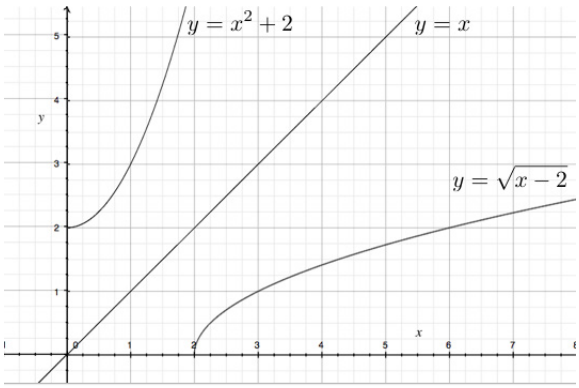
3) Find a formula for $f^{-1}(x)$;
Solution. Let $y = \sqrt{x-2}$. Then $y^2 = x - 2$ with $y \geq 0$, that is,

$$x = y^2 + 2 \text{ for } y \geq 0,$$

so that $f^{-1}(x) = x^2 + 2$ for $x \geq 0$.

4) Use 3. to recalculate $\left(f^{-1}\right)'(2)$;
Solution. Since $f^{-1}(x) = x^2 + 2$ for $x \geq 0$, $\left(f^{-1}\right)'(x) = 2x$ and thus $\left(f^{-1}\right)'(2) = 4$.

5) Sketch the graphs of f and of f^{-1}.
Solution.



Exercises

you are now prepared to work on the Practice Problems, and Homework set M1B in the manual of exercises.

1.7 Natural Exponential: definition and properties

Watch the video at

https://www.youtube.com/watch?v=zp2jv7H1qIA&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=8

Abstract

This video defines the natural exponential function as the inverse function of the natural logarithm function and explores its basic properties.

Recall that the natural logarithm function has domain $(0, \infty)$ and range $(-\infty, \infty)$ and is increasing, hence one-to-one, on its domain. Thus it admits an inverse function.

Definition 1.7.1. The *natural exponential* function, tentatively denoted exp, is the inverse function of the natural logarithm function ln. Thus it has domain $(-\infty, \infty)$ and range $(0, \infty)$ and

$$y = \exp x \iff x = \ln y,\ y > 0.$$

Since $\ln(e^x) = x \ln e = x$ by (1.1.4) and $\ln(\exp x) = x$ because ln and exp are inverse of each other, we conclude that $\ln(e^x) = \ln(\exp x)$ for all x, which ensures that

$$\exp x = e^x$$

because ln is one-to-one. We use the notation e^x from now on for the natural exponential function. Of course, we have

$$\begin{aligned} e^{\ln x} &= x \text{ for all } x > 0 \\ \ln(e^x) &= x \text{ for all } x. \end{aligned}$$

The properties of logarithm are rephrased in term of its inverse function:

Properties of exponential:

$$e^0 = 1 \tag{1.7.1}$$

$$e^{x+y} = e^x \cdot e^y \tag{1.7.2}$$

$$e^{x-y} = \frac{e^x}{e^y} \tag{1.7.3}$$

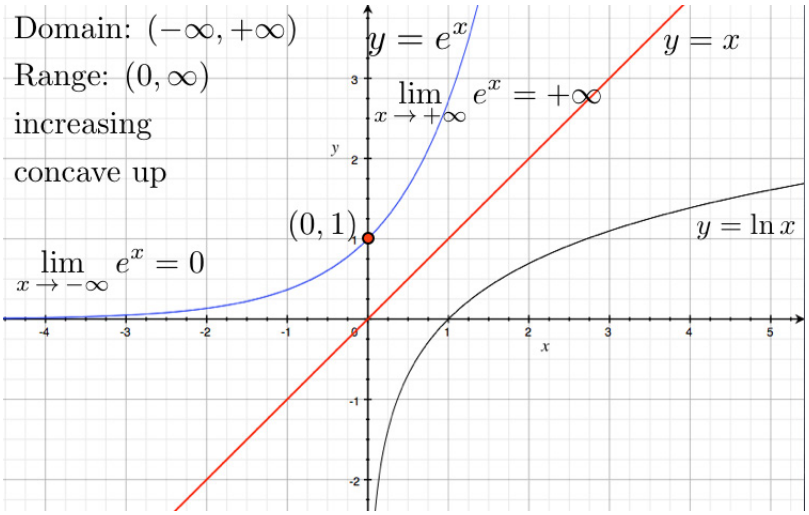
$$(e^x)^r = e^{rx}. \tag{1.7.4}$$

Moreover

$$\begin{aligned} (e^x)' &= e^x \\ \int e^x \, dx &= e^x + C. \end{aligned}$$

The graph $y = e^x$ is obtained from $y = \ln x$ by reflection about $y = x$, so that $y = e^x$ is increasing concave up (which we can also see from the fact that the first and second derivative are $e^x > 0$), with

$$\begin{aligned} \lim_{x \to \infty} e^x &= +\infty \\ \lim_{x \to -\infty} e^x &= 0 : \end{aligned}$$



1.8 Derivatives and integrals with exponentials

Watch the video at

https://www.youtube.com/watch?v=F-YEvvY6nPk&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=9

Abstract

This video goes over examples of differentiation and integration of functions defined in terms of exponential functions.

The examples below use the basic formulas

$$\begin{aligned} (e^x)' &= e^x \\ (e^u)' &= e^u \cdot u' \\ \int e^x\,dx &= e^x + C. \end{aligned}$$

Example 1.8.1. Differentiate:

1) $y = \frac{e^x}{1+x}$;

Solution. Using the Quotient Rule:

$$\frac{dy}{dx} = \frac{e^x(1+x) - e^x}{(1+x)^2} = \frac{xe^x}{(1+x)^2}.$$

2) $y = e^{u^2}\cos u$;

Solution. Using the Product Rule and the Chain Rule:

$$\frac{dy}{du} = 2ue^{u^2}\cos u - e^{u^2}\sin u.$$

3) $y = e^x \ln x$;

Solution. Using the Product Rule:

$$\frac{dy}{dx} = e^x \ln x + \frac{e^x}{x}.$$

4) $y = \cos(e^{\pi x})$.

Solution. Using the Chain Rule:

$$\frac{dy}{dx} = -\sin\left(e^{\pi x}\right) \cdot \pi e^{\pi x}.$$

EXAMPLE 1.8.2. Evaluate:

1) $\int_0^1 xe^{x^2}\,dx$;
Solution. Using substitution with $u = x^2$, so that $du = 2x\,dx$, we have

$$\int_0^1 xe^{x^2}\,dx = \frac{1}{2}\int_0^1 e^u\,du = \frac{1}{2}\left[e^u\right]_0^1 = \frac{e-1}{2}.$$

2) $\int \frac{e^{\frac{1}{x}}}{x^2}\,dx$;
Solution. Using substitution with $u = \frac{1}{x}$, so that $du = -\frac{1}{x^2}\,dx$, we have

$$\int \frac{e^{\frac{1}{x}}}{x^2}\,dx = -\int e^u\,du = -e^u + C = -e^{\frac{1}{x}} + C.$$

3) $\int e^x \sin(e^x)\,dx$.
Solution. Using substitution with $u = e^x$, so that $du = e^x\,dx$, we have

$$\int e^x \sin(e^x)\,dx = \int \sin u\,du = -\cos u + C = -\cos(e^x) + C.$$

1.9 Exponential and logarithmic equations and inequalities

Watch the video at

https://www.youtube.com/watch?v=J9ki-Vz1-Xg&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=10

Abstract

This video goes over examples of equations and inequalities involving exponential and logarithm.

Example 1.9.1. Solve:

1) $e^{2x+3} - 7 = 0$;
Solution. By applying ln on both sides of $e^{2x+3} = 7$, we obtain
$$2x + 3 = \ln 7 \iff x = \frac{\ln 7 - 3}{2}.$$

2) $\ln(5 - 2x) = -3$;
Solution. By applying the natural exponential on both sides, we obtain
$$5 - 2x = e^{-3} \iff x = \frac{5 - e^{-3}}{2}.$$

3) $e^{e^x} = 10$;
Solution. By applying ln on both sides, we have
$$e^x = \ln 10 \iff x = \ln(\ln 10).$$

4) $\ln(2x + 1) = 2 - \ln x$;
Solution. The domain of this equation is $(0, \infty)$. On this domain, we can rewrite the equation as
$$\begin{aligned} \ln(2x + 1) + \ln x &= 2 \\ \ln\left(x(2x + 1)\right) &= 2, \end{aligned}$$
so that $2x^2 + x = e^2$. This is a quadratic equation $2x^2 + x - e^2 = 0$, which we can solve with the quadratic formula to the effect that
$$x = \frac{-1 \pm \sqrt{1 + 8e^2}}{4},$$
but only $x = \frac{-1+\sqrt{1+8e^2}}{4}$ is in the domain of the equation, and is thus the only solution.

5) $7e^x - e^{2x} = 12$.

Solution. We can rewrite the equation as

$$\begin{aligned} e^{2x} - 7e^x + 12 &= 0 \\ (e^x)^2 - 7e^x + 12 &= 0 \\ u^2 - 7u + 12 &= 0 \text{ for } u = e^x \\ (u-3)(u-4) &= 0 \end{aligned}$$

so that we have $e^x = 3$ or $e^x = 4$, that is, $x = \ln 3$ or $x = \ln 4$.

EXAMPLE 1.9.2. Solve:

1) $2 < \ln x < 9$;

Solution. Since e^x is an increasing function, it preserves the direction of inequalities, so that

$$e^2 < e^{\ln x} < e^9 \iff e^2 < x < e^9$$

and the solution set is the interval (e^2, e^9).

2) $e^{2-3x} > 4$.

Solution. Since ln is an increasing function, it preserves the direction of inequalities, so that

$$\ln(e^{2-3x}) > \ln 4 \iff 2 - 3x > \ln 4 \iff \frac{2 - \ln 4}{3} > x,$$

so that the solution set is the interval $(-\infty, \frac{2-\ln 4}{3})$.

Exercises

you are now prepared to work on the Practice Problems, and Homework set M1C in the manual of exercises.

2 M2: More transcendental functions

2.1 General exponential functions

Watch the videos at

https://www.youtube.com/watch?v=0uIuLyOUqYc&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=11

and at

https://www.youtube.com/watch?v=aHckAl23AD8&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=12

Abstract

These two videos define and study exponential functions of a general base $a > 0$, $a \neq 1$.

Definition 2.1.1. Let $a > 0$ with $a \neq 1$. The *exponential function of base a* is the function

$$f(x) = a^x := e^{x \ln a}.$$

Note that $f(x) = a^x$ has domain $(-\infty, \infty)$ and range $(0, \infty)$.

For instance

$$\begin{aligned} 2^\pi &= e^{\pi \ln 2} \\ 10^{x^2} &= e^{x^2 \ln 10} \\ x^x &= e^{x \ln x}, \end{aligned}$$

and you should keep in mind that powers are really defined in terms of powers of *e*, and be comfortable rewriting them as such.

Proposition 2.1.2 (Laws of exponents). *Let x, y in* (∞, ∞) *and* $a, b > 0$. *Then*

$$\begin{aligned} a^{x+y} &= a^x \cdot a^y \\ a^{x-y} &= \frac{a^x}{a^y} \\ (a^x)^y &= a^{xy} \\ (ab)^x &= a^x \cdot b^x. \end{aligned}$$

Proposition 2.1.3. *Let* $a > 0$ *and* $a \neq 1$. *Then the exponential function of base a is differentiable on* $(-\infty, \infty)$ *and*

$$(a^x)' = a^x \cdot \ln a.$$

In other words,

$$\int a^x \, dx = \frac{a^x}{\ln a} + C.$$

Example 2.1.4. Differentiate:

1) $g(x) = x^4 \cdot 4^x$;
Solution. Using the product rule

$$g'(x) = 4x^3 \cdot 4^x + x^4 \cdot 4^x \cdot \ln 4.$$

2) $f(\theta) = 10^{\tan \theta}$;
Solution. Using the Chain Rule

$$f'(\theta) = 10^{\tan \theta} \cdot \ln 10 \cdot \sec^2 \theta.$$

3) $y = x^{\frac{1}{x}}$;

Solution. Since

$$x^{\frac{1}{x}} = e^{\frac{1}{x}\ln x},$$

we use the Chain Rule, and Product Rule, to the effect that

$$\begin{aligned}\frac{dy}{dx} &= e^{\frac{1}{x}\ln x}\cdot\left(\frac{1}{x}\ln x\right)' \\ &= x^{\frac{1}{x}}\left(-\frac{\ln x}{x^2}+\frac{1}{x^2}\right).\end{aligned}$$

4) $h(x) = (\sin x)^x$.

Solution. Since

$$(\sin x)^x = e^{x\ln(\sin x)},$$

we use the Chain Rule, and Product Rule, to the effect that

$$\begin{aligned}h'(x) &= e^{x\ln(\sin x)}\cdot(x\ln(\sin x))' \\ &= (\sin x)^x\left(\ln(\sin x)+\frac{x\cos x}{\sin x}\right).\end{aligned}$$

EXAMPLE 2.1.5. Evaluate $\int \pi^x + 2x^\pi \, dx$.

Solution.

$$\int \pi^x + 2x^\pi \, dx = \frac{\pi^x}{\ln \pi} + \frac{2x^{\pi+1}}{\pi + 1} + C.$$

Graph of exponential functions

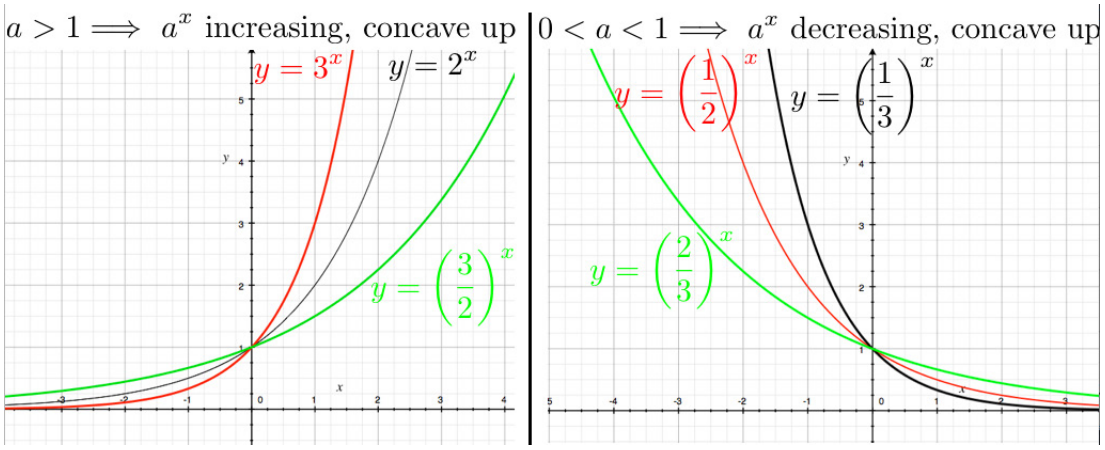
Since

$$(a^x)' = a^x \cdot \ln a$$

and $\ln a > 0$ whenever $a > 1$, $\ln a < 0$ whenever $0 < a < 1$, we conclude that a^x is increasing on its domain if $a > 1$ and decreasing on its domain if $0 < a < 1$. Moreover

$$(a^x)'' = a^x \cdot (\ln a)^2 > 0,$$

so that a^x is concave up in both cases.



2.2 General logarithm functions

Watch the videos at

https://www.youtube.com/watch?v=S4NTCdIrcJw&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=13

and at

https://www.youtube.com/watch?v=2wtpQJ_6AFA&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=14

Abstract

These two videos define and examine the properties of logarithm functions of base *a* ($a > 0$, $a \neq 1$).

We have seen that for $a > 0$, $a \neq 1$, the exponential function of base *a* is one-to-one with domain $(-\infty, \infty)$ and range $(0, \infty)$. Thus, it admits an inverse function:

Definition 2.2.1. The *logarithm function of base a*, denoted $\log_a$, is by definition the inverse function of the exponential function of base *a*. Thus, it has domain $(0, \infty)$ and range $(-\infty, \infty)$ and is defined by

$$y = \log_a x,\ x > 0 \iff a^y = x.$$

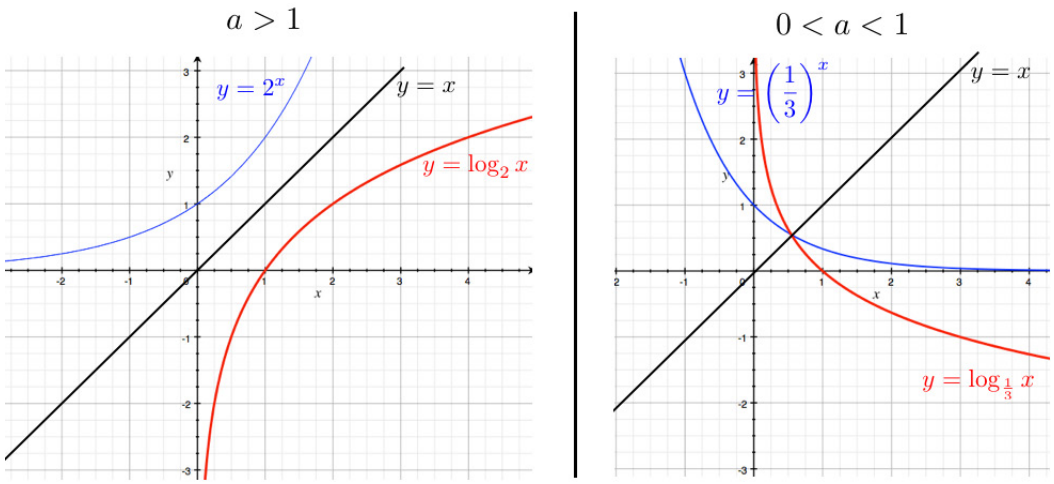
By definition,

$$\begin{aligned} a^{\log_a x} &= x \text{ for all } x > 0 \\ \log_a (a^x) &= x \text{ for all } x, \\ \log_a a &= 1 \\ \log_a 1 &= 0. \end{aligned}$$

Example 2.2.2.

$$\begin{aligned} \log_5 25 &= 2 \text{ because } 5^2 = 25 \\ \log_4 2 &= \frac{1}{2} \text{ because } 4^{\frac{1}{2}} = 2 \\ \log_2 \frac{1}{2} &= -1 \text{ because } 2^{-1} = \tfrac{1}{2}. \end{aligned}$$

Graph of logarithm functions:



PROPOSITION 2.2.3 (Change of base formula). *If* $a > 0$, $a \neq 1$, *then*

$$\log_a x = \frac{\ln x}{\ln a}.$$

COROLLARY 2.2.4 (Laws of Logarithms). *Let $a > 0$, $a \neq 1$. Let $x, y > 0$ and r be a real number. Then*

$$\begin{aligned} \log_a(xy) &= \log_a x + \log_a y \\ \log_a\left(\frac{x}{y}\right) &= \log_a x - \log_a y \\ \log_a(x^r) &= r\log_a x. \end{aligned}$$

EXAMPLE 2.2.5.

$$\begin{aligned} \log_8 320 - \log_8 5 &= \log_8\left(\frac{320}{5}\right) = \log_8(64) = 2 \text{ because } 8^2 = 64; \\ 10^{\log_{10} 4 + \log_{10} 7} &= 10^{\log_{10}(4\times 7)} = 28. \end{aligned}$$

COROLLARY 2.2.6. *If $a > 0$, $a \neq 1$, then*

$$(\log_a x)' = \frac{1}{x \ln a}.$$

EXAMPLE 2.2.7. Differentiate $f(x) = \log_5\left(\frac{x}{x-1}\right)$.

Solution. Since $f(x) = \log_5 x - \log_5(x-1)$ (by Corollary 2.2.4), we conclude from Corollary 2.2.6 that

$$f'(x) = \frac{1}{\ln 5}\left(\frac{1}{x} - \frac{1}{x-1}\right).$$

EXAMPLE 2.2.8. Evaluate:

1) $\int \frac{2^x}{2^x+1}\,dx$

Solution. Let $u = 2^x + 1$. Then $du = 2^x \ln 2\,dx$ so that

$$\int \frac{2^x}{2^x+1}\,dx = \frac{1}{\ln 2}\int \frac{du}{u} = \frac{1}{\ln 2}\ln|u| + C = \frac{\ln(2^x+1)}{\ln 2} + C.$$

2) $\int \frac{\log_2 x}{x}\,dx$

Solution. Let $u = \log_2 x$. Then $du = \frac{1}{x\ln 2}\,dx$ so that

$$\int \frac{\log_2 x}{x}\,dx = \ln 2\int u\,du = \frac{\ln 2}{2}(\log_2 x)^2 + C.$$

Exercises

you are now prepared to work on the Practice Problems, and Homework set M2A in the manual of exercises.

2.3 Inverse trig functions: arcsine

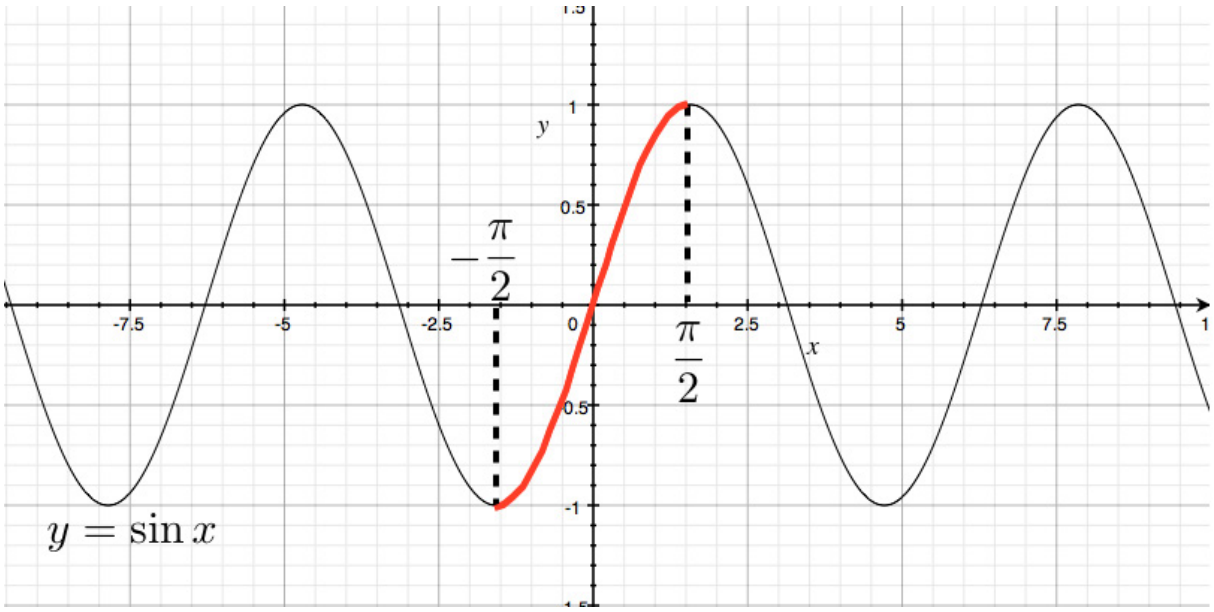
Watch the video at

https://www.youtube.com/watch?v=ELkBDmPA8VI&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=15

Abstract

This video introduces the inverse sine function and its basic properties.

The sine function is not one-to-one, but its restriction to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is:



Definition 2.3.1. The *inverse sine function,* denoted $\arcsin$, is the inverse function of the restriction of the sine function to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, that is:

$$y = \arcsin x \iff \sin y = x \text{ and } y \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right].$$

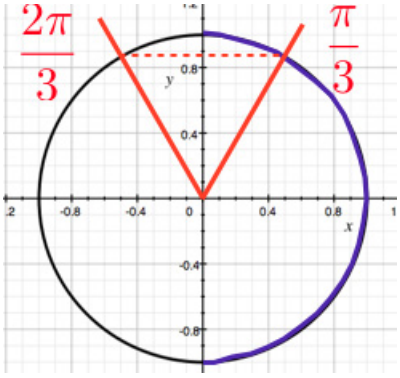
The function $\arcsin$ has thus domain $[-1, 1]$ and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Therefore

$$\begin{aligned} \arcsin(\sin x) &= x \text{ for all } x \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right] \\ \sin(\arcsin x) &= x \text{ for all } x \in [-1, 1]. \end{aligned}$$

EXAMPLE 2.3.2.

$$\begin{aligned}
\arcsin\left(\frac{1}{2}\right) &= \frac{\pi}{6} \text{ because } \sin\tfrac{\pi}{6} = \tfrac{1}{2} \text{and } \tfrac{\pi}{6} \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right];\\
\sin\left(\arcsin\frac{\sqrt{3}}{2}\right) &= \frac{\sqrt{3}}{2}\\
\arcsin\left(\sin\frac{\pi}{4}\right) &= \frac{\pi}{4} \text{ because } \tfrac{\pi}{4} \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right];\\
\arcsin\left(\sin\frac{2\pi}{3}\right) &= \arcsin\left(\sin\frac{\pi}{3}\right) = \frac{\pi}{3} \text{ because } \tfrac{\pi}{3} \text{ is the angle in } \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right] \text{with the same sine as } \tfrac{2\pi}{3}:
\end{aligned}$$



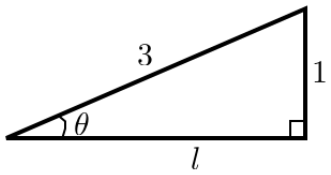
EXAMPLE 2.3.3. To calculate

$$\tan\left(\arcsin\frac{1}{3}\right)$$

note that

$$\theta := \arcsin\frac{1}{3}$$

is an angle in $(0, \frac{\pi}{2})$ with $\sin\theta = \frac{1}{3}$. This situation can be represented in the right triangle below:



Applying the Pythagorean Theorem, we have $l^2 + 1 = 9$, so that $l = \sqrt{8} = 2\sqrt{2}$. Thus

$$\tan\theta = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}.$$

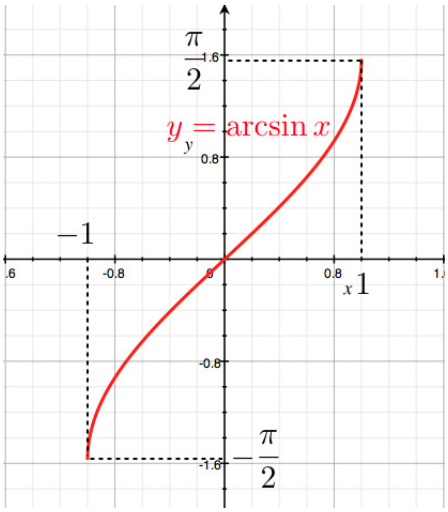
REMARK 2.3.4. The alternative notation $\sin^{-1}$ is often used for arcsin. Yet, this is in direct conflict with the standard convention to place the exponent next to the function when taking powers of the value of a trig function, as in $\sin^2 x$ to mean $(\sin x)^2$. To be coherent with this convention, $\sin^{-1} x$ should mean

$$(\sin x)^{-1} = \frac{1}{\sin x}$$

which has nothing to do with arcsin *x*! Thus we will not use this alternative notation.

Graph of arcsine:

The graph $y = \arcsin x$ is the reflection about $y = x$ of the restriction of $y = \sin x$ to $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$:



In particular, arcsin is odd:

$$\arcsin(-x) = -\arcsin x \text{ for all } x \in [-1, 1].$$

Proposition 2.3.5. *The inverse sine function is differentiable on* $(-1, 1)$ *and*

$$(\arcsin x)' = \frac{1}{\sqrt{1 - x^2}}.$$

In other words,

$$\int \frac{dx}{\sqrt{1 - x^2}} = \arcsin x + C. \tag{2.3.1}$$

2.4 Inverse trig functions: other inverse trig functions

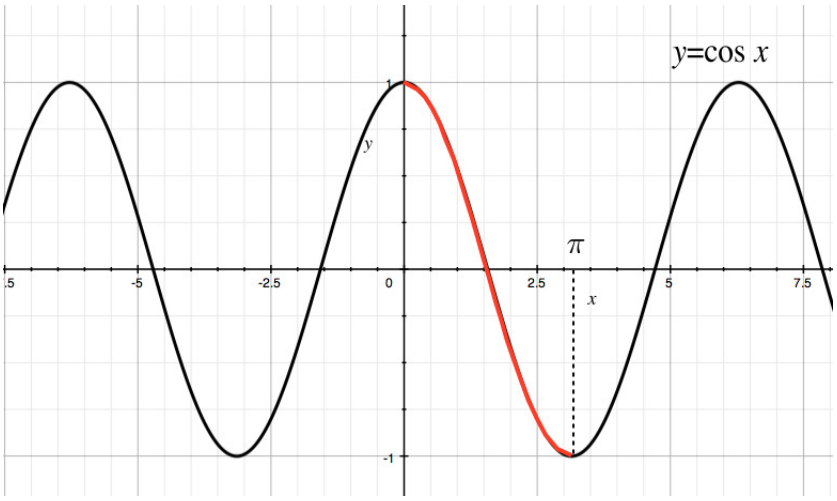
Watch the video at

https://www.youtube.com/watch?v=oKoXfFktY6A&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=16

Abstract

This video defines and studies the inverse cosine function arccos and the inverse tangent function arctan.

Inverse cosine function: The cosine function is not one-to-one, but its restriction to the interval $[0, \pi]$ is:



Definition 2.4.1. The *inverse cosine function*, denoted arccos, is the inverse function of the restriction of the cosine function to $[0, \pi]$, that is:

$$y = \arccos x \iff \cos y = x \text{ and } y \in [0, \pi].$$

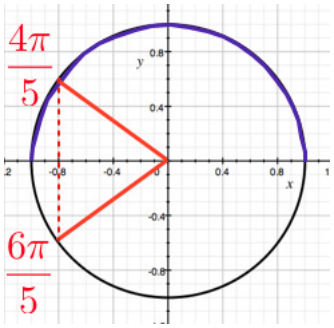
The function arccos has thus domain $[-1, 1]$ and range $[0, \pi]$.

Therefore

$$\begin{aligned} \arccos(\cos x) &= x \text{ for all } x \in [0, \pi] \\ \cos(\arccos x) &= x \text{ for all } x \in [-1, 1]. \end{aligned}$$

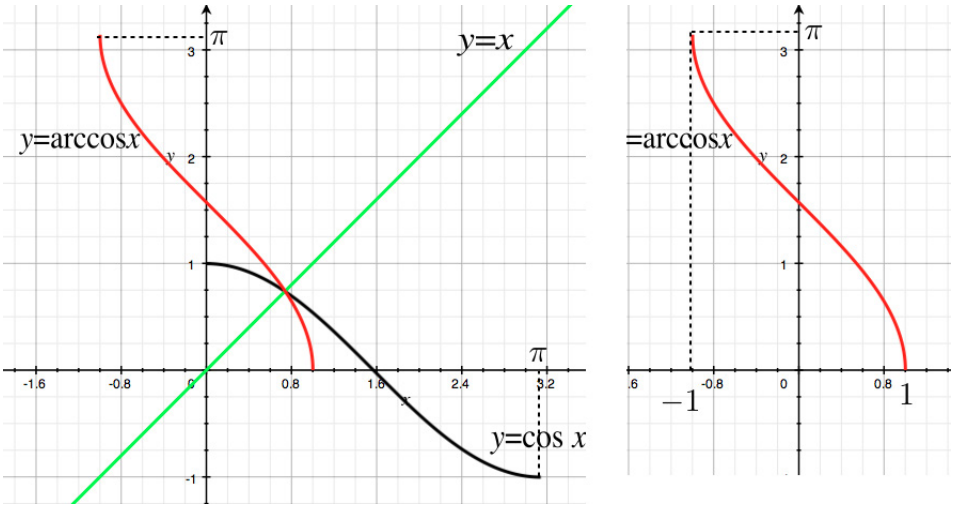
Example 2.4.2. For instance

$$\begin{aligned} \arccos\left(\cos\frac{\pi}{5}\right) &= \frac{\pi}{5} \text{ because } \tfrac{\pi}{5} \in [0, \pi] \\ \arccos\left(\cos\frac{6\pi}{5}\right) &= \arccos\left(\cos\frac{4\pi}{5}\right) \text{ because } \cos\tfrac{6\pi}{5} = \cos\tfrac{4\pi}{5} \text{ and } \tfrac{4\pi}{5} \in [0, \pi]: \end{aligned}$$



Graph of arccosine:

The graph is obtained by reflection of that of the restriction of the cosine function to $[0, \pi]$ about $y = x$:



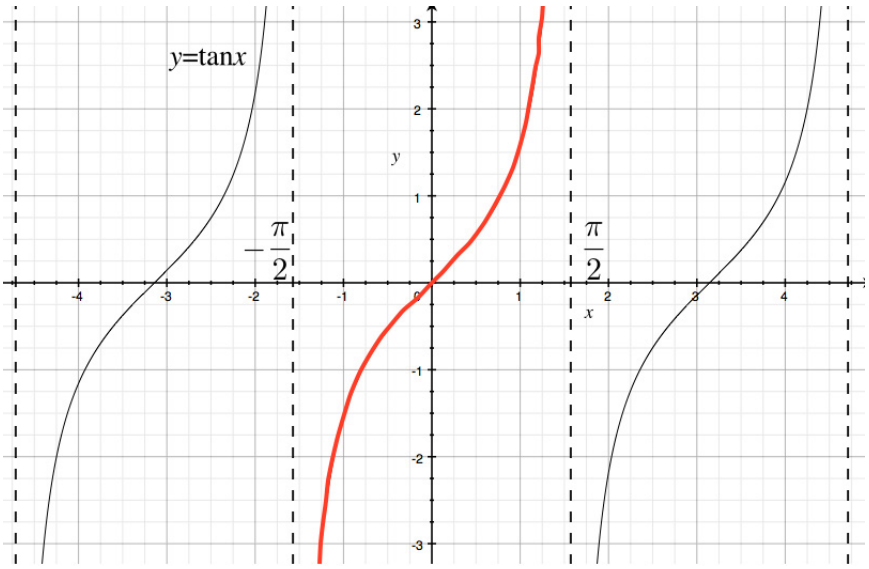
PROPOSITION 2.4.3. *The function* arccos *is differentiable on* $(-1, 1)$ *and*

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}.$$

Thus, arcsin and -arccos are two antiderivatives of $\frac{1}{\sqrt{1-x^2}}$ on $(-1, 1)$ and thus differ by a constant. More specifically:

$$\arccos x + \arcsin x = \frac{\pi}{2}.$$

Inverse tangent function: The tangent function is not one-to-one, but its restriction to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is:



DEFINITION 2.4.4. The *inverse tangent function,* denoted arctan, is the inverse function of the restriction of the tangent function to $(-\frac{\pi}{2}, \frac{\pi}{2})$, that is:

$$y = \arctan x \iff \tan y = x \text{ and } y \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right).$$

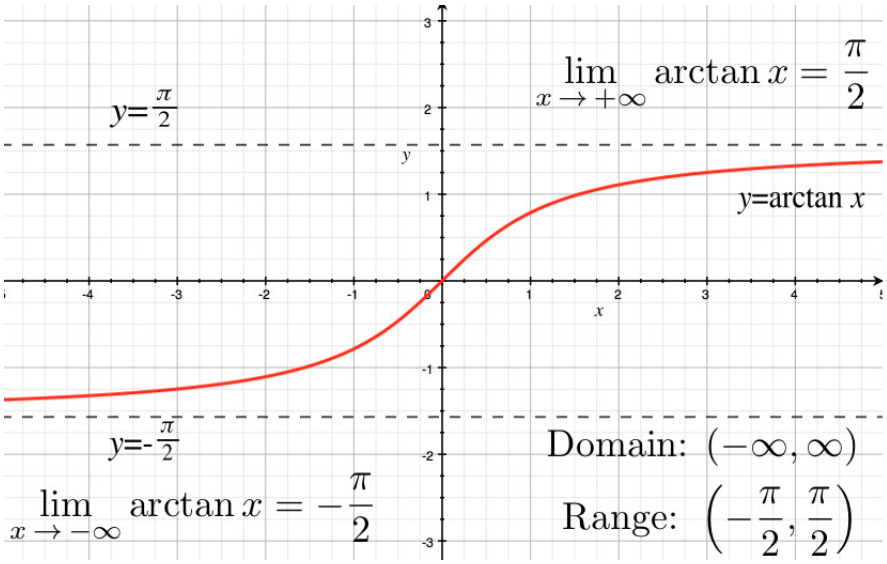
The function arctan has thus domain $(-\infty, \infty)$ and range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Therefore

$$\begin{aligned} \arctan(\tan x) &= x \text{ for all } x \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) \\ \tan(\arctan x) &= x \text{ for all } x. \end{aligned}$$

Graph of arctangent:

The graph is obtained by reflection of that of the restriction of the tangent function to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ about $y = x$:



Note in particular

$$\begin{aligned} \lim_{x \to -\infty} \arctan x &= -\frac{\pi}{2} \\ \lim_{x \to \infty} \arctan x &= \frac{\pi}{2}. \end{aligned}$$

PROPOSITION 2.4.5. *The function* arctan *is differentiable on* $(-\infty, \infty)$ *and*

$$(\arctan x)' = \frac{1}{1+x^2}.$$

In other words

$$\int \frac{dx}{1+x^2} = \arctan x + C. \tag{2.4.1}$$

2.5 Inverse trig functions: derivative and integrals

Watch the video at

https://www.youtube.com/watch?v=LlaI5eOMf7c&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=17

Abstract

This video goes over examples of derivatives and integrals involving inverse trigonometric functions.

Example 2.5.1. Differentiate:

1) $y = \sqrt{\arctan x}$;
Solution. Using the Chain Rule,
$$\frac{dy}{dx} = \frac{1}{2\sqrt{\arctan x}} \cdot \frac{1}{1+x^2}.$$

2) $f(x) = \sqrt{1-x^2}\arcsin x$;
Solution. Using the Product Rule, and the Chain Rule
$$\begin{aligned} f'(x) &= \frac{-2x}{2\sqrt{1-x^2}}\arcsin x + \sqrt{1-x^2}\cdot\frac{1}{\sqrt{1-x^2}} \\ &= 1 - \frac{x\arcsin x}{\sqrt{1-x^2}}. \end{aligned}$$

3) $g(x) = x\ln(\arctan x)$;
Solution. Using the Product Rule, and the Chain Rule
$$g'(x) = \ln(\arctan x) + \frac{x}{\arctan x}\cdot\frac{1}{1+x^2}.$$

4) $h(x) = \arccos(e^x)$.
Solution. Using the Chain Rule
$$h'(x) = -\frac{e^x}{\sqrt{1-(e^x)^2}} = -\frac{e^x}{\sqrt{1-e^{2x}}}.$$

Using substitution and the formula (2.4.1), we establish

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C. \tag{2.5.1}$$

EXAMPLE 2.5.2. Evaluate:

1) $\int \frac{4}{4+x^2}\,dx$;
Solution. Using (2.5.1),

$$\int \frac{4}{4+x^2}\,dx = \frac{4}{2}\arctan\left(\frac{x}{2}\right) + C = 2\arctan\left(\frac{x}{2}\right) + C.$$

2) $\int_0^1 \frac{4}{1+t^2}\,dt$;
Solution.

$$\int_0^1 \frac{4}{1+t^2}\,dt = 4\,[\arctan t]_0^1 = 4\cdot\frac{\pi}{4} = \pi.$$

3) $\int \frac{dt}{\sqrt{1-4t^2}}$;
Solution. Since $4t^2 = (2t)^2$, we let $u = 2t$; then $du = 2\,dt$ and

$$\int \frac{dt}{\sqrt{1-4t^2}} = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\arcsin u + C = \frac{1}{2}\arcsin(2t) + C.$$

4) $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx$;
Solution. Let $u = \cos x$. Then $du = -\sin x\,dx$ and

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx &= -\int_1^0 \frac{du}{1+u^2} \\ &= \int_0^1 \frac{du}{1+u^2} = [\arctan u]_0^1 = \frac{\pi}{4}.\end{aligned}$$

5) $\int \frac{\arctan x}{1+x^2}\,dx$;
Solution. Let $u = \arctan x$. Then $du = \frac{dx}{1+x^2}$ and

$$\int \frac{\arctan x}{1+x^2}\,dx = \int u\,du = \frac{u^2}{2} + C = \frac{(\arctan x)^2}{2} + C.$$

6) $\int \frac{e^{2x}}{\sqrt{1-e^{4x}}}\,dx$.
Solution. Since $e^{4x} = \left(e^{2x}\right)^2$, we let $u = e^{2x}$. Then $du = 2e^{2x}\,dx$, and

$$\int \frac{e^{2x}}{\sqrt{1-e^{4x}}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\arcsin u + C = \frac{1}{2}\arcsin(e^{2x}) + C.$$

Exercises

you are now prepared to work on the Practice Problems, and Homework set M2B in the manual of exercises.

2.6 Hyperbolic functions

Watch the video at

https://www.youtube.com/watch?v=DTnb2YoM8Q8&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=18

Abstract

In this video, we introduce hyperbolic functions, explore some basic properties, differentiate a number of functions defined in terms of hyperbolic functions.

Definition 2.6.1. The *hyperbolic sine function,* denoted sinh, is defined on $(-\infty, \infty)$ by

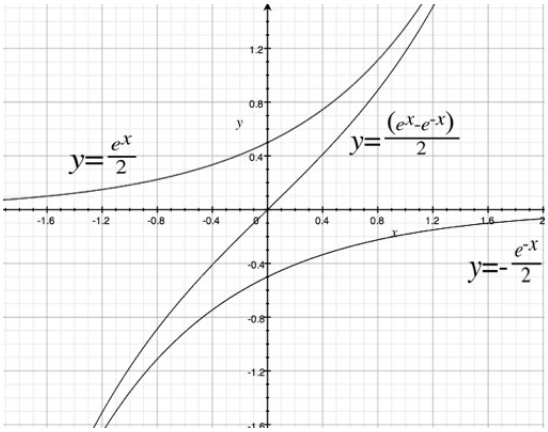
$$\sinh x := \frac{e^x - e^{-x}}{2}.$$

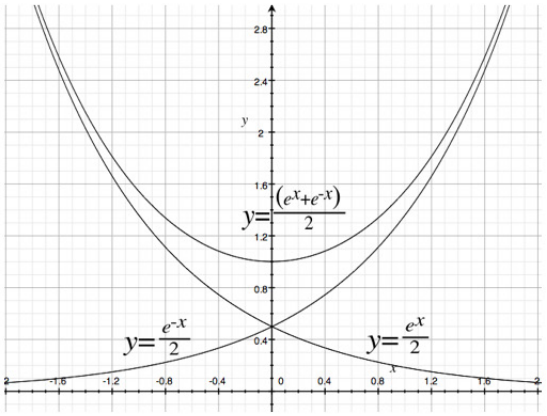
Its range is also $(-\infty, \infty)$.

The *hyperbolic cosine function,* denoted cosh, is defined on $(-\infty, \infty)$ by

$$\cosh x := \frac{e^x + e^{-x}}{2}.$$

Its range is $[1, \infty)$.





Note that sinh is odd, that is,

$$\sinh(-x) = -\sinh x \text{ for all } x$$

and that cosh is even, that is,

$$\cosh(-x) = \cosh x \text{ for all } x.$$

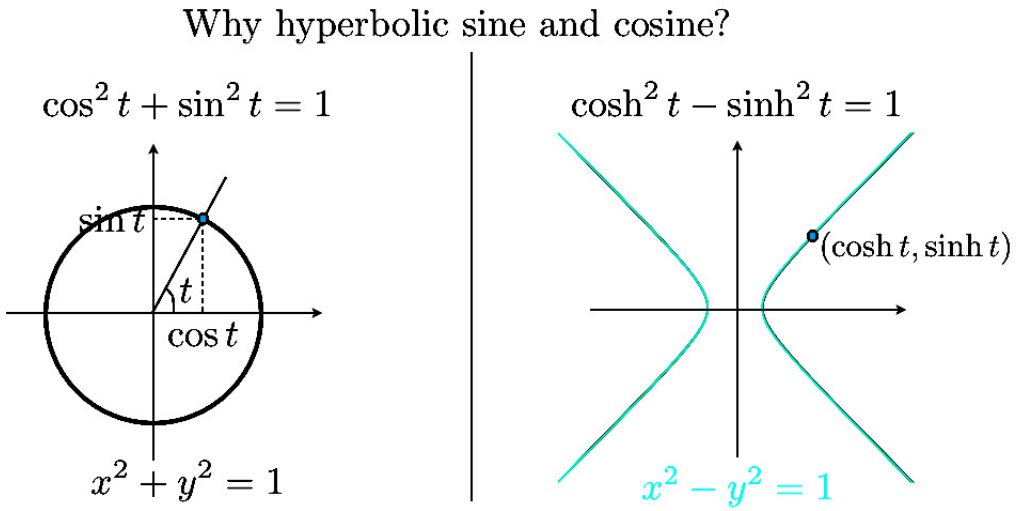
This terminology of hyperbolic sine and cosine is justified among other things by the observation that the trigonometric identity

$$\cos^2 t + \sin^2 t = 1$$

means that a point of the plane of coordinates $(\cos t, \sin t)$ lies on the unit circle $x^2 + y^2 = 1$. The corresponding identity

$$\cosh^2 t - \sinh^2 t = 1$$

means that a point of the plane of coordinates $(\cosh t, \sinh t)$ lies on the hyperbola $x^2 - y^2 = 1$.



The similarities between trigonometric and hyperbolic functions include identities. For instance:

$$\begin{aligned}
\cos(x+y) &= \cos x \cos y - \sin x \sin y \quad \text{vs.} \quad \cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y \\
\sin(x+y) &= \sin x \cos y + \cos x \sin y \quad \text{vs.} \quad \sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y \\
\sin(2x) &= 2 \sin x \cos x \quad \text{vs.} \quad \sinh(2x) = 2 \sinh x \cosh x
\end{aligned}$$

and

$$\begin{aligned}
\cos(2x) &= \cos^2 x - \sin^2 x \quad \text{vs.} \quad \cosh(2x) = \cosh^2 x + \sinh^2 x \\
&= 2\cos^2 x - 1 \qquad\qquad = 2\cosh^2 x - 1 \\
&= 1 - 2\sin^2 x \qquad\qquad = 1 + 2\sinh^2 x.
\end{aligned}$$

We define other hyperbolic functions in terms of cosh and sinh in a similar fashion to what is done for trigonometric functions, thus defining hyperbolic versions of tangent, cotangent, secant, and cosecant.

$$\begin{aligned}
\tanh x &:= \frac{\sinh x}{\cosh x} \\
\coth x &:= \frac{\cosh x}{\sinh x} \\
\operatorname{sech} x &:= \frac{1}{\cosh x} \\
\operatorname{csch} x &:= \frac{1}{\sinh x}.
\end{aligned}$$

THEOREM 2.6.2. cosh *and* sinh *are differentiable on the real line and*

$$\begin{aligned}
(\cosh x)' &= \sinh x \\
(\sinh x)' &= \cosh x.
\end{aligned}$$

Corollary 2.6.3. tanh, coth, sech, *and* csch *are differentiable on their domains and*

$$\begin{aligned}(\tanh x)' &= \operatorname{sech}^2 x \\ &= 1-\tanh^2 x \\ (\coth x)' &= -\operatorname{csch}^2 x \\ &= 1-\coth^2 x \\ (\operatorname{sech} x)' &= -\sinh x \operatorname{sech}^2 x \\ (\operatorname{csch} x)' &= -\cosh x \operatorname{csch}^2 x.\end{aligned}$$

Example 2.6.4. Differentiate

1) $f(x)=\sinh^2 x$

Solution.

$$f'(x)=2\sinh x\cosh x.$$

2) $f(t)=\ln(\sinh t)$

Solution.

$$f'(t)=\frac{\cosh t}{\sinh t}=\coth t.$$

3) $y=\sinh(\cosh x)$

Solution.

$$\frac{dy}{dx}=\cosh(\cosh x)\cdot\sinh x.$$

4) $h(x)=\tanh(4x)$

Solution.

$$h'(x)=4\operatorname{sech}^2(4x).$$

5) $u(t)=\coth\left(\sqrt{1+t^2}\right)$

Solution.

$$\begin{aligned}u'(t) &= -\operatorname{csch}^2\left(\sqrt{1+t^2}\right)\cdot\left(\sqrt{1+t^2}\right)' \\ &= -\operatorname{csch}^2\left(\sqrt{1+t^2}\right)\cdot\frac{2t}{2\sqrt{1+t^2}} \\ &= -\frac{t\operatorname{csch}^2\left(\sqrt{1+t^2}\right)}{\sqrt{1+t^2}}.\end{aligned}$$

2.7 Inverse hyperbolic functions

Watch the video at

https://www.youtube.com/watch?v=KO70S6dayk0&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=19

Abstract

In this video, we introduce the inverse hyperbolic sine and cosine functions, study their basic properties, and use them to calculate integrals.

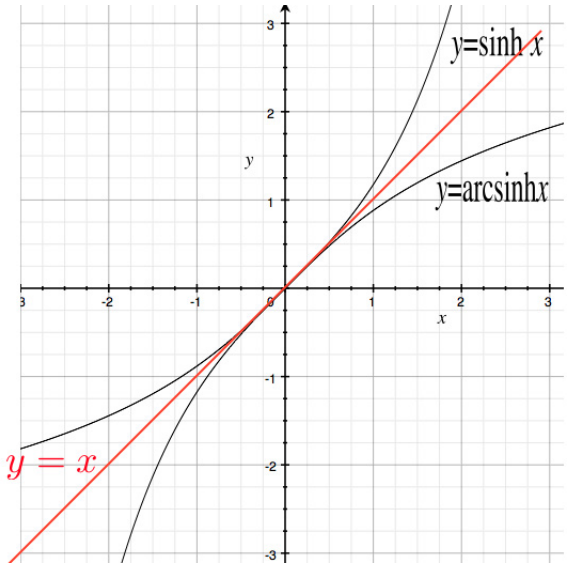
Definition 2.7.1. The *hyperbolic arcsine function* or *inverse hyperbolic sine function*, denoted $\operatorname{arcsinh}$ is defined by

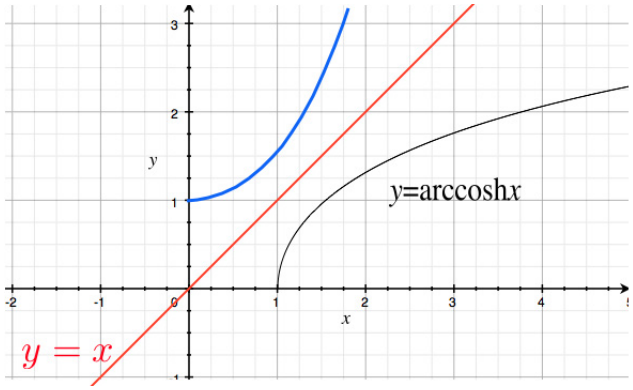
$$y = \operatorname{arcsinh} x \iff \sinh y = x$$

for all x in $(-\infty, \infty)$.

The *hyperbolic arccosine function* or *inverse hyperbolic cosine function,* denoted arccosh is the inverse function of the restriction of cosh to $[0, \infty)$. Thus, it has domain $[1, \infty)$ and range $[0, \infty)$, and

$$y = \operatorname{arccosh} x,\ x \geq 1 \iff \cosh y = x,\ y \geq 0.$$





THEOREM 2.7.2. *For all x in* $(-\infty, \infty)$

$$\operatorname{arcsinh} x = \ln\left(x + \sqrt{x^2+1}\right),$$

and for all $x \geq 1$

$$\operatorname{arccosh} x = \ln\left(x + \sqrt{x^2-1}\right).$$

COROLLARY 2.7.3. *For all x in* $(-\infty, \infty)$

$$(\operatorname{arcsinh} x)' = \frac{1}{\sqrt{x^2+1}},$$

and for all $x > 1$,

$$(\operatorname{arccosh} x)' = \frac{1}{\sqrt{x^2-1}}.$$

In integral form, this rephrases as

$$\int \frac{dx}{\sqrt{x^2+1}} = \operatorname{arcsinh} x + C \tag{2.7.1}$$

$$\int \frac{dx}{\sqrt{x^2-1}} = \operatorname{arccosh} x + C. \tag{2.7.2}$$

EXAMPLE 2.7.4. Evaluate the following integrals:

1) $\int_0^1 \frac{dt}{\sqrt{16t^2+9}}$
Solution. Let $u = \frac{4t}{3}$ in

$$\begin{aligned}\int_0^1 \frac{dt}{\sqrt{16t^2+9}} &= \int_0^1 \frac{dt}{3\sqrt{\left(\frac{4t}{3}\right)^2+1}} \\ &= \frac{1}{3}\cdot\frac{3}{4}\int_0^{\frac{4}{3}} \frac{du}{\sqrt{u^2+1}} \\ &= \frac{1}{4}\left[\operatorname{arcsinh} u\right]_0^{\frac{4}{3}} \\ &= \frac{1}{4}\operatorname{arcsinh}\frac{4}{3}.\end{aligned}$$

2) $\int \frac{dx}{\sqrt{x^2-4}}$
Solution. Let $u = \frac{x}{2}$ in

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2-4}} &= \int \frac{dx}{2\sqrt{\left(\frac{x}{2}\right)^2-1}} \\ &= \frac{2}{2}\int \frac{du}{\sqrt{u^2-1}} \\ &= \operatorname{arccosh} u + C \\ &= \operatorname{arccosh}\frac{x}{2} + C.\end{aligned}$$

Exercises

you are now prepared to work on the Practice Problems, and Homework set M2C in the manual of exercises.

3 M3: Rule of De l'Hospital

3.1 Rule of De L'Hospital: statement and proof

Watch the video at

https://www.youtube.com/watch?v=3JFFOyO2HwE&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=20

Abstract

This video states and proves the Rule of De l'Hospital.

The Rule of De l'Hospital is a tool to calculate limits that are indeterminate of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

THEOREM 3.1.1 (Rule of De l'Hospital) *Let f and g be two functions differentiable on an open interval centered at a, except possibly at a. If*

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \text{ or } \lim_{x\to a} f(x) = \pm\infty \text{ and } \lim_{x\to a} g(x) = \pm\infty,$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

if $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, or is infinite.

Additionally, the statement remains valid if a is replace by a^+, a^-, $-\infty$ or $+\infty$.

REMARK 3.1.2. We will use the symbol $\overset{H}{=}$ to indicate that two limits are equal by the Rule of De l'Hospital.

EXAMPLE 3.1.3.

$$\begin{aligned}
\lim_{x\to 1} \frac{\ln x}{x-1} &\overset{H}{=} \lim_{x\to 1} \frac{\frac{1}{x}}{1} = 1 \\
\lim_{x\to 0} \frac{2x+\sin x}{x} &\overset{H}{=} \lim_{x\to 0} \frac{2+\cos x}{1} = 3 \\
\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} &\overset{H}{=} \lim_{x\to 0} \frac{\frac{1}{2\sqrt{1+x}}}{1} = \frac{1}{2} \\
\lim_{x\to \infty} \frac{x^2-1}{2x^2-1} &\overset{H}{=} \lim_{x\to \infty} \frac{2x}{4x} \overset{H}{=} \lim_{x\to \infty} \frac{2}{4} = \frac{1}{2} \\
\lim_{x\to \infty} \frac{\ln x}{x} &\overset{H}{=} \lim_{x\to \infty} \frac{\frac{1}{x}}{1} = 0.
\end{aligned}$$

The proof of the Rule of De l'Hospital makes use of the following generalization of the Mean Value Theorem:

THEOREM 3.1.4 (Cauchy Mean Value Theorem). *Let f and g be two functions that are continuous on $[a,b]$ and differentiable on (a,b) with $g'(x) \neq 0$ on (a,b), and $g(b) \neq g(a)$. Then there exists (at least) a c in (a,b) with*

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

REMARK 3.1.5. The classical Mean Value Theorem corresponds to $g(x) = x$.

3.2 Rule of de l'Hospital: examples (quotients)

Watch the video at

https://www.youtube.com/watch?v=bRbltAEO0kk&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=21

Abstract

This video goes over standard examples of use of the Rule of De l'Hospital to evaluate limits of quotients.

Example 3.2.1.

$$\begin{aligned}
\lim_{x\to 1}\frac{x^2+x-2}{\ln x} &\overset{H}{=} \lim_{x\to 1}\frac{2x+1}{\frac{1}{x}} = 3\\
\lim_{t\to 0}\frac{e^{3t}-1}{t} &\overset{H}{=} \lim_{t\to 0}\frac{3e^{3t}}{1} = 3\\
\lim_{x\to 0}\frac{x+\tan x}{\sin x} &\overset{H}{=} \lim_{x\to 0}\frac{1+\sec^2 x}{\cos x} = 2\\
\lim_{x\to\infty}\frac{e^x}{x} &\overset{H}{=} \lim_{x\to\infty}\frac{e^x}{1} = \infty\\
\lim_{x\to 0}\frac{x+\sin x}{x+\cos x} &= \frac{0}{1} = 0 \text{ is not an indeterminate form,}
\end{aligned}$$

and thus the Rule of De l'Hospital does not apply. Applying (erroneously!) the rule in this case would result in the incorrect statement that

$$\lim_{x\to 0}\frac{x+\sin x}{x+\cos x} \overset{H}{=} \lim_{x\to 0}\frac{1+\cos x}{1-\sin x} = 2 \neq 0!$$

At times, the Rule of De l'Hospital needs to be iterated to reach a conclusion:

Example 3.2.2.

$$\begin{aligned}
\lim_{x\to\infty}\frac{x^2}{e^x} &\overset{H}{=} \lim_{x\to\infty}\frac{2x}{e^x} \overset{H}{=} \lim_{x\to\infty}\frac{2}{e^x} = 0\\
\lim_{x\to 0}\frac{\sin x - x}{x^3} &\overset{H}{=} \lim_{x\to 0}\frac{\cos x - 1}{3x^2} \overset{H}{=} \lim_{x\to 0}\frac{-\sin x}{6x} \overset{H}{=} \lim_{x\to 0}\frac{-\cos x}{6} = -\frac{1}{6}.
\end{aligned}$$

3.3 Rule of De L'Hospital: indeterminate products

Watch the video at

https://www.youtube.com/watch?v=C73cThFNg18&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=22

Abstract

This video goes over a number of examples where a limit that is indeterminate of the type $0 \cdot \infty$ is evaluated using the Rule of De l'Hospital.

Products where one factor has limit 0 and the other has an infinite limit are indeterminate forms of the type

$$0 \cdot \infty.$$

While Theorem 3.1.1 formally only applies to indeterminate forms of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$, a product of the type $0 \cdot \infty$ can easily be transformed into a quotient $\frac{0}{0}$ or $\frac{\infty}{\infty}$ because

$$\begin{aligned} \lim_{x\to a} f(x) = 0 \quad &\Longrightarrow \quad \lim_{x\to a} \frac{1}{f(x)} = \pm\infty \\ \lim_{x\to a} f(x) = \pm\infty \quad &\Longrightarrow \quad \lim_{x\to a} \frac{1}{f(x)} = 0 \end{aligned}$$

and

$$f(x) \cdot g(x) = \frac{g(x)}{\frac{1}{f(x)}}.$$

Example 3.3.1.

$$\begin{aligned} \lim_{x\to 0^+} x \ln x &= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \overset{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0 \\ \lim_{x\to -\infty} x^2 e^x &= \lim_{x\to -\infty} \frac{x^2}{e^{-x}} \overset{H}{=} \lim_{x\to -\infty} \frac{2x}{-e^{-x}} \overset{H}{=} \lim_{x\to -\infty} \frac{2}{e^{-x}} = 0 \\ \lim_{x\to 0^+} \sin x \ln x &= \lim_{x\to 0^+} \frac{\ln x}{\csc x} \overset{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\cos x \csc^2 x} \\ &= \lim_{x\to 0^+} -\frac{\sin^2 x}{x \cos x} \\ &\overset{H}{=} \lim_{x\to 0^+} -\frac{2 \sin x \cos x}{\cos x - x \sin x} = -\frac{0}{1} = 0. \end{aligned}$$

3.4 Rule of De L'Hospital: indeterminate powers

Watch the video at

https://www.youtube.com/watch?v=hx_ylpT7M7E&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=23

Abstract

This video outlines, with examples, how to evaluate indeterminate limits of the form 0^0, ∞^0 and 1^∞.

Consider limits of the form

$$\lim_{x\to a}(f(x))^{g(x)}.$$

$$\begin{aligned}
\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 &\implies \text{indeterminate form of the type } 0^0\\
\lim_{x\to a} f(x) = \infty \text{ and } \lim_{x\to a} g(x) = 0 &\implies \text{indeterminate form of the type } \infty^0\\
\lim_{x\to a} f(x) = 1 \text{ and } \lim_{x\to a} g(x) = \infty &\implies \text{indeterminate form of the type } 1^\infty,
\end{aligned}$$

but all 3 types of indeterminate forms stem from the fact that in

$$(f(x))^{g(x)} = e^{g(x)\ln(f(x))},$$

the product $g(x)\ln(f(x))$ is an indeterminate form of the type $0\cdot\infty$, which we have seen how to treat with the Rule of De l'Hospital in the previous section.

Example 3.4.1. Evaluate

1) $\lim_{x\to 0^+}(\tan(2x))^x$;
Solution. Since

$$(\tan(2x))^x = e^{x\ln(\tan(2x))},$$

we only need to find

$$\begin{aligned}\lim_{x\to 0^+} x\ln(\tan(2x)) &= \lim_{x\to 0^+}\frac{\ln(\tan(2x))}{\frac{1}{x}} \overset{H}{=} \lim_{x\to 0^+}\frac{\frac{2\sec^2(2x)}{\tan(2x)}}{-\frac{1}{x^2}}\\ &= \lim_{x\to 0^+} -\frac{2x^2}{\cos(2x)\sin(2x)}\\ &= \lim_{x\to 0^+} -\frac{4x^2}{\sin(4x)} \text{ because } \sin(4x) = 2\cos(2x)\sin(2x)\\ &\overset{H}{=} \lim_{x\to 0^+} -\frac{8x}{4\cos(4x)} = 0.\end{aligned}$$

Thus

$$\lim_{x\to 0^+}(\tan(2x))^x = e^{\lim_{x\to 0^+} x\ln(\tan(2x))} = e^0 = 1.$$

2) $\lim_{x\to\infty}(e^x+x)^{\frac{1}{x}}$;
Solution. Since

$$(e^x+x)^{\frac{1}{x}} = e^{\frac{1}{x}\ln(e^x+x)},$$

we only need to find

$$\lim_{x\to\infty}\frac{\ln(e^x+x)}{x} \overset{H}{=} \lim_{x\to\infty}\frac{\frac{e^x+1}{e^x+x}}{1} \overset{H}{=} \lim_{x\to\infty}\frac{e^x}{e^x+1} \overset{H}{=} \lim_{x\to\infty}\frac{e^x}{e^x} = 1.$$

Thus

$$\lim_{x\to\infty}(e^x+x)^{\frac{1}{x}} = e^{\lim_{x\to\infty}\frac{\ln(e^x+x)}{x}} = e^1 = e.$$

3) $\lim_{x\to\infty}\left(x+\frac{1}{x}\right)^{\frac{1}{x}}$.

Solution. Since

$$\left(x+\frac{1}{x}\right)^{\frac{1}{x}}=e^{\frac{1}{x}\ln\left(x+\frac{1}{x}\right)},$$

we only need to find

$$\begin{aligned}\lim_{x\to\infty}\frac{\ln\left(x+\frac{1}{x}\right)}{x} &\overset{H}{=} \lim_{x\to\infty}\frac{\frac{1-\frac{1}{x^2}}{x+\frac{1}{x}}}{1}\\ &= \lim_{x\to\infty}\frac{x^2-1}{x^2+1}\cdot\frac{x}{x^2}=1\cdot 0=0.\end{aligned}$$

Thus

$$\lim_{x\to\infty}\left(x+\frac{1}{x}\right)^{\frac{1}{x}}=e^{\lim_{x\to\infty}\frac{1}{x}\ln\left(x+\frac{1}{x}\right)}=e^0=1.$$

Exercises

you are now prepared to work on the Practice Problems, and Homework set M3 in the manual of exercises.

4 M4: Integration review and Integration by parts

4.1 Review of Integration: basics and completing the square

Watch the video at

https://www.youtube.com/watch?v=W88xMl9Mv28&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=24

Abstract

This video reviews basic integration formulas and explore examples where completing the square can help integrating.

Basic formulas and substitution

It is a corollary of the Fundamental Theorem of Calculus that if f is continuous on $[a, b]$ and F is an antiderivative of f on $[a, b]$ then

$$\begin{aligned} \int_a^b f(x)\,dx &= F(b) - F(a) \\ &:= [F(x)]_a^b . \end{aligned}$$

Thus, to calculate the definite integral of a continuous function over an interval, we only need to find an antiderivative of this function, that is, its indefinite integral $\int f(x)\,dx$.

Basic tools in this process are the following formulas:

$$\begin{aligned}
\int f(x)+g(x)\,dx &= \int f(x)\,dx+\int g(x)\,dx\\
\int c\cdot f(x)\,dx &= c\cdot\int f(x)\,dx \text{ for any constant } c\\
\int x^n\,dx &= \frac{x^{n+1}}{n+1}+C, \text{ for any } n\neq -1\\
\int \frac{dx}{x} &= \ln|x|+C\\
\int \cos x\,dx &= \sin x+C\\
\int \sin x\,dx &= -\cos x+C\\
\int \sec^2 x\,dx &= \tan x+C\\
\int e^x\,dx &= e^x+C\\
\int a^x\,dx &= \frac{a^x}{\ln a}+C\ (a>0,\ a\neq 1)\\
\int \frac{dx}{\sqrt{1-x^2}} &= \arcsin x+C\\
\int \frac{dx}{1+x^2} &= \arctan x+C
\end{aligned}$$

Integration by substitution is based on reversing the Chain Rule for differentiation:

$$\int f(g(x))\,g'(x)\,dx \overset{u=g(x)}{=} \int f(u)\,du,$$

and, for definite integrals

$$\int_a^b f(g(x))\,g'(x)\,dx \overset{u=g(x)}{=} \int_{g(a)}^{g(b)} f(u)\,du.$$

Using simple substitutions, we establish, for a fixed non-zero constant *k*:

$$\begin{aligned}
\int \sin(kx)\,dx &= -\frac{1}{k}\cos(kx)+C\\
\int \cos(kx)\,dx &= \frac{1}{k}\sin(kx)+C\\
\int e^{kx}\,dx &= \frac{1}{k}e^{kx}+C\\
\int \frac{dx}{k^2+x^2} &= \frac{1}{k}\arctan\left(\frac{x}{k}\right)+C.
\end{aligned}$$

EXAMPLE 4.1.1. Evaluate

1) $\int \frac{4x+3}{\sqrt{2x^2+3x-1}}\,dx$

Solution. Let $u = 2x^2 + 3x - 1$. Then $du = 4x + 3\,dx$ so that

$$\int \frac{4x+3}{\sqrt{2x^2+3x-1}}\,dx = \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{2x^2+3x-1} + C.$$

2) $\int \frac{dx}{x-\sqrt{x}}$

Solution. Note that

$$\int \frac{dx}{x-\sqrt{x}} = \int \frac{dx}{\sqrt{x}\,(\sqrt{x}-1)}$$

and that $(\sqrt{x}-1)' = \frac{1}{2\sqrt{x}}$, so that, letting $u = \sqrt{x} - 1$, we have $2du = \frac{dx}{\sqrt{x}}$ and

$$\int \frac{dx}{x-\sqrt{x}} = 2\int \frac{du}{u} = 2\ln|u| + C = 2\ln|\sqrt{x}-1| + C.$$

Completing the square

Completing the square can be useful in changing the form of an integral into one that can be handled with basic techniques described above.

Recall that to *complete the square* in $ax^2 + bx + c$, you first factor out the leading coefficient and then use the fact that

$$x^2 + 2ax = (x + a)^2 - a^2 \tag{4.1.1}$$

Example 4.1.2. Evaluate

1) $\int \frac{dx}{\sqrt{8x-x^2}}$

Solution. Note that by completing the square in the quadratic function $8x - x^2$, we have:

$$8x - x^2 = -(x^2 - 8x) = -((x-4)^2 - 16) = 16 - (x-4)^2.$$

Thus

$$\int \frac{dx}{\sqrt{8x - x^2}} = \int \frac{dx}{\sqrt{16 - (x-4)^2}} = \frac{1}{4}\int \frac{dx}{\sqrt{1 - \left(\frac{x-4}{4}\right)^2}}.$$

Letting $u = \frac{x}{4} - 1$, we have $du = \frac{1}{4}\,dx$ and

$$\int \frac{dx}{\sqrt{8x - x^2}} = \int \frac{du}{\sqrt{1 - u^2}} = \arcsin u + C = \arcsin\left(\frac{x}{4} - 1\right) + C.$$

2) $\int_2^4 \frac{2\,dx}{x^2-6x+10}$

Solution. Note that by completing the square in the quadratic function $x^2 - 6x + 10$, we have:

$$x^2 - 6x + 10 = (x-3)^2 - 9 + 10 = (x-3)^2 + 1,$$

so that

$$\int_2^4 \frac{2\,dx}{x^2 - 6x + 10} = \int_2^4 \frac{2\,dx}{1 + (x-3)^2}.$$

Let $u = x - 3$ so that $du = dx$, and we have

$$\int_2^4 \frac{2\,dx}{1 + (x-3)^2} = 2\int_{-1}^1 \frac{du}{1 + u^2} = 2\left[\arctan u\right]_{-1}^1 = 2\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right) = \pi.$$

4.2 Review of Integration: trig formulas and manipulating fractions

Watch the video at

https://www.youtube.com/watch?v=rGUSiPQ8Aqw&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=25

Abstract

This video illustrates on examples how trig formulas or algebra on fractions can change the form of an integral into one that can be handled with basic techniques.

Trigonometric identities

There are many trigonometric identities that can be useful, but one of the most commonly used is the *double angle formula:*

$$\begin{aligned} \cos(2x) &= 2\cos^2 x - 1 \\ &= 1 - 2\sin^2 x \\ &= \cos^2 x - \sin^2 x \\ \sin(2x) &= 2\sin x \cos x. \end{aligned} \tag{4.2.1}$$

Example 4.2.1. Evaluate

1) $\int_0^{\frac{\pi}{4}} \sqrt{1+\cos(4x)}\,dx$

Solution. Note that, using the double angle formula,

$$\cos(4x) = 2\cos^2(2x) - 1,$$

so that

$$\int_0^{\frac{\pi}{4}} \sqrt{1+\cos(4x)}\,dx = \int_0^{\frac{\pi}{4}} \sqrt{2\cos^2(2x)}\,dx = \sqrt{2}\int_0^{\frac{\pi}{4}} |\cos(2x)|\,dx.$$

Moreover, $0 \le x \le \frac{\pi}{4}$ so that $0 \le 2x \le \frac{\pi}{2}$ and $\cos(2x) \ge 0$ on $[0, \frac{\pi}{4}]$. Thus,

$$\int_0^{\frac{\pi}{4}} \sqrt{1+\cos(4x)}\,dx = \sqrt{2}\int_0^{\frac{\pi}{4}} \cos 2x\,dx = \sqrt{2}\left[\frac{1}{2}\sin(2x)\right]_0^{\frac{\pi}{4}} = \frac{\sqrt{2}}{2}.$$

2) $\int_0^{\pi} \sqrt{1-\cos x}\, dx$

Solution. Note that, using the double angle formula,

$$\cos x = 1 - 2\sin^2\left(\frac{x}{2}\right),$$

so that

$$\int_0^{\pi} \sqrt{1-\cos x}\, dx = \int_0^{\pi} \sqrt{2\sin^2\left(\frac{x}{2}\right)}\, dx = \sqrt{2}\int_0^{\pi} \left|\sin\left(\frac{x}{2}\right)\right| dx.$$

Since $0 \leq \frac{x}{2} \leq \frac{\pi}{2}$ if x in $[0,\pi]$, $\sin\left(\frac{x}{2}\right) \geq 0$ on this interval and

$$\int_0^{\pi} \sqrt{1-\cos x}\, dx = \sqrt{2}\int_0^{\pi} \sin\left(\frac{x}{2}\right) dx = \sqrt{2}\left[-2\cos\left(\frac{x}{2}\right)\right]_0^{\pi} = 2\sqrt{2}.$$

Transforming rational functions

The first observation is that if in a quotient of two polynomials the degree of the numerator is not less than that of the denominator, then the fraction can be simplified by *long division*, which is often useful to integrate. If you need to brush up on long division you can start at 1.50 into this video.

EXAMPLE 4.2.2. Evaluate

1) $\int \frac{3x^2-7x}{3x+2}\,dx$.

Solution. By long division, we obtain:

$$\begin{array}{r|l} & x-3 \\ \hline 3x+2 & 3x^2-7x \\ & \underline{-3x^2-2x} \\ & -9x \\ & \underline{9x+6} \\ & 6 \end{array}$$

so that

$$\frac{3x^2-7x}{3x+2} = x-3+\frac{6}{3x+2}.$$

Thus,

$$\int \frac{3x^2-7x}{3x+2}\,dx = \int x-3\,dx+6\int \frac{dx}{3x+2}.$$

Let $u=3x+2$, so that $du=3\,dx$, and

$$\begin{aligned} \int \frac{3x^2-7x}{3x+2}\,dx &= \int x-3\,dx+2\int \frac{du}{u} \\ &= \frac{x^2}{2}-3x+2\ln|3x+2|+C. \end{aligned}$$

2) $\int \frac{x}{x+1}\,dx$.

Solution. Long division in this case can be performed easily with the following trick:

$$\int \frac{x}{x+1}\,dx = \int \frac{(x+1)-1}{x+1}\,dx = \int 1-\frac{1}{x+1}\,dx = x-\ln|x+1|+C.$$

Sometimes, splitting a fraction in pieces is enough to obtain terms that can be integrated.

3) Evaluate $\int \frac{3x+2}{\sqrt{1-x^2}}\,dx$.
Solution. Note that

$$\int \frac{3x+2}{\sqrt{1-x^2}}\,dx = 3\int \frac{x}{\sqrt{1-x^2}}\,dx + 2\int \frac{dx}{\sqrt{1-x^2}}.$$

Let $u = 1-x^2$, so that $du = -2x\,dx$ and

$$\begin{aligned}\int \frac{3x+2}{\sqrt{1-x^2}}\,dx &= -\frac{3}{2}\int \frac{du}{\sqrt{u}} + 2\int \frac{dx}{\sqrt{1-x^2}}.\\ &= -3\sqrt{u} + 2\arcsin x + C\\ &= -3\sqrt{1-x^2} + 2\arcsin x + C.\end{aligned}$$

Exercises

you are now prepared to work on the Practice Problems, and Homework set M4A in the manual of exercises.

4.3 Integration by parts: indefinite integrals

Watch the videos at

https://www.youtube.com/watch?v=XwCJrJkhJP0&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=26

and

https://www.youtube.com/watch?v=_H4R2oSvujI&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL

Abstract

These videos establish the integration by parts formula for indefinite integrals and goes over a number of examples.

Turning around the Product Rule for derivatives yields the *integration by parts formula*

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

which is often more compactly written, with $u = f(x)$ and $dv = g'(x)\,dx$:

$$\int u\,dv = uv - \int v\,du. \tag{4.3.1}$$

Remark 4.3.1. When using integration by parts, you have to pick what part is interpreted as *u* and what part is interpreted as *dv*. The rule of thumb is that you chose so that $\int v\,du$ is simpler than the original integral. In other words, the derivative of *u* should not be more complicated than *u* and the integral of *dv* should not be more complicated than *dv*. This is, of course, not an absolute rule but rather a rough guideline.

Example 4.3.2. Evaluate

1) $\int x \sin x\,dx$

Solution. We pick[2] $u = x$ and $dv = \sin x\,dx$ so that $du = dx$ and $v = -\cos x$. Thus, by (4.3.1),

$$\int x \sin x\,dx = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C.$$

2) $\int xe^{-x}\,dx$

Solution. We pick $u = x$ and $dv = e^{-x}\,dx$ so that $du = dx$ and $v = -e^{-x}$. Thus, by (4.3.1),

$$\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -e^{-x}(x+1) + C.$$

3) $\int \cos x \ln(\sin x)\,dx$

Solution. Let $u = \ln(\sin x)$ and $dv = \cos x\,dx$, so that $du = \frac{\cos x}{\sin x}$ and $v = \sin x$. Thus, by (4.3.1),

$$\begin{aligned}\int \cos x \ln(\sin x)\,dx &= \sin x \ln(\sin x) - \int \sin x \cdot \frac{\cos x}{\sin x}\,dx \\ &= \sin x \ln(\sin x) - \int \cos x\,dx = \sin x\,(\ln(\sin x) - 1) + C.\end{aligned}$$

4) $\int \ln x\,dx$

Solution. Let $u = \ln x$ and $dv = dx$, so that $du = \frac{dx}{x}$ and $v = x$. Thus, by (4.3.1),

$$\int \ln x\,dx = x\ln x - \int x \cdot \frac{dx}{x} = x\ln x - \int dx = x(\ln x - 1) + C.$$

5) $\int \arcsin x\, dx$

Solution. Let $u = \arcsin x$ and $dv = dx$, so that $du = \frac{dx}{\sqrt{1-x^2}}$ and $v = x$. Thus, by (4.3.1),

$$\int \arcsin x\, dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}}\, dx.$$

By substitution with $t = 1 - x^2$, so that $dt = -2x\, dx$, we have

$$\int \arcsin x\, dx = x \arcsin x + \frac{1}{2}\int \frac{dt}{\sqrt{t}} = x \arcsin x + \sqrt{t} + C = x \arcsin x + \sqrt{1-x^2} + C.$$

6) $\int \arctan x\, dx$

Solution. Let $u = \arctan x$ and $dv = dx$, so that $du = \frac{dx}{1+x^2}$ and $v = x$. Thus, by (4.3.1),

$$\int \arctan x\, dx = x \arctan x - \int \frac{x}{1+x^2}\, dx.$$

By substitution with $t = 1 + x^2$, so that $dt = 2x\, dx$, we have

$$\int \arctan x\, dx = x \arctan x - \frac{1}{2}\int \frac{du}{u} = x \arctan x - \frac{1}{2}\ln(1+x^2) + C.$$

7) $\int t^3 e^t\, dt$

Solution. Sometimes, we need to iterate the process of integration by parts. Let $u = t^3$ and $dv = e^t\, dt$. Then $du = 3t^2\, dt$ and $v = e^t$ and

$$\int t^3 e^t\, dt = t^3 e^t - 3\int t^2 e^t\, dt. \tag{4.3.2}$$

To calculate $\int t^2 e^t\, dt$, we proceed by parts with $u = t^2$ and $dv = e^t\, dt$ so that $du = 2t\, dt$ and $v = e^t$ and

$$\int t^2 e^t\, dt = t^2 e^t - 2\int t e^t\, dt. \tag{4.3.3}$$

To calculate $\int te^t\, dt$, we proceed by parts with $u = t$ and $dv = e^t\, dt$, so that $du = dt$ and $v = e^t$. Thus

$$\int te^t\, dt = te^t - \int e^t\, dt = e^t(t-1) + C.$$

We now substitute in (4.3.3), and in turn, back in (4.3.2), to the effect that

$$\begin{aligned}\int t^3 e^t\, dt &= t^3 e^t - 3\left(t^2 e^t - 2e^t(t-1)\right) + C \\ &= e^t\left(t^3 - 3t^2 + 6t - 6\right) + C.\end{aligned}$$

4.4 Integration by parts: definite integrals

Watch the video at

https://www.youtube.com/watch?v=giqPcUV8p48&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=28

Abstract

This video states the integration by parts formula for definite integrals and goes over a few examples.

The counterpart of (4.3.1) for definite integrals is

$$\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du. \tag{4.4.1}$$

Example 4.4.1. Evaluate:

1) $\int_0^1 \left(x^2+1\right) e^{-x}\,dx$

Solution. Let $u = x^2+1$ and $dv = e^{-x}\,dx$. Then $du = 2x\,dx$ and $v = -e^{-x}$, so that, by (4.4.1),

$$\int_0^1 \left(x^2+1\right) e^{-x}\,dx = \left[-(x^2+1)e^{-x}\right]_0^1 + 2\int_0^1 xe^{-x}\,dx. \tag{4.4.2}$$

We calculate $\int_0^1 xe^{-x}\,dx$ by parts with $u = x$ and $dv = e^{-x}\,dx$, so that $du = dx$ and $v = -e^{-x}$, and, by (4.4.1),

$$\int_0^1 xe^{-x}\,dx = \left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x}\,dx = \left[-e^{-x}(x+1)\right]_0^1.$$

Substituting back in (4.4.2), we obtain:

$$\int \left(x^2+1\right) e^{-x}\,dx = \left[-e^{-x}\left(x^2+2x+3\right)\right]_0^1 = -6e^{-1}+3$$

2) $\int_1^4 \sqrt{t}\ln t\,dt$

Solution. Let $u = \ln t$ and $dv = \sqrt{t}\,dt$. Then $du = \frac{dt}{t}$ and $v = \frac{2}{3}t^{\frac{3}{2}}$, so that

$$\begin{aligned}\int_1^4 \sqrt{t}\ln t\,dt &= \left[\frac{2}{3}t^{\frac{3}{2}}\ln t\right]_1^4 - \frac{2}{3}\int_1^4 t^{\frac{3}{2}}\cdot\frac{1}{t}\,dt\\ &= \left[\frac{2}{3}t^{\frac{3}{2}}\ln t\right]_1^4 - \frac{2}{3}\int_1^4 t^{\frac{1}{2}}\,dt\\ &= \left[\frac{2}{3}t^{\frac{3}{2}}\ln t - \frac{4}{9}t^{\frac{3}{2}}\right]_1^4\\ &= \frac{16}{3}\ln 4 - \frac{32}{9} + \frac{4}{9} = \frac{16}{3}\ln 4 - \frac{28}{9}.\end{aligned}$$

3) $\int_0^1 x\,5^x\,dx$

Solution. Let $u = x$ and $dv = 5^x\,dx$. Then $du = dx$ and $v = \frac{5^x}{\ln 5}$ and

$$\begin{aligned}\int_0^1 x\,5^x\,dx &= \left[\frac{x5^x}{\ln 5}\right]_0^1 - \frac{1}{\ln 5}\int_0^1 5^x\,dx\\ &= \left[\frac{x5^x}{\ln 5} - \frac{5^x}{(\ln 5)^2}\right]_0^1 = 5\left(\frac{1}{\ln 5} - \frac{1}{(\ln 5)^2}\right) + \frac{1}{(\ln 5)^2}\\ &= \frac{5}{\ln 5} - \frac{4}{(\ln 5)^2}.\end{aligned}$$

4.5 Integration by parts: one more example

Watch the video at

https://www.youtube.com/watch?v=8W0PFO-MD5I&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=29

Abstract

This video goes over one example of integrals where two successive integrations by parts lead to an equation whose unknown is the desired integral.

Example 4.5.1. Evaluate $\int e^x \sin x \, dx$.

Solution. Let $u = \sin x$ and $dv = e^x \, dx$. Then $du = \cos x \, dx$ and $v = e^x$, so that by integration by parts,

$$\int e^x \sin x \, dx = \sin x e^x - \int e^x \cos x \, dx.$$

Integrating by parts in the new integral with $u = \cos x$ and $dv = e^x \, dx$, we have $du = -\sin x \, dx$ and $v = e^x$ so that

$$\begin{aligned} \int e^x \sin x \, dx &= \sin x e^x - \left(e^x \cos x + \int e^x \sin x \, dx \right) \\ &= e^x (\sin x - \cos x) + \int e^x \sin x \, dx. \end{aligned}$$

Solving for the desired integral $I := \int e^x \sin x \, dx$, we obtain

$$I = \frac{1}{2} e^x (\sin x - \cos x) + C.$$

Exercises

you are now prepared to work on the Practice Problems, and Homework set M4B in the manual of exercises.

Before turning to Chapter 4, you can also take Mock Test 1.

5 M5: Trigonometric integrals and trigonometric substitutions

5.1 Powers of sine and cosine

Watch the video at

https://www.youtube.com/watch?v=Lt9o_TC4F4U&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=30

Abstract

This video goes over the general technique to evaluate integrals of the form

$$\int \cos^m x \, \sin^n x \, dx$$

distinguishing different cases.

Case 1: at least one of the powers is odd

Example 5.1.1. To evaluate $\int \cos^3 x \, dx$, note that

$$\int \cos^3 x \, dx = \int \cos^2 x \, \cos x \, dx = \int (1 - \sin^2 x) \cos x \, dx,$$

so that the substitution $u = \sin x$ gives $du = \cos x \, dx$ and

$$\int \cos^3 x \, dx = \int 1 - u^2 \, du = u - \frac{u^3}{3} + C = \sin x - \frac{\sin^3 x}{3} + C.$$

This approach can be used for the general case

$$\int \cos^m x \, \sin^n x \, dx, \tag{5.1.1}$$

as long as at least one exponent is odd, say $m = 2k + 1$. Then

$$\begin{aligned}\int \cos^m x \, \sin^n x \, dx &= \int \cos^{2k} x \, \sin^n x \, \cos x \, dx \\ &= \int \left(\cos^2 x\right)^k \sin^n x \, (\cos x \, dx) \\ &= \int \left(1 - \sin^2 x\right)^k \sin^n x \, (\cos x \, dx) \\ &= \int \left(1 - u^2\right)^k u^n \, du \text{ for } u = \sin x, \end{aligned} \tag{5.1.2}$$

and the resulting integral is that of a polynomial in u. Note however that to calculate the integral, we first need to multiply things through in the integral. Thus, if both m and n are odd in (5.1.1), it is better to split the smallest power in order to minimize k in (5.1.2).

EXAMPLE 5.1.2. Evaluate $\int \cos^5 x \, \sin^3 x \, dx$.

Solution. Using the rule of thumb above, we "split" the power of sin:

$$\begin{aligned}\int \cos^5 x \, \sin^3 x \, dx &= \int \cos^5 x \, \sin^2 x \, \sin x \, dx \\ &= \int \cos^5 x \left(1 - \cos^2 x\right) \sin x \, dx \\ &= -\int u^5 (1 - u^2) \, du \text{ where } u = \cos x \\ &= \int u^7 - u^5 \, du \\ &= \frac{u^8}{8} - \frac{u^6}{6} + C = \frac{\cos^8 x}{8} - \frac{\cos^6 x}{6} + C. \end{aligned}$$

Case 2: both powers are even

In this case, we write powers as powers of

$$\begin{aligned}\cos^2 x &= \frac{1 + \cos(2x)}{2} \\ \sin^2 x &= \frac{1 - \cos(2x)}{2}, \end{aligned} \tag{5.1.3}$$

(which follows from the double angle formula (4.2.1)) to reduce the powers, and this process may need to be iterated.

EXAMPLE 5.1.3. Evaluate $\int \sin^4 x\, dx$.

Solution. In view of (5.1.3), we have

$$\begin{aligned}
\int \sin^4 x\, dx &= \int \left(\sin^2 x\right)^2\, dx \\
&= \int \left(\frac{1-\cos(2x)}{2}\right)^2 dx \\
&= \frac{1}{4}\int 1 - 2\cos(2x) + \cos^2(2x)\, dx \\
&= \frac{1}{4}\int 1 - 2\cos(2x) + \frac{1+\cos(4x)}{2}\, dx \\
&= \frac{1}{4}\int \frac{3}{2} - 2\cos(2x) + \frac{1}{2}\cos(4x)\, dx \\
&= \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C.
\end{aligned}$$

EXAMPLE 5.1.4. Evaluate $\int \cos^4 x \, \sin^2 x \, dx$.

Solution. In view of (5.1.3), we have

$$\begin{aligned}
\int \cos^4 x \, \sin^2 x \, dx &= \int \left(\cos^2 x\right)^2 \sin^2 x \, dx \\
&= \int \left(\frac{1+\cos(2x)}{2}\right)^2 \left(\frac{1-\cos(2x)}{2}\right) dx \\
&= \frac{1}{8}\int \left(1+2\cos(2x)+\cos^2(2x)\right)\left(1-\cos(2x)\right) \, dx \\
&= \frac{1}{8}\int 1+\cos(2x)-\cos^2(2x)-\cos^3(2x)\, dx \\
&= \frac{1}{8}\left(x+\frac{\sin(2x)}{2}-\int \frac{1+\cos(4x)}{2}\, dx - \int \cos^2(2x)\, \cos(2x)\, dx\right) \\
&= \frac{1}{8}\left(\frac{x}{2}+\frac{\sin(2x)}{2}-\frac{\sin(4x)}{8}-\int \left(1-\sin^2(2x)\right)\, \cos(2x)\, dx\right).
\end{aligned}$$

In the last integral, let $u = \sin(2x)$ so that $du = 2\cos(2x)\, dx$ and

$$\begin{aligned}
\int \cos^4 x \, \sin^2 x \, dx &= \frac{1}{8}\left(\frac{x}{2}+\frac{\sin(2x)}{2}-\frac{\sin(4x)}{8}-\frac{1}{2}\int \left(1-u^2\right)\, du\right) \\
&= \frac{x}{16}+\frac{\sin(2x)}{16}-\frac{\sin(4x)}{64}-\frac{\sin(2x)}{2}+\frac{\sin^3(2x)}{6}+C \\
&= \frac{x}{16}-\frac{7\sin(2x)}{16}-\frac{\sin(4x)}{64}+\frac{\sin^3(2x)}{6}+C.
\end{aligned}$$

5.2 Products of sine and cosine

Watch the video at

https://www.youtube.com/watch?v=KNwcjS2AsVw&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=31

Abstract

This video goes over evaluating integrals of one of the following types:

$$\int \sin(mx)\ \cos(nx)\,dx$$
$$\int \sin(mx)\ \sin(nx)\,dx$$
$$\int \cos(mx)\ \cos(nx)\,dx \quad .$$

This type of problem becomes easy if we transform products into half-sums with the help of the following trigonometric formulas

$$\begin{aligned}
\sin\alpha\ \cos\beta &= \frac{1}{2}\left(\sin(\alpha-\beta)+\sin(\alpha+\beta)\right)\\
\sin\alpha\ \sin\beta &= \frac{1}{2}\left(\cos(\alpha-\beta)-\cos(\alpha+\beta)\right)\\
\cos\alpha\ \cos\beta &= \frac{1}{2}\left(\cos(\alpha-\beta)+\cos(\alpha+\beta)\right).
\end{aligned}$$

Example 5.2.1. Evaluate

1) $\int \sin(3x)\ \cos x\,dx$.
Solution.

$$\begin{aligned}
\int \sin(3x)\ \cos x\,dx &= \frac{1}{2}\int \sin(2x)+\sin(4x)\,dx\\
&= -\frac{1}{4}\cos(2x)-\frac{1}{8}\cos(4x)+C.
\end{aligned}$$

2) $\int \sin(5x)\ \sin(3x)\,dx$.
Solution.

$$\begin{aligned}
\int \sin(5x)\ \sin(3x)\,dx &= \frac{1}{2}\int \cos(2x)-\cos(8x)\,dx\\
&= \frac{1}{4}\sin(2x)-\frac{1}{16}\sin(8x)+C.
\end{aligned}$$

3) $\int \cos(2x)\ \cos(7x)\,dx$.

Solution.

$$\begin{aligned}\int \cos(2x)\ \cos(7x)\,dx &= \frac{1}{2}\int \cos(5x)+\cos(9x)\,dx \\ &= \frac{1}{10}\sin(5x)+\frac{1}{18}\sin(9x)+C.\end{aligned}$$

5.3 (co)secant, (co)tangent and their powers

Watch the video at

https://www.youtube.com/watch?v=YThB05DtWJk&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=32

Abstract

This video goes over some examples of integrals of powers of sec and tan.

The following two formulas are verified:

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C \tag{5.3.1}$$

$$\begin{aligned}\int \csc x\,dx &= -\ln|\csc x + \cot x| + C,\\ &= \ln|\csc x - \cot x| + C\end{aligned} \tag{5.3.2}$$

and used in the following examples.

Example 5.3.1. Evaluate $\int \sec^3 x\,dx$.

Solution. Note that

$$\int \sec^3 x\,dx = \int \sec x\,\sec^2 x\,dx$$

can be integrated by parts, with $u = \sec x$ and $dv = \sec^2 x\,dx$, so that $du = \sin x\,\sec^2 x\,dx$ and $v = \tan x$. Thus

$$\begin{aligned}\int \sec^3 x\,dx &= \sec x\,\tan x - \int \tan x\,\sin x\,\sec^2 x\,dx\\ &= \sec x\,\tan x - \int \sin^2 x\,\sec^3 x\,dx\\ &= \sec x\,\tan x - \int (1-\cos^2 x)\sec^3 x\,dx\\ &= \sec x\,\tan x - \int \sec^3 x\,dx + \int \sec x\,dx,\end{aligned}$$

so that

$$2\int \sec^3 x\,dx = \sec x\,\tan x + \int \sec x\,dx$$

and, in view of (5.3.1),

$$\int \sec^3 x\, dx = \frac{1}{2}\left(\sec x\, \tan x + \ln|\sec x + \tan x|\right) + C.$$

EXAMPLE 5.3.2. Evaluate $\int \tan^4 x\, dx$.

Solution. Note that

$$\begin{aligned}\int \tan^4 x\, dx &= \int \tan^2 x \cdot \tan^2 x\, dx \\ &= \int \tan^2 x \left(\sec^2 x - 1\right)\, dx \\ &= \int \tan^2 x\, \sec^2 x\, dx - \int \tan^2 x\, dx.\end{aligned}$$

Using $u = \tan x$ so that $du = \sec^2 x\, dx$ in the first integral, and the identity $\tan^2 x = \sec^2 x - 1$ in the second, we have

$$\begin{aligned}\int \tan^4 x\, dx &= \int u^2\, du - \int \sec^2 x\, dx + \int dx \\ &= \frac{\tan^3 x}{3} - \tan x + x + C.\end{aligned}$$

Exercises

you are now prepared to work on the Practice Problems, and Homework set M5A in the manual of exercises.

5.4 Trig substitutions

Watch the videos at

https://www.youtube.com/watch?v=0ZAA6hlr6Hc&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=33

and

https://www.youtube.com/watch?v=GTvbWO6iYeE&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=34

and

https://www.youtube.com/watch?v=8yR5vCc2YWM&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=35

Abstract

These videos present generalities on trigonometric substitution, then go over a number of examples.

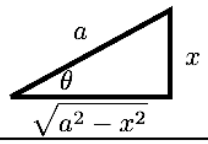
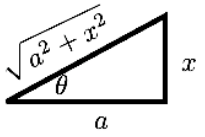
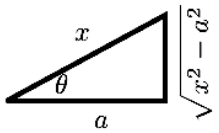
Trigonometric substitutions allow to transform expressions of the form

$$a^2 - x^2;\ a^2 + x^2 \text{ or } x^2 - a^2$$

into a single square, which often allows to simplify an integral. The basic idea is to parametrize x as

$$x = g(\theta),$$

where g is one-to-one, so that $\theta = g^{-1}(x)$ defines a regular substitution. The choice of g that will permit us to rewrite the desired expressions as one single square make use of trigonometric identities:

Expression	Substitution	Identity	triangle
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$ $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$	
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$ $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$	
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$ $0 \le \theta < \frac{\pi}{2}$ if $\frac{x}{a} \ge 1$ $\frac{\pi}{2} < \theta \le \pi$ if $\frac{x}{a} \le -1$	$\sec^2\theta - 1 = \tan^2\theta$	

The triangle is used to get back to an expression in terms of x instead of θ.

Example 5.4.1. Evaluate $\int \frac{dx}{x^2\sqrt{x^2+4}}$.

Solution. Note that there is no obvious substitution, so we may try trig substitution to get rid of the radical. We use the second row in the table above: let $x = 2\tan\theta$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, so that $dx = 2\sec^2\theta\, d\theta$ and

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = \int \frac{2\sec^2\theta}{4\tan^2\theta\sqrt{4\tan^2\theta + 4}}\, d\theta.$$

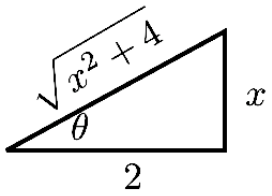
Using the fact that

$$4\tan^2\theta + 4 = 4(\tan^2\theta + 1) = 4\sec^2\theta,$$

we obtain

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{x^2+4}} &= \int \frac{\sec^2\theta}{4\tan^2\theta|\sec\theta|}\,d\theta \\ &= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta \text{ because } \sec\theta > 0 \text{ on } \left(-\tfrac{\pi}{2},\tfrac{\pi}{2}\right) \\ &= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta \\ &= \frac{1}{4}\int \frac{du}{u^2} \text{ for } u=\sin\theta \\ &= -\frac{1}{4u}+C = -\frac{1}{4\sin\theta}+C.\end{aligned}$$

To express this as a function of x, note that $\tan\theta = \frac{x}{2}$, which we represent in the following triangle:



Thus,

$$\frac{1}{\sin\theta} = \frac{\sqrt{x^2+4}}{x}$$

and

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = -\frac{\sqrt{x^2+4}}{4x}+C.$$

Remark 5.4.2. Note that trigonometric substitutions often lead to rather complicated calculations and should be considered a "last resort". Whenever possible use an alternative argument.

Example 5.4.3. While trig substitution could be use to evaluate

$$\int \frac{x}{\sqrt{x^2+4}}\,dx$$

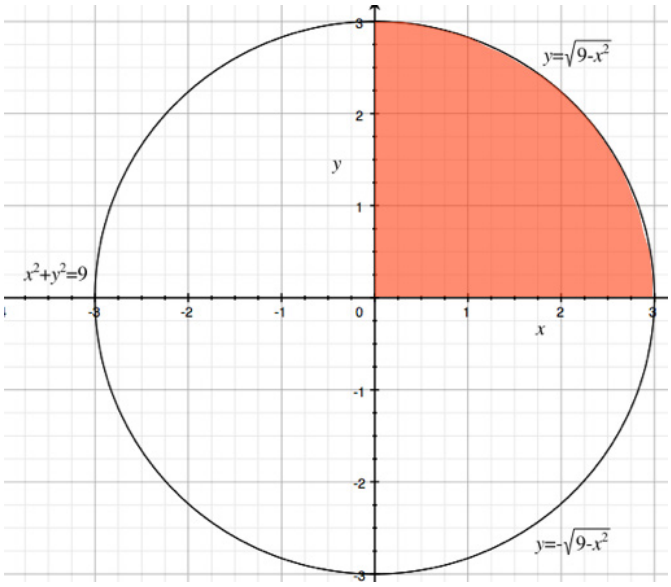
it is much more efficient to use a regular substitution with $u = x^2+4$, so $du = 2x\,dx$ and

$$\int \frac{x}{\sqrt{x^2+4}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \sqrt{u}+C = \sqrt{x^2+4}+C.$$

EXAMPLE 5.4.4. Similarly, we could use trig substitution to evaluate

$$\int_0^3 \sqrt{9-x^2}\,dx$$

but it is more efficient to note that $y = \sqrt{9-x^2}$ correspond to the upper-half of the circle $x^2 + y^2 = 9$ centered at the origin and of radius 3:



Thus

$$\int_0^3 \sqrt{9-x^2}\,dx = \frac{1}{4}\pi(3)^2 = \frac{9\pi}{4},$$

represents the area of one fourth of the disc.

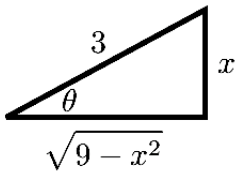
Yet in some cases, trig substitution is our only option.

EXAMPLE 5.4.5. Evaluate $\int \frac{\sqrt{9-x^2}}{x^2}\,dx$.

Solution. We proceed by trig substitution, letting $x = 3\sin\theta$ where $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. Then $dx = 3\cos\theta\,d\theta$ and

$$\begin{aligned}
\int \frac{\sqrt{9-x^2}}{x^2}\,dx &= \int \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta}\,3\cos\theta\,d\theta \\
&= \frac{1}{3}\int \frac{\sqrt{9(1-\sin^2\theta)}}{\sin^2\theta}\cos\theta\,d\theta \\
&= \int \frac{\sqrt{\cos^2\theta}}{\sin^2\theta}\cos\theta\,d\theta \\
&= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta \text{ because } \cos\theta \ge 0 \text{ on } \left[-\tfrac{\pi}{2},\tfrac{\pi}{2}\right] \\
&= \int \frac{1-\sin^2\theta}{\sin^2\theta}\,d\theta = \int \frac{1}{\sin^2\theta} - 1\,d\theta \\
&= -\cot\theta - \theta + C.
\end{aligned}$$

To rewrite this function in terms of *x*, note that $\sin\theta = \frac{x}{3}$ for $\theta \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ so that $\theta = \arcsin\left(\frac{x}{3}\right)$, and we see on the triangle below that $\cot\theta = \frac{\sqrt{9-x^2}}{x}$:



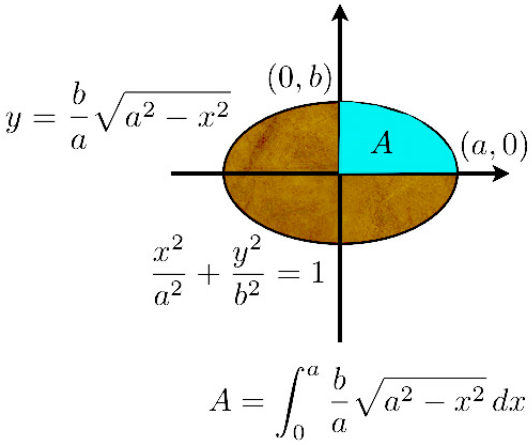
Thus

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx = -\frac{\sqrt{9-x^2}}{x} - \arcsin\left(\frac{x}{3}\right) + C.$$

EXAMPLE 5.4.6. Establish a formula for the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Solution. Solving for y leads to $y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$, so that the upper-half ellipse in the picture below



$$A = \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx$$

has equation $y = \frac{b}{a}\sqrt{a^2 - x^2}$ and the area of the ellipse is 4 times the area

$$A = \frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\, dx.$$

We can either use the idea of Example 5.4.4, and see that $\int_0^a \sqrt{a^2 - x^2}\, dx$ is the area of a quarter of a disk of radius a and conclude that

$$A = \frac{b}{a} \cdot \frac{\pi a^2}{4},$$

so that the area of the ellipse is

$$S = 4A = \pi ab,$$

or we can treat the case of the disk as a particular case of the ellipse (for $a = b$, the radius) without assuming this case known. In that latter case, we need to calculate $\int_0^a \sqrt{a^2 - x^2}\, dx$, and we proceed by trigonometric substitution with $x = a\sin\theta$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = a\cos\theta\, d\theta$, $x = 0$ for $\theta = 0$ and $x = a$ for $\theta = \frac{\pi}{2}$, so that

$$\begin{aligned}
\int_0^a \sqrt{a^2 - x^2}\, dx &= \int_0^{\frac{\pi}{2}} \sqrt{a^2(1 - \sin^2\theta)}\, a\cos\theta\, d\theta \\
&= a^2 \int_0^{\frac{\pi}{2}} \sqrt{\cos^2\theta}\cos\theta\, d\theta \\
&= a^2 \int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta \\
&= a^2 \int_0^{\frac{\pi}{2}} \frac{1 + \cos(2\theta)}{2}\, d\theta \text{ using} \qquad (5.1.3) \\
&= \frac{a^2}{2}\left[\theta + \frac{\sin(2\theta)}{2}\right]_0^{\frac{\pi}{2}} = \frac{a^2\pi}{4},
\end{aligned}$$

which leads to

$$S = 4A = 4\frac{b}{a} \cdot \frac{a^2\pi}{4} = \pi ab.$$

EXAMPLE 5.4.7. Evaluate $\int \frac{dx}{x^2\sqrt{16x^2-9}}$.

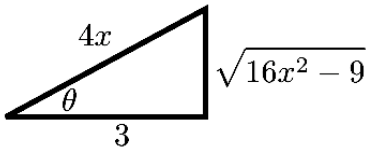
Solution. Note first that

$$\int \frac{dx}{x^2\sqrt{16x^2-9}} = \int \frac{dx}{4x^2\sqrt{x^2-\frac{9}{16}}} = \frac{1}{4}\int \frac{dx}{x^2\sqrt{x^2-\left(\frac{3}{4}\right)^2}}$$

Let now $x = \frac{3}{4}\sec\theta$ for $0 \le \theta < \frac{\pi}{2}$ (that is, for $x \ge \frac{3}{4}$). Then $dx = \frac{3}{4}\frac{\sin\theta}{\cos^2\theta}\,d\theta = \frac{3}{4}\sin\theta\,\sec^2\theta\,d\theta$, so that

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{16x^2-9}} &= \frac{1}{4}\cdot\frac{3}{4}\int \frac{\sin\theta\,\sec^2\theta}{\frac{9}{16}\sec^2\theta\sqrt{\frac{9}{16}(\sec^2\theta-1)}}\,d\theta \\
&= \frac{3}{16}\cdot\frac{16}{9}\int \frac{\sin\theta}{\frac{3}{4}\sqrt{\tan^2\theta}}\,d\theta \\
&= \frac{1}{3}\cdot\frac{4}{3}\int \frac{\sin\theta}{\tan\theta}\,d\theta \text{ because } \tan\theta \ge 0 \text{ on } [0,\tfrac{\pi}{2}) \\
&= \frac{4}{9}\int \cos\theta\,d\theta = \frac{4}{9}\sin\theta + C.
\end{aligned}$$

Moreover, $\sec\theta = \frac{4x}{3}$ so that, in view of the triangle below,



$\sin\theta = \frac{\sqrt{16x^2-9}}{4x}$ and

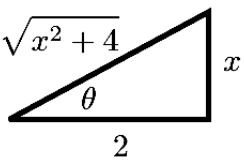
$$\int \frac{dx}{x^2\sqrt{16x^2-9}} = \frac{4}{9}\cdot\frac{\sqrt{16x^2-9}}{4x} + C = \frac{\sqrt{16x^2-9}}{9x} + C.$$

EXAMPLE 5.4.8. Evaluate $\int \frac{dx}{(x^2+4)^2}$.

Solution. We use the trig substitution $x = 2\tan\theta$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, so that $dx = 2\sec^2\theta\, d\theta$ and

$$\begin{aligned}
\int \frac{dx}{(x^2+4)^2} &= 2\int \frac{\sec^2\theta}{(4\tan^2\theta+4)^2}\, d\theta \\
&= \frac{2}{16}\int \frac{\sec^2\theta}{(\sec^2\theta)^2}\, d\theta \text{ because } \tan^2\theta + 1 = \sec^2\theta \\
&= \frac{1}{8}\int \cos^2\theta\, d\theta \\
&= \frac{1}{8}\int \frac{1+\cos(2\theta)}{2}\, d\theta \text{ using } (5.1.3) \\
&= \frac{\theta}{16} + \frac{\sin(2\theta)}{32} + C \\
&= \frac{1}{16}(\theta + \sin\theta\cos\theta) + C \text{ because } \sin(2\theta)=2\sin\theta\cos\theta.
\end{aligned}$$

Moreover, $\tan\theta = \frac{x}{2}$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ so that $\theta = \arctan\left(\frac{x}{2}\right)$ and, in view of the triangle below,



we have $\sin\theta = \frac{x}{\sqrt{x^2+4}}$ and $\cos\theta = \frac{2}{\sqrt{x^2+4}}$, so that

$$\int \frac{dx}{(x^2+4)^2} = \frac{1}{16}\left(\arctan\left(\frac{x}{2}\right) + \frac{2x}{x^2+4}\right) + C.$$

REMARK 5.4.9. In some cases, the formulas (2.7.1) and (2.7.2) might be useful to avoid a lengthy trigonometric substitution.

Exercises

you are now prepared to work on the Practice Problems, and Homework set M5B in the manual of exercises.

6 M6: Partial Fractions

6.1 Partial fractions: generalities; long division

Watch the video at

https://www.youtube.com/watch?v=zPSl1bb6tkw&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=36

Abstract

This video discusses the general strategy to integrate rational functions, and reviews long division of polynomial, which is the first step to be performed.

It is easy to integrate

$$\int \frac{3}{x-1} + \frac{2}{x+3}\,dx = 3\ln|x-1| + 2\ln|x+3| + C,$$

but not so clear at first how to integrate

$$\int \frac{5x+7}{x^2+2x-3}\,dx$$

unless you realize that

$$\frac{3}{x-1} + \frac{2}{x+3} = \frac{3(x+3)+2(x-1)}{(x-1)(x+3)} = \frac{5x+7}{x^2+2x-3}.$$

Thus, to integrate *rational functions*, i.e., quotients of polynomials, we have to be able to go the other way, that is, break down our fraction (like $\frac{5x+7}{x^2+2x-3}$) into a sum of simpler ones (like $\frac{3}{x-1} + \frac{2}{x+3}$), or *partial fractions.*

To this end, we will:

1) if the degree of the numerator is not less than that of the denominator, simplify by long division:

$$\frac{p(x)}{q(x)} = f(x) + \frac{r(x)}{q(x)},$$

where $f(x)$ is the quotient and $r(x)$ the remainder in the division, so that $r(x)$ has degree less than that of $q(x)$.[3]

2) factor $q(x)$. The factors may include:
   a) non-repeated linear factors of the type $(ax+b)$
   b) repeated linear factors of the type $(ax+b)^n$
   c) irreducible quadratic factors of the type ax^2+bx+c (irreducible because $b^2-4ac<0$)
   d) repeated irreducible quadratic factors of the type $(ax^2+bx+c)^n$.

The next steps depend on which of these four cases occurs. This will be explored in the next videos.

Example 6.1.1. To evaluate

$$\int \frac{x^3+x}{x-1}\,dx$$

we first note by long division

$$\begin{array}{r|l} & x^2 + x + 2 \\ \hline x-1 & x^3 + x \\ & -x^3 + x^2 \\ \hline & x^2 + x \\ & -x^2 + x \\ \hline & 2x \\ & -2x + 2 \\ \hline & 2 \end{array}$$

that

$$\frac{x^3+x}{x-1} = x^2 + x + 2 + \frac{2}{x-1},$$

so that

$$\int \frac{x^3+x}{x-1}\,dx = \frac{x^3}{3} + \frac{x^2}{2} + 2x - 2\ln|x-1| + C.$$

6.2 only non-repeated linear factors

Watch the videos at

https://www.youtube.com/watch?v=JFmCXnO2YSs&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=37

and

https://www.youtube.com/watch?v=ayCykwsXfkE&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=38

Abstract

These videos discuss the general methods to find the decomposition into partial fraction over a rational functions when the denominator has only non-repeated linear factors, and goes over three examples.

Recall that modulo long division, we can assume that our rational function is in lowest terms, that is, $\frac{r(x)}{q(x)}$ where the degree of r is less that the degree of q. If $q(x)$ is of the form

$$q(x) = (a_1x + b_1) \cdot (a_2x + b_2) \dots (a_nx + b_n),$$

where none of the factors is repeated, then the form of the decomposition into partial fractions is

$$\frac{r(x)}{q(x)} = \frac{A_1}{ax_1 + b_1} + \frac{A_2}{a_2x + b_2} + \ldots + \frac{A_n}{a_nx + b_n}, \tag{6.2.1}$$

and thus, we only need to find the coefficients $A_1, A_2, \ldots A_n$ to integrate $\frac{r(x)}{q(x)}$. We discuss how in the examples:

Example 6.2.1. Evaluate

$$\int \frac{x^2 + 2x - 1}{x^3 - x}\, dx.$$

Solution. The fraction is already in lowest terms, so there is no need for long division. Moreover

$$\frac{x^2 + 2x - 1}{x^3 - x} = \frac{x^2 + 2x - 1}{x(x-1)(x+1)},$$

so that, in view of (6.2.1),

$$\frac{x^2 + 2x - 1}{x^3 - x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}. \tag{6.2.2}$$

To find these coefficients:

method 1: handcover method:

To find *A*, multiply both sides by *x* and set $x = 0$:

$$\frac{x^2 + 2x - 1}{(x-1)(x+1)} = A + \frac{Bx}{x-1} + \frac{Cx}{x+1} \overset{x=0}{\Longrightarrow} 1 = A.$$

To find *B*, multiply both sides by $x - 1$ and set $x = 1$:

$$\frac{x^2 + 2x - 1}{x(x+1)} = \frac{A(x-1)}{x} + B + \frac{C(x-1)}{(x+1)} \overset{x=1}{\Longrightarrow} 1 = B$$

To find *C*, multiply both sides by $x + 1$ and set $x = -1$:

$$\frac{x^2 + 2x - 1}{x(x-1)} = \frac{A(x+1)}{x} + \frac{B(x+1)}{x-1} + C \overset{x=-1}{\Longrightarrow} -1 = C.$$

Thus

$$\int \frac{x^2 + 2x - 1}{x^3 - x}\, dx = \int \frac{1}{x} + \frac{1}{x-1} - \frac{1}{x+1}\, dx = \ln|x| + \ln|x-1| - \ln|x+1| + C.$$

method 2: system method:

To find the coefficients, we rewrite the right-hand side in (6.2.2) as one fraction, and identify the numerators:

$$\frac{x^2+2x-1}{x(x-1)(x+1)} = \frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1} = \frac{A(x-1)(x+1)+Bx(x+1)+Cx(x-1)}{x(x-1)(x+1)}$$

so that

$$\begin{aligned} x^2+2x-1 &= A(x-1)(x+1)+Bx(x+1)+Cx(x-1) \\ &= x^2(A+B+C)+x(B-C)+(-A). \end{aligned}$$

Using the fact that two polynomials are equal if and only if their coefficients of same degree are equal we obtain the system

$$\begin{cases} A+B+C=1 \\ B-C=2 \\ -A=-1 \end{cases} \iff \begin{cases} A=1 \\ B=1 \\ C=-1. \end{cases}$$

REMARK 6.2.2. While the handcover method is more efficient, it only applies to the case of non-repeated linear factors, unlike the system method, which is more universal.

Example 6.2.3. Evaluate

$$\int_0^1 \frac{x^3 - 4x - 10}{x^2 - x - 6}\, dx.$$

Solution. The fraction is not in lowest term, so we first perform long division:

$$\begin{array}{r} x\ +1 \\ x^2 - x - 6\big) \overline{\begin{array}{r} x^3\ \ \ \ \ \ -4x - 10 \\ -x^3 + x^2 + 6x \\ \hline x^2 + 2x - 10 \\ -x^2\ +x\ +6 \\ \hline 3x\ -4 \end{array}} \end{array}$$

so that

$$\int_0^1 \frac{x^3 - 4x - 10}{x^2 - x - 6}\, dx = \int_0^1 x + 1 + \frac{3x - 4}{x^2 - x - 6}\, dx.$$

The decomposition into partial fraction of $\frac{3x-4}{x^2-x-6}$ is of the form

$$\frac{3x - 4}{x^2 - x - 6} = \frac{3x - 4}{(x + 2)(x - 3)} = \frac{A}{x + 2} + \frac{B}{x - 3}.$$

The handcover method yields $A = 2$ and $B = 1$ so that

$$\begin{aligned} \int_0^1 \frac{x^3 - 4x - 10}{x^2 - x - 6}\, dx &= \int_0^1 x + 1 + \frac{2}{x + 2} + \frac{1}{x - 3}\, dx \\ &= \left[\frac{x^2}{2} + x + 2\ln|x + 2| + \ln|x - 3|\right]_0^1 \\ &= \frac{3}{2} + \ln\frac{3}{2}. \end{aligned}$$

Example 6.2.4. Evaluate

$$\int \frac{2x + 3}{x^2 + 3x - 4}\, dx.$$

Solution. Note that while we could use partial fractions, a substitution is more efficient whenever possible. In this case, for $u = x^2 + 3x - 4$, we have $du = 2x + 3\, dx$ so that

$$\int \frac{2x + 3}{x^2 + 3x - 4}\, dx = \int \frac{du}{u} = \ln|u| + C = \ln|x^2 + 3x - 4| + C.$$

Example 6.2.5. Evaluate

$$\int \frac{4x+3}{2x^2-5x-3}\,dx.$$

Solution. Since $2x^2-5x-3=(2x+1)(x-3)$, the form of the decomposition into partial fractions is

$$\frac{4x+3}{(2x+1)(x-3)}=\frac{A}{2x+1}+\frac{B}{x-3},$$

and the handcover method yields $A=-\frac{2}{7}$ and $B=\frac{15}{7}$. Thus

$$\begin{aligned}\int \frac{4x+3}{2x^2-5x-3}\,dx &= -\frac{1}{7}\int \frac{2dx}{2x+1}+\frac{15}{7}\int \frac{dx}{x-3}\\ &= -\frac{1}{7}\ln|2x+1|+\frac{15}{7}\ln|x-3|+C.\end{aligned}$$

Exercises

you are now prepared to work on the Practice Problems, and Homework set M6A in the manual of exercises.

6.3 with repeated linear factors

Watch the video at

https://www.youtube.com/watch?v=RuCuXpWCshw&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=39

Abstract

This video gives the form of the decomposition into partial fractions when the denominator contains repeated linear factors, and explains on examples how to find the corresponding coefficients, and how to integrate the resulting terms.

If the denominator $q(x)$ of a rational function $\frac{r(x)}{q(x)}$ in lowest terms contains a repeated linear factor $(ax+b)^n$ the corresponding terms in the decomposition into partial fractions of $\frac{r(x)}{q(x)}$ are

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \ldots + \frac{A_n}{(ax+b)^n}.$$

Example 6.3.1. The decomposition into partial fractions of $\frac{x}{(x-2)^3}$ has the form

$$\frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3},$$

where A, B and C are constants to be determined, while the decomposition into partial fractions of $\frac{x}{(x+1)(x-1)(x-2)^3}$ has the form

$$\frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x-2} + \frac{D}{(x-2)^2} + \frac{E}{(x-2)^3}.$$

Note that each term of the form $\frac{A}{(ax+b)^n}$ for $n > 1$ is easily integrated with the power rule:

$$\begin{aligned} \int \frac{A}{(ax+b)^n}\,dx &= \frac{A}{a}\int \frac{du}{u^n} \text{ for } u = ax+b \\ &= \frac{A}{a}\frac{u^{1-n}}{1-n} + C = \frac{A}{a(1-n)(ax+b)^{n-1}}. \end{aligned}$$

Example 6.3.2. Evaluate

$$\int \frac{x^2}{(x+1)^3}\,dx.$$

Solution. The fraction is already in lowest terms, so there is no need for long division. Moreover

$$\frac{x^2}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}. \tag{6.3.1}$$

To find A, B and C, we will present two methods: in both, we start by rewriting the right hand side in (6.3.1) as one fraction, and identify the numerators:

$$\frac{x^2}{(x+1)^3} = \frac{A(x+1)^2 + B(x+1) + C}{(x+1)^3},$$

so that

$$x^2 = A(x+1)^2 + B(x+1) + C. \tag{6.3.2}$$

In the system method we rewrite the right hand side of (6.3.2) in standard form and identify the coefficients of same degree to form a system of equations of unknowns A, B and C:

$$x^2 = Ax^2 + (B+2A)x + (A+B+C) \Longrightarrow \begin{cases} A = 1 \\ B + 2A = 0 \\ A + B + C = 0 \end{cases} \iff \begin{cases} A = 1 \\ B = -2 \\ C = 1 \end{cases}.$$

Thus,

$$\begin{aligned} \int \frac{x^2}{(x+1)^3}\,dx &= \int \frac{1}{x+1} - \frac{2}{(x+1)^2} + \frac{1}{(x+1)^3}\,dx \\ &= \ln|x+1| - \frac{2}{x+1} - \frac{1}{2(x+1)^2} + C. \end{aligned}$$

Alternatively, we can use the same idea as the handcover method, combined with differentiation, to find A, B and C from (6.3.2): Plugging in $x = -1$ yields $1 = C$. If we differentiate (6.3.2), we obtain

$$2x = 2A(x+1) + B,$$

so that plugging in $x = -1$ yields $-2 = B$. If we differentiate again, we get $2 = 2A$, that is, $A = 1$.

Example 6.3.3. Evaluate

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx.$$

Solution. By long division

$$\begin{array}{r|l} & x+1 \\ \hline x^3-x^2-x+1 & x^4 \quad -2x^2+4x+1 \\ & \underline{-x^4+x^3 \quad +x^2 \quad -x} \\ & \quad x^3 \quad -x^2+3x+1 \\ & \quad \underline{-x^3 \quad +x^2 \quad +x-1} \\ & \quad\quad\quad\quad 4x \end{array}$$

we see that

$$\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x+1+\frac{4x}{x^3-x^2-x+1}.$$

Moreover,

$$\frac{4x}{x^3-x^2-x+1} = \frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x+1)}. \tag{6.3.3}$$

To find C, we can use handcover (multiply by $x+1$ and set $x=-1$) to find $C=-1$. On the other hand, rewriting the right hand side of (6.3.3) as one fraction and identifying the numerators yields

$$4x = A(x-1)(x+1)+B(x+1)+C(x-1)^2,$$

in which $x=1$ yields $4=2B$, so that $B=2$. Differentiating gives

$$4 = A(x-1+x+1)+B+2C(x-1)$$

in which $x=1$ yields $4=2A+B$, so that $A=1$. Thus

$$\begin{aligned} \int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx &= \int x+1+\frac{1}{x-1}+\frac{2}{(x-1)^2}-\frac{1}{x+1}\,dx \\ &= \frac{x^2}{2}+x+\ln|x-1|-\frac{2}{x-1}-\ln|x+1|+C. \end{aligned}$$

6.4 with irreducible quadratic factors

Watch the video at

https://www.youtube.com/watch?v=3rszFlSL2AI&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=41

Abstract

This videos presents the form of the decomposition into partial fractions for a rational function whose denominator contains a (non-repeated) irreducible quadratic factor, goes over the method to integrate the corresponding term, and presents examples.

If the denominator $q(x)$ of a rational function $\frac{r(x)}{q(x)}$ in lowest terms contains an irreducible quadratic term $ax^2 + bx + c$ with $b^2 - 4ac < 0$, then the decomposition into partial fractions of $\frac{r(x)}{q(x)}$ contains the term

$$\frac{Ax + B}{ax^2 + bx + c},$$

where A and B are constants to be determined.

Example 6.4.1. For instance, the decomposition into partial fractions of $\frac{2x+3}{x^4+9x^2}$ has the form

$$\frac{2x+3}{x^4+9x^2} = \frac{2x+3}{x^2(x^2+9)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+9},$$

where A, B, C and D are constants to be determined. On the other hand

$$\frac{5x+4}{x^2+x+1}$$

is already a partial fraction and cannot be decomposed any further, since x^2+x+1 is irreducible.

Example 6.4.2. Evaluate

$$\int \frac{2x^2-x+4}{x^3+4x}\,dx.$$

Solution. The form of the decomposition into partial fractions is

$$\frac{2x^2-x+4}{x^3+4x} = \frac{2x^2-x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}.$$

To find A, B and C, we use the system method:

$$\begin{aligned} 2x^2-x+4 &= A(x^2+4)+Bx^2+Cx \\ &= (A+B)x^2+Cx+4A \end{aligned}$$

so that

$$\begin{cases} A+B=2 \\ C=-1 \\ 4A=4 \end{cases} \iff \begin{cases} A=1 \\ B=1 \\ C=-1 \end{cases}.$$

Thus

$$\begin{aligned} \int \frac{2x^2-x+4}{x^3+4x}\,dx &= \int \frac{1}{x} + \frac{x-1}{x^2+4}\,dx \\ &= \ln|x| + \int \frac{x}{x^2+4}\,dx - \int \frac{dx}{x^2+4}, \end{aligned}$$

and using the substitution $u = x^2+4$ ($du = 2x\,dx$) in the first integral and (2.5.1) in the second, we obtain

$$\int \frac{2x^2-x+4}{x^3+4x}\,dx = \ln|x| + \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C.$$

General method to integrate a partial fraction $\frac{Ax+B}{ax^2+bx+c}$ where ax^2+bx+c is irreducible (that is, $b^2-4ac<0$).

1) Complete the square in ax^2+bx+c:

$$\begin{aligned} ax^2+bx+c &= a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right) = a\left(\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}+\frac{c}{a}\right) \\ &= a\left(\left(x+\frac{b}{2a}\right)^2-\frac{(b^2-4ac)}{4a^2}\right). \end{aligned}$$

Because $b^2-4ac<0$, this is of the form

$$a\left(u^2+\alpha^2\right)$$

where $u := x+\frac{b}{2a}$ and $\alpha := \frac{\sqrt{4ac-b^2}}{2a}$.

2) Thus, the substitution $u := x+\frac{b}{2a}$, yields an integral of the form

$$\int \frac{Cu+D}{u^2+\alpha^2}\,du$$

3) Split the integral:

$$\int \frac{Cu+D}{u^2+\alpha^2}\,du = C\int \frac{u}{u^2+\alpha^2}\,du + D\int \frac{du}{u^2+\alpha^2}.$$

4) Calculate the first integral using the substitution $v=u^2+\alpha^2$ so that

$$\int \frac{u}{u^2+\alpha^2}\,du = \frac{1}{2}\int \frac{dv}{v} = \frac{1}{2}\ln(u^2+\alpha^2)$$

5) Calculate the second integral using (2.5.1)
6) Rewrite the result in terms of *x*.

Example 6.4.3. Evaluate

$$\int \frac{4x^2-3x+2}{4x^2-4x+3}\,dx.$$

Solution. Long division (or, equivalently, the observation that $4x^2-3x+2=(4x^2-4x+3)+(x-1)$) gives

$$\int \frac{4x^2-3x+2}{4x^2-4x+3}\,dx = \int 1+\frac{x-1}{4x^2-4x+3}\,dx.$$

Because the discriminant $\Delta = 16 - 48$ of the denominator is negative, $4x^2 - 4x + 3$ is irreducible. Completing the square, we obtain

$$4x^2 - 4x + 3 = 4\left(x^2 - x + \frac{3}{4}\right) = 4\left(\left(x - \frac{1}{2}\right)^2 - \frac{1}{4} + \frac{3}{4}\right) = 4\left(\left(x - \frac{1}{2}\right)^2 + \frac{1}{2}\right).$$

Thus

$$\int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3}\,dx = \int 1 + \frac{x - 1}{4\left(\left(x - \frac{1}{2}\right)^2 + \frac{1}{2}\right)}\,dx,$$

and, letting $u = x - \frac{1}{2}$ so that $x - 1 = u - \frac{1}{2}$, we obtain

$$\begin{aligned}
\int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3}\,dx &= x + \frac{1}{4}\int \frac{u - \frac{1}{2}}{u^2 + \frac{1}{2}}\,du \\
&= x + \frac{1}{4}\int \frac{u}{u^2 + \frac{1}{2}}\,du - \frac{1}{8}\int \frac{du}{u^2 + \frac{1}{2}} \\
&= x + \frac{1}{2}\ln\left(u^2 + \frac{1}{2}\right) - \frac{1}{8}\cdot\sqrt{2}\arctan\left(\sqrt{2}u\right) + C \\
&= x + \frac{1}{2}\ln\left(x^2 - x + \frac{3}{4}\right) - \frac{\sqrt{2}}{8}\arctan\left(\sqrt{2}\left(x - \frac{1}{2}\right)\right) + C.
\end{aligned}$$

Exercises

you are now prepared to work on the Practice Problems, and Homework set M6B in the manual of exercises.

6.5 with repeated irreducible quadratic factors

Watch the videos at

https://www.youtube.com/watch?v=LndYRZDClsY&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=42

and

https://www.youtube.com/watch?v=RtSYWe23opM&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=43

Abstract

These videos discuss the form of the decomposition into partial fraction for a rational function whose denominator contains a repeated irreducible quadratic factor, and how to integrate the corresponding terms in the decomposition. This is illustrated on examples.

If the denominator $q(x)$ of a rational function $\frac{r(x)}{q(x)}$ in lowest terms contains a repeated irreducible quadratic term $(ax^2+bx+c)^n$ with $b^2-4ac<0$, then the decomposition into partial fractions of $\frac{r(x)}{q(x)}$ contains the terms

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\ldots+\frac{A_nx+B_n}{(ax^2+bx+c)^n},$$

where $A_1,\ldots,A_n$ and $B_1,\ldots,B_n$ are constants to be determined.

Example 6.5.1. For instance, the form of the decomposition of

$$\frac{4x+5}{(x+1)(x-2)^2(x^2+4)(x^2+x+1)^2}$$

is

$$\frac{4x+5}{(x+1)(x-2)^2(x^2+4)(x^2+x+1)^2}=\frac{A}{x+1}+\frac{B}{x-2}+\frac{C}{(x-2)^2}+\frac{Dx+E}{x^2+4}+\frac{Fx+G}{x^2+x+1}+\frac{Hx+I}{(x^2+x+1)^2}.$$

General method to integrate $\frac{Ax+B}{(ax^2+bx+c)^n}$ for $b^2-4ac<0$ and $n>1$
This is in part based on the method to integrate $\frac{Ax+B}{ax^2+bx+c}$ on page 108.

1) Complete the square in ax^2+bx+c so that

$$\frac{1}{a^n}\frac{Ax+B}{\left(\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right)^n}$$

2) The substitution $u := x + \frac{b}{2a}$ yields an integral of the form

$$\int \frac{Cu + D}{(u^2 + \alpha^2)^n}\, du$$

3) Split the integral:

$$\int \frac{Cu + D}{(u^2 + \alpha^2)^n}\, du = C \int \frac{u}{(u^2 + \alpha^2)^n}\, du + D \int \frac{du}{(u^2 + \alpha^2)^n}.$$

4) The first integral can be calculated with the substitution $v = u^2 + \alpha^2$:

$$\int \frac{u}{(u^2 + \alpha^2)^n}\, du = \frac{1}{2} \int \frac{dv}{v^n} = \frac{v^{1-n}}{2(1-n)}.$$

5) The second integral can be calculated using trigonometric substitution as in Example 5.4.8: we use $u = \alpha \tan\theta$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ so that $du = \alpha \sec^2\theta\, d\theta$ and

$$\begin{aligned} \int \frac{du}{(u^2 + \alpha^2)^n} &= \alpha \int \frac{\sec^2\theta}{\left(\alpha^2(\tan^2\theta + 1)\right)^n}\, d\theta \\ &= \frac{\alpha}{\alpha^{2n}} \int \frac{\sec^2\theta}{\sec^{2n}\theta}\, d\theta \\ &= \frac{1}{\alpha^{2n-1}} \int \cos^{2(n-1)}\theta\, d\theta, \end{aligned}$$

and this can be evaluated as explained in Case 2 of Section 5.1.

6) Rewrite your answer in terms of x.

Example 6.5.2. Evaluate

$$\int \frac{1 - x + 2x^2 - x^3}{x(x^2 + 1)^2}\, dx.$$

Solution. The numerator has degree less than the denominator, so the form of the decomposition into partial fractions is

$$\begin{aligned} \frac{1 - x + 2x^2 - x^3}{x(x^2 + 1)^2} &= \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2} \\ &= \frac{A(x^2 + 1)^2 + (Bx + C)x(x^2 + 1) + (Dx + E)x}{x(x^2 + 1)^2}, \end{aligned}$$

so that, identifying numerators and reordering the terms on the right hand side, we have

$$-x^3 + 2x^2 - x + 1 = x^4(A + B) + x^3(C) + x^2(2A + B + D) + x(C + E) + A.$$

Thus, identifying the coefficients of corresponding degree:

$$\begin{cases} A+B=0 \\ C=-1 \\ 2A+B+D=2 \\ C+E=-1 \\ A=1 \end{cases} \iff \begin{cases} A=1 \\ B=-1 \\ C=-1 \\ D=1 \\ E=0 \end{cases}$$

and we have

$$\begin{aligned} \int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx &= \int \frac{1}{x}-\frac{x+1}{x^2+1}+\frac{x}{(x^2+1)^2}\,dx \\ &= \ln|x|-\int \frac{x}{x^2+1}\,dx-\int \frac{dx}{x^2+1}+\int \frac{x}{(x^2+1)^2}\,dx \\ &= \ln|x|-\frac{1}{2}\int \frac{du}{u}-\arctan x+\frac{1}{2}\int \frac{du}{u^2} \text{ for } u=x^2+1 \text{ so that } du=2x\,dx \\ &= \ln|x|-\frac{1}{2}\ln|u|-\arctan x-\frac{1}{2u}+C \\ &= \ln|x|-\frac{1}{2}\ln(x^2+1)-\arctan x-\frac{1}{2(x^2+1)}+C. \end{aligned}$$

EXAMPLE 6.5.3. Evaluate

$$\int \frac{x+1}{(x^2+x+1)^2}\,dx.$$

Solution. This is already a partial fraction of the type $\frac{Ax+B}{(ax^2+bx+c)^n}$ for $b^2-4ac<0$ as discussed above. Thus, we first need to complete the square:

$$x^2+x+1=\left(x+\frac{1}{2}\right)^2-\frac{1}{4}+1=\left(x+\frac{1}{2}\right)^2+\frac{3}{4},$$

and then substitute $u=x+\frac{1}{2}$, so that $x=u-\frac{1}{2}$ and

$$\begin{aligned}\int \frac{x+1}{(x^2+x+1)^2}\,dx &= \int \frac{u-\frac{1}{2}+1}{\left(u^2+\frac{3}{4}\right)^2}\,du\\ &= \int \frac{u}{\left(u^2+\frac{3}{4}\right)^2}\,du+\frac{1}{2}\int \frac{du}{\left(u^2+\frac{3}{4}\right)^2}.\end{aligned}$$

We evaluate the first integral by substitution, using $v=u^2+\frac{3}{4}$ so that $dv=2u\,du$ and

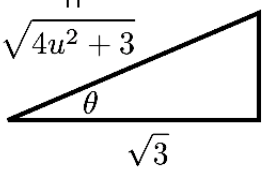
$$\int \frac{u}{\left(u^2+\frac{3}{4}\right)^2}\,du=\frac{1}{2}\int \frac{dv}{v^2}=-\frac{1}{2v}=-\frac{1}{2\left(u^2+\frac{3}{4}\right)}=-\frac{1}{2\left(x^2+x+1\right)}.$$

To evaluate the second integral, we use trigonometric substitution with $u=\frac{\sqrt{3}}{2}\tan\theta$ and $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$. Then $du=\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta$ and

$$\begin{aligned}\int \frac{du}{\left(u^2+\frac{3}{4}\right)^2} &= \frac{\sqrt{3}}{2}\int \frac{\sec^2\theta}{\left(\frac{3}{4}\left(\tan^2\theta+1\right)\right)^2}\,d\theta\\ &= \frac{\sqrt{3}}{2}\cdot\frac{16}{9}\int \frac{\sec^2\theta}{\left(\sec^2\theta\right)^2}\,d\theta\\ &= \frac{8\sqrt{3}}{9}\int \cos^2\theta\,d\theta\\ &= \frac{8\sqrt{3}}{9}\int \frac{1+\cos 2\theta}{2}\,d\theta\\ &= \frac{4\sqrt{3}}{9}\left(\theta+\frac{1}{2}\sin(2\theta)\right)+C\\ &= \frac{4\sqrt{3}}{9}\left(\theta+\sin\theta\cos\theta\right)+C \text{ by (4.2.1)}\end{aligned}$$

On the other hand, as $\tan\theta = \frac{2u}{\sqrt{3}}$, we can represent the situation on the following triangle

$$\sqrt{4\left(x+\frac{1}{2}\right)^2+3} = \sqrt{4x^2+4x+4}$$
$$||$$



$$2u = 2x+1$$

on which we read that

$$\sin\theta\cos\theta = \frac{2\sqrt{3}u}{\left(\sqrt{4u^2+3}\right)^2} = \frac{\sqrt{3}(2x+1)}{4x^2+4x+4}.$$

Given that $\theta = \arctan\left(\frac{2u}{\sqrt{3}}\right) = \arctan\left(\frac{2x+1}{\sqrt{3}}\right)$, we conclude that

$$\begin{aligned}\int \frac{du}{\left(u^2+\frac{3}{4}\right)^2} &= \frac{4\sqrt{3}}{9}\left(\theta+\sin\theta\cos\theta\right)+C \\ &= \frac{4\sqrt{3}}{9}\left(\arctan\left(\frac{2x+1}{\sqrt{3}}\right)+\frac{\sqrt{3}(2x+1)}{4x^2+4x+4}\right)+C\end{aligned}$$

and thus

$$\begin{aligned}\int \frac{x+1}{\left(x^2+x+1\right)^2}\,dx &= \int \frac{u}{\left(u^2+\frac{3}{4}\right)^2}\,du+\frac{1}{2}\int \frac{du}{\left(u^2+\frac{3}{4}\right)^2} \\ &= -\frac{1}{2\left(x^2+x+1\right)}+\frac{2\sqrt{3}}{9}\left(\arctan\left(\frac{2x+1}{\sqrt{3}}\right)+\frac{\sqrt{3}(2x+1)}{4x^2+4x+4}\right)+C.\end{aligned}$$

Exercises

you are now prepared to work on the Practice Problems, and Homework set M6C in the manual of exercises.

7 M7: Improper Integrals

7.1 Improper integrals of type I

Watch the videos at

https://www.youtube.com/watch?v=DkWSsckVrpg&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=44

and

https://www.youtube.com/watch?v=twK6qeDVEww&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=45

Abstract

These two videos introduce and define improper integrals of type I, that is,

$$\int_{-\infty}^{a} f(x)\,dx;\ \int_{a}^{\infty} f(x)\,dx;\ \int_{-\infty}^{\infty} f(x)\,dx$$

and go over a number of explicit calculations.

Definition 7.1.1 (Improper integrals of type I)

1) If $\int_a^t f(x)\,dx$ exists for every $t \geq a$, then
$$\int_a^\infty f(x)\,dx := \lim_{t\to\infty} \int_a^t f(x)\,dx$$
is called *convergent* if the limit is finite, and *divergent* if the limit does not exist.

2) If $\int_t^b f(x)\,dx$ exists for every $t \leq b$, then
$$\int_{-\infty}^b f(x)\,dx := \lim_{t\to-\infty} \int_t^b f(x)\,dx$$
is called *convergent* if the limit is finite, and *divergent* if the limit does not exist.

3) If both $\int_{-\infty}^a f(x)\,dx$ and $\int_a^\infty f(x)\,dx$ are convergent then
$$\int_{-\infty}^\infty f(x)\,dx := \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$
is *convergent*, and *divergent* otherwise.

Proposition 7.1.2. *Let p be a real number. Then*
$$\int_1^\infty \frac{dx}{x^p}$$
is convergent if and only if $p > 1$.

Example 7.1.3. Are the following integrals convergent or divergent. If convergent, find the value:

1)
$$\int_{-\infty}^0 \frac{dx}{2x-5}$$

Solution.
$$\begin{aligned}
\int_{-\infty}^0 \frac{dx}{2x-5} &= \lim_{t\to-\infty} \int_t^0 \frac{dx}{2x-5} \\
&= \lim_{t\to-\infty} \frac{1}{2} \int_{2t-5}^{-5} \frac{du}{u} \text{ for } u = 2x-5 \\
&= \lim_{t\to-\infty} \frac{1}{2} \left[\ln|u|\right]_{2t-5}^{-5} \\
&= \lim_{t\to-\infty} \frac{1}{2} \left(\ln 5 - \ln|2t-5|\right) = -\infty,
\end{aligned}$$
so that the integral is divergent.

2)

$$\int_0^\infty \frac{x}{(x^2+2)^2}\,dx$$

Solution.

$$\begin{aligned}\int_0^\infty \frac{x}{(x^2+2)^2}\,dx &= \lim_{t\to\infty}\int_0^t \frac{x}{(x^2+2)^2}\,dx \\ &= \lim_{t\to\infty}\frac{1}{2}\int_2^{t^2+2}\frac{du}{u^2} \text{ for } u=x^2+2 \\ &= \lim_{t\to\infty}\left[-\frac{1}{2u}\right]_2^{t^2+2} \\ &= \lim_{t\to\infty}\frac{1}{4}-\frac{1}{2\left(t^2+2\right)}=\frac{1}{4}.\end{aligned}$$

3)

$$\int_{-\infty}^{-1} e^{-2t}\,dt$$

Solution.

$$\begin{aligned}\int_{-\infty}^{-1} e^{-2t}\,dt &= \lim_{x\to-\infty}\int_x^{-1} e^{-2t} \\ &= \lim_{x\to-\infty}\left[-\frac{1}{2}e^{-2t}\right]_x^{-1} \\ &= \lim_{x\to-\infty}\frac{1}{2e^{2x}}-\frac{e^2}{2}=\infty,\end{aligned}$$

so that the integral is divergent.

4)

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^2}$$

Solution.

$$\begin{aligned}\int_0^\infty \frac{dx}{1+x^2} &= \lim_{t\to\infty}\int_0^t \frac{dx}{1+x^2} \\ &= \lim_{t\to\infty}\left[\arctan x\right]_0^t \\ &= \lim_{t\to\infty}\arctan t=\frac{\pi}{2}\end{aligned}$$

and

$$\int_{-\infty}^0 \frac{dx}{1+x^2}=\lim_{t\to-\infty}\int_t^0 \frac{dx}{x^2+1}=\lim_{t\to-\infty}\left[\arctan t\right]_t^0=\lim_{t\to-\infty}-\arctan t=\frac{\pi}{2}.$$

Thus

$$\int_{-\infty}^\infty \frac{dx}{1+x^2}=\int_{-\infty}^0 \frac{dx}{1+x^2}+\int_0^\infty \frac{dx}{1+x^2}=\frac{\pi}{2}+\frac{\pi}{2}=\pi.$$

5)

$$\int_0^\infty \cos^2\theta\, d\theta$$

Solution.

$$\begin{aligned}\int_0^\infty \cos^2\theta\, d\theta &= \lim_{t\to\infty}\int_0^t \frac{\cos(2\theta)+1}{2}\, d\theta \\ &= \lim_{t\to\infty}\left[\frac{\theta}{2}+\frac{1}{4}\sin(2\theta)\right]_0^t = \lim_{t\to\infty}\frac{t}{2}+\frac{1}{4}\sin(2t) = \infty,\end{aligned}$$

so that the integral is divergent.

Exercises

you are now prepared to work on the Practice Problems, and Homework set M7A in the manual of exercises.

7.2 Improper integrals of type II

Watch the videos at

https://www.youtube.com/watch?v=ISndl7-oAN0&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=46

and

https://www.youtube.com/watch?v=hXTcKdCMx0c&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=47

Abstract

These videos introduce and define improper integrals of type II, that is, integrals of functions on a closed interval, that have discontinuities. They go over several examples of calculations.

Definition 7.2.1 (Improper integrals of type II)

1) If f is continuous on $[a, b)$ and discontinuous at b then
$$\int_a^b f(x)\,dx := \lim_{t\to b^-} \int_a^t f(x)\,dx$$
is *convergent* if the limit is finite, and *divergent* if the limit does not exist.

2) If f is continuous on $(a, b]$ and discontinuous at a then
$$\int_a^b f(x)\,dx := \lim_{t\to a^+} \int_t^b f(x)\,dx$$
is *convergent* if the limit is finite, and *divergent* if the limit does not exist.

3) If f is continuous on $[a, b]$ except at $c \in (a, b)$ then
$$\int_a^b f(x)\,dx := \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$
is *convergent* if and only if both improper integrals are convergent.

EXAMPLE 7.2.2. Are the following improper integrals convergent or divergent? If convergent, find its value.

1)

$$\int_0^3 \frac{dx}{x\sqrt{x}}$$

Solution. The only discontinuity on $[0,3]$ is 0. Thus

$$\begin{aligned}\int_0^3 \frac{dx}{x\sqrt{x}} &= \lim_{t\to 0^+}\int_t^3 \frac{dx}{x\sqrt{x}} = \lim_{t\to 0^+}\int_t^3 x^{-\frac{3}{2}}\,dx \\ &= \lim_{t\to 0^+}\left[-\frac{2}{\sqrt{x}}\right]_t^3 = \lim_{t\to 0^+}\frac{2}{\sqrt{t}} - \frac{2}{\sqrt{3}} = \infty\end{aligned}$$

and the integral is divergent.

2)

$$\int_1^9 \frac{dx}{\sqrt[3]{x-9}}$$

Solution. The only discontinuity on [1, 9] is 9. Thus

$$\begin{aligned}\int_1^9 \frac{dx}{\sqrt[3]{x-9}} &= \lim_{t\to 9^-}\int_1^t \frac{dx}{\sqrt[3]{x-9}} = \lim_{t\to 9^-}\int_1^t (x-9)^{-\frac{1}{3}}\,dx \\ &= \lim_{t\to 9^-}\left[\frac{3}{2}(x-9)^{\frac{2}{3}}\right]_1^t = \lim_{t\to 9^-}\frac{3}{2}(t-9)^{\frac{2}{3}} - 6 = -6.\end{aligned}$$

3)

$$\int_0^4 \frac{dx}{x^2+x-6}$$

Solution. Since $x^2+x-6 = (x-2)(x+3)$, the only discontinuity on [0, 4] is 2. Moreover

$$\int_0^2 \frac{dx}{x^2+x-6} = \lim_{t\to 2^-}\int_0^t \frac{dx}{(x-2)(x+3)} = \lim_{t\to 2^-}\int_0^t \frac{A}{x-2} + \frac{B}{x+3}\,dx$$

and the handcover method easily yields $A = \frac{1}{5}$ and $B = -\frac{1}{5}$. Thus

$$\begin{aligned}\int_0^2 \frac{dx}{x^2+x-6} &= \lim_{t\to 2^-}\frac{1}{5}\int_0^t \frac{1}{x-2} - \frac{1}{x+3}\,dx \\ &= \lim_{t\to 2^-}\frac{1}{5}\left[\ln|x-2| - \ln|x+3|\right]_0^t = -\infty\end{aligned}$$

because $\lim_{t\to 2^-}\ln|t-2| = -\infty$. Hence $\int_0^2 \frac{dx}{x^2+x-6}$ and thus $\int_0^4 \frac{dx}{x^2+x-6}$ are divergent.

4)

$$\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$$

Solution. The only discontinuity on [0, 1] is 0. Thus

$$\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{\ln x}{\sqrt{x}}\,dx.$$

We proceed by parts with $u = \ln x$ and $dv = x^{-\frac{1}{2}}\,dx$ so that $du = \frac{dx}{x}$ and $v = 2\sqrt{x}$. Then

$$\begin{aligned}\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx &= \lim_{t\to 0^+}\left(\left[2\sqrt{x}\ln x\right]_t^1 - 2\int_t^1 \frac{dx}{\sqrt{x}}\right)\\ &= \lim_{t\to 0^+}\left[2\sqrt{x}\ln x - 4\sqrt{x}\right]_t^1\\ &= \lim_{t\to 0^+} -4 - \sqrt{t}\,(2\ln t - 4)\,.\end{aligned}$$

Moreover,

$$\lim_{t\to 0^+}\sqrt{t}\ln t = \lim_{t\to 0^+}\frac{\ln t}{\frac{1}{\sqrt{t}}} \overset{H}{=} \lim_{t\to 0^+}\frac{\frac{1}{t}}{-\frac{1}{2}t^{-\frac{3}{2}}} = \lim_{t\to 0^+} -2t^{\frac{1}{2}} = 0.$$

Thus

$$\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{t\to 0^+} -4 - \sqrt{t}\,(2\ln t - 4) = -4.$$

7.3 Comparison for improper integrals

Watch the videos at

https://www.youtube.com/watch?v=-DHgr52BHlc&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=48

and

https://www.youtube.com/watch?v=vWNqvlB0F8Y&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=49

Abstract

In these videos, the method of comparison for improper integrals is presented, general statements given, and several examples illustrate the method.

THEOREM 7.3.1 (Comparison for improper integrals (type I)) *If f and g are continuous on $[a,\infty)$ with*

$$f(x) \geq g(x) \geq 0 \text{ for all } x \geq a$$

then

1) *If $\int_a^\infty f(x)\,dx$ is convergent, so is $\int_a^\infty g(x)\,dx$.*
2) *If $\int_a^\infty g(x)\,dx$ is divergent, so is $\int_a^\infty f(x)\,dx$.*

THEOREM 7.3.2 (Comparison for improper integrals (type II)) *If f and g are continuous on $[a,b)$ and discontinuous at b with*

$$f(x) \geq g(x) \geq 0 \text{ for all } a \leq x < b$$

then

1) *If $\int_a^b f(x)\,dx$ is convergent, so is $\int_a^b g(x)\,dx$.*
2) *If $\int_a^b g(x)\,dx$ is divergent, so is $\int_a^b f(x)\,dx$.*

REMARK 7.3.3. We leave it to you to write out the obvious analogues of Theorem 7.3.1 for integrals of the type $\int_{-\infty}^b f(x)\,dx$ and of Theorem 7.3.2 for integrals of the type $\int_a^b f(x)\,dx$, where f is discontinuous at a.

EXAMPLE 7.3.4. Is $\int_0^\infty e^{-x^2}\,dx$ convergent?

Solution. For $x \geq 1$, we have $x^2 \geq x$ and thus

$$e^{-x} \geq e^{-x^2} \geq 0.$$

Moreover

$$\int_1^\infty e^{-x} = \lim_{t\to\infty}\left[-e^{-x}\right]_1^t = \lim_{t\to\infty}\frac{1}{e} - \frac{1}{e^t} = \frac{1}{e}$$

is convergent. By comparison (Theorem 7.3.1), so is $\int_1^\infty e^{-x^2}\,dx$. since

$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx$$

and $\int_0^1 e^{-x^2}\,dx$ is finite, we conclude that $\int_0^\infty e^{-x^2}\,dx$ is convergent.

EXAMPLE 7.3.5. Decide whether the following integrals are convergent or divergent.

1)
$$\int_1^\infty \frac{1+|\cos x|}{x}\,dx$$

Solution. Since
$$\frac{1+|\cos x|}{x} \geq \frac{1}{x} \geq 0$$
on $[1,\infty)$, and $\int_1^\infty \frac{dx}{x}$ is divergent by Proposition 7.1.2, we conclude from Theorem 7.3.1 that $\int_1^\infty \frac{1+|\cos x|}{x}\,dx$ is also divergent.

2)
$$\int_1^\infty \frac{2+e^{-x}}{x^2}\,dx$$

Solution. Note that for $x \geq 1$, $e^{-x} \leq e^{-1} \leq 1$ so that
$$0 \leq \frac{2+e^{-x}}{x^2} \leq \frac{3}{x^2}.$$
Moreover, $\int_1^\infty \frac{3}{x^2}\,dx = 3\int_1^\infty \frac{dx}{x^2}$ is convergent (by Proposition 7.1.2). Thus, by comparison $\int_1^\infty \frac{2+e^{-x}}{x^2}\,dx$ is also convergent.

3)
$$\int_1^\infty \frac{x}{\sqrt{1+x^6}}\,dx$$

Solution. Since $1+x^6 \geq x^6$ we conclude that $\sqrt{1+x^6} \geq \sqrt{x^6} = x^3$ so that
$$\frac{1}{x^2} = \frac{x}{x^3} \geq \frac{x}{\sqrt{1+x^6}}$$
for every $x \geq 1$. Since $\int_1^\infty \frac{dx}{x^2}$ is convergent, so is $\int_1^\infty \frac{x}{\sqrt{1+x^6}}\,dx$.

4)
$$\int_0^1 \frac{e^{-x}}{\sqrt{x}}\,dx$$

Solution. The only discontinuity on $[0,1]$ is 0. On the other hand, for $0 \leq x \leq 1$, we have
$$e^{-1} \leq e^{-x} \leq e^0 = 1$$

so that

$$\frac{e^{-1}}{\sqrt{x}} \leq \frac{e^{-x}}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}.$$

Moreover,

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{\sqrt{x}} = \lim_{t\to 0^+} \left[2\sqrt{x}\right]_t^1 = \lim_{t\to 0^+} 2 - 2\sqrt{t} = 2$$

is convergent, so that, by Theorem 7.3.2, $\int_0^1 \frac{e^{-x}}{\sqrt{x}}\, dx$ is also convergent.

Exercises

you are now prepared to work on the Practice Problems, and Homework set M7B in the manual of exercises.

Before turning to Chapter 8, you should also take Mock Test 2.

8 M8: Parametric Curves

8.1 Introduction to parametric curves

Watch the videos at

https://www.youtube.com/watch?v=5qIxW9ZvyIw&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=50

and

https://www.youtube.com/watch?v=4yaxDE-NOI0&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=51

Abstract

These two videos introduce the notion of a parametric curve and of parametric equations for such a curve. They show on examples that there are many ways to parametrize a curve. An example of parametric curve is sketched, and a parametrization is found for a curve described by mechanical motion.

A *parametric curve* in the plane is the image of *parametric equations*

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}, t \in I$$

where t is the *parameter*, ranging in an interval I. A given (parametric) curve in the plane may have a number of *parametrizations*, that is, sets of parametric equations that represent the curve. One might think of a parametric curve as the trajectory of a moving object (in the plane), and of parametric equations as describing the position at a given time, hence encoding *how* the curve is traced out (in what direction, at what speed, etc).

For instance, the unit circle $x^2 + y^2 = 1$ can be parametrized by

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases} \quad t \in [0, 2\pi) \text{(the circle is then described once counterclockwise)}$$

$$\begin{cases} x = \cos(2t) \\ y = \sin(2t) \end{cases} \quad t \in [0, \pi) \text{(the circle is then described once counterclockwise, but twice as fast)}$$

$$\begin{cases} x = \cos(2t) \\ y = \sin(2t) \end{cases} \quad t \in [0, 2\pi) \text{(the circle is then described twice counterclockwise)}$$

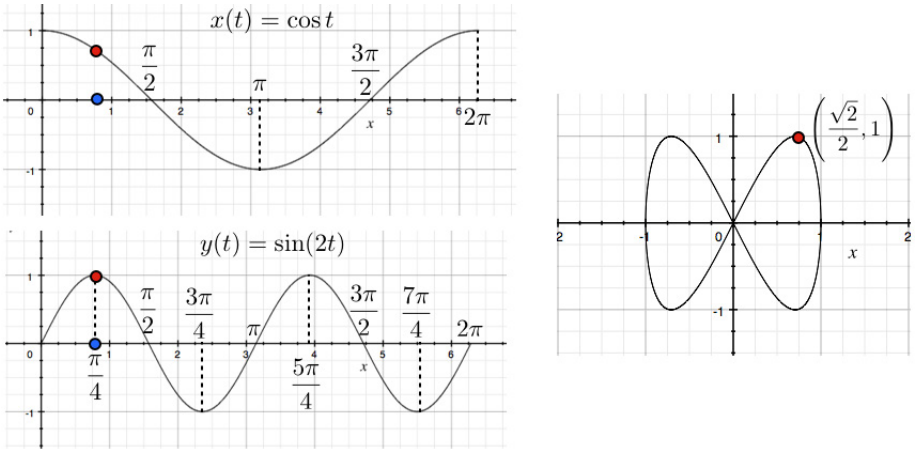
$$\begin{cases} x = \sin t \\ y = \cos t \end{cases} \quad t \in [0, 2\pi) \text{(the circle is then described once, clockwise)}$$

To sketch a parametric curve (from parametric equations), we study the variations of x and y as functions of the parameter t, and trace out the corresponding curve in the xy-plane.

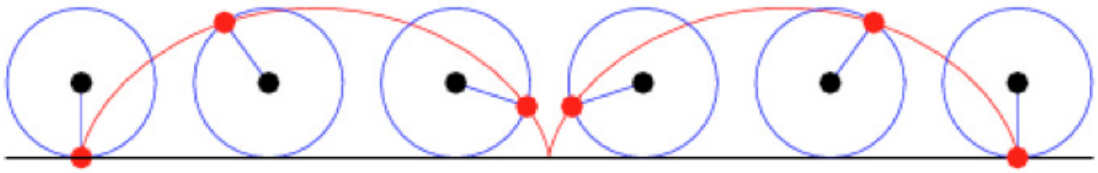
EXAMPLE 8.1.1. Sketch the curve given by

$$\begin{cases} x = \cos t \\ y = \sin(2t) \end{cases} \qquad t \in [0, 2\pi].$$

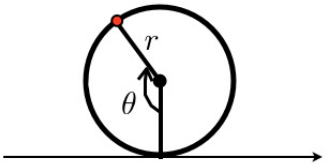
Solution. We draw $x(t)$ and $y(t)$ and then follow for various values of the parameter t the corresponding pair of coordinates (x, y). On the picture below, we have represented in particular the point on the curve corresponding to $t = \frac{\pi}{4}$.



EXAMPLE 8.1.2. The cycloid is the trajectory of a point on a circle rolling without sliding on a straight line.



We establish that, calling r the radius of the circle, and θ the angle between the radius joining the center of the circle with the point of contact of the circle with the straight line, and the radius joining the center to the point whose trajectory we study, as below,



the equations

$$\begin{cases} x = r\,(\theta - \sin\theta) \\ y = r\,(1 - \cos\theta) \end{cases} \qquad \theta \in (-\infty, \infty)$$

form parametric equations describing the cycloid.

8.2 Tangent lines to parametric curves

Watch the videos at

https://www.youtube.com/watch?v=Gv3hS7REvtg&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=52

and

https://www.youtube.com/watch?v=_bET7G94aW0&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=53

Abstract

In these videos, we study how to obtain the slope of the tangent line to a parametric curve at a point of given parameter, and how to find the points of a curve where the tangents have a given slope, are horizontal, or vertical. This is used to sketch a parametric curve.

Proposition 8.2.1. *If f and g are differentiable functions and*

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases} \quad t \in I$$

are parametric equations of a curve, the slope of the tangent line to the curve at $(f(a), g(a))$ *for* $a \in I$ *is given by*

$$\frac{dy}{dx}_{|t=a} = \frac{\frac{dy}{dt}_{|t=a}}{\frac{dx}{dt}_{|t=a}} = \frac{y'(a)}{x'(a)},$$

provided that $x'(a) \neq 0$.

Note that if $x'(a) = 0$ and $y'(a) \neq 0$ then the curve has a vertical tangent at the point of parameter *a*, while the tangent is horizontal if $x'(a) \neq 0$ and $y'(a) = 0$.

Example 8.2.2. Find an equation of the tangent line to the curve

$$\begin{cases} x = t^2 + t \\ y = t^2 - t \end{cases}$$

at the point of parameter $t = 0$.

Solution. The point of parameter 0 is $(x(0), y(0)) = (0, 0)$. The slope of the tangent line is

$$\frac{y'(0)}{x'(0)} = \frac{2t+1}{2t-1}_{|t=0} = \frac{1}{-1} = -1,$$

so that the tangent line is $y = -x$.

Example 8.2.3. Find an equation of the tangent line to the curve

$$\begin{cases} x = e^{\sqrt{t}} \\ y = t - \ln(t^2) \end{cases}$$

at the point of parameter $t = 1$.

Solution. The point of parameter 1 is $(x(1), y(1)) = (e, 1)$. The slope of the tangent line is

$$\frac{y'(1)}{x'(1)} = \frac{1 - \frac{2}{t}}{\frac{e^{\sqrt{t}}}{2\sqrt{t}}}_{|t=1} = -\frac{2}{e}.$$

Thus, an equation of the tangent line is

$$y - 1 = -\frac{2}{e}(x - e).$$

Example 8.2.4. Find the points on the curve

$$\begin{cases} x = t(t^2 - 3) \\ y = 3(t^2 - 3) \end{cases}$$

where the tangent line is horizontal or vertical. Sketch the curve.

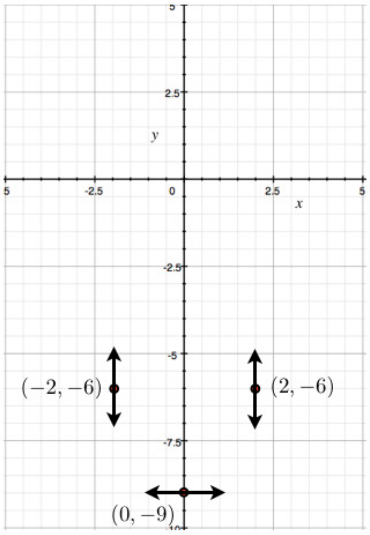
Solution. Since

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{6t}{3t^2 - 3} = \frac{2t}{(t-1)(t+1)},$$

we see that the curve has an horizontal tangent when $t = 0$, that is, at $(x(0), y(0)) = (0, -9)$, and a vertical tangent when $t = -1$ and when $t = 1$, that is, at

$$(x(1), y(1)) = (-2, -6) \text{ and } (x(-1), y(-1)) = (2, -6).$$

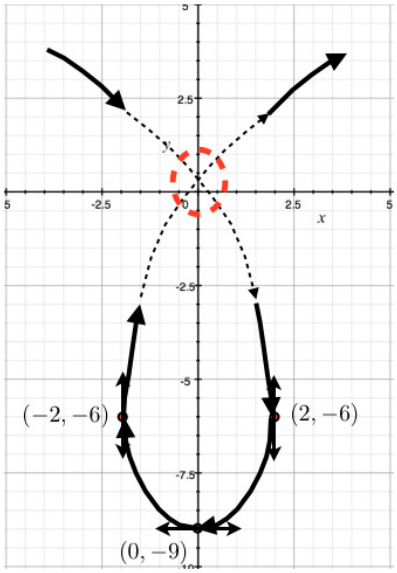
To sketch the curve, we can start with this information, combined with the variations of $x(t)$ and $y(t)$:



$$x'(t) = 3(t^2 - 1) = 3(t-1)(t+1)$$
$$y'(t) = 6t$$

t		-1		0		1	
$x'(t)$	$+$	0	$-$	0	$-$	0	$+$
$x(t)$	$-\infty$	$\nearrow$	2	$\searrow$	0	$\searrow$	-2	$\nearrow$	∞
$y'(t)$	$-$	0	$-$	0	$+$	0	$+$
$y(t)$	∞	$\searrow$	-6	$\searrow$	-9	$\nearrow$	-6	$\nearrow$	∞

Following the variations of $x(t)$ and $y(t)$, a rough sketch indicates a self-intersection:



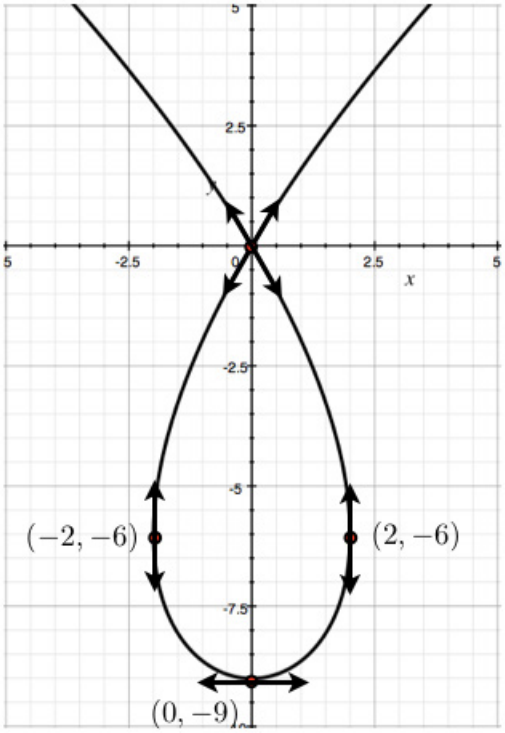
We can calculate explicitly the coordinates of this self-intersection point: two values t_1 and t_2 of the parameter correspond to the same point if

$$\begin{cases} t_1(t_1^2 - 3) = t_2(t_2^2 - 3) \\ 3(t_1^2 - 3) = 3(t_2^2 - 3) \end{cases} \underset{t_1 \neq t_2}{\Longrightarrow} t_1 = -\sqrt{3}; t_2 = \sqrt{3},$$

and this parameters correspond to the point (0, 0). Hence the slopes of the tangent lines at (0, 0) correspond to

$$\frac{y'(\sqrt{3})}{x'(\sqrt{3})} = \sqrt{3} \text{ and } \frac{y'(-\sqrt{3})}{x'(-\sqrt{3})} = -\sqrt{3},$$

so that we obtain the following sketch:



8.3 Symmetry; concavity

Watch the video at

https://www.youtube.com/watch?v=s88vR9a3t0E&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=54

Abstract

This video illustrates on the example of the astroid how to use symmetry to reduce the study of the curve to a smaller interval of the parameter, and how to study the concavity of the curve.

Symmetry

A parametric curve can admit one or more axis of symmetry, as well as center of symmetry. There are different ways this can be reflected by the parametric equations. In the example treated here the curve has parametric equations

$$\begin{cases} x(\theta) = 4\cos^3\theta \\ y(\theta) = 4\sin^3\theta \end{cases} \tag{8.3.1}$$

and the interval for θ is not specified, but we want to study the curve on an interval large enough for us to be able to draw out the whole curve.

The first observation is that both x and y are 2π-periodic, so that we can restrict the study to interval of length 2π, for instance $[-\pi, \pi]$.

The second observation is that

$$\begin{cases} x(-\theta) = x(\theta) \\ y(-\theta) = -y(\theta) \end{cases}$$

so that when a point of coordinates (x, y) is on the curve, so is the point of coordinates $(x, -y)$. In other words, the x-axis is an axis of symmetry for the curve and the transformation of the parameter $\theta \mapsto -\theta$ corresponds to this reflection about $y = 0$. Thus, we can study the curve on $[0, \pi]$ and obtain the second half of the curve by symmetry with respect to the x-axis.

Third, we observe that

$$\begin{cases} x(\pi - \theta) = -x(\theta) \\ y(\pi - \theta) = y(\theta) \end{cases}$$

so that $(-x, y)$ is on the curve whenever (x, y) is. In other words, the y-axis is also an axis of symmetry for the curve, and $\theta \mapsto \pi - \theta$ corresponds to the reflection about $x = 0$. Thus we can study the curve on $[0, \frac{\pi}{2}]$ and obtain the other half of the part of the curve corresponding to $[0, \pi]$ by symmetry with respect to the y-axis.

Concavity

Just like for graphs of functions, the concavity of the curve depends on the sign of $\frac{d^2y}{dx^2}$ along the curve. Specifically, the curve is concave up when $\frac{d^2y}{dx^2} > 0$ and concave down when $\frac{d^2y}{dx^2} < 0$. To find $\frac{d^2y}{dx^2}$ in terms of the parameter, say t, note that

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{y'(t)}{x'(t)}\right) \overset{\text{Chain Rule}}{=} \frac{d}{dt}\left(\frac{y'(t)}{x'(t)}\right) \cdot \frac{dt}{dx} = \frac{1}{x'(t)} \cdot \frac{d}{dt}\left(\frac{y'(t)}{x'(t)}\right).$$

Sketching the graph

Returning to the example (8.3.1), we have seen that we only need to study the variations of x and y over $[0, \frac{\pi}{2}]$ and then use symmetries with respect to both the x-axis and y-axis to complete the curve:

$$\begin{aligned} x'(\theta) &= -12\cos^2\theta\sin\theta \leq 0 \text{ on } \left[0, \tfrac{\pi}{2}\right] \\ y'(\theta) &= 12\sin^2\theta\cos\theta \geq 0 \text{ on } \left[0, \tfrac{\pi}{2}\right]. \end{aligned}$$

We obtain the following variations, and tangents at the end points:

θ	0		$\frac{\pi}{2}$
x'	0	$-$	0
x	4	$\searrow$	0
y'	0	$+$	0
y	0	$\nearrow$	4

$$\frac{dy}{dx} = \frac{y'}{x'} = -\frac{\sin^2\theta\cos\theta}{\cos^2\theta\sin\theta} = -\tan\theta$$

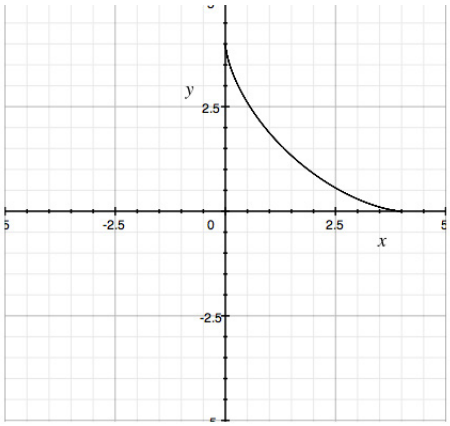
$$\Longrightarrow \frac{dy}{dx}_{|\theta=0} = 0 \Longrightarrow \text{ horizontal}$$

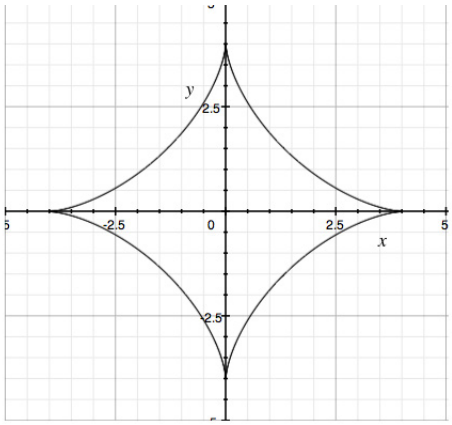
$$\Longrightarrow \frac{dy}{dx}_{|\theta=\frac{\pi}{2}} = -\infty \Longrightarrow \text{ vertical}$$

To complete the sketch, we need to know if the curve is concave up or concave down on this interval. Since

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{1}{x'(\theta)}\cdot\frac{d}{d\theta}\left(\frac{y'(\theta)}{x'(\theta)}\right) \\
&= \frac{-1}{12\cos^2\theta\sin\theta}\frac{d}{d\theta}\left(\frac{12\sin^2\theta\cos\theta}{-12\cos^2\theta\sin\theta}\right) \\
&= \frac{1}{12\cos^2\theta\sin\theta}\frac{d}{d\theta}(\tan\theta) \\
&= \frac{\sec^2\theta}{12\cos^2\theta\sin\theta} > 0 \text{ on } \left(0,\tfrac{\pi}{2}\right).
\end{aligned}$$

The sketch on $\left[0,\frac{\pi}{2}\right]$ gives the part on the left, and completing the curve by symmetry we obtain the complete curve on the right:





Exercises

you are now prepared to work on the Practice Problems, and Homework set M8A in the manual of exercises.

8.4 plane areas

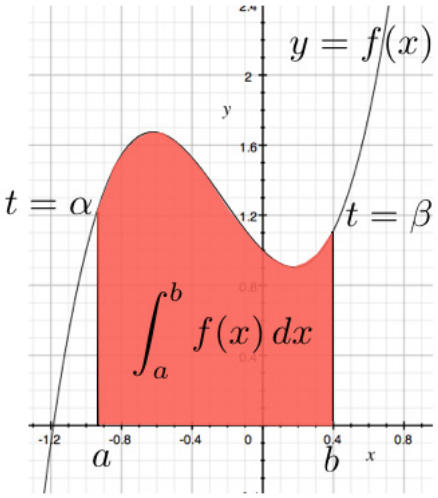
Watch the video at

https://www.youtube.com/watch?v=ZnnjO9PAnm8&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=55

Abstract

This video extrapolates from the case of the graph of a continuous positive function to obtain a formula for the area between a parametric curve and the *x*-axis. This is illustrated on the case of the astroid.

For the graph $y = f(x)$ of a non-negative continuous function *f*,



the area under the graph over $[a, b]$ is

$$\int_a^b f(x)\,dx = \int_a^b y\,dx.$$

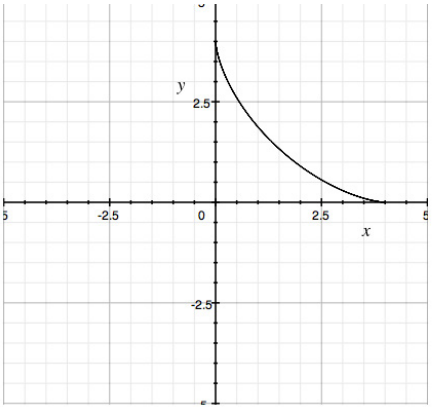
If we re-parametrize $y = f(x)$ in terms of a parameter $t \in [\alpha, \beta]$, where $x(\alpha) = a$, $x(\beta) = b$ and the curve is traversed exactly once as *t* increases from α to β, we can then rewrite the integral as

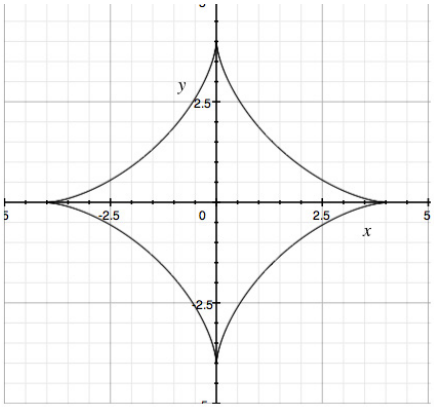
$$\int_a^b y\,dx = \int_\alpha^\beta y(t)\,x'(t)\,dt.$$

This approach can be applied to other parametric curves:

EXAMPLE 8.4.1. Find the area enclosed by the astroid $\begin{cases} x = 4\cos^3\theta \\ y = 4\sin^3\theta \end{cases}$ studied in Section 8.3.

Solution. As seen in Section 8.3, a sketch of the curve is as shown on the right hand side below





The left hand side corresponds to the part of the curve corresponding to $\theta \in \left[0, \frac{\pi}{2}\right]$, where $\theta = 0$ corresponds to the point (4, 0) and $\theta = \frac{\pi}{2}$ corresponds to (4, 0). Thus, the area under the curve on the left hand side can be represented as

$$\begin{aligned}\int_{\frac{\pi}{2}}^{0} y(\theta)x'(\theta)\,d\theta &= -\int_{0}^{\frac{\pi}{2}} 4\sin^3\theta\left(-12\cos^2\theta\sin\theta\right)\,d\theta \\ &= 48\int_{0}^{\frac{\pi}{2}} \sin^4\theta\cos^2\theta\,d\theta,\end{aligned}$$

and we use the methods discussed in Section 5.1 to evaluate this integral:

$$\begin{aligned}\int_{\frac{\pi}{2}}^{0} y(\theta)x'(\theta)\,d\theta &= 48\int_{0}^{\frac{\pi}{2}} \left(\frac{1-\cos 2\theta}{2}\right)^2\left(\frac{1+\cos 2\theta}{2}\right)\,d\theta \\ &= \frac{48}{8}\int_{0}^{\frac{\pi}{2}} 1-\cos 2\theta-\cos^2 2\theta+\cos^3 2\theta\,d\theta \\ &= 6\int_{0}^{\frac{\pi}{2}} 1-\cos 2\theta-\frac{1+\cos 4\theta}{2}\,d\theta+6\int_{0}^{\frac{\pi}{2}} \cos^2(2\theta)\,\cos(2\theta)\,d\theta \\ &= 6\left[\frac{\theta}{2}-\frac{1}{2}\sin(2\theta)-\frac{1}{8}\sin(4\theta)\right]_{0}^{\frac{\pi}{2}}+6\int_{0}^{\frac{\pi}{2}} \left(1-\sin^2(2\theta)\right)\cos(2\theta)\,d\theta \\ &= 6\cdot\frac{\pi}{4}+\frac{6}{3}\int_{0}^{0} 1-u^2\,du \text{ for } u=\sin(2\theta) \text{ and } du=2\cos(2\theta)\,d\theta \\ &= \frac{3\pi}{2}.\end{aligned}$$

Thus, by symmetry, the area enclosed by the astroid is 4 times this area, that is 6π.

8.5 arc length

Remark 8.5.1. As background, you can review the case of the graph of a function here.

Watch the videos at

https://www.youtube.com/watch?v=vnLUGQ5gdF0&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=56

and

https://www.youtube.com/watch?v=_2v7b9UHmCc&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=57

Abstract

A formula for the length of a piece of parametric curve is established and applied to a pair of examples.

Theorem 8.5.2 (Arc Length) *Let C be a curve of parametric equations* $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$ *for* $t \in [a, b]$, *where f,* f', *g, and* g' *are continuous on* [*a*, *b*] *and C is traversed exactly once as t increases from a to b. Then, the length of C is given by*

$$\int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Since the graph of a function $y = f(x)$ can be seen as a parametric curve $\begin{cases} x = t \\ y = f(t) \end{cases}$, we obtain:

Corollary 8.5.3. *If f and* f' *are continuous on* [*a*, *b*] *then the length of* $y = f(x)$ *for* $a \leq x \leq b$ *is given by*

$$\int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

Example 8.5.4. Find the length of the curve

$$\begin{cases} x = t^3 \\ y = t^2 \end{cases}, 0 \leq t \leq 4.$$

Solution. $x(t)$ and $y(t)$ are continuous with continuous derivatives, and the curve is traversed once as t in increases from 0 to 4 because $x(t)$ and $y(t)$ are one-to-one on [0, 4]. Thus the length of the curve is

$$\begin{aligned} L &= \int_0^4 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt \\ &= \int_0^4 \sqrt{(3t^2)^2 + (2t)^2}\, dt \\ &= \int_0^4 \sqrt{9t^4 + 4t^2}\, dt = \int_0^4 t\sqrt{9t^2 + 4}\, dt. \end{aligned}$$

We proceed then by substitution with $u = 9t^2 + 4$, so that $du = 18t\, dt$ and

$$L = \frac{1}{18}\int_4^{148} \sqrt{u}\, du = \frac{1}{18}\cdot\frac{2}{3}\left[u^{\frac{3}{2}}\right]_4^{148} = \frac{1}{27}\left(148^{\frac{3}{2}} - 8\right).$$

Example 8.5.5. Find the length of the astroid

$$\begin{cases} x = 4\cos^3\theta \\ y = 4\sin^3\theta \end{cases}.$$

Solution. As we have seen from Section 8.3, the length of this astroid is four times the length of the part of the astroid corresponding to $\theta \in \left[0, \frac{\pi}{2}\right]$. Moreover, $x(\theta)$ and $y(\theta)$ are continuous with continuous derivatives, and one-to-one, on $\left[0, \frac{\pi}{2}\right]$. Thus, the length is

$$\begin{aligned} L &= 4\int_0^{\frac{\pi}{2}} \sqrt{(x'(\theta))^2 + (y'(\theta))^2}\, d\theta \\ &= 4\int_0^{\frac{\pi}{2}} \sqrt{\left(-12\cos^2\theta\sin\theta\right)^2 + \left(12\sin^2\theta\cos\theta\right)^2} \\ &= 48\int_0^{\frac{\pi}{2}} \sqrt{\cos^2\theta\sin^2\theta\left(\cos^2\theta + \sin^2\theta\right)}\, d\theta \\ &= 48\int_0^{\frac{\pi}{2}} \cos\theta\sin\theta\, d\theta \text{ because } \cos^2\theta + \sin^2\theta = 1 \\ &= 24\int_0^{\frac{\pi}{2}} \sin(2\theta)\, d\theta \text{ because } \sin(2\theta) = 2\cos\theta\sin\theta \\ &= -12\left[\cos(2\theta)\right]_0^{\frac{\pi}{2}} = 24. \end{aligned}$$

8.6 Surface area of surface of revolutions

Watch the video at

https://www.youtube.com/watch?v=LxjD6kUJAlM&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=58

Abstract

Formulas for the area of a surface of revolution generated by rotating a parametric curve about an horizontal or vertical axis are established, and illustrated on examples.

THEOREM 8.6.1 (Area of a surface of revolution) *Let C be a curve of parametric equations* $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$ *for* $t \in [a, b]$, *where f,* f', *g, and* g' *are continuous on* $[a, b]$ *and C is traversed exactly once as t increases from a to b. Then the surface area of the surface generated by rotating C about the x-axis is given by*

$$A = 2\pi \int_a^b y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

and the surface area of the surface generated by rotating C about the y-axis is

$$A = 2\pi \int_a^b x(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

EXAMPLE 8.6.2. Find the surface area generated by rotating the curve

$$\begin{cases} x = e^t - t \\ y = 4e^{\frac{t}{2}} \end{cases}, t \in [0,1]$$

about the y-axis.

Solution. By Theorem 8.6.1,

$$\begin{aligned} A &= 2\pi \int_0^1 x(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= 2\pi \int_0^1 (e^t - t)\sqrt{(e^t-1)^2 + \left(2e^{\frac{t}{2}}\right)^2}\, dt \\ &= 2\pi \int_0^1 (e^t - t)\sqrt{e^{2t} + 2e^t + 1}\, dt \\ &= 2\pi \int_0^1 (e^t - t)\sqrt{(e^t+1)^2}\, dt \\ &= 2\pi \int_0^1 (e^t - t)(e^t + 1)\, dt \\ &= 2\pi \int_0^1 e^{2t} + e^t - te^t - t\, dt \\ &= 2\pi \left(\left[\frac{e^{2t}}{2} + e^t - \frac{t^2}{2}\right]_0^1 - \int_0^1 te^t\, dt \right), \end{aligned}$$

and we proceed by parts for the remaining integral, with $u = t$ and $dv = e^t\, dt$, so that $du = dt$ and $v = e^t$.

$$\begin{aligned} A &= 2\pi \left(\frac{e^2 - 1}{2} + e - \frac{3}{2} - \left[te^t - e^t\right]_0^1 \right) \\ &= \pi\left(e^2 + 2e - 6\right). \end{aligned}$$

Exercises

you are now prepared to work on the Practice Problems, and Homework set M8B in the manual of exercises.

9 M9: Polar Curves

9.1 Polar coordinates

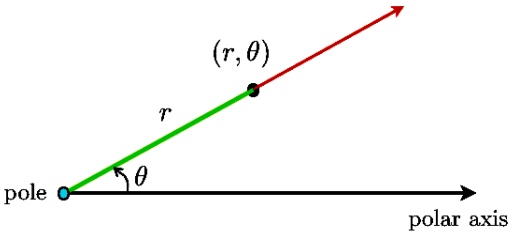
Watch the video at

https://www.youtube.com/watch?v=8AnWt-pXsqc&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=59

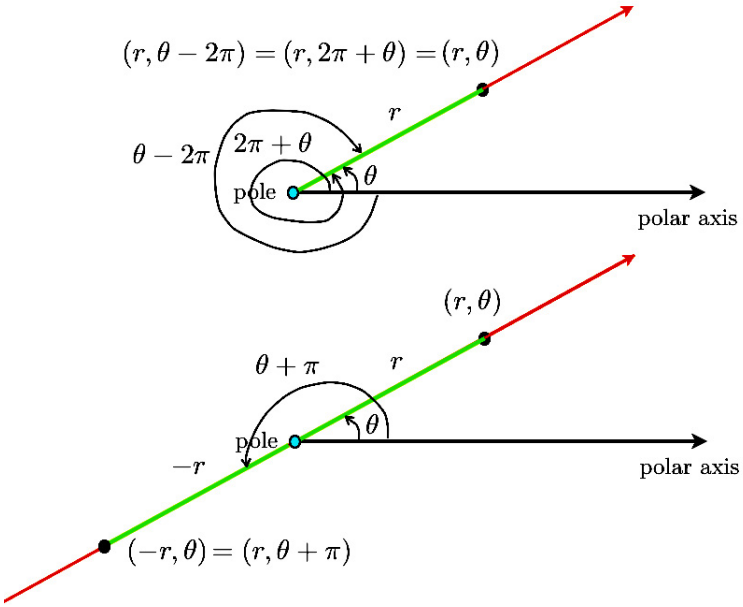
Abstract

In this video, we introduce the polar system of coordinates and how it is related to the Cartesian system.

To define a system of *polar coordinates* in the plane, we need to fix a point O, called *pole,* and a half-ray with O as origin called *polar axis*, and a unit of length. A point in the plane is then given by its *distance to the pole* r and the *angle* θ between the (positive) polar axis and the half-ray joining the pole to the point. The point is then given by the *polar coordinates* (r, θ) as shown below:

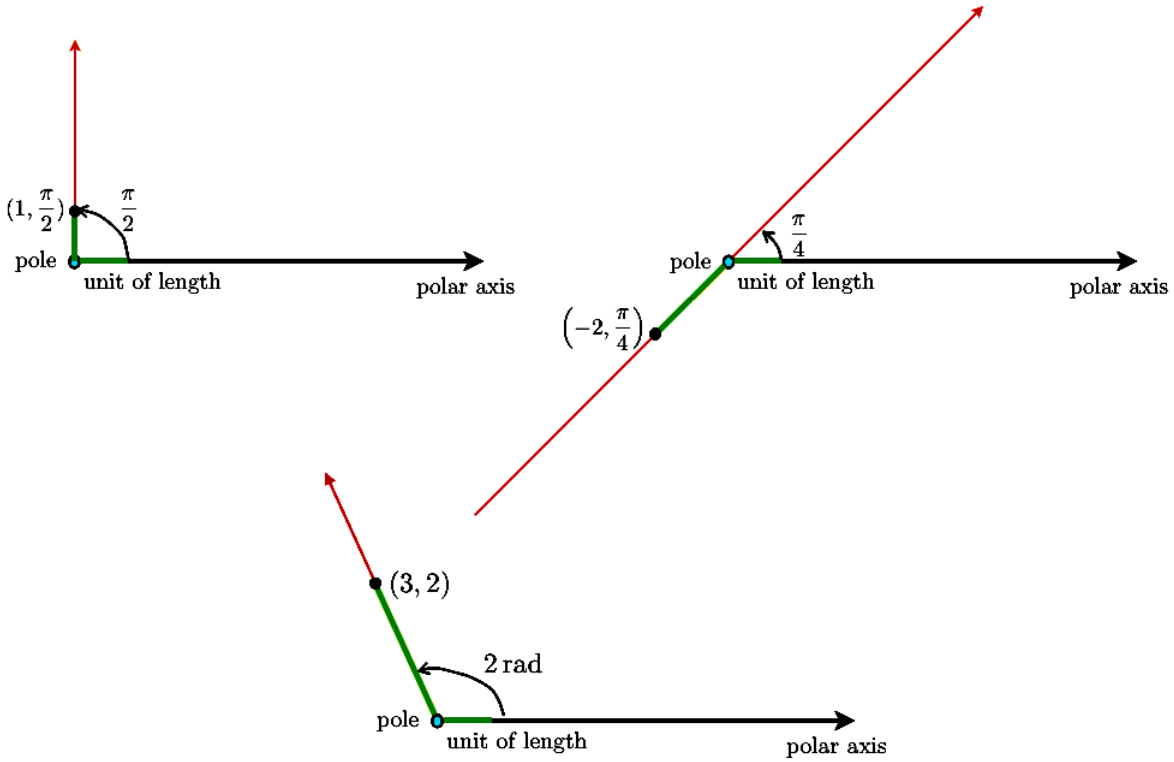


Of course, in this system the pole has an infinite number of pairs of polar coordinates, because $(0, \theta)$ represents the pole, regardless of the value of θ. On the other hand, all other points of the plane are represented by unique polar coordinates **only if we restrict ourselves to** $r \geq 0$ and $\theta \in [0, 2\pi)$. We will not always make this assumption, in which case each point has many pairs of polar coordinates:



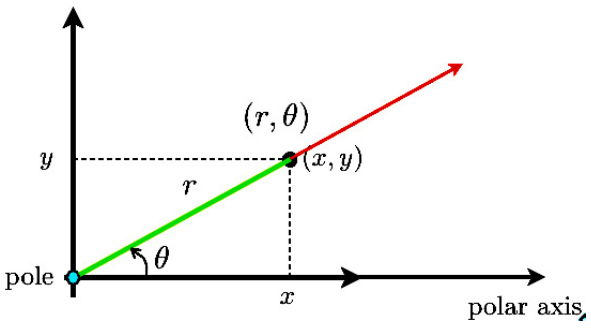
EXAMPLE 9.1.1. Plot the points of polar coordinates $\left(1, \frac{\pi}{2}\right)$, $\left(-2, \frac{\pi}{4}\right)$, $(3, 2)$.

Solution.



Polar system and Cartesian system

By convention, when a Cartesian system of coordinates is available, the *associated polar system* has the origin of the Cartesian system as pole, the positive x-axis as polar axis, and the same unit of length as the Cartesian system.



With these conventions, we have

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \tag{9.1.1}$$

$$r^2 = x^2 + y^2 \tag{9.1.2}$$

$$\tan\theta = \frac{y}{x}.$$

EXAMPLE 9.1.2. Find the Cartesian coordinates of the points of polar coordinates $\left(2\sqrt{2}, \frac{3\pi}{4}\right)$ and $\left(-1, \frac{\pi}{3}\right)$.

Solution. If the polar coordinates are $\left(2\sqrt{2}, \frac{3\pi}{4}\right)$ then

$$\begin{cases} x = r\cos\theta = 2\sqrt{2}\cos\frac{3\pi}{4} = -2 \\ y = r\sin\theta = 2\sqrt{2}\sin\frac{3\pi}{4} = 2 \end{cases}$$

and the point has Cartesian coordinates $(-2, 2)$.

If the polar coordinates are $\left(-1, \frac{\pi}{3}\right)$ then

$$\begin{cases} x = r\cos\theta = -\cos\frac{\pi}{3} = -\frac{1}{2} \\ y = r\sin\theta = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2} \end{cases}$$

and the point has Cartesian coordinates $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

9.2 Polar regions and polar curves

Watch the videos at

https://www.youtube.com/watch?v=0IK3LlokUVg&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=60

and

https://www.youtube.com/watch?v=HZx9e9DyqFI&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=61

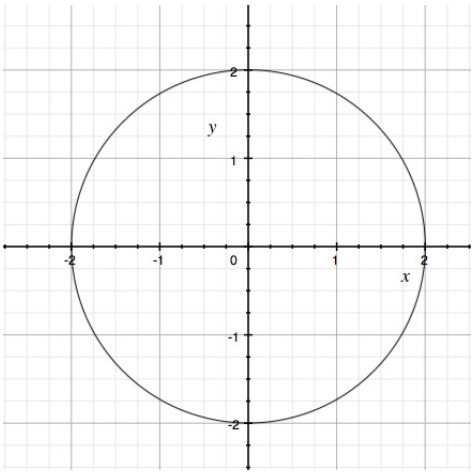
Abstract

These two videos give (sometimes rough) sketches of simple polar curves and polar regions of the plane.

Example 9.2.1. Find an equation in Cartesian coordinates of the curves given by the following equations in polar coordinates:

1) $r = 2$.
Solution. As $r^2 = x^2 + y^2$, this is the circle of radius 2 centered at the origin, of Cartesian equation

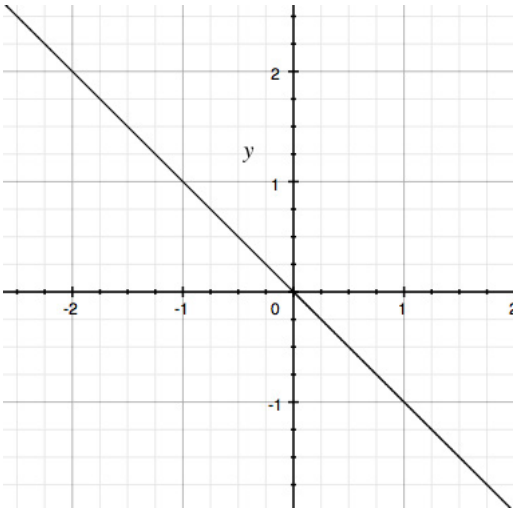
$$x^2 + y^2 = 4.$$



2) $\theta = \frac{3\pi}{4}$.

Solution. This is the set of points of the plane on the line making an angle of $\frac{3\pi}{4}$ with the positive x-axis, that is, $y = -x$. Indeed, the pole is on the curve, and for $x \neq 0$, we have $\tan\theta = \frac{y}{x}$, that is,

$$y = \tan\frac{3\pi}{4}x = -x.$$

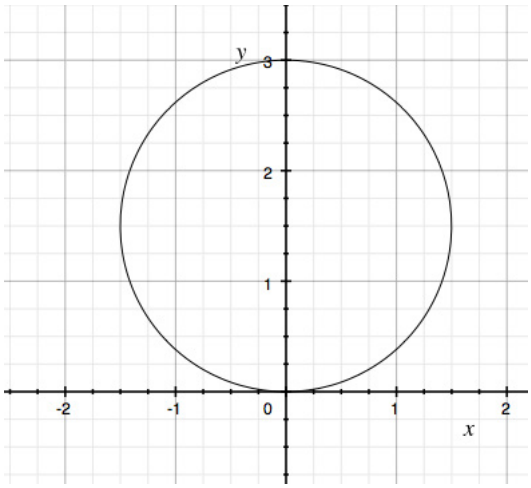


3) $r = 3\sin\theta$.

Solution. Since $\sin\theta = \frac{y}{r}$, we can rewrite this curve as $r^2 = 3y$, that is, in Cartesian coordinates

$$\begin{aligned} x^2 + y^2 = 3y &\iff x^2 + y^2 - 3y = 0 \\ &\iff x^2 + \left(y - \frac{3}{2}\right)^2 = \frac{9}{4} \end{aligned}$$

is the circle centered at $\left(0, \frac{3}{2}\right)$ of radius $\frac{3}{2}$.

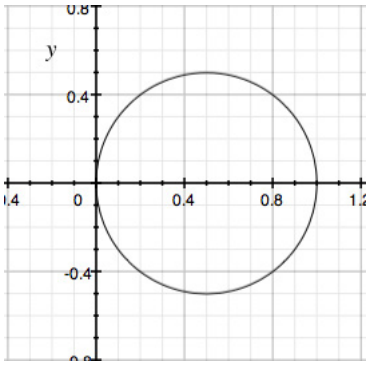


4) $r = \cos\theta$.

Solution. Since $\cos\theta = \frac{x}{r}$, we can rewrite this curve as $r^2 = x$, that is, in Cartesian coordinates

$$\begin{aligned} x^2 + y^2 = x \quad &\Longleftrightarrow \quad x^2 - x + y^2 = 0 \\ &\Longleftrightarrow \quad \left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4} \end{aligned}$$

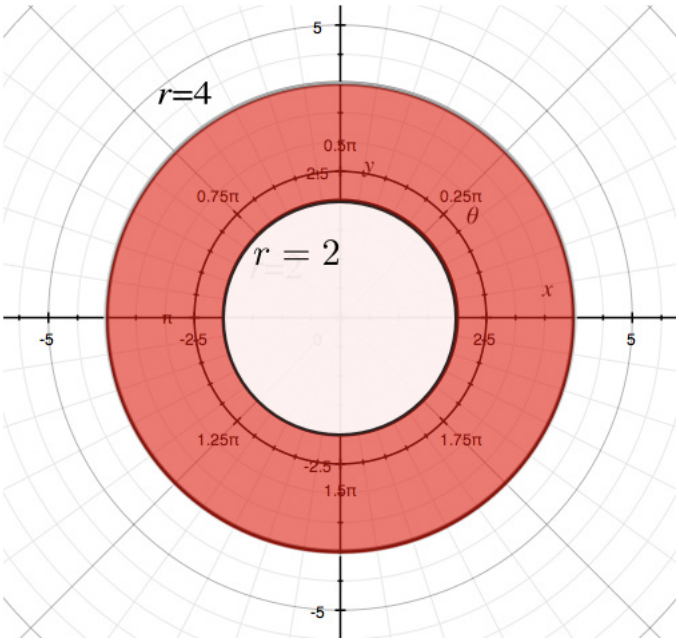
is the circle centered at $\left(\frac{1}{2}, 0\right)$ of radius $\frac{1}{2}$.



EXAMPLE 9.2.2. Sketch the following regions given in polar coordinates:

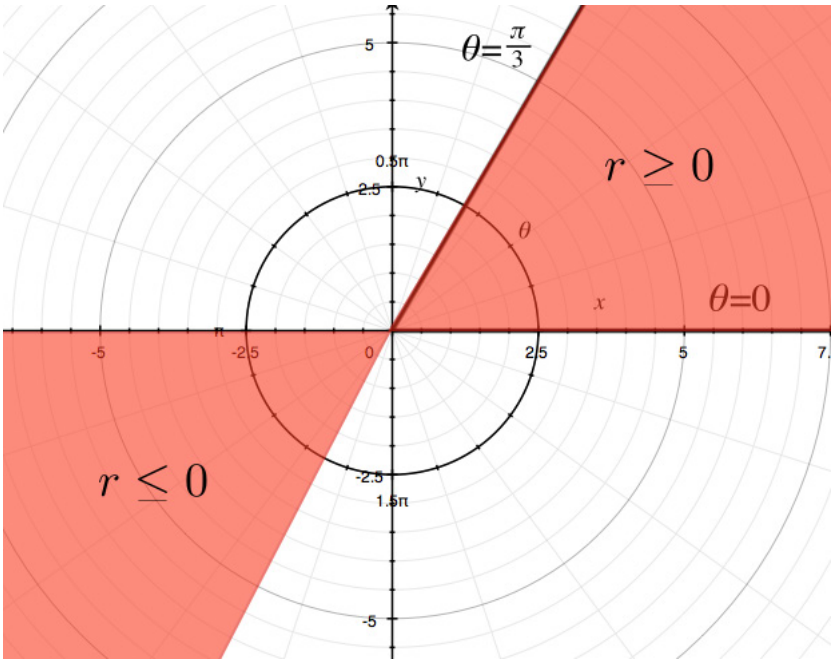
1) $2 \leq r \leq 4$.

Solution. The curves $r = 2$ and $r = 4$ are the circles centered at the origin of respective radii 2 and 4. The region $2 \leq r \leq 4$ is the annulus between them:



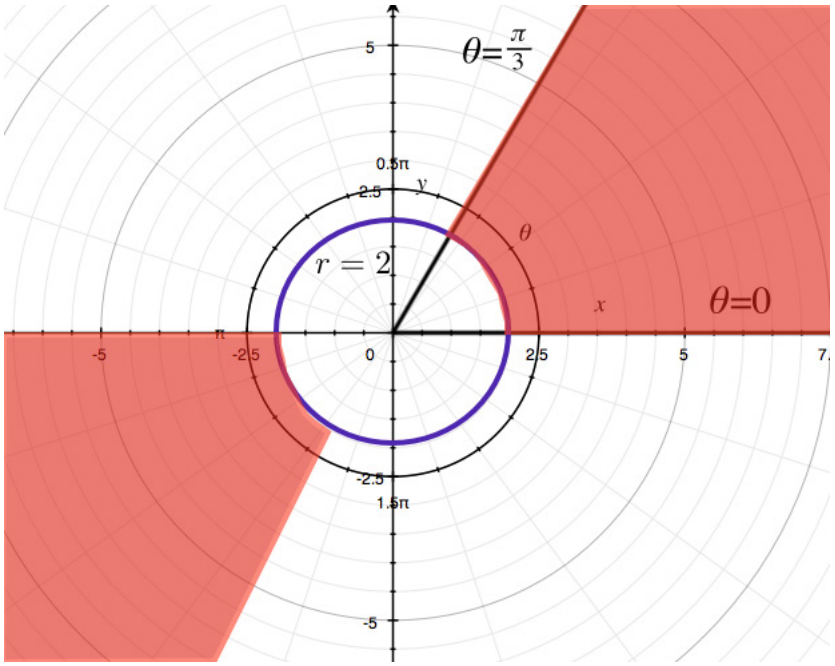
2) $0 \leq \theta \leq \frac{\pi}{3}$.

Solution. $\theta = 0$ is the x-axis and $\theta = \frac{\pi}{3}$ is the line through the origin of slope $\tan \frac{\pi}{3} = \sqrt{3}$. The region $0 \leq \theta \leq \frac{\pi}{3}$ is the sector between these two lines:



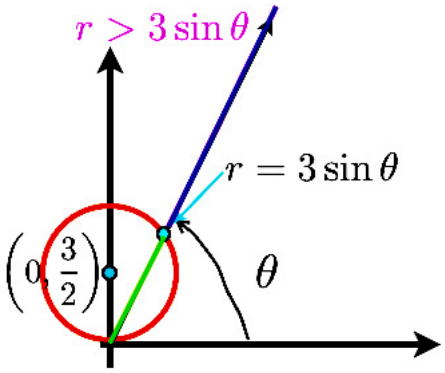
3) $0 \leq \theta \leq \frac{\pi}{3}$, $|r| \geq 2$.

Solution. To the sector above, we add the condition that $|r| \geq 2$, that is, $r \geq 2$ or $r \leq -2$. In other words, we add the condition that points are outside of the disk centered at the origin of radius 2:



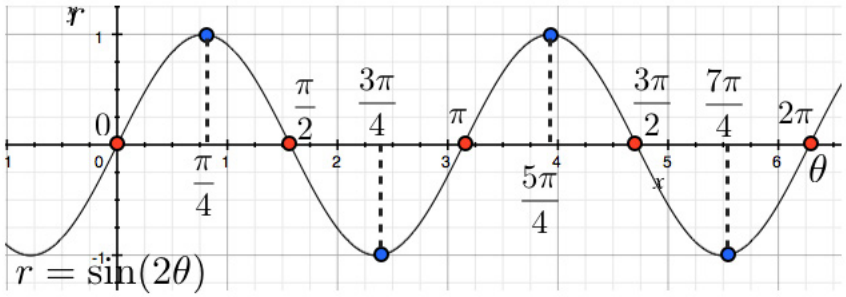
4) $0 \leq r \leq 3\sin\theta$.

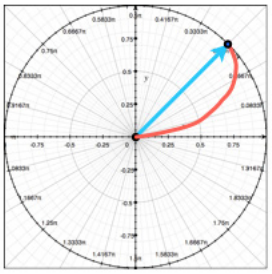
Solution. We have seen in question (2) that $r = 3\sin\theta$ is the circle centered at $(0, \frac{3}{2})$ of radius $\frac{3}{2}$. In view of the picture below, we see that the region $0 \leq r \leq 3\sin\theta$ is the corresponding disk:



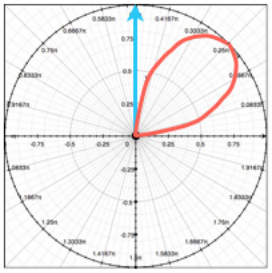
EXAMPLE 9.2.3. Sketch the polar curve $r = \sin(2\theta)$.

Solution. To sketch the curve, we first draw $r = \sin(2\theta)$ in the Cartesian coordinates θ and r in order to visualize the variations of r in terms of θ. Then we follow these variations in the polar coordinates (r, θ): to this end, for each half ray through the origin defined by a value of θ, we plot the point at (signed) distance $r(\theta)$ on this half-ray. Parts of the resulting curve traced out that way are sketched below for θ gradually increasing from 0 to $\frac{3\pi}{4}$:

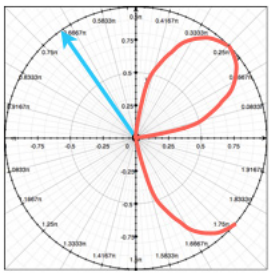




$\left[0, \frac{\pi}{4}\right]$



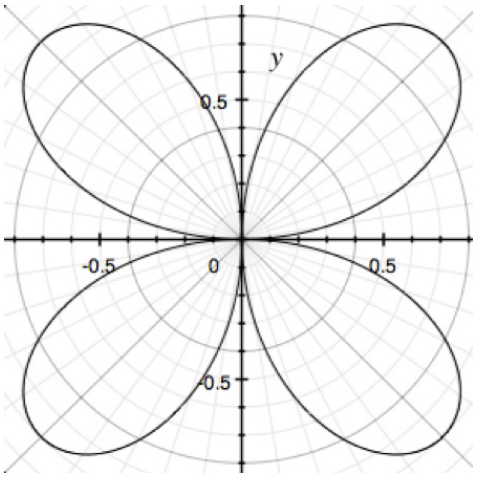
$\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$



$\left[\frac{\pi}{2}, \frac{3\pi}{4}\right], r \leq 0$

. . .

Completing the curve for $\theta \in [0, 2\pi]$, we obtain:



To be fair, to obtain such a sketch, we need to find a few tangent lines, which is what we will discuss after the exercises.

Exercises

you are now prepared to work on the Practice Problems, and Homework set M9A in the manual of exercises.

9.3 tangent lines to polar curves

Watch the videos at

https://www.youtube.com/watch?v=-Zh4SSUgRCs&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=62

and

https://www.youtube.com/watch?v=XCj9HearLzM&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=63

Abstract

In these two videos, a formula is established for the slope of the tangent line to a polar curve at a given point of the curve, tangent lines at the pole are examined, and several examples of polar curves are sketched.

Polar curves $r = f(\theta)$ can be seen, via (9.1.1), as parametric curves of the parameter θ:

$$\begin{cases} x = r\cos\theta = f(\theta)\cos\theta \\ y = r\sin\theta = f(\theta)\sin\theta \end{cases}, 0 \le \theta \le 2\pi.$$

Thus Proposition 8.2.1 regarding the slope of the tangent line to a parametric curve applies here to the effect that the slope of the tangent line to the point of $r = f(\theta)$ corresponding to $\theta = \theta_0$ is

$$\frac{dy}{dx}_{|\theta=\theta_0} = \frac{y'(\theta_0)}{x'(\theta_0)} = \frac{f'(\theta_0)\sin\theta_0 + f(\theta_0)\cos\theta_0}{f'(\theta_0)\cos\theta_0 - f(\theta_0)\sin\theta_0}. \tag{9.3.1}$$

Note also that when the curve goes through the pole for $\theta = \theta_0$, that is, when $f(\theta_0) = 0$, then

$$\frac{dy}{dx}_{|\theta=\theta_0} = \tan\theta_0$$

is the slope of the line $\theta = \theta_0$, which is the corresponding tangent line!

Example 9.3.1. Find the slope of the tangent lines to the polar curve $r = \sin(2\theta)$ (see Example 9.2.3) for $\theta = \frac{\pi}{4}$ and $\theta = \frac{\pi}{2}$.

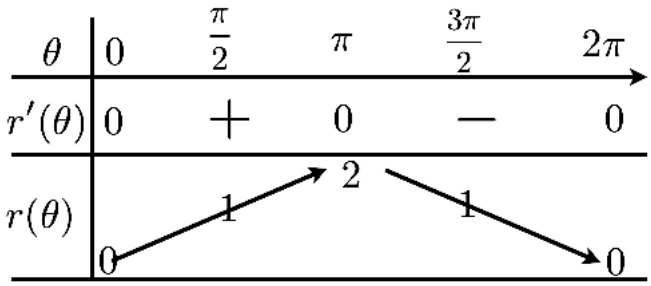
Solution. Since $r'(\theta) = 2\cos(2\theta)$, (9.3.1) gives

$$\frac{dy}{dx}_{|\theta=\frac{\pi}{4}} = \frac{2\cos\frac{\pi}{2}\sin\frac{\pi}{4} + \sin\frac{\pi}{2}\cos\frac{\pi}{4}}{2\cos\frac{\pi}{2}\cos\frac{\pi}{4} - \sin\frac{\pi}{2}\sin\frac{\pi}{4}} = \frac{\cos\frac{\pi}{4}}{-\sin\frac{\pi}{4}} = -1.$$

On the other hand, for $\theta = \frac{\pi}{2}$, $r = 0$, so that the corresponding tangent line at the pole is the vertical line $\theta = \frac{\pi}{2}$ (or $x = 0$).

EXAMPLE 9.3.2 (Cardioid) Sketch the polar curve $r = 1 - \cos\theta$.

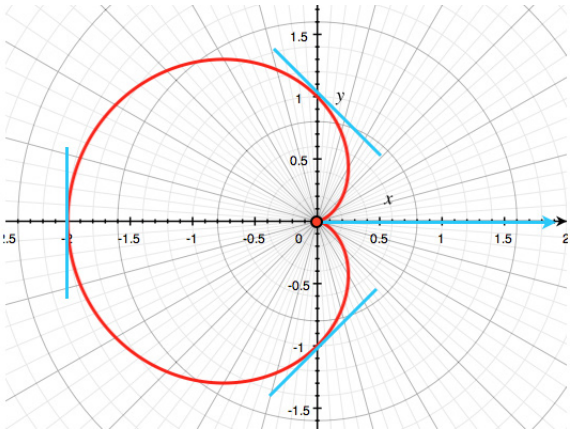
Solution. First we look at the variations of r as a function of θ. Since $r'(\theta) = \sin\theta$, we have:



Moreover, $r = 0$ for $\theta = 0$ and $\theta = 2\pi$, so that the tangent at the pole is horizontal. The tangent line at θ has slope

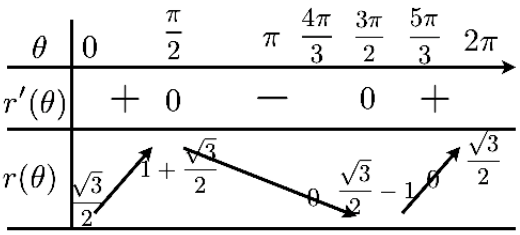
$$\frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{\sin^2\theta - \cos^2\theta + \cos\theta}{2\sin\theta\cos\theta - \sin\theta},$$

so that the slope for $\theta = \frac{\pi}{2}$ is -1, the slope at π is infinite and the tangent is therefore vertical, and the slope for $\theta = \frac{3\pi}{2}$ is 1. Taking all this into account, we obtain the following sketch:



EXAMPLE 9.3.3 (cardioid with a loop) Sketch the polar curve $r = \frac{\sqrt{3}}{2} + \sin\theta$.

Solution. First, we look at the variations of r as a function of θ. Note that $r'(\theta) = \cos\theta$, and $r = 0$ for $\sin\theta = -\frac{\sqrt{3}}{2}$, that is, for $\theta = \frac{4\pi}{3}$ or $\theta = \frac{5\pi}{3}$. Thus, we obtain the following variations:



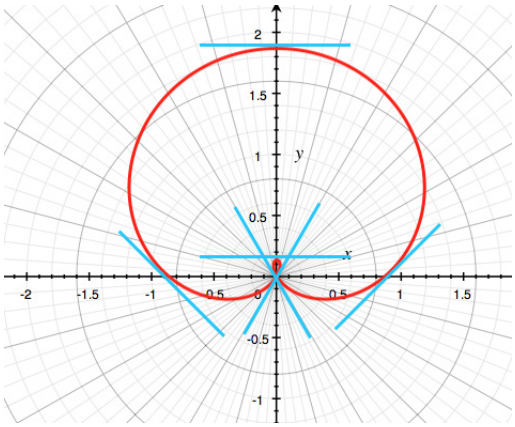
Moreover,

$$\begin{aligned}\frac{dy}{dx} &= \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} \\ &= \frac{\cos\theta\sin\theta + \left(\frac{\sqrt{3}}{2} + \sin\theta\right)\cos\theta}{\cos^2\theta - \left(\frac{\sqrt{3}}{2} + \sin\theta\right)\sin\theta},\end{aligned}$$

so that

$$\frac{dy}{dx}_{|\theta=0} = \frac{\sqrt{3}}{2} \text{ and } \frac{dy}{dx}_{|\theta=\frac{\pi}{2}} = \frac{dy}{dx}_{|\theta=\frac{3\pi}{2}} = 0.$$

Note also that $\theta = \frac{4\pi}{3}$ and $\theta = \frac{5\pi}{3}$ are tangent lines at the pole. Taking all this into account, we obtain the following sketch:



9.4 arc length for polar curves

Watch the video at

https://www.youtube.com/watch?v=phr8Ulc_N9k&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=64

Abstract

The formula for the length of a piece of parametric curve is particularized to a polar curve $r = f(\theta)$, and length of example of polar curves are calculated.

Theorem 8.5.2 is applied to the case of a curve $r = f(\theta)$, interpreted as a parametric curve

$$\begin{cases} x = f(\theta)\cos\theta \\ y = f(\theta)\sin\theta \end{cases} , 0 \leq \theta \leq 2\pi$$

to the effect that:

PROPOSITION 9.4.1. *If $r = f(\theta)$ has a continuous first derivative for $a \leq \theta \leq b$ and if the curve $r = f(\theta)$ is traced exactly once as θ runs from a to b, then the length of the curve is given by*

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

EXAMPLE 9.4.2. Find the length of the polar curve $r = 5\cos\theta$ for $0 \leq \theta \leq \frac{3\pi}{4}$.

Solution. The assumptions of Proposition 9.4.1 are satisfied, so that the length is given by

$$L = \int_0^{\frac{3\pi}{4}} \sqrt{25\cos^2\theta + 25\sin^2\theta}\, d\theta = 5\int_0^{\frac{3\pi}{4}} d\theta = 5\cdot\frac{3\pi}{4} = \frac{15\pi}{4}.$$

EXAMPLE 9.4.3. Find the length of the polar curve $r = \theta^2$ for $0 \leq \theta \leq 2\pi$.

Solution. The assumptions of Proposition 9.4.1 are satisfied, so that the length is given by

$$\begin{aligned}
L &= \int_0^{2\pi} \sqrt{r^2 + (r')^2}\, d\theta \\
&= \int_0^{2\pi} \sqrt{\theta^4 + (2\theta)^2}\, d\theta \\
&= \int_0^{2\pi} \sqrt{\theta^2\left(\theta^2 + 4\right)}\, d\theta \\
&= \int_0^{2\pi} \theta\sqrt{\theta^2 + 4}\, d\theta \\
&= \frac{1}{2}\int_4^{4\pi^2+4} \sqrt{u}\, du \text{ for } u = \theta^2 + 4 \\
&= \left[\frac{1}{3}u^{\frac{3}{2}}\right]_4^{4\pi^2+4} = \frac{1}{3}\left(\left(4\pi^2+4\right)^{\frac{3}{2}} - 8\right).
\end{aligned}$$

EXAMPLE 9.4.4. Find the length of the polar curve $r = 1 + \cos\theta$ for $0 \leq \theta \leq \pi$.

Solution. The assumptions of Proposition 9.4.1 are satisfied, so that the length is given by

$$\begin{aligned}
L &= \int_0^{\pi} \sqrt{r^2 + (r')^2}\, d\theta \\
&= \int_0^{\pi} \sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta \\
&= \int_0^{\pi} \sqrt{2(1+\cos\theta)}\, d\theta \\
&= \sqrt{2}\int_0^{\pi} \sqrt{1 + 2\cos^2\left(\frac{\theta}{2}\right) - 1}\, d\theta \\
&= (\sqrt{2})^2 \int_0^{\pi} \left|\cos\left(\frac{\theta}{2}\right)\right| d\theta \text{ and } \cos\tfrac{\theta}{2} \geq 0 \text{ on } [0,\pi] \\
&= 4\left[\sin\frac{\theta}{2}\right]_0^{\pi} = 4.
\end{aligned}$$

9.5 area enclosed by a sector of a polar curve

Watch the video at

https://www.youtube.com/watch?v=hBBjAfsyxw8&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=65

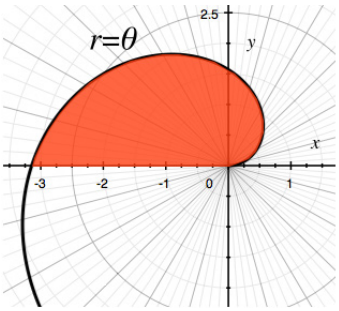
Abstract

A formula is established for the area enclosed by an angular sector and a polar curve $r = f(\theta)$, and several examples illustrate the use of the formula.

Proposition 9.5.1. *The area of the plane region bounded by two half-rays $\theta = a$ and $\theta = b$, and by a polar curve $r = f(\theta)$ is given by*

$$A = \frac{1}{2}\int_a^b (f(\theta))^2\, d\theta = \frac{1}{2}\int_a^b r^2 d\theta.$$

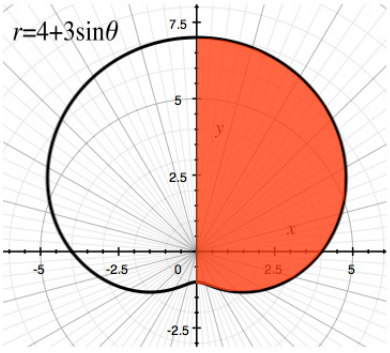
Example 9.5.2. What is the area represented below



Solution. This is the area of the plane region bounded by the half rays $\theta = 0$ and $\theta = \pi$, and the polar curve $r = \theta$. In view of Proposition 9.5.1,

$$A = \frac{1}{2}\int_0^{\pi} \theta^2\, d\theta = \frac{1}{2}\left[\frac{\theta^3}{3}\right]_0^{\pi} = \frac{\pi^3}{6}.$$

Example 9.5.3. What is the area represented below



Solution. This is the area of the plane region bounded by the half rays $\theta = -\frac{\pi}{2}$ and $\theta = \frac{\pi}{2}$, and the polar curve $r = 4 + 3\sin\theta$. In view of Proposition 9.5.1,

$$\begin{aligned} A &= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4 + 3\sin\theta)^2 \, d\theta \\ &= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 16 + 24\sin\theta + 9\sin^2\theta \, d\theta \\ &= [8\theta - 12\cos\theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{9}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta \, d\theta \\ &= 8\pi + \frac{9}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 - \cos 2\theta \, d\theta \\ &= 8\pi + \frac{9}{4}\left[\theta - \frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{41\pi}{4}. \end{aligned}$$

Exercises

you are now prepared to work on the Practice Problems, and Homework set M9B in the manual of exercises.

Before turning to Chapter 10, you should also take Mock Test 3.

10 M10: Sequences and Series

10.1 Sequences

Watch the video at

https://www.youtube.com/watch?v=tF9gna3hxDM&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=66

Abstract

In this video, various ways of defining a sequence are examined.

Formally, a *sequence of real numbers* is a real valued function defined on the set $\mathbb{N}$ of natural numbers: $s : \mathbb{N} \to \mathbb{R}$. Thus a sequence is determined by the *ordered list* of its values

$$\{s(1), s(2), \ldots, s(n), \ldots\}$$

which we denote $\{s(n)\}_{n=1}^{\infty}$, where the *index n* is an integer ranging from 1 to ∞, that is, ranging over $\mathbb{N}$. To avoid to have too many parenthesis, we write

$$s_n := s(n)$$

for the n^{th} value in the ordered list, and accordingly, the sequence is denoted

$$\{s_n\}_{n=1}^{\infty}.$$

A sequence can be defined in various ways:

- By an explicit formula giving the n^{th} term. For instance

$$\left\{\frac{n+1}{n^2+2}\right\}_{n=1}^{\infty} = \left\{\frac{2}{3}, \frac{1}{2}, \frac{4}{11}, \ldots\right\}.$$

  When a sequence is given under this form, it is easy to calculate any term in the sequence, by simply plugging in the corresponding value of *n*. For example, the 1000^{nd} term in the above sequence is $\frac{1000}{10^6+2}$.

- By the first few terms, thus implying a specific pattern. For instance, if a sequence is given by its first 5 terms $\{1, 3, 5, 7, 9, \ldots\}$, it is implicitly understood that we are considering the sequence of odd integers, which could also be given explicitly as $\{2n-1\}_{n=1}^{\infty}$.
- The ordered list may start at a number different from 1. For instance

$$\left\{\sqrt{n^2-9}\right\}_{n=3}^{\infty} = \left\{\sqrt{n^2-9}\right\}_{n\geq 3} = \left\{0, \sqrt{7}, 4, \ldots\right\}$$

is a well-defined sequence.

- By a non-numeric definition of the *n*th term. For instance, $\{p_n\}_{n=1}^{\infty}$, where p_n is the world population *n* years from today
- By induction, that is, by giving the first term or first few terms, and a rule to calculate the next term from the preceding one(s). For instance, a sequence $\{a_n\}_{n=1}^{\infty}$ can by defined inductively by

$$\begin{cases} a_1 = 1 \\ a_{n+1} = \frac{1}{1+a_n} \quad \text{for all } n. \end{cases}$$

Then

$$\begin{aligned} a_1 &= 1 \\ a_2 = a_{1+1} &= \frac{1}{1+a_1} = \frac{1}{2} \\ a_3 = a_{2+1} &= \frac{1}{1+a_2} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3} \\ a_4 = a_{3+1} &= \frac{1}{1+a_3} = \frac{1}{1+\frac{2}{3}} = \frac{3}{5} \end{aligned}$$

When a sequence is defined this way, it is computationally much more costly to calculate a given term, as you first need to calculate all the preceding terms.

- A sequence can be defined by induction by giving more than one initial term, and a relation of induction that depends on more than one preceding terms. For instance, the *Fibonacci sequence* $\{f_n\}_{n=1}^{\infty}$ is defined by

$$\begin{cases} f_1 = f_2 = 1 \\ f_{n+2} = f_{n+1} + f_n \quad \text{for all } n. \end{cases}$$

Thus, to obtain one term, you add the two preceding ones. Thus the sequence's first few terms are

$$\{1, 1, 1+1, 2+1, 3+2, 5+3, \ldots\} = \{1, 1, 2, 3, 5, 8, \ldots\}.$$

10.2 limit of sequences

Watch the video at

https://www.youtube.com/watch?v=pKrsN1fZd8k&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=67

Abstract

In this video, we define the limit of a sequence and examine limit laws and theorems for sequences.

The limit at ∞ of a real-valued function introduced in Calculus I is particularized to a sequence:

Definition 10.2.1. The *limit of a sequence* $\{a_n\}_{n=1}^{\infty}$ *is* L, in symbols

$$\lim_{n\to\infty} a_n = L,$$

if the values a_n can be made as close to L as we want by taking n sufficiently large, that is, if for every $\epsilon > 0$, there is $N \in \mathbb{N}$ such that

$$n \geq N \Longrightarrow |a_n - L| < \epsilon.$$

If a sequence has a finite limit, we say that it is *convergent*. Otherwise, we say that it is *divergent*.

As a direct result of the definition, we have:

Proposition 10.2.2. *If* $f : \mathbb{R} \to \mathbb{R}$ *and* $f(n) = a_n$ *for all n then*

$$\lim_{x\to\infty} f(x) = L \Longrightarrow \lim_{n\to\infty} a_n = L.$$

Example 10.2.3.

$$\lim_{n\to\infty} \frac{3+5n^2}{2+n+3n^2} = \lim_{x\to\infty} \frac{3+5x^2}{2+x+3x^2} = \frac{5}{3}.$$

$$\lim_{n\to\infty} \frac{\ln(n^2)}{n} = \lim_{x\to\infty} \frac{\ln(x^2)}{x} = \lim_{x\to\infty} \frac{2\ln x}{x} \overset{H}{=} \lim_{x\to\infty} \frac{\frac{2}{x}}{1} = 0.$$

$$\lim_{n\to\infty} 2 + \cos(n\pi)$$

does not exists because $2 + \cos(n\pi) = 2 + (-1)^n$ alternates between 1 and 3.

Theorem 10.2.4. *Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be two convergent sequences and let c be a constant.*

1) $\lim_{n\to\infty} c = c$;
2) $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$;
3) $\lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$;
4) *If* $\lim_{n\to\infty} b_n \neq 0$ *then*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}.$$

Proposition 10.2.5. *If* $f : \mathbb{R} \to \mathbb{R}$ *is continuous and* $\lim_{n\to\infty} a_n = L$ *then*

$$\lim_{n\to\infty} f(a_n) = f(L).$$

THEOREM 10.2.6 (Squeeze Theorem for sequences) *If for some* n_0

$$a_n \leq b_n \leq c_n \text{ for all } n \geq n_0$$

and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$$

then

$$\lim_{n\to\infty} b_n = L.$$

COROLLARY 10.2.7. *If* $\lim_{n\to\infty} |a_n| = 0$ *then* $\lim_{n\to\infty} a_n = 0$.

EXAMPLE 10.2.8.

$$\lim_{n\to\infty} (-1)^n \sin\frac{1}{n} = 0$$

because

$$\left|(-1)^n \sin\frac{1}{n}\right| = \sin\frac{1}{n}$$

and $\lim_{n\to\infty} \sin\frac{1}{n} = \sin 0 = 0$.

EXAMPLE 10.2.9. What is

$$\lim_{n\to\infty} \frac{\cos^2 n}{2^n} ?$$

Solution. Since $0 \leq \cos^2 n \leq 1$, we have

$$0 \leq \frac{\cos^2 n}{2^n} \leq \frac{1}{2^n}.$$

Moreover, $\lim_{n\to\infty} \frac{1}{2^n} = 0$, so that, by Theorem 10.2.6, $\lim_{n\to\infty} \frac{\cos^2 n}{2^n} = 0$.

EXAMPLE 10.2.10. What is

$$\lim_{n\to\infty} \frac{2^n}{n!}\,?$$

Solution. Recall that by definition

$$n! := n \cdot (n-1) \cdot (n-2) \ldots 3 \cdot 2 \cdot 1,$$

while $2^n = 2 \cdot 2 \ldots 2$ with n factors. Thus

$$\frac{2^n}{n!} = \frac{2 \cdot 2 \cdot 2 \ldots\ldots \quad 2 \cdot 2}{1 \cdot 2 \cdot 3 \ldots (n-1) \cdot n} = \frac{2 \cdot 2}{1 \cdot 2} \cdot \frac{2}{3} \cdot \frac{2}{4} \cdot \ldots \cdot \frac{2}{n-1} \cdot \frac{2}{n} \leq 2 \cdot \frac{2}{n}$$

because each one of the fractions $\frac{2}{3}, \frac{2}{4}, \ldots \frac{2}{n-1}$ is less than 1, and thus, so is their product. Thus

$$0 \leq \frac{2^n}{n!} \leq \frac{4}{n}$$

and $\lim_{n\to\infty} \frac{4}{n} = 0$, so that Theorem 10.2.6 applies to the effect that

$$\lim_{n\to\infty} \frac{2^n}{n!} = 0.$$

10.3 abstract properties of sequences

Watch the video at

https://www.youtube.com/watch?v=eXP-jUDX3KU&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=68

Abstract

This video introduces the notions of (eventually) non-decreasing and (eventually) non-increasing sequence, of monotonic sequence, of lower and upper bounds for a sequence, as well as greatest lower bound and least upper bound. It is shown that an eventually non-decreasing sequences that is bounded above is convergent and that an eventually non-increasing sequence that is bounded below is convergent.

Definition 10.3.1. A sequence $\{a_n\}_{n=1}^{\infty}$ is

- *increasing* if $a_{n+1} > a_n$ for all *n*;
- *eventually increasing* if there is n_0 with $a_{n+1} > a_n$ for all $n \geq n_0$;
- *(eventually) non-decreasing* if $a_{n+1} \geq a_n$ for all n (for all $n \geq n_0$, for some n_0)
- *(eventually) decreasing* if $a_{n+1} < a_n$ for all n (for all $n \geq n_0$, for some n_0)
- *(eventually) non-increasing* if $a_{n+1} \leq a_n$ for all n (for all $n \geq n_0$, for some n_0)
- *(eventually) monotonic* if it is either (eventually) non-decreasing or (eventually) non-increasing.

Example 10.3.2. $\{n\}_{n=1}^{\infty}$ is increasing, thus non-decreasing, thus monotonic.

$\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is decreasing, thus non-increasing, thus monotonic.

The constant sequence $\{1\}_{n=1}^{\infty}$ is both non-decreasing and non-increasing.

The sequence $\{(-1)^n\}_{n=1}^{\infty}$ is not monotonic.

Definition 10.3.3. A sequence $\{a_n\}_{n=1}^{\infty}$ is

- *bounded above* if there is a number *M* (called an upper bound) such that $a_n \leq M$ for all *n*;
- *bounded below* if there is a number *B* (called a lower bound) such that $a_n \geq B$ for all *n*;
- *bounded* if it is bounded above and below.

Example 10.3.4. The sequence $\{n\}_{n=1}^{\infty}$ is monotonic but not bounded (above).

The sequence $\left\{\cos\left(\frac{n\pi}{2}\right)\right\}_{n=1}^{\infty}$ is bounded but not monotonic.

PROPOSITION 10.3.5. *If a sequence is bounded above it has a least upper bound. If it is bounded below it has a greatest lower bound.*

The *least upper bound* $\sup\{a_n\}_{n=1}^{\infty}$ of the sequence $\{a_n\}_{n=1}^{\infty}$ satisfies

$$\text{for every } \epsilon > 0, \text{ there exists } n_\epsilon \text{with } a_{n_\epsilon} > \sup\{a_n\}_{n=1}^{\infty} - \epsilon. \tag{10.3.1}$$

Similarly, the *greatest lower bound* $\inf\{a_n\}_{n=1}^{\infty}$ of the sequence $\{a_n\}_{n=1}^{\infty}$ satisfies

$$\text{for every } \epsilon > 0, \text{ there exists } n_\epsilon \text{with } a_{n_\epsilon} < \inf\{a_n\}_{n=1}^{\infty} + \epsilon. \tag{10.3.2}$$

THEOREM 10.3.6. *If a sequence is eventually non-decreasing and bounded above, it is convergent. If a sequence is eventually non-increasing and bounded below, it is convergent.*

Note also:
PROPOSITION 10.3.7. *A convergent sequence is bounded.*

Exercises

you are now prepared to work on the Practice Problems, and Homework set M10A in the manual of exercises.

10.4 limit of sequences defined inductively

Watch the videos at

https://www.youtube.com/watch?v=2tXDcknwgFc&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=69

and

https://www.youtube.com/watch?v=DhOTjYSqm2I&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=70

Abstract

In this video, we examine through an example how to justify that a sequence defined inductively is convergent, using proofs by induction, and how to find the limit.

We are now looking at examples of sequences defined inductively by

$$\begin{cases} a_1 \\ a_{n+1} = f(a_n) \quad \text{for all } n \end{cases} \tag{10.4.1}$$

Note that if $\{a_n\}_{n=1}^{\infty}$ is convergent and f is continuous, then, in view of Proposition 10.2.5,

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} f(a_n) = f(L).$$

Thus:

PROPOSITION 10.4.1 *If $\{a_n\}_{n=1}^{\infty}$ is defined by (10.4.1) where f is continuous, then the only possible limits for $\{a_n\}_{n=1}^{\infty}$ are the* fixed points of *f, that is, values L such that*

$$f(L) = L.$$

Therefore, to find the limit (if any) of a sequence defined recursively as in (10.4.1), we first need to establish convergence, usually via Proposition 10.3.5, and then find the limit among the fixed points of f.

In order to apply Proposition 10.3.5, we will sometimes have to show that a sequence defined by (10.4.1) is increasing (decreasing), and bounded above (below), and a convenient way to do that is *by induction*:

To prove that a property $P(n)$ is true for all natural numbers n, it is enough to show that $P(1)$ is true and that for all n,

$$P(n) \Longrightarrow P(n+1). \tag{10.4.2}$$

Indeed, if $P(1)$ is true, then by (10.4.2) with $n = 1$, $P(2)$ is true, so that by (10.4.2) with $n = 2$, $P(3)$ is true, and so on. We conclude *by induction* that $P(n)$ is true for all natural number n.

EXAMPLE 10.4.2. Is the sequence defined recursively by

$$\begin{cases} a_1 = 1 \\ a_{n+1} = 3 - \frac{1}{a_n} \quad \text{for all } n \end{cases} \tag{10.4.3}$$

convergent? If yes, find its limit.

Solution. We will first show that the sequence is increasing, then that it is bounded, so that we will conclude by Proposition 10.3.5 that the sequence is convergent.

To show that it is increasing, we proceed by induction to show $a_{n+1} > a_n$ for all n. This is true for $n = 1$ for $a_1 = 1 < a_2 = 3 - \frac{1}{1} = 2$. Assume that this is true for a given n. We want to show that it is then necessarily also true for $n + 1$, that is, that $a_{n+2} > a_{n+1}$. To this end, note that

$$\begin{aligned} a_{n+1} > a_n \Longrightarrow \frac{1}{a_n} > \frac{1}{a_{n+1}} \quad &\Longrightarrow \quad 3 - \frac{1}{a_{n+1}} > 3 - \frac{1}{a_n}. \\ &\Longrightarrow \quad a_{n+2} > a_{n+1} \qquad \text{using (10.4.3) for } n \text{ and } n+1. \end{aligned}$$

We conclude by induction that $\{a_n\}_{n=1}^{\infty}$ is increasing.

Note that $0 < 1 \leq a_n$ for all n for $\{a_n\}_{n=1}^{\infty}$ is increasing and $a_1 = 1$. We show, by induction as well, that $a_n \leq 3$ for all n. It is true if $n = 1$, as $a_1 = 1 \leq 3$. Assume now that $a_n \leq 3$. Then

$$\begin{aligned} a_n \leq 3 \Longrightarrow \frac{1}{3} \leq \frac{1}{a_n} &\implies 3 - \frac{1}{a_n} \leq 3 - \frac{1}{3} \leq 3. \\ &\implies a_{n+1} \leq 3 \qquad \text{using (10.4.3).} \end{aligned}$$

Therefore $\{a_n\}_{n=1}^{\infty}$ is increasing and bounded above, hence convergent. Let $L := \lim_{n\to\infty} a_n$. By Proposition 10.4.1,

$$\begin{aligned} L = 3 - \frac{1}{L} &\iff L^2 - 3L + 1 = 0 \\ &\iff L = \frac{3 \pm \sqrt{5}}{2}. \end{aligned}$$

Since $a_n \geq 1$ for all n, $L \geq 1$ so that

$$L = \frac{3 + \sqrt{5}}{2}.$$

Example 10.4.3. Is the sequence $\{a_n\}_{n=1}^{\infty}$ defined by

$$\begin{cases} a_1 = \sqrt{2} \\ a_{n+1} = \sqrt{2 + a_n} \quad \text{for all } n \end{cases}$$

convergent? If yes, find its limit.

Solution. We show by induction that $\{a_n\}_{n=1}^{\infty}$ is increasing, that is, we show that $a_{n+1} > a_n$ for all n. This is true for $n = 1$ for

$$a_1 = \sqrt{2} < \sqrt{2 + a_1} = \sqrt{2 + \sqrt{2}} = a_2.$$

Assume now that this is true for some n, that is, $a_{n+1} > a_n$. Then

$$a_{n+1} > a_n \Longrightarrow 2 + a_{n+1} > 2 + a_n \Longrightarrow a_{n+2} = \sqrt{2 + a_{n+1}} > \sqrt{2 + a_n} = a_{n+1},$$

so that the property is true for $n + 1$. We conclude that, by induction, $a_{n+1} > a_n$ for all n, that is, $\{a_n\}_{n=1}^{\infty}$ is increasing.

We show by induction that $\{a_n\}_{n=1}^{\infty}$ is bounded above by 2, that is, that $a_n \leq 2$ for all n. This is true for $n = 1$ for $a_1 = \sqrt{2} \leq 2$. Assume that $a_n \leq 2$. Then

$$a_n \leq 2 \Longrightarrow 2 + a_n \leq 4 \Longrightarrow \sqrt{2 + a_n} \leq \sqrt{4} = 2 \iff a_{n+1} \leq 2,$$

and we conclude by induction that $a_n \leq 2$ for all n.

In view of Proposition 10.3.5, $\{a_n\}_{n=1}^{\infty}$ is convergent. Let $L := \lim_{n\to\infty} a_n$. By Proposition 10.4.1,

$$L = \sqrt{2 + L} \Longrightarrow L^2 = 2 + L \iff L^2 - L - 2 = 0 \iff (L - 2)(L + 1) = 0,$$

so that $L = 2$ or $L = -1$. But $a_n \geq \sqrt{2}$ for all n, so that $L = 2$.

10.5 fixed points and limits of sequences defined inductively

Watch the videos at

https://www.youtube.com/watch?v=MusYJvXR07A&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=71

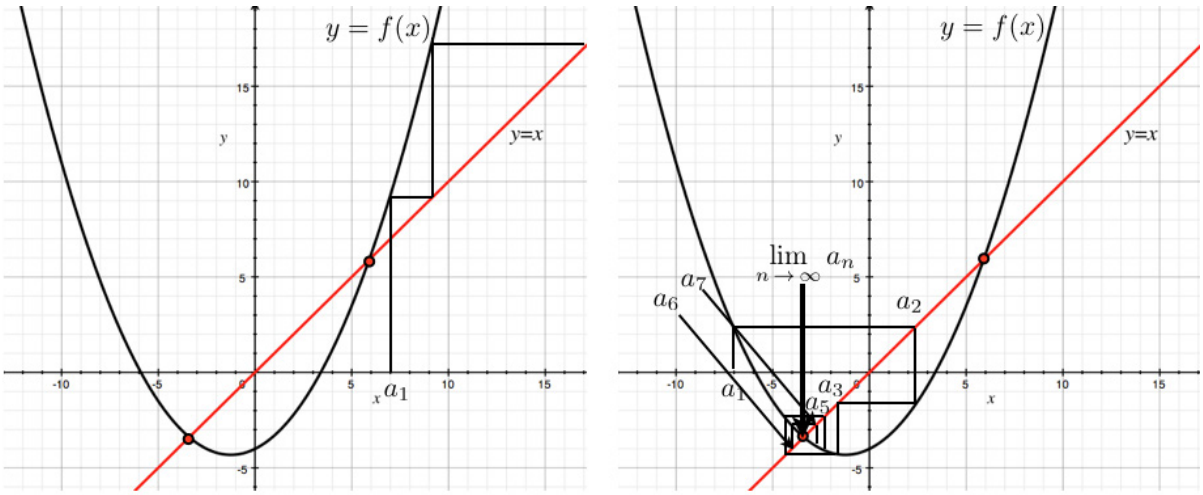
and

https://www.youtube.com/watch?v=l5c70iBiDg0&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=72

Abstract

In these videos, we examine the different possibilities for the convergence of a sequence defined by (10.4.1) where *f* has fixed points, depending on the function and the initial condition. It is also shown that a sequence converges to *L* if and only if the subsequences of odd index and of even index both converge to *L*. This is illustrated on an example.

When a sequence is defined by (10.4.1), we can track the terms of the sequence geometrically on the graph of *f*, using also the line $y = x$. Note that the fixed points of *f*, that is, the potential limits of the sequence, are obtained as intersection points of the graph $y = f(x)$ with $y = x$. Whether the sequence converges or not often depends on the initial term a_1.



As we can see on the right hand side of the picture above, the sequence may converge "spiraling" around the limit, that is, without being (eventually) monotonic, which prevents a direct application of Proposition 10.3.5. However, we see that the terms of the sequence are successively above and below the limit. Hence the subsequences indexed by odd and by even numbers are monotonic, so that Proposition 10.3.5 can be applied to these subsequences. Moreover, we establish:

PROPOSITION 10.5.1. *Let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence and* L *a real number. Then*

$$\lim_{n\to\infty} a_n = L \iff \lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n+1} = L.$$

EXAMPLE 10.5.2. Is the sequence $\{a_n\}_{n=1}^{\infty}$ defined by

$$\begin{cases} a_1 = 1 \\ a_{n+1} = \frac{1}{1+a_n} \quad \text{for all } n \end{cases}$$

convergent? If yes, find its limit.

Solution. Examining the first few terms:

$$a_1 = 1;\ a_2 = \frac{1}{1+1} = \frac{1}{2};\ a_3 = \frac{1}{1+\frac{1}{2}} = \frac{2}{3};\ a_4 = \frac{1}{1+\frac{2}{3}} = \frac{3}{5};\ a_5 = \frac{1}{1+\frac{3}{5}} = \frac{5}{8};\ a_6 = \frac{1}{1+\frac{5}{8}} = \frac{8}{13}\ldots$$

we see that $a_2 \leq a_4 \leq a_6 \ldots$ and that $a_1 \geq a_3 \geq a_5 \ldots$, and thus we set out to show, by induction, that $\{a_{2n}\}_{n=1}^{\infty}$ is non-decreasing and that $\{a_{2n+1}\}_{n=1}^{\infty}$ is non-increasing.

Thus, we want to show that $a_{2(n+1)} = a_{2n+2} \geq a_{2n}$ for all n. It is true for $n = 1$ because $a_4 \geq a_2$. Assume that it is true for n. To show that it is then necessarily true for $n+1$, note that

$$\begin{aligned} a_{2n+2} \geq a_{2n} &\implies 1 + a_{2n+2} \geq 1 + a_{2n} \\ &\implies \frac{1}{1+a_{2n}} \geq \frac{1}{1+a_{2n+2}} \\ &\implies a_{2n+1} \geq a_{2n+3} \\ &\implies \frac{1}{1+a_{2n+3}} \geq \frac{1}{1+a_{2n+1}} \\ &\implies a_{2n+4} = a_{2(n+2)} \geq a_{2n+2} = a_{2(n+1)} \end{aligned}$$

and we conclude by induction that $\{a_{2n}\}_{n=1}^{\infty}$ is non-decreasing, and a similar argument would show that $\{a_{2n+1}\}_{n=1}^{\infty}$ is non-increasing. Note that we have showed that $f(x) = \frac{1}{1+x}$ is decreasing, and thus $f \circ f$ is increasing.

We then show (by induction) that $\{a_{2n}\}_{n=1}^{\infty}$ is bounded above by 1 and that $\{a_{2n+1}\}_{n=1}^{\infty}$ is bounded below by $\frac{1}{2}$. It is true for $n = 1$ for $a_2 = \frac{1}{2} \leq 1$ and $a_3 = \frac{2}{3} \geq \frac{1}{2}$. Assume that it is true for *n*, that is,

$$a_{2n} \leq 1 \text{ and } a_{2n+1} \geq \tfrac{1}{2}.$$

Since $a_{2n+2} = f \circ f(a_{2n})$ and $a_{2n+3} = f \circ f(a_{2n+1})$, we conclude from the fact that $f \circ f$ is increasing that

$$a_{2n+2} \leq f \circ f(1) = \frac{2}{3} \leq 1 \text{ and } a_{2n+3} \geq f \circ f\left(\tfrac{1}{2}\right) = \tfrac{3}{5} \geq \tfrac{1}{2},$$

so that the property is true for $n + 1$ and we conclude by induction.

Therefore, $\{a_{2n}\}_{n=1}^{\infty}$ is non-decreasing and bounded above, hence convergent by Proposition 10.3.5, and $\{a_{2n+1}\}_{n=1}^{\infty}$ is non-increasing and bounded below, hence convergent. In view of Proposition 10.4.1 and the fact that

$$a_{2(n+1)} = f \circ f(a_{2n}) \text{ and } a_{2(n+1)+1} = f \circ f(a_{2n+1})$$

their respective limits are among the fixed points of

$$f \circ f(x) = \frac{1}{1 + \frac{1}{1+x}} = \frac{1+x}{2+x}.$$

Since

$$\frac{1+x}{2+x} = x \iff 1 + x = x^2 + 2x \iff x^2 + x - 1 = 0 \iff x = \frac{-1 \pm \sqrt{5}}{2},$$

we conclude that

$$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n+1} = \frac{-1+\sqrt{5}}{2},$$

for $\frac{-1-\sqrt{5}}{2} < 0$ cannot be a limit since $a_n > 0$ for all *n*. By Proposition 10.5.1, we conclude that

$$\lim_{n\to\infty} a_n = \frac{-1+\sqrt{5}}{2}.$$

Exercises

you are now prepared to work on the Practice Problems, and Homework set M10B in the manual of exercises.

10.6 Series

Remark 10.6.1. If you need to brush up on the sigma sign notation for sums, you can review that here.

Watch the video at

https://www.youtube.com/watch?v=cd8gf1iFfKA&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=73

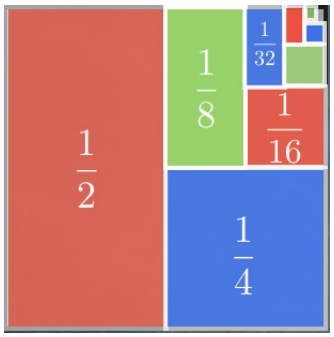
Abstract

In this video, partial sums, convergent series and divergent series are defined.

We begin with the example of the sum

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}+\ldots$$

to see whether we can make sense of an infinite sum. The geometric interpretation



seems to indicate that we should be able to make sense of this infinite sum, and that it should add up to one. To this end, we consider the sequence $\{s_n\}_{n=1}^{\infty}$ of partial sums:

$$s_n := \sum_{i=1}^{n} \frac{1}{2^i}.$$

We note that

$$s_1 = \frac{1}{2} = 1 - \frac{1}{2},\ s_2 = \frac{1}{2} + \frac{1}{4} = 1 - \frac{1}{4},\ s_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 1 - \frac{1}{8},$$

and conjecture that the formula

$$s_n = 1 - \frac{1}{2^n} \tag{10.6.1}$$

may be true for all n. This is easily shown by induction for if (10.6.1) is true for some n, then

$$s_{n+1} = \frac{1}{2} + \ldots \frac{1}{2^n} + \frac{1}{2^{n+1}} = s_n + \frac{1}{2^{n+1}} = 1 - \frac{1}{2^n} + \frac{1}{2^{n+1}} = 1 + \frac{1}{2^n}\left(-1 + \frac{1}{2}\right) = 1 - \frac{1}{2^{n+1}}.$$

Therefore

$$\lim_{n\to\infty} s_n = 1 \text{ and } \textstyle\sum_{n=1}^{\infty} \frac{1}{2^n} := \lim_{n\to\infty} s_n = 1.$$

More generally:

Definition 10.6.2. Given a sequence $\{a_n\}_{n=1}^{\infty}$ (called *sequence of terms of the series* $\sum_{n=1}^{\infty} a_n$), we define its *sequence* $\{s_n\}_{n=1}^{\infty}$ *of partial sums*

$$s_n := \sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n,$$

and define the *series of general term* a_n

$$\sum_{n=1}^{\infty} a_n := \lim_{n\to\infty} s_n,$$

which we call *convergent* if the limit exists, and *divergent* otherwise.

In the sequel, we are going to be concerned with the question of deciding whether a given series is convergent or divergent, even though we will seldom be able to calculate the exact sum.

10.7 Series: a criterion for divergence

Watch the video at

https://www.youtube.com/watch?v=T9bikTjs78A&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=74

Abstract

In this video, we establish a criterion for divergence of a series, often referred to as the *n^{th} term Test,* stating that if the sequence of terms does not converge to 0, then the series is divergent.

Theorem 10.7.1. *If* $\sum_{n=1}^{\infty} a_n$ *is convergent then* $\lim_{n\to\infty} a_n = 0$.

Equivalently:

Theorem 10.7.2 (*n^{th}* term Test). *If* $\lim_{n\to\infty} a_n \neq 0$ *then* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

Example 10.7.3. Are the following series convergent or divergent:

1) $\sum_{n=1}^{\infty} \frac{2n}{3n+1}$
Solution. $\lim_{n\to\infty} \frac{2n}{3n+1} = \frac{2}{3} \neq 0$, so that, by the *$n^{th}$* term Test, the series $\sum_{n=1}^{\infty} \frac{2n}{3n+1}$ is divergent.

2) $\sum_{n=1}^{\infty} \arctan n$
Solution. $\lim_{n\to\infty} \arctan n = \frac{\pi}{2} \neq 0$ so that, by the *n^{th}* term Test, the series $\sum_{n=1}^{\infty} \arctan n$ is divergent.

3) $\sum_{n=1}^{\infty} \frac{(n+1)^2}{n(n+2)}$
Solution. $\lim_{n\to\infty} \frac{(n+1)^2}{n(n+2)} = 1 \neq 0$ so that, by the *n^{th}* term Test, the series $\sum_{n=1}^{\infty} \frac{(n+1)^2}{n(n+2)}$ is divergent.

The converse of Theorem 10.7.2 **is false!** That is, there are sequences $\{a_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} a_n = 0$ but $\sum_{n=1}^{\infty} a_n$ is divergent. For instance, we show that for

$$a_n = \frac{1}{n}$$

the sequence $\{s_n\}_{n=1}^{\infty}$ of partial sums verifies

$$s_n \geq \int_1^{n+1} \frac{dx}{x} = \ln(n+1)$$

so that

$$\lim_{n\to\infty} s_n = \infty$$

and thus

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ is divergent.}$$

10.8 Geometric Series

Watch the videos at

https://www.youtube.com/watch?v=LczuwtRj75c&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=75

and

https://www.youtube.com/watch?v=kIEw0tdQQ10&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=76

Abstract

In these videos, we define geometric sequences and geometric series, examine when they converge, and obtain a formula for the sum of a convergent geometric series.

Definition 10.8.1. A *geometric sequence* of *common ratio r* is a sequence whose terms are of the form

$$a_1,\ a_2 = a_1 r,\ a_3 = a_2 r = a_1 r^2, \ldots, a_n = a_{n-1} r = a_1 r^{n-1}, \ldots$$

A geometric sequence is determined by its first term a and its common ratio r and can then be written in standard form as

$$\{a \cdot r^{n-1}\}_{n=1}^{\infty}.$$

Example 10.8.2. The sequence with first terms

1) $5, 10, 20, 40, 80, 160, \ldots$ is geometric with common ratio 2 and first term 5, that is, the sequence can be written

$$\{5 \cdot 2^{n-1}\}_{n=1}^{\infty} = \left\{\frac{5}{2} \cdot 2^n\right\}_{n=1}^{\infty} = \{5 \cdot 2^n\}_{n=0}^{\infty},$$

where the last two forms are easily seen to be equal to the standard form. It is sometimes convenient to change index this way to obtain powers under a certain form.

2) $4, -2, 1, -\frac{1}{2}, \frac{1}{4}, \ldots$ is geometric of common ratio $-\frac{1}{2}$ and first term 4, that is, the sequence can be written

$$\left\{4 \cdot \left(-\frac{1}{2}\right)^{n-1}\right\}_{n=1}^{\infty} = \left\{(-1)^{n-1} \cdot \frac{1}{2^{n-3}}\right\}_{n=1}^{\infty}.$$

PROPOSITION 10.8.3. *The geometric sequence* $\{a \cdot r^{n-1}\}_{n=1}^{\infty}$ *is*

1) *convergent to* 0 *if* $|r| < 1$;
2) *divergent if* $|r| > 1$;
3) *constant (and thus convergent) if* $r = 1$;
4) *divergent if* $r = -1$.

DEFINITION 10.8.4. A *geometric series* is a series whose sequence of terms is geometric. Thus it can be written in standard form as

$$\sum_{n=1}^{\infty} ar^{n-1}.$$

In view of Proposition 10.8.3 and Theorem 10.7.2, a geometric series of common ratio *r* with $|r| \geq 1$ is divergent. Moreover,

THEOREM 10.8.5. *A geometric series of common ratio r is convergent if and only if*

$$|r| < 1$$

and then

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} = \frac{\text{first term}}{1 - \text{common ratio}}.$$

EXAMPLE 10.8.6. Are the following series geometric? If yes, decide if they are convergent, and find their sum whenever possible.

1) $\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}}$
Solution.

$$\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}} = \sum_{n=1}^{\infty} e \cdot \left(\frac{e}{3}\right)^{n-1}$$

is the geometric series of first term *e* and common ratio $\frac{e}{3}$. Since $0 < \frac{e}{3} < 1$, the series is convergent and

$$\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}} = \frac{e}{1 - \frac{e}{3}} = \frac{3e}{3-e}.$$

2) $\frac{1}{8} - \frac{1}{4} + \frac{1}{2} - 1 + \ldots$
Solution. This series is geometric with first term $\frac{1}{8}$ and common ratio –2. Since $|-2| \geq 1$, the series $\sum_{n=1}^{\infty} \frac{1}{8} \cdot (-2)^{n-1}$ is divergent.

3) $1 + 0.4 + 0.16 + 0.064 + \ldots$

Solution. This series is geometric with first term 1 and common ratio 0.4. Since $|0.4| < 1$, the series is convergent and

$$\sum_{n=1}^{\infty}(0.4)^{n-1} = \frac{1}{1-0.4} = \frac{10}{6} = \frac{5}{3}.$$

4) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3^{n-1}}{4^n}$

Solution.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3^{n-1}}{4^n} = \sum_{n=1}^{\infty}\frac{1}{4}\cdot\frac{(-3)^{n-1}}{4^{n-1}} = \sum_{n=1}^{\infty}\frac{1}{4}\cdot\Big(-\frac{3}{4}\Big)^{n-1}$$

is the geometric series with first term $\frac{1}{4}$ and common ratio $-\frac{3}{4}$. Since $|-\frac{3}{4}| < 1$, the series is convergent and

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3^{n-1}}{4^n} = \frac{\frac{1}{4}}{1-(-\frac{3}{4})} = \frac{1}{7}.$$

5) $\sum_{n=1}^{\infty}3^{-2n}8^{n-1}$

Solution.

$$\sum_{n=1}^{\infty}3^{-2n}8^{n-1} = \sum_{n=1}^{\infty}\frac{8^{n-1}}{\left(3^2\right)^n} = \sum_{n=1}^{\infty}\frac{1}{9}\cdot\frac{8^{n-1}}{9^{n-1}} = \sum_{n=1}^{\infty}\frac{1}{9}\cdot\Big(\frac{8}{9}\Big)^{n-1}$$

is the geometric series of first term $\frac{1}{9}$ and common ratio $\frac{8}{9}$. Since $|\frac{8}{9}| < 1$, the series is convergent and

$$\sum_{n=1}^{\infty}3^{-2n}8^{n-1} = \frac{\frac{1}{9}}{1-\frac{8}{9}} = 1.$$

Example 10.8.7 (Decimal expansion of rational numbers and geometric series) You may know that decimal numbers with expansions that are either finite or periodic are the rationals, that is, numbers that can be written as the quotient of two integers. For a finite expansion, it is easy to see. For instance

$$3.456 = \frac{3456}{1000}.$$

What about periodic expansions? For instance, can we write

$$0.\overline{73} := 0.737373\ldots$$

as a fraction of two integers? The pattern that repeats corresponds to $\frac{73}{100}$ for the first two digits after 0, $\frac{73}{10000}$ for the next two, $\frac{73}{1000000}$ for the next two, and so on. Thus

$$0.\overline{73} := 0.737373\ldots = \frac{73}{100} + \frac{73}{(100)^2} + \frac{73}{(100)^3} + \ldots = \sum_{n=1}^{\infty}\frac{73}{100}\cdot\Big(\frac{1}{100}\Big)^{n-1}$$

is the geometric series of first term $\frac{73}{100}$ and common ratio $\frac{1}{100}$. Thus

$$0.\overline{73} = \sum_{n=1}^{\infty} \frac{73}{100} \cdot \left(\frac{1}{100}\right)^{n-1} = \frac{\frac{73}{100}}{1-\frac{1}{100}} = \frac{73}{99}.$$

Example 10.8.8. The series

$$\sum_{n=0}^{\infty} x^n$$

is the geometric series of first term 1 and common ratio x. Thus, it is convergent if and only if $-1 < x < 1$, and for these values of x, we have

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Similarly,

$$\sum_{n=1}^{\infty} x^n = \frac{x}{1-x} \text{ for } -1 < x < 1.$$

This way, we obtain representations (on (–1, 1)) of functions of x as sums of a series.

10.9 Telescoping sums

Watch the video at

https://www.youtube.com/watch?v=e6ecDWAq49g&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=77

Abstract

In this video, we obtain the exact sum for some series with telescoping sums. A result on the constant multiple and sum of convergent series is stated and applied on examples.

In general, it is difficult to find the exact sum of a convergent series. We just saw that geometric series are an exception. Another case where we can obtain exact sums is that of a series with so called *telescoping sums,* for which most terms in the partial sums cancel out.

Example 10.9.1. Is the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n}$$

convergent? If yes, find its sum.

Solution. Note that $\frac{1}{n^2+n}$ can be decomposed into partial fractions as

$$\frac{1}{n^2+n} = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Thus, the partial sum

$$s_n = \sum_{i=1}^{n} \frac{1}{i^2+i} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1} \right)$$

rewrites as

$$\begin{aligned} s_n &= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{n-1} - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1} \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

so that

$$\lim_{n\to\infty} s_n = 1 = \sum_{n=1}^{\infty} \frac{1}{n^2+n}.$$

Example 10.9.2. Is the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}$$

convergent? If yes, find its sum.

Solution. Note that

$$\frac{1}{n^2-1} = \frac{1}{(n-1)(n+1)} = \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right),$$

so that the partial sums are

$$\begin{aligned} s_n &= \frac{1}{2}\sum_{i=2}^{n} \frac{1}{i-1} - \frac{1}{i+1} \\ &= \frac{1}{2}\Big(1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \dots \\ &\quad \dots + \frac{1}{n-3} - \frac{1}{n-1} + \frac{1}{n-2} - \frac{1}{n} + \frac{1}{n-1} - \frac{1}{n+1}\Big) \\ &= \frac{1}{2}\Big(1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}\Big) \end{aligned}$$

and

$$\lim_{n\to\infty} s_n = \frac{3}{4} = \sum_{n=2}^{\infty} \frac{1}{n^2-1}.$$

Example 10.9.3. Is the series

$$\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$$

convergent? If yes, find its sum.

Solution. Note that

$$\ln\left(\frac{n}{n+1}\right) = \ln n - \ln(n+1)$$

so that the partial sum is

$$\begin{aligned} s_n &= \sum_{i=1}^{n} \ln i - \ln(i+1) \\ &= \ln 1 - \ln 2 + \ln 2 - \ln 3 + \ldots + \ln(n-1) - \ln n + \ln n - \ln(n+1) \\ &= -\ln(n+1) \end{aligned}$$

and

$$\lim_{n\to\infty} s_n = -\infty.$$

Thus $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$ is divergent.

THEOREM 10.9.4. *Let c be a real number. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two **convergent** series, then so are the series $\sum_{n=1}^{\infty} c \cdot a_n$ and $\sum_{n=1}^{\infty}(a_n + b_n)$ and we have*

$$\sum_{n=1}^{\infty} c \cdot a_n = c \cdot \sum_{n=1}^{\infty} a_n$$

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n.$$

EXAMPLE 10.9.5. Is the series

$$\sum_{n=1}^{\infty}\left(\frac{3}{n^2+n} + \frac{1}{3^n}\right)$$

convergent? If yes, find its sum.

Solution. By Example 10.9.1,

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n} = 1$$

and $\sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \frac{1}{3} \cdot \left(\frac{1}{3}\right)^{n-1}$ is a geometric series of first term $\frac{1}{3}$ and common ratio $\frac{1}{3}$ so that

$$\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}.$$

In view of Theorem 10.9.4, we conclude

$$\sum_{n=1}^{\infty}\left(\frac{3}{n^2+n} + \frac{1}{3^n}\right) = 3\sum_{n=1}^{\infty} \frac{1}{n^2+n} + \sum_{n=1}^{\infty} \frac{1}{3^n} = 3 + \frac{1}{2} = \frac{7}{2}.$$

Exercises

you are now prepared to work on the Practice Problems, and Homework set M10C in the manual of exercises.

11 M11: Integral Test and Comparison Test

1.1 Integral Test

Watch the video at

https://www.youtube.com/watch?v=E4zO0UzG4QY&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=78

Abstract

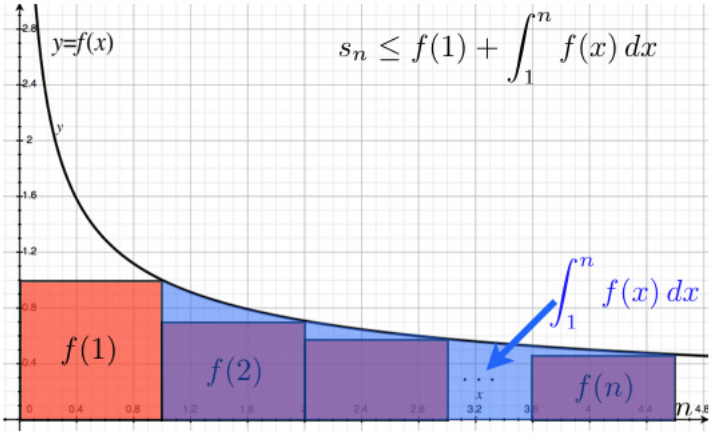
In this video, we present the Integral Test and illustrate this result on examples.

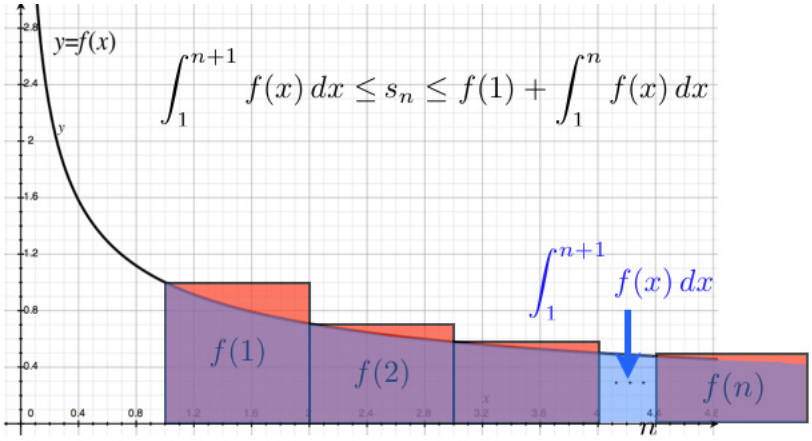
In this section, we consider the problem of convergence of a series of the form

$$\sum_{n=1}^{\infty} f(n)$$

where f is a continuous non-negative decreasing function on $[1, \infty)$.

In this case, we can represent the terms $f(1), f(2), \ldots, f(n), \ldots$ of the series as areas of rectangles, in order to compare the partial sum with integrals of f, as shown below:





Since the sequence $\{s_n\}_{n=1}^{\infty}$ of partial sums satisfies

$$\int_1^{n+1} f(x)\,dx \le s_n \le f(1) + \int_1^n f(x)\,dx, \tag{11.1.1}$$

we conclude

THEOREM 11.1.1 (Integral Test) *Let f be a continuous, non-negative, decreasing function on $[1, \infty)$. Then*

$$\sum_{n=1}^{\infty} f(n) \text{ converges} \iff \int_1^{\infty} f(x)\,dx \text{ converges.}$$

In view of (11.1.1), if $\sum_{n=1}^{\infty} f(n)$ converges, then

$$\int_1^{\infty} f(x)\,dx \le \sum_{n=1}^{\infty} f(n) \le f(1) + \int_1^{\infty} f(x)\,dx. \tag{11.1.2}$$

EXAMPLE 11.1.2. Is $\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$ convergent?

Solution. The series is of the form $\sum_{n=1}^{\infty} f(n)$ for $f(x) = xe^{-x^2}$. This function is continuous and positive on $[1, \infty)$. Moreover, it is decreasing on this interval, for

$$f'(x) = e^{-x^2} - 2x^2 e^{-x^2} = e^{-x^2}\left(1 - 2x^2\right) < 0 \text{ for } x \geq \tfrac{1}{\sqrt{2}}.$$

Thus, the Integral Test applies to the effect that $\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$ converges if and only if $\int_1^{\infty} xe^{-x^2}\,dx$ does. Moreover

$$\begin{aligned}
\int_1^{\infty} xe^{-x^2}\,dx &= \lim_{t\to\infty} \int_1^t xe^{-x^2}\,dx \\
&= \lim_{t\to\infty} -\frac{1}{2}\int_{-1}^{-t^2} e^u\,du \text{ for } u = -x^2 \\
&= \lim_{t\to\infty} \frac{1}{2}\left[e^u\right]_{-t^2}^{-1} = \lim_{t\to\infty} \frac{1}{2e} - \frac{1}{2e^{t^2}} = \frac{1}{2e},
\end{aligned}$$

so that $\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$ is convergent. Moreover, (11.1.2) yields

$$\frac{1}{2e} \leq \sum_{n=1}^{\infty} \frac{n}{e^{n^2}} \leq \frac{1}{e} + \frac{1}{2e} = \frac{3}{2e}.$$

11.2 *p*-series

Watch the video at

https://www.youtube.com/watch?v=VFuCgE8rwmg&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=79

Abstract

In this video, we establish that a *p*-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$. Additional examples of applications of the Integral Test are included.

A series of the form

$$\sum_{n=1}^{\infty} \frac{k}{n^p}$$

where *p* and *k* are fixed numbers, is called a *p-series*.

Theorem 11.2.1 (*p*-series Test). *Let p and k be fixed.*

$$\sum_{n=1}^{\infty} \frac{k}{n^p} \text{ converges} \iff p > 1.$$

Example 11.2.2. Are the following series convergent?

1. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2}}$

   Solution. The series

   $$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{3}}}$$

   is a divergent *p*-series because $p = \frac{2}{3} \leq 1$.

2. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$

   Solution. The series

   $$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$$

   is a convergent *p*-series because $p = \frac{3}{2} > 1$.

Here are additional examples illustrating the Integral Test (Theorem 11.1.1):

EXAMPLE 11.2.3. Are the following series convergent?

1. $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$

   Solution. This series is of the form $\sum_{n=1}^{\infty} f(n)$ where $f(x) = \frac{1}{x^2+4}$ is continuous non-negative and decreasing on $[1, \infty)$. By the Integral Test, the series is convergent if and only if

   $$\int_1^{\infty} \frac{dx}{x^2+4} = \lim_{t\to\infty} \left[\frac{1}{2}\arctan\frac{x}{2}\right]_1^t$$

   is convergent. Thus, $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ is convergent for

   $$\lim_{t\to\infty} \left[\frac{1}{2}\arctan\frac{x}{2}\right]_1^t = \frac{1}{2}\left(\frac{\pi}{2} - \arctan\frac{1}{2}\right).$$

2. $\sum_{n=1}^{\infty} \frac{n}{n^2+4}$

   Solution. This series is of the form $\sum_{n=1}^{\infty} f(n)$ where $f(x) = \frac{x}{x^2+4}$ is continuous non-negative, and decreasing on $[2, \infty)$ because

   $$f'(x) = \frac{x^2+4-2x^2}{(x^2+4)^2} = \frac{4-x^2}{(x^2+4)^2} < 0 \text{ for } x > 2.$$

   By the Integral Test, the series is convergent if and only if

   $$\int_2^{\infty} \frac{x}{x^2+4}\,dx$$

   is convergent. Moreover

   $$\begin{aligned}
   \int_2^{\infty} \frac{x}{x^2+4}\,dx &= \lim_{t\to\infty} \int_2^t \frac{x}{x^2+4}\,dx \\
   &= \lim_{t\to\infty} \frac{1}{2}\int_8^{t^2+4} \frac{du}{u} \text{ for } u = x^2+4 \\
   &= \lim_{t\to\infty} [\ln u]_8^{t^2+4} = \lim_{t\to\infty} \ln(t^2+4) - \ln 8 = \infty.
   \end{aligned}$$

   Thus, the integral is divergent, and, by the Integral Test, $\sum_{n=1}^{\infty} \frac{n}{n^2+4}$ is divergent.

11.3 Estimating the sum

Watch the video at

https://www.youtube.com/watch?v=iEHWR1Hc3JM&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=80

Abstract

In this video, we examine ways to control the error made in estimating the sum of a series that converges by the Integral Test by approximating it by a partial sum.

If the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} f(n)$ converges by the Integral Test (Theorem 11.1.1) then in particular $a_n \geq 0$ for all n so that the sequence $\{s_n\}_{n=1}^{\infty}$ of partial sums is non-decreasing and thus, the error

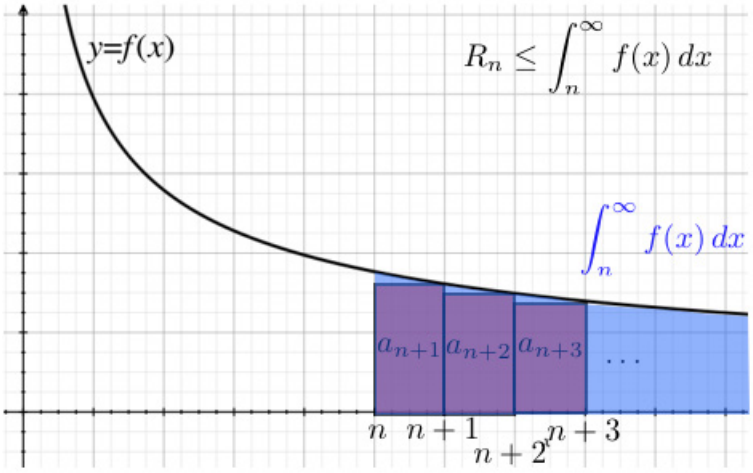
$$R_n := \sum_{i=n+1}^{\infty} a_i$$

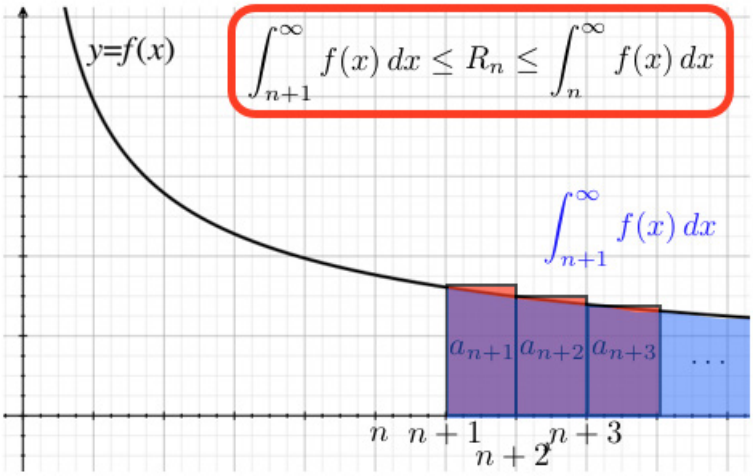
made in the approximation

$$s_n \approx \sum_{n=1}^{\infty} a_n$$

is decreasing with n and has limit 0. R_n is called n^{th} *remainder* of the series.

The considerations we used to justify Theorem 11.1.1, can be used to control R_n:





Thus,

THEOREM 11.3.1 *If a series*

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} f(n)$$

converges by the Integral Test, then the n^{th} remainder R_n satisfies

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx$$

and therefore, adding the n^{th} partial sum s_n:

$$s_n + \int_{n+1}^{\infty} f(x)dx \leq \sum_{n=1}^{\infty} a_n \leq s_n + \int_{n}^{\infty} f(x)\,dx. \tag{11.3.1}$$

EXAMPLE 11.3.2. Is $\sum_{n=1}^{\infty} \frac{1}{n^5}$ convergent? If yes, estimate the sum, exact to three decimal places.

Solution. This is a p-series for $p = 5 > 1$, and therefore $\sum_{n=1}^{\infty} \frac{1}{n^5}$ is convergent. To estimate $\sum_{n=1}^{\infty} \frac{1}{n^5}$ with three exact decimal places, we need the error to be less than 10^{-4} so as to not affect the third decimal. Recall that p-series converge by the Integral Test. Thus, we can use the estimate

$$s_n \approx \sum_{n=1}^{\infty} \frac{1}{n^5},$$

where the error R_n satisfies, according to Theorem 11.3.1,

$$R_n \leq \int_{n}^{\infty} \frac{dx}{x^5} = \lim_{t\to\infty} \left[-\frac{1}{4x^4}\right]_n^t = \frac{1}{4n^4}.$$

Thus, $R_n \leq 10^{-4}$ whenever

$$\frac{1}{4n^4} \leq 10^{-4} \iff n \geq \frac{10}{\sqrt[4]{4}} \approx 7.1.$$

Thus the partial sum

$$s_8 = 1 + \frac{1}{2^5} + \frac{1}{3^5} + \ldots + \frac{1}{8^5} = 1.036880406256885$$

approximates $\sum_{n=1}^{\infty} \frac{1}{n^5}$ with at least 3 exact decimal places.

Note that using (11.3.1), we can obtain a fourth exact decimal for

$$s_8 + \int_9^\infty \frac{dx}{x^5} \leq \sum_{n=1}^{\infty} \frac{1}{n^5} \leq s_8 + \int_8^\infty \frac{dx}{x^5}$$

$$s_8 + \frac{1}{4 \times 9^4} \leq \sum_{n=1}^{\infty} \frac{1}{n^5} \leq s_8 + \frac{1}{4 \times 8^4}$$

$$1.036918510204454 \leq \sum_{n=1}^{\infty} \frac{1}{n^5} \leq 1.036941441413135.$$

Exercises

you are now prepared to work on the Practice Problems, and Homework set M11A in the manual of exercises.

11.4 Direct Comparison Test

Watch the video at

https://www.youtube.com/watch?v=vg2q8zR4ng8&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=81

Abstract

In this video, we present the Comparison Test for series and apply it to conclude on the convergence of various examples of series.

If

$$0 \leq a_n \leq b_n$$

and

$$s_n = \sum_{i=1}^{n} a_i \text{ and } t_n = \sum_{i=1}^{n} b_n$$

then $s_n \leq t_n$ for all *n*, and both sequences of partial sums $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are non-decreasing. Thus, if $\{t_n\}_{n=1}^{\infty}$ is convergent, then $\{s_n\}_{n=1}^{\infty}$ is non-decreasing and bounded above, hence convergent. On the other hand, if $\{s_n\}_{n=1}^{\infty}$ is divergent, then $\lim_{n\to\infty} s_n = \infty$ and as $t_n \geq s_n$, $\lim_{n\to\infty} t_n = \infty$. In other words:

Theorem 11.4.1 (Comparison Test) *If*

$$0 \leq a_n \leq b_n \text{ for all } n$$

then

$$\sum_{n=1}^{\infty} b_n \text{ convergent} \Longrightarrow \sum_{n=1}^{\infty} a_n \text{ convergent}$$
$$\sum_{n=1}^{\infty} a_n \text{ divergent} \Longrightarrow \sum_{n=1}^{\infty} b_n \text{ divergent.}$$

EXAMPLE 11.4.2. Are the following series convergent or divergent?

1. $\sum_{n=1}^{\infty} \frac{1}{2^n+3}$

   Solution. Since $2^n + 3 \geq 2^n$, we conclude that

   $$\frac{1}{2^n+3} \leq \frac{1}{2^n} \text{ for all } n.$$

   Since $\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is geometric of common ratio $\frac{1}{2}$, $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent. Applying the Comparison Test, we conclude that $\sum_{n=1}^{\infty} \frac{1}{2^n+3}$ is convergent.

2. $\sum_{n=1}^{\infty} \frac{2}{n^3+4}$

   Solution. Since $n^3 + 4 \geq n^3$, we have

   $$\frac{2}{n^3+4} \leq \frac{2}{n^3} \text{ for all } n.$$

   Moreover, $\sum_{n=1}^{\infty} \frac{2}{n^3}$ is a *p*-series with $p = 3 > 1$ and is therefore convergent. By Comparison Test, the series $\sum_{n=1}^{\infty} \frac{2}{n^3+4}$ is also convergent.

3. $\sum_{n=1}^{\infty} \frac{1}{n^2+n+1}$

   Solution. Since $n^2 + n + 1 \geq n^2$, we have

   $$\frac{1}{n^2+n+1} \leq \frac{1}{n^2} \text{ for all } n.$$

   Moreover, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent *p*-series for $p = 2 > 1$. By Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{n^2+n+1}$ is also convergent.

4. $\sum_{n=2}^{\infty} \frac{1}{n-\sqrt{n}}$

   Solution. Since $n - \sqrt{n} \leq n$, we have

   $$\frac{1}{n} \leq \frac{1}{n-\sqrt{n}} \text{ for all } n.$$

   Moreover $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent *p*-series for $p = 1 \leq 1$. By Comparison Test, we conclude that $\sum_{n=1}^{\infty} \frac{1}{n-\sqrt{n}}$ is also divergent.

5. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)(n+2)}}$

Solution. Note that

$$n(n+1)(n+2) \geq n^3 \Longrightarrow \sqrt{n(n+1)(n+2)} \geq n^{\frac{3}{2}} \Longrightarrow \frac{1}{\sqrt{n(n+1)(n+2)}} \leq \frac{1}{n^{\frac{3}{2}}} \text{ for all } n.$$

Moreover, $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ is a convergent p-series for $p = \frac{3}{2} > 1$. By Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)(n+2)}}$ is also convergent.

6. $\sum_{n=1}^{\infty} \frac{5}{3^n+2}$

Solution. Since $3^n + 2 \geq 3^n$,

$$\frac{5}{3^n+2} \leq \frac{5}{3^n} \text{ for all } n,$$

and $\sum_{n=1}^{\infty} \frac{5}{3^n} = \sum_{n=1}^{\infty} 5\left(\frac{1}{3}\right)^n$ is a convergent geometric series, for its common ratio is $\frac{1}{3}$ and $|\frac{1}{3}| < 1$. By Comparison Test, $\sum_{n=1}^{\infty} \frac{5}{3^n+2}$ is also convergent.

REMARK 11.4.3. We expect the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ to behave like the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ which is geometric of common ration $\frac{1}{2}$, and thus convergent. However, direct comparison with $\sum_{n=1}^{\infty} \frac{1}{2^n}$ does not apply for

$$\frac{1}{2^n-1} \geq \frac{1}{2^n} \text{ for all } n,$$

which is not the direction of inequality we need. To address this problem, we introduce a variant of the Comparison Test in the next section.

11.5 Limit Comparison Test

Watch the video at

https://www.youtube.com/watch?v=URCmqHziYoI&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=82

Abstract

In this video, we state and prove the Limit Comparison Test for series and examine examples of applications.

THEOREM 11.5.1 (Limit Comparison Test) *Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be sequences of non-negative numbers. If*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c \text{ where } 0 < c < \infty,$$

then the series $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *either both converge or both diverge.*

EXAMPLE 11.5.2. Are the following series convergent or divergent?

1. $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

Solution. Let $a_n = \frac{1}{2^n-1}$ and $b_n = \frac{1}{2^n}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n}{2^n-1} = \lim_{n\to\infty} \frac{2^n}{2^n\left(1-\frac{1}{2^n}\right)} = \lim_{n\to\infty} \frac{1}{1-\frac{1}{2^n}} = 1 > 0.$$

Moreover, $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is a convergent geometric series, for its common ratio is $\frac{1}{2} < 1$. By the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ is convergent.

2. $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$

Solution. Let $a_n = \frac{2n^2+3n}{\sqrt{5+n^5}}$ and $b_n = \frac{n^2}{n^{\frac{5}{2}}} = \frac{1}{n^{\frac{1}{2}}}$. Then

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{n^{\frac{1}{2}}\left(2n^2+3n\right)}{(5+n^5)^{\frac{1}{2}}} = \lim_{n\to\infty} \frac{n^{\frac{5}{2}}\left(2+\frac{3}{n}\right)}{n^{\frac{5}{2}}\left(\frac{5}{n^5}+1\right)^{\frac{1}{2}}} \\ &= \lim_{n\to\infty} \frac{2+\frac{3}{n}}{\sqrt{1+\frac{5}{n^5}}} = 2 > 0. \end{aligned}$$

Moreover, $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ is a divergent p-series for $p = \frac{1}{2} \leq 1$. By the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ is also divergent.

3. $\sum_{n=1}^{\infty} \frac{1+n+n^2}{1+n^2+n^6}$

Solution. Let $a_n = \frac{1+n+n^2}{1+n^2+n^6}$ and $b_n = \frac{n^2}{n^6} = \frac{1}{n^4}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{n^4(1+n+n^2)}{1+n^2+n^6} = \lim_{n\to\infty} \frac{n^6\left(1+\frac{1}{n}+\frac{1}{n^2}\right)}{n^6\left(1+\frac{1}{n^4}+\frac{1}{n^6}\right)} = 1 > 0.$$

Moreover, $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^4}$ is a convergent p-series for $p = 4 > 1$. By the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{1+n+n^2}{1+n^2+n^6}$ is also convergent.

4. $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

Solution. Let $a_n = \sin\frac{1}{n}$ and $b_n = \frac{1}{n}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} = \lim_{x\to 0} \frac{\sin x}{x} = 1 > 0.$$

Moreover $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent p-series for $p = 1 \leq 1$. By the Limit Comparison Test, $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ is also divergent.

5. $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$

Solution. Note that

$$n^{1+\frac{1}{n}} = n \cdot n^{\frac{1}{n}} = ne^{\frac{\ln n}{n}}$$

and

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} \overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0,$$

so that

$$\lim_{n\to\infty} n^{\frac{1}{n}} = e^0 = 1.$$

Thus, letting $a_n = \frac{1}{n^{1+\frac{1}{n}}}$ and $b_n = \frac{1}{n}$, we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n^{1+\frac{1}{n}}} = \lim_{n\to\infty} \frac{1}{n^{\frac{1}{n}}} = \frac{1}{1} = 1 > 0.$$

Moreover, $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent p-series for $p = 1 \leq 1$. By the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ is divergent as well.

11.6 Estimating sums revisited

Watch the video at

https://www.youtube.com/watch?v=sXs9bjUcxbg&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=83

Abstract

In this video, we examine how to control the error made when approximating the sum of a series by a partial sum, when we used the Comparison Test to conclude on convergence.

If we established the convergence of a series $\sum_{n=1}^{\infty} a_n$ using the Comparison Test, then we had

$$0 \leq a_n \leq b_n \text{ for all } n, \tag{11.6.1}$$

and $\sum_{n=1}^{\infty} b_n$ convergent. Note that if

$$R_n = \sum_{i=n+1}^{\infty} a_i \text{ and } T_n = \sum_{i=n+1}^{\infty} b_i,$$

then

$$R_n \leq T_n$$

because of (11.6.1). In many cases, $\sum_{n=1}^{\infty} b_n$ is a *p*-series or a geometric series, and we can estimate T_n, thus R_n.

Example 11.6.1. Find n such that the n^{th} partial sum of

$$\sum_{n=1}^{\infty} \frac{1}{n^4+5}$$

has at least four exact decimal places.

Solution. Since

$$0 \leq \frac{1}{n^4+5} \leq \frac{1}{n^4}$$

and $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is a convergent *p*-series ($p = 4 > 1$), we conclude by the Comparison Test that $\sum_{n=1}^{\infty} \frac{1}{n^4+5}$ is convergent. Moreover

$$R_n = \sum_{i=n+1}^{\infty} \frac{1}{i^4+5} \leq T_n = \sum_{i=n+1}^{\infty} \frac{1}{i^4} \leq \int_n^{\infty} \frac{dx}{x^4}$$

and

$$\int_n^\infty \frac{dx}{x^4} = \lim_{t\to\infty}\left[-\frac{1}{3x^3}\right]_n^t = \frac{1}{3n^3}.$$

Thus, $R_n \le \frac{1}{3n^3}$ and we can ensure that $R_n \le 10^{-5}$ by requiring $\frac{1}{3n^3} \le 10^{-5}$ which amounts to

$$n \ge \sqrt[3]{\frac{10^5}{3}} \approx 32.2.$$

Thus

$$s_{33} = \sum_{i=1}^{33} \frac{1}{i^4+5} \approx \sum_{n=1}^{\infty} \frac{1}{n^4+5}$$

with an error of at most 10^{-5}, and thus at least 4 exact decimal places.

Example 11.6.2. Find *n* such that the n^{th} partial sum of

$$\sum_{n=1}^{\infty} \frac{2}{3^n+5}$$

has at least three exact decimal places.

Solution. Since

$$0 \le \frac{2}{3^n+5} \le \frac{2}{3^n}$$

and $\sum_{n=1}^\infty \frac{2}{3^n} = \sum_{n=1}^\infty 2\left(\frac{1}{3}\right)^n$ is a geometric series of common ratio $\frac{1}{3}$, hence convergent, we conclude by the Comparison Test that $\sum_{n=1}^\infty \frac{2}{3^n+5}$ is convergent. Moreover

$$R_n = \sum_{i=n+1}^{\infty} \frac{2}{3^i+5} \le T_n = \sum_{i=n+1}^{\infty} \frac{2}{3^i} = \frac{\frac{2}{3^{n+1}}}{1-\frac{1}{3}} = \frac{1}{3^n}.$$

Thus, we can ensure that $R_n \le 10^{-4}$ by requiring

$$\frac{1}{3^n} \le 10^{-4} \iff \log_3(10^4) \le n.$$

Since $\log_3(10^4) \approx 8.3$, we conclude that

$$s_9 = \sum_{i=1}^{9} \frac{2}{3^i+5} \approx \sum_{n=1}^{\infty} \frac{2}{3^n+5}$$

with an error of at most 10^{-4} and thus at lest 3 exact decimal places.

Exercises

you are now prepared to work on the Practice Problems, and Homework set M11B in the manual of exercises.

12 M12: Alternating Series Test

12.1 Alternating Series Test

Watch the video at

https://www.youtube.com/watch?v=INXn6LOoQNk&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=84

Abstract

In this video, alternating series are defined and examples examined. The Alternating Series Test is stated and established, and applied to a number of examples.

Definition 12.1.1. A series is alternating if its terms are alternatively positive and negative.

Example 12.1.2.

$$\sum_{n=1}^{\infty}(-1)^n\frac{n}{n+1} = -\frac{1}{2}+\frac{2}{3}-\frac{3}{4}+\frac{4}{5}-\dots$$

and

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

are both alternating series, while

$$1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \dots$$

is not, for the sign of the terms do not alternate for every *n*.

Note that an alternating series $\sum_{n=1}^{\infty} a_n$ can be written under the form

$$\sum_{n=1}^{\infty}(-1)^n\, b_n \text{ or } \sum_{n=1}^{\infty}(-1)^{n-1} b_n \text{ where } b_n = |a_n| \geq 0.$$

THEOREM 12.1.3 (Alternating Series Test) *An alternating series* $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ *or* $\sum_{n=1}^{\infty}(-1)^{n}b_n$*, where* $b_n \geq 0$ *for all n,* $\{b_n\}_{n=1}^{\infty}$ *is eventually non-increasing, and*

$$\lim_{n\to\infty} b_n = 0$$

is convergent.

EXAMPLE 12.1.4. Are the following series convergent?

1. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$

   Solution. Since $\{\frac{1}{n}\}_{n=1}^{\infty}$ is decreasing with limit 0 and the series is alternating, the Alternating Series Test applies to the effect that $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ is convergent.

2. $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{\sqrt{n}}$

   Solution. Since $\left\{\frac{1}{\sqrt{n}}\right\}_{n=1}^{\infty}$ is decreasing with limit 0 (because $\sqrt{x}$ is increasing with $\lim_{x\to\infty}\sqrt{x} = \infty$) and the series is alternating, the Alternating Series Test applies to the effect that $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{\sqrt{n}}$ is convergent.

3. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\ln(n+1)}$

   Solution. Since $\left\{\frac{1}{\ln(n+1)}\right\}_{n=1}^{\infty}$ is decreasing with limit 0 (because ln *x* is increasing with $\lim_{x\to\infty}\ln x = \infty$) and the series is alternating, the Alternating Series Test applies to the effect that $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\ln(n+1)}$ is convergent.

4. $\sum_{n=1}^{\infty}(-1)^n \cos\left(\frac{\pi}{n}\right)$

Solution. Since

$$\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right) = \cos 0 = 1,$$

$$\lim_{n\to\infty} (-1)^n \cos\left(\frac{\pi}{n}\right) \neq 0$$

and the n^{th} term Test applies to the effect that $\sum_{n=1}^{\infty}(-1)^n \cos\left(\frac{\pi}{n}\right)$ is divergent.

5. $\sum_{n=1}^{\infty}(-1)^n \frac{2n}{4n^2+1}$

Solution. Since

$$\left(\frac{2x}{4x^2+1}\right)' = \frac{2(4x^2+1)-16x^2}{(4x^2+1)^2} = \frac{2-8x^2}{(4x^2+1)^2} < 0 \text{ for } x \geq 1,$$

we conclude that $\left\{\frac{2n}{4n^2+1}\right\}_{n=1}^{\infty}$ is non-increasing. Moreover,

$$\lim_{n\to\infty} \frac{2n}{4n^2+1} = 0.$$

Thus, the Alternating Series Test applies to the effect that $\sum_{n=1}^{\infty}(-1)^n \frac{2n}{4n^2+1}$ is convergent.

6. $\sum_{n=1}^{\infty}(-1)^n \frac{2n}{4n+1}$

Solution. Since

$$\lim_{n\to\infty} \frac{2n}{4n+1} = \frac{2}{4} = \frac{1}{2} \neq 0,$$

$$\lim_{n\to\infty} (-1)^n \frac{2n}{4n+1} \neq 0$$

and the n^{th} term Test applies to the effect that $\sum_{n=1}^{\infty}(-1)^n \frac{2n}{4n+1}$ is divergent.

12.2 Absolute and conditional convergence

Watch the video at

https://www.youtube.com/watch?v=nT6AZ50CJio&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=85

Abstract

In this video, the notions of absolute convergence and of conditional convergence are introduced. It is shown that an absolutely convergent series is also convergent. Examples are examined to determined if a series is absolutely convergent, conditionally convergent, or divergent.

Definition 12.2.1. A series $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent* if $\sum_{n=1}^{\infty} |a_n|$ is convergent, and *conditionally convergent* if it is convergent but not absolutely convergent.

Example 12.2.2. The series

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$$

is absolutely convergent for

$$\sum_{n=1}^{\infty}\left|(-1)^n\frac{1}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

is a convergent *p*-series ($p = 2 > 1$). On the other hand

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$$

is conditionally convergent because $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$ is convergent by the Alternating Series Test as $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is decreasing with limit 0, but $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$ is not absolutely convergent for

$$\sum_{n=1}^{\infty}\left|(-1)^n\frac{1}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$$

is a divergent *p*-series ($p = 1 \leq 1$).

THEOREM 12.2.3. *An absolutely convergent series is convergent.*

EXAMPLE 12.2.4. Are the following series absolutely convergent, conditionally convergent or divergent?

1. $\sum_{n=1}^{\infty}\frac{\sin n}{n^3}$

   Solution. Since

   $$0 \leq \left|\frac{\sin n}{n^3}\right| \leq \frac{1}{n^3}$$

   and $\sum_{n=1}^{\infty}\frac{1}{n^3}$ is a convergent *p*-series ($p = 3 > 1$), we conclude by the Comparison Test that $\sum_{n=1}^{\infty}\left|\frac{\sin n}{n^3}\right|$ is convergent, that is, $\sum_{n=1}^{\infty}\frac{\sin n}{n^3}$ is absolutely convergent.

2. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n\sqrt{n}}$

   Solution. Since

   $$\left|\frac{(-1)^{n-1}}{n\sqrt{n}}\right| = \frac{1}{n^{\frac{3}{2}}}$$

   and $\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}$ is a convergent *p*-series ($p = \frac{3}{2} > 1$), we conclude that $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n\sqrt{n}}$ is absolutely convergent.

3. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$

Solution. Since

$$\left|\frac{(-1)^{n-1}}{\sqrt{n}}\right| = \frac{1}{n^{\frac{1}{2}}}$$

and $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ is a divergent p-series $(p = \frac{1}{2} \leq 1)$, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ is not absolutely convergent. On the other hand $\left\{\frac{1}{\sqrt{n}}\right\}_{n=1}^{\infty}$ is decreasing with limit 0. By the Alternating Series Test, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ is convergent, hence conditionally convergent.

4. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+5}$

Solution. Note that $\left|\frac{(-1)^n}{n+5}\right| = \frac{1}{n+5}$ and

$$\lim_{n\to\infty} \frac{n}{n+5} = 1 > 0,$$

so that, by the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{n+5}$ is divergent because $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent p-series $(p = 1 \leq 1)$. Thus $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+5}$ is not absolutely convergent. On the other hand $\left\{\frac{1}{n+5}\right\}_{n=1}^{\infty}$ is decreasing with limit 0. By the Alternating Series Test, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+5}$ is convergent, hence conditionally convergent.

5. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{\ln(n+1)}$

Solution. Since

$$\lim_{n\to\infty} \frac{n}{\ln(n+1)} = \lim_{x\to\infty} \frac{x}{\ln(x+1)} \overset{H}{=} \lim_{x\to\infty} \frac{1}{\frac{1}{x+1}} = \lim_{x\to\infty} x+1 = \infty,$$

the sequence $\left\{\frac{(-1)^n n}{\ln(n+1)}\right\}_{n=1}^{\infty}$ does not converge to 0. By the n^{th} term Test, $\sum_{n=1}^{\infty} \frac{(-1)^n n}{\ln(n+1)}$ is divergent.

12.3 Estimating sums with the Alternating Series Test

Watch the video at

https://www.youtube.com/watch?v=eC5_B_pIwzI&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=86

Abstract

In this video, we examine upper bounds for the error made in approximating the sum of a series that converges by the Alternating Series Test by a partial sum.

Proposition 12.3.1. *If an alternating series* $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ *converges by the Alternating Series Test (that is,* $b_n \geq 0$, $\{b_n\}_{n=1}^{\infty}$ *is eventually non-increasing and* $\lim_{n\to\infty} b_n = 0$*) then the error* $R_n = \sum_{i=n+1}^{\infty}(-1)^{i-1}b_i$ *made in approximating* $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ *by the* n^{th} *partial sum* $s_n = \sum_{i=1}^{n}(-1)^{i-1}b_i$ *satisfies*

$$|R_n| \leq b_{n+1}.$$

In other words, the first neglected term is an upper bound for the error.

EXAMPLE 12.3.2. Find

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!}$$

with 4 exact decimal places.

Solution. The sequence $\left\{\frac{1}{(2n-1)!}\right\}_{n=1}^{\infty}$ is decreasing with limit 0, so that Proposition 12.3.1 applies to the effect that the error R_n made in the approximation

$$s_n = \sum_{i=1}^{n} \frac{(-1)^{i-1}}{(2i-1)!} \approx \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!}$$

satisfies

$$|R_n| \leq \frac{1}{(2(n+1)-1)!} = \frac{1}{(2n+1)!}.$$

Examining the values of $(2n + 1)!$ as n grows we have $3! = 6$, $5! = 120$, $7! = 5040$, and $9! = 362160 > 10^5$. Thus

$$s_4 = 1 - \frac{1}{6} + \frac{1}{120} - \frac{1}{5040} \approx \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!}$$

with an error less than 10^{-5}, hence at least 4 exact decimal places.

Exercises

you are now prepared to work on the Practice Problems, and Homework set M12 in the manual of exercises.

13 M13: Ratio and Root Tests

13.1 Ratio Test (Statement and proof)

Watch the video at

https://www.youtube.com/watch?v=BOcp-elEnGk&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=87

Abstract

In this video, we state and prove the Ratio Test.

Theorem 13.1.1 (Ratio Test)

1. *If*
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$$
then $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent, hence convergent.*

2. *If*
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1 \text{ or } \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$$
then $\sum_{n=1}^{\infty} a_n$ *is divergent.*

3. *If*
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$$
the test is inconclusive.

Example 13.1.2. The Ratio Test is indeed inconclusive if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, as we can see from the fact that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$ if $a_n = \frac{1}{n}$ and if $a_n = \frac{1}{n^2}$, but $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent. Thus $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$ doesn't give any indication regarding the convergence of the series.

13.2 Ratio Test: examples

Watch the videos at

https://www.youtube.com/watch?v=ZS33eDmpD0U&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=88

and

https://www.youtube.com/watch?v=YtfOBDo4m0E&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=89

and

https://www.youtube.com/watch?v=6cm0JnpWM2I&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=90

Abstract

In these videos, the convergence of series is tested on a number of examples using the Ratio Test.

EXAMPLE 13.2.1. Are the following series convergent or divergent?

1. $\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{3^n}$

   Solution. In this case

   $$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} = \frac{1}{3} \cdot \left(\frac{n+1}{n}\right)^3$$

   so that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3} < 1$. By the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{3^n}$ is absolutely convergent, hence convergent.

2. $\sum_{n=1}^{\infty} \frac{2^n+5}{3^n}$

   Solution. In this case

   $$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}+5}{3^{n+1}} \cdot \frac{3^n}{2^n+5} = \frac{1}{3} \cdot \frac{2^n\left(2+\frac{5}{2^n}\right)}{2^n\left(1+\frac{5}{2^n}\right)}$$

   so that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{2}{3} < 1$. By the Ratio Test, $\sum_{n=1}^{\infty} \frac{2^n+5}{3^n}$ is convergent.

3. $\sum_{n=1}^{\infty} \frac{10^n}{n!}$

   Solution. Since

   $$\left|\frac{a_{n+1}}{a_n}\right| = \frac{10^{n+1}}{(n+1)!} \cdot \frac{n!}{10^n} = \frac{n!}{(n+1)!} \cdot \frac{10^{n+1}}{10^n} = \frac{10}{n+1},$$

   we conclude that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$. By the Ratio Test, $\sum_{n=1}^{\infty} \frac{10^n}{n!}$ is convergent.

4. $\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$

   Solution. Since

   $$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{(2n+2)!}{(n+1)!(n+1)!} \cdot \frac{n!n!}{2n!} = \left(\frac{n!}{(n+1)!}\right)^2 \cdot \frac{(2n+2)!}{2n!} \\ &= \frac{(2n+1)(2n+2)}{(n+1)^2},\end{aligned}$$

   we conclude that

   $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 4 > 1$$

   and, by the Ratio Test, $\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$ is divergent.

EXAMPLE 13.2.2. Are the following series convergent or divergent?

1. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

   Solution. Since

   $$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{n+1}{(n+1)^{n+1}} \cdot n^n = \left(\frac{n}{n+1}\right)^n = e^{n\ln\left(\frac{n}{n+1}\right)}$$

   and

   $$\begin{aligned}\lim_{n\to\infty} n\ln\left(\frac{n}{n+1}\right) &= \lim_{x\to\infty} \frac{\ln x - \ln(x+1)}{\frac{1}{x}} \overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{x} - \frac{1}{x+1}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{x^2}{x+1} - x \\ &= \lim_{x\to\infty} \frac{x^2 - x^2 - x}{x+1} = \lim_{x\to\infty} -\frac{x}{x+1} = -1\end{aligned}$$

   we conclude that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = e^{-1} < 1$. By the Ratio Test, $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ is convergent.

2. $\sum_{n=1}^{\infty} n!e^{-n}$

   Solution. Since

   $$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{n!} \cdot \frac{e^n}{e^{n+1}} = \frac{n+1}{e},$$

   we have $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$. By the Ratio Test, $\sum_{n=1}^{\infty} n!e^{-n}$ is divergent.

3. $\sum_{n=1}^{\infty} a_n$ where

   $$\begin{cases} a_1 = 2 \\ a_{n+1} = \frac{5n+1}{4n+3} a_n \end{cases}.$$

   Solution. By definition of $\{a_n\}_{n=1}^{\infty}$

   $$\left|\frac{a_{n+1}}{a_n}\right| = \frac{5n+1}{4n+3}$$

   so that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{5}{4} > 1$. By the Ratio Test, $\sum_{n=1}^{\infty} a_n$ is divergent.

EXAMPLE 13.2.3. Show in two different ways that $\sum_{n=1}^{\infty} ne^{-n}$ is convergent.

Solution. **Method 1:** Integral Test. The series is of the form $\sum_{n=1}^{\infty} f(n)$ where $f(x) = xe^{-x}$ is non-negative, continuous, and decreasing on $[1,\infty)$ for

$$f'(x) = e^{-x}(1-x) < 0 \text{ for } x > 1.$$

By the Integral Test, $\sum_{n=1}^{\infty} ne^{-n}$ is convergent if and only if $\int_1^{\infty} xe^{-x}\,dx$ is. moreover,

$$\int_1^{\infty} xe^{-x}\,dx = \lim_{t\to\infty} \int_1^{t} xe^{-x}\,dx$$

and we calculate $\int_1^t xe^{-x}\,dx$ by parts with $u = x$ and $dv = e^{-x}\,dx$ so that $du = dx$ and $v = -e^{-x}$, that is,

$$\int_1^t xe^{-x}\,dx = \left[-xe^{-x}\right]_1^t + \int_1^t e^{-x}\,dx = \left[-e^{-x}(x+1)\right]_1^t = \frac{2}{e} - \frac{t+1}{e^t}.$$

Moreover,

$$\lim_{t\to\infty} \frac{t+1}{e^t} \overset{H}{=} \lim_{t\to\infty} \frac{1}{e^t} = 0.$$

Thus,

$$\int_1^\infty xe^{-x}\,dx = \lim_{t\to\infty} \frac{2}{e} - \frac{t+1}{e^t} = \frac{2}{e},$$

so that $\sum_{n=1}^\infty ne^{-n}$ is convergent.

Method 2: Ratio Test. Since

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{n} \cdot \frac{e^n}{e^{n+1}} = \frac{1}{e} \cdot \frac{n+1}{n},$$

we conclude that

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{e} < 1,$$

so that $\sum_{n=1}^\infty ne^{-n}$ is convergent by the Ratio Test.

EXAMPLE 13.2.4. For what values of *x* is the sequence $\left\{\frac{x^n}{n!}\right\}_{n=1}^\infty$ convergent? When convergent, find the limit.

Solution. By the n^{th} term Test, we have

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \text{ for all } x,$$

because $\sum_{n=1}^\infty \frac{x^n}{n!}$ is convergent for all *x*. Indeed, letting $a_n = \frac{x^n}{n!}$, we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{|x|^n} = \frac{|x|}{n+1},$$

so that

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 \text{ for all } x,$$

and $\sum_{n=1}^\infty \frac{x^n}{n!}$ is absolutely convergent for all *x*, by the Ratio Test.

13.3 Root Test

Watch the video at

https://www.youtube.com/watch?v=mnry_Af3tek&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=91

Abstract

In this video, the Root Test for convergence of series is stated and proved, and applied to some examples.

Example 13.3.1. Consider the series $\sum_{n=1}^{\infty} a_n$ where

$$\begin{cases} a_n = \frac{n}{2^n} & \text{if } n \text{ is odd} \\ a_n = \frac{1}{2^n} & \text{if } n \text{ is even} \end{cases}.$$

The Ratio Test is inconclusive in this case, for

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{2^{n+1}} \cdot 2^n = \frac{n+1}{2} \quad \text{or} \quad \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{1}{2n}$$

depending on whether n is even or odd. Thus, $\left|\frac{a_{n+1}}{a_n}\right|$ does not have a limit because it has a subsequence with limit ∞ and a subsequence with limit 0. Yet, we ought to be able to conclude by "comparison" to the geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$, which is the spirit of the Ratio Test.

The alternative is:

THEOREM 13.3.2 (Root Test)

1. *If*

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$$

then $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent, hence convergent.*

2. *If*

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1 \text{ or } \lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$$

then $\sum_{n=1}^{\infty} a_n$ *is divergent.*

3. *If*

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$$

then the Test is inconclusive.

EXAMPLE 13.3.3. We see that the Root Test is inconclusive when $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$ because, given that

$$\lim_{x\to\infty} \frac{\ln x}{x} \overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0,$$

we have

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n^{\frac{1}{n}}} = \lim_{n\to\infty} \frac{1}{e^{\frac{\ln n}{n}}} = \lim_{n\to\infty} \frac{1}{e^0} = 1$$

and

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{e^{\frac{2\ln n}{n}}} = 1$$

but $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent p-series ($p = 1 \leq 1$), while $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series ($p = 2 > 1$).

EXAMPLE 13.3.4. We can use the Root Test to justify that the series of Example 13.3.1 is convergent. Indeed

$$\sqrt[n]{a_n} = \begin{cases} \frac{1}{2}^{\frac{1}{n}} & \text{if } n \text{ is odd} \\ \frac{1}{2} & \text{if } n \text{ is even} \end{cases},$$

so that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{2} < 1$$

for

$$\lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{n\to\infty} e^{\frac{\ln n}{n}} = e^0 = 1.$$

By the Root Test, $\sum_{n=1}^{\infty} a_n$ is convergent.

Note that we could also have used a comparison argument to reduce the problem to using the Ratio Test: Since

$$0 \leq a_n \leq \frac{n}{2^n}$$

and $\sum_{n=1}^{\infty} \frac{n}{2^n}$ is convergent by the Ratio Test, because

$$\frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{1}{2} \cdot \frac{n+1}{n} \underset{n\to\infty}{\longrightarrow} \frac{1}{2} < 1,$$

we conclude by the Comparison Test that $\sum_{n=1}^{\infty} a_n$ is convergent.

Example 13.3.5. Are the following series convergent?

1. $\sum_{n=1}^{\infty} \left(\frac{4n+1}{5n+3}\right)^n$

   Solution. If $a_n = \left(\frac{4n+1}{5n+3}\right)^n$ then

   $$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{4n+1}{5n+3} = \frac{4}{5} < 1,$$

   so that the series $\sum_{n=1}^{\infty} \left(\frac{4n+1}{5n+3}\right)^n$ is convergent by the Root Test.

2. $\sum_{n=1}^{\infty} \left(\frac{2n^2+1}{n^2+3}\right)^n$

   Solution. If $a_n = \left(\frac{2n^2+1}{n^2+3}\right)^n$ then

   $$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{2n^2+1}{n^2+3} = 2 > 1,$$

   so that the series $\sum_{n=1}^{\infty} \left(\frac{2n^2+1}{n^2+3}\right)^n$ is divergent by the Root Test.

3. $\sum_{n=1}^{\infty} \frac{(-1)^n}{(\arctan n)^n}$

   Solution. If $a_n = \frac{(-1)^n}{(\arctan n)^n}$ then

   $$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{1}{\arctan n} = \frac{2}{\pi} < 1,$$

   so that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{(\arctan n)^n}$ is convergent by the Root Test.

Exercises

you are now prepared to work on the Practice Problems, and Homework set M13 in the manual of exercises.

13.4 Strategies to test series for convergence (M14)

Watch the video at

https://www.youtube.com/watch?v=TdMMHGtXmwE&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=92

Abstract

In this video, we discuss strategies to decide which of the various Tests we have seen should be applied to decide on the convergence of a series.

When we want to decide if a series $\sum_{n=1}^{\infty} a_n$ is convergent, you may want to approach it following these guidelines:

1. What is $\lim_{n\to\infty} a_n$?

$$\lim_{n\to\infty} a_n \neq 0 \Longrightarrow \sum_{n=1}^{\infty} a_n \text{ is divergent by } n^{th}\text{term Test}$$

$$\lim_{n\to\infty} a_n = 0 \Longrightarrow \text{need to use other Test.}$$

2. Is the series a p-series $\sum_{n=1}^{\infty} \frac{k}{n^p}$ or a geometric series $\sum_{n=1}^{\infty} ar^{n-1}$?

   - p-series is convergent if and only if $p > 1$
   - geometric series of common ratio r is convergent if and only if $|r| < 1$, in which case

$$\sum_{n=1}^{\infty} a \cdot r^{n-1} = \frac{a}{1-r}.$$

3. Is the series "similar" to a p-series or a geometric series? Then use Direct (Theorem 11.4.1) or Limit Comparison (Theorem 11.5.1) with the appropriate p-series or geometric series. Note that Comparison only applies to series with positive terms, but if a series does not have only positive terms, these theorems can be applied to $\sum_{n=1}^{\infty} |a_n|$ to obtain absolute convergence, hence convergence.

Example 13.4.1. To conclude about the convergence of $\sum_{n=1}^{\infty} \frac{2^n}{3^{n-1}+2}$, note that

$$0 \leq \frac{2^n}{3^{n-1}+2} \leq \frac{2^n}{3^{n-1}} = 2\left(\frac{2}{3}\right)^{n-1}$$

and that $\sum_{n=1}^{\infty} 2\left(\frac{2}{3}\right)^{n-1}$ is a convergent geometric series for its common ratio $\frac{2}{3}$ is in $(-1, 1)$. By Comparison Test, $\sum_{n=1}^{\infty} \frac{2^n}{3^{n-1}+2}$ is also convergent.

EXAMPLE 13.4.2. To conclude on the convergence of $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+3n+3}}{\sqrt[3]{2n^6+n^4+1}}$, let

$$a_n := \frac{\sqrt{n^3+3n+3}}{\sqrt[3]{2n^6+n^4+1}} \text{ and } b_n := \frac{\sqrt{n^3}}{\sqrt[3]{n^6}} = \frac{1}{\sqrt{n}}$$

and note that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}\sqrt{n^3+3n+3}}{\sqrt[3]{2n^6+n^4+1}} = \lim_{n\to\infty} \frac{n^2\sqrt{1+\frac{3}{n^2}+\frac{3}{n^3}}}{n^2\sqrt[3]{2+\frac{1}{n^2}+\frac{1}{n^6}}} = \frac{1}{\sqrt{2}} > 0,$$

so that by the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+3n+3}}{\sqrt[3]{2n^6+n^4+1}}$ is divergent, for $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ is a divergent p-series ($p = \frac{1}{2} \leq 1$).

4. Is the series alternating, that is, of the form $\sum_{n=1}^{\infty}(-1)^n b_n$ or $\sum_{n=1}^{\infty}(-1)^{n-1} b_n$ where $b_n \geq 0$ for all n? If yes, the Alternating Series Test (Theorem 12.1.3) applies to the effect that the series is convergent provided that $\{b_n\}_{n=1}^{\infty}$ is eventually non-increasing and that $\lim_{n\to\infty} b_n = 0$.

5. Series with general terms defined in terms of products, quotients, powers, and particularly factorials, are usually handled by the Ratio Test (Theorem 13.1.1)

REMARK. To apply the ratio test, you need to calculate $\left|\frac{a_{n+1}}{a_n}\right|$, simplify it, and find the limit as n goes to ∞. To this end, keep in mind that.

$$\frac{a^x}{a^y} = a^{x-y} \text{ and } n! = 1 \cdot 2 \cdot 3 \ldots \cdot (n-1) \cdot n,$$

so that for instance

$$\frac{n!}{(n+2)!} = \frac{1}{(n+1)(n+2)} \text{ and } \frac{3^{2n}}{3^{n+1}} = 3^{n-1}.$$

6. If $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (b_n)^n$ use the Root Test (Theorem 13.3.2).

7. If none of the above applies, you can try to apply the Integral Test (Theorem 11.1.1) provided that

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} f(n)$$

where f is eventually continuous, non-negative, and decreasing. Then

$$\sum_{n=1}^{\infty} f(n) \text{ converges } \iff \int_1^{\infty} f(x)\,dx \text{ converges.}$$

Exercises

you are now prepared to work on the Practice Problems, and Homework set M14 in the manual of exercises.

Before turning to Chapter 14, you should also take Mock Test 4.

14 M15: Power Series and Taylor Series

14.1 Power series

Watch the video at

https://www.youtube.com/watch?v=IsZMVI4VcCU&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=93

Abstract

In this video, power series are defined and conditions for their convergence examined, leading to the definition of radius of convergence and interval of convergence.

In the sequel, we use the convention that

$$0! := 1.$$

Example 14.1.1. The series

$$\sum_{n=1}^{\infty} \frac{x^n}{n!}$$

is absolutely convergent for all x, by the Ratio Test, because

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{|x|^n} = \frac{|x|}{n+1}$$

so that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$ for all x. Thus

$$f : x \mapsto \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

is a well-defined function with domain $(-\infty, \infty)$.

In this module, we will be interested in functions defined that way, and when a given function can be represented that way.

DEFINITION 14.1.2. A *power series* (centered at 0) is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots,$$

where x is a variable and all of the c_n's are constants, called *coefficients* of the power series.

Of course

$$f(x) := \sum_{n=0}^{\infty} c_n x^n$$

is only defined for values of x making the series converge. Thus we will want to determine the set of x's making power series converge.

Note that for $x = 0$, $\sum_{n=0}^{\infty} c_n x^n = c_0$ is always convergent.

EXAMPLE 14.1.3. If $c_n = 1$ for all n, we obtain $\sum_{n=0}^{\infty} x^n$ which is a geometric series of common ratio x. Thus $\sum_{n=0}^{\infty} x^n$ converges if and only if $|x| < 1$ and then

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \text{ for all } -1 < x < 1,$$

so that $\sum_{n=0}^{\infty} x^n$ is a function with domain $(-1,1)$. We will say that $f(x) = \frac{1}{1-x}$ *has a power series representation* on $(-1,1)$.

EXAMPLE 14.1.4. For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

Solution. If $a_n := n!x^n$ then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!|x|^{n+1}}{n!|x|^n} = (n+1)\cdot|x|.$$

Thus $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$ for all $x \neq 0$ and $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$ for $x = 0$. By the Ratio Test, we conclude that $\sum_{n=0}^{\infty} n!x^n$ is divergent for all $x \neq 0$ and convergent for $x = 0$.

EXAMPLE 14.1.5. For what values of x is the series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ convergent?

Solution. If $a_n := \frac{x^n}{n}$ then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{n+1}}{n+1}\cdot\frac{n}{|x|} = |x|\cdot\frac{n}{n+1},$$

so that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x|.$$

By the Ratio Test, we conclude that $\sum_{n=0}^{\infty} \frac{x^n}{n}$ is absolutely convergent for $|x| < 1$ and divergent for $|x| > 1$. For $|x| = 1$, that is, $x = \pm 1$, we have to analyze the series separately:

When $x = 1$,

$$\sum_{n=1}^{\infty}\frac{x^n}{n} = \sum_{n=1}^{\infty}\frac{1}{n}$$

is a divergent p-series ($p = 1 \leq 1$).

When $x = -1$,

$$\sum_{n=1}^{\infty}\frac{x^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$

is an alternating series that converges by the Alternating Series Test, because $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is decreasing with limit 0. Thus $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges if and only if $x \in [-1,1)$.

Examples 14.1.1, 14.1.3, 14.1.4 and 14.1.5 suggest that a series centered at 0 converges on an interval centered at 0 that may or may not contain the endpoints.

More generally, we will consider:

DEFINITION 14.1.6. A *power series centered* at *a* is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_n(x-a)^n + \ldots$$

where *x* is a variable and all of the c_n's are constants, called *coefficients* of the power series.

THEOREM 14.1.7. *Given a power series* $\sum_{n=0}^{\infty} c_n(x-a)^n$, *there are 3 possibilities:*

1. $\sum_{n=0}^{\infty} c_n(x-a)^n$ *converges only for x = a;*
2. $\sum_{n=0}^{\infty} c_n(x-a)^n$ *converges for all x;*
3. *There is* $R > 0$ *such that* $\sum_{n=0}^{\infty} c_n(x-a)^n$ *converges absolutely if* $|x-a| < R$ *and diverges if* $|x-a| > R$.

We call the value *R* in the third case the *radius of convergence of the series* .

We include Case 1. in Theorem 14.1.7 by identifying that case with a radius of convergence $R = 0$. Similarly, allowing the radius of convergence *R* to be infinite recovers Case 2. in Theorem 14.1.7.

Thus the series converges absolutely inside an interval centered at *a* of radius *R* (where *R* may also be 0 or ∞) and diverges outside. As we have seen on the examples above, the series may or may not converge at the end points where $|x-a| = R$ and the two numerical series corresponding to $x = a + R$ and to *x*=*a*-*R* need to be tested for convergence separately.

Including endpoints whenever applicable, we then obtain the *interval of convergence*, that is, the set of values of *x* for which the power series converge.

14.2 Intervals of convergence

Watch the videos at

https://www.youtube.com/watch?v=Eq8BSG-melo&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=94

and

https://www.youtube.com/watch?v=8unc34-OqmA&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=95

and

https://www.youtube.com/watch?v=KWGOU8B7J9g&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=96

Abstract

In these videos, the interval of convergence is determined for a number of examples of power series.

Example 14.2.1. Find the interval of convergence for the series

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n^2}.$$

Solution. Let $a_n := \frac{(x-3)^n}{n^2}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-3|^{n+1}}{(n+1)^2} \cdot \frac{n^2}{|x-3|^n} = \frac{n^2}{(n+1)^2} \cdot |x-3|$$

so that

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x-3|.$$

Thus $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n^2}$ is absolutely convergent if $|x-3| < 1$ and is divergent if $|x-3| > 1$.

Moreover $|x-3| = 1$ for $x = 2$ or $x = 4$, corresponding to $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ respectively. Both are absolutely convergent for $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent *p*-series ($p = 2 > 1$). Thus, the interval of convergence is [2, 4].

Example 14.2.2. Find the interval of convergence for the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}.$$

Solution. Let $a_n := \frac{(-1)^n x^n}{n+1}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{n+1}}{n+2} \cdot \frac{n+1}{|x|^n} = \frac{n+1}{n+2} \cdot |x|$$

so that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x|$ and, by the Ratio Test, the interval of convergence is centered at 0 and of radius 1.

At $x = -1$, the series becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^n(-1)^n}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n}$$

because $(-1)^{2n} = 1$ for all *n*. Moreover, $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent *p*-series ($p = 1 \leq 1$).

At $x = 1$, the series becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$

which is convergent by the Alternating Series Test, for $\left\{\frac{1}{n+1}\right\}_{n=1}^{\infty}$ is decreasing with limit 0. Thus the interval of convergence is

$$I = (-1, 1].$$

Example 14.2.3. Find the interval of convergence for the series

$$\sum_{n=0}^{\infty} \frac{3^n x^n}{(n+1)^2}.$$

Solution. Let $a_n := \frac{3^n x^n}{(n+1)^2}$. Then,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{3^{n+1}|x|^{n+1}}{(n+2)^2} \cdot \frac{(n+1)^2}{3^n|x|^n} = 3|x| \cdot \left(\frac{n+1}{n+2}\right)^2,$$

so that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 3|x|$ and, by the Ratio Test, the interval of convergence is centered at 0 and of radius $\frac{1}{3}$. At $x = -\frac{1}{3}$, the series becomes

$$\sum_{n=0}^{\infty} \frac{3^n}{(n+1)^2} \cdot \left(-\frac{1}{3}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2},$$

which is absolutely convergent, hence convergent, for

$$\sum_{n=0}^{\infty} \left|\frac{(-1)^n}{(n+1)^2}\right| = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is a convergent p-series ($p = 2 > 1$). Thus, the series is also convergent when $x = \frac{1}{3}$, where it becomes $\sum_{n=0}^{\infty} \frac{1}{(n+1)^2}$. Therefore, the interval of convergence is

$$I = \left[-\frac{1}{3}, \frac{1}{3}\right].$$

Example 14.2.4. Find the interval of convergence for the series

$$\sum_{n=2}^{\infty} \frac{(x-1)^n}{\ln n}.$$

Solution. Let $a_n := \frac{(x-1)^n}{\ln n}$. Then,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-1|^{n+1}}{\ln(n+1)} \cdot \frac{\ln n}{|x-1|^n} = \frac{\ln n}{\ln(n+1)} \cdot |x-1|$$

and

$$\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} = \lim_{x\to\infty} \frac{\ln x}{\ln(x+1)} \overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{x+1}} = \lim_{x\to\infty} \frac{x+1}{x} = 1,$$

so that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-1|$. By the Ratio Test, the interval of convergence is centered at 1 and of radius 1. At $x = 0$, the series becomes

$$\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln n},$$

which is convergent by the Alternating Series Test, for $\{\frac{1}{\ln n}\}_{n=2}^{\infty}$ is decreasing with limit 0.

On the other hand, at $x = 2$, the series becomes

$$\sum_{n=2}^{\infty}\frac{1}{\ln n},$$

and

$$0 \le \frac{1}{n} \le \frac{1}{\ln n} \text{ for all } n \ge 2, \tag{14.2.1}$$

for $f(x) = x - \ln x > 0$ for all $x \ge 1$ and thus $n \ge \ln n$. Indeed, $f'(x) = 1 - \frac{1}{x} > 0$ for $x > 1$ and $f(1) = 1 > 0$.

In view of (14.2.1), we conclude by Comparison that $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ is divergent because $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent p-series ($p = 1 \leq 1$). Thus, the interval of convergence is

$$I = [0, 2).$$

Example 14.2.5. Find the interval of convergence for the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{(x+2)^n}{n \cdot 2^n}.$$

Solution. Let $a_n := (-1)^n \frac{(x+2)^n}{n2^n}$. Then

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{|x+2|^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n2^n}{|x+2|^n} = \frac{1}{2} \cdot \frac{n}{n+1} \cdot |x+2|,$$

so that $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{|x+2|}{2}$. By the Ratio Test, the interval of convergence is centered at –2 and of radius 2.

At x = –4, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n2^n} \cdot (-2)^n = \sum_{n=1}^{\infty} \frac{(-1)^{2n} 2^n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

which is a divergent p-series ($p = 1 \leq 1$).

At $x = 0$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which is convergent by the Alternating Series Test, for the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is decreasing with limit 0.

Therefore the interval of convergence is

$$I = (-4, 0].$$

Example 14.2.6. Find the interval of convergence for the series

$$\sum_{n=2}^{\infty} \frac{x^n}{(\ln n)^n}.$$

Solution. Let $a_n := \frac{x^n}{(\ln n)^n}$. Then

$$\sqrt[n]{|a_n|} = \frac{|x|}{\ln n},$$

so that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 0$ for all *x*. By the Root Test, the series is convergent for all *x* and the interval of convergence is

$$I = (-\infty, \infty).$$

EXAMPLE 14.2.7. Suppose that $\sum_{n=0}^{\infty} c_n x^n$ converges when $x = -4$ and diverges when $x = 6$. Can we conclude about the convergence of:

1. $\sum_{n=0}^{\infty} c_n$

   Solution. Since this is a power series centered at 0, its interval of convergence is centered at 0. Because it converges for $x = -4$, it s radius of convergence *R* is at least 4. Because it diverges for $x = 6$, *R* is at most 6:

   $$4 \leq R \leq 6.$$

   Now

   $$\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} c_n x^n \text{ for } x = 1$$

   and 1 is in the interval of convergence for $R \geq 4$. Thus $\sum_{n=0}^{\infty} c_n$ is absolutely convergent.

2. $\sum_{n=0}^{\infty} c_n 8^n$

   Solution. Note that

   $$\sum_{n=0}^{\infty} c_n 8^n = \sum_{n=0}^{\infty} c_n x^n \text{ for } x = 8$$

   which is divergent because $R \leq 6 < 8$.

3. $\sum_{n=0}^{\infty} c_n(-3)^n$

Solution. Note that

$$\sum_{n=0}^{\infty} c_n(-3)^n = \sum_{n=0}^{\infty} c_n x^n \text{ for } x = -3$$

which is convergent because $R \geq 4$.

4. $\sum_{n=0}^{\infty}(-1)^n c_n 9^n$

Solution. Note that

$$\sum_{n=0}^{\infty}(-1)^n c_n 9^n = \sum_{n=0}^{\infty} c_n x^n \text{ for } x = -9$$

which is divergent because $R \leq 6 < 9$.

Exercises

you are now prepared to work on the Practice Problems, and Homework set M15A in the manual of exercises.

14.3 Representation of functions as power series

Watch the video at

https://www.youtube.com/watch?v=tAQa4brg3F4&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=97

Abstract

In this video, we use the sum of a geometric series formula to find power series representations for various functions.

Recall that a geometric series of common ratio x and first term 1 converges if and only if $|x| < 1$, and then

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \text{ for all } x \text{ with } |x| < 1. \tag{14.3.1}$$

We can see this formula as providing a *power series representation* of the function $f(x) = \frac{1}{1-x}$ on the interval $(-1, 1)$.

This basic observation can be used to find power series representation for more functions.

Example 14.3.1. Find power series representations (on some interval to be specified) for the functions

1. $f(x) = \frac{1}{1+x}$

   Solution. Note that

   $$f(x) = \frac{1}{1-(-x)} \overset{(14.3.1)}{=} \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n \text{ for all } x \text{ with } |x| < 1.$$

2. $f(x) = \frac{1}{1+x^2}$

   Solution. Note that

   $$f(x) = \frac{1}{1-(-x^2)} \overset{(14.3.1)}{=} \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

   for all x with $|x^2| < 1$, that is, $|x| < 1$.

3. $f(x) = \frac{1}{x-5}$

Solution. Note that

$$f(x) = -\frac{1}{5} \cdot \frac{1}{1-\frac{x}{5}} \overset{(14.3.1)}{=} -\frac{1}{5} \sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^n = -\sum_{n=0}^{\infty} \frac{1}{5^{n+1}} x^n$$

for all x with $|\frac{x}{5}| < 1$, that is, $|x| < 5$.

4. $f(x) = \frac{1+x^2}{1-x^2}$

Solution. Since

$$\begin{aligned} f(x) &= (1+x^2) \cdot \frac{1}{1-x^2} \\ &\overset{(14.3.1)}{=} (1+x^2) \cdot \sum_{n=0}^{\infty} x^{2n} \end{aligned}$$

for $|x^2| < 1$, that is, for $|x| < 1$. Thus, for $x \in (-1, 1)$, we have

$$f(x) = \sum_{n=0}^{\infty} x^{2n} + x^2 \sum_{n=0}^{\infty} x^{2n} = \sum_{n=0}^{\infty} x^{2n} + \sum_{n=0}^{\infty} x^{2n+2}.$$

5. $f(x) = \frac{7x-1}{3x^2+2x-1}$

Solution. The decomposition into partial fraction for this function is:

$$\begin{aligned} \frac{7x-1}{3x^2+2x-1} &= \frac{7x-1}{(x+1)(3x-1)} = \frac{2}{x+1} + \frac{1}{3x-1} \\ &= 2 \cdot \frac{1}{1-(-x)} - \frac{1}{1-3x} \\ &\overset{(14.3.1)}{=} 2\sum_{n=0}^{\infty} (-x)^n - \sum_{n=0}^{\infty} (3x)^n \end{aligned}$$

provided that $|-x| < 1$ and $|3x| < 1$. Thus, the equality above is true for $|x| < \frac{1}{3}$ and then

$$\begin{aligned} f(x) &= 2\sum_{n=0}^{\infty} (-1)^n x^n - \sum_{n=0}^{\infty} 3^n x^n \\ &= \sum_{n=0}^{\infty} \left((-1)^n \cdot 2 - 3^n\right) x^n \text{ for } -\tfrac{1}{3} < x < \tfrac{1}{3}. \end{aligned}$$

14.4 term-by-term differentiation and integration of power series

Watch the video at

https://www.youtube.com/watch?v=8WwilmZqkGs&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=98

Abstract

In this video, we state a theorem for term-by-term differentiation and integration of power series and apply it to find more power series representations, and estimate numerical integrals.

If a power series

$$\sum_{n=0}^{\infty} c_n(x-a)^n$$

has radius of convergence *R*, then it is absolutely convergent for $|x-a|<R$, and defines on this interval a function

$$f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n.$$

It is natural to ask whether the function *f* is differentiable, and whether it can be differentiated "like a polynomial", yielding a power series representation for f' . Dually, we ask whether *f* is integrable on this interval, and whether *f* can be integrated "like a polynomial", yielding a power series representation for an antiderivative of *f*. The answers are all positive:

Theorem 14.4.1. *If the power series* $\sum_{n=0}^{\infty} c_n(x-a)^n$ *has radius of convergence* $R>0$, *then* $f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n$ *is deifined on* $(a-R,a+R)$ *and*

1. *f is differentiable on* $(a-R,a+R)$ *and*

$$f'(x)=\sum_{n=1}^{\infty} n\cdot c_n(x-a)^{n-1}=c_1+2c_2(x-a)+3c_3(x-a)^2+\dots$$

and the series representing f' *has radius of convergence R.*

2. *f admits antiderivatives on* $(a-R,a+R)$ *and on this interval*

$$\int f(x)\,dx=C+\sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$$

and the series on the right hand side has radius of convergence R.

We can restate this result as

$$\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right)' = \sum_{n=0}^{\infty} c_n\left((x-a)^n\right)' \text{ on } (a-R, a+R)$$

$$\int \sum_{n=0}^{\infty} c_n(x-a)^n\,dx = \sum_{n=0}^{\infty} c_n \int (x-a)^n\,dx \text{ on } (a-R, a+R).$$

EXAMPLE 14.4.2. Find a power series representation for $\int \frac{dx}{1+x^4}$ on $(-1,1)$ and find an estimate with at least 4 exact decimal places of $\int_0^{\frac{1}{2}} \frac{dx}{1+x^4}$.

Solution. Note that

$$\begin{aligned}\frac{1}{1+x^4} &= \frac{1}{1-(-x^4)} \overset{(14.3.1)}{=} \sum_{n=0}^{\infty}(-x^4)^n \text{ for } |-x^4|<1\\ &= \sum_{n=0}^{\infty}(-1)^n x^{4n} \text{ for } |x|<1.\end{aligned}$$

Using Theorem 14.4.1,

$$\int \frac{dx}{1+x^4} = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{4n+1} x^{4n+1} \text{ on } (-1,1).$$

In particular,

$$\int_0^{\frac{1}{2}} \frac{dx}{1+x^4} = \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{4n+1}\, x^{4n+1}\right]_0^{\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+1)2^{4n+1}}$$

is converging by the Alternating Series Test, so that, in view of Proposition 12.3.1, the error made in approximating $\int_0^{\frac{1}{2}} \frac{dx}{1+x^4}$ by

$$s_n := \sum_{i=0}^{n} \frac{(-1)^i}{(4i+1)\, 2^{4i+1}}$$

satisfies

$$|R_n| \leq \frac{1}{(4(n+1)+1)2^{4(n+1)+1}} = \frac{1}{(4n+5)2^{4n+5}}.$$

Thus, for $n = 2$,

$$|R_n| \leq \frac{1}{13 \times 2^{13}} \approx 9.10^{-6}$$

and

$$s_2 = \frac{1}{2} - \frac{1}{160} + \frac{1}{4608} \approx \int_0^{\frac{1}{2}} \frac{dx}{1+x^4}$$

with at least 4 exact decimal places.

14.5 more power series representations

Watch the video at

https://www.youtube.com/watch?v=_AK0pH4W9i4&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=99

Abstract

In this video, we use term-by-term differentiation and integration of power series and the sum of a geometric series to find power series representations.

Example 14.5.1. Find a power series representation for $f(x) = \ln(5 - x)$.

Solution. Since

$$\begin{aligned} f'(x) &= -\frac{1}{5-x} = -\frac{1}{5}\cdot\frac{1}{1-\frac{x}{5}} \\ &\overset{(14.3.1)}{=} -\frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n \text{ for } |\tfrac{x}{5}| < 1 \\ &= -\sum_{n=0}^{\infty}\frac{x^n}{5^{n+1}} \text{ for } |x| < 5, \end{aligned}$$

and f is an antiderivative of f', we deduce from Theorem 14.4.1 that for x in $(-5, 5)$, f has the form

$$f(x) = C - \sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)5^{n+1}}. \tag{14.5.1}$$

Moreover, $f(0) = \ln 5$ because $f(x) = \ln(5 - x)$, and $f(0) = C$ because of (14.5.1), so that

$$f(x) = \ln 5 - \sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)5^{n+1}} \text{ for } -5 < x < 5.$$

EXAMPLE 14.5.2. Find a power series representation for $f(x) = \frac{1}{(1+x)^2}$.

Solution. Since for $u = 1 + x$,

$$\begin{aligned}
\int f(x)\,dx &= \int \frac{du}{u^2} = -\frac{1}{1+x} \\
&= -\frac{1}{1-(-x)} \\
&\overset{(14.3.1)}{=} -\sum_{n=0}^{\infty}(-x)^n \text{ for } |x| < 1 \\
&= \sum_{n=0}^{\infty}(-1)^{n+1}x^n \text{ for } |x| < 1,
\end{aligned}$$

we conclude by Theorem 14.4.1 that for x in $(-1, 1)$

$$f(x) = \left(\int f(x)\,dx\right)' = \sum_{n=1}^{\infty}(-1)^{n+1}nx^{n-1} = \sum_{n=0}^{\infty}(-1)^n(n+1)x^n.$$

14.6 Power series and sums of numerical series

Watch the video at

https://www.youtube.com/watch?v=rQ9RmZ2BNyQ&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=100

Abstract

In this video, we use power series representations of functions to find the exact sums of various numerical series.

Example 14.6.1. Find:

1. the sum of $\sum_{n=1}^{\infty} nx^{n-1}$ for $|x| < 1$

   Solution. Since

   $$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \text{ for } |x| < 1,$$

   we have by term-by-term differentiation that

   $$\sum_{n=1}^{\infty} nx^{n-1} = \left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2} \text{ for } |x| < 1.$$

2. the sum of $\sum_{n=1}^{\infty} nx^n$ for $|x| < 1$

   Solution. By the previous question, for $|x| < 1$,

   $$\sum_{n=1}^{\infty} nx^n = x\sum_{n=1}^{\infty} nx^{n-1} = \frac{x}{(1-x)^2}.$$

3. the sum of the numerical series $\sum_{n=1}^{\infty} \frac{n}{2^n}$

   Solution. Note that for $x = \frac{1}{2}$, which satisfies $|x| < 1$, we have

   $$\sum_{n=1}^{\infty} nx^n = \sum_{n=1}^{\infty} \frac{n}{2^n}.$$

   Thus, in view of the previous question, $\sum_{n=1}^{\infty} \frac{n}{2^n}$ is the value of the function $\frac{x}{(1-x)^2}$ at $x = \frac{1}{2}$, that is,

   $$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{\frac{1}{2}}{(1-\frac{1}{2})^2} = 2.$$

4. the sum of $\sum_{n=2}^{\infty} n(n-1)x^n$ for $|x| < 1$

   Solution. We have seen in the first question that for $|x| < 1$,

   $$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}.$$

   Differentiating term-by-term, we obtain

   $$\sum_{n=2}^{\infty} n(n-1)x^{n-2} = \left(\frac{1}{(1-x)^2}\right)' = \frac{2}{(1-x)^3},$$

   so that

   $$\sum_{n=2}^{\infty} n(n-1)x^n = x^2 \cdot \sum_{n=2}^{\infty} n(n-1)x^{n-2} = \frac{2x^2}{(1-x)^3}$$

   for all x in $(-1, 1)$.

5. the sum of $\sum_{n=2}^{\infty} \frac{n^2-n}{2^n}$

   Solution. Note that for $x = \frac{1}{2}$,

   $$\sum_{n=2}^{\infty} n(n-1)x^n = \sum_{n=2}^{\infty} \frac{n^2-n}{2^n},$$

   so that, in view of the previous question, $\sum_{n=2}^{\infty} \frac{n^2-n}{2^n}$ is the value at $x = \frac{1}{2}$ of the function $\frac{2x^2}{(1-x)^3}$, that is,

   $$\sum_{n=2}^{\infty} \frac{n^2-n}{2^n} = \frac{2 \cdot \frac{1}{4}}{(1-\frac{1}{2})^3} = 4.$$

6. then sum of $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$.

   Solution. Note that we have obtained

   $$\sum_{n=1}^{\infty} \frac{n}{2^n} = 2 \text{ and } \sum_{n=2}^{\infty} \frac{n^2-n}{2^n} = 4.$$

   Thus,

   $$\begin{aligned} \sum_{n=1}^{\infty} \frac{n^2}{2^n} &= \sum_{n=1}^{\infty} \frac{n^2-n}{2^n} + \sum_{n=1}^{\infty} \frac{n}{2^n} \\ &= 0 + \sum_{n=2}^{\infty} \frac{n^2-n}{2^n} + \sum_{n=1}^{\infty} \frac{n}{2^n} = 4 + 2 = 6. \end{aligned}$$

Exercises

you are now prepared to work on the Practice Problems, and Homework set M15B in the manual of exercises.

14.7 Taylor and MacLaurin series

Watch the video at

https://www.youtube.com/watch?v=sDTfZOcYk_o&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=101

THEOREM 14.7.1. *If a function f has a representation as a power series centered at a, it is necessarily represented by the series*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

where $f^{(n)}$ *denotes the* n^{th} *derivative of f and we use the convention that* $0! = 1$.

DEFINITION 14.7.2. The series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is called *Taylor series of f at a.* When $a = 0$, we obtain the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, x^n,$$

called *MacLaurin series of f.*

EXAMPLE 14.7.3. Find the MacLaurin series of $f(x) = e^x$.

> *Solution.* Since $f'(x) = f(x)$, we have by an immediate induction that $f^{(n)}(x) = f(x) = e^x$ for all *n*. Thus
>
> $$\frac{f^{(n)}(0)}{n!} = \frac{e^0}{n!} = \frac{1}{n!},$$
>
> and the MacLaurin series of e^x is
>
> $$\sum_{n=0}^{\infty} \frac{x^n}{n!},$$
>
> which has interval of convergence $(-\infty, \infty)$ as easily seen from the Ratio Test (see e.g., Example 13.2.4).

Note that even though the series converges, we have not yet established that it converges to e^x. We will consider this question in Section 14.9.

14.8 Examples of Taylor Series

Watch the video at

https://www.youtube.com/watch?v=E9kjOiwbpV8&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=102

Abstract

In this video, we find Taylor series explicitly for a number of examples.

Example 14.8.1. Find the Taylor series of $f(x) = 1 + x + x^2$ at $a = 2$.

Solution. Since $f'(x) = 1 + 2x$, $f''(x) = 2$ and $f^{(n)}(x) \equiv 0$ for all $n \geq 3$, there Taylor series of f at 2 is

$$f(0) + f'(0)(x-2) + \frac{f''(0)}{2}(x-2)^2 = 7 + 5(x-2) + (x-2)^2,$$

and is in fact a polynomial.

Of course, the Taylor series of a polynomial of degree *n* is a polynomial of degree *n*: the same polynomial but written in terms of powers of $(x-a)$ rather than in standard form.

EXAMPLE 14.8.2. Find the Taylor series of $f(x)=\frac{1}{x}$ at $a=1$.

Solution. Note that $f(x)=x^{-1}$ so that

$$\begin{aligned} f^{(0)}(x) &= x^{-1} \\ f^{(1)}(x) &= -x^{-2} \\ f^{(2)}(x) &= (-1)(-2)x^{-3} \\ f^{(3)}(x) &= (-1)(-2)(-3)x^{-4} \\ &\vdots \\ f^{(n)}(x) &= (-1)(-2)\dots(-n)x^{-(n+1)} \\ &= (-1)^n n!\, x^{-(n+1)}. \end{aligned}$$

Thus

$$\frac{f^{(n)}(1)}{n!}=\frac{(-1)^n n!}{n!}=(-1)^n \text{ for all } n\geq 0$$

and the Taylor series of *f* at 1 is

$$\sum_{n=0}^{\infty}(-1)^n(x-1)^n.$$

Note that we could also have observed that

$$f(x)=\frac{1}{x}=\frac{1}{1-(1-x)}=\sum_{n=0}^{\infty}(1-x)^n=\sum_{n=0}^{\infty}(-1)^n(x-1)^n \text{ for } |x-1|<1,$$

so that *f* has a power series representation centered at 1, which is therefore its Taylor series.

EXAMPLE 14.8.3. Find the Taylor series of $f(x)=\sin x$ at $a=\frac{\pi}{4}$.

Solution. Note that

$$\begin{aligned} f^{(0)}(x) &= \sin x \\ f^{(1)}(x) &= \cos x \\ f^{(2)}(x) &= -\sin x \\ f^{(3)}(x) &= -\cos x \\ f^{(4)}(x) &= \sin x = f^{(0)}(x), \end{aligned}$$

so that this pattern then repeats: $f^{(5)}(x) = f^{(1)}(x)$, $f^{(6)}(x) = f^{(2)}(x)$, and so on. In particular, derivatives of even order are of the form $\pm \sin x$ and those of odd order are of the form $\pm \cos x$. More specifically, we have

$$f^{(2n)}(x) = (-1)^n \sin x \text{ and } f^{(2n+1)}(x) = (-1)^n \cos x.$$

Thus

$$\frac{f^{(n)}\left(\frac{\pi}{4}\right)}{n!} = \pm\frac{\sqrt{2}}{2} \cdot \frac{1}{n!},$$

where two positive signs alternate with two negative signs periodically. Thus the Taylor series is of the form

$$\frac{\sqrt{2}}{2}\left(1 + \left(x - \frac{\pi}{4}\right) - \frac{\left(x - \frac{\pi}{4}\right)^2}{2!} - \frac{\left(x - \frac{\pi}{4}\right)^3}{3!} + \frac{\left(x - \frac{\pi}{4}\right)^4}{4!} + \frac{\left(x - \frac{\pi}{4}\right)^5}{5!} - \dots\right).$$

EXAMPLE 14.8.4. Find the MacLaurin series of $f(x) = \sin x$.

Solution. We have already established in the previous example that

$$f^{(2n)}(x) = (-1)^n \sin x \text{ and } f^{(2n+1)}(x) = (-1)^n \cos x.$$

Thus $f^{(2n)}(0) = 0$ for all *n*, and the MacLaurin series reduces to

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

It is easily verified by the Ratio Test that this series has interval of convergence $(-\infty, \infty)$, for

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{|x|^{2n+1}} = \frac{|x|^2}{(2n+2)(2n+3)},$$

so that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$ for all *x*. however, we cannot yet justify that

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1},$$

even though we will in Section 14.9.

14.9 Convergence of Taylor Series

Watch the video at

https://www.youtube.com/watch?v=JboE2aHr0hE&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=103

Abstract

In this video, we state and establish the Theorem of Taylor-Lagrange giving an explicit formula for the remainder in a Taylor series. We deduce Taylor's inequality and apply it to prove equality of e^x and sin x with their MacLaurin series.

Definition 14.9.1. The n^{th} *Taylor polynomial of* f *at* a is

$$T_n(x) := \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!} (x-a)^i.$$

We denote by $R_n(x)$ the error made in approximating f by $T_n(x)$, that is,

$$R_n(x) := f(x) - T_n(x).$$

Obviously $\lim_{n\to\infty} T_n(x) = f(x)$ if and only if $\lim_{n\to\infty} |R_n(x)| = 0$.

THEOREM 14.9.2 (Taylor-Lagrange) *If f and its first n derivatives are continuous on* $[a, b]$ *and* $f^{(n)}$ *is differentiable on* (a, b), *then there exists c in* (a, b) *such that*

$$f(b) = T_n(b) + \frac{f^{(n+1)}(c)}{(n+1)!}\,(b-a)^{n+1},$$

so that

$$R_n(b) = \frac{f^{(n+1)}(c)}{(n+1)!}\,(b-a)^{n+1}.$$

COROLLARY 14.9.3 (Taylor's Inequality) *If there is M such that for every x with* $|x-a| \le d$,

$$|f^{(n+1)}(x)| \le M,$$

then

$$|R_n(x)| \le \frac{M}{(n+1)!} \cdot |x-a|^{n+1}$$

for all x with $|x-a| \le d$.

EXAMPLE 14.9.4. Let $f(x) = e^x$ and consider its MacLaurin series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, as obtained in Example 14.7.3. Since f is increasing and $f^{(n)} = f$ for all n,

$$|x| \le d \Longrightarrow |f^{(n+1)}(x)| = e^x \le e^d,$$

so that, in view of Taylor's Inequality, if $|x| \le d$ then

$$|R_n(x)| \le \frac{e^d |x|^{n+1}}{(n+1)!}.$$

Since $\lim_{n\to\infty} \frac{e^d|x|^{n+1}}{(n+1)!} = 0$, we conclude by the Squeeze Theorem that $\lim_{n\to\infty} |R_n(x)| = 0$ if $|x| \le d$, that is, $\lim_{n\to\infty} T_n(x) = e^x$. Since this is true for every d, we conclude that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ for all } x.$$

EXAMPLE 14.9.5. We have seen in Example 14.8.4 that the MacLaurin series of $f(x) = \sin x$ is $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$. Moreover, the successive derivatives of f are of the form $\pm\cos x$ or $\pm\sin x$. At any rate,

$$|f^{(n+1)}(x)| \leq 1 \text{ for all } x,$$

so that, in view of Taylor's Inequality,

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}.$$

Since $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$ for all x, we conclude that $\lim_{n\to\infty} |R_n(x)| = 0$ for all x, that is

$$\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} \text{ for all } x.$$

14.10 More examples of Taylor Series

Watch the video at

https://www.youtube.com/watch?v=aByBBTG5AEM&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=104

Abstract

In this video, equalities between functions and their MacLaurin series on the appropriate intervals are established for more functions: $\cos(x)$, $\ln(1+x)$ and arctan *x*.

To summarize the standard power series representations established:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \text{ for } -1 < x < 1 \tag{14.10.1}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ for all } x \tag{14.10.2}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \text{ for all } x \tag{14.10.3}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \text{ for all } x \tag{14.10.4}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n \text{ for } -1 < x \leq 1 \tag{14.10.5}$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \text{ for } -1 \leq x \leq 1. \tag{14.10.6}$$

Exercises

you are now prepared to work on the Practice Problems, and Homework set M15C in the manual of exercises.

15 M16: Applications of power series

15.1 Power series and sums of numerical series

Watch the video at

https://www.youtube.com/watch?v=hxHkazlw6F8&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=105

Abstract

In this video, standard power series representations of functions are used to obtain exact values of various numerical series as values of the functions represented.

Example 15.1.1. Find the exact values of the following series:

1. $\sum_{n=0}^{\infty} \frac{3^n}{5^n n!}$

   Solution. Recall from (14.10.2) that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x. Thus

   $$\sum_{n=0}^{\infty} \frac{3^n}{5^n n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{3}{5}\right)^n = e^{\frac{3}{5}}.$$

2. $\sum_{n=0}^{\infty} \frac{(-\pi^2)^n}{(36)^n (2n)!}$

   Solution. Recall from (14.10.4) that $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$. Thus

   $$\sum_{n=0}^{\infty} \frac{(-\pi^2)^n}{(36)^n (2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(6^2)^n (2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot \left(\frac{\pi}{6}\right)^n = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$

3. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$

   Solution. Recall from (14.10.6) that $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ for $x \in [-1, 1]$. Thus,

   $$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \arctan 1 = \frac{\pi}{4}.$$

4. $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{4n}}{n!}$

Solution.

$$\sum_{n=0}^{\infty} \frac{(-1)^n 2^{4n}}{n!} = \sum_{n=0}^{\infty} \frac{(-16)^n}{n!} = e^{-16}.$$

5. $1 - \ln 2 + \frac{(\ln 2)^2}{2!} - \frac{(\ln 2)^3}{3!} + \dots$

Solution.

$$1 - \ln 2 + \frac{(\ln 2)^2}{2!} - \frac{(\ln 2)^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{(-\ln 2)^n}{n!} = e^{-\ln 2} = e^{\ln(2^{-1})} = \frac{1}{2}.$$

6. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3^n n}$

Solution. Recall from (14.10.5) that $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$ for $-1 < x \leq 1$. Thus

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3^n n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{1}{3}\right)^n = \ln\left(1 + \frac{1}{3}\right) = \ln\left(\frac{4}{3}\right).$$

15.2 Estimating integrals

Watch the video at

https://www.youtube.com/watch?v=x4kG01QCZvE&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=106

Abstract

In this video, we use power series representations for antiderivatives to estimate definite integrals.

Example 15.2.1. Estimate $\int_0^1 e^{-x^2}\,dx$ with 2 exact decimal places.

Solution. We cannot obtain an antiderivative of e^{-x^2} in closed form, even though antiderivatives exist since the function is continuous. However, we can find a power series representation for the antiderivative in order to use the Fundamental Theorem of Calculus. Indeed, plugging in $-x^2$ in (14.10.2), we obtain

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{\left(-x^2\right)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$

for all x. Integrating term-by-term with Theorem 14.4.1,

$$\int e^{-x^2}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!(2n+1)},$$

so that, by the Fundamental Theorem of Calculus,

$$\int_0^1 e^{-x^2}\,dx = \left[\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!(2n+1)}\right]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)}.$$

This series is convergent by the Alternating Series Test for $\left\{\frac{1}{n!(2n+1)}\right\}_{n=0}^{\infty}$ is decreasing with limit 0. Thus, in view of Proposition 12.3.1, the error committed in the approximation

$$s_n \approx \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} = \int_0^1 e^{-x^2}\,dx$$

satisfies

$$|R_n| \leq \frac{1}{(n+1)!(2n+3)}.$$

To ensure two exact decimal places, we want $|R_n| \leq 10^{-3}$ and $(n+1)!(2n+3) \geq 10^3$ for $n \geq 4$. Thus

$$s_4 = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} \approx \int_0^1 e^{-x^2}\,dx$$

with at least two exact decimal places.

EXAMPLE 15.2.2. Estimate $\int_0^1 \sin(x^2)\,dx$ with 3 exact decimal places.

Solution. We proceed similarly: plugging in x^2 in (14.10.3),

$$\sin(x^2) = \sum_{n=0}^{\infty}(-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+2}}{(2n+1)!},$$

so that, integrating term-by-term via Theorem 14.1.1, we obtain

$$\int \sin(x^2)\,dx = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+3}}{(2n+1)!(4n+3)}.$$

By the Fundamental Theorem of Calculus

$$\int_0^1 \sin(x^2)\,dx = \left[\sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+3}}{(2n+1)!(4n+3)}\right]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!(4n+3)}.$$

This series is converging by the Alternating Series Test for $\left\{\frac{1}{(2n+1)!(4n+3)}\right\}_{n=1}^{\infty}$ is decreasing with limit 0. Thus, in view of Proposition 12.3.1, the error committed in the approximation

$$s_n \approx \int_0^1 \sin(x^2)\,dx$$

satisfies

$$|R_n| \leq \frac{1}{(2n+3)!(4n+7)}.$$

To ensure that the estimate has 3 exact decimal places, we want $|R_n| \leq 10^{-4}$, which is achieved for $n = 2$ for $7! \times 15 = 75600 > 10^4$. Thus

$$s_2 = \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} \approx \int_0^1 \sin(x^2)\,dx$$

with at least 3 exact decimal places.

15.3 Calculating limits

Watch the video at

https://www.youtube.com/watch?v=-KkU1DmvHNY&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=107

Abstract

In this video, we use power series representations to evaluate limits.

Example 15.3.1. Evaluate

$$\lim_{x\to 0}\frac{\sin x - x + \frac{x^3}{6}}{x^5}.$$

Solution. This limit is an indeterminate form of the type $\frac{0}{0}$, so we could use the Rule of De l'Hospital, but we would have to iterate it five times. Alternatively, we can use the power series representation (14.10.3) of $\sin x$:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \Longrightarrow \sin x - x + \frac{x^3}{6} = \frac{x^5}{5!} - \frac{x^7}{7!} + \dots,$$

so that

$$\sin x - x + \frac{x^3}{6} = x^5\left(\frac{1}{5!} - \frac{x^2}{7!} + \frac{x^4}{9!} - \dots\right)$$

and

$$\lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5} = \lim_{x\to 0} \frac{1}{5!} - \frac{x^2}{7!} + \frac{x^4}{9!} - \dots = \frac{1}{5!} = \frac{1}{120}.$$

Example 15.3.2. Evaluate

$$\lim_{x\to 0} \frac{x - \arctan x}{x^3}.$$

Solution. Since this is a limit as x approaches 0, we can use the power series representation (14.10.6) of arctan, which is valid on $[-1, 1]$. Since

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots,$$

we have

$$x - \arctan x = \frac{x^3}{3} - \frac{x^5}{5} + \dots = x^3\left(\frac{1}{3} - \frac{x^2}{5} + \dots\right)$$

so that

$$\lim_{x\to 0} \frac{x - \arctan x}{x^3} = \lim_{x\to 0} \frac{1}{3} - \frac{x^2}{5} + \frac{x^4}{7} - \dots = \frac{1}{3}.$$

Example 15.3.3. Evaluate

$$\lim_{x\to 0} \frac{1 - \cos x}{1 + x - e^x}.$$

Solution. In view of (14.10.4),

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots \Longrightarrow 1 - \cos x = \frac{x^2}{2} - \frac{x^4}{4!} + \dots$$

On the other hand, in view of (14.10.2),

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots \Longrightarrow 1 + x - e^x = -\frac{x^2}{2} - \frac{x^3}{6} - \dots$$

Thus

$$\lim_{x\to 0} \frac{1 - \cos x}{1 + x - e^x} = \lim_{x\to 0} \frac{x^2\left(\frac{1}{2} - \frac{x^2}{4!} + \frac{x^4}{6!} - \dots\right)}{x^2\left(-\frac{1}{2} - \frac{x}{6} - \frac{x^2}{4!} - \dots\right)} = \frac{\frac{1}{2}}{-\frac{1}{2}} = -1.$$

15.4 More power series: products

Watch the video at

https://www.youtube.com/watch?v=bxxAXuMCsMA&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=108

Abstract

In this video, we state a theorem on the power series representation of the product of two absolutely convergent power series and apply it to obtain the first few terms of the power series representation for more functions.

THEOREM 15.4.1. *If* $\sum_{n=0}^{\infty} a_n x^n$ *and* $\sum_{n=0}^{\infty} b_n x^n$ *both converge absolutely for* $|x| < R$, *then for every* x *in* $(-R, R)$, *we have*

$$\sum_{n=0}^{\infty} a_n x^n \cdot \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} c_n x^n$$

where

$$c_n = \sum_{k=0}^{n} a_k \cdot b_{n-k} = a_0 b_n + a_1 b_{n-1} + \ldots + a_{n-1} b_1 + a_n b_0,$$

and the series $\sum_{n=0}^{\infty} c_n x^n$ *is absolutely convergent for* $|x| < R$.

EXAMPLE 15.4.2. Find the first 3 terms of a power series representation of

$$\frac{\ln(1+x)}{1-x} = \frac{1}{1-x} \cdot \ln x.$$

Solution. Since $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$ by (14.10.1) and $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$ by (14.10.5) for $|x| < 1$, we conclude from Theorem 15.4.1 that $\frac{\ln(1+x)}{1-x}$ has a power series representation for $|x| < 1$, which has the form

$$\left(1 + x + x^2 + x^3 + x^4 + \ldots\right) \cdot \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right),$$

and thus starts with the terms

$$x + \left(1 - \frac{1}{2}\right) x^2 + \left(-\frac{1}{2} + 1 + \frac{1}{3}\right) x^3 + \ldots = x + \frac{x^2}{2} + \frac{5x^3}{6} + \ldots$$

As we usually only use the first few terms of a power series representation when estimating integrals or limits, this can be used in applications.

EXAMPLE 15.4.3. Evaluate

$$\lim_{x\to 0}\frac{(1+x-e^x)\sin x}{\cos x-1+\frac{x^2}{2}}.$$

Solution. In view of (14.10.2) and (14.10.3), we have for every x,

$$\begin{aligned}(1+x-e^x)\sin x &= -\sum_{n=2}^{\infty}\frac{x^n}{n!}\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}\\ &= -\left(\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\dots\right)\cdot\left(x-\frac{x^3}{6}+\frac{x^5}{120}-\dots\right)\\ &= -\frac{x^3}{2}-\frac{x^4}{6}+x^5\left(\frac{1}{12}+\frac{1}{24}\right)+\dots\\ &= -\frac{x^3}{2}-\frac{x^4}{6}+\frac{x^5}{8}+\dots\end{aligned}$$

On the other hand, in view of (14.10.4),

$$\cos x-1+\frac{x^2}{2}=\frac{x^4}{4!}-\frac{x^6}{6!}+\dots.$$

Thus,

$$\frac{(1+x-e^x)\sin x}{\cos x-1+\frac{x^2}{2}}=\frac{x^3\left(-\frac{1}{2}-\frac{x}{6}+\frac{x^2}{8}+\dots\right)}{x^4\left(\frac{1}{4!}-\frac{x^2}{6!}+\dots\right)},$$

so that

$$\lim_{x\to 0^+}\frac{(1+x-e^x)\sin x}{\cos x-1+\frac{x^2}{2}}=-\infty$$

for $\lim_{x\to 0^+}\frac{1}{x}=+\infty$ and $\lim_{x\to 0}\frac{-\frac{1}{2}-\frac{x}{6}+\frac{x^2}{8}+\dots}{\frac{1}{4!}-\frac{x^2}{6!}+\dots}=-\frac{4!}{2}<0$.

15.5 More power series: Binomial series

Watch the video at

https://www.youtube.com/watch?v=K07xwIg7orQ&list=PLm168eGEcBjnS6ecJflh7BTDaUB6jShIL&index=109

Abstract

In this video, we state the binomial theorem giving a power series representation of $(1+x)^r$, where r is not a positive integer, and we examine applications. In particular, we derive a power series representation for arcsin.

Theorem 15.5.1 (Binomial Theorem) *Let r be a real number that is not a positive integer. Then for every x in $(-1, 1)$,*

$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n$$

where

$$\binom{r}{0} := 1 \text{ and } \binom{r}{1} = r.$$

is called binomial coefficient *and by convention*

$$\binom{r}{0} := 1 \text{ and } \binom{r}{1} = r.$$

Moreover, the binomial series $\sum_{n=0}^{\infty} \binom{r}{n} x^n$ *is absolutely convergent for* $|x| < 1$.

EXAMPLE 15.5.2. Obtain the first 4 terms of power series representations on appropriate intervals of

1. $\sqrt{1+x}$

   Solution. For $-1 < x < 1$, Theorem 15.5.1 applies to the effect that

$$\begin{aligned}\sqrt{1+x} &= (1+x)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^n \\ &= 1 + \frac{x}{2} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \dots \\ &= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \dots\end{aligned}$$

2. $\sqrt{1-x^2}$

   Solution. Note that $|x^2| < 1$ if and only if $|x| < 1$. Thus, for $-1 < x < 1$, Theorem 15.5.1 applies to the effect that

$$\begin{aligned}\sqrt{1-x^2} &= \left(1 + \left(-x^2\right)\right)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \left(-x^2\right)^n \\ &= \sum_{n=0}^{\infty} (-1)^n \binom{\frac{1}{2}}{n} x^{2n} \\ &= 1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16} - \dots\end{aligned}$$

EXAMPLE 15.5.3. Obtain a power series representation of $\arcsin$ on an appropriate interval.

Solution. Since

$$\begin{aligned}(\arcsin x)' &= \frac{1}{\sqrt{1-x^2}} = \left(1 + \left(-x^2\right)\right)^{-\frac{1}{2}} \\ &= \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \left(-x^2\right)^n \text{ for } |-x^2| < 1 \\ &= \sum_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} x^{2n} \text{ for } |x| < 1,\end{aligned}$$

we conclude by term-by-term integration via Theorem 14.4.1 that for $|x| < 1$,

$$\arcsin x = C + \sum_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} \frac{x^{2n+1}}{2n+1}.$$

Moreover, $\arcsin 0 = 0 = C$. Thus

$$\arcsin x = \sum_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} \frac{x^{2n+1}}{2n+1} = x + \frac{x^3}{6} + \frac{3}{40}x^5 + \dots$$

for all x in $(-1, 1)$.

Exercises

you are now prepared to work on the Practice Problems, and Homework set M16 in the manual of exercises.

Once you are done, you are now ready to take the Sample Final Exam.

16 Notations

(r, θ)	polar coordinates, page 112
$0 \cdot \infty$	type of indeterminate limit, page 48
0^0	type of indeterminate limit, page 49
1^∞	type of indeterminate limit, page 49
arccos	arcsine or inverse cosine function, page 34
arccosh	inverse hyperbolic cosine, page 43
arcsin	arcsine or inverse sine function, page 31
arcsinh	inverse hyperbolic sine, page 43
arctan	arctangent or inverse tangent function, page 36
cosh	hyperbolic cosine function, page 40
coth	hyperbolic cotangent, page 41
$\frac{0}{0}$	type of indeterminate limit, page 45
$\frac{\infty}{\infty}$	type of indeterminate limit, page 45
∞^0	type of indeterminate limit, page 49
$\int_a^\infty$	improper integral over $[a, \infty)$, page 89
$\int_{-\infty}^\infty$	improper integral over $(-\infty, \infty)$, page 89
$\int_{-\infty}^b$	improper integral over $(-\infty, b]$, page 89
$\{s_n\}_{n=1}^\infty$	sequence, page 128
$\lim_{n\to\infty} a_n$	limit of the sequence $\{a_n\}_{n=1}^\infty$, page 130
ln	natural logarithm function, page 8
$\log_a$	logarithm of base a, page 28
csch	hyperbolic cosecant, page 41
sech	hyperbolic secant, page 41
$\overset{H}{=}$	equality of limits by use of the Rule of De l'Hospital, page 45
sinh	hyperbolic sine function, page 40
$\sum_{n=1}^\infty a_n$	series of general term a_n, or infinite sum, page 140
tanh	hyperbolic tangent, page 41
a^x	exponential of base a: $e^x \ln a$, page 25
e	Euler number, page 9
e^x	natural exponential function, page 19
$f^{(n)}$	n^{th} derivative of f, page 188
f^{-1}	inverse of f, page 14
$n!$	page 167

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18 Endnotes

1. The *discriminant* of a quadratic equation $ax^2 + bx + c = 0$ is $\Delta := b^2 - 4ac$. If $\Delta > 0$, then the equation has two real solutions $\frac{-b \pm \sqrt{\Delta}}{2a}$. If $\Delta = 0$, the equation has a unique solution $-\frac{b}{2a}$. If $\Delta < 0$, the equation has no real solution.
2. Trying to apply the rule of thumb, you see that x is simplified by differentiation but would be made more complicated by integration, whereas $\sin x$ gives $\pm \cos x$ depending on integration or differentiation, that is, it leads to the same complexity. Thus we prefer to take $u = x$ and $dv = \sin x\, dx$.
3. This step is sometimes enough, as we have seen with Example 4.2.2. If you need to brush up on long division you can start at 1.50 into this video.