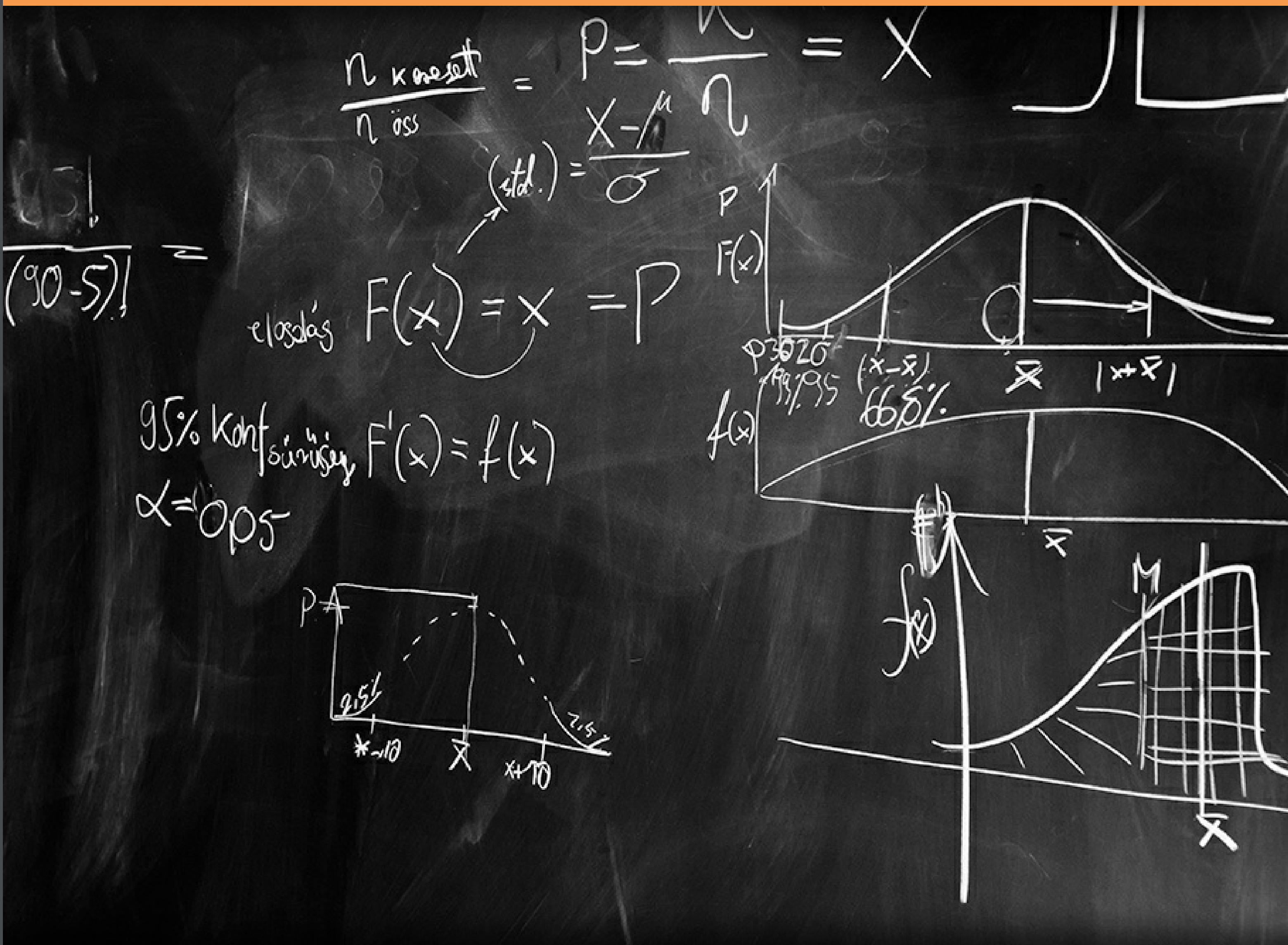
A youtube Calculus Workbook (Part I)

Frédéric Mynard



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A youtube Calculus Workbook (Part I)
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Contents

Preface

With the explosion of resources available on the internet, virtually anything can be learned on your own, using free online resources. Or can it, really? If you are looking for instructional videos to learn Calculus, you will probably have to sort through thousands of hits, navigate through videos of inconsistent quality and format, jump from one instructor to another, all this without written guidance.

This free e-book is a guide through a playlist of Calculus instructional videos. The format, level of details and rigor, and progression of topics are consistent with a semester long college level first Calculus course, or equivalently an AP Calculus AB course. The continuity of style should help you learn the material more consistently than jumping around the many options available on the internet. The book further provides simple summary of videos, written definitions and statements, worked out examples–even though fully step by step solutions are to be found in the videos – and an index.

The playlist and the book are divided into 15 thematic learning modules. At the end of each learning module, one or more quiz with full solutions is provided. Every 3 or 4 modules, a mock test on the previous material, with full solutions, is also provided. This will help you test your knowledge as you go along.

The present book is a guide to instructional videos, and as such can be used for self study, or as a textbook for a Calculus course following the flipped classroom model.

To the reader who would like to complement it with a more formal, yet free, textbook I would recommend a visit to Paul Hawkins' Calculus I pages at http://tutorial.math.lamar.edu/Classes/CalcI/CalcI.aspx, where a free e-book and a more extensive supply of practice problems are available.

For future reference, the play list of all the videos, as well as a Calculus II playlist, are available at: https://www.youtube.com/user/calculusvideos.

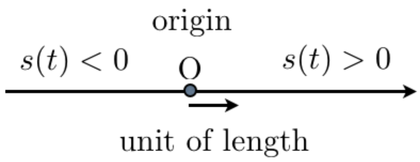
1 M1: Limits

1.1 Why limits?

Watch the video at
http://www.youtube.com/watch?v=Vl1xdx12BRs&list=PL265CB737C01F8961&index=1.

Abstract This video presents motivations for limits, and examines in particular the relation between average velocity and instantaneous velocity. The need for a concept of limit to properly define the tangent line to the graph of a function is also examined. It is further shown that the instantaneous velocity problem is nothing but an instance of the tangent line problem.

Motion In the study of motion along a straight line, the position at time t is given by a number $s(t)$ on a line, given by the signed distance to a fixed origin, relative to a fixed unit of length.



Definition 1.1.1. The *average velocity* over a given interval time, to t_1 to t_2, is given by

$$v_{[t_1,t_2]} = \frac{\text{displacement}}{\text{time}} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}.$$

Definition 1.1.2. The *instantaneous velocity* at time t_0 is approximated by the average velocities over interval of time containing t_0 and of time span getting smaller and smaller:

$$v(t_0) := \lim_{t \to t_0} v_{[t,t_0]} = \lim_{t \to t_0} \frac{s(t) - s(t_0)}{t - t_0},$$

where the precise meaning of

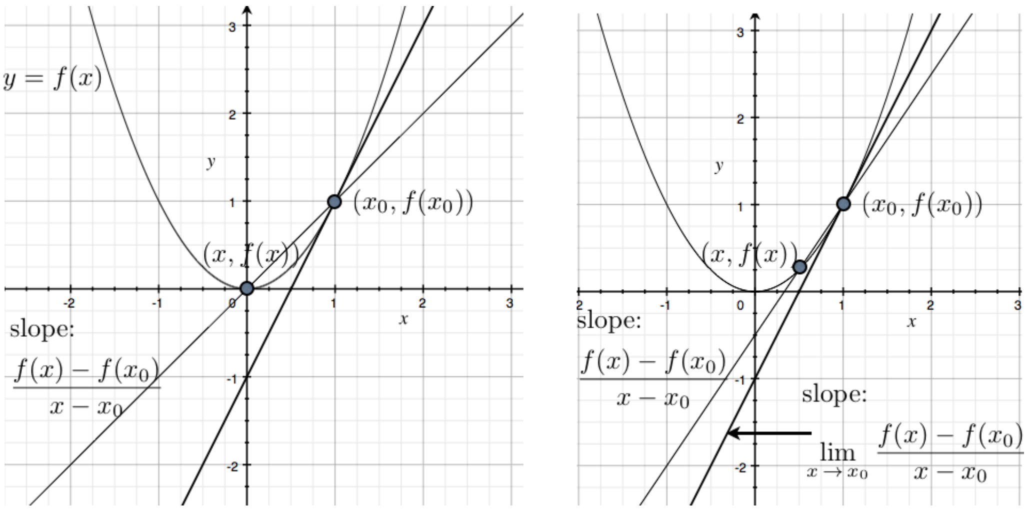
$$\lim_{t \to t_0}$$

is to be determined.

Tangent Line. The tangent line to the graph $y = f(x)$ of a function through a given point $(x_0, f(x))$ is either vertical (i.e., $x = x_0$) or is entirely determined by its slope. To define its slope, first note that the *slope* of the line through $(x_0, f(x_0))$ and another point $(x, f(x))$ on the graph of f is given by

$$\frac{f(x) - f(x_0)}{x - x_0}.$$

This secant line is not the desired tangent line, but it becomes a better approximation of it as x approaches x_0:



Definition 1.1.3. The *tangent line to $y = f(x)$ at* $(x_0, f(x_0))$ is the line through $(x_0, f(x_0))$ of slope

$$m := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

where the exact meaning of

$$\lim_{x \to x_0}$$

is to be clarified.

An *equation of the line* is then given by

$$y - f(x_0) = m(x - x_0). \tag{1.1.1}$$

1.2 Definition of the limit of a function

Watch the video at
http://www.youtube.com/watch?v=drGBIdD6gD0&list=PL265CB737C01F8961&index=2.

Abstract This video presents a (informal) definition of the limit of a function at a given value, examines a more formal re-interpretation of the definition (the so-called ϵ-δ-definition), and illustrates the fact that the limit of a function may or may not exist.

Let *f* be a real-valued function on the real line, and let *a* and *L* be two real numbers.

Definition 1.2.1. *(*Informal*)* The *limit of a function f at a is L,* in symbols

$$\lim_{x\to a} f(x) = L,$$

if the values of *f*(*x*) can be made as close to *L* as desired, by taking *x* sufficiently close to *a*, but not equal to *a*.

The formal re-interpretation is as follows (see video):

Definition 1.2.2. (formal) The limit of a function *f at a is L*, in symbols

$$\lim_{x\to a} f(x) = L,$$

if for every $\epsilon > 0$, there is $\delta > 0$ such that

$$0 < |x-a| < \delta \Longrightarrow |f(x) - L| < \epsilon.$$

1.3 Limit laws

Watch the video at
http://www.youtube.com/watch?v=Koh6kBB7xws&list=PL265CB737C01F8961&index=3.

Abstract This video presents and proves simple limit laws (thus illustrating the use of the ϵ-δ-definition of limits).

If

$$\begin{aligned} \lim_{x\to a} f(x) &= L \\ \lim_{x\to a} g(x) &= M \end{aligned}$$

exist, and c is a fixed real number, then

$$\begin{aligned} \lim_{x\to a} c &= c \\ \lim_{x\to a} x &= a \\ \lim_{x\to a} c\cdot f(x) &= c\cdot \lim_{x\to a} f(x) \\ \lim_{x\to a} (f+g)(x) &= \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \\ \lim_{x\to a} (f\cdot g)(x) &= \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x) \\ \lim_{x\to a} \left(\frac{f}{g}\right)(x) &= \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \text{ IF } \lim_{x\to a} g(x) \neq 0. \end{aligned}$$

1.4 Evaluating limits

Watch the video at
http://www.youtube.com/watch?v=BkyD2jT3crw&list=PL265CB737C01F8961&index=4.

Abstract This video examines how to evaluate limits using the limit laws from the previous section, and in cases of where the limit laws do not apply. In such cases, techniques of *factoring* and of using the *conjugate* are presented.

Using the limit laws to obtain the following limits

$$\begin{aligned}\lim_{x\to 2} 5x^3 - 2x^2 + 3x - 2 &= 36 \\ \lim_{x\to 1} \frac{x^2 - 2x + 1}{x + 9} &= 0\end{aligned}$$

illustrates the following:

Corollary 1.4.1. *If f is*

1. *a polynomial function then*

   $$\lim_{x\to a} f(x) = f(a)$$

   for any $a \in (-\infty, \infty)$*;*

2. *a rational function then*

   $$\lim_{x\to a} f(x) = f(a)$$

   for any a ***in the domain of*** *f.*

In the case of a limit of the form

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

where $f(x) = g(x) = 0$, we have an *indeterminate form of the type* $\frac{0}{0}$. Two types of instances are examined, each with its technique to solve the indetermination.

- **If f and g are both polynomial functions**: we use the fact that for a polynomial f we have

  $$f(a)=0 \iff f(x)=(x-a)q(x)$$

  where $q(x)$ is another polynomial. In other words, $f(a)=g(a)=0$ means that $(x - a)$ is a common factor in f and g. Thus, we factor $(x - a)$ out of f and g and cancel it (without changing the limit at a, because the limit does not depend on the value of the function at a). For instance:

  $$\begin{aligned}\lim_{x\to 2}\frac{2x^2+x-10}{x^2+x-6} &= \lim_{x\to 2}\frac{(x-2)(2x+5)}{(x-2)(x+3)}\\ &= \lim_{x\to 2}\frac{2x+5}{x+3}\\ &= \frac{4+5}{2+3}=\frac{9}{5}.\end{aligned}$$

- **If f or g is of the form $\sqrt{\square}\pm\bigcirc$:** we use the *conjugate quantity* $\sqrt{\square}\mp\bigcirc$ and the formula

  $$(a-b)(a+b)=a^2-b^2 \tag{1.4.1}$$

  to get read of the radicals, and then apply the previous factoring technique. For instance, to evaluate the indeterminate form $\frac{0}{0}$ below

  $$\lim_{t\to -2}\frac{\sqrt{11+t}-3}{t^2+t-2}$$

  we multiply top and bottom by the conjugate of $\sqrt{11+t}-3$, which is $\sqrt{11+t}+3$. This does not change the fraction, therefore not the limit. Using (1.4.1), we obtain:

  $$\begin{aligned}\lim_{t\to -2}\frac{\sqrt{11+t}-3}{t^2+t-2} &= \lim_{t\to -2}\frac{(\sqrt{11+t}-3)(\sqrt{11+t}+3)}{(t^2+t-2)(\sqrt{11+t}+3)}\\ &= \lim_{t\to -2}\frac{11+t-9}{(t^2+t-2)(\sqrt{11+t}+3)}\\ &= \lim_{t\to -2}\frac{t+2}{(t+2)(t-1)(\sqrt{11+t}+3)}\\ &= \lim_{t\to -2}\frac{1}{(t-1)(\sqrt{11+t}+3)}\\ &= \frac{1}{(-3)(\sqrt{9}+3)}=-\frac{1}{18}.\end{aligned}$$

1.5 Squeeze Theorem

Watch the video at
https://www.youtube.com/watch?v=2QxaVfc9KNA&list=PL265CB737C01F8961&index=5.

Abstract This video presents the Squeeze Theorem as stated below and two simple examples of applications.

The basic observation that:

Proposition 1.5.1. *If* $f(x) \leq g(x)$ *for all x in an open interval centered at a, except possibly at a, and both f and g have a limit at a then*

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x).$$

leads to the important result:

Theorem 1.5.2 (Squeeze Theorem) *If* $h(x) \leq f(x) \leq g(x)$ *for all x in an open interval centered at a, except possibly at a, and if*

$$\lim_{x\to a} h(x) = \lim_{x\to a} g(x) = L$$

then

$$\lim_{x\to a} f(x) = L.$$

To illustrate how to apply this Theorem, consider:

Example 1.5.3. Evaluate

$$\lim_{x\to 0} x^2 \cos\left(\frac{\pi}{x}\right).$$

It is unclear how $\cos\left(\frac{\pi}{x}\right)$ behaves as *x* approaches 0, but what we know is that

$$-1 \leq \cos\left(\frac{\pi}{x}\right) \leq 1$$

as long as $\cos\left(\frac{\pi}{x}\right)$ is defined, that is, for all $x \neq 0$. Multiplying each term by x^2 (which preserves inequalities because $x^2 \geq 0$), we obtain

$$-x^2 \leq x^2 \cos\left(\frac{\pi}{x}\right) \leq x^2$$

for all $x \neq 0$. Note moreover that

$$\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0$$

Thus, the Squeeze Theorem applies to the effect that

$$\lim_{x\to 0} x^2 \cos\left(\frac{\pi}{x}\right) = 0.$$

Example 1.5.4. If

$$3x \leq f(x) \leq x^3 + 2$$

for all $x \in (0, 2)$, what is

$$\lim_{x\to 1} f(x)?$$

We don't have a lot of information on *f*, yet this is sufficient to conclude. Indeed,

$$\lim_{x\to 1} 3x = \lim_{x\to 1} x^3 + 2 = 3,$$

so that the Squeeze Theorem applies to the effect that

$$\lim_{x\to 1} f(x) = 3.$$

1.6 Applications

Watch the video at
https://www.youtube.com/watch?v=V6h3L_DkoNA&list=PL265CB737C01F8961&index=6.

Abstract This video presents examples of applications of the techniques previously seen to calculate limits to the initial motivating problems of finding the tangent line to the graph of a function at given point, and of finding instantaneous velocities.

More specifically, the following examples are considered:

Example 1.6.1. Find (an equation of) the tangent line to

$$y = 2x^2 + 3x - 1$$

at $x = 2$.

Solution. Let $f(x) = 2x^2 + 3x - 1$. In view of Definition 1.1.3, the tangent line is the line through

$$(2, f(2)) = (2, 13)$$

of slope

$$\begin{aligned}
\lim_{x\to 2} \frac{f(x) - f(2)}{x-2} &= \lim_{x\to 2} \frac{2x^2 + 3x - 1 - 13}{x-2} \\
&= \lim_{x\to 2} \frac{2x^2 + 3x - 14}{x-2} \\
&= \lim_{x\to 2} \frac{(x-2)(2x+7)}{(x-2)} \\
&= \lim_{x\to 2} 2x + 7 = 11.
\end{aligned}$$

Thus, in view of (1.1.1), an equation of the tangent line is

$$y - 13 = 11(x - 2).$$

Example 1.6.2. An elephant falls off a 144 feet tall building. After t seconds he is

$$s(t) = 144 - 16t^2$$

feet from the ground.

1. What is its velocity after 2 seconds?

   Solution. Since $s(t)$ gives the position at time t, the velocity after 2 seconds is given by

$$\begin{aligned}
\lim_{t\to 2}\frac{s(t)-s(2)}{t-2} &= \lim_{t\to 2}\frac{144-16t^2-80}{t-2}\\
&= \lim_{t\to 2}\frac{64-16t^2}{t-2}\\
&= \lim_{t\to 2}\frac{(t-2)(-16t-32)}{t-2}\\
&= \lim_{t\to 2}-(16t+32)\\
&= -64\, ft/sec.
\end{aligned}$$

2. When does it reach the ground?

   Solution. It reaches the ground when the distance $s(t)$ to the ground is 0:

$$\begin{aligned}
s(t)=0 &\iff 144-16t^2=0\\
&\iff t^2=\frac{144}{16}=9\\
&\iff t=3
\end{aligned}$$

   because in this problem, $t \geq 0$. Thus, the elephant strikes the ground after 3 seconds.

3. With what speed does it strike the ground?

   Solution. This is the velocity after 3 seconds:

$$\begin{aligned}
\lim_{t\to 3}\frac{s(t)-s(3)}{t-3} &= \lim_{t\to 3}\frac{144-16t^2-0}{t-3}\\
&= \lim_{t\to 3}\frac{(t-3)(-16t-48)}{t-3}\\
&= \lim_{t\to 3}-(16t+48)\\
&= -96\, ft/sec.
\end{aligned}$$

1.7 M1 Sample Quiz

1. Evaluate the following limits

   a) $\lim\limits_{x \to 4} \dfrac{x^2 - x + 4}{x + 4}$

   b) $\lim\limits_{x \to -1} \dfrac{2x^2 + x - 1}{x^2 - x - 2}$

   c) $\lim\limits_{x \to 3} \dfrac{x + 2}{x^2 + x - 12}$

   d) $\lim\limits_{x \to 2} \dfrac{4 - \sqrt{10 + 3x}}{x^2 + x - 6}$

2. Find the tangent line to

   $$f(x) = 2x^2 - x + 3$$

   at $x = 1$.

1.8 Solutions to M1 sample Quiz

1. Evaluate the following limits

a) $$\lim_{x\to 4}\frac{x^2-x+4}{x+4} = \lim_{x\to 4}\frac{16-4+4}{4+4} = \frac{16}{8}=2$$

b) $$\begin{aligned}\lim_{x\to -1}\frac{2x^2+x-1}{x^2-x-2} &= \lim_{x\to -1}\frac{(x+1)(2x-1)}{(x+1)(x-2)}\\ &= \lim_{x\to -1}\frac{(2x-1)}{(x-2)}\\ &= \frac{-3}{-3}=1\end{aligned}$$

c) $$\lim_{x\to 3}\frac{x+2}{x^2+x-12}$$

does not exist because $x+2=5\neq 0$ when $x = 3$, while $x^2+x-12=0$ when $x = 3$.

d) $$\begin{aligned}\lim_{x\to 2}\frac{4-\sqrt{10+3x}}{x^2+x-6} &= \lim_{x\to 2}\frac{(4-\sqrt{10+3x})}{(x^2+x-6)}\cdot\frac{(4+\sqrt{10+3x})}{(4+\sqrt{10+3x})}\\ &= \lim_{x\to 2}\frac{16-(10+3x)}{(x^2+x-6)(4+\sqrt{10+3x})}\\ &= \lim_{x\to 2}\frac{6-3x}{(x^2+x-6)(4+\sqrt{10+3x})}\\ &= \lim_{x\to 2}\frac{-3(x-2)}{(x-2)(x+3)(4+\sqrt{10+3x})}\\ &= \lim_{x\to 2}\frac{-3}{(x+3)(4+\sqrt{10+3x})}\\ &= \frac{-3}{5(4+4)}=-\frac{3}{40}.\end{aligned}$$

2. Find the tangent line to

   $f(x) = 2x^2 - x + 3$

   at $x = 1$.

   The point of tangency is

   $(1, f(1)) = (1, 4)$.

   The slope of the tangent line is

$$\begin{aligned}
\lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} &= \lim_{x\to 1} \frac{2x^2 - x + 3 - 4}{x - 1} \\
&= \lim_{x\to 1} \frac{2x^2 - x - 1}{x - 1} \\
&= \lim_{x\to 1} \frac{(x-1)(2x+1)}{x - 1} \\
&= \lim_{x\to 1} 2x + 1 = 3.
\end{aligned}$$

   Therefore the tangent line goes through $(1, 4)$ and has slope 3. An equation of the line is given by

   $y - 4 = 3(x - 1)$.

2 M2: One-sided limits; infinite limits and limits at infinity

2.1 one-sided limits: definition

Watch the video at
https://www.youtube.com/watch?v=dmi6_ex20aM&list=PL265CB737C01F8961&index=7.

Abstract This video presents the definition, both informal and formal, of one-sided limit of a function.

Definition 2.1.1 (Informal)

1. The *limit of f as x is approaching a from the left is L*, in symbols,
$$\lim_{x\to a^-} f(x) = L,$$
if the values of $f(x)$ can be made as close to L as wanted, by taking x sufficiently close to a, **and less than** a.

2. The *limit of f as x is approaching a from the right is L*, in symbols,
$$\lim_{x\to a^+} f(x) = L,$$
if the values of $f(x)$ can be made as close to L as wanted, by taking x sufficiently close to a, **and greater than** a.

Definition 2.1.2 (Formal)

1. The *limit of f as x is approaching a from the left is L*, in symbols,
$$\lim_{x\to a^-} f(x) = L,$$
if for every $\epsilon > 0$ there is $\delta > 0$ such that
$$0 < a - x < \delta \Longrightarrow |f(x) - L| < \epsilon.$$

2. The *limit of f as x is approaching a from the right is L*, in symbols,
$$\lim_{x\to a^+} f(x) = L,$$
if for every $\epsilon > 0$ there is $\delta > 0$ such that
$$0 < x - a < \delta \Longrightarrow |f(x) - L| < \epsilon.$$

Note that a function may have one-sided limits, but no limit in the usual sense. More specifically:

Proposition 2.1.3.

$$\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L.$$

This means that for the limit to exist, both one-sided limits have to exist, and they have to be equal.

2.2 one-sided limits: examples

Watch the video at
https://www.youtube.com/watch?v=YFs3hdMEUFY&list=PL265CB737C01F8961&index=8.

Abstract This video considers the existence of limits for functions defined piecewise, or in terms of absolute values. The examples below are explained.

Example 2.2.1. Consider the function

$$f(x) = \begin{cases} x^2+2 & \text{if } x<2 \\ 5 & \text{if } x=2 \\ 4x-3 & \text{if } x>2. \end{cases}$$

What is $\lim_{x\to 2} f(x)$?

Solution. Because the function is defined differently on the left and on the right of 2, we consider one-sided limit, in order to apply the criterion of Proposition 2.1.3. Specifically

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2+2 = 6$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 4x-3 = 5.$$

Since $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$, we conclude that $\lim_{x\to 2} f(x)$ does not exist.

Example 2.2.2. Consider the function

$$f(x) = \begin{cases} \sqrt{x+3} & \text{if } x\leq 1 \\ x^2+1 & \text{if } 1<x<3 \\ 2x+1 & \text{if } x\geq 3. \end{cases}$$

Find $\lim_{x\to 1} f(x)$, $\lim_{x\to 2} f(x)$ and $\lim_{x\to 3} f(x)$.

Solution. Because the function is defined differently on the left and on the right of 1 and of 3, we consider one-sided limits at 1 and 3, in order to apply the criterion of Proposition 2.1.3. Specifically

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sqrt{x+3} = 2$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2+1 = 2$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} x^2+1 = 10$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} 2x+1 = 7.$$

We conclude from Proposition 2.1.3 that $\lim_{x\to 1} f(x) = 2$ but that $\lim_{x\to 3} f(x)$ does not exist. On the other hand, $f(x) = x^2 + 1$ on an open interval containing $x = 2$, so that

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} x^2 + 1 = 5.$$

Example 2.2.3. Find

$$\lim_{x\to 2} \frac{|x-2|}{x^2+x-6}.$$

Solution. Note that this is an indeterminate form of the type $\frac{0}{0}$, but we cannot use factorization because the numerator is not a polynomial. To get rid of the absolute value, recall that

$$|a| := \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0. \end{cases}$$

Thus $|x-2| = x-2$ if $x > 2$ and $|x-2| = -(x-2)$ if $x < 2$. We should therefore consider one-sided limits at 2, because the expression for the function is different on both sides of 2. Specifically:

$$\begin{aligned}
\lim_{x\to 2^-} \frac{|x-2|}{x^2+x-6} &= \lim_{x\to 2^-} \frac{-(x-2)}{x^2+x-6} \\
&= \lim_{x\to 2^-} \frac{-(x-2)}{(x-2)(x+3)} \\
&= \lim_{x\to 2^-} -\frac{1}{x+3} = -\frac{1}{5}; \\
\lim_{x\to 2^+} \frac{|x-2|}{x^2+x-6} &= \lim_{x\to 2^+} \frac{x-2}{x^2+x-6} \\
&= \lim_{x\to 2^+} \frac{x-2}{(x-2)(x+3)} \\
&= \lim_{x\to 2^+} \frac{1}{x+3} = \frac{1}{5}. \\
&= \lim_{x\to 2^+} \frac{x-2}{(x-2)(x+3)}
\end{aligned}$$

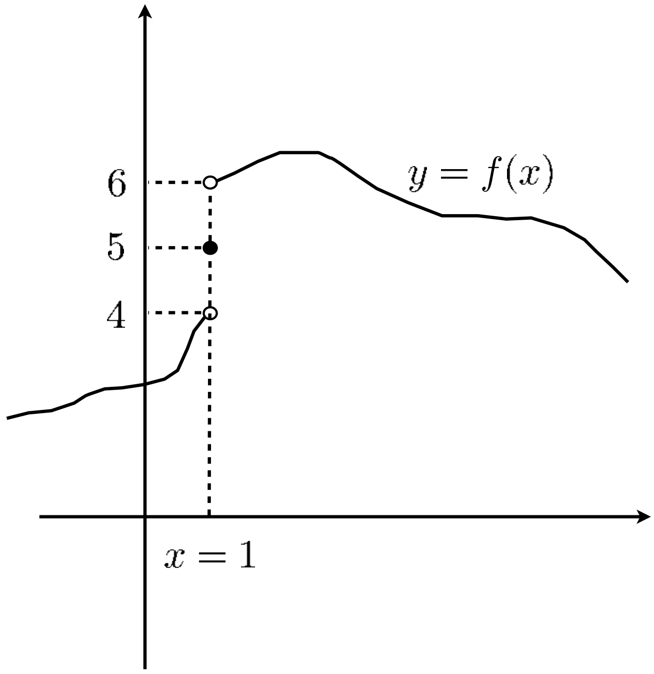
Since

$$\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x),$$

we conclude that $\lim_{x\to 2} f(x)$ does not exist.

2.3 M2 Sample Quiz 1: one-sided limits

1. Match $\lim_{x\to 1^-} f(x)$, $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1} f(x)$ and $f(1)$ with their values, if $f(x)$ is the function represented below:



2. Do the following limits exist? Justify your answers

a) $\lim_{x\to -1} f(x)$ where

$$f(x) = \begin{cases} 2x+3 & \text{if } x < -1 \\ x^2 - x & \text{if } x > -1 \end{cases}.$$

b) $\lim_{x\to 2} f(x)$ where

$$f(x) = \begin{cases} 1-x & \text{if } x < 2 \\ x^2 - 3x + 1 & \text{if } x > 2 \end{cases}.$$

c) $\displaystyle\lim_{x\to -2} \frac{x^2+x-2}{|x+2|}$

d) $\displaystyle\lim_{x\to 3} \frac{\sqrt{x+1}-2}{|-x^2-x+12|}$

2.4 Solutions to the M2 sample Quiz 1

1. $$\begin{aligned} \lim_{x\to 1^-} f(x) &= 4 \\ \lim_{x\to 1^+} f(x) &= 6 \\ f(1) &= 5 \\ \lim_{x\to 1} f(x) & \quad \text{does not exist.} \end{aligned}$$

2) Do the following limits exist? Justify your answers

a) $\lim_{x\to -1} f(x)$ where

$$f(x) = \begin{cases} 2x+3 & \text{if } x < -1 \\ x^2 - x & \text{if } x > -1 \end{cases}.$$

Solution.

$$\begin{aligned} \lim_{x\to -1^-} f(x) &= \lim_{x\to -1^-} 2x+3 = 1 \\ \lim_{x\to -1^+} f(x) &= \lim_{x\to -1^+} x^2 - x = 2. \end{aligned}$$

Thus,

$$\lim_{x\to -1^-} f(x) \neq \lim_{x\to -1^+} f(x),$$

and $\lim_{x\to -1} f(x)$ does not exist.

b) $\lim_{x\to 2} f(x)$ where

$$f(x) = \begin{cases} 1-x & \text{if } x<2 \\ x^2-3x+1 & \text{if } x>2 \end{cases}.$$

Solution.

$$\begin{aligned} \lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-} 1-x = -1 \\ \lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+} x^2-3x+1 = -1, \end{aligned}$$

so that $\lim_{x\to 2} f(x) = -1$.

c) $\displaystyle\lim_{x\to -2} \frac{x^2+x-2}{|x+2|}$

Solution.

$$\begin{aligned} \lim_{x\to -2^-} \frac{x^2+x-2}{|x+2|} &= \lim_{x\to -2^-} \frac{(x+2)(x-1)}{-(x+2)} \\ &= \lim_{x\to -2^-} -(x-1) = 3 \\ \lim_{x\to -2^+} \frac{x^2+x-2}{|x+2|} &= \lim_{x\to -2^+} \frac{(x+2)(x-1)}{(x+2)} \\ &= \lim_{x\to -2^+} x-1 = -3 \end{aligned}$$

so that

$$\lim_{x\to -2^-} \frac{x^2+x-2}{|x+2|} \neq \lim_{x\to -2^+} \frac{x^2+x-2}{|x+2|}$$

and $\lim_{x\to -2} \frac{x^2+x-2}{|x+2|}$ does not exist.

d) $\displaystyle\lim_{x\to 3} \frac{\sqrt{x+1}-2}{|-x^2-x+12|}$

Solution. Note that

$$|-x^2-x+12| = |-(x-3)(x+4)| = |x-3|\cdot|x+4|.$$

Thus

$$
\begin{aligned}
\lim_{x\to 3^-} \frac{\sqrt{x+1}-2}{|-x^2-x+12|} &= \lim_{x\to 3^-} \frac{\sqrt{x+1}-2}{-(x-3)(x+4)} \\
&= \lim_{x\to 3^-} \frac{\sqrt{x+1}-2}{-(x-3)(x+4)} \cdot \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} \\
&= \lim_{x\to 3^-} \frac{(x+1)-4}{-(x-3)(x+4)(\sqrt{x+1}+2)} \\
&= \lim_{x\to 3^-} -\frac{x-3}{(x-3)(x+4)(\sqrt{x+1}+2)} \\
&= \lim_{x\to 3^-} -\frac{1}{(x+4)(\sqrt{x+1}+2)} = -\frac{1}{28}
\end{aligned}
$$

and

$$
\begin{aligned}
\lim_{x\to 3^+} \frac{\sqrt{x+1}-2}{|-x^2-x+12|} &= \lim_{x\to 3^+} \frac{\sqrt{x+1}-2}{(x-3)(x+4)} \\
&= \lim_{x\to 3^+} \frac{\sqrt{x+1}-2}{(x-3)(x+4)} \cdot \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} \\
&= \lim_{x\to 3^+} \frac{x-3}{(x-3)(x+4)(\sqrt{x+1}+2)} \\
&= \lim_{x\to 3^+} \frac{1}{(x+4)(\sqrt{x+1}+2)} = \frac{1}{28}.
\end{aligned}
$$

Therefore, $\lim_{x\to 3} \frac{\sqrt{x+1}-2}{|-x^2-x+12|}$ does not exist.

2.5 Definition of infinite limits

Watch the video at
https://www.youtube.com/watch?v=p8dX3e79owI&list=PL265CB737C01F8961&index=9.

Abstract This video presents informal and formal definitions of $\lim_{x\to a} f(x) = +\infty$, $\lim_{x\to a} f(x) = -\infty$, and $\lim_{x\to a^\pm} f(x) = \pm\infty$. The geometric interpretation in terms of vertical asymptote is also presented.

Infinite limits specify the behavior of the function when the limit does not exist. An infinite limit is one of the ways the limit (in the usual sense) can fail to exist.

Definition 2.5.1 The limit of f at a is:

1. $+\infty$, in symbols
   $\lim\limits_{x\to a} f(x) = +\infty,$

   if the values of $f(x)$ can be made as large as we want by taking x sufficiently close to a, but not equal to a.

2. $-\infty$, in symbols
   $\lim\limits_{x\to a} f(x) = -\infty,$

   if the values of $f(x)$ can be made as *negative* large as we want by taking x sufficiently close to a, but not equal to a.

Definition 2.5.2 (Formal) The limit of f at a is:

1. $+\infty$, in symbols
   $\lim\limits_{x\to a} f(x) = +\infty,$

   if for every $M > 0$ there is $\delta > 0$ such that

   $$0 < |x - a| < \delta \Longrightarrow f(x) > M.$$

2. $-\infty$, in symbols
   $\lim\limits_{x\to a} f(x) = -\infty,$

if for every $M > 0$ there is $\delta > 0$ such that

$$0 < |x-a| < \delta \Longrightarrow f(x) < -M.$$

Of course, there are one-sided counterparts to these notions:

Definition 2.5.3 (Formal)

1. The *limit of f at a from the right* is:
   a) $+\infty$, in symbols
   $$\lim_{x\to a^+} f(x) = +\infty,$$
   if for every $M > 0$ there is $\delta > 0$ such that
   $$0 < x-a < \delta \Longrightarrow f(x) > M.$$
   b) $-\infty$, in symbols
   $$\lim_{x\to a^+} f(x) = -\infty,$$
   if for every $M > 0$ there is $\delta > 0$ such that
   $$0 < x-a < \delta \Longrightarrow f(x) < -M.$$
2. The *limit* of *f* at *a* from the left is:
   a) $+\infty$, in symbols
   $$\lim_{x\to a^-} f(x) = +\infty,$$
   if for every $M > 0$ there is $\delta > 0$ such that
   $$0 < a-x < \delta \Longrightarrow f(x) > M.$$
   b) $-\infty$, in symbols
   $$\lim_{x\to a^-} f(x) = -\infty,$$
   if for every $M > 0$ there is $\delta > 0$ such that
   $$0 < a-x < \delta \Longrightarrow f(x) < -M.$$

Example 2.5.4. Find

$$\lim_{x\to 2^-} \frac{3}{x-2}$$

specifying $+\infty$ or $-\infty$ if applicable.

Solution. Because $\lim\limits_{x\to 2} x-2=0$, the values of $\left|\frac{3}{x-2}\right|$ grow without bounds when x approaches 2. Since $x\to 2^-$, $x<2$, that is, $x-2<0$. Thus

$$\frac{3}{x-2}<0.$$

Therefore

$$\lim_{x\to 2^-} \frac{3}{x-2}=-\infty.$$

Definition 2.5.5. A line $x = a$ is a *vertical asymptote for* f if at least one one-sided limit of f at a is infinite (either $+\infty$ or $-\infty$).

2.6 Finding vertical asymptotes

Watch the video at
https://www.youtube.com/watch?v=XX7AxZRz8ck&list=PL265CB737C01F8961&index=10.

Abstract This video presents how to find the vertical asymptotes for a given function, going over the examples below.

Example 2.6.1. Find the vertical asymptotes of

$$f(x) = \frac{1}{2x^2 + x - 3}.$$

Solution. We are looking for values a such that $\lim_{x\to a^\pm} f(x) = \pm\infty$. Since f is a rational function, this can only happen at values outside the domain, that is, at zeros of the denominator. Moreover

$$f(x) = \frac{1}{(x-1)(2x+3)}$$

so that f has infinite one-sided limits at 1 and at $-\frac{3}{2}$. Thus the lines $x = 1$ and $x = -\frac{3}{2}$ are vertical asymptotes.

Example 2.6.2. Find the vertical asymptotes of

$$f(x) = \frac{2x^2 + 3x - 5}{x^2 + x - 2}.$$

Solution. Again, the only potential vertical asymptotes are $x = a$, where a is a zero of the denominator. Here

$$f(x) = \frac{2x^2 + 3x - 5}{(x-1)(x+2)} = \frac{(x-1)(2x+5)}{(x-1)(x+2)}.$$

We factor *both* numerator and denominator to see at what zero of the denominator the function has an infinite (one-sided) limit. Here $x = -2$ is a vertical asymptote, but $x = 1$ is **not**, because

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{2x+5}{x+2} = \frac{7}{3}$$

is finite.

Example 2.6.3. Find the vertical asymptotes of

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4x + 4}.$$

Solution. Again, the only *potential* vertical asymptotes are $x = a$, where a is a zero of the denominator. Here

$$f(x) = \frac{(x-2)(x+3)}{(x-2)^2}$$

so that

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x+3}{x-2}$$

is infinite. Thus $x = 2$ is the only vertical asymptote.

2.7 Limits at infinity and horizontal asymptotes

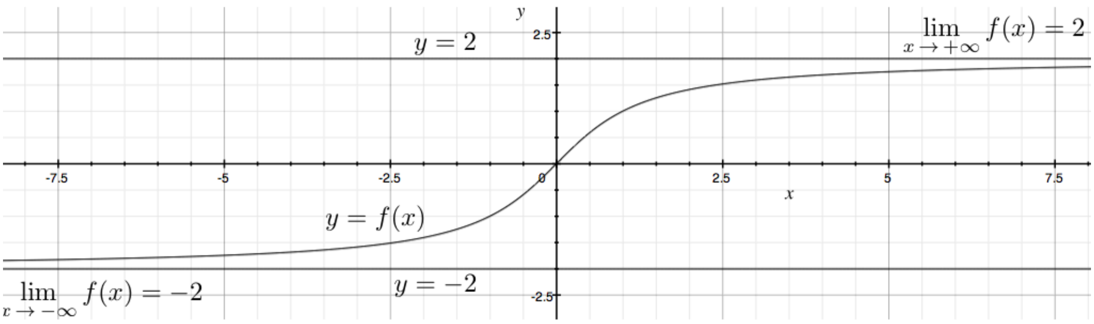
Watch the video at
https://www.youtube.com/watch?v=vQDACWVf1l0&list=PL265CB737C01F8961&index=11.

Abstract This video examines the "end behavior" of a function, introduces the notion of limit at $+\infty$ and at $-\infty$ and the notion of horizontal asymptote.

Definition 2.7.1. We say that

1. the limit at $+\infty$ is L, in symbols
$$\lim_{x\to+\infty} f(x) = L,$$
if the values of $f(x)$ can be made as close to L as we want by taking x sufficiently large; formally, if for every $\epsilon > 0$, there is $M > 0$ such that
$$x > M \Longrightarrow |f(x) - L| < \epsilon.$$

2. the limit at $-\infty$ is L, in symbols
$$\lim_{x\to-\infty} f(x) = L,$$
if the values of $f(x)$ can be made as close to L as we want by taking x sufficiently negative large; formally, if for every $\epsilon > 0$, there is $M > 0$ such that
$$x < -M \Longrightarrow |f(x) - L| < \epsilon.$$



Definition 2.7.2. The line $y = L$ is a *horizontal asymptote* of f if
$$\lim_{x\to+\infty} f(x) = L \text{ or } \lim_{x\to-\infty} f(x) = L.$$

2.8 Finding horizontal asymptotes

Watch the video at
https://www.youtube.com/watch?v=COTm4zRBYaY&list=PL265CB737C01F8961&index=12.

Abstract This video presents how to quickly find horizontal asymptotes for rational functions, using the degree of the numerator and denominator.

The key observation is that if c is a constant and $r > 0$ then

$$\lim_{x\to\pm\infty} \frac{c}{x^r} = 0.$$

This simple observation is used to deduce the following general rule:

Theorem 2.8.1. *If $p(x)$ and $q(x)$ are two polynomial functions of respective degrees $d^\circ p$ and $d^\circ q$ and*

1. $d^\circ p < d^\circ q$ *then*

$$\lim_{x\to\pm\infty} \frac{p(x)}{q(x)} = 0,$$

and $y = 0$ is a horizontal asymptote of $f(x) = \frac{p(x)}{q(x)}$;

2. $d^\circ p = d^\circ q$ *then*

$$\lim_{x\to\pm\infty} \frac{p(x)}{q(x)} = \frac{\text{leading coeff of } p}{\text{leading coeff of } q} = m,$$

and $y = m$ is a horizontal asymptote of $f(x) = \frac{p(x)}{q(x)}$;

3. $d^\circ p > d^\circ q$ *then*

$$\lim_{x\to\pm\infty} \frac{p(x)}{q(x)}$$

is infinite and $f(x) = \frac{p(x)}{q(x)}$ has no horizontal asymptote.

Example 2.8.2. Find the horizontal and vertical asymptotes of

$$\begin{aligned} a) \quad f(x) &= \frac{2x^2+x+1}{3x^2+4} \\ b) \quad f(x) &= \frac{3x}{x^2-9} \\ c) \quad f(x) &= \frac{3x^3+2x+1}{x^2+x-6}. \end{aligned}$$

Solution. a) The denominator $3x^2 + 4$ is never 0 and therefore *f* has no vertical asymptote. Numerator and denominator have the same degree, so that, according to Theorem 2.8.1(2),

$$\lim_{x\to\pm\infty} f(x) = \frac{2}{3}$$

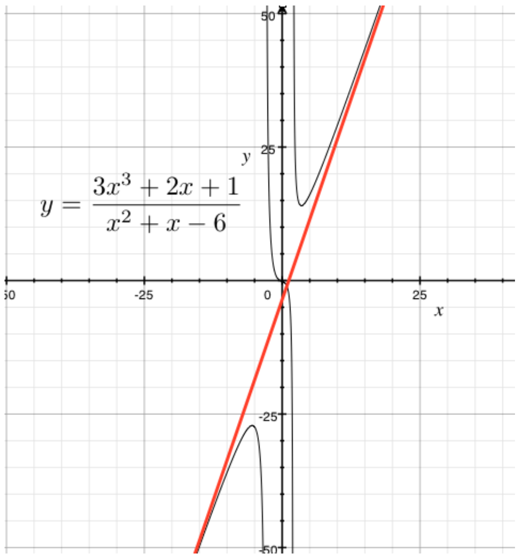
so that $y = \frac{2}{3}$ is a horizontal asymptote.

b) The denominator $x^2 - 9$ takes the value zero at -3 and 3. Moreover, the numerator does not take the value 0 at ± 3. Thus $x = -3$ and $x = 3$ are vertical asymptotes. Moreover, the degree of the denominator is greater than that of the numerator, so that

$$\lim_{x\to\pm\infty} f(x) = 0$$

by Theorem 2.8.1(1), and we conclude that $y = 0$ is a horizontal asymptote.

c) The denominator $x^2 + x - 6 = (x - 2)(x + 3)$ takes the value zero at 2 and -3, but the numerator does not. Hence $x = -3$ and $x = 2$ are vertical asymptotes. On the other hand, the degree of the numerator is greater than that of the denominator so that *f* does not have any horizontal asymptote, according to Theorem 2.8.1(3). However, the graph of the function indicates that it has a slant asymptote, which is the subject of the next section.



2.9 Slant asymptotes

Watch the video at
https://www.youtube.com/watch?v=iju4GxstffI&list=PL265CB737C01F8961&index=13.

Abstract This video defines slant asymptotes and examines how to find slant asymptotes in the case of rational functions, using long division. A review of the algorithm for long division is included.

Definition 2.9.1. A line of equation $y = ax + b$ is a *slant asymptote* of a function f if

$$\lim_{x \to \pm\infty} (f(x) - (ax + b)) = 0.$$

In the case where

$$f(x) = \frac{p(x)}{q(x)}$$

is a rational function (i.e., quotient of two polynomial functions), f has a slant asymptote if and only if the degree of the numerator is one more than the degree of the denominator, that is,

$$d^\circ p = d^\circ q + 1.$$

We then find the equation of the slant asymptote by long division of p by q. The quotient is then of degree one, that is, of the form $ab + b$, and the asymptote has equation $y = ax + b$.

Example 2.9.2. Find the asymptotes of

$$f(x) = \frac{2x^2 - 5x - 2}{x - 3}.$$

Solution. Since the numerator is not 0 at 3, the line $x = 3$ is a vertical asymptote. Because the degree of the numerator is greater than that of the denominator, there is no horizontal asymptote, but because it s greater by 1, there is a slant asymptote. The long division yields

$$\begin{array}{r|l} & 2x + 1. \\ \hline x - 3 & 2x^2 - 5x - 2 \\ & -2x^2 + 6x \\ \cline{2-2} & x - 2 \\ & -x + 3 \\ \cline{2-2} & 1 \end{array}$$

Therefore, the line $y = 2x + 1$ is a (slant) asymptote.

2.10 M2 sample Quiz 2: infinite limits, limits at infinity, asymptotes

1. Find all the asymptotes of
$$f(x) = \frac{2x+1}{x^2+x-6}$$
2. Find all the asymptotes of
$$f(x) = \frac{x^2+x-2}{3x^2+x-4}$$
3. Find all the asymptotes of
$$f(x) = \frac{x^3+x+4}{x^2+3x+2}$$
4. Evaluate the following limit (specify $-\infty$ or $+\infty$ if applicable):
$$\lim_{x\to 3^-} \frac{3x+2}{x^2+x-12}.$$

2.11 Solutions to the M2 sample Quiz 2

1. Find all the asymptotes of
$$f(x) = \frac{2x+1}{x^2+x-6}$$

Solution. Since
$$f(x) = \frac{2x+1}{x^2+x-6} = \frac{2x+1}{(x-2)(x+3)}$$

and $2x + 1$ is non-zero at $x = 2$ and $x = -3$, we see that the vertical lines $x = 2$ and $x = -3$ are asymptotes. On the other hand, $\lim_{x\to\infty} f(x) = 0$ because the degree of the denominator is higher than that of the numerator. Therefore, the horizontal line $y = 0$ is also an asymptote.

2. Find all the asymptotes of
$$f(x) = \frac{x^2+x-2}{3x^2+x-4}.$$

Solution. Since
$$f(x) = \frac{(x-1)(x+2)}{(x-1)(3x+4)},$$

we see that $\lim_{x\to-\frac{4}{3}} f(x)$ is infinite, so that $x = -\frac{4}{3}$ is a vertical asymptote, but
$$\lim_{x\to1} f(x) = \lim_{x\to1} \frac{x+2}{3x+4} = \frac{3}{7}$$

is finite, so that $x = 1$ is **not** an asymptote. On the other hand, numerator and denominator have the same degree, so that the limit of f at ∞ is the quotient of leading coefficients:
$$\lim_{x\to\infty} f(x) = \frac{1}{3}.$$

Therefore, the horizontal line $y = \frac{1}{3}$ is an asymptote.

3. Find all the asymptotes of
$$f(x) = \frac{x^3+x+4}{x^2+3x+2}$$

Solution. Since
$$f(x) = \frac{x^3+x+4}{(x+1)(x+2)}$$

and the numerator is non-zero at -1 and -2, so that the vertical lines $x = -1$ and $x = -2$ are asymptotes. On the other hand, f has no horizontal asymptote because the degree of the numerator is larger than the degree of the denominator. But it is larger by exactly 1, so f has a slant asymptote. Long division of $x^3 + x + 4$ by $x^2 + 3x + 2$ yields

$$f(x) = x - 3 + \frac{8x + 10}{x^2 + 3x + 2},$$

so that the line $y = x - 3$ is an asymptote.

4. Evaluate the following limit (specify $-\infty$ or $+\infty$ if applicable):

$$\lim_{x\to 3^-} \frac{3x+2}{x^2+x-12}$$

Solution. Note that

$$\frac{3x+2}{x^2+x-12} = \frac{3x+2}{(x-3)(x+4)}.$$

If $x < 3$ and x is close to 3, then $3x + 2$ is close to 11 and $x + 4$ is close to 7. On the other hand, $x - 3 < 0$ and close to 0. Therefore, in this situation

$$\frac{3x+2}{(x-3)(x+4)} < 0$$

and

$$\lim_{x\to 3^-} \frac{3x+2}{x^2+x-12} = -\infty.$$

3 M3: Continuity and Derivatives

3.1 Continuity: definition

Watch the video at
https://www.youtube.com/watch?v=0txz6kN-v2Q&list=PL265CB737C01F8961&index=14.

Abstract This video defines continuity at a point and on an interval, removable and non-removable discontinuity, and examines general laws for continuity.

Definition 3.1.1. A function f is *continuous at a* if

$$\lim_{x \to a} f(x) = f(a),$$

that is, $\lim_{x \to a} f(x)$ and $f(a)$ both exist, and they are equal.

Definition 3.1.2. If f is not continuous at a, we say that a is a *discontinuity* of f. If a is a discontinuity such that $\lim_{x \to a} f(x)$ exists, we say that a is a *removable discontinuity.* Otherwise, the discontinuity is *non-removable.*

Theorem 3.1.3. *Polynomial functions are continuous at each point of* $(-\infty, \infty)$*. Rational functions are continuous at each point of their domain. The cosine and sine function are continuous at each point of* $(-\infty, \infty)$.

Theorem 3.1.4. *If f and g are two functions that are continuous at a and k is a constant then* $k \cdot f$, $f + g$ *and* $f \cdot g$ *are continuous at a. If moreover,* $g(a) \neq 0$ *then* $\frac{f}{g}$ *is also continuous at a.*

Theorem 3.1.5. *If*

$$\lim_{x \to a} f(x) = L$$

and g is continuous at L, then

$$\lim_{x \to a} g \circ f(x) = g(L).$$

Definition 3.1.6. If *f* is

1. continuous at each point *x* of (*a, b*), we say that *f* is *continuous on* (*a, b*). Here *a* or *b* or both could be infinite;
2. continuous on (a, b) and

   $$\lim_{x \to a^+} f(x) = f(a)$$

   we say that *f* is *continuous on* [*a, b*);

3. continuous on (a, b) and

   $$\lim_{x \to b^-} f(x) = f(b)$$

   we say that *f* is *continuous on* (*a, b*];

4. continuous on [*a, b*) and on (*a, b*], we say that *f* is *continuous on* [*a, b*].

3.2 Finding discontinuities

Watch the video at
https://www.youtube.com/watch?v=AFNpO42F2hM&list=PL265CB737C01F8961&index=15.

Abstract This video presents examples of functions for which discontinuities are identified, and found to be removable or non-removable.

Example 3.2.1. Find the discontinuities of

$$f(x) = \frac{x^3 + x^2 - 2}{x^2 + 3x - 4}$$

and state for each if it is removable or not.

Solution. Since f is a rational function, the discontinuity are the points outside of its domain (see Theorem 3.1.3), that is, the zeros of the denominator

$$x^2 + 3x - 4 = (x - 1)(x + 4)$$

that is, f is discontinuous at 1 and at –4. Moreover

$$\begin{aligned} \lim_{x\to 1} \frac{x^3 + x^2 - 2}{x^2 + 3x - 4} &= \lim_{x\to 1} \frac{(x-1)(x^2 + 2x + 2)}{(x-1)(x+4)} \\ &= \lim_{x\to 1} \frac{x^2 + 2x + 2}{x + 4} = 1 \end{aligned}$$

so that $x = 1$ is a removable discontinuity. On the other hand, $\lim_{x\to -4} f(x)$ is infinite because the numerator is not 0 at $x = -4$. Thus –4 is non-removable.

Example 3.2.2. Find the discontinuities of

$$f(x) = \begin{cases} 2x + 1 & \text{if } x < -1 \\ x^2 & \text{if } -1 \le x < 3 \\ 10 & \text{if } x = 3 \\ 4x - 3 & \text{if } 3 < x < 5 \\ x^2 - x - 3 & \text{if } x \ge 5 \end{cases}$$

and state for each if it is removable or not.

Solution. Since f is polynomial on $(-\infty, -1)$, on $(-1, 3)$, on $(3, 5)$ and on $(5, \infty)$, f is continuous on each one of these intervals. Thus the only possible discontinuities are -1, 3 and 5. Let us examine each one.

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} 2x+1 = -1$$

and

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} x^2 = 1$$

so that $\lim_{x\to -1} f(x)$ does not exist, because

$$\lim_{x\to -1^-} f(x) \neq \lim_{x\to -1^+} f(x).$$

Thus -1 is a non-removable discontinuity. Since $f(3) = 10$ and

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} x^2 = 9 = \lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} 4x-3,$$

we conclude that 3 is a removable discontinuity. Finally, f is continuous at 5 because

$$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-} 4x-3 = 17$$

and

$$f(5) = 17 = \lim_{x \to 5^+} x^2 - x - 3 = \lim_{x \to 5^+} f(x).$$

Example 3.2.3. For what value or values of c is the function

$$f(x) = \begin{cases} 3x + c & \text{if } x \leq c \\ x^2 + 3x - 2 & \text{if } x > c \end{cases}$$

continuous on $(-\infty, \infty)$?

Solution. Since f is polynomial on $(-\infty, c)$ and on (c, ∞), the only possible discontinuity is at c. At c, we have:

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^-} 3x + c = 4c = f(c)$$

and

$$\lim_{x \to c^+} f(x) = \lim_{x \to c^+} x^2 + 3x - 2 = c^2 + 3c - 2.$$

Thus f is continuous at c if and only if

$$4c = c^2 + 3c - 2 \iff c^2 - c - 2 = 0 \iff (c+1)(c-2) = 0,$$

so that f is continuous on $(-\infty, \infty)$ when $c = -1$ and when $c = 2$.

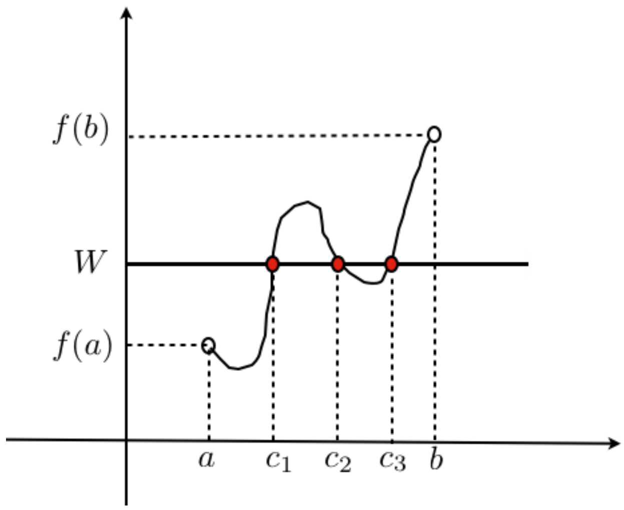
3.3. The Intermediate Value Theorem

Watch the video at
https://www.youtube.com/watch?v=3x3tgjoAjVw&list=PL265CB737C01F8961&index=16.

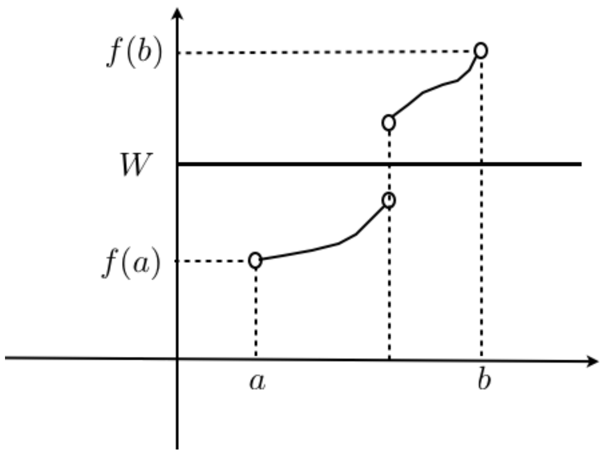
Abstract This video presents the Intermediate Value Theorem, shows that continuity is an essential assumption, and examines application to the existence of solutions to an equation, as well as the bisection method.

Theorem 3.3.1 (Intermediate Value Theorem). *If f is a continuous function on a closed interval $[a, b]$ and W is a value between $f(a)$ and $f(b)$ then there exists at least one c in (a, b) with*

$$f(c) = W.$$



The assumption of continuity is essential, as can be seen on this graph:



Example 3.3.2. Show the the equation

$$\cos x = x \tag{3.3.1}$$

has a solution.

Solution. Note that the question is not to solve the equation but only to justify the existence of a solution! Clearly, a solution of (3.3.1) is a zero of the function

$$f(x) = \cos x - x$$

which is continuous on $(-\infty, \infty)$. Note that $f(0)$ $1 > 0$ and $f(\frac{\pi}{2}) = -\frac{\pi}{2} < 0$ so that $W = 0$ is between $f(0)$ and $f(\frac{\pi}{2})$. Moreover, f is continuous on $[0, \frac{\pi}{2}]$. Thus the Intermediate Value Theorem applies to f on $[0, \frac{\pi}{2}]$ to the effect that there is c in $(0, \frac{\pi}{2})$ with $f(c) = 0$. Clearly, c is a solution to (3.3.1).

Bisection method. Iterating the process outlined in Example 3.3.2, can be a way to find a solution numerically, with a given accuracy. Indeed, we know now that there is a solution between 0 and $\frac{\pi}{2}$. So in a sense, we have found the solution, but with an error of up to $\frac{\pi}{2}$, which is not a very good approximation. But we can do better. Note that

$$f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} - \frac{\pi}{4} < 0,$$

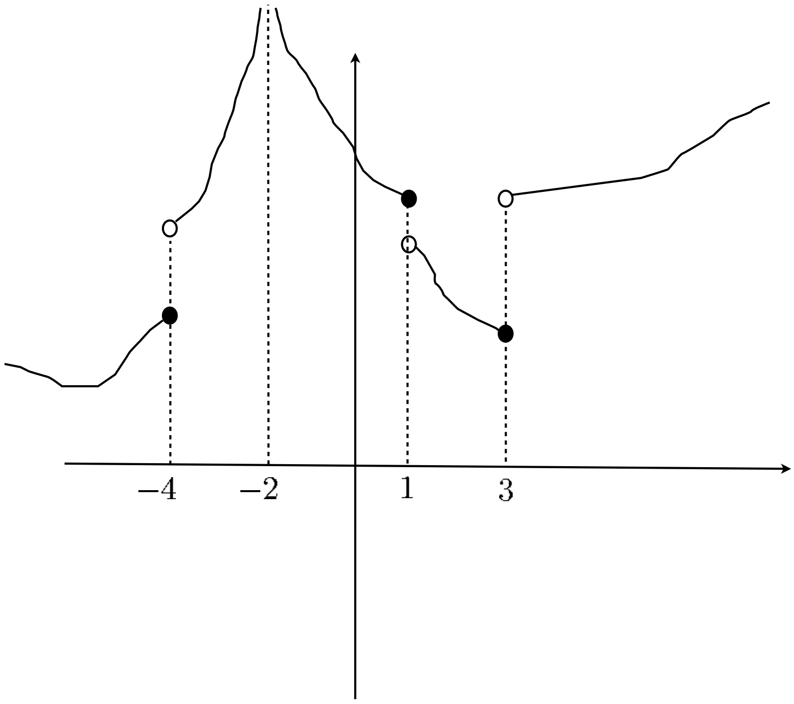
so that we can conclude from the Intermediate Value Theorem applied to f on $[0, \frac{\pi}{4}]$ that there is a zero in $(0, \frac{\pi}{4})$. Now we have doubled our accuracy. Iterating the process of evaluating at the midpoint to decide in which half of the interval our solution is, we double the accuracy at each step, and can therefore approximate the solution as many exact decimal places we want, by iterating sufficiently many times. Here, we find that

$$\begin{aligned} f\left(\frac{\pi}{8}\right) > 0 &\implies c \in \left(\frac{\pi}{8}, \frac{\pi}{4}\right) \\ f\left(\frac{3\pi}{16}\right) > 0 &\implies c \in \left(\frac{3\pi}{16}, \frac{\pi}{4}\right) \\ f\left(\frac{7\pi}{32}\right) > 0 &\implies c \in \left(\frac{7\pi}{32}, \frac{\pi}{4}\right), \end{aligned}$$

and so on. Noting that $\left|\frac{\pi}{4} - \frac{7\pi}{32}\right| < 0.1$, we see that we already have an estimate of the solution with an error less than 0.1.

3.4 M3 Sample Quiz 1: continuity

1. What are the intervals of continuity of the function whose graph is represented below?



2. What are the discontinuities of the following function. Are any of them removable?

$$f(x) = \frac{x^3 - 2x - 4}{x^2 + x - 6}.$$

3. What are the discontinuities of the following function. Are any of them removable?

$$f(x) = \begin{cases} 3x+3 & \text{if } x < -1 \\ 2x+1 & \text{if } -1 \le x < 2 \\ 7 & \text{if } x = 2 \\ x^2+1 & \text{if } 2 < x < 3 \\ 5x-5 & \text{if } x \ge 3 \end{cases}$$

4. For what value(s) of c is the function below continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} 1-2x & \text{if } x < c \\ cx-2 & \text{if } x \ge c \end{cases}.$$

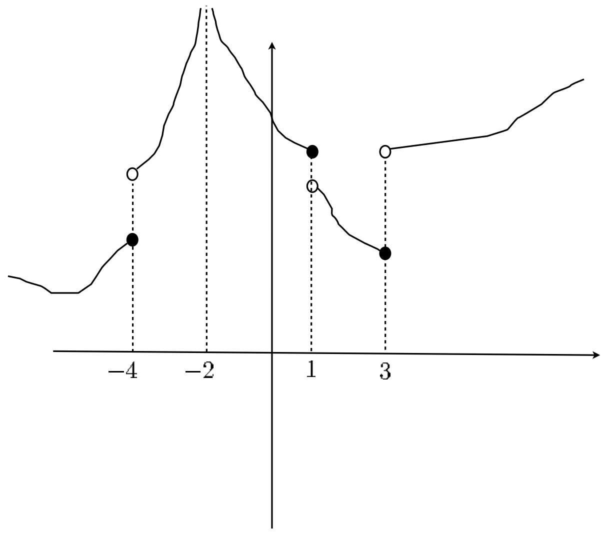
5. Show that the equation

$$x^5 + 3x^2 - x - 2 = 0$$

has a solution.

3.5 M3 Sample Quiz 1 Solutions

1. What are the intervals of continuity of the function whose graph is represented below?



Solution. f is continuous on $(-\infty,-4]$ and on $(-4,-2)$ and on $(-2,1]$ and on $(1,3]$ and on $(3,\infty)$.

2. What are the discontinuities of the following function. Are any of them removable?

$$f(x)=\frac{x^3-2x-4}{x^2+x-6}.$$

Solution. f is a rational function, so its only discontinuities are zeros of its denominator. Since

$$f(x)=\frac{x^3-2x-4}{(x-2)(x+3)}=\frac{(x-2)(x^2+2x+2)}{(x-2)(x+3)},$$

f has two discontinuities: 2 and –3. Moreover,

$$\lim_{x\to 2}f(x)=\lim_{x\to 2}\frac{x^2+2x+2}{x+3}=2 \text{ exists},$$

but $\lim_{x\to -3}f(x)$ is infinite. Therefore, 2 is a removable discontinuity while –3 is non-removable.

3. What are the discontinuities of the following function. Are any of them removable?

$$f(x)=\begin{cases}3x+3 & \text{if } x<-1\\ 2x+1 & \text{if } -1\le x<2\\ 7 & \text{if } x=2\\ x^2+1 & \text{if } 2<x<3\\ 5x-5 & \text{if } x\ge 3\end{cases}$$

Solution. The function *f* is continuous on each of the open intervals $(-\infty, -1)$, (–1, 2), (2, 3) and (3, ∞) because on each such interval, it coincides with a polynomial function. Hence the only possible discontinuities are –1, 2 and 3. At –1

$$\begin{aligned}\lim_{x\to -1^-} f(x) &= \lim_{x\to -1^-} 3x+3=0\\ \lim_{x\to -1^+} f(x) &= \lim_{x\to -1^+} 2x+1=-1\end{aligned}$$

so that $\lim_{x\to -1} f(x)$ does not exist, and –1 is a non-removable discontinuity. At 2

$$\begin{aligned}\lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-} 2x+1=5\\ \lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+} x^2+1=5\end{aligned}$$

so that

$$\lim_{x\to 2} f(x)=5\neq f(2)=7.$$

Therefore, 2 is a removable discontinuity. At 3

$$\begin{aligned}\lim_{x\to 3^-} f(x) &= \lim_{x\to 3^-} x^2+1=10\\ \lim_{x\to 3^+} f(x) &= \lim_{x\to 3^+} 5x-5=10\\ f(3) &= 10\end{aligned}$$

so that *f* is continuous at 3.

4. For what value(s) of *c* is the function below continuous on (– ∞, ∞)?

$$f(x)=\begin{cases}1-2x & \text{if } x<c\\ cx-2 & \text{if } x\geq c\end{cases}.$$

Solution. The function *f* coincides with polynomial functions on (–∞, *c*) and on (*c*, ∞), so that the only possible discontinuity is *c*. At *c*

$$\begin{aligned}\lim_{x\to c^-} f(x) &= \lim_{x\to c^-} 1-2x=1-2c\\ \lim_{x\to c^+} f(x) &= \lim_{x\to c^+} cx-2=c^2-2\\ f(c) &= c^2-2\end{aligned}$$

so that f is continuous at c if $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)$, that is, if

$$\begin{aligned} 1-2c &= c^2-2 \\ 0 &= c^2+2c-3 \\ 0 &= (c-1)(c+3). \end{aligned}$$

Hence f is continuous on $(-\infty,\infty)$ if $c = -3$ or if $c = 1$.

5. Show that the equation

$$x^5+3x^2-x-2=0$$

has a solution.

Solution. The function $f(x)=x^5+3x^2-x-2$ is polynomial, hence continuous on $(-\infty,\infty)$. Moreover,

$$f(0)=-2<0<f(1)=1.$$

Since f is continuous on [0, 1], the Intermediate Value Theorem applies to f on this interval to the effect that there exists $c\in(0,1)$ with $f(c)=0$. This number c is a solution of the equation.

3.6 Definition of the derivative

Watch the video at
https://www.youtube.com/watch?v=I7nK7zSbLg4&list=PL265CB737C01F8961&index=17.

Abstract This video revisits the initial motivations for the introduction of limits, and introduces the derivative of a function at a point as the concept that encapsulates these examples.

Definition 3.6.1. The *derivative of f at a* is([1])

$$f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided that the limit exists. If it does, we say that f is *differentiable at a*.

Geometrically, the derivative $f'(a)$ of f at a is the *slope of the tangent line* to $y = f(x)$ at $(a, f(a))$. It can alternatively be interpreted as the *instantaneous rate of change of f* with respect to when $x = a$. For example, if $f(t)$ gives the position at time t of a solid in motion along a straight line, then $f'(a)$ represents the instantaneous rate of change of the position with respect to time, that is, *velocity*, at $t = a$.

3.7 Derivative as a function

Watch the video at
https://www.youtube.com/watch?v=Exbz2SrBeN8&list=PL265CB737C01F8961&index=18.

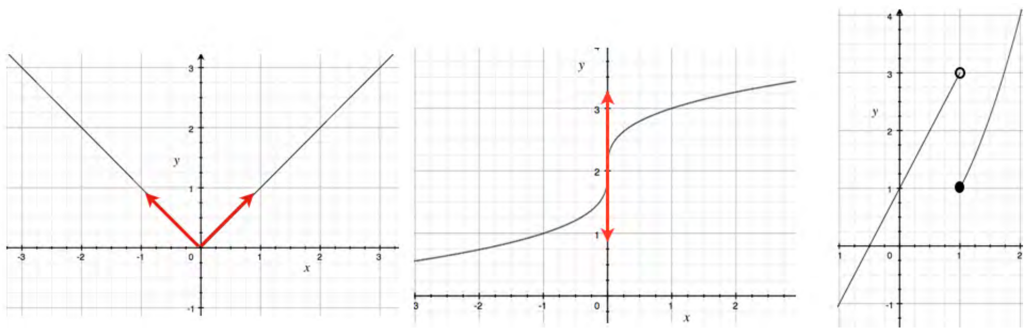
Abstract This video examines the situations in which a function fails to be differentiable at a given point (two half-tangents, vertical tangent, discontinuity). The fact that the domain of the derivative function may be smaller than that of the function is illustrated, and the relationship between the graph of f and the graph of f' is examined.

The pictures below illustrate situations were a function may fail to be differentiable at a point a. This means that the defining limit

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

does not exist. Standard situations leading to this are:

- there are two different one-sided limits, corresponding to two different half-tangents (cusp);
- the limit is infinite, corresponding to a vertical tangent;
- the function is not continuous at a.



In particular note that continuity is a necessary condition for differentiability:

Proposition 3.7.1. *If f is differentiable at a then f is continuous at a.*

However, for each x where $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ exists, we can consider the derivative number $f'(x)$, so that f' defines a function, called *derivative function of f*. Since we have seen that a function that is defined at a may fail to be differentiable at a, that means that the domain of f' may be smaller than the domain of f.

3.8 Derivative: Examples and applications

Watch the videos at
https://www.youtube.com/watch?v=yBxISod6b4Y&list=PL265CB737C01F8961&index=19

and
https://www.youtube.com/watch?v=THSys0FJ2ks&list=PL265CB737C01F8961&index=20

Abstract These two videos provide examples of step by step calculations of derivative numbers for various functions, and derivative functions. Applications to finding tangent lines and instantaneous velocities are provided.

Example 3.8.1. Find $f'(2)$ if

$$f(x) = 3x^2 + 2x - 1.$$

Deduce an equation of the tangent line to $y = f(x)$ at $x = 2$.

Solution. By definition

$$\begin{aligned} f'(2) &= \lim_{x\to 2} \frac{f(x) - f(2)}{x-2} \\ &= \lim_{x\to 2} \frac{3x^2 + 2x - 1 - 15}{x-2} \\ &= \lim_{x\to 2} \frac{3x^2 + 2x - 16}{x-2} \\ &= \lim_{x\to 2} \frac{(x-2)(3x+8)}{x-2} \\ &= \lim_{x\to 2} 3x + 8 = 14. \end{aligned}$$

This can be interpreted as the slope of the tangent line to $y = f(x)$ at

$$(2, f(2)) = (2, 15).$$

Thus an equation of this line is

$$y - 15 = 14(x - 2).$$

Example 3.8.2. What if for the same function

$$f(x) = 3x^2 + 2x - 1$$

we need the tangent lines at 2, 0, -1, 1 and 3?

In this case, we want to calculate $f'(x)$ for an unspecified *x*, and then plug the different values of *x* to obtain the slope of each tangent. To this end:

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{3(x+h)^2+2(x+h)-1-\left(3x^2+2x-1\right)}{h} \\ &= \lim_{h\to 0}\frac{6xh+3h^2+2h}{h} \\ &= \lim_{h\to 0}\frac{h(6x+2+3h)}{h} \\ &= \lim_{h\to 0} 6x+2+3h = 6x+2. \end{aligned}$$

With this we obtain:

x	$f(x)=3x^2+2x-1$	$f'(x)=6x+2$	tangent line
2	15	14	$y-15=14(x-2)$
0	-1	2	$y+1=2x$
-1	0	-4	$y=-4(x+1)$
1	4	8	$y-4=8(x-1)$
3	32	20	$y-32=20(x-3)$

Example 3.8.3. Find $f'(x)$ if

$$f(x)=\sqrt{2x+3}$$

and find an equation of the tangent line to $y=f(x)$ at *x* = 3.

Solution. By definition

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{\sqrt{2(x+h)+3}-\sqrt{2x+3}}{h} \\ &= \lim_{h\to 0}\frac{\left(\sqrt{2(x+h)+3}-\sqrt{2x+3}\right)\left(\sqrt{2(x+h)+3}+\sqrt{2x+3}\right)}{h\left(\sqrt{2(x+h)+3}+\sqrt{2x+3}\right)} \end{aligned}$$

$$\begin{aligned}
&= \lim_{h\to 0} \frac{2(x+h)+3-(2x+3)}{h\left(\sqrt{2(x+h)+3}+\sqrt{2x+3}\right)} \\
&= \lim_{h\to 0} \frac{2h}{h\left(\sqrt{2(x+h)+3}+\sqrt{2x+3}\right)} \\
&= \lim_{h\to 0} \frac{2}{\left(\sqrt{2(x+h)+3}+\sqrt{2x+3}\right)} \\
&= \frac{2}{2\sqrt{2x+3}} = \frac{1}{\sqrt{2x+3}}.
\end{aligned}$$

The tangent line at $x = 3$ is the line through

$$(3, f(3)) = (3, 3)$$

of slope

$$f'(3) = \frac{1}{3}$$

and has therefore equation

$$y - 3 = \frac{1}{3}(x-3).$$

Example 3.8.4. Find $f'(x)$ if

$$f(x) = \frac{2x+1}{3x-4}.$$

Solution. By definition

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{2(x+h)+1}{3(x+h)-4} - \frac{2x+1}{3x-4}}{h} \\
&= \lim_{h\to 0} \frac{\frac{(2x+2h+1)(3x-4)-(2x+1)(3x+3h-4)}{(3x+3h-4)(3x-4)}}{h} \\
&= \lim_{h\to 0} \frac{(2x+1)(3x-4-3x+4-3h)+2h(3x-4)}{h\,(3x+3h-4)\,(3x-4)} \\
&= \lim_{h\to 0} \frac{h\,(-3(2x+1)+2(3x-4))}{h\,(3x+3h-4)\,(3x-4)} \\
&= \lim_{h\to 0} \frac{-11}{(3x+3h-4)\,(3x-4)} \\
&= \frac{-11}{(3x-4)^2}.
\end{aligned}$$

Example 3.8.5. What is the velocity of two seconds after a solid moving along a straight line whose position function is given by

$$s(t) = t^2 + \sqrt{t},$$

where *s* is measured in feet and *t* in seconds.

Solution. The velocity is the instantaneous rate of change of position, so that the velocity after 2 second is given by

$$\begin{aligned}
s'(2) &= \lim_{t\to 2} \frac{s(t) - s(2)}{t-2} \\
&= \lim_{t\to 2} \frac{t^2 + \sqrt{t} - (4 + \sqrt{2})}{t-2} \\
&= \lim_{t\to 2} \frac{t^2 - 4}{t-2} + \frac{\sqrt{t} - \sqrt{2}}{t-2} \\
&= \lim_{t\to 2} \frac{(t-2)(t+2)}{t-2} + \frac{\left(\sqrt{t} - \sqrt{2}\right)\left(\sqrt{t} + \sqrt{2}\right)}{(t-2)\left(\sqrt{t} + \sqrt{2}\right)} \\
&= \lim_{t\to 2} t + 2 + \frac{t-2}{(t-2)(\sqrt{t} + \sqrt{2})} \\
&= \lim_{t\to 2} t + 2 + \frac{1}{\sqrt{t} + \sqrt{2}} \\
&= 4 + \frac{1}{2\sqrt{2}} = 4 + \frac{\sqrt{2}}{4}\, ft/s.
\end{aligned}$$

3.9 M3 Sample Quiz 2: derivative

General comment: You may know formulas to obtain derivatives in a much faster way. For now, we have not seen these formulas yet, and therefore you should train to calculate derivatives from the definition. When we have seen the formulas, you will be able to use them, without going back to the definition in terms of limit.

1. Find $f'(3)$ if $f(x) = 2x^2 - 4x + 3$.
2. Find $f'(x)$ if $f(x) = \sqrt{3x+1}$ and give equations the tangent line to the graph of f at $x = 1$ and at $x = 0$.
3. Find $f'(x)$ if $f(x) = \frac{4}{3x+1}$.
4. A stone is dropped from the top of a 176.4 meters tall tower and is
$$s(t) = 176.4 - 4.9t^2$$
meters from the ground after t seconds. When does it reach the ground and with what velocity?

3.10 M3 Sample Quiz 2 Solutions

1. Find $f'(3)$ if $f(x) = 2x^2 - 4x + 3$.

Solution.

$$\begin{aligned}
f'(3) &= \lim_{x\to 3} \frac{f(x) - f(3)}{x - 3} \\
&= \lim_{x\to 3} \frac{2x^2 - 4x + 3 - 9}{x - 3} \\
&= \lim_{x\to 3} \frac{2x^2 - 4x - 6}{x - 3} \\
&= \lim_{x\to 3} \frac{(x-3)(2x+2)}{(x-3)} \\
&= \lim_{x\to 3} 2x + 2 = 8.
\end{aligned}$$

2. Find $f'(x)$ if $f(x) = \sqrt{3x+1}$ and give equations the tangent line to the graph of f at $x = 1$ and at $x = 0$.

Solution.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{3(x+h)+1} - \sqrt{3x+1}}{h} \\
&= \lim_{h\to 0} \frac{\left(\sqrt{3(x+h)+1} - \sqrt{3x+1}\right)\left(\sqrt{3(x+h)+1} + \sqrt{3x+1}\right)}{h\left(\sqrt{3(x+h)+1} + \sqrt{3x+1}\right)} \\
&= \lim_{h\to 0} \frac{3(x+h)+1-(3x+1)}{h\left(\sqrt{3(x+h)+1} + \sqrt{3x+1}\right)} \\
&= \lim_{h\to 0} \frac{3h}{h\left(\sqrt{3(x+h)+1} + \sqrt{3x+1}\right)} \\
&= \lim_{h\to 0} \frac{3}{\sqrt{3(x+h)+1} + \sqrt{3x+1}} \\
&= \frac{3}{2\sqrt{3x+1}}.
\end{aligned}$$

Therefore the tangent line to the graph of f at $x = 1$ is the line of slope $f'(1) = \frac{3}{4}$ through $(1, f(1)) = (1, 2)$. An equation of this line is

$$y - 2 = \frac{3}{4}(x - 1).$$

Similarly, the tangent line to the graph of f at $x = 0$ is the line of slope $f'(0) = \frac{3}{2}$ through $(0, f(0)) = (0, 1)$. An equation of this line is

$$y - 1 = \frac{3}{2}x.$$

3. Find $f'(x)$ if $f(x) = \frac{4}{3x+1}$.

   Solution.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{4}{3(x+h)+1} - \frac{4}{3x+1}}{h} \\
&= \lim_{h\to 0} \frac{\frac{4(3x+1)-4(3(x+h)+1)}{(3(x+h)+1)(3x+1)}}{h} \\
&= \lim_{h\to 0} \frac{12x + 4 - 12x - 12h - 4}{h(3(x+h)+1)(3x+1)} \\
&= \lim_{h\to 0} \frac{-12h}{h(3(x+h)+1)(3x+1)} \\
&= \lim_{h\to 0} -\frac{12}{(3(x+h)+1)(3x+1)} \\
&= -\frac{12}{(3x+1)^2}.
\end{aligned}$$

4. A stone is dropped from the top of a 176.4 meters tall tower and is

$$s(t) = 176.4 - 4.9t^2$$

   meters from the ground after t seconds. When does it reach the ground and with what velocity?

   Solution. It reaches the ground when the distance to the ground $s(t)$ is 0, that is, if

$$176.4 - 4.9t^2 = 0 \iff t^2 = \frac{176.4}{4.9} = 36.$$

   Since t represents time for $t \geq 0$, this means that the stone reaches the ground after 6 seconds. At that time, its velocity is given by

$$\begin{aligned}
s'(6) &= \lim_{t\to 6} \frac{s(t) - s(6)}{t - 6} \\
&= \lim_{t\to 6} \frac{(176.4 - 4.9t^2) - (176.4 - 4.9 \times 36)}{t - 6} \\
&= \lim_{t\to 6} \frac{-4.9(t^2 - 36)}{t - 6} \\
&= \lim_{t\to 6} -4.9\frac{(t-6)(t+6)}{t - 6} \\
&= \lim_{t\to 6} -4.9(t+6) = -4.9 \times 12 = -58.8\, m/s.
\end{aligned}$$

4 Review for the first 3 modules

4.1 MOCK TEST 1

Instructions: *Do the following test, without your notes, in limited time (75 minutes top). Then grade yourself using the solutions provided separately. It is important that you show all your work and justify your answers. Carefully read the solutions to see how you should justify answers.*

In question 1 to 6, find the limit and specify $+\infty$ or $-\infty$ if applicable.

1. [5pts]
$$\lim_{x\to 2^+} \frac{3}{2-x}.$$

2. [5pts]
$$\lim_{x\to 3} \frac{x^2-3x}{x^2-4x-8}$$

3. [10 pts]
$$\lim_{x\to -2} \frac{x^2-x-6}{2x^2+3x-2}$$

4. [10pts]
$$\lim_{t\to 2} \frac{\sqrt{7+t}-3}{t^2+2t-8}$$

5. [5pts]
$$\lim_{x\to 1} \frac{|x-1|}{x^2-x}$$

6. [5pts]
$$\lim_{x\to 0} x^6 \cos(\frac{2\pi}{x})$$

7. [5pts] Show that the equation $x^5+3x^3+x-3=0$ has a solution.

8. [10 pts] Find the discontinuities of $g(x)=\frac{x^2-x-12}{x^2-16}$. Is any of them removable? If yes, find a continuous extension of *g* at this value.

9. [10pts] For what value(s) of c is the function

$$f(x) = \begin{cases} x^2 + cx + 1 \text{ if } x < c \\ x + 2c \text{ if } x \geq c \end{cases}$$

continuous?

10. Find all the asymptotes of

a) [10pts]

$$f(x) = \frac{2x^3 + x^2 - 3}{x^2 + 2x - 3}.$$

b) [5pts]

$$g(x) = \frac{2x^2 + 1}{x^2 - 9}.$$

11. [10pts] Find an equation of the tangent line to $y = \sqrt{3x - 3}$ at (4, 3).

12. [10pts] Find the slope of the tangent line to $y = f(x)$ at $x = a$ if $f(x) = \frac{1}{2x+1}$. For what values of a does the tangent line exist?

13. [5pts] If $s(t) = t^2 + 1$ gives the position (in feet, with respect to a fixed origin) at time t (in seconds) of a particle moving along a straight line, find the instantaneous speed of the particle at time t = 3.

4.2 Solutions to Mock Test 1

In question 1 to 6, find the limit and specify $+\infty$ or $-\infty$ if applicable.

1. [5pts]

$$\lim_{x\to 2^+} \frac{3}{2-x}.$$

If $x \to 2^+$ then $x > 2$ so that $2-x<0$. Hence

$$\lim_{x\to 2^+} \frac{3}{2-x} = -\infty.$$

2. [5pts]

$$\lim_{x\to 3} \frac{x^2-3x}{x^2-4x-8} = \frac{0}{-11} = 0$$

3. [10 pts]

$$\lim_{x\to -2} \frac{x^2-x-6}{2x^2+3x-2} = \lim_{x\to -2} \frac{(x+2)(x-3)}{(x+2)(2x-1)} = \lim_{x\to -2} \frac{x-3}{2x-1} = \frac{-5}{-5} = 1$$

4. [10pts]

$$\lim_{t\to 2} \frac{\sqrt{7+t}-3}{t^2+2t-8}$$

$$= \lim_{t\to 2} \frac{(\sqrt{7+t}-3)(\sqrt{7+t}+3)}{(t-2)(t+4)(\sqrt{7+t}+3)}$$
$$= \lim_{t\to 2} \frac{7+t-9}{(t-2)(t+4)(\sqrt{7+t}+3)}$$
$$= \lim_{t\to 2} \frac{t-2}{(t-2)(t+4)(\sqrt{7+t}+3)} = \lim_{t\to 2} \frac{1}{(t+4)(\sqrt{7+t}+3)} = \frac{1}{36}.$$

5. [5pts]

$$\lim_{x\to 1} \frac{|x-1|}{x^2-x}$$

$$\lim_{x\to 1^+} \frac{|x-1|}{x^2-x} = \lim_{x\to 1^+} \frac{x-1}{x(x-1)} = \lim_{x\to 1^+} \frac{1}{x} = 1$$

because $|x-1| = x-1$ if $x>1$. On the other hand,

$$\lim_{x\to 1^-} \frac{|x-1|}{x^2-x} = \lim_{x\to 1^-} \frac{1-x}{x(x-1)} = \lim_{x\to 1^+} \frac{-1}{x} = -1$$

because $|x-1| = 1-x$ if $x<1$. Hence $\lim_{x\to 1} \frac{|x-1|}{x^2-x}$ does not exist, because $\lim_{x\to 1^+} \frac{|x-1|}{x^2-x} \neq \lim_{x\to 1^-} \frac{|x-1|}{x^2-x}$.

6. [5pts]

$$\lim_{x\to 0} x^6 \cos\left(\frac{2\pi}{x}\right)$$

For every $x \neq 0$,

$$-1 \le \cos\left(\frac{2\pi}{x}\right) \le 1,$$

so that

$$-x^6 \le x^6 \cos\left(\frac{2\pi}{x}\right) \le x^6.$$

But $\lim_{x\to 0} -x^6 = \lim_{x\to 0} -x^6 = 0$ so that by the Squeeze Theorem,

$$\lim_{x\to 0} x^6 \cos\left(\frac{2\pi}{x}\right) = 0.$$

7. [5pts] Show that the equation $x^5 + 3x^3 + x - 3 = 0$ has a solution.
   The function $f(x) = x^5 + 3x^3 + x - 3$ is polynomial, hence continuous on $(-\infty, \infty)$. Moreover $f(0) = -3 < 0$ and $f(1) = 2 > 0$. As f is continuous on $[0, 1]$ and $f(0) < 0 < f(1)$, the Intermediate Value Theorem applies to the effect that there is a number c in the interval $(0, 1)$ such that $f(c) = 0$. Hence the equation has (at least) one solution.

8. [10 pts] Find the discontinuities of $g(x) = \frac{x^2-x-12}{x^2-16}$. Is any of them removable? If yes, find a continuous extension of g at this value.
   g is a rational function, hence it is continuous on its domain. Therefore the only discontinuities are -4 and 4. Moreover, $g(x) = \frac{(x-4)(x+3)}{(x-4)(x+4)}$, so that $x = -4$ is a vertical asymptote for g and -4 is not a removable discontinuity. however,

$$\lim_{x\to 4} \frac{(x-4)(x+3)}{(x-4)(x+4)} = \lim_{x\to 4} \frac{x+3}{x+4} = \frac{7}{8},$$

   so that the function F defined by $F(x) = g(x)$ if $x \neq 4$ and $F(4) = \frac{7}{8}$ is a continuous extension of g at 4.

9. [10pts] For what value(s) of c is the function

$$f(x) = \begin{cases} x^2 + cx + 1 \text{ if } x < c \\ x + 2c \text{ if } x \geq c \end{cases}$$

   continuous?

   The function f is continuous on $(-\infty, c)$ and on $(c, +\infty)$ because it is polynomial on these intervals. The only possible discontinuity is $x = c$. Note that

$$\lim_{x\to c^-} f(x) = c^2 + c^2 + 1 = 2c^2 + 1$$

   and that

$$\lim_{x\to c^+} f(x) = f(c) = c + 2c = 3c.$$

   Hence, f is continuous at c if and only if $2c^2 + 1 = 3c$, equivalently if $(c-1)(2c-1) = 0$, that is, if $c = 1$ or $c = \frac{1}{2}$.

10. Find all the asymptotes of
    a) [10pts]

$$f(x) = \frac{2x^3 + x^2 - 3}{x^2 + 2x - 3}.$$

By long division, we obtain that $f(x) = 2x - 3 + \frac{12x-12}{x^2+2x-3}$ so that $y = 2x - 3$ is an asymptote for *f*. Moreover,

$$f(x) = \frac{(x-1)(2x^2+2x+3)}{(x-1)(x+3)}$$

so that $x = -3$ is a vertical asymptote, but $x = 1$ is not.

b) [5pts]

$$g(x) = \frac{2x^2+1}{x^2-9}.$$

The line $y = 2$ is an asymptote because $\lim_{x\to\infty} g(x) = 2$. Moreover, $x = 3$ and $x = -3$ are also asymptotes.

11. [10pts] Find an equation of the tangent line to $y = \sqrt{3x-3} = f(x)$ at (4, 3).
The slope of this line is given by

$$\begin{aligned}\lim_{x\to 4} \frac{f(x)-f(4)}{x-4} &= \lim_{x\to 4} \tfrac{\sqrt{3x-3}-3}{x-4} \\ &= \lim_{x\to 4} \tfrac{(\sqrt{3x-3}-3)(\sqrt{3x-3}+3)}{(x-4)(\sqrt{3x-3}+3)} \\ &= \lim_{x\to 4} \tfrac{3x-3-9}{(x-4)(\sqrt{3x-3}+3)} \\ &= \lim_{x\to 4} \tfrac{3(x-4)}{(x-4)(\sqrt{3x-3}+3)} \\ &= \lim_{x\to 4} \tfrac{3}{(\sqrt{3x-3}+3)} = \tfrac{1}{2}.\end{aligned}$$

The tangent line goes through (4, 3) and has slope $\frac{1}{2}$, so that an equation is given by

$$y - 3 = \frac{1}{2}(x-4).$$

12. [10pts] Find the slope of the tangent line to $y = f(x)$ at $x = a$ if $f(x) = \frac{1}{2x+1}$. For what values of *a* does the tangent line exist?The slope of this line is given by

$$\begin{aligned}\lim_{x\to a} \frac{f(x)-f(a)}{x-a} &= \lim_{x\to a} \tfrac{\frac{1}{2x+1}-\frac{1}{2a+1}}{x-a} \\ &= \lim_{x\to a} \tfrac{\left(\frac{2a+1-(2x+1)}{(2x+1)(2a+1)}\right)}{x-a} \\ &= \lim_{x\to a} \tfrac{-2(x-a)}{(x-a)(2x+1)(2a+1)} \\ &= \lim_{x\to a} \tfrac{-2}{(2x+1)(2a+1)} \\ &= -\tfrac{2}{(2a+1)^2}.\end{aligned}$$

Hence the tangent line exists for every $a \neq -\frac{1}{2}$.

13. [5pts] If $s(t) = t^2 + 1$ gives the position (in feet, with respect to a fixed origin) at time t (in seconds) of a particle moving along a straight line, find the instantaneous speed of the particle at time $t = 3$.

The instantaneous speed at $t = 3$ is given by

$$\begin{aligned}\lim_{h\to 0}\frac{s(3+h)-s(3)}{h} &= \lim_{h\to 0}\tfrac{(3+h)^2+1-10}{h}\\ &= \lim_{h\to 0}\tfrac{9+6h+h^2+1-10}{h}\\ &= \lim_{h\to 0}\tfrac{h(6+h)}{h}\\ &= \lim_{h\to 0} 6+h = 6.\end{aligned}$$

Hence the instantaneous speed of this particle after 3 seconds is 6 ft/s.

5 M4: Differentiation Rules

5.1 Power Rule for differentiation

Watch the video at
https://www.youtube.com/watch?v=sUVmB-U3BPc&list=PL265CB737C01F8961&index=21.

Abstract This video presents the power rule for differentiation, and basic examples.

Theorem 5.1.1 (Power Rule) *Let n be a real number.*

$$(x^n)' = nx^{n-1}.$$

Example 5.1.2. In particular

$$\begin{aligned}
\left(x^{97}\right)' &= 97x^{96} \\
\left(\frac{1}{x}\right)' &= \left(x^{-1}\right)' = -x^{-2} = -\frac{1}{x^2} \\
\left(\sqrt{x}\right)' &= \left(x^{\frac{1}{2}}\right)' = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \\
\left(\sqrt[3]{x^4}\right)' &= \left(x^{\frac{4}{3}}\right)' = \frac{4}{3}x^{\frac{1}{3}} = \frac{4}{3}\sqrt[3]{x} \\
\left(x^{\pi}\right)' &= \pi x^{\pi-1}.
\end{aligned}$$

5.2 Constant multiple and Sum Rules for derivatives

Watch the video at
https://www.youtube.com/watch?v=zHKyMiBFfYA&list=PL265CB737C01F8961&index=22.

Abstract This video presents the constant multiple and sum rules for derivatives, with basic examples of applications.

Theorem 5.2.1 (Constant multiple and Sum Rules) *If f and g are two differentiable functions and c is a constant then*

$$\begin{aligned} (c \cdot f)'(x) &= c \cdot f'(x) \\ (f+g)'(x) &= f'(x) + g'(x). \end{aligned}$$

Example 5.2.2. Using the Power Rule, the constant multiple Rule and the Sum Rule, we can now differentiate any polynomial function (and more):

$$\begin{aligned} \left(5x^4 - 3x^2 + 2x - 5\right)' &= \left(5x^4\right)' + \left(-3x^2\right)' + (2x)' + (-5)' \text{ using the Sum Rule} \\ &= 5\left(x^4\right)' - 3\left(x^2\right)' + 2(x)' + 0 \text{ using the constant multiple Rule} \\ &= 5 \cdot 4x^3 - 3 \cdot 2x + 2 \text{ using the Power Rule} \\ &= 20x^3 - 6x + 2. \end{aligned}$$

Of course, from now on we are going to give the result directly when differentiating such simple functions, as these are simple steps.

5.3 Product Rule for differentiation

Watch the video at
https://www.youtube.com/watch?v=qBQoUEu9enw&list=PL265CB737C01F8961&index=23.

Abstract This video presents the Product Rule for derivatives, and one simple example.

Theorem 5.3.1 (Product Rule) *If f and g are two differentiable functions then*

$$(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x). \tag{5.3.1}$$

The way to read this is that when you differentiate a product of two factors, you differentiate the first and multiply by the second factor unchanged, then add the result of multiplying the first factor unchanged by the derivative of the second factor, or symbolically

$$(AB)' = A'B + AB'.$$

Example 5.3.2. Differentiate

$$f(x) = \left(3x^2 + 2x + 4\right)\left(2\sqrt{x} + 3x + 2\right).$$

Solution. Here $\left(3x^2 + 2x + 4\right)$ is one factor, whose role in the formula (5.3.1) is played by f, and $(2\sqrt{x} + 3x + 2)$ is the other factor, whose role in the formula (5.3.1) is played by g. Thus

$$\begin{aligned} f'(x) &= \left(3x^2 + 2x + 4\right)'\left(2\sqrt{x} + 3x + 2\right) + \left(3x^2 + 2x + 4\right)\left(2\sqrt{x} + 3x + 2\right)' \\ &= (6x + 2)\left(2\sqrt{x} + 3x + 2\right) + \left(3x^2 + 2x + 4\right)\left(\frac{1}{\sqrt{x}} + 3\right). \end{aligned}$$

Note that we could simplify further, but for now, we are mostly concerned with applying the formulas correctly.

5.4 Quotient Rule for derivatives

Watch the video at
https://www.youtube.com/watch?v=zeq-Dl7B5Pg&list=PL265CB737C01F8961&index=24.

Abstract This video presents the Quotient Rule for derivatives and a couple of examples illustrating how to apply it.

Theorem 5.4.1 (Quotient Rule) *Let f and g be two differentiable functions. Then for each x with $g(x) \neq 0$, we have*

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x) \cdot g(x) - g'(x) \cdot f(x)}{(g(x))^2}. \tag{5.4.1}$$

In other words, when differentiating a fraction, you differentiate the top and multiply it by the bottom unchanged, then *subtract* the product of the derivative of the bottom by the top unchanged, and divide the whole thing by the square of the bottom. Symbolically

$$\left(\frac{A}{B}\right)' = \frac{A'B - B'A}{B^2}.$$

Example 5.4.2. Differentiate

$$f(x) = \frac{3x^4 - 2x^3 + 3x - 4}{2x^2 - x + 1}.$$

Solution. Here $3x^4 - 2x^3 + 3x - 4$ corresponds to *f* in (5.4.1), while $2x^2 - x + 1$ corresponds to *g*. Thus

$$\begin{aligned} f'(x) &= \frac{\left(3x^4 - 2x^3 + 3x - 4\right)' \left(2x^2 - x + 1\right) - \left(2x^2 - x + 1\right)' \left(3x^4 - 2x^3 + 3x - 4\right)}{\left(2x^2 - x + 1\right)^2} \\ &= \frac{\left(12x^3 - 6x^2 + 3\right)\left(2x^2 - x + 1\right) - (4x - 1)\left(3x^4 - 2x^3 + 3x - 4\right)}{\left(2x^2 - x + 1\right)^2}. \end{aligned}$$

Note that we could simplify further, but for now, we are mostly concerned with applying the formulas correctly.

Formulas can of course be combined. For instance:

Example 5.4.3. Differentiate

$$f(x) = \frac{\left(\sqrt{x} + \frac{1}{x}\right)\left(x^2 + x + 1\right)}{2x^3 + \frac{2}{x^2}}.$$

Solution. Here, we have a quotient, but the top part is a product. Hence we are going to start with the quotient rule, and use the product rule to explicit the part involving the derivative of the top.

$$\begin{aligned} f'(x) &= \frac{\left(\left(\sqrt{x} + \frac{1}{x}\right)\left(x^2 + x + 1\right)\right)' \left(2x^3 + \frac{2}{x^2}\right) - \left(2x^3 + \frac{2}{x^2}\right)' \left(\sqrt{x} + \frac{1}{x}\right)\left(x^2 + x + 1\right)}{\left(2x^3 + \frac{2}{x^2}\right)^2} \\ &= \frac{\left(\left(\sqrt{x} + \frac{1}{x}\right)' \left(x^2 + x + 1\right) + \left(\sqrt{x} + \frac{1}{x}\right)\left(x^2 + x + 1\right)'\right)\left(2x^3 + \frac{2}{x^2}\right) - \left(6x^2 - \frac{4}{x^3}\right)\left(\sqrt{x} + \frac{1}{x}\right)\left(x^2 + x + 1\right)}{\left(2x^3 + \frac{2}{x^2}\right)^2} \\ &= \frac{\left(\left(\frac{1}{2\sqrt{x}} - \frac{1}{x^2}\right)\left(x^2 + x + 1\right) + \left(\sqrt{x} + \frac{1}{x}\right)(2x + 1)\right)\left(2x^3 + \frac{2}{x^2}\right) - \left(6x^2 - \frac{4}{x^3}\right)\left(\sqrt{x} + \frac{1}{x}\right)\left(x^2 + x + 1\right)}{\left(2x^3 + \frac{2}{x^2}\right)^2} \end{aligned}$$

Note that we could simplify further, but for now, we are mostly concerned with applying the formulas correctly.

5.4 Differentiation Rules, examples and applications

Watch the video at
https://www.youtube.com/watch?v=qzNAi_kk4Ho&list=PL265CB737C01F8961&index=25.

Abstract This video differentiate step by step a number of different functions, using the rules seen in this Chapter. It concludes with applications to finding tangent lines, and finding instantaneous velocity.

Example 5.5.1. Differentiate the following functions:

1. $f(x) = x^4 - \dfrac{4}{3x} + \sqrt{x} + 4x + 1;$

2. $g(x) = \dfrac{x^2 + x + 1}{x};$

3. $h(x) = \dfrac{x^2 + x + 1}{x + 1};$

4. $y = \left(x^2 + \sqrt{x}^3 + 1\right)\left(2 + 2x + x^5\right).$

Solution.

$$\begin{aligned} f'(x) &= 4x^3 - \frac{4}{3}\cdot\left(-\frac{1}{x^2}\right) + \frac{1}{2\sqrt{x}} + 4 \\ &= 4x^3 + \frac{4}{3x^2} + \frac{1}{2\sqrt{x}} + 4. \end{aligned}$$

$$g(x) = \frac{x^2+x+1}{x} = x + 1 + \frac{1}{x},$$

so that

$$g'(x) = 1 - \frac{1}{x^2}.$$

$$h'(x) = \frac{(2x+1)(x+1) - (x^2+x+1)}{(x+1)^2} = \frac{x^2+2x}{(x+1)^2}.$$

$$\frac{dy}{dx} = \left(2x + \frac{3}{2}x^{\frac{1}{2}}\right)\left(2+2x+x^5\right) + \left(x^2 + x^{\frac{3}{2}} + 1\right)\left(2+5x^4\right).$$

Example 5.5.2. Find the tangent line to the graph of

$$f(x) = \frac{2x^2+x+1}{4x-2}$$

at $x = 1$.

Solution. The tangent line is the line through $(1, f(1)) = (1,2)$ of slope $f'(1)$. Moreover,

$$f'(x) = \frac{(4x+1)(4x-2) - 4\left(2x^2+x+1\right)}{(4x-2)^2},$$

so that

$$f'(1) = \frac{5\times 2 - 4\times 4}{2^2} = -\frac{3}{2}.$$

Thus an equation of the tangent line is given by

$$y - 2 = -\frac{3}{2}(x-1).$$

Example 5.5.3. A particle is moving along a straight line and its position at time t (in seconds) is given by

$$s(t) = \left(t^3+t+1\right)\left(1-\sqrt{t}\right)$$

measured in meters. What is its velocity after 4 seconds?

Solution. The desired velocity is $s'(4)$. Moreover,

$$s'(t) = (3t^2+1)(1-\sqrt{t}) + (t^3+t+1)\left(-\frac{1}{2\sqrt{t}}\right)$$

so that

$$s'(4) = 49 \times (-1) + 69 \times (-\frac{1}{4}) = -\frac{265}{4}\, m/s.$$

5.6 M4 Sample Quiz

1. Differentiate the following functions (you do not need to simplify):

   a) $f(x) = 3x^7 - 4x^3 + \frac{3}{2}x^3 - x + 2$

   b) $f(x) = \frac{2}{x^2} - \sqrt{x} + \frac{x^2}{\sqrt[3]{x}}$

   c) $h(t) = \left(t^3 - 2t + 3\right)\left(\sqrt{t} - t\right)$

   d) $g(x) = \frac{x^2 + 4x - 1}{2x^3 + x + 3}$

   e) $f(x) = \left(\frac{x^2 + \sqrt{x}}{3x + 2}\right)\left(4x^3 + 3x^2 - 5x + 2\right).$

2. Find an equation of the tangent line to

$$y = 5x^2 + 2\sqrt{x} - 3,$$

   at $x = 1$.

3. A particle is moving along a straight line with law of motion

$$s(t) = 2t^3 - t^2 + 3,$$

   where $s(t)$ is measured in meters and t in seconds. Fin the instantaneous velocity of the particle after two seconds.

4. The quantity of charge Q in Coulombs (C) that has passed through a section of a wire up to time t (in seconds) is given by

$$Q(t) = t^3 - 2t^2 + 6t + 2.$$

   The current $I(t)$ (measured in Ampere (A) where $1A = 1C/s$) is by definition the rate of change of the quantity of charge with respect to time.

   a) Find the current after 1 second.
   b) At what time is the current lowest?

5.7 M4 Sample Quiz Solutions

1. Differentiate the following functions:

a) $f(x) = 3x^7 - 4x^3 + \frac{3}{2}x^3 - x + 2$

Solution.

$$f'(x) = 21x^6 - 12x^2 + \frac{9}{2}x^2 - 1.$$

b) $f(x) = \frac{2}{x^2} - \sqrt{x} + \frac{x^2}{\sqrt[3]{x}}$

Solution.

$$\begin{aligned} f(x) &= 2x^{-2} - x^{\frac{1}{2}} + x^{2-\frac{1}{3}} \\ &= 2x^{-2} - x^{\frac{1}{2}} + x^{\frac{5}{3}} \\ \Longrightarrow f'(x) &= -4x^{-3} - \frac{1}{2}x^{-\frac{1}{2}} + \frac{5}{3}x^{\frac{2}{3}} \\ &= -\frac{3}{x^3} - \frac{1}{2\sqrt{x}} + \frac{5}{3}\sqrt[3]{x^2}. \end{aligned}$$

c) $h(t) = \left(t^3 - 2t + 3\right)\left(\sqrt{t} - t\right)$

Solution. Using the product rule, we get:

$$\begin{aligned} h'(t) &= (t^3 - 2t + 3)'\left(\sqrt{t} - t\right) + (t^3 - 2t + 3)\left(\sqrt{t} - t\right)' \\ &= (3t^2 - 2)\left(\sqrt{t} - t\right) + \left(t^3 - 2t + 3\right)\left(\frac{1}{2\sqrt{t}} - 1\right). \end{aligned}$$

d) $g(x) = \frac{x^2 + 4x - 1}{2x^3 + x + 3}$

Solution. Using the quotient rule, we get:

$$\begin{aligned} g'(x) &= \frac{(x^2 + 4x - 1)'\left(2x^3 + x + 3\right) - (2x^3 + x + 3)'\left(x^2 + 4x - 1\right)}{(2x^3 + x + 3)^2} \\ &= \frac{(2x + 4)(2x^3 + x + 3) - (6x^2 + 1)(x^2 + 4x - 1)}{(2x^3 + x + 3)^2} \end{aligned}$$

e) $f(x) = \left(\frac{x^2 + \sqrt{x}}{3x + 2}\right)\left(4x^3 + 3x^2 - 5x + 2\right).$

Solution. We start with the product rule:

$$f'(x) = \left(\frac{x^2 + \sqrt{x}}{3x + 2}\right)'\left(4x^3 + 3x^2 - 5x + 2\right) + \left(\frac{x^2 + \sqrt{x}}{3x + 2}\right)\left(4x^3 + 3x^2 - 5x + 2\right)',$$

and we use the quotient rule to calculate the derivative in the first term of the sum:

$$f'(x) = \frac{(2x + \frac{1}{2\sqrt{x}})(3x+2) - 3(x^2+\sqrt{x})}{(3x+2)^2} \cdot (4x^3+3x^2-5x+2) + \left(\frac{x^2+\sqrt{x}}{3x+2}\right)(12x^2+6x-5x)\,.$$

2. Find an equation of the tangent line to

$$y = 5x^2 + 2\sqrt{x} - 3,$$

at $x = 1$.

Solution. The point on the graph corresponding to $x = 1$ is $(1, 4)$. The slope of the tangent at this point is the value of $\frac{dy}{dx}$ at $x = 1$. Since,

$$\frac{dy}{dx} = 10x + \frac{2}{2\sqrt{x}},$$

we conclude that $\frac{dy}{dx}_{|x=1} = 11.$ Hence the tangent line is the line through$(1, 4)$ of slope 11 and has therefore equation

$$y - 4 = 11(x-1).$$

3. A particle is moving along a straight line with law of motion

$$s(t) = 2t^3 - t^2 + 3,$$

where $s(t)$ is measured in meters and t in seconds. Fin the instantaneous velocity of the particle after two seconds.

Solution. The instantaneous velocity at time t is

$$v(t) = s'(t) = 6t^2 - 2t$$

so that $v(2) = 20\,m/s$.

4. The quantity of charge Q in Coulombs (C) that has passed through a section of a wire up to time t (in seconds) is given by

$$Q(t) = t^3 - 2t^2 + 6t + 2.$$

The current $I(t)$ (measured in Ampere (A) where $1A = 1C/s$) is by definition the rate of change of the quantity of charge with respect to time.

a) Find the current after 1 second.

Solution. $I(t)$ is the rate of change of $Q(t)$ with respect to time, that is,

$$I(t) = Q'(t) = 3t^2 - 4t + 6\,(A).$$

Therefore, after one second, the current has an intensity of $I(1) = 5\ A$.

b) At what time is the current lowest?

Solution. The function $I(t)$ is a quadratic function with a positive coefficient for t^2. Therefore, its graph is a parabola opening upward, so that it has a unique minimum at its vertex. The tangent line at the vertex is horizontal, that is, has slope zero. Hence, the vertex can be located by solving

$$\begin{aligned} I'(t) &= 0 \\ 6t - 4 &= 0 \\ t &= \frac{2}{3}. \end{aligned}$$

Hence, the current is lowest after $\frac{2}{3}$ of a second.

6 M5: Derivatives of Trigonometric functions; Chain Rule

6.1 $\lim_{x\to 0} \frac{\sin x}{x}$

Watch the video at
https://www.youtube.com/watch?v=EhCBYGfWNN4&list=PL265CB737C01F8961&index=27.

Abstract This video establishes with a geometric argument that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1,$$

and explains how it relates to the derivative of sin x.

Theorem 6.1.1.

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0} \frac{\cos x - 1}{x} = 0.$$

Example 6.1.2. Evaluate

$$\lim_{x\to 0} \frac{\sin(5x)}{\sin(3x)}.$$

Solution. Let us rewrite the expression as

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin(5x)}{\sin(3x)} &= \lim_{x\to 0} \frac{\sin(5x)}{5x} \cdot \frac{5x}{3x} \cdot \frac{3x}{\sin(3x)} \\
&= \frac{5}{3} \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \cdot \lim_{\alpha\to 0} \frac{\alpha}{\sin\alpha} \text{ with } \theta = 5x \text{ and } \alpha = 3x \\
&= \frac{5}{3} \cdot 1 \cdot \lim_{\alpha\to 0} \frac{1}{\frac{\sin\alpha}{\alpha}} \\
&= \frac{5}{3}.
\end{aligned}$$

6.2 Derivatives of trig functions

Watch the video at
https://www.youtube.com/watch?v=gLJE5oIVfj4&list=PL265CB737C01F8961&index=28.

Abstract This video establishes derivatives for all 6 standard trigonometric functions.

Theorem 6.2.1.

$$\begin{aligned}
(\sin x)' &= \cos x \\
(\cos x)' &= -\sin x \\
(\tan x)' &= \sec^2 x = 1 + \tan^2 x \\
(\cot x)' &= -\csc^2 x = -1 - \cot^2 x \\
(\csc x)' &= -\cos x \csc^2 x = -\csc x \cot x \\
(\sec x)' &= \sin x \sec^2 x = \sec x \tan x.
\end{aligned}$$

6.3 Derivatives of trig functions: Examples

Watch the video at
https://www.youtube.com/watch?v=hsr55DWQyJw&list=PL265CB737C01F8961&index=29.

Abstract This video provide step by step differentiation of various functions defined in terms of trig functions. Tangent lines and instantaneous velocities are among the applications.

Example 6.3.1. Differentiate

1) $f(x) = x^2 + 3\cos x$
2) $g(x) = \sec x - 3\cot x$
3) $h(x) = \dfrac{\cos x + x}{2x^2 - \sin x}$
4) $r(x) = \sqrt{\cos x \cdot \sec x}$
5) $y = x^2 \cos x + 3\tan x.$

Solution. In view of Theorem 6.2.1 and differentiation formulas of Chapter 5, we have

$$f'(x) = 2x - 3\sin x$$

$$g'(x) = \sin x \sec^2 x + 3\csc^2 x$$

$$h'(x) = \frac{(-\sin x + 1)\left(2x^2 - \sin x\right) - (4x - \cos x)(\cos x + x)}{\left(2x^2 - \sin x\right)^2}$$

Note that $r(x) = \sqrt{1} = 1$ so that $r'(x) = 0$. Finally,

$$\frac{dy}{dx} = 2x\cos x - x^2 \sin x + 3\sec^2 x.$$

Example 6.3.2. Find an equation of the tangent line to

$$y = \cos x + 2x\sin x$$

at $x = \frac{\pi}{2}$.

Solution. The tangent line is the line through

$$\left(\frac{\pi}{2}, \cos\frac{\pi}{2} + 2\frac{\pi}{2}\sin\frac{\pi}{2}\right) = \left(\frac{\pi}{2}, \pi\right)$$

of slope $\frac{dy}{dx}_{|x=\frac{\pi}{2}}$. Moreover,

$$\frac{dy}{dx} = -\sin x + 2\sin x + 2x\cos x$$

so that

$$\frac{dy}{dx}_{|x=\frac{\pi}{2}} = 1$$

and the line has equation

$$y - \pi = x - \frac{\pi}{2}.$$

Example 6.3.3. A particle moves along a straight line with position

$$s(t) = (2t+1)\cos t - \sin t$$

at time t, where t is measured in seconds and s in meters. What is its initial velocity (at $t = 0$)?

Solution. The required velocity is $s'(0)$. Moreover,

$$\begin{aligned} s'(t) &= 2\cos t - (2t+1)\sin t - \cos t \\ &= \cos t - (2t+1)\sin t \end{aligned}$$

so that

$$s'(0) = 1\, m/s.$$

6.4 M5 Sample Quiz 1: derivatives of trig functions

1. Differentiate the following functions (you do not need to simplify):

   a) $f(x) = 3x^2 + 2\cos x - 3\sin x$

   b) $f(x) = (2\cos x + \tan x)(x^5 + 4x^2 + 1)$

   c) $h(x) = 2\sec x - \cot x$

   d) $g(x) = \dfrac{x\cos x}{2x^3 + \sin x}$

   e) $f(t) = \sqrt{\cos t \cdot \sec t}$.

2. Find an equation of the tangent line to

   $$y = 2x\sin x,$$

   at $x = \frac{\pi}{2}$.

3. A mass attached to a spring hanging from the ceiling oscillates up and down. Its vertical position at time t (in seconds) relative to its rest position is given by

   $$f(t) = 4\sin t$$

   (in centimeters). Find its velocity at time t. When and where is it at rest?

6.5 M5 Sample Quiz 1 Solutions

1. Differentiate the following functions (you do not need to simplify):

a) $f(x) = 3x^2 + 2\cos x - 3\sin x$

Solution.

$$f'(x) = 6x - 2\sin x - 3\cos x.$$

b) $f(x) = (2\cos x + \tan x)(x^5 + 4x^2 + 1)]$

Solution.

$$f'(x) = (-2\sin x + \sec^2 x)(x^5 + 4x^2 + 1) + (2\cos x + \tan x)(5x^4 + 8x).$$

c) $h(x) = 2\sec x - \cot x$

Solution.

$$h'(x) = 2\sin x \sec^2 x + \csc^2 x.$$

d) $g(x) = \dfrac{x\cos x}{2x^3 + \sin x}$

Solution.

$$g'(x) = \frac{(\cos x - x\sin x)(2x^3 + \sin x) - (6x^2 + \cos x)x\cos x}{(2x^3 + \sin x)^2}.$$

$$f(t) = \sqrt{\cos t \cdot \sec t}.$$

Solution.

$$f(t) = \sqrt{\cos t \cdot \frac{1}{\cos t}} = \sqrt{1} = 1.$$

Thus $f'(t) = 0$.

2. Find an equation of the tangent line to

$$y = 2x\sin x,$$

at $x = \frac{\pi}{2}$.

Solution. When $x = \frac{\pi}{2}$, $y = \pi$ so the point of tangency is $(\frac{\pi}{2}, \pi)$. The slope of the tangent line is $\frac{dy}{dx}_{|x=\frac{\pi}{2}}$.

$$\frac{dy}{dx} = 2\sin x + 2x\cos x$$

using the product rule. Thus $\frac{dy}{dx}_{|x=\frac{\pi}{2}} = 2$ and an equation of the line is given by

$$y - \pi = 2(x - \frac{\pi}{2}) \iff y = 2x.$$

3. A mass attached to a spring hanging from the ceiling oscillates up and down. Its vertical position at time t (in seconds) relative to its rest position is given by

   $f(t) = 4\sin t$

   (in centimeters). Find its velocity at time t. When and where is it at rest?

   Solution. The velocity at time t is the derivative of the position:

   $$v(t) = f'(t) = 4\cos t \ (cm/s).$$

   The mass is at rest if $v(t) = 0$, which happens when $\cos t = 0$, that is, if

   $$t = \frac{\pi}{2} + k\pi$$

   where k is an arbitrary integer. At those times $\sin t = \pm 1$ so that $f(t) = \pm 4\,(cm)$.

6.6 Chain Rule

Watch the video at
https://www.youtube.com/watch?v=o14A4KGO-YE&list=PL265CB737C01F8961&index=30.

Abstract This video reviews composite functions and establishes the Chain Rule to differentiate composite functions. Finally the rule is applied in a couple of instances.

Recall that the composite $g \circ f$ of f with g is defined by

$$g \circ f(x) := g(f(x)).$$

Theorem 6.6.1 (Chain Rule) *If f is differentiable at a and g is differentiable at $f(a)$, then $g \circ f$ is differentiable at a and*

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a).$$

Alternatively, we can write

$$y = g \circ f(x) = g(u) \text{ where } u = f(x)$$

and differentiate with respect to x as follows:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Example 6.6.2. Differentiate

$$f(x) = \sqrt{x^2 + 1}.$$

In this instance, we want to differentiate $y = \sqrt{u}$ where $u = x^2 + 1$ with respect to x. According to the Chain Rule

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \frac{1}{2\sqrt{u}} \cdot \frac{du}{dx} \\ &= \frac{1}{2\sqrt{x^2+1}} \cdot 2x \\ &= \frac{x}{\sqrt{x^2+1}}. \end{aligned}$$

Example 6.6.3. Differentiate

$$y = \cos(x^3 + 1).$$

We want to differentiate $y = \cos u$ where $u = x^3 + 1$. Applying the Chain Rule:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\
&= -\sin u \cdot \frac{du}{dx} \\
&= -\sin(x^3 + 1) \cdot 3x^2.
\end{aligned}$$

6.7 Examples using the Chain Rule

Watch the videos at
https://www.youtube.com/watch?v=y-qPXj-Mlx0&list=PL265CB737C01F8961&index=31

and
https://www.youtube.com/watch?v=VQ8IqzWBlLM&list=PL265CB737C01F8961&index=32

Abstract These videos go over a number of examples of differentiation of functions, each involving the Chain Rule. Additionally, an application to finding the tangent line to a graph is provided, as well as a more concrete problem.

Example 6.7.1. Differentiate the following functions

$$\begin{aligned} f(x) &= \left(3x^3+4x+2\right)^5 \\ g(x) &= x\cos(x^3) \\ h(x) &= \frac{\sin(x^2)}{(x^2+1)^4} \\ r(x) &= \sqrt{x\sin x} \\ y &= \cos(\sin x)+\tan(5x). \end{aligned}$$

Solution.

$$\begin{aligned} f'(x) &= 5\left(3x^3+4x+2\right)^4\cdot\left(9x^2+4\right) \\ g'(x) &= \cos(x^3)-3x^3\sin(x^3) \\ h'(x) &= \frac{2x\cos(x^2)(x^2+1)^4 = 8x(x^2+1)^3\sin(x^2)}{(x^2+1)^8} \\ r'(x) &= \frac{1}{2\sqrt{x\sin x}}(\sin x + x\cos x) \\ \frac{dy}{dx} &= -\sin(\sin x)\cos x+5\sec^2(5x). \end{aligned}$$

Example 6.7.2. Find an equation of the tangent line to the graph of

$$f(x)=\sqrt{2x^2+x+4}$$

at $x = 3$.

Solution. The tangent line is the line through

$$(3, f(3)) = (3, 5)$$

of slope $f'(3)$. Moreover,

$$f'(x) = \frac{4x+1}{2\sqrt{2x^2+x+4}},$$

so that $f'(3) = \frac{13}{10}$. Thus, an equation of the tangent line is

$$y - 5 = \frac{13}{10}(x-3).$$

Example 6.7.3. An environmental study of a certain suburban community suggests that the average daily level of carbon monoxide in the air will be

$$C(p) = \sqrt{0.5p^2 + 17}$$

parts per million when the population is p thousand.

On the other hand it is estimated that t years from now, the population of that community will be

$$p(t) = 3.1 + 0.1t^2$$

thousands. At what rate will the carbon monoxide level be changing with respect to time 3 years from now.

Solution. We are looking for the rate of change of C with respect to t, that is,

$$\begin{aligned} \frac{dC}{dt} &= \frac{dC}{dp} \cdot \frac{dp}{dt} \\ &= \frac{0.5 \times 2p}{2\sqrt{0.5p^2+17}} \cdot (0.2t) \\ &= \frac{0.1tp}{\sqrt{0.5p^2+17}}, \end{aligned}$$

so that when $t = 3$, we have

$$\frac{dC}{dt}_{|t=3} = \frac{0.1 \times 3 \times (3.1 + 0.1 \times 3^2)}{2\sqrt{0.5(3.1 + 0.1 \times 3^2) + 17}} = 0.24$$

in parts per million per year.

6.8 M5 Sample Quiz 2: Chain Rule

1. Differentiate the following functions (you do not need to simplify):

   a) $f(x) = \left(4x^5 + \sqrt{x} + \cos x\right)^3$
   b) $f(x) = x\tan(x^2 + 1)$
   c) $h(t) = \sin(4t)\sqrt{t^3 + 3}$
   d) $g(x) = \dfrac{\cos(x^2)}{\sin x}$
   e) $f(x) = \left(\dfrac{\tan(3x)}{\sin(\cos x)}\right)\left(x^2 + 1\right)^2.$

2. Find an equation of the tangent line to

$$y = \cos^3 x,$$

   at $x = \frac{\pi}{4}$.

3. A particle is moving along a straight line with law of motion

$$s(t) = \cos(5t),$$

   where $s(t)$ is measured in meters and t in seconds. Find the instantaneous velocity of the particle after $\frac{\pi}{10}$ seconds.

4. The quantity of charge Q in Coulombs (C) that has passed through a section of a wire up to time t (in seconds) is given by

$$Q(t) = \sin(\pi t).$$

   The current $I(t)$ (measured in Ampere (A) where $1A = 1C/s$) is by definition the rate of change of the quantity of charge with respect to time. Find the current after 1 second.

6.9 M5 Sample Quiz 2 Solutions

1. Differentiate the following functions (you do not need to simplify):

   a) $f(x) = \left(4x^5 + \sqrt{x} + \cos x\right)^3$

   Solution.

   $$f'(x) = 5\left(4x^5 + \sqrt{x} + \cos x\right)^2 \cdot \left(20x^4 + \frac{1}{2\sqrt{x}} - \sin x\right).$$

   b) $f(x) = x\tan(x^2 + 1)$

   Solution.

   $$f'(x) = \tan(x^2+1) + x \cdot \sec^2(x^2+1) \cdot 2x = \tan(x^2+1) + 2x\sec^2(x^2+1).$$

c) $h(t) = \sin(4t)\sqrt{t^3+3}$

Solution.

$$\begin{aligned} h'(t) &= (\sin(4t))' \cdot \sqrt{t^3+3} + \sin(4t) \cdot \left(\sqrt{t^3+3}\right)' \\ &= 4\cos(4t)\sqrt{t^3+3} + \sin(4t) \cdot \frac{3t^2}{2\sqrt{t^3+3}}. \end{aligned}$$

d) $g(x) = \dfrac{\cos(x^2)}{\sin x}$

Solution.

$$\begin{aligned} g'(x) &= \frac{\left(\cos(x^2)\right)' \cdot \sin x - (\sin x)' \cdot \cos(x^2)}{\sin^2 x} \\ &= \frac{-2x\sin(x^2)\sin x - \cos x\cos(x^2)}{\sin^2 x} \end{aligned}$$

e) $f(x) = \left(\dfrac{\tan(3x)}{\sin(\cos x)}\right)\left(x^2+1\right)^2.$

Solution.

$$\begin{aligned} f'(x) &= \left(\frac{\tan(3x)}{\sin(\cos x)}\right)'\left(x^2+1\right)^2 + \left(\frac{\tan(3x)}{\sin(\cos x)}\right)\left(\left(x^2+1\right)^2\right)' \\ &= \frac{(\tan(3x))' \cdot \sin(\cos x) - (\sin(\cos x))' \cdot \tan(3x)}{\sin^2(\cos x)}\left(x^2+1\right)^2 + \left(\frac{\tan(3x)}{\sin(\cos x)}\right) \cdot 2(x^2+1) \cdot 2x \\ &= \frac{3\sec^2(3x)\sin(\cos x) - \cos(\cos x) \cdot (-\sin x) \cdot \tan(3x)}{\sin^2(\cos x)}\left(x^2+1\right)^2 + 4x(x^2+1) \cdot \frac{\tan(3x)}{\sin(\cos x)} \\ &= \frac{3\sec^2(3x)\sin(\cos x) + \cos(\cos x) \cdot \sin x\tan(3x)}{\sin^2(\cos x)}\left(x^2+1\right)^2 + 4x(x^2+1) \cdot \frac{\tan(3x)}{\sin(\cos x)} \end{aligned}$$

2. Find an equation of the tangent line to

$$y = \cos^3 x,$$

at $x = \frac{\pi}{4}$.

Solution. The tangent line is the line through

$$\left(\frac{\pi}{4}, \cos^3\left(\frac{\pi}{4}\right)\right) = \left(\frac{\pi}{4}, \left(\frac{\sqrt{2}}{2}\right)^3\right)$$

of slope

$$\begin{aligned} \frac{dy}{dx}_{|x=\frac{\pi}{4}} &= 3\cos^2 x \cdot (-\sin x)_{|x=\frac{\pi}{4}} \\ &= -3\left(\frac{\sqrt{2}}{2}\right)^3. \end{aligned}$$

Hence an equation of the tangent line is

$$y - \left(\frac{\sqrt{2}}{2}\right)^3 = -3\left(\frac{\sqrt{2}}{2}\right)^3\left(x - \frac{\pi}{4}\right).$$

3. A particle is moving along a straight line with law of motion

$$s(t) = \cos(5t),$$

where $s(t)$ is measured in meters and t in seconds. Find the instantaneous velocity of the particle after $\frac{\pi}{10}$ seconds.

Solution. The velocity after $\frac{\pi}{10}$ seconds is $s'\left(\frac{\pi}{10}\right)\ m/s$.

$$s'(t) = -5\sin(5t),$$

so that

$$s'\left(\frac{\pi}{10}\right) = -5\sin\left(\frac{\pi}{2}\right) = -5\,m/s.$$

4. The quantity of charge Q in Coulombs (C) that has passed through a section of a wire up to time t (in seconds) is given by

$$Q(t) = \sin(\pi t).$$

The current $I(t)$ (measured in Ampere (A) where $1A = 1C/s$) is by definition the rate of change of the quantity of charge with respect to time. Find the current after 1 second.

Solution. We are looking for $I(1) = \frac{dQ}{dt}_{|t=1}\,A$. Since $Q'(t) = \pi\cos(\pi t)$, we have

$$I(1) = Q'(1) = -\pi\,A.$$

7 M6: Implicit Differentiation; Related Rates Problems

7.1 Implicit Differentiation

Watch the video at
https://www.youtube.com/watch?v=6XLM4IFD0LI&list=PL265CB737C01F8961&index=33.

Abstract This video present the general idea behind implicit differentiation to find the slope of a tangent line to a general curve of the plane, and goes over two examples.

A general curve of the plane might have an equation given under the form $f(x,y) = g(x,y)$. The slope of the tangent line to such a curve at a point (x_0, y_0) of the curve is the rate of change

$$\frac{dy}{dx}_{|x=x_0;\, y=y_0}$$

of y with respect to x, along the curve, at (x_0, y_0). To obtain this rate of change, we differentiate both sides of the equation **with respect to** x **considering** y **as a function of** x of derivative $\frac{dy}{dx}$, and then solve for $\frac{dy}{dx}$ in the resulting equation.

Example 7.1.1. Find the tangent line to

$$x^2 + y^2 = 4$$

at $(1, \sqrt{3})$.

Solution. The tangent line is the line through $(1, \sqrt{3})$ of slope $\frac{dy}{dx}_{|x=1;\, y=\sqrt{3}}$. To find this, we differentiate both sides of the equation implicitly, and solve for $\frac{dy}{dx}$:

$$\begin{aligned} 2x + 2y \cdot \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{x}{y}. \end{aligned}$$

Therefore

$$\frac{dy}{dx}_{|x=1;\, y=\sqrt{3}} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

and an equation of the tangent line is

$$y - \sqrt{3} = -\frac{\sqrt{3}}{3}(x-1).$$

Example 7.1.2. Find an equation of the tangent line to

$$x^3 + y^3 = 4xy + 1$$

at (1, 2).

Solution. The tangent line is the line through (1, 2) of slope $\frac{dy}{dx}_{|x=1;\,y=2}$. To find this, we differentiate implicitely the equation of the curve, and solve for $\frac{dy}{dx}$:

$$\begin{aligned} 3x^2 + 3y^2 \cdot \frac{dy}{dx} &= 4y + 4x \cdot \frac{dy}{dx} \\ \frac{dy}{dx}\left(3y^2 - 4x\right) &= 4y - 3x^2 \\ \frac{dy}{dx} &= \frac{4y - 3x^2}{3y^2 - 4x}, \end{aligned}$$

so that

$$\frac{dy}{dx}_{|x=1;\,y=2} = \frac{5}{8}$$

and an equation of the desired tangent line is

$$y - 2 = \frac{5}{8}\left(x - 1\right).$$

7.2 Implicit Differentiation: Examples

Watch the video at
https://www.youtube.com/watch?v=qGoAAVRxuVM&list=PL265CB737C01F8961&index=34.

Abstract This video goes through step by step calculations of $\frac{dy}{dx}$ along various plane curves, applying this to finding tangent lines in some instances.

Example 7.2.1. Find $\frac{dy}{dx}$ along the curve

$$3xy^3 - 4x = 10y^2.$$

Solution. To find this, we differentiate both sides of the equation of the curve with respect to x, considering y as a function of x:

$$\begin{aligned} 3y^3 + 9xy^2\frac{dy}{dx} - 4 &= 20y\frac{dy}{dx} \\ \frac{dy}{dx}\left(9xy^2 - 20y\right) &= 4 - 3y^3 \\ \frac{dy}{dx} &= \frac{4 - 3y^3}{9xy^2 - 20y}. \end{aligned}$$

Example 7.2.2. Find $\frac{dy}{dx}$ along the curve

$$\sin(xy) = x^2 - 3.$$

Solution. To find this, we differentiate both sides of the equation of the curve with respect to x, considering y as a function of x:

$$\begin{aligned} \cos(xy)\cdot\left(y + x\frac{dy}{dx}\right) &= 2x \\ x\cos(xy)\frac{dy}{dx} &= 2x - y\cos(xy) \\ \frac{dy}{dx} &= \frac{2x - y\cos(xy)}{x\cos(xy)}. \end{aligned}$$

Example 7.2.3. Find $\frac{dy}{dx}$ along the curve

$$x\cos y - 3y\sin x = 1.$$

Solution. To find this, we differentiate both sides of the equation of the curve with respect to x, considering y as a function of x:

$$\begin{aligned}\cos y - x\sin y\cdot\frac{dy}{dx} - 3\sin x\cdot\frac{dy}{dx} - 3y\cos x &= 0\\ \frac{dy}{dx}\left(-x\sin y - 3\sin x\right) &= 3y\cos x - \cos y\\ \frac{dy}{dx} &= \frac{\cos y - 3y\cos x}{x\sin y + 3\sin x}.\end{aligned}$$

Example 7.2.4. Find an equation of the tangent line to

$$x^3y^2 = -3xy$$

at (–1, –3).

Solution. The tangent line is the line through (–1, –3) of slope $\frac{dy}{dx}_{|x=-1;\,y=-3}$. To find $\frac{dy}{dx}$ we differentiate the equation of the curve implicitly:

$$\begin{aligned}3x^2y^2 + 2x^3y\frac{dy}{dx} &= -3y - 3x\frac{dy}{dx}\\ \frac{dy}{dx}\left(2x^3y + 3x\right) &= -3y - 3x^2y^2,\end{aligned}$$

so that, when $x = -1$ and $y = -3$, we have

$$\frac{dy}{dx}(6-3) = 9 - 27 \iff \frac{dy}{dx}_{|x=-1;\,y=-3} = -\frac{18}{3} = -6.$$

Therefore, an equation of the tangent line is

$$y + 3 = -6(x+1).$$

Example 7.2.5. Find all the horizontal and vertical tangents to the curve

$$xy^2 - 2y = 2. \tag{7.2.1}$$

Solution. We are looking for points of coordinate (x, y) on the curve where $\frac{dy}{dx} = 0$ or $\frac{dy}{dx}$ is infinite. First we calculate $\frac{dy}{dx}$ along the curve, by implicit differentiation:

$$\begin{aligned}y^2 + 2xy\frac{dy}{dx} - 2\frac{dy}{dx} &= 0\\ \frac{dy}{dx}(2xy - 2) &= -y^2\\ \frac{dy}{dx} &= \frac{y^2}{2 - 2xy}.\end{aligned}$$

Thus $\frac{dy}{dx}$ can only be 0 if $y = 0$, but the curve $xy^2 - 2y = 2$ does not intersect the line $y = 0$. Thus the curve has no horizontal tangent. On the other hand, $\frac{dy}{dx}$ is infinite if $2 - 2xy = 0$, that is, if $xy = 1$. If a point(x, y) is on the curve and satisfies $xy = 1$, then, substituting in (7.2.1), we obtain $1 \cdot y - 2y = 2$, that is, $y = -2$, which forces $x = -\frac{1}{2}$ because $xy = 1$. Thus the only point of the curve that has a vertical tangent is $\left(-\frac{1}{2}, -2\right)$.

7.3 M6 Sample Quiz 1: Implicit Differentiation

1. Find $\frac{dy}{dx}$ along the following curve

$$y^2x + \frac{y}{x} = 3x^2 + y.$$

2. Find $\frac{dy}{dx}$ along the following curve

$$\cos(xy) + xy = x + y - \sin(xy).$$

3. Find an equation of the tangent line to

$$2x^3 + y^3 = 2 + 4xy$$

at (1, 2).

4. Find an equation of the tangent line to

$$x \cos y + y \sin x = 0$$

at $(\pi, \frac{\pi}{2})$.

7.4 M6 Sample Quiz 1 Solutions

1. Find $\frac{dy}{dx}$ along the following curve

$$y^2x + \frac{y}{x} = 3x^2 + y.$$

Solution. Differentiating both sides of the equation with respect to x yields

$$2xy\frac{dy}{dx} + y^2 + \frac{x\frac{dy}{dx} - y}{x^2} = 6x + \frac{dy}{dx}.$$

Solving for $\frac{dy}{dx}$ gives

$$\begin{aligned}\frac{dy}{dx}\left(2xy + \frac{1}{x} - 1\right) &= 6x - y^2 + \frac{y}{x^2}\\ \frac{dy}{dx} &= \frac{6x^3 - y^2x^2 + y}{2x^3y + x - x^2}.\end{aligned}$$

2. Find $\frac{dy}{dx}$ along the following curve

$$\cos(xy) + xy = x + y - \sin(xy).$$

Solution. Differentiating both sides of the equation with respect to x yields

$$\left(y + x\frac{dy}{dx}\right)(-\sin(xy) + 1) = 1 + \frac{dy}{dx} - \cos(xy)\left(y + x\frac{dy}{dx}\right).$$

Solving for $\frac{dy}{dx}$ gives

$$\begin{aligned}\frac{dy}{dx}(-x\sin(xy) + x - 1 + x\cos(xy)) &= y\sin(xy) - y + 1 - y\cos(xy)\\ \frac{dy}{dx} &= \frac{y(\sin(xy) - \cos(xy) - 1) + 1}{x(\cos(xy) - \sin(xy) + 1) - 1}.\end{aligned}$$

3. Find an equation of the tangent line to

$$2x^3 + y^3 = 2 + 4xy$$

at (1, 2).

Solution. The slope of the tangent line is $\frac{dy}{dx}_{|x=1;y=2}$. To calculate $\frac{dy}{dx}$ we differentiate both sides of the equation with respect to x, then solve for $\frac{dy}{dx}$:

$$\begin{aligned} 6x^2 + 3y^2 \frac{dy}{dx} &= 4y + 4x \frac{dy}{dx} \\ \frac{dy}{dx}\left(3y^2 - 4x\right) &= 4y - 6x^2 \\ \frac{dy}{dx} &= \frac{4y - 6x^2}{3y^2 - 4x}. \end{aligned}$$

Plugging in $x = 1$ and $y = 2$ yields the slope

$$m = \frac{2}{8} = \frac{1}{4}.$$

Therefore, an equation of the tangent line is

$$y - 2 = \frac{1}{4}(x - 1).$$

4. Find an equation of the tangent line to

$$x \cos y + y \sin x = 0$$

at $(\pi, \frac{\pi}{2})$.

Solution. The slope of the tangent line is $\frac{dy}{dx}_{|x=\pi;\, y=\frac{\pi}{2}}$. To calculate $\frac{dy}{dx}$ we differentiate both sides of the equation with respect to x, then solve for $\frac{dy}{dx}$:

$$\begin{aligned} \cos y - x \sin y \frac{dy}{dx} + \frac{dy}{dx} \sin x + y \cos x &= 0 \\ \frac{dy}{dx}(\sin x - x \sin y) &= -(y \cos x + \cos y) \\ \frac{dy}{dx} &= \frac{y \cos x + \cos y}{x \sin y - \sin x}. \end{aligned}$$

Plugging in $x = \pi$ and $y = \frac{\pi}{2}$ yields the slope

$$m = \frac{-\frac{\pi}{2}}{\pi} = -\frac{1}{2}.$$

Therefore, an equation of the tangent line is

$$y - \frac{\pi}{2} = -\frac{1}{2}(x - \pi).$$

7.5 Related Rates: first problems

Watch the video at
https://www.youtube.com/watch?v=T3JSvqRmK1A&list=PL265CB737C01F8961&index=35.

Abstract This video introduces related rates problem and guidelines to solve them, examining two examples.

Example 7.5.1. A stone is thrown into a pond causing a circular ripple to spread. If the front wave moves away from the entry point of the stone at 0.5 meter per second, how fast is the area of the circular ripple increasing when the front wave is 30 meters away from the entry point of the stone?

Solution. If the front wave moves at $0.5\,m/s$ and we call R the radius of the outer circle of the ripple, we have $\frac{dR}{dt} = 0.5\,m/s$. Let us call A the area of the circular ripple. We are looking for $\frac{dA}{dt}$ when $R = 30\ m$. Since

$$A = \pi R^2$$

and A and R are functions of time, we obtain by differentiation

$$\frac{dA}{dt} = 2\pi R\,\frac{dR}{dT},$$

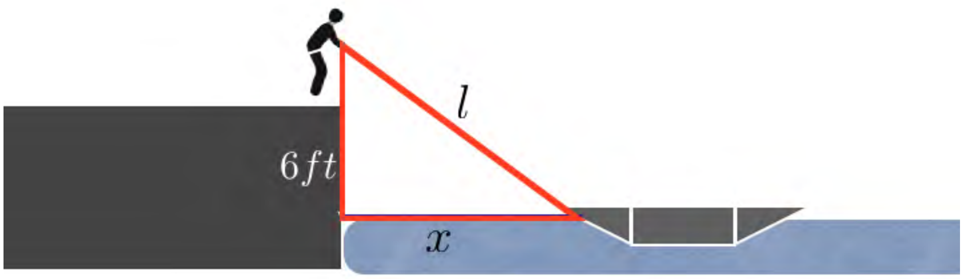
so that

$$\frac{dA}{dt} = 2\pi \cdot 30 \cdot \frac{1}{2} = 30\pi\, m^2/s$$

is the desired rate.

Example 7.5.2. Suppose you stand on the edge of a dock, 6 feet above water level, and pull, at a constant rate of 2 feet per second, a rope attached (at water level) to a boat. Assuming the boat remains at water level, at what speed is the boat approaching the dock when it is 20 feet away from the dock? When it is 10 feet away?

Solution. Consider the situation



where we called x the distance between the boat and the dock, and l the length of rope between the boat and the hand of the man. Both are functions of time. We are given the rate of change $\frac{dl}{dt} = -2\, ft/sec$, and we look for $\frac{dx}{dt}$. In the right triangle, we have, by the Pythagorean Theorem

$$l^2 = 36 + x^2,$$

so that, differentiating with respect to time:

$$2l\,\frac{dl}{dt} = 2x\,\frac{dx}{dt},$$

and

$$\frac{dx}{dt} = \frac{l}{x}\,\frac{dl}{dt} = \frac{\sqrt{36 + x^2}}{x}\,\frac{dl}{dt}.$$

Thus, when the boat is 20 feet away, that is, if $x = 20$, then

$$\frac{dx}{dt} = \frac{\sqrt{436}}{20}(-2) \approx -2.09\, ft/sec$$

and when $x = 10$, then

$$\frac{dx}{dt} = \frac{\sqrt{136}}{10}(-2) \approx -2.33\, ft/sec.$$

Thus the boat is accelerating as it is approaching the dock, even though the rope is pulled at a constant rate.

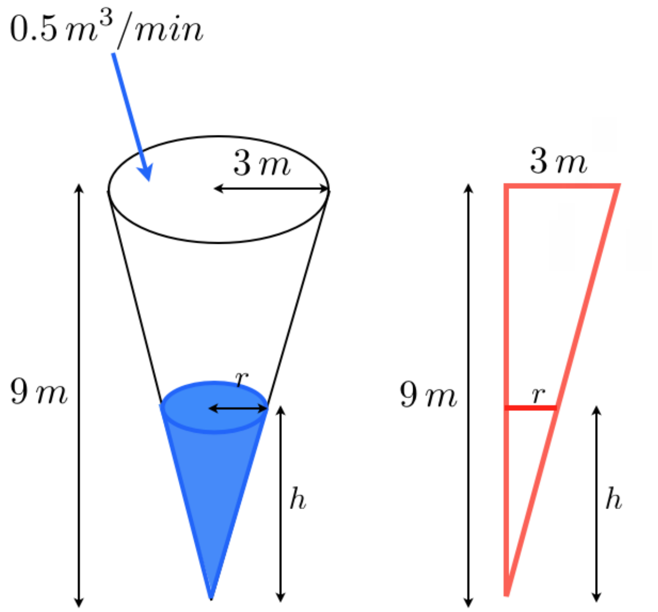
7.6 Related Rates: filling up a tank

Watch the video at
https://www.youtube.com/watch?v=TKoBaOfFlck&list=PL265CB737C01F8961&index=36.

Abstract This video goes over a problem in which a conical tank is being filled at a constant given rate, and we look for the rate of change of the water level inside the tank.

Example 7.6.1. An inverted conical tank, 9 meters high, with radius 3 meters at the top, is being filled at a constant rate of $0.5\,m^3/min$. At what rate is the water level rising when the water in the tank is 4 meters deep?

Solution. Let us first represent the situation:



where h is the depth of water in the tank, and r the radius at the base of the cone of water inside the tank. We call V the volume of water in the tank. Note that h, r and V are functions of time, and the rate of change of V is the rate at which water is being pumped inside the tank, that is, $\frac{dV}{dt} = 0.5\,m^3/min$. We look for $\frac{dh}{dt}$ when $h = 4m$. The volume V of the cone of water, of height h and radius r is given by

$$V = \frac{1}{3}\pi r^2 h.$$

Moreover, using similar triangles as shown above, we have $\frac{r}{h} = \frac{3}{9}$ so that $r = \frac{h}{3}$. Thus

$$V = \frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h = \frac{\pi}{27}h^3.$$

Differentiating with respect to time, we get

$$\frac{dV}{dt} = \frac{\pi}{9} h^2 \frac{dh}{dt},$$

so that

$$\frac{dh}{dt} = \frac{9}{\pi h^2} \frac{dV}{dt}.$$

Thus, when $h = 4$, we have

$$\frac{dh}{dt} = \frac{9}{16\pi} 0.5 = \frac{9}{32\pi} \approx 0.09\, m/min.$$

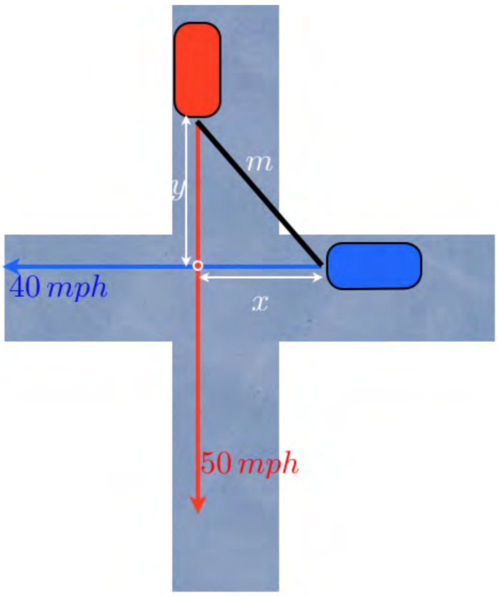
7.7 Related Rates: Radar gun

Watch the video at
https://www.youtube.com/watch?v=Alo7SkW50WI&list=PL265CB737C01F8961&index=37.

Abstract This video goes over a problem in which two cars are approaching an intersection at given speeds, and we want to know the rate at which they are approaching each other.

Example 7.7.1. A car is traveling south at 50 mph. When this car is half a mile north of an intersection, a police car traveling west at 40 mph and located a quarter of a mile east of the intersection uses a radar-gun, which measures the rate at which the distance between the two cars is decreasing. What is the reading on the radar gun?

Solution. We first draw a picture to sketch the situation:



Here, we have introduced the relevant variables: x is the distance between the police car and the intersection, y is the distance between the other car and the intersection, and m is the distance between the two cars. What is given to us are the rate of change of the positions of the two cars, which we can interpret as distances to the intersection. In other words, we have

$$\frac{dx}{dt} = -40\,mph\,; \; \frac{dy}{dt} = -50\,mph$$

and we want to find $\frac{dm}{dt}$ when $y = \frac{1}{2}$ and $x = \frac{1}{4}$. By the Pythagorean Theorem, we have

$$m^2 = x^2 + y^2$$

so that, differentiating with respect to time, we obtain:

$$2m\,\frac{dm}{dt} = 2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt},$$

that is,

$$\begin{aligned}\frac{dm}{dt} &= \frac{1}{m}\left(x\,\frac{dx}{dt} + y\,\frac{dy}{dt}\right)\\ &= \frac{1}{\sqrt{x^2+y^2}}\left(x\,\frac{dx}{dt} + y\,\frac{dy}{dt}\right).\end{aligned} \tag{7.7.1}$$

Thus, when $y = \frac{1}{2}$ and $x = \frac{1}{4}$, we have

$$\frac{dm}{dt} = \frac{1}{\sqrt{\frac{1}{4}+\frac{1}{16}}}\left(\frac{1}{4}\cdot(-40) + \frac{1}{2}\cdot(-50)\right) = -\frac{140}{\sqrt{5}} \approx -62.6\,mph.$$

Remark 7.7.2. You probably noticed that the radar-gun reading of about 63 mph is much higher than the actual speed of 50 mph of the car. This is because the police car is in motion, and not on the side of the road. Indeed, (7.7.1) gives a good approximation of $\frac{dy}{dt}$ if x and $\frac{dx}{dt}$ are small, with equality $\frac{dm}{dt} = \frac{dy}{dt}$ if $x = 0$, but this approximation becomes worse as x and/or $\frac{dx}{dt}$ increase.

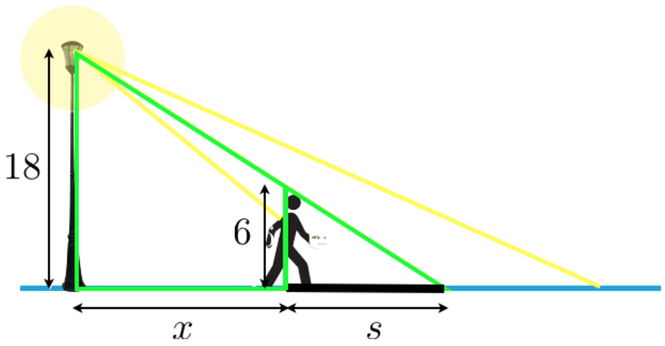
7.8 Related Rates: moving shadow

Watch the video at
https://www.youtube.com/watch?v=Anse8yylLwM&list=PL265CB737C01F8961&index=38.

Abstract This video goes over a problem in which we look for the rate of change of the length of the shadow of a man walking away from a lamppost, given the height of the lamppost, that of the man, and his speed.

Example 7.8.1. If at night a 6 feet tall man is walking away at 3 feet per second from a 18 feet tall lamppost, how fast is the length of his shadow changing?

Solution. We first draw a picture of the situation, and introduce the relevant variables:



Here x is the distance between the man and the lamppost, which changes at the rate

$$\frac{dx}{dt} = 3\,ft/s,$$

and s is the length (in feet) of his shadow. We want to find $\frac{ds}{dt}$. Using similar triangles, we see that

$$\frac{18}{x+s} = \frac{6}{s} \iff 18s = 6x + 6s \iff s = \frac{x}{2}.$$

Thus,

$$\frac{ds}{dt} = \frac{1}{2}\frac{dx}{dt} = \frac{3}{2}\,ft/s.$$

7.9 M6 Sample Quiz 2: Related Rates

1. A car traveling north at 60 mph and a pickup traveling west at 50 mph leave an intersection at the same time. At what rate is the distance between them changing two hours later?
2. A circular oil slick spreads in such a way that its radius is increasing at the rate of 20ft/hour. How fast is the area of the slick changing when the radius is 200 feet?
3. In an adiabatic chemical process, there is no change of heat. If a container of oxygen is subjected to such a process, the pressure P and the volume V satisfy

$$PV^{1.4} = C,$$

where C is a constant. At a certain time, $V = 5\,m^3,\;\; P = 0.6\,kg/m^2$ and P is increasing at $0.23\,kg/m^2$ per second. What is the rate of change of the volume at this instant? Is V increasing or decreasing?

7.10 M6 Sample Quiz 2 Solutions

1. A car traveling north at 60 mph and a pickup traveling west at 50 mph leave an intersection at the same time. At what rate is the distance between them changing two hours later?

   Solution. Let w be the distance between the intersection and the pickup after t hours, and let n be the distance between the intersection and the car after t hours. Finally, let D be the distance between the pickup and the car after t hours. Since the car travels at 60 mph, $\frac{dn}{dt} = 60\,mph$. Since the pickup travels at 50 mph, $\frac{dw}{dt} = 50\,mph$. We are looking for $\frac{dD}{dt}$. Since the triangle whose vertices are the car, the pickup and the intersection is a right triangle (at the intersection), the Pythagorean Theorem applies to the effect that

   $$w^2 + n^2 = D^2.$$

   All 3 quantities are functions of time, and, differentiating with respect to time, we obtain

   $$2w \cdot \frac{dw}{dt} + 2n \cdot \frac{dn}{dt} = 2D \cdot \frac{dD}{dt}.$$

   Solving for the unknown rate, we have

   $$\begin{aligned} \frac{dD}{dt} &= \frac{1}{D}\left(w \cdot \frac{dw}{dt} + n \cdot \frac{dn}{dt}\right) \\ &= \frac{1}{\sqrt{w^2+n^2}}\left(w \cdot \frac{dw}{dt} + n \cdot \frac{dn}{dt}\right). \end{aligned}$$

   After 2 hours, we have $w = 100$ and $n = 120$ so that

   $$\frac{dD}{dt} = \frac{1}{\sqrt{24400}}(100 \cdot 50 + 120 \cdot 60) \sim 78.1\,mph.$$

2. A circular oil slick spreads in such a way that its radius is increasing at the rate of 20 ft/hour. How fast is the area of the slick changing when the radius is 200 feet?

   Solution. Let R denote the radius of the slick at time t, and let A denote its area. We know that $\frac{dR}{dt} = 20\,ft/h$, and we want $\frac{dA}{dt}$ when $R = 200$. Since the are of a disk of radius R is $A = \pi \cdot R^2$, we can differentiate with respect to time to the effect that

   $$\frac{dA}{dt} = 2\pi R \cdot \frac{dR}{dt}.$$

   Thus, when $R = 200$, we have

   $$\frac{dA}{dt} = 2\pi \cdot 200 \cdot 20 \sim 25132.7\,ft^2/h.$$

3. In an adiabatic chemical process, there is no change of heat. If a container of oxygen is subjected to such a process, the pressure P and the volume V satisfy

$$PV^{1.4} = C,$$

where C is a constant. At a certain time, $V = 5\,m^3,\ \ P = 0.6\,kg/m^2$ and P is increasing at $0.23\,kg/m^2$ per second. What is the rate of change of the volume at this instant? Is V increasing or decreasing?

Solution. We know $\frac{dP}{dt} = 0.23$ and we want $\frac{dV}{dt}$. Differentiating $PV^{1.4} = C$ with respect to time, we obtain

$$V^{1.4} \cdot \frac{dP}{dt} + P \cdot 1.4 \cdot V^{0.4} \cdot \frac{dV}{dt} = 0.$$

Solving for the unknown rate:

$$\frac{dV}{dt} = -\frac{V^{1.4}}{1.4PV^{0.4}} \cdot \frac{dP}{dt} = -\frac{V}{1.4P} \cdot \frac{dP}{dt}.$$

Thus, at this particular instant, we have:

$$\frac{dV}{dt} = -\frac{5}{1.4 \times 0.6} \cdot (0.23) \sim -1.37\,m^3/sec.$$

As the rate of change of the volume is negative, we see that the volume is decreasing.

8 Review on modules M4 to M6

8.1 MOCK TEST 2

Instructions: *Do the following test, without your notes, in limited time (75 minutes top). Then grade yourself using the solutions provided separately. It is important that you show all your work and justify your answers. Carefully read the solutions to see how you should justify answers.*

1. [6×5pts] Differentiate the following functions (you do not need to simplify!):
   a) $f(x) = 3x^6 - 2x^3 + \frac{1}{2}x^2 + x - 1$.
   b) $h(t) = (t^3 + t)(3t^{-1} + \sqrt{t} + 1)$.
   c) $y = \frac{x^3-2x+2}{x^2-3x+1}$.
   d) $y = (5 - 2\theta^3)(\cos\theta - \sin\theta)$.
   e) $f(x) = (4x^3 + 2\sqrt{x})^4$.
   f) $f(x) = \sin(\tan x) + \frac{\cos^4 x}{\sin^2(5x)}$.

2. a) [10pts] Find the equation of the tangent line to $y = 2\sin x + 3\cos x$ at $(\frac{\pi}{2}, 2)$.
   b) [10pts] Find the equation of the tangent line to $y = \left(3x^2 - 4x\right)^3$ at (1, –1).

3. a) [10pts] Find $\frac{dy}{dx}$ if
   $$3x^2 + 2y = xy + y^3x^2$$
   b) [10pts] Find the equation of the tangent line to the curve defined by
   $$x^3 - y^3 = 19,$$
   at (3, 2).

4. [10pts] Find the second derivative of $f(x) = \sqrt{3x^2 + 1}$.

5. [5pts]

   When air expands adiabatically (without gaining or losing heat), its pressure *P* and volume V are related by the equation $PV^{1.4} = C$, where *C* is a constant. Suppose that at a certain instant the pressure is $40\ kPa$ and the volume is $200\ cm^3$ and is increasing at a rate of $10cm^3/min$. At what rate is the pressure changing at this instant? Is it increasing or decreasing?

6. [10pts]

   Water is being pumped into an inverted conical tank at a constant rate. The tank has height 8 m and the diameter at the top is 2 m. If the water level is rising at a rate of $0.1m/min$ when the height of the water is 2 m, find the rate at which is water is being pumped into the tank.

7. [10pts] A boat is being pulled towards a dock by pulling a rope attached to the bow of the boat, through a pulley on the dock that is 5 meters above the level of the bow. If the rope is pulled at 1 meter per second, at what speed is the boat approaching the dock when the boat is 10 meters away from the dock?

8.2 MOCK TEST 2 Solutions

1. Differentiate the following functions (you do not need to simplify!):

a) $f(x) = 3x^6 - 2x^3 + \frac{1}{2}x^2 + x - 1$.

Solution.

$$f'(x) = 18x^5 - 6x^2 + x + 1.$$

b) $h(t) = (t^3 + t)(3t^{-1} + \sqrt{t} + 1)$.

Solution.

$$h'(t) = \left(3t^2 + 1\right)\left(3t^{-1} + \sqrt{t} + 1\right) + \left(t^3 + t\right)\left(-\frac{3}{t^2} + \frac{1}{2\sqrt{t}}\right).$$

c) $y = \frac{x^3-2x+2}{x^2-3x+1}$.

Solution.

$$\frac{dy}{dx} = \frac{\left(3x^2 - 2\right)\left(x^2 - 3x + 1\right) - (2x - 3)\left(x^3 - 2x + 2\right)}{(x^2 - 3x + 1)^2}.$$

d) $y = (5 - 2\theta^3)(\cos\theta - \sin\theta)$.

Solution.

$$\frac{dy}{dx} = -6\theta^2\left(\cos\theta - \sin\theta\right) + \left(5 - 2\theta^3\right)\left(-\sin\theta - \cos\theta\right).$$

e) $f(x) = (4x^3 + 2\sqrt{x})^4$.

Solution.

$$f'(x) = 4\left(4x^3 + 2\sqrt{x}\right)^3\left(12x^2 + \frac{1}{\sqrt{x}}\right).$$

f) $f(x) = \sin(\tan x) + \frac{\cos^4 x}{\sin^2(5x)}$.

Solution.

$$\begin{aligned} f'(x) &= \cos(\tan x)\cdot\sec^2 x + \frac{4\cos^3 x\cdot(-\sin x)\cdot\sin^2(5x) - 2\sin(5x)\cdot\cos(5x)\cdot 5\cdot\cos^4 x}{\sin^4(5x)} \\ &= \cos(\tan x)\cdot\sec^2 x - \frac{4\cos^3 x\sin x\sin^2(5x) + 10\sin(5x)\cos(5x)\cos^4 x}{\sin^4(5x)}. \end{aligned}$$

2. a) Find the equation of the tangent line to $y = 2\sin x + 3\cos x$ at $(\frac{\pi}{2}, 2)$.

   Solution. The tangent line is the line through $(\frac{\pi}{2}, 2)$ of slope $\frac{dy}{dx}_{|x=\frac{\pi}{2}}$. Since

   $$\frac{dy}{dx} = 2\cos x - 3\sin x,$$

   we conclude that the slope is $m = 2 \cdot 0 - 3 \cdot 1 = -3$, so that an equation of the line is

   $$y - 2 = -3\left(x - \frac{\pi}{2}\right).$$

   b) Find the equation of the tangent line to $y = \left(3x^2 - 4x\right)^3$ at $(1, -1)$.
   Solution. The tangent line is the line through (1, –1) of slope $\frac{dy}{dx}_{|x=1}$. Since

   $$\frac{dy}{dx} = 4\left(3x^2 - 4x\right)^2 \cdot (6x - 4),$$

   we conclude that the slope is $m = 4 \cdot (-1)^2 \cdot (2) = 8$, so that an equation of the line is

   $$y + 1 = 8(x - 1).$$

3. a) Find $\frac{dy}{dx}$ if

   $$3x^2 + 2y = xy + y^3x^2.$$

   Solution. Differentiating implicitly with respect to *x*, we obtain:

   $$6x + 2\frac{dy}{dx} = y + x\frac{dy}{dx} + 3y^2x^2\frac{dy}{dx} + 2y^3x.$$

   Solving for $\frac{dy}{dx}$ gives:

   $$\begin{aligned} \frac{dy}{dx}\left(2 - x - 3x^2y^2\right) &= y + 2y^3x - 6x \\ \frac{dy}{dx} &= \frac{y + 2y^3x - 6x}{2 - x - 3x^2y^2}. \end{aligned}$$

   b) Find the equation of the tangent line to the curve defined by

   $$x^3 - y^3 = 19,$$

   at (3, 2).

Solution. The tangent line is the line through (3, 2) of slope $\frac{dy}{dx}_{|x=3;y=2}$. Differentiating implicitly with respect to x, we have:

$$\begin{aligned} 3x^2 - 3y^2\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{3x^2}{3y^2} = \left(\frac{x}{y}\right)^2 \end{aligned}$$

so that the slope is $m = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$ and an equation of the line is

$$y - 2 = \frac{9}{4}(x-3).$$

4. Find the second derivative of $f(x) = \sqrt{3x^2+1}$.

Solution. The first derivative is (using the Chain Rule):

$$f'(x) = \frac{6x}{2\sqrt{3x^2+1}} = 3x\left(3x^2+1\right)^{-\frac{1}{2}}.$$

To find the second derivative, we use the product rule combined with the Chain rule, to the effect that

$$\begin{aligned} f''(x) &= 3\left(3x^2+1\right)^{-\frac{1}{2}} - \frac{3x}{2}\left(3x^2+1\right)^{-\frac{3}{2}} \cdot 6x \\ &= \frac{3}{\sqrt{3x^2+1}}\left(1-\frac{3x^2}{3x^2+1}\right). \end{aligned}$$

5. When air expands adiabatically (without gaining or losing heat), its pressure *P* and volume *V* are related by the equation $PV^{1.4} = C$, where *C* is a constant. Suppose that at a certain instant the pressure is 40 *kPa* and the volume is $200\ cm^3$ and is increasing at a rate of $10cm^3/min$. At what rate is the pressure changing at this instant? Is it increasing or decreasing?

   Solution. Both pressure *P* and volume *V* are functions of time *t*. Differentiating with respect to *t*, we have:

   $$\frac{dP}{dt} \cdot V^{1.4} + P \cdot (1.4)V^{0.4} \cdot \frac{dV}{dt} = 0,$$

   so that, solving for the unknown rate $\frac{dP}{dt}$, we obtain:

   $$\begin{aligned} \frac{dP}{dt} &= -1.4\frac{PV^{0.4}}{V^{1.4}} \cdot \frac{dV}{dt} \\ &= -1.4\frac{P}{V} \cdot \frac{dV}{dt} \\ &= -1.4\frac{40}{200} \cdot 10 = -2.8\,kPa/min \end{aligned}$$

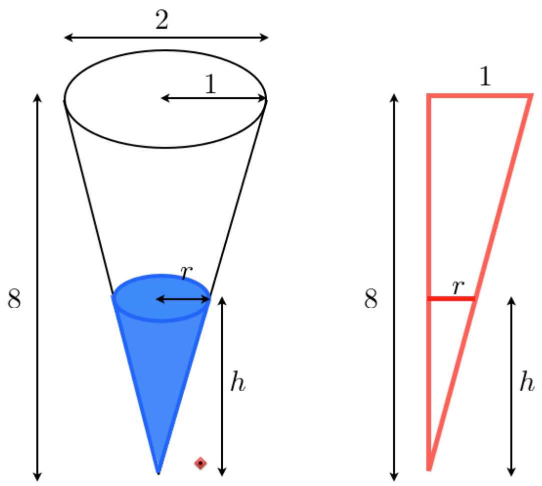
   so that the pressure is decreasing.

6. Water is being pumped into an inverted conical tank at a constant rate. The tank has height 8 *m* and the diameter at the top is 2 *m*. If the water level is rising at a rate of 0.1*m/min* when the height of the water is 2 *m*, find the rate at which is water is being pumped into the tank.

   Solution. The depth of water *h* is increasing at a given rate $\frac{dh}{dt} = 0.1\,m/min$ when *h* = 2*m*. We look for the rate of change $\frac{dV}{dt}$ of the volume *V* of water at that instant. The cone of water inside the tank has volume

   $$V = \frac{1}{3}\pi r^2 h,$$

   where *r* is the radius of the base of the cone. Using similar triangles as shown below, the ratio of the radius at the base of the tank (which is 1 meter) to the radius *r* is the same as the ratio of heights: $\frac{8}{h}$.



Thus $r = \frac{h}{8}$ so that

$$V = \frac{1}{3}\pi\frac{h^2}{64}h = \frac{\pi}{192}h^3.$$

In this expression, both V and h are functions of the time t. Differentiating with respect to t we obtain:

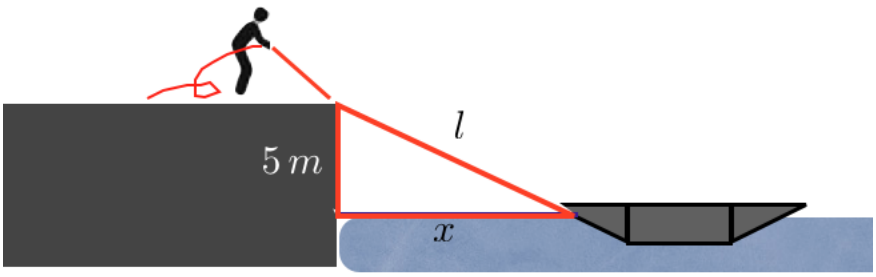
$$\frac{dV}{dt} = \frac{\pi}{192} \cdot 3h^2\frac{dh}{dt},$$

so that at the instant considered

$$\frac{dV}{dt} = \frac{\pi}{64}(2)^2 \cdot 0.1 = \frac{\pi}{160}\, m^3/min.$$

7. A boat is being pulled towards a dock by pulling a rope attached to the bow of the boat, through a pulley on the dock that is 5 meters above the level of the bow. If the rope is pulled at 1 meter per second, at what speed is the boat approaching the dock when the boat is 10 meters away from the dock?

Solution. The length l of rope between the pulley and the boat is decreasing at $\frac{dl}{dt} = -1\, m/s$. The distance x between the dock and the boat is decreasing as a result, at a rate $\frac{dx}{dt}$ to be determined, which is the speed at which the boat is approaching the dock:



By the Pythagorean Theorem

$$x^2 + 25 = l^2,$$

so that, differentiating with respect to time, we obtain:

$$2x\frac{dx}{dt} = 2l\frac{dl}{dt}.$$

Solving for the unknown rate $\frac{dx}{dt}$, we obtain

$$\begin{aligned} \frac{dx}{dt} &= \frac{l}{x}\frac{dl}{dt} \\ &= \frac{\sqrt{x^2+25}}{x} \cdot \frac{dl}{dt}. \end{aligned}$$

Thus, when the boat is 10 meters away from the dock ($x = 10$), we have

$$\frac{dx}{dt} = \frac{\sqrt{125}}{10} \cdot (-1) \approx -1.12\, m/s.$$

9 M7: Extreme Values of a function

9.1 Extrema

Watch the video at
https://www.youtube.com/watch?v=ORi1_m60RB8&list=PL265CB737C01F8961&index=39.

Abstract This video introduces the concepts of absolute maximum and *absolute minimum* of a function on an interval and states the Extreme Value Theorem.

Definition 9.1.1. Let f be a function defined on an interval D. We say that f has an *absolute maximum* on D at c, and that $f(c)$ is the *absolute maximum* of f on D if

$$f(c) \geq f(x) \text{ for all } x \in D.$$

Definition 9.1.2. Let f be a function defined on an interval D. We say that f has an *absolute minimum* on D at c, and that $f(c)$ is the *absolute minimum* of f on D if

$$f(c) \leq f(x) \text{ for all } x \in D.$$

An *absolute extremum* is either an absolute maximum or an absolute minimum.

Note (see video) that absolute extrema may or may not exist, depending on the function, and on the interval on which we consider the function. However, we have:

Theorem 9.1.3 (Extreme Value Theorem) *A continuous function on a closed interval attains both an absolute minimum and an absolute maximum on this interval.*

9.2 local extrema and critical values

Watch the video at
https://www.youtube.com/watch?v=RVi_k3wDZMc&list=PL265CB737C01F8961&index=40.

Abstract This video defines local extrema and critical values and establishes Fermat's theorem relating them. A number of examples are treated to find critical values of a function.

Definition 9.2.1. $f(c)$ is a *local maximum of f* if $f(c) \geq f(x)$ for all x in an open interval containing c. We say that *f has a local maximum at c.*

$f(c)$ is a local minimum of f if $f(c) \leq f(x)$ for all x in an open interval containing c. We say that *f has a local minimum at c.*

A local *extremum* is either a local maximum or local minimum.

Definition 9.2.2. A number c in the domain of a function f is a *critical value of f* if the derivative of f at c is either 0 or undefined.

Theorem 9.2.3 (Fermat) *If f has a local extremum at c then c is a critical value of f.*

The converse is false: not every critical value corresponds to a local extremum.

For instance $f(x) = x^3$ has one critical value (0) because

$$f'(x) = 3x^2 = 0 \iff x = 0$$

but it has no local extremum.

Example 9.2.4. Find the critical values of the following functions:

1. $f(x) = \frac{2x^2}{x+2}$;

   Solution. This is a rational function. Thus it is differentiable on its domain (Note that –2 is not in the domain, and therefore cannot be critical). Moreover

   $$f'(x) = \frac{4x(x+2) - 2x^2}{(x+2)^2} = \frac{2x(x+4)}{(x+2)^2},$$

   so that

   $$f'(x) = 0 \iff x = 0 \text{ or } x = -4,$$

   which are the critical values of *f*.

2. $f(x) = -x^2 + 4x + 2$;

   Solution. The function is polynomial, hence differentiable on the real line. Moreover

   $$f'(x) = -2x + 4$$

   so that

   $$f'(x) = 0 \iff x = 2,$$

   which is the only critical value of *f*.

3. $f(x) = x^4 + 6x^2 - 2$;

Solution. The function is polynomial, hence differentiable on the real line. Moreover

$$f'(x) = 4x^3 + 12x = 4x(x^2+3) = 4x(x-\sqrt{3})(x+\sqrt{3}),$$

so that the critical values of *f* are $-\sqrt{3}$, 0 and $\sqrt{3}$.

4. $f(x) = \frac{x^2-x+4}{x-1}$;

Solution. This is a rational function. Thus it is differentiable on its domain. Moreover

$$\begin{aligned} f'(x) &= \frac{(2x-1)(x-1)-(x^2-x+4)}{(x-1)^2}, \\ &= \frac{(x+1)(x-3)}{(x-1)^2}, \end{aligned}$$

so that the critical values of *f* are –1 and 3.

5. $f(x) = x^{\frac{7}{3}} - 28x^{\frac{1}{3}}$.

Solution. This function is defined on the real line, but not differentiable at 0. Thus 0 is critical. Moreover, for $x \neq 0$, we have

$$f'(x) = \frac{7}{3}x^{\frac{4}{3}} - \frac{28}{3}x^{-\frac{2}{3}} = \frac{7}{3}x^{-\frac{2}{3}}\left(x^2-4\right) = \frac{7}{3}x^{-\frac{2}{3}}(x-2)(x+2).$$

Thus the critical values of *f* are –2, 0 and 2.

9.3 Closed Interval Method

Watch the video at
https://www.youtube.com/watch?v=LkeRk2A57SI&list=PL265CB737C01F8961&index=41.

Abstract This video discusses the Closed Interval Method to find absolute extrema of a continuous function on a closed interval.

Theorem 9.3.1 (Closed Interval Method) *Let f be a continuous function on a closed interval* $[a, b]$. *Then the absolute minimum and maximum exist and occur either at a critical value of f in (a, b), or at an endpoint (a or b).*

Thus to find the absolute extrema, you need to find the critical values of f **in** $[a, b]$ and evaluate f at a, at b and at these critical values. The largest value in that list is the absolute maximum; the smallest is the absolute minimum.

Example 9.3.2. Find the absolute maximum and minimum of the following functions on the specified interval:

1. $f(x) = x^3 - 3x + 1$ on [–3, 2];

   Solution. f is continuous on a closed interval. Since

   $$f'(x) = 3x^2 - 3 = 3(x-1)(x+1),$$

   the critical values of *f* in the interval are –1 and 1. Moreover

   $$f(-3) = -17\,;\ f(-1) = 3\,;\ f(1) = -1\,;\ f(2) = 3,$$

   so that the absolute minimum is –17 and occurs at –3, and the absolute maximum is 3 and occurs at –1 and at 2.

2. $f(x) = 2x^3 + 3x^2 + 4$ on [–2, 1];

   Solution. f is continuous on a closed interval. Since

   $$f'(x) = 6x^2 + 6x = 6x(x+1),$$

   the critical values of *f* in the interval are –1 and 0. Moreover

   $$f(-2) = 0\,;\ f(-1) = 5\,;\ f(0) = 4\,;\ f(1) = 9,$$

   so that the absolute minimum is 0 and occurs at –2, and the absolute maximum is 9 and occurs at 1.

3. $f(x) = \frac{x}{x^2+1}$ on [0, 2];

   Solution. f is continuous on a closed interval, because $x^2 + 1 \neq 0$ for all *x*. Since

   $$f'(x) = \frac{x^2 + 1 - 2x^2}{(x^2+1)^2} = \frac{(1-x)(1+x)}{(x^2+1)^2},$$

   the critical values of *f* are –1 and 1, but only 1 lies in the interval. Moreover

   $$f(0) = 0\,;\ f(1) = \frac{1}{2}\,;\ f(2) = \frac{2}{5},$$

   so that the absolute minimum is 0 and occurs at 0, and the absolute maximum is $\frac{1}{2}$ and occurs at 1.

4. $f(x) = x^{\frac{2}{3}}$ on [–1, 3];

Solution. *f* is continuous on a closed interval. Since

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}},$$

f is not differentiable at 0, which is the only critical value. Moreover

$$f(-1) = 1\,;\; f(0) = 0\,;\; f(3) = \sqrt[3]{9} \approx 2.1,$$

so that the absolute minimum is 0 and occurs at 0, and the absolute maximum is $\sqrt[3]{9}$ and occurs at 3.

5. $f(x) = x^{\frac{2}{3}}$ *on* [–4, –2].

Solution. *f* is continuous on a closed interval, but has no critical value in [–4, –2], because 0 does not lie in this interval. Moreover

$$f(-4) = \sqrt[3]{16} \approx 2.5\,;\; f(-2) = \sqrt[3]{4} \approx 1.6,$$

so that the absolute minimum is $\sqrt[3]{4}$ and occurs at –2, and the absolute maximum is $\sqrt[3]{16}$ and occurs at –4.

9.4 M7 Sample Quiz

1. Find all the critical values for the following functions:

   a) $f(x) = 2x^3 - 3x^2 - 12x + 5$

   b) $f(x) = x - \sqrt{x}$

2. Find the absolute maximum and absolute minimum of the following functions on the specified interval:

   a) $f(x) = 2x^3 + 3x^2 - 36x + 1$ on [−2, 3].

   b) $f(x) = \frac{x+1}{x^2+3}$ on [0, 2].

9.5 M7 Sample Quiz Solutions

1. Find all the critical values for the following functions:

   a) $f(x) = 2x^3 - 3x^2 - 12x + 5$

   Solution.

   $$f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x+1)(x-2),$$

   so that the critical values of f are -1 and 2.

   b) $f(x) = x - \sqrt{x}$

   Solution.
   $$f'(x) = 1 - \frac{1}{2\sqrt{x}},$$

   so that 0 is critical because f is not differentiable at 0, even though 0 is in the domain of f. Moreover,

   $$\begin{aligned} f'(x) = 0 \quad &\Longleftrightarrow \quad 1 - \frac{1}{2\sqrt{x}} = 0 \\ &\Longleftrightarrow \quad \sqrt{x} = \frac{1}{2} \\ &\Longleftrightarrow \quad x = \frac{1}{4}. \end{aligned}$$

   Thus the critical values of f are 0 and $\frac{1}{4}$.

2. Find the absolute maximum and absolute minimum of the following functions on the specified interval:

   a) $f(x) = 2x^3 + 3x^2 - 36x + 1$ on $[-2, 3]$.

   Solution.

   $$f'(x) = 6x^2 + 6x - 36 = 6(x^2 + x - 6) = 6(x-2)(x+3),$$

so that –3 and 2 are critical. But only 2 belongs to [–2, 3]. Moreover

$$\begin{aligned} f(-2) &= 69 \\ f(2) &= -43 \\ f(3) &= -26 \end{aligned}$$

so that the absolute maximum is 69 and occurs at –2 and the absolute minimum is –43 and occurs at 2.

b) $f(x) = \frac{x+1}{x^2+3}$ on [0, 2].

Solution.

$$\begin{aligned} f'(x) &= \frac{x^2+3-2x(x+1)}{(x^2+3)^2} \\ &= \frac{-x^2-2x+3}{(x^2+3)^2} \\ &= \frac{-(x-1)(x+3)}{(x^2+3)^2} \end{aligned}$$

so that the critical values are –3 and 1. Only 1 belongs to [0, 2]. Since,

$$\begin{aligned} f(0) &= \frac{1}{3} \\ f(1) &= \frac{1}{2} \\ f(2) &= \frac{3}{7} \end{aligned}$$

we see that the absolute maximum is $\frac{1}{2}$ and occurs at 1, while the absolute minimum is $\frac{1}{3}$ and occurs at 0.

10 M8: the Mean Value Theorem and first derivative Test

10.1 Rolle's Theorem

Watch the video at
https://www.youtube.com/watch?v=K1nEzRGF0aU&list=PL265CB737C01F8961&index=42.

Abstract This video presents Rolle' s Theorem and its proof, shows that the assumptions are all essentials, and examines examples.

Theorem 10.1.1 (Rolle) *Suppose that f is continuous on* $[a, b]$*, differentiable on* (a, b)*, and that* $f(a) = f(b)$. *Then there exists c in* (a, b) *with*

$$f'(c) = 0.$$

10.2 The Mean Value Theorem

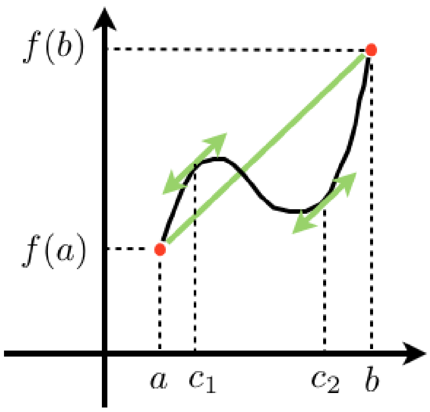
Watch the video at
https://www.youtube.com/watch?v=KbsICOVX4gA&list=PL265CB737C01F8961&index=43.

Abstract This video presents the Mean Value Theorem and its proof, and shows that the assumptions are essentials.

Theorem 10.2.1 (Mean Value Theorem) *Suppose f is continuous on* $[a, b]$ *and differentiable on* (a, b). *Then there exists c in* (a, b) *with*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

In other words, under these assumptions, the instantaneous rate of change of f takes as a value the average rate of change of f over the interval.



10.3 Applications of the Mean Value Theorem

Watch the video at
https://www.youtube.com/watch?v=gAB5mUHeB0A&list=PL265CB737C01F8961&index=44.

Abstract This video examines a few examples of applications of the Mean Value Theorem.

Proposition 10.3.1. *If f is continuous on* [*a*, *b*], *differentiable on* (*a*, *b*) *and the equation* $f(x)=0$ *has two solutions in* [*a*, *b*], *then the equation* $f'(x)=0$ *has at least one solution in* (*a*, *b*).

Example 10.3.2. Show that the equation

$$2x^5+3x^3+2x-5=0$$

has a unique solution.

Solution. The **existence** of solutions follows from the Intermediate Value Theorem 3.3.1. Indeed, the function $f(x)=2x^5+3x^3+2x-5$ is continuous on [0, 1] and

$$f(0)=-5<0<f(1)=2,$$

so that the Intermediate Value Theorem applies to the effect that there exists *c* in (0, 1) with $f(c)=0$.

The **uniqueness** of solution follows from Proposition 10.3.1. Indeed, if there were two different solutions to $f(x)=0$, then there would be at least a solution to $f'(x)=0$. But

$$f'(x)=10x^4+9x^2+2\geq 2,$$

so that $f'(x)=0$ has no solution. Thus $f(x)=0$ has at most one solution.

More generally:

Proposition 10.3.3. *If f is continuous on* [*a*, *b*], *differentiable on* (*a*, *b*) *and the equation* $f(x)=0$ *has n solutions in* [*a*, *b*], *then the equation* $f'(x)=0$ *has at least* (*n* – 1) *solution in* (*a*, *b*).

Example 10.3.4. Let *f* be a function that is differentiable on $(-\infty,\infty)$, and satisfies $f(1)=10$ and $f'(x)\geq 2$ for all *x* in (1, 4). How small can $f(4)$ be?

Solution. Since the function is differentiable on $(-\infty, \infty)$, it is also continuous on this interval, and thus so is its restriction to [1, 4], and it is differentiable on (1, 4). Thus the Mean Value Theorem applies to *f* on [1, 4] to the effect that there is $c \in (1, 4)$ such that

$$f'(c) = \frac{f(4) - f(1)}{3} \iff f(4) = 3f'(c) + f(1).$$

Since $f'(c) \geq 2$, we conclude that

$$f(4) \geq 3 \times 2 + 10 = 16.$$

Example 10.3.5. Find, if possible, a function *f* differentiable on $(-\infty, \infty)$ such that $f(0) = -1$, $f(2) = 4$ and $f'(x) \leq 2$ for all *x*.

Solution. Such a function does not exist, for otherwise, it would be continuous on [0, 2] and differentiable on (0, 2), so that, by the Mean Value Theorem, there would be $c \in (0, 2)$ such that

$$f'(c) = \frac{f(2) - f(0)}{2} = \frac{5}{2} > 2$$

which is not compatible with the assumption that $f'(x) \leq 2$ for all *x*.

Example 10.3.6. A *fixed point* of a function *f* is a number *a* such that $f(a) = a$. Find, if possible, a differentiable function *f* with at least two fixed points that satisfies $f'(x) \neq 1$ for all *x*.

Solution. Such a function does not exist. Indeed, if *s* < *t* are two different fixed points of *f*, then by the Mean Value Theorem applied to *f* on [*s, t*] (which applies because *f* is supposed differentiable on $(-\infty, \infty)$, hence continuous on [*s, t*] and differentiable on (*s, t*)), there would be *c* in (*s, t*)) such that

$$f'(c) = \frac{f(t) - f(s)}{t - s} = \frac{t - s}{t - s} = 1,$$

in contradiction to the assumption that $f'(x) \neq 1$ for all *x*.

10.4 M8 Sample Quiz 1: Mean Value Theorem

1. Does Rolle' s Theorem apply to the function

$$f(x) = \frac{1}{x^2} + 1$$

on [–1, 1]? Explain. If yes, give all the values c that satisfy the conclusion of Rolle' s Theorem.

2. Does the Mean Value Theorem apply to

$$f(x) = |x - 2|$$

on [0, 3]? Explain. If yes, give all the values c that satisfy the conclusion of the Mean Value Theorem.

3. Does the Mean Value Theorem apply to

$$f(x) = 3x^2 + 2x + 5$$

on [–1, 1]? Explain. If yes, give all the values c that satisfy the conclusion of the Mean Value Theorem.

4. If $f(3) = 5$ and $f'(x) \geq 3$ for every x in [3, 9], how small can $f(9)$ be?
5. Justify that the equation $3x^5 + 2x^3 + 3x - 5 = 0$ has a unique solution.

10.5 M8 Sample Quiz 1 Solutions

1. Does Rolle' s Theorem apply to the function

$$f(x) = \frac{1}{x^2} + 1$$

on [–1, 1]? Explain. If yes, give all the values *c* that satisfy the conclusion of Rolle' s Theorem.

Solution. The function *f* has a discontinuity at $0 \in [-1, 1]$. Thus *f* does *not* satisfy the assumptions of Rolle' s theorem.

2. Does the Mean Value Theorem apply to

$$f(x) = |x - 2|$$

on [0, 3]? Explain. If yes, give all the values *c* that satisfy the conclusion of the Mean Value Theorem.

Solution. The function *f* is not differentiable at $2 \in [0, 3]$. Thus *f* does *not* satisfy the assumptions of the Mean Value Theorem.

3. Does the Mean Value Theorem apply to

$$f(x) = 3x^2 + 2x + 5$$

on [–1, 1]? Explain. If yes, give all the values *c* that satisfy the conclusion of the Mean Value Theorem.

Solution. The function *f* is differentiable on the real line, so that its restriction to [–1, 1] is continuous and differentiable on (–1, 1). Thus the Mean Value Theorem applies to *f* on [–1, 1] to the effect that there is $c \in (-1, 1)$ such that

$$f'(c) = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{10 - 6}{2} = 2.$$

Since $f'(x) = 6x + 2$, values *c* satisfying the conclusion of the theorem are solutions of $6x + 2 = 2$, that is, *x* = 0.

4. If $f(3)=5$ and $f'(x)\geq 3$ for every *x* in [3, 9], how small can $f(9)$ be?

   Solution. Since f' is defined on [3, 9], *f* is continuous on [3, 9] and differentiable on (3, 9) so that the Mean Value Theorem applies to *f* on [3, 9] to the effect that there is $c\in(3,9)$ with

   $$f'(c)=\frac{f(9)-f(3)}{9-3}=\frac{f(9)-5}{6}$$

   so that $f(9)=6f'(c)+5$. Since $f'(c)\geq 3$, we conclude that $f(9)\geq 6\times 3+5=23$.

5. Justify that the equation $3x^5+2x^3+3x-5=0$ has a unique solution.

   Solution. Let $f(x)=3x^5+2x^3+3x-5$. We want to justify that $f(x)=0$ has a unique solution. Note that

   $$f(0)=-5<0<f(1)=3$$

   and that *f* is continuous on [0, 1]. By the Intermediate Value Theorem, there is (at least) one solution to $f(x)=0$ in the interval (0, 1). Moreover, if the equation had two different solutions, then the equation $f'(x)=0$ would have at least one solution. But

   $$f'(x)=15x^4+6x^2+3\geq 3$$

   does not take the value 0. Hence $f(x)=0$ has a unique solutions.

10.6 Intervals of increase and decrease

Watch the videos at
https://www.youtube.com/watch?v=n5SSxOmiPek&list=PL265CB737C01F8961&index=45

and
https://www.youtube.com/watch?v=EhGfkxCiDtQ&list=PL265CB737C01F8961&index=46

Abstract These two videos define increasing and decreasing functions, present a criterion of increase or decrease on an interval for differentiable functions, describes how to find the intervals of increase and decrease of a function on an example, and examine the consequences in terms of local extrema, culminating in the statement of the First Derivative Test.

Definition 10.6.1. Let I be an open interval. A function f is *increasing on* I if for every x_1, x_2 in I

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2);$$

and *decreasing on* I if for every x_1, x_2 in I

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2):$$

Theorem 10.6.2. *Let I be an open interval, and let f be a differentiable function on I. Then:*

1. *If* $f'(x) > 0$ *for all x in I, then f is constant on I;*
2. *If* $f'(x) < 0$ *for all x in I, then f is constant on I;*
3. *If* $f'(x) = 0$ *for all x in I, then f is constant on I.*

Example 10.6.3. Find the intervals of increase and decrease of $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$.

Solution. This depends on the sign of

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x+1)(x-2).$$

The sign of the product depends on the sign of each factor, and can be obtained via a sign chart:

x	$(-\infty,-1)$	$(-1,0)$	$(0,2)$	$(2,\infty)$
$12x$	$-$	$-$	$+$	$+$
$x+1$	$-$	$+$	$+$	$+$
$x-2$	$-$	$-$	$-$	$+$
$f'(x)$	$-$	$+$	$-$	$+$
$f(x)$	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$

Thus, *f* is decreasing on $(-\infty, -1)$ and on (0, 2) and is increasing on (–1, 0) and on $(2, \infty)$.

The chart visually indicates that the function *f* has a local minimum at –1, which is $f(-1) = 0$, and at 2, which is $f(2) = -27$, and that *f* has a local maximum at 0, which is $f(0) = 5$. This is formalized in the Theorem below:

Theorem 10.6.4 (First Derivative Test) *Let c be a critical value of a differentiable function f.*

1. *If* f' *changes from positive to negative at c, then f has a local maximum at c;*
2. *If* f' *changes from negative to positive at c, then f has a local minimum at c;*
3. *If* f' *does not change sign at c, then f has no local extremum at c.*

10.7 First Derivative Test: further examples

Watch the video at
https://www.youtube.com/watch?v=CvfOHIRuytA&list=PL265CB737C01F8961&index=47.

Abstract This video goes over examples of finding intervals of increase and decrease, as well as local extrema, for various functions.

Example 10.7.1. Find the intervals of increase and decrease, and the local extrema, for the function

$$f(x) = x^3 - 12x + 1.$$

Solution. The intervals of increase and decrease depend on the sign of

$$f'(x) = 3x^2 - 12 = 3(x-2)(x+2),$$

so that the sign is obtained in the following sign chart:

x	$(-\infty,-2)$	$(-2,2)$	$(2,\infty)$
$x-2$	$-$	$-$	$+$
$x+2$	$-$	$+$	$+$
$f'(x)$	$+$	$-$	$+$
$f(x)$	$\nearrow$	$\searrow$	$\nearrow$

Thus f is increasing on $(-\infty,-2)$ and on $(2,\infty)$, and is decreasing on $(-2,2)$. It has a local maximum at −2, which is $f(-2) = 17$, and a local minimum at 2, which is $f(2) = -15$.

Example 10.7.2. Find the intervals of increase and decrease, and the local extrema, for the function

$$f(x) = x^4 - 6x^2.$$

Solution. The intervals of increase and decrease depend on the sign of

$$f'(x) = 4x^3 - 12x = 4x(x^2-3) = 4x(x-\sqrt{3})(x+\sqrt{3}),$$

so that the sign is obtained in the following sign chart:

x	$(-\infty,-\sqrt{3})$	$(-\sqrt{3},0)$	$(0,\sqrt{3})$	$(\sqrt{3},\infty)$
$4x$	$-$	$-$	$+$	$+$
$x-\sqrt{3}$	$-$	$-$	$-$	$+$
$x+\sqrt{3}$	$-$	$+$	$+$	$+$
$f'(x)$	$-$	$+$	$-$	$+$
$f(x)$	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$

Thus f is decreasing on $(-\infty, -\sqrt{3})$ and on $(0, \sqrt{3})$, and is increasing on $(-\sqrt{3}, 0)$ and on $(\sqrt{3}, \infty)$. It has a local minimum at $-\sqrt{3}$ and at $\sqrt{3}$, which is $f(\sqrt{3}) = f(-\sqrt{3}) = -9$; and it has a local maximum at 0 which is $f(0) = 0$.

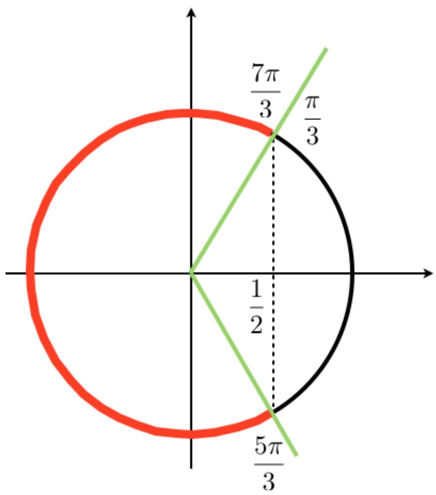
Example 10.7.3. Find the intervals of increase and decrease, and the local extrema, for the function

$$f(x) = x - 2\sin x \text{ for } x \in [0, 3\pi].$$

Solution. The intervals of increase and decrease depend on the sign of

$$f'(x) = 1 - 2\cos x.$$

Note that $f'(x) > 0 \iff \cos x < \frac{1}{2}$. Inspecting the trigonometric circle



we obtain the following sign chart:

x	$(0, \frac{\pi}{3})$	$(\frac{\pi}{3}, \frac{5\pi}{3})$	$(\frac{5\pi}{3}, \frac{7\pi}{3})$	$(\frac{7\pi}{3}, 3\pi)$
$f'(x)$	$-$	$+$	$-$	$+$
$f(x)$	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$

Thus f is decreasing on $(0, \frac{\pi}{3})$ and on $(\frac{5\pi}{3}, \frac{7\pi}{3})$ and is increasing on $(\frac{\pi}{3}, \frac{5\pi}{3})$ and on $(\frac{7\pi}{3}, 3\pi)$. Moreover, f has a local minimum at $\frac{\pi}{3}$ which is $f(\frac{\pi}{3}) = \frac{\pi}{3} - \sqrt{3}$ and at $\frac{7\pi}{3}$, which is $f(\frac{7\pi}{3}) = \frac{7\pi}{3} - \sqrt{3}$. It has a local maximum at $\frac{5\pi}{3}$ which is $f(\frac{5\pi}{3}) = \frac{5\pi}{3} + \sqrt{3}$.

10.8 M8 Sample Quiz 2: Intervals of increase and decrease

1. Find the intervals of increase and decrease and the local extrema of the function
$$f(x) = 2x^3 + 3x^2 - 36x + 6.$$
2. Find the interval of increase and decrease and the local extrema of the function
$$f(x) = \frac{2x-4}{x+3}.$$
3. Find the intervals of increase and decrease and the local extrema of the function
$$f(x) = \frac{x^3}{x^2+x-2}.$$
4. Find the intervals of increase and decrease and the local extrema of the function
$$f(x) = x - 2\cos x$$
on the interval $[0, 2\pi]$.

10.9 M8 Sample Quiz 2 Solutions

1. Find the intervals of increase and decrease and the local extrema of the function

$$f(x) = 2x^3 + 3x^2 - 36x + 6.$$

Solution. The derivative of *f* is

$$f'(x) = 6x^2 + 6x - 36 = 6(x^2 + x - 6) = 6(x-2)(x+3).$$

Thus the sign of f' depends on the sign of $x - 2$ and the sign of $x + 3$:

x	$(-\infty, -3)$	$(-3, 2)$	$(2, \infty)$
$x-2$	$-$	$-$	$+$
$x+3$	$-$	$+$	$+$
f'	$+$	$-$	$+$
f	$\nearrow$	$\searrow$	$\nearrow$

In other words, *f* is increasing on $(-\infty, -3)$ and on $(2, \infty)$ and decreasing on $(-3, 2)$. As a consequence, *f* has a local maximum at $x = -3$, which is $f(-3) = 87$, and a local minimum at $x = 2$, which is $f(2) = -38$.

2. Find the interval of increase and decrease and the local extrema of the function

$$f(x) = \frac{2x-4}{x+3}.$$

Solution. *f* is defined on $(-\infty, -3) \cup (-3, \infty)$ and has a vertical asymptote $x = -3$. Its derivative is

$$f'(x) = \frac{2(x+3) - 1(2x-4)}{(x+3)^2} = \frac{10}{(x+3)^2} > 0.$$

Thus $f'(x) > 0$ whenever f' is defined:

x	$(-\infty, -3)$	$(-3, \infty)$
f'	$+$	$+$
f	$\nearrow$	$\nearrow$

Thus *f* is increasing on $(-\infty, -3)$ and on $(-3, \infty)$ and has no local extremum.

3. Find the intervals of increase and decrease and the local extrema of the function

$$f(x) = \frac{x^3}{x^2+x-2}.$$

Solution. The domain of *f* is $(-\infty,-2)\cup(-2,1)\cup(1,\infty)$ because $x^2+x-2=(x-1)(x+2)$. Since $x^3 \neq 0$ for *x* = –2 and *x* = 1, the function *f* has 2 vertical asymptotes *x* = –2 and *x* = 1. Its derivative is

$$\begin{aligned} f'(x) &= \frac{3x^2(x^2+x-2)-(2x+1)x^3}{(x^2+x-2)^2} \\ &= \frac{3x^4+3x^3-6x^2-2x^4-x^3}{(x^2+x-2)^2} \\ &= \frac{x^4+2x^3-6x^2}{(x^2+x-2)^2} \\ &= \frac{x^2(x^2+2x-6)}{(x^2+x-2)^2}. \end{aligned}$$

Since x^2 and $(x^2+x-2)^2$ are non-negative, the sign of f' is the sign of x^2+2x-6. Using the quadratic formula, we find the zeros of x^2+x-6:

$$\begin{aligned} x_1 &= \frac{-2-\sqrt{28}}{2} = -1-\sqrt{7} \\ x_2 &= -1+\sqrt{7} \end{aligned}$$

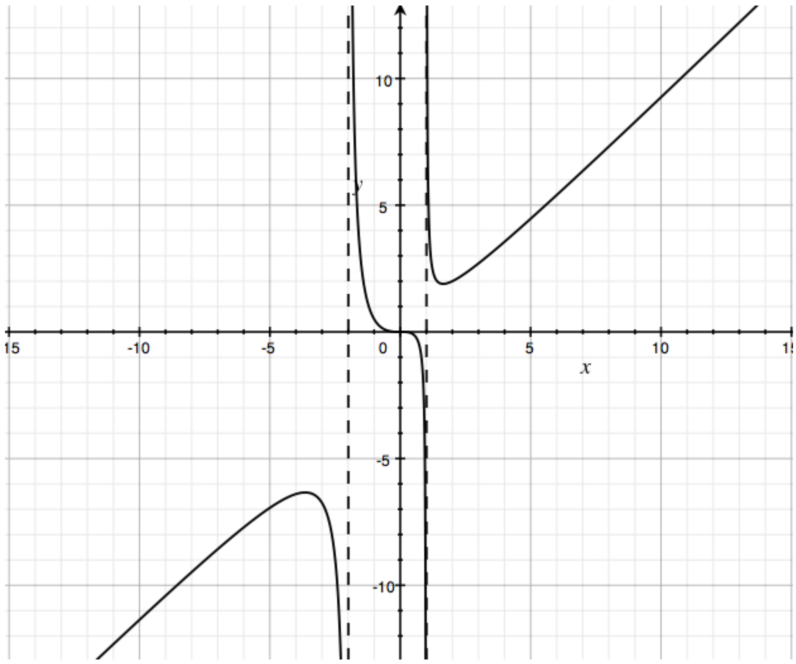
and conclude that

$$f'(x) = \frac{x^2(x-(-1-\sqrt{7}))(x+(-1+\sqrt{7})}{(x^2+x-2)^2}.$$

Note that $-1-\sqrt{7} < -2$ and $-1+\sqrt{7} > 1$. Hence we obtain the sign of f':

x	$(-\infty,-1-\sqrt{7})$	$(-1-\sqrt{7},-2)$	$(-2,1)$	$(1,-1+\sqrt{7})$	$(-1+\sqrt{7},\infty)$
$x-(-1-\sqrt{7})$	$-$	$+$	$+$	$+$	$+$
$x-(-1+\sqrt{7})$	$-$	$-$	$-$	$-$	$+$
f'	$+$	$-$	$-$	$-$	$+$
f	$\nearrow$	$\searrow$	$\searrow$	$\searrow$	$\nearrow$

Thus *f* is increasing on $(-\infty,-1-\sqrt{7})$ and on $(-1+\sqrt{7},\infty)$ and decreasing on $(-1-\sqrt{7},-2)$, on $(-2,1)$ (with an horizontal tangent at *x* = 0), and on $(1,-1+\sqrt{7})$. As a consequence, *f* has a local maximum at $x=-1-\sqrt{7}$, which is $f(-1-\sqrt{7}) \approx -6.34$ and a local minimum at $x=-1+\sqrt{7}$ which is $f(-1+\sqrt{7}) \approx 1.89$. For the sake of completeness, here is the graph, with vertical asymptotes dotted:



4. Find the intervals of increase and decrease and the local extrema of the function

$$f(x) = x - 2\cos x$$

on the interval $[0, 2\pi]$.

Solution. f is defined on the real line. Its derivative is

$$f'(x) = 1 + 2\sin x.$$

Thus $f'(x) > 0$ if $\sin x > -\frac{1}{2}$. Inspecting the trig circle, we see that $\sin x = -\frac{1}{2}$ in $[0, 2\pi]$ at $x = \frac{7\pi}{6}$ and $x = \frac{11\pi}{6}$, and that $\sin x > -\frac{1}{2}$ for x in $[0, \frac{7\pi}{2})$ and in $(\frac{11\pi}{6}, 2\pi]$. Hence we have:

x	$[0, \frac{7\pi}{6})$	$(\frac{7\pi}{6}, \frac{11\pi}{6})$	$(\frac{11\pi}{6}, 2\pi]$
f'	$+$	$-$	$+$
f	$\nearrow$	$\searrow$	$\nearrow$

Thus f is increasing on $(0, \frac{7\pi}{6})$ and on $(\frac{11\pi}{6}, 2\pi)$ and decreasing on $(\frac{7\pi}{6}, \frac{11\pi}{6})$. As a consequence, f has a local maximum at $x = \frac{7\pi}{6}$, which is $f(\frac{7\pi}{6}) = \frac{7\pi}{6} + \sqrt{3}$ and a local minimum at $x = \frac{11\pi}{6}$, which is $f(\frac{11\pi}{6}) = \frac{11\pi}{6} - \sqrt{3}$.

11 M9: Curve Sketching

11.1 Concavity and inflection points

Watch the video at
https://www.youtube.com/watch?v=VmTJ2DXsodw&list=PL265CB737C01F8961&index=48.

Abstract This video introduces the notion of concavity and inflection point of a function, and examines on an example how to find the intervals of a concavity and inflection points of a function, and how to use this information to sketch the graph.

Definition 11.1.1. A (differentiable) function f is *concave up* on an interval I if f' is increasing on I, and *concave down* on I if f' is decreasing on I.

Definition 11.1.2. An *inflection point* of a function f is a point **of the graph** of f where the concavity of f changes.

Proposition 11.1.3. *A twice differentiable function f is:*

1. *concave up on an interval I if and only if $f''(x) > 0$ for all $x \in I$;*
2. *concave down on an interval I if and only if $f''(x) < 0$ for all $x \in I$.*

Example 11.1.4. Find the intervals of increase and decrease, local extrema, intervals of concavity and inflection points and sketch the graph of

$$f(x) = x^3 - 12x + 1.$$

Solution. We have already obtained the intervals of increase and decrease and local extrema in Example 10.7.1:

x	$(-\infty, -2)$	$(-2, 2)$	$(2, \infty)$
$x-2$	$-$	$-$	$+$
$x+2$	$-$	$+$	$+$
$f'(x)$	$+$	$-$	$+$
$f(x)$	$\nearrow$	$\searrow$	$\nearrow$

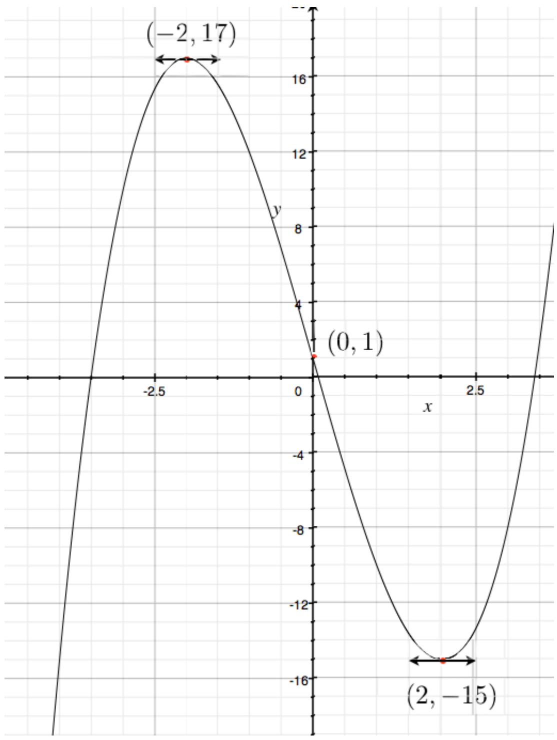
It has a local maximum at –2, which is $f(-2) = 17$, and a local minimum at 2, which is $f(2) = -15$. The intervals of concavity depend on the sign of

$$f''(x) = 6x,$$

so that *f* is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$ and has an inflection point at 0, which is

$$(0, f(0)) = (0, 1).$$

Using this information to sketch the graph, we obtain:



11.2 Second derivative Test

Watch the video at
https://www.youtube.com/watch?v=VUZybqQROig&list=PL265CB737C01F8961&index=49.

Abstract This video states the Second Derivative Test and examines an example.

Theorem 11.2.1 (Second Derivative Test) *Let c be a critical value of f where* $f'(c) = 0$, *and assume that f is twice differentiable and that* f'' *is continuous at c. Then:*

1. *If* $f''(c) > 0$, *f has a local minimum at c;*
2. *If* $f''(c) < 0$, *f has a local maximum at c.*

Note that the Theorem is inconclusive is $f''(c) = 0$, in which case the First Derivative Test (Theorem 10.6.4) should be used.

This theorem is used when you need the local extrema of a function, without needing its intervals of increase and decrease (for otherwise, you might as well use the First Derivative Test).

Example 11.2.2. Find the local extrema of $f(x) = 4x^2 - x^4$.

Solution. The function is polynomial so that its critical values are the zeros of

$$f'(x) = 8x - 4x^3 = 4x(2 - x^2) = 4x(\sqrt{2} - x)(\sqrt{2} + x).$$

We can test each critical value ($-\sqrt{2}$, 0, and $\sqrt{2}$) with

$$f''(x) = 8 - 12x^2.$$

Specifically, $f''(\pm\sqrt{2}) = -16 < 0$, so that *f* has local maxima at $-\sqrt{2}$ and at $\sqrt{2}$, which is $f(\pm\sqrt{2}) = 4$; and $f''(0) = 8 > 0$, so that *f* has a local minimum at 0 which is $f(0) = 0$.

11.3 Curve Sketching: Examples

Watch the videos at
https://www.youtube.com/watch?v=m3PsZ53rWPM&list=PL265CB737C01F8961&index=50

and
https://www.youtube.com/watch?v=bIwhIKLxfvo&list=PL265CB737C01F8961&index=51

and
https://www.youtube.com/watch?v=6XCVnejBpJE&list=PL265CB737C01F8961&index=52

and
https://www.youtube.com/watch?v=2-SGzyKOsfQ&list=PL265CB737C01F8961&index=53

and
https://www.youtube.com/watch?v=obN2I9FLtQA&list=PL265CB737C01F8961&index=54

and
https://www.youtube.com/watch?v=1iTPSKkbYn0&list=PL265CB737C01F8961&index=55

Abstract These videos introduce general guidelines to sketch the graph of a function, and go over a number of examples.

General Guidelines to sketch the graph of a function:

1. Find the domain of the function;
2. Find all asymptotes (review how in Sections 2.6, 2.8 and 2.9);
3. Find the intervals of increase and decrease (sign of the derivative), as in Section 10.6. Mark vertical asymptotes in the chart!
4. Find the local extrema (First Derivative Test: see Section 10.7);
5. Find the intervals of concavity (sign of the second derivative) as in Section 11.1;
6. Find inflection points;
7. To sketch the graph:
   a) draw asymptotes if any;
   b) plot local extrema and inflection points;
   c) connect them respecting the information on asymptotes, increase/decrease, and concavity.

Example 11.3.1. Sketch the graph of

$$f(x) = 2 - 15x + 9x^2 - x^3.$$

Solution. The function is polynomial, hence its domain is $(-\infty, \infty)$ and it has no asymptote. The intervals of increase and decrease depend on the sign of

$$f'(x) = -15 + 18x - 3x^2 = -3(x^2 - 6x + 5) = 3(1 - x)(x - 5),$$

so that we obtain the sign in the following chart:

x	$(-\infty, 1)$	$(1, 5)$	$(5, \infty)$
$1 - x$	$+$	$-$	$-$
$x - 5$	$-$	$-$	$+$
f'	$-$	$+$	$-$
f	$\searrow$	$\nearrow$	$\searrow$

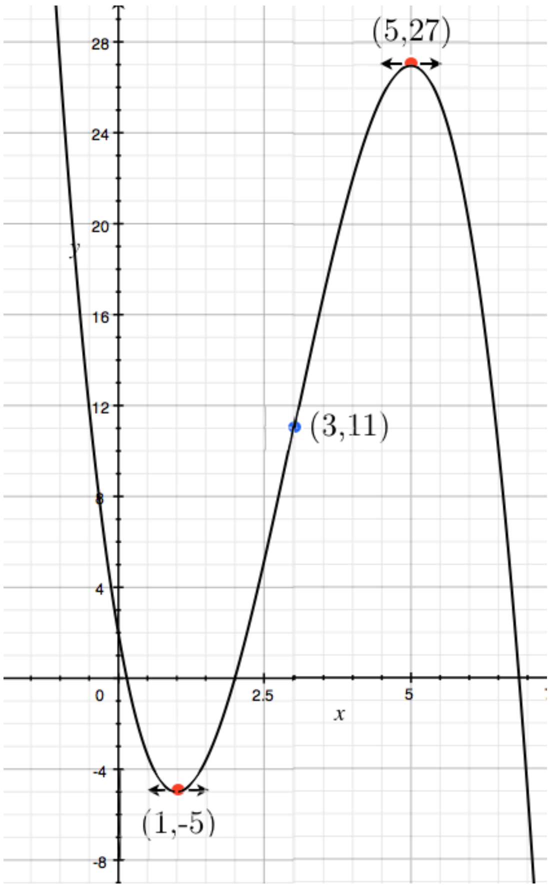
Thus f has a local minimum at 1, which is $f(1) = -5$, and a local maximum at 5 which is $f(5) = 27$. Intervals of concavity depend on the sign of

$$f''(x) = 18 - 6x > 0 \iff 3 > x,$$

so that *f* is concave up on $(-\infty, 3)$ and concave down on $(3, \infty)$, and has an inflection at 3:

$$(3, f(3)) = (3, 11).$$

Taking all this into account, we obtain the following sketch:



Example 11.3.2. Sketch the graph of

$$f(x) = \frac{x}{x-1}.$$

Solution. The domain is $(-\infty, 1) \cup (1, \infty)$ and *f* admits $x = 1$ as a vertical asymptote, and $y = 1$ as a horizontal asymptote, because $\lim_{x\to\infty} f(x) = 1$.

The intervals of increase and decrease depend on the sign of

$$f'(x) = \frac{x-1-x}{(x-1)^2} = -\frac{1}{(x-1)^2} < 0,$$

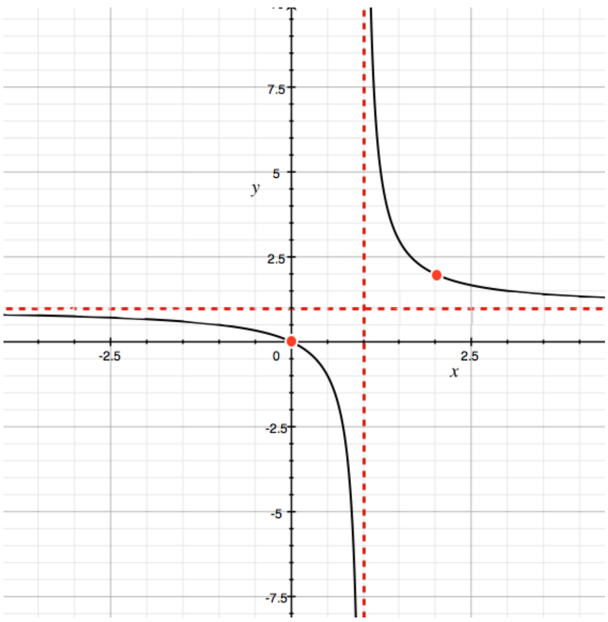
so that f is decreasing on $(-\infty, 1)$ and on $(1, \infty)$:

x	$(-\infty, 1)$	$(1, \infty)$
f'	$-$	$-$
f	$\searrow$	$\searrow$

Therefore, it has no local extremum.
The intervals of concavity depend on the sign of

$$f''(x) = 2(x-1)^{-3},$$

which is of the sign of $x - 1$, so that the function is concave down on $(-\infty, 1)$ and concave up on $(1, \infty)$. Thus it has no inflection point as the only change of concavity occurs at a vertical asymptote. Taking all this into account, we obtain the following sketch:



Example 11.3.3. Sketch the graph of

$$f(x) = x - 2\sin x \text{ on } [0, 3\pi].$$

Solution. The domain is $[0, 3\pi]$ and there is no asymptote. We have already obtained the intervals of increase and decrease in Example 10.7.3:

x	$(0, \frac{\pi}{3})$	$(\frac{\pi}{3}, \frac{5\pi}{3})$	$(\frac{5\pi}{3}, \frac{7\pi}{3})$	$(\frac{7\pi}{3}, 3\pi)$
$f'(x)$	$-$	$+$	$-$	$+$
$f(x)$	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$

and f has a local minimum at $\frac{\pi}{3}$ which is $f(\frac{\pi}{3}) = \frac{\pi}{3} - \sqrt{3}$ and at $\frac{7\pi}{3}$, which is $f(\frac{7\pi}{3}) = \frac{7\pi}{3} - \sqrt{3}$. It has a local maximum at $\frac{5\pi}{3}$ which is $f(\frac{5\pi}{3}) = \frac{5\pi}{3} + \sqrt{3}$.

The intervals of concavity depend on the sign of

$$f''(x) = 2\sin x,$$

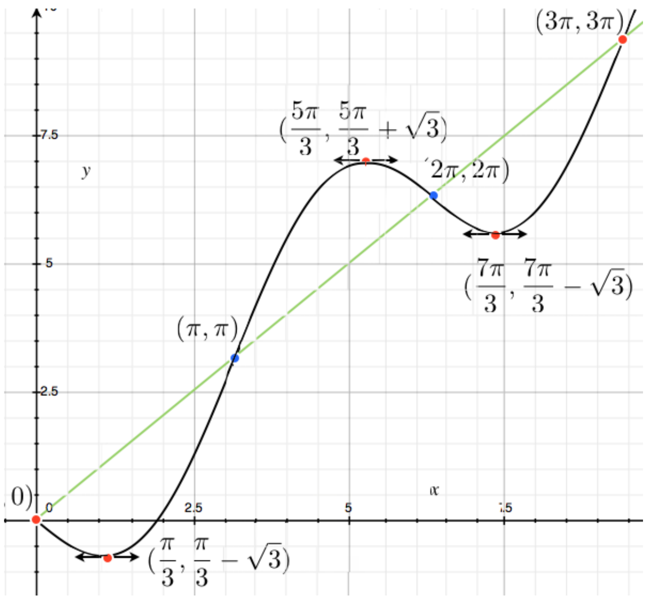
so that the concavity is given by:

x	$(0,\pi)$	$(\pi,2\pi)$	$(2\pi,3\pi)$
f''	$+$	$-$	$+$
concavity	$\cup$	$\cap$	$\cup$

Thus f has inflection points at π and 2π:

$$(\pi, f(\pi)) = (\pi,\pi) \text{ and } (2\pi, f(2\pi)) = (2\pi, 2\pi).$$

Taking all this into account, we obtain the following sketch:



Example 11.3.4. Sketch the graph of

$$f(x) = \frac{x}{x^2+1}.$$

Solution. Since $x^2 + 1 \neq 0$ for all x, the domain of f is $(-\infty, \infty)$. Since $\lim_{x\to\infty} f(x) = 0$, the line $y = 0$ is a horizontal asymptote. It is the only asymptote of f.

The intervals of increase and decrease depend on the sign of

$$f'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} = \frac{(1-x)(1+x)}{(x^2+1)^2},$$

so that the sign is obtained in the following sign chart:

x	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
$1-x$	$+$	$+$	$-$
$1+x$	$-$	$+$	$+$
f'	$-$	$+$	$-$
f	$\searrow$	$\nearrow$	$\searrow$

Thus, f has a local minimum at -1, which is $f(-1) = -\frac{1}{2}$, and a local maximum at 1, which is $f(1) = \frac{1}{2}$.

The intervals of concavity depend on the sign of

$$\begin{aligned} f''(x) &= \frac{-2x(x^2+1)^2-(1-x^2)4x(x^2+1)}{(x^2+1)^4} \\ &= \frac{2x(x^2-3)}{(x^2+1)^3} = \frac{2x(x-\sqrt{3})(x+\sqrt{3})}{(x^2+1)^3}, \end{aligned}$$

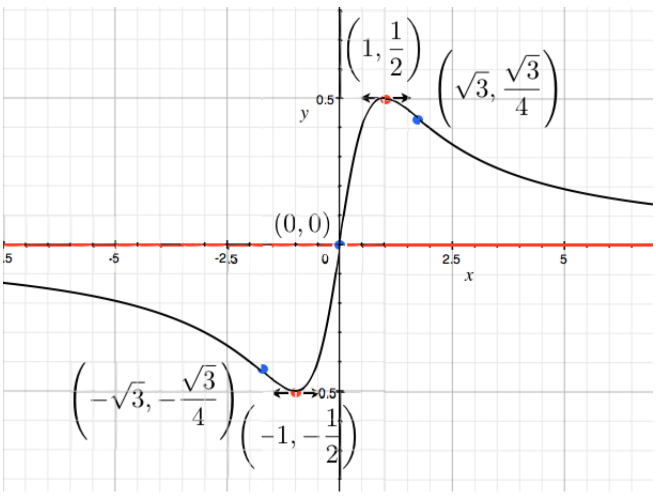
so that the concavity is given by:

x	$(-\infty,-\sqrt{3})$	$(-\sqrt{3},0)$	$(0,\sqrt{3})$	$(\sqrt{3},\infty)$
$2x$	$-$	$-$	$+$	$+$
$x-\sqrt{3}$	$-$	$-$	$-$	$+$
$x+\sqrt{3}$	$-$	$+$	$+$	$+$
f''	$-$	$+$	$-$	$+$
concavity	$\cap$	$\cup$	$\cap$	$\cup$

Thus *f* has inflection points at $-\sqrt{3}$, at 0, and at $\sqrt{3}$:

$$\left(-\sqrt{3}, f(-\sqrt{3})\right) = \left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right); \; (0, f(0)) = (0,0); \; \left(\sqrt{3}, f(\sqrt{3})\right) = \left(\sqrt{3}, \frac{\sqrt{3}}{4}\right).$$

Taking all this into account, we obtain the following sketch:



Example 11.3.5. Sketch the graph of

$$f(x) = \frac{x^3}{x^2+3x-10}.$$

Solution. Since

$$f(x) = \frac{x^3}{(x-2)(x+5)},$$

the domain is $(-\infty,-5)\cup(-5,2)\cup(2,\infty)$ and the lines $x = -5$ and $x = 2$ are vertical asymptotes. Moreover, the degree of the numerator is one more than that of the denominator, so that f has a slant asymptote. Long division gives:

$$\begin{array}{r|l}
 & \quad x \quad -3 \\ \hline
x^2+3x-10 & \quad x^3 \\
 & -x^3-3x^2+10x \\ \hline
 & \quad -3x^2+10x \\
 & \quad 3x^2 \ \ +9x-30 \\ \hline
 & \qquad\quad 19x-30
\end{array}$$

so that $y = x - 3$ is an asymptote. The intervals of increase and decrease depend on the sign of

$$\begin{aligned} f'(x) &= \frac{3x^2(x^2+3x-10)-(2x+3)x^3}{(x^2+3x-10)^2} \\ &= \frac{x^2(x^2+6x-30)}{(x^2+3x-10)^2} \\ &= \frac{x^2\left(x-(-3-\sqrt{39})\right)\left(x+(-3+\sqrt{39})\right)}{(x^2+3x-10)^2}, \end{aligned}$$

finding the zeros of $x^2+6x-30$ with the quadratic formula. Since squares are non-negative, we obtain the following sign chart:

x	$(-\infty, -3-\sqrt{39})$	$(-3-\sqrt{39}, -5)$	$(5, 2)$	$(2, -3+\sqrt{39})$	$(-3+\sqrt{39}, \infty)$
$x-(-3-\sqrt{39})$	$-$	$+$	$+$	$+$	$+$
$x-(-3+\sqrt{39})$	$-$	$-$	$-$	$-$	$+$
f'	$+$	$-$	$-$	$-$	$+$
f	$\nearrow$	$\searrow$	$\searrow$	$\searrow$	$\nearrow$

Thus f has a local maximum at $-3-\sqrt{39}$, which is $f(-3-\sqrt{39}) \approx -16.6$ and a local minimum at $-3+\sqrt{39}$, which is $f(-3+\sqrt{39}) \approx 3.3$.

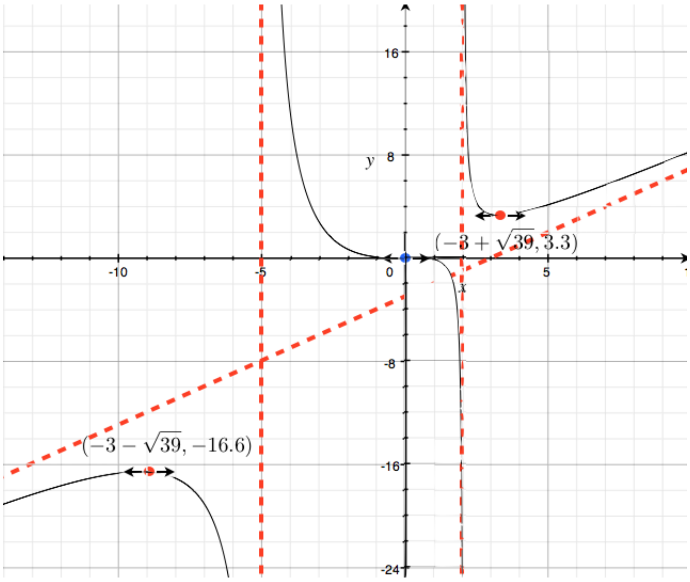
The intervals of concavity depend on the sign of

$$\begin{aligned} f''(x) &= \frac{(4x^3+18x^2-60x)(x^2+3x-10)^2-2(x^2+3x-10)(2x+3)(x^4+6x^3-30x^2)}{(x^2+3x-10)^4} \\ &= \frac{x(38x^2-180x+600)}{(x^2+3x-10)^3}. \end{aligned}$$

Since the discriminant of $38x^2-180x+600$ is negative, this quadratic term has no zero. Thus f'' is of the sign of $\frac{x}{(x^2+3x-10)^3}$, which is also the sign of $x(x+5)(x-2)$. Therefore the concavity is given by:

x	$(-\infty, -5)$	$(-5, 0)$	$(0, 2)$	$(2, \infty)$
x	$-$	$-$	$+$	$+$
$x+5$	$-$	$+$	$+$	$+$
$x-2$	$-$	$-$	$-$	$+$
f''	$-$	$+$	$-$	$+$
concavity	$\cap$	$\cup$	$\cap$	$\cup$

Thus f has only one inflection point at 0 (because the other changes in concavity occur at vertical asymptotes), which is $(0, f(0)) = (0, 0)$. Taking all this into account, we obtain the following sketch:



11.4 M9 Sample Quiz: Curve Sketching

1. Use the second derivative test to the find the relative maxima and minima of the function

$$f(x) = 2x^3 + 3x^2 - 12x - 7.$$

2. Sketch the graph (i.e., find asymptotes, intervals of increase and decrease, local extrema, intervals of concavity, inflection points, then sketch the graph) of

$$f(x) = x^4 - 6x^2.$$

3. Sketch the graph (i.e., find asymptotes, intervals of increase and decrease, local extrema, intervals of concavity, inflection points, then sketch the graph) of

$$f(x) = \frac{x^3}{x^2+1}.$$

4. Sketch the graph (i.e., find asymptotes, intervals of increase and decrease, local extrema, intervals of concavity, inflection points, then sketch the graph) of

$$f(x) = \frac{\sin x}{1+\cos x}.$$

11.5 M9 Sample Quiz Solutions

1. Use the second derivative test to the find the relative maxima and minima of the function

$$f(x) = 2x^3 + 3x^2 - 12x - 7.$$

Solution. Because *f* is differentiable, critical points are the zeros of the derivative, that is, of

$$f'(x) = 6x^2 + 6x - 12 = 6(x^2 + x - 2) = 6(x-1)(x+2),$$

so that critical points are –2 and 1. To test them with the second derivative test, we use

$$f''(x) = 12x + 6.$$

Specifically, $f''(-2) = -18 < 0$ so that *f* has a local maximum at –2, which is $f(-2) = 13$; and $f''(1) = 18 > 0$ so that *f* has a local minimum at 1, which is $f(1) = -14$.

2. Sketch the graph (i.e., find asymptotes, intervals of increase and decrease, local extrema, intervals of concavity, inflection points, then sketch the graph) of

$$f(x) = x^4 - 6x^2.$$

Solution. Because *f* is polynomial, its domain is the real line and it has no asymptote. Intervals of increase and decrease depend on the sign of the first derivative

$$f'(x) = 4x^3 - 12x = 4x(x^2 - 3) = 4x(x - \sqrt{3})(x + \sqrt{3}).$$

The sign of this product depends on the sign of each factor:

x	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, 0)$	$(0, \sqrt{3})$	$(\sqrt{3}, \infty)$
$4x$	$-$	$-$	$+$	$+$
$x-\sqrt{3}$	$-$	$-$	$-$	$+$
$x+\sqrt{3}$	$-$	$+$	$+$	$+$
f'	$-$	$+$	$-$	$+$
f	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$

This gives us the intervals of increase and decrease, and the fact that *f* has

- a local minimum at $-\sqrt{3}$ which is $f(-\sqrt{3}) = -9$;
- local maximum at 0 which is $f(0) = 0$;
- local minimum at $\sqrt{3}$ which is $f(\sqrt{3}) = -9$.

Intervals of concavity depend on the sign of the second derivative

$$f''(x) = 12x^2 - 12 = 12(x^2 - 1) = 12(x-1)(x+1).$$

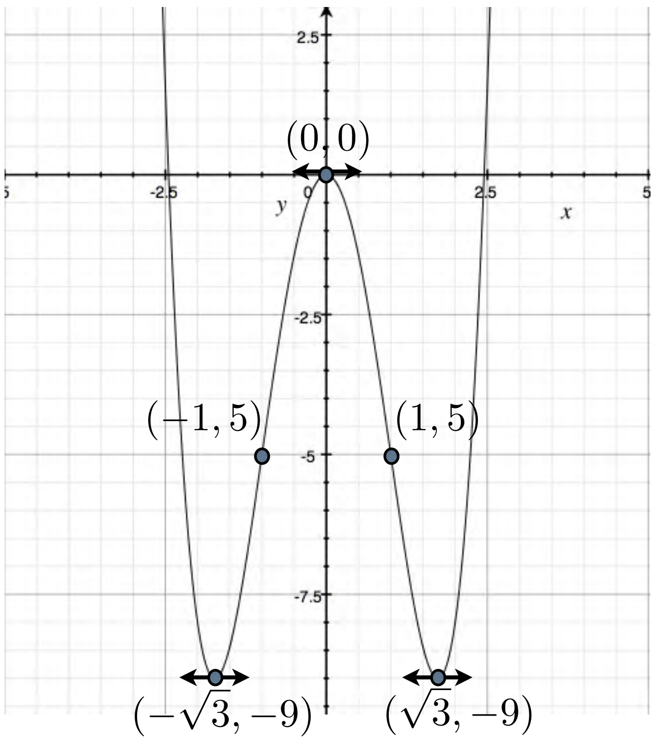
The sign of this product depends on the sign of each factor:

x	$(-\infty,-1)$	$(-1,1)$	$(1,\infty)$
$12(x-1)$	$-$	$-$	$+$
$x+1$	$-$	$+$	$+$
f''	$+$	$-$	$+$
concavity	$\bigcup$	$\bigcap$	$\bigcup$

Thus, the function f has an inflection point at $x = -1$ and another at $x = 1$ which are

$$\begin{aligned} (-1, f(-1)) &= (-1,-5) \\ (1, f(1)) &= (1,-5). \end{aligned}$$

Based on this information, we can sketch the graph:



3. Sketch the graph (i.e., find asymptotes, intervals of increase and decrease, local extrema, intervals of concavity, inflection points, then sketch the graph) of

$$f(x) = \frac{x^3}{x^2+1}.$$

Solution. Since $x^2+1\neq 0$ for all x, the domain of f is the real line and f has no vertical asymptotes. Since the degree of the numerator is higher than the degree of the denominator, $\lim_{x\to\infty} f(x)=\infty$ and f has no horizontal asymptote. However, since the degree is higher by one, it has a slant asymptote, whose equation we find by long division:

$$\begin{array}{r|l} & \quad\quad x \\ \hline x^2+1 & \quad x^3 \\ & -x^3-x \\ \hline & \quad\quad -x \end{array}$$

This shows that $y = x$ is a slant asymptote.

The intervals of increase and decrease depend on the sign of

$$\begin{aligned} f'(x) &= \frac{3x^2(x^2+1)-2x\cdot x^3}{(x^2+1)^2} \\ &= \frac{x^4+3x^2}{(x^2+1)^2}\geq 0 \end{aligned}$$

Hence the function is increasing on its domain $(-\infty,\infty)$ (with an horizontal tangent at $x = 0$). Thus f does not have any local extremum.

Intervals of concavity depend on the sign of

$$\begin{aligned} f''(x) &= \frac{(4x^3+6x)(x^2+1)^2 - 4x(x^2+1)(x^4+3x^2)}{(x^2+1)^4} \\ &= \frac{x(x^2+1)\left[(4x^2+6)(x^2+1) - 4x^4 - 12x^2\right]}{(x^2+1)^4} \\ &= \frac{x\left[6-2x^2\right]}{(x^2+1)^3} \\ &= \frac{2x(\sqrt{3}-x)(\sqrt{3}+x)}{(x^2+1)^3}, \end{aligned}$$

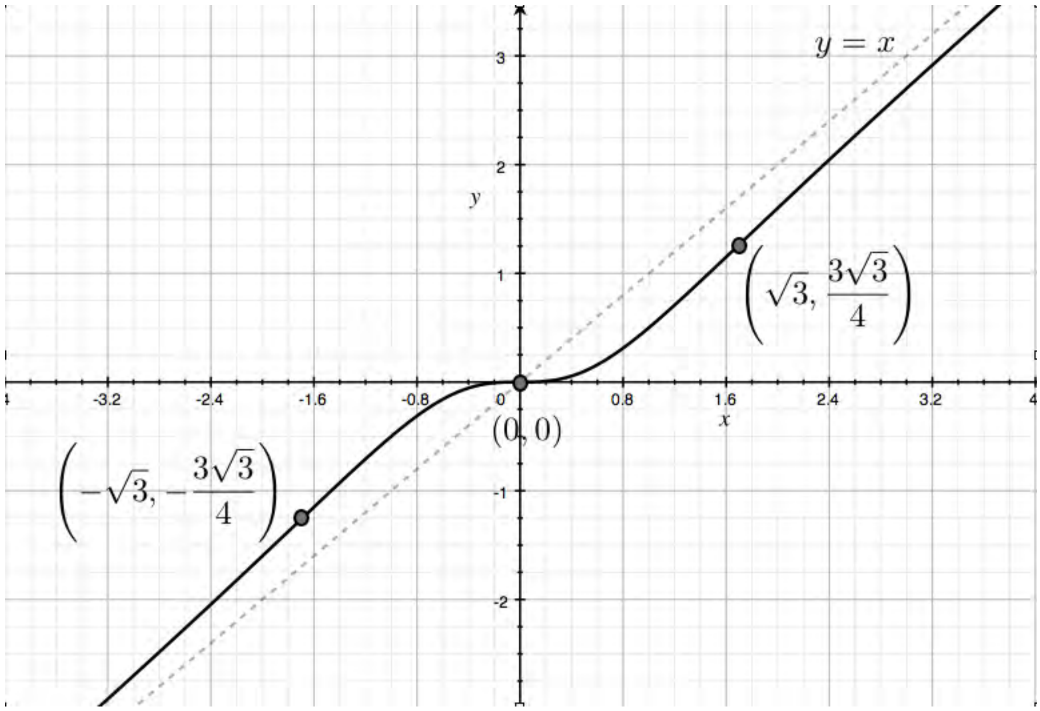
which we obtain from the sign of each factor, taking into account that $(x^2+1)^3 \geq 1$ does not change sign:

x	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, 0)$	$(0, \sqrt{3})$	$(\sqrt{3}, \infty)$
$2x$	$-$	$-$	$+$	$+$
$\sqrt{3}-x$	$+$	$+$	$+$	$-$
$\sqrt{3}+x$	$-$	$+$	$+$	$+$
f''	$+$	$-$	$+$	$-$
concavity	$\cup$	$\cap$	$\cup$	$\cap$

Thus, *f* has 3 points of inflection:

$$\begin{aligned} (-\sqrt{3}, f(-\sqrt{3})) &= (-\sqrt{3}, -\frac{3\sqrt{3}}{4}) \\ (0, f(0)) &= (0,0) \\ (\sqrt{3}, f(\sqrt{3})) &= (\sqrt{3}, \frac{3\sqrt{3}}{4}). \end{aligned}$$

Based on all this information, we can sketch the graph:



4. Sketch the graph (i.e., find asymptotes, intervals of increase and decrease, local extrema, intervals of concavity, inflection points, then sketch the graph) of

$$f(x) = \frac{\sin x}{1 + \cos x}.$$

Solution. The functions sin x and cos x are defined for all real numbers. Hence all x are in the domain of f, except those for which cos $x = -1$, that is, except for odd multiples of π; in other words

$$\mathrm{Dom} f = (-\infty, \infty) \setminus \{(2k+1)\pi : \ k \in \mathbb{Z}\},$$

where $\mathbb{Z}$ denotes the set of all integers. Because sin x and cos x are both periodic of period 2π, so is f. Therefore, it is enough to study the function on the period $(-\pi, \pi)$, taking into account that $x = -\pi$ and $x = \pi$ are vertical asymptotes for f.

The intervals of increase and decrease of *f* depend on the sign of

$$
\begin{aligned}
f'(x) &= \frac{\cos x(1+\cos x)+\sin^2 x}{(1+\cos x)^2} \\
&= \frac{\cos x+\cos^2 x+\sin^2 x}{(1+\cos x)^2} \\
&= \frac{1+\cos x}{(1+\cos x)^2} \\
&= \frac{1}{1+\cos x} > 0
\end{aligned}
$$

for all $x \in (-\pi, \pi)$ because $|\cos x| < 1$ on that interval. Thus *f* is increasing on $(-\pi, \pi)$ and has no local extremum.

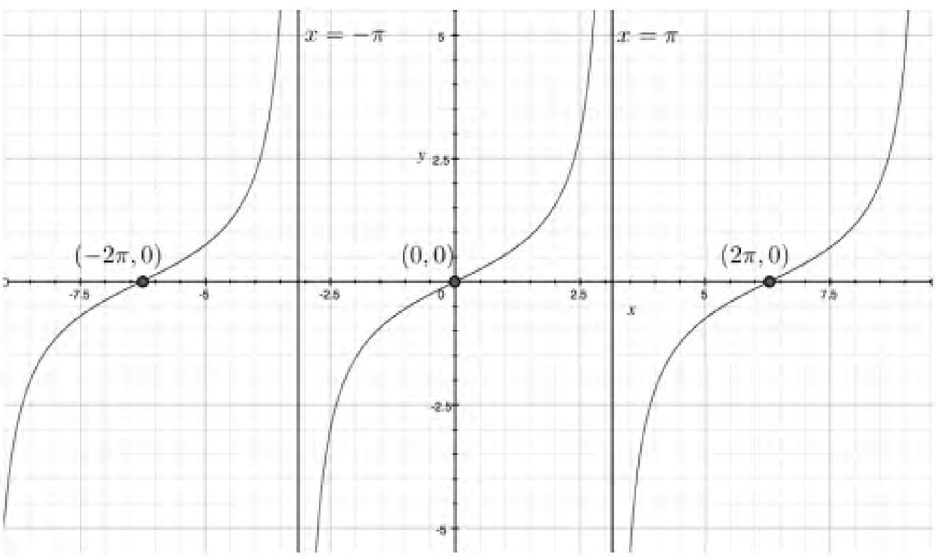
Intervals of concavity depend on the sign of

$$f''(x) = \frac{\sin x}{(1+\cos x)^2},$$

which has the same sign as sin *x*. Thus f'' changes sign at *x* = 0 so that $(0, f(0)) = (0,0)$ is an inflection point.

x	$(-\pi, 0)$	$(0, \pi)$
f'	$+$	
f	$\nearrow$	
f''	$-$	$+$
concavity	$\cap$	$\cup$

Based on this information, we can sketch the graph:



12 M10: Optimization

12.1 Optimization: First examples and general method

Watch the video at
https://www.youtube.com/watch?v=vau5NjKa1pE&list=PL265CB737C01F8961&index=56.

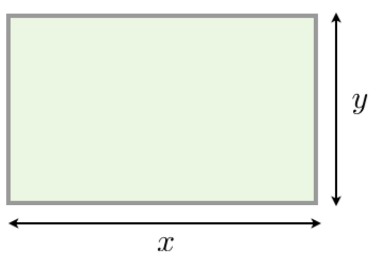
Abstract This video goes over two elementary examples of optimization problems and establishes some general guidelines.

General Guidelines for Optimization Problems

1. Understand the problem:
   a) What are you looking for (what is the objective function, to maximize or minimize)?
   b) What is given (what are the constraints)?
   c) Draw a picture if relevant
2. Introduce notations: Assign a symbol to the quantity to be maximized or minimized (the objective function). Assign symbols to other unknown quantities and label the picture with those symbols
3. Express the objective function and the constraint(s) in terms of other variables.
4. Use the constraint(s) to eliminate all but one variable in the objective function
5. Use the methods of module 8 to find the absolute minimum or maximum of the objective function on the domain defined by the constraints.

Example 12.1.1. You have 40 feet of fencing to enclose a rectangular space for a garden. Find the dimensions of the largest possible garden.

Solution. We are looking for the dimensions x and y of a rectangle



whose area

$$A = xy$$

we want to maximize, under the constraint that the perimeter is 40 feet, that is,

$$2x + 2y = 40 \iff x + y = 20.$$

Since under this constraint $y = 20 - x$, we substitute in A and obtain

$$A = x(20 - x) = 20x - x^2,$$

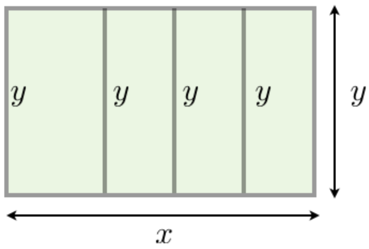
which we want to maximize on [0, 20], because as x represents a dimension of the rectangle, it cannot take values outside of this range. The function A is continuous on the closed interval [0, 20], so we can use the Closed Interval Method (Section 9.3) to find its maximum. Since

$$A'(x) = 20 - 2x,$$

the only critical value is $x = 10$. Since $A(0) = A(20) = 0$, A reaches its maximum for $x = 10\,ft$. Then $y = 20 - x = 10\,ft$. Thus the largest possible garden is $10\,ft \times 10\,ft$.

Example 12.1.2. A farmer with 750 feet of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible total area for the four pens?

Solution. Let x and y denote the outside dimensions of the rectangle, and assume that the pens are obtained with fencing parallel to the side of length y:



We want to maximize the area

$$A = xy$$

of the outer rectangle, under the constraint that we use 750 feet of fencing, that is,

$$2x + 5y = 750.$$

Under this constraint, $y = \frac{1}{5}(750 - 2x)$ so that

$$A(x) = xy = \frac{1}{5}\left(750x - 2x^2\right),$$

which we want to maximize on [0, 375], as one dimension of the rectangle cannot take values outside of this range. Since A is continuous on [0, 375], we can use the Closed Interval Method (Section 9.3) to find its maximum. Since

$$A'(x) = 150 - \frac{4}{5}x,$$

the only critical value is $x = \frac{750}{4}$. Since $A(0) = A(375) = 0$, A reaches its maximum for $x = \frac{750}{4}$ and the maximal area is

$$A\left(\frac{750}{4}\right) = 14062.5\, ft^2.$$

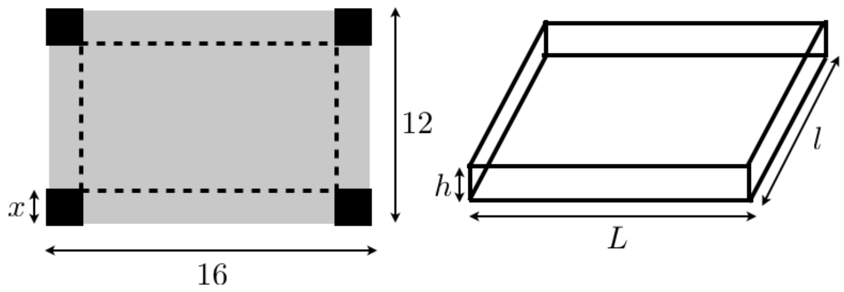
12.2 Example: an open box

Watch the video at
https://www.youtube.com/watch?v=AIxOcXcZiqg&list=PL265CB737C01F8961&index=57.

Abstract This video goes over an example of optimization problem in which we want to find the dimensions of a rectangular box without top of maximal volume.

Example 12.2.1. A box with no top is to built by taking a 16 inches by 12 inches rectangular piece of cardboard, cutting an identical square out of each corner, and folding up the sides. Find the dimensions of the box of maximum volume.

Solution. Let L, l and h denote the desired dimensions of the box, and let x denote the side of the square to be cut out of each corner:



Note that

$$L = 16 - 2x;\ l = 12 - 2x;\ h = x.$$

Thus the volume of the box is

$$V = L \cdot l \cdot h = (16 - 2x)(12 - 2x)x,$$

which we want to maximize on the interval [0, 6], for cutting out a square of side more than 6 inches out of each corner is impossible. As V is continuous on $[0, 6]$, we can use the Closed Interval Method (Section 9.3) to find its maximum. As

$$V(x) = 4x^3 - 56x^2 + 192x,$$

we conclude that

$$V'(x) = 12x^2 - 112x + 192,$$

whose roots are obtained by the quadratic formula to be $\frac{14\pm2\sqrt{13}}{3}$. Since $\frac{14+2\sqrt{13}}{3} > 6$, the only critical value in the interval is $\frac{14-2\sqrt{13}}{3}$ and $V(0) = V(6) = 0$. Thus V reaches its maximum for $x = \frac{14-2\sqrt{13}}{3}\, in$. The dimensions of the corresponding box are

$$\begin{aligned} h &= x = \frac{14-2\sqrt{13}}{3}\, in \\ L &= 16 - 2x = \frac{20+4\sqrt{13}}{3}\, in \\ l &= 12 - 2x = \frac{8+4\sqrt{13}}{3}\, in. \end{aligned}$$

12.3 Example: the best poster

Watch the video at
https://www.youtube.com/watch?v=z8TPs-UcL2w&list=PL265CB737C01F8961&index=58.

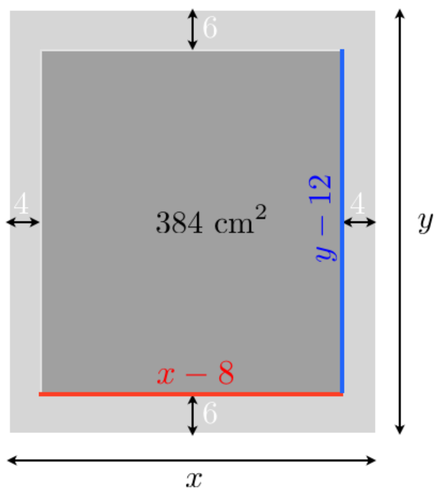
Abstract This video goes over an example of optimization problem in which we want to find the dimensions of a poster of minimal area for fixed margins and a fixed printed rectangular area.

Example 12.3.1. The top and bottom margins of a rectangular poster are 6 centimeters each, and the side margins are 4 centimeters each. If the area printed on the posted is fixed at $384\,cm^2$, find the dimensions of the poster using the least amount of paper.

Solution. Let x and y be the dimensions of the rectangular posted. We want to minimize the amount of paper, that is, the surface area

$$A = xy$$

of the poster, under the constraint that the printed area is $384\,cm^2$.



Taking into consideration the margins, we see that the printed area is

$$(x-8)(y-12) = 384.$$

Under this constraint, $y = 12 + \frac{384}{x-8}$, and, substituting in A, we have

$$\begin{aligned} A &= x\left(12 + \frac{384}{x-8}\right) \\ &= 12x + \frac{384x}{x-8}, \end{aligned}$$

which we want to minimize on $(8, \infty)$, as x cannot be smaller or equal to 8 *cm* for the given margins. to this end, we study the variations of A:

$$A'(x) = 12 + \frac{384(x-8) - 384x}{(x-8)^2} = 12 - \frac{3072}{(x-8)^2},$$

so that

$$A'(x) > 0 \iff (x-8)^2 > \frac{3072}{12} = 256 \iff x > 24.$$

In other words,

x	$(8, 24)$	$(24, \infty)$
f'	$-$	$+$
f	$\searrow$	$\nearrow$

and f reaches its absolute minimum on $(8, \infty)$ at $x = 24$, and the dimensions of the smallest poster are

$$\begin{aligned} x &= 24\,cm \\ y &= 12 + \frac{384}{24-8} = 36\,cm. \end{aligned}$$

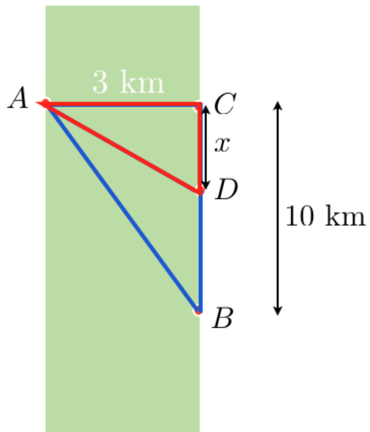
12.4 Example: across the marshes

Watch the video at
https://www.youtube.com/watch?v=1PYbcVxlDUY&list=PL265CB737C01F8961&index=59.

Abstract This video goes over an example of optimization problem in which a man wants to reach a point across marshes as soon as possible, given the position of this point, the width of the marshes, his speed across the marshes, and his speed outside of the marsh.

Example 12.4.1. A man stands on the edge of a 3 kilometers wide band of marsh land and needs to reach as soon as possible a point B situated across the marshes but 10 kilometers south of the point (C) directly across his current position (A). If he can walk at 2 kilometers per hour in the marshes and run at 8 kilometers per hour on normal terrain, where should he emerge from the marshes to reach B as soon as possible?

Solution. We want to minimize the time required to reach B, in terms of the position of the point where he emerges form the marshes, which we locate by its relative position to point C, that is, we introduce the distance x between C and the point D where he emerges from the marsh land:



We now have to express this time in terms of x. The time needed to reach B is the sum of the time to travel from A to D and the time to travel from D to B. On the first part of the trip, the man travels at the constant speed of 2 km/h, while on the second part of the trip, he travels at the constant speed of 8 km/h. Thus, the time required is

$$\begin{aligned} T &= \frac{AD}{2} + \frac{DB}{8} \\ &= \frac{\sqrt{9+x^2}}{2} + \frac{10-x}{8}, \end{aligned}$$

where the length AD is obtained via the Pythagorean Theorem applied in the right triangle ACD.

We now want to minimize

$$T(x)=\frac{\sqrt{9+x^2}}{2}+\frac{10-x}{8} \text{ on } [0{,}10],$$

for any value of x outside of this range would result in a longer time to reach B than $x = 0$ or $x = 10$. To this end, we can use the Closed Interval Method (Section 9.3).

$$T'(x)=\frac{1}{2}\cdot\frac{2x}{2\sqrt{9+x^2}}-\frac{1}{8}=\frac{x}{2\sqrt{9+x^2}}-\frac{1}{8},$$

so that

$$T'(x)=0 \iff \frac{x}{\sqrt{9+x^2}}=\frac{1}{4}=\frac{\sqrt{15}}{5}.$$

Moreover

$$T(0)=2.75\,h;\;T\left(\frac{\sqrt{15}}{5}\right)\approx 2.7\,h;\;T(10)\approx 5.2\,h$$

so that the absolute minimum of T on [0, 10] occurs for $x=\frac{\sqrt{15}}{5}$. Thus, the man should emerge from the marsh land $\frac{\sqrt{15}}{5}\,km$ south of A.

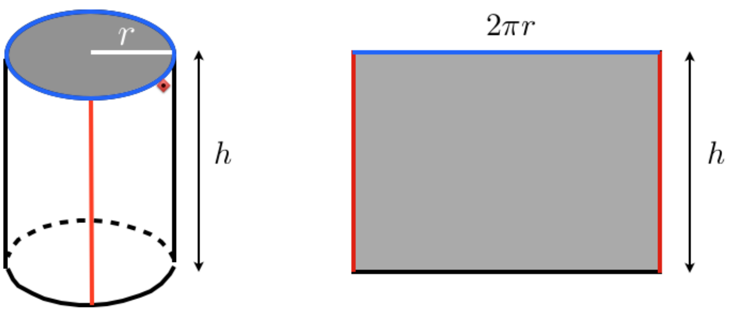
12.5 Example: the best soda can

Watch the video at
https://www.youtube.com/watch?v=XOPK6YFR518&list=PL265CB737C01F8961&index=60.

Abstract This video goes over an example of optimization problem in which we look for the dimensions of a soda can holding a fixed volume of liquid that uses the minimal amount of material.

Example 12.5.1. A cylindrical soda can is to hold 12 fluid ounces (that is, $355\,cm^3$). Suppose that the bottom and top of the can are twice as thick as the side. Find the dimensions of the can that minimize the amount of material used.

Solution. The dimensions for a cylinder are its radius *r* and height *h*.



Its volume is fixed at $355\,cm^3$, that is,

$$V = \pi r^2 h = 355. \tag{12.5.1}$$

We want to minimize the amount of material used, which is proportional to the outer surface area of the can, counting top and bottom twice to account for the thickness. The surface area of the top (or bottom) is πr^2, and the surface area of the side is that of a rectangle of length $2\pi r$ and width *h*. Thus the amount of material is proportional to

$$A = 4\pi r^2 + 2\pi r h.$$

Using the constraint (12.5.1), we have $h = \frac{355}{\pi r^2}$, so that

$$A = 4\pi r^2 + \frac{710\pi r}{\pi r^2} = 4\pi r^2 + \frac{710}{r},$$

which we want to minimize on $(0,\infty)$, as the only constraint on r is to be positive. To this end, we study the variations of the function $A(r)$:

$$A'(r) = 8\pi r - \frac{710}{r^2},$$

so that

$$A'(r) > 0 \iff r^3 > \frac{710}{8\pi} \iff r > \sqrt[3]{\frac{355}{4\pi}} \approx 3.04\,cm$$

and

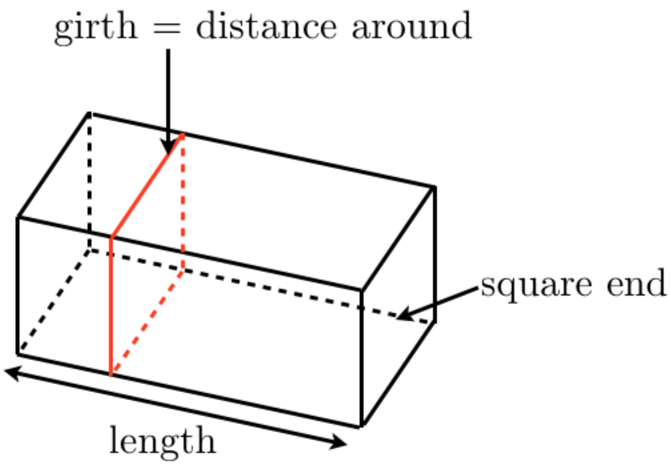
x	$(0, \sqrt[3]{\frac{355}{4\pi}})$	$(\sqrt[3]{\frac{355}{4\pi}}, \infty)$
$A'(r)$	$-$	$+$
$A(r)$	$\searrow$	$\nearrow$

Thus, A reaches its absolute minimum on $(0,\infty)$ at $r_0 = \sqrt[3]{\frac{355}{4\pi}}$, so that the dimensions of the optimal can are

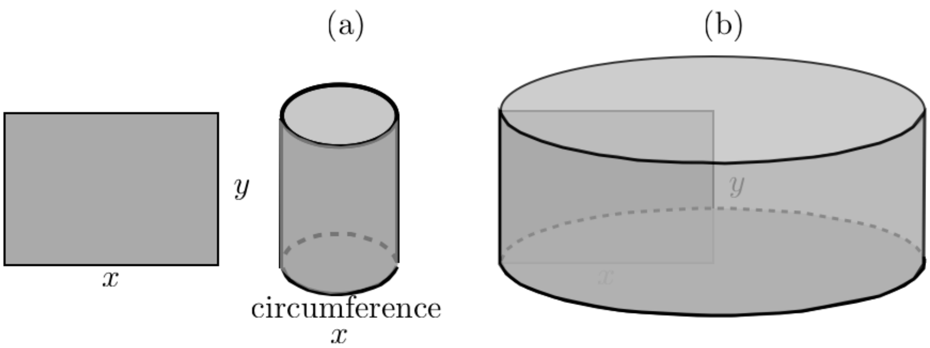
$$\begin{aligned} r &= \sqrt[3]{\frac{355}{4\pi}} \approx 3.04\,cm \\ h &= \frac{355}{\pi\left(\frac{355}{4\pi}\right)^{\frac{2}{3}}} \approx 12.2\,cm. \end{aligned}$$

12.6 M10 Sample Quiz: optimization

1. A 216 m^2 rectangular pea patch is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of the sides. What dimensions for the outer rectangle will require the smallest length of fence. How much fence will be needed?
2. The U.S. Postal Services will accept a box for domestic shipment only if the sum of its length and *girth* (distance around) does not exceed 108 inches. What dimensions will give a box with a square end the largest possible volume?



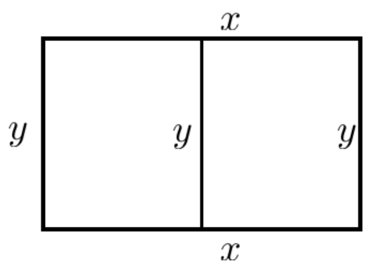
3. A rectangular sheet of perimeter 36 cm and dimensions x and y (in centimeters) is to be rolled into a cylinder as shown in part (a) of the figure below.
   a) What values of x and y give the largest volume?
   b) The same sheet is to be revolved about one of the sides of length y to sweep out the cylinder shown in part (b) of the figure below. What values of x and y give the largest volume?



12.7 M10 sample Quiz Solutions

1. A 216 m^2 rectangular pea patch is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of the sides. What dimensions for the outer rectangle will require the smallest length of fence. How much fence will be needed?

   Solution. Let us call *x* and *y* the dimensions (in meters) of the outer rectangle, where the divider is parallel to the side of length *y*, as in this figure:

   The constraint that the area of the patch is 216 m^2 means that

   $$xy = 216.$$

   We want to minimize the required length of fence, which is $L = 3y + 2x$. Using the constraint to express one variable in terms of the other, that is,

   $$y = \frac{216}{x},$$

   we obtain L as a function of *x*:

   $$L = 3 \times \frac{216}{x} + 2x = \frac{648}{x} + 2x.$$

   We want to minimize this function on $(0, \infty)$. The derivative of L

   $$L'(x) = -\frac{648}{x^2} + 2$$

   is positive if

   $$-\frac{648}{x^2} + 2 > 0 \iff x^2 > \frac{648}{2} = 324,$$

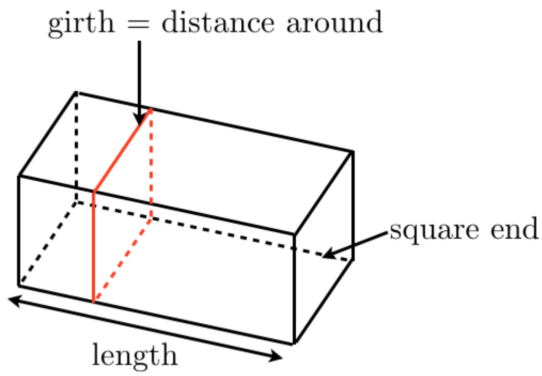
so that the only critical point in $(0,\infty)$ is $\sqrt{324}=18$ and the behavior of the function L is given by

x	$(0,18)$	$(18,\infty)$
$L'(x)$	$-$	$+$
$L(x)$	$\searrow$	$\nearrow$

Thus L has an absolute minimum at $x=18\,m$. The corresponding other dimension is $y=\frac{216}{18}=12\,m$. The optimal dimensions of the rectangle are thus 12×18 meters and the amount of fence required is

$$L=3y+2x=3\times 12+2\times 18=72\,m.$$

2. The U.S. Postal Services will accept a box for domestic shipment only if the sum of its length and girth (distance around) does not exceed 108 inches. What dimensions will give a box with a square end the largest possible volume?



Solution. Let x be the side of the square end. Then the girth of the box is $4x$. Calling the length y, the constraint is

$$4x + y \leq 108,$$

and we want to maximize the volume, which is

$$V = x^2 y.$$

Evidently, the maximum volume will be obtain when the upper bound for the constraint is reached, that is, for $4x + y = 108$. Then $y = 108 - 4x$, so that, substituting in V, we obtain the volume as a function of one variable:

$$V(x) = x^2(108 - 4x).$$

We now want to maximize V on $[0, 27]$. Since $V(0) = V(27) = 0$, the maximum is obtained at a critical point for V in $(0, 27)$. To find the critical point, we consider the derivative

$$V'(x) = 216x - 12x^2 = 12x(18 - x),$$

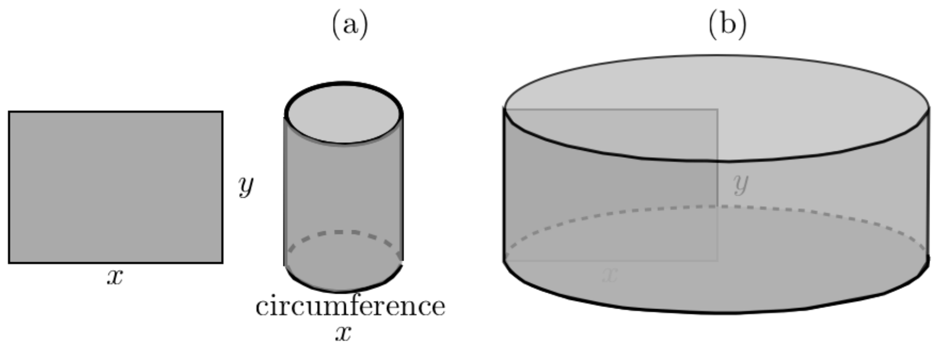
so that $x = 12$ and $x = 18$ are critical values. Moreover

$$\begin{aligned} V(12) &= 8640 \\ V(18) &= 11664 \end{aligned}$$

so that the maximum volume of 11664 cubic inches is reached when the side x of the square end is 18 inches. The corresponding length is then

$$y = 108 - 4 \times 18 = 36 \ in^3.$$

3. A rectangular sheet of perimeter 36 cm and dimensions x and y (in centimeters) is to be rolled into a cylinder as shown in part (a) of the figure below.



a) What values of x and y give the largest volume?
Solution. The constraint that the perimeter is 36 cm rephrases as

$$2x+2y=36 \iff x+y=18.$$

The volume of a circular cylinder of radius r and height h is $\pi r^2 h$. Here the circumference $2\pi r$ is x so that $r=\frac{x}{2\pi}$, and the height is y. Thus the volume that we want to maximize is

$$V=\pi\left(\frac{x}{2\pi}\right)^2 y=\frac{x^2y}{4\pi},$$

which taking into account the constraint $y=18-x$, can be rewritten

$$V=\frac{x^2(18-x)}{4\pi}.$$

We want to find the maximum of $V(x)$ on $[0,18]$. Note that $V(0)=V(18)=0$ so that the maximum is reached at a critical point of V in $(0,18)$. To find these critical value, we calculate

$$V'(x)=\frac{1}{4\pi}(36x-3x^2)=\frac{3}{4\pi}x(12-x)$$

so that $x=0$ and $x=12$ are critical. Thus the maximum volume is attained when $x=12\,cm$, in which case $y=18-12=6\,cm$.

b) The same sheet is to be revolved about one of the sides of length y to sweep out the cylinder shown in part (b) of the figure below. What values of x and y give the largest volume?

Solution. The constraint $x + y = 18$ is unchanged. The cylinder has now radius x and height y so that we want to maximize

$$\begin{aligned} V &= \pi x^2 y \\ &= \pi x^2 (18 - x) \end{aligned}$$

on $[0, 18]$. $V(0) = V(18) = 0$ so that the maximum is reached at a critical point of V in $(0, 18)$. To find critical values, we calculate

$$V'(x) = 36\pi x - 3\pi x^2 = 3\pi x(12 - x)$$

so that $x = 0$ and $x = 12$ are critical, and the volume is maximal if $x = 12\,cm$, in which case $y = 18 - 12 = 6\,cm$.

13 Review on modules 7 through 10

13.1 MOCK TEST 3

Instructions: *Do the following test, without your notes, in limited time (75 minutes top). Then grade yourself using the solutions provided separately. It is important that you show all your work and justify your answers. Carefully read the solutions to see how you should justify answers.*

1. [10 points] Find the local maxima and minima of $f(x) = x - 3x^{\frac{1}{3}}$ by use of the second derivative test if possible and by the first derivative test whenever needed.
2. [10 points] If $f(3) = 5$ and $f'(x) \geq 3$ for every *x* in [3, 9], how small can $f(9)$ be?
3. [10 points] Find the absolute minimum and maximum of $f(x) = x^5 - x^4 - 4$ on [–3, 3].
4. [10 points] Show that the equation $x^7 + 4x^3 + x + 5 = 0$ has exactly one real root.
5. [5 points] Find the critical numbers for $f(x) = 2x^3 + 3x^2 - 36x + 6.$
6. [15 points] Sketch the graph (2) of

$$f(x) = 2 - 15x + 9x^2 - x^3.$$

7. [10 points] Find all the asymptotes (3) of

$$f(x) = \frac{x^3 - 2x + 1}{x^2 - x - 2}.$$

8. [10 points] Sketch the graph of

$$f(x) = \frac{2x}{x - 3}.$$

9. [15 points] A man launches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 10 km downstream on the opposite bank, as quickly as possible. He could row his boat directly across the river to point C and then run to B, or he could row directly to B, or he could row to some point D between C and B and then run to B. If he can row at 6 km/h and run at 9 km/h, where should he land to reach B as soon as possible?
10. [5 points] Find the dimensions of a rectangle with perimeter 100 meters and whose area is as large as possible.

13.2 MOCK TEST 3 Solutions

1. [10 points] Find the local maxima and minima of $f(x)=x-3x^{\frac{1}{3}}$ by use of the second derivative test if possible and by the first derivative test whenever needed.

   Solution. The domain of the function is $(-\infty,\infty)$, and it is differentiable everywhere but at 0. Thus, 0 is a critical point. Moreover, for $x\neq 0$,

   $$f'(x)=1-x^{-\frac{2}{3}},$$

   so that $f'(x)=0$ if $x=-1$ or $x=1$. The critical value 0 cannot be tested by the second derivative test, but it is clear that f' does not change sign at 0, so that there is no local extremum at 0 (First Derivative Test, Theorem 10.6.4). The other two critical values can be tested by the second derivative test: $f''(x)=\frac{2}{3}x^{-\frac{5}{3}}$ so that $f''(1)>0$ and $f''(-1)<0$ and, by the Second Derivative Test (Theorem 11.2.1), f has a local minimum at 1, which is $f(1)=-2$, and a local maximum at –1, which is $f(-1)=2$.

2. [10 points] If $f(3)=5$ and $f'(x)\geq 3$ for every x in [3, 9], how small can $f(9)$ be?

   Solution. Since f' is defined at each $x\in[3,9]$, f is continuous on [3, 9] and differentiable on (3, 9) so that the Mean Value Theorem (Theorem 10.2.1) applies to f on [3, 9] to the effect that there is $c\in(3,9)$ with

   $$f'(c)=\frac{f(9)-f(3)}{9-3}(9)=6f'(c)+f(3)\geq 18+5=23,$$

   because $f'(c)\geq 3$. Thus $f(9)\geq 23$.

3. [10 points] Find the absolute minimum and maximum of $f(x)=x^5-x^4-4$ on [–3, 3].

   Solution. Since f is continuous on the closed interval [–3, 3], we use the Closed Interval Method (Section 9.3). Since

   $$f'(x)=5x^4-4x^3=x^3(5x-4),$$

   the critical values of f are 0 and $\frac{4}{5}$, which are both in [–3, 3]. Moreover,

   $$f(-3)=-328;\ f(0)=-4;\ f(\frac{4}{5})\approx -4.08;\ f(3)=158,$$

   so that the absolute minimum is –328 and occurs at –3, while the absolute maximum is 158 and occurs at 3.

4. [10 points] Show that the equation $x^7 + 4x^3 + x + 5 = 0$ has exactly one real root.

Solution. Let $f(x) = x^7 + 4x^3 + x + 5$. Note that

$$f(-1) = -1 < 0 < f(0) = 5$$

and that *f* is polynomial, hence continuous on $[-1, 0]$. By the Intermediate Value Theorem (Theorem 3.3.1), there is *c* in $(-1, 0)$ such that $f(c) = 0$, that is, the equation has at least one solution. Moreover, the solution is unique for if there were two different solutions $x_1 < x_2$ then Rolle's Theorem (Theorem 10.1.1) would apply to *f* on $[x_1, x_2]$ to the effect that a value *d* would exist where $f'(d) = 0$. But

$$f'(x) = 7x^6 + 12x^2 + 1 \geq 1$$

does not take the value 0; a contradiction.

5. [5 points] Find the critical numbers for $f(x) = 2x^3 + 3x^2 - 36x + 6$.

Solution. Since the function is polynomial, the critical values are the zeros of

$$f'(x) = 6x^2 + 6x - 36 = 6(x-2)(x+3),$$

that is, 2 and −3.

6. [15 points] Sketch the graph of

$$f(x) = 2 - 15x + 9x^2 - x^3.$$

Solution. This is Example 11.3.1.

7. [10 points] Find all the asymptotes of

$$f(x) = \frac{x^3 - 2x + 1}{x^2 - x - 2}.$$

Solution. Note that

$$f(x) = \frac{x^3 - 2x + 1}{x^2 - x - 2} = \frac{x^3 - 2x + 1}{(x+1)(x-2)}$$

and that $x^3 - 2x + 1 \neq 0$ for $x = -1$ and for $x = 2$. Thus $x = -1$ and $x = 2$ are vertical asymptotes. Moreover, we see by long division

$$\begin{array}{r|l} & x+1 \\ \hline x^2 - x - 2 & x^3 \qquad - 2x + 1 \\ & -x^3 + x^2 + 2x \\ \hline & \quad x^2 \qquad + 1 \\ & \quad -x^2 + x + 2 \\ \hline & \qquad x + 3 \end{array}$$

that $y = x + 1$ is a slant asymptote.

8. [10 points] Sketch the graph of

$$f(x) = \frac{2x}{x-3}.$$

Solution. The domain of f is $(-\infty, 3) \cup (3, \infty)$ and $x = 3$ is a vertical asymptote. Additionally, $\lim_{x\to\infty} f(x) = 2$ so that $y = 2$ is a horizontal asymptote. The intervals of increase and decrease depend on the sign of

$$f'(x) = \frac{2(x-3) - 2x}{(x-3)^2} = -\frac{6}{(x-3)^2} < 0$$

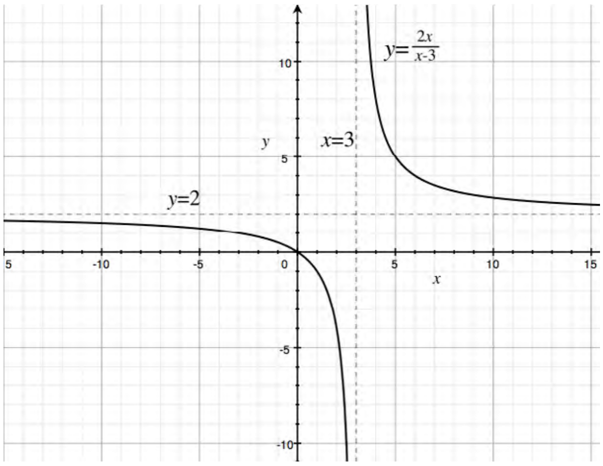
so that *f* is decreasing on $(-\infty, 3)$ and on $(3, \infty)$, and thus has no local extremum:

x	$(-\infty, 3)$	$(3, \infty)$
$f'(x)$	$-$	$-$
$f(x)$	$\searrow$	$\searrow$

The concavity depends on the sign of

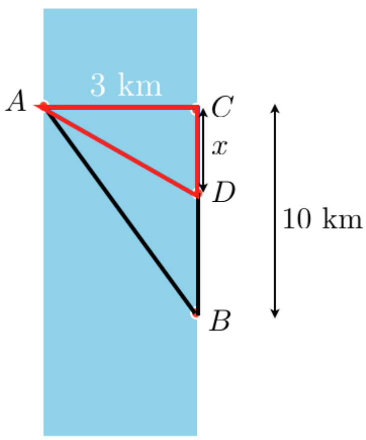
$$f''(x) = 12(x-3)^{-3},$$

that is, of the sign of $x-3$. Thus the function is concave down on $(-\infty, 3)$ and concave up on $(3, \infty)$ and has no inflection point. Taking all this into account, we obtain the following sketch:



9. [15 points] A man launches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 10 km downstream on the opposite bank, as quickly as possible. He could row his boat directly across the river to point C and then run to B, or he could row directly to B, or he could row to some point D between C and B and then run to B. If he can row at 6 km/h and run at 9 km/h, where should he land to reach B as soon as possible?

Solution. We represent the situation as follows, introducing the distance *x* between *C* and *D*:



From A to D, the man travels at the constant speed of 6 km/h so that the time required to go from A to D is

$$\frac{AD}{6}=\frac{\sqrt{9+x^2}}{6},$$

applying the Pythagorean Theorem in the right triangle ACD to find the distance AD as a function of x. Similarly, from D to B, the man travels at the constant speed of 9 km/h so that the time required to go from D to B is

$$\frac{DB}{9}=\frac{10-x}{9}.$$

Thus, the time required to go from A to B is

$$T(x)=\frac{\sqrt{9+x^2}}{6}+\frac{10-x}{9},$$

which we want to minimize on [0, 10], by the Closed Interval Method (Section 9.3). Since

$$T'(x)=\frac{1}{6}\cdot\frac{2x}{2\sqrt{9+x^2}}-\frac{1}{9},$$

we have

$$\begin{aligned} T'(x)=0 \quad \Longleftrightarrow \quad & \frac{x}{\sqrt{9+x^2}}=\frac{2}{3} \\ \Longrightarrow \quad & x^2=\frac{4}{9}(9+x^2) \\ \Longrightarrow \quad & x^2=\frac{9}{5}\times 4 \\ \Longrightarrow \quad & x=\frac{6}{\sqrt{5}}=\frac{6\sqrt{5}}{5}\in[0,10]. \end{aligned}$$

Moreover,

$$T(0)\approx 1.6\,h;\; T\left(\frac{6\sqrt{5}}{5}\right)\approx 1.5\,h;\; T(10)\approx 1.7\,h,$$

so that $T(x)$ reaches its absolute minimum on [0, 10] for $x=\frac{6\sqrt{5}}{5}$. Thus the man should land $\frac{6\sqrt{5}}{5}$ km downstream.

10. [5 points] Find the dimensions of a rectangle with perimeter 100 meters and whose area is as large as possible.

Solution. Let x and y denote the dimensions of the rectangle. We want to maximize its area

$$A = xy$$

under the constraint that the perimeter is 100 meters, that is,

$$2x + 2y = 100 \iff x + y = 50.$$

Thus $y = 50 - x$ and

$$A(x) = x(50 - x) = 50x - x^2,$$

which we want to maximize on [0, 50], by the Closed Interval Method. Since

$$A'(x) = 50 - 2x = 0 \iff x = 25,$$

and $A(0) = A(50) = 0,$ the area is maximal when $x = 25\,m$ and $y = 50 - x = 25\,m$, that is, for a square of side 25 meters.

14 M11: Definite Integral

14.1 Preliminaries: Sums

Watch the video at
https://www.youtube.com/watch?v=zIrGWV_aqQE&list=PL265CB737C01F8961&index=61.

Abstract This video introduces the sigma notation for sums, goes over examples, and establishes formulas for the sum of consecutive integers, and for the sum of consecutive squares.

To write large sums more compactly, we introduce the notation

$$\sum_{i=1}^{n} f(i) := f(1) + f(2) + \ldots + f(n-1) + f(n),$$

in which the index i takes all integer values ranging from the initial value given at the bottom of the sigma sign to the final value given at the top of the sigma sign, and all the resulting values are added together.

The following two formulas are established :

$$\begin{aligned} \sum_{i=1}^{n} i &= \frac{n(n+1)}{2} \\ \sum_{n=1}^{n} i^2 &= \frac{n(n+1)(2n+1)}{6}. \end{aligned}$$

14.2 The area problem

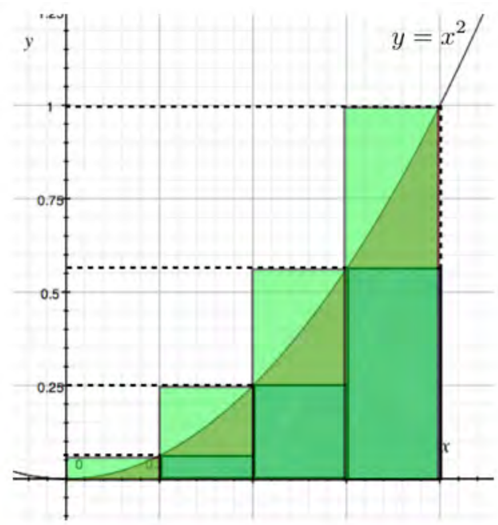
Watch the videos at
https://www.youtube.com/watch?v=E42fMALg5Dc&list=PL265CB737C01F8961&index=62

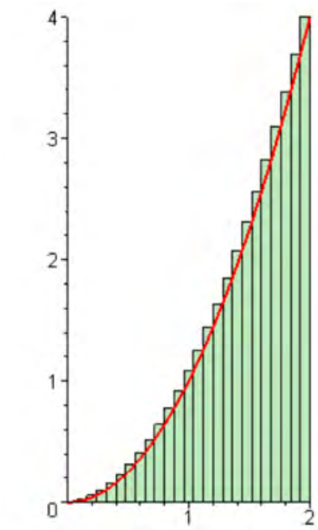
and
https://www.youtube.com/watch?v=3zfEHx2kzAw&list=PL265CB737C01F8961&index=63

Abstract These two videos introduce the problem of calculating the area under the graph of a continuous positive functions over a closed interval. On a specific example, it presents the approach by approximation of the desired area by a sum of areas of rectangles. Specifically, it is established that way that the area of the region

$$\{(x, y) : 0 \leq x \leq 1; 0 \leq y \leq x^2\}$$

is $\frac{1}{3}$ by approximating the area from below and from above by areas of rectangles.





14.3 Formal definition of the definite integral

Watch the videos at
https://www.youtube.com/watch?v=Zvyg0d3mXro&list=PL265CB737C01F8961&index=64

and
https://www.youtube.com/watch?v=51qBG3lLsDU&list=PL265CB737C01F8961&index=65

Abstract These two videos present the formal definition of the definite integral of a function over a closed interval.

Definition 14.3.1. A *partition* P of an interval $[a, b]$ is a finite sequence

$$P := (x_0 = a < x_1 < x_2 < \ldots < x_n = b)\,,$$

and a *pointed partition* of $[a, b]$ is a partition together with a choice of a point x_i^* in each interval $[x_{i-1}, x_i]$ for $i = 1 \ldots n$.

The *parameter* $\delta(P)$ of a partition $P = (x_0 = a < x_1 < \ldots < x_n = b)$ of $[a, b]$ is by definition the largest length of the subintervals $[x_{i-1}, x_i]$:

$$\delta(P) = \max_{i=1\ldots n} \Delta x_i := \max_{i=1\ldots n} x_i - x_{i-1}.$$

Definition 14.3.2. Given a pointed partition P of $[a, b]$ and a function f on $[a, b]$, the *Riemann sum* for f and P is

$$R(f, P) := \sum_{i=1}^{n} f(x_i^*)\Delta x_i.$$

Definition 14.3.3. A function f is called *integrable on* $[a, b]$ if the limit of $R(f, P)$ as $\delta(P)$ approaches 0 exists, that is, if there exists a number I such that for every $\epsilon > 0$, there exists $r > 0$ such that

$$\delta(P) < r \Longrightarrow |R(f, P) - I| < \epsilon. \tag{14.3.1}$$

In that case, the number I is called *definite integral* of f on $[a, b]$ and is denoted

$$\int_a^b f(x)\,dx,$$

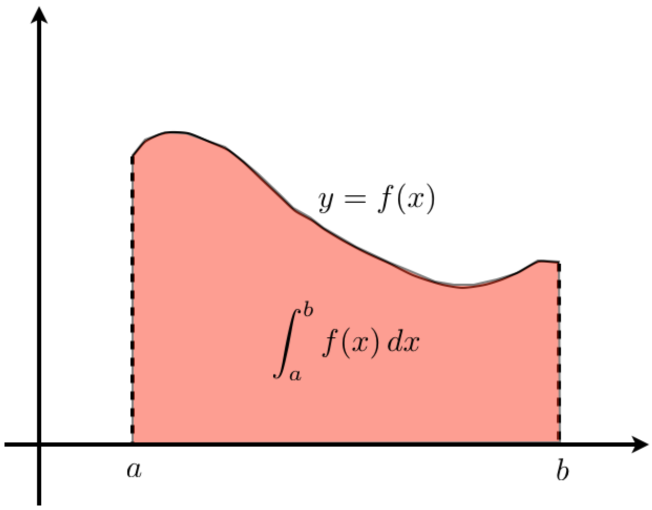
where $\int$ is called *integral sign*, a is called *lower bound of integration* and b is called *upper bound of integration*. Symbolically,

$$\int_a^b f(x)\,dx := \lim_{\delta(P)\to 0} R(f, P),$$

where the limit is defined modulo (14.3.1).

Proposition 14.3.4. *If f is continuous on $[a, b]$ then $\int_a^b f(x)\,dx$ exists. If additionally $f(x) \geq 0$ for all x in $[a, b]$ then*

$$\begin{aligned} \int_a^b f(x)\,dx &= \text{area under the graph of } f \text{ over } [a, b] \\ &= \text{area of } \{(x, y) : a \leq x \leq b;\ 0 \leq y \leq f(x)\}. \end{aligned}$$

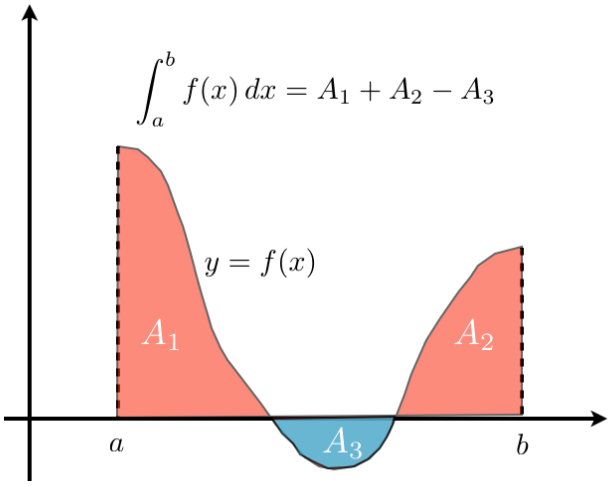


14.4 First examples of definite integrals

Watch the video at
https://www.youtube.com/watch?v=HRXD9OwT6W8&list=PL265CB737C01F8961&index=66.

Abstract This video focuses on the geometric interpretation of the definite integrals to calculate the values of a few examples.

Proposition 14.3.4 provides a geometric interpretation for the definite integral of a continuous non-negative function on $[a, b]$. A similar interpretation is available if we drop the assumption that the function is non-negative:



Using geometric interpretations, the following values of definite integrals are obtained:

$$\begin{aligned}
\int_0^1 \sqrt{1-x^2}\,dx &= \frac{\pi}{4} \\
\int_0^3 x-1\,dx &= \frac{3}{2} \\
\int_a^b x\,dx &= \frac{b^2-a^2}{2}.
\end{aligned}$$

14.5 Properties of integrals

Watch the videos at
https://www.youtube.com/watch?v=VZONHbQ-ilg&list=PL265CB737C01F8961&index=67

and
https://www.youtube.com/watch?v=qUrS5SXL5zw&list=PL265CB737C01F8961&index=68.

Abstract These videos present various properties of the definite integrals, illustrates how to use them, introduces the average value, and examines examples.

The following properties of integrals are established:

$$\int_a^a f(x)\,dx = 0 \tag{14.5.1}$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \tag{14.5.2}$$

$$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx \tag{14.5.3}$$

$$\int_a^b c \cdot f(x)\,dx = c \cdot \int_a^b f(x)\,dx \tag{14.5.4}$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \tag{14.5.5}$$

$$\int_a^b c\,dx = c\cdot(b-a) \tag{14.5.6}$$

$$\int_a^b x\,dx = \frac{b^2-a^2}{2} \tag{14.5.7}$$

$$f(x)\geq 0 \text{ on } [a,b] \implies \int_a^b f(x)\,dx \geq 0 \tag{14.5.8}$$

$$f(x)\geq g(x) \text{ on } [a,b] \implies \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx. \tag{14.5.9}$$

In particular:

$$m \leq f(x) \leq M \text{ on } [a,b] \Longrightarrow m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a). \tag{14.5.10}$$

Definition 14.5.1. The *average value of f on* $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

Example 14.5.2. Find:

1. $\displaystyle\int_1^1 x^2\cos x\,dx$;

   Solution. By (14.5.1),

$$\int_1^1 x^2\cos x\,dx = 0.$$

2. $\displaystyle\int_9^4 \sqrt{t}\,dt$

   given that $\int_4^9 \sqrt{x}\,dx = \frac{38}{3}$;

   Solution. In view of (14.5.2)

$$\int_9^4 \sqrt{t}\,dt = \int_9^4 \sqrt{x}\,dx = -\int_4^9 \sqrt{x}\,dx = -\frac{38}{3}.$$

3. $\displaystyle\int_0^1 5-6x^2\,dx$

   given that $\int_0^1 x^2\,dx = \frac{1}{3}$;

Solution.

$$\begin{aligned} \int_0^1 5-6x^2\,dx &= \int_0^1 5dx - 6\int_0^1 x^2\,dx \text{ by (14.5.3) and (14.5.4)} \\ &= 5\times 1+6\times\frac{1}{3}=7 \quad \text{by (14.5.6) and (14.5.7).} \end{aligned}$$

4. $\displaystyle\int_1^4 f(x)\,dx$

given that $\int_1^5 f(x)\,dx = 12$ and $\int_4^5 f(x)\,dx = 3.6$.

Solution. By (14.5.5),

$$\int_1^5 f(x)\,dx = \int_1^4 f(x)\,dx + \int_4^5 f(x)\,dx \Longrightarrow \int_1^4 f(x)\,dx = 12-3.6 = 8.4.$$

Example 14.5.3 (inequalities)

1. Justify that

$$\int_1^2 \sqrt{5-x}\,dx \geq \int_1^2 \sqrt{x+1}\,dx;$$

Solution. If $1\leq x\leq 2$ then $5-2\leq 5-x\leq 5-1$ so that

$$\sqrt{3}\leq\sqrt{5-x}\leq 2$$

because $\sqrt{x}$ is an increasing function. On the other hand, $2\leq x+1\leq 3$, so that

$$\sqrt{2}\leq\sqrt{x+1}\leq\sqrt{3}.$$

Hence

$$\sqrt{x+1}\leq\sqrt{5-x} \text{ for all } x \text{in } [1,2],$$

so that, by (14.5.9),

$$\int_1^2 \sqrt{5-x}\,dx \geq \int_1^2 \sqrt{x+1}\,dx.$$

2. Justify that

$$\frac{\pi}{6}\leq\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin x\,dx \leq \frac{\pi}{3}.$$

Solution. Note that $\sin x$ is increasing on $(\frac{\pi}{6}, \frac{\pi}{2})$ so that

$$\frac{1}{2} = \sin\frac{\pi}{6} \leq \sin x \leq \sin\frac{\pi}{2} = 1 \text{ for all } x \text{in } \left[\tfrac{\pi}{6}, \tfrac{\pi}{2}\right]$$

and, by (14.5.10), we conclude that

$$\frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{6}\right) \leq \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin x\, dx \leq \frac{\pi}{2} - \frac{\pi}{6}$$

that is,

$$\frac{\pi}{6} \leq \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin x\, dx \leq \frac{\pi}{3}.$$

Example 14.5.4. Find the average value of $f(x) = \sqrt{4 - x^2}$ on [–2, 2].

Solution. The graph of *f* is the upper half of the circle centered at the origin and of radius 2. Thus $\int_{-2}^{2} \sqrt{4 - x^2}\, dx$ is half the area of the corresponding disk, that is,

$$\int_{-2}^{2} \sqrt{4 - x^2}\, dx = \frac{4\pi}{2} = 2\pi.$$

Thus the average value of *f* over [–2, 2] is

$$\frac{1}{4}\int_{-2}^{2} \sqrt{4 - x^2}\, dx = \frac{\pi}{2}.$$

14.6 M11 Sample Quiz

1. Find the following sums:

$$\sum_{j=2}^{5}(2j+1) = ?$$
$$\sum_{k=1}^{200} k = ?$$
$$\sum_{i=1}^{50} i^2 = ?$$

2. Evaluate the following integrals

$$\int_3^3 \sin(x^2)\,dx$$
$$\int_0^4 \sqrt{16-x^2}\,dx$$
$$\int_0^4 2x-1\,dx$$
$$\int_0^4 \frac{\sqrt{16-x^2}}{2}+2x-1\,dx$$

3. If f is integrable on [1, 5], $\int_1^3 f(x)\,dx = 7$ and $\int_1^5 f(x)\,dx = 16$, what are the values of $\int_3^5 f(x)\,dx$ and of $\int_5^1 f(x)\,dx$?
4. What is the average value of $f(x) = \sqrt{5-x^2}$ on $[0, \sqrt{5}]$?

14.7 M11 Sample Quiz Solutions

1. Find the following sums:

$$\begin{aligned}
\sum_{j=2}^{5}(2j+1) &= 5+7+9+11=32\\
\sum_{k=1}^{200}k &= \frac{200\times 201}{2}=20100\\
\sum_{i=1}^{50}i^2 &= \frac{50\times 51\times 101}{6}=42925,
\end{aligned}$$

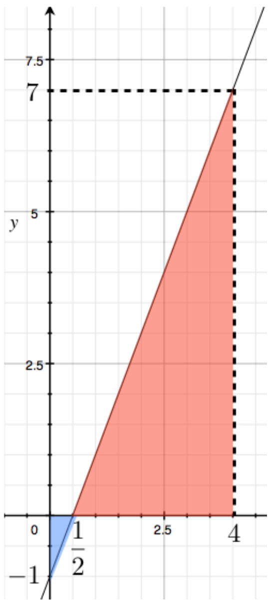
using the formulas

$$\begin{aligned}
\sum_{k=1}^{n}k &= \frac{n(n+1)}{2}\\
\sum_{k=1}^{n}k^2 &= \frac{n(n+1)(2n+1)}{6}.
\end{aligned}$$

2. Evaluate the following integrals

$$\begin{aligned}
\int_3^3 \sin(x^2)\,dx &= 0 \text{ b/c } \textstyle\int_a^a f(x)\,dx = 0 \\
\int_0^4 \sqrt{16-x^2}\,dx &= \frac{16\pi}{4} = 4\pi \text{ , the area of } \tfrac{1}{4}\text{of a disk of radius 4.} \\
\int_0^4 2x-1\,dx &= \frac{49}{4} - \frac{1}{4} = \frac{48}{4} = 12 \text{ (see figure below)} \\
\int_0^4 \frac{\sqrt{16-x^2}}{2} + 2x - 1\,dx &= \frac{1}{2}\int_0^4 \sqrt{16-x^2}\,dx + \int_0^4 2x-1\,dx = \frac{4\pi}{2} + 12 = 2\pi + 12
\end{aligned}$$

In the picture below, the blue triangle has area $\frac{1}{2}\cdot\frac{1}{2}\cdot 1 = \frac{1}{4}$ and the red triangle has area $\frac{1}{2}\cdot\frac{7}{2}\cdot 7 = \frac{49}{4}$.



3. If f is integrable on [1, 5], $\int_1^3 f(x)\,dx = 7$ and $\int_1^5 f(x)\,dx = 16$, what are the values of $\int_3^5 f(x)\,dx$ and of $\int_5^1 f(x)\,dx$?

$$\int_1^3 f(x)\,dx + \int_3^5 f(x)\,dx = \int_1^5 f(x)\,dx \iff \int_3^5 f(x)\,dx = \int_1^5 f(x)\,dx - \int_1^3 f(x)\,dx = 16-7 = 9.$$

$$\int_5^1 f(x)\,dx = -\int_1^5 f(x)\,dx = -16.$$

4. What is the average value of $f(x) = \sqrt{5 - x^2}$ on $[0, \sqrt{5}]$?

By definition the average value is

$$\frac{1}{\sqrt{5}} \int_0^{\sqrt{5}} \sqrt{5 - x^2}\, dx = \frac{5\pi}{4\sqrt{5}} = \frac{\sqrt{5}\pi}{4},$$

taking into account that $\int_0^{\sqrt{5}} \sqrt{5 - x^2}\, dx$ is one fourth of the area of a disk of radius $\sqrt{5}$ because $y = \sqrt{5 - x^2}$ is the upper half of the circle centered at the origin and of radius $\sqrt{5}$.

15 M12: Indefinite Integral

15.1 Antiderivatives

Watch the video at
https://www.youtube.com/watch?v=oV0uEntR7Gc&list=PL265CB737C01F8961&index=69.

Abstract This video introduces the notion of antiderivative, and the integral notation, and establishes a first set of formulas for indefinite integrals.

Definition 15.1.1. A function $F(x)$ is an *antiderivative* of another function *f* on an interval *I* if

$$F'(x) = f(x) \text{ for all } x \text{ in } I.$$

Proposition 15.1.2. *If F is an antiderivative of f on I, then all antiderivative of f on I are obtained under the form*

$$F(x) + k$$

where k is a constant that can range over all real numbers.

The notation

$$\int f(x)\,dx$$

denotes the family of all antiderivative of *f* on an interval, and is called *indefinite integral of f*:

$$\int f(x)\,dx := F(x) + k;\ k \in (-\infty, \infty),$$

where *F* is an antiderivative of *f*.

The following formulas are established, where c denotes a constant:

$$\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx \tag{15.1.1}$$

$$\int c \cdot f(x)\,dx = c \cdot \int f(x)\,dx \tag{15.1.2}$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c;\ \text{for any } n \neq -1. \tag{15.1.3}$$

$$\int \cos x\,dx = \sin x + c \tag{15.1.4}$$

$$\int \sin x\,dx = -\cos x + c \tag{15.1.5}$$

$$\int \sec^2 x\,dx = \tan x + c. \tag{15.1.6}$$

15.2 Antiderivatives: Examples

Watch the videos at
https://www.youtube.com/watch?v=3Nn35QJg_o0&list=PL265CB737C01F8961&index=70

and
https://www.youtube.com/watch?v=3JF7BpmGMCQ&list=PL265CB737C01F8961&index=71

Abstract These videos go over examples to illustrate how to use the formulas obtained in Section 15.1 to calculate antiderivatives. Applications to the study of free fall are also considered.

Example 15.2.1. Find the general form of an antiderivative of:

1. $f(x) = 6x^2 - 8x + 3$;

   Solution. By (15.1.1) and (15.1.2), we have

   $$\int 6x^2 - 8x + 3\,dx = 6\int x^2\,dx - 8\int x\,dx + 3\int dx,$$

   so that, using (15.1.3), we obtain:

   $$\begin{aligned} \int 6x^2 - 8x + 3\,dx &= 6 \cdot \frac{x^3}{3} - 8 \cdot \frac{x^2}{2} + 3x + c \\ &= 2x^3 - 4x^2 + 3x + c. \end{aligned}$$

2. $f(x) = 5x^{\frac{1}{4}} - 7x^{\frac{3}{4}}$;

   Solution. By (15.1.1) and (15.1.2), we have

   $$\int 5x^{\frac{1}{4}} - 7x^{\frac{3}{4}}\,dx = 5\int x^{\frac{1}{4}}\,dx - 7\int x^{\frac{3}{4}}\,dx,$$

   so that, using (15.1.3), we obtain:

   $$\begin{aligned} \int 5x^{\frac{1}{4}} - 7x^{\frac{3}{4}}\,dx &= 5 \cdot \frac{x^{\frac{5}{4}}}{\frac{5}{4}} - 7\frac{x^{\frac{7}{4}}}{\frac{7}{4}} + c \\ &= 4\left(x^{\frac{5}{4}} - x^{\frac{7}{4}}\right) + c. \end{aligned}$$

3. $f(\theta) = \cos\theta - 2\tan^2\theta - 2$;

Solution. By (15.1.1) and (15.1.2), we have:

$$\int \cos\theta - 2\tan^2\theta - 2\,d\theta = \int \cos\theta\,d\theta - 2\int \tan^2\theta + 1\,d\theta.$$

Since $\tan^2\theta + 1 = \sec^2\theta$, we can use (15.1.4) and (15.1.6) to the effect that:

$$\begin{aligned}\int \cos\theta - 2\tan^2\theta - 2\,d\theta &= \int \cos\theta\,d\theta - 2\int \sec^2\theta\,d\theta.\\ &= \sin\theta - 2\tan\theta + c.\end{aligned}$$

4. $f(t) = \sqrt{t} + \sqrt[3]{t} - \frac{1}{t^2} + 1$;

Solution. By (15.1.1) and (15.1.3) we have:

$$\begin{aligned}\int \sqrt{t} + \sqrt[3]{t} - \frac{1}{t^2} + 1\,dt &= \int t^{\frac{1}{2}} + t^{\frac{1}{3}} - t^{-2} + 1\,dt\\ &= \int t^{\frac{1}{2}}\,dt + \int t^{\frac{1}{3}}\,dt - \int t^{-2}\,dt + \int dt\\ &= \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + \frac{t^{\frac{4}{3}}}{\frac{4}{3}} + t^{-1} + t + c\\ &= \frac{2}{3}\sqrt{t^3} + \frac{3}{4}\sqrt[3]{t^4} + \frac{1}{t} + t + c.\end{aligned}$$

5. $f(x) = x^2(1+3x^3)$;

Solution. We have no formula for antiderivatives of products or quotients, but

$$f(x) = x^2(1+3x^3) = x^2 + 3x^5,$$

so that

$$\int f(x)\,dx = \int x^2 + 3x^5\,dx = \frac{x^3}{3} + \frac{1}{2}x^6 + c.$$

6. $f(x) = \frac{2x^2+x-2}{x^4}$;

Solution. Since

$$f(x) = \frac{2x^2}{x^4} + \frac{x}{x^4} - \frac{2}{x^4} = 2x^{-2} + x^{-3} - 2x^{-4},$$

we have, via (15.1.1), (15.1.2) and (15.1.3)

$$\begin{aligned}\int f(x)\,dx &= \int 2x^{-2}+x^{-3}-2x^{-4}\,dx\\ &= -2x^{-1}-\frac{1}{2}x^{-2}+\frac{2}{3}x^{-3}+c\\ &= -\frac{2}{x}-\frac{1}{2x^2}+\frac{2}{3x^3}+c.\end{aligned}$$

7. $f(x)=\frac{\sin(2x)}{\sin x}$;

   Solution. Since

$$f(x)=\frac{\sin(2x)}{\sin x}=\frac{2\sin x\cos x}{\sin x}\underset{\sin x\neq 0}{=}2\cos x,$$

   we conclude from (15.1.4) that

$$\int f(x)\,dx=2\sin x+c,$$

   on an interval where *f* is defined, that is, on an interval where $\sin x$ does not take the value 0.

8. $f(t)=\sqrt{2t}$.

   Solution. Since

$$f(t)=\sqrt{2t}=\sqrt{2}\cdot\sqrt{t}=\sqrt{2}t^{\frac{1}{2}},$$

   we conclude from (15.1.2) and (15.1.3) that

$$\begin{aligned}\int f(t)\,dt &= \sqrt{2}\cdot\frac{t^{\frac{3}{2}}}{\frac{3}{2}}+c\\ &= \frac{2\sqrt{2}}{3}\sqrt{t^3}+c.\end{aligned}$$

Example 15.2.2. Find:

1. the function *f* given that $f''(x)=6x+12x^2$, $f'(0)=3$ and $f(0)=-5$;

   Solution. Since f' is an antiderivative of f'', we have

$$f'(x)=\int f''(x)\,dx=3x^2+4x^3+c.$$

Moreover, evaluating at $x = 0$, we have $f'(0) = c$. But we are given $f'(0) = 3$, so that $c = 3$. Thus

$$f'(x) = 3x^2 + 4x^3 + 3.$$

Since *f* is an antiderivative of f', we have

$$f(x) = \int f'(x)\,dx = x^3 + x^4 + 3x + d.$$

Moreover, evaluating at $x = 0$, we have $f(0) = d$. But we are given $f(0) = -5$, so that $d = -5$ and

$$f(x) = x^3 + x^4 + 3x - 5.$$

2. the function *f* given that $f''(x) = 6x + 6$, $f(0) = 4$ and $f(1) = 3$.

   Solution. Since f' is an antiderivative of f'', we have

   $$f'(x) = \int f''(x)\,dx = 3x^2 + 6x + c.$$

   Since *f* is an antiderivative of *f*, we have

   $$f(x) = \int f'(x)\,dx = x^3 + 3x^2 + cx + d.$$

   Moreover, evaluating at $x = 0$, we have $f(0) = d$, but $f(0) = 4$ is given, so that $d = 4$. Evaluating now at $x = 1$, we obtain $f(1) = 4 + c + 4 = 8 + c$. But $f(1) = 3$ is given so that $8 + c = 3$, that is, $c = -5$. Thus,

   $$f(x) = x^3 + 3x^2 - 5x + 4.$$

Example 15.2.3. A stone is dropped from the 450 meters high CN tower.

1. Find the distance of the stone to the ground *t* seconds later, given than acceleration from gravity is $g = -9.8\,m/s^2$.

   Solution. The rate of change of this distance *d* with respect to time is the velocity $v(t)$ whose rate of change is the constant acceleration *g*. Thus

   $$v(t) = \int g\,dt = gt + c.$$

   Since the stone is dropped with velocity 0, $v(0) = 0 = c$, that is, $v(t) = gt$. Thus

$$d(t) = \int v(t)\,dt = \frac{g}{2}t^2 + k.$$

Moreover, the distance to the ground at time 0 is $d(0) = 450 = k$, so that

$$d(t) = -4.9t^2 + 450\,m.$$

2. How long will it take the stone to reach the ground?

   Solution. The stone reaches the ground when $d(t) = 0$, that is,

$$-4.9t^2 + 450 = 0 \iff t^2 = \frac{450}{4.9} \underset{t \geq 0}{\iff} t = \sqrt{\frac{450}{4.9}} \approx 9.6\,s,$$

   so that the stone reaches the ground after approximately 9.6 seconds.

3. With what velocity does it strike the ground?
   Solution. The speed at impact is

$$\left| v\left(\sqrt{\frac{450}{4.9}}\right) \right| = 9.8 \times \sqrt{\frac{450}{4.9}} \approx 94\,m/s.$$

4. If the stone is thrown downward with a speed of $5\,m/s$, how long does it take for the stone to reach the ground?

   Solution. We proceed the same way, expect that $v(0) = -5\,m/s$, so that $v(t) = gt - 5$, and

$$d(t) = \int v(t)\,dt = \frac{g}{2}t^2 - 5t + k,$$

   where $k = d(0) = 450$, so that

$$d(t) = -4.9t^2 - 5t + 450.$$

   The stone hits the ground when $d(t) = 0$, that is,

$$-4.9t^2 - 5t + 450 = 0,$$

   which is a quadratic equation in *t*, whose solutions are obtained via the quadratic formula:

$$t = \frac{5 \pm \sqrt{25 + 4 \times 4.9 \times 450}}{-9.8},$$

   only one of which is positive:

$$t = \frac{-5 + \sqrt{8845}}{9.8} \approx 9.1\, s,$$

so that the stone reaches the ground after approximately 9.1 seconds.

15.3 M12 Sample Quiz: indefinite integrals

1. Evaluate the following integrals:

   a) $\displaystyle\int 5x^3 + 3x^2 - 13\,dx.$

   b) $\displaystyle\int 3\cos x - 2\sin x\,dx$

   c) $\displaystyle\int 3\sqrt[3]{x^2} - 2\sqrt{x^3}\,dx$

   d) $\displaystyle\int \frac{5+4t^3}{t^2}\,dt$

2. Find the function $f(x)$ given $f(0)=1$, $f(1)=3$ and $f''(x) = 2x^2 - 3x + 1$.
3. A particle is moving with a velocity (in meters per second)

   $$v(t) = 2\sin t + \cos t$$

   at time t (in seconds) and with initial position $s(0)=0$. Find the position $s(t)$ at time t.

15.4 M12 Sample Quiz Solutions

1. Evaluate the following integrals:

2. $\displaystyle\int 5x^3 + 3x^2 - 13\,dx = \frac{5}{4}x^4 + x^3 - 13x + C$

   using the power rule $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ and linearity of integrals.

   a) $\displaystyle\int 3\cos x - 2\sin x\,dx = 3\sin x + 2\cos x + C$

   using $\int \cos x\,dx = \sin x + C$, $\int \sin x\,dx = -\cos x + C$ and linearity of integrals.

   b)
$$\begin{aligned}\int 3\sqrt[3]{x^2} - 2\sqrt{x^3}\,dx &= \int 3x^{\frac{2}{3}} - 2x^{\frac{3}{2}}\,dx\\ &= 3\frac{3}{5}x^{\frac{5}{3}} - 2\frac{2}{3}x^{\frac{5}{2}} + C\\ &= \frac{9}{5}x^{\frac{5}{3}} - \frac{4}{3}x^{\frac{5}{2}} + C\\ &= \frac{9}{5}\sqrt[3]{x^5} - \frac{4}{3}\sqrt{x^5} + C.\end{aligned}$$

   c)
$$\begin{aligned}\int \frac{5+4t^3}{t^2}\,dt &= \int \frac{5}{t^2} + 4t\,dt\\ &= \int 5t^{-2} + 4t\,dt\\ &= -5t^{-1} + 2t^2 + C\\ &= -\frac{5}{t} + 2t^2 + C.\end{aligned}$$

3. Find the function $f(x)$ given $f(0) = 1$, $f(1) = 3$ and $f''(x) = 2x^2 - 3x + 1$.

   f' is an antiderivative of f'' so that f' is of the form

$$f'(x) = \int 2x^2 - 3x + 1\,dx = \frac{2}{3}x^3 - \frac{3}{2}x^2 + x + C.$$

   Similarly, f is an antiderivative of f' and has the form

$$\begin{aligned}f(x) &= \int f'(x)\,dx\\ &= \int \frac{2}{3}x^3 - \frac{3}{2}x^2 + x + C\,dx\\ &= \frac{1}{6}x^4 - \frac{1}{2}x^3 + \frac{x^2}{2} + Cx + D.\end{aligned}$$

   Plugging in $x = 0$ gives $f(0) = D$. Since $f(0)$ is given to be 1, we conclude that $D = 1$. Plugging in $x = 1$ gives

$$f(1) = \frac{1}{6} - \frac{1}{2} + \frac{1}{2} + C + 1 = C + \frac{7}{6}.$$

Since $f(1)=3$, we conclude that $C+\frac{7}{6}=3$, that is, $C=\frac{11}{6}$. Thus

$$f(x)=\frac{1}{6}x^4-\frac{1}{2}x^3+\frac{1}{2}x^2+\frac{11}{6}x+1.$$

4. A particle is moving with a velocity (in meters per second)

$$v(t)=2\sin t+\cos t$$

at time *t* (in seconds) and with initial position $s(0)=0$. Find the position $s(t)$ at time *t*.

The position $s(t)$ is an antiderivative of the velocity $v(t)$ and has therefore the form

$$s(t)=\int 2\sin t+\cos t\,dt=-2\cos t+\sin t+C.$$

Plugging in $t=0$, we have $s(0)=-2+C$. Since $s(0)=0$ is given, we conclude that $-2+C=0$, that is, $C=2$. Thus

$$s(t)=-2\cos t+\sin t+2.$$

16 M13: Calculating Integrals

16.1 Fundamental Theorem of Calculus

Watch the videos at
https://www.youtube.com/watch?v=MkeATgzdVKs&list=PL265CB737C01F8961&index=72

and
https://www.youtube.com/watch?v=8RLF1pPRjEE&list=PL265CB737C01F8961&index=73

Abstract These video states the Fundamental Theorem of Calculus and goes over a few basic examples of applications.

Theorem 16.1.1 (Fundamental Theorem of Calculus) *Let f be a continuous function on* $[a, b]$. *The function defined by*

$$g(x) = \int_a^x f(t)\,dt$$

is continuous on $[a, b]$, *differentiable on* (a, b) *and*

$$g'(x) = f(x).$$

Corollary 16.1.2. *If f is continuous on* $[a, b]$ *and F is an antiderivative of f on* $[a, b]$ *then*

$$\int_a^b f(x)\,dx = [F(x)]_a^b := F(b) - F(a).$$

Thus, to calculate a definite integral of a continuous function, you only need to find an antiderivative, that is, find the indefinite integral, which justifies, *a posteriori,* the terminology.

Example 16.1.3. Evaluate the following integrals:

1. $\displaystyle\int_0^1 x^3 + x - 1\,dx$

   Solution. $f(x) = x^3 + x - 1$ is continuous on $[0, 1]$ so that Corollary 16.1.2 applies, via (15.1.3), to the effect that

$$\int_0^1 x^3 + x - 1\,dx = \left[\frac{x^4}{4} + \frac{x^2}{2} - x\right]_0^1 = \left(\frac{1}{4} + \frac{1}{2} - 1\right) - 0 = -\frac{1}{4}.$$

2. $\displaystyle\int_0^2 x^2(1-x^3)\,dx$

Solution. The function $f(x) = x^2(1-x^3) = x^2 - x^5$ is continuous on $[0,2]$ so that Corollary 16.1.2 applies, via (15.1.3), to the effect that

$$\begin{aligned}\int_0^2 x^2(1-x^3)\,dx &= \int_0^2 x^2 - x^5\,dx \\ &= \left[\frac{x^3}{3} - \frac{x^6}{6}\right]_0^2 = \frac{8}{3} - \frac{32}{3} = -\frac{24}{3} = -8.\end{aligned}$$

3. $\displaystyle\int_0^\pi \cos\theta\,d\theta$

Solution. The cosine function is continuous on $[0,\pi]$ so that Corollary 16.1.2 applies to the effect that

$$\int_0^\pi \cos\theta\,d\theta = [\sin\theta]_0^\pi = \sin\pi - \sin 0 = 0,$$

using (15.1.4).

4. $\displaystyle\int_0^{\frac{\pi}{4}} \sec^2 t\,dt$

Solution. The function $f(t) = \sec^2 t$ is continuous on $[0,\frac{\pi}{4}]$ so that Corollary 16.1.2 applies, using (15.1.6), to the effect that

$$\int_0^{\frac{\pi}{4}} \sec^2 t\,dt = [\tan t]_0^{\frac{\pi}{4}} = \tan\frac{\pi}{4} - \tan 0 = 1.$$

Note that Corollary 16.1.2 would **not** apply on, say, $[0,\frac{3\pi}{4}]$ because $f(t) = \sec^2 t$ is discontinuous at $\frac{\pi}{2}$.

5. $\displaystyle\int_0^1 3 + x\sqrt{x}\,dx$

Solution. Since $f(x) = 3 + x\sqrt{x} = 3 + x^{\frac{3}{2}}$ is continuous on $[0,1]$:

$$\int_0^1 3 + x\sqrt{x}\,dx = \left[3x + \frac{2}{5}x^{\frac{5}{2}}\right]_0^1 = 3 + \frac{2}{5} = \frac{17}{5}.$$

6. $\displaystyle\int_{-\pi}^{\pi} f(x)\,dx$ where $f(x) = \begin{cases} x & \text{if } -\pi \leq x \leq 0 \\ \sin x & \text{if } 0 < x \leq \pi \end{cases}$

Solution. In view of (14.5.5),

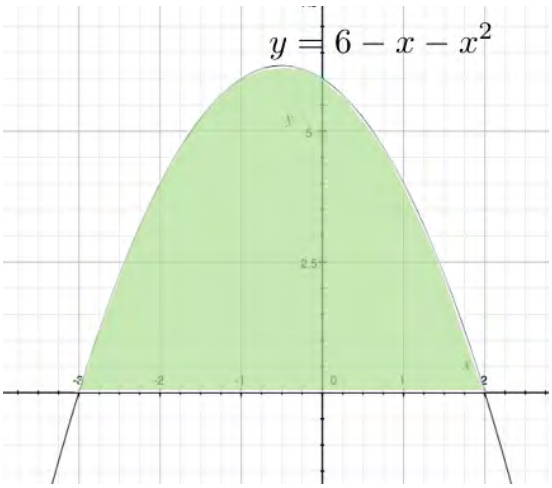
$$\begin{aligned}
\int_{-\pi}^{\pi} f(x)\,dx &= \int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx \\
&= \int_{-\pi}^{0} x\,dx + \int_{0}^{\pi} \sin x\,dx \\
&= \left[\frac{x^2}{2}\right]_{-\pi}^{0} + [-\cos x]_0^{\pi} \\
&= -\frac{\pi^2}{2} - \cos\pi + \cos 0 = 2 - \frac{\pi^2}{2}.
\end{aligned}$$

Example 16.1.4. Find the area of the region bounded by the *x*-axis and the parabola $6 - x - x^2$.

Solution. The function $f(x) = 6 - x - x^2$ is a parabola opening downward, that intersects the *x*-axis if

$$x^2 + x - 6 = 0 = (x-2)(x+3),$$

that is, for $x = -3$ and $x = 2$.



Thus the desired area is the area under the graph of *f* over the interval $[-3, 2]$, where *f* is a non-negative function. In view of Proposition 14.3.4, the area is given by

$$\begin{aligned}
\int_{-3}^{2} 6 - x - x^2\,dx &= \left[6x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-3}^{2} \\
&= \left(12 - 2 - \frac{8}{3}\right) - \left(-18 - \frac{9}{2} + 9\right) \\
&= \frac{125}{6}.
\end{aligned}$$

16.2 Proof of the Fundamental Theorem of Calculus

Watch the video at
https://www.youtube.com/watch?v=DNQs7gHjIuc&list=PL265CB737C01F8961&index=74.

Abstract This video goes over a proof of the Fundamental Theorem of Calculus (Theorem 16.1.1).

16.3 M13 Sample Quiz 1: FTC applied

1. Evaluate the following integrals:

   a) $\displaystyle\int_0^1 4x^3 - 2x^2 + x - 4\,dx$

   b) $\displaystyle\int_1^2 2x^2(x^3+3)\,dx$

   c) $\displaystyle\int_0^{\frac{\pi}{6}} 2\sin x - \cos x\,dx$

   d) $\displaystyle\int_1^4 2x^2\sqrt{x} - \frac{1}{x^3}\,dx$

   e) $\displaystyle\int_0^{2\pi} |\cos x|\,dx$

2. Find the area enclosed by $y = 4 - (x-1)^2$ and $y = 0$.

16.4 M13 Sample Quiz 1 Solutions

1. Evaluate the following integrals:

a) $\displaystyle\int_0^1 4x^3 - 2x^2 + x - 4\,dx = \left[x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2 - 4x\right]_0^1 = 1 - \frac{2}{3} + \frac{1}{2} - 4 = -\frac{13}{6}.$

b)
$$\begin{aligned}\int_1^2 2x^2(x^3+3)\,dx &= \int_1^2 2x^5 + 6x^2\,dx\\ &= \left[\frac{x^6}{3} + 2x^3\right]_1^2\\ &= \left(\frac{2^6}{3} + 2^4\right) - \left(\frac{1}{3} + 2\right)\\ &= 35.\end{aligned}$$

c)
$$\begin{aligned}\int_0^{\frac{\pi}{6}} 2\sin x - \cos x\,dx &= [-2\cos x - \sin x]_0^{\frac{\pi}{6}}\\ &= \left(-2\cdot\frac{\sqrt{3}}{2} - \frac{1}{2}\right) - (-2-0)\\ &= \frac{3-2\sqrt{3}}{2}.\end{aligned}$$

d)
$$\begin{aligned}\int_1^4 2x^2\sqrt{x} - \frac{1}{x^3}\,dx &= \int_1^4 2x^{\frac{5}{2}} - x^{-3}\,dx\\ &= \left[2\cdot\frac{2}{7}\cdot x^{\frac{7}{2}} - \frac{x^{-2}}{-2}\right]_1^4\\ &= \left[\frac{4}{7}x^{\frac{7}{2}} + \frac{1}{2x^2}\right]_1^4\\ &= \left(\frac{4}{7}\cdot 2^7 + \frac{1}{32}\right) - \left(\frac{4}{7} + \frac{1}{2}\right)\\ &= \frac{508}{7} - \frac{15}{32} = \frac{16151}{224}\end{aligned}$$

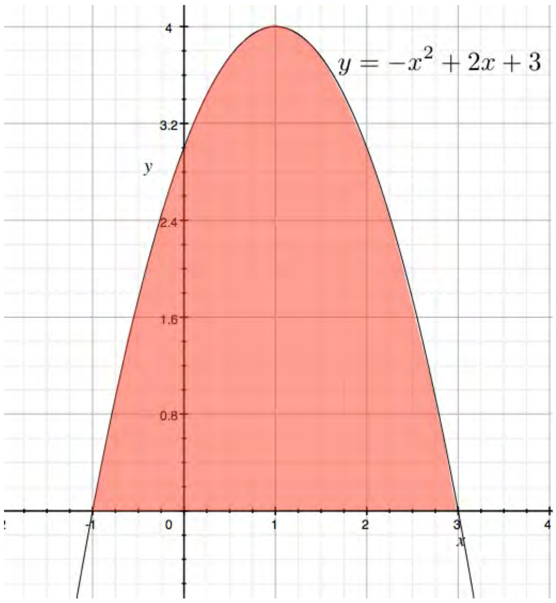
e)
$$\begin{aligned}\int_0^{2\pi} |\cos x|\,dx &= \int_0^{\frac{\pi}{2}} |\cos x|\,dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} |\cos x|\,dx + \int_{\frac{3\pi}{2}}^{2\pi} |\cos x|\,dx\\ &= \int_0^{\frac{\pi}{2}} \cos x\,dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x\,dx + \int_{\frac{3\pi}{2}}^{2\pi} \cos x\,dx\\ &= [\sin x]_0^{\frac{\pi}{2}} - [\sin x]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} + [\sin x]_{\frac{3\pi}{2}}^{2\pi}\\ &= 1 - (-1-1) + (0-(-1))\\ &= 4.\end{aligned}$$

2. Find the area enclosed by $y = 4 - (x-1)^2$ and $y = 0$.
   $y = 4 - (x-1)^2 = -x^2 + 2x + 3$ is a parabola that open downwards. It intersects the x-axis $y = 0$ at solutions to

$$-x^2 + 2x + 3 = 0 = (3 - x)(x + 1),$$

that is, at $x = -1$ and $x = 3$. Thus the area we consider is the area under the graph of $f(x) = -x^2 + 2x + 3$ over $[-1, 3]$ as shown below, that is,

$$\begin{aligned}\int_{-1}^{3} -x^2 + 2x + 3\,dx &= \left[-\frac{x^3}{3} + x^2 + 3x\right]_{-1}^{3} \\ &= (-9 + 9 + 9) - \left(\frac{1}{3} + 1 - 3\right) \\ &= \frac{32}{3}.\end{aligned}$$



16.5 Substitution for indefinite integrals

Watch the videos at
https://www.youtube.com/watch?v=RnRRtxVIi6A&list=PL265CB737C01F8961&index=75

and
https://www.youtube.com/watch?v=EAZlrzEj-Kc&list=PL265CB737C01F8961&index=76

Abstract These videos explain the principal behind substitution and go over a number of examples, in the case of indefinite integrals.

Theorem 16.5.1 *If $u = f(x)$ is a differentiable function whose range is an interval I and g is continuous on I then*

$$\int g(f(x)) \cdot f'(x)\,dx = \int g(u)\,du.$$

Example 16.5.2. Evaluate the following indefinite integrals:

1. $\displaystyle\int 2x\sqrt{1+x^2}\,dx$

   Solution. Let $u = 1 + x^2$. Then $du = 2x\,dx$. Thus

   $$\int 2x\sqrt{1+x^2}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{\frac{3}{2}} + c = \frac{2}{3}\left(1+x^2\right)^{\frac{3}{2}} + c.$$

2. $\displaystyle\int x^3\cos(x^4+2)\,dx$

   Solution. Let $u = x^4 + 2$. Then $du = 4x^3\,dx$ and $x^3\,dx = \frac{1}{4}du$. Thus

   $$\begin{aligned}\int x^3\cos(x^4+2)\,dx &= \frac{1}{4}\int \cos u\,du \\ &= \frac{1}{4}\sin u + c \\ &= \frac{1}{4}\sin(x^4+2) + c.\end{aligned}$$

3. $\displaystyle\int x^2(x^3+5)^9\,dx$

   Solution. Let $u = x^3 + 5$. Then $du = 3x^2\,dx$ and $x^2\,dx = \frac{1}{3}du$. Thus

   $$\begin{aligned}\int x^2(x^3+5)^9\,dx &= \frac{1}{3}\int u^9\,du \\ &= \frac{1}{30}u^{10} + c = \frac{(x^3+5)^{10}}{30} + c.\end{aligned}$$

4. $\displaystyle\int \frac{x}{(x^2+1)^2}\,dx$

Solution. Let $u = x^2+1$. Then $du = 2x\,dx$ and $x\,dx = \frac{1}{2}du$. Thus

$$\begin{aligned}\int \frac{x}{(x^2+1)^2}\,dx &= \frac{1}{2}\int \frac{du}{u^2}\\ &= -\frac{1}{2u} + c = -\frac{1}{2(x^2+1)} + c.\end{aligned}$$

5. $\displaystyle\int \cos^4\theta \sin\theta\,d\theta$

Solution. Let $u = \cos\theta$. Then $du = -\sin\theta\,d\theta$. Thus

$$\int \cos^4\theta\sin\theta\,d\theta = -\int u^4\,du = -\frac{u^5}{5} + c = -\frac{\cos^5\theta}{5} + c.$$

6. $\displaystyle\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$

Solution. Let $u = \sqrt{x}$. Then $du = \frac{dx}{2\sqrt{x}}$ and $\frac{dx}{\sqrt{x}} = 2du$. Thus

$$\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx = 2\int \sin u\,du = -2\cos u + c = -2\cos(\sqrt{x}) + c.$$

7. $\displaystyle\int \sin(\pi t)\,dt$

Solution. Let $u = \pi t$. Then $du = \pi\,dt$ and $dt = \frac{1}{\pi}du$. Thus

$$\int \sin(\pi t)\,dt = \frac{1}{\pi}\int \sin u\,du = -\frac{1}{\pi}\cos u + c = -\frac{1}{\pi}\cos(\pi t) + c.$$

8. $\displaystyle\int \frac{\cos\left(\frac{\pi}{x}\right)}{x^2}\,dx$

Solution. Let $u = \frac{\pi}{x}$. Then $du = -\frac{\pi}{x^2}\,dx$ and $\frac{dx}{x^2} = -\frac{1}{\pi}du$. Thus

$$\int \frac{\cos\left(\frac{\pi}{x}\right)}{x^2}\,dx = -\frac{1}{\pi}\int \cos u\,du = -\frac{1}{\pi}\sin u + c = -\frac{1}{\pi}\sin\left(\frac{\pi}{x}\right) + c.$$

16.6 Substitution for definite integrals

Watch the videos at
https://www.youtube.com/watch?v=rp1pmWktDHE&list=PL265CB737C01F8961&index=77

and
https://www.youtube.com/watch?v=lfUz2OhP9Bw&list=PL265CB737C01F8961&index=78

Abstract These videos present the substitution rule for definite integrals and goes over a number of examples.

Theorem 16.6.1(Substitution for definite integrals) *If f' is continuous on* $[a, b]$ *and* g *is continuous on the range of* $u = f(x)$, *then*

$$\int_a^b g(f(x)) \cdot f'(x)\, dx = \int_{f(a)}^{f(b)} g(u)\, du.$$

Example 16.6.2. Evaluate the following integrals

1. $\displaystyle\int_0^{\sqrt{\pi}} x\cos(x^2)\,dx$

   Solution. Note that $f(x) = x\cos(x^2)$ is continuous on $[0, \sqrt{\pi}]$, so we will be able to use Corollary 16.1.2. Let $u = x^2$. Then $du = 2x\,dx$ and $x\,dx = \frac{1}{2}du$. Moreover, when $x = 0$, $u = 0$ and when $x = \sqrt{\pi}$, $u = \pi$. Thus

$$\int_0^{\sqrt{\pi}} x\cos(x^2)\,dx = \frac{1}{2}\int_0^{\pi}\cos u\,du = \frac{1}{2}\left[\sin u\right]_0^{\pi} = 0.$$

2. $\displaystyle\int_0^1 \frac{dx}{(2x-3)^2}$

   Solution. Note that $f(x) = \frac{1}{(2x-3)^2}$ is continuous on $[0,1]$, so we will be able to use Corollary 16.1.2, even though we would not be able to do so on, say, $[0,2]$ because f is discontinuous at $\frac{3}{2}$. Let $u = 2x-3$. Then $du = 2\,dx$ and $dx = \frac{1}{2}du$. Moreover, when $x = 0$, $u = -3$ and when $x = 1$, $u = -1$. Thus

$$\int_0^1 \frac{dx}{(2x-3)^2} = \frac{1}{2}\int_{-3}^{-1}\frac{du}{u^2} = \frac{1}{2}\left[-\frac{1}{u}\right]_{-3}^{-1} = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}.$$

3. $\displaystyle\int_0^{\frac{\pi}{2}} \cos x\sin(\sin x)\,dx$

   Solution. Let $u = \sin x$. Then $du = \cos x\,dx$ and when $x = 0$, $u = \sin 0 = 0$, and when $x = \frac{\pi}{2}$, $u = \sin\frac{\pi}{2} = 1$. Thus

$$\int_0^{\frac{\pi}{2}} \cos x\sin(\sin x)\,dx = \int_0^1 \sin u\,du = \left[-\cos u\right]_0^1 = 1 - \cos 1.$$

4. $\displaystyle\int_0^7 \sqrt{4+3x}\,dx$

   Solution. Let $u = 4+3x$. Then $du = 3\,dx$ and when $x = 0$, $u = 4$, and when $x = 7$, $u = 25$. Thus

$$\int_0^7 \sqrt{4+3x}\,dx = \frac{1}{3}\int_4^{25}\sqrt{u}\,du = \frac{1}{3}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_4^{25} = \frac{2}{9}(125-8) = 26.$$

5. $\displaystyle\int_0^4 \frac{x}{\sqrt{1+2x}}\,dx$

   Solution. Let $u = 1+2x$. Then $du = 2\,dx$ and when $x = 0$, $u = 1$, and when $x = 4$, $u = 9$. Moreover, $x = \frac{u-1}{2}$. Thus

$$\begin{aligned}\int_0^4 \frac{x}{\sqrt{1+2x}}\,dx &= \frac{1}{2}\int_1^9 \frac{\frac{u-1}{2}}{\sqrt{u}}\,du = \frac{1}{4}\int_1^9 \sqrt{u} - \frac{1}{\sqrt{u}}\,du \\ &= \frac{1}{4}\left[\frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right]_1^9 = \frac{10}{3}.\end{aligned}$$

16.7 Integrals and symmetry

Watch the video at
https://www.youtube.com/watch?v=LOnBQbPn8Kc&list=PL265CB737C01F8961&index=79

Abstract This video examines the particular case of definite integrals of odd or of even functions over an interval centered at the origin.

Definition 16.7.1. A function f is *odd* if whenever x is in the domain, so is $-x$, and

$$f(-x) = -f(x)$$

for all x in the domain; equivalently, the graph of f admits the origin as a center of symmetry.

A function is *even* if whenever x is in the domain, so is $-x$, and

$$f(-x) = f(x)$$

for all x in the domain; equivalently, the graph of f admits the y-axis as an axis of symmetry.

Theorem 16.7.2. *Let* $a > 0$.

1. *If f is odd then*
$$\int_{-a}^{a} f(x)\,dx = 0;$$
2. *If f is even then*
$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

Example 16.7.3. Evaluate the following integrals:

1. $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2 \sin x}{1+x^4}\,dx$

   Solution. The function $f(x) = \frac{x^2\sin x}{1+x^4}$ is odd because $f(-x) = \frac{(-x)^2\sin(-x)}{1+(-x)^4} = -f(x)$ because sin is odd. Thus, by Theorem 16.7.2,
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2 \sin x}{1+x^4}\,dx = 0.$$

2. $\displaystyle\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \tan^3\theta\,d\theta$

   Solution. The function $f(\theta) = \tan^3\theta$ is odd, because both the tangent function and the cubic function are. Thus, by Theorem 16.7.2,
$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \tan^3\theta\,d\theta = 0.$$

3. $\displaystyle\int_{-3}^{3} x\sqrt{x^2+9}\,dx$

   Solution. Even though we could proceed by substitution, it is faster to observe that $f(x) = x\sqrt{x^2+9}$ is odd, so that
$$\int_{-3}^{3} x\sqrt{x^2+9}\,dx = 0.$$

4. $\displaystyle\int_{-2}^{2} x^2+1\,dx$

   Solution. The function $f(x) = x^2+1$ is even so that
$$\int_{-2}^{2} x^2+1\,dx = 2\int_{0}^{2} x^2+1\,dx = 2\left[\frac{x^3}{3}+x\right]_0^2 = \frac{28}{3}.$$

16.8 M13 Sample Quiz 2: substitution

Evaluate the following integrals:

1. $\displaystyle\int x^2 \cos(x^3)\,dx$
2. $\displaystyle\int (4x+1)\sqrt{2x^2+x+1}\,dx$
3. $\displaystyle\int \frac{\sec^2(\sqrt{x})}{\sqrt{x}}\,dx$
4. $\displaystyle\int_1^5 \sqrt{5x+11}\,dx$
5. $\displaystyle\int_0^{\frac{\pi}{2}} \sin x \cos(\cos x)\,dx$
6. $\displaystyle\int_{-2}^{2} \frac{x^6 \sin x}{1+x^4}\,dx$

16.9 M13 Sample Quiz 2 Solutions

1. $$\begin{aligned}\int x^2\cos(x^3)\,dx &= \frac{1}{3}\int \cos(u)\,du \\ &= \frac{1}{3}\sin u + C \\ &= \frac{1}{3}\sin(x^3)+C,\end{aligned}$$

   using $u = x^3$, hence $du = 3x^2\,dx$.

2. $$\begin{aligned}\int (4x+1)\sqrt{2x^2+x+1}\,dx &= \int \sqrt{u}\,du \\ &= \frac{2}{3}u^{\frac{3}{2}} + C \\ &= \frac{2}{3}\left(2x^2+x+1\right)^{\frac{3}{2}} + C,\end{aligned}$$

   using $u = 2x^2+x+1$, hence $du = (4x+1)\,dx$.

3. $$\begin{aligned}\int \frac{\sec^2(\sqrt{x})}{\sqrt{x}}\,dx &= 2\int \sec^2(u)\,du \\ &= 2\tan(u)+C \\ &= 2\tan(\sqrt{x})+C,\end{aligned}$$

   using $u = \sqrt{x}$ hence $du = \frac{1}{2\sqrt{x}}\,dx$.

4. $$\begin{aligned}\int_1^5 \sqrt{5x+11}\,dx &= \frac{1}{5}\int_{16}^{36}\sqrt{u}\,du \\ &= \frac{1}{5}\cdot\frac{2}{3}\cdot\left[u^{\frac{3}{2}}\right]_{16}^{36} \\ &= \frac{2}{15}\left(36^{\frac{3}{2}} - 16^{\frac{3}{2}}\right) \\ &= \frac{2}{15}(216-64) = \frac{304}{15},\end{aligned}$$

   using $u = 5x+11$, hence $du = 5\,dx$. Note that when $x=1$ then $u=16$ and when $x=5$ then $u=36$.

5. $$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin x\cos(\cos x)\,dx &= -\int_1^0 \cos u\,du \\ &= \int_0^1 \cos u\,du \\ &= [\sin u]_0^1 \\ &= \sin(1)-\sin(0) = \sin(1),\end{aligned}$$

   using $u=\cos x$ and thus $du = -\sin x\,dx$, and noting that $u=1$ when $x=0$ and $u=0$ when $x=\frac{\pi}{2}$.

6. $\displaystyle\int_{-2}^{2} \frac{x^6 \sin x}{1+x^4}\, dx = 0$

because the function $f(x) = \frac{x^6 \sin x}{1+x^4}$ is odd and $[-2,2]$ is centered at the origin. Indeed,

$$f(-x) = \frac{(-x)^6 \sin(-x)}{1+(-x)^4} = -\frac{x^6 \sin x}{1+x^4}$$

because $\sin(-x) = -\sin x$.

17 M14: areas and other applications

171 Area between two curves

Watch the videos at
https://www.youtube.com/watch?v=2CYpuS0PoJI&list=PL265CB737C01F8961&index=80

and
https://www.youtube.com/watch?v=_uJxA1eo_2A&list=PL265CB737C01F8961&index=81

and
https://www.youtube.com/watch?v=83GvmDv1CGc&list=PL265CB737C01F8961&index=82

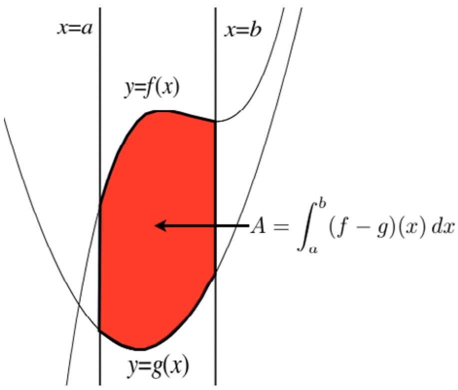
Abstract These videos apply the previous results on calculating definite integrals to calculate area of various plane regions.

Theorem 17.1.1. *Let f and g be two continuous functions on* $[a, b]$ *satisfying*

$$f(x) \geq g(x) \text{ for all } x \text{ in } [a, b].$$

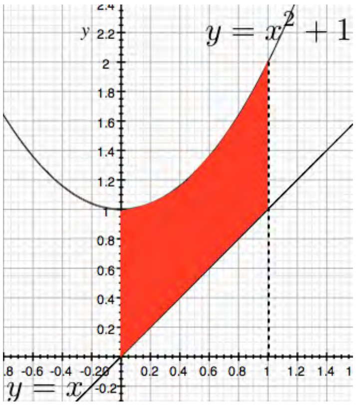
Then the area of the plane region bounded by $x = a$, $x = b$, $y = g(x)$, *and* $y = f(x)$ *is*

$$A = \int_a^b (f - g)(x)\, dx.$$



Example 17.1.2. Find the area of the plane region bounded by $y = x^2 + 1$, $y = x$, $x = 0$ and $x = 1$.

Solution. Since $x \leq 1 \leq 1 + x^2$ for x in $[0, 1]$, the region considered is



Thus the desired area is given by

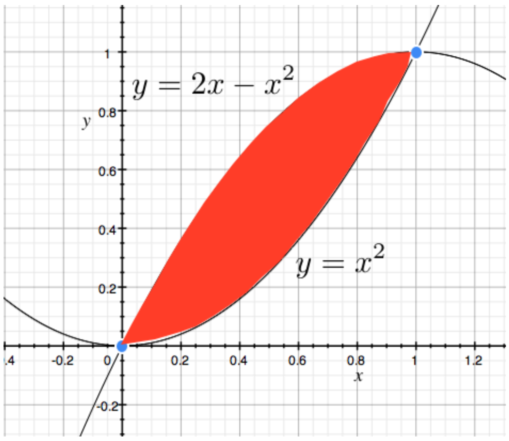
$$\int_0^1 x^2 + 1 - x\,dx = \left[\frac{x^3}{3} + x - \frac{x^2}{2}\right]_0^1 = \frac{1}{3} + 1 - \frac{1}{2} = \frac{5}{6}.$$

Example 17.1.3. Find the area of the plane region bounded by $y = x^2$ and $y = 2x - x^2$.

Solution. The two curves intersect when

$$x^2 = 2x - x^2 \iff 2x^2 - 2x = 2x(x-1) = 0,$$

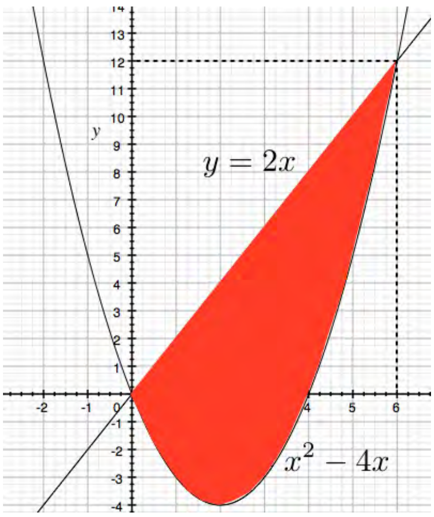
that is, $x = 0$ and $x = 1$. Moreover, $y = x^2$ is a parabola opening upward while $y = 2x - x^2$ is a parabola opening downward. Thus the region considered is sketched as follows:



Thus the desired area is given by

$$\int_0^1 2x - x^2 - x^2\,dx = \int_0^1 2x - 2x^2\,dx = \left[x^2 - \frac{2x^3}{3}\right]_0^1 = 1 - \frac{2}{3} = \frac{1}{3}.$$

Example 17.1.4. Find the area of the plane region sketched below:



Solution. The area is given by

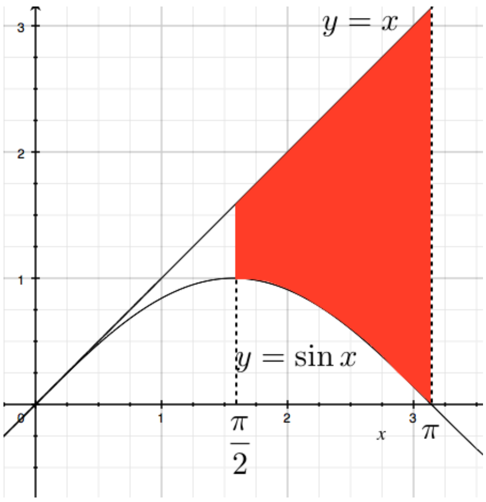
$$\int_0^6 2x - (x^2 - 4x)\,dx = \int_0^6 6x - x^2\,dx = \left[3x^2 - \frac{x^3}{3}\right]_0^6 = 36.$$

Example 17.1.5. Find the area of the plane region bounded by $y = \sin x$, $y = x$, $x = \frac{\pi}{2}$, $x = \pi$.

Solution. Here

$$x \geq \frac{\pi}{2} \geq 1 \geq \sin x$$

so that the region considered is sketched as below:



Thus, the desired area is given by

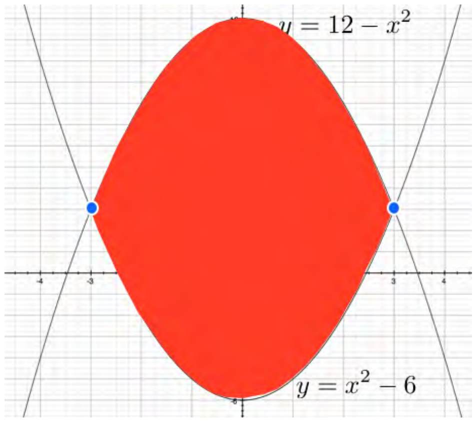
$$\int_{\frac{\pi}{2}}^{\pi} x - \sin x\,dx = \left[\frac{x^2}{2} + \cos x\right]_{\frac{\pi}{2}}^{\pi} = \left(\frac{\pi^2}{2} - 1\right) - \frac{\pi^2}{8} = \frac{3\pi^2}{8} - 1.$$

Example 17.1.6. Find the area of the plane region bounded by $y = 12 - x^2$ and $y = x^2 - 6$.

Solution. The two curves intersect when

$$12 - x^2 = x^2 - 6 \iff x^2 = 9 \iff x = \pm 3,$$

and $y = 12 - x^2$ is a parabola opening downward, while $y = x^2 - 6$ is a parabola opening upward. We obtain the following sketch:

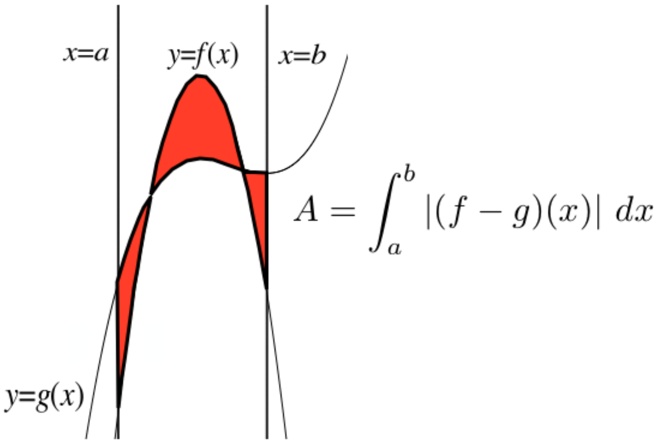


Thus the desired area is given by

$$\begin{aligned}
\int_{-3}^{3} 12 - x^2 - (x^2 - 6)\,dx &= \int_{-3}^{3} 18 - 2x^2\,dx \\
&= 4\int_{0}^{3} 9 - x^2\,dx \text{ because } 9 - x^2 \text{ is even} \\
&= 4\left[9x - \frac{x^3}{3}\right]_0^3 = 4 \times 18 = 72.
\end{aligned}$$

Theorem 17.1.7. *Let f and g be two continuous functions on* $[a, b]$. *The area of the plane region bounded by* $x = a$, $x = b$, $y = g(x)$, *and* $y = f(x)$ *is*

$$A = \int_a^b |(f - g)(x)| \, dx.$$



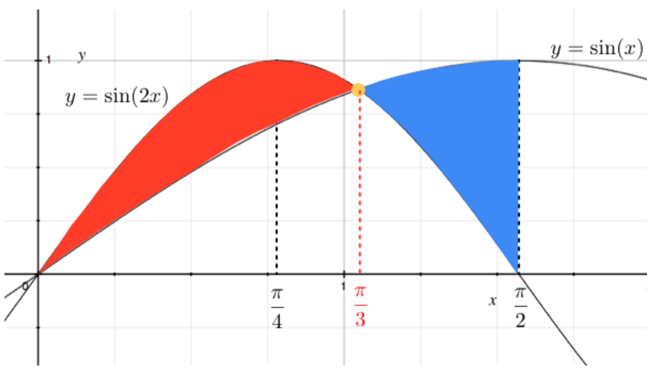
In practice, we need to find the points of intersections of the two curves and calculate the area of each piece of the region.

Example 17.1.8. Find the area of the plane region enclosed by $y = \sin x$, $y = \sin(2x)$, $x = 0$, and $x = \frac{\pi}{2}$.

Solution. The two curves intersect when

$$\sin x = \sin(2x) = 2 \sin x \cos x \iff \sin x \,(1 - 2\cos x) = 0,$$

that is, if $\sin x = 0$, which, in $[0, \frac{\pi}{2}]$, only happens at $x = 0$, or if $\cos x = \frac{1}{2}$, which, in $[0, \frac{\pi}{2}]$, only happens at $x = \frac{\pi}{3}$. We obtain the following sketch:



Thus the desired area is the sum of the red and blue parts, that is

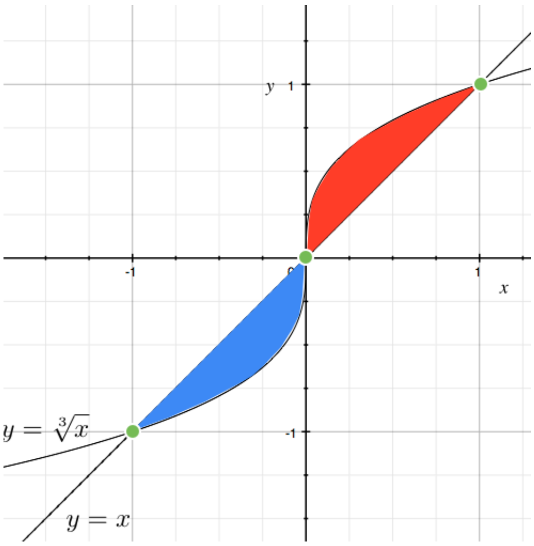
$$\begin{aligned} A &= \int_0^{\frac{\pi}{3}} \sin(2x) - \sin x \, dx + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x - \sin(2x) \, dx \\ &= \left[-\frac{1}{2}\cos(2x) + \cos x \right]_0^{\frac{\pi}{3}} + \left[-\cos x + \frac{1}{2}\cos(2x) \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\ &= \frac{1}{2}. \end{aligned}$$

Example 17.1.9. Find the area of the plane region enclosed by $y = x$, $y = \sqrt[3]{x}$.

Solution. The two curves intersect when

$$x = x^{\frac{1}{3}} \iff x^3 - x = 0 \iff x(x-1)(x+1) = 0,$$

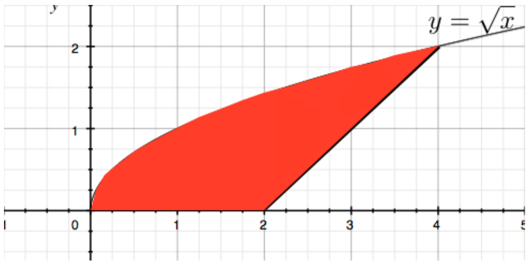
that is, if $x = -1$, $x = 0$ or $x = 1$. Thus we obtain the following sketch of the region:



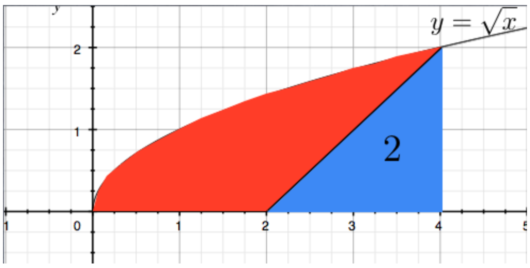
We see that from –1 to 0 $x \geq \sqrt[3]{x}$, while $\sqrt[3]{x} \geq x$ on [0, 1]. Thus, the desired area is the sum of the blue and red area, that is,

$$\begin{aligned} A &= \int_{-1}^{0} x - \sqrt[3]{x} \, dx + \int_0^1 \sqrt[3]{x} - x \, dx \\ &= \left[\frac{x^2}{2} - \frac{3}{4}x^{\frac{4}{3}} \right]_{-1}^{0} + \left[\frac{3}{4}x^{\frac{4}{3}} - \frac{x^2}{2} \right]_0^1 \\ &= \frac{3}{4} - \frac{1}{2} + \frac{3}{4} - \frac{1}{2} = \frac{1}{2}. \end{aligned}$$

Example 17.1.10. Find the area sketched below:



Solution. We could calculate the equation of the segment of line and proceed as before, or observe that the desired area is the area under the graph of $\sqrt{x}$ on $[0,4]$ minus the area of the blue rectangle below, which has area 2:



Thus, the desired area is given by

$$\begin{aligned} A &= \int_0^4 \sqrt{x}\,dx - 2 \\ &= \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^4 - 2 \\ &= \frac{16}{3} - 2 = \frac{10}{3}. \end{aligned}$$

17.2 M14 Sample Quiz 1: areas

Find the value of the area of the plane region bounded by

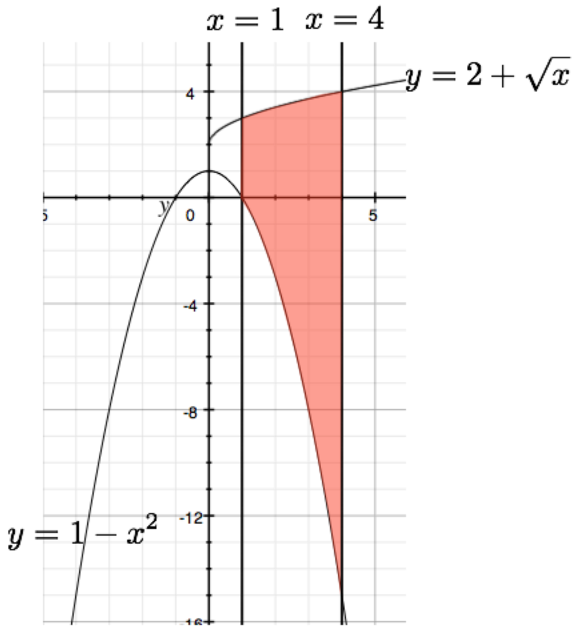
1. $x = 1\,,\ x = 4\,,\ y = 2 + \sqrt{x}$ and $y = 1 - x^2$
2. $y = 0$ and $y = x^3 - 6x^2 + 9x$.
3. $y = 4x - x^2$ and $y = 8x - 2x^2$.
4. $y = \cos x\,,\ y = \sin x\,,\ x = 0$ and $x = \frac{\pi}{2}$.

17.3 M14 Sample Quiz 1 Solutions

Find the value of the area of the plane region bounded by

1. $x = 1$, $x = 4$, $y = 2 + \sqrt{x}$ and $y = 1 - x^2$

   Solution. Here is a sketch of the graphs considered, and the corresponding region:



   Accordingly, the area of the shaded region is given by the integral

$$
\begin{aligned}
A &= \int_1^4 (2 + \sqrt{x}) - (1 - x^2)\, dx \\
&= \int_1^4 x^2 + \sqrt{x} + 1\, dx \\
&= \left[\frac{x^3}{3} + \frac{2}{3}x^{\frac{3}{2}} + x\right]_1^4 \\
&= \left(\frac{64 + 16 + 12}{3}\right) - \left(\frac{1}{3} + \frac{2}{3} + 1\right) \\
&= \frac{86}{3}.
\end{aligned}
$$

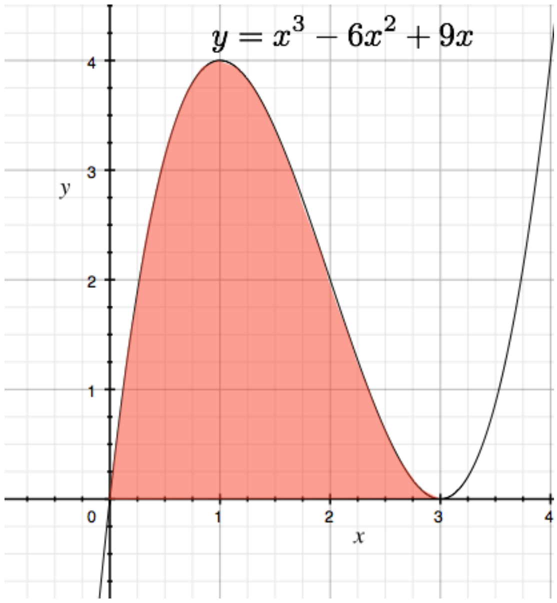
2. $y = 0$ and $y = x^3 - 6x^2 + 9x$.

   Solution. The two curves intersect when

$$
\begin{aligned}
x^3 - 6x^2 + 9x = 0 &\iff x(x^2 - 6x + 9) = 0 \\
&\iff x(x - 3)^2 = 0 \\
&\iff x = 0 \text{ or } x = 3.
\end{aligned}
$$

Moreover, $f(x) = x^3 - 6x^2 + 9x$ is positive between 0 and 3. Hence, the desired region is the area under the graph *f* over the interval $[0,3]$:

$$\begin{aligned} A &= \int_0^3 x^3 - 6x^2 + 9x\,dx \\ &= \left[\frac{x^4}{4} - 2x^3 + \frac{9x^2}{2}\right]_0^3 \\ &= \left(\frac{81}{4} - 54 + \frac{81}{2}\right) - 0 \\ &= \frac{27}{4}. \end{aligned}$$

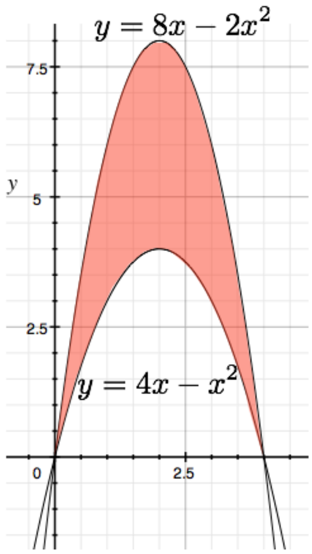


3. $y = 4x - x^2$ and $y = 8x - 2x^2$.

Solution. The curves intersect if

$$4x - x^2 = 8x - 2x^2 \iff x^2 - 4x = x(x-4) = 0,$$

that is, when $x = 0$ and when $x = 4$. Here is a sketch of the region considered:



Thus, the area of the region considered is given by

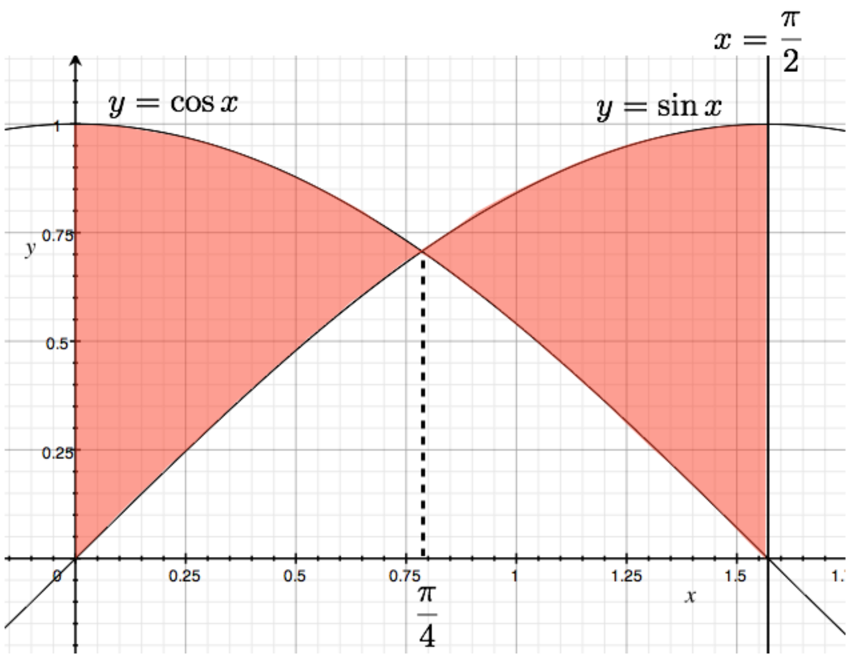
$$\begin{aligned} A &= \int_0^4 (8x - 2x^2) - (4x - x^2)\,dx \\ &= \int_0^4 4x - x^2\,dx \\ &= \left[2x^2 - \frac{x^3}{3}\right]_0^4 \\ &= 32 - \frac{64}{3} = \frac{32}{3}. \end{aligned}$$

4. $y = \cos x$, $y = \sin x$, $x = 0$ and $x = \frac{\pi}{2}$.

Solution. On this interval,

$$\cos x = \sin x \iff x = \frac{\pi}{4}$$

as shown below:



On $[0, \frac{\pi}{4}]$, we have $\cos x \geq \sin x$ and on $[\frac{\pi}{4}, \frac{\pi}{2}]$, we have $\sin x \geq \cos x$. Thus, the area shaded above is given by

$$\begin{aligned} A &= \int_0^{\frac{\pi}{4}} \cos x - \sin x \, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x - \cos x \, dx \\ &= [\sin x + \cos x]_0^{\frac{\pi}{4}} + [-\cos x - \sin x]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\ &= (\sqrt{2} - 1) + \left(-1 + \sqrt{2}\right) \\ &= 2\sqrt{2} - 2. \end{aligned}$$

17.4 Arc Length

Watch the videos at
https://www.youtube.com/watch?v=Piawwy3Scys&list=PL265CB737C01F8961&index=83

Abstract This video establishes a formula for the arc length of the graph of a differentiable function over a closed interval.

Theorem 17.4.1. *Let f be a differentiable function. The length of the graph* $y = f(x)$ *for* $a \leq x \leq b$ *is given by*

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

Example 17.4.2. Find the length of the curve

$$y = x^{\frac{3}{2}} \text{ for } 0 \leq x \leq 1.$$

Solution. Applying the formula from Theorem 17.4.1, we obtain

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + \left(\frac{3}{2}x^{\frac{1}{2}}\right)^2}\, dx \\ &= \int_0^1 \sqrt{1 + \frac{9}{4}x}\, dx. \end{aligned}$$

Letting $u = 1 + \frac{9}{4}x$, we have $du = \frac{9}{4}\, dx$ and when $x = 0$, $u = 1$, and when $x = 1$, $u = \frac{13}{4}$. Thus

$$L = \frac{4}{9}\int_1^{\frac{13}{4}} \sqrt{u}\, du = \frac{4}{9}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^{\frac{13}{4}} = \frac{8}{27}\left(\left(\frac{13}{4}\right)^{\frac{3}{2}} - 1\right).$$

17.5 Work

Watch the videos at
https://www.youtube.com/watch?v=kGE_y9PicEg&list=PL265CB737C01F8961&index=84

and
https://www.youtube.com/watch?v=CL9SaB7N2so&list=PL265CB737C01F8961&index=85

Definition 17.5.1. The *work* done if a body moves a distance d along a straight line by applying a constant force of magnitude F in the direction of motion is

$$W = F \cdot d,$$

where the force is measured in Newton (N) and the distance in meters (m) and the work in Newton.meters, or Joules (J), or force is measured in pound (lb) and the distance in meters (ft) and the work in $lb \cdot ft$.

If the magnitude of the force is no longer constant, we obtain:

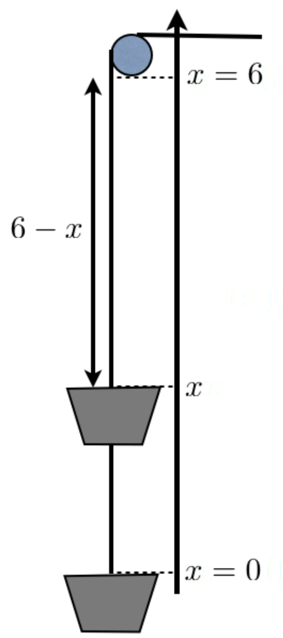
Definition 17.5.2. The *work* done if a body moves along a straight line from $x = a$ to $x = b$ by applying a force of magnitude $F(x)$ that depends on the position, in the direction of motion is

$$W = \int_a^b F(x)\,dx,$$

and the units are as before.

Example 17.5.3. A 2 kg bucket is lifted from the ground into the air by pulling in 6 meters of rope at a constant speed. The rope weighs 250 grams per meter. How much work is done in lifting the rope and bucket?

Solution. The force applied when the bucket is *x* meters from the ground compensates the combined weight of the bucket and the rope.



Since 250 grams is 0.25 kg, the weight of rope when the bucket is *x* meters from the ground is

$$m \cdot g = (6 - x) \cdot 0.25 \cdot g$$

because the length of rope is $6 - x$ meters. Thus the combined weight is

$$F = (2 + (6 - x) \cdot 0.25) \cdot g.$$

Thus using $g = 9.8\,m/s^2$, the work is given by

$$\begin{aligned} W &= \int_0^6 F(x)\,dx = 9.8\int_0^6 2 + \frac{3}{2} - \frac{x}{4}\,dx \\ &= 9.8\int_0^6 \frac{7}{2} - \frac{x}{4}\,dx = 9.8\left[\frac{7}{2}x - \frac{x^2}{8}\right]_0^6 \\ &= 161.7\,J. \end{aligned}$$

Hooke's Law: The force it takes to stretch or compress a spring *x* unit of length from its natural length is proportional to *x*, that is,

$$F = k \cdot x,$$

where *k* is a constant that depends on the spring, called *spring constant.*

Example 17.5.4. A spring has a natural length of 1 meter, and a force of 24 *N* stretches it to a length of 1.8 meters.

1. Find the spring constant;
   Solution. If the spring is stretched from 1 meter to 1.8 meters, the elongation *x* is 0.8 *m*. Here, this is the result of applying a 24 *N* force so that, according to Hooke's Law:

   $$24 = 0.8\,k \iff k = \frac{24}{0.8} = 30\,N/m.$$

2. How much work would it take to stretch this spring from its natural length to a length of 2 meters;
   Solution. Stretching the spring to a length of 2 meters corresponds to an elongation of 1 meter. Thus

   $$W = \int_0^1 F(x)\,dx = \int_0^1 30x\,dx = \left[15x^2\right]_0^1 = 15\,J.$$

3. How far will a 5 kilograms mass attached to the spring stretch it?
   Solution. The force exerted is the weight

   $$m \cdot g = 5 \cdot 9.8 = 49\,N$$

   of the 5 *kg* mass. Since the spring constant is $k = 30\,N/m$, we have

   $$F = kx \iff 49 = 30x \iff x = \frac{49}{30} = 1.63\,m,$$

   so that this force stretches the spring 1.63 meters from its natural length, that is, stretches it to a length of 2.63 meters.

17.6 M14 Sample Quiz 2: applications

1. Find the length of the curve $y = 2\sqrt{x^3} + 2$ for $1 \leq x \leq 2$.
2. What is the work necessary to lift a 100 kilograms package by 50 cm?
3. A 200 pounds cable (with homogeneous density) is 100 feet long and hang vertically from the top of a tall building. How much work is required to lift the cable to the top of the building?
4. A spring has a natural length of 20 cm. If a 25 N force is required to keep it stretched to a length of 30 cm, how much work is required to stretch it from 20 cm to 25 cm?

17.7 M14 Sample Quiz 2 Solutions

1. Find the length of the curve $y = 2\sqrt{x^3} + 2$ for $1 \le x \le 2$.
   Solution. The arc-length formula ensures that the desired length is given by

$$\begin{aligned} L &= \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \int_1^2 \sqrt{1 + \left(2 \cdot \frac{3}{2} \cdot x^{\frac{1}{2}}\right)^2}\, dx \\ &= \int_1^2 \sqrt{1+9x}\, dx \\ &= \frac{1}{9}\int_{10}^{19} \sqrt{u}\, du \\ &= \frac{1}{9}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_{10}^{19} \\ &= \frac{2}{27}\left(19^{\frac{3}{2}} - 10^{\frac{3}{2}}\right), \end{aligned}$$

   using the substitution $u = 1 + 9x$ for which $du = 9\, dx$ and $u = 10$ when $x = 1$ and $u = 19$ when $x = 2$.

2. What is the work necessary to lift a 100 kilograms package by 50 cm (taking $g = 9.8\, m/s^2$ for the acceleration from gravity)?

   Solution. In this case, a constant vertical force compensating the weight does the work, and

$$\begin{aligned} W &= F \cdot d \\ &= (100 \times 9.8) \times 0.5 \\ &= 490\, J \end{aligned}$$

   taking into account that $50\, cm = 0.5\, m$.

3. A 200 pounds cable (with homogeneous density) is 100 feet long and hang vertically from the top of a tall building. How much work is required to lift the cable to the top of the building?

   Solution. The density of the cable is 2 pounds per foot. If *x* feet of cable have already been lifted to the top, the remaining weight of cable is

$$F(x) = 2(100 - x),$$

so that the work needed is given by

$$\begin{aligned} W &= \int_0^{100} 2(100-x)\,dx \\ &= \left[200x - x^2\right]_0^{100} \\ &= 10000\,lb\cdot ft. \end{aligned}$$

4. A spring has a natural length of 20 cm. If a 25 N force is required to keep it stretched to a length of 30 cm, how much work is required to stretch it from 20 cm to 25 cm?

   Solution. Since a force of 25 N is needed to stretch the spring 10 cm from its natural length, the spring constant is (using meters instead of centimeters, hence $10\,cm = 0.1\,m$)

$$k = \frac{F}{x} = \frac{25}{0.1} = 250\,N/m.$$

   The work to stretch the spring from its natural length ($x = 0$) to $5\,cm = 0.05\,m$ from its natural length is therefore

$$\begin{aligned} W &= \int_0^{0.05} 250x\,dx \\ &= \left[125x^2\right]_0^{0.05} \\ &= 125\times(0.05)^2 = 0.3125\,J. \end{aligned}$$

18 M15: Volumes

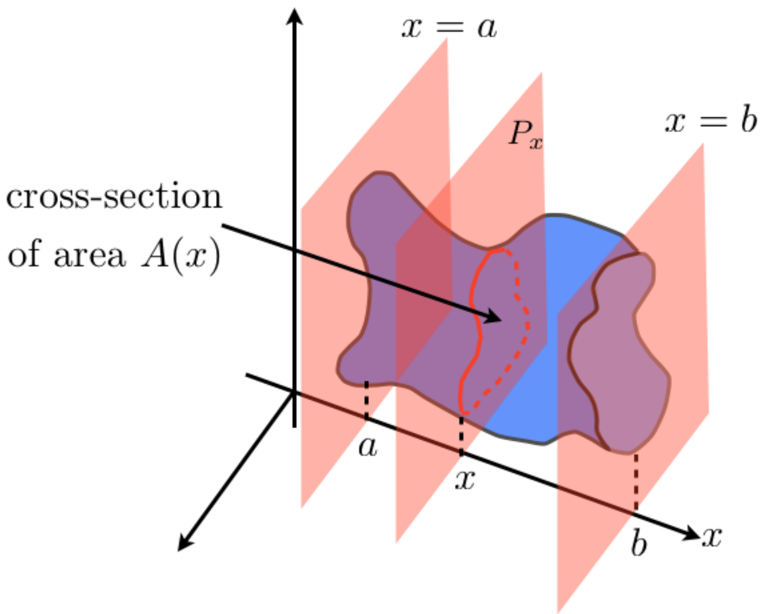
18.1 Volume by cross-section

Watch the videos at
https://www.youtube.com/watch?v=lr4depygEjo&list=PL265CB737C01F8961&index=86

and
https://www.youtube.com/watch?v=VJqNP4yCB00&list=PL265CB737C01F8961&index=87

Abstract These videos present the method of calculating volumes by cross-section, establish a formula, and examines examples that are not solids of revolution.

We look for the volume of the solid *S* that is bounded by two vertical planes $x = a$ and $x = b$, knowing, for each *x* in [*a*, *b*], the area $A(x)$ of each cross-section by a plane P_x of equation $x = cst$, perpendicular to the *x*-axis ([4]):

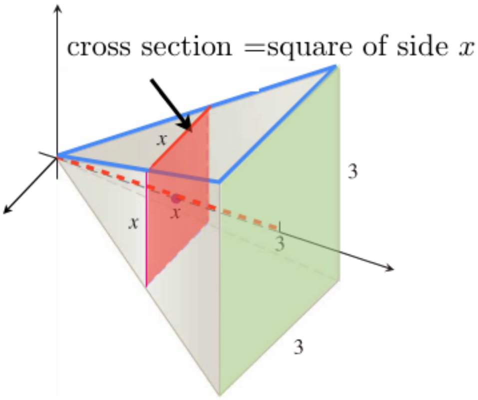


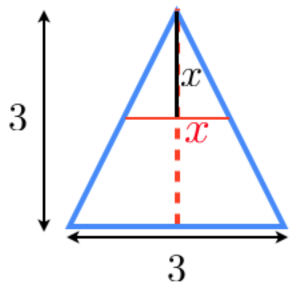
Then, if $A(x)$ is continuous on [*a*, *b*], the volume of the solid is given by

$$V = \int_a^b A(x)\,dx. \tag{18.1.1}$$

Example 18.1.1. Find the volume of a 3 meters high pyramid with a square base of side 3 meters.

Solution. The geometry of the problem is as follows, if we use the axis of symmetry of the pyramid as x-axis, with the origin placed at the vertex:



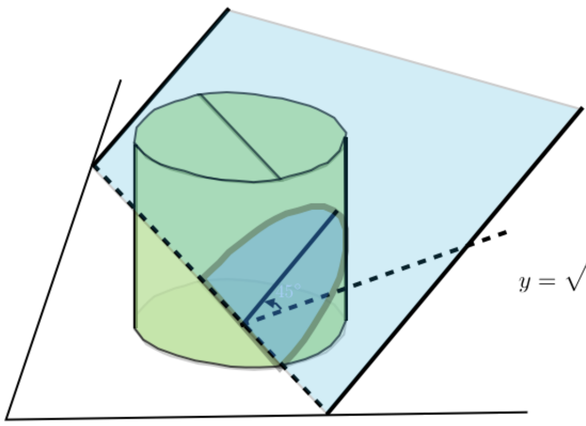


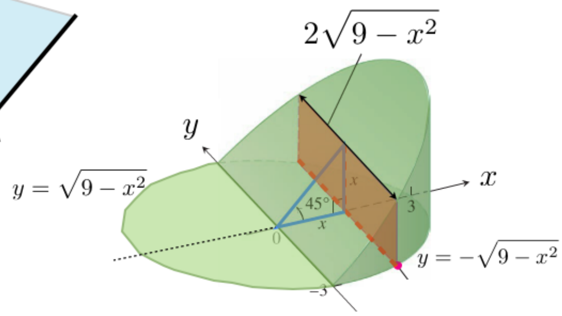
Thus, the cross section is a square of side x, that is, $A(x) = x^2$. In view of (18.1.1), the volume of the pyramid is given by

$$V = \int_0^3 x^2\,dx = \left[\frac{x^3}{3}\right]_0^3 = 9\,m^3.$$

Example 18.1.2. Find the volume of a wedge cut out of a circular cylinder of radius 3 by one plane perpendicular to the axis of the cylinder, and by another crossing the first plane at a $45°$ angle along a diameter of the disk forming the cross-section of the cylinder by the first plane.

Solution. We see from the geometry of the problem





that the cross section is a rectangle whose sides are of length x and $2\sqrt{9-x^2}$ respectively. Thus the area of the cross section is

$$A(x) = 2x\sqrt{9-x^2}$$

and, by (18.1.1), the volume of the wedge is

$$V = \int_0^3 2x\sqrt{9-x^2}\,dx$$

which we calculate by substitution with $u = 9 - x^2$, so that $du = -2x\,dx$:

$$V = -\int_9^0 \sqrt{u}\,du = \int_0^9 \sqrt{u}\,du = \left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^9 = 18\,(\text{unit of length})^3.$$

18.2 Volume by cross-section: solids of revolution

Watch the videos at
https://www.youtube.com/watch?v=t-vV_Wj0gx0&list=PL265CB737C01F8961&index=88

and
https://www.youtube.com/watch?v=wy_179P9EWc&list=PL265CB737C01F8961&index=89

and
https://www.youtube.com/watch?v=P3WUSOW0Sq4&list=PL265CB737C01F8961&index=90

and
https://www.youtube.com/watch?v=Xh0AhuftFTc&list=PL265CB737C01F8961&index=91

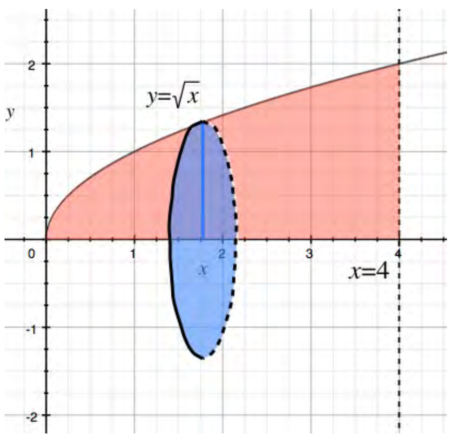
Abstract These videos go over examples of calculations of volumes of solids of revolution using the cross-section method, so that the cross-section is either a disk or a washer. Both cases are examined.

Definition 18.2.1. A *solid of revolution* is a solid obtained by revolving a plane region about a line in that plane.

In this section, we focus on solids obtained by revolving plane regions about horizontal line, as the cross-section method is most efficient in this context. We consider also an example where the rotation is about a vertical line.

Example 18.2.2. Find the volume of the solid of revolution obtained by revolving the plane region bounded by $x = 0$, $x = 4$, $y = 0$ and $y = \sqrt{x}$ about the x-axis.

Solution. First, we sketch the plane region to be rotated, the intersection of $x = cst$ with this region, and the cross-section it generates by rotation about the x-axis:



The cross-section is a disk of radius $\sqrt{x}$, and thus its area is

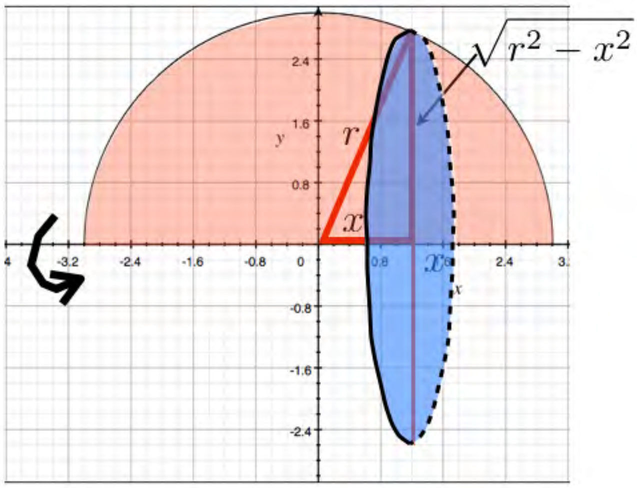
$$A(x) = \pi\left(\sqrt{x}\right)^2 = \pi x,$$

so that, by (18.1.1), the volume of the solid of revolution is

$$V = \int_0^4 \pi x\,dx = \pi\left[\frac{x^2}{2}\right]_0^4 = 8\pi.$$

Example 18.2.3. Establish a formula for the volume of a ball of radius r.

Solution. A solid ball of radius r can be obtained by rotating the area under the graph of $y = \sqrt{r^2 - x^2}$ over $[-r, r]$ about the x-axis:



The cross-section is then a disc of radius $\sqrt{r^2 - x^2}$, so that its area is

$$A(x) = \pi(r^2 - x^2).$$

Thus the volume of the ball is

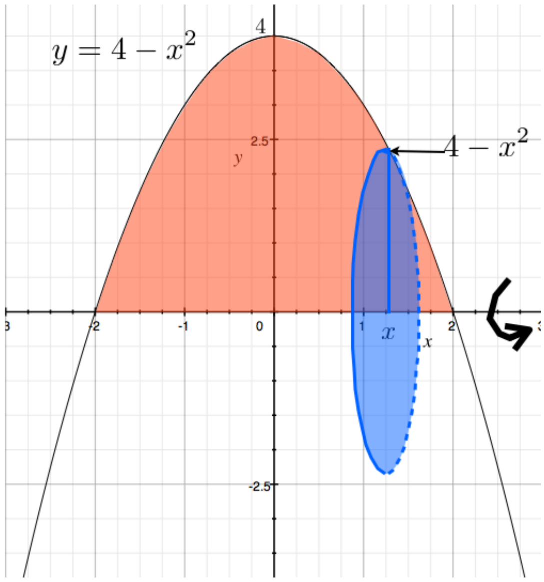
$$V = \int_{-r}^{r} \pi(r^2 - x^2)\,dx = \pi\left[r^2x - \frac{x^3}{3}\right]_{-r}^{r} = \frac{4}{3}\pi r^3.$$

Example 18.2.4. Find the volume of the solid of revolution obtained by revolving the plane region bounded by $y = 0$ and $y = 4 - x^2$ about the x-axis.

Solution. The two curves intersects when

$$4 - x^2 = 0 \iff x = 2 \text{ or } x = -2.$$

We first sketch the plane region to be rotated, the intersection with $x = cst$, and the cross-section generated by rotating it about the *x*-axis:



Thus the cross-section is a disk of radius $4 - x^2$ and has area

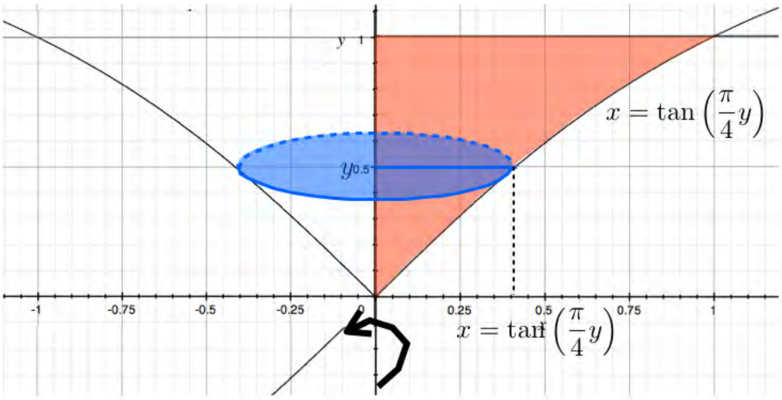
$$A(x) = \pi\left(4 - x^2\right)^2 = \pi\left(x^4 - 8x^2 + 16\right),$$

so that the solid of revolution has volume

$$V = \int_{-2}^{2} \pi\left(x^4 - 8x^2 + 16\right) dx = \pi\left[\frac{x^5}{5} - \frac{8x^3}{3} + 16x\right]_{-2}^{2} = \frac{512\pi}{15}.$$

Example 18.2.5. Find the volume of the solid of revolution obtained by revolving the plane region bounded by $x = 0$, $y = 1$ and $x = \tan\left(\frac{\pi}{4}\right) y$ about the *y*-axis $x = 0$.

Solution. We first sketch the plane region to be rotated, the intersection with $y = cst$, and the cross-section generated by rotating it about the *y*-axis:



Thus the cross section is a disk of radius $x = \tan\left(\frac{\pi}{4}y\right)$ and has area

$$A(y) = \pi \tan^2\left(\frac{\pi}{4}y\right),$$

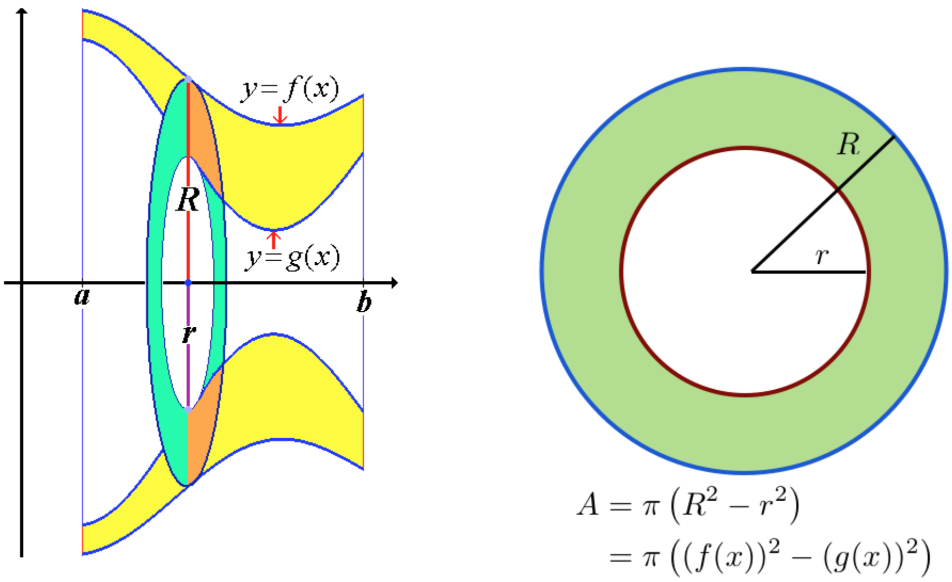
so that the volume of the resulting solid of revolution is

$$\begin{aligned} V &= \int_0^1 \pi \tan^2\left(\frac{\pi}{4}y\right) dy \\ &= \pi \int_0^1 \left(\tan^2\left(\frac{\pi}{4}y\right)+1\right) - 1\, dy \\ &= \pi \int_0^1 \sec^2\left(\frac{\pi}{4}y\right) - 1\, dy. \end{aligned}$$

Using the substitution $u = \frac{\pi}{4}y$ (so that $du = \frac{\pi}{4}\,dy$), we obtain:

$$V = \pi \cdot \frac{4}{\pi} \int_0^{\frac{\pi}{4}} \sec^2 u - 1\, du = 4\left[\tan u - u\right]_0^{\frac{\pi}{4}} = 4 - \pi.$$

In all of the previous examples, the cross-section turns out to be a disk, because the region to be rotated shares a boundary with the axis of rotation. When this is not the case, the resulting cross-section is a *washer,* that is, a a disk with a smaller concentric disk removed from it ([5]):

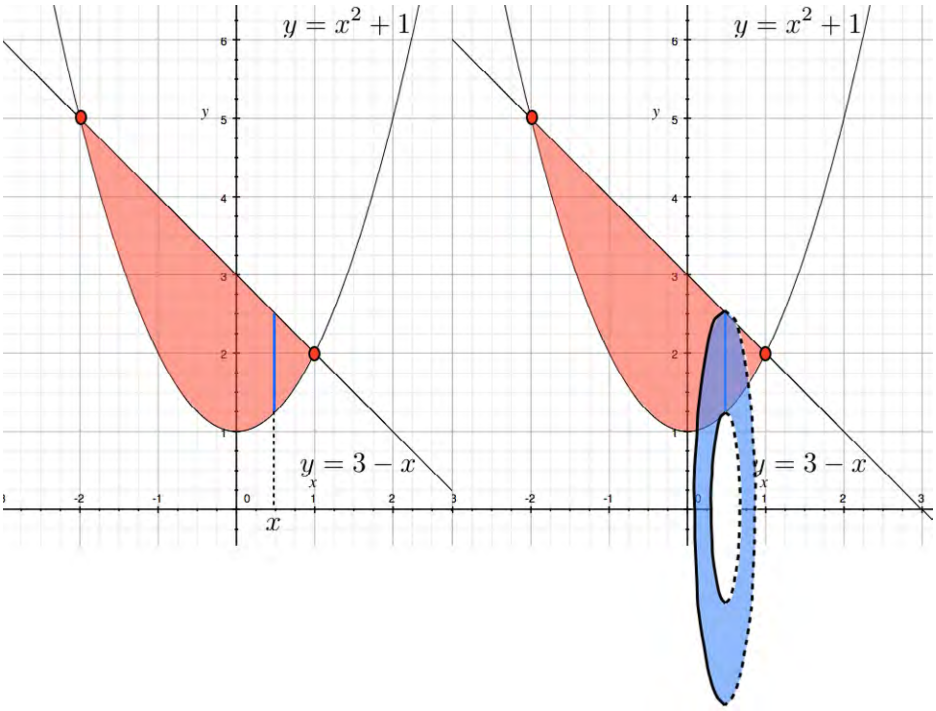


Example 18.2.6. Find the volume of the solid of revolution obtained by revolving the plane region bounded by $y = x^2 + 1$ and $y = 3 - x$ about the line $y = 0$.

Solution. We first sketch the plane region to be rotated. To this end, we need to determine where the curves intersect, that is, to solve:

$$x^2 + 1 = 3 - x \iff x^2 + x - 2 = (x-1)(x+2) = 0.$$

Thus the curves intersect for $x = -2$ and $x = 1$ and we obtain the following sketch for the region, its intersection with $x = cst$, and, for the right-hand side, the cross-section generated by rotating this intersection about the x-axis:



Thus the cross-section is a washer of outer radius $3 - x$ and inner radius $x^2 + 1$ so that the area is

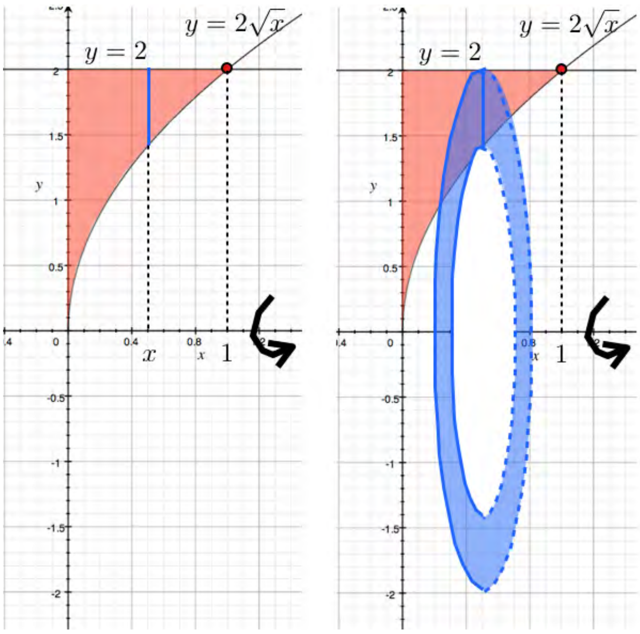
$$\begin{aligned} A(x) &= \pi\left((3-x)^2 - \left(x^2+1\right)^2\right) \\ &= \pi\left(-x^4 - x^2 - 6x + 8\right). \end{aligned}$$

Therefore the volume of the resulting solid of revolution is

$$V = \pi \int_{-2}^{1} -x^4 - x^2 - 6x + 8\, dx = \pi\left[-\frac{x^5}{5} - \frac{x^3}{3} - 3x^2 + 8x\right]_{-2}^{1} = \frac{117\pi}{5}.$$

Example 18.2.7. Find the volume of the solid of revolution obtained by revolving the plane region bounded by $y = 2\sqrt{x}$, $y = 2$ and $x = 0$ about the line $y = 0$.

Solution. We first sketch the plane region to be rotated, together with its intersection with $x = cst$, and, for the right-hand side, the cross-section generated by rotating this intersection about the x-axis. Note that $y = 2\sqrt{x}$ and $y = 2$ intersect exactly when $x = 1$:



Thus the cross-section is a washer of inner radius $2\sqrt{x}$ and outer radius 2, and has therefore area

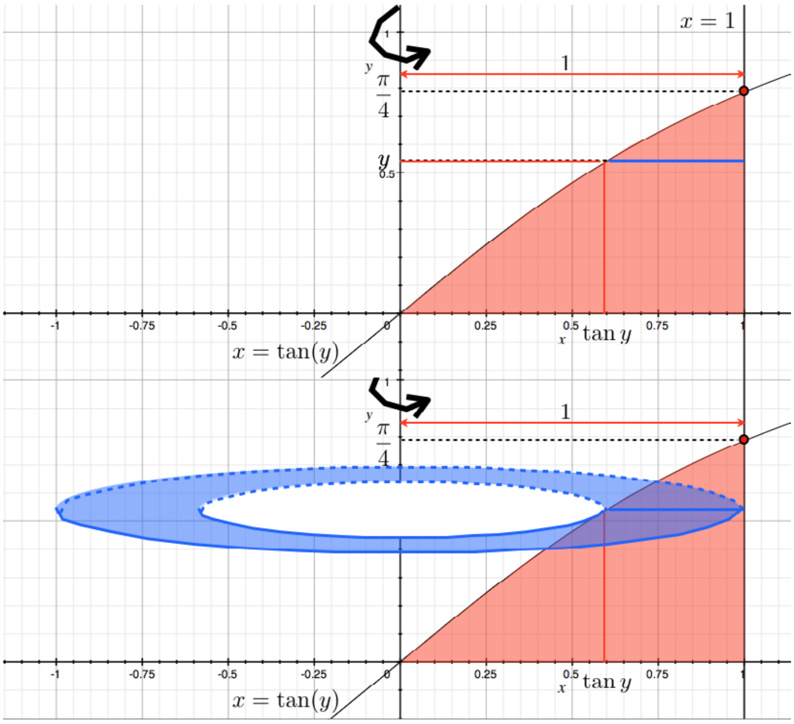
$$A(x) = \pi\left(2^2 - (2\sqrt{x})^2\right) = 4\pi(1-x),$$

and the volume of the resulting solid of revolution is

$$V = 4\pi\int_0^1 1 - x\,dx = 4\pi\left[x - \frac{x^2}{2}\right]_0^1 = 2\pi.$$

Example 18.2.8. Find the volume of the solid of revolution obtained by revolving the plane region bounded by $x = \tan y$, $y = 0$ and $x = 1$ about the line $x = 0$.

Solution. We first sketch the plane region to be rotated, together with its intersection with $y = cst$, and, for the picture below, the cross-section generated by rotating this intersection about the *y*-axis. Note that $\tan y = 1$ when $y = \frac{\pi}{4}$, so that $x = 1$ and $x = \tan y$ intersect for $y = \frac{\pi}{4}$:



Thus the cross-section is a washer of inner radius $\tan y$ and outer radius 1, so that its area is

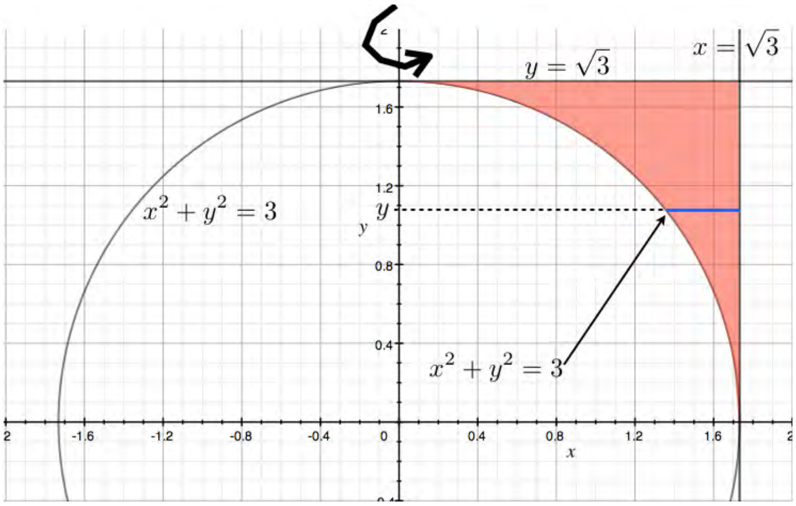
$$A(y) = \pi\left(1 - \tan^2 y\right),$$

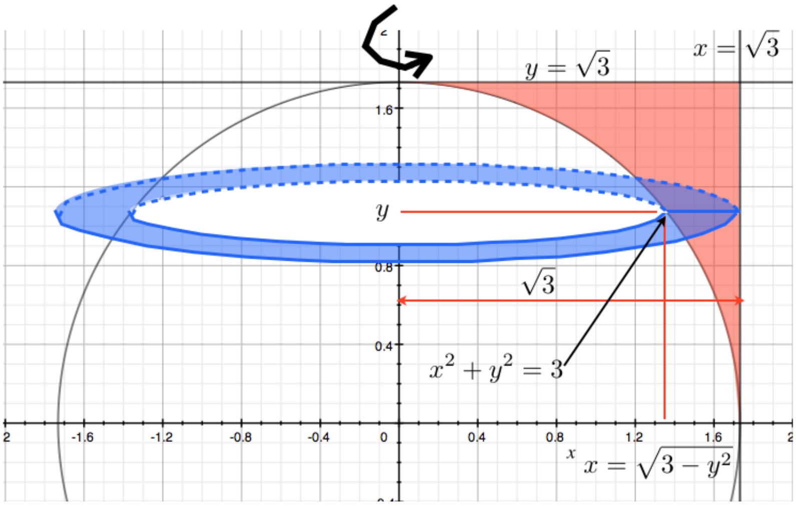
and the volume of the resulting solid of revolution is given by

$$\begin{aligned} V &= \pi \int_0^{\frac{\pi}{4}} 1 - \tan^2 y\, dy = \pi \int_0^{\frac{\pi}{4}} 2 - (1 + \tan^2 y)\, dy \\ &= \pi \left[2y - \tan y\right]_0^{\frac{\pi}{4}} = \frac{\pi^2}{2} - \pi. \end{aligned}$$

Example 18.2.9. Find the volume of the solid of revolution obtained by revolving the first quadrant plane region bounded on the left by $x^2 + y^2 = 3$, on the right by $x = \sqrt{3}$, and above by $y = \sqrt{3}$, about the line $x = 0$.

Solution. We first sketch the plane region to be rotated, together with its intersection with $y = cst$, and, for the picture below, the cross-section generated by rotating this intersection about the *y*-axis:





Thus the cross-section is a washer of inner radius $\sqrt{3-y^2}$ and outer radius $\sqrt{3}$ and has therefore area

$$A(y) = \pi\left((\sqrt{3})^2 - \left(\sqrt{3-y^2}\right)^2\right) = \pi y^2,$$

so that the volume of the resulting solid of revolution is given by

$$V = \pi \int_0^{\sqrt{3}} y^2\, dy = \pi \left[\frac{y^3}{3}\right]_0^{\sqrt{3}} = \pi\sqrt{3}.$$

18.3 Volume by cylindrical Shells

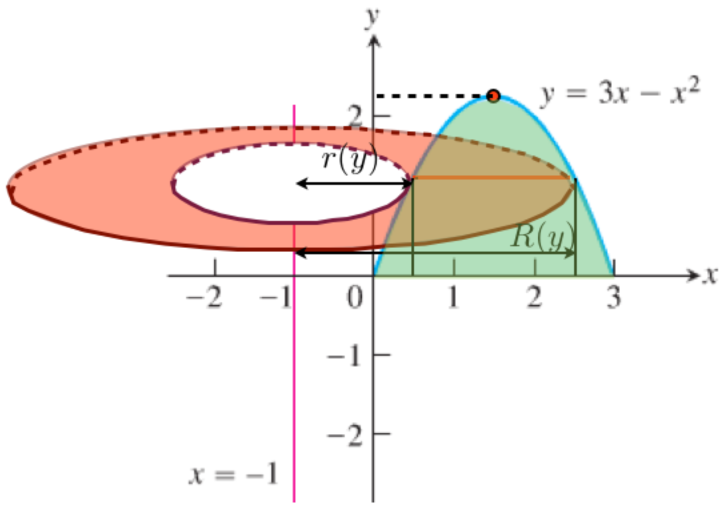
Watch the videos at
https://www.youtube.com/watch?v=cMWkUeKtVrM&list=PL265CB737C01F8961&index=92

and
https://www.youtube.com/watch?v=sV-7Atmm0SY&list=PL265CB737C01F8961&index=93

and
https://www.youtube.com/watch?v=Gq13TKPa6Vc&list=PL265CB737C01F8961&index=94

Abstract The first video establishes a formula for calculating the volume of a solid of revolution by cylindrical shell, while the other go over examples.

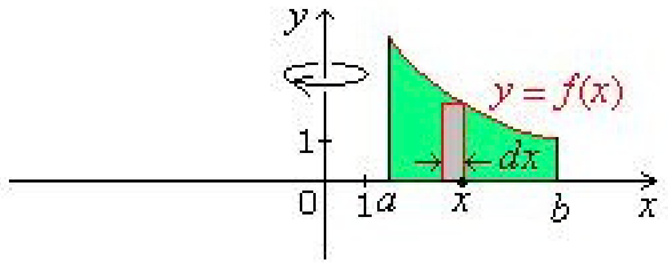
When a region of the plane is rotated about a vertical line, the method by cross-section to determine the volume of the resulting solid of revolution can be very cumbersome. Consider for instance the region bounded by $y = 0$ and $y = 3x - x^2$ rotated about the vertical line $x = -1$, as represented below:

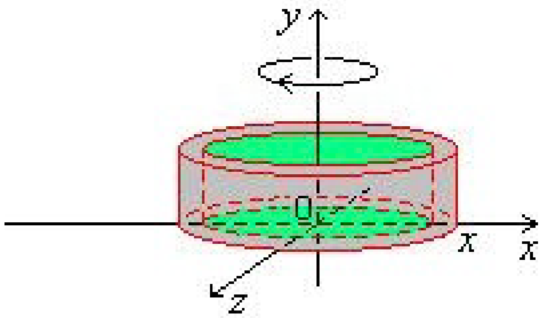


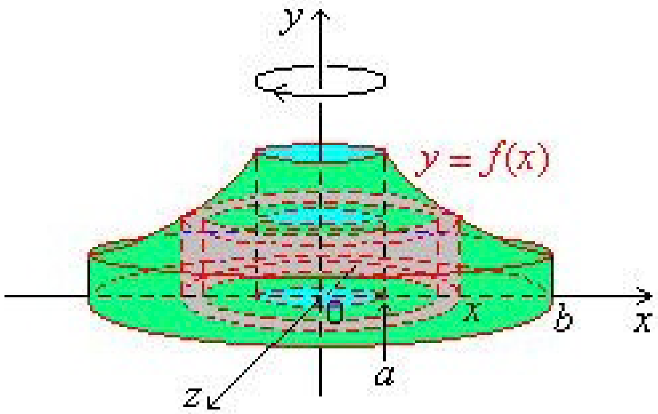
To use the cross-section method, we would need to:

- find the maximum of the function $y = 3x - x^2$ to determine the interval of y values to use in the integral of $A(y)$;
- solve the equation $3x - x^2 = y$ for x in terms of the parameter y.

This could be done, but is cumbersome. For a more complicated function, this might become difficult or even impossible. Thus, we need another, more efficient, method when we generate solids of revolution via a rotation about a vertical line. Approximating the area by rectangle, we approximate the volume of the solid of revolution by *cylindrical shells* ([6]):





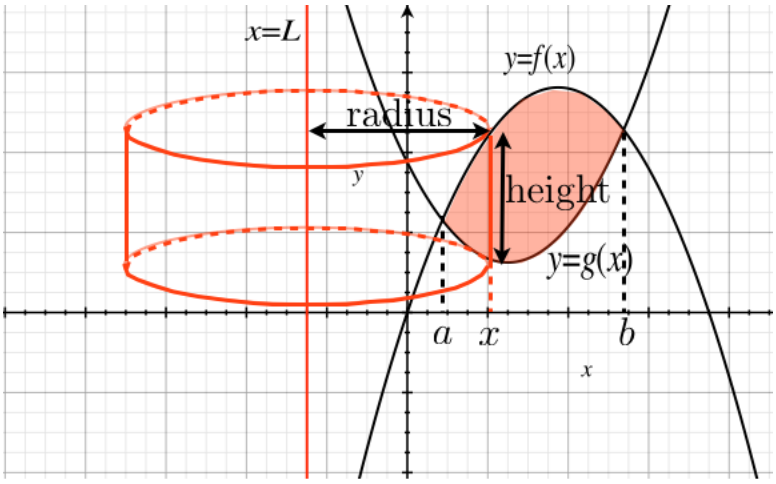


With this method we obtain:

Theorem 18.3.1 (Volume by cylindrical shells) *If a plane region bounded by* $x = a$, $x = b$, $y = f(x)$ *and* $y = g(x)$ *where f and g are continuous, is rotated about a vertical line* $x = L$ *(to simplify, we assume that L is not in* (a, b)*) the resulting solid of revolution has volume*

$$V = 2\pi \int_a^b \text{height} \cdot \text{radius}\, dx,$$

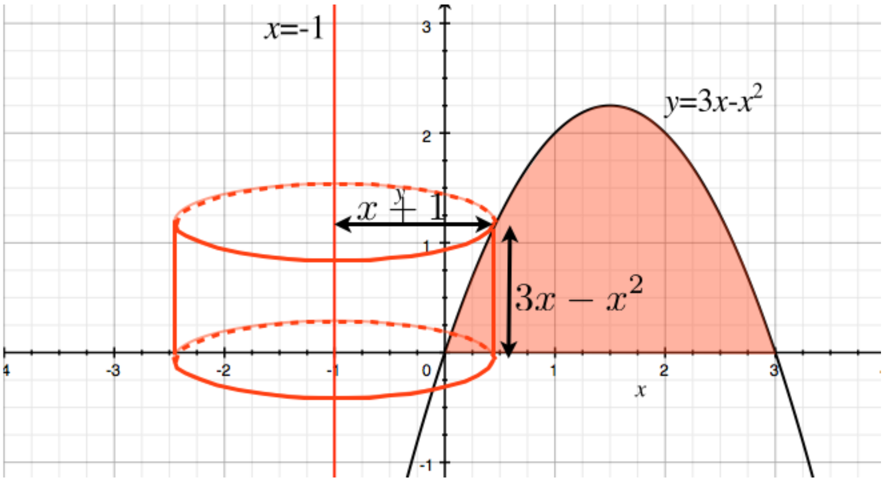
where height and radius are those of the cylinder generated by rotating the intersection of $x = cst$ with the plane region about the axis of rotation $x = L$:



Going back to the motivating example:

Example 18.3.2. Find the volume of the solid of revolution obtained by rotating the region bounded by $y = 0$ and $y = 3x - x^2$ about the line $x = -1$.

Solution. We represent the region to be rotated and a typical cylindrical shell:

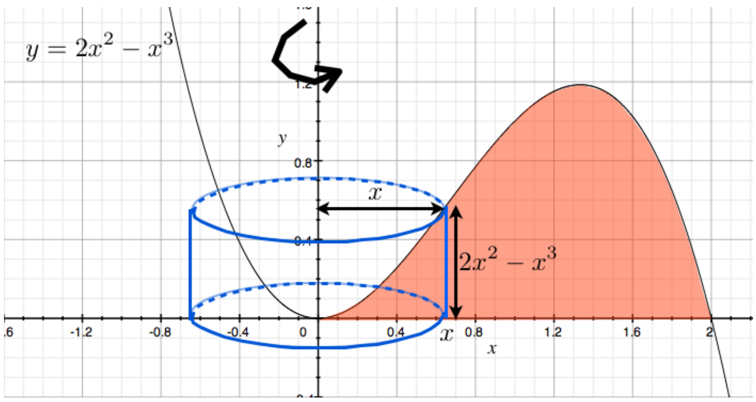


Thus a typical shell has height $3x - x^2$ and radius $x+1$, so that the volume of the solid of revolution is

$$\begin{aligned} V &= 2\pi \int_0^3 (3x-x^2)(x+1)\,dx \\ &= 2\pi \int_0^3 -x^3+2x^2+3x\,dx \\ &= 2\pi \left[-\frac{x^4}{4} + \frac{2x^3}{3} + \frac{3x^2}{2} \right]_0^3 = \frac{45\pi}{2}. \end{aligned}$$

Example 18.3.3. Find the volume of the solid of revolution obtained by revolving the plane region bounded by $y = 2x^2 - x^3$ and $y = 0$ about the line $x = 0$.

Solution. To represent the region to be rotated, we first note that $y = 2x^2 - x^3 = x^2(2-x)$ intersects $y = 0$ for $x = 0$ and $x = 2$. Representing the region to be rotated together with a typical shell, we obtain:

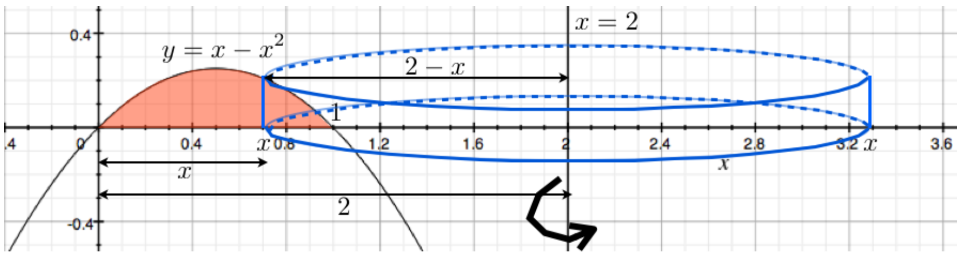


Thus the radius is x and the height is $2x - x^3$, so that the volume of the resulting solid of revolution is

$$\begin{aligned} V &= 2\pi \int_0^2 x(2x^2 - x^3)\,dx = 2\pi \int_0^2 2x^3 - x^4\,dx \\ &= 2\pi \left[\frac{x^4}{2} - \frac{x^5}{5}\right]_0^2 = \frac{16\pi}{5}. \end{aligned}$$

Example 18.3.4. Find the volume of the solid of revolution obtained by revolving the plane region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.

Solution. To represent the region to be rotated, we first note that $y = x - x^2 = x(1 - x)$ intersects $y = 0$ for $x = 0$ and $x = 1$. Representing the region to be rotated together with a typical shell, we obtain:



Thus we see that a typical shell has radius $2-x$ and height $x-x^2$, so that the volume of the resulting volume of revolution is

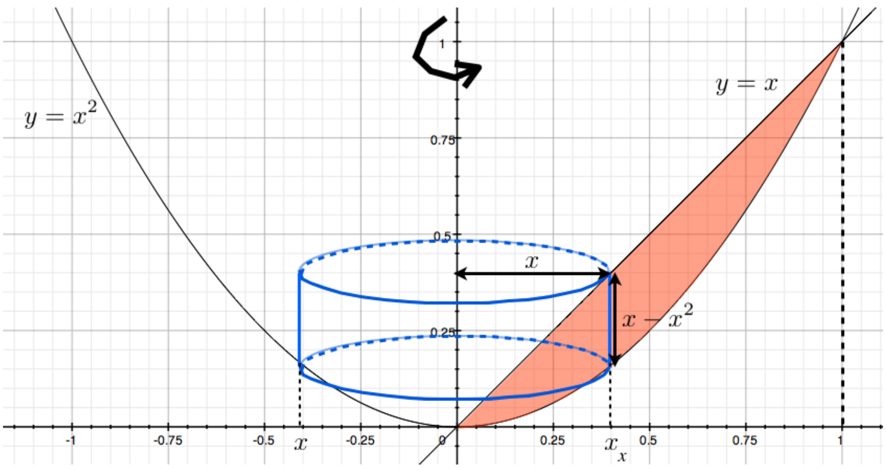
$$\begin{aligned} V &= 2\pi\int_0^1 (2-x)(x-x^2)\,dx \\ &= 2\pi\int_0^1 x^3-3x^2+2x\,dx \\ &= 2\pi\left[\frac{x^4}{4}-x^3+x^2\right]_0^1=\frac{\pi}{2}. \end{aligned}$$

Example 18.3.5. Find the volume of the solid of revolution obtained by revolving the plane region bounded by $y=x$ and $y=x^2$ about the line $x=0$.

Solution. To represent the region to be rotated, we first note that the two curves intersect when

$$x^2=x \iff x(x-1)=0,$$

that is, when $x=0$ or $x=1$. Representing the region to be rotated together with a typical shell, we obtain:



Thus, we see that a typical shell has radius *x* and height $x-x^2$, so that the volume of the resulting solid of revolution is

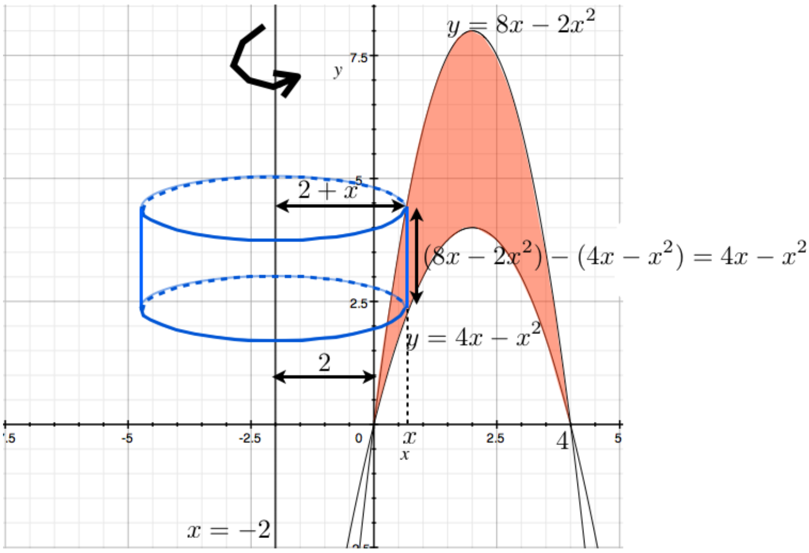
$$V=2\pi\int_0^1 x(x-x^2)\,dx=2\pi\int_0^1 x^2-x^3\,dx=2\pi\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1=\frac{\pi}{6}.$$

Example 18.3.6. Find the volume of the solid of revolution obtained by revolving the plane region bounded by $y=4x-x^2$ and $y=8x-2x^2$ about the line $x=-2$.

Solution. To represent the region to be rotated, we first note that the two curves intersect when

$$4x - x^2 = 8x - 2x^2 \iff 4x - x^2 = x(4-x) = 0,$$

that is, when $x = 0$ or $x = 4$. Representing the region to be rotated together with a typical shell, we obtain:



Thus, a typical shell has radius $2 + x$ and height $4x - x^2$, so that the volume of the resulting solid of revolution is

$$\begin{aligned} V &= 2\pi \int_0^4 (2+x)(4x - x^2)\, dx \\ &= 2\pi \int_0^4 -x^3 + 2x^2 + 8x\, dx \\ &= 2\pi \left[-\frac{x^4}{4} + \frac{2x^3}{3} + 4x^2 \right]_0^4 = \frac{256\pi}{3}. \end{aligned}$$

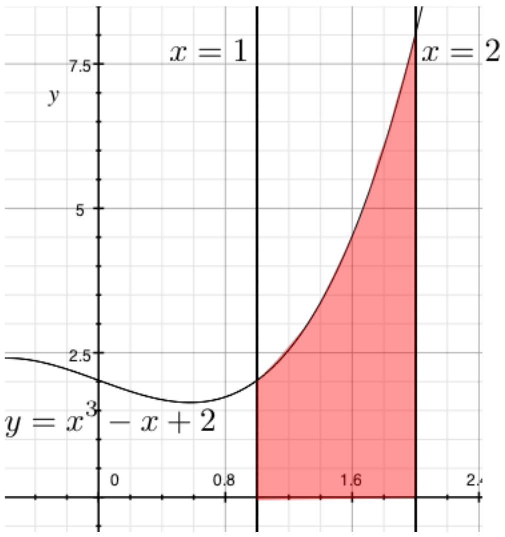
18.4 M15 Sample Quiz: volumes

1. Find the volume of the solid of revolution generated by revolving the plane region bounded by $x = 1$, $x = 2$, $y = 0$ and $y = x^3 - x + 2$ about the x-axis $y = 0$.
2. Find the volume of the solid of revolution generated by revolving the plane region bounded by $x = 1$, $x = 2$, $y = 0$ and $y = x^3 - x + 2$ about the y-axis $x = 0$.
3. Find the volume of the solid of revolution generated by revolving the plane region enclosed by $y = (x-1)^2 + 1$ and $y = 6 - 2x$ about the line $y = -1$.
4. Find the volume of the solid of revolution generated by revolving the plane region enclosed by $y = (x-1)^2 + 1$ and $y = 6 - 2x$ about the line $x = -3$.

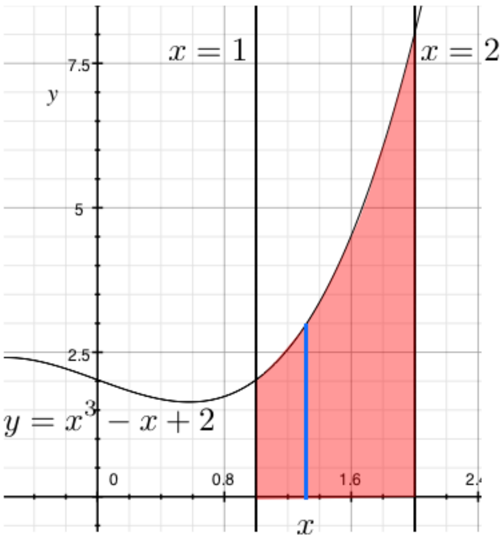
18.5 M15 Sample Quiz Solutions

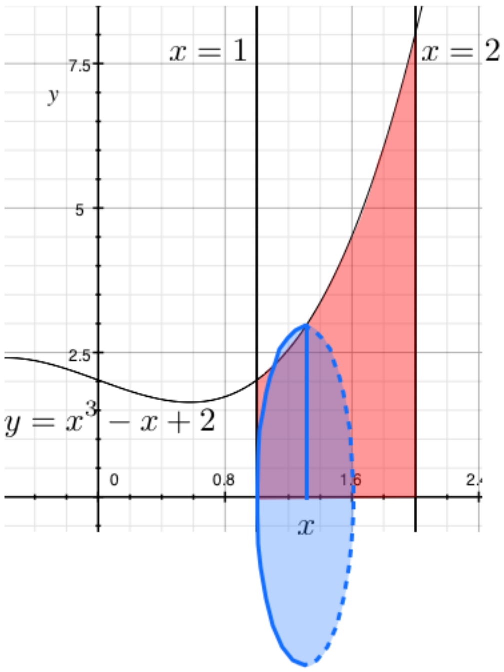
1. Find the volume of the solid of revolution generated by revolving the plane region bounded by $x = 1$, $x = 2$, $y = 0$ and $y = x^3 - x + 2$ about the *x*-axis $y = 0$.

Solution. First, we sketch the plane region:



We revolve the region about the x -axis, hence we use the cross-section method:





Thus the cross-section by $x = cst$ is a disk of radius $(x^3 - x + 2)$. Therefore the area of the cross-section is

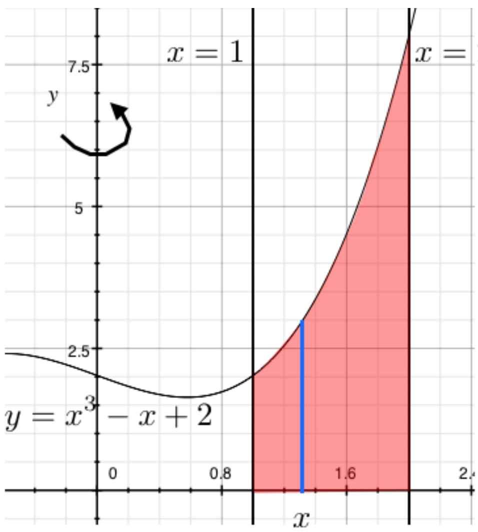
$$A(x) = \pi(x^3 - x + 2)^2 = \pi(x^6 - 2x^4 + 4x^3 + x^2 - 4x + 4)$$

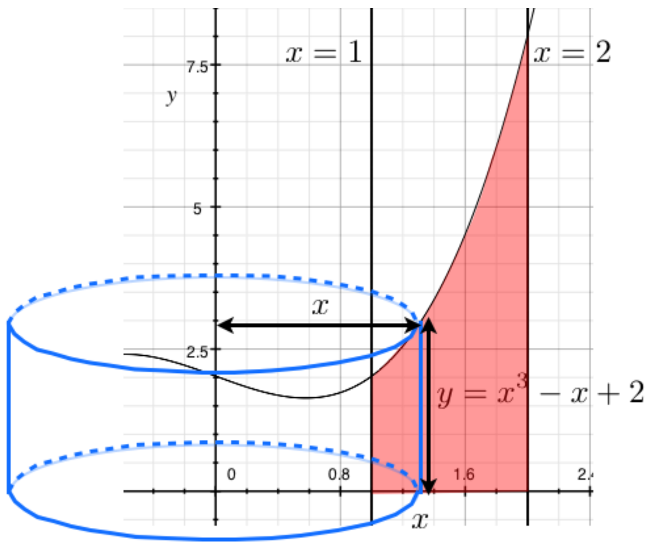
and the volume of the corresponding solid of revolution is

$$\begin{aligned} V &= \int_1^2 A(x)\,dx \\ &= \pi \int_1^2 x^6 - 2x^4 + 4x^3 + x^2 - 4x + 4\,dx \\ &= \pi \left[\frac{x^7}{7} - \frac{2}{5}x^5 + x^4 + \frac{x^3}{3} - 2x^2 + 4x \right]_1^2 \\ &= \pi \left(\left(\frac{2536}{105} \right) - \left(\frac{323}{105} \right) \right) \\ &= \frac{2213\pi}{105}. \end{aligned}$$

2. Find the volume of the solid of revolution generated by revolving the plane region bounded by $x = 1$, $x = 2$, $y = 0$ and $y = x^3 - x + 2$ about the y-axis $x = 0$.

Solution. The plane region considered is the same has in the previous question, but this time, we revolve it about $x = 0$. Thus we use cylindrical shells, has shown below:



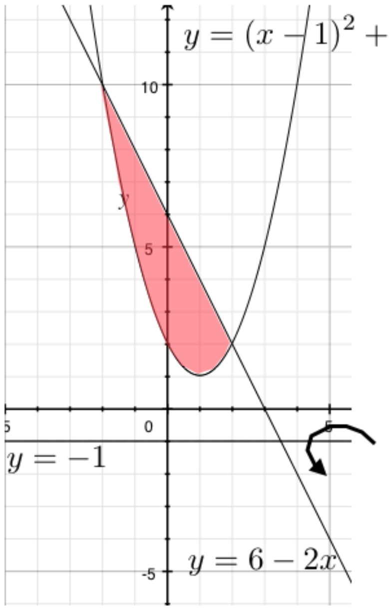


As shown on the picture, a typical cylindrical shell has radius x and height $x^3 - x + 2$. Thus the volume of the resulting solid of revolution is

$$\begin{aligned} V &= 2\pi \int_1^2 (x^3 - x + 2)x\,dx \\ &= 2\pi \int_1^2 x^4 - x^2 + 2x\,dx \\ &= 2\pi \left[\frac{x^5}{5} - \frac{x^3}{3} + x^2\right]_1^2 \\ &= 2\pi \left(\left(\frac{32}{5} - \frac{8}{3} + 4\right) - \left(\frac{1}{5} - \frac{1}{3} + 1\right)\right) \\ &= \frac{206\pi}{15}. \end{aligned}$$

3. Find the volume of the solid of revolution generated by revolving the plane region enclosed by $y = (x-1)^2 + 1$ and $y = 6 - 2x$ about the line $y = -1$.

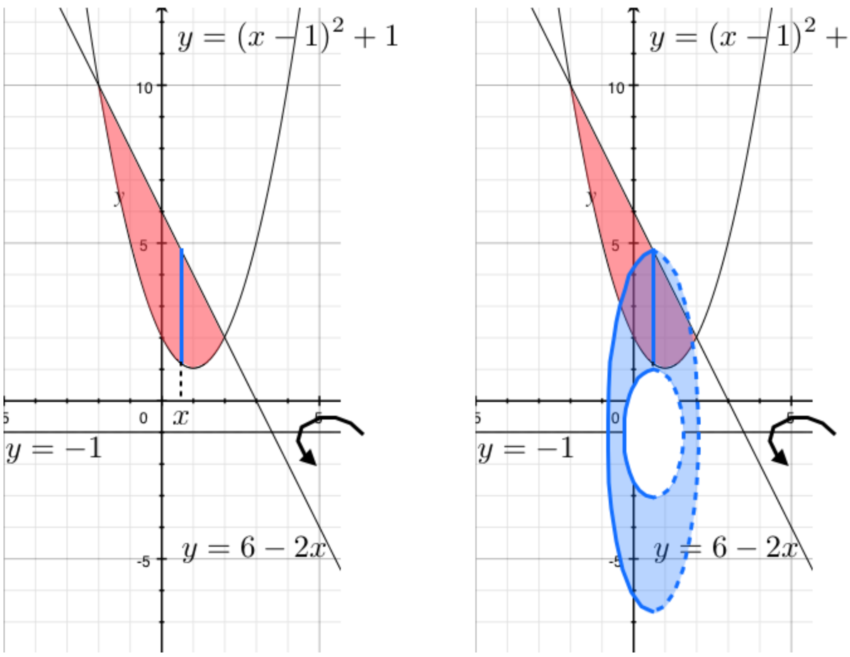
Solution. First, we sketch the plane region:



Note that the two curves intersect if

$$
\begin{aligned}
6 - 2x = (x-1)^2 + 1 \quad &\Longleftrightarrow \quad x^2 - 2x + 2 - (6 - 2x) = 0 \\
&\Longleftrightarrow \quad x^2 - 4 = 0 \\
&\Longleftrightarrow \quad x = -2 \text{ or } x = 2,
\end{aligned}
$$

so that the interval of x-values to consider is $[-2, 2]$. Because we revolve the region about an horizontal line, we use cross-section, as shown below:



The cross-section is a washer with inner radius

$$(x-1)^2+1+1=x^2-2x+4,$$

which is the vertical distance between the curve $y=-1$ and the curve $y=(x-1)^2+1$, and outer radius

$$6-2x+1=7-2x,$$

which is the vertical distance between the curve $y=-1$ and the curve $y=6-2x$. Thus the area of the cross-section is

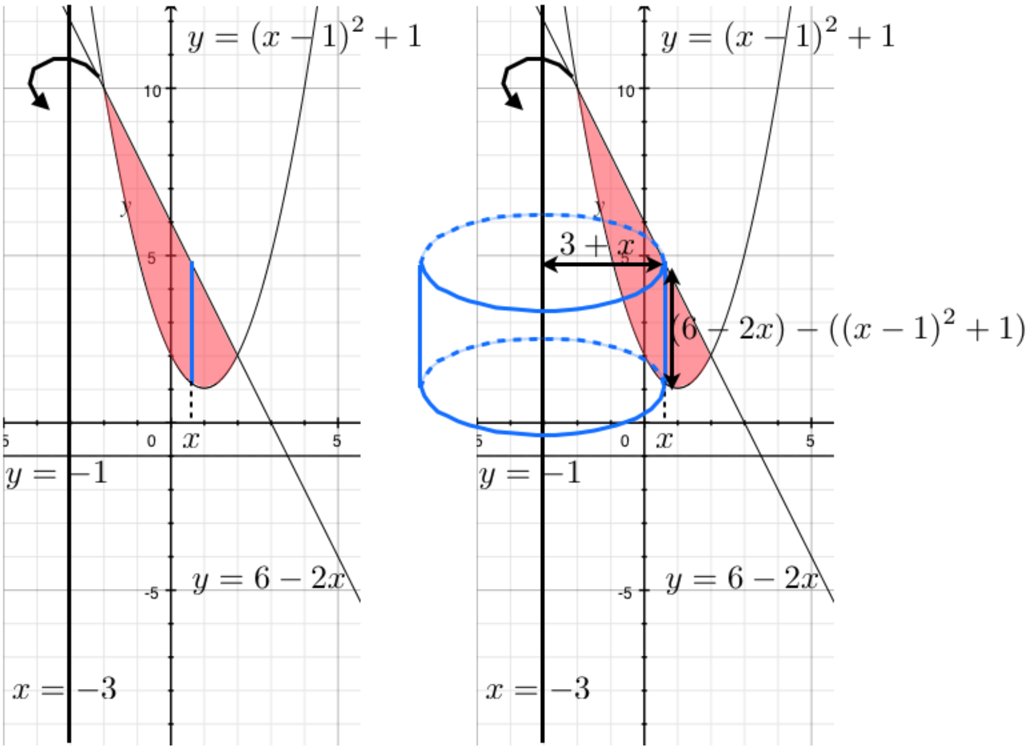
$$\begin{aligned} A(x) &= \pi\left((7-2x)^2-\left(x^2-2x+4\right)^2\right) \\ &= \pi\left(4x^2-28x+49-\left(x^4-4x^3+12x^2-16x+16\right)\right) \\ &= \pi\left(-x^4+4x^3-8x^2-12x+33\right). \end{aligned}$$

The volume of the resulting solid of revolution is then

$$\begin{aligned} V &= \int_{-2}^{2} A(x)\,dx \\ &= \pi\int_{-2}^{2} -x^4+4x^3-8x^2-12x+33\,dx \\ &= \pi\left[-\frac{x^5}{5}+x^4-\frac{8}{3}x^3-6x^2+33x\right]_{-2}^{2} \\ &= \pi\left(\left(-\frac{32}{5}-\frac{64}{3}+58\right)-\left(\frac{32}{5}+\frac{64}{3}-74\right)\right) \\ &= \frac{1148\pi}{15}. \end{aligned}$$

4. Find the volume of the solid of revolution generated by revolving the plane region enclosed by $y=(x-1)^2+1$ and $y=6-2x$ about the line $x=-3$.

Solution. The plane region considered is the same has in the previous question, but this time, we revolve it about $x = -3$. Thus we use cylindrical shells, has shown below:



A typical cylindrical shell has radius $3 + x$ (the distance between the axis of revolution $x = -3$ and a line $x = cst$) and height

$$6 - 2x - ((x-1)^2 + 1) = 4 - x^2.$$

Thus the volume of the corresponding solid of revolution is given by

$$\begin{aligned} V &= 2\pi \int_{-2}^{2} (4 - x^2)(3 + x)\, dx \\ &= 2\pi \int_{-2}^{2} -x^3 - 3x^2 + 4x + 12\, dx \\ &= 2\pi \left[-\frac{x^4}{4} - x^3 + 2x^2 + 12x \right]_{-2}^{2} \\ &= 2\pi\,(20 - (-12)) \\ &= 64\pi. \end{aligned}$$

19 Review on Modules 11 through 15

19.1 MOCK TEST 4

Instructions: *Do the following test, without your notes, in limited time (75 minutes top). Then grade yourself using the solutions provided separately. It is important that you show all your work and justify your answers. Carefully read the solutions to see how you should justify answers.*

1. [10] A stone is dropped from the CN tower (450 m tall). If acceleration from gravity is $g = 9.8\,m/s^2$, determine how long it will take for the stone to reach the ground, and at what speed it strikes the ground.
2. $[8 \times 5 = 40]$ Estimate the following integrals:

a) $\int_0^1 \left(2x^3 - 2x^2 + 3\right) dx$

b) $\int_{-1}^1 \tan(3x^5 - 6x^3) dx$

c) $\int x^3 \sin\left(x^4\right) dx$

d) $\int_1^1 x^7 \sin^5(x^\pi + 5x) dx$

e) $\int \frac{2\sqrt{x^3} - 2}{x^2} dx$

f) $\int (12x - 4)\sqrt{3x^2 - 2x + 1} dx$

g) $\int_0^{\frac{\pi}{2}} 5 \sin x \cos(\cos x) dx$

h) $\int_0^1 x^2(5 + 2x^3)^4 dx$

3. [5] Find

$$\frac{d}{dt}\left(\int_t^1 \frac{y\sqrt{y}\cos y}{y^5+3y^3+1}\,dy\right).$$

4. [10] Find the area enclosed by the curves $y = 2 - x^2$ and $y = -x$.
5. [15] Find the volume of the solid obtained by rotating the region enclosed by $y = x^2 - 5x + 1$, $y = -x^2 + x - 3$ about the line $y = 0$, then about $x = 0$.
6. [15] Find the volume of the solid obtained by rotating the region enclosed by $y = 2$, $y = 2\sin x$, $x = 0$ and $x = \frac{\pi}{2}$ about the $y = 2$.
7. [10] A spring has natural length 20cm. If a force of 50N is required to keep it stretched to a length of 25cm, how much work is required to stretch it from 25cm to 35cm?

10.2 MOCK TEST 4 Solutions

1. [10] A stone is dropped from the CN tower (450 m tall). If acceleration from gravity is $g = 9.8\,m/s^2$, determine how long it will take for the stone to reach the ground, and at what speed it strikes the ground.

Solution. Let $s(t)$ denote the distance of the stone from the ground at time *t*, and $v(t)$ its velocity. Since the stone is dropped from a 450 meters high tower, we have $s(0) = 450$ and $v(0) = 0$. The only force exerted on the stone is gravity, and therefore acceleration is $a(t) = -9.8\,m/s^2$. Velocity is an antiderivative of acceleration, that is,

$$v(t) = \int a(t)\,dt = -9.8t + C.$$

Thus $v(0) = C$, and we have observed that $v(0) = 0$, so that $C = 0$. Therefore

$$v(t) = -9.8t\,(m/s).$$

The position $s(t)$ is an antiderivative of $v(t)$:

$$s(t) = \int v(t)\,dt = -4.9t^2 + D.$$

Thus $s(0) = D$ and we have observed that $s(0) = 450$, so that $D = 450$. Therefore

$$s(t) = 450 - 4.9t^2\,(m).$$

The stone strikes the ground when $s(t) = 0$, that is, when

$$\begin{aligned} 450 - 4.9t^2 = 0 \quad &\iff \quad t^2 = \frac{450}{4.9} \\ &\iff \quad t = \sqrt{\frac{450}{4.9}} \approx 9.58\,s, \end{aligned}$$

because we only consider positive time. Thus the stone reaches the ground after 9.58 seconds, at a speed of

$$\left| v\left(\sqrt{\frac{450}{4.9}}\right)\right| \approx 93.9\,m/s.$$

2. $[8 \times 5 = 40]$ Estimate the following integrals:

a) $\displaystyle\int_0^1 \left(2x^3 - 2x^2 + 3\right) dx$

Solution.

$$\begin{aligned} \int_0^1 \left(2x^3 - 2x^2 + 3\right) dx &= \left[\frac{x^4}{2} - \frac{2x^3}{3} + 3x\right]_0^1 \\ &= \frac{1}{2} - \frac{2}{3} + 3 = \frac{17}{6}. \end{aligned}$$

b) $\displaystyle\int_{-1}^1 \tan(3x^5 - 6x^3) dx$

Solution. $\int_{-1}^1 \tan(3x^5 - 6x^3) dx = 0$ because the interval is centered at the origin and the function is odd.

c) $\displaystyle\int x^3 \sin\left(x^4\right) dx$

Solution.

$$\begin{aligned} \int x^3 \sin\left(x^4\right) dx &= \frac{1}{4}\int \sin u \, du \\ &= -\frac{1}{4}\cos u + C \\ &= -\frac{1}{4}\cos(x^4) + C, \end{aligned}$$

using $u = x^4$ and thus $du = 4x^3\, dx$.

d) $\displaystyle\int_1^1 x^7 \sin^5(x^\pi + 5x) dx$

Solution. $\int_1^1 x^7 \sin^5(x^\pi + 5x) dx = 0$ because we integrate over an interval of width 0.

e) $\displaystyle\int \frac{2\sqrt{x^3} - 2}{x^2} dx$

Solution.

$$\begin{aligned} \int \frac{2\sqrt{x^3} - 2}{x^2} dx &= \int 2x^{-\frac{1}{2}} - 2x^{-2}\, dx \\ &= 4x^{\frac{1}{2}} + 2x^{-1} + C \\ &= 4\sqrt{x} + \frac{2}{x} + C. \end{aligned}$$

f) $\int (12x-4)\sqrt{3x^2-2x+1}dx$

Solution. Let $u = 3x^2 - 2x + 1$. Then $du = (6x-2)\,dx$ and $(12x-4)\,dx = 2du$. Thus

$$\begin{aligned}\int (12x-4)\sqrt{3x^2-2x+1}dx &= 2\int \sqrt{u}\,du \\ &= 2\cdot\frac{2}{3}u^{\frac{3}{2}} + C \\ &= \frac{4}{3}\left(3x^2-2x+1\right)^{\frac{3}{2}} + C.\end{aligned}$$

g) $\int_0^{\frac{\pi}{2}} 5\sin x\cos(\cos x)dx$

Solution. Let $u = \cos x$. Then $du = -\sin x\,dx$. Note that $u = 1$ when $x = 0$ and $u = 0$ when $x = \frac{\pi}{2}$. Thus

$$\begin{aligned}\int_0^{\frac{\pi}{2}} 5\sin x\cos(\cos x)dx &= -5\int_1^0 \cos u\,du \\ &= 5\int_0^1 \cos u\,du \\ &= 5\left[\sin u\right]_0^1 \\ &= 5\sin(1).\end{aligned}$$

h) $\int_0^1 x^2(5+2x^3)^4dx$

Solution. Let $u = 5+2x^3$. Then $du = 6x^2\,dx$ and $x^2\,dx = \frac{1}{6}\,du$. Note also that $u = 5$ when $x = 0$ and $u = 7$ when $x = 1$. Thus

$$\begin{aligned}\int_0^1 x^2(5+2x^3)^4dx &= \frac{1}{6}\int_5^7 u^4\,du \\ &= \frac{1}{6}\left[\frac{u^5}{5}\right]_5^7 \\ &= \frac{7^5-5^5}{30}.\end{aligned}$$

3. [5] Find

$$\frac{d}{dt}\left(\int_t^1 \frac{y\sqrt{y}\cos y}{y^5+3y^3+1}\,dy\right).$$

Solution. Recall that

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

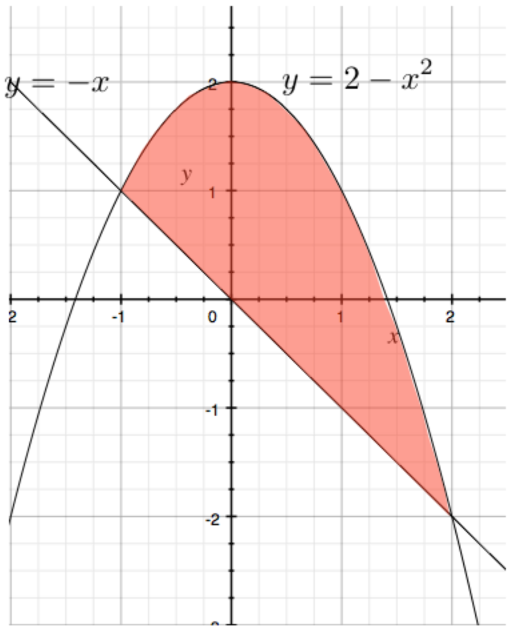
provided that f is continuous, according to the Fundamental Theorem of Calculus. Hence

$$\begin{aligned}
\frac{d}{dt}\left(\int_t^1 \frac{y\sqrt{y}\cos y}{y^5+3y^3+1}\,dy\right) &= \frac{d}{dt}\left(-\int_1^t \frac{y\sqrt{y}\cos y}{y^5+3y^3+1}\,dy\right)\\
&= -\frac{d}{dt}\left(\int_1^t \frac{y\sqrt{y}\cos y}{y^5+3y^3+1}\,dy\right)\\
&= -\frac{t\sqrt{t}\cos t}{t^5+3t^3+1},
\end{aligned}$$

on an interval where this function is continuous.

4. [10] Find the area enclosed by the curves $y = 2 - x^2$ and $y = -x$.

Solution. Here is a sketch of the region:



The two curves intersect when

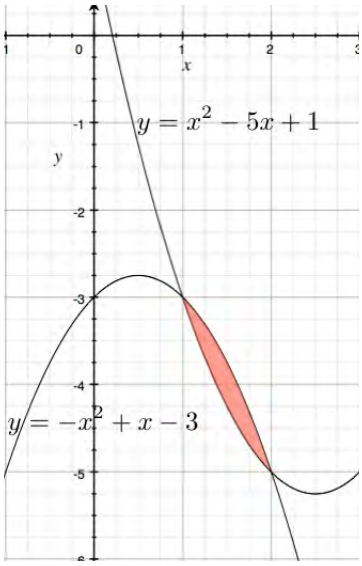
$$\begin{aligned} 2-x^2=-x &\iff x^2-x-2=0 \\ &\iff (x+1)(x-2)=0 \\ &\iff x=-1 \text{ or } x=2. \end{aligned}$$

Thus the area we are looking for is given by the integral

$$\begin{aligned} A &= \int_{-1}^{2} (2-x^2)-(-x)\,dx \\ &= \int_{-1}^{2} -x^2+x+2\,dx \\ &= \left[-\frac{x^3}{3}+\frac{x^2}{2}+2x\right]_{-1}^{2} \\ &= \left(-\frac{8}{3}+2+4\right)-\left(\frac{1}{3}+\frac{1}{2}-2\right) \\ &= \frac{9}{2}. \end{aligned}$$

5. [15] Find the volume of the solid obtained by rotating the region enclosed by $y=x^2-5x+1$, $y=-x^2+x-3$ about the line $y=0$, then about $x=0$.

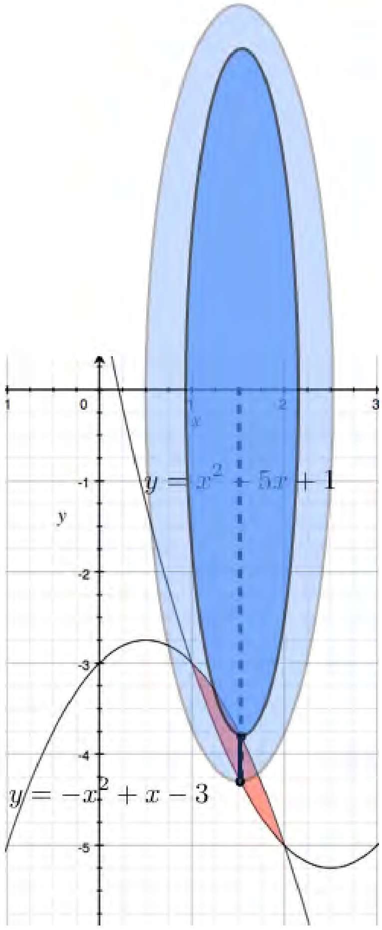
Solution. Here is a sketch of the region to be rotated:



The two curves intersect when

$$\begin{aligned} x^2 - 5x + 1 = -x^2 + x - 3 \quad &\Longleftrightarrow \quad 2x^2 - 6x + 4 = 0 \\ &\Longleftrightarrow \quad 2\left(x^2 - 3x + 2\right) = 0 \\ &\Longleftrightarrow \quad 2(x-1)(x-2) = 0 \\ &\Longleftrightarrow \quad x = 1 \text{ or } x = 2. \end{aligned}$$

When rotating about $y = 0$, we use the washer method, as shown below



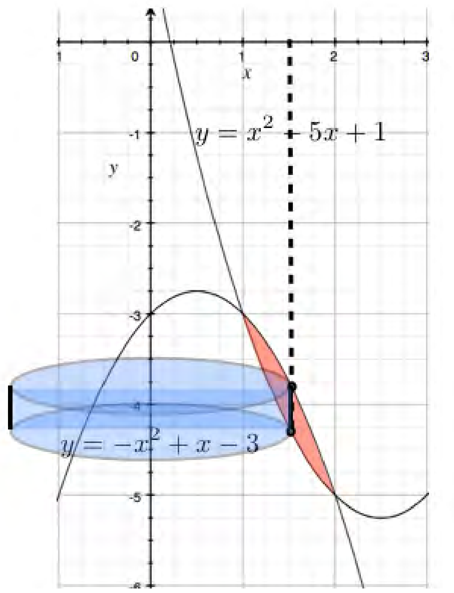
The cross-section of the solid of revolution generated by rotating the region about $y = 0$ is a washer of inner radius $-x^2 + x - 3$ and outer radius $x^2 - 5x + 1$ and has thus area

$$
\begin{aligned}
A(x) &= \pi\left(\left(x^2 - 5x + 1\right)^2 - \left(-x^2 + x - 3\right)^2\right) \\
&= \pi\left(x^4 - 10x^3 + 27x^2 - 10x + 1 - \left(x^4 - 2x^3 + 7x^2 - 6x + 9\right)\right) \\
&= \pi\left(-8x^3 + 20x^2 - 4x - 8\right).
\end{aligned}
$$

Thus the volume of the corresponding solid of revolution is given by

$$\begin{aligned} V &= \int_1^2 A(x)\,dx \\ &= \pi\int_1^2 -8x^3+20x^2-4x-8\,dx \\ &= \pi\left[-2x^4+\frac{20}{3}x^3-2x^2-8x\right]_1^2 \\ &= \pi\left(\left(-32+\frac{160}{3}-8-16\right)-\left(\frac{20}{3}-12\right)\right) \\ &= \frac{8}{3}\pi. \end{aligned}$$

When rotating about $x=0$, we use the shell method as shown below



A typical shell has radius *x* and height

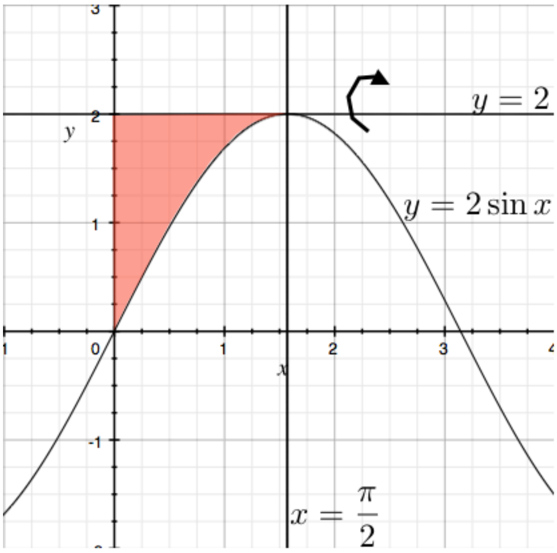
$$-x^2+x-3-\left(x^2-5x+1\right)=-2x^2+6x-4.$$

Thus the volume of the corresponding solid of revolution is given by

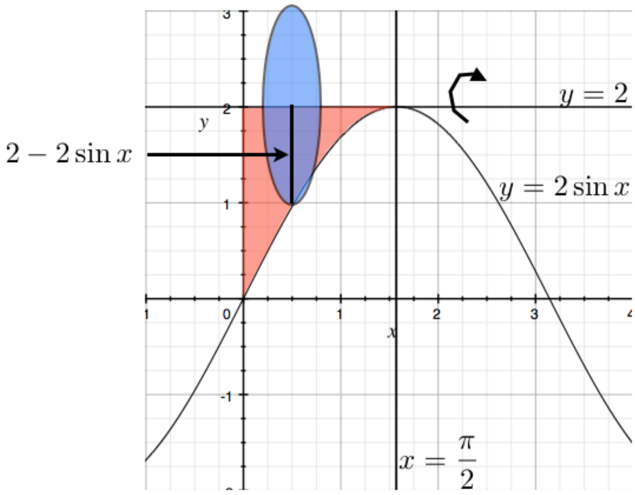
$$\begin{aligned} V &= \int_1^2 2\pi x\left(-2x^2+6x-4\right)\,dx \\ &= 2\pi\int_1^2 -2x^3+6x^2-4x\,dx \\ &= 2\pi\left[-\frac{x^4}{2}+2x^3-2x^2\right]_1^2 \\ &= 2\pi\left((-8+16-8)-\left(-\frac{1}{2}+2-2\right)\right) \\ &= 2\pi\times\frac{1}{2}=\pi. \end{aligned}$$

6. [15] Find the volume of the solid obtained by rotating the region enclosed by $y = 2$, $y = 2\sin x$, $x = 0$ and $x = \frac{\pi}{2}$ about the $y = 2$.

Solution. Here is a sketch of the region



Because we rotate about an horizontal line, $y = 2$, we use the cross-section method. Here, as shown below, the cross-section is a disk of radius $2 - 2\sin x$.



Thus the volume of the corresponding volume of revolution is given by

$$\begin{aligned} V &= \int_0^{\frac{\pi}{2}} \pi (2 - 2\sin x)^2 \, dx \\ &= 4\pi \int_0^{\frac{\pi}{2}} 1 - 2\sin x + \sin^2 x \, dx \\ &= 4\pi \int_0^{\frac{\pi}{2}} 1 - 2\sin x + \frac{1 - \cos(2x)}{2} \, dx \text{ using } \cos(2x) = 1 - 2\sin^2 x \\ &= 4\pi \int_0^{\frac{\pi}{2}} \frac{3}{2} - 2\sin x - \frac{1}{2}\cos(2x) \, dx \\ &= 4\pi \left[\frac{3}{2}x + 2\cos x - \frac{1}{4}\sin(2x) \right]_0^{\frac{\pi}{2}} \\ &= 4\pi \left(\frac{3\pi}{4} - 2 \right) = 3\pi^2 - 8\pi. \end{aligned}$$

7. [10] A spring has natural length 20cm. If a force of 50*N* is required to keep it stretched to a length of 25cm, how much work is required to stretch it from 25cm to 35cm?

Solution. A force of 50*N* is required to increase the length of 5cm. Thus the spring constant is

$$k = \frac{F}{x} = \frac{50}{5} = 10 \, N/cm$$

(or $\frac{50}{0.05} = 1000 \, N/m$)The work required to stretch the spring from 25cm to 35cm, that is, to stretch from 5 cm more than its natural length to 15 cm more than its natural length is given by

$$W = \int_5^{15} 10x \, dx = \left[5x^2\right]_5^{15} = 1000 \, N \cdot cm$$

or

$$\int_{0.05}^{0.15} 1000x \, dx = 10 \, N \cdot m = 10 \, J.$$

21 Index

22 Endnotes

1. the second equality is simply a rephrasing, letting $h = x - a$.
2. i.e., find the asymptotes, intervals of increase and decrease, local extrema, intervals of concavity, inflection points and then sketch the graph.
3. you do not need to sketch the graph.
4. Some of the pictures in this section are reworked versions of examples in this book
5. half of this picture is taken from this website
6. The picture is borrowed from this site.