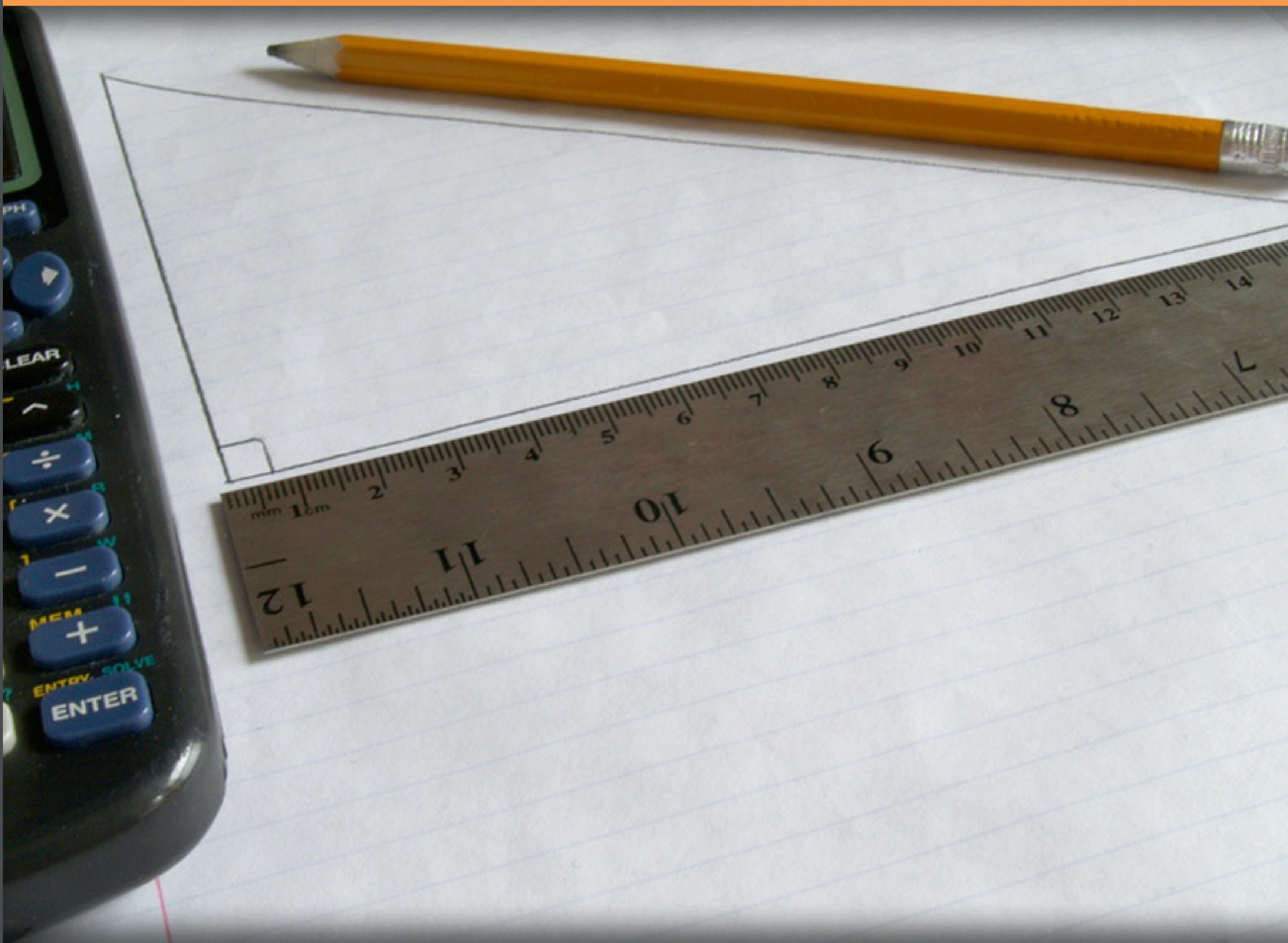
Linear algebra c-1

Linear equations, matrices and determinants

Leif Mejlbro



Leif Mejlbro

Linear Algebra Examples c-1

Linear Equations, Matrices and Determinants

Linear Algebra Examples c-1 – Linear Equations, Matrices and Determinants

ISBN 978-87-7681-506-6

Indholdsfortegnelse

Introduction

Here we collect all tables of contents of all the books on mathematics I have written so far for the publisher. In the rst list the topics are grouped according to their headlines, so the reader quickly can get an idea of where to search for a given topic.In order not to make the titles too long I have in the numbering added

a for a compendium

b for practical solution procedures (standard methods etc.)

c for examples.

The ideal situation would of course be that all major topics were supplied with all three forms of books, but this would be too much for a single man to write within a limited time.

After the rst short review follows a more detailed review of the contents of each book. Only Linear Algebra has been supplied with a short index. The plan in the future is also to make indices of every other book as well, possibly supplied by an index of all books. This cannot be done for obvious reasons during the rst couple of years, because this work is very big, indeed.

It is my hope that the present list can help the reader to navigate through this rather big collection of books.

Finally, since this list from time to time will be updated, one should always check when this introduction has been signed. If a mathematical topic is not on this list, it still could be published, so the reader should also check for possible new books, which have not been included in this list yet.

Unfortunately errors cannot be avoided in a rst edition of a work of this type. However, the author has tried to put them on a minimum, hoping that the reader will meet with sympathy the errors which do occur in the text.

Leif Mejlbro
5th October 2008

1 Linear equations

Example 1.1 *Solve the system of equations*

$$
\begin{array}{ccccccccccc}
x_1 & + & x_2 & - & x_3 & & & & & = & 1 \\
 & & x_2 & + & x_3 & - & x_4 & & & = & 1 \\
 & & & & x_3 & + & x_4 & - & x_5 & = & 1 \\
 & - & x_2 & + & x_3 & + & x_4 & & & = & 1 \\
 & & & - & x_3 & + & x_4 & + & x_5 & = & 1.
\end{array}
$$

Adding the second and the fourth equation we get $2x_3 = 2$, hence $x_3 = 1$.
Adding the third and the fifth equation we get $2x_4 = 2$, hence $x_4 = 1$.
When these values are put into the second equation we get $x_2 = 1$. Analogously, we obtain from the fifth equation that $x_5 = 1$.
Finally, it follows from the first equation and $x_2 = x_3 = 1$ that $x_1 = 1$.

The only possibility of solution is $x_1 = \cdots = x_5 = 1$, and a check shows immediately that $\mathbf{x} = (1,1,1,1,1)$ is a solution.

Alternatively we perform a Gauss elimination. We keep the first and the second equation. Adding the second and the fourth equation we get as before that $2x_3 = 2$, so the third equation is replaced by $x_3 = 1$. This is put into the old third and fifth equation, giving after a reduction that

$$
\begin{array}{l} x_4 - x_5 = 0, \\ x_4 + x_5 = 2, \end{array} \qquad \text{dvs.} \qquad \begin{array}{r} x_4 - x_5 = 0, \\ x_5 = 1. \end{array}
$$

Then the scheme of Gauß elimination becomes

$$\begin{array}{rcrcrcrcrcl} x_1 & + & x_2 & - & x_3 & & & & & = & 1 \\ & & x_2 & + & x_3 & - & x_4 & & & = & 1 \\ & & & & x_3 & & & & & = & 1 \\ & & & & & & x_4 & - & x_5 & = & 0 \\ & & & & & & & & x_5 & = & 1. \end{array}$$

By solving this system backwards we get

$$x_5 = 1, \quad x_4 = x_5 = 1, \quad x_3 = 1, \quad x_2 = 1 \text{ and } x_1 = 1,$$

and the unique solution is $\mathbf{x} = (1, 1, 1, 1, 1)$.

Example 1.2 *Find the complete solution of the system of equations*

$$\begin{array}{rcrcrcrcl} x_1 & + & 2x_1 & + & 3x_3 & + & 4x_4 & = & 0 \\ x_1 & - & 2x_2 & + & 3x_3 & - & 4x_4 & = & 0 \\ 3x_1 & + & 2x_2 & + & 9x_3 & + & 4x_4 & = & 0 \\ 4x_1 & - & 4x_2 & + & 12x_3 & - & 8x_4 & = & 0. \end{array}$$

It follows from the two first equations that

$$x_1 + 3x_3 = \pm(2x_2 + 4x_4) = \pm 2(x_2 + 2x_4),$$

which is only possible if

$$x_2 + 2x_4 = 0, \qquad \text{and thus} \qquad x_1 + 3x_3 = 0,$$

hence

$$x_1 = -3x_3 \qquad \text{and} \qquad x_2 = -2x_4.$$

When these results are put into the latter two equations of the system we get

$$0 = +3x_1 + 2x_2 + 9x_3 + 4x_4 = -9x_3 - 4x_4 + 9x_3 + 4x_4 = 0$$

and

$$0 = 4x_1 - 4x_2 + 12x_3 - 8x_4 = -12x_3 + 8x_4 + 12x_3 - 8x_4 = 0,$$

and the system of equations is satisfied if $x_1 = -3x_3$ and $x_2 = -2x_4$. The complete solution is in its parametric form given as

$$\{(-3s, -2t, s, t) \mid s, t \in \mathbb{R}\}.$$

Alternatively we apply Gauß elimination. It follows from the first equation

$$x_1 + 2x_2 + 3x_3 + 4x_4 = 0$$

that

$$x_1 = -2x_2 - 3x_3 - 4x_4.$$

Then by insertion into the latter three equations,

$$\begin{array}{l} -4x_2 - 8x_4 = 0, \\ -4x_2 - 8x_4 = 0, \qquad \text{hence} \qquad x_2 + 2x_4 = 0, \\ -4x_2 - 8x_4 = 0, \end{array}$$

and the system is reduced to

$$\begin{array}{ccccccccc} x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & = 0 & \\ & & x_2 & & & + & 2x_4 & = & 0, \end{array}$$

because the latter three equations are now identical. This is again split into

$$x_1 + 3x_3 = 0 \qquad \text{and} \qquad x_2 + 2x_4 = 0.$$

If we choose the parameters $x_3 = s$ and $x_4 = t$, we obtain the complete solution

$$\{(-3s, -2t, s, t) \mid x, t \in \mathbb{R}\}.$$

Example 1.3 *Solve the system of equations*

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & 2x_3 & = & 3 \\ 2x_1 & - & x_2 & + & 4x_3 & = & 0 \\ x_1 & + & 3x_2 & - & 2x_3 & = & 3 \\ -3x_1 & - & 2x_2 & + & x_3 & = & 0. \end{array}$$

Here, we have four equations in only three unknowns, so we may *expect* that the system is over determined (hence no solution). This argument is of course no proof in itself, only an indication, so we shall start with e.g. Gauß elimination.

It follows from the first equation that

$$x_1 = -x_2 - 2x_3 + 3,$$

which gives by insertion into the remaining three equations successively

$$\begin{array}{rcl} 0 & = & 2x_1 - x_2 + 4x_3 \\ & = & -2x_2 - 4x_3 + 6 - x_2 + 4x_3 \\ & = & -3x_2 + 6, \end{array}$$

$$\begin{array}{rcl} 3 & = & x_1 + 3x_2 - 2x_3 \\ & = & -x_2 - 2x_3 + 3 + 3x_2 - 2x_3 \\ & = & 3 + 2x_2 - 4x_3, \end{array}$$

$$\begin{aligned} 0 &= -3x_1 - 2x_2 + x_3 \\ &= 3x_2 + 6x_3 - 9 - 2x_2 + x_3 \\ &= x_2 + 7x_3 - 9. \end{aligned}$$

Summing up we have

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & 2x_3 & = & 3 \\ & & x_2 & & & = & 2 \\ & - & 2x_2 & + & 4x_3 & = & 0 \\ & & x_2 & + & 7x_3 & = & 9. \end{array}$$

We immediately exploit the second equation, $x_2 = 2$, and the system is reduced to

$$\begin{array}{rcrcl} x_1 & + & 2x_3 & = & 1 \\ & & 4x_3 & = & 4 \\ & & 7x_3 & = & 7. \end{array}$$

Hence fås $x_3 = 1$, and whence $x_1 = -1$.

The only possible solution is now

$$x_1 = -1, \qquad x_2 = x, \qquad x_3 = 1.$$

Here, a check *must* be carried out:

$$\begin{array}{rcrcrcrcrcrcrcl} x_1 & + & x_2 & + & 2x_3 & = & -1 & + & 2 & + & 2 & = & 3, & \text{OK,} \\ 2x_1 & - & x_2 & + & 4x_3 & = & -2 & - & 2 & + & 4 & = & 0, & \text{OK,} \\ x_1 & + & 3x_2 & - & 2x_3 & = & -1 & + & 6 & - & 2 & = & 3, & \text{OK,} \\ -3x_1 & - & 2x_2 & + & x_3 & = & 3 & - & 4 & + & 1 & = & 0, & \text{OK.} \end{array}$$

This shows that we in spite of our previous indication has one solution,

$$\mathbf{x} = (-1, 2, 1).$$

Example 1.4 *Solve the system of equations*

$$\begin{array}{rcrcrcrcr} 8x_1 & + & 6x_2 & - & x_3 & + & 3x_4 & = & -9 \\ x_1 & + & 2x_2 & - & 2x_3 & + & 11x_4 & = & -28 \\ 2x_1 & + & 2x_2 & - & x_3 & + & 5x_4 & = & -13 \\ & & 2x_2 & - & 3x_3 & + & 17x_4 & = & -43. \end{array}$$

Here, nothing is obvious, so we just use the ordinary Gauß elimination. It follows from the second equation that

$$x_1 = -2x_2 + 2x_3 - 11x_4 - 28,$$

which gives by insertion into the first equation,

$$\begin{aligned} -9 &= 8x_1 + 6x_2 - x_3 + 3x_4 \\ &= -16x_2 + 16x_3 - 88x_4 - 224 + 6x_2 - x_3 + 3x_4 \\ &= -10x_2 + 15x_3 - 85x_4 - 224, \end{aligned}$$

thus

$$10x_2 - 15x_3 + 85x_4 = -224 + 9 = -215,$$

and hence

$$2x_2 - 3x_3 + 17x_4 = -43.$$

Analogously we obtain for the third equation

$$\begin{aligned} -13 &= 2x_1 + 2x_2 - x_3 + 5x_4 \\ &= -4x_2 + 4x_3 - 22x_4 - 56 + 2x_2 - x_3 + 5x_4 \\ &= -2x_2 + 3x_3 - 17x_4 - 56, \end{aligned}$$

and we get

$$2x_2 - 3x_3 + 17x_4 = -56 + 13 = -43.$$

The fourth equation does not contain x_1, and since the second and the third and the fourth equation are identical in the new system, the system is reduced to the following under determined system

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & + & 2x_3 & - & 11x_4 & = & -28 \\ & & 2x_2 & - & 3x_3 & + & 17x_4 & = & -43. \end{array}$$

The set of solutions is a 2-parametric set. By choosing $x_3 = s$ and $x_4 = t$ as parameters we get

$$2x_2 = 3x_3 - 17x_4 - 43 = \frac{3}{2}\,s - \frac{17}{2}\,t - \frac{43}{2}$$

and

$$\begin{aligned} x_1 &= -2x_2 - 2x_3 + 11x_4 - 28 \\ &= -3x_3 + 17x_4 + 43 - 2x_3 + 11x_4 - 28 \\ &= -5x_3 + 28x_4 + 15 = -5s + 28t + 15, \end{aligned}$$

and the set of solutions becomes

$$\left\{ \left(-5s + 28t + 15, \frac{3}{2}\,s - \frac{17}{2}\,t - \frac{43}{2}, s, t \right) \;\middle|\; s, t \in \mathbb{R} \right\}.$$

Example 1.5 *Find the complete solution of the system of equations*

$$\begin{array}{rcrcrcrcr} x_1 & - & x_2 & + & 2x_3 & + & x_4 & = & 1 \\ -4x_1 & - & 5x_2 & + & 7x_3 & - & 7x_4 & = & -7 \\ 2x_1 & + & x_2 & - & x_3 & + & 3x_4 & = & 3 \\ -x_1 & - & 5x_2 & + & 8x_3 & - & 3x_4 & = & -3. \end{array}$$

Here, nothing is obvious. One may try to add the first and the fourth equation, the first and the third equation, and the third and the fourth equation, however, with no obvious result. Hence we use the Gauß elimination instead. It follows from the first equation that

$$x_1 = x_2 - 2x_3 - x_4 + 1.$$

By insertion into the second equation we get

$$\begin{aligned} -7 &= -4x_1 - 5x_2 + 7x_3 - 7x_4 \\ &= -4x_2 + 8x_3 + 4x_4 - 4 - 5x_2 + 7x_3 - 7x_4 \\ &= -9x_2 + 15x_3 - 3x_4 - 4, \end{aligned}$$

which is reduced to

$$3x_2 - 5x_3 + x_4 = 1.$$

By insertion into the third equation we get

$$\begin{aligned} 3 &= 2x_1 + 2x_2 - x_3 + 3x_4 \\ &= 2x_2 - 4x_3 - 2x_4 + 2 + x_2 - x_3 + 3x_4 \\ &= 3x_2 - 5x_3 + x_4 + 2, \end{aligned}$$

thus

$$3x_2 - 5x_3 + x_4 = 1$$

as above.

By insertion into the fourth equation we get

$$\begin{aligned}-3 &= -x_1 - 5x_2 + 8x_3 - 3x_4 \\ &= -x_2 + 2x_3 + x_4 - 1 - 5x_2 + 8x_3 - 3x_4 \\ &= -6x_2 + 10x_3 - 2x_4 - 1,\end{aligned}$$

which is reduced to

$$3x_2 - 5x_3 + x_4 = 1.$$

The total system is reduced to

$$\begin{array}{l} x_1 = x_2 - 2x_3 - x_4 + 1, \\ 3x_2 - 5x_3 + x_4 = 1, \qquad \text{dvs.} \qquad x_4 = -3x_2 + 5x_3 + 1. \end{array}$$

When we eliminate x_4 in the first equation, we get

$$x_1 = x_2 - 2x_3 + 3x_2 - 5x_3 - 1 + 1 = 4x_2 - 7x_3.$$

Using $x_2 = s$ and $x_3 = t$ as parameters we finally get the complete solution

$$\{(4s - 7t, s, t, -3s + 5t + 1) \mid s.t \in \mathbb{R}\}.$$

Example 1.6 *Find the complete solution of the system of equations below in five unknowns,*

$$\begin{array}{rcrcrcrcrcl} x_1 & + & x_2 & - & x_3 & - & x_4 & + & x_5 & = & 0 \\ x_1 & + & 2x_2 & - & 2x_3 & - & x_4 & + & x_5 & = & 0 \\ 2x_1 & + & 3x_2 & - & 3x_3 & - & 2x_4 & + & 3x_5 & = & 0. \end{array}$$

We have three equations in five unknown, so we can expect at least a 2-parametric set of solutions. By subtracting the first equation from the second equation and twice the first equation from the third equation we obtain the equivalent system

$$\begin{array}{rcrcrcrcrcl} x_1 & + & x_2 & - & x_3 & - & x_4 & + & x_5 & = & 0 \\ & & x_2 & - & x_3 & & & & & = & 0 \\ & & x_2 & - & x_3 & & & + & x_5 & = & 0. \end{array}$$

This is only possible, if $x_5 = 0$ and $x_3 = x_2$, and the system is reduced to $x_1 - x_4 = 0$. Choosing the parameters $x_1 = s$ and $x_2 = t$ we get the complete set of solutions

$$\{(s, t, t, s, 0) \mid s, t \in \mathbb{R}\}.$$

Example 1.7 *Find the complete solution of the system of equations below,*

$$\begin{array}{rcrcrcrcl}
x_1 & & & - & x_3 & + & x_4 & = & 0\\
x_1 & + & x_2 & + & x_3 & + & x_4 & = & 1\\
4x_1 & + & 4x_2 & + & 4x_3 & + & 3x_4 & = & 5.
\end{array}$$

We have three equations in four unknowns, so we may expect a 1-parametric set of solutions. Subtracting four times the second equation from the third equation, and the first equation from the second equation we get the equivalent system

$$\begin{array}{rcrcrl}
x_1 & & - & x_3 & + \; x_4 & = 0\\
 & x_2 & + & 2x_3 & & = 1\\
 & & & & - \; x_4 & = 1,
\end{array}$$

hence $x_4 = -1$ and

$$x_1 = x_3 + 1, \qquad x_2 = -2x_3 + 1 \quad \text{and} \quad x_4 = -1.$$

Choosing $x_3 = s$ as the parameter the set of solutions becomes

$$\{(s+1, -2s+1, s, -1) \mid s \in \mathbb{R}\}.$$

Example 1.8 *1. Explain why the inhomogeneous linear system of equations*

$$(1)\qquad \begin{array}{rcrcrcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & + & 5x_5 & = & 1 \\ 2x_1 & + & 3x_2 & + & 4x_3 & + & 5x_4 & + & x_5 & = & 2 \\ 3x_1 & + & 4x_2 & + & 5x_3 & + & 6x_4 & - & 3x_5 & = & 3 \end{array}$$

has infinitely many solutions and find a parametric description of the complete solution.

2. Find the complete solution of the homogeneous *system corresponding to (1).*

1. We have three equations in five unknowns, so we may expect at least a 2-parametric set of solutions. By subtracting the second equation from the third one, and the first equation from the second one we obtain the equivalent system

$$\begin{array}{rcrcrcrcrcl} x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & + & 5x_5 & = & 1 \\ x_1 & + & x_2 & + & x_3 & + & x_4 & - & 4x_5 & = & 1 \\ x_1 & + & x_2 & + & x_3 & + & x_4 & - & 4x_5 & = & 1, \end{array}$$

hence by subtracting in the new system the second equation from the first one and then remove the superfluous third equation,

$$\begin{array}{rcrcrcrcrcl} x_1 & + & x_2 & + & x_3 & + & x_4 & - & 4x_5 & = & 1 \\ & & x_2 & + & 2x_3 & + & 3x_4 & + & 9x_5 & = & 0, \end{array}$$

which again is equivalent to

$$\begin{array}{rcrcrcrcrcl} x_1 & & & - & x_3 & - & 2x_4 & - & 13x_5 & = & 1 \\ & & x_2 & + & 2x_3 & + & 3x_4 & + & 9x_5 & = & 0. \end{array}$$

Choosing the three parameters

$$x_3 = s, \qquad x_4 = t, \qquad x_5 = u,$$

we get the 3-parametric set of solution

$$\{(s + 2t + 13u + 1, -2s - 3t - 9u, s, t, u) \mid s, t, u \in \mathbb{R}\}.$$

2. We obtain the complete set of solution of the corresponding homogeneous system by removing the constant 1 in the x_1 coordinate above,

$$\{(s + 2t + 13u, -2s - 3t - 9u, s, t, u) \mid s, t, u \in \mathbb{R}\}.$$

Example 1.9 *In an ordinary rectangular coordinate system in space, four planes α_1, α_2, α_3 and α_4 are given by the equations*

$$\begin{array}{lrcrcrcl} \alpha_1: & x & + & y & - & 2z & = & 0 \\ \alpha_2: & 2x & - & y & + & z & = & 1 \\ \alpha_3: & 7x & + & y & - & 4z & = & 2 \\ \alpha_4: & x & - & 2y & + & 3z & = & 1. \end{array}$$

Prove that the four planes α_1, α_2, α_3 and α_4 have a straight line ℓ in common and derive a parametric description of ℓ.

Any point of intersection must fulfil all four equations. If we eliminate x by α_1, we get the equivalent system

$$\begin{array}{rcrcrcl} x & + & y & - & 2z & = & 0, \\ & - & 3y & + & 5z & = & 1, \\ & - & 6y & + & 10z & = & 2, \\ & - & 3y & + & 5z & = & 1, \end{array}$$

where we see that the latter three equations are equivalent. Thus, it suffices to consider the system

$$\begin{array}{rcrcrcl} x & + & y & - & 2z & = & 0 \\ & & y & - & \frac{5}{3}\,z & = & -\frac{1}{3} \end{array} \qquad \text{thus} \qquad \begin{array}{l} x = -y + 2z = -\frac{5}{3}\,z + \frac{1}{3} + 2z = \frac{1}{3}\,z + \frac{1}{3} \\ y = \frac{5}{3}\,z - \frac{1}{3}. \end{array}$$

Thus the points of intersection are given by

$$\left\{ \left(\frac{1}{3}\,z + \frac{1}{3}, \frac{5}{3}\,z - \frac{1}{3}, z \right) \;\middle|\; z \in \mathbb{R} \right\} = \left(\frac{1}{3}, -\frac{1}{3}, 0 \right) + \{(s, 5s, 3s) \mid s \in \mathbb{R}\}$$

where $z = 3s$. This is a parametric description of a line ℓ through $\left(\frac{1}{3}, -\frac{1}{3}, 0 \right)$ in the direction of $(1, 5, 3)$.

Example 1.10 *Find the values of the parameter a for which the homogeneous linear system of equations*

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & - & (a+1)x_3 & = & 0 \\ (2+a)x_1 & + & 4x_2 & - & 2x_3 & = & 0 \\ 3x_1 & + & (6+2a)x_2 & - & 3x_3 & = & 0 \end{array}$$

has nontrivial solutions. Find for any of these values of a the complete solution of the system.

First solution. If we allow ourselves to apply determinants, which formally have not yet been introduced, the task is very simple. The condition is that the corresponding determinant is 0. We compute

$$\begin{aligned} \begin{vmatrix} 1 & 2 & -a-1 \\ 2+a & 4 & -2 \\ 3 & 6+2a & -3 \end{vmatrix} &= 2 \begin{vmatrix} -a & 1 & -a-1 \\ a & 2 & -2 \\ 0 & 3+a & -3 \end{vmatrix} = 2a \begin{vmatrix} -1 & 1 & -a \\ 1 & 2 & 0 \\ 0 & 3+a & a \end{vmatrix} \\ &= 2a^2 \begin{vmatrix} -1 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 3+a & 1 \end{vmatrix} = 2a^2\{-2-3-a-1\} = -2a^2(a+6) = 0. \end{aligned}$$

Thus, $a = 0$ and $a = -6$.

Second solution. Elimating x_1 by means of the first equation we obtain the equivalent system

$$\begin{array}{ccccccc} x_1 & + & 2x_2 & - & (a+1)x_3 & = & 0 \\ & - & 2ax_2 & + & (a^2+3a)x_3 & = & 0 \\ & & 2ax_2 & + & 3ax_3 & = & 0, \end{array}$$

whence by Gauß elimination,

$$\begin{array}{ccccccc} x_1 & + & 2x_2 & - & (a+1)x_3 & = & 0 \\ & & 2ax_2 & + & 3ax_3 & = & 0 \\ & & & & (a^2+6a)x_3 & = & 0. \end{array}$$

If $a^2 + 6a = a(a+6) \neq 0$, i.e. $a \neq 0$ and $a \neq -6$, then $x_3 = 0$, which implies that $x_2 = 0$ and thus $x_1 = 0$, and we get no nontrivial solution.

If $a = 0$, then x_2 and x_3 can be chosen freely, while $x_1 = -2x_2 + x_3$, hence the set of solution becomes

$$\{(-2s + t, s, t) \mid s, t \in \mathbb{R}\} \qquad \text{for } a = 0.$$

If $a = -6$, then x_3 can be chosen freely. Since $a \neq 0$, it follows that $x_2 = \frac{3}{2}\,x_3$ and

$$x_1 = -2x_2 + (a+1)x_3 = -3x_3 + (-6+1)x_3 = -8x_3.$$

Introducing the parameter $s = \frac{1}{2}\,x_3$ the set of solutions becomes

$$\{(-16s, 32, 2s) \mid s \in \mathbb{R}\}.$$

Example 1.11 *Find all values of a, for which the system of equations*

$$\begin{array}{rcrcrcc} x_1 & + & 2x_2 & + & x_3 & = & a \\ 3x_1 & + & 4x_2 & + & 2x_3 & = & a-3 \\ -4x_1 & + & 2x_2 & + & x_3 & = & 2 \end{array}$$

has a solution and find for everyone of these values of a the complete solution.

We note that the group $2x_2 + x_3$ occurs in all three equations (disguised as $2 \cdot (2x_2 + x_3) = 4x_2 + 2x_3$ in the second equation). Using the first equation to eliminate this group from the second and the third equation we get the equivalent system

$$\begin{array}{rcrcrcc} x_1 & + & 2x_2 & + & x_3 & = & a \\ x_1 & & & & & = & -a-3 \\ -5x_1 & & & & & = & -a+2. \end{array}$$

It follows from the second and the third equation of the new system that

$$5x_1 = -5a - 15 = a - 2, \text{ thus } 6a = -13, \text{ i.e. } a = -\frac{13}{6}.$$

A necessary condition of a solution is that $a = -\frac{13}{6}$. In this case,

$$x_1 = -a - 3 = \frac{13}{6} - 3 = -\frac{5}{6},$$

and $2x_2 + x_3 = a - x_1 = -\frac{13}{6} + \frac{5}{6} = -\frac{8}{6} = -\frac{4}{3}$. Using the parameter $s = x_2$ we get the complete set of solutions

$$\left\{ \left(-\frac{5}{6}, s, -\frac{4}{3} - 2s \right) \;\middle|\; s \in \mathbb{R} \right\}, \qquad \text{for } a = -\frac{13}{6},$$

and the set of solutions is empty for any other value of a.

Example 1.12 *Solve the system of equations*

$$\begin{array}{rcrcrcl} a^2x_1 & + & 5x_2 & + & x_3 & = & b \\ ax_1 & + & (a+3)x_2 & + & 3x_3 & = & 0 \\ x_1 & + & 2x_2 & + & x_3 & = & 0, \end{array}$$

where a and b are real numbers.

This example can be treated more or less elegant. First note that the coefficients of x_3 do not contain a or b. We therefore start by eliminating x_3 from the first and second equation by means of the thirt equation. Hence we obtain the equivalent system

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & x_3 & = & 0 \\ (a^2-1)x_1 & + & 3x_2 & & & = & b \\ (a-3)x_1 & + & (a-3)x_2 & & & = & 0. \end{array}$$

1. If $a = 3$, the system of equations is reduced to

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & x_3 & = & 0 \\ 8x_1 & + & 3x_2 & & & = & b. \end{array}$$

   Choosing $x_2 = s$ as the parameter we get $x_1 = \dfrac{b}{8} - \dfrac{3}{8}\,s$ and

$$x_3 = -x_1 - 2x_2 = -\frac{b}{8} + \frac{3}{8}\,s - 2s = -\frac{b}{8} - \frac{13}{8}\,s,$$

   and the set of solutions is

$$\left\{ \left(\frac{b}{8} - \frac{3s}{8}, s, -\frac{b}{8} - \frac{13s}{8} \right) \;\middle|\; s \in \mathbb{R} \right\} \qquad \text{for } a = 3.$$

2. If $a \neq 3$, the system of equations is reduced to

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & x_3 & = & 0 \\ x_1 & + & x_2 & & & = & 0 \\ (a^2-1)x_1 & + & 3x_2 & & & = & b \end{array}$$

   It follows from the first two equations that $x_2 + x_3 = 0$ and $x_1 + x_2 = 0$, thus $x_1 = x_3 = -x_2$, and hence

$$b = (a^2-1)x_1 + 3x_2 = (a^2-4)x_1.$$

   Then we have two possibilities:

   (a) If $a \neq \pm 2$, then $x_1 = \dfrac{b}{a^2-4}$, hence the set of solutions is one single point

$$\left(\frac{b}{a^2-4}, -\frac{b}{a^2-4}, \frac{b}{a^2-4} \right).$$

   (b) If $a^2 = 4$, the system can only be solved for $b = 0$. In this case the set of solution becomes

$$\{(s,-s,s) \mid s \in \mathbb{R}\} \qquad a = \pm 2 \text{ and } b = 0.$$

Example 1.13 *Given the system of equations*

$$\begin{array}{rcrcrcrcr}
x_1 & + & (a-1)x_2 & + & 2x_3 & + & (a+2)x_4 & = & a+b \\
x_1 & + & 2ax_2 & & & + & ax_4 & = & 2a+b \\
& & -(a+1)x_2 & + & (2a+2)x_3 & & & = & 2a+b \\
& & (2a+2)x_2 & + & (4a-4)x_3 & & (a^2+a-8)x_4 & = & 4a+ab+b,
\end{array}$$

where a and b are real numbers.

1. *Find the rank of the total matrix of the system of equations for every pair $(a,b) \in \mathbb{R}^2$.*
2. *Find the pairs (a,b), for which the system of equations has*
   - (a) *no solution,*
   - (b) *one solution,*
   - (c) *infinitely many solutions.*
3. *Solve the system of equations for $(a,b) = (-1,1)$.*

1. The total matrix is equivalent to

$$\left(\begin{array}{cccc|c}
1 & a-1 & 2 & a+2 & a+b \\
1 & 2a & 0 & a & 2a+b \\
0 & -(a+1) & 2a+2 & 0 & 0 \\
0 & 2a+2 & 4a-4 & a^2+a-8 & 4a+ab+b
\end{array}\right) \sim \begin{array}{l} R_1 := R_2 \\ R_2 := R_1 - R_2 \end{array}$$

$$\left(\begin{array}{cccc|c}
1 & 2a & 0 & a & 2a+b \\
0 & -(a+1) & 2 & 2 & -a \\
0 & -(a+1) & 2a+2 & 0 & 0 \\
0 & 2a+2 & 4a-4 & a^2+a-8 & 4a+ab+b
\end{array}\right) \sim \begin{array}{l} R_1 := -R_2 \\ R_3 := R_3 - R_2 \\ R_4 := R_4 + 2R_2 \end{array}$$

$$\left(\begin{array}{cccc|c}
1 & 2a & 0 & a & 2a+b \\
0 & a+1 & -2 & -2 & a \\
0 & 0 & 2a & -2 & a \\
0 & 0 & 4a & a^2+a-4 & 2a+ab+b
\end{array}\right) \sim R_4 := R_4 - 2R_3$$

$$\left(\begin{array}{cccc|c}
1 & 2a & 0 & a & 2a+b \\
0 & a+1 & -2 & -2 & a \\
0 & 0 & 2a & -2 & a \\
0 & 0 & 0 & a(a+1) & (a+1)b
\end{array}\right)$$

It follows that the rank is 4, if $a \neq 0$ and $a \neq -1$.

When $a = 0$, the total matrix is equivalent to

$$\left(\begin{array}{cccc|c}
1 & 0 & 0 & 0 & b \\
0 & 1 & -2 & -2 & 0 \\
0 & 0 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & b
\end{array}\right).$$

This is of rank 4, if $b \neq 0$, and of rank 3, if $b = 0$.

When $a = -1$, the total matrix is equivalent to

$$\left(\begin{array}{cccc|c}
1 & -2 & 0 & -1 & b-2 \\
0 & 0 & -2 & -2 & -1 \\
0 & 0 & -2 & -2 & -1 \\
0 & 0 & 0 & 0 & 0
\end{array}\right).$$

The rank is 3 for every $b \in \mathbb{R}$.

2. (a) It follows from the above that if

$$(a,b) \in \{(0,b) \mid b \neq 0\},$$

then the system of equations has no solution.

(b) Since the matrix of coefficients has rank 4 for $a \neq 0$ and $a \neq 1$, we get precisely one solution in these cases, i.e. in the parametric set

$$\{(a,b) \mid a \neq 0, -1 \text{ og } b \in \mathbb{R}\}.$$

(c) It follows from the above that we have infinitely many solutions when $a = 0$ and $b = 0$, or when $a = -1$ and $b \in \mathbb{R}$, thus in the set of parameters

$$\{(0,0)\} \cup \{(-1,b) \mid b \in \mathbb{R}\}.$$

3. When we choose $(a,b) = (-1,1)$, then the total matrix is by 1) equivalent to

$$\left(\begin{array}{rrrr|c} 1 & -2 & 0 & -1 & b-2 \\ 0 & 0 & -2 & -2 & -1 \\ 0 & 0 & -2 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} \sim \\ R_3 := R_2 - R_3 \\ R_2 := -R_2 \end{array} \quad \left(\begin{array}{rrrr|c} 1 & -2 & 0 & -1 & b-2 \\ 0 & 0 & +2 & +2 & +1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

corresponding to the system of equations

$$\begin{array}{ccccccc} x_1 & - & 2x_2 & & - & x_4 & = & b-2 \\ & & & x_3 & + & x_4 & = & \dfrac{1}{2}. \end{array}$$

Choosing $x_2 = s$ and $x_4 = t$ as our parameters we obtain the set of solutions

$$\left\{ \left(2s + t - 2 + b, s, \frac{1}{2} - t, t\right) \;\middle|\; s, t \in \mathbb{R} \right\}.$$

Example 1.14 *Given the linear system of equations*

$$\begin{array}{rcrcrcrcrcr}
x_1 & + & ax_2 & + & 2x_3 & - & x_4 & - & 2x_5 & = & -a \\
-x_1 & + & 2x_2 & - & 2x_3 & + & x_4 & - & x_5 & = & -2 \\
-2x_1 & + & 4x_2 & + & (a-5)x_3 & + & 2x_4 & - & 2x_5 & = & a-4 \\
x_1 & - & 2x_2 & + & (a+1)x_3 & + & (a+1)x_4 & + & x_5 & = & a+2.
\end{array}$$

1. *Find for every $a \in \mathbb{R}$ the rank of the matrix of coefficients and the total matrix of the system of equations.*

2. *Find for every $a \in \mathbb{R}$ the complete solution of the system of equations.*

1. Since the matrix of coefficients is contained in the total matrix, the best strategy is first to discuss the total matrix. This is equivalent to

$$\left(\begin{array}{rrrrr|r}
1 & a & 2 & -1 & -2 & -a \\
-1 & 2 & -2 & 1 & -1 & -2 \\
-2 & 4 & a-5 & 2 & -2 & a-4 \\
1 & -2 & a+1 & a+1 & 1 & a+2
\end{array}\right) \sim \begin{array}{l} R_1 := -R_2 \\ R_2 := R_1 + R_2 \\ R_3 := R_3 - 2R_2 \\ R_4 := R_2 + R_4 \end{array}$$

$$\left(\begin{array}{rrrrr|r}
1 & -2 & 2 & -1 & 1 & 2 \\
0 & a+2 & 0 & 0 & -3 & -a-2 \\
0 & 0 & a-1 & 0 & 0 & a \\
0 & 0 & a-1 & a & 0 & a
\end{array}\right) \begin{array}{l} \sim \\ R_4 := R_4 - R_3 \end{array}$$

$$\left(\begin{array}{rrrrr|r}
1 & -2 & 2 & -1 & 1 & 2 \\
0 & a+2 & 0 & 0 & -3 & -a-2 \\
0 & 0 & a-1 & 0 & 0 & a \\
0 & 0 & 0 & a & 0 & 0
\end{array}\right).$$

If $a \neq \{-2, 0, 1\}$, then both the matrix of coefficients and the total matrix have rank 4.

If $a = -2$, then both the matrix of coefficients and the total matrix have rank 4.

If $a = -2$, then the system of equation becomes

$$\begin{array}{rcrcrcrcrcr} x_1 & - & 2x_2 & + & 2x_3 & - & x_4 & + & x_5 & = & 2 \\ & & & & & & & - & 3x_5 & = & 0 \\ & & & - & 3x_3 & & & & & = & -2 \\ & & & & & - & 2x_4 & & & = & 0. \end{array}$$

It follows immediately that $x_5 = x_4 = 0$ and $x_3 = \dfrac{2}{3}$, thus the latter equation is reduced to

$$x_1 - 2x_2 = 2 - \frac{4}{3} = \frac{2}{3}.$$

Using $x_2 = s$ as our parameter the complete solution becomes

$$\left\{ \left(2s + \frac{2}{3}, s, \frac{2}{3}, 0, 0 \right) \;\middle|\; s \in \mathbb{R} \right\} \qquad \text{for } a = -2.$$

Finally, if $a \neq 0$, -1, 2, then we get the system of equations

$$\begin{array}{rcrcrcrcrcr} x_1 & - & 2x_2 & + & 2x_3 & - & x_4 & + & x_5 & = & 2 \\ & & (a+2)x_2 & & & & & - & 3x_5 & = & -(a+2) \\ & & & & (a-1)x_3 & & & & & = & a \\ & & & & & & ax_4 & & & = & 0 \end{array}$$

If $a = 0$, then both the matrix of coefficients and the total matrix have rank 3.

Finally, if $a = -1$, then the matrix of coefficients has rank 3 (there are only zeros in the third row), while the total matrix has rank 4.

2. If $a = -1$, then it follows from the above that the system of equations does not have any solution.

   If $a = 0$, then the system of equations is written

$$\begin{array}{rcrcrcrcrcr} x_1 & - & 2x_2 & + & 2x_3 & - & x_4 & + & x_5 & = & 2 \\ & & 2x_2 & & & & & - & -3x_5 & = & -2 \\ & & & - & x_3 & & & & & = & 0 \end{array}$$

   thus $x_3 = 0$ and

$$\begin{array}{rcrcrcr} x_1 & - & x_4 & - & 2x_5 & = & 0 \\ 2x_2 & & & - & 3x_5 & = & -2 \end{array}$$

   Using $x_4 = s$ and $x_5 = 2t$ as parameters we get the solution

$$\{(s + 4t, 3t - 1, 0, s, 2t) \mid s, t \in \mathbb{R}\} \qquad \text{for } a = 0.$$

   It follows immediately that $x_4 = 0$ and $x_3 = \dfrac{a}{a-1} = 1 + \dfrac{1}{a-1}$, hence

$$x_1 - 2x_2 + x_5 = 2 - 2 - \frac{2}{a-1} = -\frac{2}{a-1}$$

and

$$(a+2)x_2 - 3x_5 = -(a+2).$$

If we choose $x_2 = s$ as parameter, we get $x_5 = \frac{1}{3}(a+2)(s+1)$ and

$$x_1 = 2x_2 - x_5 - \frac{1}{a-1} = 2s - \frac{1}{3}(a+2)(s+1) - \frac{2}{a-1}.$$

Hence for $a \neq 0, -2, 1$, the solution becomes

$$\left\{ \left(2s - \frac{1}{3}(a+2)(s+1) - \frac{2}{a-1}, s, 1 + \frac{1}{a-1}, 0, \frac{1}{3}(a+2)(s+1) \right) \;\middle|\; s \in \mathbb{R} \right\}.$$

Example 1.15 *Let in an ordinary rectangular coordinate system in space for every value of $a \in \mathbb{R}$ the planes α and β be given by the equations*

$$\begin{array}{rrcrcrcl} \alpha: & x & + & a^2y & + & az & = & 2a-1 \\ \beta: & ax & + & ay & + & z & = & 1. \end{array}$$

Check for every $a \in \mathbb{R}$, if $\alpha \cap \beta$ is empty or a line or a plane.

The corresponding total matrix is equivalent to

$$\left(\begin{array}{ccc|c} 1 & a^2 & a & 2a-1 \\ a & a & 1 & 1 \end{array} \right) \begin{array}{l} \sim \\ R_1 := R_2 \\ R_2 := aR_2 - R_1 \end{array}$$
$$\left(\begin{array}{ccc|c} a & a & 1 & 1 \\ a^2-1 & 0 & 0 & -a+1 \end{array} \right).$$

When $a \neq \pm 1$, then both the matrix of coefficients and the total matrix are of rank 2.
When $a = 1$, then both the matrix of coefficients and the total matrix are of rank 1.
When $a = -1$, then the matrix of coefficients is of rank 1, while the total matrix is of rank 2.
If $a = -1$, the set of solutions is empty which also follows from

$$\begin{array}{rrcrcrcl} \alpha: & x & + & y & - & z & = & -3 \\ \beta: & -\;x & - & y & + & z & = & 1 \end{array} \quad \text{dvs.} \quad \begin{array}{l} x+y-1=-3 \\ x+y-z=-1. \end{array}$$

If $a = 1$, the set of solutions forms the plane $\alpha = \beta$, which also follows from

$$\begin{array}{rl} \alpha: & x+y+z=1 \\ \beta: & x+y+z=1. \end{array}$$

If $a \neq \pm 1$, then $\alpha \cap \beta$ is a straight line. The corresponding system of equations is equivalent to

$$\begin{array}{rcrcrcl} ax & + & ay & + & z & = & 1, \\ (a+1)x & & & & & = & -1, \end{array} \quad \text{dvs. } x = -\frac{1}{a+1},$$

and

$$\alpha \cap \beta : \left\{ \left(-\frac{1}{a+1}, s, 1 - as + \frac{a}{a+1} \right) \;\middle|\; s \in \mathbb{R} \right\} = \left\{ \left(-\frac{1}{a+1}, 0, 2 - \frac{1}{a+1} \right) + s(0,1,-a) \;\middle|\; s \in \mathbb{R} \right\}.$$

Example 1.16 *Consider for every real number a the system of equations.*

$$\begin{array}{rcrcrcl} x_1 & + & (a+1)x_2 & + & a^2x_3 & = & a^3 \\ (1-a)x_1 & + & (1-2a)x_2 & + & & = & a^3 \\ x_1 & + & (a+1)x_2 & + & ax^3 & = & a^2. \end{array}$$

1. *Find the solution for $a = -1$.*
2. *Find the values of a, for which we get infinitely many solutions.*

1. If $a = -1$ then the system becomes

$$\begin{array}{rcrcrcr} x_1 & + & & + & x_3 & = & -1, \\ 2x_1 & + & 3x_2 & & & = & -1, \\ x_1 & & & - & x_3 & = & 1. \end{array}$$

We get from the first and the third equation that $x_1 = 0$ and $x_3 = -1$, hence $x_2 = -\dfrac{1}{3}$, and the solution is

$$\mathbf{x} = \left(0, -\frac{1}{3}, -1\right).$$

2. The total matrix is equivalent to

$$\left(\begin{array}{ccc|c} 1 & a+1 & a^2 & a^3 \\ 1-a & 1-2a & 0 & a^3 \\ 1 & a+1 & a & a^2 \end{array}\right) \begin{array}{l} \sim \\ R_3 := R_1 - R_3 \\ R_1 := R_3 \end{array}$$
$$\left(\begin{array}{ccc|c} 1 & a+1 & a & a^2 \\ 1-a & 1-2a & 0 & a^3a^3 - a^2 \\ 0 & 0 & a^2 - a & \end{array}\right) \underset{R_2 := R_2 + (a-1)R_1}{\sim}$$
$$\left(\begin{array}{ccc|c} 1 & a+1 & a & a^2 \\ 0 & a^2 - 2a & a^2 - a & 2a^3 - a^2 \\ 0 & 0 & a^2 - a & a^3 - a^2 \end{array}\right) \underset{R_2 := R_2 - R_3}{\sim}$$
$$\left(\begin{array}{ccc|c} 1 & a+1 & a & a^2 \\ 0 & a(a-2) & 0 & a^3 \\ 0 & 0 & a(a-1) & a^2(a-1) \end{array}\right)$$

When $a \neq \{0, 1, 2\}$, then both the matrix of coefficients and the total matrix are of rank 3, so the solution exists and is unique.

When $a = 0$, then both the matrix of coefficients and the total matrix are of rank 1, so we have infinitely many solutions.

When $a = 0$, the system is equivalent to

$$x_1 + x_2 = 0.$$

Choosing the parameters $x_1 = s$ and $x_3 = t$ the set of solutions is given by

$$\{(s, -s, t) \mid s, t \in \mathbb{R}\}.$$

When a = 1, then both the matrix of coefficients and the total matrix are of rank 2, so we have infinitely many solutions.

The system is for $a = 1$ equivalent to

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & x_3 & = & 1, \\ & - & x_2 & & & = & 1, \end{array}$$

thus $x_2 = -1$ and $x_1 + x_3 = 3$. Choosing the parameter $x_1 = s$ the set of solutions is described by

$$\{(s, -1, 3 - s) \mid s \in \mathbb{R}\}.$$

When $a = 2$, then the matrix of coefficients is of rank 2, while the total matrix is of rank 3- We therefore get no solution.

We get infinitely many solutions for $a = 0$ and $a = 1$.

Example 1.17 *Find numbers a_0, a_1, a_2 and a_3, such that the curve of the equation*

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

goes through the four points of the coordinates

$$(-2,16), \quad (-1,0), \quad (1,-8), \quad (3,-24).$$

When (x, y) is replaced successively by the four points, we get the four linear equations in the four unknowns a_0, a_1, a_2 og a_3,

$$\begin{array}{rcrcrcrcr}
a_0 & - & 2a_1 & + & 4a_2 & - & 8a_3 & = & 16, \\
a_0 & + & a_1 & + & a_2 & - & a_3 & = & 0, \\
a_0 & + & a_1 & + & a_2 & + & a_3 & = & -8, \\
a_0 & + & 3a_1 & + & 9a_2 & + & 27a_3 & = & -24.
\end{array}$$

The total matrix is equivalent to

$$\left(\begin{array}{rrrr|r}
1 & -2 & 4 & -8 & 16 \\
1 & -1 & 1 & -1 & 0 \\
1 & 1 & 1 & 1 & -8 \\
1 & 3 & 9 & 27 & -24
\end{array}\right)
\begin{array}{l}
\sim \\
R_1 := R_3 \\
R_2 := R_3 - R_2 \\
R_3 := R_3 - R_1 \\
R_4 := R_4 - R_3
\end{array}$$

$$\left(\begin{array}{rrrr|r}
1 & 1 & 1 & 1 & -8 \\
0 & 2 & 0 & 2 & -8 \\
0 & 3 & -3 & 9 & -24 \\
0 & 2 & 8 & 26 & -16
\end{array}\right)
\begin{array}{l}
\sim R_2 := R_2/2 \\
R_3 := R_3/3 \\
R_4 := R_4/2
\end{array}$$

$$\left(\begin{array}{rrrr|r}
1 & 1 & 1 & 1 & -8 \\
0 & 1 & 0 & 1 & -4 \\
0 & 1 & -1 & 3 & -8 \\
0 & 1 & 4 & 13 & -8
\end{array}\right)
\begin{array}{l}
\sim \\
R_3 := R_2 - R_3 \\
R_4 := R_4 - R_2 \\
R_1 := R_1 - R_2
\end{array}$$

$$\left(\begin{array}{rrrr|r}
1 & 0 & 1 & 0 & -4 \\
0 & 1 & 0 & 1 & -4 \\
0 & 0 & 1 & -2 & 4 \\
0 & 0 & 4 & 12 & -4
\end{array}\right)
\begin{array}{c}
\sim \\
R_4 := R_4/4
\end{array}$$

$$\left(\begin{array}{rrrr|r}
1 & 0 & 1 & 0 & -4 \\
0 & 1 & 0 & 1 & -4 \\
0 & 0 & 1 & -2 & 4 \\
0 & 0 & 1 & 3 & -1
\end{array}\right)
\begin{array}{c}
\sim \\
R_4 := R_4 - R_4
\end{array}$$

$$\left(\begin{array}{rrrr|r}
1 & 0 & 1 & 0 & -4 \\
0 & 1 & 0 & 1 & -4 \\
0 & 0 & 1 & -2 & 4 \\
0 & 0 & 0 & 5 & -5
\end{array}\right)
\begin{array}{c}
\sim \\
R_4 := R_4/5
\end{array}$$

$$\left(\begin{array}{rrrr|r}
1 & 0 & 1 & 0 & -4 \\
0 & 1 & 0 & 1 & -4 \\
0 & 0 & 1 & -2 & 4 \\
0 & 0 & 0 & 1 & -1
\end{array}\right)$$

This corresponds to the system of equations

$$\begin{array}{rcrcrcr}
a_0 & & + & a_2 & & = & -4, \\
& a_1 & & & + \; a_3 & = & -4, \\
& & & a_2 & - \; 2a_3 & = & 4, \\
& & & & a_3 & = & -1.
\end{array}$$

When we solve this system from below and upwards, we get successively $a_3 = -1$, $a_2 = 2$, $a_1 = -3$ and $a_6 = -6$. Thus the solution is

$$y = -6 - 3x + 2x^2 - x^3.$$

It is left to the reader to check the results, i.e. proving that all points

$$(x, y) \in \{(-2, 16), (-1, 0), (1, -8), (3, -24)\}$$

fulfil the equation.

Example 1.18 *Find the numbers a_0, a_1 and a_2, such that the parabola of the equation*

$$y = a_0 + a_1 x + a_2 x^2$$

passes through the points of coordinates $(2, -3)$, $(9, 4)$ and $(t, 4)$.
Find for every given t the number of solutions.

Replacing (x, y) by the coordinates above we get three linear equations in the unknowns a_0, a_1 and a_2, and the parameter $t \in \mathbb{R}$,

$$\begin{array}{ccccccc} a_0 & + & 2a_1 & + & 4a_2 & = & -3, \\ a_0 & + & 9a_1 & + & 81a_2 & = & 4, \\ a_0 & + & ta_1 & + & t^2 a_2 & = & 4. \end{array}$$

The total matrix is then equivalent to

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & -3 \\ 1 & 9 & 81 & 4 \\ 1 & t & t^2 & 4 \end{array}\right) \begin{array}{l} \sim \\ R_2 := R_2 - R_1 \\ R_3 := R_3 - R_2 \end{array}$$
$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & -3 \\ 0 & 7 & 77 & 7 \\ 0 & t-9 & t^2-81 & 0 \end{array}\right) \begin{array}{c} \sim \\ R_2 := R_2/7 \end{array}$$
$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & -3 \\ 0 & 1 & 11 & 1 \\ 0 & t-9 & t^2-81 & 0 \end{array}\right) \begin{array}{c} \sim \\ R_3 := R_3 - (t-9)R_2 \end{array}$$
$$\left(\begin{array}{ccc|c|c} 1 & 2 & 4 & 4 & -3 \\ 0 & 1 & 11 & 11 & 1 \\ 0 & 0 & (t-2)(t-9) & (t-2)(t-9) & -(t-9) \end{array}\right).$$

If $t = 2$, then we have no solution, corresponding to the fact that a parabola of this particular form cannot possibly pass through all three points $(2, -3)$, $(2, 4)$ and $(9, 4)$.

If $t = 9$, then both the matrix of coefficients and the total matrix are of rank 2, so we have infinitely many solutions. This is also reasonable, because $(9, 4)$ and $(t, 4)$ coincide for $t = 9$.

The set of solutions is derived from

$$\begin{array}{ccccccccc} a_0 & + & 2a_1 & = & - & 3 & - & 4a_2, \\ & & a_1 & = & + & 1 & - & 11a_2, \end{array}$$

so by choosing $a_2 = s$ as parameter we get

$$(a_0, a_1, a_2) \in \{(-5+18s, 1-11s, s) \mid s \in \mathbb{R}\}.$$

When $t \neq 2, 9$, the solution is uniquely determined by the equations

$$\begin{array}{rcrcrcr} a_0 & + & 2a_1 & + & 4a_2 & = & -3, \\ & & a_1 & + & 11a_2 & = & 1, \\ & & & & (t-2)a_2 & = & -1, \end{array}$$

hence $a_2 = -\dfrac{1}{t-2}$ and

$$a_1 = 1 + \frac{11}{t-2} = \frac{t+9}{t-2}$$

and

$$a_0 = -3 - \frac{2t+18}{t-2} + \frac{4}{t-2} = -3 - \frac{2t+14}{t-2} = -\frac{5t+8}{t-2}.$$

Thus, if $t \neq 2, 9$, then

$$(a_0, a_1, a_2) = \left(-\frac{5t+8}{t-2}, \frac{t+9}{t-2}, -\frac{1}{t-2}\right) = \left(-5 - \frac{18}{t-2}, 1 + \frac{11}{t-2}, -\frac{1}{t-2}\right).$$

Example 1.19 *Given the two following systems of linear equations in the unknowns* x_1, x_2, x_3, x_4, x_5*:*

$$(2)\quad \begin{cases} x_1 + 2x_2 + 3x_3 + 2x_4 + x_5 = 0, \\ x_1 - 2x_2 + 3x_3 - 3x_4 + x_5 = 0, \\ x_1 + 2x_2 - 3x_3 + 2x_4 + x_5 = 0, \\ x_2 + x_4 = 0, \end{cases}$$

$$(3)\quad \begin{cases} x_1 = x_5, \\ x_2 = x_4. \end{cases}$$

The complete solution of (2) is denoted by L_1*, while the complete solution of (3) is denoted by* L_2*.*

1. *Find the parametric descriptions of* L_1 *and* L_2*.*
2. *Describe the intersection* $L_1 \cap L_2$*, and write the vector* $(1, 2, 3, 4, 5)$ *as a sum of two vectors, one from* L_1 *and the other one from* L_2*.*
3. *Find the set of solutions of the inhomogeneous linear system of equations*

$$\begin{cases} x_1 = x_5 + 5, \\ x_2 = x_4 + 4. \end{cases}$$

1. First reduce the corresponding matrix of (2),

$$\left(\begin{array}{rrrrr|r} 1 & 2 & 3 & 2 & 1 & 0 \\ 1 & -2 & 3 & -2 & 1 & 0 \\ 1 & 2 & -3 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{rrrrr|r} 1 & 2 & 3 & 2 & 1 & 0 \\ 0 & 4 & 0 & 4 & 0 & 0 \\ 0 & 0 & -6 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right)$$
$$\sim \left(\begin{array}{rrrrr|r} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The rank is 3, so the null space is of dimension $5 - 3 = 2$. Choosing $x_1 = s$ and $x_2 = t$ as parameters we get

$$\begin{aligned} L_1 &= \{s(1, 0, 0, 0, -1) + t(0, 1, 0, -1, 0) \mid s,\, t \in \mathbb{R}\} \\ &= \{(s, t, 0, -t, -s) \mid s,\, t \in \mathbb{R}\}. \end{aligned}$$

In L_2 we choose $x_1 = s$, $x_2 = t$ and $x_3 = u$ as parameters, giving

$$\begin{aligned} L_2 &= \{s(1, 0, 0, 0, 1) + t(0, 1, 0, 1, 0) + u(0, 0, 1, 0, 0) \mid s,\, t,\, u \in \mathbb{R}\} \\ &= \{(s, t, u, t, s) \mid s,\, t,\, u \in \mathbb{R}\} \end{aligned}$$

which is of dimension 3.

2. If $\mathbf{x} \in L_1 \cap L_2$, then we have the two descriptions

$$\mathbf{x} = (s_1, t_1, 0, -t_1, -s_1) = (s_2, t_2, u_2, t_2, s_2),$$

from which $s_1 = s_2 = 0$, $t_1 = t_2 = 0$ and $u_2 = 0$, hence

$$L_1 \cap L_2 = \{\mathbf{0}\}.$$

It follows that $\mathbb{R}^5 = L_1 \otimes L_2$, which again implies that

$$\begin{array}{rcrcr} (1,2,3,4,5) & = & (-2,0,0,0,2) & + & (3,0,0,0,3) \\ & & +(0,-1,0,-1,0) & + & (0,3,0,3,0) \\ & & & + & (0,0,3,0,0) \\ & = & (-2,-1,0,1,2) & + & (3,3,3,3,3), \end{array}$$

where $(-2,-1,0,1,2) \in L_1$ and $(3,3,3,3,3) \in L_2$.
This description is of course unique.

3. The solution set is of course

$$L = \{(5+s, 4+t, u, t, s) \mid s,\ t,\ u \in \mathbb{R}\}.$$

2 Matrices

Example 2.1 *Form the product matrix* $\mathbf{AB}$*, where*

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 2 \\ 3 & 4 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 3 & 1 & 0 & 1 & 2 \\ 1 & 2 & -2 & 3 & 1 \end{pmatrix}.$$

It follows by direct computation that

$$\begin{aligned} \mathbf{AB} &= \begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 3 & 1 & 0 & 1 & 2 \\ 1 & 2 & -2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 6+1 & 2+2 & 0-2 & 2+3 & 4+1 \\ 3-1 & 1-2 & 0+2 & 1-3 & 2-1 \\ 0+2 & 0+4 & 0-4 & 0+6 & 0+2 \\ 9+4 & 3+8 & 0-8 & 3+12 & 6+4 \end{pmatrix} \\ &= \begin{pmatrix} 7 & 4 & -2 & 5 & 5 \\ 2 & -1 & 2 & -2 & 1 \\ 2 & 4 & -4 & 6 & 2 \\ 13 & 11 & -8 & 15 & 10 \end{pmatrix}. \end{aligned}$$

Example 2.2 *Compute the matrix products* $\mathbf{AB}$ *and* $\mathbf{BA}$*, given that*

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 2 & -3 \\ -2 & 3 \\ 2 & -3 \end{pmatrix}.$$

By direct computations,

$$\mathbf{AB} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ -2 & 3 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} 2-4+2 & -3+6-3 \\ 2-6+4 & -3+9-6 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{aligned} \mathbf{BA} &= \begin{pmatrix} 2 & -3 \\ -2 & 3 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 2-3 & 4-9 & 2-6 \\ -2+3 & -4+9 & -2+6 \\ 2-3 & 4-9 & 2-6 \end{pmatrix} \\ &= \begin{pmatrix} -1 & -5 & -4 \\ 1 & 5 & 4 \\ -1 & -5 & -4 \end{pmatrix}. \end{aligned}$$

The idea of the example is of course partly that $\mathbf{AB} \neq \mathbf{BA}$, and partly that one of the products may be the $\mathbf{0}$ matrix, while the other product is not the $\mathbf{0}$ matrix.

Example 2.3 *Compute the product matrix* $\mathbf{C} = \mathbf{AB}$ *of the two square matrices*

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix} \quad \textit{and} \quad \mathbf{B} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 3 \\ 1 & -1 & 2 \end{pmatrix}.$$

By a direct computation,

$$\begin{aligned} \mathbf{C} = \mathbf{AB} &= \begin{pmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 3 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1+4+1 & -1+0-1 & 1+6+2 \\ 3+4+1 & -3+0-1 & 3+6+2 \\ 1+6+2 & -1+0-2 & 1+9+4 \end{pmatrix} \\ &= \begin{pmatrix} 6 & -2 & 9 \\ 8 & -4 & 11 \\ 9 & -3 & 14 \end{pmatrix}. \end{aligned}$$

Example 2.4 *Compute the product matrices* $\mathbf{AB}$ *and* $\mathbf{BA}$ *of the two square matrices*

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 5 & 4 \\ 0 & 2 & 1 \end{pmatrix} \quad \textit{and} \quad \mathbf{B} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 3 \\ 1 & -1 & 2 \end{pmatrix}.$$

By direct computations,

$$\begin{aligned} \mathbf{AB} &= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 5 & 4 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 3 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1+4+3 & -1+0-3 & 1+6+6 \\ 3+10+4 & -3+0-4 & 3+15+8 \\ 0+4+1 & 0+0-1 & 0+6+2 \end{pmatrix} \\ &= \begin{pmatrix} 8 & -4 & 7 \\ 17 & -7 & 26 \\ 5 & -1 & 8 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \mathbf{BA} &= \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 3 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 5 & 4 \\ 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1-3+0 & 2-5+2 & 3-4+1 \\ 2+0+0 & 4+0+6 & 6+0+3 \\ 1-3+0 & 2-5+4 & 3-4+2 \end{pmatrix} \\ &= \begin{pmatrix} -2 & -1 & 0 \\ 2 & 10 & 9 \\ -2 & 1 & 1 \end{pmatrix}. \end{aligned}$$

We see that we also in this case have $\mathbf{AB} \neq \mathbf{BA}$.

Example 2.5 *Given*

$$\mathbf{A} = \begin{pmatrix} -3 & -8 & 12 \\ 3 & 7 & -9 \\ 1 & 2 & -2 \end{pmatrix} \quad \textit{and} \quad \mathbf{B} = \mathbf{A} - \mathbf{I}.$$

Find $\mathbf{A}^2$, $\mathbf{AB}$ *and* $\mathbf{B}^2$.

First compute

$$\begin{aligned}
\mathbf{A}^2 &= \begin{pmatrix} -3 & -8 & 12 \\ 3 & 7 & -9 \\ 1 & 2 & -2 \end{pmatrix}\begin{pmatrix} -3 & -8 & 12 \\ 3 & 7 & -9 \\ 1 & 2 & -2 \end{pmatrix} \\
&= \begin{pmatrix} 9-24+12 & 24-56+24 & -36+72-24 \\ -9+21-9 & -24+49-18 & 36-63+18 \\ -3+6-2 & -8+14-4 & 12-18+4 \end{pmatrix} \\
&= \begin{pmatrix} -3 & -8 & 12 \\ 3 & 7 & -9 \\ 1 & 2 & -2 \end{pmatrix} = \mathbf{A}.
\end{aligned}$$

If $\mathbf{A}^2 = \mathbf{A}$, we say that $\mathbf{A}$ is *idempotent*. This condition implies by induction that $\mathbf{A}^n = \mathbf{A}$ for every $n \in \mathbb{N}$.

One may of course compute the matrix product $\mathbf{AB}$ by first computing $\mathbf{B}$ and then use the definition of the matrix product. Here it is far easier to apply the rules of calculations,

$$\mathbf{AB} = \mathbf{A}(\mathbf{A}-\mathbf{I}) = \mathbf{A}^2 - \mathbf{A} = \mathbf{A} - \mathbf{A} = \mathbf{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Analogously,

$$\begin{aligned}
\mathbf{B}^2 &= (\mathbf{A}-\mathbf{I})(\mathbf{A}-\mathbf{I}) = \mathbf{A}^2 - 2\mathbf{A} + \mathbf{I} = \mathbf{A} - 2\mathbf{A} + \mathbf{I} \\
&= \mathbf{I} - \mathbf{A} = -\mathbf{B} = \begin{pmatrix} 4 & 8 & -12 \\ -3 & -6 & 9 \\ -1 & -2 & 3 \end{pmatrix}.
\end{aligned}$$

Example 2.6 *Find all the matrix solutions of the matrix equation*

$$\mathbf{X}^2 = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix},$$

where a is any number different from 0.

If we put

$$\mathbf{X} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix},$$

then

$$\begin{aligned}
\mathbf{X}^2 &= \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} x_{11}^2 + x_{12}x_{21} & x_{11}x_{12} + x_{12}x_{22} \\ x_{21}x_{11} + x_{22}x_{21} & x_{21}x_{12} + x_{22}^2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

We get in particular by identifying the diagonal elements,

$$x_{11}^2 + x_{12}x_{21} = 1 = x_{21}x_{22}^2,$$

hence $x_{11}^2 = x_{22}^2$.

Furthermore,

$$x_{12}(x_{11}+x_{22}) = a \quad \text{and} \quad x_{21}(x_{11}+x_{22}) = 0.$$

It follows from $a \neq 0$ that $x_{12} \neq 0$ and $x_{22} \neq -x_{11}$. Since $x_{11}^2 = x_{22}^2$, we must have $x_{22} = x_{11} \neq 0$, and the equations are reduced to

$$2x_{11} \cdot x_{12} = a \quad \text{and} \quad 2x_{11} \cdot x_{21} = 0,$$

hence $x_{21} = 0$. It follows from the diagonal element that

$$x_{11}^2 + x_{12} \cdot x_{21} = x_{11}^2 = 1, \qquad \text{thus } x_{11} = \pm 1.$$

For $x_{11} = 1$ we get the solution

$$\mathbf{X} = \begin{pmatrix} 1 & a/2 \\ 0 & 1 \end{pmatrix},$$

and for $x_{11} = -1$ we get

$$\mathbf{X} = \begin{pmatrix} -1 & -a/2 \\ 0 & -1 \end{pmatrix}.$$

Example 2.7 *Let* $\mathbf{X} = \begin{pmatrix} x & y \\ z & u \end{pmatrix}$. *Solve the matrix equation*

1. $\mathbf{X}^2 = \mathbf{0}$, *where* $\mathbf{0}$ *denotes the* (2×2) *zero matrix.*
2. $\mathbf{X}^2 = \mathbf{I}$, *where* $\mathbf{I}$ *denotes the* (2×2) *unit matrix.*
3. $\mathbf{X}^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

We have in general

$$\mathbf{X}^2 = \begin{pmatrix} x & y \\ z & u \end{pmatrix}\begin{pmatrix} x & y \\ z & u \end{pmatrix} = \begin{pmatrix} x^2+yz & y(x+u) \\ z(x+u) & yz+u^2 \end{pmatrix}.$$

1. When $\mathbf{X}^2 = \mathbf{0}$, we get the two possibilities
   (a) $u = -x$,
   (b) $y = z = 0$.

   (a) If $u = -x$, and x, y are used as parameters, we get for $y \neq 0$ that $z = -x^2/y$, thus

   $$\left\{ \begin{pmatrix} x & y \\ -\dfrac{x^2}{y} & -x \end{pmatrix} \;\middle|\; x \in \mathbb{R},\, y \in \mathbb{R} \setminus \{0\} \right\}.$$

   (b) If $y = z = 0$, then also $x = u = 0$, and we must add the trivial solution.

   $$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

2. If $\mathbf{X}^2 = \mathbf{I}$, we get the same two possibilities:
   (a) $u = -x$,
   (b) $y = z = 0$.

   (a) If $u = -x$, we have the diagonal elements

   $$x^2 + yz = 1, \qquad \text{thus } yz = 1 - x^2.$$

   Choosing x and $y \neq 0$ as parameters we get

   $$\left\{ \begin{pmatrix} x & y \\ \dfrac{1-x^2}{y} & -x \end{pmatrix} \;\middle|\; x \in \mathbb{R},\, y \in \mathbb{R} \setminus \{0\} \right\}.$$

   (b) If $y = z = 0$, then $\mathbf{X}^2$ is reduced to $\begin{pmatrix} x^2 & 0 \\ 0 & u^2 \end{pmatrix}$, hence $x = \pm 1$ and $u = \pm 1$, and we add the four trivial possibilities

   $$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

3. In this case we get the equations

$$\begin{array}{ll} x^2+yz=0, & y(x+u)=1, \\ z(x+u)=0, & yz+u^2=0. \end{array}$$

We conclude from $y(x+u)=1$ that $y\neq 0$ and $x+u\neq 0$. Now, we conclude by the zero rule from $z(x+u)=0$ that $z=0$. Then the system is reduced to

$$x^2=0, \quad y(x+u)=1 \quad \text{and} \quad u^2=0,$$

hence $x=0$ and $u=0$. However, this system has no solution, because the derived necessary condition was that $x+u\neq 0$.

Example 2.8 *Find the rank of the matrix*

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 4 & 5 \\ 2 & 1 & 0 & 6 & 8 & 9 \\ -1 & 3 & 1 & 2 & 3 & 4 \end{pmatrix}.$$

The matrix is equivalent to

$$\begin{array}{ll}
\begin{pmatrix} 1 & 0 & 0 & 3 & 4 & 5 \\ 2 & 1 & 0 & 6 & 8 & 9 \\ -1 & 3 & 1 & 2 & 3 & 4 \end{pmatrix} & \begin{array}{l} \sim \\ R_2 := R_2 - 2R_1 \\ R_3 := R_3 + R_1 \end{array} \\
\begin{pmatrix} 1 & 0 & 0 & 3 & 4 & 5 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 3 & 1 & 5 & 7 & 9 \end{pmatrix} & \begin{array}{c} \\ \sim \\ R_3 := R_3 - 3R_2 \end{array} \\
\begin{pmatrix} 1 & 0 & 0 & 3 & 4 & 5 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 5 & 7 & 12 \end{pmatrix}. &
\end{array}$$

It is obvious that the rank is 3, which we also could expect.

Example 2.9 *Find the rank of the matrices*

$$1)\quad \begin{pmatrix} 1 & 2 & -1 & 3 & 4 \\ 1 & 3 & 1 & 2 & 6 \\ 0 & 1 & 3 & 0 & 3 \\ 1 & 1 & -3 & 4 & 2 \end{pmatrix}, \qquad 2)\quad \begin{pmatrix} 1 & 0 & 2 & -1 & 3 \\ 2 & 1 & 3 & 2 & 1 \\ 0 & 1 & -1 & 0 & 2 \\ 1 & 2 & -3 & 0 & 1 \end{pmatrix}.$$

1. The matric is equivalent to

$$\begin{pmatrix} 1 & 2 & -1 & 3 & 4 \\ 1 & 3 & 1 & 2 & 6 \\ 0 & 1 & 3 & 0 & 3 \\ 1 & 1 & -3 & 4 & 2 \end{pmatrix} \begin{matrix} \sim \\ R_n := R_3 \\ R_3 := R_2 - R_1 \\ R_4 := R_1 - R_4 \end{matrix}$$
$$\begin{pmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 1 & 3 & 0 & 3 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & 1 & 2 & -1 & 2 \end{pmatrix} \begin{matrix} \sim \\ R_3 := R_2 - R_3 \\ R_4 := R_3 - R_4 \end{matrix}$$
$$\begin{pmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 1 & 3 & 0 & 3 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

from which follows that the rank is 3.

2. The matrix is equivalent to

$$\begin{pmatrix} 1 & 0 & 2 & -1 & 3 \\ 2 & 1 & 3 & 2 & 1 \\ 0 & 1 & -1 & 0 & 2 \\ 1 & 2 & -3 & 0 & 1 \end{pmatrix} \begin{matrix} \sim \\ R_2 := R_2 - 2R_1 \\ R_4 := R_4 - R_1 \end{matrix}$$
$$\begin{pmatrix} 1 & 0 & 2 & -1 & 3 \\ 0 & 1 & -1 & 4 & -5 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 2 & -5 & 1 & -2 \end{pmatrix} \begin{matrix} \sim \\ R_2 := R_3 \\ R_3 := R_4 - 2R_3 \\ R_4 := R_2 - R_3 \end{matrix}$$
$$\begin{pmatrix} 1 & 0 & 2 & -1 & 3 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & -3 & 1 & -6 \\ 0 & 0 & 0 & 4 & -7 \end{pmatrix},$$

from which follows that the rank is 4.

Example 2.10 *Given the matrix* $\mathbf{A} = \begin{pmatrix} 1 & 1 & a \\ -a & -1 & 1 \end{pmatrix}$, *where* $a \in \mathbb{R}$.

1. *Find the rank of* $\mathbf{A}$.
2. *Solve the matrix equation* $\mathbf{A}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.
3. *Find for every* a *and* b, *where* $a, b \in \mathbb{R}$, *the set of solutions of the matrix equation*

$$\mathbf{A}^T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}.$$

1. The matrix is equivalent to

$$\begin{array}{l}
\begin{pmatrix} 1 & 1 & a \\ -a & -1 & 1 \end{pmatrix} \quad \underset{R_2 := aR_1 + R_2}{\sim} \\
\begin{pmatrix} 1 & 1 & a \\ 0 & a-1 & a^2+1 \end{pmatrix} \quad \underset{S_3 := S_3 - aS_2}{\sim} \\
\begin{pmatrix} 1 & 1 & 0 \\ 0 & a-1 & a+1 \end{pmatrix} \left(\underset{S_2 := S_3 - S_2}{\sim} \right) \\
\begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & a+1 \end{pmatrix},
\end{array}$$

so the rank is 2.

2. Writing the equation,

$$\mathbf{A}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & a \\ -a & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + ax_3 \\ -ax_1 - x_2 + x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

it follows that the total matrix is equivalent to

$$\left(\begin{array}{ccc|c} 1 & 1 & a & 0 \\ -a & -1 & 1 & 0 \end{array}\right) \underset{R_2 := R_1 + r_2}{\sim} \left(\begin{array}{ccc|c} 1 & 1 & a & 0 \\ 1-a & 0 & a+1 & 0 \end{array}\right).$$

Choosing $x_2 = s$ as parameter, this system is also written

$$x_1 + ax_3 = -s, \qquad (1-a)x_1 + (a+1)x_3 = 0,$$

from which $[R_2 := R_2 - (1-a)R_1]$

$$x_1 + ax_3 = -s, \qquad (a^2+1)x_3 = (1-a)s,$$

hence $x_3 = \dfrac{1-a}{1+a^2}\,s$ and $x_1 = -s - a \cdot \dfrac{1-a}{1+a^2}\,s = -\dfrac{1+a}{1+a^2}\,s$. Thus, the solution is

$$\mathbf{x} \in \left\{ \left(-\frac{1+a}{1+a^2}\,s, s, \frac{1-a}{1+a^2}\,s \right) \;\middle|\; s \in \mathbb{R} \right\}.$$

3. We first compute the left hand side,

$$\mathbf{A}^T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -a \\ 1 & -1 \\ a & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 - ax_2 \\ x_1 - x_2 \\ ax_1 + x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}.$$

It follows that $x_1 = ax_2$ and $x_2 = -ax_1$, hence $x_1 = -a^2x_1$ and $x_2 = -a^2x_2$. This is only possible for $x_1 = x_2 = 0$, forcing us to put $b = 0$, if the set of solutions is not empty.

If $b = 0$ and $a \in \mathbb{R}$, then the solution is $\mathbf{x} = (0, 0)$.

If $b \in \mathbb{R} \setminus \{0\}$, then the set of solutions is empty.

Example 2.11 *Solve the matrix equation* $\mathbf{AX} = \mathbf{0}$, *where*

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

and $\mathbf{0}$ *is the* (3×3) *zero matrix.*
Find the rank of any solution matrix $\mathbf{X}$.

Since the zero matrix consists of three zero columns, we consider the equivalent matrix of the matrix of coefficients

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{matrix} \sim \\ R_2 := R_2 - R_1 \\ R_3 := R_3 - R_2 \end{matrix}$$
$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} \begin{matrix} \sim \\ R_2 := R_2/3 \\ R_3 := R_2 - R_3 \end{matrix}$$
$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \underset{R := R_1 - R_2}{\sim}$$
$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \underset{R_1 := R_1 - R_2}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence a particular column in the $\mathbf{X}$ matrix must necessarily satisfy

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 0, \\ & & x_2 & + & 2x_3 & = & 0. \end{array}$$

Using $x_3 = s$ as parameter we get $x_2 = -2s$ and $x_1 = s$. Each column is of the same structure, though the value of the parameter may be changed. This gives the set of solutions

$$\left\{ \begin{pmatrix} s & t & u \\ -2s & -2t & -2u \\ s & t & u \end{pmatrix} \;\middle|\; s,\, t,\, u \in \mathbb{R} \right\}.$$

When $s = t = u = 0$, we get the zero matrix as a solution of rank 0.

If just one of the parameters s, t, u is $\neq 0$, then the rank is 1, because the columns are multiples of $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

Example 2.12 *Given the matrices*

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 4 & 3 \\ 2 & 4 & 3 & 6 \\ 3 & 6 & 2 & 9 \end{pmatrix}, \quad \mathbf{b}_1 = \begin{pmatrix} 7 \\ 4 \\ 1 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 7 \\ 4 \\ 2 \end{pmatrix} \ \textit{og}\ \mathbf{B} = \begin{pmatrix} 7 & 7 \\ 4 & 4 \\ 1 & 2 \end{pmatrix}.$$

1. *Find the rank of* **A**.
2. *Find the set of solutions of each of the following three matrix equations,*

$$\text{(a)}\quad \mathbf{AX} = \mathbf{b}_1, \qquad \text{(b)}\quad \mathbf{AX} = \mathbf{b}_2, \qquad \text{(c)}\quad \mathbf{AX} = \mathbf{B}.$$

1. Since **A** is equivalent to

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 4 & 3 \\ 2 & 4 & 3 & 6 \\ 3 & 6 & 2 & 9 \end{pmatrix} \begin{matrix} \sim \\ S_2 := S_2 - 2S_1 \\ S_4 := S_4 - 3S_1 \end{matrix} \begin{pmatrix} 1 & 0 & 4 & 0 \\ 2 & 0 & 3 & 0 \\ 3 & 0 & 2 & 0 \end{pmatrix},$$

it follows without further reduction that the rank is 2.

2. (a) We shall solve the equation

$$\mathbf{AX} = \begin{pmatrix} 1 & 2 & 4 & 3 \\ 2 & 4 & 3 & 6 \\ 3 & 6 & 2 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \mathbf{b}_1 = \begin{pmatrix} 7 \\ 4 \\ 1 \end{pmatrix}.$$

The total matrix is equivalent to

$$\left(\begin{array}{cccc|c} 1 & 2 & 4 & 3 & 7 \\ 2 & 4 & 3 & 6 & 4 \\ 3 & 6 & 2 & 9 & 1 \end{array}\right) \begin{matrix} \sim \\ R_2 := 2R_1 - R_2 \\ R_3 := 3R_1 - R_3 \end{matrix} \left(\begin{array}{cccc|c} 1 & 2 & 4 & 3 & 7 \\ 0 & 0 & 5 & 0 & 10 \\ 0 & 0 & 10 & 0 & 20 \end{array}\right),$$

corresponding to the system

$$\begin{array}{ccccccccc} x_1 & + & 2x_2 & + & 4x_3 & + & 3x_4 & = & 7, \\ & & & & x_3 & & & = & 2, \end{array}$$

thus $x_3 = 2$ and $x_1 + 2x_2 + 3x_4 = -2$. Choosing $x_2 = s$ and $x_4 = t$, the set of solutions is

$$\{(-1 - 2s - 3t, s, 2, t) \mid s,\, t \in \mathbb{R}\}.$$

(b) In this case the total matrix is equivalent to

$$\left(\begin{array}{cccc|c} 1 & 2 & 4 & 3 & 7 \\ 2 & 4 & 3 & 6 & 4 \\ 3 & 6 & 2 & 9 & 2 \end{array}\right) \begin{matrix} \sim \\ R_2 := 2R_1 - R_2 \\ R_3 := 3R_1 - R_3 \end{matrix} \left(\begin{array}{cccc|c} 1 & 2 & 4 & 3 & 7 \\ 0 & 0 & 5 & 0 & 10 \\ 0 & 0 & 10 & 0 & 19 \end{array}\right),$$

and the set of solutions is the empty set.

(c) Since $\mathbf{B} = (\mathbf{b}_1 \quad \mathbf{b}_2)$, the set of solutions is also empty for this system.

Example 2.13 *Find the complete solution of the equation*

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$$

Then solve the matrix equation

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \mathbf{Y} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

The total matrix is equivalent to

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 1 & 2 & -1 \\ 1 & 0 & 1 & -1 \end{array}\right) \underset{\substack{R_1 := R_3 \\ R_2 := R_1 - R_3 \\ R_3 := R_2 - 2R_3}}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 2 & 0 & 2 \\ 0 & 1 & 0 & 1 \end{array}\right),$$

hence the system of equations is equivalent to

$$x_1 + x_3 = -1 \qquad \text{and} \qquad x_2 = 1.$$

Choosing $x_1 = t$ as parameter, the solution becomes

$$\mathbf{x} = (t, 1, -t-1), \qquad t \in \mathbb{R}.$$

If the latter column in the total matrix is the zero column, then we get instead

$$x_1 + x_3 = 0 \qquad \text{and} \qquad x_2 = 0,$$

and

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has the solution

$$\mathbf{y} = (s, 0, -s), \qquad s \in \mathbb{R}.$$

Since the zero column occurs twice, the complete solution is

$$\mathbf{Y} = \begin{pmatrix} s & t & u \\ 0 & 1 & 0 \\ -s & -t-1 & -u \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix} + \begin{pmatrix} s & t & u \\ 0 & 0 & 0 \\ -s & -t & -u \end{pmatrix},$$

where s, t, $u \in \mathbb{R}$ are arbitrary constants.

Example 2.14 *Given the matrices*

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & a & 1 \\ 2 & a+2 & a-2 \\ 1 & a+1 & a-1 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 1 & -1 \\ -2 & 2 \\ b & b \\ b-1 & 2b+1 \end{pmatrix}$$

and

$$\mathbf{C} = \begin{pmatrix} 1 & 1 & -1 & 1 & -1 \\ 0 & a & 1 & -2 & 2 \\ 2 & a+2 & a-2 & b & b \\ 1 & a+1 & a-1 & b-1 & 2b+1 \end{pmatrix}.$$

1. *Find the rank of* $\mathbf{A}$ *for every* $a \in \mathbb{R}$, *and the rank of* $\mathbf{C}$ *for every pair* $(a, b) \in \mathbb{R}^2$.
2. *Find the sets of* $(a, b) \in \mathbb{R}^2$, *for which the matrix equation* $\mathbf{AX} = \mathbf{B}$ *does have solutions*
3. *Solve the matrix equation for* $(a, b) = (2, 0)$.

1. Since $\mathbf{A}$ is identical with the first three columns in $\mathbf{C}$, it suffices to find the equivalent matrices of $\mathbf{C}$,

$$\mathbf{C} = \left(\begin{array}{ccc|cc} 1 & 1 & -1 & 1 & -1 \\ 0 & a & 1 & -2 & 2 \\ 2 & a+2 & a-2 & b & b \\ 1 & a+1 & a-1 & b-1 & 2b+1 \end{array}\right) \begin{array}{l} \sim \\ R_3 := R_3 - 2R_1 \\ R_4 := R_4 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|cc} 1 & 1 & -1 & 1 & -1 \\ 0 & a & 1 & -2 & 2 \\ 0 & a & a & b-2 & b+2 \\ 0 & a & a & b-2 & 2b+2 \end{array}\right) \begin{array}{l} \sim \\ R_3 := R_3 - R_2 \\ R_4 := R_4 - R_3 \end{array}$$

$$\left(\begin{array}{ccc|cc} 1 & 1 & -1 & 1 & -1 \\ 0 & a & 1 & -2 & 2 \\ 0 & 0 & a-1 & b & b \\ 0 & 0 & 0 & 0 & b \end{array}\right).$$

If $a \neq 0$ and $a \neq 1$, then $\mathbf{A}$ is of rank 3.

If $a = 0$ or $a = 1$, then $\mathbf{A}$ is of rank 2.

If $a \neq 0$ and $a \neq 1$ and $b \neq 0$, then $\mathbf{C}$ is of rank 4.

If $a \neq 0$ and $a \neq 1$ and $b = 0$, the $\mathbf{C}$ is of rank 3.

If $a = 0$ or $a = 1$ and $b \neq 0$, then $\mathbf{C}$ is of rank 3.

If $a = 1$ and $b = 0$, then $\mathbf{C}$ is of rank 2.

If $a = 0$ and $b = 0$, then $\mathbf{C}$ is equivalent to

$$\begin{pmatrix} 1 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \sim \\ R_1 := R_1 - R_3 \\ R_2 := -R_3 \\ R_3 := R_2 + R_3 \end{matrix}$$

$$\begin{pmatrix} 1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \sim \\ R_1 := R_1 - R_3/2 \\ R_3 := -R_3/2 \end{matrix}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

from which follows that $\mathbf{C}$ is of rank 3.

2. We have solutions of the matrix equation $\mathbf{AX} = \mathbf{B}$, if $\mathbf{A}$ and $\mathbf{C}$ have the same rank, i.e. if

$$a \in \mathbb{R} \setminus \{0, 1\} \quad \text{and} \quad b = 0,$$

or

$$a = 1 \qquad \text{and} \qquad b = 0.$$

Summarizing for

$$a \in \mathbb{R} \setminus \{0\} \quad \text{and} \quad b = 0.$$

3. If $(a, b) = (2, 0)$, the solution is found by 2).
According to 1) the total matrix $\mathbf{C}$ is equivalent to

$$\left(\begin{array}{ccc|cc} 1 & 1 & -1 & 1 & -1 \\ 0 & 2 & 1 & -2 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{matrix} \sim \\ R_1 := R_1 + R_3 \\ R_2 := (R_2 - R_3)/2 \end{matrix}$$

$$\left(\begin{array}{ccc|cc} 1 & 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{matrix} \sim \\ R_1 := R_1 - R_2 \end{matrix}$$

$$\left(\begin{array}{ccc|cc} 1 & 0 & 0 & 2 & -2 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

corresponding to the solution

$$\mathbf{X} = \begin{pmatrix} 2 & -2 \\ -1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Since rank $\mathbf{A} = 3$, this is the only solution.

Example 2.15 *Given the matrix*

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & a^2-a \\ a & a & 1 & a^3-2a^2+a \\ -1 & 3a & 0 & 2a^2-2a \\ a & 2a & 1 & a^3-a \end{pmatrix}, \qquad \textit{where } a \in \mathbb{R}.$$

1. *Find the rank of* $\mathbf{A}$ *for every* $a \in \mathbb{R}$.

2. *Find for every* $a \in \mathbb{R}$ *the complete solution of the matrix equation*

   $$\mathbf{AX} = \mathbf{A}.$$

1. First notice that $\mathbf{A}$ is equivalent to

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & a^2-a \\ a & a & 1 & a^3-2a^2+a \\ -1 & 3a & 0 & 2a^2-2a \\ a & 2a & 1 & a^3-a \end{pmatrix} \begin{array}{l} \sim \\ R_2 := R_2 - aR_1 \\ R_3 := R_3 + R_1 \\ R_4 := R_4 - R_2 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 & a^2-a \\ 0 & a & 1 & -a^2+a \\ 0 & 3a & 0 & 3a^2-3a \\ 0 & a & 0 & 2a^2-2a \end{pmatrix} \begin{array}{l} \sim \\ R_3 := R_3/3 \\ R_4 := R_4 - R_2 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 & a^2-a \\ 0 & a & 1 & -a^2+a \\ 0 & a & 0 & a^2-a \\ 0 & 0 & -1 & 2a^2-2a \end{pmatrix} \begin{array}{c} \sim \\ R_2 := R_2 + R_4 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 & a^2-a \\ 0 & a & 0 & a^2-a \\ 0 & a & 0 & a^2-a \\ 0 & 0 & -1 & 2a^2-2a \end{pmatrix} \begin{array}{l} \sim \\ R_3 := -R_4 \\ R_4 := R_3 - R_2 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 & a^2-a \\ 0 & a & 0 & a^2-a \\ 0 & 0 & 1 & -2a^2+2a \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We conclude that the rank is 3 for $a \neq 0$, and 2 for $a = 0$.

2. Since $\mathbf{A}$ is a square matrix, the matrix equation $\mathbf{AX} = \mathbf{A}$ is equivalent to

$$\mathbf{A}(\mathbf{X} - \mathbf{I}) = \mathbf{0}, \quad \text{thus} \quad \mathbf{AY} = \mathbf{0} \text{ and } \mathbf{X} = \mathbf{Y} + \mathbf{I}.$$

Putting the zero vector on the right hand side we obtain the system

$$\begin{array}{rcrcl} y_1 & + & a(a-1)y_4 & = & 0, \\ ay_2 & + & a(a-1)y_4 & = & 0, \\ y_3 & - & 2a(a-1)y_4 & = & 0, \end{array}$$

so $y_4 = s$ is a free parameter, and

$$y_1 = -a(a-1)s, \quad ay_2 = -a(a-1)s, \quad y_3 = 2a(a-1)s.$$

If $a \neq 0$, then it follows that $y_2 = -(a-1)s$, and the solutions are

$$\mathbf{Y} = \begin{pmatrix} -a(a-1)s & -a(a-1)t & -a(a-1)u & -a(a-1)v \\ -(a-1)s & -(a-1)t & -(a-1)u & -(a-1)v \\ 2a(a-1)s & 2a(a-1)t & 2a(a-1)u & 2a(a-1)v \\ s & t & u & v \end{pmatrix}$$

for s, t, u and $v \in \mathbb{R}$, i.e.

$$\mathbf{X} = \begin{pmatrix} 1-a(a-1)s & -a(a-1)t & -a(a-1)u & -a(a-1)v \\ -(a-1)s & 1-(a-1)t & -(a-1)u & -(a-1)v \\ 2a(a-1)s & 2a(a-1)t & 1+2a(a-1)u & 2a(a-1)v \\ s & t & u & 1+v \end{pmatrix}.$$

If $a=0$, we can freely choose y_2, while $y_1=y_3=0$. Hence

$$\mathbf{Y}=\begin{pmatrix}0&0&0&0\\ s_1&t_1&u_1&v_1\\ 0&0&0&0\\ s_2&t_2&u_2&v_2\end{pmatrix} \text{ og } \mathbf{X}=\begin{pmatrix}1&0&0&0\\ s_1&1+t_1&u_1&v_1\\ 0&0&1&0\\ s_2&t_2&u_2&1+v_2\end{pmatrix},$$

where s_1, s_2, t_1, t_2, u_1, u_2, v_1 and $v_2\in\mathbb{R}$ are arbitrary constants.

Example 2.16 *Given the matrix*

$$\mathbf{A}=\begin{pmatrix}0&1&1\\ 1&-1&0\\ 1&0&1\end{pmatrix}.$$

1. *Find the complete solution of the system of equations*

$$\mathbf{A}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}.$$

2. *Find the complete solution of the matrix equation*

$$\mathbf{AX}=\mathbf{A}^2.$$

(One may benefit from the fact that the matrix $\mathbf{A}$ *is a particular solution of this matrix equation).*

1. The matrix of coefficients is equivalent to

$$\mathbf{A}=\begin{pmatrix}0&1&1\\ 1&-1&0\\ 1&0&1\end{pmatrix}\begin{array}{l}\sim\\ R_1:=R_3\\ R_2:=R_3-R_2\\ R_3:=R_1+R_2-R_3\end{array}\begin{pmatrix}1&0&1\\ 0&1&1\\ 0&0&0\end{pmatrix},$$

corresponding to the system of equations

$$x_1+x_3=0 \quad\text{and}\quad x_2+x_3=0,$$

thus

$$x_1=x_2=-x_3.$$

Choosing $x_3=-s$ as parameter, the complete solution is

$$\left\{\begin{pmatrix}s\\ s\\ -s\end{pmatrix}\;\middle|\; s\in\mathbb{R}\right\}.$$

2. It follows immediately that the equation is fulfilled for $\mathbf{X} = \mathbf{A}$, so this is a *particular solution*. We shall add all solution of the homogeneous equation to this particular solution. These are easily constructed from 1), so the complete solution becomes

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} s & t & u \\ s & t & u \\ -s & -t & -u \end{pmatrix}, \qquad s,\, t,\, u \in \mathbb{R}.$$

Example 2.17 *Given a* (3×3)*-matrix* $\mathbf{A}$ *satisfying*

1. $\mathbf{A}$ *is of rank 2,*
2. *the last column of* $\mathbf{A}$ *is equal to the sum of the first two columns.*

Find all matrices $\mathbf{X}$*, for which* $\mathbf{AX} = \mathbf{A}$.

Putting $\mathbf{Y} = \mathbf{X} - \mathbf{I}$, it follows that it suffices first to solve $\mathbf{AY} = \mathbf{0}$ and then put $\mathbf{X} = \mathbf{Y} + \mathbf{I}$. According to 2) the structure of $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{11} + a_{12} \\ a_{21} & a_{22} & a_{21} + a_{22} \\ a_{31} & a_{32} & a_{31} + a_{32} \end{pmatrix},$$

where (a_{11}, a_{21}, a_{31}) and (a_{12}, a_{22}, a_{32}) are linearly independent by 1). The system of equation for one single column is

$$\begin{aligned} a_{11}(y_1 + y_3) + a_{12}(y_2 + y_3) &= 0, \\ a_{21}(y_1 + y_3) + a_{22}(y_2 + y_3) &= 0, \\ a_{31}(y_1 + y_3) + a_{32}(y_2 + y_3) &= 0. \end{aligned}$$

The system is of rank 2, hence $y_1 + y_3 = 0$ and $y_2 + y_3 = 0$. Choosing e.g. $y_1 = s$ as parameter we obtain the solution $\mathbf{y} = s(1, 1, -1)$, $s \in \mathbb{R}$. Since the three columns are independent, the complete solution is given by

$$\mathbf{Y} = \begin{pmatrix} s & t & u \\ s & t & u \\ -s & -t & -u \end{pmatrix}, \qquad s, t, u \in \mathbb{R},$$

and hence

$$\mathbf{X} = \mathbf{I} + \mathbf{Y} = \begin{pmatrix} s+1 & t & u \\ s & t+1 & u \\ -s & -t & 1-u \end{pmatrix}, \qquad s, t, u \in \mathbb{R}.$$

Remark 2.1 The matrix of Example 2.16 is precisely of this type. $\Diamond$

Example 2.18 *Find the solution of each of the following two matrix equations*

$$\mathbf{AX} = \mathbf{B} \quad \textit{and} \quad \mathbf{XA} = \mathbf{B},$$

når

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \quad \textit{og} \quad \mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 2 & -1 \\ 1 & 1 & 2 \end{pmatrix}.$$

1. First consider the equation $\mathbf{AX} = \mathbf{B}$.
   The total matrix is equivalent to

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 2 & 3 \\ 1 & 1 & 2 & 4 & 2 & -1 \\ 1 & 1 & 1 & 1 & 1 & 2 \end{array}\right) \begin{array}{l} \sim \\ R_1 := R_1 - R_2 \\ R_2 := R_2 + R_3 - R_1 \\ R_3 := R_2 - R_3 \end{array}$$
$$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & -3 & 0 & 4 \\ 0 & +1 & +3 & +4 & +1 & -2 \\ 0 & 0 & 1 & 3 & 1 & -3 \end{array}\right) \begin{array}{l} \sim \\ R_1 := R_1 + 2R_3 \\ R_2 := R_2 - 3R_3 \end{array}$$
$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -5 & -2 & 7 \\ 0 & 0 & 1 & 3 & 1 & -3 \end{array}\right),$$

   hence the (unique) solution is given by

$$\mathbf{X} = \begin{pmatrix} 3 & 0 & 2 \\ -5 & -2 & 7 \\ 3 & 1 & -3 \end{pmatrix}.$$

2. We get by transposing $\mathbf{A}^T\mathbf{X}^T = \mathbf{B}^T$, the total matrix of which is equivalent to

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 2 & 2 & 1 \\ 0 & 2 & 1 & 3 & -1 & 2 \end{array}\right) \begin{array}{l} \sim \\ R_1 := R_1 - R_2 \\ R_2 := 2R_2 - R_1 \end{array}$$
$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 1 & 1 & 3 & 0 & 1 \\ 0 & 2 & 1 & 3 & -1 & 2 \end{array}\right) \begin{array}{l} \sim \\ R_2 := R_3 - R_2 \\ R_3 := 2R_2 - R_3 \end{array}$$
$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 3 & 1 & 0 \end{array}\right).$$

   Hence the transposed solution is

$$\mathbf{X}^T = \begin{pmatrix} -1 & 2 & 0 \\ 0 & -1 & 1 \\ 3 & 1 & 0 \end{pmatrix},$$

   so

$$\mathbf{X} = \begin{pmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Remark 2.2 Note that the two solutions are not identical. On the other hand, this could not be expected, because the matrix product is not commutative. $\Diamond$

Example 2.19 *For every real number a there are given the matrices*

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & a \\ a & 1-a^2 & 1 \end{pmatrix} \quad \textit{and} \quad \mathbf{B} = \begin{pmatrix} 1 & 1 \\ a & 1 \end{pmatrix}.$$

1. *Find for every $a \in \mathbb{R}$ the solution of the matrix equation*

$$\mathbf{AX} = \mathbf{B}.$$

2. *Does there exist a matrix $\mathbf{X}$, such that*

$$\mathbf{XA} = \mathbf{B}?$$

1. Since $\mathbf{A}$ is a (2×3) matrix, and $\mathbf{B}$ is a (2×2) matrix, we must have that $\mathbf{X}$ is a (3×2) matrix, if it exists. The total matrix is equivalent to

$$\left(\begin{array}{ccc|cc} 1 & 0 & a & 1 & 1 \\ a & 1-a^2 & 1 & a & 1 \end{array}\right) \underset{R_2 := R_2 - aR_1}{\sim} \left(\begin{array}{ccc|cc} 1 & 0 & a & 1 & 1 \\ 0 & 1-a^2 & 0 & 0 & 1-a \end{array}\right),$$

If $a = -1$, then the matrix of coefficients is of rank 1, while the total matrix is of rank 2, and there is no solution.

If $a = 1$, then both the matrix of coefficients and the total matrix are of rank 1.

If $a \neq \pm 1$, then both the matrix of coefficients and the total matrix are of rank 2.

If $a = 1$, then $x_1 + x_3 = 1$, and x_2 is a free parameter for both columns, hence the solution is given by

$$\mathbf{X} = \begin{pmatrix} s & t \\ u & v \\ 1-s & 1-t \end{pmatrix}, \qquad s, t, u, v \in \mathbb{R}.$$

If $a \neq \pm 1$, then $x_1 + x_3 = 1$ for both columns, and $x_{21} = 0$, $x_{22} = 1 - a$, thus the solution becomes

$$\mathbf{X} = \begin{pmatrix} s & t \\ 0 & 1-a \\ 1-s & 1-t \end{pmatrix}, \qquad s, t \in \mathbb{R}.$$

2. The answer is "no" of dimensional reasons- In fact, if $\mathbf{XA}$ is defined, then the result must have three columns, and the right hand side $\mathbf{B}$ has only got two columns.

Example 2.20 *Given the matrices*

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -1 & 3 \end{pmatrix} \quad \textit{and} \quad \mathbf{B} = \begin{pmatrix} 1 & 1 & 3 \\ 3 & -1 & 5 \end{pmatrix}.$$

Find the complete solution of each of the matrix equations

$$\mathbf{AX} = \mathbf{B} \quad \textit{and} \quad \mathbf{YA} = \mathbf{B}.$$

1. The total matrix $(\mathbf{A} \,|\, \mathbf{B})$ is equivalent to

$$(\mathbf{A} \,|\, \mathbf{B}) = \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 3 & -1 & 5 \end{array}\right) \begin{array}{l} \sim \\ R_1 := R_2 - R_1 \\ R_2 := R_1 \end{array}$$
$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 2 & -2 & 2 \\ 1 & -1 & 1 & 1 & 1 & 3 \end{array}\right) \begin{array}{c} \sim \\ R_2 := R_1 - R_2 \end{array}$$
$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 2 & -2 & 2 \\ 0 & 1 & 1 & 1 & -3 & -1 \end{array}\right),$$

corresponding to the equations

$$\begin{cases} x_{11} + 2x_{31} = 2, \\ x_{21} + x_{31} = 1, \end{cases} \qquad \begin{cases} x_{12} + 2x_{32} = -2, \\ x_{22} + x_{32} = -3, \end{cases} \qquad \begin{cases} x_{13} + 2x_{33} = 2, \\ x_{23} + x_{33} = -1. \end{cases}$$

Choosing $x_{31} = s_1$, $x_{32} = s_2$ and $x_{33} = s_3$ as parameters we get the solution

$$\mathbf{X} = \begin{pmatrix} 2-2s_1 & -2-2s_2 & 2-2s_3 \\ 1-s_1 & -3-s_2 & -1-s_3 \\ s_1 & s_2 & s_3 \end{pmatrix}, \qquad s_1,\, s_2,\, s_3 \in \mathbb{R}.$$

2. Since $\mathbf{YA} = \mathbf{B}$ is equivalent to $\mathbf{A}^T\mathbf{Y}^T = \mathbf{B}^T$, where $\mathbf{Y}$ is a (2×2) matrix, the total matrix $(\mathbf{A}^T \,|\, \mathbf{B}^T)$ is equivalent to

$$(\mathbf{A}^T \,|\, \mathbf{B}^T) = \left(\begin{array}{rr|rr} 1 & 2 & 1 & 3 \\ -1 & -1 & 1 & -1 \\ 1 & 3 & 3 & 5 \end{array}\right) \begin{array}{l} \sim \\ R_2 := R_1 + R_2 \\ R_3 := R_3 - R_1 \end{array}$$
$$\left(\begin{array}{rr|rr} 1 & 2 & 1 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 2 \end{array}\right) \begin{array}{l} \sim \\ R_1 := R_1 - 2R_2 \\ R_3 := R_2 - R_3 \end{array}$$
$$\left(\begin{array}{rr|rr} 1 & 0 & -3 & -1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

which is of rank 2. It follows that

$$\mathbf{Y}^T = \begin{pmatrix} -3 & -1 \\ 2 & 2 \end{pmatrix}, \quad \text{thus} \quad \mathbf{Y} = \begin{pmatrix} -3 & 2 \\ -1 & 2 \end{pmatrix}.$$

Remark 2.3 There are infinitely many solutions of 1, (and only one to 2). Furthermore, $\mathbf{X}$ is a (3×3) matrix, while $\mathbf{Y}$ is a (2×2) matrix. $\Diamond$

Example 2.21 *Given the matrices*

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & a+3 & 5 \\ -1 & a-3 & a+2 \end{pmatrix} \quad og \quad \mathbf{B} = \begin{pmatrix} 1 & -1 & 2 \\ 4 & 1 & 4 \\ 1 & 4 & a+4 \end{pmatrix}, \quad a \in \mathbb{R}.$$

Find the values of a, such that the matrix equation $\mathbf{AX} = \mathbf{B}$ has at least one solution.
Solve the equation for $a = 0$.

The total matrix is equivalent to

$$(\mathbf{A} \,|\, \mathbf{B}) = \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & -1 & 2 \\ 2 & a+3 & 5 & 4 & 1 & 4 \\ -1 & a-3 & a+2 & 1 & 4 & a+4 \end{array}\right) \begin{array}{l} \sim \\ R_2 := R_2 - R_3 \\ R_3 := R_3 + R_1 \end{array}$$
$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & -1 & 2 \\ 3 & 6 & 3-a & 3 & -3 & -a \\ 0 & a-1 & a+5 & 2 & 3 & a+6 \end{array}\right) \begin{array}{l} \sim \\ R_2 := R_3 \\ R_3 := R_2 - 3R_1 \end{array}$$
$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & -1 & 2 \\ 0 & a-1 & a+5 & 2 & 3 & a+6 \\ 0 & 0 & -a-6 & 0 & 0 & -a-6 \end{array}\right) \underset{R_2 := R_2 + R_3}{\sim}$$
$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & -1 & 2 \\ 0 & a-1 & -1 & 2 & 3 & 0 \\ 0 & 0 & -a-6 & 0 & 0 & a-6 \end{array}\right).$$

The matrix of coefficients and the total matrix are both of rank 3, when $a \neq 1, -6$, so in this case there is precisely one solution.
If $a = 1$, then the matrix of coefficients is of rank 2, while the total matrix is of rank 3, hence there is no solution.

If $a = -6$, then both the matrix of coefficients and the total matrix are of rank 2, and we conclude that there are infinitely many solutions.
Summing up we have at least one solution, when $a \neq 1$.
If $a = 0$, then the total matrix is equivalent to

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & -1 & 2 \\ 0 & -1 & -1 & 2 & 3 & 0 \\ 0 & 0 & -6 & 0 & 0 & -6 \end{array}\right) \begin{array}{l} \sim \\ R_1 := R_1 + 2R_2 \\ R_2 := -R_2 \\ R_3 := -R_3/6 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 5 & 5 & 2 \\ 0 & 1 & 1 & -2 & -3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \sim \\ R_1 := R_1 - R_3 \\ R_2 := R_2 - R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & 5 & 1 \\ 0 & 1 & 0 & -2 & -3 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

from which it immediately follows that the unique solution is

$$\mathbf{X} = \begin{pmatrix} 5 & 5 & 1 \\ -2 & -3 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Example 2.22 *Given the matrices*

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 4 & 3 \end{pmatrix} \quad \textit{and} \quad \mathbf{B} = \begin{pmatrix} 3 & 7 \\ 7 & 10 \\ a & 13 \end{pmatrix}, \quad \textit{where } a \in \mathbb{R}.$$

Find for every $a \in \mathbb{R}$ the complete solution of the matrix equation

$$\mathbf{AX} = \mathbf{B}.$$

The total matrix is equivalent to

$$(\mathbf{A} \,|\, \mathbf{B}) = \left(\begin{array}{ccc|cc} 1 & 2 & 1 & 5 & 7 \\ 1 & 3 & 2 & 7 & 10 \\ 1 & 4 & 3 & a & 13 \end{array}\right) \begin{array}{l} \sim \\ R_2 := R_2 - R_1 \\ R_3 := R_3 - R_2 \end{array}$$

$$\left(\begin{array}{ccc|cc} 1 & 2 & 1 & 5 & 7 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & a-7 & 3 \end{array}\right) \begin{array}{l} \sim \\ R_3 := R_3 - 2R_2 \\ R_1 := R_1 - R_2 \end{array}$$

$$\left(\begin{array}{ccc|cc} 1 & 0 & -1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 0 & a-9 & 0 \end{array}\right).$$

The matrix of coefficients is of rank 2. The total matrix is of rank 2 when $a = 9$, and of rank 3 when $a \neq 9$. Hence we have no solution for $a \neq 9$.
If $a = 9$, the solutions are

$$\mathbf{X} = \begin{pmatrix} 1+s & 1+t \\ 2-s & 3-t \\ s & t \end{pmatrix}, \qquad s, t \in \mathbb{R}.$$

Example 2.23 *Given the matrices*

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 0 \\ 2 & -3 & 0 \\ 3 & -4 & a^2-1 \end{pmatrix} \quad \textit{and} \quad \mathbf{B} = \begin{pmatrix} 3 & -1 \\ 2 & -1 \\ 1 & a-2 \end{pmatrix}, \qquad \textit{where } a \in \mathbb{R}.$$

Find for every $a \in \mathbb{R}$ *the complete solution of the matrix equation*

$$\mathbf{AX} = \mathbf{X}.$$

The total matrix is equivalent to

$$(\mathbf{A}\,|\,\mathbf{B}) = \left(\begin{array}{ccc|cc} 1 & -2 & 0 & 3 & -1 \\ 2 & -3 & 0 & 2 & -1 \\ 3 & -4 & a^2-1 & 1 & a-2 \end{array}\right) \begin{array}{l} \sim \\ R_2 := R_2 - R_1 \\ R_3 := R_3 - R_2 \end{array}$$

$$\left(\begin{array}{ccc|cc} 1 & -2 & 0 & 3 & -1 \\ 1 & -1 & 0 & -1 & 0 \\ 1 & -1 & a^2-1 & -1 & a-1 \end{array}\right) \begin{array}{l} \sim \\ R_2 := R_2 - R_1 \\ R_3 := R_3 - R_2 \end{array}$$

$$\left(\begin{array}{ccc|cc} 1 & -2 & 0 & 3 & -1 \\ 0 & 1 & 0 & -4 & 1 \\ 0 & 0 & a^2-1 & 0 & a-1 \end{array}\right) \begin{array}{c} \sim \\ R_1 := R_1 + 2R_2 \end{array}$$

$$\left(\begin{array}{ccc|cc} 1 & 0 & 0 & -5 & 1 \\ 0 & 1 & 0 & -4 & 1 \\ 0 & 0 & a^2-1 & 0 & a-1 \end{array}\right)$$

If $a \neq \pm 1$, then both the matrix of coefficients and the total matrix are of rank 3. The unique solution is

$$\mathbf{X} = \begin{pmatrix} -5 & 1 \\ -4 & 1 \\ 0 & \dfrac{1}{a+1} \end{pmatrix}.$$

If $a = 1$, then both the matrix of coefficients and the total matrix are of rank 2. We have infinitely many solutions

$$\mathbf{X} = \begin{pmatrix} -5 & 1 \\ -4 & 1 \\ s & t \end{pmatrix}, \qquad s,\, t \in \mathbb{R}.$$

If $a = -1$, then the matrix of coefficients is of rank 2 and the total matrix is of rank 3, and the set of solutions iks empty.

Example 2.24 *Given the matrices*

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -2 \\ -1 & -1 & 1 \\ -1 & a & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} -1 & -3 \\ a+1 & 3 \\ a+1 & 4 \end{pmatrix}, \quad a \in \mathbb{R}.$$

1. *Find all values of a, for which the matrix equation $\mathbf{AX} = \mathbf{B}$ has precisely one solution.*
2. *Solve the matrix equation, when $a = 0$.*

1. The condition of precisely one solution is that the matrix of coefficients is of rank 3. The total matrix is equivalent to

$$(\mathbf{A} \,|\, \mathbf{B}) = \left(\begin{array}{ccc|cc} 1 & 2 & -2 & -1 & -3 \\ -1 & -1 & 1 & a+1 & 3 \\ -1 & a & 1 & a+1 & 4 \end{array}\right) \begin{array}{l} \sim \\ R_2 := R_1 + R_2 \\ R_3 := R_3 - R_2 \end{array}$$
$$\left(\begin{array}{ccc|cc} 1 & 2 & -2 & -1 & -3 \\ 0 & 1 & -1 & a & 0 \\ 0 & a+1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \sim \\ R_1 := R_1 - 2R_2 \\ R_3 := R_3 - (a+1)R_2 \end{array}$$
$$\left(\begin{array}{ccc|cc} 1 & 0 & 0 & -1-2a & -3 \\ 0 & 1 & -1 & a & 0 \\ 0 & 0 & a+1 & -a(a+1) & 1 \end{array}\right).$$

It follows that the matrix equation has precisely one solution, when $a \neq -1$.

2. If $a = 0$, then the total matrix is equivalent to

$$\left(\begin{array}{ccc|cc} 1 & 0 & 0 & -1 & -3 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right) \underset{R_2 := R_2 + R_3}{\sim} \left(\begin{array}{ccc|cc} 1 & 0 & 0 & -1 & -3 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right).$$

It follows that the unique solution is

$$\mathbf{X} = \begin{pmatrix} -1 & -3 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Example 2.25 *Given the matrices*

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -2 \\ 1 & 4 & a \\ 0 & -2 & 1 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 2 & 6 \\ b & 13+a \\ -2 & -4 \end{pmatrix},$$

where a and b are arbitrary real numbers.

1. *Find the pairs (a,b), for which the matrix equation $\mathbf{AX} = \mathbf{B}$ has precisely one solution.*
2. *Find the complete solution of the matrix equation for $(a,b) = (-3,4)$.*

1. The total matrix is equivalent to

$$(\mathbf{A}\,|\,\mathbf{B}) = \left(\begin{array}{rrr|rr} 1 & 2 & -2 & 2 & 6 \\ 1 & 4 & a & b & 13+a \\ 0 & -2 & 1 & -2 & -4 \end{array}\right) \begin{array}{c} \sim \\ R_2 := R_2 - R_1 \end{array}$$
$$\left(\begin{array}{rrr|rr} 1 & 2 & -2 & 2 & 6 \\ 0 & 2 & a+2 & b-2 & 7+a \\ 0 & -2 & 1 & -2 & -4 \end{array}\right) \begin{array}{c} \sim \\ R_3 := R_2 + R_3 \end{array}$$
$$\left(\begin{array}{rrr|rr} 1 & 2 & -2 & 2 & 6 \\ 0 & 2 & a+2 & b-2 & 7+a \\ 0 & 0 & a+3 & b-4 & 3+a \end{array}\right)$$

It follows that the matrix of coefficients is of rank 3 for $a \neq -3$, hence there exists precisely one solution, when $a \neq -3$ and $b \in \mathbb{R}$.

2. If $(a,b) = (-3,4)$, then the total matrix is equivalent to

$$\left(\begin{array}{rrr|rr} 1 & 2 & -2 & 2 & 6 \\ 0 & 2 & -1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \sim \\ R_2 := R_2/2 \\ R_1 := R_1 - R_2 \end{array}$$
$$\left(\begin{array}{rrr|rr} 1 & 0 & -1 & 0 & 2 \\ 0 & 1 & -1/2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Choosing $x_{3i} = s_i$, $i = 1, 2$, as parameters it follows that the complete solution is

$$\begin{pmatrix} s_1 & 2+s_2 \\ 1+s_1/2 & 2+s_2/2 \\ s_1 & s_2 \end{pmatrix}, \qquad s_1,\, s_2 \in \mathbb{R}.$$

Example 2.26 *Given the matrix*

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 5 \\ 2 & 5 & 6 \end{pmatrix}.$$

1. *Solve the matrix equation* $\mathbf{XA} = \mathbf{0}$*, where* $\mathbf{0}$ *is the zero matrix in* $\mathbb{R}^{3\times 3}$*.*
2. *Then solve the matrix equation* $\mathbf{XA} = \mathbf{A}^T\mathbf{A}$*.*

1. By transposing the equation is transferred into the standard form $\mathbf{A}^T\mathbf{X}^T = \mathbf{0}$ with the unknown matrix $\mathbf{X}^T$ to the right of $\mathbf{A}$. The matrix of coefficients $\mathbf{A}^T$ is equivalent to

$$\mathbf{A}^T = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 5 \\ 1 & 5 & 6 \end{pmatrix} \begin{matrix} \sim \\ R_2 := R_2 - 2R_1 \\ R_3 := R_3 - R_1 \end{matrix}$$
$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 4 & 4 \end{pmatrix} \begin{matrix} \sim \\ R_3 := R_3 - 4R_2 \\ R_1 := R_1 - R_2 \end{matrix}$$
$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence, every column $\mathbf{y}$, for which $\mathbf{A}^T\mathbf{y}$ is the zero column must therefore be of the form $s(1,1,-1)^T$, where s is a parameter, thus

$$\mathbf{X}^T = \begin{pmatrix} s & t & u \\ s & t & u \\ -s & -t & -u \end{pmatrix}, \qquad s,\, t,\, u \in \mathbb{R},$$

and we get by transposing

$$\mathbf{X} = \begin{pmatrix} s & s & -s \\ t & t & -t \\ u & u & -y \end{pmatrix}, \qquad s,\, t,\, u \in \mathbb{R}.$$

2. Now, $\mathbf{X} = \mathbf{A}^T$ is a particular solution, hence the complete solution is

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 5 \\ 1 & 5 & 6 \end{pmatrix} + \begin{pmatrix} s & s & -s \\ t & t & -t \\ u & u & -u \end{pmatrix} = \begin{pmatrix} s+1 & s+1 & 2-s \\ t+2 & t+3 & 5-t \\ u+1 & u+5 & 6-u \end{pmatrix},$$

where $s,\, t,\, u \in \mathbb{R}$ are arbitrary constants.

Example 2.27 *Given the matrix*

$$\mathbf{A} = \begin{pmatrix} 0,1 & 0,2 \\ 0,3 & 0,2 \end{pmatrix}.$$

Find $(\mathbf{I} - \mathbf{A})^{-1}$, *and compare the result with the matrix* $\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3$.

We first note that $(\mathbf{I} - \mathbf{A}, \mathbf{I})$ is equivalent to

$$\left(\begin{array}{cc|cc} \frac{9}{10} & \frac{8}{10} & 1 & 0 \\ \frac{7}{10} & \frac{8}{10} & 0 & 1 \end{array}\right) \begin{array}{l} \sim \\ R_1 := \frac{10}{9} R_1 \\ R_2 := 9R_2 - 7R_1 \end{array}$$

$$\left(\begin{array}{cc|cc} 1 & \frac{8}{9} & \frac{10}{9} & 0 \\ 0 & \frac{8}{5} & -7 & 9 \end{array}\right) \begin{array}{l} \sim \\ R_1 := R_1 - \frac{5}{9} R_2 \\ R_2 := \frac{5}{8} R_2 \end{array}$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 5 & -5 \\ 0 & 1 & -\frac{35}{8} & \frac{45}{8} \end{array}\right),$$

hence

$$(\mathbf{I} - \mathbf{A})^{-1} = \begin{pmatrix} 5 & -5 \\ -\frac{35}{8} & \frac{45}{8} \end{pmatrix}.$$

It follows from

$$\mathbf{A} = \frac{1}{10}\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$

that

$$\mathbf{A}^2 = \frac{1}{100}\begin{pmatrix} 7 & 6 \\ 9 & 10 \end{pmatrix} \quad \text{og} \quad \mathbf{A}^3 = \frac{1}{1000}\begin{pmatrix} 25 & 26 \\ 39 & 38 \end{pmatrix},$$

thus

$$\begin{aligned} \mathbf{I}+\mathbf{A}+\mathbf{A}^2+\mathbf{A}^3 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{10}\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} + \frac{1}{100}\begin{pmatrix} 7 & 6 \\ 9 & 10 \end{pmatrix} + \frac{1}{1000}\begin{pmatrix} 25 & 26 \\ 39 & 38 \end{pmatrix} \\ &= \frac{1}{1000}\begin{pmatrix} 1000+100+70+25 & 0+200+60+26 \\ 0+300+90+39 & 1000+200+10+38 \end{pmatrix} \\ &= \frac{1}{1000}\begin{pmatrix} 1195 & 286 \\ 429 & 1248 \end{pmatrix}. \end{aligned}$$

Remark 2.4 Note that $(\mathbf{I}-\mathbf{A})^{-1}$ contains negative elements, while $\mathbf{A}^n$ never contains a negative number. The same must be true for any finite sum

$$\mathbf{I}+\mathbf{A}+\cdots+\mathbf{A}^n,$$

so even if

$$(1-t)^{-1} = 1+t+t^3+\cdots, \qquad \text{for } |t|<1,$$

a corresponding result does *not* hold for this particular matrix $\mathbf{A}$, even if $(\mathbf{I}-\mathbf{A})^{-1}$ exists. $\Diamond$

Example 2.28 *Given the matrix*

$$\mathbf{A} = \begin{pmatrix} 0 & a & 0 & 0 \\ a & 0 & a & 0 \\ 0 & a & 0 & a \\ 0 & 0 & a & 0 \end{pmatrix},$$

where a is a real number $\neq 0$. Find $\mathbf{A}^{-1}$.

If we put

$$\mathbf{B} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

then $\mathbf{A} = a\mathbf{B}$. If the inverse matrices exist, then $\mathbf{A}^{-1} = \dfrac{1}{a}\mathbf{B}^{-1}$, hence it suffices to find $\mathbf{B}^{-1}$. We reduce the total matrix

$$
\begin{array}{rcl}
(\mathbf{B}\,|\,\mathbf{I}) & = & \left(\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \sim \\ R_1 := R_2 - R_4 \\ R_2 := R_1 \\ R_3 := R_4 \\ R_4 := R_3 - R_1 \end{array} \\
& \sim & \left(\begin{array}{cccc|rrrr} 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \end{array}\right).
\end{array}
$$

Then

$$
\mathbf{B}^{-1} = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix}
$$

and

$$
\mathbf{A}^{-1} = \frac{1}{a}\,\mathbf{B}^{-1} = \begin{pmatrix} 0 & \frac{1}{a} & 0 & -\frac{1}{a} \\ \frac{1}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{a} \\ -\frac{1}{a} & 0 & \frac{1}{a} & 0 \end{pmatrix}, \qquad a \neq 0.
$$

Example 2.29 *Given the matrix*

$$
\mathbf{A} = \begin{pmatrix} 1 & 3 \\ -1 & 0 \\ 2 & -2 \end{pmatrix}.
$$

1. *Compute* $\mathbf{A}^T\mathbf{A}$.
2. *Check, if* $\mathbf{A}^T\mathbf{A}$ *has an inverse.*

1. By a direct computation,

$$
\mathbf{A}^T\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ -1 & 0 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 1+1+4 & 3+0-4 \\ 3+0-4 & 9+0+4 \end{pmatrix} = \begin{pmatrix} 6 & -1 \\ -1 & 13 \end{pmatrix}.
$$

2. Clearly, $\mathbf{A}^T\mathbf{A}$ is regular, so $\mathbf{A}^T\mathbf{A}$ has an inverse. It follows from the reduction

$$
\begin{array}{l}
\left(\begin{array}{rr|rr} 6 & -1 & 1 & 0 \\ -1 & 13 & 0 & 1 \end{array}\right) \begin{array}{c} \sim \\ R_1 := R_1 + 5R_2 \end{array} \\
\left(\begin{array}{rr|rr} 1 & 64 & 1 & 5 \\ -1 & 13 & 0 & 1 \end{array}\right) \begin{array}{c} \sim \\ R_2 := R_1 + R_2 \end{array} \\
\left(\begin{array}{rr|rr} 1 & 64 & 1 & 5 \\ 1 & 77 & 1 & 6 \end{array}\right) \begin{array}{c} \sim \\ R_1 := R_1 - R_2 \end{array} \\
\left(\begin{array}{rr|rr} 1 & -13 & 0 & -1 \\ 0 & 77 & 1 & 6 \end{array}\right) \begin{array}{l} \sim \\ R_1 := R_1 + \frac{13}{77}\,R_2 \\ R_2 := \frac{1}{77}\,R_2 \end{array} \\
\left(\begin{array}{rr|rr} 1 & 0 & \frac{13}{77} & \frac{1}{77} \\ 0 & 1 & \frac{1}{77} & \frac{6}{77} \end{array}\right),
\end{array}
$$

that the inverse is given by

$$\frac{1}{77}\begin{pmatrix} 13 & 1 \\ 1 & 6 \end{pmatrix}.$$

Example 2.30 *Given the two matrices*

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 1 & -2 & 1 \end{pmatrix} \quad og \quad \mathbf{B} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Find $\mathbf{A}^{-1}$, $\mathbf{B}^{-1}$ *and* $(\mathbf{AB})^{-1}$.

It follows from the reductions

$$(\mathbf{A}\,|\,\mathbf{I}) = \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \sim \\ R_3 := R_3 - R_1 \\ R_1 := R_1 + 2R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \begin{array}{l} \sim \\ R_1 := R_1 - 2R_3 \\ R_2 := R_2 - R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 2 & -2 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

that

$$\mathbf{A}^{-1} = \begin{pmatrix} 3 & 2 & -2 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}.$$

It is trivial that

$$\mathbf{B}^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}.$$

Finally,

$$\begin{aligned} (\mathbf{AB})^{-1} &= \mathbf{B}^{-1}\mathbf{A}^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 3 & 2 & -2 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -3 & -2 & 2 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{3} & 0 & -\frac{1}{3} \end{pmatrix} = \frac{1}{6}\begin{pmatrix} -18 & -12 & 12 \\ 3 & 3 & -3 \\ -2 & 0 & -2 \end{pmatrix}. \end{aligned}$$

ALTERNATIVELY,

$$\mathbf{AB} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 1 & -2 & 2 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} -1 & -4 & 0 \\ 0 & 2 & 3 \\ -1 & -4 & 3 \end{pmatrix}.$$

Then we perform the reductions

$$
(\mathbf{AB}\,|\,\mathbf{I}) = \left(\begin{array}{rrr|rrr} -1 & -4 & 0 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ -1 & -4 & 3 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \sim \\ R_1 := R_1 + 2R_2 \\ R_3 := R_3 - R_1 \end{array}
$$

$$
\left(\begin{array}{rrr|rrr} -1 & 0 & 6 & 1 & 2 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & 3 & -1 & 0 & 1 \end{array}\right) \begin{array}{l} \sim \\ R_1 := R_1 - 2R_3 \\ R_2 := R_2 - R_3 \end{array}
$$

$$
\left(\begin{array}{rrr|rrr} -1 & 0 & 0 & 3 & 2 & -2 \\ 0 & 2 & 0 & 1 & 1 & -1 \\ 0 & 0 & 3 & -1 & 0 & 1 \end{array}\right) \begin{array}{l} \sim \\ R_1 := -R_1 \\ R_2 := R_2/2 \\ R_3 := R_3/3 \end{array}
$$

$$
\left(\begin{array}{rrr|rrr} 1 & 0 & 0 & -3 & -2 & +2 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{3} & 0 & -\frac{1}{3} \end{array}\right),
$$

hence

$$
(\mathbf{AB})^{-1} = \begin{pmatrix} -3 & -2 & 2 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{3} & 0 & -\frac{1}{3} \end{pmatrix} = \frac{1}{6}\begin{pmatrix} -18 & -12 & 12 \\ 3 & 3 & -3 \\ -2 & 0 & -2 \end{pmatrix}.
$$

Example 2.31 *Given the columns*

$$\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \qquad \textit{and} \qquad \mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

Let the matrices $\mathbf{A}$ *and* $\mathbf{B}$ *be given by*

$$\mathbf{A} = \mathbf{u}\mathbf{v}^T \qquad \textit{og} \qquad \mathbf{B} = \mathbf{I} + \mathbf{A}.$$

1. *Compute the matrices* $\mathbf{A}$ *and* $\mathbf{A}^2$*.*
2. *Find the rank of* $\mathbf{A}$*, and prove that* $\mathbf{B}$ *is regular.*
3. *Find the inverse matrix* $\mathbf{B}^{-1}$ *of the matrix* $\mathbf{B}$*.*

1. It follows by direct computations that

$$\mathbf{A} = \mathbf{u}\mathbf{v}^T = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} (2 \;\; 1 \;\; 2) = \begin{pmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ -4 & -2 & -4 \end{pmatrix}$$

and

$$\begin{aligned} \mathbf{A}^2 &= \mathbf{u}\mathbf{v}^T\mathbf{u}\mathbf{v}^T = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} (2 \;\; 1 \;\; 2) \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} (2 \;\; 1 \;\; 2) \\ &= \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \cdot 0 \cdot (2 \;\; 1 \;\; 2) = \mathbf{0}. \end{aligned}$$

2. and the rank of $\mathbf{A}$ is clearly 1.
3. Then $\mathbf{B} = \mathbf{I} + \mathbf{A}$, hence

$$\mathbf{B}(\mathbf{I} - \mathbf{A}) = (\mathbf{I} + \mathbf{A})(\mathbf{I} - \mathbf{A}) = \mathbf{I} - \mathbf{A}^2 = \mathbf{I} = (\mathbf{I} - \mathbf{A})\mathbf{B},$$

from which we conclude that

$$\mathbf{B}^{-1} = \mathbf{I} - \mathbf{A} = \begin{pmatrix} -1 & -1 & -2 \\ -4 & -1 & -4 \\ 4 & 2 & 5 \end{pmatrix}.$$

Example 2.32 *Given the two matrices*

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 3 \end{pmatrix} \quad og \quad \mathbf{B} = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

1. *Find* $(\mathbf{BA})^{-1}$.
2. *Show that* $\mathbf{AB}$ *is not regular.*

1. We first compute

$$\mathbf{BA} = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 10 \\ 5 & 8 \end{pmatrix},$$

which clearly is regular. Then by the reductions

$$(\mathbf{BA} \,|\, \mathbf{I}) = \left(\begin{array}{cc|cc} 4 & 10 & 1 & 0 \\ 5 & 8 & 0 & 1 \end{array} \right) \begin{array}{c} \sim \\ R_1 := R_2 - R_1 \\ R_2 := R_1/4 \end{array}$$

$$\left(\begin{array}{cc|cc} 1 & -2 & -1 & 1 \\ 1 & \frac{5}{2} & \frac{1}{4} & 0 \end{array} \right) \begin{array}{c} \sim \\ R_2 := R_2 - R_1 \end{array}$$

$$\left(\begin{array}{cc|cc} 1 & -2 & -1 & 1 \\ 0 & \frac{9}{2} & \frac{5}{18} & -\frac{2}{9} \end{array} \right) \begin{array}{c} \sim \\ R_1 := R_1 + 2R_2 \end{array} \left(\begin{array}{cc|cc} 1 & 0 & -\frac{8}{18} & \frac{5}{9} \\ 0 & 1 & \frac{5}{18} & -\frac{2}{9} \end{array} \right),$$

thus the inverse is given by

$$(\mathbf{BA})^{-1} = \begin{pmatrix} -\frac{4}{9} & \frac{5}{9} \\ \frac{5}{18} & -\frac{2}{9} \end{pmatrix} = \frac{1}{18}\begin{pmatrix} -8 & 10 \\ 5 & -4 \end{pmatrix}.$$

2. Since $\mathbf{A}$ is of rank 2, it follows that $\mathbf{AB}$ is at most of rank 2. Now, $\mathbf{AB}$ is a (3×3) matrix, hence $\mathbf{AB}$ cannot be regular.

Example 2.33 *Given the matrices*

$$\mathbf{A} = \begin{pmatrix} \alpha & \alpha & 0 & 1 \\ 2\alpha & 2\alpha & \alpha & 1 \\ 2\alpha & 3\alpha & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \quad \textit{and} \quad \mathbf{b} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix},$$

where α, a, b, c and d are real numbers.

1. *Solve the system of equations*

$$\mathbf{A}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \mathbf{b}$$

for every α, a, b, c, d.

2. *Find the values of α, for which $\mathbf{A}$ is regular, and find $\mathbf{A}^{-1}$ (apply e.g. the result of 1.).*

1. Here we get the following reductions of the total matrix

$$\left(\begin{array}{cccc|c} \alpha & \alpha & 0 & 1 & a \\ 2\alpha & 2\alpha & \alpha & 1 & b \\ 2\alpha & 3\alpha & 0 & 1 & c \\ 1 & 1 & 0 & 0 & d \end{array}\right) \begin{array}{l} \sim \\ R_1 := R_1 - \alpha R_4 \\ R_2 := R_2 - 2R_1 \\ R_3 := R_3 - R_2 \end{array}$$

$$\left(\begin{array}{cccc|c} 0 & 0 & 0 & 1 & a - \alpha d \\ 0 & 0 & \alpha & -1 & b - 2a \\ 0 & \alpha & -\alpha & 0 & c - d \\ 1 & 1 & 0 & 0 & d \end{array}\right) \begin{array}{l} \sim \\ R_1 := R_4 \\ R_2 := R_3 + R_2 + R_1 \\ R_3 := R_2 + R_1 \\ R_4 := R_1 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & d \\ 0 & \alpha & 0 & 0 & -a + b + c - (1+\alpha)d \\ 0 & 0 & \alpha & 0 & -a + b - \alpha d \\ 0 & 0 & 0 & 1 & a - \alpha d \end{array}\right).$$

When $\alpha = 0$, the matrix of coefficients is of rank 2. while the total matrix only is of the same rank 2, if furthermore

$$-a + b + c - d = 0 \quad \text{and} \quad -a + b = 0,$$

thus if $a = b$ and $c = d$. In this case the equations are reduced to

$$x_1 + x_2 = c \qquad \text{and} \qquad x_4 = a.$$

If $\alpha = 0$ and $\mathbf{b} = (a \;\; a \;\; c \;\; c)^T$, then the complete solution is

$$\{(s, c - s, t, a) \mid s,\, t \in \mathbb{R}\}.$$

If $\alpha \neq 0$, then the matrix is regular, and the unique solution is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} d \\ -\dfrac{a}{\alpha} + \dfrac{b}{\alpha} + \dfrac{c}{\alpha} - \left(1 + \dfrac{1}{\alpha}\right) d \\ -\dfrac{a}{\alpha} + \dfrac{b}{\alpha} - d \\ a - \alpha d \end{pmatrix}.$$

2. It was mentioned above that $\mathbf{A}$ is regular for $\alpha \neq 0$. We find the inverse by applying the solution of 1), where we put $a = 1$, $b = c = d = 0$ in the first one, $b = 1$ and $a = c = d = 0$ etc. in the second one. In this way,

$$\mathbf{A}^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -\frac{1}{\alpha} & \frac{1}{\alpha} & \frac{1}{\alpha} & -\frac{1+\alpha}{\alpha} \\ -\frac{1}{\alpha} & \frac{1}{\alpha} & 0 & -1 \\ 1 & 0 & 0 & -\alpha \end{pmatrix} = \frac{1}{\alpha} \begin{pmatrix} 0 & 0 & 0 & \alpha \\ -1 & 1 & 1 & -1-\alpha \\ -1 & 1 & 0 & -\alpha \\ \alpha & 0 & 0 & \alpha^2 \end{pmatrix}.$$

Example 2.34 *Given the matrices*

$$\mathbf{A} = \begin{pmatrix} 0 & 2 & 2 \\ -2 & 0 & 2 \end{pmatrix} \quad \textit{and} \quad \mathbf{B} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix}.$$

Prove that $\mathbf{B}$ *is regular, and find* $\mathbf{B}^{-1}$.
Then solve the matrix equation

$$\mathbf{XB} = \mathbf{A}.$$

First reduce the total matrix

$$(\mathbf{B} \mid \mathbf{I}) = \left(\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ -1 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \sim \\ R_2 := R_1 + R_2 \\ R_3 := R_1 + R_3 \end{array}$$
$$\left(\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 1 \end{array}\right) \begin{array}{l} \sim \\ R_2 := R_2/2 \\ R_3 = R_3/2 \end{array}$$
$$\left(\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} \end{array}\right) \begin{array}{l} \sim \\ R_1 := R_1 - R_2 \\ R_2 := R_2 - R_3 \end{array}$$
$$\left(\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} \end{array}\right).$$

We conclude from this that

$$\mathbf{B}^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}.$$

It follows from $\mathbf{XB} = \mathbf{A}$ that

$$\begin{aligned} \mathbf{X} &= \mathbf{XBB}^{-1} = \mathbf{AB}^{-1} = \begin{pmatrix} 0 & 2 & 2 \\ -2 & 0 & 2 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}. \end{aligned}$$

Example 2.35 *Let*

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 & -1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Prove that the matrix equation $\mathbf{AYA} = \mathbf{I}$ *has precisely one solution, and find this solution.*

We see that $\mathbf{A}$ is an upper triangular matrix of diagonal elements $\neq 0$. Then clearly $\mathbf{A}$ is regular, and $\mathbf{A}^{-1}$ exists- Then

$$\mathbf{Y} = \mathbf{A}^{-1}\mathbf{AYAA}^{-1} = \mathbf{A}^{-1}\mathbf{IA}^{-1} = (\mathbf{A}^2)^{-1}.$$

We compute

$$\mathbf{A}^2 = \begin{pmatrix} 1 & -1 & 2 & -1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 & -1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -3 & 0 & 1 \\ 0 & 4 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Finally, by reducing the total matrix,

$$\begin{aligned} (\mathbf{A}^2 \,|\, \mathbf{I}) &= \left(\begin{array}{cccc|cccc} 1 & -3 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \sim \\ R_1 := R_1 + R_2 \\ R_4 := R_4 + 2R_3 \\ R_2 := R_2/4 \end{array} \\ &\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 & 2 & 1 \end{array}\right) \begin{array}{l} \sim \\ R_1 := R_1 - R_2 \\ R_4 := -R_4/3 \end{array} \\ &\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{7}{4} & 1 & \frac{3}{4} & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -\frac{2}{3} & -\frac{1}{3} \end{array}\right) \begin{array}{l} \sim \\ R_1 := R_1 - \frac{7}{4}R_4 \\ R_2 := R_2 - \frac{1}{4}R_4 \\ R_3 := R_3 + 2R_4 \end{array} \\ &\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & \frac{3}{4} & \frac{7}{6} & \frac{7}{12} \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{6} & \frac{1}{12} \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & 1 & 0 & 0 & -\frac{2}{3} & -\frac{1}{3} \end{array}\right), \end{aligned}$$

thus

$$\mathbf{Y}=(\mathbf{A}^2)^{-1}=\begin{pmatrix} 1 & \frac{3}{4} & \frac{7}{6} & \frac{7}{12} \\ 0 & \frac{1}{4} & \frac{1}{6} & \frac{1}{12} \\ 0 & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & -\frac{2}{3} & -\frac{1}{3} \end{pmatrix}=\frac{1}{12}\begin{pmatrix} 12 & 9 & 14 & 7 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & -4 & -8 \\ 0 & 0 & -8 & -4 \end{pmatrix}.$$

Example 2.36 *Given the matrices*

$$\mathbf{A}=\begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 3 & 5 \end{pmatrix} \quad \text{and} \quad \mathbf{b}=\begin{pmatrix} -8 \\ 11 \\ c \end{pmatrix},$$

where $c \in \mathbb{R}$.

1. *Solve for any given c the system of equations $\mathbf{Ax}=\mathbf{b}$.*
2. *Find the matrix $\mathbf{A}^T\mathbf{A}$, and prove that it is regular. Then solve the system of equations $\mathbf{A}^T\mathbf{Ax}=\mathbf{A}^T\mathbf{b}$ for any given constant c.*

1. The matrix of coefficients is of rank 2, hence we only get solutions, when the total matrix also is of rank 2. We reduce,

$$(\mathbf{A}\,|\,\mathbf{b})=\left(\begin{array}{cc|c} 1 & 3 & -8 \\ 2 & 5 & 11 \\ 3 & 5 & c \end{array}\right) \begin{array}{l} \sim \\ R_2:=2R_1-R_2 \\ R_3:=R_3-3R_1 \end{array}$$
$$\left(\begin{array}{cc|c} 1 & 3 & -8 \\ 0 & 1 & -27 \\ 0 & -4 & c+24 \end{array}\right) \begin{array}{l} \sim \\ R_3:=4R_2+R_3 \\ R_1:=R_1-3R_2 \end{array}$$
$$\left(\begin{array}{cc|c} 1 & 0 & 73 \\ 0 & 1 & -27 \\ 0 & 0 & c-84 \end{array}\right).$$

If $c \neq 84$, then the set of solutions is empty. If $c=84$, then $x_1=73$ and $x_2=-27$.

2. We first compute

$$\mathbf{A}^T\mathbf{A}=\begin{pmatrix} 1 & 2 & 3 \\ 3 & 5 & 5 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 3 & 5 \end{pmatrix}=\begin{pmatrix} 14 & 28 \\ 28 & 59 \end{pmatrix},$$

which obviously is regular.

Then

$$\mathbf{A}^T\mathbf{b}=\begin{pmatrix} 1 & 2 & 3 \\ 3 & 5 & 5 \end{pmatrix}\begin{pmatrix} -8 \\ 11 \\ c \end{pmatrix}=\begin{pmatrix} 14+3c \\ 31+5c \end{pmatrix}.$$

We now reduce the total matrix:

$$
\begin{aligned}
(\mathbf{A}^T\mathbf{A} \,|\, \mathbf{A}^T\mathbf{b}) = \left(\begin{array}{cc|c} 14 & 28 & 14+3c \\ 28 & 59 & 31+5c \end{array}\right) & \underset{R_2 := R_2 - 2R_1}{\sim} \\
\left(\begin{array}{cc|c} 14 & 28 & 14+3c \\ 0 & 3 & 3-c \end{array}\right) & \underset{R_2 := R_2/3}{\sim} \\
\left(\begin{array}{cc|c} 14 & 28 & 14+3c \\ 0 & 1 & 1-\frac{1}{3}\,c \end{array}\right) & \underset{R_1 := R_1 - 28r_2}{\sim} \\
\left(\begin{array}{cc|c} 14 & 0 & -14+\frac{37}{3}\,c \\ 0 & 1 & 1-\frac{1}{3}\,c \end{array}\right) & \underset{R_1 := \frac{1}{14}\,R_1}{\sim} \\
\left(\begin{array}{cc|c} 1 & 0 & -1+\frac{37}{42}\,c \\ 0 & 1 & 1-\frac{1}{3}\,c \end{array}\right), &
\end{aligned}
$$

and the unique solution is

$$
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{37}{42}\,c - 1 \\ 1 - \frac{1}{3}\,c \end{pmatrix}.
$$

Remark 2.5 If $c = 84$ the two solutions must agree. It follows from

$$
\begin{pmatrix} \frac{37}{42}\cdot 84 - 1 \\ 1 - \frac{1}{3}\cdot 84 \end{pmatrix} = \begin{pmatrix} 74 - 1 \\ 1 - 28 \end{pmatrix} = \begin{pmatrix} 73 \\ -27 \end{pmatrix},
$$

that this is true. $\Diamond$

Example 2.37 *Given the matrices*

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix} \quad \textit{and} \quad \mathbf{B} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

1. *Prove that* $\mathbf{A}$ *is regular, and solve the matrix equation* $\mathbf{AX} = \mathbf{B}$.
2. *Solve the matrix equation* $\mathbf{AY} = \mathbf{BA}^{-1}\mathbf{B}$.

1. We start by reducing the total matrix

$$(\mathbf{A}\,|\,\mathbf{B}) = \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & -1 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \sim \\ R_2 := R_2 - R_1 \\ R_3 := R_3 - 2R_1 \end{array}$$
$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & -2 & 0 & -2 \end{array}\right) \begin{array}{l} \sim \\ R_1 := R_1 + R_3 \\ R_2 := R_2 - R_3 \end{array}$$
$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -2 & 0 & -1 \end{array}\right).$$

It follows that 1) $\mathbf{A}$ is regular, and that 2) the solution of $\mathbf{AX} = \mathbf{B}$ is given by

$$\mathbf{X} = \begin{pmatrix} -1 & 0 & -1 \\ 1 & 1 & 1 \\ -2 & 0 & -1 \end{pmatrix} = \mathbf{A}^{-1}\mathbf{B}.$$

2. When we multiply the equation $\mathbf{AY} = \mathbf{BA}^{-1}\mathbf{B}$ from the left by $\mathbf{A}^{-1}$, then

$$\begin{aligned} \mathbf{Y} &= \mathbf{A}^{-1}\mathbf{AY} = (\mathbf{A}^{-1}\mathbf{B})(\mathbf{A}^{-1}\mathbf{B}) = \mathbf{X}^2 \\ &= \begin{pmatrix} -1 & 0 & -1 \\ 1 & 1 & 1 \\ -2 & 0 & -2 \end{pmatrix}\begin{pmatrix} -1 & 0 & -1 \\ 1 & 1 & 1 \\ -2 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 3 \\ -2 & 1 & -2 \\ 6 & 0 & 6 \end{pmatrix}. \end{aligned}$$

Example 2.38 *For every real number a we define the matrix*

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 5 \\ 4 & 3 & 7 \\ -2 & 3 & a \end{pmatrix}.$$

1. *Find a lower triangular unit matrix* $\mathbf{L}$ *and an upper triangular matrix* $\mathbf{U}$*, such that* $\mathbf{A} = \mathbf{LU}$.
2. *Find all* $a \in \mathbb{R}$*, such that* $\mathbf{A}$ *is regular.*

1. By using Gauß elimination row by row we reduce $\mathbf{A}$ to an upper triangular matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 5 \\ 4 & 3 & 7 \\ -3 & 3 & a \end{pmatrix} \begin{matrix} \sim \\ R_2 := R_2 - c_{221}R_1 = R_2 - 2R_1 \\ R_3 := R_3 - c_{31}R_1 = R_3 + R_1 \end{matrix}$$
$$\begin{pmatrix} 2 & 1 & 5 \\ 0 & 1 & -3 \\ 0 & 4 & 5+a \end{pmatrix} \begin{matrix} \sim \\ R_3 := R_3 - c_{32}R_2 = R_3 - 4R_2 \end{matrix}$$
$$\begin{pmatrix} 2 & 1 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 17+a \end{pmatrix} = \mathbf{U},$$

from which we get

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ c_{21} & 1 & 0 \\ c_{31} & c_{32} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 4 & 1 \end{pmatrix}.$$

2. It follows from the structure of $\mathbf{U}$ that $\mathbf{A}$ is regular for $a \neq 17$.

Example 2.39 *Given the matrix*

$$\mathbf{A} = \begin{pmatrix} 2 & 4 & 6 \\ 6 & 12 & 19 \\ 4 & 11 & 18 \end{pmatrix}.$$

1. *Find* $\mathbf{X} = \mathbf{A}^{-1}$ *by solving the matrix equation* $\mathbf{AX} = \mathbf{I}$*, and find at the same time the factorization* $\mathbf{PA} = \mathbf{LU}$.

2. *Rewrite this factorization to*

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{LDV}$$

(i.e. rewrite $\mathbf{P}^{-1}$, $\mathbf{D}$ *and* $\mathbf{V}$*).*

3. *Find the inverse matrix by using*

$$\mathbf{A}^{-1} = \mathbf{V}^{-1}\mathbf{D}^{-1}\mathbf{L}^{-1}\mathbf{P},$$

where we first compute the inverses of the factors and then multiply.

1. Reduce the total matrix,

$$(\mathbf{A}\,|\,\mathbf{I}) = \left(\begin{array}{ccc|ccc} 2 & 4 & 6 & 1 & 0 & 0 \\ 6 & 12 & 19 & 0 & 1 & 0 \\ 4 & 11 & 18 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \sim \\ R_2 := R_2 - c_{21}R_1 = R_2 - 3R_1 \\ R_3 := R_3 - c_{31}R_1 = R_3 - 2R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 2 & 4 & 6 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 1 & 0 \\ 0 & 3 & 6 & -2 & 0 & 1 \end{array}\right) \begin{array}{l} \sim \\ R_2 := R_3 \\ R_3 := R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 2 & 4 & 6 & 1 & 0 & 0 \\ 0 & 3 & 6 & -2 & 0 & 1 \\ 0 & 0 & 1 & -3 & 1 & 0 \end{array}\right).$$

It follows that $\mathbf{PA} = \mathbf{LU}$, where

$$\mathbf{U} = \begin{pmatrix} 2 & 4 & 6 \\ 0 & 3 & 6 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix},$$

where we have interchanged c_{21} and c_{31}. Furthermore,

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

where $\mathbf{PA} = \mathbf{LU}$, which is easily checked.

Returning to the computation of $\mathbf{A}^{-1}$ it follows from the above that

$$
\begin{aligned}
(\mathbf{A}\,|\,\mathbf{I}) \sim \left(\begin{array}{ccc|ccc} 2 & 4 & 6 & 1 & 0 & 0 \\ 0 & 3 & 6 & -2 & 0 & 1 \\ 0 & 0 & 1 & -3 & 1 & 0 \end{array}\right) \begin{array}{c} \sim \\ R_1 := R_1/2 \\ R_2 := R_2/3 \end{array} \\
\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 2 & -\frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & -3 & 1 & 0 \end{array}\right) \begin{array}{c} \sim \\ R_1 := R_1 - 3R_3 \\ R_2 - 2R_3 \end{array} \\
\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{19}{2} & -3 & 0 \\ 0 & 1 & 0 & \frac{16}{3} & -2 & \frac{1}{3} \\ 0 & 0 & 1 & -3 & 1 & 0 \end{array}\right) \begin{array}{c} \sim \\ R_1 := R_1 - 2R_2 \end{array} \\
\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{7}{6} & 1 & -\frac{2}{3} \\ 0 & 1 & 0 & \frac{16}{3} & -2 & \frac{1}{3} \\ 0 & 0 & 1 & -3 & 1 & 0 \end{array}\right),
\end{aligned}
$$

thus

$$
\mathbf{A}^{-1} = \begin{pmatrix} -\frac{7}{6} & 1 & -\frac{2}{3} \\ \frac{16}{3} & -2 & \frac{1}{3} \\ -3 & 1 & 0 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} -7 & 6 & -4 \\ 32 & -12 & 2 \\ -18 & 6 & 0 \end{pmatrix}.
$$

These computations are fairly complicated, so it is recommended to check the result. This was actually done in the draft, though not included in the text.

2. So far we have obtained

$$
\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \mathbf{P}^{-1},
$$

where the latter equality sign follows from the fact that due to $\mathbf{P}$ the second and the third row are interchanged, hence $\mathbf{P}^2 = \mathbf{I}$, and we get $\mathbf{P}^{-1} = \mathbf{P}$.

Finally,

$$
\mathbf{U} = \begin{pmatrix} 2 & 4 & 6 \\ 0 & 3 & 6 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{DV} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix},
$$

from which

$$
\mathbf{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{V} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.
$$

The factorization is then

$$
\begin{aligned}
\mathbf{A} &= \mathbf{P}^{-1}\mathbf{LDV} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned}
$$

3. It follows from

$$(\mathbf{V}\,|\,\mathbf{I}) = \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{c} \sim \\ R_1 := R_1 - 3R_3 \\ R_2 := R_2 - 2R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & -3 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{c} \sim \\ R_1 := R_1 - 2R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right),$$

that

$$\mathbf{V}^{-1} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Furthermore,

$$(\mathbf{L}\,|\,\mathbf{I}) = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{c} \sim \\ R_2 := R_2 - 2R_1 \\ R_3 := R_3 - 3R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right),$$

so

$$\mathbf{L}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}.$$

Finally, it follows from

$$\mathbf{D}^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{P}^{-1} = \mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

that

$$\begin{aligned} \mathbf{A}^{-1} &= \mathbf{V}^{-1}\mathbf{D}^{-1}\mathbf{L}^{-1}\mathbf{P} \\ &= \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} & -\frac{2}{3} & 1 \\ 0 & \frac{1}{3} & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 0 & 1 \\ -3 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{7}{6} & 1 & -\frac{2}{3} \\ \frac{16}{3} & -2 & \frac{1}{3} \\ -3 & 1 & 0 \end{pmatrix} \\ &= \frac{1}{6} \begin{pmatrix} -7 & 6 & -4 \\ 32 & -12 & 2 \\ -18 & 6 & 0 \end{pmatrix} \end{aligned}$$

and we have obtained the same result as previously.

Remark 2.6 If one does not use a pocket calculator or a computer, one cannot claim that the method above is persuasive. Its importance, however, can be seen in the applications in Numerical Analysis. $\Diamond$

Example 2.40 *Given a matrix* $\mathbf{A}$ *by*

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 & -2 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Find the complete solution of the homogeneous linear system of equations $\mathbf{Ax} = \mathbf{0}$.

We notice that $\mathbf{A}$ is of the form $\mathbf{A} = \mathbf{LU}$. Since $\mathbf{L}$ is regular, it follows that $\mathbf{Ax} = \mathbf{L}(\mathbf{Ux}) = \mathbf{0}$ is equivalent to $\mathbf{Ux} = \mathbf{0}$.
Choosing $x_4 = s$, $s \in \mathbb{R}$, as parameter, we reduce $\mathbf{Ux} = \mathbf{0}$ to

$$\begin{array}{rcrcrcrcl} 2x_1 & + & 3x_2 & + & x_3 & = & 2x_4 & = & 2s, \\ & - & 3x_2 & + & x_3 & = & 0, & & \\ & & & & x_3 & = & -3s, & & \end{array}$$

thus

$$x_2 = -\frac{1}{3}\,x_3 = s$$

and

$$x_1 = s - \frac{3}{2}\,x_2 - \frac{1}{2}\,x_3 = s - \frac{3}{2}\,s + \frac{3}{2}\,s = s.$$

The complete solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = s \begin{pmatrix} 1 \\ 1 \\ -3 \\ 1 \end{pmatrix}, \qquad s \in \mathbb{R}.$$

Example 2.41 *Given the matrix*

$$\mathbf{A} = \begin{pmatrix} 3 & 7 & 2 \\ 6 & 14 & a \\ -9 & 4 & 19 \end{pmatrix}, \qquad \textit{where } a \in \mathbb{R}.$$

Find a matrix of permutations **P***, and a lower triangular unit matrix* **L** *and a diagonal matrix* **D** *and and an upper unit triangular matrix* **V***, such that*

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{LDV}.$$

We reduce **A** to,

$$\mathbf{A} = \begin{pmatrix} 3 & 7 & 2 \\ 6 & 14 & a \\ -9 & 4 & 19 \end{pmatrix} \begin{matrix} \sim \\ R_2 := R_2 - 2R_1 \\ R_3 := R_3 + 3R_1 \end{matrix} \begin{pmatrix} 3 & 7 & 2 \\ 0 & 0 & a-4 \\ 0 & 25 & 25 \end{pmatrix}.$$

It follows from

$$\mathbf{P} = \mathbf{P}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

that

$$\mathbf{PA} = \begin{pmatrix} 3 & 7 & 2 \\ -9 & 4 & 19 \\ 6 & 14 & a \end{pmatrix} \begin{matrix} \sim \\ R_2 := R_2 - c_{21}R_1 = R_2 + 3R_1 \\ R_3 := R_3 - c_{31}R_1 = R_3 - 2R_1 \end{matrix} \begin{pmatrix} 3 & 7 & 2 \\ 0 & 25 & 25 \\ 0 & 0 & a-4 \end{pmatrix} = \mathbf{U},$$

hence

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ c_{21} & 1 & 0 \\ c_{31} & c_{32} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix},$$

because $c_{32} = 0$. Furthermore,

$$\mathbf{U} = \mathbf{D}\mathbf{V} = \begin{pmatrix} 3 & 7 & 2 \\ 0 & 25 & 25 \\ 0 & 0 & a-4 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & a-4 \end{pmatrix} \begin{pmatrix} 1 & \frac{7}{3} & \frac{2}{3} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

hence

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{L}\mathbf{D}\mathbf{V} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & a-4 \end{pmatrix} \begin{pmatrix} 1 & \frac{7}{3} & \frac{2}{3} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

We conclude that $\mathbf{A}$ is regular, if and only if $a \neq 4$.

Example 2.42 *Given*

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{L}\mathbf{U} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 7 & 7 \\ 0 & 0 & -8 \end{pmatrix}$$

and

$$\mathbf{b} = \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix}.$$

Find the set of solutions of the linear system of equations $\mathbf{A}\mathbf{x} = \mathbf{b}$.

Since $\mathbf{A}$ is composed of regular matrices, $\mathbf{A}$ is also regular, and the solution is unique. Using that $\mathbf{P}^{-1} = \mathbf{P}$, it follows that $\mathbf{A}\mathbf{x} = \mathbf{b}$ is equivalent to

$$\mathbf{L}\mathbf{U}\mathbf{x} = \mathbf{P}\mathbf{b} = \begin{pmatrix} -2 \\ -1 \\ 4 \end{pmatrix}.$$

Putting $\mathbf{y} = \mathbf{U}\mathbf{x}$, we solve $\mathbf{L}\mathbf{y} = \mathbf{P}\mathbf{b}$. The total matrix is reduced to

$$(\mathbf{L} \,|\, \mathbf{P}\mathbf{b}) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 4 & 1 & 0 & -1 \\ 2 & 0 & 1 & 4 \end{array}\right) \begin{array}{c} \sim \\ R_1 := R_2 - 4R_1 \\ R_3 := R_3 - 2R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 8 \end{array}\right),$$

thus

$$\mathbf{y} = \mathbf{U}\mathbf{x} = \begin{pmatrix} -2 \\ 7 \\ 8 \end{pmatrix} = \mathbf{c}.$$

The total matrix is here

$$(\mathbf{U} \,|\, \mathbf{c}) = \left(\begin{array}{ccc|c} 1 & -2 & 3 & -2 \\ 0 & 7 & 7 & 7 \\ 0 & 0 & -8 & 8 \end{array}\right) \begin{array}{c} \sim \\ R_2 := R_2/7 \\ R_3 := -R_3/8 \end{array}$$
$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right) \begin{array}{c} \sim \\ R_1 := R_1 + 2R_2 \\ R_2 := R_2 - R_3 \end{array}$$
$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right) \begin{array}{c} \sim \\ R_1 := R_1 - 5R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right).$$

Hence, the unique solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}.$$

Example 2.43 *Given the matrix*

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & a \\ 1 & 4 & a-2 \\ 3 & 5 & 3a \end{pmatrix}, \qquad \textit{where } a \in \mathbb{R}.$$

1. *Find the LU factorization for* $\mathbf{A}$.
2. *Prove that* $\mathbf{A}$ *is regular for all* $a \in \mathbb{R}$, *and find by the method of complements the element* $(\mathbf{A}^{-1})_{12}$.

1. We first reduce by applying simple Gauß elimination,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & a \\ 1 & 4 & a-2 \\ 3 & 5 & 3a \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & a \\ 0 & 2 & -2 \\ 0 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & a \\ 0 & 2 & -2 \\ 0 & 0 & -1 \end{pmatrix} = \mathbf{U}$$

where

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & -\frac{1}{2} & 1 \end{pmatrix}, \quad \text{dvs. } \mathbf{LU} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & a \\ 0 & 2 & -2 \\ 0 & 0 & -1 \end{pmatrix} = \mathbf{A}.$$

2. It follows from

$$\det \mathbf{A} = \det \mathbf{L} \cdot \det \mathbf{U} = \det \mathbf{U} = 1 \cdot 2 \cdot (-1) = -2 \neq 0,$$

that $\mathbf{A}$ is regular for all $a \in \mathbb{R}$.

By using the method of complements (delete the first column and the second row) we get

$$(\mathbf{A}^{-1})_{12} = \frac{(-1)^{1+2}}{\det \mathbf{A}} \begin{vmatrix} 2 & a \\ 5 & 3a \end{vmatrix} = \frac{-1}{-2}(6a - 5a) = \frac{a}{2}.$$

Example 2.44 *Given the matrices*

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{pmatrix} \quad \textit{and } \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 3 & -1 \\ 1 & a \end{pmatrix}, \qquad \textit{hvor } a \in \mathbb{R}.$$

1. *Find for every a all solutions of the matrix equation*

   $$\mathbf{AX} = \mathbf{B}.$$

2. *Given a linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ (with respect to the usual basis) by the matrix equation*

   $$\mathbf{y} = \mathbf{Ax}.$$

   Find the dimension of the range $f(\mathbb{R}^3)$ and an orthonormal basis (with respect to the usual scalar product) for this range.

   Find all $a \in \mathbb{R}$ for which the vector $(0, -1, a)$ belongs to the range.

1. We get by reduction,

   $$\begin{aligned}(\mathbf{A} \mid \mathbf{B}) &= \left(\begin{array}{rrr|rr} 1 & 0 & -1 & 1 & 0 \\ 1 & 1 & 1 & 3 & -1 \\ -1 & 1 & 3 & 1 & a \end{array}\right) \sim \left(\begin{array}{rrr|rr} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 2 & -1 \\ 0 & 1 & 2 & 2 & a \end{array}\right) \\ &\sim \left(\begin{array}{rrr|rr} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 2 & -1 \\ 0 & 0 & 0 & 0 & a+1 \end{array}\right).\end{aligned}$$

   It follows that if $a \neq -1$, then we have no solution, because the total matrix is of rank 3, while the matrix of coefficients is only of rank 2.

   If $a = -1$, then we have infinitely many solutions

   $$\begin{pmatrix} 1+s & t \\ 2-2s & -1-2t \\ s & t \end{pmatrix}, \qquad s, t \in \mathbb{R}.$$

2. It follows from 1) that $\mathbf{A}$ is of rank 2, so the dimension of $f(\mathbb{R}^3)$ is 2. A basis is e.g. given by $\mathbf{v}_1 = (1, 1, -1)$ and $\mathbf{v}_2 = (0, 1, 1)$, because these are linearly independent vectors of the range.

   Now, $\|\mathbf{v}_1\| = \sqrt{3}$ and $\|\mathbf{v}_2\| = \sqrt{2}$ and $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$, so an orthonormal basis of $f(\mathbb{R}^3)$ is given by

   $$\mathbf{q}_1 = \frac{1}{\sqrt{3}}(1, 1, -1), \qquad \mathbf{q}_2 = \frac{1}{\sqrt{2}}(0, 1, 1).$$

   It follows from 1) that $(0, -1, a)$ does only belong to the range when $a = -1$.

Example 2.45 *Given the matrices*

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & 5 & 3 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 3 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 2 & 4 & 0 \\ 4 & -3 & 1 \\ 0 & 1 & -5 \end{pmatrix}.$$

Check for everyone of the matrices $\mathbf{A}$, $\mathbf{B}$ *and* $\mathbf{C}$, *if it can be diagonalized.*

1. Since

   $$\det(\mathbf{A} - \lambda \mathbf{I}) = -(\lambda - 1)(\lambda - 2)(\lambda - 3)$$

   has three different simple real roots, it follows that $\mathbf{A}$ can be diagonalized.

2. We cannot use the same argument in this case, because the characteristic polynomial has the double root $\lambda = 2$. It follows by reduction that

   $$\mathbf{B} - 2\mathbf{I} = \begin{pmatrix} -1 & 3 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} -1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

   is of rank 2. The eigenspace is of dimension $3 - 2 = 1 \neq 2$, hence the geometric multiplicity and the algebraic multiplicity of $\lambda = 2$ do not agree, so $\mathbf{B}$ cannot be diagonalized.

3. Since $\mathbf{C}$ is symmetric, $\mathbf{C}$ can be diagonalized.

Example 2.46 *Given in* $\mathbb{R}^4$ *the vectors*

$$\mathbf{u}_1 = (1,1,1,1),\ \mathbf{u}_2 = (3,1,1,3),\ \mathbf{u}_3 = (2,0,-2,4),\ \mathbf{u}_4 = (1,1,-1,3),$$

and let $U = \operatorname{span}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,\mathbf{u}_4)$.

1. *Prove that* $(\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3)$ *is a basis of* U*, and find the coordinate matrix of* $\mathbf{u}_4$ *with respect to this basis.*
2. *Let* $\mathbb{R}^4$ *be equipped with the usual scalar product.*
   Find an orthonormal basis of U.

1. We get by reduction,

$$(\mathbf{u}_1\ \ \mathbf{u}_2\ \ \mathbf{u}_3 \mid \mathbf{u}_4) = \left(\begin{array}{rrr|r} 1 & 3 & 2 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & -2 & -1 \\ 1 & 3 & 4 & 3 \end{array}\right) = \left(\begin{array}{rrr|r} 1 & 3 & 2 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & -2 & -4 & -2 \\ 0 & 0 & 2 & 2 \end{array}\right)$$
$$\sim \left(\begin{array}{rrr|r} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The matrix of coefficients $(\mathbf{u}_1\ \ \mathbf{u}_2\ \ \mathbf{u}_3)$ is of rank 3, hence $(\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3)$ are linearly independent. Since the total matrix is also of rank 3, it follows that

$$\mathbf{u}_4 = 2\mathbf{u}_1 - \mathbf{u}_2 + \mathbf{u}_3$$

is a linear combination of $(\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3)$ with the coordinates $(2,-1,1)$.

2. Since $\|\mathbf{u}_1\| = \sqrt{1+1+1+1} = 2$, we choose $\mathbf{v}_1 = \frac{1}{2}\mathbf{u}_1 = \frac{1}{2}(1,1,1,1)$. Then by the Gram-Schmidt method,

$$\begin{aligned} \mathbf{u}_2 - \frac{1}{\|\mathbf{u}_1\|^2}\langle \mathbf{u}_1,\mathbf{u}_2\rangle \mathbf{u}_1 &= (3,1,1,3) - \frac{1}{4}(3+1+1+3)\cdot(1,1,1,1) \\ &= (3,1,1,3) - (2,2,2,2) = (1,-1,-1,1) \end{aligned}$$

which also is of length 2. We therefore choose $\mathbf{v}_2 = \frac{1}{2}(1,-1,-1,1)$.

By applying the Gram-Schmidt method once more,

$$\begin{aligned} \mathbf{u}_3 - \langle \mathbf{u}_3,\mathbf{v}_1\rangle \mathbf{v}_1 - \langle \mathbf{u}_3,\mathbf{v}_1\rangle \mathbf{v}_2 &= (1,1,-1,3) - \frac{1}{4}(1+1-1+3)(1,1,1,1) - \frac{1}{4}(1-1+1+3)(1,-1,-1,1) \\ &= (1,1,-1,3) - (1,1,1,1) - (1,-1,-1,1) = (1,1,-1,3) - (2,0,0,2) = (-1,1,-1,1). \end{aligned}$$

Its length is 2, so we choose $\mathbf{v}_3 = \frac{1}{2}(-1,1,-1,1)$.

An orthonormal basis of U is then given by

$$\mathbf{v}_1 = \frac{1}{2}(1,1,1,1), \quad \mathbf{v}_2 = \frac{1}{2}(1,-1,-1,1), \quad \mathbf{v}_3 = \frac{1}{2}(-1,1,-1,1).$$

3 Determinants

Example 3.1 *Compute for every n the determinant*

$$\begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ n+1 & n+2 & n+3 & \cdots & 2n \\ 2n+1 & 2n+2 & 2n+3 & \cdots & 3n \\ \vdots & \vdots & \vdots & & \vdots \\ (n-1)n+1 & (n-1)n+2 & (n-1)n+3 & \cdots & n^2 \end{vmatrix}.$$

If $n = 1$, then $|1| = 1$.
If $n = 2$, then

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2.$$

If $n = 3$, then

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 0 & 0 & 0 \end{vmatrix} = 0.$$

If $n \geq 3$, then the second row minus the first row and the third row minus the second row both give $(n, n, \dots, n)$, thus

$$R_3 - 2R_2 + R_1 = \mathbf{0}.$$

Hence the determinant is always equal to 0, when $n \geq 3$.

Example 3.2 *Compute the determinant*

$$\begin{vmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 2 & 7 \\ 2 & 4 & -3 & 2 \\ 3 & 0 & 15 & 3 \end{vmatrix}.$$

We reduce by using some row operations and then expand after the first column in order to get

$$\begin{aligned} \begin{vmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 2 & 7 \\ 2 & 4 & -3 & 2 \\ 3 & 0 & 15 & 3 \end{vmatrix} &= \begin{vmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & -9 & 4 \\ 0 & -6 & 6 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 7 \\ 0 & -9 & 4 \\ -6 & 6 & 6 \end{vmatrix} = 6 \begin{vmatrix} 1 & 2 & 7 \\ 0 & -9 & 4 \\ -1 & 1 & 1 \end{vmatrix} \\ &= 6 \begin{vmatrix} 1 & 2 & 7 \\ 0 & -9 & 4 \\ 0 & 3 & 8 \end{vmatrix} = 6 \begin{vmatrix} -9 & 4 \\ 3 & 8 \end{vmatrix} \\ &= 6(-72 - 12) = -6 \cdot 84 = -504. \end{aligned}$$

Example 3.3 *Compute the determinant*

$$\begin{vmatrix} 0 & 2 & 3 & 4 \\ 2 & 0 & 4 & 3 \\ 3 & 4 & 0 & 2 \\ 4 & 3 & 2 & 0 \end{vmatrix}.$$

We get by expanding after the first column that

$$\begin{aligned}
\begin{vmatrix} 0 & 2 & 3 & 4 \\ 2 & 0 & 4 & 3 \\ 3 & 4 & 0 & 2 \\ 4 & 3 & 2 & 0 \end{vmatrix} &= -2\begin{vmatrix} 2 & 3 & 4 \\ 4 & 0 & 2 \\ 3 & 2 & 0 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 & 4 \\ 0 & 4 & 3 \\ 3 & 2 & 0 \end{vmatrix} - 4\begin{vmatrix} 2 & 3 & 4 \\ 0 & 4 & 3 \\ 4 & 0 & 2 \end{vmatrix} \\
&= -2\{18+12-8\} + 3\{27-48-18\} - 4\{16+36-64\} \\
&= -2\cdot 22 - 3\cdot 39 + 4\cdot 12 \\
&= -44 - 117 + 48 = -113.
\end{aligned}$$

Example 3.4 *Find the roots of*

$$\begin{vmatrix} x^2+x-6 & 4x-8 & 14 \\ x^2-3 & 3x-5 & 7 \\ x-3 & x-3 & 6 \end{vmatrix}.$$

We compute the determinant by starting with the subtraction $R_1 - R_2$:

$$\begin{aligned}
\begin{vmatrix} x^2+x-6 & 4x-8 & 14 \\ x^2-3 & 3x-5 & 7 \\ x-3 & x-3 & 6 \end{vmatrix} &= \begin{vmatrix} x-3 & x-3 & 7 \\ x^2-3 & 3x-5 & 7 \\ x-3 & x-3 & 6 \end{vmatrix} \underset{R_1 := R_1 - R_3}{=} \\
&= \begin{vmatrix} 0 & 0 & 1 \\ x^2-3 & 3x-5 & 7 \\ x-3 & x-3 & 6 \end{vmatrix} = \begin{vmatrix} x^2-3 & 3x-5 \\ x-3 & x-3 \end{vmatrix} = (x-3)\begin{vmatrix} x^2-3 & 3x-5 \\ 1 & 1 \end{vmatrix} \\
&= (x-3)(x^2-3-3x+5) = (x-3)(x^2-3x+2) \\
&= (x-1)(x-2)(x-3).
\end{aligned}$$

The roots are $x \in \{1, 2, 3\}$.

Example 3.5 *Find value of the determinant below of order $2n$:*

$$\begin{vmatrix} a & 0 & 0 & \cdots & 0 & 0 & b \\ 0 & a & 0 & \cdots & 0 & b & 0 \\ 0 & 0 & a & \cdots & b & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & b & \cdots & a & 0 & 0 \\ 0 & b & 0 & \cdots & 0 & a & 0 \\ b & 0 & 0 & \cdots & 0 & 0 & a \end{vmatrix}.$$

Denote the determinant of order $2n$ by Δ_n. Then we get by expanding after the first column followed by some obvious reductions,

$$\begin{aligned}\Delta_n &= a\begin{vmatrix} a & 0 & \cdots & b & 0 \\ 0 & a & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ b & 0 & \cdots & a & 0 \\ 0 & 0 & \cdots & 0 & a \end{vmatrix} - b\begin{vmatrix} 0 & 0 & \cdots & 0 & b \\ a & 0 & \cdots & b & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & b & \cdots & 0 & 0 \\ b & 0 & \cdots & a & 0 \end{vmatrix} \\ &= a^2\Delta_{n-1} - b^2\Delta_{n-1} = (a^2-b^2)\Delta_{n-1}.\end{aligned}$$

If $n = 1$, then

$$\Delta_1 = \begin{vmatrix} a & b \\ b & a \end{vmatrix} = a^2 - b^2,$$

hence by recursion (or by induction),

$$\Delta_n = (a^2-b^2)^n, \qquad n \in \mathbb{N}.$$

Example 3.6 *Compute the determinant below of order n:*

$$\begin{vmatrix} 1-n & 1 & 1 & \cdots & 1 \\ 1 & 1-n & 1 & \cdots & 1 \\ 1 & 1 & 1-n & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \\ 1 & 1 & 1 & \cdots & 1-n \end{vmatrix}.$$

When we subtract the latter row from all the others, we get

$$\begin{vmatrix} 1-n & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1-n & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1-n & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1-n & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1-n \end{vmatrix} = \begin{vmatrix} -n & 0 & 0 & \cdots & 0 & n \\ 0 & -n & 0 & \cdots & 0 & n \\ 0 & 0 & -n & \cdots & 0 & n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -n & n \\ 1 & 1 & 1 & \cdots & 1 & 1-n \end{vmatrix}$$

$$= n^{n-1} \begin{vmatrix} -1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & -1 & 0 & \cdots & 0 & 1 \\ 0 & 0 & -1 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \\ 1 & 1 & 1 & \cdots & 11-n & \end{vmatrix} = n^{n-1} \begin{vmatrix} -1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & -1 & 0 & \cdots & 0 & 1 \\ 0 & 0 & -1 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{vmatrix}$$

$$= 0,$$

where we in the latter determinant have out

$$R_n := R_1 + R_2 + \cdots + R_n,$$

which gives the zero row.

Example 3.7 *Compute the determinant below of order n:*

$$\begin{vmatrix} 1 & 2 & 3 & 4 & \cdots & n-1 & n \\ 1 & 3 & 3 & 4 & \cdots & n-1 & n \\ 1 & 2 & 5 & 4 & \cdots & n-1 & n \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 2 & 3 & 4 & \cdots & 2n-3 & n \\ 1 & 2 & 3 & 4 & \cdots & n-1 & 2n-1 \end{vmatrix}.$$

In we first step we apply the column operation $S_j := S_j - jS_1$, $j = 2, \dots, n$. Then

$$\begin{vmatrix} 1 & 2 & 3 & 4 & \cdots & n-1 & n \\ 1 & 3 & 3 & 4 & \cdots & n-1 & n \\ 1 & 2 & 5 & 4 & \cdots & n-1 & n \\ \vdots & \vdots & \vdots & \vdots & & 2n-3 & n \\ 1 & 2 & 3 & 4 & \cdots & n-1 & 2n-1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & n-2 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & n-1 \end{vmatrix} = (n-1)!,$$

because we have reduced to a lower triangular determinant.

Example 3.8 *Compute the determinant below of order n:*

$$\begin{vmatrix} 1 & 2 & 3 & 4 & \cdots & n-1 & n \\ 2 & 1 & 2 & 3 & \cdots & n-2 & n-1 \\ 3 & 2 & 1 & 2 & \cdots & n-3 & n-2 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ n & n-1 & n-2 & n-3 & \cdots & 2 & 1 \end{vmatrix}.$$

Here the procedure is the following: In the first step we perform the row operations $R_j := R_j - R_{j-1}$, $j = 2, \dots, n$.
In the next step we apply the column operations $S_j := S_j + S_1$. In the third step we expand after the last column:

$$\begin{vmatrix} 1 & 2 & 3 & 4 & \cdots & n-1 & n \\ 2 & 1 & 2 & 3 & \cdots & n-2 & n-1 \\ 3 & 2 & 1 & 2 & \cdots & n-3 & n-2 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ n-1 & n-2 & n-3 & n-4 & \cdots & 1 & 2 \\ n & n-1 & n-2 & n-3 & \cdots & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 & \cdots & n-1 & n \\ 1 & -1 & -1 & -1 & \cdots & -1 & -1 \\ 1 & 1 & -1 & -1 & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & -1 & -1 \\ 1 & 1 & 1 & 1 & \cdots & 1 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 3 & 4 & 5 & \cdots & n & n+1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 2 & 2 & 2 & \cdots & 0 & 0 \\ 1 & 2 & 2 & 2 & \cdots & 2 & 0 \end{vmatrix} = (-1)^{n+1}(n+1)2^{n-2}.$$

Example 3.9 *Compute the determinant below of order n:*

$$\begin{vmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{vmatrix}.$$

Here we must assume that $n \geq 2$. First compute

$$\Delta_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \quad \text{and} \quad \Delta_3 = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = 0.$$

In general, we first expand after the first column and then after the first row. Then

$$\Delta_n=(-1)\begin{vmatrix}1&0&0&\cdots&0&0\\1&0&1&\cdots&0&0\\0&1&0&\cdots&0&0\\\vdots&\vdots&\vdots&&\vdots&\vdots\\0&0&0&\cdots&0&1\\0&0&\cdots&1&1\end{vmatrix}=(-1)^1\begin{vmatrix}0&1&0&\cdots&0&0\\1&0&1&\cdots&0&0\\0&1&0&\cdots&0&0\\\vdots&\vdots&\vdots&&\vdots&\vdots\\0&0&0&\cdots&0&1\\0&0&0&\cdots&1&0\end{vmatrix}=(-1)^1\Delta_{n-2},$$

because the final determinant has the same structure as Δ_n, only with two rows and two columns fewer.

There is a leap of 2 in the indices, hence, we get for $n=2m+1$, $m\in\mathbb{N}$, odd,

$$\Delta_{2m+1}=(-1)^1\Delta_{2(m-1)+1}=\cdots=(-1)^{m-1}\cdot\Delta_3=0,$$

and for $n=2m$, $m\in\mathbb{N}$, even we get

$$\Delta_{2m}=(-1)^1\Delta_{2(m-1)}=(-1)^2\Delta_{2(m-2)}=\cdots=(-1)^{m-1}\Delta_{2\cdot 1}=(-1)^{m-1}.$$

Example 3.10 *Compute the determinant below of order n:*

$$\begin{vmatrix}0&1&0&0&\cdots&0&0\\1&0&1&0&\cdots&0&0\\1&1&0&1&\cdots&0&0\\\vdots&\vdots&\vdots&\vdots&&\vdots&\vdots\\1&1&1&1&\cdots&0&1\\1&1&1&1&\cdots&1&0\end{vmatrix}.$$

When we expand after the first row, we get

$$\Delta_n=\left|\begin{matrix}0&1&0&0&\cdots&0&0\\1&0&1&0&\cdots&0&0\\1&1&0&1&\cdots&0&0\\\vdots&\vdots&\vdots&\vdots&&\vdots&\vdots\\1&1&1&1&\cdots&0&1\\1&1&1&1&\cdots&1&0\end{matrix}\right|_n=(-1)\left|\begin{matrix}1&1&0&0&\cdots&0&0\\1&0&1&0&\cdots&0&0\\1&1&0&1&\cdots&0&0\\\vdots&\vdots&\vdots&\vdots&&\vdots&\vdots\\1&1&1&1&\cdots&0&1\\1&1&1&1&\cdots&1&0\end{matrix}\right|_{n-1}\qquad S_1:=S_1-S_2$$

$$=(-1)\left|\begin{matrix}0&1&0&0&\cdots&0&0\\1&0&1&0&\cdots&0&0\\0&1&0&1&\cdots&0&0\\\vdots&\vdots&\vdots&\vdots&&\vdots&\vdots\\0&1&1&1&\cdots&0&1\\0&1&1&1&\cdots&1&0\end{matrix}\right|_{n-1}=(+1)\left|\begin{matrix}1&0&0&\cdots&0&0\\1&0&1&\cdots&0&0\\1&1&0&\cdots&0&0\\\vdots&\vdots&\vdots&&\vdots&\vdots\\1&1&1&\cdots&0&1\\1&1&1&\cdots&1&0\end{matrix}\right|_{n-2}=\Delta_{n-3}.$$

In particular,

$$\Delta_2=\begin{vmatrix}0&1\\1&0\end{vmatrix}=-1,\qquad\Delta_3=\begin{vmatrix}0&1&0\\1&0&1\\1&1&0\end{vmatrix}=-\begin{vmatrix}1&1\\1&0\end{vmatrix}=1,$$

$$\Delta_4 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = - \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = 0.$$

Then by recursion (or induction)

$$\Delta_{3n-1} = -1, \qquad \Delta_{3n} = 1, \qquad \Delta_{3n+1} = 0, \qquad \text{for } n \in \mathbb{N}.$$

Example 3.11 *Assume that all elements of the diagonal are equal to a in an $(n \times n)$ matrix, while all other elements are equal to b. Find the determinant of this matrix.*

If $n = 1$, then $\Delta_1 = a$.
If $n = 2$, then $\Delta_2 = \begin{vmatrix} a & b \\ b & a \end{vmatrix} = a^2 - b^2 = (a-b)(a+b)$.
If $n = 3$, then

$$\begin{aligned}\Delta_3 &= \begin{vmatrix} a & b & b \\ b & a & b \\ b & b & a \end{vmatrix} = \begin{vmatrix} a & b & b \\ b-a & a-b & 0 \\ b-a & 0 & a-b \end{vmatrix} = (a-b)^2 \begin{vmatrix} a & b & b \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} \\ &= (a-b)^2(a+b+b) = (a-b)(a+2b).\end{aligned}$$

If $n = 4$, then

$$\begin{aligned}\Delta_4 &= \begin{vmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{vmatrix} = \begin{vmatrix} a & b & b & b \\ b-a & a-b & 0 & 0 \\ b-a & 0 & a-b & 0 \\ b-a & 0 & 0 & a-b \end{vmatrix} \\ &= (a-b)^3 \begin{vmatrix} a & b & b & b \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{vmatrix} = (a-b)^3 \begin{vmatrix} a+3b & b & b & b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \\ &= (a-b)^3(a+3b).\end{aligned}$$

We see now that the computations are following some pattern. It is obvious that if $a = b$ and $n \geq 2$, then $\Delta_n = 0$, so we may expect a factor $a-b$ in some power (it looks like the exponent should be $n-1$, because Δ_n in general is a polynomial of degree n in (a, b)). Then the latter factor should possibly be of the form $a + (n-1)b$. We shall now prove this assumption for $n \geq 2$. The first operation is of the form $R_j := R_j - R_1$ for $j \geq 2$, and the following operation is $S_1 = S_1 + S_2 + \cdots + S_n$. Then we get

$$\begin{aligned}\Delta_n &= \begin{vmatrix} a & b & b & \cdots & b & b \\ b & a & b & \cdots & b & b \\ b & b & a & \cdots & b & b \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ b & b & b\cdots & a & b & \\ b & b & b & \cdots & b & a \end{vmatrix} = \begin{vmatrix} a & b & b & \cdots & b & b \\ b-a & a-b & 0 & \cdots & 0 & 0 \\ b-a & 0 & a-b & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ b-a & 0 & 0 & \cdots & a-b & 0 \\ b-a & 0 & 0 & \cdots & 0 & a-b \end{vmatrix} \\ &= \begin{vmatrix} a+(n-1)b & b & b & \cdots & b & b \\ 0 & a-b & 0 & \cdots & 0 & 0 \\ 0 & 0 & a-b & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a-b & 0 \\ 0 & 0 & 0 & \cdots & 0a-b & \end{vmatrix} \\ &= (a-b)^{n-1}\{a+(n-1)b\},\end{aligned}$$

proving our conjecture.

Example 3.12 *Prove that the determinant D_n corresponding to the $(n \times n)$ matrix*

$$\begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix},$$

where all elements of the diagonal are 0, while all other elements are 1, has the value $D_n = (-1)^{n-1}(n-1)$.

First variant. If we put $a = 0$ and $b = 1$ into Example 3.11, then it follows immediately that

$$B_n = (a-b)^{n-1}\{a + (n-1)b\} = (-1)^{n-1}(n-1).$$

Second variant. If we subtract the first row from all the others, $S_j := S_j - S_1$, for $j \geq 2$, followed by the column operation $S_1 := S_1 + S_2 + \cdots + S_n$, then

$$\begin{aligned} D_n &= \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & -1 & 0 \\ 1 & 0 & 0 & \cdots & 0 & -1 \end{vmatrix} \\ &= \begin{vmatrix} n-1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 0 & \cdots 0 & -1 & \end{vmatrix} \\ &= (-1)^{n-1}(n-1). \end{aligned}$$

Example 3.13 *Given the matrices*

$$\mathbf{A} = \begin{pmatrix} 3 & 4 & 4 \\ 1 & a & 2 \\ 2 & 3 & 3 \end{pmatrix} \quad \textit{and} \quad \mathbf{B} = \begin{pmatrix} 2 & -1 & 0 \\ -5 & 3 & -1 \\ 0 & 1 & a \end{pmatrix}, \qquad \textit{where } a \in \mathbb{R}.$$

1. *Find* $\det \mathbf{A}$ *and* $\det \mathbf{B}$.
2. *Find* $\det(\mathbf{AB})$ *and* $\det((\mathbf{A}^T\mathbf{B})^4)$.
3. *Find all as, for which* $\mathbf{A}$ *is regular, and then compute for these values of a the determinant* $\det(\mathbf{A}^{-1})$.

1. We get

$$\det \mathbf{A} = \begin{vmatrix} 3 & 4 & 4 \\ 1 & a & 2 \\ 2 & 3 & 3 \end{vmatrix} = 9a + 16 + 12 - 8a - 12 - 18 = a - 2$$

and

$$\det \mathbf{B} = \begin{vmatrix} 2 & -1 & 0 \\ -5 & 3 & -1 \\ 0 & 1 & a \end{vmatrix} = 6a - 5a + 2 = a + 2.$$

2. By the rules of calculations,
$$\det(\mathbf{AB}) = \det \mathbf{A} \cdot \det \mathbf{B} = (a-2)(a+2) = a^2 - 4,$$
and
$$\det((\mathbf{A}^T\mathbf{B})^4) = \{\det(\mathbf{A}^T\mathbf{B})\}^4 = \{\det \mathbf{A} \cdot \det \mathbf{B}\}^4 = (a^2-4)^4.$$

3. Since $\mathbf{A}$ is regular for $\det \mathbf{A} \neq 0$, we get the condition that $a \neq 2$. In this case,
$$\det(\mathbf{A}^{-1}) = \frac{1}{a-2}.$$

Example 3.14 *Given the matrix*
$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 7 & 0 \\ 4 & 0 & 5 \end{pmatrix}.$$
Find $\det(\mathbf{A}^{-1}\mathbf{A}^T\mathbf{A})$.

Since
$$\det \mathbf{A} = \begin{vmatrix} 2 & 0 & 3 \\ 0 & 7 & 0 \\ 4 & 0 & 5 \end{vmatrix} = 7 \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = -14 \neq 0,$$
it follows that $\mathbf{A}$ is regular, thus $\mathbf{A}^{-1}$ exists and
$$\det(\mathbf{A}^{-1}\det A^T\mathbf{A}) = (\det \mathbf{A})^{-1} \cdot \det \mathbf{A} \cdot \det \mathbf{A} = \det \mathbf{A} = -14.$$

Example 3.15 *Given the matrices*

$$\mathbf{A}=\begin{pmatrix} a & 0 & 0 & 1 \\ 0 & a & 1 & 0 \\ 0 & 1 & a & 0 \\ 1 & 0 & 0 & a \end{pmatrix} \quad \text{and} \quad \mathbf{B}=\begin{pmatrix} a & 0 & 0 & 0 & 1 \\ 0 & a & 0 & 1 & 0 \\ 0 & 0 & b & 0 & 0 \\ 0 & 1 & 0 & a & 0 \\ 1 & 0 & 0 & 0 & a \end{pmatrix},$$

where a and b are real number.

1. *Find* $\det \mathbf{A}$*, and also the rank of* $\mathbf{A}$ *for every* a*.*
2. *Find* $\det \mathbf{B}$*, and also the rank of* $\mathbf{B}$ *for every* a *and* b*.*

1. First apply the row operations $R_1 := R_1 - R_4$ and $R_2 := R_2 - R_3$ and then the column operations $S_1 := S_1 + S_4$ and $S_2 := S_2 + S_3$. This gives

$$\begin{aligned}\det \mathbf{A} &= \begin{vmatrix} a & 0 & 0 & 1 \\ 0 & a & 1 & 0 \\ 0 & 1 & a & 0 \\ 1 & 0 & 0 & a \end{vmatrix} = \begin{vmatrix} a-1 & 0 & 0 & -(a-1) \\ 0 & a-1 & -(a-1) & 0 \\ 0 & 1 & a & 0 \\ 1 & 0 & 0 & a \end{vmatrix} \\ &= (a-1)^2 \begin{vmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & a & 0 \\ 1 & 0 & 0 & a \end{vmatrix} = /a-1)^2 \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & a+1 & a & 0 \\ a+1 & 0 & 0 & a \end{vmatrix} \\ &= (-1)^2(a-1)^2(a+1)^2 \begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} = -(a^2-1)^2.\end{aligned}$$

   If $a \neq \pm 1$, then $\det \mathbf{A} \neq 0$, hence the rank is 4.
   If $a = \pm 1$, it follows by inspection that the rank is 2.

2. When we expand after the third row we get

$$\det \mathbf{B} = b \cdot \det \mathbf{A} = -b \cdot (a^2-1)^2.$$

   The rank of $\mathbf{B}$ is 5, unless either $a = \pm 1$ or $b = 0$.
   If $a = \pm 1$ and $b = 0$, then $\varrho(\mathbf{B}) = 2$.
   If $a = \pm 1$ and $b \neq 0$, then $\varrho(\mathbf{B}) = 3$.
   If $a \neq \pm 1$ and $b = 0$, then $\varrho(\mathbf{B}) = 4$.
   If $a \neq \pm 1$ and $b \neq 0$, then $\varrho(\mathbf{B}) = 5$.

Example 3.16 *Given a and b real numbers, and given the matrix*

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ a & 2 & 3 & 4 \\ 4 & 3 & 2 & b \end{pmatrix}.$$

Compute $\det \mathbf{A}$*, and find the rank of* $\varrho(\mathbf{A})$ *for every pair* $(a, b) \in \mathbb{R}^2$*.*

We get by the row operations $R_1 := R_1 - R_3$ and $R_4 := R_4 - R_2$ that

$$\begin{aligned} \det \mathbf{A} &= \begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ a & 2 & 3 & 4 \\ 4 & 3 & 2 & b \end{vmatrix} = \begin{vmatrix} 1-a & 0 & 0 & 0 \\ 4 & 3 & 2 & 1 \\ a & 2 & 3 & 4 \\ 0 & 0 & 0 & b-1 \end{vmatrix} \\ &= (1-a)/(b-1) \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 5(1-a)(b-1). \end{aligned}$$

If $a \neq 1$ and $b \neq 1$, then $\varrho(\mathbf{A}) = 4$.
If $a = 1$ and $b \neq 1$, then $\varrho(\mathbf{A}) = 3$.
If $a \neq 1$ and $b = 1$, then $\varrho(\mathbf{A}) = 3$.
If $a = 1$ and $b = 1$, then $\varrho(\mathbf{A}) = 2$.

Example 3.17 *Given the matrix*

$$\mathbf{A} = \begin{pmatrix} x+y & 3 & 2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & x-y & 3 \\ 0 & 3 & 0 & 1 \end{pmatrix}, \qquad \textit{where } (x, y) \in \mathbb{R}^2.$$

Find all pairs $(x, y) \in \mathbb{R}^2$*, for which* $\det \mathbf{A} = 1$*. (Sketch the set of solutions).*

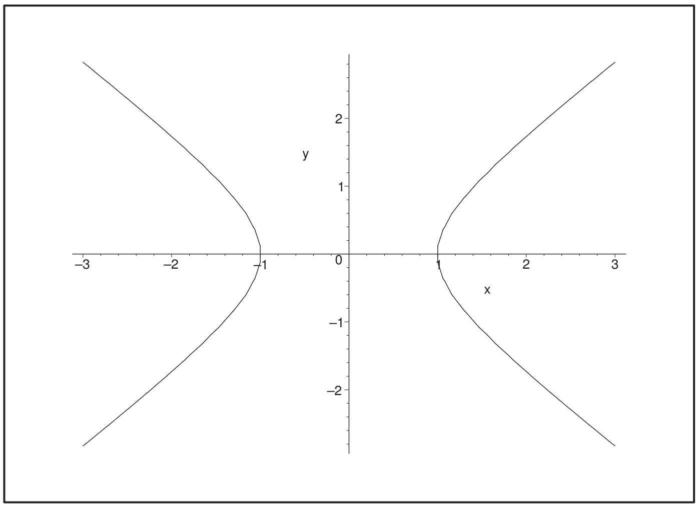
We first compute the determinant,

$$\begin{aligned} \det \mathbf{A} &= \begin{vmatrix} x+y & 3 & 2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & x-y & 3 \\ 0 & 3 & 0 & 1 \end{vmatrix} = (x+y) \begin{vmatrix} 1 & 0 & 0 \\ 2 & x-y & 3 \\ 3 & 0 & 1 \end{vmatrix} \\ &= (x+y)(x-y) \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} = x^2 - y^2. \end{aligned}$$

It follows that $\det \mathbf{A} = 1$, when (x, y) lies on the hyperbola $x^2 - y^2 = 1$.

Example 3.18 *Given* $\mathbf{A}$ *a regular matrix, the elements of which are integers. When are the elements of* $\mathbf{A}^{-1}$ *also integers?*

Assume that the elements of $\mathbf{A}$ and $\mathbf{A}^{-1}$ are all integers. Then both $\det \mathbf{A}$ and $\det \mathbf{A}^{-1} = \dfrac{1}{\det \mathbf{A}}$ are integers. This is only possible, if $\det \mathbf{A} = \pm 1$.
In on the contrary, $\mathbf{A}$ has only integers as elements, and $\det \mathbf{A} = \pm 1$, then $\mathbf{A}^{-1}$ has clearly also only integers as elements, because the complementary matrix has only integers as elements.



Example 3.19 *Assume that* $\mathbf{A}$ *is a square matrix, and denote by* $\mathbf{K}_{\mathbf{A}}$ *its complementary matrix.*

1. *Prove that if* $\mathbf{A}$ *is regular, then*

$$\det \mathbf{K}_{\mathbf{A}} = (\det \mathbf{A})^{n-1} \quad \textit{and} \quad \mathbf{K}_{\mathbf{K}_{\mathbf{A}}} = (\det \mathbf{A})^{n-2}\mathbf{A}.$$

2. *Prove that if* $\mathbf{A}$ *and* $\mathbf{B}$ *are regular of the same type, then*

$$\mathbf{K}_{\mathbf{AB}} = \mathbf{K}_{\mathbf{A}}\mathbf{K}_{\mathbf{B}}.$$

3. *Try also to prove the claims from 1) and 2) for singular matrices.*

1. It follows from

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\,(\mathbf{K}_{\mathbf{A}})^T$$

that

$$\mathbf{K}_{\mathbf{A}} = \det \mathbf{A} \cdot \left(\mathbf{A}^{-1}\right)^T,$$

hence

$$\det \mathbf{K}_{\mathbf{A}} = \{\det \mathbf{A}\}^n \cdot \det\left(\mathbf{A}^{-1}\right) = (\det \mathbf{A})^{n-1}.$$

Furthermore,

$$\mathbf{K}_{\mathbf{A}}^{-1} = \frac{1}{\det \mathbf{K}_{\mathbf{A}}} \cdot \{\mathbf{K}_{\mathbf{K}_{\mathbf{A}}}\}^T = \frac{1}{\det \mathbf{A}} \cdot \left\{\left(\mathbf{A}^{-1}\right)^T\right\}^{-1} = \frac{1}{\det \mathbf{A}}\,\mathbf{A}^T.$$

When we solve in $\mathbf{K}_{\mathbf{K}_{\mathbf{A}}}$, we get

$$\mathbf{K}_{\mathbf{K}_{\mathbf{A}}} = \frac{\det \mathbf{K}_{\mathbf{A}}}{\det \mathbf{A}}\,\mathbf{A} = \frac{(\det \mathbf{A})^{n-1}}{\det \mathbf{A}} \cdot \mathbf{A} = (\det \mathbf{A})^{n-2}\mathbf{A}.$$

2. By the definition,

$$(\mathbf{AB})^{-1} = \frac{1}{\det(\mathbf{AB})} (\mathbf{K_{AB}})^T,$$

hence

$$\begin{aligned}\mathbf{K_{AB}} &= \det\mathbf{A}\cdot\det\mathbf{B}\cdot\left(\mathbf{B}^{-1}\mathbf{A}^{-1}\right)^T \\ &= \det\mathbf{A}\cdot\left(\mathbf{A}^{-1}\right)^T\cdot\det\mathbf{B}\cdot\left(\mathbf{B}^{-1}\right)^T = \mathbf{K_A}\cdot\mathbf{K_B}.\end{aligned}$$

3. The question is not precisely formulated. We should at least require that $n \geq 2$. If $n = 2$ and $\det\mathbf{A} = 0$, then

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{med} \quad \mathbf{K_A} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{and} \quad \mathbf{K_{K_A}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathbf{A}.$$

It follows that

$$\det\mathbf{A} = \det\mathbf{K_A} = 0 = (\det\mathbf{A})^{2-1},$$

hence the first formula is correct.

If we interpret $(\det\mathbf{A})^{n-2} := 1$ for $n = 2$, then the second claim is also true for $n = 2$ and $\det\mathbf{A} = 0$.

In general, the claims can now be proved by a continuity argument known from Calculus.

Consider $\det \mathbf{A}$ as a continuous function with the n^2 elements as variables. In each term, a_{ij} occurs at most of degree 1, hence $\det \mathbf{A}$ is of class C^∞. The set

$$\{\mathbf{A} \mid \det \mathbf{A} = 0\}$$

is closed and of dimension $n^2 - 1$, and since $\det \mathbf{A}$ is C^∞, we can to every fixed $\mathbf{A}$, for which $\det \mathbf{A} = 0$, find a set $\{\mathbf{A}_\varepsilon \mid \varepsilon > 0\}$, such that $\mathbf{A}_\varepsilon$ is continuous in $\varepsilon > 0$, and

$$\lim_{\varepsilon \to 0+} \mathbf{A}_\varepsilon = \mathbf{A} \qquad \text{and} \qquad \det \mathbf{A}_\varepsilon = \varepsilon > 0.$$

Now, 1) and 2) hold for each of the matrices $\mathbf{A}_\varepsilon$ (and $\mathbf{B}_\varepsilon$). The formation of $\mathbf{K}_\mathbf{A}$ is also a continuous function in the n^2 variables, and we have continuity in ε. Thus we get for $n \geq 2$ that

$$\det \mathbf{K}_\mathbf{A} = \lim_{\varepsilon \to 0+} \det \mathbf{K}_{\mathbf{A}_\varepsilon} = \lim_{\varepsilon \to 0+} (\det \mathbf{A}_\varepsilon)^{n-1} = (\det \mathbf{A})^{n-1} = 0,$$

and

$$\mathbf{K}_{\mathbf{K}_\mathbf{A}} = \lim_{\varepsilon \to 0+} \mathbf{K}_{\mathbf{K}_{\mathbf{A}_\varepsilon}} = \lim_{\varepsilon \to 0+} (\det \mathbf{A}_\varepsilon)^{n-2} \mathbf{A}_\varepsilon = (\det \mathbf{A})^{n-2} \mathbf{A},$$

where vi for $n = 2$ have

$$(\det \mathbf{A}_\varepsilon)^{n-2} = \varepsilon^0 = 1 \to 1 \qquad \text{for } \varepsilon \to 0+.$$

If $n > 2$, then we get instead

$$(\det \mathbf{A}_\varepsilon)^{n-2} \to 0^{n-2} = 0 \qquad \text{for } \varepsilon \to 0+.$$

In the same way we get

$$\mathbf{K}_{\mathbf{AB}} = \lim_{\varepsilon \to 0+} \mathbf{K}_{bfA_\varepsilon \mathbf{B}_\varepsilon} = \mathbf{K}_\mathbf{A} \mathbf{K}_\mathbf{B}.$$

Example 3.20 *Find all a, for which the determinant of the matrix*

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & -1 & 0 \\ 1 & -1 & -1 & 1 \\ -1 & -1 & a & -1 \\ -1 & -1 & -1 & 3-a \end{pmatrix}$$

is equal to 0. Find for the largest and the smallest of these values of a the complete complete solution of the system

$$(x_1 \;\; x_2 \;\; x_3 \;\; x_4)\mathbf{A} = (2 \;\; 2 \;\; 2 \;\; 0).$$

When we add all rows we get after some reductions,

$$\begin{aligned} \det \mathbf{A} &= \begin{vmatrix} -1 & a & -1 & 0 \\ a & -1 & -1 & 1 \\ -1 & -1 & a & -1 \\ -1 & -1 & -1 & 3-a \end{vmatrix} = \begin{vmatrix} a-3 & a-3 & a-3 & -(a-3) \\ a & -1 & -1 & 1 \\ -1 & -1 & a & -1 \\ -1 & -1 & -1 & 3-a \end{vmatrix} \\ &= (a-3) \begin{vmatrix} 1 & 1 & 1 & -1 \\ a & -1 & -1 & 1 \\ -1 & -1 & a & -1 \\ -1 & -1 & -1 & 3-a \end{vmatrix} = (a-3) \begin{vmatrix} 1 & 1 & 1 & -1 \\ 0 & -a-1 & -a-1 & a+1 \\ 0 & 0 & a+1 & -2 \\ 0 & 0 & 0 & 2-a \end{vmatrix} \\ &= (a+1)^2(a-2)(a-3). \end{aligned}$$

This expression is 0 for $a \in \{-1,2,3\}$, where $a=-1$ is a double root.

The system of equations is transposed into

$$\begin{pmatrix} -1 & a & -1 & -1 \\ a & -1 & -1 & -1 \\ -1 & -1 & a & -1 \\ 0 & 1 & -1 & 3-a \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 0 \end{pmatrix}.$$

For $a=-1$ we get the reductions

$$\left(\begin{array}{rrrr|r} -1 & -1 & -1 & -1 & 2 \\ -1 & -1 & -1 & -1 & 2 \\ -1 & -1 & -1 & -1 & 2 \\ 0 & 1 & -1 & 4 & 0 \end{array}\right) \sim \left(\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 4 & 0 \end{array}\right) \sim \left(\begin{array}{rrrr|r} 1 & 0 & 2 & -3 & -2 \\ 0 & 1 & -1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

of rank 2. Choosing $x_3 = s$ and $x_4 = t$ as parameters we get

$$x_1 = -2-2s+3t \qquad \text{and} \qquad x_2 = s-4t,$$

and the complete solution is

$$\mathbf{x} = (-2,0,0,0) + s(-2,1,1,0) + t(3,-4,0,1), \qquad s,\, t \in \mathbb{R}.$$

For $a=3$ we get by reduction

$$\left(\begin{array}{rrrr|r} -1 & 3 & -1 & -1 & 2 \\ 3 & -1 & -1 & -1 & 2 \\ -1 & -1 & 3 & -1 & 2 \\ 0 & 1 & -1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{rrrr|r} 1 & -3 & 1 & 1 & -2 \\ 0 & 8 & -4 & -4 & 8 \\ 0 & -4 & 4 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{rrrr|r} 1 & -3 & 1 & 1 & -2 \\ 0 & 2 & -1 & -1 & 2 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$
$$\sim \left(\begin{array}{rrrr|r} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

which is of rank 3. Choosing $x_2 = s$ as parameter, we get

$$x_1 = s, \qquad x_3 = s \quad \text{and} \quad x_4 = -2+s,$$

so the complete solution is

$$\mathbf{x} = (0,0,0,-2) + s(1,1,1,1), \qquad s \in \mathbb{R}.$$

Example 3.21 *Denote by* $\mathbf{A}$ *the matrix*

$$\mathbf{A} = \begin{pmatrix} 2 & 4 & 0 \\ 2 & 10 & -4 \\ 4 & -4 & 13 \end{pmatrix}.$$

1. *Find a lower triangular unit matrix* $\mathbf{L}$ *and an upper triangular matrix* $\mathbf{U}$*, such that* $\mathbf{A} = \mathbf{LU}$.
2. *Prove that* $\mathbf{A}^T$ *is regular, and find* $\det\left(\left(\mathbf{A}^T\right)^{-1}\right)$.
3. *Solve the system of equations*

$$\mathbf{Ly} = \begin{pmatrix} 1 \\ 7 \\ 10 \end{pmatrix},$$

and then also

$$\mathbf{Ax} = \begin{pmatrix} 1 \\ 7 \\ 10 \end{pmatrix}.$$

1. We get by a simple Gauß elimination,

$$\mathbf{A} = \begin{pmatrix} 2 & 4 & 0 \\ 2 & 10 & -4 \\ 4 & -4 & 13 \end{pmatrix} \sim \begin{pmatrix} 2 & 4 & 0 \\ 0 & 6 & -4 \\ 0 & -12 & 13 \end{pmatrix} \sim \begin{pmatrix} 2 & 4 & 0 \\ 0 & 6 & -4 \\ 0 & 0 & 5 \end{pmatrix} = \mathbf{U},$$

hence

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix},$$

and whence

$$\mathbf{LU} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 & 0 \\ 0 & 6 & -4 \\ 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 0 \\ 2 & 10 & -4 \\ 4 & -4 & 13 \end{pmatrix} = \mathbf{A}.$$

2. It follows from $\det \mathbf{A}^T = \det \mathbf{A} = \det \mathbf{U} = 2 \cdot 6 \cdot 5 = 60 \neq 0$ that $\mathbf{A}^T$ is regular, and

$$\det\left(\left(\mathbf{A}^T\right)^{-1}\right) = \frac{1}{\det \mathbf{A}^T} = \frac{1}{\det \mathbf{A}} = \frac{1}{60}.$$

3. It follows from

$$\mathbf{Ly} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_1 + y_2 \\ 2y_1 - 2y_2 + y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \\ 10 \end{pmatrix}$$

that $y_1 = 1$, $y_2 = 6$ and $y_3 = 10 - 2 + 12 = 20$, hence

$$\mathbf{y} = (1, 6, 20).$$

Then $\mathbf{Ax} = \mathbf{L}(\mathbf{Ux}) = (1 \;\; 7 \;\; 10)^T$, when

$$\mathbf{Ux} = \begin{pmatrix} 2 & 4 & 0 \\ 0 & 6 & -4 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{y} = \begin{pmatrix} 1 \\ 6 \\ 20 \end{pmatrix},$$

so $x_3 = 4$, $6x_2 = 6 + 16$, i.e. $x_2 = \dfrac{11}{3}$ and $2x_1 = 1 - \dfrac{44}{3} = -\dfrac{41}{3}$, and therefore

$$\mathbf{x} = \left(-\frac{41}{3}, \frac{11}{3}, 4 \right).$$

CHECK. The numbers are apparently untypical compared with the usual examples, so one should indeed check the results- We get

$$-\frac{41}{3} + \frac{44}{3} = 1, \quad 22 - 16 = 6, \quad 5 \cdot 4 = 20,$$

and the results are correct. $\Diamond$

Example 3.22 *Given the matrix*

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & a \\ -1 & -1 & a \end{pmatrix}, \qquad \textit{where } a \in \mathbb{R}.$$

1. *Find a lower triangular unit matrix* $\mathbf{L}$ *and an upper triangular matrix* $\mathbf{U}$*, so* $\mathbf{A} = \mathbf{LU}$*.*
2. *Prove that* $\mathbf{A}$ *is regular for all* $a \in \mathbb{R}$*, and compute* $\det\left(\mathbf{A}\mathbf{A}^T\mathbf{A}^{-1}\right)$*.*

1. It follows by a simple Gauß elimination that

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & a \\ -1 & -1 & a \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & a-2 \\ 0 & 1 & a+1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & a-2 \\ 0 & 0 & 3 \end{pmatrix} = \mathbf{U}$$

where

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix},$$

and

$$\mathbf{LU} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & a-2 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & a \\ -1 & -1 & a \end{pmatrix} = \mathbf{A}.$$

2. It follows from $\det \mathbf{A} = \det \mathbf{U} = 1 \cdot 1 \cdot 3 = 3 \neq 0$ that $\mathbf{A}$ is regular for all $a \in \mathbb{R}$.

   Furthermore,

$$\det\left(\mathbf{A}\mathbf{A}^T\mathbf{A}^{-1}\right) = \det \mathbf{A} \cdot \det \mathbf{A}^T \cdot \left(\det \mathbf{A}^{-1}\right) = \det \mathbf{A} = 3.$$

Example 3.23 *Given the matrix*

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{pmatrix}.$$

1. *Prove that* $\mathbf{A}$ *is regular and find* $\mathbf{A}^{-1}$.
2. *Given the vectors* $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ *by the coordinates*

$$(1,1,2), \qquad (1,0,3) \quad \textit{and} \quad (1,-2,1)$$

in an ordinary rectangular coordinate system of positive orientation.
Find the coordinates of the vectors

$$\vec{w}_1 = \vec{v}_2 \times \vec{v}_3, \quad \vec{w}_2 = \vec{v}_3 \times \vec{v}_1 \quad \textit{and} \quad \vec{w}_3 = \vec{v}_1 \times \vec{v}_2.$$

3. *Denote by* $\mathbf{W}$ *the matrix, where the coordinates of* $\vec{w}_j$ *are given by the* j*-h column,* $j = 1, 2, 3$. *Prove that*

$$\mathbf{W} = (\det \mathbf{A})\, \mathbf{A}^{-1}.$$

1. It follows from

$$\det \mathbf{A} = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & -3 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ -3 & -1 \end{vmatrix} = 4 \neq 0,$$

that $\mathbf{A}$ is regular.

Then compute the complements,

$$A_{11} = \begin{vmatrix} 0 & 3 \\ -2 & 1 \end{vmatrix} = 6, \qquad -A_{12} = -\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = +2, \qquad A_{13} = \begin{vmatrix} 1 & 0 \\ 1 & 0 \\ 1 & -2 \end{vmatrix} = -2,$$

$$-A_{21} = -\begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = -5, \qquad A_{22} = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1, \qquad -A_{23} = -\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = 3,$$

$$A_{31} = \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = 3, \qquad -A_{32} = -\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = -1, \qquad A_{33} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1,$$

så

$$\mathbf{A}^{-1} = \frac{1}{4} \begin{pmatrix} 6 & -5 & 3 \\ 2 & -1 & -1 \\ -2 & 3 & -1 \end{pmatrix}.$$

Check. These examples are full of details, so one should always check one's computations. In the present case we get

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{pmatrix} \frac{1}{4} \begin{pmatrix} 6 & -5 & 3 \\ 2 & -1 & -1 \\ -2 & 3 & -1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \mathbf{I},$$

and we have checked our solution. $\Diamond$

2. Then by a computation,

$$\vec{w}_1 = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{vmatrix} = (6, 2, -2),$$

$$\vec{w}_2 = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & -2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = (-5, -1, 3),$$

$$\vec{w}_3 = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & 1 & 2 \\ 1 & 0 & 3 \end{vmatrix} = (3, -1, -1).$$

3. It follows immediately that

$$\mathbf{W} = \begin{pmatrix} 6 & -5 & 3 \\ 2 & -1 & -1 \\ -2 & 3 & -1 \end{pmatrix} = 4\mathbf{A}^{-1} = (\det \mathbf{A})\mathbf{A}^{-1}.$$

Example 3.24 *Given the matrix*

$$\mathbf{A} = \begin{pmatrix} 0 & a & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & a & 0 \end{pmatrix}, \qquad \textit{where } a \in \mathbb{R}.$$

1. *Find* $\det \mathbf{A}$ *and* $\det\left(\mathbf{A}^{12}\right)$.
2. *Find the rank of* $\mathbf{A}$.
3. *Solve the linear system*

$$\mathbf{A}\mathbf{x} = \begin{pmatrix} b \\ 0 \\ 0 \\ b \end{pmatrix},$$

where $b \in \mathbb{R}$.

1. We find

$$\det \mathbf{A} = \begin{vmatrix} 0 & a & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & a & 0 \end{vmatrix} = -\begin{vmatrix} a & 0 & 0 \\ 1 & 0 & 1 \\ 0 & a & 0 \end{vmatrix} = -a \begin{vmatrix} 0 & 1 \\ a & 0 \end{vmatrix} = a^2,$$

and

$$\det\left(\mathbf{A}^{12}\right) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & a & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ a & 0 \end{vmatrix} = -a.$$

2. If $a \neq 0$, then $\det \mathbf{A} \neq 0$, hence the rank is 4.
   If $a = 0$, then it follows immediately that the rank is 2.

3. Then by a reduction we get for $a \neq 0$ that

$$
\begin{aligned}
(\mathbf{A} \mid \mathbf{b}) &= \left(\begin{array}{cccc|c} 0 & a & 0 & 0 & b \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & a & 0 & b \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{b}{a} \\ 0 & 0 & 1 & 0 & \frac{b}{a} \\ 0 & 0 & 0 & 1 & -\frac{b}{a} \end{array}\right) \\
&\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -\frac{b}{a} \\ 0 & 1 & 0 & 0 & \frac{b}{a} \\ 0 & 0 & 1 & 0 & \frac{b}{a} \\ 0 & 0 & 0 & 1 & -\frac{b}{a} \end{array}\right),
\end{aligned}
$$

   thus

$$
\mathbf{x} = \left(-\frac{b}{a}, \frac{b}{a}, \frac{b}{a}, -\frac{b}{a}\right) \qquad \text{for } a \neq 0.
$$

   If $a = 0$ and $b \neq 0$, then there are no solutions.

Finally, if $a = b = 0$, then

$$x_1 = s, \quad x_2 = t, \quad x_3 = -s, \quad x_4 = -t,$$

thus

$$\mathbf{x} = s(1,0,-1,0) + t(0,1,0,-1) = (s,t,-s,-t), \qquad s,\, t \in \mathbb{R}.$$

Example 3.25 *Given the matrices*

$$\mathbf{A} = \begin{pmatrix} -3 & -6 & -2 \\ 2 & 5 & 2 \\ -2 & -6 & 3 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} -2 & 4 & 2 \\ 2 & -3 & -1 \\ -3 & 3 & 0 \end{pmatrix}.$$

Compute $\mathbf{A}^2$ *and* $\mathbf{AB}$. *Then find* $\mathbf{A}^{-1}$ *and* $\det \mathbf{B}$.

We obtain by mechanical computations

$$\mathbf{A}^2 = \begin{pmatrix} -3 & -6 & -2 \\ 2 & 5 & 2 \\ -2 & -6 & -3 \end{pmatrix} \begin{pmatrix} -3 & -6 & -2 \\ 2 & 5 & 2 \\ -2 & -6 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\mathbf{AB} = \begin{pmatrix} -3 & -6 & -2 \\ 2 & 5 & 2 \\ -2 & -6 & -3 \end{pmatrix} \begin{pmatrix} -2 & 4 & 2 \\ 2 & -3 & -1 \\ -3 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 1 & 1 & 2 \end{pmatrix}.$$

It follows from the former expression that $\mathbf{A}^{-1} = \mathbf{A}$. In particular, $\det \mathbf{A} = \pm 1$.

It follows from the latter expression that

$$0 = \det(\mathbf{AB}) = \det \mathbf{A} \cdot \det \mathbf{B},$$

hence $\det \mathbf{B} = 0$.

Example 3.26 *Given the matrices*

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ a & 2 & 1 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} b & 1 & 0 \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -a & -2-a & 1 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 0 & -1 & -1 \\ 1 & b & b \\ 1 & b-1 & b \end{pmatrix},$$

where a and b are real numbers.

1. *Compute* $\mathbf{AC}$, $\mathbf{BD}$, $\mathbf{DC}$, *and* $\det \mathbf{A}$ *and* $\det \mathbf{B}$.
2. *Compute* $\mathbf{A}^{-1}$, $\mathbf{B}^{-1}$, $(\mathbf{AB})^{-1}$ *and* $\det(\mathbf{DC})$.

1. By mechanical computations,

$$\begin{aligned}
\mathbf{AC} &= \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ a & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -a & -2-a & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
\mathbf{BD} &= \begin{pmatrix} b & 1 & 0 \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & -1 \\ 1 & b & b \\ 1 & b-1 & b \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
\mathbf{DC} &= \begin{pmatrix} 0 & -1 & -1 \\ 1 & b & b \\ 1 & b-1 & b \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -a & -2-a & 1 \end{pmatrix} = \begin{pmatrix} a & a+1 & -1 \\ 1-ab & 1-b-ab & b \\ 1-ab & -b-ab & b \end{pmatrix}, \\
\det \mathbf{A} &= \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ a & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1, \\
\det \mathbf{B} &= \begin{vmatrix} b & 1 & 0 \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} b & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} b & 1 \\ -1 & 0 \end{vmatrix} = 1.
\end{aligned}$$

2. It follows from 1) that $\mathbf{A}^{-1} = \mathbf{C}$ and $\mathbf{B}^{-1} = \mathbf{D}$, and

$$(\mathbf{AB}^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} = \mathbf{DC},$$

and

$$\det(\mathbf{DC}) = \det(\mathbf{AB})^{-1} = \{\det \mathbf{A} \cdot \det \mathbf{B}\}^{-1} = \frac{1}{1 \cdot 1} = 1.$$

Example 3.27 *Given the matrix*

$$\mathbf{A} = \begin{pmatrix} 1 & -4 & -4 \\ 0 & 4 & 3 \\ 4 & -4 & -7 \end{pmatrix}.$$

1. *Find a lower triangular unit matrix* $\mathbf{L}$ *and an upper triangular matrix* $\mathbf{U}$*, such that* $\mathbf{A} = \mathbf{LU}$*.*
2. *Solve the matrix equation* $\mathbf{AX} = \mathbf{0}$*, where* $\mathbf{0}$ *denotes the zero matrix in* $\mathbb{R}^{3\times 3}$*.*
3. *Solve the matrix equation* $\mathbf{AX} = \mathbf{A}^2 + 3\mathbf{A}$*.*
4. *Find all* $a \in \mathbb{R}$*, for which the matrix equation* $\mathbf{AX} = a\mathbf{X}$*, where* $\mathbf{X} \in \mathbb{R}^{3\times 3}$*, has proper solutions.*

1. We get by a simple Gauß elimination,

$$\mathbf{A} = \begin{pmatrix} 1 & -4 & -4 \\ 0 & 4 & 3 \\ 4 & -4 & -7 \end{pmatrix} \sim \begin{pmatrix} 1 & -4 & -4 \\ 0 & 4 & 3 \\ 0 & 12 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & -4 & -4 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{U},$$

whence

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 3 & 1 \end{pmatrix}.$$

CHECK:

$$\mathbf{LU} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & -4 & -4 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -4 & -4 \\ 0 & 4 & 3 \\ 4 & -4 & -7 \end{pmatrix} = \mathbf{A}. \quad \diamond$$

2. This can be done in several ways.

(a) It follows that

$$\mathbf{U} = \begin{pmatrix} 1 & -4 & -4 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \end{pmatrix},$$

so $x_1 = x_3$ and $3x_1 = -4x_2$, which e.g. is fulfilled for $(4, -3, 4)$. This gives

$$\mathbf{X} = \begin{pmatrix} 4s & 4t & 4u \\ -3s & -3t & -3u \\ 4s & 4t & 4u \end{pmatrix}, \qquad s,\, t,\, u \in \mathbb{R}.$$

(b) ALTERNATIVELY, $\mathbf{AX} = \mathbf{LUX} = \mathbf{0}$ is equivalent to $\mathbf{UX} = \mathbf{0}$, because $\det \mathbf{L} \neq 0$. Then continue by a variant of the above.

3. We reformulate the equation to

$$\mathbf{A}(\mathbf{X}-\mathbf{A}-3\mathbf{I})=\mathbf{0},$$

from which follows that this is in principle solved in 2), hence

$$\begin{aligned}\mathbf{X} &= \mathbf{A}+3\mathbf{I}+\begin{pmatrix} 4s & 4t & 4u \\ -3s & -3t & -3u \\ 4s & 4t & 4u \end{pmatrix}=\begin{pmatrix} 4 & -4 & -4 \\ 0 & 7 & 3 \\ 4 & -4 & -4 \end{pmatrix}+\begin{pmatrix} 4s & 4t & 4u \\ -3s & -3t & -3u \\ 4s & 4t & 4u \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}+\begin{pmatrix} 4s_1 & 4t_1 & 4u_1 \\ -3s_1 & -3t_1 & -3u_1 \\ 4s_1 & 4t_1 & 4u_1 \end{pmatrix}, \qquad s_1,\, t_1,\, u_1 \in \mathbb{R}.\end{aligned}$$

4. The characteristic polynomial is

$$\begin{aligned}\det(\mathbf{A}-\lambda\mathbf{I}) &= \begin{vmatrix} 1-\lambda & -4 & -4 \\ 0 & 4-\lambda & 3 \\ 4 & -4 & -7-\lambda \end{vmatrix} = -\begin{vmatrix} \lambda-1 & 4 & 4 \\ 0 & \lambda-4 & -3 \\ -4 & 4 & \lambda+7 \end{vmatrix} \\ &= -\{(\lambda-1)(\lambda-4)(\lambda+7)+48+16(\lambda-4)+12(\lambda-1)\} \\ &= -\left\{\left(\lambda^2-5\lambda+4\right)(\lambda+7)+28\lambda+48-64-12\right\} \\ &= -\left\{\lambda^3+2\lambda^2-31\lambda+28+28\lambda-28\right\} \\ &= -\lambda\left(\lambda^2+2\lambda-3\right)=-\lambda(\lambda+3)(\lambda-1).\end{aligned}$$

The answer is $a=0$, $a=1$ and $a=-3$.

Remark 3.1 For completeness we add the solutions, which are not requested in the example. If $a=0$, then we see that e.g. $\mathbf{v}_1=(4,-3,4)$ is a eigenvector.

If $a=1$, then we get the reductions

$$\mathbf{A}-\mathbf{I}=\begin{pmatrix} 0 & -4 & -4 \\ 0 & 3 & 3 \\ 4 & -4 & -8 \end{pmatrix}\sim\begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\sim\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

thus an eigenvector is e.g. $\mathbf{v}_2=(1,-1,1)$.

If $a=-3$, we get the reduction

$$\mathbf{A}+3\mathbf{I}=\begin{pmatrix} 4 & -4 & -4 \\ 0 & 7 & 3 \\ 4 & -4 & -4 \end{pmatrix}\sim\begin{pmatrix} 1 & -1 & 1 \\ 0 & 7 & 3 \\ 0 & 0 & 0 \end{pmatrix},$$

so an eigenvector is e.g. $\mathbf{v}_3=(10,3,-7)$.

In each of the three cases the solution is constructed by means of columns,

$$\mathbf{X}=(s\mathbf{v}_i\ \ t\mathbf{v}_i\ \ u\mathbf{v}_i), \qquad \text{where } s,\, t,\, u \in \mathbb{R}. \quad \Diamond$$

Example 3.28 *Given the matrix*

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & -1 \\ 3 & -1 & 1 \\ -3 & 1 & -1 \end{pmatrix}.$$

1. *Find all* (3×3) *matrices* $\mathbf{X}$, *which satisfies*

   $$\mathbf{AX} = \mathbf{0}.$$

2. *Find the characteristic polynomial of* $\mathbf{A}$ *in the form*

   $$R(\lambda) = a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0,$$

   and prove that

   $$(4) \qquad a_3\mathbf{A}^3 + a_2\mathbf{A}^2 + a_1\mathbf{A} + a_0\mathbf{I} = \mathbf{0}.$$

3. *Apply e.g. (4) to find a particular solution of the matrix equation*

   $$(5) \qquad \mathbf{AX} = -\mathbf{A}^3 + 2\mathbf{A}.$$

4. *Find the complete solution of the matrix equation (5).*

1. We get by reduction

$$\begin{aligned} \mathbf{A} &= \begin{pmatrix} 1 & -1 & -1 \\ 3 & -1 & 1 \\ -3 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

An element of $\ker f$ is e.g. $(1, 2, -1)$, so $\ker f$ has the dimension 1.
The complete solution of $\mathbf{AX} = \mathbf{0}$ is given by

$$\mathbf{X} = \begin{pmatrix} s & t & u \\ 2s & 2t & 2u \\ -s & -t & -u \end{pmatrix}, \qquad s,\, t,\, u \in \mathbb{R}.$$

2. The characteristic polynomial is

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} 1-\lambda & -1 & -1 \\ 3 & -1-\lambda & 1 \\ -3 & 1 & -1-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & -1 & -1 \\ 0 & -\lambda & -\lambda \\ -3 & 1 & -1-\lambda \end{vmatrix} \\ &= -\lambda \begin{vmatrix} 1-\lambda & -1 & -1 \\ 0 & 1 & 1 \\ -3 & 1 & -1-\lambda \end{vmatrix} = -\lambda \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1 & 1 \\ -3 & 1 & -1-\lambda \end{vmatrix} \\ &= -\lambda(1-\lambda) \begin{vmatrix} 1 & 1 \\ 1 & -1-\lambda \end{vmatrix} = -\lambda(1-\lambda)(-\lambda-2) \\ &= -\lambda(\lambda-1)(\lambda+2) = -\left\{\lambda^3 + \lambda^2 - 2\lambda\right\}, \end{aligned}$$

so apart from the sign,

$$R(\lambda) = \lambda^3 + \lambda^2 - 2\lambda$$

where

$$a_3 = 1, \quad a_2 = 1, \quad a_1 = -2, \quad a_0 = 0.$$

Then we compute

$$\begin{aligned}
\mathbf{A}^2 &= \begin{pmatrix} 1 & -1 & -1 \\ 3 & -1 & 1 \\ -3 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ 3 & -1 & 1 \\ -3 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 \\ -3 & -1 & -5 \\ 3 & 1 & 5 \end{pmatrix}, \\
\mathbf{A}^3 &= \begin{pmatrix} 1 & -1 & -1 \\ 3 & -1 & 1 \\ -3 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ -3 & -1 & -5 \\ 3 & 1 & 5 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 \\ 9 & -1 & 7 \\ -9 & 1 & -7 \end{pmatrix},
\end{aligned}$$

thus

$$\begin{aligned} R(\mathbf{A}) &= \mathbf{A}^3+\mathbf{A}^2-2\mathbf{A} \\ &= \begin{pmatrix} 1 & -1 & -1 \\ 9 & -1 & 7 \\ -9 & 1 & -7 \end{pmatrix} + \begin{pmatrix} 1 & -1 & -1 \\ -3 & -1 & -5 \\ 3 & 1 & 5 \end{pmatrix} - \begin{pmatrix} 2 & -2 & -2 \\ 6 & -2 & 2 \\ -6 & 2 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 1+1-2 & -1-1+2 & -1-1+2 \\ 9-3-6 & -1-1+2 & 7-5-2 \\ -9+3+6 & 1+1-2 & -7+5+2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{0}. \end{aligned}$$

3. It follows from $\mathbf{A}^3+\mathbf{A}^2-2\mathbf{A}=\mathbf{0}$ that

$$-\mathbf{A}^3+2\mathbf{A}=\mathbf{A}^2=\mathbf{A}\mathbf{A},$$

hence a particular solution is $\mathbf{X}_0=\mathbf{A}$.

4. Combining 1) and 3) we obtain the complete solution

$$\mathbf{X}=\begin{pmatrix} 1 & -1 & -1 \\ 3 & -1 & 1 \\ -3 & 1 & -1 \end{pmatrix} + \begin{pmatrix} s & t & u \\ 2s & 2t & 2u \\ -s & -t & -u \end{pmatrix}, \qquad s,\,t,\,u\in\mathbb{R}.$$

Example 3.29 *Given the matrices*

$$\mathbf{A}=\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \mathbf{B}=\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \mathbf{C}=\begin{pmatrix} 2 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 4 \end{pmatrix}.$$

1. *Compute* $\det\left(\mathbf{A}^2\mathbf{B}^{-1}\left(\mathbf{A}^T\right)^{-1}\right)$.
2. *Find the LU factorization of* $\mathbf{A}$.
3. *Can* $\mathbf{A}$ *be diagonalized?*
4. *Check if* $\mathbf{A}$ *and* $\mathbf{B}$ *are similar matrices.*
5. *Is* $\mathbf{C}$ *positive definite?*

1. Since $\det\mathbf{A}=4$ and $\det\mathbf{B}=4$, it follows by the rules of calculations that

$$\det\left(\mathbf{A}^2\mathbf{B}^{-1}\left(\mathbf{A}^T\right)^{-1}\right)=(\det\mathbf{A})^2(\det\mathbf{B})^{-1}(\det\mathbf{A})^{-1}=4^2\cdot4^{-1}\cdot4^{-1}=1.$$

2. This question is almost trivial,

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{LU}.$$

3. Since $\lambda = 2$ is of algebraic multiplicity 2 and geometrical multiplicity 1, we cannot diagonalize $\mathbf{A}$. In fact, we get by reduction,

$$\mathbf{A} - 2\mathbf{I} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

which is of rank 2, so the dimension of the eigenspace is $3 - 2 = 1$.

4. Both $\mathbf{A}$ and $\mathbf{B}$ have the characteristic polynomial

$$-(\lambda - 1)(\lambda - 2)^2,$$

and an inspection shows that $\lambda = 2$ in both cases have the geometric multiplicity 1. Hence they are both similar with the matrix

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

thus

$$\mathbf{A} \;\; s \;\; \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\; s \;\; \mathbf{B}.$$

Since similarity is an equivalence relation, we conclude that $\mathbf{A}$ and $\mathbf{B}$ are similar.

5. The matrix $\mathbf{C}$ is symmetric and it has the characteristic polynomial

$$\begin{aligned} \det(\mathbf{C} - \lambda\mathbf{I}) &= \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 4-\lambda & 0 \\ 1 & 0 & 4-\lambda \end{vmatrix} \\ &= \begin{vmatrix} 1 & 4-\lambda \\ 1 & 0 \end{vmatrix} + (4-\lambda) \begin{vmatrix} 2-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix} \\ &= (4-\lambda)\{-1 + (\lambda-2)(\lambda-4) - 1\} \\ &= -(\lambda-4)\left\{\lambda^2 - 6\lambda + 6\right\} = -(\lambda-4)\left\{(\lambda-3)^3 - 3\right\}. \end{aligned}$$

The roots $\lambda = 4$ and $\lambda = 3 \pm \sqrt{3}$ are all positive, so $\mathbf{C}$ is positive definite.

Index