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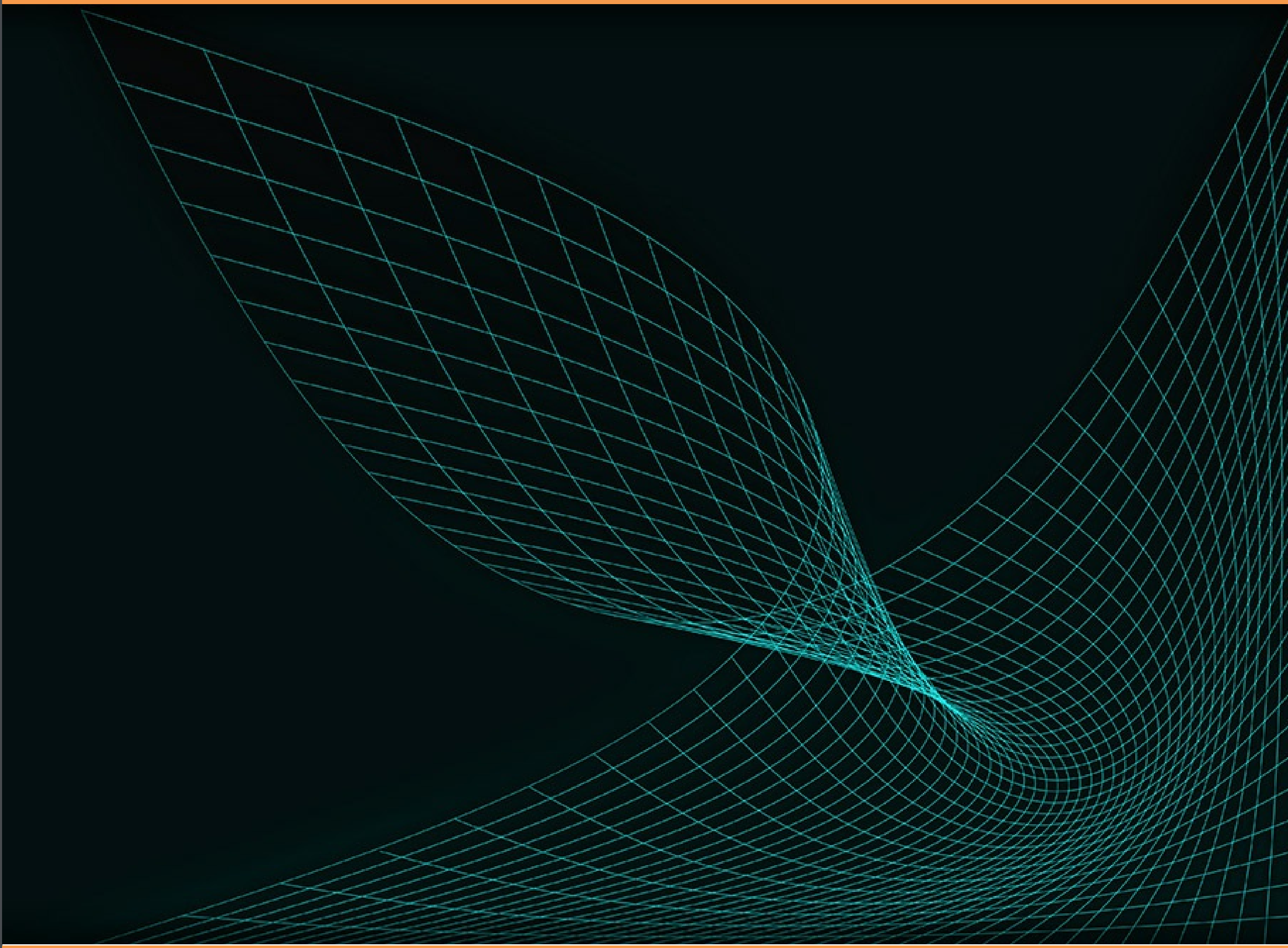
Spectral Theory

Functional Analysis Examples c-4

Leif Mejlbro



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Example 1.1 *Define, for $h \in \mathbb{R}$, the operator τ_h on $L^2(\mathbb{R})$ by*

$$\tau_h f(x) = f(x-h).$$

Show that τ_h is bounded.

Obviously, τ_h is linear, and it follows from

$$\|\tau_h f\|_2^2 = \int_{-\infty}^{+\infty} |f(x-h)|^2 dx = \int_{-\infty}^{+\infty} |f(x)|^2 dx = \|f\|_2^2,$$

that $\|Tf\|_2 = \|f\|_2$ for all $f \in L^2(\mathbb{R})$, hence $\|T\| = 1$.

Remark 1.1 Here we add that τ_h is also regular. In fact, if $\tau_h f = 0$, then $f(x-h) = 0$ for all $x \in \mathbb{R}$, thus $f \equiv 0$. This shows that τ_h is injective, hence the inverse operator exists. Then we get by the change of variable $y = x - h$, i.e. $x = y + h$, that $\tau_h f(x+h) = f(x)$, and we infer that

$$(\tau_h)^{-1} f(x) = f(x+h) = \tau_{-h} f(x),$$

so also $\|(\tau_h)^{-1}\| = 1$, and we have proved that τ_h is regular for every $h \in \mathbb{R}$. $\Diamond$

Example 1.2 *Consider in $L^2(\mathbb{R})$ the operator Q defined by*

$$Qf(x) = x\,f(x),$$

with

$$D(Q) = \{f \in L^2(\mathbb{R}) \mid Qf \in L^2(\mathbb{R})\}.$$

Determine $\varrho(Q)$ and $\sigma_p(Q)$.

A qualified guess is that $\varrho(Q) = \mathbb{C} \setminus \mathbb{R}$. Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. We shall prove that $Q_\lambda = Q - \lambda I$ is regulær. Write $\lambda = \xi + i\,\eta$, where ξ, $\eta \in \mathbb{R}$ and $\eta \neq 0$. It follows from the equation

$$Q_\lambda f(x) = Qf(x) - \lambda\,f(x) = (x-\lambda)f(x) = g(x),$$

that

$$Q_\lambda^{-1} g(x) = f(x) = \frac{g(x)}{x-\lambda} = \frac{g(x)}{(x-\xi)+i\,\eta}.$$

It follows for $\eta \neq 0$ that

$$\left|Q_\lambda^{-1} g(x)\right|^2 = \frac{|g(x)|^2}{|(x-\xi)+i\,\eta|^2} \leq \frac{1}{|\eta|^2}\,|g(x)|^2,$$

and we infer that Q_λ^{-1} is defined on all of $L^2(\mathbb{R})$, and

$$\left\|Q_\lambda^{-1}\right\|_2 \leq \frac{1}{|\eta|}\,\|g\|_2.$$

Hence,

$$\left\|Q_\lambda^{-1}\right\| \leq \frac{1}{|\eta|} = \frac{1}{|\text{Im }\lambda|},$$

and we have proved that $\mathbb{C} \setminus \mathbb{R} \subseteqq \varrho(Q)$.

Then let $\lambda \in \mathbb{R}$. As before, Q_λ^{-1} is defined by

$$Q_\lambda^{-1} g(x) = \frac{g(x)}{\lambda - x},$$

only the domain is now given by

$$D\left(Q_\lambda^{-1}\right) = \left\{ g \in L^2(\mathbb{R}) \;\middle|\; \frac{g(x)}{\lambda - x} \in L^2(\mathbb{R}) \right\}.$$

Due to the singularity at $x = \lambda$, the inverse Q_λ^{-1} is not defined in all of $L^2(\mathbb{R})$. However, it is easily seen that the subspace

$$U = \{f \in L^2(\mathbb{R}) \mid \exists\,\varepsilon > 0\,\forall\,x \in [\lambda-\varepsilon, \lambda+\varepsilon] : f(x) = 0\}$$

of $L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, so we conclude from $U \subseteqq D\left(Q_\lambda^{-1}\right)$ that Q_λ^{-1} is densely defined and unbounded, hence $\lambda \in \sigma_c(Q)$ for every $\lambda \in \mathbb{R}$. Utilizing that the splitting of the spectral sets is disjoint, we conclude that

$$\varrho(Q) = \mathbb{C} \setminus \mathbb{R}, \quad \sigma_p(Q) = \emptyset, \quad \sigma_c(Q) = \mathbb{R}, \quad \sigma_r(Q) = \emptyset.$$

Example 1.3 *Let (e_n) denote an orthonormal basis in a Hilbert space H, and consider the operator*

$$T\left(\sum_{k=1}^{\infty} a_k e_k\right) = \sum_{k=1}^{\infty} a_k e_{k+1}.$$

Determine $\|T\|$ and $\sigma(T)$.

It is well-known that T is called the *shift operator.* We first analyze $T_\lambda = T - \lambda\, I$, thus

$$T_\lambda x = T_\lambda\left(\sum_{k=1}^{+\infty} a_k e_k\right) = \sum_{k=1}^{+\infty} a_k e_{k+1} - \sum_{k=1}^{+\infty} \lambda\, a_k e_k = -\lambda\, a_1 e_1 + \sum_{k=2}^{+\infty} \{a_{k-1} - \lambda\, a_k\}\, e_k.$$

Hence, if $T_\lambda x = 0$, then

$$\lambda\, a_1 = 0 \qquad \text{and} \qquad \lambda\, a_k = a_{k-1}, \quad k \geq 2.$$

We have two possibilities:

1) If $\lambda = 0$, then $a_1 = \lambda a_2 = 0$, and $a_{k-1} = \lambda\, a_k = 0$, thus $x = 0$, and $T_0 = T$ is injective, so $\lambda = 0$ is not an eigenvalue.

2) If $\lambda \neq 0$, then $a_1 = 0$ and $a_k = \dfrac{1}{\lambda}\, a_{k-1}$, hence we get by recursion that all $a_k = 0$, which means that $x = 0$. This proves that every T_λ is injective.

Summing up we have proved that T_λ^{-1} exists for every $\lambda \in \mathbb{C}$, så $\sigma_p(T) = \emptyset$.

It follows from

$$\|Tx\|^2 = \left\|T\left(\sum_{k=1}^{+\infty} a_k e_k\right)\right\|^2 = \left\|\sum_{k=1}^{+\infty} a_k e_{k+1}\right\|^2 = \sum_{k=1}^{+\infty} |a_k|^2 = \|x\|^2$$

for all x that $\|T\| = 1$, hence

$$\varrho(T) \supseteqq \{\lambda \in \mathbb{C} \mid |\lambda| > 1\}.$$

Let $\lambda \neq 0$, $|\lambda| < 1$ and

$$y = \sum_{k=1}^{+\infty} b_k e_k \in H.$$

We shall try to solve the equation $T_\lambda x = y$. It follows immediately from the above that

$$-\lambda x_1 = b_1 \qquad \text{and} \qquad x_{k-1} - \lambda\, x_k = b_k, \quad k \geq 2,$$

thus

$$x_1 = -\frac{b_1}{\lambda} \qquad \text{and} \qquad x_k = \frac{1}{\lambda}\, x_{k-1} - \frac{1}{\lambda}\, b_k, \quad k \geq 2,$$

from which e.g. $x_2 = -\dfrac{b_1}{\lambda^2} - \dfrac{b_2}{\lambda}$. Choosing in particular $y = e_1$ we get $x_1 = -\dfrac{1}{\lambda}$, $x_2 = -\dfrac{1}{\lambda^2}$, and in general,

$$x_n = -\frac{1}{\lambda^n}, \qquad n \in \mathbb{N}.$$

From $0 < |\lambda| < 1$ follows that $|x_n| \to +\infty$ for $n \to +\infty$, so the only possible solution is

$$x = \sum_{n=1}^{+\infty} x_n e_n = -\sum_{n=1}^{+\infty} \frac{1}{\lambda^n}\, e_n \notin H,$$

which, however, does *not* belong to H. This shows that

$$e_1 \notin T_\lambda\left(D\left(T_\lambda\right)\right) = T_\lambda(H).$$

Hence we conclude that T_λ^{-1} exists, but it is unbounded, when $0 < |\lambda| < 1$, so

$$\{\lambda \in \mathbb{C} \mid 0 < |\lambda| < 1\} \subsetneqq \sigma(T).$$

The set $\sigma(T)$ is closed, so it follows from $\sigma(T) \cap \varrho(T) = \emptyset$ that

$$\sigma(T) = \{\lambda \in \mathbb{C} \mid |\lambda| \le 1\} \qquad \text{and} \qquad \varrho(T) = \{\lambda \in \mathbb{C} \mid |\lambda| > 1\}.$$

Example 1.4 *Consider in ℓ^2 the operator*

$$(x_1, x_2, x_3, \dots) \mapsto \left(x_1, \frac{1}{2}(x_1+x_2), \frac{1}{4}(x_1+x_2+x_3), \dots, \frac{1}{2^{n-1}}(x_1+x_2+\cdots+x_n), \dots\right).$$

Show that the operator is bounded and not surjective.
Let (e_n) denote an orthonormal basis in a Hilbert space H, and consider the operator

$$T\left(\sum_{k=1}^{\infty} a_k e_k\right) = \sum_{k=2}^{\infty} \sqrt{k}\, a_k e_{k-1}.$$

Determine the spectrum $\sigma(T)$, and find for each eigenvalue the corresponding eigenvectors.

Assume that

$$Tx = \left(x_1, \frac{1}{2}(x_1+x_2), \frac{1}{4}(x_1+x_2+x_3), \dots\right) = (0,0,0,\dots).$$

Then $x_1 = 0$, $\dfrac{1}{2}\, x_2 = 0$, thus $x_2 = 0$, and we get by induction that $x_n = 0$ for all $n \in \mathbb{N}$. It follows that $Tx = 0$ implies that $x = 0$, hence T is injective.

Then we get

$$\|Tx\|_2^2 = \sum_{n=1}^{+\infty} \frac{1}{4^{n-1}}\, |x_1 + x_2 + \cdots + x_n|^2 \le \sum_{n=1}^{+\infty} \frac{1}{4^{n-1}} \sum_{j=1}^{n} n^2\, |x_j|^2 \le \sum_{n=1}^{+\infty} \frac{n^2}{4^{n-1}}\, \|x\|_2^2,$$

from which we conclude that

$$\|T\| \le \sqrt{\sum_{n=1}^{+\infty} \frac{n^2}{4^{n-1}}} < +\infty,$$

and T is bounded.

If

$$y_0 = 0 \qquad \text{and} \qquad y_n = \frac{1}{2^{n-1}}\left(x_1 + x_2 + \cdots + x_n\right),$$

then

$$x_1 + x_2 + \cdots + x_n = 2^{n-1} y_n, \quad \text{thus} \quad x_n = 2^{n-1} y_n - 2^{n-2} y_{n-1}, \quad n \in \mathbb{N}.$$

Choose in particular, $y = \dfrac{1}{n}$, $n \in \mathbb{N}$. Then $(y_n) \in \ell^2$ with $\|y\| = \dfrac{\pi}{\sqrt{6}}$, while

$$x_n = \frac{2^{n-1}}{n} - \frac{2^{n-2}}{n-1} = 2^{n-2} \cdot \frac{n-2}{n(n-1)} \to +\infty,$$

according to the rule of magnitudes. In particular, the necessary condition of convergence of $\sum |x_n|^2$ is not fulfilled. We conclude that T is not surjective, $T\ell^2 \neq \ell^2$, hence T is singular.

Let us first find the point spectrum, i.e. let $\lambda \in \sigma_p(T)$ be an eigenvalue. Then there exists a vector $x \neq 0$, such that $Tx = \lambda\, x$, which can also be written

$$T\left(\sum_{k=1}^{+\infty} x_k e_k\right) = \sum_{k=2}^{+\infty} \sqrt{k} \cdot x_k e_{k-1} = \sum_{k=1}^{+\infty} \sqrt{k-1} \cdot x_{k+1} e_k = \sum_{k=1}^{+\infty} \lambda\, x_k e_k.$$

Then

$$x_{k+1} = \frac{\lambda}{\sqrt{k+1}}\, x_k = \cdots = \frac{\lambda^k}{\sqrt{(k+1)!}} \cdot x_1.$$

Choosing $x_1 = 1$ we see that *if* x is an eigenvector with $x_1 = 1$, *then* x necessarily has the form

$$x = \sum_{k=1}^{+\infty} \frac{\lambda^{k-1}}{\sqrt{k!}}\, e_k.$$

It only remains to check if the constructed x belongs to H. We get

$$\|x\|^2 = \sum_{k=1}^{+\infty} |x_k|^2 = \sum_{k=1}^{+\infty} \frac{\left|\lambda^2\right|^{k-1}}{k!} = \frac{1}{|\lambda|^2}\left\{e^{|\lambda|^2} - 1\right\},$$

because the series is convergent for all $\lambda \in \mathbb{C}$, and the sum function above has a removable singularity for $\lambda = 0$. (Notice that e_1 is an eigenvector corresponding to $\lambda = 0$). We infer that

$$\sigma(T) = \sigma_p(T) = \mathbb{C},$$

and the given linear operator has every complex $\lambda \in \mathbb{C}$ as an eigenvalue.

Example 1.5 *Let* (e_n) *denote an orthonormal basis in a Hilbert space* H. *We define the sequence* $(f_k)_{k\in\mathbb{Z}}$ *by*

$$\begin{aligned} f_0 &= e_1, \\ f_k &= e_{2k+1} \qquad \textit{for } k > 0, \\ f_k &= e_{-2k} \qquad \textit{for } k < 0. \end{aligned}$$

In this way $(f_k)_{k\in\mathbb{Z}}$ *is an orthonormal basis. We define the* double sided shift operator S *by*

$$S\left(\sum_{k=-\infty}^{\infty} a_k f_k\right) = \sum_{k=-\infty}^{\infty} a_k f_{k+1}.$$

Show that S *is a bounded operator and show that* S *has no eigenvalues.*

First notice that

$$\sum_{k=-\infty}^{+\infty} a_k f_k = \sum_{k=0}^{+\infty} a_k e_{2k+1} + \sum_{k=1}^{+\infty} a_{-k} e_{2k},$$

and

$$T\left(\sum_{k=-\infty}^{+\infty} a_k f_k\right) = \sum_{k=-\infty}^{+\infty} a_k f_{k+1} = \sum_{k=-\infty}^{infty} a_{k-1} f_k = \sum_{k=0}^{+\infty} a_{k-1} e_{2k+1} + \sum_{k=1}^{+\infty} a_{-k-1} e_{2k}.$$

From $(f_k)_{k\in\mathbb{Z}}$ being an orthonormal basis follows that

$$\left\|T\left(\sum_{k=-\infty}^{+\infty} a_k f_k\right)\right\|^2 = \left\|\sum_{k=-\infty}^{+\infty} a_k f_{k+1}\right\|^2 = \sum_{k=-\infty}^{+\infty} |a_k|^2 = \left\|\sum_{k=-\infty}^{+\infty} a_k f_k\right\|^2,$$

from which $\|T\| = 1$ and $T \in B(H)$.

Assume that the equation $Tx = \lambda\, x$ is fulfilled. It follows from the above that

$$\lambda\, a_k = a_{k-1} \quad \text{for } k \in \mathbb{N}_0, \quad \text{and} \quad \lambda\, a_{-k} = a_{-k-1} \quad \text{for } k \in \mathbb{N}.$$

If $\lambda = 0$, then $Tx = 0$, and we get from $\|Tx\| = \|x\| = 0$ that $x = 0$, hence $\lambda = 0 \notin \sigma_p(T)$.

If $\lambda \neq 0$, then we get by recursion,

$$a_k = \frac{1}{\lambda^{k+1}}\, a_{-1} \quad \text{for } k \in \mathbb{N}_0, \quad \text{and} \quad a_{-k-1} = \lambda^k a_{-1} \quad \text{for } k \in \mathbb{N}.$$

Thus, if $a_{-1} \neq 0$, then all possible $a_k \neq 0$, and we get

$$\begin{aligned} \sum_{k=-\infty}^{+\infty} |a_k|^2 &= \sum_{k=0}^{+\infty} \frac{1}{\left|\lambda^2\right|^{k+1}}\, |a_{-1}|^2 + \sum_{k=1}^{+\infty} \left|\lambda^2\right|^k \cdot |a_{-1}|^2 \\ &= |a_{-1}|^2 \sum_{k=-\infty}^{+\infty} \left|\lambda^2\right|^k, \end{aligned}$$

which of course is divergent for every $\lambda \in \mathbb{C}$. We conclude that T does not have eigenvalues, hence $\sigma_p(T) = \emptyset$.

Example 1.6 *Define, for $h \in \mathbb{R}_+$, the operator τ_h on $L^2(\mathbb{R})$ by*

$$\tau_h f(x) = f(x-h).$$

Show that τ_h has no eigenvalues and that

$$\sigma\left(\tau_h\right) \subset \{z \in \mathbb{C} \mid |z| = 1\}.$$

(It is in fact true that $\sigma\left(\sigma_h\right) = \{z \in \mathbb{C} \mid |z| = 1\}$.)

Remark 1.2 Note that if $h = 0$, then $\tau_0 = I$, and $\lambda = 1$ is trivially an eigenvalue with all of $L^2(\mathbb{R})$ as its eigenspace. For that reason we assume that $h > 0$. $\Diamond$

It follows from

$$\|\tau_h f\|_2^2 = \int_{-\infty}^{+\infty} |f(x-h)|^2 dx = \int_{-\infty}^{+\infty} |f(x)|^2 dx = \|f\|_2^2,$$

that $\|\tau_h\| = 1$, hence

$$\sigma\left(\tau_h\right) \subseteqq \{z \in \mathbb{C} \mid |z| \leq 1\}.$$

Assume that

$$\tau_h f(x) = f(x-h) = \lambda\, f(x), \qquad \text{where } |\lambda| \leq \|\tau_h\| = 1.$$

If $|\lambda| = 1$, then $|f(x-h)| = |f(x)|$, $h > 0$. Thus the function $|f(x)|$ is periodic of period $h > 0$, hence

$$\|f\|_2^2 = \int_{-\infty}^{+\infty} |f(x)|^2 dx = \sum_{n=-\infty}^{+\infty} \int_0^h |f(x)|^2 dx < +\infty.$$

This is of course only possible, if $\int_0^h |f(x)|^2 dx = 0$, i.e. if $f(x) = 0$ for almost every $x \in [0,h]$, and hence for almost every $x \in \mathbb{R}$. Then x is represented by the zero function, and we infer that no $\lambda \in \mathbb{C}$ satisfying $|\lambda| = 1$ can be an eigenvalue.

It has previously been proven in Example 1.1 that $(\tau_h)^{-1} = \tau_{-h}$. Of course, this can also be proved directly,

$$\tau_{-h}\tau_h f(x) = \tau_{-h} f(x-h) = f(x-h+h) = f(x) = I f(x),$$

and

$$\tau_h \tau_{-h} f(x) = \tau_h f(x+h) = f(x+h-h) = f(x) = I f(x).$$

It is also obvious that $\left\|(\tau_h)^{-1}\right\| = \|\tau_{-h}\| = 1$, and $(\tau_h)^{-1} \in B(H)$. Thus if $|\lambda| < 1$, then

$$(\tau_h - \lambda I)^{-1} = (\tau_h)^{-1} \left(I - \lambda (\tau_h)^{-1}\right)^{-1} \in B(H),$$

because $\left\|\lambda\left(\tau_h\right)^{-1}\right\| = |\lambda| < 1$. Therefore, $\{\lambda \in \mathbb{C} \mid |\lambda| < 1\} \subset \varrho(\tau_h)$, and thus

$$\varrho\left(\tau_h\right) \supseteqq \{\lambda \in \mathbb{C} \mid |\lambda| \neq 1\},$$

which implies that

$$\sigma\left(\tau_h\right) \subseteqq \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}.$$

Finally,

$$\sigma_p(\tau_h) \subseteqq \sigma\left(\tau_h\right) \quad \text{og} \quad \sigma_p\left(\tau_h\right) \cap \{\lambda \in \mathbb{C} \mid |\lambda| = 1\} = \emptyset,$$

from which follows that $\sigma_p\left(\tau_h\right) = \emptyset$.

Example 1.7 *Given below some closed linear operators from ℓ^2 into ℓ^2. Check in each case if the operator is singular.*

1) $T_1 x = (x_2, x_3, \dots)$.

2) $T_2 z = \left(\frac{1}{2}\, x_1, \frac{1}{2^2}\, x_2, \frac{1}{2^3}\, x_3, \dots\right)$.

3) $T_3 x = (0, x_1, x_2, \dots)$.

4) $T_4 x = (0, x_2, x_3, \dots)$.

A linear operator is singular, if at least one of the following three conditions if satisfied:

1) There exists an $f \in D(T) \setminus \{0\}$, such that $Tf = 0$.

2) The inverse T^{-1} exists, and $\overline{D\left(T^{-1}\right)} = \overline{TD(T)} = Y$, while T^{-1} itself is unbounded.

3) The inverse T^{-1} exists, but it is not densely defined in Y, thus $\overline{TD(T)} \neq Y$.

We shall below check these three conditions.

1) It follows by choosing $x = (1, 0, 0, \dots) \neq 0$ that $T_1 x = 0$, hence T_1 is singular of type (1). This means that $0 \in \sigma_p(T_1)$, i.e. 0 is an eigenvalue of T_1.

2) Clearly, $T_2 x = 0$ implies that $x = 0$, so T_2 is injective and the inverse exists. Then we solve the equation $T_2 x = y$, thus

$$T_2 x = \left(\frac{1}{2}\, x_1, \frac{1}{2^2}\, x_2, \frac{1}{2^3}\, x_3, \dots\right) = (y_1, y_2, y_2, \dots) = y.$$

When we identify the coordinates we get $\frac{1}{2^n}\, x_n = y_n$, hence $x_n = 2^n y_n$, and the inverse operator T_2^{-1} is given by

$$T_2^{-1} y = \left(2y_1, 2^2 y_2, 2^3 y_3, \dots\right)$$

for

$$y \in D\left(T_2^{-1}\right) = \left\{ y \in \ell^2 \;\middle|\; \sum_{n=1}^{+\infty} 2^{2n} |y_n|^2 < +\infty \right\} \subset \ell^2.$$

Let U be the subspace consisting of all sequences which are 0 eventually. Then clearly,

$$U \subset D\left(T_2^{-1}\right) \subset \ell^2.$$

The subspace U is dense in ℓ^2, so this is also the case for the larger subspace $D\left(T_2^{-1}\right)$. Furthermore, it follows from the definition of the inverse T_2^{-1} that it is unbounded, i.e. T_2 is singular of type (2). This means that $0 \in \sigma_c\left(T_2\right)$ lies in the continuous spectrum for T_2.

3) It is obvious that T_3 is injective and that

$$T_3^{-1} y = (y_2, y_2, y_4, \dots)$$

for

$$y \in D\left(T_3^{-1}\right))\{y \in \ell^2 \mid y_1 = 0\}.$$

Clearly, T_3^{-1} is bounded, though not densely defined, so T_3 is of type (3), corresponding to that $0 \in \sigma_r(T_3)$ lies in the residual spectrum for T_3.

4) We infer from $T_4 x = 0$ for $x = (1, 0, 0, \dots) \neq 0$ that 0 is an eigenvalue, $0 \in \sigma_p\left(T_4\right)$, hence T_4 is singular of type (1).

Example 1.8 *Let V denote the Banach space $(C([0,1]), \|\cdot\|_\infty)$, and let the operator T be given by*

$$Tf(x) = \int_0^x f(t)\,dt, \qquad f \in V.$$

Check if T is regular.

The inverse operator of T is the differential operator $\mathcal{D}$, given by

$$D(\mathcal{D}) = \{f \in C^1([0,1]) \mid f(0) = 0\},$$

$$\mathcal{D}f = \frac{df}{dx} = f' \qquad \text{for } f \in C^1([0,1]), \quad f(0) = 0.$$

It is easily seen (e.g. by using Weierstraß's Approximation Theorem) that $D(\mathcal{D})$ is dense in V. On the other hand, $\mathcal{D}$ is unbounded. In fact, choose

$$f_n(x) = \sin(\pi n x), \qquad x \in [0,1], \qquad f_n \in D(\mathcal{D}).$$

then

$$\mathcal{D}f_n(x) = \pi\, n \cdot \cos(\pi n x), \qquad x \in [0,1],$$

hence $\|f_n\|_\infty = 1$ and $\|\mathcal{D}f_n\|_\infty = \pi n$.

Remark 1.3 A simpler example is of course $g_n(x) = x^n$, $x \in [0,1]$. However, the f_n occur very frequently as an example in other cases, so we have chosen to present it here. $\Diamond$

We have proved that T is singular of type (2), i.e. $0 \in \sigma_c(T)$ lies in the continuous spectrum for T.

Example 1.9 *Let $H = L^2(\mathbb{R})$, and let g be a bounded continuous real function defined on $\mathbb{R}$. Prove that the operator T given by*

$$Tf(x) = g(x)f(x), \qquad f \in L^2(\mathbb{R}),$$

belongs to $B(H)$.
Find a necessary and sufficient condition on g that T is regular.

When g is bounded, $\|g\|_\infty < +\infty$, then

$$\|Tf\|_2^2 = \int_{-\infty}^{+\infty} g(x)^2 |f(x)|^2 dx \le \|g\|_\infty^2 \int_{-\infty}^{+\infty} |f(x)|^2 dx = \|g\|_\infty^2 \cdot \|f\|_2^2,$$

hence $\|Tf\|_2 \le \|g\|_\infty \cdot \|f\|_2$ for all $f \in H$, and we infer that $T \in B(H)$ with $\|T\| \le \|g\|_\infty$.

Then we shall find when T is regular, i.e. when T fulfils the following three conditions:

1) The equation $Tf = 0$ has only the trivial solution $f = 0$, so T^{-1} exists.
2) The inverse operator T^{-1} is densely defined, i.e.

   $$D\left(T^{-1}\right) = T\left(L^2(\mathbb{R})\right)$$

   is dense in $L^2(\mathbb{R})$.
3) The inverse operator T^{-1} is bounded.

We now check each of these conditions:

1) It follows from $Tf(x) = g(x) \cdot f(x)$ that $Tf = 0$, if and only if $g(x) \cdot f(x) = 0$ for almost every $x \in \mathbb{R}$. Therefore, if we want always to conclude that $f = 0$ (in $L^2(\mathbb{R})$), then we must assume that $g(x) \neq 0$ for almost every $x \in \mathbb{R}$.
2) Then we want that T^{-1} is bounded. It follows from $Tf(x) = g(x)f(x) = h(x)$ that

   $$f(x) = T^{-1}h(x) = \frac{1}{g(x)}\, h(x),$$

   and then the same consideration as above shows that we must require that

   $$\left\|\frac{1}{g}\right\|_\infty < +\infty.$$

3) Based on the conditions above, assume that

   $$0 < b \le |g(x)| < a < +\infty, \qquad \text{for all } x \in \mathbb{R}.$$

   Then clearly all three conditions are fulfilled, so these conditions are sufficient that both T and $T^{-1} \in B(H)$.

Example 1.10 *Let* (e_k) *denote an orthonormal basis in a Hilbert space* H, *and let the operator* T *be defined by*

$$T\left(\sum_{k=1}^{+\infty} a_k e_k\right) = \sum_{k=2}^{+\infty} a_k e_{k-1}.$$

Prove that λ *is an eigenvalue for* T, *if and only if* $|\lambda| < 1$.
Find $\sigma(T)$ *and* $\varrho(T)$.

Assume that $\lambda \in \sigma_p(T)$, thus there exists

$$x = \sum_{k=1}^{+\infty} x_k e_k, \qquad \text{where} \qquad 0 < \sum_{k=1}^{+\infty} |x_k|^2 < +\infty,$$

such that $Tx = \lambda x$, i.e.

$$\sum_{k=2}^{+\infty} x_k e_{k-1} = \sum_{k=1}^{+\infty} x_{k+1} e_k = \lambda \sum_{k=1}^{+\infty} x_k e_k.$$

When we identify the coordinates we get

$$x_{k+1} = \lambda x_k, \qquad k \in \mathbb{N}.$$

Choosing $x_1 = 1$, we get by either induction or by recursion – both methods can be applied – that $x_k = \lambda^{k-1}$, and an eigenvector corresponding to the eigenvalue λ must *necessarily* be of the form

$$x = x_1 \sum_{k=1}^{+\infty} \lambda^{k-1} e_k.$$

This *candidate* belongs to the Hilbert space, if and only if

$$\sum_{k=1}^{+\infty} \left|\lambda^{k-1}\right|^2 = \sum_{k=0}^{+\infty} |\lambda|^{2k} < +\infty,$$

i.e. if and only if $|\lambda| < 1$. We infer that

$$\sigma_p(T) \subseteqq \{\lambda \in \mathbb{C} \mid |\lambda| < 1\}.$$

If on the other hand $\lambda \in \mathbb{C}$ satisfies $|\lambda| < 1$, then we get by insertion that $x = \sum_{k=1}^{+\infty} \lambda^{k-1} e_k$ is an eigenvector, so $\lambda \in \sigma_p(T)$, and we have proved that

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\}.$$

Then assume that $\lambda \in \mathbb{C}$ satisfies $|\lambda| > 1$. We shall prove that $\lambda \in \varrho(T)$.

Remark 1.4 If here one tries directly to find the inverse operator T_λ^{-1}, thus try to solve the equation $T_\lambda x = y$ with respect to $x \in H$ for given $yn \in H$, then we end up with an unpleasant infinite system of equations of the form

$$(1) \quad x_{k+1} - \lambda\, x_k = y_k, \qquad k \in \mathbb{N},$$

where the solution also must satisfy

$$\sum_{k=1}^{+\infty} |x_k|^2 < +\infty.$$

Even this is possible, it is very difficult to solve this system of equations. Hence we search an alternative method of solution. $\Diamond$

We note that

$$\|Tx\| = \left\| \sum_{k=1}^{+\infty} x_{k+1} e_k \right\| \le \|x\|,$$

where we get equality, when $x_1 = 0$. This shows that $\|T\| = 1$.

It follows from

$$T_\lambda = T - \lambda\, I = -\lambda\, I \left(I - \frac{1}{\lambda}\, T \right), \qquad |\lambda| > 1,$$

and

$$\left\| \frac{1}{\lambda} T \right\| = \frac{1}{|\lambda|} < 1,$$

by using the Neumann series that

$$T_\lambda^{-1} = -\frac{1}{\lambda} \left(I - \frac{1}{\lambda} T \right)^{-1} \in B(H).$$

Remark 1.5 The explicit solution is given by the Neumann series

$$x = T_\lambda^{-1} y = -\frac{1}{\lambda} \sum_{j=0}^{+\infty} \frac{1}{\lambda^j} T^j y,$$

which can also be found directly, if we work on (1). However, the precise solution is not so interesting in this connection. $\Diamond$

We infer that

$$\{\lambda \in \mathbb{C} \mid |\lambda| > 1\} \subseteqq \varrho(T).$$

Now, $\sigma(T)$ is *closed* and disjoint from $\varrho(T)$, and

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\} \subseteqq \sigma(T),$$

hence

$$\sigma(T) = \{\lambda \in \mathbb{C} \mid |\lambda| \le 1\} \qquad \text{og} \qquad \varrho(T) = \{\lambda \in \mathbb{C} \mid |\lambda| > 1\}.$$

Example 1.11 *Consider the Banach space $(C([0,1]), \|\cdot\|_\infty)$. Let $v \in C([0,1])$ be real, and let the operator T be defined by*

$$Tf(x) = v(x) f(x).$$

Find $\sigma(T)$ and $\varrho(T)$.

We conclude from

$$\|Tf\|_\infty = \|v(x)f(x)\|_\infty \le \|v\|_\infty \|f\|_\infty,$$

where we get equality by choosing $f = v$, that $\|T\| = \|v\|_\infty$. Then it follows that

$$\sigma(T) \subseteqq \{\lambda \in \mathbb{C} \mid |\lambda| \le \|v\|_\infty\}.$$

Now, v is continuous, and $[0,1]$ is compact, hence $v([0,1])$ is also compact. Let $\lambda \notin v([0,1])$. Then there exists a $b_\lambda > 0$, such that

$$|v(x) - \lambda| \ge b_\lambda \qquad \text{for all } x \in [0,1].$$

Then

$$T_\lambda f(x) = \{v(x) - \lambda\} f(x) = g(x) \in C([0,1])$$

for

$$f(x) = T_\lambda^{-1} g(x) = \frac{g(x)}{v(x) - \lambda} \in C([0,1]).$$

It follows that $\|T_\lambda\| \le \dfrac{1}{b_\lambda}$, hence $T_\lambda \in B(C([0,1]))$, and

$$\varrho(T) \supseteqq \mathbb{C} \setminus v([0,1]) \quad \text{and} \quad \sigma(T) \subseteqq v([0,1]).$$

If conversely $\lambda \in v([0,1])$, then there exists an $x_0 \in [0,1]$, such that $v(x_0) = \lambda$. Then the equation $T_\lambda f = g$ cannot be solved for any g, for which $f(x_0) \neq 0$, because then the candidate f then will not be continuous at x_0. Hence we finally get

$$\sigma(T) = v([0,1]) \qquad \text{and} \qquad \varrho(T) = \mathbb{C} \setminus v([0,1]).$$

Example 1.12 *Consider in the Banach space ℓ^∞ the operator T given by*

$$T(x_1, x_2, \dots) = (x_2, x_3, \dots).$$

Find $\varrho(T)$, $\sigma_p(T)$, $\sigma_c(T)$ and $\sigma_r(T)$.

We get from $\|Tx\|_\infty \le \|x\|_\infty$ with equality for

$$|x_1| \le \sup_{i \ge 2} |x_i|,$$

that $\|T\| = 1$, hence $\sigma(T) \subseteqq \{\lambda \in \mathbb{C} \mid |\lambda| \le 1\}$.

Therefore, if $\lambda \in \sigma_p(T)$, then $|\lambda| \le 1$, and there exists an $x \neq 0$, such that $Tx = \lambda\, x$, i.e.

$$x_{k+1} = \lambda\, x_k = \cdots = \lambda k x_1.$$

We can therefor put $x_1 = 1$ for an eigenvector, and thus any eigenvector has the form of a constant times

$$\left(1, \lambda, \lambda^2, \dots, \lambda^{n-1}, \dots\right).$$

It follows by insertion that this candidate indeed is an eigenvector, *if it belongs to ℓ^∞*, i.e. if $|\lambda| \le 1$. We conclude that

$$\sigma_p(T) = \sigma(T) = \{\lambda \in \mathbb{C} \mid |\lambda| \le 1\},$$

and

$$\varrho(T) = \{\lambda \in \mathbb{C} \mid |\lambda| > 1\},$$

and $\sigma_c(T) = \sigma_r(T) = \emptyset$.

Example 1.13 *Let $T : \ell^2 \to \ell^2$ denote the operator*

$$T\left(x_1, x_2, \ldots, x_n, \ldots\right) = \left(x_2, x_4, \ldots, x_{2n}, \ldots\right).$$

Find $\|T\|$.
Find all eigenvalues for T.
Show that the eigenspace corresponding to any eigenvalue is infinite dimensional.
Determine the operators $T^\star$, $TT^\star$ and $T^\star T$.
Determine $\sigma(T)$ and $\varrho(T)$.

1) We infer from

$$\|Tx\|^2 = \sum_{n=1}^{+\infty} \left|x_{2n}\right|^2 \le \sum_{n=1}^{+\infty} \left|x_n\right|^2 = \|x\|^2$$

for every $x \in \ell^2$ that $\|T\| \le 1$.

For $x = (0, x_2, 0, x_4, 0, x_6, 0, \ldots)$ we get in particular that

$$\|Tx\|^2 = \|T(0, x_2, 0, x_4, 0, x_6, \ldots)\|^2 = \sum_{n=1}^{+\infty} \left|x_{2n}\right|^2 = \sum_{n=1}^{+\infty} \left|x_n\right|^2 = \|x\|^2,$$

and we conclude that $\|T\| = 1$.

2) Assume that $\lambda \in \sigma_p(T)$. Then there exists an $x \in \ell^2 \setminus \{0\}$, such that $Tx = \lambda\, x$. We get for the n-th coordinate of this equation that

(2) $x_{2n} = \lambda\, x_n, \qquad n \in \mathbb{N}.$

If $\lambda = 0$, then we get the conditions $x_{2n} = 0$, $n \in \mathbb{N}$. It follows that if

$$\sum_{n=0}^{+\infty} |x_{2n+1}|^2 < +\infty,$$

then $(x_1, 0, x_3, 0, x_5, 0, \dots)$ is an eigenvector corresponding to the eigenvalue $\lambda = 0$, hence $0 \in \sigma_p(T)$, and the eigenspace corresponding to $\lambda = 0$ is spanned by $\{e_{2n-1} \mid n \in \mathbb{N}\}$, hence it is infinite dimensional, cf. the third question.

Assume that $\lambda \in \sigma_p(T) \setminus \{0\}$. Then it follows from (2) with $n = 2^{m-1}q$ that

$$x_{2^m q} = \lambda\, x_{x^{n-1}q} = \lambda^2\, x_{2^{m-2}q} = \cdots = \lambda^m\, x_q, \qquad m \in \mathbb{N}.$$

We get in particular for $q = 1$,

$$x_{2^m} = \lambda^m\, x_1.$$

If we put $x_1 = 1$ and $x^r = 0$, when r is not of the form 2^n, we get an eigenvector, if and only if

$$\sum_{n=0}^{+\infty} |\lambda^n|^2 < +\infty.$$

This condition is fulfilled if and only if $|\lambda| < 1$. Hence we conclude that the point spectrum is given by

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\}.$$

3) Assume that $\lambda \in \sigma_p(T)$, so $|\lambda| < 1$. Then we get by a simple computation that every odd index $2q+1$, $q \in \mathbb{N}_0$, determines an eigenvector x by

$$x_{(2q+1)\cdot 2^n} = \lambda^n, \quad n \in \mathbb{N}_0, \qquad \text{og} \qquad x_r = 0 \text{ otherwise.}$$

All these eigenvectors are linearly independent, so we conclude that the eigenspace corresponding to an eigenvalue $\lambda \in \sigma_p(T)$ is infinite dimensional.

4) Now, $T \in B\left(\ell^2\right)$, and $\|T\| = 1$, so $T^\star \in B\left(\ell^2\right)$ and $\|T^\star\| = 1$.

We have for every $x \in \ell^2$ and every $y \in \ell^2$ that

$$\begin{aligned} (Tx, y) &= ((x_2, x_4, x_6, \dots), (y_1, y_2, y_3, \dots)) = \sum_{n=1}^{+\infty} x_{2n}\, \overline{y_n} \\ &= ((0, x_2, 0, x_4, 0, \dots), (0, y_1, 0, y_2, 0, \dots)) = (x, T^\star y), \end{aligned}$$

so we infer that

$$T^\star y = T^\star(y_1, y_2, y_3, \dots) = (0, y_1, 0, y_2, 0, y_3, \dots), \qquad y \in \ell^2.$$

Furthermore,

$$TT^\star x = T\left(T^\star(x_1, x_2, x_3, \dots)\right) = T(0, x_1, 0, x_2, 0, x_3, \dots) = (x_1, x_2, x_3, \dots) = x,$$

i.e. $TT^\star = I$, and

$$T^\star T x = T^\star\left(T(x_1, x_2, x_3, \dots)\right) = T^\star(x_2, x_4, x_6, \dots) = (0, x_2, 0, x_4, 0, x_6, \dots),$$

proving that $T^\star T = P$ is the projection onto the subspace of ℓ^2 which is spanned by $\{e_{2n} \mid n \in \mathbb{N}\}$.

5) It follows from $\|T| \ = 1$ that

$$\sigma(T) \subseteqq \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|T\|\} = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}.$$

Furthermore,

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\} \subseteqq \sigma(T) \subseteqq \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\},$$

and the spectrum is closed, hence

$$\sigma(T) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\} \quad \text{og} \quad \varrho(T) = \mathbb{C} \setminus \sigma(T) = \{\lambda \in \mathbb{C} \mid |\lambda| > 1\}.$$

Remark 1.6 It is also easy to prove that

$$\sigma_p\left(T^\star\right) = \emptyset.$$

In fact, we get from $T^\star y = \lambda\, y$ that

$$(0, y_1, 0, y_2, 0, y_3, \dots) = \lambda\,(y_1, y_2, y_3, y_4, y_5, y_6, \dots).$$

If $\lambda = 0$, then the right hand side is 0, and this implies that $y_n = 0$, thus $y = 0$, and $0 \notin \sigma_p\left(T^\star\right)$. If $\lambda \neq 0$, then

$$0 = \lambda\, y_{2n+1}, \quad n \in \mathbb{N}_0, \quad \text{and} \quad y_n = \lambda\, y_{2n}, \quad n \in \mathbb{N}.$$

The former equation gives $y_{2n+1} = 0$, which is then inserted into the latter (follows by an iteration, when n is even) to give $y_{2n} = 0$, hence $y = 0$, and we have proved that $\sigma_p\left(T^\star\right) = \emptyset$.

Now, $\sigma_p\left(T^\star\right) =$, hence also $\sigma_r(T) = \emptyset$. Since $\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$ is a disjoint splitting of the spectrum, we conclude that

$$\begin{aligned}
\varrho(T) &= \{\lambda \in \mathbb{C} \mid |\lambda| > 1\},\\
\sigma(T) &= \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\},\\
\sigma_p(T) &= \{\lambda \in \mathbb{C} \mid |\lambda| < 1\},\\
\sigma_c(T) &= \{\lambda \in \mathbb{C} \mid |\lambda| = 1\},\\
\sigma_r(T) &= \emptyset. \qquad \Diamond
\end{aligned}$$

Example 1.14 *Let X denote the Banach space of $C([-1,1])$-functions equipped with the usual* sup-*norm $\|\cdot\|_\infty$, and let $T \in B(X)$ be given by*

$$Tf = f(0) + f.$$

1) Find the norm of T.

2) Determine the resolvent set $\varrho(T)$ for T and find

$$T_\lambda^{-1} = (T - \lambda I)^{-1}$$

for all $\lambda \in \varrho(T)$.

3) Show that the spectrum for T is a pure point spectrum and find all eigenvalues and corresponding eigenfunctions.

4) Show that all $f \in X$ can be written as a sum of eigenfunctions belonging to different eigenspaces, and show that this decomposition is unique.

1) Clearly,

$$\|Tf\|_\infty \le |f(0)| + \|f\|_\infty \le \|f\|_\infty + \|f\|_\infty = 2\|f\|_\infty,$$

where we obtain equality if e.g. f is a real function with maximum at 0, i.e. $\|T\| = 2$.

2) Then we shall check when it is possible for all $g \in X$ to solve the equation

$$(T - \lambda I)g = g, \qquad f \in X.$$

We get

(3) $g(x) = Tf(x) - \lambda\, f(x) = f(0) + f(x) - \lambda\, f(x).$

In particular for $x = 0$,

$$g(0) = f(0) + f(0) - \lambda\, f(0) = (2 - \lambda)\, f(0).$$

Now, the solution f must be continuous, so this equation *cannot* be solved for arbitrary $g \in X$, when $\lambda = 2$, hence $2 \in \sigma(T)$.

If $\lambda \ne 2$, then

$$f(0) = \frac{1}{2-\lambda}\, g(0),$$

which gives by insertion into (3),

$$g(x) - \frac{1}{2-\lambda}\, g(0) = (1 - \lambda)\, f(x).$$

Hence, if $\lambda = 1$, then this equation *cannot* be solved for an arbitrary $g \in X$, so $1 \in \sigma(T)$. If we assume that $\lambda \ne 1$, then we get the candidate of the solution

$$f(x) = T_\lambda^{-1} g(x) = \frac{1}{1-\lambda}\, g(x) - \frac{1}{(1-\lambda)(2-\lambda)}\, g(0),$$

which is clearly continuous, when g is continuous. Finally,

$$\left\|T_\lambda^{-1} g\right\|_\infty \le \left\{\frac{1}{|1-\lambda|}+\frac{1}{|1-\lambda|\cdot|2-\lambda|}\right\}\|g\|_\infty = C(\lambda)\,\|g\|_\infty .$$

This implies that $\varrho(T) \supseteqq \mathbb{C}\setminus\{1,2\}$, and because we have proved above that $\{1,2\} \subseteqq \sigma(T)$, it follows that

$$\varrho(T) = \mathbb{C}\setminus\{1,2\} \qquad \text{and} \qquad \sigma(T) = \{1,2\}.$$

3) Here we shall prove that $\lambda = 1$ and $\lambda = 2$ are eigenvalues, i.e. we shall prove that the equation

$$Tf = f(0) + f(x) = \lambda\, f(x)$$

has non-trivial solutions for $\lambda = 1$ and $\lambda = 2$.

If $\lambda = 1$, then a check gives

$$f(0) + f(x) = f(x),$$

and the condition becomes $f(0) = 0$. Any function $f \in C([-1,1])$, for which $f(0) = 0$, is therefore an eigenfunction corresponding to the eigenvalue $\lambda = 1$.

If $\lambda = 2$, then

$$f(0) + f(x) = 2\, f(x),$$

and we get the condition $f(x) = f(0)$ for all $x \in [-1,1]$. This shows that every constant function $f(x) = c$ is an eigenfunction corresponding to the eigenvalue $\lambda = 2$, and we have proved that

$$\sigma(T) = \{1,2\} = \sigma_p(T).$$

4) Let $f \in C([-1,1])$. Then we have the following splitting of f,

$$f(x) = \{f(x) - f(0)\} + f(0) = g(x) + h(x),$$

where $g(x) = f(x) - f(0)$ satisfies $g(0) = 0$, so g belongs to the eigenspace corresponding to $\lambda = 1$, and where $h(x) = f(0)$ is constant, hence $h(x)$ belongs to the eigenspace of the eigenvalue. This proves the existence.

If conversely

$$f(x) = g(x) + h(x)$$

is such a splitting, then

$$Tf(x) = f(x) + f(0) = Tg(x) + Th(x) = g(x) + 2h(x),$$

and we get the two equations

$$\left\{ \begin{array}{ccccc} g(x) & + & 2h(x) & = & f(x) + f(0), \\ g(x) & + & h(x) & = & f(x), \end{array} \right.$$

from which we get $h(x) = f(0)$ by subtraction, and then

$$g(x) = f(x) - h(x) = f(x) - f(0),$$

and we have proved the uniqueness.

Example 1.15 *Let H denote a Hilbert space and let $T \in B(H)$. Assume that we have for some $m \in \mathbb{N}$ that $T^m = 0$.*
Show that

$$(I - \lambda T)^{-1} = \sum_{n=0}^{m-1} \lambda^n T^n \in B(H),$$

and deduce that $\mathbb{C} \setminus \{0\} \subset \varrho(T)$.
Show next that $\sigma(T) = \sigma_p(T) = \{0\}$.

We have $T^m = 0$, and

$$\begin{aligned} (I - \lambda T) \sum_{n=0}^{m-1} \lambda^n T^n &= \sum_{n=0}^{m-1} \lambda^n T^n - \sum_{n=0}^{m-1} \lambda^{n+1} T^{n+1} = I + \sum_{n=1}^{m-1} \lambda^n T^n - \sum_{n=1}^{m} \lambda^n T^n \\ &= I - \lambda^m T^m = I, \end{aligned}$$

and analogously because T is defined everywhere,

$$\sum_{n=0}^{m-1} \lambda^n T^n (I - \lambda T) = I.$$

We therefore conclude that

$$\sum_{n=0}^{m-1} \lambda^n T^n = I + \sum_{n_1}^{m-1} \lambda^n T^n = (I - \lambda T)^{-1} \qquad \text{for every } \lambda \in \mathbb{C}.$$

If $\mu \neq 0$, then

$$(T - \mu I)^{-1} = -\frac{1}{\mu}\left(I - \frac{1}{\mu} T\right)^{-1} = -\frac{1}{\mu} \sum_{n=0}^{m-1} \frac{1}{\mu^n} T^n \in B(H),$$

proving that $\varrho(T) \supseteqq \mathbb{C} \setminus \{0\}$.

Clearly, $T^m = 0$ implies that $T^m f = T\left(T^{m-1} f\right) = 0$ for every $f \in H$. Hence if $T^{m-1} f \neq 0$ for some $f \in H$, then $T^{m-1} f$ is an eigenvector for T, corresponding to $\lambda = 0$.

First find the smallest $m \in \mathbb{N}$, such that $T^m = 0$ and $T^{m-1} \neq 0$. It follows from this that

$$\sigma(T) = \sigma_p(T) = \{0\},$$

and hence

$$\varrho(T) = \mathbb{C} \setminus \{0\}.$$

Example 1.16 *Let E be a Banach space and let $P \in B(E)$ satisfy $P^2 = P$.*

1) Show that $P - \lambda I$ is injective for $\lambda \in \mathbb{C} \setminus \{0, 1\}$.

2) Show that $P - \lambda I$ is surjective for $\lambda \in \mathbb{C} \setminus \{0, 1\}$, and find $(P - \lambda I)^{-1}$.

3) Show that $\sigma(P) = \sigma_p(P) = \{0, 1\}$.

Remark 1.7 The latter claim of the example is not true, if $P = 0$ or I. In fact, it is well-known that

$$\sigma(0) = \sigma_p(0) = \{0\} \quad \text{and} \quad \sigma(I) = \sigma_p(I) = \{1\},$$

and it is obvious that both $0^2 = 0$ and $I^2 = I$. Of a similar reason we must assume in (2) that $\lambda \notin \{0, 1\}$, while (1) also holds for 0 and I. $\Diamond$

1) Let $\lambda \in \mathbb{C} \setminus \{0, 1\}$, and assume that

$$(P - \lambda I)x = Px - \lambda x = 0,$$

i.e. $Px = \lambda x$. Then also

$$Px = P^2 x = \lambda Px.$$

Because $\lambda \neq 1$, we must have $Px = 0$, and since also $\lambda \neq 0$, we get

$$x = \frac{1}{\lambda} Px = 0,$$

and we have proved that $P - \lambda I$ is injective.

2) Let again $\lambda \in \mathbb{C} \setminus \{0,1\}$. Because $P^2 = P$, the formal Neumann series for $(P - \lambda I)^{-1}$ can in principle be reduced to $\mu P - \frac{1}{\lambda} I$, where we shall find μ and then prove that this is indeed the inverse operator. A check gives

$$\begin{aligned}\left(\mu P - \frac{1}{\lambda} I\right)(P - \lambda I) &= (P - \lambda I)\left(\mu P - \frac{1}{\lambda} I\right) = I + \mu P^2 - \lambda\mu P - \frac{1}{\lambda} P \\ &= I + \left\{\mu - \lambda\mu - \frac{1}{\lambda}\right\} P = I + \left\{\mu(1-\lambda) - \frac{1}{\lambda}\right\} P.\end{aligned}$$

Choosing $\mu = \dfrac{1}{\lambda(1-\lambda)}$ we get that the inverse operator is given by

$$(P - \lambda I)^{-1} = \frac{1}{\lambda(1-\lambda} P - \frac{1}{\lambda} I \in B(E)$$

and that in particular, $P - \lambda I$ is surjective.

3) It follows from (2) that $\varrho(P) \supseteq \mathbb{C} \setminus \{0,1\}$, hence $\sigma(P) \subseteq \{0,1\}$. We have also assumed that $P \neq 0$ and $P \neq I$, hence

$$\{x \in M \mid Px = 0\} \neq \{0\},\, M,$$

and

$$\{x \in M \mid Px = x\} \neq \{0\},\, M,$$

are the eigenspaces corresponding to $\lambda = 0$ and $\lambda = 1$, respectively, hence

$$\sigma(P) = \sigma_p(P) = \{0,1\}.$$

2 The adjoint of a bounded operator

Example 2.1 *Let $T \in B(H)$ where H is a Hilbert (or just Banach) space. Show that $\|R_\lambda(T)\| \to 0$ for $|\lambda| \to \infty$.*

Since $T \in B(H)$, we see that $R_\lambda(T) = (T - \lambda I)^{-1}$ exists for every $\lambda \in \mathbb{C}$, for which $|\lambda| > \|T\|$. Then by the Neumann series,

$$R_\lambda(T) = (T - \lambda I)^{-1} = -\frac{1}{\lambda}\left(I - \frac{1}{\lambda}T\right)^{-1} = -\frac{1}{\lambda}\sum_{n=0}^{+\infty}\frac{1}{\lambda^n}T^n.$$

We get the estimate

$$\|R_\lambda(T)\| \le \frac{1}{|\lambda|}\sum_{n=0}^{+\infty}\left\{\frac{\|T\|}{|\lambda|}\right\}^n = \frac{1}{|\lambda|}\cdot\frac{1}{1-\dfrac{\|T\|}{|\lambda|}} \to 0 \qquad \text{for } |\lambda| \to +\infty,$$

and the claim is proved.

Example 2.2 *Let T be a self adjoint operator in a Hilbert space H. Show that if $D(T) = H$, then T is bounded.*

When T is self adjoint, then T is closed, and since $D(T) = H$ is closed, it follows from the *Closed Graph Theorem* that T is bounded.

Example 2.3 *Let T be a bounded operator on a Hilbert space H and assume that N and M are closed subspaces of H. Show that*

$$T(M) \subset N \qquad \textit{if and only if} \qquad T^\star\left(N^\perp\right) \subset M^\perp.$$

Show moreover that

$$\ker(T) = T^\star(H)^\perp \qquad \textit{and} \qquad \ker(T)^\perp = \overline{T^\star(H)}.$$

We assume that $T(M) \subseteq N$, and we shall prove that $T^\star\left(N^\perp\right) \subseteq M^\perp$.

Let $x \in M$ and $y \in N^\perp$. By the assumption, $Tx \in N$, thus

$$0 = (Tx, y) = (x, T^\star y).$$

Now, $x \in M$ was arbitrary, so it follows that $T^\star y \in M^\perp$. This holds for every $y \in N^\perp$, hence

$$T^\star\left(N^\perp\right) \subseteq M^\perp.$$

Then by iteration, $T^{\star\star}\left(M^{\perp\perp}\right) \subseteq N^{\perp\perp}$. However, $T^{\star\star} = T$ and $M^{\perp\perp} = M$, and $N^{\perp\perp} = N$, so we conclude that

$$T(M) \subseteq N \qquad \text{if and only if} \qquad T^\star\left(N^\perp\right) \subseteq M^\perp.$$

If $x \in \ker(T)$, then $Tx = 0$, and $\ker(T)$ is a closed subspace. Then put $M = \ker(T)$ and $N = \{0\}$, and it follows from the above that

$$T^\star\left(N^\perp\right) = T^\star(H) \subseteq \ker(T)^\perp, \qquad \text{thus} \qquad \{T^\star(H)\}^\perp \supseteq \ker(T).$$

If conversely $x \in \{T^\star(H)\}^\perp$, then for every $y \in H$,

$$0 = (x, T^\star y) = (Tx, y),$$

so $Tx = 0$, and we have $x \in \ker(T)$. We have proved that

$$\ker(T) = \{T^\star(H)\}^\perp.$$

Finally, it follows from this equation that

$$\ker(T)^\perp = \{T^\star(H)\}^{\perp\perp} = \overline{T^\star(H)},$$

where the bar means the closure of the set.

Example 2.4 *Let T be a bounded operator on a Hilbert space H with $\|T\| = 1$, and assume that we can find $x_0 \in H$ such that $Tx_0 = x_0$. Show that also $T^\star x_0 = x_0$.*

First we get

$$\begin{aligned}
0 &\leq \left\|T^\star x_0 - x_0\right\|^2 = (T^\star x_0 - x_0, T^\star x_0 - x_0) \\
&= (T^\star x_0, T^\star x_0) - (x_0, T^\star x_0) - (T^\star x_0, x_0) + (x_0, x_0) \\
&= \left\|T^\star x_0\right\|^2 - (Tx_0, x_0) - (x_0, Tx_0) + \left\|x_0\right\|^2 \\
&= \left\|T^\star x_0\right\|^2 - (x_0, x_0) - (x_0, x_0) + \left\|x_0\right\|^2 \\
&= \left\|T^\star x_0\right\|^2 - \left\|x_0\right\|^2,
\end{aligned}$$

from which $\|T^\star x_0\|^2 \geq \|x_0\|^2$, or

$$\|x_0\| \leq \|T^\star x_0\| \leq \|T^\star\| \cdot \|x_0\| = \|T\| \cdot \|x_0\| = \|x_0\| .$$

Thus we must have equality everywhere, and therefore in particular, $\|x_0\| = \|T^\star x_0\|$, hence by insertion,

$$\|T^\star x_0 - x_0\|^2 = \|T^\star x_0\|^2 - \|x_0\|^2 = \|x_0\|^2 - \|x_0\|^2 = 0.$$

This shows that $T^\star x_0 - x_0$, or after a rearrangement, $T^\star x_0 = x_0$.

Example 2.5 *Let (e_n) denote an orthonormal basis in a Hilbert space H, and consider the operator*

$$T\left(\sum_{k=1}^{\infty} a_k e_k\right) = \sum_{k=1}^{\infty} a_k e_{k+1}.$$

Find the adjoint $T^\star$ and show that $T^\star$ is an extension of T^{-1}.

Put

$$x = \sum_{k=1}^{+\infty} x_k e_k \in H \qquad \text{and} \qquad y = \sum_{k=1}^{+\infty} y_k e_k \in D\left(T^\star\right) = H.$$

then

$$\begin{aligned}(Tx, y) &= \left(\sum_{k=1}^{+\infty} x_k e_{k+1}, \sum_{k=1}^{+\infty} y_k e_k\right) = \left(\sum_{k=2}^{+\infty} x_{k-1} e_k, \sum_{k=1}^{+\infty} y_k e_k\right) = \sum_{k=2}^{+\infty} x_{k-1}\overline{y_k} = \sum_{k=1}^{+\infty} x_k \overline{y_{k+1}} \\ &= \left(\sum_{k=1}^{+\infty} x_k e_k, \sum_{k=1}^{+\infty} y_{k+1} e_k\right) = \left(x, T^\star y\right),\end{aligned}$$

from which

$$T^\star y = T^\star\left(\sum_{k=1}^{+\infty} y_k e_k\right) = \sum_{k=1}^{+\infty} y_{k+1} e_k.$$

It follows from $D\left(T^{-1}\right) = \{e_1\}^\perp$ and

$$T^{-1} y = T^{-1}\left(\sum_{k=2}^{+\infty} y_k e_k\right) = \sum_{k=1}^{+\infty} y_{k+1} e_k \qquad \text{for } y \in D\left(T^{-1}\right),$$

that $T^{-1} y = T^\star y$ for all $y \in D\left(T^{-1}\right) \subset H$, hence $T^{-1} \subset T^\star$.

Finally, we notice that $T^\star e_1 = 0$, thus $T^{-1} \neq T^\star$.

Example 2.6 *Let* (e_n) *denote an orthonormal basis in a Hilbert space* H*, and consider the operator*

$$T\left(\sum_{k=1}^{\infty} a_k e_k\right) = \sum_{k=2}^{\infty} \sqrt{k-1}\, a_k e_{k-1}.$$

Show that T *is a densely defined unbounded operator, and find* $T^{\star}$*.*

It follows from $\|e_n\| = 1$ and

$$\|Te_n\| = \sqrt{n-1} \to +\infty \qquad \text{for } n \to +\infty,$$

that T is unbounded.

Put

$$x = \sum_{k=1}^{+\infty} x_k e_k \qquad \text{and} \qquad y = \sum_{n=1}^{+\infty} y_n e_n.$$

Then

$$\begin{aligned}(Tx, y) &= \left(\sum_{k=1}^{+\infty} \sqrt{k}\, x_{k+1} e_k, \sum_{n=1}^{+\infty} y_n e_n\right) = \sum_{n=1}^{+\infty} \sqrt{n} \cdot x_{n+1} \overline{y_n} \\ &= (x, T^{\star} y) = \left(\sum_{n=1}^{+\infty} x_{n+1} e_{n+1}, \sum_{k=1}^{+\infty} \sqrt{k} \cdot y_k e_{k+1}\right) = \left(x, \sum_{k=2}^{+\infty} \sqrt{k-1} \cdot y_{k-1} e_k\right),\end{aligned}$$

and we infer that

$$T^{\star} y = T^{\star}\left(\sum_{k=1}^{+\infty} y_k e_k\right) = \sum_{k=2}^{+\infty} \sqrt{k-1} \cdot y_{k-1} e_k.$$

Then we shall explain that the formal computations above are legal. Thus, we shall prove that

$$D(T) = \left\{ x \in H \;\middle|\; \sum_{k=2}^{+\infty} k\, |a_k|^2 < +\infty \right\}$$

is dens in H. Let $x \in H$ be arbitrary. To any $\varepsilon > 0$ there exists an N, such that

$$\sum_{k=N+1}^{+\infty} |a_k|^2 < \varepsilon^2.$$

Choose $x_N = (a_1, a_2, \dots, a_N, 0, 0, \dots) \in D(T)$. Then $\|x - x_N\| < \varepsilon$. This proves that $D(T)$ is dense in H, thus $T^{\star}$ exists and the formal computations above are correct, when $x \in D(T)$ and $y \in D(T^{\star})$.

We infer from

$$\|T^{\star} y\|^2 = \sum_{k=2}^{+\infty} (k-1)\, |y_{k-1}|^2 = \sum_{k=1}^{+\infty} k\, |y_k|^2 \quad (= \|Ty\|^2),$$

that $D(T^{\star}) = D(T)$.

Example 2.7 *Consider the operator* $T : \ell^2 \to \ell^2$ *given by*

$$T(x_1, x_n, \dots, x_n, \dots) = \left(\frac{1}{2}\,x_2, \frac{2}{3}\,x_3, \dots, \frac{n}{n+1}\,x_n, \dots\right).$$

1) Determine $\|T\|$*.*

2) Find all eigenvalues $\sigma_p(T)$ *and corresponding eigenvectors.*

3) Determine the adjoint $T^\star$ *and* $\sigma_p(T^\star)$ *and the resolvent* $\varrho(T)$*.*

1) It is obvious that $\|Tx\| \le \|x\|$. Then it follows from

$$\|T(e_n)\| = \frac{n}{n+1} \to 1 \qquad \text{for } n \to +\infty,$$

that $\|T\| = 1$.

2) Assume that $\lambda \in \sigma_p(T)$ is an eigenvalue, and let $x \in \ell^2$ be a corresponding eigenvector. Then we get for the coordinates,

$$\lambda\, x_n = \frac{n}{n+1}\, x_{n+1}, \qquad n \in \mathbb{N},$$

hence by a rearrangement and recursion,

$$x_{n+1} = \lambda \cdot \frac{n+1}{n}\, x_n = \cdots = \lambda^n \cdot \frac{n+1}{n}\,\frac{n}{n-1} \cdots \frac{2}{1} \cdot x_1 = \lambda^n (n+1) x_1,$$

hence

$$x_n = n \cdot \lambda^{n-1} x_1, \qquad n \in \mathbb{N}.$$

It follows that

$$\sum_{n=1}^{+\infty} |x_n|^2 = \sum_{n=1}^{+\infty} n^2 |\lambda|^{2(n-1)}\, |x_1|^2 = |x_1|^2 \sum_{n=1}^{+\infty} n^2 |\lambda|^{2(n-1)},$$

where the series is convergent, if and only if $|\lambda| < 1$, thus

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\},$$

and a corresponding eigenvector is

$$x_\lambda = \left(1, 2\lambda, 3\lambda^2, \ldots, n\lambda^{n-1}, \ldots\right).$$

3) We see that $T^\star$ exists in $B\left(\ell^2\right)$, so

$$\begin{aligned} (x, T^\star y) &= (Tx, y) = \sum_{n=1}^{+\infty} \frac{n}{n+1}\, x_{n+1} \overline{y_n} = \sum_{n=2}^{+\infty} x_n \cdot \frac{n-1}{n}\, \overline{y_{n-1}} \\ &= \left((x_1, x_2, \ldots, x_n, \ldots), \left(0, \frac{1}{2}\, y_1, \frac{2}{3}\, y_2, \ldots, \frac{n-1}{n}\, y_{n-1}, \ldots \right)\right), \end{aligned}$$

and we get

$$T^\star y = \left(0, \frac{1}{2}\, y_1, \frac{2}{3}\, y_2, \ldots, \frac{n-1}{n}\, y_{n-1}, \ldots\right), \qquad y \in \ell^2.$$

Assume that $\lambda \in \sigma_p\left(T^\star\right)$ is an eigenvalue for $T^\star$. Then

$$\lambda y_1 = 0, \qquad \lambda y_n = \frac{n-1}{n}\, y_{n-1}, \quad n \in \mathbb{N} \setminus \{1\}.$$

We have two possibilities: Either $\lambda = 0$, or $y_1 = 0$.

(a) $\lambda = 0$**.** It follows from the latter equation that $y_{n-1} = 0$ for $n \in \mathbb{N} \setminus \{1\}$, meaning that $y = 0$, and we conclude that $0 \notin \sigma_p\left(T^\star\right)$.

(b) $\lambda \neq 0$**.** In this case, $y_1 = 0$, and then it follows by induction on

$$y_n = \frac{n-1}{n\lambda}\, y_{n-1}, \qquad n \in \mathbb{N} \setminus \{1\},$$

that $y_n = 0$, and hence $y = 0$. We conclude that $\lambda \notin \sigma_p\left(T^\star\right)$.

Summing up,

$$\sigma_p\left(T^\star\right) = \emptyset.$$

Hence $\sigma_r(T) = \emptyset$. Furthermore,

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\} \subseteqq \sigma(T) \subseteqq \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|T\| = 1\},$$

and because $\sigma(T)$ is closed, we must have

$$\sigma(T) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}.$$

Utilizing that

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup_r (T) = \sigma_p(T) \cup \sigma_c(T)$$

is a disjoint splitting, we finally find the continuous spectrum

$$\sigma_c(T) = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\},$$

and the resolvent set

$$\varrho(T) = \{\lambda \in \mathbb{C} \mid |\lambda| > 1\}.$$

Example 2.8 *Let $T : \ell^2 \to \ell^2$ be the linear operator given by*

$$T(x_1, x_2, \dots, x_n, \dots) = (x_1 + x_2, x_2 + x_3, \dots, x_n + x_{n+1}, \dots).$$

1) Find the point spectrum $\sigma_p(T)$ and determine all eigenvectors associated to $\lambda \in \sigma_p(T)$.

2) Determine $\|T\|$.

3) Determine the adjoint $T^\star$ and find also the point spectrum $\sigma_p\left(T^\star\right)$.

4) Let $S = T - I$. Determine $\|S\|$.

5) Find $\sigma_c(T)$ and $\sigma_r(T)$ with the help of S above.

1) We shall find the non-trivial solutions of the equation

$$Tx = \lambda\, x.$$

The coordinate equation of this equation becomes

$$x_n + x_{n+1} = \lambda\, x_n, \qquad n \in \mathbb{N},$$

thus

$$(4)\quad x_{n+1} = (\lambda - 1)x_n, \qquad n \in \mathbb{N}.$$

If $\lambda = 1$, then $x_{n+1} = 0$, so we can only choose $x_1 \neq 0$. On the other hand, e_1 is clearly an eigenvector and $1 \in \sigma_p(T)$.

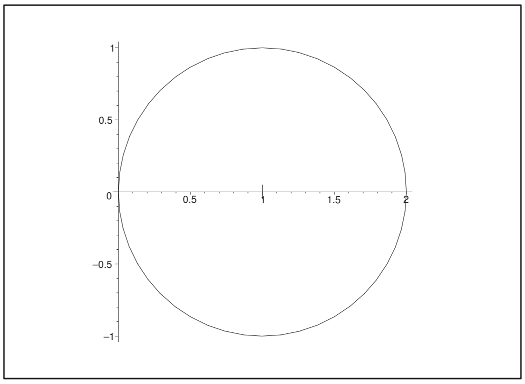

Figure 1: The point spectrum $\sigma_p(T)$ is the open set inside the circle.

If $\lambda \neq 1$, then we can divide (4) by $(\lambda-1)^{n+1} \neq 0$. Then it follows by recursion that

$$\frac{x_{n+1}}{(\lambda-1)^{n+1}} = \frac{x_n}{(\lambda-1)^n} = \cdots = \frac{x_1}{\lambda-1},$$

so $x_n = (\lambda)^{n-1}x_1$. Choosing $x_1 = 1$ we see that one *candidate* of an eigenvector is given by its coordinates $x_n = (\lambda-1)^{n-1}$. Because

$$\sum_{n=1}^{+\infty} |x_n|^2 = \sum_{n=1}^{+\infty} |\lambda-1|^{2(n-1)} = \sum_{n=0}^{+\infty} |\lambda-1|^{2n}$$

is convergent, if and only if $|\lambda-1| < 1$, it follows that

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid |\lambda-1| < 1\}$$

with the eigenvectors

$$\left(1, \lambda-1, (\lambda-1)^2, \dots, (\lambda-1)^{n-1}, \dots\right), \qquad \text{for } |\lambda-1| < 1.$$

We notice for $\lambda = 1$ that we get precisely $(1, 0, 0, \dots)$.

2) From

$$2 \in \sigma(T) \subseteqq \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|T\|\},$$

and a consideration of the figure, it follows that $\|T\| \geq 2$.

On the other hand, an application of Minkowski's inequality gives

$$\|Tx\| = \|x + (0, x_1, x_2, \dots)\| \leq \|x\| + \|x\| = 2\,\|x\|,$$

proving that $\|T\| \leq 2$.

Summing up, $\|T\| = 2$.

3) It follows from

$$
\begin{aligned}
(Tx,y) &= \sum_{n=1}^{+\infty}(x_n+x_{n+1})\,\overline{y_n} = \sum_{n=1}^{+\infty} x_n\,\overline{y_n} + \sum_{n=2}^{+\infty} x_n\,\overline{y_{n-1}} \\
&= x_1\,\overline{y_1} + \sum_{n=2}^{+\infty} x_n\overline{(y_{n-1}+y_n)} = (x,T^\star y) = \sum_{n=1}^{+\infty} x_n\overline{(T^\star y)_n},
\end{aligned}
$$

that

$$T^\star y = (y_1, y_1+y_2, y_2+y_3, \dots, y_{n-1}+y_n, \dots),$$

or written in coordinates,

$$(T^\star y)_1 = y_1, \qquad (T^\star y)_n = y_{n-1}+y_n \quad \text{for } n \geq 2.$$

The equation $T^\star y = \lambda\, y$ is written in coordinates as

$$y_1 = \lambda\, y_1 \qquad \text{and} \qquad y_{n-1}+y_n = \lambda\, y_n \quad \text{for } n \geq 2,$$

thus

$$(\lambda-1) = y_1 = 0 \qquad \text{and} \qquad (\lambda-1)y_n = y_{n-1} \quad \text{for } n \geq 2.$$

We get from the first equation that either $\lambda = 1$ or $y_1 = 0$. If $\lambda = 1$, then it follows from the last equations that $y_{n-1} = 0$ for all $n \geq 2$, hence $y = 0$, and $\lambda = 1$ is not an eigenvalue for $T^\star$.

If $\lambda \neq 1$ and $y_1 = 0$, then we see by recursion on

$$y_n = \frac{1}{\lambda - 1}\, y_{n-1}$$

that the only solution is $y = 0$.

Summing up, $\sigma_p\left(T^\star\right) = \emptyset$.

Then of course, $\sigma_r(T) = \emptyset$.

4) Because

$$(Sx)_n = (Tx)_n - x_n = x_{n+1},$$

and $\|Sx\| \leq \|x\|$ with equality for $x_1 = 0$, it follows immediately that $\|S\| = 1$.

5) We get from $T = S + I$ that $T - \lambda I = S - (\lambda - 1)I$, so

$$\begin{array}{llcll} \lambda \in \sigma_p(T) & \text{if and only if} & \lambda - 1 \in \sigma_p(S), & \text{thus.} & \sigma_p(T) = 1 + \sigma_p(S), \\ \lambda \in \sigma_c(T) & \text{if and only if} & \lambda - 1 \in \sigma_c(S), & \text{thus} & \sigma_c(T) = 1 + \sigma_c(S), \\ \lambda \in \sigma_r(T) & \text{if and only if} & \lambda - 1 \in \sigma_r(S), & \text{thus} & \sigma_r(T) = 1 + \sigma_r(S). \end{array}$$

It is not surprising that the various parts of the spectrum for is obtained by translating the corresponding parts of the spectrum for S. We now conclude from

$$\sigma_p(S) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\},$$

and

$$\sigma_r(S) = \emptyset, \qquad \text{(because } \sigma_r(T) = \emptyset\text{)},$$

and from $\sigma(S)$ being closed, and

$$\sigma_p(S) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\} \subseteqq \sigma(S) \subseteqq \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|S|\} = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\},$$

that

$$\sigma(S) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\},$$

and hence that

$$\sigma_c(S) = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}.$$

Finally, by utilizing the translation, we get

$$\begin{array}{rcl} \sigma(T) & = & \{\lambda \in \mathbb{C} \mid |\lambda - 1| \leq 1\}, \\ \sigma_p(T) & = & \{\lambda \in \mathbb{C} \mid |\lambda - 1| < 1\}, \\ \sigma_c(T) & = & \{\lambda \in \mathbb{C} \mid |\lambda - 1| = 1\} \\ \sigma_r(T) & = & \emptyset. \end{array}$$

Example 2.9 *We consider in* ℓ^2 *the operator*

$$T(x_1, x_2, \dots, x_n, \dots) = \left(2x_2, \frac{3}{2}\, x_3, \dots, \frac{n+1}{n}\, x_{n+1}, \dots\right).$$

1) Find $\|T\|$.

2) Find $\sigma_p(T)$ *and find the eigenspace associated to all* $\lambda \in \sigma_p(T)$.

3) Determine the adjoint $T^\star$.

4) Determine $\sigma_r(T)$.

5) Let $\lambda \notin \sigma_p(T) \cup \sigma_r(T)$. *For* $k \in \mathbb{N}$ *we define an operator* I_k *on* ℓ^2 *by*

$$I_k\left((x_1, x_2, \dots, x_k, x_{k+1}, \dots)\right) = (0, 0, \dots, 0, x_k, x_{k+1}, \dots),$$

and we define $T_k = I_k T$. *Show that there is a* $k \in \mathbb{N}$ *such that*

$$\|T_k\| < \lambda.$$

Use this to solve the equation

$$(T_k - \lambda\, I_k)\, x = y$$

for a given $y \in \ell^2$. *Finally, show that the equation*

$$(T - \lambda\, I)x = y$$

has a solution $x = (T - \lambda\, T)^{-1}y$ *for all* $y \in \ell^2$.

6) Find $\sigma(T)$ *and* $\varrho(T)$ *(e.g. by use of the Closed Graph Theorem).*

1) From $1 + \dfrac{1}{n} \le 2$ for all $n \in \mathbb{N}$, follows for every $x \in \ell^2$ that

$$\|Tx\|^2 = \sum_{n=1}^{+\infty} \left(1 + \frac{1}{n}\right)^2 |x_{n+1}|^2 \le 2^2 \sum_{n=1}^{\infty} |x_{n+1}|^2 \le \{2\,\|x\|\}^2,$$

proving that $\|T\| \le 2$.

On the other hand,

$$T(0, 1, 0, 0, \dots) = (2, 0, 0, 0, \dots),$$

and we infer that $\|T\| = 2$.

2) Assume that $Tx = \lambda\, x$, thus

$$\frac{n+1}{n}\, x_{n+1} = \lambda\, x_n, \qquad n \in \mathbb{N}.$$

For $\lambda = 0$ we get $x = (1, 0, 0, \dots)$ as an eigenvector, and 0 is an eigenvalue, $0 \in \sigma_p(T)$.

If $\lambda \neq 0$, then a multiplication by $n\,\lambda^{-(n+1)}$ follows by a recursion gives that

$$(n+1)\,\lambda^{-(n+1)}\,x_{n+1} = n\,\lambda^{-n}\,x_n = \cdots = 1 \cdot \lambda^{-1}\,x_1,$$

and we get the coordinates of the candidate

$$x_n = \frac{1}{n}\,\lambda^{n-1}\,x_1, \qquad n \in \mathbb{N}.$$

the corresponding sequence lies in ℓ^2 for $x_1 \neq 0$, if and only if

$$\sum_{n=1}^{+\infty} \frac{1}{n^2}\,|\lambda|^{2(n-1)} < +\infty.$$

It is well-known that $\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < +\infty$, so this condition is equivalent to $|\lambda| \leq 1$, and we conclude that

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\},$$

and an eigenvector corresponding to $\lambda \in \sigma_p(T)$ is given by

$$x_1 \left(1, \frac{\lambda}{2}, \frac{\lambda^2}{3}, \dots, \frac{\lambda^{n-1}}{n}, \dots \right).$$

3) If x, $y \in \ell^2$, then

$$(Tx, y) = \sum_{n=1}^{+\infty} (Tx)_n\,\overline{y_n} = \sum_{n=1}^{+\infty} \frac{n+1}{n}\,x_{n+1}\,\overline{y_n} = \sum_{n=2}^{+\infty} x_n \cdot \overline{\frac{n}{n-1}\,y_{n-1}} = (x, T^\star y)\,,$$

hence

$$T^\star\,(y_1, y_2, \dots, y_n, \dots) = \left(0, 2y_1, \frac{3}{2}\,y_2, \dots, \frac{n}{n-1}\,y_{n-1}, \dots \right),$$

or written in coordinates,

$$\begin{cases} (T^\star y)_1 = 0, & \text{for } n = 1, \\[2mm] (T^\star y)_n = \dfrac{n}{n-1}\,y_{n-1}, & \text{for } n \in \mathbb{N} \setminus \{1\}. \end{cases}$$

4) We prove that $\sigma_p\,(T^\star) = \emptyset$, because this will imply that $\sigma_r(T) = \emptyset$.

Assume that $\lambda \in \sigma_p\,(T^\star)$. It follows from the equation $T^\star y = \lambda y$ that

$$\begin{cases} 0 = \lambda\,y_1, & \text{for } n = 1, \\[2mm] \dfrac{n}{n-1}\,y_{n-1} = \lambda\,y_n, & \text{for } n \in \mathbb{N} \setminus \{1\}. \end{cases}$$

If $\lambda = 0$, then clearly $y = 0$, so $0 \notin \sigma_p(T^\star)$.

If $\lambda \neq 0$, then $y_1 = 0$. Multiply the last coordinate equation by $\dfrac{1}{n}\lambda^{n-1}$. Then it follows by recursion that

$$\frac{\lambda^n}{n}\, y_n = \frac{\lambda^{n-1}}{n-1}\, y_{n-1} = \cdots = \frac{\lambda}{1}\, y_1 = 0,$$

from which $y_n = 0$ for all $n \in \mathbb{N}$, and there is no eigenvectors. Hence, $\sigma_p(T^\star) = \emptyset$, and therefore $\sigma_r(T) = \emptyset$.

5) If

$$\lambda \notin \sigma_p(T) \cup \sigma_r(T) = \sigma_p(T) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\},$$

then $|\lambda| > 1$. It follows from

$$\|T_k x\|^2 = \sum_{n=k}^{+\infty} \left(\frac{n+1}{n}\right)^2 |x_{n+1}|^2 \leq \left(\frac{k+1}{k}\right)^2 \|x\|^2,$$

that

$$\|T_k\| \leq \frac{k+1}{k} = 1 + \frac{1}{k},$$

where we can obtain equality, so

$$\|T_k\| = 1 + \frac{1}{k}.$$

Because $|\lambda| > 1$, we can choose k so big that

$$\|T_k\| = 1 + \frac{1}{k} < |\lambda|.$$

Now, $\lambda \notin \sigma_p(T) \cup \sigma_r(T)$, so $(T - \lambda I)^{-1}$ exists and is densely defined.

It follows from $\|T_k\| < |\lambda|$, that

$$(T_k - \lambda I_k)^{-1} \in B\left(I_k \ell^2\right(,$$

where $I_k \ell^2$ is a Hilbert space which is isomorphic to ℓ^2.

The equation

$$Tx - \lambda x = y, \qquad y \in \ell^2,$$

has the coordinate form

$$\frac{n+1}{n} x_{n+1} - \lambda x_n = y_n, \qquad n \in \mathbb{N}.$$

Thus is follows from $\lambda \neq 0$ that

$$\begin{cases} x_n = \dfrac{1}{\lambda} \left(\dfrac{n+1}{n} x_{n+1} - y_n \right), & n \in \{1, \dots, k-1\}, \\[2ex] T_k x - \lambda I_k x = I_k y. \end{cases}$$

It follows from the above that the latter equation can be solved,

$$I_k x = (T_k - \lambda I_k)^{-1} I_k y \qquad \text{for all } y \in \ell^2.$$

Hence for a given $y \in \ell^2$,

$$I_k x = (0, \dots, 0, x_k, x_{k+1}, \dots) = (T_k - \lambda I_k)^{-1} I_k y$$

is uniquely determined. The recursion formula

$$x_n = \frac{1}{\lambda} \left\{ \frac{n+1}{n} x_{n+1} - y_n \right\}, \qquad \text{for } n \in \{1, \dots, k-1\},$$

determines the remaining elements of x, so $(T - \lambda I)^{-1}$ is defined everywhere.

6) If $|\lambda| > 1$, then it follows from the above that $(T - \lambda I)^{-1}$ is defined everywhere. Now, $T - \lambda I$ is closed, so $(T - \lambda I)^{-1}$ is also closed. Then it follows from the Closed Graph Theorem that $\lambda \in \varrho(T)$ for every $\lambda \in \mathbb{C}$, for which $|\lambda| > 1$. Hence

$$\sigma(T) = \sigma_p(T) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}, \qquad \sigma_r(T) = \sigma_c(T) = \emptyset,$$

and

$$\varrho(T) = \{\lambda \in \mathbb{C} \mid |\lambda| > 1\}.$$

Remark 2.1 This example shows that it is possible that every λ for which $\lambda \in \mathbb{C}$ med $|\lambda| = \|T\|$ belongs to the resolvent set, $\varrho \in \varrho(T)$. So far we have only seen examples, in which there is always at least one $\lambda \in \sigma(T)$, such that $|\lambda| = \|T\|$. This is not the case in the present example. $\Diamond$

3 Self adjoint operators

Example 3.1 *Let $T \in B(H)$. Show that we can write T as*

$$T = A + i\,B,$$

where A and B are uniquely determined, bounded self adjoint operators.

First assume that T can be written in the form $T = A + i\,B$, where A and B are self adjoint. Then

$$\begin{aligned}(Tx, y) &= (Ax + i\,Bx, y) = (Ax, y) + i\,(Bx, y) \\ &= (x, Ay) + i\,(x, By) = (x, Ay - i\,By) = (x, (A - i\,B)y) = (x, T^\star y)\,,\end{aligned}$$

and it follows that if

$$T = A + i\,B \qquad \text{then} \qquad T^\star = A - i\,B.$$

We get by simple addition or subtraction,

$$A = \frac{1}{2}\,(T + T^\star) \qquad \text{and} \qquad B = \frac{1}{2i}\,(T - T^\star)\,.$$

Conversely, if

$$A = \frac{1}{2}\,(T + T^\star) \qquad \text{and} \qquad B = \frac{1}{2i}\,(T - T^\star)\,,$$

then clearly, $T = A + i\,B$. Furthermore, A and B are obviously linear and

$$\|A\| \le \frac{1}{2}\,\{\|T\| + \|T^\star\|\} = \|T\|, \qquad \|B\| \le \frac{1}{2}\,\{\|T\| + \|T^\star\|\} = \|T\|,$$

so A and B are bounded. Finally,

$$(Ax, y) = \left(\frac{1}{2}\{T + T^\star\}x, y\right) = \left(x, \frac{1}{2}\{T^\star + T^{\star\star}\}y\right) = \left(x, \frac{1}{2}\{T + T^\star\}y\right) = (x, Ay),$$

and

$$(Bx, y) = \left(\frac{1}{2i}\{T - T^\star\}x, y\right) = \left(x, -\frac{1}{2i}\{T^\star - T^{\star\star}\}y\right) = \left(x, \frac{1}{2i}\{T - T^\star\}y\right) = (x, By),$$

shows that both A and B are self adjoint.

Example 3.2 *Show that $T \in B(H)$ is self adjoint if and only if one of the following conditions is satisfied:*

$$(Tx, x) = (x, Tx) \qquad \textit{for all } x \in H,$$

and

$$(Tx, x) \in \mathbb{R} \qquad \textit{for all } x \in H.$$

We assume implicitly that H is a complex Hilbert space.

We have $T \in B(H)$, thus T is self adjoint if and only if $T^\star = T$, thus if and only if

(5) $(Tx, y) = (x, Ty) \qquad$ for all $x, y \in H$.

Choosing $y = x$ in (5) we get in particular the first condition above, thus

(6) $(Tx, x) = (x, Tx) \qquad$ for all $x \in H$.

This condition is equivalent with

$$(Tx, x) = (x, Tx) = \overline{(Tx, x)} \quad (\in \mathbb{R}),$$

and it follows that the two conditions are equivalent. It only remains to prove that (6) implies that T is self adjoint.

Assume (6). We shall prove (5). We get by replacing x in (6) by $x + y$ that

$$\begin{array}{ccccc} (T(x+y), x+y) & = & (Tx, x) & +(Tx, y) + (Ty, x) + & (Ty, y), \\ (x+y, T(x+y)) & = & (x, Tx) & +(x, Ty) + (y, Tx) + & (y, Ty), \\ \star & & \star & & \star \end{array}$$

It follows from the assumption (6) that the three columns marked with a $\star$ inside each column are mutually equal. Hence by a subtraction and a rearrangement,

(7) $(Tx, y) + (Ty, x) = (x, Ty) + (y, Tx)$.

If we write $x + iy$ in (6) instead of x, then we get analogously,

$$\begin{array}{ccccc} (T(x+iy), x+iy) & = & (Tx, x) & -i(Tx, y) + i(Ty, x) + & (Ty, y), \\ (x+iy, T(x+iy)) & = & (x, Tx) & -i(x, Ty) + i(y, Tx) + & (y, Ty), \\ \star & & \star & & \star \end{array}$$

We conclude as before by utilizing that the columns marked with a $\star$ by the assumption (6) are identical that

(8) $(Tx, y) - (Ty, x) = (x, Ty) - (y, Tx)$.

We get by adding (7) and (8), followed by a division by 2,

$$(Tx, y) = (x, Ty).$$

This is true for all $x, y \in H$, so we have proved (5), thus T is self adjoint.

Example 3.3 *Let S and T be bounded, self adjoint operators on a Hilbert space. Show that $ST+TS$ and $i(ST-TS)$ are self adjoint.*

The proof is simple, because $S, T \in B(H)$ and

$$(ST+TS)^\star = (ST)^\star + (TS)^\star = T^\star S^\star + S^\star T^\star = ST+TS,$$

and

$$\{i(ST-TS)\}^\star = -i\{(ST)^\star - (TS)^\star\} = -i\{T^\star S^\star - S^\star T^\star\} = i(ST-TS).$$

Example 3.4 *Let T be a bounded self adjoint operator. Define the numbers*

$$m = \inf\{(Tx, x) \mid \|x\| = 1\},$$

and

$$M = \sup\{(Tx, x) \mid \|x\| = 1\}.$$

Show that $\sigma(T) \subset [m, M]$, and show that both m and M belong to $\sigma(T)$.
Show that $\|T\| = \max\{|m|, |M|\}$.

We deduce from the definitions of m and M that

$$m\,\|x\|^2 \le (Tx, x) \le M\,\|x\|^2 \qquad \text{for all } x \in H.$$

Now, $T \in B(H)$ is self adjoint, so $\sigma(T) \subseteq \mathbb{R}$. Choose $\lambda \in \mathbb{R} \setminus [m, M]$. We shall prove that $\lambda \in \varrho(T)$. First assume that $\lambda < m$. Then

$$\begin{aligned} \|(T - \lambda I)x\|^2 &= (Tx - \lambda x, Tx - \lambda x) \\ &= (Tx - mx + (m - \lambda)x, Tx - mx + (m - \lambda)x) \\ &= \|Tx - mx\|^2 + (m - \lambda)^2\|x\|^2 + 2\{m - \lambda\}\,(Tx - mx, x). \end{aligned}$$

It follows from $m - \lambda > 0$ and $(Tx - mx, x) = (Tx, x) - m(x, x) \le 0$ that we have the estimate,

$$\|(T - \lambda I)x\|^2 \ge 0 + (m - \lambda)^2\|x\|^2 + 0 = (m - \lambda)^2\|x\|^2,$$

which implies that $T - \lambda I$ is injective, and $(T - \lambda I)^{-1}$ exists and is bounded. Then

$$\lambda \in \varrho(T) \cup \sigma_r(T).$$

Because T is self adjoint, the residual spectrum is $\sigma_r(T) = \emptyset$, hence $\lambda \in \varrho(T)$.

If instead $\lambda > M$, then we get analogously

$$\begin{aligned} \|(T - \lambda I)x\|^2 &= (Tx - \lambda x, Tx - \lambda x) \\ &= (Mx - Tx + (\lambda - M)x, Mx - Tx + (\lambda - M)x) \\ &= \|Mx - Tx\|^2 + (\lambda - M)^2\|x\|^2 + 2\{\lambda - M\}\,(Mx - Tx, x) \\ &\ge (\lambda - M)^2\|x\|^2, \end{aligned}$$

because $\lambda - M > 0$ and $(Mx - Tx, x) = M\|x\|^2 - (Tx, x) \ge 0$. As before we infer that $(T - \lambda I)^{-1}$ exists and is bounded. We have proved that $\mathbb{C} \setminus [m, M] \subseteq \varrho(T)$, and it follows that $\sigma(T) \subseteq [m, M]$.

Using a well-known formula we get

$$\|T\| = \sup\{|(Tx, x)| \mid \|x\| = 1\} = \max\{|m|, |M|\}.$$

Assume e.g. that $\|T\| = |M| = M \ge 0$, and let $\lambda = M$. Then

$$M \in \sigma_p(T) \cup \sigma_c(T) \cup \varrho(T).$$

We shall prove that $M \notin \varrho(T)$. This is done INDIRECTLY.

Assume that $M \in \varrho(T)$, thus $(T - M\,I)^{-1} \in B(H)$. Then there exists a $c > 0$, such that

$$\|(T - M\,I)^{-1}x\| \le \frac{1}{c}\,\|x\| \qquad \text{for all } x \in H.$$

If we put $y = (T - M\,I)^{-1}x$, then $x = (T - M\,I)y$, hence

$$\|(T - M\,I)y\| \ge c\|y\| \qquad \text{for all } y \in H.$$

This implies that $\|T - M\,I\| \ge c > 0$.

From $M = \sup\{(Tx, x) \mid \|x\| = 1\}$ follows the existence of a sequence x_n, $\|x_n\| = 1$, of unit vectors, such that

$$(Tx_n, x_n) \to M = \|T\| \qquad \text{for } n \to +\infty,$$

and we conclude from

$$(Tx_n, x_n) \le \|Tx_n\| \cdot \|x_n\| = \|Tx_n\| \le \|T\| = M,$$

that also $\|Tx_n\| \to M$. Then for every such sequence,

$$\begin{aligned} 0 &\le \|(T - M\,I)x_n\|^2 = (Tx_n - Mx_n, Tx_n - Mx_n) \\ &= \|Tx_n\|^2 + M^2\|x_n\|^2 - 2M(Tx_n, x_n) \\ &\to M^2 + M^2 - 2M^2 = 0, \end{aligned}$$

which shows that the estimate $\|(T - M\,I)x_n\| \ge c\|x_n\| = c > 0$ is not true, and we have derived a contradiction. Therefore, $M \notin \varrho(T)$, i.e. $M \in \sigma(T)$.

An analogous argument shows that if $\|T\| = |m| = -m$, then $m \in \sigma(T)$.

Finally, assume that $|m| = -m < M$. It follows from the above that $M \in \sigma(T)$. We shall prove that also $m \in \sigma(T)$. First notice that $T - M\,I$ of course is self adjoint. Then it follows from

$$((T - M\,I)x, x) = (Tx, x) - M\,\|x\|^2,$$

and

$$m\,\|x\|^2 \le (Tx, x) \le M\,\|x\|^2,$$

that

$$(m - M)\|x\|^2 \le ((T - M\,I)x, x) \le (M - M)\|x\|^2 = 0,$$

and

$$\inf\{((T - M\,I)x, x) \mid \|x\| = 1\} = m - M < 0.$$

Then from the above, $m - M \in \sigma(T - M\,I)$, which means that

$$(T - M\,I) - (m - M)I = T - m\,I$$

is not regular, so $m \in \sigma(T)$.

Example 3.5 *Consider in $L^2(\mathbb{R})$ the operator Q defined by*

$$Qf(x) = x\,f(x),$$

with

$$D(Q) = \{f \in L^2(\mathbb{R}) \mid Qf \in L^2(\mathbb{R})\}.$$

Show that Q is self adjoint.

Let $f,\, g \in D(Q)$, thus $f,\, g \in L^2(\mathbb{R})$ and $x \cdot f(x),\; x \cdot g(x) \in L^2(\mathbb{R})$. Because Q is densely defined, we get

$$(Qf, g) = \int_{-\infty}^{+\infty} x\,f(x)\overline{g(x)}\,dx = \int_{-\infty}^{+\infty} f(x) \cdot \overline{x\,g(x)}\,dx = (f, Qg),$$

proving that Q is symmetric, $Q \subseteqq Q^\star$. It remains to prove that $D(Q) = D\left(Q^\star\right)$. To do this it suffices to prove that Q is a closed operator.

Assume that $(f_n) \subset D(Q)$ and $f_n \to f \in L^2(\mathbb{R})$, and $x\,f_n \to g \in L^2(\mathbb{R})$. We shall prove that $g(x) = x \cdot f(x)$ almost everywhere. We find

$$\|g - x\,f\|_2^2 = \int_{-1}^{1} |g(x) - x\,f(x)|^2 dx + \left\{\int_{-\infty}^{-1} + \int_{1}^{+\infty} |g(x) - x\,f(x)|^2 dx\right\}.$$

Here, $\int_{-1}^{1} |g(x) - x\,f(x)|^2 dx = 0$, because $f \in L^2([-1,1])$ implies that also $x \cdot f \in L^2([-1,1])$, noting that the interval is bounded. This means that $g(x) = x \cdot f(x)$ for almost every $x \in [-1,1]$. If $|x| \geq 1$, then we get $f_n \to f$ and $f_n \to \dfrac{g(x)}{x}$, both in the sense of L^2, because

$$\int_{|x|\geq 1} \left|\frac{g(x)}{x}\right|^2 dx \leq \int_{|x|\geq 1} |g(x)|^2 dx < +\infty.$$

The limit value is unique, hence $f(x) = \dfrac{g(x)}{x}$ almost everywhere for $|x| \geq 1$. Hence we conclude that $g(x) = x\,f(x)$ for almost every $x \in \mathbb{R}$.

This proves that Q is closed, which again implies by the above that $Q = Q^\star$, and we have proved that Q is self adjoint.

Example 3.6 *Show that the set of self adjoint operators is closed in $B(H)$.*

We shall only prove that if $(T_n) \subset B(H)$ is a sequence of self adjoint operators, converging towards $T \in B(H)$, then T is also self adjoint. The condition $T_n \to T$ for $n \to +\infty$ means that

$$Tx = \lim_{n\to+\infty} T_n x \qquad \text{for all } x \in H.$$

Therefore, if $x,\, y \in H$, then

$$(Tx, y) = \lim_{n\to+\infty} (T_n, y) = \lim_{n\to+\infty} (x, T_n y) = (x, Ty),$$

proving that $T \subseteqq T^\star$. Because $D(T) = H$, we have $T = T^\star$, hence T is self adjoint.

Example 3.7 *Let (e_n) denote an orthonormal basis in a Hilbert space H, and let (r_k) be all the rational numbers in $]0,1[$, arranged as a sequence. Consider the operator*

$$T\left(\sum_{k=1}^{\infty} a_k e_k\right) = \sum_{k=1}^{\infty} r_k a_k e_k .$$

Show that T is self adjoint and that $\|T\| = 1$. Find $\varrho(T)$ and determine the point spectrum and the continuous spectrum for T.

First note that

$$\|Tx\|^2 = \sum_{k=1}^{+\infty} r_k^2 \left|x_k\right|^2 \le \sum_{k=1}^{+\infty} \left|x_k\right|^2 = \|x\|^2,$$

thus $T \in B(H)$ and $\|T\| \le 1$. Furthermore,

$$(Tx,y) = \sum_{k=1}^{+\infty} r_k x_k \overline{y_k} = \sum_{k=1}^{+\infty} x_k \overline{r_k y_k} = (x,Ty),$$

proving that T is self adjoint. This implies that the residual spectrum is empty, $\sigma_r(T) = \emptyset$.

From $Te_k = r_k e_k$ follows that every $r_k \in \sigma_p(T)$, and we concluder further from $0 < r_k \le \|T\|$ that

$$\|T\| \ge \sup_{k \in \mathbb{N}} r_k = 1,$$

hence $\|T\| = 1$.

Conversely, if $Tx = \lambda\, x$, then

$$T x - \lambda\, x = \sum_{k=1}^{+\infty} (r_k - \lambda) x_k e_k = 0,$$

so either $\lambda = r_k$ or $x_k = 0$. This shows that

$$\sigma_p(T) = \mathbb{Q} \cap]0,1[= \{r_k \mid k \in \mathbb{N}\}.$$

Assume that $\lambda < 0$. Then

$$\|Tx - \lambda\, x\|^2 = \left\| \sum_{k=1}^{+\infty} (r_k + |\lambda|)\, x_k e_k \right\|^2 = \sum_{k=1}^{+\infty} (r_k + |\lambda|)^2\, |x_k|^2 \geq |\lambda|^2 \sum_{k=1}^{+\infty} |x_k|^2 = |\lambda|^2 \cdot \|x\|^2,$$

from which we infer that $\|Tx - \lambda\, x\| \geq |\lambda| \cdot \|x\|$, hence $\lambda \in \varrho(T)$. It follows that

$$\varrho(T) \supseteqq \mathbb{C} \setminus [0,1].$$

On the other hand, $\sigma(T)$ is closed, so it follows from

$$\sigma(T) \supseteqq \sigma_p(T) = \mathbb{Q} \cap]0,1[,$$

that $\sigma(T) \supseteqq [0,1]$. From $\varrho(T)$ and $\sigma(T)$ being disjoint we conclude that

$$\varrho(T) = \mathbb{C} \setminus [0,1] \qquad \text{and} \qquad \sigma(T) = [0,1].$$

Now, $\sigma_r(T) = \emptyset$ for self adjoint operators, and $\sigma_p(T) = \mathbb{Q} \cap]0,1[$, hence the continuous spectrum is

$$\sigma_c(T) = \sigma(T) \setminus \sigma_p(T) = ([0,1] \setminus \mathbb{Q}) \cup \{0,1\}.$$

Example 3.8 *Let (e_k) be an orthonormal basis in a Hilbert space H, and assume that $T \in B(H)$ has the matrix representation $\mathbf{T} = (t_{jk})$ with respect to the orthonormal basis (e_k) (see* Ventus, Hilbert spaces, Example 2.7*). Derive a necessary and sufficient condition on the t_{jk} that T is self adjoint.*

In Ventus, Hilbert spaces, Example 2.7 we derived that $t_{jk} = (Te_j, e_k)$, and

$$T\left(\sum_{j=1}^{+\infty} x_j e_j \right) = \sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} x_j t_{jk} e_k = \sum_{k=1}^{+\infty} \left\{ \sum_{j=1}^{+\infty} x_j t_{jk} \right\} e_k.$$

If

$$x = \sum_{j=1}^{+\infty} x_j e_j \qquad \text{and} \qquad y = \sum_{k=1}^{+\infty} y_k e_k,$$

then

$$\begin{aligned}(Tx, y) &= \left(\sum_{k=1}^{+\infty} \left\{ \sum_{j=1}^{+\infty} x_j t_{jk} \right\} e_k, \sum_{k=1}^{+\infty} y_k e_k \right) = \sum_{k=1}^{+\infty} \sum_{j=1}^{+\infty} x_j t_{jk} \overline{y_k} = \sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} \overline{y_k}\, t_{jk} x_j \\ &= \left(\sum_{j=1}^{+\infty} x_j e_j, \sum_{j=1}^{+\infty} \left\{ \sum_{k=1}^{+\infty} \overline{t_{jk}} y_k \right\} e_j \right) = (x, T^\star y)\,.\end{aligned}$$

Hence

$$T^\star y = \sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty} y_k t_{jk}^\star e_j = \sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty} y_k \overline{t_{jk}} e_j,$$

so $\mathbf{T}^\star = \left(t_{jk}^\star\right) = \left(\overline{t_{kj}}\right)$. This means that $\mathbf{T}^\star$ is obtained from $\mathbf{T}$ by taking the transpose and apply complex conjugating.

It follows from the above that T is self adjoint if and only if $\mathbf{T}^\star = \mathbf{T}$, i.e. if and only if

$$\overline{t_{kj}} = t_{jk} \qquad \text{for all } j,\, k = 1,\, 2,\, 3,\, \dots .$$

Note that

$$\overline{t_{kj}} = \overline{(Te_k, e_j)} = (e_j, Te_k)\,,$$

so the example shows that in this case T is self adjoint, if

$$(Te_j, e_k) = (e_j, Te_k) \qquad \text{for all } j,\, k \in \mathbb{N},$$

and there is nothing new in that statement.

Example 3.9 *Let $H = L^2(\mathbb{R})$, and let V denote a bounded real continuous function. We define the operator T by*

$$Tf(x) = V(x)\cdot f(x), \qquad f \in L^2(\mathbb{R}).$$

Prove that T is a bounded self adjoint operator.
In Quantum Mechanics the operator T is called a potential operator.

It follows from $\|V\|_\infty < +\infty$ that

$$\begin{aligned}\|Tf\|_2^2 &= \int_{-\infty}^{+\infty} V(x)f(x)\cdot\overline{V(x)f(x)}\,dx = \int_{-\infty}^{+\infty} V(x)^2|f(x)|^2\,dx \\ &\le \|V\|_\infty^2 \int_{-\infty}^{+\infty} |f(x)|^2 dx = \|V\|_\infty^2\cdot\|f\|_2^2,\end{aligned}$$

hence

$$\|Tf\|_2 \le \|V\|_\infty \cdot \|f\|_2 \qquad \text{for ethvert } f \in L^2(\mathbb{R}).$$

We conclude that $T \in B(V)$ and $\|T\| \le \|V\|_\infty$.

Utilizing that $V(x)$ is real we see that

$$(Tf, g) = \int_{-\infty}^{+\infty} V(x)f(x)\cdot\overline{g(x)}\,dx = \int_{-\infty}^{+\infty} f(x)\cdot\overline{V(x)g(x)}\,dx = (f, Tg),$$

which shows that T is self adjoint.

Example 3.10 *Let H denote a Hilbert space. Introduce in the set of all self adjoint operators from $B(H)$ a relation $\leq$ by*

$$S \leq T, \qquad \text{if } T - S \geq 0,$$

cf. EXAMPLE 6.1. *Prove that $\leq$ is a partial relation.*

It follows from $S - S = 0 \geq 0$ that $S \leq S$.

Assume that $S \leq T$ and $T \leq U$, thus $T - S \geq 0$ and $U - T \geq 0$.
We shall prove that $S \leq U$, i.e. that $U - S \geq 0$.
We have

$$\begin{aligned} ((U-S)x, x) &= ((U-T) + (T-S)x, x) \\ &= ((U-T)x, x) + ((T-S)x, x) \geq 0. \end{aligned}$$

This holds for every $x \in H$, hence the claim is proved.

Example 3.11 *Let H be a Hilbert space and let $T \in B(H)$ be positive and self adjoint. Show that*

$$\|(Tx, y)\|^2 \leq (Tx, x)\,(Ty, y),$$

for all x, $y \in H$.

We shall here be aware of two possible obstacles. First, (Tx, y) could be a complex number, and secondly (Tx, x) could be 0, so we must never divide by (Tx, x).

Let x, $y \in H$ be given, and choose $\alpha \in \mathbb{R}$ such that

$$(Tx, y) = |(Tx, y)|\, e^{i\alpha}.$$

Using the assumption it follows for any $\lambda \in \mathbb{C}$ that

$$\begin{aligned} 0 &\leq (T(\lambda x + y), \lambda x + y) \\ &= |\lambda|^2 (Tx, x) + \lambda (Tx, y) + \overline{\lambda}\,(Ty, x) + (Ty, y) \\ &= |\lambda|^2 (Tx, x) + \lambda (Tx, y) + \overline{\lambda}\,(y, Tx) + (Ty, y) \\ &= |\lambda|^2 (Tx, x) + 2\,\mathrm{Re}\{\lambda (Tx, y)\} + (Ty, y), \end{aligned}$$

where we have used that T is self adjoint, hence

$$(Ty, x) = (t, Tx) = \overline{(Tx, y)}.$$

Choosing in particular $\lambda = \mu\, e^{-i\alpha}$, $\mu \in \mathbb{R}$, then

$$\mu^2 (Tx, x) + 2\mu |(Tx, y)| + (Ty, x) \geq 0 \qquad \text{for all } \mu \in \mathbb{R}.$$

All coefficients are real, so the condition of the discriminant $B^2 - AC \leq 0$ holds, thus

$$|(Tx, y)|^2 \leq (Tx, x)\,(Ty, y) \qquad \text{for all } x,\ y \in H,$$

and the claim is proved.

Example 3.12 *1) Let V denote a normed space. Show that*

$$\|x-y\| \geq |\,\|x\|-\|y\|\,| \qquad \textit{for all } x,\, y \in V.$$

2) Let T be a bounded, linear and self adjoint operator on a Hilbert space. Assume that T is surjective and show that T is then injective.

3) Assume that T is a closed linear operator on a normed space X. Show that $\ker(T)$ *is closed in X.*

4) Let H denote a Hilbert space and assume that (x_n) and (y_n) are two sequences in the closed unit ball of H such that $(x_n, y_n) \to 1$. Show that $\|x_n - y_n\| \to 0$.

5) Let (x_n) and (y_n) denote two orthonormal sequences in a Hilbert space H, and assume that

$$\sum_{n=1}^{\infty} \|x_n - y_n\|^2 < 1.$$

Show that if (x_n) is an orthonormal basis, then so is (y_n).

1) It follows from the triangle inequality that

$$\|x\| = \|(x-y)+y\| \leq \|x-y\| + \|y\|,$$

and analogously (or just by interchanging letters)

$$\|y\| \leq \|x-y\| + \|x\|.$$

By a rearrangement,

$$\left.\begin{array}{l}\|x\|-\|y\| \\ \|y\|-\|x\|\end{array}\right\} \le \|x-y\|,$$

hence

$$\|x-y\| \ge |\,\|x\|-\|y\|\,|.$$

2) We shall prove that if $Tx=0$, then $x=0$. We get for every $y \in H$ that

$$0=(0,y)=(Tx,y)=(x,Ty).$$

From T being surjective follows that the image of T is all of H, so x is perpendicular to H, thus $x=0$, and T is injective.

3) Let T be closed, thus the graph $\mathcal{G}(T)$ is closed as a subset of $X \times X$. Let $(x_n) \subset \ker(T)$ denote a convergent sequence in X, i.e. $x_n \to x$. Then $((x_n,0)) \subset \mathcal{G}(T)$, and

$$(x_n,0) \to (x,0) \in \overline{\mathcal{G}(T)} = \mathcal{G}(T),$$

which shows that $x \in \ker(T)$.

4) Here,

$$\begin{aligned}\|x_n-y_n\|^2 &= (x_n-y_n,x_ny_n)=(x_n,x_n)-(y_n,x_n)-(x_n,y_n)+(y_n,y_n) \\ &= \|x_n\|^2+\|y_n\|^2-2\,\mathrm{Re}\,\{(x_n,y_n)\}\,,\end{aligned}$$

and since all x_n and y_n belong to the unit ball, we have

$$0 \le \|x_n-y_n\|^2 \le 1+1-2\,\mathrm{Re}\,\{(x_n,y_n)\} \to 2-2=0 \qquad \text{for } n\to\infty,$$

proving that

$$\|x_n-y_n\| \to 0 \qquad \text{for } n \to \infty.$$

5) Let $x \in H$ be perpendicular to all y_n. From (x_n) being an orthonormal basis and $(x,y_n)=0$ we get

$$x=\sum_{n=1}^{\infty}(x,x_n)\,x_n=\sum_{n=1}^{\infty}\{(x,y_n)+(x,x_n-y_n)\}\,x_n=\sum_{n=1}^{\infty}(x,x_n-y_n)\,x_n.$$

This implies the estimate, when we apply that (x_n) is orthonormal and the Cauchy-Schwarz inequality,

$$\|x\|=\sum_{n=1}^{\infty}|(x,x_n-y_n)|^2 \le \sum_{n=1}^{\infty}\|x\|^2\cdot\|x_n-y_n\|^2=\|x\|^2\sum_{n=1}^{\infty}\|x_n-y_n\|^2.$$

It follows from the assumption that $\sum_{n=1}^{\infty}\|x_n-y_n\|^2<1$, so the only possibility for this inequality is when $x=0$, hence $x=0$ is the only vector in H, which is perpendicular on all y_n. This shows that (y_n) is an orthonormal basis.

Example 3.13 *Let $(x_n) \subset \ell^2$ and define the sequence $y = (y_n)$ by*

$$y_n = x_{n+1} + n\,x_n + x_{n-1},$$

where we put $x_0 = 0$ whenever it is necessary.

1. *Show that $y \in \ell^2$ if and only if $(n\,x_n) \in \ell^2$.*

Let

$$D = \{x \in \ell^2 \mid (n\,x_n) \in \ell^2\},$$

and define a linear operator $T : D \to \ell^2$ by $Tx = y$, where y is given above.

2. *Show that D is dense in ℓ^2.*

3. *Show that T is self adjoint.*

1) It follows from

$$(y_n) = (x_{n+1}) + (n\,x_n) + (x_{n-1}),$$

and that ℓ^2 is a vector space that if (x_n) and $n\,x_n) \in \ell^2$, then $(y_n) \in \ell^2$.

If conversely (x_n) and $(y_n) \in \ell^2$, then it follows from

$$(n\,x_n) = (y_n) - (x_{n+1}) - (x_{n-1}),$$

that $(n\,x_n) \in \ell^2$.

Alternatively, we have the following possible, though not very brilliant variant,

$$\begin{aligned}
\sum_{n=1}^{+\infty} y_n^2 &= \sum_{n=1}^{+\infty} \left(x_{n+1} + n\,x_n + x_{n-1}\right)^2 \\
&= \sum_{n=1}^{+\infty} x_{n+1}^2 + \sum_{n=1}^{+\infty} (n\,x_n)^2 + \sum_{n=1}^{+\infty} x_{n-1}^2 + 2\sum_{n=1}^{+\infty} x_{n+1} n\,x_n + 2\sum_{n=1}^{+\infty} n\,x_n x_{n-1} + 2\sum_{n=1}^{+\infty} x_{n+1}x_{n-1} \\
&\le \|x\|_2^2 + \sum_{n=1}^{+\infty} (n\,x_n)^2 + 2\|x\|_2 \left\{\sum_{n=1}^{+\infty} (n\,x_n)^2\right\}^{\frac{1}{2}} + 2\left\{\sum_{n=1}^{+\infty} (n\,x_n)^2\right\}^2 \|x\|_2 + 2\|x\|_2\|x\|_2 \\
&= 4\,\|x\|_2^2 + 4\,\|x\| \left\{\sum_{n=1}^{+\infty} (n\,x_n)^2\right\}^{\frac{1}{2}} + \sum_{n=1}^{+\infty} (n\,x_n)^2 \left(\left\{\sum_{n=1}^{+\infty} (n\,x_n)^2\right\}^{\frac{1}{2}} + 2\,\|x\|_2\right)^2.
\end{aligned}$$

Hence, if $\sum_{n=1}^{+\infty} (n\,x_n)^2 < +\infty$, then $\sum_{n=1}^{+\infty} y_n^2 < +\infty$, so $y \in \ell^2$.

Conversely, if $y \in \ell^2$, then by a rearrangement,

$$n\,x_n = y_n - x_{n+1} - x_{n-1},$$

hence

$$\begin{aligned}
\sum_{n=1}^{+\infty}(n\,x_n)^2 &= \sum_{n=1}^{+\infty}\left(y_n - x_{n+1} - x_{n-1}\right)^2 \\
&= \sum_{n=1}^{+\infty} y_n^2 + \sum_{n=1}^{+\infty} x_{n+1}^2 + \sum_{n=1}^{+\infty} x_{n-1}^2 - 2\sum_{n=1}^{+\infty} y_n x_{n+1} - 2\sum_{n=1}^{+\infty} y_n x_{n-1} + 2\sum_{n=1}^{+\infty} x_{n+1}x_{n-1} \\
&\le \|y\|_2^n + \|x\|_2^2 + \|x\|_2^2 + 2\|y\|_2\|x\|_2 + 2\|y\|_2\|x\|_2 + 2\|x\|_2\|x\|_2 \\
&= \|y\|_2^2 + 4\,\|y\|_2\|x\|_2 + 4\,\|x\|_2^2 = \left\{\|y\|_2 + 2\,\|x\|_2\right\}^2 < +\infty.
\end{aligned}$$

We conclude that $(n\,x_n) \in \ell^2$.

2) Let $D = \{x \in \ell^2 \mid (n\,x_n) \in \ell^2\}$, and let $z \in \ell^2$ be arbitrary, i.e. $\sum_{n=1}^{+\infty} z_n^2 < +\infty$. To any $\varepsilon > 0$ there exists an $N \in \mathbb{N}$, such that

$$\sum_{n=N+1}^{+\infty} z_n^2 < \varepsilon^2.$$

Define $x = (x_n)$ by

$$x_n = \begin{cases} z_n & \text{for } n = 1,\, 2,\, \dots\, ,\, N, \\ 0 & \text{for } n > N. \end{cases}$$

Then

$$\sum_{n=1}^{+\infty}(n\,x_n)^2 = \sum_{n=1}^{N} n^2 x_n^2 < +\infty,$$

because the sum is finite, so $x \in D$, and

$$\|z-x\|_2 = \left\{\sum_{n=1}^{+\infty}(z_n - x_n)^2\right\}^{\frac{1}{2}} = \left\{\sum_{n=N+1}^{+\infty} z_n^2\right\}^{\frac{1}{2}} < \left(\varepsilon^2\right)^{\frac{1}{2}} = \varepsilon,$$

which shows that x approximates z, and we get that D is dense in ℓ^2. Clearly, D is a subspace, because (x_n), (y_n), $(n\,x_n)$, $(n\,y_n) \in \ell^2$ for every $\lambda \in \mathbb{R}$ imply that $(x_n + \lambda y_n)$ and $(n(x_n + \lambda y_n)) = (n\,x_n + \lambda n\,y_n) \in \ell^2$. Finally, it is obvious that T is linear.

3) Because T is densely defined, the adjoint $T^\star$ exists. Let $x \in D$, and let $y \in \mathcal{D}\left(T^\star\right)$. Then

$$(Tx, y) = (x, T^\star y)\,,$$

thus

$$
\begin{aligned}
(Tx, y) &= \sum_{n=1}^{+\infty} (x_{n+1} + n\, x_n + x_{n-1})\, y_n \\
&= \sum_{n=1}^{+\infty} x_{n+1} y_n + \sum_{n=1}^{+\infty} n\, x_n y_n + \sum_{n=1}^{+\infty} x_{n-1} y_n \\
&= \sum_{n=2}^{+\infty} x_n y_{n-1} + \sum_{n=1}^{+\infty} x_n n\, y_n + \sum_{n=0}^{+\infty} x_n y_{n+1} \\
&= \sum_{n=1}^{+\infty} x_n y_{n-1} + \sum_{n=1}^{+\infty} x_n n\, y_n + \sum_{n=1}^{+\infty} x_n y_{n+1} \\
&= \sum_{n=1}^{+\infty} x_n \left(y_{n+1} + n\, y_n + y_{n-1}\right) = (x, T^\star y)\,.
\end{aligned}
$$

The splitting of the sums in the second equality is legal, because each of the three series on the right hand side is absolutely convergent by the Cauchy-Schwarz inequality. Hence we conclude that

$$T^\star y = (y_{n+1} + n\, y_n + y_{n-1})\,,$$

thus $D \subseteqq \mathcal{D}(T^\star)$, and $T \subseteqq T^\star$, so T is at least symmetric.

It follows from the result of (1) that $(y_{n+1} + n\, y_n + y_{n-1}) \in \ell^2$, when $(y_n) \in \ell^2$, if and only if $(n\, y_n) \in \ell^2$. Hence $\mathcal{D}(T^\star) = D$, and $T = T^\star$, and we have proved that T is self adjoint.

4 Isometric operators

Example 4.1 *Let $T \in B(H)$. An operator is called* isometric *if $\|Tx\| = \|x\|$ for all $x \in H$. Show that the following conditions are equivalent for $T \in B(H)$.*

1) T is isometric.

2) $T^\star T = I$.

3) $(Tx, Ty) = (x, y)$ for all x, $y \in H$.

(3) $\Rightarrow$ (2). This is almost trivial:

$$(x, y) = (Tx, Ty) = (T^\star Tx, y) \qquad \text{for all } x,\ y \in H,$$

thus $T^\star Tx = x$ for all $x \in H$, and hence $T^\star T = I$.

(2) $\Rightarrow$ (1). If $T^\star T = I$, then

$$\|Tx\|^2 = (Tx, Tx) = (T^\star Tx, x) = (x, x) = \|x\|^2,$$

proving that T is isometric.

(1) $\Rightarrow$ (3). If T is isometric, we get as above,

$$(T^\star Tx, x) = (Ix, x), \quad \text{thus} \quad ((T^\star T - I)x, x) = 0,$$

for all $x \in H$. Then it follows from EXAMPLE 1.8 in VENTUS, FUNCTIONAL ANALYSIS, HILBERT SPACES that $T^\star T - I = 0$, if H is a complex Hilbert space, hence $T^\star T = I$.

Example 4.2 *Let $T \in B(H)$ be an isometric operator. Show that $T(H)$ is a closed subspace. Show that $T(H) = H$ if H is finite dimensional. Give an example of an isometric operator with $T(H) \neq H$.*

1) When $T \in B(H)$ is isometric, i.e. $\|Tx\| = \|x\|$ for all $x \in H$, then in particular T is injective, thus $T^{-1} : T(H) \to H$ exists.
   Put $y = Tx$. Then it follows from the above that $\left\|T^{-1}t\right\| = \|y\|$, and T^{-1} is continuous (though not necessarily defined in all of H).
   Now, H is closed, so $T(H) = \left(T^{-1}\right)^{-1}(H)$ is also closed.

2) Let H be finite dimensional, $\dim H = n$, and denote by $\{e_1, \dots, e_n\}$ a basis of H.
   When T is isometric, then T is injective. In fact, $0 = \|Tx\| = \|x\|$ implies trivially that $x = 0$.
   We claim that the images $\{Te_1, \dots, Te_n\}$ of the basis vectors are linearly independent. Assume that

   $$0 = \lambda_1 Te_1 + \cdots + \lambda_n Te_n \qquad (= T(\lambda_1 e_1 + \cdots + \lambda_n e_n)).$$

   The operator T is injective, so also $\lambda_1 e_1 + \cdots + \lambda_n e_n = 0$. Here $\{e_1, \dots, e_n\}$ is a basis, so $\lambda_1 = \cdots = \lambda_n = 0$. It follows that $Te_1, \dots, Te_n$ are linearly independent, so $n \leq \dim T(H) \leq n$, thus $\dim T(H) = n$. This is only possible, if $T(H) = H$, because $T : H \to H$.

3) Let $(e_k)_{k \in \mathbb{N}}$ denote an orthonormal basis in an infinite dimensional Hilbert space. Define $T \in B(H)$ by

$$Tx = T\left(\sum_{k=1}^{\infty} x_k e_k\right) = \sum_{k=1}^{\infty} x_k e_{k+1}.$$

Then clearly T is isometric, $\|Tx\| = \|x\|$ for all $x \in H$, and

$$T(H) = \{e_1\}^{\perp} \neq H.$$

Example 4.3 *Let $T \in B(H)$ be an isometric operator and let M and N denote closed subspaces of the Hilbert space H. Show that*

$$T(M) = N \quad \Longrightarrow \quad T\left(M^{\perp}\right) \subset N^{\perp}.$$

Show that T is isometric if and only if for any orthonormal basis (e_k), (Te_k) is an orthonormal sequence.

Assume that $T \in B(H)$ is isometric, and let M and $N \subseteqq H$ be closed subspaces, and assume that $T(M) = N$. We shall prove that for every $x \in M^{\perp}$ and for every $y \in N$ we have that $(Tx, y) = 0$.

From $y \in N = T(M)$ follows that there exists a $z \in M$, such that $y = Tz$, and then we get from Example 4.1, (3) that

$$(Tx, y) = (Tx, Tz) = (x, z) = 0,$$

because $x \in M^{\perp}$ and $z \in M$. It follows that $T\left(M^{\perp}\right) \subseteqq N^{\perp}$.

Let (e_k) denote an orthonormal basis, and assume that T is isometric. We get again from Example 4.1, (3) that

$$(Te_j, Te_k) = (e_j, e_k) = \delta_{jk},$$

(Kronecker symbol), which shows that (Te_k) is an orthonormal *sequence*. Of course (Te_k) needs *not* be a basis. An example is given in Example 4.2.

If conversely there exists an orthonormal basis (e_k), such that (Te_k) is an orthonormal sequence, then

$$Tx = \sum_{k=1}^{+\infty} x_k Te_k, \qquad \text{thus} \qquad \|Tx\|^2 = \sum_{k=1}^{+\infty} |x_k|^2 = \|x\|^2,$$

and T is isometric.

Remark 4.1 The answer of the latter question above shows that if there is just one orthonormal basis (e_k), such that (Te_k) is an orthonormal sequence, then every orthonormal basis has this property. $\Diamond$

Example 4.4 *Let $T \in B(H)$ be an isometric operator. Show that $TT^\star$ is a projection and determine its range.*

Assume that $T \in B(H)$ is isometric. We shall prove that $TT^\star$ is a projection, i.e. $TT^\star$ must satisfy the two conditions,

$$(TT^\star x, y) = (x, TT^\star y) \qquad \text{for all } x,\, y \in H,$$

and

$$(TT^\star)^2 = TT^\star.$$

We get

$$(TT^\star x, y) = (T^\star x, T^\star y) = (x, TT^\star y)\,,$$

and the first condition is fulfilled. Then apply the result $T^\star T = I$ from Example 4.1, (2),

$$(TT^\star)^2 = TT^\star TT^\star = T\,(T^\star T)\,T^\star = TIT^\star = TT^\star,$$

and it follows that $P = TT^\star$ is a projection.

The range of the projection $P = TT^\star$ is given by $Px = TT^\star x = x$, i.e. $TT^\star H$. Now,

$$T^\star(H) = \overline{T^\star(H)} = \ker(T)^\perp,$$

thus $TT^\star(H) = T\left(\ker(T)^\perp\right)$. It follows from

$$H = \ker(T) \oplus \ker(T)^\perp,$$

that

$$TT^\star(H) = T\left(\ker(T)^\perp\right) = T\left(\ker(T) \oplus \ker(T)^\perp\right) = T(H),$$

and the range is TH.

Example 4.5 *Consider the Hilbert space $L^2([0,\infty))$. Let $h > 0$ and define the operator T by*

$$\begin{array}{ll} Tf(x) = 0 & \text{for } 0 \le x < h, \\ Tf(x) = f(x-h) & \text{for } h \le x. \end{array}$$

Show that T is isometric and determine $T^\star$. Find $TT^\star$ and $T^\star T$.

First notice that

$$\|Tf\|_2^2 = \int_0^{+\infty} |Tf(x)|^2 dx = \int_h^{+\infty} |f(x-h)|^2 dx = \int_0^{+\infty} |f(x)|^2 dx = \|f\|_2^2,$$

which shows that T is isometric. Then it follows from Example 4.1, (2) that $T^\star T = I$.

Let $f, g \in H$. Then

$$\begin{aligned}(Tf, g) &= \int_0^{+\infty} Tf(x)\overline{g(x)}\,dx = \int_h^{+\infty} f(x-h)\overline{g(x)}\,dx \\ &= \int_0^{+\infty} f(x)\overline{g(x+h)}\,dx = (f, T^\star g)\,,\end{aligned}$$

and we conclude that

$$T^\star g(x) = g(x+h) \qquad \text{for } x \in [0, +\infty[.$$

Then finally we get

$$TT^\star g(x) = Tg(x+h) = \begin{cases} g(x+h-h) = g(x) & \text{for } x \in [h, +\infty[, \\ 0 & \text{for } x \in [0, h[, \end{cases}$$

thus $TT^\star g = 1_{[h,+\infty[} \cdot g$.

5 Unitary and normal operators

Example 5.1 *An operator $T \in B(H)$ is called* unitary *if it is isometric and surjective. Show that the following conditions are equivalent for an operator $T \in B(H)$,*

(a) *T is unitary.*

(b) *T is bijective and $T^{-1} = T^\star$.*

(c) *$T^\star = TT^\star = I$.*

(d) *T and $T^\star$ are isometric.*

(e) *T is isometric and $T^\star$ is injective.*

(f) *$T^\star$ is unitary.*

(a) $\Rightarrow$ (b). Assume that T is unitary, thus $T(H) = H$, and $\|Tx\| = \|x\|$ for $x \in H$. Clearly, $Tx = 0$ implies that $x = 0$, so T is injective, and T^{-1} exists and is continuous with $\|T\|^{-1} = 1$. (Sketch of proof: Put $y = Tx$, etc.) From $D\left(T^{-1}\right) = T(H) = H$, we even get that $T^{-1} \in B(H)$, and we conclude that T is bijective.

Then it follows from Example 4.1, (2) that $T^\star T = I$, and from the definition of T^{-1} we get $T^{-1}T = I$. Hence,

$$0 = \left(T^\star - T^{-1}\right) T, \qquad \text{thus} \qquad \left(T^\star - T^{-1}\right) T(H) = \{0\}.$$

From $T(H) = H$ follows that $T^\star - T^{-1}$ is identically 0 on all of H, thus $T^\star = T^{-1}$.

(b) $\Rightarrow$ (c). Assume that T is bijective and that $T^{-1} = T^\star$. Then

$$T^\star T = T^{-1}T = I \qquad \text{and} \qquad TT^\star = TT^{-1} = I.$$

(c) $\Rightarrow$ (d). Let $T^\star T = TT^\star = I$. It follows from Example 4.1, (2) that T is isometric. Then we conclude from

$$(TT^\star)^\star = (T^\star)^\star T^\star = I^\star = I,$$

that $T^\star$ is also isometric by Example 4.1, (2).

(d) $\Rightarrow$ (e). If T and $T^\star$ are isometric, then $T^\star$ is in particular injective.

(e) $\Rightarrow$ (a). Assume that T is isometric and that $T^\star$ is injective. We shall prove (a), so it only remains to prove that $T(H) = H$.

Because $T(H)$ is closed, it suffices to prove that if

$$(Ty, x) = 0 \qquad \text{for all } y \in H,$$

then $x = 0$. We have

$$0 = (Ty, x) = (y, T^\star x) \qquad \text{for all } y \in H.$$

When we in particular choose $y = T^\star x$, then

$$(T^\star x, T^\star x) = \|T^\star x\|^2 = 0, \qquad \text{thus} \qquad T^\star x = 0.$$

Now, $T^\star$ is injective, so $x = 0$.

Summing up we have proved that (a)–(e) are equivalent. We shall only prove that we can add (f) to this family of equivalent conditions.

(a) $\wedge$ (d) $\Rightarrow$ (f). If T is unitary, then $T^\star$ and $T^{\star\star} = T$ are isometric, so $T^\star$ is unitary by (d).

(f) $\wedge$ (d)$\Rightarrow$ (a). If $T^\star$ is unitary, then $T^\star$ and $T^{\star\star} = Y$ are isometric, and T is unitary by (d).

Example 5.2 *Let (e_k) denote an orthonormal basis in a Hilbert space H and let $T \in B(H)$ be given by*

$$T\left(\sum_{k=1}^{\infty} a_k e_k\right) = \sum_{k=1}^{\infty} \lambda_k a_k e_k.$$

Show that T is unitary if and only if $|\lambda_k| = 1$ for all k.

We conclude from

$$\|Tx\|^2 = \left\|\sum_{k=1}^{\infty} \lambda_k x_k e_k\right\|^2 = \sum_{k=1}^{\infty} |\lambda_k|^2 \, |x_k|^2 \,,$$

that if $|\lambda_k| = 1$ for all k, then $\|Tx\| = \|x\|$, hence T is isometric.

If there exists a k, such that $|\lambda_k| \neq 1$, then $\|Te_k\| = |\lambda_k| \neq 1 = \|e_k\|$, and T is not isometric.

We have proved that T is isometric, if and only if $|\lambda_k| = 1$ for all $k \in \mathbb{N}$. We shall only prove that *if* $|\lambda_k| = 1$ for all $k \in \mathbb{N}$, *then* $T(H) = H$, because this implies by Example 5.1 that T is unitary.

Let $y \in H$, i.e.

$$y = \sum_{k=1}^{\infty} y_k e_k \qquad \text{and} \qquad \sum_{k=1}^{\infty} |y_k|^2 < \infty.$$

If there exists an $x \in H$, such that $Tx = y$, then

$$\sum_{k=1}^{\infty} \lambda_k x_k e_k = \sum_{k=1}^{\infty} y_k e_k \qquad \text{and} \qquad \sum_{k=1}^{\infty} |x_k|^2 < \infty.$$

It is seen by the identification that since $\lambda_k \cdot \overline{\lambda_k} = |\lambda_k|^2 = 1$, we have only the possibility that $\lambda_k x_k = y_k$, thus

$$x_k = \frac{y_k}{\lambda_k} = \overline{\lambda_k} y_k.$$

We shall only prove that the *candidate*

$$x = \sum_{k=1}^{\infty} \overline{\lambda_k} y_k e_k$$

belongs to H. This is trivial, because

$$\sum_{k=1}^{\infty} |x_k|^2 = \sum_{k=1}^{\infty} |\overline{\lambda_k}|^2 |y_k|^2 = \sum_{k=1}^{\infty} |y_k|^2 = \|y\|^2 < \infty,$$

so $x \in H$, and $Tx = y$. This proves that $T(H) = H$, and it then follows from EXAMPLE 5.1 that T is unitary.

Example 5.3 *Let $T \in B(H)$ be unitary. Show that*

$$\sigma(T) \subset \{z \in \mathbb{C} \mid |z| = 1\}.$$

Let $|\lambda| \neq 1$. Because T is unitary, we get in particular that $\|T\| = \|x\|$, hence

$$\|Tx - \lambda\, x\| \geq |\, \|Tx\| - \|\lambda\, x\|\, | = |1 - |\lambda|\,| \cdot \|x\|.$$

It follows that $(T - \lambda\, I)^{-1}$ exists for every $\lambda \in \mathbb{C}$, for which $|\lambda| \neq 1$. We shall finish the proof by showing that $(T - \lambda\, I)^{-1}$ is densely defines in H, because then

$$\varrho(T) \supseteqq \mathbb{C} \setminus \{z \in \mathbb{C} \mid |z| = 1\} \quad \text{and} \quad \sigma(T) \subseteqq \{z \in \mathbb{C} \mid |z| = 1\}.$$

Assume that $(T - \lambda\, I)^{-1}$ is not densely defined for some $\lambda \in \mathbb{C}$. Then there exists an $y \neq 0$, such that

$$y \perp (T - \lambda\, I)D(T - \lambda\, I) = (T - \lambda\, I)(H),$$

thus

$$0 = ((T - \lambda\, I)x, y) = \left(x, \left(T^\star - \overline{\lambda}\, I\right) y\right) = (x, 0) \quad \text{for all } x \in H.$$

We conclude that $T^\star y - \overline{\lambda}\, y = 0$, hence $\overline{\lambda}$ is even an eigenvalue for $T^\star = T^{-1}$.

By EXAMPLE 5.1, $T^\star$ is also unitary, thus $|\overline{\lambda}| = 1$, and hence also $|\lambda| = 1$. Then it follows by contraposition that *if* $|\lambda| \neq 1$, *then* $(T - \lambda\, I)^{-1}$ is densely defined. Then

$$\varrho(T) \supseteqq \mathbb{C} \setminus \{z \in \mathbb{C} \mid |z| = 1\} \quad \text{and} \quad \sigma(T) \subseteqq \{z \in \mathbb{C} \mid |z| = 1\}.$$

Example 5.4 *An operator $T \in B(H)$ is* normal *if*

$$TT^\star = T^\star T.$$

Show that T is normal if and only if $\|T^\star x\| = \|Tx\|$ for all $x \in H$.

If $T \in B(H)$ is normal, i.e. $T^\star T = TT^\star$, then

$$\|T^\star x\|^2 = (T^\star x, T^\star x) = (TT^\star x, x) = (T^\star Tx, x) = (Tx, Tx) = \|Tx\|^2,$$

and we conclude that $\|T^\star x\| = \|Tx\|$ for all $x \in H$.

Assume conversely that $\|T^\star x\| = \|Tx\|$ for all $x \in H$. Then

$$0 = \|T^\star x\|^2 - \|Tx\|^2 = (T^\star x, T^\star x) - (Tx.Tx) = (TT^\star x, x) - (T^\star Tx, x) = ((TT^\star - T^\star T)\, x, x)\,.$$

The space H is complex. so it follows that $TT^\star - T^\star T = 0$, hence $T^\star T = TT^\star$ as required.

Example 5.5 *Let $T \in B(H)$ be normal. Show that*

$$\|(T - \lambda I)x\| = \left\|\left(T^\star - \overline{\lambda}\, I\right) x\right\|$$

for all $x \in H$. Show that $\sigma_r(T)$ is empty.

If T is normal, then $T^\star T = TT^\star$, and we get

$$\begin{aligned}
\|(T - \lambda T)x\|^2 &= ((T - \lambda I)x, (T - \lambda I)x \\
&= (Tx, Tx) - \lambda(x.Tx) - \overline{\lambda}(Tx, x) + |\lambda|^2(x, x) \\
&= (T^\star Tx, x) - \lambda\,(T^\star x, x) - \overline{\lambda}\,(x, T^\star x) + |\lambda|^2(x, x) \\
&= (TT^\star x, x) - \left(T^\star x, \overline{\lambda}\, x\right) - \left(\overline{\lambda}\, x, T^\star x\right) + \left(\overline{\lambda}\, x, \overline{\lambda}\, x\right) \\
&= (T^\star x, T^\star x) - \left(T^\star x, \overline{\lambda}\, x\right) - \left(\overline{\lambda}\, x, T^\star x\right) + \left(\overline{\lambda}\, x, \overline{\lambda}\, x\right) \\
&= \left(\left(T^\star - \overline{\lambda}\, I\right) x, \left(T^\star - \overline{\lambda}\, I\right) x\right) = \left\|\left(T^\star - \overline{\lambda}\, I\right) x\right\|^2,
\end{aligned}$$

and the first claim is proved.

It follows that λ is an eigenvalue for T (of eigenvector x), if and only if $\overline{\lambda}$ is an eigenvalue for $T^\star$ (the same eigenvector x), thus

$$\sigma_p\left(T^\star\right) = \overline{\sigma_p(T)}.$$

On the other hand, $\sigma_r(T) \subseteq \overline{\sigma_p\left(T^\star\right)} = \sigma_p(T)$, and because $\sigma_r(T)$ and $\sigma_p(T)$ are disjoint, we must have $\sigma_r(T) = \emptyset$.

Example 5.6 *Let $H = L^2([0,1])$ and consider the operator*

$$Tf(x) = \sqrt{3}\, x\, f\left(x^3\right).$$

1) Show that $T \in B(H)$ and find $\|T\|$.

2) Show that T^{-1} exists and that $T^{-1} \in B(H)$.
Determine $T^{-1}g(y)$ for $g \in H$, and find $\left\|T^{-1}\right\|$.

3) Show that $\sigma(T) \subset \{\lambda \in \mathbb{C} \mid |\lambda| = \|T\|\}$.

1) The operator T is obviously linear.

Then by the change of variable $y = x^3$,

$$\|Tf\|_2^2 = \int_0^1 |Tf(x)|^2\, dx = \int_0^1 3x^2 \left|f\left(x^3\right)\right|^2\, dx = \int_0^1 |f(y)|^2\, dy = \|f\|_2^2,$$

hence T is isometric ($\|Tf\|_2 = \|f\|_2$), thus $T \in B(H)$ and $\|T\| = 1$.

2) We shall prove that the equation

$$Tf(x) = g(x), \qquad g \in L^2([0,1]),$$

always has a uniquely determined solution, thus $T^{-1} : H \to H$. It follows by the definition that we shall solve

$$Tf(x) = \sqrt{3}\, x\, f\left(x^3\right) = g(x).$$

Utilizing the monotone change of variable $x = \sqrt[3]{y}$, we get

$$f(y) = \frac{1}{\sqrt{3}} \cdot \frac{1}{x}\, g(x) = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt[3]{y}} \cdot g\left(\sqrt[3]{y}\right) = T^{-1}g(y),$$

hence

$$T^{-1}g(x) = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt[3]{x}}\, g\left(\sqrt[3]{x}\right), \qquad g \in H \in L^2([0,1]).$$

We get from the computation in (1) that $Tf = g$ and $f = T^{-1}g$ that

$$\|Tf\|_2 = \|g\|_2 = \|f\|_2 = \left\|T^{-1}g\right\|_2, \qquad T^{-1} \in B(H),$$

and T^{-1} is also isometric, $\left\|T^{-1}g\right\|_2 = \|g\|_2$, and $\left\|T^{-1}\right\| = 1$.
We say that T is *unitary*, cf. Example 5.1.

3) This has already been proved in Example 5.3. However, let us do it again. If $|\lambda| > 1$, then

$$T - \lambda I = -\lambda \left(I - \frac{1}{\lambda}\, T\right), \qquad \text{where } \left\|\frac{1}{\lambda}\, T\right\| = \frac{1}{|\lambda|} < 1,$$

thus $(T - \lambda I)^{-1} \in B(H)$, and $(T - \lambda\, I)^{-1}$ is given by the Neumann series

$$(T - \lambda\, I)^{-1} = -\frac{1}{\lambda} \sum_{n=0}^{+\infty} \frac{1}{\lambda^n}\, T^n.$$

Then let $|\lambda| < 1$. From $T^{-1} \in B(H)$ follows that $T - \lambda I = T\left(I - \lambda T^{-1}\right)$. From $\left\|\lambda T^{-1}\right\| = |\lambda| < 1$ follows by a Neumann series that

$$(T - \lambda I)^{-1} = \left(I - \lambda T^{-1}\right)^{-1} T^{-1} = \left(\sum_{n=0}^{+\infty} \lambda^n \left(T^{-1}\right)^n\right) = \sum_{n=0}^{+\infty} \lambda^n \left(T^{-1}\right)^{n+1},$$

hence $(T - \lambda I)^{-1} \in B(H)$, and we conclude that

$$\varrho(T) \supseteqq \{\lambda \in \mathbb{C} \mid |\lambda| \neq 1\} \quad \text{and} \quad \sigma(T) \subseteqq \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}.$$

6 Positive operators and projections

Example 6.1 *An operator $T \in B(H)$ is* positive *if*

$$(Tx, x) \geq 0 \qquad \textit{for all } x \in H,$$

and we write $T \geq 0$.
Prove the following:

1) $T \geq 0$ implies that T is self adjoint.

2) If S, $T \geq 0$, $\alpha \geq 0$, then $S + \alpha T \geq 0$.

3) If $T \geq 0$ and $S \in B(H)$, then $S^\star TS \geq 0$.

4) If $T \in B(H)$ then $T^\star T \geq 0$,

5) If T is an orthogonal projection then $T \geq 0$.

1) Assume that $T \in B(H)$ is positive, i.e. $(Tx, x) \geq 0$ for every $x \in H$. Then

$$(T^\star x, x) = (x, Tx) = \overline{(Tx, x)} = (Tx, x) \geq 0,$$

and $T^\star$ is also positive, and

$$((T^\star - T)\, x, x) = 0 \qquad \text{for every } x \in H.$$

Then assume that the vector space is complex. Then it follows that $T^\star - T = 0$, i.e. $T^\star = T$, and we have proved that T is self adjoint.

2) This is trivial: For every $x \in H$,

$$((S + \alpha T)x, x) = (Sx, x) + \alpha(Tx, x) \geq 0 + \alpha \cdot 0 = 0.$$

3) It follows from $Sx \in H$ for every $x \in H$ that

$$(S^\star TSx, x) = (T(Sx), Sx) \geq 0.$$

4) This is again trivial. In fact, for every $x \in H$,

$$(T^\star Tx, x) = (Tx, Tx) = \|Tx\|^2 \geq 0.$$

5) Let T denote an orthogonal projection. Then

$$T^\star = T \qquad \text{and} \qquad T^2 = T.$$

It follows from (4) that

$$T^\star T = TT = T^2 = T$$

is positive, hence $T \geq 0$.

Example 6.2 *Let P_M and P_N denote the orthogonal projections of the closed subspaces M and N of a Hilbert space H. Show that $M \subset N$ implies that $P_M \leq P_N$.*

If $M \subseteq N$, then

$$H = N \oplus N^{\perp} = M \oplus \left(M^{\perp} \cap N\right) \oplus N^{\perp},$$

which means that every element $x \in H$ has a unique decomposition

$$x = x_M + x_N + x^{\perp}, \qquad \text{where } x_m \in M, \quad x_N \in M^{\perp} \cap N, \quad x^{\perp} \in N^{\perp}.$$

Then

$$P_M x = P_M \left(x_M + x_N + x^{\perp}\right) = x_M \quad \text{and} \quad P_N x = P_N \left(x_M + x_N + x^{\perp}\right) = x_M + x_N.$$

It follows that

$$\begin{aligned} \left(\left(P_N - P_M\right) x, x\right) &= \left(x_M + x_n - x_M, x_M + x_N + x^{\perp}\right) = \left(x_N, x_M + x_N + x^{\perp}\right) \\ &= \left(x_N, x_M\right) + \left(x_N, x_N\right) + \left(x_N, x^{\perp}\right) \\ &= 0 + \|x_N\|^2 + 0 = \|x_N\|^2 \geq 0, \end{aligned}$$

hence $P_N - P_M \geq 0$, and whence $P_M \leq P_N$.

Example 6.3 *An operator $T \in B(H)$ is called a* contraction

$$\|Tx\| \le \|x\| \qquad \text{for all } x \in H.$$

Show that the following conditions are equivalent for an operator $T \in B(H)$:

1) T is a contraction,

2) $\|T\| \le 1$,

3) $T^\star T \le I$,

4) $TT^\star \le I$,

5) $T^\star$ is a contraction,

6) $T^\star T$ is a contraction.

(1) $\Rightarrow$ (2). Let $T \in B(H)$ denote a contraction, thus $\|Tx\| \le \|x\|$ for all $x \in H$. Then

$$\|T\| = \sup\{\|Tx\| \mid \|x\| \le 1\} \le \sup\{\|x\| \mid \|x\| \le 1\} = 1,$$

and we have proved (2).

(2) $\Rightarrow$ (3). Assume that $\|T\| \le 1$. Then

$$(9)\ ((I - T^\star T)\,x, x) \;=\; (x,x) - (T^\star Tx, x) = \|x\|^2 - (Tx, Tx)$$
$$= \; \|x\|^2 - \|Tx\|^2 \ge \|x\|^2 - 1 \cdot \|x\|^2 = 0,$$

and we have proved that $I = T^\star T \ge 0$, hence $T^\star T \le I$, and we have proved that (3).

(3) $\Rightarrow$ (1). Assume that $T^\star T \le I$. By repeating (9) we see that $\|x\|^2 - \|Tx\|^2 \ge 0$, thus $\|Tx\| \le \|x\|$, and we have proved (1).

It follows from the above that the former three conditions (1)–(3) are equivalent.

(1) $\Leftrightarrow$ (5). If T is a contraction, then by (2), $\|T^\star\| = \|T\| \le 1$, and we infer that $T^\star$ is a contraction.

If conversely $T^\star$ is a contraction, then $T^{\star\star} = T$ is contraction.

We have proved that the conditions (1)–(3) and (5) are equivalent.

(1) $\Leftrightarrow$ (4). If (1) is fulfilled, then also (3) and (5), and it follows that (5) is equivalent with

$$(T^\star)^\star\, T^\star = TT^\star \le I,$$

and (1)–(5) are all equivalent.

(1) $\Rightarrow$ (6). If T is a contraction, then we have proved that $\|T^\star\| = \|T\| \le 1$, and it follows that

$$\|TT^\star\| \le \|T^\star\| \cdot \|T\| \le 1^1 = 1,$$

thus $T^\star T$ is a contraction by (2), and we have proved (6).

(6) ⇒ (1). If $T^{\star}T$ is a contraction, then

$$\|T^{\star}Tx\| \leq \|x\| \qquad \text{for all } x \in H,$$

hence by the Cauchy-Schwarz inequality

$$\|Tx\|^2 = (Tx, Tx) = (T^{\star}Tx, x) \leq \|T^{\star}Tx\| \cdot \|x\| \leq \|x\|^2.$$

We infer that $\|Tx\| \leq \|x\|$ for every $x \in H$, and T is by the definition a contraction.

We have proved that the six conditions (1)–(6) are equivalent.

7 Compact operators

Example 7.1 *Let S and T be linear and bounded operators and assume that S is compact. Show that ST and TS are compact.*

According to the definition, $S \in B(H)$ is compact, if $\overline{S(X)}$ is compact for every bounded set $X \subset H$.

Consider S, $T \in B(H)$, and let S be compact. If X is bounded, then $T(X)$ is also bounded. In fact, if

$$M = \sup\{\|x\| \mid \|x\| \in X\},$$

then

$$\|Tx\| \le \|T\| \cdot \|x\| \le \|T\| \cdot M \qquad \text{for all } x \in X.$$

It follows that $\overline{ST(X)} = \overline{S(T(X))}$ is compact, hence the composite operator ST is compact.

Since T is continuous, it follows that $\overline{TS(X)} \subseteq T\left(\overline{S(X)}\right)$. Now, $\overline{S(X)}$ is compact for every bounded set X, and T is continuous, hence $T\left(\overline{S(X)}\right)$ is also compact. Now every closed subset of a compact set is compact, hence $\overline{TS(X)}$ is compact, and the composite operator TS is compact.

Example 7.2 *Let S and T be compact operators in $B(H)$, and let $\alpha \in \mathbb{C}$. Show that $S + \alpha T$ is compact.*

Denote by X a bounded set. Then $\overline{S(X)}$ and $\overline{T(X)}$ are both compact sets, because S and T are compact operators. Choose any sequence $(x_n) \subseteq (S + \alpha T)(X)$. Then we can find other sequences $(y_n) \subseteq X$ and $(z_n) \subseteq X$, such that

$$x_n = Sy_n + \alpha T z_n.$$

The set $\overline{S(X)}$ is compact, hence there exists a subsequence (y_{n_j}), such that $Sy_{n_j} \to y$, and we obtain the subsequence (x_{n_j}) by

$$x_{n_j} = Sy_{n_j} + \alpha T z_{n_j}.$$

If $\alpha = 0$, there is nothing to prove. If $\alpha \ne 0$, it follows by a rearrangement that

$$Tz_{n_j} = \frac{1}{\alpha} x_{n_j} - \frac{1}{\alpha} Sy_{n_j} \in T(X).$$

The set $\overline{T(X)}$ is compact, so there is a subsequence (n_{j_k}), such that $Tz_{n_{j_k}} \to z$. This implies that the subsequence $\left(x_{n_{j_k}}\right)$ is convergent,

$$x_{n_{j_k}} = Sy_{n_{j_k}} + \alpha T z_{n_{j_k}} \to y + \alpha z.$$

We have proved that any sequence (x_n) from $(S+\alpha T)(X)$ has a convergent subsequence, hence $\overline{(S + \alpha T)(X)}$ is compact. Furthermore, X is any bounded set in H, so we infer that $S + \alpha T$ is compact.

Remark 7.1 This result shows that the set of compact operators in $B(H)$ is a subspace of $B(H)$. Then it follows from the result of EXAMPLE 7.1 that the subspace of compact operators is even a so-called two-sided ideal in $B(H)$ with the composition of operators as multiplication. $\Diamond$

Example 7.3 *Let (e_k) denote an orthonormal basis in a Hilbert space H, and define the operator T by*

$$T\left(\sum_{k=1}^{\infty} a_k e_k\right) = \sum_{k=2}^{\infty} \frac{1}{k}\, a_k e_{k-1}.$$

Show that T is compact and find $T^\star$. Find $\sigma_p(T)$ and $\sigma_p\left(T^\star\right)$.

Define T_n, $n \geq 2$, by

$$T_n\left(\sum_{k=1}^{+\infty} a_k e_k\right) = \sum_{k=2}^{n} \frac{1}{k}\, a_k e_{k-1}.$$

Then T_n is of finite rank, thus also compact. It follows from

$$\left(T - T_n\right)\left(\sum_{k=1}^{\infty} a_k e_k\right) = \sum_{k=n+1}^{+\infty} \frac{1}{k}\, a_k e_{k-1},$$

that

$$\left\|\left(T - T_n\right)\left(\sum_{n=1}^{+\infty} a_k e_k\right)\right\|^2 = \sum_{k=n+1}^{+\infty} \frac{1}{k^2}\, \left|a_k\right|^2 \leq \frac{1}{(n+1)^2} \sum_{k=n+1}^{+\infty} \left|a_k\right|^2 \leq \frac{1}{(n+1)^2} \left\|\sum_{k=1}^{+\infty} a_k e_k\right\|^2.$$

thus $\left\|\left(T - T_n\right) x\right\| \leq \dfrac{1}{n+1}\, \|x\|$ for all $x \in H$, and we have proved that $\left\|T - T_n\right\| \leq \dfrac{1}{n+1}$, hence $\left\|T - T_n\right\| \to 0$ for $n \to +\infty$. It follows that T is compact.

Then we check when $T_\lambda = T - \lambda\, I$ is injective. It follows by recursion from

$$T_\lambda\left(\sum_{k=1}^{+\infty} a_k e_k\right) = \sum_{k=1}^{+\infty} \left\{\frac{1}{k+1}\, a_{k+1} - \lambda\, a_k\right\} e_k = 0,$$

that

$$a_{k+1} = (k+1)\lambda\, a_k = \cdots = (k+1)!\lambda^k a_1, \qquad k \in \mathbb{N}.$$

If $\lambda \neq 0$, then

$$\sum_{n=1}^{+\infty} \left|a_k\right|^2 = \sum_{k=1}^{+\infty} \left|a_1\right|^2 \left(k!|\lambda|^{k-1}\right)^2.$$

Now, $\left(k!|\lambda|^{k-1}\right)^2 \to +\infty$ for $k \to +\infty$, thus this series is only convergent, if $a_1 = 0$, and hence all $a_k = 0$. Therefore, when $\lambda \neq 0$, then $T_\lambda x = 0$ implies that $x = 0$, thus T_λ is injective for $\lambda \neq 0$. In particular we get for the point spectrum $\sigma_p(T) \subseteqq \{0\}$. On the other hand $Te_1 = 0 = 0 \cdot e_1$, thus 0 is an eigenvalue, and $\sigma_p(T) = \{0\}$.

Then we search the adjoint operator $T^\star$. Let

$$x = \sum_{k=1}^{+\infty} x_k e_k \qquad \text{og} \qquad y = \sum_{k=1}^{+\infty} y_k e_k.$$

Then

$$(Tx, y) = \left(\sum_{k=2}^{+\infty} \frac{1}{k}\, x_k e_{k-1}, \sum_{n=1}^{+\infty} y_n e_n \right) = \sum_{k=2}^{+\infty} \frac{1}{k}\, x_k \cdot \overline{y_{k-1}} = \left(\sum_{k=2(1)}^{+\infty} x_k e_k, \sum_{n=2}^{+\infty} \frac{1}{n}\, y_{n-1} e_n \right) = (x, T^\star y)\,,$$

from which

$$T^\star \left(\sum_{n=1}^{+\infty} y_n e_n \right) = \sum_{n=2}^{+\infty} \frac{1}{n}\, y_{n-1} e_n = \sum_{n=1}^{+\infty} \frac{1}{n+1}\, y_n e_{n+1}.$$

Assume that $\mu \in \sigma_p(T^\star)$ is an eigenvalue for $T^\star$. Then there is a $y = \sum_{n=1}^{+\infty} y_n e_n \neq 0$, for which

$$(10)\quad (T^\star - \mu\, I) \left(\sum_{n=1}^{+\infty} y_n e_n \right) = -\mu\, y_1 e_1 + \sum_{n=2}^{+\infty} \left\{ \frac{1}{n}\, y_{n-1} - \mu\, y_n \right\} e_n = 0.$$

Here we derive the conditions

$$\mu\, y_1 = 0 \qquad \text{and} \qquad \frac{1}{n}\, y_{n-1} = \mu\, y_n, \quad n \geq 2.$$

If $\mu = 0$, then it follows immediately from (10) that $y = 0$, thus $0 \notin \sigma_p(T^\star)$.
If $\mu \neq 0$, then

$$y_1 = 0 \qquad \text{and} \qquad y_n = \frac{1}{n\mu}\, y_{n-1}, \quad n \geq 2,$$

and it follows by either induction or by recursion that $y = 0$, contradiction the assumption. We therefore conclude that $\sigma_p(T^\star) = \emptyset$. This implies that the residual spectrum for T is empty, $\sigma_r(T) = \emptyset$.

Remark 7.2 It is also possible here to find $\sigma(T)$ and $\sigma(T^\star)$, though this is not an easy task. For completeness the derivations are given in the following.

It follows immediately from the expressions of T and $T^\star$ that

$$\|T\| = \|T^\star\| = \frac{1}{2},$$

hence

$$\sigma(T) \subseteqq \left\{ z \in \mathbb{C} \;\middle|\; |z| \leq \frac{1}{2} \right\} \quad \text{and} \quad \sigma(T^\star) \subseteqq \left\{ z \in \mathbb{C} \;\middle|\; |z| \leq \frac{1}{2} \right\}.$$

It follows from the expression of $T^\star$,

$$T^\star\left(\sum_{n=1}^{+\infty} y_n e_n\right) = \sum_{n=1}^{+\infty} \frac{1}{n+1}\, y_n e_{n+1},$$

that $T^\star$ is injective, so $(T^\star)^{-1}$ exists. Then from $e_1 \perp T^\star D(T^\star)$ follows that $(T^\star)^{-1}$ is not densely defined. This means that $0 \in \sigma_r(T^\star)$.

It follows from $T^\star \in B(H)$ and $T \in B(H)$, that $T^{\star\star} = \overline{T} = T$. We have already proved that

$$\sigma_p(T) = \sigma_p(T^{\star\star}) = \{0\}.$$

so it follows by contraposition that $\sigma_r(T^\star) = \{0\}$. We have proved

$$\sigma_p(T) = \{0\}, \quad \sigma_r(T) = \emptyset, \quad \sigma_p(T^\star) = \emptyset, \quad \sigma_r(T^\star) = \{0\}.$$

Then we *claim* that

(11) $\sigma_c(T) = \sigma_c(T^\star) = \emptyset.$

First notice that if (11) holds, then it easily follows that

$$\sigma(T) = \sigma(T^\star) = \{0\} \quad \text{and} \quad \varrho(T) = \varrho(T^\star) = \mathbb{C} \setminus \{0\}.$$

In order to prove (11) we shall need the following theorem:

Theorem 7.1 *Assume that $T \in B(H)$ is compact, and choose $\lambda \neq 0$. If $T_\lambda = T - \lambda I$ is injective, then the range $(T - \lambda I)(H)$ is closed.*

First assume that Theorem 7.1 holds. Let $\lambda \in \sigma_c(T)$. Then $\sigma_p(T) = \{0\}$, and because $\sigma_p(T)$ and $\sigma_c(T)$ are disjoint, we must have $\lambda \neq 0$. Then it follows from the definition of $\sigma_c(T)$ that $T - \lambda I$ is injective and that $(T - \lambda I)(H)$ is dense in H. Theorem 7.1 shows that $(T - \lambda I)(H)$ is closed, hence $(T - \lambda I)(H) = H$, and whence $(T - \lambda I)^{-1}$ is bounded by the theorem of bounded inverse. This means that $\lambda \in \varrho(T)$, contradicting the assumption that $\lambda \in \sigma_c(T)$. We conclude that $\sigma_c(T) = \emptyset$.

The proof of $\sigma_c(T^\star) = \emptyset$ is apart from a very small modification exactly the same as that above. This modification is that we this time shall use that because $\sigma_r(T^\star) = \{0\}$, we must have $\lambda \neq 0$ for any possible $\lambda \in \sigma_c(T)$. $\Diamond$

Proof of theorem 7.1. Let $y = \lim_{n \to +\infty} y_n$, where $y_n = (T - \lambda I)x_n$.

1) Assume that (x_n) has a bounded subsequence. Because T is compact, there must exist another subsequence (x_{n_i}) such that the image sequence (Tx_{n_i}) is convergent. From follows

$$x_{n_i} = \frac{1}{\lambda}\left(Tx_{n_i} - y_{n_i}\right),$$

that $x_{n_i} \to x$ and $y = (T - \lambda I)x$, hence $y \in (T - \lambda I)(H)$, and we have proved that $(T - \lambda I)(H)$ is closed in this case.

2) Then assume that (x_n) does *not* have any bounded subsequence. Then $\|x_n\| \to +\infty$. We define

$$z_n = \frac{x_n}{\|x_n\|}, \qquad \|z_n\| = 1,$$

thus $(T - \lambda I)z_n \to 0$. There is a subsequence (z_{n_i}), such that (Tz_{n_i}) is convergent. However, $\left(z_{n_i} - \frac{1}{\lambda}Tz_{n_i}\right)$ is convergent, so $z_{n_i} \to z$, where $\|z\| = 1$ and $(T - \lambda I)z = 0$, contradicting that $T - \lambda I$ is injective. Hence the sequence (x_n) must have a bounded subsequence, and we are back in case (1) above, and the claim is proved. $\square$

Example 7.4 *Let T be a bounded operator on a Hilbert space H. Show that:*

1) If T is compact, then $T^\star$ is also compact.

2) If $T^\star T$ is compact, then T is compact.

3) If T is self adjoint and T^n is compact for some n, then T is compact.

1) Assume that T is compact. Let X be a bounded set, and let $(y_n) \subsetneqq T^\star(X)$ be any sequence, thus there exists a sequence $(x_n) \subsetneqq X$, such that $y_n = T^\star x_n$.

We shall prove that there exists a subsequence $\left(x_{n_j}\right)$, such that $\left(T^\star x_{n_j}\right)$ is convergent. This is done indirectly. Assume that $T^\star$ is not compact. Then there exists a bounded sequence (φ_n), which converges weakly towards φ, such that $(T^\star\varphi_n)$ does not converge strongly towards $T^\star\varphi$, thus there exist a subsequence (f_n) and an $\eta > 0$, such that

$$\|T^\star f_n - T^\star\varphi\| > \eta \qquad \text{for all } n \in \mathbb{N},$$

hence

$$\eta \le \|T^\star f_n - T^\star\varphi\| \le \|T^\star\| \cdot \|f_n - \varphi\| \qquad (< M),$$

and whence

$$\|f_n - \varphi_n\| \ge \frac{\eta}{\|T^\star\|}.$$

Now, $(T^\star f_n - T^\star\varphi)$ is bounded and it converges weakly towards 0, hence $TT^\star f_n$ converges strongly towards $TT^\star\varphi$, i.e.

$$\eta^2 \le \|T^\star(f_n-\varphi)\|^2 = (TT\star(f_n-\varphi), f_n-\varphi) \le \|TT^\star(f_n-\varphi)\| \cdot \|f_n-\varphi\| \to 0$$

for $n \to +\infty$. This gives a contradiction, $\eta > 0$ being fixed, and our assumption that $T^\star$ is not compact, must be wrong. We therefore conclude that $T^\star$ is compact as claimed above.

2) It follows trivially from EXAMPLE 7.1 that if T is compact, then $T^\star T$ is also compact.

Assume that $T^\star T$ is compact, and also *assume* (thus an INDIRECT proof) that T is *not* compact. Then there exists a bounded sequence (φ_n), which converges weakly towards φ, such that (cf. (1))

$$\|T\varphi_n - T\varphi\| \ge \eta \qquad \text{for all } n \in \mathbb{N}.$$

Because $(\varphi_n - \varphi)$ is bounded and weakly convergent, it follows that $(T^\star T\varphi_n - T^\star T\varphi)$ is strongly convergent, and we get

$$\begin{aligned}
\eta^2 &\le \|T(\varphi_n-\varphi)\|^2 = (T(\varphi_n-\varphi), T(\varphi_n-\varphi)) \\
&= (T^\star T(\varphi_n-\varphi), \varphi_n-\varphi) \le \|T^\star T(\varphi_n-\varphi)\| \cdot \|\varphi_n-\varphi\| \\
&\le \|T^\star T(\varphi_n-\varphi)\| \cdot M \to 0 \qquad \text{for } n \to +\infty,
\end{aligned}$$

which is a contradiction, because $\eta > 0$ is a given constant. We therefore conclude that T is compact.

3) Finally, assume that T is self adjoint, $T^\star = T$, and that T^n is compact for some given $n \in \mathbb{N}$.

If $n = 2m$ is even, then it follows from T being self adjoint that

$$T^n = T^{2m} = (T^m)^\star (T^m)$$

is compact. Then we infer from (2) that T^m is compact, where $m = \dfrac{n}{2} < n$.

If instead $n = 2m-1$ is odd, then

$$T^{n+1}T^nT = T^{2m} = (T^m)^\star (T^m)$$

is compact, cf. EXAMPLE 7.1, and we infer as above that T^m is compact, where $m = \dfrac{n+1}{2} < n$, when $n > 1$.

By recursion we get after a finite number of steps that T^3 is compact, and hence that $T^2 = T\star T$ is also compact, which by (2) implies that T is compact.

Example 7.5 *Let $T : \ell^2 \to \ell^2$ be the linear operator given by*

$$T(x_1, x_2, \dots, x_{2n-1}, x_{2n}, dots) = \left(x_2, x_1, \frac{1}{2}\, x_4, \frac{1}{2}\, x_3, \dots, \frac{1}{n}\, x_{2n}, \frac{1}{n}\, x_{2n-1}, \dots \right).$$

1) *Find $\|T\|$.*
2) *Find $T^\star$.*
3) *Prove that T is compact.*
4) *Find the spectrum and resolvent set for T, and determine a set of basis vectors for the eigenspace associated to $\lambda \in \sigma_p(T)$.*

1) In general,

$$\|Tx\|^2 = \sum_{n=1}^{+\infty} \frac{1}{n^2} \left\{ |x_{2n}|^2 + |x_{2n-1}|^2 \right\} \le \sum_{n=1}^{+\infty} |x_n|^2 = \|x\|^2,$$

thus $\|T\| \le 1$.

On the other hand,

$$\|Te_1\| = \|e_2\| = 1 = \|e_1\| \qquad \text{and} \qquad \|Te_2\| = \|e_1\| = 1 = \|e_2\|,$$

so $\|T\| = 1$, and $T \in B\left(\ell^2\right)$.

2) Because $T \in B\left(\ell^2\right)$, we also have $T^\star \in B\left(\ell^2\right)$, and $\|T^\star\| = \|T\|$. Then

$$\begin{aligned} (Tx, y) &= \sum_{n=1}^{+\infty} \left\{ \frac{1}{n}\, x_{2n}\, \overline{y_{2n-1}} + \frac{1}{n}\, x_{2n-1}\, \overline{y_{2n}} \right\} \\ &= \sum_{n=1}^{+\infty} \left\{ x_{2n-1}\, \overline{\frac{1}{n}\, y_{2n}} + x_{2n}\, \overline{\frac{1}{n}\, y_{2n-1}} \right\} = (x, T^\star y) = (x, Ty), \end{aligned}$$

hence $T = T^\star$, and T is self adjoint.

3) We get that T is compact from $T_n \to T$, where

$$T_n(x_1, x_2, \dots) = \left(x_2, x_1, \frac{1}{2}\, x_4, \frac{1}{2}\, x_3, \dots, \frac{1}{n}\, x_{2n}, \frac{1}{n}\, x_{2n-1}, 0, 0, \dots \right)$$

is of finite rank, thus compact, and where

$$\|(T - T_n)\, x\|^n = \sum_{k=n+1}^{+\infty} \frac{1}{k^2} \left\{ |x_{2k}|^2 + |x_{2k-1}|^2 \right\} \le \frac{1}{(n+1)!}\, \|x\|^2,$$

i.e.

$$\|T - T_n\| \le \frac{1}{n+1} \to 0 \qquad \text{for } n \to +\infty.$$

4) Because T is self adjoint and compact, we can apply the main theorem, thus

$$\sigma_p(T) = \{\lambda_n \mid n \in \mathbb{N}\}.$$

Now $Tx = 0$ implies that $x = 0$, hence $0 \notin \sigma_p(T)$, which means that $\sigma_c(T) = \{0\}$ and $\sigma_r(T) = \emptyset$, because T is self adjoint.

The eigenvalue problem $Tx = \lambda x$, $\lambda \neq 0$, is now written in coordinates

$$\left\{\begin{array}{rcl} \dfrac{1}{n}\,x_{2n} & = & \lambda\,x_{2n-1}, \\[2ex] \dfrac{1}{n}\,x_{2n-1} & = & \lambda\,x_{2n}, \end{array}\right. \qquad \text{i.e.} \qquad \left\{\begin{array}{rcl} -\lambda\,x_{2n-1} + \dfrac{1}{n}\,x_{2n} & = & 0, \\[2ex] \dfrac{1}{n}\,x_{2n-1} - \lambda\,x_{2n} & = & 0, \end{array}\right. \qquad n \in \mathbb{N},$$

which has non-trivial solutions, if and only if there exists an $n \in \mathbb{N}$, such that

$$\begin{vmatrix} -\lambda & \dfrac{1}{n} \\[2ex] \dfrac{1}{n} & -\lambda \end{vmatrix} = 0, \qquad \text{i.e.} \qquad \lambda^2 = \frac{1}{n^2}.$$

We get the eigenvalues $\lambda = \pm \dfrac{1}{n}$, $n \in \mathbb{N}$, corresponding to e.g. the eigenvectors

$$\left\{\begin{array}{lll} e_{2n-1} + e_{2n}, & \lambda_n = \dfrac{1}{n}, & \\ & & n \in \mathbb{N}. \\ e_{2n-1} - e_{2n}, & \lambda_{-n} = -\dfrac{1}{n}, & \end{array}\right.$$

We finally get

$$\sigma_p(T) = \left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{Z} \setminus \{0\} \right\}, \qquad \sigma_c(T) = \{0\}, \qquad \sigma_r(T) = \emptyset,$$

and

$$\varrho(T) = \mathbb{C} \setminus \left(\{0\} \cup \left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{Z} \setminus \{0\} \right\} \right).$$

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