Real Functions in Several Variables: Volume VI

Antiderivatives and Plane Integrals

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Preface

The topic of this series of books on *"Real Functions in Several Variables"* is very important in the description in e.g. *Mechanics* of the real 3-dimensional world that we live in. Therefore, we start from the very beginning, modelling this world by using the coordinates of $\mathbb{R}^3$ to describe e.g. a motion in space. There is, however, absolutely no reason to restrict ourselves to $\mathbb{R}^3$ alone. Some motions may be rectilinear, so only $\mathbb{R}$ is needed to describe their movements on a line segment. This opens up for also dealing with $\mathbb{R}^2$, when we consider plane motions. In more elaborate problems we need higher dimensional spaces. This may be the case in *Probability Theory* and *Statistics*. Therefore, we shall in general use $\mathbb{R}^n$ as our abstract model, and then restrict ourselves in examples mainly to $\mathbb{R}^2$ and $\mathbb{R}^3$.

For rectilinear motions the familiar *rectangular coordinate system* is the most convenient one to apply. However, as known from e.g. *Mechanics*, circular motions are also very important in the applications in engineering. It becomes natural alternatively to apply in $\mathbb{R}^2$ the so-called *polar coordinates* in the plane. They are convenient to describe a circle, where the rectangular coordinates usually give some nasty square roots, which are difficult to handle in practice.

Rectangular coordinates and polar coordinates are designed to model each their problems. They supplement each other, so difficult computations in one of these coordinate systems may be easy, and even trivial, in the other one. It is therefore important always in advance carefully to analyze the geometry of e.g. a domain, so we ask the question: Is this domain best described in rectangular or in polar coordinates?

Sometimes one may split a problem into two subproblems, where we apply rectangular coordinates in one of them and polar coordinates in the other one.

It should be mentioned that in *real life* (though not in these books) one cannot always split a problem into two subproblems as above. Then one is really in trouble, and more advanced mathematical methods should be applied instead. This is, however, outside the scope of the present series of books.

The idea of polar coordinates can be extended in two ways to $\mathbb{R}^3$. Either to *semi-polar* or *cylindric coordinates*, which are designed to describe a cylinder, or to *spherical coordinates*, which are excellent for describing spheres, where rectangular coordinates usually are doomed to fail. We use them already in daily life, when we specify a place on Earth by its longitude and latitude! It would be very awkward in this case to use rectangular coordinates instead, even if it is possible.

Concerning the contents, we begin this investigation by modelling point sets in an n-dimensional Euclidean space E^n by $\mathbb{R}^n$. There is a subtle difference between E^n and $\mathbb{R}^n$, although we often identify these two spaces. In E^n we use *geometrical methods* without a coordinate system, so the objects are independent of such a choice. In the coordinate space $\mathbb{R}^n$ we can use ordinary calculus, which in principle is not possible in E^n. In order to stress this point, we call E^n the "abstract space" (in the sense of calculus; not in the sense of geometry) as a warning to the reader. Also, whenever necessary, we use the colour black in the "abstract space", in order to stress that this expression is theoretical, while variables given in a chosen coordinate system and their related concepts are given the colours blue, red and green.

We also include the most basic of what mathematicians call *Topology*, which will be necessary in the following. We describe what we need by a function.

Then we proceed with limits and continuity of functions and define continuous curves and surfaces, with parameters from subsets of $\mathbb{R}$ and $\mathbb{R}^2$, resp..

Continue with (partial) differentiable functions, curves and surfaces, the chain rule and Taylor's formula for functions in several variables.

We deal with maxima and minima and extrema of functions in several variables over a domain in $\mathbb{R}^n$. This is a very important subject, so there are given many worked examples to illustrate the theory.

Then we turn to the problems of integration, where we specify four different types with increasing complexity, *plane integral, space integral, curve (or line) integral and surface integral.*

Finally, we consider *vector analysis*, where we deal with vector fields, Gauß's theorem and Stokes's theorem. All these subjects are very important in theoretical Physics.

The structure of this series of books is that each subject is *usually* (but not always) described by three successive chapters. In the first chapter a brief theoretical theory is given. The next chapter gives some practical guidelines of how to solve problems connected with the subject under consideration. Finally, some worked out examples are given, in many cases in several variants, because the standard solution method is seldom the only way, and it may even be clumsy compared with other possibilities.

I have as far as possible structured the examples according to the following scheme:

A *Awareness*, i.e. a short description of what is the problem.

D *Decision*, i.e. a reflection over what should be done with the problem.

I *Implementation*, i.e. where all the calculations are made.

C *Control*, i.e. a test of the result.

This is an ideal form of a general procedure of solution. It can be used in any situation and it is not linked to Mathematics alone. I learned it many years ago in the Theory of Telecommunication in a situation which did not contain Mathematics at all. The student is recommended to use it also in other disciplines.

From high school one is used to immediately to proceed to **I**. *Implementation.* However, examples and problems at university level, let alone situations in real life, are often so complicated that it in general will be a good investment also to spend some time on the first two points above in order to be absolutely certain of what to do in a particular case. Note that the first three points, **ADI**, can always be executed.

This is unfortunately not the case with **C** *Control*, because it from now on may be difficult, if possible, to check one's solution. It is only an extra securing whenever it is possible, but we cannot include it always in our solution form above.

I shall on purpose not use the logical signs. These should in general be avoided in Calculus as a shorthand, because they are often (too often, I would say) misused. Instead of $\wedge$ I shall either write "and", or a comma, and instead of $\vee$ I shall write "or". The arrows $\Rightarrow$ and $\Leftrightarrow$ are in particular misunderstood by the students, so they should be totally avoided. They are not telegram short hands, and from a logical point of view they usually do not make sense at all! Instead, write in a plain language what you mean or want to do. This is difficult in the beginning, but after some practice it becomes routine, and it will give more precise information.

When we deal with multiple integrals, one of the possible pedagogical ways of solving problems has been to colour variables, integrals and upper and lower bounds in blue, red and green, so the reader by the colour code can see in each integral what is the variable, and what are the parameters, which do not enter the integration under consideration. We shall of course build up a hierarchy of these colours, so the order of integration will always be defined. As already mentioned above we reserve the colour black for the theoretical expressions, where we cannot use ordinary calculus, because the symbols are only shorthand for a concept.

The author has been very grateful to his old friend and colleague, the late Per Wennerberg Karlsson, for many discussions of how to present these difficult topics on real functions in several variables, and for his permission to use his textbook as a template of this present series. Nevertheless, the author has felt it necessary to make quite a few changes compared with the old textbook, because we did not always agree, and some of the topics could also be explained in another way, and then of course the results of our discussions have here been put in writing for the first time.

The author also adds some calculations in MAPLE, which interact nicely with the theoretic text. Note, however, that when one applies MAPLE, one is forced first to make a geometrical analysis of the domain of integration, i.e. apply some of the techniques developed in the present books.

The theory and methods of these volumes on "Real Functions in Several Variables" are applied constantly in higher Mathematics, Mechanics and Engineering Sciences. It is of paramount importance for the calculations in *Probability Theory*, where one constantly integrate over some point set in space.

It is my hope that this text, these guidelines and these examples, of which many are treated in more ways to show that the solutions procedures are not unique, may be of some inspiration for the students who have just started their studies at the universities.

Finally, even if I have tried to write as careful as possible, I doubt that all errors have been removed. I hope that the reader will forgive me the unavoidable errors.

Leif Mejlbro
March 21, 2015

Introduction to volume VI,
Integration of a Function in Several Variables

This is the sixth volume in the series of books on *Real Functions in Several Variables.* We start the investigation of how to integrate a real function in several variables. First we introduce the so-called *"gradient fields"*, which are linked to *conservative forces* in Physics. We mention that restricted to two dimensions this theory is also closely connected with the theory of *analytic functions* in *Complex Functions Theory.* However, we shall not go into the realm of complex functions in this volume.

In Chapter 20, we introduce the *plane integral.* For completeness we start with a flow diagram of how all the following concepts of integration are connected. The basic theory is the plane integral (in two dimensions over a domain in $\mathbb{R}^2$), which in *rectangular coordinates* is reduced to a double integral, each in one variable, so the well-known integration theory from *Real Functions in One Variable* can be applied twice. In general, the innermost integral will have limits which are depending on the variable in the outer integral, so one must be careful in the calculations.

What is new here, is that one must always start with a careful analysis of the plane domain, before we can set up the double integral. In rectangular coordinates we fix, what is going to be the "outer variable" and then find the bounds of the "inner variable" for this fixed "outer variable". Then we first integrate with respect to the "inner variable" to get a result, which after the integration only depends on the "outer variable". Then we perform the second integration with respect to the "outer variable".

In order to visualize this procedure we introduce a colour code. Blue (and later also green) integrals are *abstract integrals* in the sense that they cannot be computed directly by some integration technique known for one real variable. We may in special cases find their values by a geometrical argument, but we cannot rely on this. Then the hierarchy is that one should start with the red integral, which is always the inner integral. Its bounds are functions in the black "outer variable", indicating that they are playing the role of a constant with respect to this first red integration. Occasionally, when the bounds are constants, we shall also colour them in red. When the inner inner integration has been performed, the result must be a function in the black "outer" variable alone, and the red colour must not occur at this step. Finally, we calculate the outer black integral.

There are two versions here. Either we start by integrating *vertically*, in which case y is the red "inner" variable, and x is the black "outer" variable. Or we start by integrating *horizontally*, where x is the red "inner" variable, and y is the black "outer" variable. Clearly, whenever possible one should always sketch a figure of the domain of integration.

Then we turn to the case of *polar coordinates* in plane. This becomes more abstract than the rectangular case, because the area element $\varrho\,\mathrm{d}\varphi\,\mathrm{d}\varrho$ contains a *weight function* ϱ. The integration domain B, in which we apply the polar coordinates, is pulled back to the *parameter domain* $(\varrho, \varphi) \in D$, which must *not* be confused with the original domain B itself. For the price of introducing the weight function ϱ we obtain that the abstract integration in B in polar coordinates is transformed into an abstract integration of another function (namely including the weight function as a factor) over D, where we can apply the methods of setting up the corresponding double integral as in the case of rectangular coordinates. Again, there are here two cases. Either ϱ is the red "inner" variable and φ is the black "outer variable", or φ is the red "inner" variable and ϱ is the black "outer variable".

Whenever convenient we have supplied the calculations with a comparison with the corresponding results, when we apply MAPLE. We must still perform the geometrical analysis of the domain in order to get the variables right, and then the definition of the bounds of the "inner" variable is also interior in the MAPLE command, i.e. before the specification of the bounds of the "outer" variable.

Once this geometrical analysis has been applied, the MAPLE calculations are usually faster than the old-fashioned ones by pen and pencil, but occasionally we meet cases, which MAPLE apparently does not like, if we are not to supply with some further help

In the next volume we continue with the space integrals, which in principle are handled in the same way, only there are formally six versions of the triple integrals in rectangular coordinates, depending on the order of the variables. Furthermore, we also get six versions when we apply semi-polar coordinates, as well as in the case of spherical coordinates. When applying semi-polar or spherical coordinates we also get som weight function, which is connected with the chosen coordinate system.

19 Antiderivatives of functions in several variables

19.1 The theory of antiderivatives of functions in several variables

When we are going to discuss integration of functions in several variables, we naturally start with writing down, what is known already in 1 dimension, and what we should expect in the simplest situation in more variables, before we proceed to more general cases.

We begin with the well-known theorem that if $f : I \to \mathbb{R}$ is a continuous function in *one variable*, $x \in I$, where $I \subseteq \mathbb{R}$ is an interval, then we by an integration can find all differentiable functions F, for which the derivative is f, i.e. such that

$$F'(x) = f(x), \qquad \text{for } x \in I \subseteq \mathbb{R}.$$

This can be reformulated in the following way, where we use differentials instead,

$$\mathrm{d}F = f(x)\,\mathrm{d}x.$$

As already mentioned above, this problem can always be solved in 1 dimension, and the solutions are here called the *antiderivatives* of f. It can be expressed as an *integral* with the *variable* x as the upper bound, and an *arbitrary constant* a as the lower bound, i.e.

$$F(x) = F_a(x) := \int_a^x f(\xi)\,\mathrm{d}\xi, \qquad \text{where } a \in \mathbb{R} \text{ is an arbitrary constant.}$$

It is customary also to write this in the following way,

$$F(x) = \int f(x)\,\mathrm{d}x + c,$$

where the variable x now occurs under the integral sign, and where c is some arbitrary constant.

When we consider higher dimensional spaces we first note that we have previously seen (cf. Chapter 9) that the *gradient* in some sense is the generalisation of the differential quotient in 1 dimension. Therefore, the generalised problem should be phrased in the following way:

Problem 19.1 *Given a* continuous vector field $\mathbf{f}$ *on an open set* $A \subseteq \mathbb{R}^m$*. When is it possible to find a* C^1*-function* $F : A \to \mathbb{R}$*, such that*

$$\nabla F(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \qquad \textit{for all } \mathbf{x} \in A?$$

The answer to this question is that if $m \geq 2$, then this is far from always possible. We shall therefore introduce the following new concepts. *If* such a function F exists, *then* we call it an *antiderivative* of the *vector field* $\mathbf{f}$. Since $\mathbf{f}$ in this case can be written as a gradient of a C^1-function, we call $\mathbf{f}$ a *gradient field*.

Since gradient fields in particular are important in *Physics*, we shall in the rest of this chapter give a brief description of them. Note that if $m = 2$, then gradient fields may also be interpreted as analytic functions, so the reader has the possibility of also consulting the books on *Complex Function Theory*.

If F is an antiderivative of the gradient field $\mathbf{f}$, then it is trivial that so is also $F + c$ for every arbitrary constant c. Conversely, if A is *connected*, then $F + c$, $c \in \mathbb{R}$, are describing all possible antiderivatives. In fact, let F and G be two antiderivatives of the same gradient fielt $\mathbf{f}$. Then by a trivial subtraction

$$\nabla(F - G) = \nabla F - \nabla G = \mathbf{f} - \mathbf{f} = \mathbf{0}.$$

Referring again to Chapter 9 we see that this is only possible, when the difference $F - G$ is a constant, and the claim follows.

Let us again compare with the 1-dimensional case. The analogue of $\mathrm{d}F = f(x)\,\mathrm{d}x$ in one variable can only be

(19.1) $\quad \mathrm{d}F = \mathbf{f}(\mathbf{x}) \cdot \mathrm{d}\mathbf{x}, \qquad \text{for } \mathbf{x} \in A.$

In fact, both $\mathbf{f}(\mathbf{x})$ and $\mathrm{d}\mathbf{x}$ must enter more or less as already indicated, but they are of dimension m, while $\mathrm{d}F$ is 1-dimensional. We can only obtain the right dimension by introducing the dot product.

An expression $\mathbf{f}(\mathbf{x}) \cdot \mathrm{d}\mathbf{x}$ like the one on the right hand side of (19.1), is called a *differential form.* If furthermore there exists an antiderivative F, such that (19.1) indeed holds, then the right hand side $\mathbf{f}(\mathbf{x}) \cdot \mathrm{d}\mathbf{x}$ is called an *exact differential form.* Clearly, not all differential forms are also exact, when $m \geq 2$.

In order to become more familiar with these new concepts we restricts ourselves in the following to the "simple" case of just two variables (x, y).

Problem 19.2 *Given a continuous 2-dimensional vector field (f, g) in an open set $A \subseteq \mathbb{R}^2$ in the plane. What are the conditions on the functions (f, g) and (so it turns up) on the domain A, in order that (f, g) is a gradient field with a function F as its antiderivative, and how do we explicitly construct F, when we have proved that it exists?*

We shall first derive a necessary condition, so we assume that (f,g) is a gradient field with the antiderivative F. This means that

$$\frac{\partial F}{\partial x} = f \qquad \text{and} \qquad \frac{\partial F}{\partial y} = g \qquad \text{in } A.$$

We assume furthermore that (f,g) is a C^1 vector field. In the practical applications, where this theory is applied, this assumption is no obstacle at all. Then $F \in C^2(A)$, so we can differentiate F with respect to x and y and then interchange the order of differentiation. This gives

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left\{\frac{\partial F}{\partial x}\right\} = \frac{\partial}{\partial x}\left\{\frac{\partial F}{\partial y}\right\} = \frac{\partial g}{\partial x},$$

so we have derived the *necessary condition*

$$(19.2) \qquad \frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$$

for the C^1 vector field (f,g) to be a gradient field.

Without further assumptions we can only use (19.2) in the negative way:

Theorem 19.1 *If the C^1 vector field (f,g) does not fulfil (19.2) $\left(\text{in } A \subseteq \mathbb{R}^2\right)$, i.e. if*

$$\frac{\partial f}{\partial y} \neq \frac{\partial g}{\partial x},$$

then $(f(x,y), g(x,y))$ is not *a gradient field.*

Trivial as Theorem 19.1 may seem, there are lots of applications of this result.

In general, (19.2) is *not* sufficient to conclude that (f,g) is a gradient field. It will be shown in an example in the following that the vectorfield

$$(f(x,y), g(x,y)) := \left(\frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2}\right) \qquad \text{for } (x,y) \neq (0,0),$$

does satisfy (19.2), and yet (f,g) is not a gradient field in all of $A = \mathbb{R}^2 \setminus \{(0,0)\}$.

Theorem 19.2 *Assume that $(f(x,y), g(x,y)$ is a C^2 vector field in an* open simply connected domain *$A \subseteq \mathbb{R}^2$, which satisfies the necessary condition (19.2). Then (f,g) is a gradient field.*

We recall that the simply connected sets were introduced in Section 1.5. These sets are connected sets "without holes". If $A \subseteq \mathbb{R}^2$ is a plane simply connected set, then for every closed curve Γ lying entirely in A all points inside Γ also lie in A. In the sketched example above the unit circle lies in A, and the point $(0,0)$ inside Γ does not belong to A, so A is not simply connected.

When we add the assumption of simply connectedness to (19.2) we get a *sufficient*, though not necessary condition. Consider e.g. a gradient field (f,g) on a simply connected set A. Then (f,g) remains a gradient field on every subset of A. Choose any subset of A which is not simply connected, and we see that the assumption of being simply connected is not necessary.

Proof of Theorem 19.2. The first part of the proof is done by brute force by simply constructing an antiderivative F, where we at the same time get a template of how to find F in practice. The only problem is that we finally shall check that we have obtained the right solution. We assume that (19.2) holds and that A is simply connected.

We *define* a function

$$F_1(x,y) := \int f(x,y)\,\mathrm{d}x,$$

where we consider y as a parameter. Then clearly

$$\frac{\partial F_1}{\partial x} = f(x,y),$$

so the first equation is fulfilled.

If F_1 also satisfies

$$\text{(19.3)} \qquad \frac{\partial F_1}{\partial y} = g(x,y),$$

then $F_1(x,y)$ is our antiderivative, and the problem is solved.

If F_1 does not satisfy (19.3), then we add a function $F_2(y)$ depending only of y and derive an equation, which F_2 should fulfil. So we define

$$F(x,y) := F_1(x,y) + F_2(y).$$

Then

$$\frac{\partial F}{\partial x} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial x} = f(x,y),$$

because $F_2(y)$ does not depend on x.

Concerning the second condition, we want

$$\frac{\partial\left(F_1+F_2\right)}{\partial y} = \frac{\partial F_1}{\partial y} + \frac{\mathrm{d}F_2}{\mathrm{d}y} = g(x,y),$$

a condition, which we rewrite as

$$\text{(19.4)} \qquad \frac{dF_2}{dy}(y) = g(x,y) - \frac{\partial F_1}{\partial y}(x,y).$$

If the right hand side of (19.4) is independent of x, then this is just an ordinary integration problem in the variable y alone, so $F_2(y)$ can be found, and the claim follows.

In this part of the proof it only remains to prove that the right hand side of (19.4) is independent of x. When we differentiate it with respect to x, we get

$$\frac{\partial}{\partial x}\left(g - \frac{\partial F_1}{\partial y}\right) = \frac{\partial g}{\partial x} - \frac{\partial^2 F_1}{\partial x \partial y} = \frac{\partial g}{\partial x} - \frac{\partial^2 F_1}{\partial y \partial x} = \frac{\partial g}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial x}\right) = \frac{\partial f}{\partial y} - \frac{\partial f}{\partial y} = 0.$$

So, where does the assumption of the simply connectedness enter the solution?

We must analyze the situation once more to see why we have not yet finished the proof. We have above proved that when y is kept fixed, then $g - \dfrac{\partial F_1}{\partial y}$ is independent of x in a horizontal subset of A. This construction holds for every y, but the problem is that the set $A \cap (\mathbb{R} \times \{y\})$ is not necessarily connected, but could consist of a union of some disjoint x-intervals, so the actual solution could differ by a constant on the different x-intervals.

Therefore, we have by the argument above only proved that when $F_1(x,y)$ and $F_2(y)$ are fixed by the procedure described above, then

$$F(x,y) = F_1(x,y) + F_2(y), \qquad (x,y) \in A_1,$$

is an antiderivative of (f,g) in a subset $A_1 \subseteq A$.

We note that if $D \subseteq A$ is an open axiparallel rectangle, then the construction above combined with the continuity of $F(x,y)$ shows that $F(x,y)$ is an antiderivative in D.

Since all open rectangles contained in A allow $F(x,y)$ as an antiderivative, we can find a maximal open simply connected subdomain $A_1 \subseteq A$, such that $F(x,y)$ is an antiderivative on A_1. Such a subregion exists $\neq \emptyset$. We shall prove that $A_1 = A$.

Contrariwise. Assume that $A_1 \neq A$. Then we can find a point $(x,y) \in A \cap \partial A_1$ and an open axiparallel rectangle D, such that $(x,y) \in D$, and such that $A_1 \cup D$ is simply connected. Since $A_1 \cap D \neq \emptyset$, because D is an open neighbourhood of the boundary point (x,y) of A_1, we are forced to use the same antiderivative in D, and we have shown that (f,g) has an antiderivative in the larger set $A_1 \cup D$, which is not possible, because A_1 was assumed to be maximal.

This means that our assumption that $A_1 \neq A$ is wrong, so we conclude that $A_1 = A$, and the theorem is proved. $\Diamond$

In practice we just use the procedure given above in the proof. We list a short version of the two-dimensional case in the following.

1) First calculate

$$F_1(x,y) := \int f(x,y)\,\mathrm{d}x, \qquad y \text{ fixed}.$$

2) Then check that

$$g(x,y) - \frac{\partial F_1}{\partial y}(x,y)$$

is independent of x. If not, then either $(f(x,y), g(x,y))$ is not a gradient field, or we have made a miscalculation. (Check!)

3) Calculate

$$F_1(y) := \int \left\{ g(x,y) - \frac{\partial F_1}{\partial y}(x,y) \right\} \mathrm{d}y.$$

4) Finally, check if

$$F(x,y) := F_1(x,y) + F_2(y)$$

really is an antiderivative, i.e. check the two equations

$$\frac{\partial F}{\partial x}(x,y) = f(x,y) \qquad \text{and} \qquad \frac{\partial F}{\partial y}(x,y) = g(x,y).$$

We may of course, whenever convenient, interchange x and y in the procedure above.

A template for the three-dimensional case is described in Section 19.2.vsi

Remark 19.1 If A is not simply connected, we can still use the method above on a simply connected subdomain $A_1 \subset A$. However, if A_1 is maximal, then the proof above will not give us another nontrivial *simply connected set* $A_1 \cup D$, and then we may even be forced to choose two different (local) antiderivatives on D, where these differ by a constant $\neq 0$. This is of course not possible. $\Diamond$

Example 19.1 The simplest possible example is given by the vector field $(f(x), g(y))$, where f is continuous in the interval I_1, and g is continuous in the interval I_2. In fact,

$$F(x,y) = \int f(x)\,\mathrm{d}x + \int g(y)\,\mathrm{d}y, \qquad (x,y) \in I_1 \times I_2,$$

is an antiderivative, because we immediately get $\nabla F(x,y) = (f(x), g(y))$. It is in this case no need to assume that $f \in C^1(I_1)$ ad $g \in C^1(I_2)$ in their respective variables x and y, because trivially

$$\frac{\partial f}{\partial y} = 0 = \frac{\partial g}{\partial x}.$$

The corresponding differential form is

$$\mathrm{d}f = f(x)\,\mathrm{d}x + g(y)\,\mathrm{d}y,$$

and the integration of this is called integration by separating the variables. $\Diamond$

Example 19.2 Then let us see what happens, when we interchange the variables in Example 19.1. We assume that $f \in C_1(I_2)$ and $g \in C^1(I_1)$ and consider the vector field

$$(f(y), g(x)), \qquad (x,y) \in D = I_1 \times I_2.$$

Clearly, $D = I_1 \times I_2$ is simply connected, so the condition of $(f(y), g(x))$ being a gradient field is that

$$f'(y) = g'(x).$$

The right hand side does not depend on y, and since the left hand side only depends (at most) on y, it must be a constant, $f'(y) = c_0 (= g'(x))$, so when this is the case, we get by integration,

$$(f(y), g(x)) = (c_0 y + c_1, c_0 x + c_2) = c_0(y,x) + (c_1, c_2).$$

This equation describes all possible functions $f(y) = c_0 y + c_1$ and $g(x) = c_0 x + c_2$, if $(f(y), g(x))$ is going to be a gradient field. When this is the case, we get by inspection that

$$(f(y), g(x)) = c_0(y,x) + (c_1, c_2) = \bigtriangledown (c_0 xy + c_1 x + c_2 y + c_3),$$

from which we immediately derive that all antiderivatives are given by

$$F(x,y) = c_0 xy + c_1 x + c_2 y + c_3. \qquad \Diamond$$

Example 19.3 Given the plane C^∞ vector field

$$(f(x,y), g(x,y)) = \left(\frac{y}{\sqrt{y^2+2xy}}, \frac{x+y}{\sqrt{y^2+2xy}} + 2y \right).$$

This vector field is only defined, when $0 < y^2 + 2xy = y(y+2x)$, i.e. it is only defined in $A = A_1 \cup A_2$, where

$$A_1 := \{(x,y) \mid y > 0 \text{ and } y+2x > 0\} \quad \text{and} \quad A_2 := \{(x,y) \mid y < 0 \text{ and } y+2x < 0\},$$

cf. Figure 19.1.

The strategy is to proceed directly to the solution procedure *without first checking the necessary condition*, because this will give us some nasty computations. So we shall directly find the *candidates* of a possible antiderivative. Finally, we shall of course check these candidates in order to see, if we indeed have a gradient field.

Inspecting Figure 19.1 we see that for $y > 0$ the horizontal integration is performed in the interval $\left]-\frac{y}{2}, +\infty\right[$, while the horizontal integration for fixed $y < 0$ is taking place over the interval $\left]-\infty, -\frac{y}{2}\right[$. In each of these cases we get for fixe $y \neq 0$ the primitive

$$F_1(x,y) = \int f(x,y)\,\mathrm{d}x = \int \frac{y}{\sqrt{y^2+2xy}}\,\mathrm{d}x = \sqrt{y^2+2xy}.$$

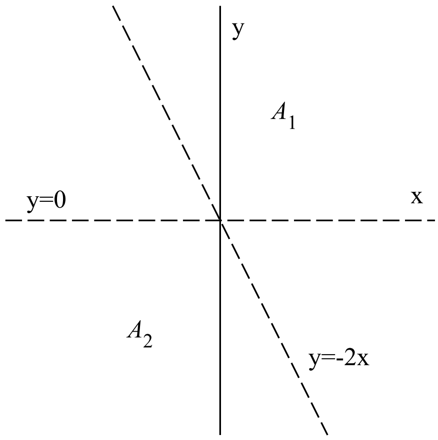


Figure 19.1: The domain $A = A_1 \cup A_2$ of the vector field of Example 19.2. Note that A has two connected components, bounded by the lines $y = 0$ and $y + 2x = 0$.

Then we compute the correction term,

$$\frac{\partial F_1}{\partial y}(x,y) = \frac{y+x}{\sqrt{y^2+2xy}},$$

so by subtracting this from $g(x, y)$ we get

$$g(x,y) - \frac{\partial F_1}{\partial y}(x,y) = 2y.$$

We note that if the result had depended on x, then either we had made an error in our calculations (check!), or (f, g) is not a gradient field.

We obtain by another integration,

$$F_2(y) = \int \left\{ g(x,y) - \frac{\partial F_1}{\partial y}(x,y) \right\} \mathrm{d}y = \int 2y \, \mathrm{d}y = y^2,$$

so the candidates of the antiderivatives are

$$F(x,y) = F_1(x,y) + F_2(y) = \begin{cases} \sqrt{y^2+2xy} + y^2 + c_1, & (x,y) \in A_1, \\ \sqrt{y^2+2xy} + y^2 + c_2, & (x,y) \in A_2, \end{cases}$$

where c_1 and c_2 are arbitrary constants. It follows immediately that

$$\bigtriangledown F(x,y) = (f(x,y), g(x,y)) \qquad \text{for } (x,y) \in A = A_1 \cup A_2.$$

Note that $A = A_1 \cup A_2$ is not simply connected.

Alternatively we get by *inspection* in A that

$$(f(x,y),g(x,y))\cdot(\,\mathrm{d}x,\,\mathrm{d}y)=\frac{y}{\sqrt{y^2+2xy}}\,\mathrm{d}x+\left(\frac{x+y}{\sqrt{y^2+2xy}}-2y\right)\mathrm{d}y$$

$$=\frac{y\,\mathrm{d}x+x\,\mathrm{d}y}{\sqrt{y^2+2xy}}+\frac{y\,\mathrm{d}y}{\sqrt{y^2+2xy}}+2y\,\mathrm{d}y=\frac{\mathrm{d}(xy)}{\sqrt{y^2+2xy}}+\frac{1}{2}\frac{\mathrm{d}\left(y^2\right)}{\sqrt{y^2+2xy}}+\mathrm{d}\left(y^2\right)$$

$$=\frac{1}{2}\frac{\mathrm{d}\left(y^2+2xy\right)}{\sqrt{y^2+2xy}}+\mathrm{d}\left(y^2\right)=\mathrm{d}\left(\sqrt{y^2+2xy}\right)+\mathrm{d}\left(y^2\right)=\mathrm{d}\left(\sqrt{y^2+2xy}+y^2\right),$$

which proves that (f,g) is a gradient field, and that one of its antiderivatives is

$$F(x,y)=\sqrt{y^2+2xy}+y^2.$$

Then continue by discussing the situation in each of the two connected subdomains A_1 and A_2. $\Diamond$

The following two examples are classical. They are given in every textbook on real functions in several real variables. In both cases the domain $A=\mathbb{R}^2\setminus\{(0,0)\}$ is not simply connected.

Example 19.4 Consider the C^∞ vector field

$$(f(x,y),g(x,y))=\left(\frac{x}{\sqrt{x^2+y^2}},\frac{y}{\sqrt{x^2+y^2}}\right),\qquad\text{for } (x,y)\neq(0,0).$$

We proceed directly to the calculation of the primitive of the first coordinate $f(x,y)$ with respect to for first variable,

$$F_1(x,y)=\int\frac{x}{\sqrt{x^2+y^2}}\,\mathrm{d}x=\sqrt{x^2+y^2},\qquad\text{for } (x,y)\neq(0,0).$$

Then we compute the correction term,

$$\frac{\partial F_1}{\partial y}(x,y)=\frac{y}{\sqrt{x^2+y^2}},\qquad\text{for } (x,y)\neq(0,0).$$

We get that already $F_1(x,y)=\sqrt{x^2+y^2}$ is an antiderivative, and (f,g) is a gradient field.

Alternatively we may also argue directly *by inspection* on the corresponding differential form,

$$(f(x,y),g(x,y))\cdot(\,\mathrm{d}x,\,\mathrm{d}y)=\frac{x}{\sqrt{x^2+y^2}}\,\mathrm{d}x+\frac{y}{\sqrt{x^2+y^2}}\,\mathrm{d}y$$

$$=\frac{1}{2}\left\{\frac{\mathrm{d}\left(x^2\right)}{\sqrt{x^2+y^2}}+\frac{\mathrm{d}\left(y^2\right)}{\sqrt{x^2+y^2}}\right\}=\frac{\mathrm{d}\left(x^2+y^2\right)}{\sqrt{x^2+y^2}}=\mathrm{d}\left(\sqrt{x^2+y^2}\right),$$

and we see that (f,g) has the antiderivatives

$$F(x,y)=\sqrt{x^2+y^2}+c,\qquad\text{for } (x,y)\neq(0,0),\ \text{and } c \text{ arbitrary.}\Diamond$$

Example 19.5 Then we consider the vector field

$$(f(x,y),g(x,y)) = \left(\frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2}\right), \qquad \text{for } (x,y) \neq (0,0).$$

Using the same method as in Example 19.4 we get for $y \neq 0$,

$$F_1(x,y) = \int \frac{y}{x^2+y^2}\,dx = \int \frac{1}{1+\left(\frac{x}{y}\right)^2}\,d\left(\frac{x}{y}\right) = \text{Arctan}\left(\frac{x}{y}\right).$$

When $y \neq 0$, the correction term becomes

$$\frac{\partial F_1}{\partial y}(x,y) = \frac{1}{1+\left(\frac{x}{y}\right)^2}\,\frac{-x}{y} = \frac{-x}{x^2+y^2} = g(x,y),$$

so we conclude hat (f,g) is a gradient field in the two simply connected subdomains of A defined by $y > 0$ and $y < 0$. The antiderivatives are therefore

$$\begin{cases} F_+(x,y) = \text{Arctan}\left(\frac{x}{y}\right) + c_1, & \text{for } y > 0, \\[2mm] F_-(x,y) = \text{Arctan}\left(\frac{x}{y}\right) + c_2, & \text{for } y < 0, \end{cases}$$

where c_1 and c_2 are arbitrary constants.

Then we investigate what happens for $y = 0$ and $x \neq 0$.

Let $x < 0$ be fixed. Then

$$\lim_{y \to 0+} F_+(x,y) = -\frac{\pi}{2} + c_1, \quad \text{and} \quad \lim_{y \to 0-} F_-(x,y) = +\frac{\pi}{2} + c_1.$$

We extend both F_+ and F_- by continuity to the negative x-axis so that they agree. This requires that $c_2 = c_1 - \pi$. Then

$$F(x,y) = \begin{cases} F_+(x,y) = \operatorname{Arctan}\left(\dfrac{x}{y}\right) + c_1, & \text{for } y > 0 \text{ and } x \in \mathbb{R}, \\[2mm] c_1 - \dfrac{\pi}{2}, & \text{for } y = 0 \text{ and } x < 0, \\[2mm] F_-(x,y) = \operatorname{Arctan}\left(\dfrac{x}{y}\right) + c_1 - \pi, & \text{for } y < 0 \text{ and } x \in \mathbb{R}, \end{cases}$$

is a (continuous) antiderivative of the vector field (f, g) in the open, simply connected domain $\mathbb{R}^2 \setminus \{(x,0) \mid x \geq 0\}$.

However, when $x > 0$, then

$$\lim_{y \to 0+} F_+(x,y) = +\frac{\pi}{2} + c_1, \quad \text{and} \quad \lim_{y \to 0-} F_-(x,y) = -\frac{\pi}{2} + c_1,$$

so the limits from and below differ by the constant 2π, and we cannot extend the antiderivative $F(x,y)$ to all of $A = \mathbb{R}^2 \setminus \{(0,0)\}$.

Alternatively we could also here have used *inspection* in the calculations. If as above we assume that $y \neq 0$, then

$$(f(x,y), g(x,y)) \cdot (\mathrm{d}x,\, \mathrm{d}y) = \frac{y}{x^2+y^2}\,\mathrm{d}x + \frac{-x}{x^2+y^2}\,\mathrm{d}y = \frac{1}{x^2+y^2}\{y\,\mathrm{d}x - x\,\mathrm{d}y\}$$

$$= \frac{y^2}{x^2+y^2}\left\{\frac{1}{y}\,\mathrm{d}x - \frac{x}{y^2}\,\mathrm{d}y\right\} = \frac{1}{1+\left(\dfrac{x}{y}\right)^2}\,\mathrm{d}\left(\frac{x}{y}\right) = \operatorname{Arctan}\left(\frac{x}{y}\right),$$

and then we proceed as above. $\Diamond$

19.2 Templates for gradient fields and antiderivatives of functions in three variables

As the main case we consider vector fields in $\mathbb{R}^3$, i.e.

$$\mathbf{V}(x, y, z) = (f(x, y, z), g(x, y, z), h(x, y, z)),$$

which is assumed to be of class C^1 in an open domain $A \subseteq \mathbb{R}^3$. Whenever necessary we shall mention the modifications to $\mathbb{R}^2$.

Problem 19.3 *Check whether the vector field* $\mathbf{V} = (f, g, h)$ *is a* gradient field*. When this is the case, find an* antiderivative F*. This means that the function* F *satisfies the equation*

$$\nabla F = \mathbf{V}.$$

Remark 19.2 The problem is tricky, because there exist so many solution methods that one may be confused the first time one is confronted with this situation. Furthermore, there exist *necessary* conditions which are not sufficient, and *sufficient* conditions which are not necessary. Finally, the standard procedure assumes some knowledge of line integrals, which is not always the case in every textbook, the first time this problem is encountered. It will, however, be known at the end of any course dealing with functions in several variables. ♢

Procedure:

Existence. This section is not necessary, if only one remembers to *check* the solution in the next section. The considerations of this section may, however, be useful in some particular situations.

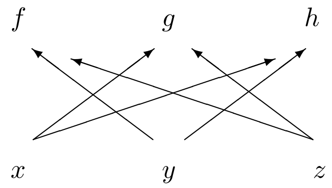


Figure 19.2: Diagram for "cross differentiation".

1) Check that f, g, $h \in C^2(A)$ satisfy

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \qquad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}, \qquad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}.$$

One may call these the "cross derivatives". In case of *2 dimensions* we only use the first equation.

a) If these equations are *not* fulfilled, then $\mathbf{V}(x, y, z)$ is *not* a gradient field, and the problem does not have an antiderivative.

b) If the equations *are* satisfied, then $\mathbf{V}(x, y, z)$ is indeed a gradient field in every *simply connected region* of A.

Remark 19.3 Note the extra condition that we only consider *simply connected regions* A. This is a sufficient condition, though not necessary. $\Diamond$

2) Suppose that the equations of 1) are satisfied. Check whether A is a *simply connected region*. If "yes", then we have proved the existence. If "no", construct a candidate by means of one of the methods in the next section and *check it*, i.e. check the equation

$$\bigtriangledown F = \mathbf{V}.$$

Construction of a possible antiderivative. We shall describe four methods, of which the former two have intrinsically built a check into them, while the latter two do *not* contain such a check! For that reason the latter two methods may be tricky, because their simple formulæ usually give some results, *even when no such antiderivative exists!* A reasonable strategy is therefore to skip the investigation in the section above and instead start by constructing a *candidate* F of an antiderivative and then as a rule *always* perform a *check*, i.e. check whether the *candidate* really satisfies the equation

$$\bigtriangledown F = \mathbf{V}.$$

1) **Indefinite integration.**

a) Write the differential form

$$\mathbf{V}(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} = f(x,y,z)\,\mathrm{d}x + g(x,y,z)\,\mathrm{d}y + h(x,y,z)\,\mathrm{d}z.$$

b) Choose the simplest looking of the three terms in a), e.g. $f(x,y,z)\,\mathrm{d}x$. Then calculate

$$F_1(x,y,z) = \int f(x,y,z)\,\mathrm{d}x, \qquad y,\, z \text{ are here considered as constants.}$$

c) **Check** the result, i.e. calculate

$$\mathrm{d}F_1 = \frac{\partial F_1}{\partial x}\,\mathrm{d}x + \frac{\partial F_1}{\partial y}\,\mathrm{d}y + \frac{\partial F_1}{\partial z}\,\mathrm{d}z,$$

and compare this with

$$\mathbf{V} \cdot \mathrm{d}\mathbf{x} = f\,\mathrm{d}x + g\,\mathrm{d}y + h\,\mathrm{d}z.$$

i) If $\frac{\partial F_1}{\partial x} \neq f$, then we have made an error in our calculations. There is only one thing to do: Start from the very beginning!

ii) If $g_1 = g - \frac{\partial F_1}{\partial y}$, or $h_1 = h - \frac{\partial F_1}{\partial z}$ depends on x, then we have two possibilities: Either

(*) we have made an error in our calculations above,

or

(**) $\mathbf{V}(x,y,z)$ is *not* a gradient field.

Note that both possibilities may occur, so in this case one should check one's calculations an extra time.

iii) When neither g_1 nor h_1 depend on x, (which loosely speaking has been integrated in the first process, and therefore should have disappeared from the reduced problem), then we have

$$\mathbf{V} \cdot \mathrm{d}\mathbf{x} = \mathrm{d}F_1 + g_1(x,y)\,\mathrm{d}y + h_1(y,z)\,\mathrm{d}z.$$

In this case we repeat the process above on the *reduced* form

$$g_1(y,z)\,\mathrm{d}y + h_1(y,z)\,\mathrm{d}z.$$

d) After *at most* three repetitions of this process we either get

$$\mathbf{V}(x,y,z) \text{ is } \textit{not} \text{ a gradient field (in which case the task is finished),}$$

or

$$\mathbf{V}(\mathbf{x})\cdot \mathrm{d}\mathbf{x} = \mathrm{d}F_1(x,y,z) + \mathrm{d}F_2(y,z) + \mathrm{d}F_3(z),$$

or something similar. The essential thing is that $\mathrm{d}F_1$ depends on all three variables, that $\mathrm{d}F_2$ only depends on two of them, and that $\mathrm{d}F_3$ only depends on one variable. Since all terms on the right hand side are "put under the d-sign", it follows that $\mathbf{V}(\mathbf{x})$ is a *gradient field*. One gets an *antiderivative* by removing the d-sign in all three terms,

$$F(x,y,z) = F_1(x,y,z) + F_2(y,z) + F_3(z).$$

Finally, we get *all* possible antiderivatives by adding an arbitrary constant.

2) **The method of inspection.**

This method is often called the "method of guessing", but this is misleading, because it uses systematically the well-known rules of differentiation, read in the opposite direction of what one is used to from the reader's previous education:

Linearity: $\mathrm{d}f + \alpha\,\mathrm{d}g = \mathrm{d}(f + \alpha\, g)$, α constant,

Product: $g\,\mathrm{d}f + f\,\mathrm{d}g = \mathrm{d}(f \cdot g)$,

Quotient: $$g\,\mathrm{d}f - f\,\mathrm{d}g = \begin{cases} g^2\,\mathrm{d}\left(\dfrac{f}{g}\right), & g \neq 0, \\ -f^2\,\mathrm{d}\left(\dfrac{g}{f}\right), & f \neq 0, \end{cases}$$

Composition: $F'(f)\,\mathrm{d}f = \mathrm{d}(F \circ f)$.

These rules are all what we need, so learn them in this form!

a) Apply the rules of differentiation above to put as much as possible under the d-sign:

$$\mathbf{V}(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} = \mathrm{d}F_1 + \mathbf{V}_1(\mathbf{x}) \cdot \mathrm{d}\mathbf{x}.$$

b) If one by this process obtains that $\mathbf{V}_1(\mathbf{x}) = \mathbf{0}$, then $\mathbf{V}(\mathbf{x})$ is indeed a *gradient field*,

$$\mathbf{V}(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} = \mathrm{d}F_1 = \nabla F_1(\mathbf{x}) \cdot \mathrm{d}\mathbf{x},$$

and $F_1(\mathbf{x})$ is an *antiderivative*.

c) If one *cannot* obtain an equation of the form $\mathbf{V}_1(\mathbf{x}) = \mathbf{0}$, then either $\mathbf{V}(\mathbf{x})$ is *not* a gradient field, or one has run out of ideas of further reductions. In this case one chooses another possible $\mathbf{V}_1(\mathbf{x})$, which is *simpler* than $\mathbf{V}(\mathbf{x})$, and uses one of the other methods on the reduced form $\mathbf{V}_1(\mathbf{x}) \cdot \mathrm{d}\mathbf{x}$.

Even when $\mathbf{V}$ is not a gradient field, it is often quite useful to remove a term of the form $\mathrm{d}F_1(\mathbf{x})$, because later calculations of e.g. line integrals will be considerably easier to perform on the residual vector field. This technique may be useful in practical calculations in e.g. Thermodynamics.

3) **Standard method; line integration along a curve consisting of axis parallel lines.**

Once the *tangential line integral* has been introduced, and $\mathbf{V}(\mathbf{x})$ is defined in $\mathbb{R}^3$, (or in some region which allows curves consisting of axis parallel lines as e.g. described in the following), it is easy to calculate a *candidate* to an antiderivative by *integration along such a curve* like e.g.

$$(0,0,0) \longrightarrow (x,0,0) \longrightarrow (x,y,0) \longrightarrow (x,y,z).$$

a) Start by writing down the differential form

$$\mathbf{V}(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} = f(x,y,z)\,\mathrm{d}x + g(x,y,z)\,\mathrm{d}y + h(x,y,z)\,\mathrm{d}z.$$

b) Integrate this differential form along the curve mentioned above,

$$F_0(x,y,z) = \int_0^x f(t,0,0)\,\mathrm{d}t + \int_0^y g(x,t,0)\,\mathrm{d}t + \int_0^z h(x,y,t)\,\mathrm{d}t.$$

c) **Check the result!** This means that one should check the equation

$$\triangledown F_0 = \mathbf{V}(\mathbf{x}).$$

If this is *not* fulfilled, then $\mathbf{V}(\mathbf{x})$ is *not* a gradient field, *not even in the case where the candidate* $F_0(x,y,z)$ *exists!* It is *not* an antiderivative in this case.

d) If on the other hand $F_0(\mathbf{x})$ is an antiderivative, then we get all antiderivatives by adding an arbitrary constant.

4) **Radial integration.**

In this case we integrate along the line

$$(0,0,0) \longrightarrow (x,y,z).$$

a) Be extremely careful when x is replaced by tx, and y by ty, and z by tz in $\mathbf{V}(\mathbf{x})$. By this process we get $\mathbf{V}(tx,ty,tz)$.

Remark 19.4 Warning! This seemingly simple process is far more difficult to perform than one would believe at the first sight! ♢

b) Calculate

$$F_0(x,y,z) = (x,y,z) \cdot \int_0^1 \mathbf{V}(tx,ty,tz)\,\mathrm{d}t.$$

Note that the dot product is used here.

c) **Check the result!** This means that one should check the equation

$$\triangledown F_0(\mathbf{x}) = \mathbf{V}(\mathbf{x}).$$

Remark 19.5 The method of *radial integration* is only mentioned here, because it may be found in some textbooks. I shall here strongly advise against the use of it, partly because the transform to $\mathbf{x} \to t\,\mathbf{x}$ is far more difficult to perform than one would believe, and partly because the integral which is used in the calculation of $F_0(x,y,z)$ in general is far more complicated than the analogous integral where we integrate along a simple curve consisting of straight lines parallel with one of the axis. ♢

19.3 Examples of gradient fields and antiderivatives

Example 19.6 *Find for every of the given vector fields first the domain and then every indefinite integral, whenever such an integral exists.*

1) $\mathbf{V}(x,y) = (x,y)$.

2) $\mathbf{V}(x,y) = (y,x)$.

3) $\mathbf{V}(x,y) = \left(\dfrac{1}{x+y}, \dfrac{-x}{y(x+y)}\right)$.

4) $\mathbf{V}(x,y) = (3x^2+2y^2, 2xy)$.

5) $\mathbf{V}(x,y) = \left(3x^2+y^2+\dfrac{y}{1+x^2y^2}, 2xy-4+\dfrac{x}{1+x^2y^2}\right)$.

6) $\mathbf{V}(x,y) = \left(\begin{array}{c} \dfrac{-2x}{2-x^2-2y^2}+\dfrac{-x}{\sqrt{2-x^2-2y^2}} \\ \dfrac{-4y}{2-x^2-2y^2}+\dfrac{-2y}{\sqrt{2-x^2-2y^2}} \end{array}\right)$.

7) $\mathbf{V}(x,y) = \left(\dfrac{x}{(x-y)^2}, \dfrac{-x^2}{y(x-y)^2}\right)$.

8) $\mathbf{V}(x,y) = \left(\dfrac{2x(1-e^y)}{(1+x^2)^2}, \dfrac{e^y}{1+x^2}\right)$.

9) $\mathbf{V}(x,y) = (\sin y + y\sin x + x, \cos x + x\cos y + y)$.

A Gradient fields; integrals.

D First find the domain. Then check if we are dealing with a differential, or use indefinite integration. Another alternative is to integrate along a step line within the domain.

I 1) The vector field $\mathbf{V}(x,y) = (x,y)$ is defined in the whole of $\mathbb{R}^2$.

a) First method. We get by only using the rules of calculation,

$$\mathbf{V}(x,y)\cdot(\,\mathrm{d}x,\,\mathrm{d}y) = x\,\mathrm{d}x + y\,\mathrm{d}y = \,\mathrm{d}\left\{\frac{1}{2}(x^2+y^2)\right\},$$

which shows that $\mathbf{V}(x,y)$ has an indefinite integral,

$$F(x,y) = \frac{1}{2}(x^2+y^2).$$

b) Second method. We get by indefinite integration,

$$F_1(x,y) = \int x\,\mathrm{d}x = \frac{1}{2}\,x^2,$$

hence

$$y - \frac{\partial}{\partial y}F_1(x,y) = y, \qquad \text{i.e. } F_2(x,y) = \frac{1}{2}\,y^2.$$

An integral is

$$F(x,y) = F_1(x,y) + F_2(x,y) = \frac{1}{2}(x^2+y^2).$$

c) THIRD METHOD. When we integrate along the step line

$$C: \quad (0,0) \longrightarrow (x,0) \longrightarrow (x,y),$$

which lies in the domain, we get the candidate

$$\int_C \mathbf{V} \cdot (\,\mathrm{d}x,\, \mathrm{d}y) = \int_0^x t\,\mathrm{d}t + \int_0^y t\,\mathrm{d}t = \frac{1}{2}\,x^2 + \frac{1}{2}\,y^2.$$

d) CHECK. The check is *always* mandatory by the latter method; though it is not necessary in the two former ones, it is nevertheless highly recommended. Obviously,

$$\nabla F(x,y) = (x,y) = \mathbf{V}(x,y),$$

and we have checked our result.

2) The vector field $\mathbf{V}(x,y) = (y,x)$ is defined in $\mathbb{R}^2$.

a) FIRST METHOD. It follows by the rules of calculations that

$$\mathbf{V}(x,y) \cdot (\,\mathrm{d}x,\, \mathrm{d}y) = y\,\mathrm{d}x + x\,\mathrm{d}y = \,\mathrm{d}(xy),$$

which shows that $\mathbf{V}(x,y)$ has an integral

$$F(x,y) = xy.$$

b) SECOND METHOD. We get by indefinite integration

$$F_1(x,y) = \int y\,dx = xy,$$

thus

$$x - \frac{\partial}{\partial y} F_1(x,y) = x - x = 0, \qquad \text{i.e. } F_2(x,y) = 0.$$

An integral is

$$F(x,y) = xy.$$

c) THIRD METHOD. When we integrate along the step line

$$C: \quad (0,0) \longrightarrow (x,0) \longrightarrow (x,y),$$

which lies in the domain, then

$$\int_C \mathbf{V} \cdot (\,\mathrm{d}x,\, \mathrm{d}y) = \int_0^x 0\,\mathrm{d}t + \int_0^y x\,\mathrm{d}t = xy.$$

d) CHECK (which is mandatory by the third method). Clearly, $\bigtriangledown F = (y,x)$, so the calculations are all right.

3) The vector field $\mathbf{V}(x,y) = \left(\dfrac{1}{x+y}, \dfrac{-x}{y(x+y)} \right)$ is defined in the set

$$A = \{(x,y) \mid y \neq 0,\, y \neq -x\}.$$

This set is the union of four angular spaces, where one considers each of these separately when we solve the problem.

a) FIRST METHOD. Here we get by some clever reductions,

$$\frac{1}{x+y}\,\mathrm{d}x - \frac{x}{y(x+y)} = \frac{y^2}{y(x+y)} \left(\frac{1}{y}\,\mathrm{d}x - \frac{x}{y^2}\,\mathrm{d}y \right) = \frac{1}{1+\dfrac{x}{y}}\,\mathrm{d}\left(\frac{x}{y}\right) = \mathrm{d}\ln\left|1+\frac{x}{y}\right|,$$

so an integral in each of the four domains is

$$F(x,y) = \ln\left|1+\frac{x}{y}\right| = \ln|x+y| - \ln|y|.$$

b) SECOND METHOD. We get by indefinite integration,

$$F_1(x,y) = \int \frac{1}{x+y}\,\mathrm{d}x = \ln|x+y|,$$

thus

$$-\frac{x}{y(x+y)} - \frac{\partial F_1}{\partial y} = -\frac{x}{y(x+y)} - \frac{1}{x+y} = -\frac{1}{y}\,\frac{x+y}{x+y} = -\frac{1}{y}.$$

Hence by integration, $F_2(x,y) = -\ln|y|$, so an integral is

$$F(x,y) = F_1(x,y) + F_2(x,y) = \ln|x+y| - \ln|y| = \ln\left|1+\frac{x}{y}\right|.$$

c) Third method. In this case the integration along a step line is fairly complicated, because we shall choose a point and a step curve in each of the four angular spaces. It is possible to go through this method of solution, but since it is fairly long, we shall here leave it to the reader.

d) Check. Here

$$\nabla F(x,y) = \left(\frac{1}{x+y}, \frac{1}{x+y} - \frac{1}{y}\right) = \left(\frac{1}{x+y}, -\frac{x}{y(x+y)}\right) = \mathbf{V}(x,y),$$

so our calculations are correct.

4) The vector field $\mathbf{V}(x,y) = (3x^2+2y^2, 2xy)$ is defined in $\mathbb{R}^2$.

a) First method Since

$$\begin{aligned}\mathbf{V}(x,y)\cdot(\,\mathrm{d}x,\,\mathrm{d}y) &= (3x^2+2y^2)\,\mathrm{d}x + 2xy\,\mathrm{d}y \\ &= \mathrm{d}(x^3) + y^2\,\mathrm{d}x + (y^2\,\mathrm{d}x + x\,\mathrm{d}(y^2)) \\ &= \mathrm{d}(x^3+xy^2) + \underline{y^2\,\mathrm{d}x},\end{aligned}$$

cannot be written as a differential, we conclude that $\mathbf{V}(x,y)$ is *not* a gradient field and no integral exists.

b) Second method. We get by indefinite integration,

$$F_1(x,y) = \int (3x^2+2y^2)\,\mathrm{d}x = x^3 + 2xy^2,$$

and accordingly,

$$2xy - \frac{\partial F_1}{\partial y} = 2xy - 4xy = -2xy.$$

This expression depends on x, which it should not if the field is a gradient field. Therefore, we conclude that the field is *not* a gradient field, and also that there does *not* exist any integral.

If one does not immediately see the above, we get by the continuation,

$$F_2(x,y) = -\int 2xy\,\mathrm{d}y = -xy^2,$$

so a *candidate* of the integral is

$$F(x,y) = F_1(x,y) + F_2(x,y) = x^3 + xy^2.$$

Then the check below will prove that this is not an integral.

c) Third method. Integration along the step curve

$$C: \quad (0,0) \longrightarrow (x,0) \longrightarrow (x,y),$$

in the domain gives

$$\int_C \mathbf{V}\cdot(\,\mathrm{d}x,\,\mathrm{d}y) = \int_0^x (3t^2+0)\,\mathrm{d}t + \int_0^y 2xt\,\mathrm{d}t = x^3 + xy^2,$$

in other words the same candidate as by the second method.

d) Check. We find

$$\nabla F(x,y) = (3x^2+y^2, 2xy) \neq \mathbf{V}(x,y),$$

so the check is not successful. The field is *not* a gradient field.

5) The vector field

$$\mathbf{V}(x,y) = \left(3x^2+y^2+\frac{y}{1+x^2y^2}, 2xy-4+\frac{x}{1+x^2y^2}\right)$$

is defined in $\mathbb{R}^2$.

a) First method. Here

$$\begin{aligned}\mathbf{V}\cdot d\mathbf{x} &= 3x^2\,dx + (y^2\,dx + 2xy\,dy) - 4\,dy + \frac{1}{1+x^2y^2}(y\,dx + x\,dy)\\ &= d(x^3) + d(xy^2) - d(4t) + \frac{1}{1+x^2y^2}\,d(xy)\\ &= d\{x^3+xy^2-4y+\text{Arctan}(xy)\},\end{aligned}$$

so $\mathbf{V}(x,y)$ has an integral given by

$$F(x,y) = x^3+xy^2-4y+\text{Arctan}(xy).$$

b) Second method. We get by an indefinite integration,

$$F_1(x,y) = \int\left\{3x^2+y^2+\frac{y}{1+x^2y^2}\right\}dx = x^3+xy^2+\text{Arctan}(xy),$$

hence

$$\frac{\partial F_1}{\partial y} = 2xy + \frac{x}{1+x^2y^2},$$

and whence

$$2xy-4+\frac{x}{1+x^2y^2} - \frac{\partial F_1}{\partial y} = -4.$$

It follows immediately that $F_2(y) = -4y$. The vector field is a gradient field with an integral

$$F(x,y) = F_1(x,y) + F_2(y) = x^3+xy^2-4y+\text{Arctan}(xy).$$

c) Third method. If we integrate along the step curve

$$C: \quad (0,0) \longrightarrow (x,0) \longrightarrow (xy),$$

entirely in the domain, we get

$$\begin{aligned}\int_C \mathbf{V}\cdot d\mathbf{x} &= \int_0^x (3t^2+0+0)\,dt + \int_0^y \left\{2xt-4+\frac{x}{1+x^2t^2}\right\}dt\\ &= x^3 + \{xy^2-4y+\text{Arctan}(xy)\}.\end{aligned}$$

As mentioned above one *shall always* check the result by this method! The check is not necessary in the two former methods, but it is nevertheless highly recommended.

d) Check. By some routine calculations,

$$\bigtriangledown F(x,y)=\left(3x^2+y^2-0+\frac{y}{1+(xy)^2},0+2xy-4+\frac{x}{1+(xy)^2}\right)=\mathbf{V}(x,y).$$

We get the correct answer, so $\mathbf{V}(x,y)$ is a gradient field and an integral is

$$F(x,y)=x^3+xy^2-4y+\ \text{Arctan}(xy).$$

6) The vector field

$$\mathbf{V}(x,y)=\begin{pmatrix}\dfrac{-2x}{2-x^2-2y^2}+\dfrac{-x}{\sqrt{2-x^2-2y^2}}\\[2mm] \dfrac{-4y}{2-x^2-2y^2}+\dfrac{-2y}{\sqrt{2-x^2-2y^2}}\end{pmatrix}$$

is defined in the open ellipsoidal disc

$$A=\left\{(x,y)\;\middle|\;\left(\frac{x}{\sqrt{2}}\right)^2+y^2<1\right\}$$

of centrum $(0,0)$ and half axes $\sqrt{2}$ and 1.

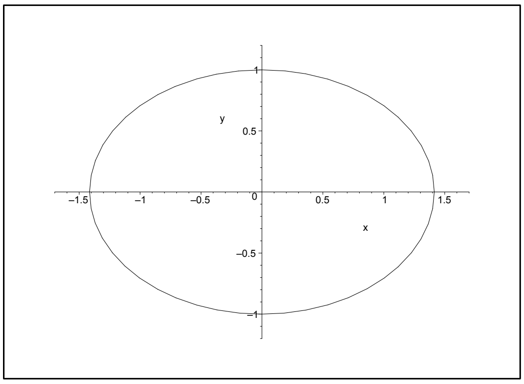


Figure 19.3: The open domain of 6).

a) First method. By collecting terms which look more or less the same we get

$$\begin{aligned}\mathbf{V}\cdot \mathrm{d}\mathbf{x} &= \frac{1}{2-x^2-2y^2}(-2x\,\mathrm{d}x-4y\,\mathrm{d}y)+\frac{1}{\sqrt{2-x^2-2y^2}}(-x\,\mathrm{d}x-2y\,\mathrm{d}y)\\ &= \frac{1}{2-x^2-2y^2}\,\mathrm{d}(2-x^2-2y^2)+\frac{1}{2}\,\frac{1}{\sqrt{2-x^2-2y^2}}\,\mathrm{d}(2-x^2-2y^2)\\ &= \mathrm{d}\left(\ln|2-x^2-2y^2|\right)+\,\mathrm{d}\left(\sqrt{2-x^2-2y^2}\right)\\ &= \mathrm{d}\left\{\ln(2-x^2-2y^2)+\sqrt{2-x^2-2y^2}\right\}\qquad \text{for } (x,y)\in A.\end{aligned}$$

This vector field is a gradient field, an an integral in A is given by

$$F(x,y)=\ln(2-x^2-2y^2)+\sqrt{2-x^2-2y^2}.$$

b) Second method. We get by indefinite integration,

$$\begin{aligned}F_1(x,y) &= \int\left\{\frac{-2x}{2-x^2-2y^2}+\frac{-x}{\sqrt{2-x^2-2y^2}}\right\}\mathrm{d}x\\ &= \ln|2-x^2-2y^2|+\sqrt{2-x^2-2y^2}\\ &= \ln(2-x^2-2y^2)+\sqrt{2-x^2-2y^2},\qquad (x,y)\in A,\end{aligned}$$

hence

$$\frac{\partial F_1}{\partial y}=\frac{-4y}{2-x^2-2y^2}+\frac{-2y}{\sqrt{2-x^2-2y^2}}=g(x,y).$$

An integral in A is given by

$$F(x,y)=F_1(x,y)=\ln(2-x^2-2y^2)+\sqrt{2-x^2-2y^2},$$

and the vector field is a gradient field.

c) Since A is convex and symmetric about e.g. the X axis, the step curve

$$C:\quad (0,0)\longrightarrow(x,0)\longrightarrow(x,y)$$

lies totally inside A for each $(x,y) \in A$. By an integration along this step curve we get

$$\begin{aligned}
\int_C \mathbf{V}\cdot \mathrm{d}\mathbf{x} &= \int_0^x \left\{ \frac{-2t}{2-t^2-0} + \frac{-t}{\sqrt{2-t^2-0}} \right\} \mathrm{d}t \\
&\quad + \int_0^y \left\{ \frac{-4t}{2-x^2-2t^2} + \frac{-2t}{\sqrt{2-x^2-2t^2}} \right\} \mathrm{d}t \\
&= \{\ln(2-x^2) - \ln 2\} + \{\sqrt{2-x^2} - \sqrt{2}\} \\
&\quad + \{\ln(2-x^2-2y^2) - \ln(2-x^2)\} \\
&\quad + \{\sqrt{2-x^2-2y^2} - \sqrt{2-x^2}\} \\
&= \ln(2-x^2-2y^2) + \sqrt{2-x^2-2y^2} - \ln 2 - \sqrt{2}.
\end{aligned}$$

Here we can of course neglect the constant $-\ln 2 - \sqrt{2}$.

d) CHECK. We get by standard calculations

$$\bigtriangledown F = \left(\frac{-2x}{2-x^2-2y^2} + \frac{-x}{\sqrt{2-x^2-2y^2}}, \frac{-4y}{2-x^2-2y^2} + \frac{-2y}{\sqrt{2-x^2-2y^2}} \right) = \mathbf{V}(x,y).$$

The check is OK, and $\mathbf{V}(x,y)$ is a gradient field with the obtained function $F(x,y)$ as an integral.

7) The vector field

$$\mathbf{V}(x,y) = \left(\frac{x}{(x-y)^2}, \frac{-x^2}{y(x-y)^2} \right)$$

is defined in the set

$$A = \{(x,y) \mid y \neq 0,\, y \neq x\},$$

with four angular spaces as its components.

a) FIRST METHOD. Since $y \neq 0$ in A, it seems natural to put y^2 outside the denominator. Then

$$\begin{aligned}
\mathbf{V}\cdot \mathrm{d}\mathbf{x} &= \frac{x}{(x-y)^2}\,\mathrm{d}x - \frac{x^2}{y(x-y)^2}\,\mathrm{d}y = \frac{\frac{x}{y}}{\left(\frac{x}{y}-1\right)^2} \cdot \frac{1}{y}\,\mathrm{d}x + \frac{\frac{x}{y}}{\left(\frac{x}{y}-1\right)^2} \left(-\frac{x}{y^2}\right) \mathrm{d}y \\
&= \frac{\frac{x}{y}}{\left(\frac{x}{y}-1\right)^2} \left\{ \frac{1}{y}\,\mathrm{d}x + x\,\mathrm{d}\left(\frac{1}{y}\right) \right\} = \frac{\frac{x}{y}-1+1}{\left(\frac{x}{y}-1\right)^2}\,\mathrm{d}\left(\frac{x}{y}\right) \\
&= \left\{ \frac{1}{\frac{x}{y}-1} + \frac{1}{\left(\frac{x}{y}-1\right)} \right\} \mathrm{d}\left(\frac{x}{y}\right) = \mathrm{d}\left\{ \ln\left|\frac{x}{y}-1\right| - \frac{1}{\frac{x}{y}-1} \right\}.
\end{aligned}$$

It follows that $\mathbf{V}(x,y)$ has an integral, e.g.

$$F(x,y) = \ln\left|\frac{x-y}{y}\right| + \frac{y}{y-x},$$

defined in each of the four connected components of A. Furthermore, $\mathbf{V}(x,y)$ is a gradient field.

b) Second method. We get by indefinite integration in A that

$$\begin{aligned} F_1(x,y) &= \int \frac{x}{(x-y)^2}\,\mathrm{d}x = \int \frac{x-y+y}{(x-y)^2}\,\mathrm{d}x = \int \frac{1}{x-y}\,\mathrm{d}x + y\int \frac{1}{(x-y)^2}\,\mathrm{d}x \\ &= \ln|x-y| - \frac{y}{x-y}, \end{aligned}$$

whence

$$\frac{\partial F_1}{\partial y} = \frac{-1}{x-y} - \frac{1}{x-y} - \frac{y}{(x-y)^2} = \frac{-2x+2y-y}{(x-y)^2} = \frac{-2x+y}{(x-y)^2},$$

so

$$-\frac{x^2}{y(x-y)^2} - \frac{\partial F_1}{\partial y} = -\frac{-x^2+2xy-y^2}{y(x-y)^2} = \frac{-(x-y)^2}{y(x-y)^2} = -\frac{1}{y},$$

and hence by an integration, $F_2(y) = -\ln|y|$, so an integral is

$$F(x,y) = \ln|x-y| - \frac{y}{x-y} - \ln|y| = \ln\left|\frac{x-y}{y}\right| - \frac{y}{x-y}.$$

c) The third method can also be applied here but it is fairly difficult due to the structure of A, so here follows only a short description of the method. Choose a point in each of the four connected components. Then a geometric analysis shows that one should in the two angular spaces in which the angle is acute first integrate horizontally and then vertically, so the integration path has here the form

$$C:\quad (x_0,y_0) \longrightarrow (x,y_0) \longrightarrow (x,y),$$

because we have to stay inside the connected component.
In the other two angular spaces this path of integration may go beyond the connected component (sketch examples on a figure), so we introduce instead the following path of integration

$$C:\quad (x_0,y_0) \longrightarrow (x_0,y) \longrightarrow (x,y),$$

i.e. we first integrate vertically and then horizontally.
This is clearly a tedious procedure, and on the top of it one should also check the result before we can recognize the result as being correct.

d) Check.

$$\begin{aligned} \bigtriangledown F(x,y) &= \left(\frac{\frac{1}{y}}{\frac{x-y}{y}} + \frac{y}{(x-y)^2}, \frac{1}{\frac{x-y}{y}}\cdot\left(-\frac{x}{y^2}\right) - \frac{x-y+y}{(x-y)^2}\right) \\ &= \left(\frac{x-y}{(x-y)^2} + \frac{y}{(x-y)^2}, -\frac{x}{y(x-y)} - \frac{x}{(x-y)^2}\right) \\ &= \left(\frac{x}{(x-y)^2}, \frac{-x}{y(x-y)^2}(x-y+y)\right) \\ &= \left(\frac{x}{(x-y)^2}, -\frac{x^2}{y(x-y)^2}\right) = \mathbf{V}(x,y), \end{aligned}$$

so we have checked that $\mathbf{V}(x,y)$ is indeed a gradient field with $F(x,y)$ as its integral.

8) The vector field

$$\mathbf{V}(x,y) = \left(\frac{2x(1-e^y)}{(1+x^2)^2}, \frac{e^y}{1+x^2} \right)$$

is defined in $\mathbb{R}^2$.

a) First method. We get by some clever manipulation

$$\begin{aligned} \mathbf{V}\cdot \mathrm{d}\mathbf{x} &= \frac{1-e^y}{(1+x^2)^2}\cdot 2x\,\mathrm{d}x + \frac{1}{1+x^2}\,e^y\,\mathrm{d}y = (1-e^y)\,\frac{1}{(1+x^2)^2}\,\mathrm{d}(x^2) + \frac{1}{1+x^2}\,\mathrm{d}(e^y) \\ &= (e^y-1)\,\mathrm{d}\left(\frac{1}{1+x^2}\right) + \frac{1}{1+x^2}\,\mathrm{d}(e^y-1) = \mathrm{d}\left(\frac{e^y-1}{1+x^2}\right), \end{aligned}$$

so an integral is

$$F(x,y) = \frac{e^y-1}{1+x^2},$$

and $\mathbf{V}(x,y)$ is a gradient field.

b) Second method. We get by indefinite integration,

$$F_1(x,y) = \int \frac{2x(1-e^y)}{(1+x^2)^2}\,\mathrm{d}x = (1-e^y)\int \frac{\mathrm{d}(1+x^2)}{(1+x^2)^2} = \frac{e^y-1}{1+x^2},$$

where

$$\frac{\partial F_1}{\partial y} = \frac{e^y}{1+x^2} = g(x,y),$$

and $\mathbf{V}(x,y)$ is seen to be a gradient field.

c) Third method. Here there is plenty of space to integrate along the step curve

$$C: \quad (0,0) \longrightarrow (x,0) \longrightarrow (x,y),$$

no matter where $(x,y) \in \mathbb{R}^2$ lies. Then

$$\int_C \mathbf{V}\cdot \mathrm{d}\mathbf{x} = \int_0^x 0\,\mathrm{d}t + \int_0^y \frac{e^t}{1+x^2}\,\mathrm{d}t = \frac{e^y-1}{1+x^2},$$

which is the *candidate*, which should be checked.

d) Check.

$$\nabla F(x,y) = \left(-\frac{2x(e^y-1)}{(1+x^2)^2}, \frac{e^y}{1+x^2}\right) = \mathbf{V}(x,y).$$

We conclude that $\mathbf{V}(x,y)$ is a gradient field and an integral is $F(x,y)$.

9) The vector field

$$\mathbf{V}(x,y) = (\sin y + y\sin x + x, \cos x + x\cos y + y)$$

is defined in $\mathbb{R}^2$.

a) First method. We get by some simple manipulations

$$\begin{aligned}\mathbf{V}\cdot \mathrm{d}\mathbf{x} &= \sin y\,\mathrm{d}x + y\sin x\,\mathrm{d}x + x\,\mathrm{d}x + \cos x\,\mathrm{d}y + x\cos y\,\mathrm{d}y + y\,\mathrm{d}y \\ &= \frac{1}{2}\,\mathrm{d}\left(x^2+y^2\right) + (\sin y\,\mathrm{d}x + x\,\mathrm{d}\sin y) + (-y\,\mathrm{d}\cos x + \cos x\,\mathrm{d}y) \\ &= \mathrm{d}\left\{\frac{1}{2}\,x^2 + \frac{1}{2}\,y^2 + x\sin y - y\cos x\right\} + \underline{2\cos x\,\mathrm{d}y},\end{aligned}$$

which clearly is *not* a differential, so the field is *not* a gradient field.

b) Second method. Indefinite integration gives

$$F_1(x,y) = \int (\sin y + y\sin x + x)\,\mathrm{d}x = x\sin y - y\cos x + \frac{x^2}{2},$$

where

$$\frac{\partial F_1}{\partial y} = x\cos y - \cos x,$$

thus

$$\cos x + x\cos y + y - \frac{\partial F_1}{\partial y} = 2\cos x + y.$$

This expression depends on x, so we conclude that the vector field is *not* a gradient field.

c) Third method. Choose the step curve

$$C: \quad (0,0) \longrightarrow (x,0) \longrightarrow (x,y)$$

as the path of integration in $\mathbb{R}^2$. Then

$$\begin{aligned}\int_C \mathbf{V}\cdot \mathrm{d}\mathbf{x} &= \int_0^x (0+0+t)\,\mathrm{d}t + \int_0^y (\cos x + x\cos t + t)\,\mathrm{d}t \\ &= \frac{1}{2}\,x^2 + y\cos x + x\sin y + \frac{1}{2}\,y^2 \\ &= F(x,y),\end{aligned}$$

which is the *candidate*, and we *must* check it.

d) Check. It follows from

$$\begin{aligned}\triangledown F(x,y) &= (x - y\sin x + \sin y, \cos x + x\cos y + y) \\ &= \mathbf{V}(x,y) - (2y\sin x, 0) \neq \mathbf{V}(x,y),\end{aligned}$$

that $\mathbf{V}(x,y)$ is not a gradient field.

Example 19.7 *Sketch the domain A of the vector field*

$$\mathbf{V}(x,y) = \left(\frac{4x-y}{x(3x-y)}, -\frac{1}{3}\left(\frac{1}{y} + \frac{1}{3x-y}\right)\right).$$

Prove that $\mathbf{V}$ is a gradient field and find all its integrals.
(Consider first a connected subset of A).

A Gradient field; integral.

D If there exists an integral, it can be found by one of the following three standard methods:

1) First method. Rules of calculation for differentials.
2) Second method. Indefinite integration.
3) Third method. Integration along some curve, typically a step curve. Notice that the *check is mandatory* by this method, because one may get "false solutions" by this method.

I The vector field is defined in the set

$$A = \{(x,y) \mid x \neq 0,\ y \neq 0,\ y \neq 3x\},$$

which is the union of six connected components. We shall in the following consider any one of these.

Aside. Before we start on the calculations it will be quite convenient in advance to perform the following simple decomposition:

$$\frac{4x-y}{x(3x-y)} = \frac{x + (3x-y)}{x(3x-y)} = \frac{1}{x} + \frac{1}{3x-y}, \qquad (x,y) \in A.$$

The vector field can then also be written

$$\mathbf{V}(x,y) = \left(\frac{1}{x} + \frac{1}{3x-y}, -\frac{1}{3}\left(\frac{1}{y} + \frac{1}{3x-y}\right)\right),$$

which we shall exploit in the following. $\diamond$

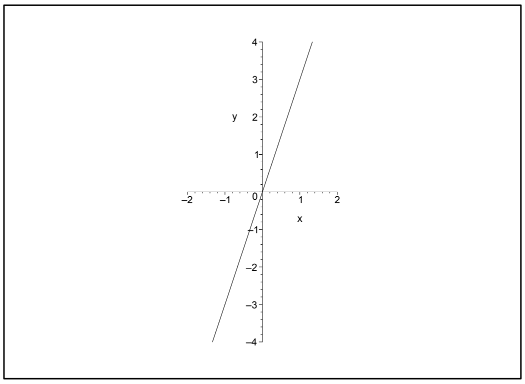


Figure 19.4: The six connected components of the domain of $\mathbf{V}(x, y)$.

1) FIRST METHOD. The ides is that *if* $F(x, y)$ is an integral, then we can write

$$\mathrm{d}F = \mathbf{V}(x, y) \cdot (\,\mathrm{d}x,\, \mathrm{d}y).$$

The task is therefore to prove that $\mathbf{V}(x, y) \cdot (\,\mathrm{d}x,\, \mathrm{d}y)$ can be written as a differential, $\mathrm{d}F$, from which we directly get the integral $F(x, y)$.

We get in this case, where we always pair terms which are similar,

$$\begin{aligned}
\mathbf{V} \cdot \mathrm{d}\mathbf{x} &= \left(\frac{1}{x} + \frac{1}{3x-y}\right) \mathrm{d}x - \frac{1}{3}\left(\frac{1}{y} + \frac{1}{3x-y}\right) \mathrm{d}y \\
&= \frac{1}{x}\,\mathrm{d}x - \frac{1}{3}\,\frac{1}{y}\,\mathrm{d}y + \frac{1}{3} \cdot \frac{1}{3x-y}(3\,\mathrm{d}x - \,\mathrm{d}y) \\
&= \mathrm{d}\ln|x| - \frac{1}{3}\,\mathrm{d}\ln|y| + \frac{1}{3} \cdot \frac{1}{3x-y}\,\mathrm{d}(3x-y) \\
&= \frac{1}{3}\,\ln\left|\frac{x^3}{y}\right| + \frac{1}{3}\,\ln|3x-y| = \frac{1}{3}\,\ln\left|\frac{x^3(3x-y)}{y}\right|.
\end{aligned}$$

Since a possible check only consists of doing the same calculations in the reverse order, we conclude that all the integrals in A_i, $i = 1, \ldots, 6$, are given by

$$F(x, y) = \frac{1}{3}\,\ln\left|\frac{x^3(3x-y)}{y}\right| + C_i,$$

where $C_i \in \mathbb{R}$ for $(x, y) \in A_i$, $i = 1, \ldots, 6$.

2) SECOND METHOD. Indefinite integration. It follows from the form of the expression that it is most convenient to perform indefinite integration on the *latter* coordinate of the vector field. (It is not an error to choose the former coordinate instead; it is only that the calculations become somewhat more difficult in that case).

$$F_1(x, y) = -\frac{1}{3}\int \left(\frac{1}{y} + \frac{1}{3x-y}\right) dy = -\frac{1}{3}\,\ln|y| + \frac{1}{3}\,\ln|3x-y|.$$

Hence

$$\frac{\partial F_1}{\partial x} = \frac{1}{3} \cdot \frac{3}{3x-y} = \frac{1}{3x-y},$$

so

$$V_1(x,y) - \frac{\partial F_1}{\partial x} = \frac{4x-y}{x(3x-y)} - \frac{1}{3x-y} = \frac{4x-y-x}{x(3x-y)} = \frac{1}{x}.$$

As a weak control we note that this expression no longer depends on y, which is the variable which should have been removed by the integration above.

We get by another integration,

$$F_2(x) = \int \frac{1}{x}\,dx = \ln|x|.$$

We get in any connected component A_i, $i = 1, \dots, 6$, the integral

$$F(x,y) = F_1(x,y) + F_2(x) = -\frac{1}{3}\ln|y| + \frac{1}{3}\ln|3x-y| + \ln|x| = \frac{1}{3}\ln\left|\frac{x^3(3x-y)}{y}\right|,$$

and all integrals in a connected component A_i, $i = 1, \dots, 5$ is

$$F(x,y) = \frac{1}{3}\ln\left|\frac{x^3(3x-y)}{y}\right| + C_i, \quad C_i \in \mathbb{R}, \quad (x,y) \in A_i, \quad i = 1, \dots, 6.$$

3) Third method. Because A is the union of six connected components, and since none of these have a natural starting point, the third method becomes somewhat complicated, so we shall leave this to the reader. In principle the calculations can be performed.

4) CHECK. It follows from

$$F(x,y) = -\frac{1}{3}\ln|y| + \frac{1}{3}\ln|3x-y| + \ln|x| + C_1 \quad \text{in } A_i$$

that

$$\begin{aligned}\nabla F(x,y) &= \left(\frac{1}{3}\cdot\frac{3}{3x-y}+\frac{1}{x}, -\frac{1}{3}\cdot\frac{1}{y}-\frac{1}{3}\cdot\frac{1}{3x-y}\right)\\ &= \left(\frac{4x-y}{x(3x-y)}, -\frac{1}{3}\left(\frac{1}{y}+\frac{1}{3x-y}\right)\right) = \mathbf{V}(x,y).\end{aligned}$$

which shows that $\mathbf{V}$ is a gradient field.

Example 19.8 *Let A denote the point set which is obtained by removing the origo and the positive part of the Y axis from the plane $\mathbb{R}^2$,*

$$A = \{(x,y) \mid x \neq 0 \text{ or } (x=0 \text{ and } y<0)\}.$$

We define a function $f : A \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 0, & y<0,\\ y^2, & x>0 \text{ and } y \geq 0,\\ -y^2, & x<0 \text{ and } y \geq 0.\end{cases}$$

Prove that f is a C^1-function, and that its partial derivative $\frac{\partial f}{\partial x}$ is zero everywhere in A.

A A C^1-function, which is not identically 0, and where nevertheless $\frac{\partial f}{\partial x} = 0$ everywhere.

D Apply the definition of differentiability itself (and not one of the weaker rules of calculations) to prove that f is of class C^1. Then calculate $\frac{\partial f}{\partial x}$ by going to the limit in the difference quotient.

I Clearly, A is open, and $f(x,y)$ is continuous across the X axis (with the exception of $(0,0)$, which is not included in the domain). Furthermore, $f(x,y)$ is of class C^∞, when $(x,y) \in A$ does not lie on the X axis.

Let us consider a point $(x_0,0)$, $x_0 \neq 0$, on the X axis minus $(0,0)$. If $x_0 < 0$, then

$$f(x,y) - f(x_0,0) = \begin{cases} -y^2, & \text{for } y>0,\\ 0, & \text{for } y \leq 0,\end{cases} \qquad x<0,$$

thus

$$|f(x,y)-f(x_0,0)| \leq |y|^2 = 0 + \sqrt{(x-x_0)^2+y^2}\cdot\varepsilon(\sqrt{(x-x_0)^2+y^2}).$$

We get analogously for $x_0 > 0$ that

$$f(x,y) - f(x_0,0) = \begin{cases} y^2, & \text{for } y>0,\\ 0, & \text{for } y \leq 0,\end{cases} \qquad x>0,$$

i.e.

$$f(x,y)-f(x_0,y)| \leq |y|^2 = 0+\sqrt{(x-x_0)^2+y^2}\cdot\varepsilon(\sqrt{(x-x_0)^2+y^2}).$$

It follows from these considerations that f is of class C^1 on the set of points on the X axis which also is included in A.

It is finally trivial that $\dfrac{\partial f}{\partial x} = 0$ everywhere in A.

Example 19.9 *Given the vector field*

$$\mathbf{V}(x,y) = \left(\frac{2x^3+2x+2xy}{1+x^2}, \frac{x^2y^2+2y+x^2}{1+y^2} \right), \qquad (x,y) \in \mathbb{R}^2.$$

1) Prove that $\mathbf{V}$ *is a gradient field, and find all its integrals.*

2) Explain why any of the integrals has the range $\mathbb{R}$.

A Gradient field.

D Prove that $\omega = \mathbf{V} \cdot \mathrm{d}\mathbf{x}$ is a differential. (Reduce!)

I 1) When we compute ω we get

$$\begin{aligned} \omega &= \mathbf{V} \cdot \mathrm{d}\mathbf{x} = \frac{2x^3y+2x+2xy}{1+x^2}\,\mathrm{d}x + \frac{x^2y^2+2y+x^2}{1+y^2}\,\mathrm{d}y \\ &= \frac{2x}{1+x^2}\,\mathrm{d}x + 2xy\,\mathrm{d}x + \frac{2y}{1+y^2}\,\mathrm{d}y + x^2\,\mathrm{d}y \\ &= \mathrm{d}\ln(1+x^2) + \mathrm{d}\ln(1+y^2) + \{2xy\,\mathrm{d}x + x^2\,\mathrm{d}y\} \\ &= \mathrm{d}\left\{\ln\left(1+x^2\right) + \ln\left(1+y^2\right) + x^2y\right\}. \end{aligned}$$

This proves that $\mathbf{V}$ is a gradient field, and that all integrals are given by

$$F_C(x,y) = \ln\left(1+x^2\right) + \ln\left(1+y^2\right) + x^2y + C, \qquad C \in \mathbb{R}.$$

2) It follows from the rules of magnitude that

$$F_C(x,1) \to +\infty \qquad \text{for } x \to \pm\infty,$$

and that

$$F_C(x,-1) \to -\infty \qquad \text{for } x \to \pm\infty.$$

Since $F_C(x,y)$ is continuous in $(x,y) \in \mathbb{R}^2$ for every fixed $C \in \mathbb{R}$, we conclude that the range is all of $\mathbb{R}$.

20 Integration in the plane

20.1 An overview of integration in the plane and in the space

Consider the abstract integral $\int_A f(\mathbf{x})\,\mathrm{d}\mu$, where $f : A \to \mathbb{R}$ always is assumed to be continuous. When we want to classify the type of integration, we should go through the following flow diagram.

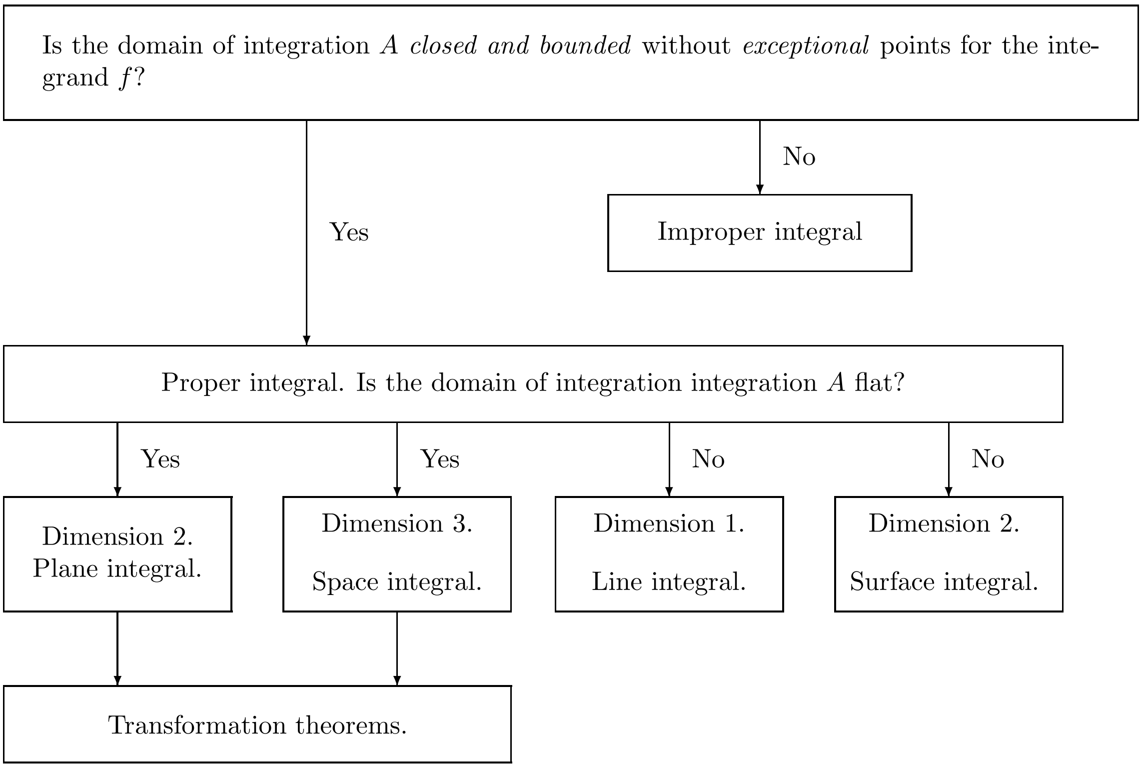


Figure 20.1: Flow diagram for types of integration.

These are ordered according to their difficulties in the following way:

1) **Plane integral** (rectangular, polar)

2) **Space integral** (rectangular, semi-polar, spherical)

3) **Line integral** (rectangular, polar; parametrical representation)

4) **Surface integral** (rectangular, semi-polar, spherical; parametrical representation)

5) **Transformation theorems** (plane, space)

6) **Improper integral** (bounded or unbounded domain).

20.2 Introduction

We shall here extend integration in $\mathbb{R}$ to integration in $\mathbb{R}^2$ and $\mathbb{R}^3$, from which it will be easy for the reader to generalize to integration in $\mathbb{R}^n$. As expected, there are lots of theoretical problems, when we rigorously define integration in higher dimensions. One would of course expect that in particular the geometry of the domain of integration interferes in a profound way, and it surely does. However, in order not to stray into lots of theoretical considerations of the Riemann and Lebesgue integrals we shall make this introduction as short as possible.

In the first analysis we shall, whenever nothing else is stated, assume that $f : A \to \mathbb{R}$ is a continuous function on a bounded and closed domain $A \subset \mathbb{R}^2$, or $\subset \mathbb{R}^3$. Then we let the symbol

$$\int_A f(\mathbf{x})\, \mathrm{d}\mu$$

denote the (abstract concept of the) integral of f over the set A. This is interpreted as the limit (in some vaguely described sense) of the mean of f, written as

$$\sum_{i=1}^{n} f\left(\mathbf{x}_i\right)\, \mathrm{d}\mu\left(A_i\right), \qquad \text{for } \mathbf{x}_i \in A_i,$$

where $\mu\left(A_i\right)$ denotes the (Riemann or Lebesque) measure of A_i, and where

$$A = A_1 \cup \cdots \cup A_n, \qquad \mu\left(A_i \cap A_j\right) = 0 \text{ for } i \neq j,$$

is an (almost disjoint) measurable subdivision of A, where $\mathrm{diam}(A_i) \to 0$ for $n \to +\infty$, or something similar, which shall not be specified here.

Note that we write $\mathrm{d}\mu$ for the Lebesgue measure, and not $\mathrm{d}\mathbf{x}$, as one would expect.

Later on we shall also take a closer look on the situation, when A is not bounded or closed – or when f is not continuous everywhere.

The above describes the intuitive idea. It is easy to comprehend, and yet difficult to get theoretically right. We ought here to mention that there exist sets, which are *not* measurable. Fortunately, these cannot be materialized in the real world, because otherwise we might come across strange problems like cutting a 3-dimensional body into nonmeasurable subsets, which then in a jigsaw puzzle can be put together in a new way, such that the initial body is doubled, and yet no point is missing! Clearly, this is against the practical experience. We therefore relegate these sets into the strange world of Mathematics, and from now on we tacitly assume that all sets are measurable.

We shall use the following *rules of integration* over and over again.

1) *Linearity.* Let $f,\, g \in C(A)$, and let $c \in \mathbb{R}$ be a constant. Then

$$\int_A \{f(\mathbf{x}) + c\, g(\mathbf{x})\}\,\mathrm{d}\mu = \int_A f(\mathbf{x})\,\mathrm{d}\mu + c\int g(\mathbf{x})\,\mathrm{d}\mu.$$

2) *Splitting of sets.* Let $f \in C(A \cup B)$. Then

$$\int_{A\cup B} f(\mathbf{x})\,\mathrm{d}\mu + \int_{A\cap B} f(\mathbf{x})\,\mathrm{d}\mu = \int_A f(\mathbf{x})\,\mathrm{d}\mu + \int_B f(\mathbf{x})\,\mathrm{d}\mu.$$

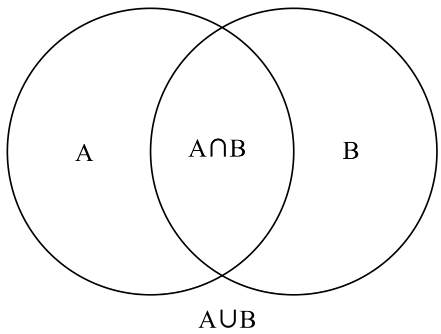


Figure 20.2: Venn diagram for $A \cup B$. The points in $A \cap B$ lie in both A and in B, so they enter both $\int_A \cdot\,\mathrm{d}\mu$ and $\int_B \cdot\,\mathrm{d}\mu$. We compensate for this by adding $\int_{A\cup B} \cdot\,\mathrm{d}\mu$.

3) *The area/volume of A.*

$$\int_A 1\,\mathrm{d}\mu = \mu(A) = \mathrm{vol}(A).$$

4) *Integration respects the (weak) order relation.* Assume that $\mu(A) > 0$ and that $f(\mathbf{x}) \le g(\mathbf{x})$ for all $\mathbf{x} \in A$. Then

$$\int_A f(\mathbf{x})\,\mathrm{d}\mu \le \int_A g(\mathbf{x})\,\mathrm{d}\mu.$$

5) *Absolute value.* If $f : A \to \mathbb{R}$ is continuous on the closed and bounded set A, then

$$\left|\int_A f(\mathbf{x})\,\mathrm{d}\mu\right| \le \int_A |f(\mathbf{x})|\,\mathrm{d}\mu.$$

6) *Integration over nullsets always gives 0.* Let A be a (Lebesgue) nullset, i.e. $\mu(A) = 0$. Then for all continuous functions $f \in C(A)$,

$$\int_A f(\mathbf{x})\,\mathrm{d}\mu = 0.$$

This rule does not hold for the *Dirac measure* "$\delta(t)''$, but this is an atomic measure and not a true function in the usual sense.

7) *Extension to the closure.* Let $f \in C(A)$, where A is bounded, and *the boundary ∂A is a nullset*, i.e. $\mu(\partial A) = 0$. Then

$$\int_A f(\mathbf{x})\,\mathrm{d}\mu = \int_{A^\circ} f(\mathbf{x})\,\mathrm{d}\mu = \int_{\overline{A}} f(\mathbf{x})\,\mathrm{d}\mu.$$

We note here that it is possible to construct sets A, such that ∂A is *not* a nullset, so $\mu(\partial A) > 0$. In the applications such a "space filling curve" as boundary may e.g. enter the problem of *Brownian motion* in the plane. We shall of course in the following not consider such "strange sets", so we shall always in the following assume that $\partial A = 0$.

8) *The mean value.* When $\mu(A) > 0$, we define the *mean value* as

$$\frac{1}{\mu(A)} \int_A f(\mathbf{x})\,\mathrm{d}\mu.$$

The mean value theorem. If A, where $\mu(A) > 0$, is closed and bounded and connected, then there is an $\mathbf{u} \in A$, such that

$$\int_A f(\mathbf{x})\,\mathrm{d}\mu = f(\mathbf{u})\mu(A).$$

9) *Integration of a vector function in $\mathbb{R}^m$.* This is done for each coordinate,

$$\int_A \mathbf{f}(\mathbf{x})\,\mathrm{d}\mu := \left(\int_A f_1(\mathbf{x})\,\mathrm{d}\mu, \cdots, \int_A f_m(\mathbf{x})\,\mathrm{d}\mu \right).$$

10) *Rules of calculation for a function defined by an integral.* Let $f \in C^n(A \times B)$, $n \geq 1$ and $A \subseteq \mathbb{R}^k$ and $B \subseteq \mathbb{R}$. We define a function $f; B \to \mathbb{R}$ by

$$F(u) := \int_A f(\mathbf{x}, u)\, \mathrm{d}\mu, \qquad \text{for } u \in B.$$

Then $F \in C^n(B)$, and we can differentiate the integral with respect to u under the integration sign,

$$F'(u) = \int_A f_u'(\mathbf{x}, u)\, \mathrm{d}\mu, \qquad \text{for } u \in B.$$

We shall in the following go through various cases of integration. The program follows the list in Section 20.1, where they have been listed according to their complexity. However, we first give some examples which show the importance of integration in space in Physics.

Example 20.1 Let f denote the *density* of a given mass spread over the set A. Then the total mass over A is

$$M = \int_A f(\mathbf{x})\, \mathrm{d}\mu.$$

The *centre of the mass*, or the *barycentre*, ξ is defined by the vector function

$$M\,\xi = \int_A \mathbf{x}\, f(\mathbf{x})\, \mathrm{d}\mu.$$

It can be proved that the barycentre of A does not change, if we use another coordinate system, so ξ is an invariant.

We note in particular that if the mass is spread evenly over A, then the density becomes a constant f_0, and we get

$$M = f_0\, \mu(A), \qquad \text{and} \qquad \mu(A)\xi = \int_A \mathbf{x}\, \mathrm{d}\mu.$$

Note that even if ξ is called the *centre of mass* this does not imply that $\xi \in A$. Let e.g. A describe a ring of constant density. Then the centre of mass (i.e. the barycentre) lies at the centre of the ring, i.e. outside the ring. $\Diamond$

Example 20.2 Integration in space is also used in the *Theory of Electricity.* If f denotes the "density" of the electrical charge, then the total electric charge of A is

$$Q = \int_A f(\mathbf{x})\, \mathrm{d}\mu.$$

We note that the electrical "density" can be positive, negative and zero. In most other applications a density f satisfies that $f(\mathbf{x}) \geq 0$.

Another application in the *Theory of Electricity* is the definition of the *electrical dipole moment*,

$$\mathbf{p} := \int_A \mathbf{x}\, f(\mathbf{x})\, \mathrm{d}\mu.$$

Furthermore, a *distribution of current* is defined by its vectorial *"density"* $\mathbf{f}(\mathbf{x})$. If furthermore, $A \subseteq \mathbb{R}^3$, we also introduce the *magnetic dipole moment*,

$$\mathbf{m} := \frac{1}{2} \int_A \mathbf{x} \times \mathbf{f}(\mathbf{x})\, \mathrm{d}\mu. \qquad \Diamond$$

Example 20.3 In *Mechanics* we define the *angular momentum* of a rotation of a solid around a line in the following way

$$I := \int_A \{R(\mathbf{x})\}^2 f(\mathbf{x})\, \mathrm{d}\mu,$$

where $R(\mathbf{x})$ denotes the distance from the point $\mathbf{x}$ to the line ℓ (the axis of rotation), and where f is the density function of the body.

If ℓ is one of the coordinate axes, this expression becomes very simple. If e.g. ℓ is the z-axis, and we use rectangular coordinates (x, y, z), then R is the distance between (x, y, z) and $(0, 0, z)$, so $R^2 = x^2 + y^2$ and

$$I = \int_A \left(x^2 + y^2\right)^2 f(x, y, z)\, \mathrm{d}\mu. \qquad \Diamond$$

20.3 The plane integral in rectangular coordinates

The simplest of the new cases of extension of the integral is the *plane integral*. The domain of integration $B \subseteq \mathbb{R}^2$ is a set in the (x,y)-plane, and $\mathrm{d}\mu$ is replaced by the *element of area* $\mathrm{d}S$, and the integrand is written $f(x,y)$, so the generally used notation of the plane integral is

$$(20.1) \quad \int_B f(x,y)\,\mathrm{d}S \qquad \text{or} \qquad \int_B f(x,y)\,\mathrm{d}S,$$

where we shall later explain the colour green. Sometimes one writes in the applications

$$\int\int_B f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

to emphasize that we are dealing with a 2-dimensional integration.

We consider (20.1) as an *abstract notation* of the plane integral, sometimes coloured in blue or green in the following to signalize that one cannot calculate it directly, but must use some reduction like in Theorem 20.1 below. In these reductions the integral is broken down to a series of ordinary 1-dimensional integrals, which are then sometimes for pedagogical reasons given the colours red, black or blue. We need Theorem 20.1, before we can explain, what is meant by this. Cf. also Section 20.3.2 on the colour code.

We note that the *area* of a plane set B, written $\mathrm{area}(B)$, is found by integrating the constant 1, thus

$$\mathrm{area}(B) := \int_B 1\,\mathrm{d}S = \int_B \mathrm{d}S, \qquad \text{for } B \subset \mathbb{R}^2.$$

When we reduce (20.1) to a computable form, we shall in general rewrite (20.1) as a concatenation of two integrals with respect to the chosen coordinates. For plane sets we have introduced in Section 1.4,

1) rectangular coordinates,

2) polar coordinates.

We shall in the following subsections show the reduction theorems in the two two types of coordinate systems.

20.3.1 Reduction in rectangular coordinates

As mentioned earlier we shall not bother too much with the proofs, but only sketch the ideas. So given a plane domain $B \subset \mathbb{R}^2$ in rectangular coordinates.

The optimum situation is when we can describe B as either

$$B = \left\{(x,y) \in \mathbb{R}^2 \mid a \le x \le b,\, Y_1(x) \le y \le Y_2(x)\right\},$$

where $Y_1(x)$ and $Y_2(x)$, $x \in [a,b]$, are continuous functions, which describe the lower and the upper boundary curves, or as

$$B = \{(x,y) \mid c \le y \le d,\, X_1(y) \le x \le X_2(y)\},$$

where $X_1(y)$ and $X_2(y)$, $y \in [c,d]$, are continuous functions, which describe the left and the right boundary curves, cf. Figure 20.3, where the left hand side refers to the first case, and the right hand side refers to the second case.

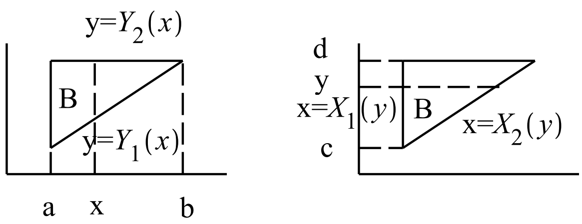


Figure 20.3: Analysis of the domain B of integration in rectangular coordinates.

In the first case we integrate *vertically* for every fixed $x \in [a, b]$ over the interval $y \in [Y_1(x), Y_2(x)]$. One may think of this process as if we collect all "mass" lying in B on the vertical line above x and take it as a value $F(x)$ of a new function,

$$F(x) := \int_{Y_1(x)}^{Y_2(x)} f(x, y)\, \mathrm{d}y, \qquad \text{for } x \in [a, b].$$

When we afterwards integrate $F(x)$ horizontally along the x-axis, we collect all "mass" from B, so we get

$$\int_B f(x, y)\, \mathrm{d}S = \int_a^b F(x)\, \mathrm{d}x = \int_a^b \left\{ \int_{Y_1(x)}^{Y_2(x)} f(x, y)\, \mathrm{d}y \right\} \mathrm{d}x.$$

The latter double integral can now by computed by two successive ordinary 1-dimensional integrations, where one must be very careful to integrate in the right order. In order to explain this order we shall sometimes colour these integrals, so

$$\int_B f(x, u)\, \mathrm{d}S = \int_a^b \left\{ \int_{Y_1(x)}^{Y_2(x)} f(x, y)\, \mathrm{d}y \right\} \mathrm{d}x,$$

which we interpret in the following way: The blue integral to the left is an *abstract symbol*, which cannot be computed in this form. It is, however, equal to the double integral on the right hand side, which has been coloured in red and black. The procedure is then that we first integrate the red integral, where all other colours (here only black) are considered as constants. Once this has been done, the result must no longer contain the red y as a variable. If it does, when we have made an error and must start from the beginning. When the red integral has been calculated, then we continue with the black integral and reduce.

Similarly, if instead $B = \{(x,y) \mid c \le y \le d,\, X_1(y) \le x \le X_2(y)\}$.

We state the results as a theorem.

Theorem 20.1 Reduction theorem of a plane integral in rectangular coordinates. *Assume that $B \subset \mathbb{R}^2$ is bounded and closed, and that $f : B \to \mathbb{R}$ is a continuous function.*

1) Assume that B can be written

$$B = \left\{(x,y) \in \mathbb{R}^2 \mid a \le x \le b,\, Y_1(x) \le y \le Y_2(x)\right\},$$

where

$$Y_1,\, Y_2 \in C^0([a,b]) \qquad \text{and} \qquad Y_1(x) \le Y_2(x) \text{ for } x \in [a,b].$$

Then

$$\int_B f(x,u)\, dS = \int_a^b \left\{ \int_{Y_1(x)}^{Y_2(x)} f(x,y)\, dy \right\} dx.$$

2) Assume instead that

$$B = \left\{(x,y) \in \mathbb{R}^2 \mid c \le y \le d,\, X_1(y) \le x \le X_2(y)\right\},$$

where

$$X_1,\, X_2 \in C^0([c,d]) \qquad \text{and} \qquad X_1(y) \le X_2(y) \text{ for } y \in [c,d].$$

Then

$$\int_B f(x,y)\, dS = \int_c^d \left\{ \int_{X_1(y)}^{X_2(y)} f(x,y)\, dx \right\} dy.$$

Theorem 20.1 gives us some methods for explicitly calculating a plane integral. It is formulated in its simplest form. When the geometry of the domain B is more complicated, we may split it up into simpler subdomains,

$$B = B_1 \cup \cdots \cup B_n, \qquad \text{where } \mu\left(B_i \cap B_j\right) = 0 \quad \text{for } i \ne j,$$

where the reduction theorem can be used on every B_i, $i = 1, \dots, n$.

Note that we do not require a disjoint splitting of B. The condition $\mu\left(B_i \cap B_j\right) = 0$ for $i \ne j$ means that the intersection $B_i \cap B_j$ is a nullset – a point, or a straight line in RR^2, or a plane in $\mathbb{R}^3$. This may be convenient in order to put the bounds of the integrals right.

One should note that even if both reductions can be used, like in Figure 20.3, the resulting two double integrals may lead to calculations of different difficulties, so one should carefully choose the variant, which gives the simplest calculations. This will be illustrated in some of the examples in the following.

20.3.2 The colour code, and a procedure of calculating a plane integral

In this section we give a more thorough treatment, because it is the basic form which all other reductions are referring to.

The colour code: We let blue or green integrals denote abstract concepts, which cannot be computed without further reduction. These colours signal that a reduction should be carried out. Whenever possible, we shall prefer the use of the blue colour, because the green colour may be difficult to read. It is only in $\mathbb{R}^3$ that we shall need four colours.

Red and black integrals are calculated by elementary calculus methods. When we calculate a red integral, i.e. a red variable, we consider any other colour as representing constants with respect to this integration.

Blue integrals represent in $\mathbb{R}^2$ the abstract integral, while in space integrals in $\mathbb{R}^3$ it will be convenient to let the blue integrals represent a simpler (2-dimensional) abstract integral, and use the green colour for the full abstract integral in 3 dimensions.

The chosen hierarchy of colours is therefore

green — blue — black — red ,

which means that we whenever possible establish the integrals in this order. In the calculations we start from behind with first integrating the black variable, then the red one, and then finally the blue integral. A weak check is that after the "black" integration, the black variable has disappeared. If we still have the black variable somewhere in the expression, then this is an indication of that we have made an error, so we must go back to the analysis of the domain.

The problems of understanding what is going on here is caused by the *historical* notation of an integral in the form

$$\int \cdots \, \mathrm{d}\mu,$$

which says that the integral is actually written as an advanced form of a pair of brackets, where $\int$ is the first bracket and $\mathrm{d}\mu$ is the second bracket. Other alternative notations are, however, possible but they have all some other disadvantages, so we shall stick to the familiar notation.

Geometry.

Whenever possible, one should always start by sketching a figure of the *parametric domain*. In its basic form (the rectangular version) the parametric domain is identical with the domain itself, so one does not distinguish between whether some of the coordinates are lying in the domain itself, while other coordinates at the same time are considered as lying in the parametric domain. Therefore, the same letter may occur both as a variable and as a parameter, depending on the actual situation. This may at a first glance look very confusing. In other words, one lacks a notation, by which one can distinguish between a variable and a parameter. Some experiments have been made by using different colours for the different aspects with some success. It has been included here in some of the examples.

In the rectangular case there are two *basic geometrical forms*:

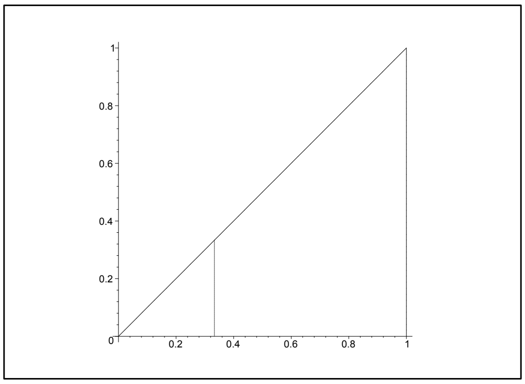


Figure 20.4: First version, where one in the inner (i.e. the first) integral integrates vertically. The bounds are here $Y_1(x) = 0$ and $Y_2(x) = x$.

The actual figure is divided into sub-figures of one of these two types. Then each sub-figure is treated separately. We use the following principles:

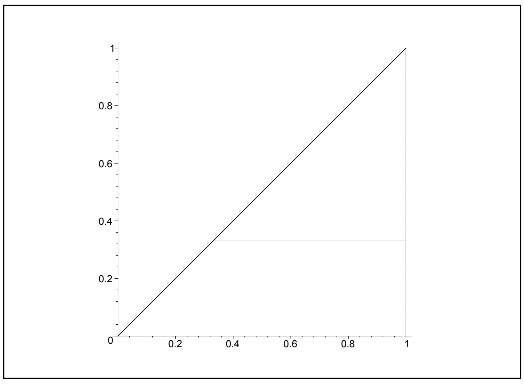


Figure 20.5: Second version, where one in the inner (i.e. the first) integral integrates horizontally. The bounds are here $X_1(y) = y$ and $X_2(y) = 1$.

First version. The variable x lies in an interval $[a, b]$, where a and b are constants. Note that x is considered as a *parameter* in the first (i.e. the inner) integration. We colour x black to indicate that x is considered as a constant, when we integrate with respect to y, which is here coloured red. The variable y lies between the two curves of equations $y = Y_1(x)$ and $y = Y_2(x)$, where the x fixes both $Y_1(x)$ and $Y_2(x)$ as constants, so they are both black.

Second version. The variable y lies in an interval $[c, d]$, where c and d are constants. Note that y is considered as a *parameter* in the first (i.e. the inner) integration. We colour y black to indicate that y is considered as a constant, when we integrate with respect to y, which is here coloured red. The variable x lies between the two curves of equations $x = X_1(y)$ and $x = X_2(y)$. Since y is fixed, the numbers are both fixed, therefore coloured black.

Remark 20.1 Some figures, like e.g. axis-parallel rectangles, can be described equally well in the two geometrical versions. Then it depends on the structure of the *integrand* whether one should choose the first or the second version, because the integrations themselves do not be equally easy in the two versions. Whenever one is in trouble with the calculations in one of the two versions, one may try the other one instead. In some cases these integrations can in fact be calculated. ♢

Problem 20.1 *Calculate the abstract plane integral*

$$\int_B f(x, y)\, dS,$$

by reduction in rectangular coordinates.

Procedure:

1) Sketch the domain of integration B. If necessary, divide B into sub-domains of type 1 or type 2, described above.

2) *If* B (or some sub-domain) is of type 1, then write

$$B = \{(x, y) \mid a \leq x \leq b,\, Y_1(x) \leq y \leq Y_2(x)\},$$

and set up the reduction formula

$$\int_B f(x,y)\,\mathrm{d}S = \int_a^b \left\{ \int_{Y_1(x)}^{Y_2(x)} f(x,y)\,\mathrm{d}y \right\} \mathrm{d}x.$$

According to the figure of the first version, one first keeps x fixed and integrates y *vertically*. This can be interpreted as if we are collecting the "mass" of the vertical line above the point x in this point on the x-axis. Then the total "mass" is afterwards obtained by integrating the partial results from the inner integrations after x.

3) Perform a *separate* calculation of the inner integral:

$$\varphi(x) = \int_{Y_1(x)}^{Y_2(x)} f(x,y)\,\mathrm{d}y,$$

where x is considered as a constant.

When one has obtained the necessary training in this technique, one may of course skip this point.

4) Insert and calculate the integral by methods from elementary calculus

$$\int_B f(x,y)\,\mathrm{d}S = \int_a^b \varphi(x)\,\mathrm{d}x.$$

5) If B (or a subdomain of B) is of type 2, then write

$$B = \{(x,y) \mid c \le y \le d,\, X_1(y) \le x \le X_2(y)\},$$

and apply the formula of reduction

$$\int_B f(x,y)\,\mathrm{d}S = \int_c^d \left\{ \int_{X_1(y)}^{X_2(y)} f(x,y)\,\mathrm{d}x \right\} \mathrm{d}y.$$

According to the figure, 2. version, we here keep y fixed and then integrate horizontally after x. By this we collect the "mass" along the horizontal line at the point y on the y-axis. The total "mass" is then obtained by integrating the subresults after y.

6) Compute separately the inner integral:

$$\psi(y) = \int_{X_1(y)}^{X_2(y)} f(x,y)\,\mathrm{d}x,$$

where y is considered as a constant.

When one has obtained the necessary training in this technique, one may of course skip this point.

7) Calculate the integral after insertion using simple methods known from elementary Calculus.

$$\int_B f(x,y)\,\mathrm{d}S = \int_c^d \psi(y)\,\mathrm{d}y.$$

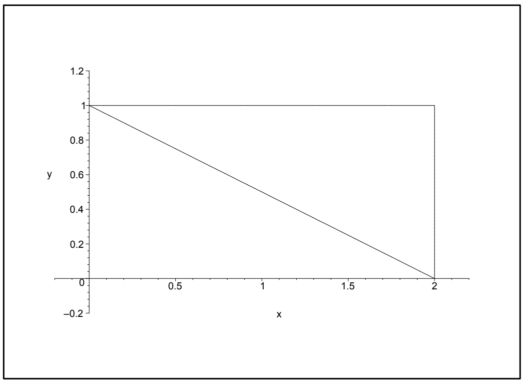


Figure 20.6: The domain B.

20.4 Examples of the plane integral in rectangular coordinates

Example 20.4 **A.** *Compute* $\int_B xy \, \mathrm{d}S$, *where* B *is the set given on Figure 20.6.*

D. Here we can reduce in two different ways,

D1. We first integrate (i.e. the inner integral) vertically.

D2. We first integrate (i.e. the inner integral) horizontally.

In order to compare the two possibilities they are both included here.

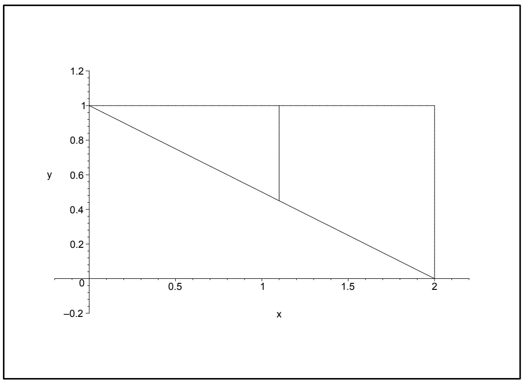


Figure 20.7: The domain B with a vertical path of integration from $y = 1 - \frac{1}{2}x$ to $y = 1$.

D 1. We shall first integrate vertically.

I 1. In this case we write the domain in the following way,

$$B = \{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 2, \, 1 - \frac{1}{2}x \le y \le 1\}.$$

One derives this in the following way. The *"outer"* variable of integration x must always lie between two *constants*, here $0 \le x \le 2$. Then we analyze the figure for every fixed x to find the interval of integration for den *"inner"* variable of integration y, her $1 - \frac{1}{2}\,x \le y \le 1$.

Then we set up the double integral

$$(20.2)\qquad \int_B xy\,\mathrm{d}S = \int_0^2 \left\{ \int_{1-\frac{1}{2}x}^{1} xy\,\mathrm{d}y \right\} \mathrm{d}x = \int_0^2 x \left\{ \int_{1-\frac{1}{2}x}^{1} y\,\mathrm{d}y \right\} \mathrm{d}x.$$

We compute the inner integral,

$$\int_{1-\frac{1}{2}x}^{1} y\,\mathrm{d}y = \left[\frac{1}{2}\,y^2\right]_{1-\frac{1}{2}x}^{1} = \frac{1}{2}\left\{1 - \left(1 - \frac{1}{2}\,x\right)^2\right\} \frac{1}{2}\left\{1 - \frac{1}{4}\,x^2\right\} = \frac{1}{2}\,x - \frac{1}{8}\,x^2.$$

By insertion into (20.2),

$$\int_B xy\,\mathrm{d}S = \int_0^2 x\left\{\frac{1}{2}\,x - \frac{1}{8}\,x^2\right\}\,\mathrm{d}x = \int_0^2 \left\{\frac{1}{2}\,x^2 - \frac{1}{8}\,x^3\right\}\,\mathrm{d}x = \left[\frac{1}{6}\,x^3 - \frac{1}{32}\,x^4\right]_0^2 = \frac{8}{6} - \frac{1}{2} = \frac{5}{6}.$$

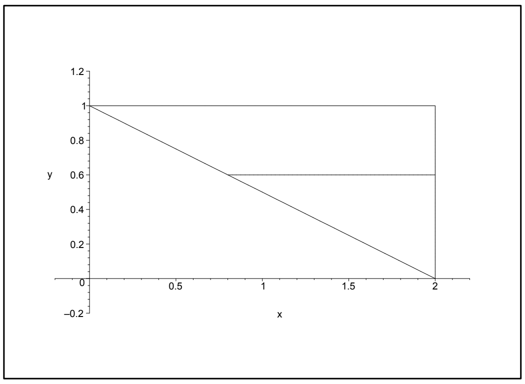


Figure 20.8: The domain B with the horizontal path of integrations from $x = 2 - 2y$ to $x = 2$.

D 2. In the second variant we first integrate horizontally, cf. Figure 20.8

I 2. The set is here written

$$B = \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le 1,\; 2 - 2y \le x \le 2\},$$

because $y \in [0,1]$ here is the *outer* variable of integration (between two constants), and where $2 - 2y \le x \le 2$ for the *inner* variable of integration x for every fixed y.

The double integral becomes

$$(20.3)\quad \int_B xy\,\mathrm{d}S = \int_0^1 \left\{\int_{2-2y}^2 xy\,\mathrm{d}x\right\}\,\mathrm{d}y = \int_0^1 \left\{\int_{2-2y}^2 x\,\mathrm{d}x\right\}\,\mathrm{d}y.$$

First compute the inner integral,

$$\int_{2(1-y)}^2 x\,\mathrm{d}x = \left[\frac{1}{2}\,x^2\right]_{2(1-y)}^2 = \frac{1}{2}\cdot 2^2\left\{1^2 - (1-y)^2\right\} = 2(2y - y^2) = 4y - 2y^2.$$

When this is inserted into (20.3), we get

$$\int_B xy\,dS = \int_0^1 y(4y - 2y^2)\,\mathrm{d}y = \int_0^1 (4y^2 - 2y^3)\,\mathrm{d}y = \left[\frac{4}{3}\,y^3 - \frac{1}{2}\,y^4\right]_0^1 = \frac{5}{6}. \qquad \Diamond$$

Example 20.5 We get the area of a domain B by integrating the constant 1 over B. In particular, when B is lying between two graphs,

$$B = \left\{(x,y) \in \mathbb{R}^2 \mid a \le x \le b,\, Y_1(x) \le y \le Y_2(x)\right\},$$

then

$$\text{area}(B) = \int_B 1\, \mathrm{d}S = \int_a^b \{Y_2(x) - Y_1(x)\}\, \mathrm{d}x,$$

because the inner integral is trivially

$$\int_{Y_1(x)}^{Y_2(x)} 1\, \mathrm{d}y = Y_2(x) - Y_1(x). \qquad \diamond$$

Example 20.6 Another simple observation is the following: Assume that $D = [a,b] \times [c,d]$ is an axiparallel rectangle, and that the integrand $f(x,y) = F(x)G(y)$ has separated variables. Then by first integrating vertically, so $F(x)$ is considered to be a constant in this inner integration and hence can be moved outside to the outer black integral,

$$\int_B F(x)G(y)\, \mathrm{d}S = \int_a^b \left\{\int_c^d F(x)G(y)\, \mathrm{d}y\right\} \mathrm{d}x = \int_a^b \left\{\int_c^d G(y)\, \mathrm{d}y\right\} \mathrm{d}x$$

$$= \left\{\int_a^b F(x)\, \mathrm{d}x\right\} \left\{\int_c^d G(y)\, \mathrm{d}y\right\},$$

and we see that we in this case can separate the integrations. $\diamond$

Example 20.7 Let the domain be the square $\left[0, \frac{\pi}{2}\right] \times \left[0, \frac{\pi}{2}\right]$ and the integrand $\cos(x-y)$. Then by the reduction theorem,

$$\begin{aligned}\int_B \cos(x-y)\, dS &= \int_0^{\frac{\pi}{2}} \left\{\int_0^{\frac{\pi}{2}} \cos(x-y)\, dy\right\} dx = \int_0^{\frac{\pi}{2}} [-\sin(x-y)]_{y=0}^{y=\frac{\pi}{2}}\, dx \\ &= \int_0^{\frac{\pi}{2}} \left\{-\sin\left(x - \frac{\pi}{2}\right) + \sin x\right\} dx = \int_0^{\frac{\pi}{2}} (\cos x + \sin x)\, dx \\ &= \int_0^{\frac{\pi}{2}} \cos x\, dx + \int_0^{\frac{\pi}{2}} \sin x\, dx = 1 + 1 = 2. \qquad \diamond\end{aligned}$$

Example 20.8 **A.** *Compute* $\int_B x \exp\left(y^3\right) \, \mathrm{d}S$, *where* B *is sketched on Figure 20.9.*

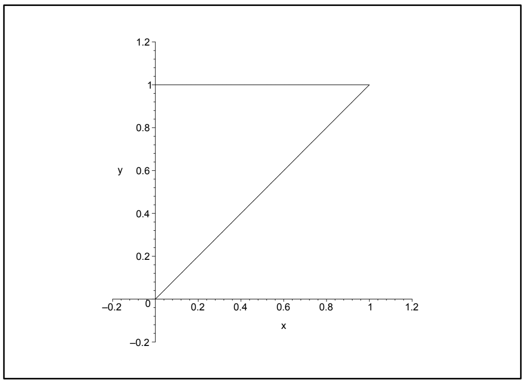


Figure 20.9: The domain B.

D. We shall check the two possibilities of order of integration.

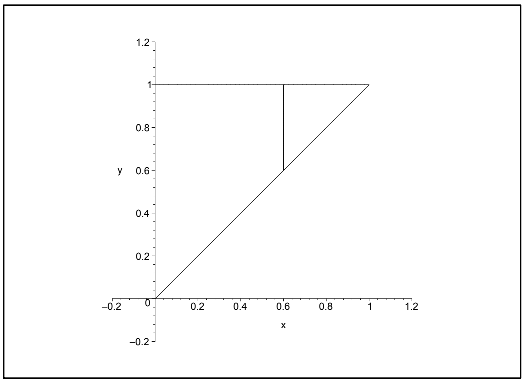


Figure 20.10: The domain B with a vertical path of integration from $y = x$ to $y = 1$.

D 1. We shall first try to integrate vertically for fixed x.

I 1. The domain is here written (note the order of x and y):

$$B = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1, \, x \le y \le 1\}.$$

Then we can set up the double integral,

$$\int_B x \exp\left(y^3\right) \, \mathrm{d}S = \int_0^a \left\{ \int_x^1 x \exp\left(y^3\right) \, \mathrm{d}y \right\} \mathrm{d}x = \int_0^1 x \left\{ \int_x^1 \exp\left(y^3\right) \, \mathrm{d}y \right\} \mathrm{d}x.$$

The inner integral,

$$\int_x^1 \exp\left(y^3\right) \, \mathrm{d}y,$$

does not look nice, so at this point one should take a look at the alternative way of computing.

Apparently, the simplest application of MAPLE does not solve the problem either:

with(Student[MultivariateCalculus]):

$$\text{MultiInt}\left(x \cdot e^{y^3}, y = x..1, x = 0..1\right)$$

$$\int_0^1 \int_x^1 x\, e^{y^3}\, \mathrm{d}y\, \mathrm{d}x$$

where we cannot go further. Then one could try

$$\text{int}\left(e^{y^3}, y = x..1\right)$$

which returns

$$\int_x^1 e^{y^3}\, \mathrm{d}y$$

which is of no help either. Finally, the indefinite integral

$$\int e^{y^3}\, \mathrm{d}y$$

gives the following result

$$-\frac{1}{3}(-1)^{2/3}\left(\frac{2}{3}\,\frac{y(-1)^{1/3}\pi\sqrt{3}}{\Gamma\left(\frac{2}{3}\right)(-y^3)^{1/3}}-\frac{y(-1)^{1/3}\Gamma\left(\frac{1}{3},-y^3\right)}{(-y^3)^{1/3}}\right)$$

a result which would require a course in *Special Functions* to understand.

The conclusion is that it in principle is possible to apply MAPLE, but this application is in this particular case more difficult than the following method, where we just interchange the order of x and y.

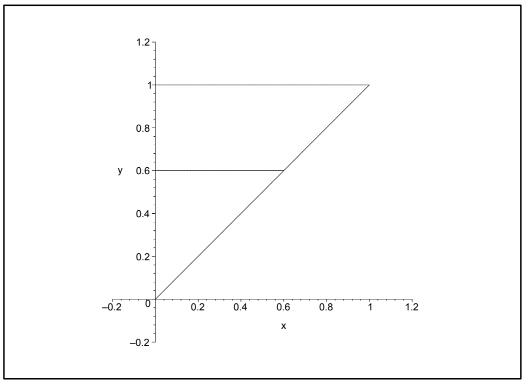


Figure 20.11: The domain B with a horizontal path of integration from $x=0$ to $x=y$ for fixed y.

D 2. We write the set in the following way (note the order of x and y):

$$B=\{(x,y)\in\mathbb{R}^2 \mid 0\le y\le 1,\, 0\le x\le y\}.$$

Then we set up the double integral

$$(20.4)\qquad \int_B x\,\exp\left(y^3\right)\,\mathrm{d}S=\int_0^1\left\{\int_0^y x\exp\left(y^3\right)\,\mathrm{d}x\right\}\mathrm{d}y=\int_0^1\exp\left(y^3\right)\left\{\int_0^y x\,\mathrm{d}x\right\}\mathrm{d}y.$$

The inner integral is straightforward,

$$\int_0^y x\,\mathrm{d}x=\left[\frac{1}{2}\,x^2\right]_0^y=\frac{1}{2}\,y^2.$$

By an insertion into (20.4) followed by the substitutionn $t=y^3$ and $\mathrm{d}t=3y^2\,\mathrm{d}y$, where $y^2\,dy$ already is present under the sign of integration, we finally get

$$\begin{aligned}\int_B x\,\exp\left(y^3\right)\,\mathrm{d}S &= \int_0^1\exp\left(y^3\right)\cdot\frac{1}{2}\,y^2\,\mathrm{d}y=\int_0^1\exp(t)\cdot\frac{1}{2}\cdot\frac{1}{3}\,\mathrm{d}t\\ &= \frac{1}{6}\left[e^t\right]_0^1=\frac{1}{6}\,\{e-1\}\,.\end{aligned}$$

Remark 20.2 We have here an example, in which one of the possible methods cannot be applied, and where MAPLE does not help much in this variant. On the other hand, by interchanging the order of integration we were nevertheless able to compute this plane integral. ◊

Example 20.9 A luminous element of area $\mathrm{d}S$ contributes to the lighting of another element of area $\mathrm{d}S'$ with the amount

$$\frac{L\cos\alpha\cos\beta}{R^2}\,\mathrm{d}S,$$

where R is the distance between the two elements of area, α is the angle between the normal of $\mathrm{d}S$ and the direction from $\mathrm{d}S$ to $\mathrm{d}S'$, and β is the angle between the normal of $\mathrm{d}S'$ and the dierection from $\mathrm{d}S'$ to $\mathrm{d}S$. Finally, L describes the emission of light from $\mathrm{d}S$.

The lighting E onto $\mathrm{d}S'$ from the shining part of B of a plane can then be written as a plane integral. When L is a constant, and $B = [0,a]\times[0,b]$ is a rectangle, and $\mathrm{d}S'$ is perpendicular to the shining plane, situated at some distance on the z-axis, it can be shown that

$$E = \frac{L\cos\alpha\cos\beta}{R^2}\,\mathrm{d}S = Lz\int_0^a\left\{\int_0^b \frac{y}{\left(x^2+y^2+z^2\right)^2}\,\mathrm{d}y\right\}\mathrm{d}x.$$

A. *Find the value of*

$$E = Lz\int_B \frac{y}{(x^2+y^2+z^2)^2}\,\mathrm{d}S, \quad \textit{where } B=[0,a]\times[0,b] \textit{ and } z>0 \textit{ a constant.}$$

D. We can expect a lot of trouble in this example no matter the version, we choose. This is due to the fact that the integrand invites to a description in polar coordinates, while the domain B in the (x,y)-plane is best described in rectangular coordinates. Experience shows that whenever we have a mixed problem of rectangular and polar coordinates, then the calculations are in general very difficult. Unfortunately, this is often the case in practical applications.

Then note that *if* we first in the inner integral integrate with respect to x, then already the first integral

$$\int_0^a \frac{\mathrm{d}x}{(x^2+y^2+z^2)^2}$$

will also cause some trouble. It is possible to use this variant, but the computations are far from easy to perform. However, if we instead start by integrating y, we obtain some easier computations, because $y\,\mathrm{d}y = \frac{1}{2}\,\mathrm{d}\left(y^2\right)$, and y occurs only as y^2 in the remaining of the integrand. We therefore choose this variant, so we first integrate with respect to y in the inner integral.

I. The domain is an axiparallel rectangle, so we can go straight to setting up the double integral,

$$(20.5)\quad E = Lz\int_B \frac{y}{(x^2+y^2+z^2)^2}\,\mathrm{d}S = Lz\int_0^a\left\{\int_0^b \frac{y}{(x^2+y^2+z^2)^2}\,\mathrm{d}y\right\}\mathrm{d}x.$$

We put $c = x^2+z^2 = \text{constant}$ in the inner integral, and we apply the substitution $t = y^2$ with $\mathrm{d}t = 2y\,\mathrm{d}y$, where the group $y\,\mathrm{d}y$ already occurs under the sign of integration, i.e. $y\,\mathrm{d}y = \frac{1}{2}\,\mathrm{d}t$.

Then

$$\begin{aligned}\int_0^b \frac{y}{(x^2+y^2+z^2)^2}\,\mathrm{d}y &= \int_0^b \frac{1}{(y^2+c)^2}\,y\,\mathrm{d}y = \frac{1}{2}\int_0^{b^2} \frac{\mathrm{d}t}{(t+c)^2} = \frac{1}{2}\left[-\frac{1}{t+c}\right]_{t=0}^{b^2} \\ &= \frac{1}{2}\left(\frac{1}{c}-\frac{1}{b^2+c}\right) = \frac{1}{2}\left(\frac{1}{x^2+z^2}-\frac{1}{x^2+b^2+z^2}\right).\end{aligned}$$

This result is then inserted into (20.5),

$$(20.6)\quad E = \frac{Lz}{2}\int_0^a \left(\frac{1}{x^2+z^2}-\frac{1}{x^2+b^2+z^2}\right)\mathrm{d}x.$$

At this place it is absolutely *not* a good idea to "reduce" it by writing these two fractions as one, so we keep the form above.

In the next step we prove that if $k^2>0$, then it follows by the change of variable $t=\frac{x}{k}$, that

$$\int \frac{\mathrm{d}x}{x^2+k^2} = \frac{1}{k}\int \frac{1}{1+\left(\frac{x}{k}\right)^2}\,\frac{1}{k}\,\mathrm{d}x = \frac{1}{k}\int \frac{\mathrm{d}t}{1+t^2} = \frac{1}{k}\,\text{Arctan}\,t = \frac{1}{k}\,\text{Arctan}\left(\frac{x}{k}\right).$$

The trick here is by a division to obtain the constant 1 plus a square in the denominator.

Using the above we get

$$k_1 = z \qquad \text{and} \qquad k_2 = \sqrt{b^2+z^2},$$

so by insertion into (20.6),

$$\begin{aligned}E &= \frac{Lz}{2}\left[\frac{1}{k_1}\,\text{Arctan}\left(\frac{x}{k_1}\right)-\frac{1}{k_2}\,\text{Arctan}\left(\frac{x}{k_2}\right)\right]_0^a \\ &= \frac{Lz}{2}\left\{\frac{1}{z}\,\text{Arctan}\left(\frac{a}{z}\right)-\frac{1}{\sqrt{b^2+z^2}}\,\text{Arctan}\left(\frac{a}{\sqrt{b^2+z^2}}\right)\right\} \\ &= \frac{L}{2}\left\{\text{Arctan}\left(\frac{a}{z}\right)-\frac{z}{\sqrt{b^2+z^2}}\,\text{Arctan}\left(\frac{a}{\sqrt{b^2+z^2}}\right)\right\}. \qquad \Diamond\end{aligned}$$

Example 20.10 *Calculate in each of the following cases the given plane integral by applying the theorem of reduction for rectangular coordinates. Sketch first the domain of integration B.*

1) $\int_B \frac{1}{(x+y)^2}\,\mathrm{d}S$, *where* $B = \{(x,y) \mid 1 \le x \le 2$ *and* $0 \le y \le x^3\}$.

2) $\int_B \frac{x}{1+xy}\,\mathrm{d}S$, *where* $B = [0,1]\times[0,1]$.

3) $\int_B (x\sin y - ye^x)\,\mathrm{d}S$, *where* $B = [-1,1]\times\left[0,\frac{\pi}{2}\right]$.

4) $\int_B \sqrt{|y-x^2|}\,\mathrm{d}S$, *where* $B = [-1,1]\times[0,2]$.

5) $\int_B (x^2y^2+x)\,\mathrm{d}S$, *where* $B = [0,2]\times[-1,0]$.

6) $\int_B |y|\cos\frac{\pi x}{4}\,\mathrm{d}S$, *where* $B = [0,2]\times[-1,0]$.

7) $\int_B \frac{x^2}{(1+x+y)^2}\,\mathrm{d}S$, *where* $B = \{(x,y) \mid 0 \le x,\ 0 \le y,\ x+y \le 1\}$.

8) $\int_B (4-y)\,\mathrm{d}S$, *where* $B = \{(x,y) \mid 0 \le x,\ 0 \le y,\ x^2+y^2 \le 2\}$.

9) $\int_B (\sqrt{x}-y^2)\,\mathrm{d}S$, *where* B *is the bounded set in the first quadrant, which is bounded by the curves* $y = x^2$ *and* $x = y^4$.

10) $\int_B x\cos(x+y)\,\mathrm{d}S$, *where* B *is the triangle of the vertices* $(0,0)$, $(0,0)$, $(\pi,0)$ *and* (π,π).

11) $\int_B x\sqrt[3]{1+y-y^2+\frac{1}{3}y^3}\,\mathrm{d}S$, *where* $B = \{(x,y) \mid 0 \le x,\ 0 \le y,\ x+y \le 1\}$.

12) $\int_B (3y^2+2xy)\,\mathrm{d}S$, *where* $B = \{(x,y) \mid 0 \le x,\ 0 \le y,\ x+y \le 1\}$.

A Plane integrals in rectangular coordinates.

D Sketch the domain and apply the theorem of reduction.

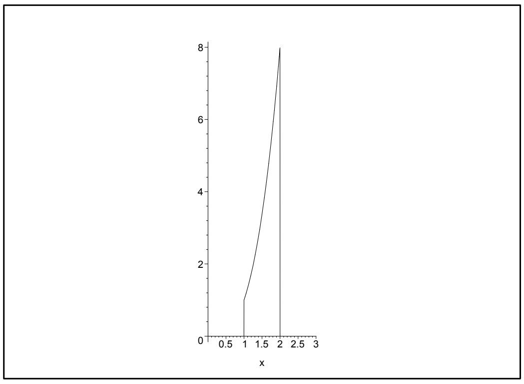


Figure 20.12: The domain B of **Example 20.10.1**.

I 1) We get by the theorem of reduction,

$$\begin{aligned}
\int_B \frac{1}{(x+y)^2}\,\mathrm{d}S &= \int_1^2 \left\{ \int_0^{x^3} \frac{1}{(x+y)^2}\,\mathrm{d}y \right\} \mathrm{d}x = \int_1^2 \left[-\frac{1}{x+y} \right]_{y=0}^{x^3} \mathrm{d}x \\
&= \int_1^2 \left\{ -\frac{1}{x+x^3} + \frac{1}{x} \right\} \mathrm{d}x = \int_1^2 \left\{ -\frac{1}{x} + \frac{x}{1+x^2} + \frac{1}{x} \right\} \mathrm{d}x \\
&= \left[\frac{1}{2} \ln(1+x^2) \right]_1^2 = \frac{1}{2} \ln\left(\frac{5}{2}\right).
\end{aligned}$$

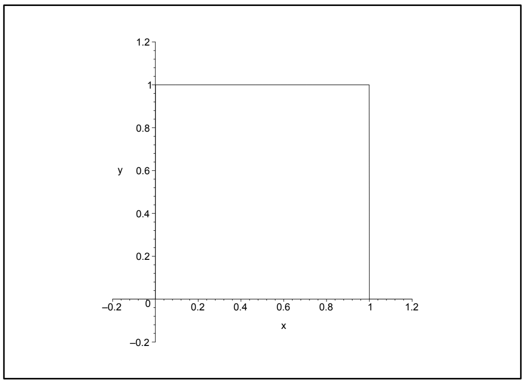


Figure 20.13: The domain B of **Example 20.10.2**.

MAPLE. We get by the commands with(Student[MultivariateCalculus]):

$$\text{MultiInt}\left(\frac{1}{(x+y)^2}, y = 0..x^3, x = 1..2\right)$$

$$-\frac{1}{2}\ln(2) + \frac{1}{2}\ln(5)$$

2) We get by the theorem of reduction,

$$\begin{aligned}\int_B \frac{x}{1+xy}\,\mathrm{d}S &= \int_0^1 \left\{\int_0^1 \frac{x}{1+xy}\,\mathrm{d}y\right\}\mathrm{d}x = \int_0^1 [\ln(1+xy)]_{y=0}^1\,\mathrm{d}x \\ &= \int_0^1 1\cdot\ln(1+x)\,\mathrm{d}x = [x\ln(1+x)]_0^1 - \int_0^1 \frac{x}{1+x}\,\mathrm{d}x \\ &= \ln 2 - \int_0^1 \left\{1 - \frac{1}{1+x}\right\}\mathrm{d}x = \ln 2 - 1 + \ln 2 \\ &= 2\ln 2 - 1.\end{aligned}$$

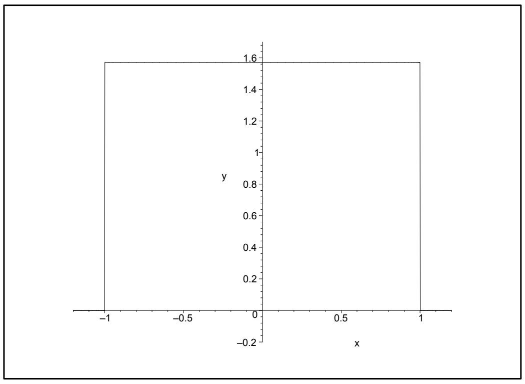


Figure 20.14: The domain B of **Example 20.10.3**.

MAPLE. We get by the commands with(Student[MultivariateCalculus]):

$$\text{MultiInt}\left(\frac{x}{1+x\cdot y}, y = 0..1, x = 0..1\right)$$

$$-1 + 2\ln(2)$$

3) We get by the theorem of reduction,

$$\int_B (x\sin y - ye^x)\,\mathrm{d}S = \int_0^{\frac{\pi}{2}} \left\{\int_{-1}^1 (x\sin y - ye^x)\,\mathrm{d}x\right\}\mathrm{d}y$$

$$= 0 - \int_0^{\frac{\pi}{2}} y\cdot\left(e - \frac{1}{e}\right)\mathrm{d}y = -\frac{1}{2}(e - e^{-1})\left[y^2\right]_0^{\frac{\pi}{2}}$$

$$= -\frac{\pi^2}{4}\sinh 1 \qquad \left(= -\frac{\pi^2(e^2-1)}{8e}\right),$$

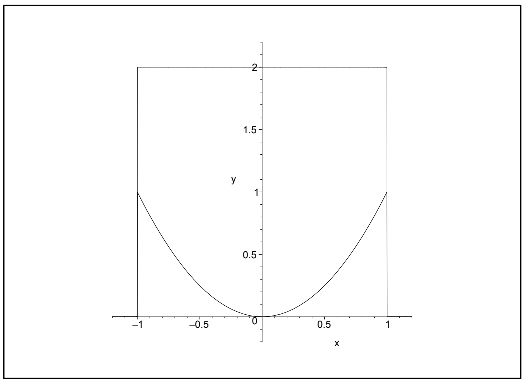


Figure 20.15: The domain B of **Example 20.10.4**.

where we first integrate with respect to x and then with respect to y.
MAPLE. We get by the commands with(Student[MultivariateCalculus]):

$$\text{MultiInt}\left(x\cdot\sin(y)-y\cdot e^x, x=-1..1, y=0..\frac{\pi}{2}\right)$$

$$\frac{1}{8}\left(-e+e^{-1}\right)\pi^2$$

4) Here, the curve $y=x^2$ may cause some troubles. For symmetric reasons

$$\int_B \sqrt{|y-x^2|}\,\mathrm{d}S = \int_{-1}^{1}\left\{\int_0^2 \sqrt{|y-x^2|}\,\mathrm{d}y\right\}\mathrm{d}x$$

$$= 2\int_0^1\left\{\int_0^{x^2}\sqrt{x^2-y}\,\mathrm{d}y+\int_{x^2}^2\sqrt{y-x^2}\,\mathrm{d}y\right\}\mathrm{d}x$$

$$= 2\int_0^1\left\{\left[-\frac{2}{3}(x^2-y)^{\frac{3}{2}}\right]_{y=0}^{x^2}+\left[\frac{2}{3}(y-x^2)^{\frac{3}{2}}\right]_{y=x^2}^{2}\right\}\mathrm{d}x$$

$$= 2\int_0^1\left\{\frac{2}{3}(x^2)^{\frac{3}{2}}+\frac{2}{3}(2-x^2)^{\frac{3}{2}}\right\}\mathrm{d}x = \frac{4}{3}\int_0^1 x^3\,\mathrm{d}x+\frac{4}{3}\cdot 2\sqrt{2}\int_0^1\left\{1-\left(\frac{x}{\sqrt{2}}\right)^2\right\}^{\frac{3}{2}}\mathrm{d}x$$

$$= \frac{1}{3}+\frac{16}{3}\int_0^{\frac{\sqrt{2}}{2}}\{1-t^2\}^{\frac{3}{2}}\,\mathrm{d}t=\frac{1}{3}+\frac{16}{3}\int_0^{\frac{\pi}{4}}\{1-\sin^2 u\}^{\frac{3}{2}}\cos u\,\mathrm{d}u=\frac{1}{3}+\frac{16}{3}\int_0^{\frac{\pi}{4}}\cos^4 u\,\mathrm{d}u$$

$$= \frac{1}{3}+\frac{16}{3}\int_0^{\pi}4\left(\frac{1+\cos 2u}{2}\right)^2\mathrm{d}u = \frac{1}{3}+\frac{4}{3}\int_0^{\frac{\pi}{4}}\left\{1+2\cos 2u+\frac{1+\cos 4u}{2}\right\}\mathrm{d}u$$

$$= \frac{1}{3}+\frac{4}{3}\left[\frac{3}{2}u+\sin 2u+\frac{1}{8}\sin 4u\right]_0^{\frac{\pi}{4}} = \frac{1}{3}+\frac{\pi}{2}+\frac{4}{3}=\frac{5}{3}+\frac{\pi}{2}.$$

MAPLE. We get by the commands with(Student[MultivariateCalculus]):

$$\text{MultiInt}\left(\sqrt{|y-x^2|}, y=0..2, x=-1..1\right)$$

$$\frac{5}{3}+\frac{1}{2}\pi$$

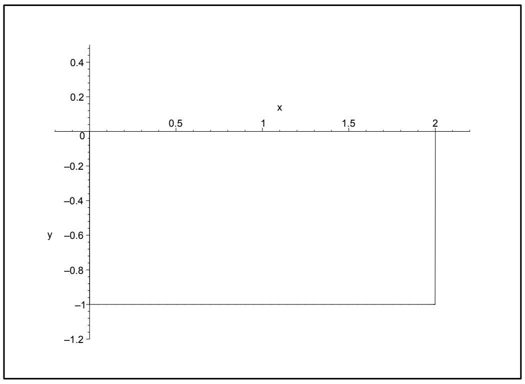


Figure 20.16: The domain B of **Example 20.10.5** and of **Example 20.10.6**.

5) Here,

$$\int_B (x^2y^2+x)\,dS = \int_0^2 \left\{ \int_{-1}^0 (x^2y^2+x)\,dy \right\} dx$$
$$= \int_0^2 \left[\frac{1}{3}x^2y^3 + xy \right]_{y=-1}^0 dx = \int_0^2 \left\{ \frac{1}{3}x^2 + x \right\} dx$$
$$= \left[\frac{1}{9}x^3 + \frac{1}{2}x^2 \right]_0^2 = \frac{8}{9} + \frac{4}{2} = 2 + \frac{8}{9} = \frac{26}{9}.$$

MAPLE. We get by the commands with(Student[MultivariateCalculus]):

MultiInt $\left(x^2 \cdot y^2 + x, y = -1..0, x = 0..2\right)$

$$\frac{26}{9}$$

6) The domain is identical with that of **Example 20.10.5**. It follows that

$$\int_B |y| \cos\frac{\pi x}{4}\,dS = \int_{-1}^0 (-y)\,dy \cdot \int_0^2 \cos\frac{\pi x}{4}\,dx = \left[-\frac{y^2}{2} \right]_{-1}^0 \cdot \frac{4}{\pi} \left[\sin\frac{\pi x}{4} \right]_0^2 = \frac{2}{\pi}.$$

MAPLE. We get by the commands with(Student[MultivariateCalculus]):

MultiInt $\left(|y| \cdot \cos\left(\frac{\pi}{4} \cdot x\right), y = -1..0, x = 0..2\right)$

$$\frac{2}{\pi}$$

7) Here,

$$\int_B \frac{x^2}{(1+x+y)^2}\,dS = \int_0^1 \left\{ \int_0^{1-x} \frac{x^2}{(1+x+y)^2}\,dy \right\} dx = \int_0^1 \left[-\frac{x^2}{1+x+y} \right]_{y=0}^{1-x} dx$$
$$= \int_0^1 \left\{ \frac{x^2}{1+x} - \frac{x^2}{2} \right\} dx = \int_0^1 \left\{ x - 1 + \frac{1}{x+1} - \frac{x^2}{2} \right\} dx$$
$$= \left[\frac{x^2}{2} - x + \ln(1+x) - \frac{x^3}{6} \right]_0^1 = \frac{1}{2} - 1 + 2\ln 2 - \frac{1}{6} = \ln 2 - \frac{2}{3}.$$

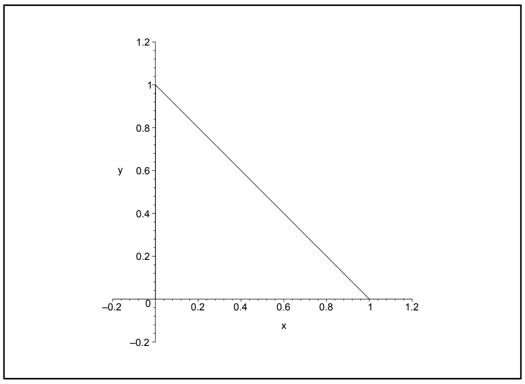


Figure 20.17: The domain B of **Example 20.10.7**.

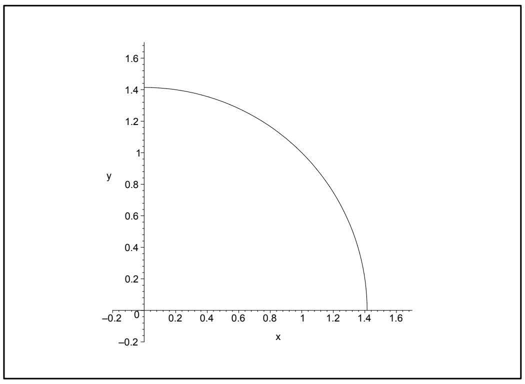


Figure 20.18: The domain B of **Example 20.10.8**.

MAPLE. We get by the commands with(Student[MultivariateCalculus]):

$$\text{MultiInt}\left(\frac{x^2}{(1+x+y)^2}, y=0..1-x, x=0..1\right)$$

$$-\frac{2}{3}+\ln(2)$$

8) The domain is a quarter of a disc in the first quadrant, hence by combining the method of identifying obvious areas and the theorem of reduction in rectangular coordinates,

$$\int_B (4-y)\,\mathrm{d}S = 4\,\text{area}(B) - \int_0^{\sqrt{2}}\left\{\int_0^{\sqrt{2-x^2}} y\,\mathrm{d}y\right\}\mathrm{d}x = 4\cdot\frac{1}{4}\pi(\sqrt{2})^2 - \int_0^{\sqrt{2}}\left[\frac{1}{2}\,y^2\right]_{y=0}^{\sqrt{2-x^2}}\mathrm{d}x$$

$$= 2\pi - \frac{1}{2}\int_0^{\sqrt{2}}(2-x^2)\,\mathrm{d}x = 2\pi - \sqrt{2} + \frac{1}{6}(\sqrt{2})^3 = 2\pi - \frac{2}{3}\sqrt{2}.$$

ALTERNATIVELY we get by using polar coordinates instead, cf. **Example 20.29.1**,

$$\int_B (4-y)\,\mathrm{d}S = 4\,\text{area}(B) - \int_0^{\frac{\pi}{2}} \left\{ \int_0^{\sqrt{2}} \varrho \sin\varphi \cdot \varrho\,\mathrm{d}\varrho \right\} \mathrm{d}\varphi$$

$$= 2\pi + [\cos\varphi]_0^{\frac{\pi}{2}} \cdot \left[\frac{\varrho^3}{3}\right]_0^{\sqrt{2}} = 2\pi - \frac{2}{3}\sqrt{2}.$$

MAPLE. We get by the commands with(Student[MultivariateCalculus]):

$$\text{MultiInt}\left(4-y, y=0..\sqrt{2-x^2}, x=0..\sqrt{2}\right)$$

$$2\pi - \frac{2}{3}\sqrt{3}$$

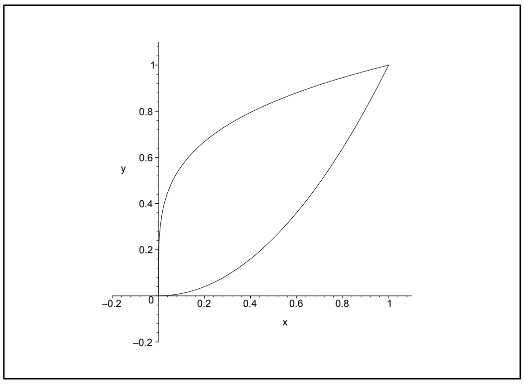


Figure 20.19: The domain B of **Example 20.10.9**.

9) When $x = y^4$ in the first quadrant, the inverse function is given by $y = \sqrt[4]{x}$, and it follows by the theorem of reduction that

$$\int_B (\sqrt{x} - y^2)\,\mathrm{d}S = \int_0^1 \left\{ \int_{x^2}^{\sqrt[4]{x}} (\sqrt{x} - y^2)\,\mathrm{d}y \right\} \mathrm{d}x = \int_0^1 \left[y\sqrt{x} - \frac{1}{3}\,y^3 \right]_{y=x^2}^{\sqrt[4]{x}} \mathrm{d}x$$

$$= \int_0^1 \left\{ x^{\frac{3}{4}} - \frac{1}{3}\,x^{\frac{3}{4}} - x^{\frac{5}{2}} + \frac{1}{3}\,x^6 \right\} \mathrm{d}x = \left[\frac{2}{3} \cdot \frac{4}{7}\,x^{\frac{7}{4}} - \frac{2}{7}\,x^{\frac{7}{2}} + \frac{2}{1}\,x^7 \right]_0^1 = \frac{8}{21} - \frac{2}{7} + \frac{1}{21}$$

$$= \frac{1}{7}.$$

MAPLE. We get by the commands with(Student[MultivariateCalculus]):

$$\text{MultiInt}\left(\sqrt{x} - y^2, y = x^2..\sqrt[4]{x}, x = 0..1\right)$$

$$\frac{1}{7}$$

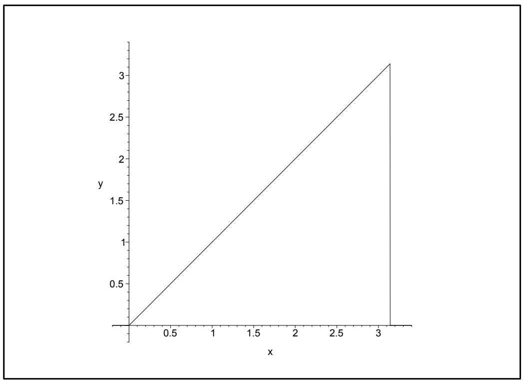


Figure 20.20: The domain B of **Example 20.10.10**.

10) The domain is the triangle bounded by the X-axis, the line $x = \pi$ and the line $y = x$. We get by the theorem of reduction,

$$\int_B x\cos(x+y)\,\mathrm{d}S = \int_0^{\pi} \left\{ \int_0^x x\cos(x+y)\,\mathrm{d}y \right\} \mathrm{d}x = \int_0^{\pi} [x\sin(x+y)]_{y=0}^{x}\,\mathrm{d}x$$

$$= \int_0^{\pi} \{x\sin 2x - x\sin x\}\,\mathrm{d}x = \left[-x \cdot \frac{1}{2}\,\cos 2x + x\cos x \right]_0^{\pi} + \int_0^{\pi} \left\{ \frac{1}{2}\,\cos 2x - \cos x \right\} \mathrm{d}x$$

$$= -\frac{\pi}{2} - \pi + \left[\frac{1}{4}\,\sin 2x - \sin x \right]_0^{\pi} = -\frac{3\pi}{2}.$$

MAPLE. We get by the commands with(Student[MultivariateCalculus]):

$$\text{MultiInt}\,(x \cdot \cos(x+y), y = 0..x, x = 0..\pi)$$

$$-\frac{3}{2}\pi$$

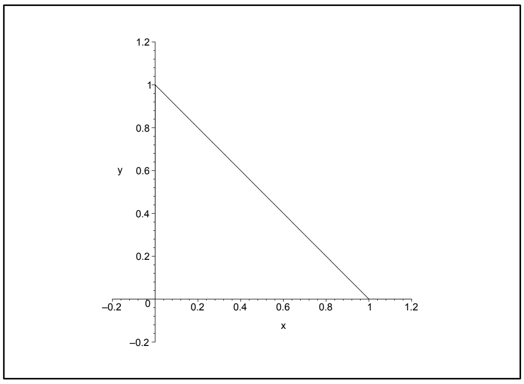


Figure 20.21: The domain B of **Example 20.10.11** and of **Example 20.10.12**.

11) Here, the idea of first (i.e. innermost) integrating with respect to y for fixed x is stillborn, so we interchange the order of integration. We shall therefore first (innermost) integrate with respect to x and then outermost with respect to y.

$$\int_B x\sqrt[3]{1+y-y^2+\frac{1}{3}\,y^3}\,\mathrm{d}S = \int_0^2\left\{\int_0^{1-y} x\left\{1+y-y^2+\frac{1}{3}\,y^3\right\}^{\frac{1}{3}}\mathrm{d}x\right\}\mathrm{d}y$$

$$= \frac{1}{2}\int_0^2\left\{1+y-y^2+\frac{1}{3}\,y^3\right\}^{\frac{1}{3}}(1-y)^2\,\mathrm{d}y$$

$$= \frac{1}{2}\int_0^2\left\{\frac{4}{3}+\frac{1}{3}(y^3-3y^2+3y-1)\right\}^{\frac{1}{3}}(y-1)^2\,\mathrm{d}y$$

$$= \frac{1}{2}\int_{y=0}^2\left\{\frac{4}{3}+\frac{1}{3}(y-1)^3\right\}^{\frac{1}{3}}\mathrm{d}\left(\frac{1}{3}(y-1)^3\right) = \frac{1}{2}\cdot\frac{3}{4}\left[\left(\frac{4}{3}+\frac{1}{3}(y-1)^3\right)^{\frac{4}{3}}\right]_{y=0}^2$$

$$= \frac{3}{8}\left\{\left(\frac{5}{3}\right)^{\frac{4}{3}}-\left(\frac{4}{3}-\frac{1}{3}\right)^{\frac{4}{3}}\right\} = \frac{3}{8}\left\{\left(\frac{4}{3}\right)^{\frac{5}{3}}-1\right\} = \frac{5}{8}\sqrt[3]{\frac{5}{3}}-\frac{3}{8}.$$

MAPLE. We get by the commands with(Student[MultivariateCalculus]):

$$\mathrm{MultiInt}\left(x\cdot\sqrt[3]{1+y-y^2+\frac{1}{3}y^3}, x=0..1-y, y=0..2\right)$$

$$-\frac{3}{8}+\frac{5}{24}\sqrt[3]{45}$$

12) The sketch of B is identical with **Example 20.10.11**. We get by the theorem of reduction,

$$\int_B(3y^2+2xy)\,\mathrm{d}S = \int_0^1\left\{\int_0^{1-y}(3y^2+2xy)\,\mathrm{d}x\right\}\mathrm{d}y$$

$$= \int_0^1\left\{3y^2(1-y)+y(1-y)^2\right\}\mathrm{d}y = \int_0^1\left\{3y^2-3y^2+y-2y^2+y^3\right\}\mathrm{d}y$$

$$= \int_0^1\left(y+y^2-2y^3\right)\mathrm{d}y = \frac{1}{2}+\frac{1}{3}-2\cdot\frac{1}{4} = \frac{1}{3}.$$

MAPLE. We get by the commands with(Student[MultivariateCalculus]):

MultiInt $\left(3y^2+2x\cdot y, x=0..1-y, y=0..1\right)$

$\frac{1}{3}$

Example 20.11 *Let B be the rectangle $[0,2\pi]\times\left[\frac{5}{4},\frac{5}{3}\right]$. Reduce the plane integral*

$$\int_B \frac{1}{y+\sin x}\,\mathrm{d}S$$

in two ways, and then show the formula

$$\int_0^{2\pi}\ln\left(\frac{5+3\sin x}{5+4\sin x}\right)\mathrm{d}x = 2\pi\,\ln\left(\frac{9}{8}\right).$$

A Plane integral.

D Reduce the plane integral in two different ways as double integrals, and then just compute.

I First note that the domain of integration is given by

$$y+\sin x>0 \qquad \text{and} \qquad y\geq\frac{5}{4}>1.$$

Then we reduce the plane integral in two different ways as double integrals,

$$\int_B \frac{1}{y+\sin x}\,dS = \int_0^{2\pi}\left\{\int_{\frac{5}{4}}^{\frac{5}{3}} \frac{1}{y+\sin x}\,dy\right\}dx = \int_{\frac{5}{4}}^{\frac{5}{3}}\left\{\int_0^{2\pi}\frac{1}{y+\sin x}\,dx\right\}dy.$$

When we use that $\sin x$ is periodic, and then introduce the substitution $t=\tan\frac{x}{2}$, we get

$$\begin{aligned}
&\int_{\frac{5}{4}}^{\frac{5}{3}}\left\{\int_0^{2\pi}\frac{1}{y+\sin x}\,dx\right\}dy = \int_{\frac{5}{4}}^{\frac{5}{3}}\left\{\int_{-\pi}^{\pi}\frac{1}{y+\sin x}\,dx\right\}dy\\
&= \int_{\frac{5}{4}}^{\frac{5}{3}}\left\{\int_{-\pi}^{\pi}\frac{1}{y\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}+y\cos^2\frac{x}{2}}\,dx\right\}dy\\
&= 2\int_{\frac{5}{4}}^{\frac{5}{3}}\left\{\int_{-\infty}^{+\infty}\frac{1}{yt^2+2y+y}\,dt\right\}dy = 2\int_{\frac{5}{4}}^{\frac{5}{3}}\frac{1}{y}\left\{\int_{-\infty}^{+\infty}\frac{1}{u^2+\frac{2}{y}+1}\,du\right\}dy\\
&= 2\int_{\frac{5}{4}}^{\frac{5}{3}}\frac{1}{y}\left\{\int_{-\infty}^{+\infty}\frac{1}{\left(u+\frac{1}{y}\right)^2+1-\frac{1}{y^2}}\,du\right\}dy\\
&= 2\int_{\frac{5}{4}}^{\frac{5}{3}}\frac{1}{y}\,\frac{1}{\sqrt{1-\frac{1}{y^2}}}\left[\operatorname{Arctan}\left(\frac{u+\frac{1}{y}}{\sqrt{1-\frac{1}{y^2}}}\right)\right]_{u=-\infty}^{+\infty}dy = 2\pi\int_{\frac{5}{4}}^{\frac{5}{3}}\frac{1}{\sqrt{y^2-1}}\,dy\\
&= 2\pi\left[\ln\left(y+\sqrt{y^2-1}\right)\right]_{\frac{5}{4}}^{\frac{5}{3}} = 2\pi\left\{\ln\left(\frac{5}{3}+\sqrt{\left(\frac{5}{3}\right)^2-1}\right)-\ln\left(\frac{5}{4}+\sqrt{\left(\frac{5}{4}\right)^2-1}\right)\right\}\\
&= 2\pi\left\{\ln\left(\frac{5}{3}+\frac{4}{3}\right)-\ln\left(\frac{5}{4}+\frac{3}{4}\right)\right\} = 2\pi\{\ln 3-\ln 2\} = 2\pi\ln\left(\frac{3}{2}\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_0^{2\pi}\left\{\int_{\frac{5}{4}}^{\frac{5}{3}}\frac{1}{y+\sin x}\,dy\right\}dx = \int_0^{2\pi}[\ln(y+\sin x)]_{y=\frac{5}{4}}^{\frac{5}{3}}\,dx = \int_0^{2\pi}\ln\left(\frac{\frac{5}{3}+\sin x}{\frac{5}{4}+\sin x}\right)dx\\
&= \int_0^{2\pi}\left\{\ln\left(\frac{4}{3}\right)+\ln\left(\frac{5+3\sin x}{5+4\sin x}\right)\right\}dx = 2\pi\ln\left(\frac{4}{3}\right)+\int_0^{2\pi}\ln\left(\frac{5+3\sin x}{5+4\sin x}\right)dx.
\end{aligned}$$

As a conclusion we get

$$\int_B\frac{1}{y+\sin x}\,dS = 2\pi\ln\left(\frac{4}{3}\right)+\int_0^{2\pi}\ln\left(\frac{5+3\sin x}{5+4\sin x}\right)dx = 2\pi\ln\left(\frac{3}{2}\right).$$

Finally, by a rearrangement

$$\int_0^{2\pi}\ln\left(\frac{5+3\sin x}{5+4\sin x}\right)dx = 2\pi\left\{\ln\left(\frac{3}{2}\right)-\ln\left(\frac{4}{3}\right)\right\} = 2\pi\ln\left(\frac{9}{8}\right),$$

as required.

MAPLE. One should note that the order of integration is also of importance in MAPLE. We get by the commands with(Student[MultivariateCalculus]):

$$\text{MultiInt}\left(\frac{1}{y+\sin(x)}, y = \frac{5}{4}..\frac{5}{3}, x = 0..2\pi\right)$$

$$-2\pi\ln(2) + 2\ln(3)\pi - 2I\pi arctan\left(\frac{1}{3}\right) + 2I\pi\arctan\left(\frac{1}{2}\right) + 2I\pi\arctan(2) - 2I\pi\arctan(3)$$

while

$$\text{MultiInt}\left(\frac{1}{y+\sin(x)}, x = 0..2\pi, y = \frac{5}{4}..\frac{5}{3}\right)$$

gives the correct answer, which was also found above,

$$2\ln(3)\pi - 2\pi\ln(2)$$

Example 20.12 *The unit square $E = [0,1] \times [0,1]$ is divided by the straight line of equation $y = x$ into two triangles: T_1 given by $y \le x$, and T_2 given by $y > x$. We define a function $f : E \to \mathbb{R}$ in the following way:*

$$f(x,y) = \begin{cases} x^2 + 2y, & (x,y) \in T_1, \\ 1 + 3y^2, & (x,y) \in T_2. \end{cases}$$

Compute the plane integral $\int_E f(x,y)\,dS$.

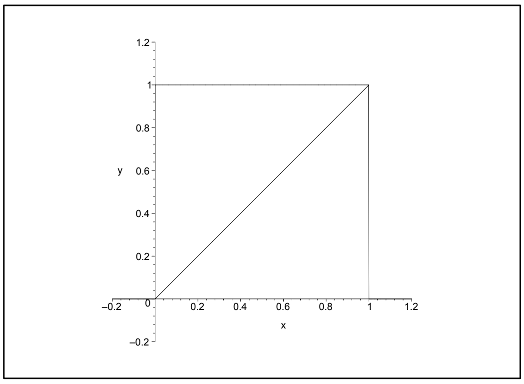


Figure 20.22: The triangle T_1 has an edge along the X-axis, and the triangle T_2 has an edge along the Y-axis.

A Plane integral.

D Reduce over each of the sets T_1 and T_2. The plane integral can be reduced to double integrals in $2 \times 2 = 4$ different ways, of which we only show one.

I From

$$\begin{aligned} T_1 &= \{(x,y) \mid 0 \le x \le 1,\, 0 \le y \le x\} \\ T_2 &= \{(x,y) \mid 0 \le y \le 1,\, 0 \le x \le y\}, \end{aligned}$$

follows (note the two different successions of the order of integration)

$$\begin{aligned} \int_E f(x,y)\,\mathrm{d}S &= \int_{T_1} f(x,y)\,\mathrm{d}S + \int_{T_2} f(x,y)\,\mathrm{d}S \\ &= \int_0^1 \left\{ \int_0^x (x^2+2y)\,\mathrm{d}y \right\} \mathrm{d}x + \int_0^1 \left\{ \int_0^y (1+3y^2)\,\mathrm{d}x \right\} \mathrm{d}y \\ &= \int_0^1 \left[x^2y+y^2\right]_{y=0}^{x} \mathrm{d}x + \int_0^1 \left[x+3y^2x\right]_{x=0}^{y} \mathrm{d}y \\ &= \int_0^1 (x^3+x^2)\,\mathrm{d}x + \int_0^1 (y+3y^3)\,\mathrm{d}y \\ &= \int_0^1 (4t^3+t^2+t)\,\mathrm{d}t = \left[t^4+\frac{1}{3}\,t^3+\frac{1}{2}\,t^2\right]_0^1 = 1+\frac{1}{3}+\frac{1}{2} = \frac{11}{6}. \end{aligned}$$

MAPLE. We get by the commands with(Student[MultivariateCalculus]):

$$\text{MultiInt}\left(x^2+2y, y=0..x, x=0..1\right) + \text{MultiInt}\left(1+3y^2, x=0..y, y=0..1\right)$$

$$\frac{11}{6}$$

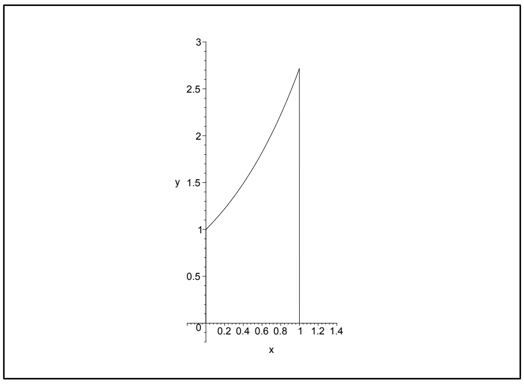
Example 20.13 *Let D be the set which is bounded by the curve $y = e^x$, and the line $x = 1$, and the coordinate axes. Sketch D, and compute the plane integral*

$$\int_D \frac{1}{(1+y)^2 \cosh x}\,\mathrm{d}S.$$

A Plane integral in rectangular coordinates.

D Sketch the domain and apply the theorem of reduction.

I When we reduce the plane integral, introduce the substitution $u = e^x$, and apply a decomposition,



we get

$$\int_D \frac{1}{(1+y)^2\cosh x}\,\mathrm{d}S = \int_0^1 \frac{1}{\cosh x}\left\{\int_0^{e^x} \frac{1}{(1+y)^2}\,\mathrm{d}y\right\}\mathrm{d}x = \int_0^1 \frac{2e^x}{e^{2x}+1}\left[-\frac{1}{1+y}\right]_{y=0}^{e^x}\mathrm{d}x$$

$$= \int_0^1 \left\{\frac{2e^x}{e^{2x}+1} - \frac{2e^x}{e^{2x}+1}\cdot\frac{1}{e^x+1}\right\}\mathrm{d}x = \int_1^e \left\{\frac{2}{u^2+1} - \frac{2}{(u^2+1)(u+1)}\right\}\mathrm{d}u$$

$$= \int_1^e \left\{\frac{2}{u^2+1} - \frac{1}{u+1} - \frac{2}{(u^2+1)(u+1)} + \frac{1}{u+1}\right\}\mathrm{d}u$$

$$= \int_1^e \left\{\frac{2}{u^2+1} - \frac{1}{u+1} + \frac{u^2+1-2}{(u^2+1)(u+1)}\right\}\mathrm{d}u$$

$$= \int_1^e \left\{\frac{2}{u^2+1} - \frac{1}{u+1} + \frac{u}{u^2+1} - \frac{1}{u^2+1}\right\}\mathrm{d}u$$

$$= \int_1^e \left\{\frac{1}{u^2+1} + \frac{u}{u^2+1} - \frac{1}{u+1}\right\}\mathrm{d}u = \left[\operatorname{Arctan} u + \frac{1}{2}\ln(u^2+1) - \frac{1}{2}\ln(u+1)\right]_1^e$$

$$= \operatorname{Arctan} e - \frac{\pi}{4} + \frac{1}{2}\ln\left(\frac{e^2+1}{(e+1)^2}\right) - \frac{1}{2}\ln\left(\frac{1+1}{(1+1)^2}\right)$$

$$= \operatorname{Arctan} e - \frac{\pi}{4} - \frac{1}{2}\ln\left(e^2+1\right) + \frac{1}{2}\ln 2,$$

where we also can obtain the equivalent results

$$\int_D \frac{1}{(1+y)^2\cosh x}\,\mathrm{d}S = \operatorname{Arctan} e - \frac{\pi}{4} + \frac{1}{2}\ln\left(\frac{2(e^2+1)}{(e+1)^2}\right)$$

$$= \operatorname{Arctan} e - \frac{\pi}{4} + \ln\left(\frac{\cosh 1}{\cosh^2\frac{1}{2}}\right)$$

$$= \operatorname{Arctan} e - \frac{\pi}{4} + \ln\left(\frac{2\cosh 1}{1+\cosh 1}\right).$$

MAPLE. We get by the commands with(Student[MultivariateCalculus]):

$$\text{MultiInt}\left(\frac{1}{(1+y)^2\cdot\cosh(x)}, y=0..e^x, x=0..1\right)$$

$$\frac{1}{2}\ln(2)+\frac{1}{2}\left(e^{-1}+e\right)-\ln\left(e^{-\frac{1}{2}}+e^{\frac{1}{2}}\right)-\arctan\left(\frac{e^{-\frac{1}{2}}-e^{\frac{1}{2}}}{e^{-\frac{1}{2}}+e^{\frac{1}{2}}}\right)$$

which then should be reduced further.

Example 20.14 *The function* $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ *is given by*

$$f(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}.$$

Note that the domain of the function is the open *first quadrant. By the computations of integrals we shall whenever necessary use a continuous extension to the axes.*

1) Compute the double integrals

$$I_1 = \int_0^1 \left\{ \int_0^1 f(x, y)\, \mathrm{d}y \right\} \mathrm{d}x \quad \textit{and} \quad I_2 = \int_0^1 \left\{ \int_0^1 f(x, y)\, \mathrm{d}x \right\} \mathrm{d}y.$$

2) It follows from 1) that $I_1 \neq I_2$*. Make a comment on this result by considering the plane integral of the function* f *over the unit square* $[0, 1] \times [0, 1]$*.*

A Double integrals.

D Compute I_1, and apply that $I_2 = -I_1$ by an argument of symmetry.

I 1) We get when $x \neq 0$,

$$\int_0^1 f(x, y)\, \mathrm{d}y = \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2}\, \mathrm{d}y = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}y} \left(\frac{y}{x^2 + y^2} \right) \mathrm{d}y = \frac{1}{1 + x^2},$$

so

$$I_1 = \int_0^1 \left\{ \int_0^1 f(x, y)\, \mathrm{d}y \right\} \mathrm{d}x = \int_0^1 \frac{1}{1 + x^2}\, \mathrm{d}x = \text{ Arctan } 1 = \frac{\pi}{4}.$$

From $f(y, x) = -f(x, y)$ follows by interchanging the letters and by a small argument of symmetry that

$$\begin{aligned} I_2 &= \int_0^1 \left\{ \int_0^1 f(x, y)\, \mathrm{d}x \right\} \mathrm{d}y = \int_0^1 \left\{ \int_0^1 f(y, x)\, \mathrm{d}y \right\} \mathrm{d}x \\ &= -\int_0^1 \left\{ \int_0^1 f(x, y)\, \mathrm{d}y \right\} = -I_1 = -\frac{\pi}{4} \neq I_1. \end{aligned}$$

2) The plane integral $\int_{[0,1]^2} f(x, y)\, \mathrm{d}x\, \mathrm{d}y$ is improper at $(0, 0)$, and it is *not* convergent. If e.g.

$$D = \left\{ (x, y) \in [0, 1]^2 \;\middle|\; y < \frac{1}{2}\, x \right\},$$

then

$$\begin{aligned} \int_D \frac{x^2 - y^2}{(x^2 + y^2)^2}\, \mathrm{d}x\, \mathrm{d}y &\geq \int_D \frac{x^2 - \frac{1}{4}\, x^2}{(x^2 + \frac{1}{4}\, x^2)^2}\, \mathrm{d}x\, \mathrm{d}y \\ &= \int_0^1 \frac{\frac{3}{4}\, x^2}{(\frac{5}{4})^2 x^4} \cdot \frac{1}{2}\, x\, \mathrm{d}x = \frac{3}{4} \cdot \frac{4^2}{5^2} \cdot \frac{1}{2} \int_0^1 \frac{1}{x}\, \mathrm{d}x = +\infty, \end{aligned}$$

and $D \subset [0, 1] \times [0, 1]$.

Clearly, MAPLE does not like these integrals. Nothing happens, when we use the commands

with(Student[MultivariateCalculus]):

$$\text{MultiInt}\left(\frac{x^2-y^2}{(x^2+y^2)^2}, y=0..1, x=0..1\right)$$

Not even a message from MAPLE. It just carried on computing, so the author had to stop the operation.

Example 20.15 *Find the domain* B *for*

$$f(x,y)=\sqrt{1-x^2-y^2}+\sqrt{x^2y}.$$

Then find the range $f(B)$ *and the* plane *integral*

$$\int_B f(x,y)\,\mathrm{d}S.$$

A Domain, range and plane integral.

D Use the standard methods. When we calculate the plane integral we neglect the zero set.

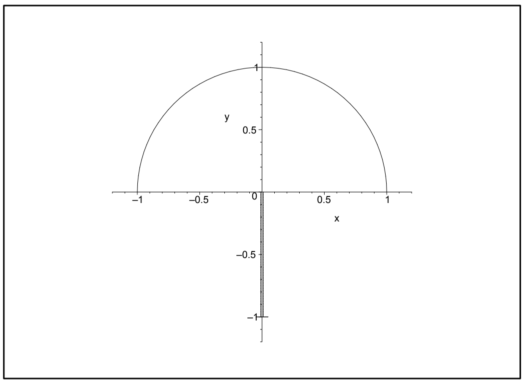


Figure 20.23: The domain B. Note the interval on the negative Y-axis.

I The function is defined and continuous when $x^2+y^2\le 1$ and $x^2y\ge 0$. From the first condition follows that B is contained in the closed unit disc. From the second condition follows that if $x\ne 0$, then $y\ge 0$; however, if $x=0$, then $x^2y=0$ for every y, so the latter term is defined in union of the closed upper half plane and the y-axis.

The domain is the intersection of these closed domains, i.e. union of the closed half disc in the upper half plane and the interval $[-1,0]$ on the y-axis, cf. the figure.

Since f is continuous in B, and B is closed and bounded and connected, then f has a maximum value S and a minimum value M in B (second main theorem), and by the first main theorem the range is connected, so $f(B)=[M,S]$.

We shall search the maximum and the minimum among:

1) the interior points, where f is not differentiable (the exceptional points: $x = 0$ and $0 < y < 1$),
2) the interior stationary points (i.e. inside the set $x^2 + y^2 < 1$, $y > 0$, $x \neq 0$),
3) the boundary points.

1) The restriction of f to $x = 0$ and $y \in]0, 1[$ is

$$\varphi(y) = \sqrt{1 - y^2}, \qquad y \in]0, 1[.$$

This function is decreasing and of the range $]0, 1[$, so it has neither a minimum value nor a maximum value.

2) If (x, y) is a stationary point in the open quarter disc in the first quadrant, then $(-x, y)$ is clearly a stationary point in the open quarter disc in the second quadrant, and *vice versa*. Now, f only contains x in the form x^2, so the value is the same, $f(x, y) = f(-x, y)$. It will therefore suffice to consider the quarter disc

$$\{(x, y) \mid x > 0,\, y > 0,\, x^2 + y^2 < 1\}$$

in the first quadrant. We have in this subdomain,

$$f(x, y) = \sqrt{1 - x^2 - y^2} + x\sqrt{y}.$$

The equations of possible stationary points are here

$$\begin{cases} \dfrac{\partial f}{\partial x} = -\dfrac{x}{\sqrt{1 - x^2 - y^2}} + \sqrt{y} = 0, \\ \dfrac{\partial f}{\partial y} = -\dfrac{y}{\sqrt{1 - x^2 - y^2}} + \dfrac{1}{2}\,\dfrac{x}{\sqrt{y}} = 0, \end{cases}$$

and it follows from $x > 0$ and $y > 0$ that

$$\frac{xy}{\sqrt{1 - x^2 - y^2}} = y\sqrt{y} = \frac{1}{2}\,\frac{x^2}{\sqrt{y}}.$$

Hence $y^2 = \dfrac{1}{2}\,x^2$, so $y = +\dfrac{1}{\sqrt{2}}\,x$. Then

$$y = \frac{1}{\sqrt{2}}\,x = \frac{x^2}{1 - x^2 - y^2} = \frac{x^2}{1 - \frac{3}{2}\,x^2},$$

hence by a rearrangement,

$$x^2 + 2\,\frac{\sqrt{2}}{3}\,x - \frac{2}{3} = 0.$$

The solutions are $x = -\sqrt{2}$ (must be rejected because we are only considering points of the unit disc in the first quadrant) and $x = \dfrac{\sqrt{2}}{3}$, corresponding to $y = \dfrac{1}{\sqrt{2}}\,x = \dfrac{1}{3}$. Clearly, $\left(\dfrac{\sqrt{2}}{3}, \dfrac{1}{3}\right)$ is an inner point of the domain, so it is a stationary point. Then by the above, $\left(-\dfrac{\sqrt{2}}{3}, \dfrac{1}{3}\right)$ is also a stationary point, and these two points are the only stationary points. The value of the functions is here

$$f\left(\pm\frac{\sqrt{2}}{3}, \frac{1}{3}\right) = \sqrt{1 - \frac{2}{9} - \frac{1}{9}} + \frac{\sqrt{2}}{3}\sqrt{\frac{1}{3}} = \sqrt{\frac{2}{3}} + \frac{1}{3}\sqrt{\frac{2}{3}} = \frac{4}{3}\sqrt{\frac{2}{3}}.$$

3) The examination of the boundary is split into

   a) The circular arc, $x^2 + y^2 = 1$, $x \in [-1, 1]$, $y \in [0, 1]$.

   b) The line segment on the X-axis, $y = 0$, $x \in [-1, 1]$.

   c) The line segment on the Y-axis, $x = 0$, $y \in [-1, 0]$.

   a) Since $f(-x, y) = f(x, y)$, it suffices to consider the quarter circular arc $x^2 = 1 - y^2$, $x \geq 0$, $y \geq 0$. The restriction of f becomes

$$\varphi(y) = \sqrt{(1-y^2)y} = \sqrt{y - y^3}, \qquad y \in [0, 1].$$

Since φ and $\Phi(y) = \varphi(y)^2 = y - y^3$ attain their maximum value and minimum value at the same points we compute

$$\Phi'(y) = 1 - 3y^2, \qquad \text{hence } \Phi'(y) = 0 \text{ for } y = \frac{1}{\sqrt{3}}.$$

Correspondingly, $x = \pm\sqrt{\dfrac{2}{3}}$, and

$$f\left(\pm\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}\right) = \sqrt{\frac{2}{3} \cdot \frac{1}{\sqrt{3}}} = \frac{1}{3}\sqrt{2\sqrt{3}}.$$

At the end points

$$[f(-1,0) =] \quad f(1,0) = 0 \quad \text{and} \quad f(0,1) = 0.$$

b) When $y = 0$ and $x \in [-1, 1]$, the restriction of f is given by

$$f(x, 0) = \sqrt{1 - x^2}, \qquad x \in [-1, 1],$$

which clearly has its maximum value $f(0, 0) = 1$ and its minimum value $f(-1, 0) = f(1, 0) = 0$.

c) When $x = 0$ and $y \in [-1, 0]$, we get

$$f(0, y) = \sqrt{1 - y^2}, \qquad y \in [-1, 0],$$

with the maximum value $f(0, 0) = 1$ and the minimum value $f(0, 1) = 0$.

It follows by a numerical comparison that the minimum value is attained at the boundary points

$$M = f(1, 0) = f(0, 1) = f(-1, 0) = f(+, -1) = 0,$$

and the maximum value is attained at the stationary points,

$$S = f\left(\frac{\sqrt{2}}{3}, \frac{1}{3}\right) = f\left(-\frac{\sqrt{2}}{3}, \frac{1}{2}\right) = \frac{4\sqrt{2}}{3\sqrt{3}}.$$

According to the first main theorem for continuous functions the range of the function is connected, thus

$$f(B) = [M, S] = \left[0, \frac{4\sqrt{2}}{3\sqrt{3}}\right].$$

We shall finally compute a plane integral. Since $f(x, y)$ is continuous on B, and the interval on the Y-axis in the lower half plane is a null set, the integral is zero over this part.

Let $\tilde{B}$ denote the closed half disc in the upper half plane. Then we get by reduction in polar coordinates

$$\begin{aligned}
\int_B f(x, y)\, \mathrm{d}S &= \int_{\tilde{B}} \left\{\sqrt{1 - x^2 - y^2} + \sqrt{x^2 y}\right\} \mathrm{d}S \\
&= \int_0^{\pi} \left\{\int_0^1 \left(\sqrt{1 - \varrho^2} + \sqrt{\varrho^2 \cos^2\varphi \cdot \varrho \sin\varphi}\right) \varrho\, \mathrm{d}\varrho\right\} \mathrm{d}\varphi \\
&= \frac{\pi}{2} \int_0^1 \left(1 - \varrho^2\right)^{\frac{1}{2}} 2\varrho\, \mathrm{d}\varrho + 2 \int_0^{\frac{\pi}{2}} |\cos\varphi| \sqrt{\sin\varphi}\, \mathrm{d}\varphi \cdot \int_0^1 \varrho^{\frac{5}{2}}\, \mathrm{d}\varrho \\
&= \frac{\pi}{2}\left[-\frac{2}{3}\left(1 - \varrho^2\right)^{\frac{3}{2}}\right]_0^1 + 2\left[\frac{2}{3}(\sin\varphi)^{\frac{3}{2}}\right]_{\varphi=0}^{\frac{\pi}{2}} \cdot \left[\frac{2}{7}\varrho^{\frac{7}{2}}\right]_{\varrho=0}^1 \\
&= \frac{\pi}{2} \cdot \frac{2}{3} + 2 \cdot \frac{2}{3} \cdot \frac{2}{7} = \frac{\pi}{3} + \frac{8}{21}.
\end{aligned}$$

MAPLE. We get by the commands

with(Student[MultivariateCalculus]):

$$\text{MultiInt}\left(\left(\sqrt{1 - r^2} + \sqrt{r^2 \cdot \cos(t)^2 \cdot r \cdot \sin(t)}\right) r, r = 0..1, t = 0..\pi\right)$$

$$\frac{1}{3}\pi + \frac{8}{21}$$

Example 20.16 *Calculate the plane integral*

$$\int_B 3xy\,\mathrm{d}x\,\mathrm{d}y,$$

where B is the closed set in the first quadrant, which is bounded by the parabola of the equation $y = 4 - 4x^2$ and the coordinate axes.

A Plane integral.

D Sketch the domain and compute the plane integral.

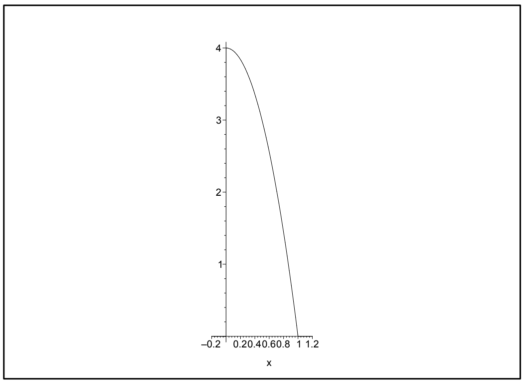


Figure 20.24: The domain of integration B.

I We get immediately,

$$\begin{aligned}\int_B 3xy\,\mathrm{d}y\,\mathrm{d}x &= 3\int_0^1 x\left\{\int_0^{4-4x^2} y\,\mathrm{d}y\right\}\mathrm{d}x = \frac{3}{2}\int_0^1 x\left(4-4x^2\right)^2\,\mathrm{d}x \\ &= \frac{3}{2}\cdot 16\cdot\frac{1}{2}\int_0^1 (1-t)^2\,\mathrm{d}t = 12\int_0^1 u^2\,du = 4.\end{aligned}$$

MAPLE. We get by the commands
with(Student[MultivariateCalculus]):

MultiInt $\left(3x\cdot y, y = 0..4-4x^2, x = 0..1\right)$

4

Example 20.17 *Let B denote the bounded set in the (X,Y)-plane, which is bounded by the line $y=x$ and the parabola $y=x^2$. Compute the plane integral*

$$\int_B x^2 y \, \mathrm{d}x \, \mathrm{d}y.$$

A Plane integral.

D First sketch B.

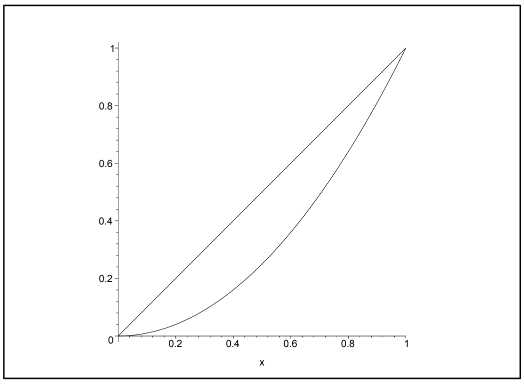


Figure 20.25: The domain B.

I Since

$$B\left\{(x,y) \mid 0 \le x \le 1,\, x^2 \le y \le x\right\},$$

the plane integral is reduced to

$$\int_B x^2 y \, \mathrm{d}x \, \mathrm{d}y = \int_0^1 x^2 \left\{ \int_{x^2}^{x} y \, \mathrm{d}y \right\} \mathrm{d}x = \frac{1}{2} \int_0^1 x^2 \left[y^2\right]_{x^2}^{x} \mathrm{d}x = \frac{1}{2} \int_0^1 \left(x^4 - x^6\right) \mathrm{d}x = \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right) = \frac{1}{35}.$$

MAPLE. We get by the commands
with(Student[MultivariateCalculus]):

$$\text{MultiInt}\left(x^2 \cdot y, y = x^2..x, x = 0..1\right)$$

$$\frac{1}{35}$$

Example 20.18 *Let the set B be given by the inequalities*

$$x \geq 0, \qquad y \geq 0, \qquad \frac{x}{a} + \frac{y}{h} \leq 1.$$

where a and h are positive constants. Sketch B, and then compute the plane integral

$$J = \int_B x^3 y \, \mathrm{d}S.$$

A Plane integral.

D Follow the guidelines and apply one of the theorems of reduction.

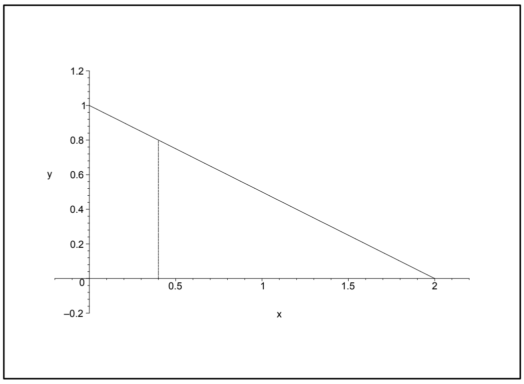


Figure 20.26: The domain B when $a = 2$ and $h = 1$.

I Since the integrand contains y of lower exponent than x, it will be easier first (i.e. innermost) to integrate vertically with respect to y, i.e. for fixed x,

$$0 \leq y \leq h\left(1 - \frac{x}{a}\right), \qquad 0 \leq x \leq a.$$

Then by means of the theorem of reduction in rectangular coordinates,

$$\begin{aligned} J &= \int_B x^3 y \, \mathrm{d}S = \int_0^a x^3 \left(\int_0^{h(1-\frac{x}{a})} y \, \mathrm{d}y \right) = \int_0^a x^3 \cdot \frac{h^2}{2} \left(1 - \frac{x}{a}\right)^2 \mathrm{d}x \\ &= \frac{h^2}{2} \int_0^a x^3 \left(1 - \frac{2}{a}\,x + \frac{1}{a^2}\,x^2\right) \mathrm{d}x = \frac{h^2}{2} \int_0^a \left(x^3 - \frac{2}{a}\,x^4 + \frac{1}{a^2}\,x^5\right) \mathrm{d}x \\ &= \frac{h^2}{2} \left[\frac{x^4}{4} - \frac{2}{5a}\,x^5 + \frac{1}{6a^2}\,x^6 \right]_0^a = \frac{h^2}{2} \left(\frac{a^2}{4} - \frac{2}{5}\,a^4 + \frac{1}{6}\,a^4 \right) \\ &= \frac{h^2 a^4}{2} \left(\frac{1}{4} - \frac{2}{5} + \frac{1}{6} \right) = \frac{h^2 a^4}{2} \cdot \frac{15 - 24 + 10}{60} = \frac{1}{120}\,h^2 a^4. \end{aligned}$$

If we ALTERNATIVELY first integrate horizontally with respect to x, i.e.

$$0 \leq x \leq a\left(1 - \frac{y}{h}\right), \qquad 0 \leq y \leq h,$$

then we get by another theorem of reduction in rectangular coordinates, where we apply the substitution $t = 1 - \dfrac{y}{h}$, $y = h(1-t)$ and $\mathrm{d}y = -h\,\mathrm{d}t$,

$$\begin{aligned} J &= \int_B x^3 y\,\mathrm{d}S = \int_0^h y\left(\int_0^{a\left(1-\frac{y}{h}\right)} x^3\,\mathrm{d}x\right)\mathrm{d}y = \int_0^h y\cdot\frac{a^4}{4}\cdot\left(1-\frac{y}{h}\right)^4\mathrm{d}y \\ &= \int_0^1 \frac{a^4}{4}\cdot h(1-t)\cdot t^4\cdot h\,\mathrm{d}t = \frac{a^4h^2}{4}\int_0^1\left(t^4-t^5\right)\mathrm{d}t = \frac{a^4h^2}{4}\left(\frac{1}{5}-\frac{1}{6}\right) = \frac{1}{120}\,a^4h^2. \end{aligned}$$

MAPLE. Here MAPLE is in trouble. We get by the commands
with(Student[MultivariateCalculus]):

$$\text{MultiInt}\left(x^3\cdot y, y = 0..h\left(1-\frac{x}{a}\right), x = 0..a\right)$$

$$\int_0^a \frac{1}{2}\,x^3 h\left(1-\frac{x}{a}\right)^2\mathrm{d}x$$

Example 20.19 *In each of the following cases a plane integral of a continuous function $f : B \to \mathbb{R}$ is written as a double integral. Sketch in each case the set B, and set up the double integral, or the sum of double integrals, which occur by interchanging the order or integration.*

1) $\int_0^1 \left\{\int_{x^2}^x f(x,y)\,\mathrm{d}y\right\}\,\mathrm{d}x.$

2) $\int_1^e \left\{\int_0^{\ln x} f(x,y)\,\mathrm{d}y\right\}\,\mathrm{d}x.$

3) $\int_1^2 \left\{\int_{2-x}^{\sqrt{2x-x^2}} f(x,y)\,\mathrm{d}y\right\}\,\mathrm{d}x.$

4) $\int_0^2 \left\{\int_{-\sqrt{2x-x^2}}^{0} f(x,y)\,\mathrm{d}y\right\}\,\mathrm{d}x.$

5) $\int_0^3 \left\{\int_{\frac{4y}{3}}^{\sqrt{25-y^2}} f(x,y)\,\mathrm{d}x\right\}\,\mathrm{d}y.$

6) $\int_{-6}^2 \left\{\int_{\frac{y^2-4}{4}}^{1-y} f(x,y)\,\mathrm{d}x\right\}\,\mathrm{d}y.$

7) $\int_0^1 \left\{\int_{-\sqrt{1-y^2}}^{1-y} f(x,y)\,\mathrm{d}x\right\}\,\mathrm{d}y.$

8) $\int_0^3 \left\{\int_0^{\sqrt{25-y^2}} f(x,y)\,\mathrm{d}x\right\}\,\mathrm{d}y.$

A Interchange of the order of integrations in double integrals.

D Sketch the set B and set up the double integral in the reverse order. Note that a nice description in one case does not imply a nice description in the reverse order.

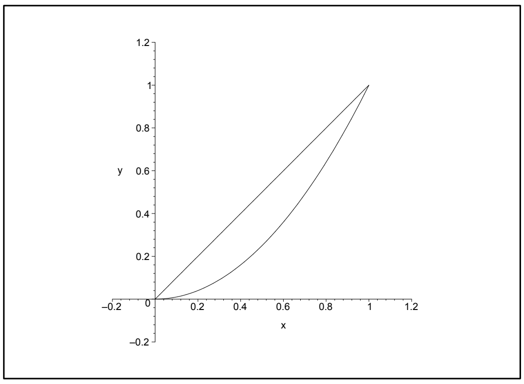


Figure 20.27: The domain B of **Example 20.19.1**.

I 1) The domain is given by

$$B = \{(x,y) \mid 0 \le x \le 1,\, x^2 \le y \le x\} = \{(x,y) \mid 0 \le y \le 1,\, y \le x \le \sqrt{y}\}.$$

In fact, it follows from the inner integral that $x^2 \le y \le x$, from which it is easy to derive

$$y \le x \le \sqrt{y}.$$

By interchanging the order of integration we get

$$\int_0^1 \left\{ \int_{x^2}^{x} f(x,y)\,\mathrm{d}y \right\} \mathrm{d}x = \int_0^1 \left\{ \int_y^{\sqrt{y}} f(x,y)\,\mathrm{d}x \right\} \mathrm{d}y.$$

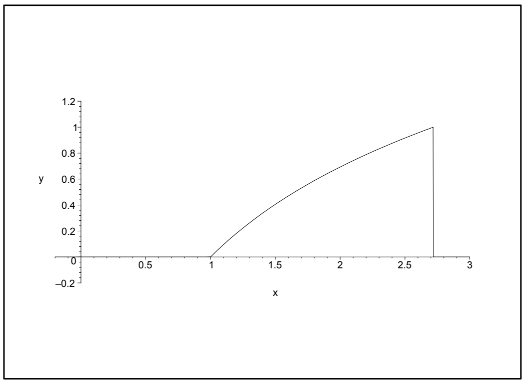


Figure 20.28: The domain B of **Example 20.19.2**.

2) The domain is found in the same way as in **Example 20.19.1**. It is given by

$$B = \{(x,y) \mid 1 \le x \le e,\, 0 \le y \le \ln x\} = \{(x,y) \mid 0 \le y \le 1,\, e^y \le x \le e\},$$

hence by interchanging the order of integration,

$$\int_1^e \left\{ \int_0^{\ln x} f(x,y)\,\mathrm{d}y \right\} \mathrm{d}x = \int_0^1 \left\{ \int_{e^y} f(x,y)\,\mathrm{d}x \right\} \mathrm{d}y.$$

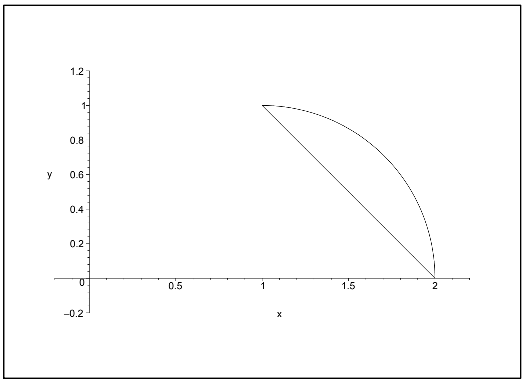


Figure 20.29: The domain B of **Example 20.19.3**.

3) This domain is bounded by the circle $(x-1)^2 + y^2 = 1$ and the straight line $y = 2 - x$, hence

$$\begin{aligned} B &= \{(x,y) \mid 1 \le x \le 2,\, 2-x \le y \le \sqrt{2x - x^2}\} \\ &= \{(x,y) \mid 0 \le y \le 1,\, 2-y \le x \le 1 + \sqrt{1-y^2}\}. \end{aligned}$$

When we interchange the order of integration we get

$$\int_1^2 \left\{ \int_{2-x}^{\sqrt{2x-x^2}} f(x,y)\,\mathrm{d}y \right\} \mathrm{d}x = \int_0^1 \left\{ \int_{2-y}^{1+\sqrt{1-y^2}} f(x,y)\,\mathrm{d}x \right\} \mathrm{d}y.$$

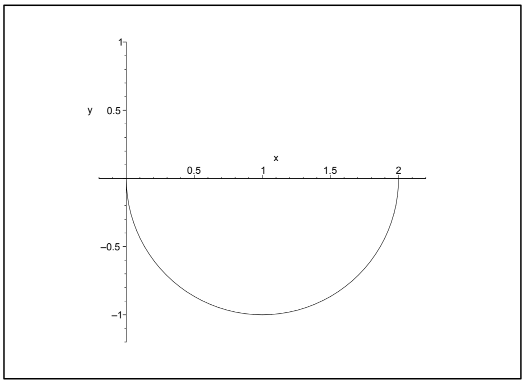


Figure 20.30: The domain B of **Example 20.19.4**.

4) The domain is that part of the disc $(x-1)^2 + y^2 \leq 1$ of centrum$(1,0)$ and radius 1, which lies in the fourth quadrant, thus below the X-axis, so

$$\begin{aligned} B &= \{(x,y) \mid 0 \leq x \leq 2,\, -\sqrt{2x-x^2} \leq y \leq 0\} \\ &= \{(x,y) \mid -1 \leq y \leq 0,\, 1 - \sqrt{1-y^2} \leq x \leq 1 + \sqrt{1-y^2}\}. \end{aligned}$$

When we interchange the order of integration we get

$$\int_0^2 \left\{ \int_{-\sqrt{2x-x^2}}^{0} f(x,y)\,\mathrm{d}y \right\} \mathrm{d}x = \int_{-1}^0 \left\{ \int_{1-\sqrt{1-y^2}}^{1+\sqrt{1-y^2}} f(x,y)\,\mathrm{d}x \right\} \mathrm{d}y.$$

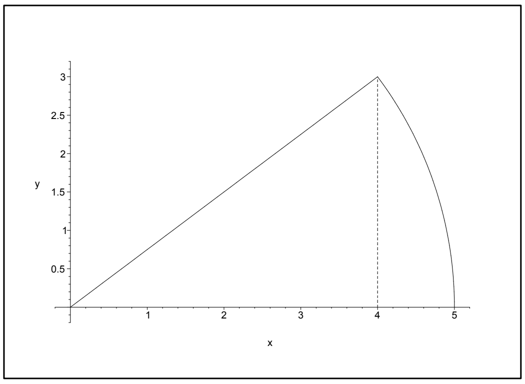


Figure 20.31: The domain B of **Example 20.19.5**.

5) The domain is bounded by the circle $x^2 + y^2 = 5^2$ and the lines $y = 0$ and $y = \frac{3}{4}x$. By the alternative description we must cut the domain by the dotted line $x = 4$. Then we get the two possible descriptions:

$$\begin{aligned} B &= \left\{ (x,y) \;\middle|\; 0 \le y \le 3,\, \frac{4y}{3} \le x \le \sqrt{25-y^2} \right\} \\ &= \left\{ (x,y) \;\middle|\; 0 \le x \le 4,\, 0 \le y \le \frac{3x}{4} \right\} \cup \{(x,y) \mid 4 \le x \le 5,\, 0 \le y \le \sqrt{25-x^2}\}. \end{aligned}$$

When we interchange the order of integration we obtain the following complicated expression

$$\int_0^3 \left\{ \int_{\frac{4y}{3}}^{\sqrt{25-y^2}} f(x,y)\,\mathrm{d}x \right\} \mathrm{d}y = \int_0^4 \left\{ \int_0^{\frac{3x}{4}} f(x,y)\,\mathrm{d}y \right\} \mathrm{d}x + \int_4^5 \left\{ \int_0^{\sqrt{25-x^2}} f(x,y)\,\mathrm{d}y \right\} \mathrm{d}x.$$

In this case we get the sum of two double integrals by interchanging the order of integration.

REMARK. It follows from the form of the domain that it would be far more reasonable here to use polar coordinates, because B in these is described by

$$B = \left\{ (\varrho,\varphi) \mid 0 \le \varrho \le 5,\, 0 \le \varphi \le \operatorname{Arctan} \frac{3}{4} \right\},$$

and the integral is transformed into

$$\int_0^{\operatorname{Arctan} \frac{3}{4}} \left\{ \int_0^5 \tilde{f}(\varrho,\varphi)\varrho\,\mathrm{d}\varrho \right\} \mathrm{d}\varphi. \qquad \Diamond$$

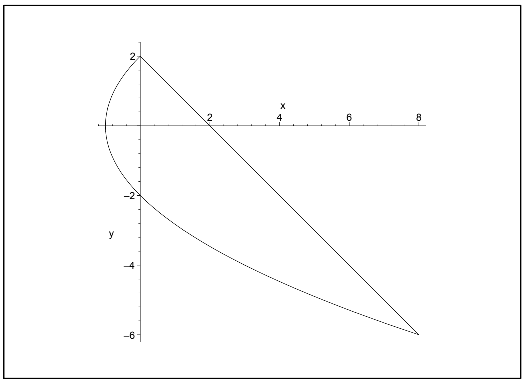


Figure 20.32: The domain B of **Example 20.19.6**.

6) By inspection of the integral we see that the domain is given by

$$B=\left\{(x,y)\;\middle|\; -6\le y\le 2,\ \frac{y^2-4}{4}\le x\le 2-y\right\}.$$

It follows from the inequality $\dfrac{y^2-4}{4}\le x$ that $y^2\le 4(x+1)$, and likewise we get from $x\le 2-y$ that $y\le 2-x$. Whenever the square root occurs (here by $|y|\le 2\sqrt{x+1}$), we must be very careful! The figure shows that we have to split by the line $x=0$, so B is written as a union of two sets which do not have the same structure,

$$\begin{aligned} B &= \{(x,y)\mid -1\le x\le 0,\ -2\sqrt{x+1}\le y\le 2\sqrt{x+1}\}\\ &\quad\cup\{(x,y)\mid 0\le x\le 8,\ -2\sqrt{x+1}\le y\le 2-x\}. \end{aligned}$$

When we interchange the order of the integration we get a sum of two double integrals,

$$\int_{-6}^{2}\left\{\int_{\frac{y^2-4}{4}}^{2-y} f(x,y)\,\mathrm{d}x\right\}\mathrm{d}y=\int_{-1}^{0}\left\{\int_{-2\sqrt{x+1}}^{2\sqrt{x+1}} f(x,y)\,\mathrm{d}y\right\}\mathrm{d}x+\int_{0}^{8}\left\{\int_{-2\sqrt{x+1}}^{2-x} f(x,y)\,\mathrm{d}y\right\}\mathrm{d}x.$$

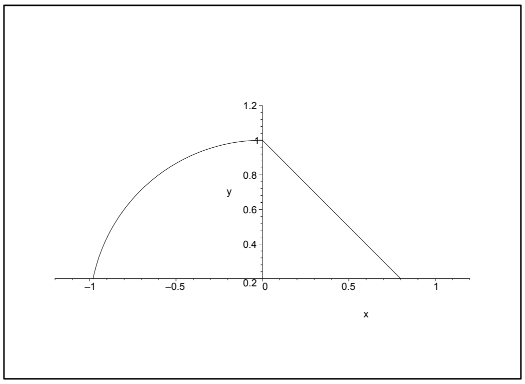


Figure 20.33: The domain B of **Example 20.19.7**.

7) The domain is bounded by the unit circle in the second quadrant, by the X-axis and by the line $y+x=1$. It is natural to split in the two subdomains along the Y-axis, thus

$$\begin{aligned} B &= \{(x,y) \mid 0 \le y \le 1,\, -\sqrt{1-y^2} \le x \le 1-y\} \\ &= \{(x,y) \mid -1 \le x \le 0,\, 0 \le y \le \sqrt{1-x^2}\} \cup <, \{(x,y) \mid 0 \le x \le 1,\, 0 \le y \le 1-x\}. \end{aligned}$$

Then by interchanging the order of integration,

$$\int_0^1 \left\{ \int_{-\sqrt{1-y^2}}^{1-y} f(x,y)\,\mathrm{d}x \right\} \mathrm{d}y = \int_{-1}^0 \left\{ \int_0^{\sqrt{1-x^2}} f(x,y)\,\mathrm{d}y \right\} \mathrm{d}x + \int_0^1 \left\{ \int_0^{1-x} f(x,y)\,\mathrm{d}y \right\} \mathrm{d}x.$$

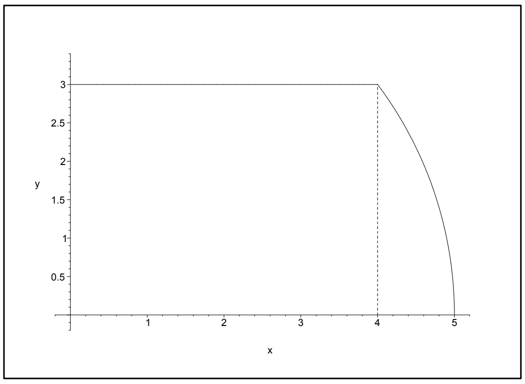


Figure 20.34: The domain B of **Example 20.19.8**.

8) The domain is described by

$$B = \{(x,y) \mid 0 \le x \le \sqrt{25-y^2},\, 0 \le y \le 3\},$$

thus B is that part of the quarter disc in the first quadrant of centrum $(0,2,)$ and radius 5, which also lies below the line $y=3$. When we interchange the coordinates we must cut the domain by the line $x=4$. Then B is written as the union of the two sets,

$$B = \{(x,y) \mid 0 \le y \le \sqrt{25-x^2},\, 4 \le x \le 5\} \cup \{(x,y) \mid 0 \le x \le 4,\, 0 \le y \le 3\}.$$

Then by interchanging the order of integration,

$$\int_0^3 \left\{ \int_0^{\sqrt{25-y^2}} f(x,y)\,\mathrm{d}x \right\} \mathrm{d}y = \int_0^4 \left\{ \int_0^3 f(x,y)\,\mathrm{d}y \right\} \mathrm{d}x + \int_4^5 \left\{ \int_0^{\sqrt{25-x^2}} f(x,y)\,\mathrm{d}y \right\} \mathrm{d}x.$$

Example 20.20 *Sketch the point sets*

$$B = \{(x,y) \mid 0 \le x \le 2,\, 0 \le y \le 2,\, xy \ge 2\}$$

and

$$D = \{(x,y) \mid 1 \le x,\, 1 \le y,\, xy \le 2\}.$$

Then compute the plane integrals

$$\int_B \frac{1}{xy}\,\mathrm{d}S \qquad \textit{and} \qquad \int_D \frac{1}{xy}\,\mathrm{d}S.$$

A and D Sketch of a domain; computation of a plane integral.

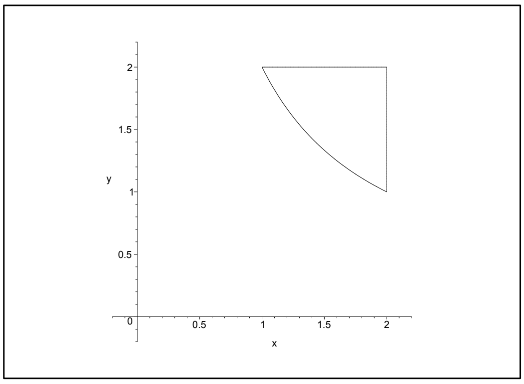


Figure 20.35: The domain B.

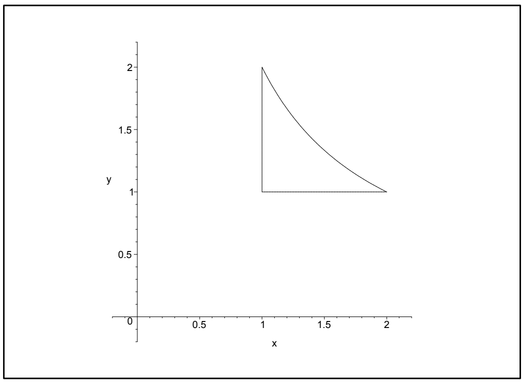


Figure 20.36: The domain D.

I The domains are sketched on the two figures. We see that

$$B \cup D = [1,2] \times [1,2],$$

which may be exploited in one of the variants, because B and D have just one boundary curve in common and are otherwise disjoint; cf. the alternative below.

From

$$B = \left\{ (x,y) \;\middle|\; 1 \le x \le 2,\; \frac{2}{x} \le y \le 2 \right\},$$

follows that

$$\int_B \frac{1}{xy}\,\mathrm{d}S = \int_1^2 \left\{ \int_{\frac{2}{x}}^2 \frac{1}{xy}\,\mathrm{d}y \right\} \mathrm{d}x = \int_1^2 \frac{1}{x}\,[\ln y]_{\frac{2}{x}}^2\,\mathrm{d}x = \int_1^2 \frac{1}{x}\,\ln x\,dx = \frac{1}{2}(\ln 2)^2.$$

MAPLE. We get by the commands
with(Student[MultivariateCalculus]):

$$\text{MultiInt}\left(\frac{1}{x\cdot y}, y = \frac{2}{x}..2, x = 1..2\right)$$

$$\frac{1}{2}\ln(2)^2$$

From

$$D=\left\{(x,y)\;\middle|\;1\le x\le 2,\,1\le y\le \frac{2}{x}\right\},$$

we get analogously

$$\begin{aligned}\int_D \frac{1}{xy}\,\mathrm{d}S &= \int_1^2\left\{\int_1^{\frac{2}{x}}\frac{1}{xy}\,\mathrm{d}y\right\}\mathrm{d}x=\int_1^2\frac{1}{x}\,[\ln y]_1^{\frac{2}{x}}\,\mathrm{d}x\\ &= \int_1^2\frac{1}{x}\{\ln 2-\ln x\}\,\mathrm{d}x=\left[\ln 2\cdot\ln x-\frac{1}{2}(\ln x)^2\right]_1^2=(\ln 2)^2-\frac{1}{2}(\ln 2)^2=\frac{1}{2}(\ln 2)^2.\end{aligned}$$

ALTERNATIVELY,

$$\int_{B\cup D}\frac{1}{xy}\,\mathrm{d}S=\int_1^2\frac{\mathrm{d}x}{x}\cdot\int_1^2\frac{\mathrm{d}y}{y}=(\ln 2)^2=\int_B\frac{1}{xy}\,\mathrm{d}S+\int_D\frac{1}{xy}\,\mathrm{d}S=\frac{1}{2}(\ln 2)^2+\int_D\frac{1}{xy}\,\mathrm{d}S,$$

hence

$$\int_D\frac{1}{xy}\,\mathrm{d}S=(\ln 2)^2-\frac{1}{2}(\ln 2)^2=\frac{1}{2}(\ln 2)^2.$$

Example 20.21 *Let the point set B be given by*

$$B=\left\{(x,y)\in\mathbb{R}^2\;\middle|\;0\le x\le\frac{\pi}{4},\,x\le y\le\frac{1}{\cos x}\right\}.$$

Find the value of the plane integral

$$\int_B y\,\mathrm{d}S.$$

A Plane integral.

D Sketch the domain B and reduce to a double integral.

I By the reduction to a double integral we get

$$\begin{aligned}\int_B y\,\mathrm{d}S &= \int_0^{\frac{\pi}{4}}\left\{\int_x^{1/\cos x}y\,\mathrm{d}y\right\}\mathrm{d}x=\int_0^{\frac{\pi}{4}}\left[\frac{1}{2}\,y^2\right]_x^{1/\cos x}\mathrm{d}x=\frac{1}{2}\int_0^{\frac{\pi}{4}}\left\{\frac{1}{\cos^2 x}-x^2\right\}\mathrm{d}x\\ &=\frac{1}{2}\left[\tan x-\frac{1}{3}\,x^3\right]_0^{\frac{\pi}{4}}=\frac{1}{2}\left\{1-\frac{1}{3}\cdot\frac{\pi^3}{64}\right\}=\frac{1}{2}-\frac{\pi^3}{384}.\end{aligned}$$

MAPLE. We get by the commands
with(Student[MultivariateCalculus]):

$$\mathrm{MultiInt}\left(y,y=x..\frac{1}{\cos(x)},x=0..\frac{\pi}{4}\right)$$

$$-\frac{\pi^3}{384}+\frac{1}{2}$$

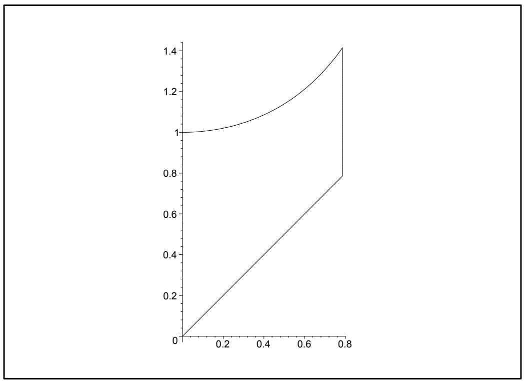


Figure 20.37: The domain B.

20.5 The plane integral in polar coordinates

Some point sets in $\mathbb{R}^2$ are more conveniently described in polar coordinates that in rectangular coordinates, where the transformation formula is

$$x = \varrho \cos \varphi \qquad \text{and} \qquad y = \varrho \sin \varphi,$$

and $\varrho = \sqrt{x^2 + y^2}$, while the angle can be more difficult to specify. We note that

$$\tan \varphi = \frac{y}{x}, \text{ if } x \neq 0, \qquad \text{and} \qquad \cot \varphi = \frac{x}{y}, \text{ if } y \neq 0.$$

When $(x, y) = (0, 0)$, then $\varrho = 0$, and φ is unspecified. Referring to Figure 20.38 we see that if $\varrho > 0$,

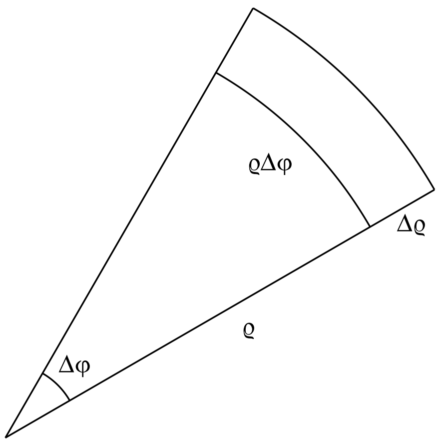


Figure 20.38: Analysis of the area element in polar coordinates.

and $\Delta\varrho$ and $\Delta\varphi$ are small, then the area element is almost a rectangle (not likely on Figure 20.38) of area $\varrho\Delta\varphi \cdot \Delta\varrho = \varrho\Delta\varrho \cdot \Delta\varphi$, so in the limit $\mathrm{d}x\,\mathrm{d}y$ should be expected to be replaced by $\varrho\,\mathrm{d}\varrho\,\mathrm{d}\varphi$. This is actually true, though we shall not go into details of the proof of this claim. Then we quote, also without proof.

Theorem 20.2 Reduction theorem of a plane integral in polar coordinates. *Let $B \subseteq \mathbb{R}^2$ be a closed and bounded set, and let $f : B \to \mathbb{R}$ be a continuous function.*

1) Assume that B is described in polar coordinates by

$$B = \{(\varrho, \varphi) \mid \alpha \le \varphi \le \beta,\, P_1(\varphi) \le \varrho \le P_2(\varphi)\},$$

where

$$P_1,\, P_2 \in C^0([\alpha, \beta]) \quad \text{and} \quad 0 \le P_1(\varphi) < P_2(\varphi) \quad \text{for } \varphi \in]\alpha, \beta[, \quad \beta - \alpha \le 2\pi.$$

Then the plane integral of f over B is reduced to a double integral in the following way,

$$\int_B f(x,y)\,\mathrm{d}S = \int_\alpha^\beta \left\{ \int_{P_1(\varphi)}^{P_2(\varphi)} f(\varrho\cos\varphi, \varrho\sin\varphi)\varrho\, d\varrho \right\} \mathrm{d}\varphi.$$

2) Assume that B is described in polar coordinates by

$$B = \{(\varrho, \varphi) \mid a \le \varrho \le b,\, \Phi_1(\varrho) \le \varphi \le \Phi_2(\varphi)\},$$

where

$$\Phi_1,\, \Phi_2 \in C^0([a, b]), \quad a \ge 0 \quad \text{and} \quad 0 \le \Phi_1(\varrho) < \Phi_2(\varrho) \le 2\pi \quad \text{for } \varrho \in]a, b[.$$

Then the plane integral of f over B is reduced to a double integral in the following way,

$$\int_B f(x,y)\,\mathrm{d}S = \int_a^b \left\{ \int_{\Phi_1(\varrho)}^{\Phi_2(\varrho)} f(\varrho\cos\varphi, \varrho\sin\varphi)\,\mathrm{d}\varphi \right\} \varrho\,\mathrm{d}\varrho.$$

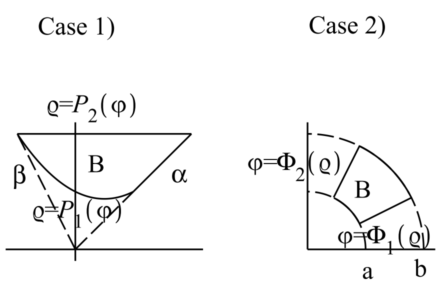


Figure 20.39: Examples of domains B in Theorem 20.2 of type 1) to the left, and of type 2) to the right.

A reduction in polar coordinates is in particular easy to perform, when both the domain B and the integrand have simpler descriptions in polar coordinates than in rectangular coordinates.

If, however, the integrand is better described in polar coordinates, and the domain in rectangular coordinates, or *vice versa*, then we may expect some hard calculations.

Example 20.22 A classical application is the calculation of the value of the *Gaussian integral*,

$$I = \int_{-\infty}^{+\infty} \exp\left(-x^2\right) \, dx.$$

The domain $\mathbb{R}$ is unbounded, but the integrand is > 0 everywhere, so *if* the integral was divergent (it is not!), then we could see this by getting the value $+\infty$. The trick is to compute I^2 as a plane integral, first in rectangular coordinates, and then we switch to polar coordinates. This is done in the following way,

$$\begin{aligned} I^2 &= \int_{-\infty}^{+\infty} \exp\left(-x^2\right) \, dx \cdot \int_{-\infty}^{+\infty} \exp\left(-y^2\right) \, dy = \int\!\!\int_{\mathbb{R}^2} \exp\left(-\left(x^2+y^2\right)\right) \, dx \, dy \\ &= \int\!\!\int_{\mathbb{R}_+ \times [0,2\pi]} \exp\left(-\varrho^2\right) \varrho \, d\varrho \, d\varphi = 2\pi \int_0^{+\infty} \exp\left(-\varrho^2\right) \varrho \, d\varrho = \pi \left[-\exp\left(-\varrho^2\right)\right]_0^{+\infty} = \pi, \end{aligned}$$

because $\varphi \in [0, 2\pi]$ does not enter the integrand, and because $\varrho = \sqrt{x^2+y^2} \geq 0$. By taking the square root we finally get

$$I = \int_{-\infty}^{+\infty} \exp\left(-x^2\right) \, dx = \sqrt{\pi}. \qquad \Diamond$$

Example 20.23 **A.** *Compute the plane integral*

$$I = \int_B (x^2 + y^2)\,\mathrm{d}S,$$

where B *in* polar *coordinates is described by the parameter domain*

$$A = \left\{ (\varrho, \varphi) \;\middle|\; a \le \varrho \le 2a,\, \frac{\varrho}{2a} \le \varphi \le \frac{\varphi}{a} \right\}.$$

Note that B *has a curved form in the* (x, y) *plane, while the parameter domain* A *in the* (ϱ, φ) *plane is "straightened out", so we can use the* rectangular *version in the* (ϱ, φ) *plane. The price of obtaining this convenience is that we must add the weight function* ϱ *as a factor in the integrand.*

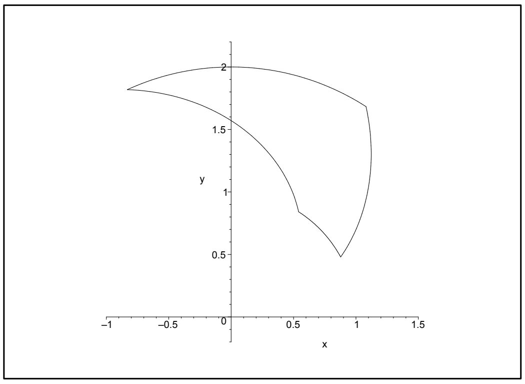


Figure 20.40: The domain B in the (x, y) plane.

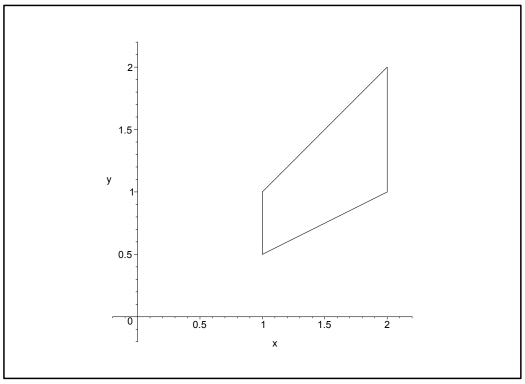


Figure 20.41: The parameter domain A in the (ϱ, φ) plane.

D. Set up the reduction formula in its second version, i.e. with the inner φ integral. This means that we start with a vertical integration in the parameter domain!

I. When we use the reduction formula in its second version, where we multiply the integrand by the weight function ϱ, we get the calculation,

$$\int_B (x^2+y^2)\,\mathrm{d}S = \int_a^{2a}\left\{\int_{\frac{\varrho}{2a}}^{\frac{\varrho}{a}} \varrho^2\,\mathrm{d}\varphi\right\}\varrho\,\mathrm{d}\varrho = \int_a^{2a}\varrho^3\left\{\int_{\frac{\varrho}{2a}}^{\frac{\varrho}{a}}\mathrm{d}\varphi\right\}\mathrm{d}\varrho.$$

The value of the inner integral is then calculated,

$$\int_{\frac{\varrho}{2a}}^{\frac{\varrho}{a}}\mathrm{d}\varphi = \frac{\varrho}{a}-\frac{\varrho}{2a}=\frac{\varrho}{2a},$$

which is the length of the φ interval.

Then by insertion,

$$\begin{aligned}\int_B (x^2+y^2)\,\mathrm{d}S &= \int_a^{2a}\varrho^3\cdot\frac{\varrho}{2a}\,\mathrm{d}\varrho = \frac{1}{2a}\int_a^{2a}\varrho^4\,\mathrm{d}\varrho = \frac{1}{2a}\left[\frac{1}{5}\,\varrho^5\right]_1^{2a}\\ &= \frac{1}{10a}\left\{(2a)^5-a^5\right\} = \frac{1}{10a}\left\{32\,a^5-a^5\right\} = \frac{31}{10}\,a^4.\end{aligned}$$

Example 20.24 **A.** *Calculate the plane integral*

$$I = \int_B (x+y)\,\mathrm{d}S,$$

where B is described in polar *coordinates (for $a > 0$) by the parameter domain*

$$A = \left\{ (\varrho, \varphi) \;\middle|\; -\frac{\pi}{2} \le \varphi \le \frac{\pi}{4},\, 0 \le \varrho \le a \right\},$$

i.e. A is a rectangle in the (ϱ, φ) plane.

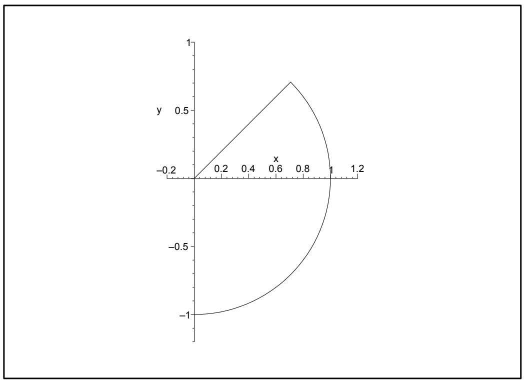


Figure 20.42: The domain B in the (x, y)-plane when $a = 1$.

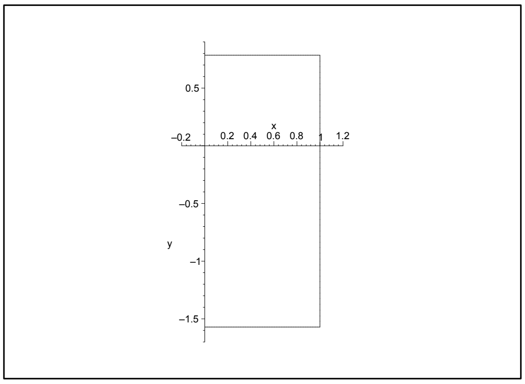


Figure 20.43: The parameter domain A in the (ϱ, φ) plane for $a = 1$.

D. In this case both reduction formulæ are applicable, so we give two solutions.

D 1. Apply reduction formula 1); *do not forget the weight function ϱ!.*

I 1. Since $x = \varrho\cos\varphi$ and $y = \varrho\sin\varphi$, the first version, where we start in the inner integral by integrating horizontally with respect to ϱ, gives

$$I = \int_B (x+y)\,\mathrm{d}S = \int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \left\{ \int_0^a (\varrho\cos\varphi + \varrho\sin\varphi)\,\varrho\,\mathrm{d}\varrho \right\} \mathrm{d}\varphi.$$

Then we calculate the inner integral,

$$\int_0^a (\varrho\cos\varphi + \varrho\sin\varphi)\,\varrho\,d\varrho = (\cos\varphi+\sin\varphi)\int_0^a \varrho^2\,d\varrho = \frac{a^3}{3}\,(\cos\varphi+\sin\varphi).$$

Finally, by insertion of this result,

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \frac{a^3}{3}\,(\cos\varphi+\sin\varphi)\,d\varphi = \frac{a^3}{3}\,[\sin\varphi-\cos\varphi]_{-\frac{\pi}{2}}^{\frac{\pi}{4}} = \frac{a^3}{3}.$$

D 2. Then we use the reduction formula 2)I; *do not forget the weight function ϱ!*

I 2. In the second version we only interchance the order of integration. The limits are constants, and the integrand is factorized, so we can split the integral in the product of two integrala. This gives

$$\begin{aligned} I &= \int_0^a \left\{\int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} (\varrho\,\cos\varphi + \varrho\,\sin\varphi)\,d\varphi\right\}\varrho\,d\varrho \\ &= \int_0^a \varrho^2\,d\varrho\cdot\int_{-\frac{\pi}{2}}^{\frac{\pi}{4}}(\cos\varphi+\sin\varphi)\,d\varphi = \frac{a^3}{3}\cdot[\sin\varphi-\cos\varphi]_{-\frac{\pi}{2}}^{\frac{\pi}{4}} = \frac{a^3}{3}. \end{aligned}$$

Example 20.25 **A.** *Compute the plane integral* $I = \int_B x\,dS$, *where* $B = K\left(\left(\frac{a}{2},0\right);\frac{a}{2}\right)$, $a > 0$.

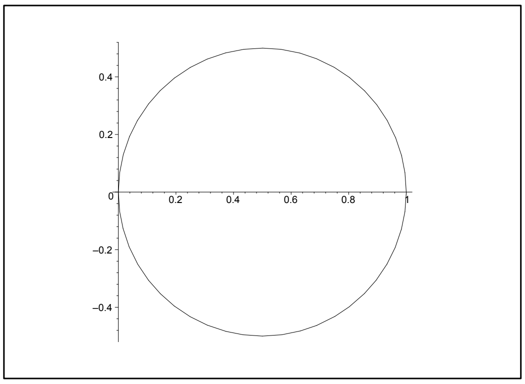


Figure 20.44: The domain B when $a = 1$, i.e. $-\sqrt{x-x^2} \le y \le \sqrt{x-x^2}$ for $0 \le x \le 1$.

D. In this case it is possible to calculate the plane integral both in rectangular and in polar coordinates, so we give two variants.

D 1. The domain B is in rectangular coordinates described by

$$B = \{(x,y) \mid 0 \le 0 \le a,\, -\sqrt{ax-x^2} \le y \le \sqrt{ax-x^2}\}.$$

I 1. The *rectangular* double integral is

$$I = \int_B x\,dS = \int_0^a x\left\{\int_{-\sqrt{ax-x^2}}^{\sqrt{ax-x^2}} dy\right\}dx = \int_a 2a\sqrt{ax-x^2}\,dx.$$

The trick in tasks like this is to give the "bad term" a new name. We put

$$t := ax - x^2, \qquad \mathrm{d}t = (a-2x)\,\mathrm{d}x.$$

The next trick is to add something and then sbtract it again. This is here done in the following way,

$$\begin{aligned} I &= \int_0^a 2x\sqrt{ax-x^2}\,\mathrm{d}x = -\int_0^a (a-2x-a)\sqrt{ax-x^2}\,\mathrm{d}x \\ &= -\int_0^a \sqrt{ax-x^2}\cdot(a-2x)\,\mathrm{d}x + a\int_0^a \sqrt{ax-x^2}\,\mathrm{d}x \\ &= -\int_{x=0}^a \sqrt{t}\,\mathrm{d}t + a\int_0^a \sqrt{ax-x^2}\,\mathrm{d}x \\ &= -\left[\frac{2}{3}\left(ax-x^2\right)^{\frac{3}{2}}\right]_0^a + a\int_0^a \sqrt{ax-x^2}\,\mathrm{d}x = 0 + a\int_0^a \sqrt{ax-x^2}\,\mathrm{d}x. \end{aligned}$$

The integral $\int_0^a \sqrt{ax-x^2}\,\mathrm{d}x$ looks nasty. However, by a geometrical consideration we see that the integral must be equal to the area of the domain between the x axis and the graph of

$$y = +\sqrt{ax-x^2}.$$

This is (cf. the figure) the area of the half of a disc of radius $\frac{a}{2}$. Hence,

$$I = a\cdot\left\{\frac{1}{2}\cdot\pi\left(\frac{a}{2}\right)^2\right\} = \frac{\pi a^3}{8}.$$

D 2. The polar version; *do not forget the weight function* ϱ*!*

I 2. When we put $x = \varrho\cos\varphi$ and $y = \varrho\sin\varphi$, then the equation of the boundary curve is transformed into

$$0 = x^2+y^2-ax = \varrho^2 - a\,\varrho\cos\varphi = \varrho(\varrho - a\cos\varphi).$$

Since $\varrho = 0$ corresponds to the point $(0,0)$, we conclude that the boundary curve is described by

$$\varrho = a\cos\varphi \qquad \text{with } -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}.$$

Thus the *parameter domain* A corresponding to B is given by

$$A = \left\{(\varrho,\varphi)\;\middle|\; -\frac{\pi}{2}\le\varphi\le\frac{\pi}{2},\; 0\le\varrho\le a\cos\varphi\right\}.$$

The reduction formula in its first version gives

$$\int_B x\,\mathrm{d}S = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left\{\int_0^a \cos\varphi\,\varrho\cos\varphi\cdot\varrho\,\mathrm{d}\varrho\right\}\mathrm{d}\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\varphi\left\{\int_0^{a\cos\varphi}\varrho^2\,\mathrm{d}\varrho\right\}\mathrm{d}\varphi.$$

We first calculate the inner integral,

$$\int_0^{a\cos\varphi}\varrho^2\,\mathrm{d}\varrho = \left[\frac{1}{3}\varrho^3\right]_0^{a\cos\varphi} = \frac{a^3}{3}\cos^3\varphi.$$

Then by insertion,

$$\int_B x\,\mathrm{d}S = \frac{a^3}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4\varphi\,\mathrm{d}\varphi = 2\cdot\frac{a^3}{3}\int_0^{\frac{\pi}{2}}\cos^4\varphi\,\mathrm{d}\varphi,$$

where we have used than the even function $\cos^4\varphi$ is integrated over a symmetric interval.

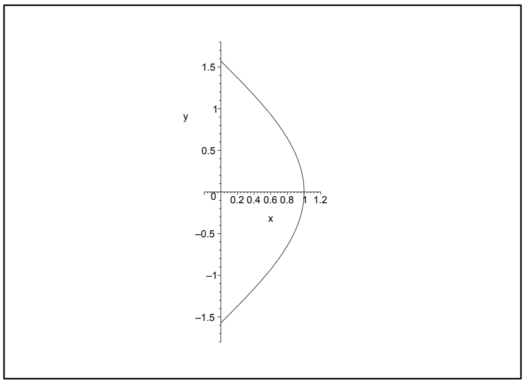


Figure 20.45: The parameter domain A in the (ϱ, φ) plane.

This is a trigonometric integral, where the integrand is of *even* order. The trick is to use the double angle as a new variable,

$$\begin{aligned}\cos^4 x &= \left\{\cos^2 x\right\}^2 = \left\{\frac{1}{2}(1+\cos 2x)\right\}^2 = \frac{1}{4}\left\{1+2\cos 2x+\cos^2 2x\right\}\\ &= \frac{1}{4}\left\{1+2\cos 2x+\frac{1}{2}(1+\cos 4x)\right\}\\ &= \frac{3}{8}+\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x.\end{aligned}$$

Then by insertion

$$\int_B x\,\mathrm{d}S = \frac{2}{3}a^3\int_0^{\frac{\pi}{2}}\left\{\frac{3}{8}+\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x\right\}\mathrm{d}x = \frac{2}{3}a^3\cdot\frac{3}{8}\,\frac{\pi}{2}+0+0 = \frac{\pi a^3}{8}.$$

20.6 Procedure of reduction of the plane integral; polar version

Geometri.
When a domain B is bounded by radial half lines from $(0,0)$, or circular arcs (with or without its centrum at $(0,0)$), then the calculations will often be easier in polar coordinates,

$$x = \varrho\cos\varphi \qquad \text{and} \qquad y = \varrho\sin\varphi,$$

than rectangular coordinates. When we apply this technique, then the reader must be aware of that the parameter domain $\tilde{B}$ is *not* equal to the original domain B. The price is that we must not forget the weight function, which in the case of polar coordinates is ϱ. But then we can reduce as in the rectangular case with respect to (ϱ, φ).

This is formally expressed by the abstract element of area $\mathrm{d}S$. In the case of polar coordinates this is identified with $\varrho\,\mathrm{d}\varrho\,\mathrm{d}\varphi$, calculated in the rectangular (ϱ, φ) domain,

$$\mathrm{d}S = \varrho\,\mathrm{d}\varrho\,\mathrm{d}\varphi,$$

or, written as an integral,

$$\begin{aligned}\int_B f(x,y)\,\mathrm{d}S &= \int_{\tilde{B}} f(x,y)\text{ weight function }\,\mathrm{d}\varrho\,\mathrm{d}\varphi \\ &= \int_{\tilde{B}} f(\varrho\cos\varphi,\varrho\sin\varphi)\,\varrho\,\mathrm{d}\varrho\,\mathrm{d}\varphi.\end{aligned}$$

Thus we have reduced the problem to the rectangular case, so we could in principle stop here. However, we shall not do this, because we ought to mention some reductions, which are not immediate, and which may be of some help in the following.

Just like in the rectangular case there are here two main cases.

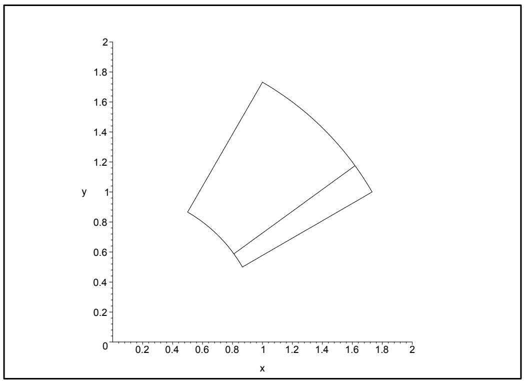


Figure 20.46: First version. We integrate over B, first radially with respect to ϱ.

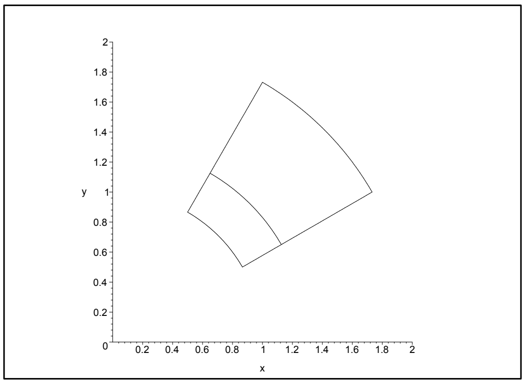


Figure 20.47: Second version. We integrate over B, first along the circular arc with respect to φ.

Procedure.

1) Sketch B, and split if necessary B into subdomains of type 1 or type 2.

2) If B is of type 1 (i.e. B lies in some angular space), then sketch, if necessary the parameter domain $\tilde{B}$, which we here write in the form

$$\tilde{B} = \{(\varphi, \varrho) \mid \alpha \leq \varphi \leq \beta, P_1(\varphi) \leq \varrho \leq P_2(\varphi)\}.$$

Note that e.g. P_1 is read "big rho 1", because it is the Greek letter.

Then the reduction formula becomes

$$\begin{aligned} \int_B f(x,y)\,\mathrm{d}S &= \int_{\tilde{B}} f(\varrho\cos\varphi, \varrho\sin\varphi)\,\varphi\,\mathrm{d}\varphi\,\mathrm{d}\varrho \\ &= \int_\alpha^\beta \left\{ \int_{P_1(\varphi)}^{P_2(\varphi)} f(\varrho\cos\varphi, \varrho\sin\varphi\,\varrho\,d\varrho) \right\} \mathrm{d}\varphi. \end{aligned}$$

3) Calculate separately the inner integral, i.e. keep φ fixed and integrate with respect to ϱ:

$$F(\varphi)=\int_{P_1(\varphi)}^{P_2(\varphi)} f(\varrho\cos\varphi,\varrho\sin\varphi)\,\varrho\,\mathrm{d}\varrho.$$

4) Insert the result and then calculate

$$\int_B f(x,y)\,\mathrm{d}S=\int_\alpha^\beta F(\varphi)\,\mathrm{d}\varphi.$$

5) If instead B is of type 2 (i.e. lies between two circular arcs of centrum at $(0,0)$), then we sketch, if necessary, the parameter domain $\tilde{B}$, which is written

$$\tilde{B}=\{(\varphi,\varrho)\mid a\le\varrho\le b,\ \Phi_1(\varrho)\le\varphi\le\Phi_2(\varrho)\}.$$

Then the reduction formula becomes

$$\begin{aligned}\int_B f(x,y)\,\mathrm{d}S &= \int_{\tilde{B}} f(\varrho\,\cos\varphi,\varrho\,\sin\varphi)\,\varrho\,\mathrm{d}\varphi\,\mathrm{d}\varrho\\ &= \int_a^b\left\{\int_{\Phi_1(\varrho)}^{\Phi_2(\varrho)} f(\varrho\,\cos\varphi,\varrho\,\sin\varphi)\,\mathrm{d}\varphi\right\}\varrho\,\mathrm{d}\varrho.\end{aligned}$$

6) Calculate separately the inner integral, where ϱ is kept fixed, and then integrate with respect to φ:

$$G(\varrho)=\int_{\Phi_1(\varrho)}^{\Phi_2(\varrho)} f(\varrho\,\cos\varphi,\varrho\,\sin\varphi)\,\mathrm{d}\varphi.$$

7) Insert the result and calculate

$$\int_B f(x,y)\,\mathrm{d}S=\int_a^b G(\varrho)\,\varrho\,\mathrm{d}\varrho,$$

where we must not forget the weight function ϱ in the integrand.

20.7 Examples of the plane integral in polar coordinates

Example 20.26

A. *Calculate*

$$I = \int_b (x^2 + y^2)\,\mathrm{d}S,$$

where B *is described in* polar *coordinates by*

$$A = \left\{ (\varrho, \varphi) \;\middle|\; a \le \varrho \le 2a,\ \frac{\varrho}{2a} \le \varphi \le \frac{\varphi}{a} \right\}.$$

Note that B has a "weird" form in the (x, y)-plane, while the parameter domain A in the (ϱ, φ)-plane is "straightened out", so one can apply the *rectangular* version in the $(\varrho.\varphi)$-plane. The price for this is that one must add the weight function ϱ to the integrand.

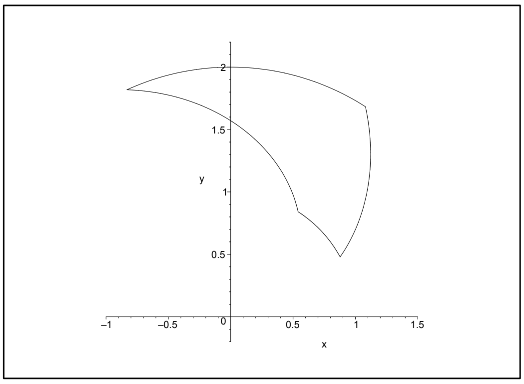


Figure 20.48: The domain B in the (x, y)-plane.

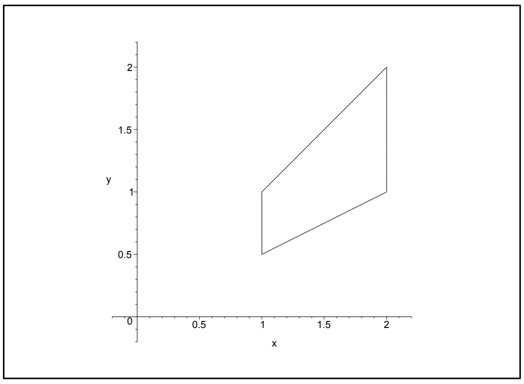


Figure 20.49: The parameter domain A in the (ϱ, φ)-plane.

D. Apply the reduction formula in the second version, i.e. where the φ-integral is the inner integral. This means that we first integrate vertically in the parameter domain.

I. By the reduction formula in its second version we get with the weight function ϱ

$$\int_B (x^2+y^2)\,\mathrm{d}S = \int_a^{2a}\left\{\int_{\frac{\varrho}{2a}}^{\frac{\varrho}{a}} \varrho^2\,\mathrm{d}\varphi\right\}\varrho\,\mathrm{d}\varrho = \int_a^{2a}\varrho^3\left\{\int_{\frac{\varrho}{2a}}^{\frac{\varrho}{a}}\mathrm{d}\varphi\right\}\mathrm{d}\varrho.$$

First calculate the inner integral,

$$\int_{\frac{\varrho}{2a}}^{\frac{\varrho}{a}}\mathrm{d}\varphi = \frac{\varrho}{a}-\frac{\varrho}{2a}=\frac{\varrho}{2a},$$

which is seen to be the length of the φ-interval. Then by insertion,

$$\begin{aligned}\int_B (x^2+y^2)\,\mathrm{d}S &= \int_a^{2a}\varrho^3\cdot\frac{\varrho}{2a}\,\mathrm{d}\varrho = \frac{1}{2a}\int_a^{2a}\varrho^4\,\mathrm{d}\varrho = \frac{1}{2a}\left[\frac{1}{5}\varrho^5\right]_1^{2a}\\ &= \frac{1}{10a}\left\{(2a)^5-a^5\right\} = \frac{1}{10a}\left\{32\,a^5-a^5\right\} = \frac{31}{10}\,a^4.\end{aligned}$$

MAPLE. We get by the commands

with(Student[MultivariateCalculus]):

$$\text{MultiInt}\left(r^2\cdot r, t=\frac{r}{2a}..\frac{r}{a}, r=a..2a\right)$$

$$\frac{31}{10}\,a^4 \qquad \diamond$$

Example 20.27

A. *Calculate*

$$I = \int_B (x+y)\,\mathrm{d}S,$$

where B is described in polar *coordinate (for $a>0$) by*

$$A = \left\{(\varrho,\varphi) \;\middle|\; -\frac{\pi}{2} \le \varphi \le \frac{\pi}{4},\, 0 \le \varrho \le a\right\},$$

i.e. A is a rectangle in the (ϱ,φ) plane.

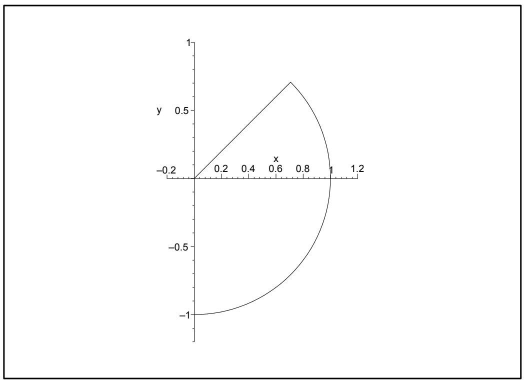


Figure 20.50: The domain B for $a=1$ in the (x,y)-plane.

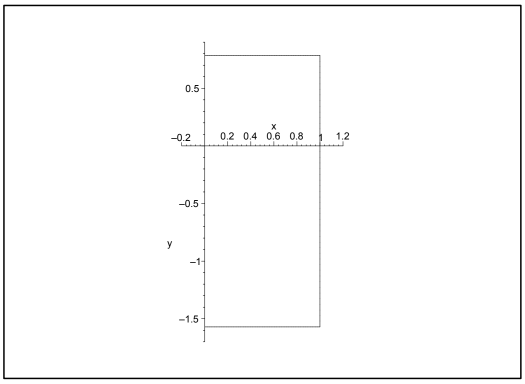


Figure 20.51: The parametric domain A for $a=1$ in the (ϱ,φ)-plane.

D. Here we can apply both reduction formulæ, so we give two solutions.

D 1. Apply the first reduction formula; do not forget the weight function ϱ.

I 1. From $x = \varrho\cos\varphi$ and $y = \varrho\sin\varphi$, we get in the first version, where we start by integrating horizontally after ϱ, that

$$I = \int_B (x+y)\,\mathrm{d}S = \int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \left\{ \int_0^a (\varrho\cos\varphi + \varrho\sin\varphi)\,\varrho\,\mathrm{d}\varrho \right\} \mathrm{d}\varphi.$$

Then calculate the inner integral,

$$\int_0^a (\varrho\cos\varphi + \varrho\sin\varphi)\,\varrho\,\mathrm{d}\varrho = (\cos\varphi+\sin\varphi)\int_0^a \varrho^2\,\mathrm{d}\varrho = \frac{a^3}{3}\,(\cos\varphi+\sin\varphi).$$

By insertion of this result we finally get

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \frac{a^3}{3}\,(\cos\varphi+\sin\varphi)\,\mathrm{d}\varphi = \frac{a^3}{3}\,[\sin\varphi-\cos\varphi]_{-\frac{\pi}{2}}^{\frac{\pi}{4}} = \frac{a^3}{3}.$$

D 2. Apply the second reduction formula. Again, do not forget the weight function ϱ.

I 2. In the second version we just interchange the order of integration. Since the bounds are constants, and the variables can be separated in the integrand, we can split the integral into a product of two integrals. Then

$$\begin{aligned} I &= \int_0^a \left\{ \int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} (\varrho\,\cos\varphi + \varrho\,\sin\varphi)\,\mathrm{d}\varphi \right\} \varrho\,\mathrm{d}\varrho \\ &= \int_0^a \varrho^2\,\mathrm{d}\varrho \cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} (\cos\varphi+\sin\varphi)\,\mathrm{d}\varphi = \frac{a^3}{3}\cdot[\sin\varphi-\cos\varphi]_{-\frac{\pi}{2}}^{\frac{\pi}{4}} = \frac{a^3}{3}. \end{aligned}$$

MAPLE. We get by the commands

with(Student[MultivariateCalculus]):

$$\text{MultiInt}\left((r\cdot\cos(t)+r\cdot\sin(t))\cdot r, r=0..a, t=-\frac{\pi}{2}..\frac{\pi}{4}\right)$$

$$\frac{1}{3}a^3 \qquad \Diamond$$

Example 20.28

A. *Calculate* $I = \int_B x \, \mathrm{d}S$, *where* $B = K\left(\left(\frac{a}{2}, 0\right); \frac{a}{2}\right)$, $a > 0$.

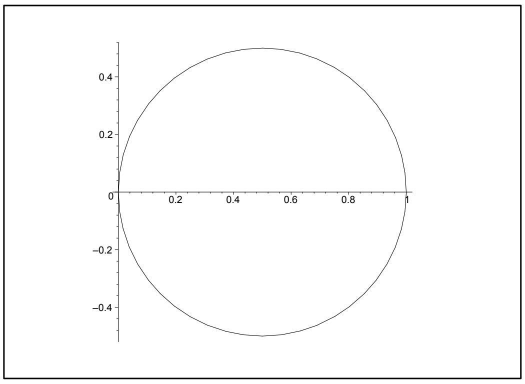


Figure 20.52: The domain B for $a = 1$, i.e. $-\sqrt{x - x^2} \le y \le \sqrt{x - x^2}$ for $0 \le x \le 1$.

D. In this case it is possible to calculate the integral by using either rectangular or polar coordinates.

D 1. In rectangular coordinates the domain B is described by

$$B = \{(x, y) \mid 0 \le 0 \le a, \, -\sqrt{ax - x^2} \le y \le \sqrt{ax - x^2}\}.$$

I 1. The *rectangular* double integral is given by

$$I = \int_B x \, \mathrm{d}S = \int_0^a x \left\{ \int_{-\sqrt{ax-x^2}}^{\sqrt{ax-x^2}} \mathrm{d}y \right\} \mathrm{d}x = \int_a 2a \sqrt{ax - x^2} \, \mathrm{d}x.$$

The trick in problems of this type is to call the "ugly" part something different. We put

$$t = ax - x^2, \qquad \mathrm{d}t = (a - 2x) \, \mathrm{d}x.$$

Then by adding the right term and subtract it again we get

$$\begin{aligned} I &= \int_0^a 2x \sqrt{ax - x^2} \, \mathrm{d}x = -\int_0^a (a - 2x - a)\sqrt{ax - x^2} \, \mathrm{d}x \\ &= -\int_0^a \sqrt{ax - x^2} \cdot (a - 2x) \, \mathrm{d}x + a \int_0^a \sqrt{ax - x^2} \, \mathrm{d}x \\ &= -\int_{x=0}^a \sqrt{t} \, \mathrm{d}t + a \int_0^a \sqrt{ax - x^2} \, \mathrm{d}x \\ &= -\left[\frac{2}{3}\left(ax - x^2\right)^{\frac{3}{2}}\right]_0^a + a \int_0^a \sqrt{ax - x^2} \, dx = 0 + a \int_0^a \sqrt{ax - x^2} \, \mathrm{d}x. \end{aligned}$$

The integral $\int_0^a \sqrt{ax - x^2} \, \mathrm{d}x$ does not look nice; but the geometrical interpretation helps a lot: The integral is the area of the domain between the x-axis and the curve

$$y = +\sqrt{ax - x^2},$$

i.e. (cf. the figure) the area of a half-disc of radius $\dfrac{a}{2}$. Therefore,

$$I = a \cdot \left\{ \frac{1}{2} \cdot \pi \left(\frac{a}{2} \right)^2 \right\} = \frac{\pi a^3}{8}.$$

D 2. The polar version; do not forget the weight function ϱ.

I 2. When we put $x = \varrho \cos\varphi$ and $y = \varrho \sin\varphi$, the equation of the boundary curve becomes

$$0 = x^2 + y^2 - ax = \varrho^2 - a\,\varrho \cos\varphi = \varrho(\varrho - a \cos\varphi).$$

Since $\varrho = 0$ corresponds to the point $(0,0)$, it follows that the boundary curve is described by

$$\varrho = a \cos\varphi \qquad \text{with } -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}.$$

The parametric domain A corresponding to B is therefore

$$A = \left\{ (\varrho, \varphi) \;\middle|\; -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2},\ 0 \le \varrho \le a \cos\varphi \right\}.$$

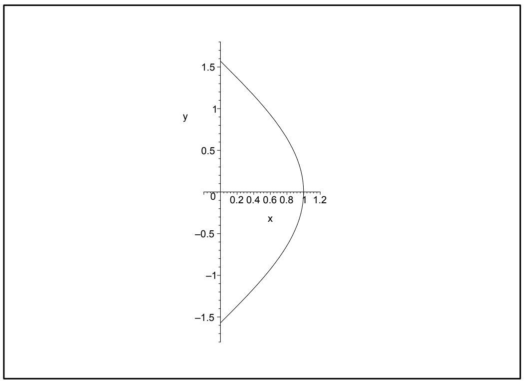


Figure 20.53: The parametric domain A in the (ϱ, φ) plane.

When we use the first version of the reduction formula we get

$$\int_B x\,\mathrm{d}S = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ \int_0^{a\,\cos\varphi} \varrho\cos\varphi \cdot \varrho\,\mathrm{d}\varrho \right\} \mathrm{d}\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi \left\{ \int_0^{a\,\cos\varphi} \varrho^2\,\mathrm{d}\varrho \right\} \mathrm{d}\varphi.$$

When we calculate the inner integral we get

$$\int_0^{a\,\cos\varphi} \varrho^2\,\mathrm{d}\varrho = \left[\frac{1}{3}\,\varrho^3\right]_0^{a\,\cos\varphi} = \frac{a^3}{3}\,\cos^3\varphi.$$

Then by insertion

$$\int_B x\,\mathrm{d}S = \frac{a^3}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\varrho\,\mathrm{d}\varphi = 2 \cdot \frac{a^3}{3} \int_0^{\frac{\pi}{2}} \cos^4\varphi\,d\varphi,$$

where we use that the even function $\cos^4\varphi$ is integrated over a symmetric interval.

When we shall calculate a trigonometric integral, where the integrand is of *even* order, we change variables to the double angle:

$$\begin{aligned}
\cos^4 x &= \left\{\cos^2 x\right\}^2 = \left\{\frac{1}{2}(1+\cos 2x)\right\}^2 = \frac{1}{4}\left\{1 + 2\cos 2x + \cos^2 2x\right\} \\
&= \frac{1}{4}\left\{1 + 2\cos 2x + \frac{1}{2}(1+\cos 4x)\right\} \\
&= \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x.
\end{aligned}$$

Finally, by insertion,

$$\int_B x\,\mathrm{d}S = \frac{2}{3}\,a^3 \int_0^{\frac{\pi}{2}} \left\{\frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x\right\} \mathrm{d}x = \frac{2}{3}\,a^3 \cdot \frac{3}{8}\,\frac{\pi}{2} + 0 + 0 = \frac{\pi a^3}{8}.$$

MAPLE. We get by the commands
with(Student[MultivariateCalculus]):

$$\text{MultiInt}\left(r\cdot\cos(t)\cdot r, r = 0..a\cdot\cos(t), t = -\frac{\pi}{2}..\frac{\pi}{2}\right)$$

$\frac{1}{8}\pi a^3$ $\Diamond$

Example 20.29 *Compute in each of the following cases the given plane integral by applying a theorem of reduction for polar coordinates. First sketch the domain of integration B.*

1) $\int_B (4-y)\,\mathrm{d}S$, *where B is given by $x \ge 0$, $y \ge 0$, and $x^2+y^2 \le 2$.*

2) $\int_B (a+y)\,\mathrm{d}S$, *where B is given by $0 \le \varphi \le \dfrac{\pi}{2}$ and $0 \le \varrho \le a\cos\varphi$.*

3) $\int_B \sqrt{a^2-x^2-y^2}\,\mathrm{d}S$, *where B is given by $-\dfrac{\pi}{2} \le \varphi \le \dfrac{\pi}{2}$ and $0 \le \varrho \le a\cos\varphi$.*

4) $\int_B xy\,\mathrm{d}S$, *where B is given by $0 \le \varphi \le \dfrac{\pi}{3}$ and $2\cos\varphi \le \varrho \le \dfrac{4}{1+\cos\varphi}$.*

5) $\int_B \dfrac{x(x+y)}{(2x^2+y^2)(x^2+y^2)^{\frac{3}{2}}}\,\mathrm{d}S$, *where B is given by $0 \le \varphi \le \dfrac{\pi}{4}$ and $\cos\varphi \le \varrho \le \cos\varphi+\sin\varphi$.*

6) $\int_B \dfrac{1}{\sqrt{a^2+x^2+y^2}}\,\mathrm{d}S$, *where B is the disc $\overline{K}((0,0);a)$.*

7) $\int_B \dfrac{x}{(x^2+y^2)^{\frac{3}{2}}}\,\mathrm{d}S$, *where B is given by $-\pi \le \varphi \le \pi$ and $b\exp(a\cos\varphi) \le \varrho \le 1$, and where furthermore $b < e^{-a}$.*

8) $\int_B \dfrac{x}{(x^2+y^2)^{\frac{3}{2}}}\,\mathrm{d}S$, *where B is given by $-\pi \le \varphi \le \pi$ and $1 \le \varrho \le b\exp(a\cos\varphi)$, and where furthermore $b > e^{a}$.*

9) $\int_B (x^2-y^2)\,\mathrm{d}S$, *where B is given by $-\dfrac{\pi}{4} \le \varphi \le \dfrac{\pi}{2}$ and $0 \le \varrho \le a$.*

10) $\int_B \sqrt{x^2+y^2}\,\mathrm{d}S$, *where B is given by $-\dfrac{\pi}{2} \le \varphi \le \dfrac{\pi}{2}$ and $0 \le \varrho \le a\,\cos\varphi$.*

11) $\int_B xy\,\mathrm{d}S$, *where B is given by $0 \le \varphi \le \dfrac{\pi}{4}$ and $a \le \varrho \le 2a\,\cos^2\varphi$.*

A Plane integral in polar coordinates.

D Sketch the domain and apply the theorem of reduction.

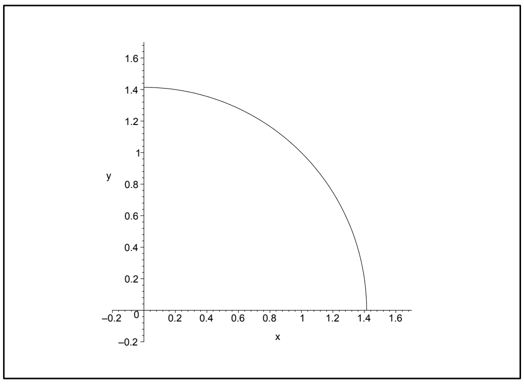


Figure 20.54: The domain B of **Example 20.29.1**.

I 1) This example is the same as **Example 20.10.8**. We shall, however, use polar coordinates in the present case.
In polar coordinates B is described by

$$0 \le \varphi \le \frac{\pi}{2}, \qquad 0 \le \varrho \le \sqrt{2}.$$

From the theorem of reduction in polar coordinates follows that

$$\int_B (4-y)\,\mathrm{d}S = 4\,\text{area}(B) - \int_0^{\frac{\pi}{2}} \left\{ \int_0^{\sqrt{2}} \varrho \sin\varphi \cdot \varrho\,\mathrm{d}\varrho \right\} \mathrm{d}\varphi$$

$$= 4 \cdot \frac{1}{4}(\sqrt{2})^2\pi + [\cos\varphi]_0^{\frac{\pi}{2}} \cdot \left[\frac{1}{3}\varrho^3\right]_0^{\sqrt{2}} = 2\pi + \frac{2\sqrt{3}}{3}.$$

MAPLE. We get by the commands
with(Student[MultivariateCalculus]):

$$\text{MultiInt}\left(r \cdot \sin(t) \cdot r, r = 0..\sqrt{2}, t = 0..\frac{\pi}{2}\right)$$

$$\frac{2}{3}\sqrt{2}$$

which gives the value of the integral.

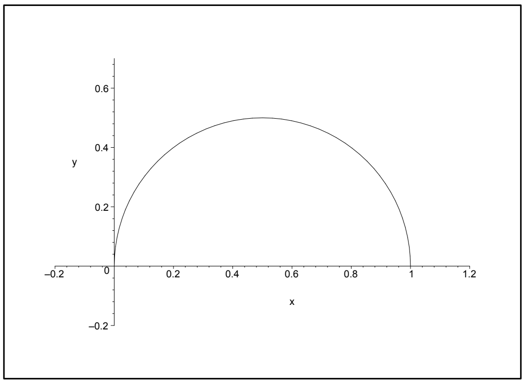


Figure 20.55: The domain B of **Example 20.29.2**.

2) From $0 \le \varrho \le a\,\cos\varphi$ follows that

$$0 \le \varrho^2 = x^2 + y^2 = a\varrho\cos\varphi = ax,$$

so the domain is a half disc in the first quadrant of centrum $\left(\frac{a}{2}, 0\right)$ and radius $\frac{a}{2}$. By the reduction formula in polar coordinates,

$$\int_B (a+y)\,\mathrm{d}S = a \cdot \text{area}(B) + \int_B y\,\mathrm{d}S = a \cdot \frac{1}{2} \cdot \pi \left(\frac{a}{2}\right)^2 + \int_0^{\frac{\pi}{2}} \left\{ \int_0^{a\,\cos\varphi} \varrho \sin\varphi \cdot \varrho\,\mathrm{d}\varrho \right\} \mathrm{d}\varphi$$

$$= \pi \cdot \frac{a^3}{8} + \int_0^{\frac{\pi}{2}} \left[\frac{1}{3}\varrho^3 \sin\varphi\right]_{\varrho=0}^{a\,\cos\varphi} \mathrm{d}\varphi = \frac{a^3\pi}{8} + \frac{a^3}{3}\int_0^{\frac{\pi}{2}} \cos^3\varphi \cdot \sin\varphi\,\mathrm{d}\varphi$$

$$= \frac{\pi a^3}{8} - \frac{a^3}{12}\left[\cos^4\varphi\right]_0^{\frac{\pi}{2}} = a^3\left(\frac{\pi}{8} + \frac{1}{12}\right).$$

MAPLE. We get by the commands
with(Student[MultivariateCalculus]):

$$\text{MultiInt}\left(r\cdot\sin(t)\cdot r, r=0..a\cdot\cos(t), t=0..\frac{\pi}{2}\right)$$

$$\frac{1}{12}\,a^3$$

which gives the value of the integral.

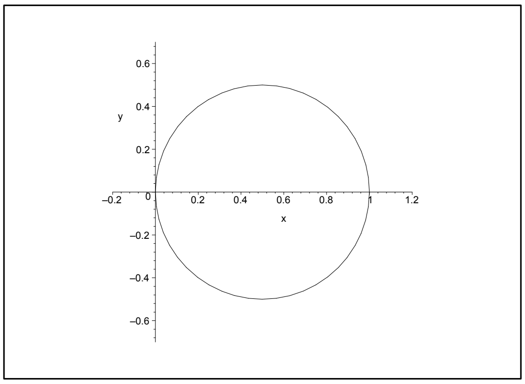


Figure 20.56: The domain B of **Example 20.29.3**.

3) Here B is the disc of centrum $\left(\frac{a}{2},0\right)$ and radius $\frac{a}{2}$, cf. **Example 20.29.2**. From the reduction formula in polar coordinates follows that

$$\int_B \sqrt{a^2-x^2-y^2}\,\mathrm{d}S = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left\{\int_0^{a\,\cos\varphi}\sqrt{a^2-\varrho^2}\cdot\varrho\,\mathrm{d}\varrho\right\}\mathrm{d}\varphi$$

$$= 2\int_0^{\frac{\pi}{2}}\left[-\frac{1}{3}(a^2-\varrho^2)^{\frac{3}{2}}\right]_{\varrho=0}^{a\,\cos\varphi}d\varphi = \frac{2}{3}\int_0^{\frac{\pi}{2}}\left\{(a^2)^{\frac{3}{2}}-(a^2-a^2\cos^2\varphi)^{\frac{3}{2}}\right\}d\varphi$$

$$= \frac{2}{3}\int_0^{\frac{\pi}{2}}\{a^3-a^3(1-\cos^2\varphi)\sin\varphi\}\,d\varphi = \frac{2}{3}\,a^3\left\{\frac{\pi}{2}+\int_{\varphi=0}^{\frac{\pi}{2}}(1-\cos^2\varphi)\,d\cos\varphi\right\}$$

$$= \frac{\pi a^3}{3}+\frac{2}{3}\,a^3\left[\cos\varphi-\frac{1}{3}\,\cos^3\varphi\right]_{\varphi=0}^{\frac{\pi}{2}} = \frac{\pi a^3}{3}-\frac{4}{9}\,a^3 = \frac{a^3}{9}(3\pi-4).$$

MAPLE. We get by the commands
with(Student[MultivariateCalculus]):

$$\text{MultiInt}\left(\sqrt{a^2-r^2}\cdot r, r=0..a\cdot\cos(t), t=-\frac{\pi}{2}..\frac{\pi}{2}\right)$$

$$\frac{1}{3}\,a^3\,\text{csgn}(a)\pi-\frac{4}{9}\,a^3\,\text{csgn}(a)$$

where csgn(a) denotes the sign of a.

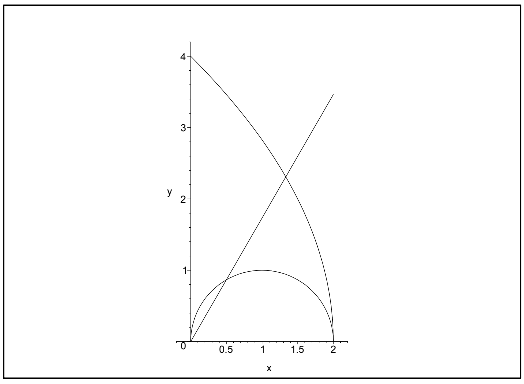


Figure 20.57: The domain B of **Example 20.29.4**.

4) From $2\cos\varphi \le \varrho$ follows

$$2x = 2\varrho\cos\varphi \le \varrho^2 = x^2+y^2,$$

which is rewritten as the inequality $(x-1)^2+y^2 \ge 1$ for the complementary set of the disc of centrum $(1,0)$ and radius 1.
From $\varrho \le \dfrac{4}{1+\cos\varphi}$ follows $\varrho+\varrho\cos\varphi = \varrho+x \le 4$, i.e. $\varrho \le 4-x$, so $x \le 4$. Under this assumption we get by a squaring that $\varrho^2 = x^2+y^2 \le (4-x)^2$, hence

$$y^2 \le (4-x)^2 - x^2 = 4(4-2x) = 8(2-x),$$

from which follows that we shall also require that $x \le 2$, because $y^2 \ge 0$.
The domain is bounded by the parabola $y^2 = 16-8x$ and the circle $(x-1)^2+y^2=1$ and the tow lines $\varphi = 0$ and $\varphi = \dfrac{\pi}{3}$.

Then by the theorem of reduction in polar coordinates followed by the substitution $u=\cos\varphi$,

$$\begin{aligned}
\int_B xy\,\mathrm{d}S &= \int_0^{\frac{\pi}{3}}\left\{\int_{2\cos\varphi}^{\frac{4}{1+\cos\varphi}} \varrho^3 \sin\varphi\cdot\cos\varphi\,\mathrm{d}\varrho\right\}\mathrm{d}\varphi \\
&= \frac{1}{4}\int_0^{\frac{\pi}{3}} \sin\varphi\cdot\cos\varphi\left\{\frac{4^4}{(1+\cos\varphi)^4} - 2^4\cos^4\varphi\right\}\mathrm{d}\varphi \\
&= \int_0^{\frac{\pi}{3}}\left\{\frac{64\cos\varphi}{(1+\cos\varphi)^4} - 4\cos^5\varphi\right\}\sin\varphi\,\mathrm{d}\varphi = \int_{\frac{1}{2}}^1 \left\{\frac{64(u+1-1)}{(u+1)^4} - 4u^5\right\}\mathrm{d}u \\
&= \int_{\frac{1}{2}}^1\left\{\frac{64}{(u+1)^3} - \frac{64}{(u+1)^4} - 4u^5\right\}\mathrm{d}u \\
&= \left[-\frac{1}{2}\cdot\frac{64}{(u+1)^2} + \frac{1}{3}\cdot\frac{64}{(u+1)^3} - \frac{4}{6}u^6\right]_{\frac{1}{2}}^1 \\
&= -\frac{32}{4} + \frac{1}{3}\cdot\frac{64}{8} - \frac{2}{3} + \frac{32}{\frac{9}{4}} - \frac{1}{3}\cdot\frac{64}{\frac{27}{8}} + \frac{2}{3}\cdot\frac{1}{64} \\
&= -8 + \frac{8}{3} - \frac{2}{3} + \frac{128}{9} - \frac{512}{81} + \frac{1}{3\cdot 32} = -6 + \frac{1252-512}{81} + \frac{1}{3\cdot 32} \\
&= \frac{640-486}{81} + \frac{1}{3\cdot 32} = \frac{154}{81} + \frac{1}{3\cdot 32} = \frac{154\cdot 32+27}{32\cdot 81} = \frac{4955}{2592}.
\end{aligned}$$

MAPLE. We get by the commands
with(Student[MultivariateCalculus]):

$$\mathrm{MultiInt}\left(r^3\cdot\sin(t)\cdot\cos(t), r = 2\cos(t)..\frac{4}{1+\cos(t)}, t = 0..\frac{\pi}{3}\right)$$

$$\frac{4955}{2592}$$

5) Here the condition $\cos\varphi \le \varrho$ implies that

$$\varrho\cos\varphi = x \le \varrho^2 = x^2+y^2,$$

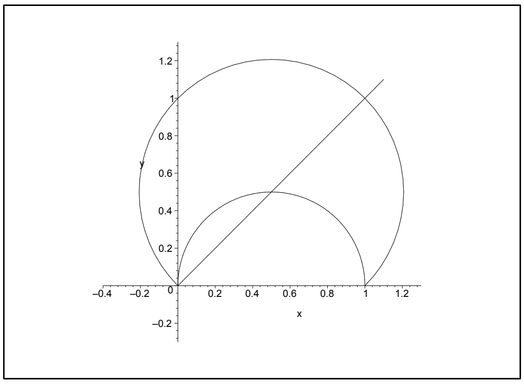


Figure 20.58: The domain B of **Example 20.29.5**.

which we rewrite as

$$\left(x-\frac{1}{2}\right)^2+y^2 \ge \frac{1}{4}=\left(\frac{1}{2}\right)^2,$$

and we are describing the complementary set of a disc of centrum $\left(\frac{1}{2},0\right)$ and radius $\frac{1}{2}$.
The condition $\varrho \le \cos\varphi+\sin\varphi$ means that

$$\varrho^2=x^2+y^2\le \varrho\cos\varphi+\varrho\sin\varphi=x+y,$$

which is rewritten as

$$\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2\le\frac{1}{2}=\left(\frac{1}{\sqrt{2}}\right)^2.$$

This inequality represents a disc of centrum $\left(\frac{1}{2},\frac{1}{2}\right)$ and radius $\frac{1}{\sqrt{2}}$. As also $0\le\varphi\le\frac{\pi}{4}$, it is now easy to sketch the domain B.

Then by the theorem of reduction in polar coordinates,

$$\begin{aligned}
&\int_B \frac{x(x+y)}{(2x^2+y^2)(x^2+y^2)^{\frac{3}{2}}}\,dS \\
&\quad=\int_0^{\frac{\pi}{4}}\left\{\int_{\cos\varphi}^{\cos\varphi+\sin\varphi}\frac{\varrho^2(\cos\varphi+\sin\varphi)\cos\varphi}{\varrho^2(2\cos^2\varphi+\sin^2\varphi)\varrho^3}\cdot\varrho\,d\varrho\right\}d\varphi \\
&\quad=\int_0^{\frac{\pi}{4}}\frac{(\cos\varphi+\sin\varphi)\cos\varphi}{2\cos^2\varphi+\sin^2\varphi}\left[-\frac{1}{\varrho}\right]_{\varrho=\cos\varphi}^{\cos\varphi+\sin\varphi}d\varphi \\
&\quad=\int_0^{\frac{\pi}{4}}\left\{\frac{\cos\varphi+\sin\varphi}{2\cos^2\varphi+\sin^2\varphi}-\frac{\cos\varphi}{2\cos^2\varphi+\sin^2\varphi}\right\}d\varphi=\int_0^{\frac{\pi}{4}}\frac{\sin\varphi}{\cos^2\varphi+1}\,d\varphi \\
&\quad=\left[-\operatorname{Arctan}(\cos\varphi)\right]_0^{\frac{\pi}{4}}=\operatorname{Arctan}1-\operatorname{Arctan}\left(\frac{\sqrt{2}}{2}\right)=\frac{\pi}{4}-\operatorname{Arctan}\left(\frac{\sqrt{2}}{2}\right).
\end{aligned}$$

MAPLE. We get by the commands

with(Student[MultivariateCalculus]):

$$\text{MultiInt}\left(\frac{r^2\cdot(\cos(t)+\sin(t))\cdot\cos(t)}{r^2\cdot(2\cos(t)^2+\sin(t)^2(\cdot r^3}\cdot r, r=\cos(t)..\cos(t)+\sin(t), t=0..\frac{\pi}{4}\right)$$

$$\frac{1}{4}\pi-\arctan\left(\frac{1}{2}\sqrt{2}\right)$$

6) The disc $\overline{K}((0,0);a)$ is described in polar coordinates by

$$-\pi\le\varphi\le\pi,\qquad 0\le\varrho\le a.$$

We shall here omit the sketch of the well-known disc of centrum $(0,0)$ and radius a.

Then by an application of the theorem of reduction in polar coordinates,

$$\int_B\frac{1}{\sqrt{a^2+x^2+y^2}}\,\mathrm{d}S=\int_{-\pi}^{\pi}\left\{\int_0^a\frac{\varrho}{\sqrt{a^2+\varrho^2}}\,\mathrm{d}\varrho\right\}\mathrm{d}\varphi=2\pi\left[\sqrt{a^2+\varrho^2}\right]_0^a=2\pi a(\sqrt{2}-1).$$

MAPLE. We get by the commands
with(Student[MultivariateCalculus]):

$$\text{MultiInt}\left(\frac{r}{\sqrt{a^2+r^2}}, r=0..a, t=-\pi..\pi\right)$$

$$-2\text{csgn}(a)a\pi+2\sqrt{2}\text{csgn}(a)a\pi$$

where $\text{csgn}(a)$ denotes the sign of a.

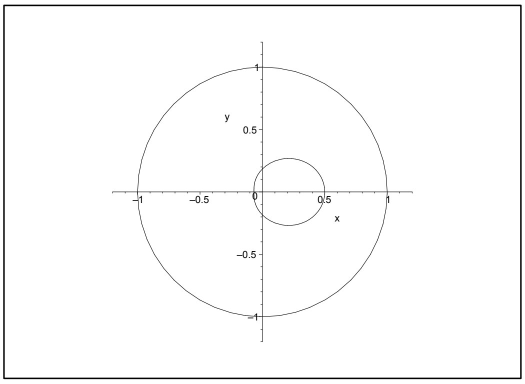


Figure 20.59: The domain B of **Example 20.29.7**, when $a=1$ and $b=\dfrac{1}{2e}$.

7) The set is an annulus shaped domain which is neither nice in a rectangular description nor in a polar description.

When we reduce the plane integral it is fairly simple to get

$$\int_B\frac{x}{(x^2+y^2)^{\frac{3}{2}}}\,\mathrm{d}S=\int_{-\pi}^{\pi}\left\{\int_{b\exp(a\cos\varphi)}^1\frac{\varrho\cos\varphi}{\varrho^3}\cdot\varrho\,\mathrm{d}\varrho\right\}\mathrm{d}\varphi=\int_{-\pi}^{\pi}\cos\varphi\cdot[\ln\varrho]_{b\exp(a\cos\varphi)}^1\,\mathrm{d}\varphi$$

$$=\int_{-\pi}^{\pi}\cos\varphi\{-\ln b-a\cos\varphi\}\,\mathrm{d}\varphi=-a\int_{-\pi}^{\pi}\cos^2\varphi\,\mathrm{d}\varphi=-a\pi.$$

MAPLE. Here MAPLE needs some help, because we get by the commands with(Student[MultivariateCalculus]):

$$\text{MultiInt}\left(\frac{r\cdot\cos(t)}{r^3}\cdot r, r=b\cdot e^{a\cdot\cos(t)}..1, t=-\pi..\pi\right)$$

$$\int_{-\pi}^{\pi}\int_{b\,e^{a\cos(t)}}^{1}\frac{\cos(t)}{r}\,\mathrm{d}r\,\mathrm{d}t$$

so we must split the double integral into two separate integrals.

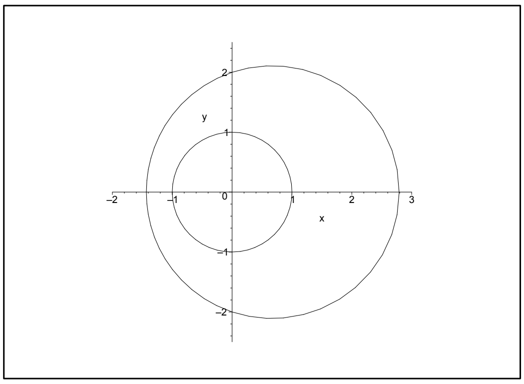


Figure 20.60: The domain B of **Example 20.29.8**, when $a=\dfrac{1}{3}$ and $b=2$.

8) This case is similar to **Example 20.29.7**. We get

$$\int_B \frac{x}{(x^2+y^2)^{\frac{3}{2}}}\,\mathrm{d}S = \int_{-\pi}^{\pi}\left\{\int_1^{b\exp(a\cos\varphi)} \frac{\cos\varphi}{\varrho}\,\mathrm{d}\varrho\right\}\mathrm{d}\varphi = +a\pi,$$

because, apart from the change of sign, the computations are formally the same as in **Example 20.29.7**.

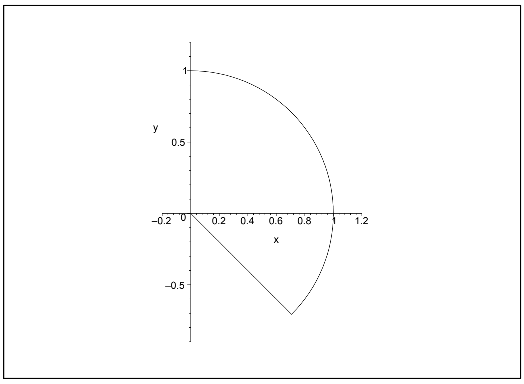


Figure 20.61: The domain B of **Example 20.29.9**.

9) The set B is a circular sector as shown on the figure.

Then by the theorem of reduction,

$$\int_B (x^2-y^2)\,\mathrm{d}S = \int_{-\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\int_0^a \left\{\varrho^2\cos^2\varphi - \varrho^2\sin^2\varphi\right\}\varrho\,\mathrm{d}\varrho\right)\mathrm{d}\varphi$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\int_0^a \cos 2\varphi\cdot\varrho^3\,\mathrm{d}\varrho\right)\mathrm{d}\varphi = \left[\frac{1}{2}\sin 2\varphi\right]_{-\frac{\pi}{4}}^{\frac{\pi}{2}}\cdot\frac{a^4}{4} = \frac{1}{2}\{0-(-1)\}\cdot\frac{a^4}{4} = \frac{a^4}{8}.$$

MAPLE. We get by the commands
with(Student[MultivariateCalculus]):

$$\mathrm{MultiInt}\left((r^2\cdot\cos(t)^2 - r^2\cdot\sin(t)^2)\cdot r, r = 0..a, t = -\frac{\pi}{4}..\frac{\pi}{2}\right)$$

$$\frac{1}{8}a^4$$

10) From $0 \le \varrho \le a\,\cos\varphi$ follows that

$$0 \le \varrho^2 = x^2+y^2 = a\varrho\cos\varphi = ax,$$

so B is the closed disc of centrum $\left(\frac{a}{2},0\right)$ and radius $\frac{a}{2}$.

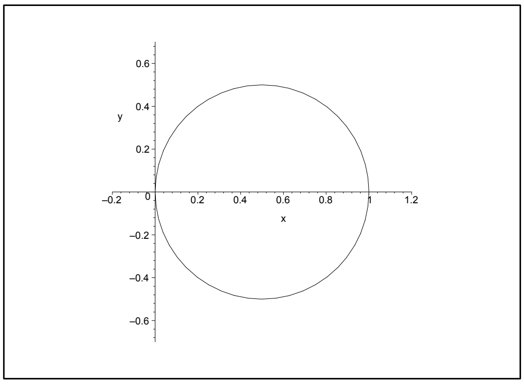


Figure 20.62: The domain B of **Example 20.29.10**.

Then by the theorem of reduction,

$$\begin{aligned}\int_B \sqrt{x^2+y^2}\,\mathrm{d}S &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{\int_0^{a\cos\varphi} \varrho\cdot\varrho\,\mathrm{d}\varrho\right\}\mathrm{d}\varphi = \frac{a^3}{3}\int_{-\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^3\varphi\,\mathrm{d}\varphi \\ &= \frac{a^3}{3}\int_{-\frac{\pi}{3}}^{\frac{\pi}{2}} (1-\sin^2\varphi)\cos\varphi\,\mathrm{d}\varphi = \frac{a^3}{3}\left[\sin\varphi - \frac{1}{3}\sin^3\varphi\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= \frac{a^3}{3}\cdot 2\left(1-\frac{1}{3}\right) = \frac{4a^3}{9}.\end{aligned}$$

MAPLE. We get by the commands
with(Student[MultivariateCalculus]):

$$\text{MultiInt}\left(r\cdot r, r = 0..a\cdot\cos(t), t = -\frac{\pi}{2}..\frac{\pi}{2}\right)$$

$$\frac{4}{9}\,a^3$$

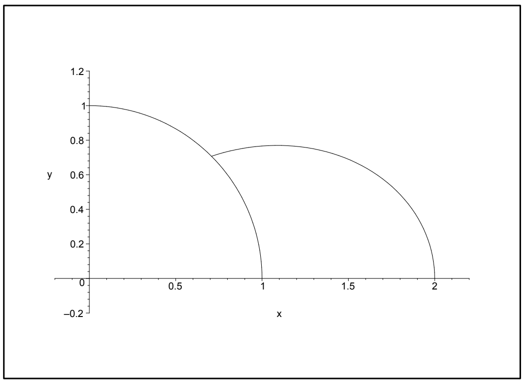


Figure 20.63: The domain B of **Example 20.29.11**.

11) There is no nice rectangular description of the domain. It follows by the theorem of reduction,

$$\begin{aligned}
\int_B xy\,\mathrm{d}S &= \int_0^{\frac{\pi}{4}} \left\{ \int_a^{2a\cos^2\varphi} \varrho\cos\varphi \cdot \varrho\sin\varphi \cdot \varrho\,\mathrm{d}\varrho \right\} \mathrm{d}\varphi \\
&= \int_0^{\frac{\pi}{4}} \cos\varphi\sin\varphi \left\{ \int_a^{2a\cos^2\varphi} \varrho^3\,\mathrm{d}\varrho \right\} \mathrm{d}\varphi, \\
&= \frac{1}{4}\int_0^{\frac{\pi}{4}} \cos\varphi\cdot\sin\varphi\left[(2a)^4\cos^8\varphi - a^4\right]\mathrm{d}\varphi \qquad\qquad (t=\cos\varphi) \\
&= \frac{a^4}{4}\int_{\frac{1}{\sqrt{2}}}^1 \left\{16t^9 - t\right\}\mathrm{d}t = \frac{a^4}{4}\left[\frac{16}{10}t^{10} - \frac{1}{2}t^2\right]_{\frac{1}{\sqrt{2}}}^1 \\
&= \frac{a^4}{4}\left\{\frac{8}{5} - \frac{1}{2} - \frac{8}{5}\cdot\frac{1}{32} + \frac{1}{2}\cdot\frac{1}{2}\right\} = \frac{a^4}{4}\left\{\frac{31}{20} - \frac{1}{4}\right\} \\
&= \frac{a^4}{4}\cdot\frac{26}{20} = \frac{13}{40}a^4.
\end{aligned}$$

MAPLE. We get by the commands
with(Student[MultivariateCalculus]):

$$\mathrm{MultiInt}\left(r\cdot\cos(t)\cdot r\cdot\sin(t)\cdot r, r = a..2a\cdot\cos(t)^2, t = 0..\frac{\pi}{4}\right)$$

$$\frac{13}{40}a^4$$

Example 20.30 *Let B be the domain in the first quadrant, which is bounded by the curves of the equations*

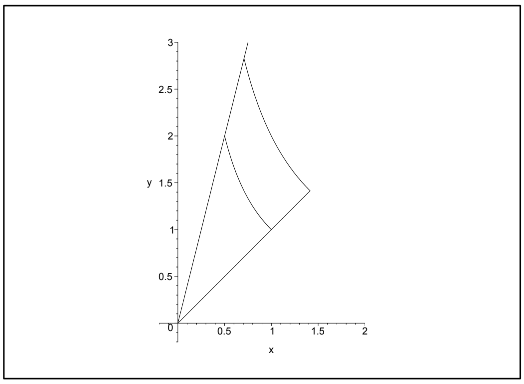
$$y = x, \qquad y = 4x, \qquad xy = 1, \qquad xy = 2.$$

Describe B in polar coordinates and then compute the plane integral

$$\int_B x^2 \exp(xy) \ln\left(\frac{y}{x}\right) \, dS.$$

A Plane integral reduced by polar coordinates.

D Sketch B. Then describe B in polar coordinates.



I In polar coordinates the line $y = 4x$ is described by $\varphi = \text{ Arctan } 4$, and the line $y = x$ by

$$\varphi = \text{ Arctan } 1 = \frac{\pi}{4}.$$

Since

$$xy = \varrho^2 \sin\varphi \cos\varphi,$$

the hyperbola $xy = 1$ is described by

$$\varrho = \frac{1}{\sqrt{\sin\varphi\cos\varphi}}, \qquad \varphi \in \left[\frac{\pi}{4}, \text{Arctan } 4\right],$$

and the hyperbola $xy = 2$ by

$$\varrho = \frac{2}{\sqrt{\sin\varphi\cos\varphi}}, \qquad \varphi \in \left[\frac{\pi}{4}, \text{Arctan } 4\right].$$

Summarizing we get by the reduction of the plane integral in polar coordinates that

$$\int_B x^2 \exp(xy) \ln\left(\frac{y}{x}\right) \, dS$$

$$(20.7) \qquad = \int_{\text{Arctan } 1}^{\text{Arctan } 4} \left\{ \int_{1/\sqrt{\sin\varphi\cos\varphi}}^{2/\sqrt{\sin\varphi\cos\varphi}} \varrho^2 \cos^2\varphi \cdot \exp\left(\varrho^2 \sin\varphi\cos\varphi\right) \ln(\tan\varphi) \varrho \, d\varrho \right\} d\varphi.$$

First calculate the inner integral by using the substitution $t = \varrho^2 \sin\varphi\cos\varphi$, where φ is kept fixed. This gives

$$\int_{1/\sqrt{\sin\varphi\cos\varphi}}^{2/\sqrt{\sin\varphi\cos\varphi}} \varrho^2 \cos^2\varphi \cdot \exp\left(\varrho^2 \sin\varphi\cos\varphi\right) \ln(\tan\varphi)\varrho\,d\varrho$$

$$= \frac{1}{2}\,\frac{\ln(\tan\varphi)}{\sin^2\varphi}\int_1^2 t\,e^t\,dt = \frac{1}{2}\,\frac{\ln(\tan\varphi)}{\sin^2\varphi}\left[t\,e^t - e^t\right]_1^2 = \frac{e^2}{2}\,\frac{\ln(\tan\varphi)}{\sin^2\varphi}.$$

When this result is put into (20.7), it follows, when we use the substitution $u = \tan\varphi$ that

$$\begin{aligned}
\int_B x^2\exp(xy)\ln\left(\frac{y}{x}\right)\,dS &= \frac{e^2}{2}\int_{\text{Arctan }1}^{\text{Arctan }4} \frac{\ln(\tan\varphi)}{\sin^2\varphi}\,d\varphi \\
&= \frac{e^2}{2}\int_{\text{Arccot }1}^{\text{Arccot }\frac{1}{4}} (+\ln(\cot\varphi))\cdot\frac{-1}{\sin^2\varphi}\,d\varphi \\
&= \frac{e^2}{2}\int_1^{\frac{1}{4}} \ln u\,du = \frac{e^2}{2}\left[u\ln u - u\right]_1^{\frac{1}{4}} \\
&= \frac{e^2}{2}\left(\frac{1}{4}\cdot 2\ln\frac{1}{2} - \frac{1}{4} + 1\right) = \frac{e^2}{8}(3 - 2\ln 2).
\end{aligned}$$

MAPLE makes a mess here, which we shall not show.

ALTERNATIVELY one may introduce the new variables

$$(u,v) = \left(xy, \frac{y}{x}\right).$$

This transformation is considered in all details in Example 29.4, so we shall just mention the main points, namely

$$D = \{(u,v) \mid 1 \le u \le 2,\, 1 \le v \le 4\} = [1,2]\times[1,4],$$

and

$$x(u,v) = \sqrt{\frac{u}{v}} \qquad \text{and} \qquad y(u,v) = \sqrt{uv},$$

and that the Jacobian is $\dfrac{1}{2v}$.
By the transformation of the plane integral

$$\int_B x^2\exp(xy)\ln\left(\frac{y}{x}\right)\,dS = \int_D \frac{u}{v}\cdot e^u \ln v\cdot\frac{1}{2v}\,du\,dv = \frac{1}{2}\int_1^2 ue^u\,du\cdot\int_1^4 \frac{1}{v^2}\ln v\,dv$$

$$= \frac{1}{2}\left[ue^u - e\right]_1^2\cdot\left[-\frac{\ln v}{v} - \frac{1}{v}\right]_1^4 = \frac{1}{2}e^2\left\{1 - \frac{\ln 4}{4} - \frac{1}{4}\right\} = \frac{e^2}{8}(3 - 2\ln 2),$$

which is far easier than the method above. $\Diamond$

Example 20.31 *Find the domain D of the function*

$$f(x,y)=\sqrt{a^2-x^2-y^2},$$

where a is a positive constant. Then compute the plane integral

$$\int_D \{f(x,y)\}^2\,\mathrm{d}x\,\mathrm{d}y.$$

A Domain of a function, plane integral.

D Analyze f. Compute the plane integral by using polar coordinates.

I It follows immediately that

$$D=\{(x,y)\mid x^2+y^2\le a^2\}=\overline{K}(\mathbf{0};a),$$

and

$$\int_D \{f(x,y)\}^2\,\mathrm{d}x\,\mathrm{d}y=\int_{\overline{K}(\mathbf{0};a)}\{a^2-x^2-y^2\}\,\mathrm{d}x\,\mathrm{d}y$$
$$= a^2\cdot\mathrm{area}(\overline{K}(\mathbf{0};a))-2\pi\int_0^a \varrho^2\cdot\varrho\,\mathrm{d}\varrho=a^2\cdot\pi a^2-2\pi\cdot\frac{a^4}{4}=\frac{\pi}{2}\,a^4.$$

Example 20.32 *Compute the plane integral*

$$\int_B yx^2\,\mathrm{d}S,$$

where B er is the quarter disc given by the inequalities

$$1\le x,\qquad 0\le y,\qquad x^2+y^2\le 2x.$$

A Plane integral.

D There are at least three different solutions:

1) Reduction in rectangular coordinates.
2) Reduction in polar coordinates.
3) Reduction in a translated polar coordinate system.

I First method. *Reduction in rectangular coordinates.*
The set B is described in rectangular coordinates by

$$B=\{(x,y)\mid 0\le y\le\sqrt{2x-x^2},\;x\in[1,2]\}.$$

Hence

$$\int_B yx^2\,\mathrm{d}S = \int_1^2\left\{\int_0^{\sqrt{2x-x^2}} yx^2\,\mathrm{d}y\right\}\mathrm{d}x=\frac{1}{2}\int_1^2 x^2\{2x-x^2\}\,\mathrm{d}x=\frac{1}{2}\int_1^2\left\{2x^3-x^4\right\}\,\mathrm{d}x$$
$$=\frac{1}{2}\left[\frac{x^4}{2}-\frac{x^5}{5}\right]_1^2=\frac{1}{4}\{16-1\}-\frac{1}{10}\{32-1\}=\frac{75-62}{20}=\frac{13}{20}.$$

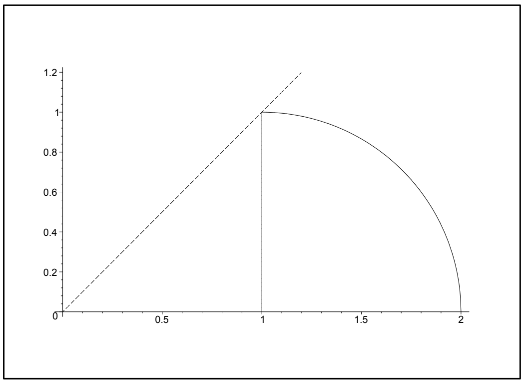


Figure 20.64: The quarter disc B.

Second method. *Reduction in polar coordinates.*

It follows from the figure that every point in B lies in the angular space $\varphi \in \left[0, \frac{\pi}{4}\right]$ (den dotted oblique line). We get the lower ϱ-limit from $a \le x = \varrho \cos\varphi$,

$$\frac{1}{\cos\varphi} \le \varrho.$$

From $\varrho^2 = x^2 + y^2 \le 2x = 2\varrho\cos\varphi$ we get the upper ϱ-limit $\varrho \le 2\cos\varphi$.
Summarizing, B is described in polar coordinates by

$$\left\{(\varrho,\varphi) \;\middle|\; \frac{1}{\cos\varphi} \le \varrho \le 2\cos\varphi,\, \varphi \in \left[0,\frac{\pi}{4}\right]\right\}.$$

Hence by reduction in polar coordinates,

$$\begin{aligned}
\int_B yx^2\,\mathrm{d}S &= \int_0^{\frac{\pi}{4}} \left\{\int_{\frac{1}{\cos\varphi}}^{2} \cos\varphi\,\varrho\sin\varphi\cdot\{\varrho\cos\varphi\}^2\cdot\varrho\,\mathrm{d}\varrho\right\}\mathrm{d}\varphi \\
&= \int_0^{\frac{\pi}{4}} \sin\varphi\cdot\cos^2\varphi\left\{\int_{\frac{1}{\cos\varphi}}^{2\cos\varphi}\varrho^4\,\mathrm{d}\varrho\right\}\mathrm{d}\varphi = \int_0^{\frac{\pi}{4}}\sin\varphi\cdot\cos^2\varphi\left[\frac{1}{5}\varrho^5\right]_{\frac{1}{\cos\varphi}}^{2\cos\varphi}\mathrm{d}\varphi \\
&= \frac{1}{5}\int_0^{\frac{\pi}{4}}\left\{32\cos^7\varphi - \frac{1}{\cos^3\varphi}\right\}\sin\varphi\,\mathrm{d}\varphi = \frac{1}{5}\left[-32\cdot\frac{1}{8}\cos^8\varphi - \frac{1}{2}\cdot\frac{1}{\cos^2\varphi}\right]_0^{\frac{\pi}{4}} \\
&= \frac{1}{5}\left\{4\left(-\cos^8\frac{\pi}{4}+1\right)+\frac{1}{2}\left(-\frac{1}{\cos^2\frac{\pi}{2}}+1\right)\right\} \\
&= \frac{1}{5}\left\{4\left(-\frac{1}{16}+1\right)+\frac{1}{2}(-2+1)\right\} = \frac{1}{5}\left\{\frac{15}{4}-\frac{1}{2}\right\} = \frac{1}{5}\cdot\frac{13}{4} = \frac{13}{20}.
\end{aligned}$$

Third method. *Translated polar coordinate system.*
As $x^2+y^2 \le 2x$ can also be written $(x-1)^2+y^2 \le 1$, the set B can be described by

$$\left\{(x,y) \;\middle|\; x = 1+\varrho\cos\varphi,\, y = \varrho\sin\varphi,\, \varrho\in[0,1],\, \varphi\in\left[0,\frac{\pi}{2}\right]\right\},$$

where the pole lies in $(x,y) = (1,0)$. Then we get the plane integral

$$\begin{aligned}
\int_B yx^2\,\mathrm{d}S &= \int_0^{\frac{\pi}{2}}\left\{\int_0^1 \varrho\sin\varphi\cdot\{1+\varrho\cos\varphi\}^2\varrho\,\mathrm{d}\varrho\right\}\mathrm{d}\varphi \\
&= \int_0^{\frac{\pi}{2}}\left\{\int_0^1 \varrho^2\left\{1+2\varrho\cos\varphi+\varrho^2\cos^2\varphi\right\}\mathrm{d}\varrho\right\}\sin\varphi\,\mathrm{d}\varphi \\
&= \int_0^{\frac{\pi}{2}}\left[\frac{\varrho^3}{3}+\frac{\varrho^4}{2}\cos\varphi+\frac{\varrho^5}{5}\cos^2\varphi\right]_{\varrho=0}^1\sin\varphi\,\mathrm{d}\varphi \\
&= \int_0^{\frac{\pi}{2}}\left\{\frac{1}{3}+\frac{1}{2}\cos\varphi+\frac{1}{5}\cos^2\varphi\right\}\sin\varphi\,\mathrm{d}\varphi \\
&= \left[-\frac{1}{3}\cos\varphi-\frac{1}{4}\cos^2\varphi-\frac{1}{15}\cos^3\varphi\right]_0^{\frac{\pi}{2}} \\
&= \frac{1}{3}+\frac{1}{4}+\frac{1}{15} = \frac{1}{4}+\frac{6}{15} = \frac{1}{4}+\frac{2}{5} = \frac{13}{20}.
\end{aligned}$$

MAPLE. For completeness we add the commands in MAPLE,
with(Student[MultivariateCalculus]):

$$\text{MultiInt}\left(y\cdot x^2, y = 0..\sqrt{2x-x^2}, x = 1..2\right)$$

$$\frac{13}{20}$$

20.8 Examples of area in polar coordinates

Example 20.33 *Let A be the plane point set which in polar coordinates is bounded by the inequalities*

$$-\pi \le \varphi \le \pi, \qquad 0 \le \varrho \le 1+\cos\varphi;$$

the boundary curve ∂A is a cardioid. Let B be the disc which is bounded by $0 \le \varrho \le 1$. Find the area of the intersection $A \cap B$.

A Area of a set given in polar coordinates.

D Sketch the boundary curves. Then set up the integrals of the area and compute.

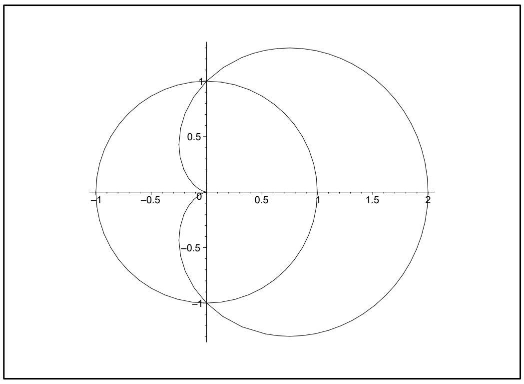


Figure 20.65: The intersection of the unit disc and the cardioid.

I By examining the figure we set up the formula of the area where we have a half disc in the right half plane,

$$\begin{aligned}
\text{area}(A\cap B) &= \frac{1}{2}\,\pi\cdot 1^2 + 2\int_{\frac{\pi}{2}}^{\pi}\left\{\int_0^{1+\cos\varphi}\varrho\,d\varrho\right\}d\varphi = \frac{\pi}{2}+2\int_{\frac{\pi}{2}}^{\pi}\frac{1}{2}\,(1+\cos\varphi)^2\,d\varphi \\
&= \frac{\pi}{2}+\int_{\frac{\pi}{2}}^{\pi}\left\{1+2\cos\varphi+\frac{1}{2}\,(1+\cos 2\varphi)\right\}d\varphi \\
&= \frac{\pi}{2}+\frac{3\pi}{2}\cdot\frac{1}{2}+[2\sin\varphi]_{\frac{\pi}{2}}^{\pi}+\frac{1}{4}\,[\sin 2\varphi]_{\frac{\pi}{2}}^{\pi} = \frac{5\pi}{4}-2.
\end{aligned}$$

MAPLE. We add the commands in MAPLE,

with(Student[MultivariateCalculus]):

$$\frac{\pi}{2}+2\cdot\text{MultiInt}\left(r, r=0..1+\cos(t), t=\frac{\pi}{2}..\pi\right)$$

$$\frac{5}{4}\,\pi-2$$

Example 20.34 *In each of the following cases a plane and bounded point set B is given by the boundary curve ∂B given in polar coordinates. Sketch B and find the area of B.*

1) The cardiod,

$$\varrho = a(1+\cos\varphi), \qquad \varphi \in [-\pi,\pi].$$

2) (A part of) Descartes's leaf,

$$\varrho = \frac{3a\sin\varphi\cos\varphi}{\sin^3\varphi+\cos^3\varphi}, \qquad \varphi \in \left[0,\frac{\pi}{2}\right].$$

3) (Part of) Maclaurin's trisectrix,

$$\varrho = 4a\cdot\cos\varphi - \frac{1}{\cos\varphi}, \qquad \varphi \in \left[-\frac{\pi}{3},\frac{\pi}{3}\right].$$

A Sketches of curves given in polar coordinates. Area by a plane integral.

D Sketch the boundary curve. Then apply the theorem of reduction.

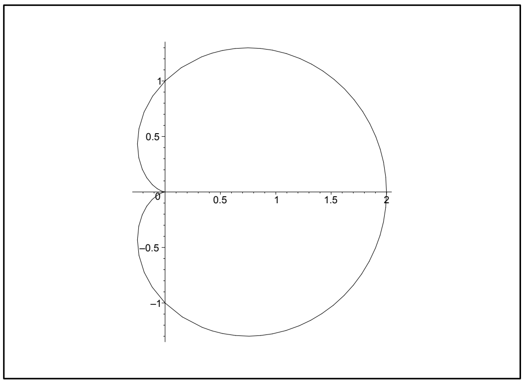


Figure 20.66: The cardioid.

I 1) Cardioid, from Greek "$\eta\ \kappa\alpha\varrho\delta\iota\alpha$ = the heart", because the curve has the shape of a heart. The area is given by

$$\begin{aligned}\int_B \mathrm{d}S &= \int_{-\pi}^{\pi}\left\{\int_0^{a(1+\cos\varphi)}\varrho\,\mathrm{d}\varrho\right\}\mathrm{d}\varphi = \int_{-\pi}^{\pi}\frac{1}{2}a^2(1+\cos\varphi)^2\,\mathrm{d}\varphi\\ &= \frac{1}{2}a^2\int_{-\pi}^{\pi}\left\{1+2\cos\varphi+\frac{1+\cos 2\varphi}{2}\right\}\mathrm{d}\varphi = \frac{1}{2}a^2\cdot\frac{3}{2}\cdot 2\pi = \frac{3}{2}a^2\pi.\end{aligned}$$

MAPLE. We add the commands in MAPLE,
with(Student[MultivariateCalculus]):

MultiInt $(r, r = 0..a\cdot(1+\cos(t)), t = -\pi..\pi)$

$$\frac{3}{2}\pi a^2$$

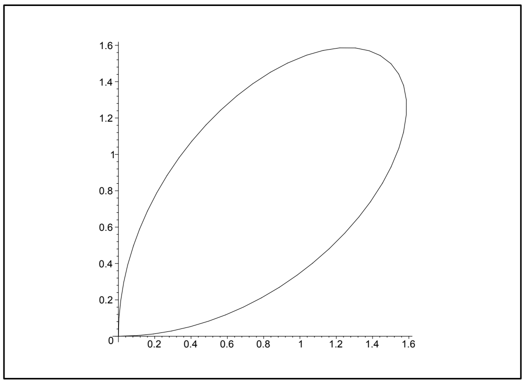


Figure 20.67: Part of Descartes's leaf.

2) The area is here computed in the following way

$$\begin{aligned}
\int_B \mathrm{d}S &= \int_0^{\frac{\pi}{2}} \left\{ \int_0^{\frac{3a\sin\varphi\cos\varphi}{\sin^3\varphi+\cos^3\varphi}} \varrho \,\mathrm{d}\varrho \right\} \mathrm{d}\varphi = \frac{1}{2}\int_0^{\frac{\pi}{2}} 9a^2 \cdot \frac{\sin^2\varphi\cos^2\varphi}{(\sin^3\varphi+\cos^3\varphi)^2}\,\mathrm{d}\varphi \\
&= \frac{9}{2}a^2 \int_0^{\frac{\pi}{2}} \frac{\tan^2\varphi\cdot\cos^4\varphi}{\cos^6\varphi(1+\tan^3\varphi)^2}\,\mathrm{d}\varphi = \frac{9a^2}{2}\int_{u=\tan\varphi=0}^{+\infty} \frac{u^2}{(1+u^3)^2}\,\mathrm{d}u \\
&= \frac{3}{2}a^2\left[-\frac{1}{1+u^3}\right]_0^{+\infty} = \frac{3}{2}a^2.
\end{aligned}$$

MAPLE. We add the commands in MAPLE,
with(Student[MultivariateCalculus]):

$$\frac{\pi}{2} + 2 \cdot \text{MultiInt}\left(r, r = 0..\frac{3a \cdot \sin(t) \cdot \cos(t)}{\sin(t)^3 + \cos(t)^3}, t = 0..\pi\right)$$

$$\frac{3}{2}\, a^2$$

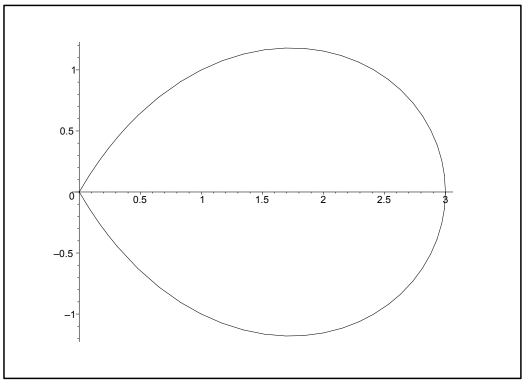


Figure 20.68: A part of Maclaurin's trisectrix.

3) By the usual reduction the area is here computed in the following way,

$$\begin{aligned}\int_B \mathrm{d}S &= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2}\left\{\frac{a}{\cos\varphi} - 4a\cos\varphi\right\}^2 \mathrm{d}\varphi = \frac{a^2}{2}\cdot 2\int_0^{\frac{\pi}{3}}\left\{\frac{1}{\cos^2\varphi} - 8 + 8 + 8\cos 2\varphi\right\}\mathrm{d}\varphi \\ &= a^2[\tan t + 4\sin 2\varphi]_0^{\frac{\pi}{3}} = a^2\left\{\tan\frac{\pi}{3} + 4\sin\frac{2\pi}{3}\right\} = a^2\left(\sqrt{3} + 4\cdot\frac{\sqrt{3}}{2}\right) = 3\sqrt{3}\,a^2.\end{aligned}$$

Example 20.35 *Find the area of the plane domain B, which is bounded by (i) a part of* Archimedes's spiral *given in polar coordinates by*

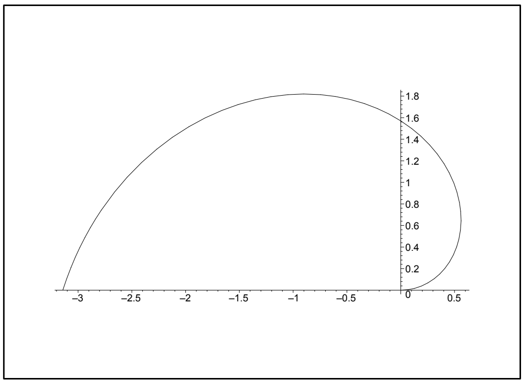
$$\varrho = a\varphi, \qquad \varphi \in [0, \pi],$$

and (ii) the part of the negative X-axis given by

$$(y = 0 \text{ and } x \in [-\pi a, 0]), \qquad \text{or} \qquad (\varphi = \pi \text{ and } \varrho \in [0, \pi a]).$$

A Area in polar coordinates.

D Sketch the domain; compute the area by reduction in polar coordinates.



I The area is

$$\int_B \mathrm{d}S = \int_0^{\pi} \left\{ \int_0^{a\varphi} \varrho \, \mathrm{d}\varrho \right\} \mathrm{d}\varphi = \int_0^{\pi} \frac{1}{2} a^2 \varphi^2 \, \mathrm{d}\varphi = \frac{1}{6} a^2 \pi^3.$$

21 Formulæ

Some of the following formulæ can be assumed to be known from high school. It is highly recommended that one *learns most of these formulæ in this appendix by heart.*

21.1 Squares etc.

The following simple formulæ occur very frequently in the most different situations.

$$
\begin{array}{ll}
(a+b)^2 = a^2+b^2+2ab, & a^2+b^2+2ab = (a+b)^2, \\
(a-b)^2 = a^2+b^2-2ab, & a^2+b^2-2ab = (a-b)^2, \\
(a+b)(a-b) = a^2-b^2, & a^2-b^2 = (a+b)(a-b), \\
(a+b)^2 = (a-b)^2+4ab, & (a-b)^2 = (a+b)^2-4ab.
\end{array}
$$

21.2 Powers etc.

Logarithm:

$$
\begin{array}{rcll}
\ln|xy| &=& \ln|x|+\ln|y|, & x,y\neq 0, \\
\ln\left|\dfrac{x}{y}\right| &=& \ln|x|-\ln|y|, & x,y\neq 0, \\
\ln|x^r| &=& r\ln|x|, & x\neq 0.
\end{array}
$$

Power function, fixed exponent:

$$(xy)^r = x^r\cdot y^r, x,y>0 \quad \text{(extensions for some } r\text{)},$$

$$\left(\frac{x}{y}\right)^r = \frac{x^r}{y^r}, x,y>0 \quad \text{(extensions for some } r\text{)}.$$

Exponential, fixed base:

$$a^x\cdot a^y = a^{x+y}, \quad a>0 \quad \text{(extensions for some } x,\, y\text{)},$$
$$(a^x)^y = a^{xy}, a>0 \quad \text{(extensions for some } x,\, y\text{)},$$

$$a^{-x} = \frac{1}{a^x}, a>0, \quad \text{(extensions for some } x\text{)},$$

$$\sqrt[n]{a} = a^{1/n}, a\geq 0, \qquad n\in\mathbb{N}.$$

Square root:

$$\sqrt{x^2} = |x|, \qquad x\in\mathbb{R}.$$

Remark 21.1 It happens quite frequently that students make errors when they try to apply these rules. They must be mastered! In particular, as one of my friends once put it: "If you can master the square root, you can master everything in mathematics!" Notice that this innocent looking square root is one of the most difficult operations in Calculus. Do not forget the *absolute value!* ◊

21.3 Differentiation

Here are given the well-known rules of differentiation together with some rearrangements which sometimes may be easier to use:

$$\{f(x) \pm g(x)\}' = f'(x) \pm g'(x),$$

$$\{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x) = f(x)g(x)\left\{\frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}\right\},$$

where the latter rearrangement presupposes that $f(x) \neq 0$ and $g(x) \neq 0$.
If $g(x) \neq 0$, we get the usual formula known from high school

$$\left\{\frac{f(x)}{g(x)}\right\}' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

It is often more convenient to compute this expression in the following way:

$$\left\{\frac{f(x)}{g(x)}\right\} = \frac{d}{dx}\left\{f(x) \cdot \frac{1}{g(x)}\right\} = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2} = \frac{f(x)}{g(x)}\left\{\frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}\right\},$$

where the former expression often is *much easier* to use in practice than the usual formula from high school, and where the latter expression again presupposes that $f(x) \neq 0$ and $g(x) \neq 0$. Under these assumptions we see that the formulæ above can be written

$$\frac{\{f(x)g(x)\}'}{f(x)g(x)} = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)},$$

$$\frac{\{f(x)/g(x)\}'}{f(x)/g(x)} = \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}.$$

Since

$$\frac{d}{dx}\ln|f(x)| = \frac{f'(x)}{f(x)}, \qquad f(x) \neq 0,$$

we also name these the *logarithmic derivatives*.

Finally, we mention the rule of **differentiation of a composite function**

$$\{f(\varphi(x))\}' = f'(\varphi(x)) \cdot \varphi'(x).$$

We first differentiate the function itself; then the insides. This rule is a 1-dimensional version of the so-called *Chain rule*.

21.4 Special derivatives.

Power like:

$$\frac{d}{dx}(x^\alpha) = \alpha \cdot x^{\alpha-1}, \qquad \text{for } x > 0, \ \text{(extensions for some } \alpha).$$

$$\frac{d}{dx}\ln|x| = \frac{1}{x}, \qquad \text{for } x \neq 0.$$

Exponential like:

$$\frac{d}{dx}\exp x = \exp x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\left(a^x\right) = \ln a \cdot a^x, \qquad \text{for } x \in \mathbb{R} \text{ and } a > 0.$$

Trigonometric:

$$\frac{d}{dx}\sin x = \cos x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\cos x = -\sin x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\tan x = 1 + \tan^2 x = \frac{1}{\cos^2 x}, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, p \in \mathbb{Z},$$

$$\frac{d}{dx}\cot x = -\left(1 + \cot^2 x\right) = -\frac{1}{\sin^2 x}, \qquad \text{for } x \neq p\pi, p \in \mathbb{Z}.$$

Hyperbolic:

$$\frac{d}{dx}\sinh x = \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\cosh x = \sinh x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\tanh x = 1 - \tanh^2 x = \frac{1}{\cosh^2 x}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\coth x = 1 - \coth^2 x = -\frac{1}{\sinh^2 x}, \qquad \text{for } x \neq 0.$$

Inverse trigonometric:

$$\frac{d}{dx}\text{ Arcsin } x = \frac{1}{\sqrt{1-x^2}}, \qquad \text{for } x \in\,]-1,1\,[,$$

$$\frac{d}{dx}\text{ Arccos } x = -\frac{1}{\sqrt{1-x^2}}, \qquad \text{for } x \in\,]-1,1\,[,$$

$$\frac{d}{dx}\text{ Arctan } x = \frac{1}{1+x^2}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\text{ Arccot } x = \frac{1}{1+x^2}, \qquad \text{for } x \in \mathbb{R}.$$

Inverse hyperbolic:

$$\frac{d}{dx}\text{ Arsinh } x = \frac{1}{\sqrt{x^2+1}}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\text{ Arcosh } x = \frac{1}{\sqrt{x^2-1}}, \qquad \text{for } x \in\,]\,1,+\infty\,[,$$

$$\frac{d}{dx}\text{ Artanh } x = \frac{1}{1-x^2}, \qquad \text{for } |x| < 1,$$

$$\frac{d}{dx}\text{ Arcoth } x = \frac{1}{1-x^2}, \qquad \text{for } |x| > 1.$$

Remark 21.2 The derivative of the trigonometric and the hyperbolic functions are to some extent exponential like. The derivatives of the inverse trigonometric and inverse hyperbolic functions are power like, because we include the logarithm in this class. ◊

21.5 Integration

The most obvious rules are dealing with linearity

$$\int \{f(x)+\lambda g(x)\}\,dx = \int f(x)\,dx + \lambda \int g(x)\,dx, \qquad \text{where } \lambda \in \mathbb{R} \text{ is a constant,}$$

and with the fact that differentiation and integration are "inverses to each other", i.e. modulo some arbitrary constant $c \in \mathbb{R}$, which often tacitly is missing,

$$\int f'(x)\,dx = f(x).$$

If we in the latter formula replace $f(x)$ by the product $f(x)g(x)$, we get by reading from the right to the left and then differentiating the product,

$$f(x)g(x) = \int \{f(x)g(x)\}'\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx.$$

Hence, by a rearrangement

The rule of partial integration:

$$\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx.$$

The differentiation is moved from one factor of the integrand to the other one by changing the sign and adding the term $f(x)g(x)$.

Remark 21.3 This technique was earlier used a lot, but is almost forgotten these days. It must be revived, because MAPLE and pocket calculators apparently do not know it. It is possible to construct examples where these devices cannot give the exact solution, unless you first perform a partial integration yourself. $\Diamond$

Remark 21.4 This method can also be used when we estimate integrals which cannot be directly calculated, because the antiderivative is not contained in e.g. the catalogue of MAPLE. The idea is by a succession of partial integrations to make the new integrand smaller. $\Diamond$

Integration by substitution:

If the integrand has the special structure $f(\varphi(x))\cdot\varphi'(x)$, then one can change the variable to $y=\varphi(x)$:

$$\int f(\varphi(x))\cdot\varphi'(x)\,dx = \text{“}\int f(\varphi(x))\,d\varphi(x)\text{''} = \int_{y=\varphi(x)} f(y)\,dy.$$

Integration by a monotonous substitution:

If $\varphi(y)$ is a *monotonous* function, which maps the y-interval *one-to-one* onto the x-interval, then

$$\int f(x)\,dx = \int_{y=\varphi^{-1}(x)} f(\varphi(y))\varphi'(y)\,dy.$$

Remark 21.5 This rule is usually used when we have some "ugly" term in the integrand $f(x)$. The idea is to put this ugly term equal to $y=\varphi^{-1}(x)$. When e.g. x occurs in $f(x)$ in the form $\sqrt{x}$, we put $y=\varphi^{-1}(x)=\sqrt{x}$, hence $x=\varphi(y)=y^2$ and $\varphi'(y)=2y$. $\Diamond$

21.6 Special antiderivatives

Power like:

$$\int \frac{1}{x}\,dx = \ln|x|, \qquad \text{for } x \neq 0. \quad \text{(Do not forget the numerical value!)}$$

$$\int x^{\alpha}\,dx = \frac{1}{\alpha+1}x^{\alpha+1}, \qquad \text{for } \alpha \neq -1,$$

$$\int \frac{1}{1+x^2}\,dx = \text{Arctan}\,x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{1-x^2}\,dx = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|, \qquad \text{for } x \neq \pm 1,$$

$$\int \frac{1}{1-x^2}\,dx = \text{Artanh}\,x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{1-x^2}\,dx = \text{Arcoth}\,x, \qquad \text{for } |x| > 1,$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \text{Arcsin}\,x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = -\,\text{Arccos}\,x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \text{Arsinh}\,x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \ln(x+\sqrt{x^2+1}), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{x}{\sqrt{x^2-1}}\,dx = \sqrt{x^2-1}, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \text{Arcosh}\,x, \qquad \text{for } x > 1,$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \ln|x+\sqrt{x^2-1}|, \qquad \text{for } x > 1 \text{ eller } x < -1.$$

There is an error in the programs of the pocket calculators TI-92 and TI-89. The *numerical signs* are missing. It is obvious that $\sqrt{x^2-1} < |x|$ so if $x<-1$, then $x+\sqrt{x^2-1}<0$. Since you cannot take the logarithm of a negative number, these pocket calculators will give an error message.

Exponential like:

$$\int \exp x \, dx = \exp x, \qquad \text{for } x \in \mathbb{R},$$

$$\int a^x \, dx = \frac{1}{\ln a} \cdot a^x, \qquad \text{for } x \in \mathbb{R}, \text{ and } a > 0, a \neq 1.$$

Trigonometric:

$$\int \sin x \, dx = -\cos x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \cos x \, dx = \sin x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \tan x \, dx = -\ln|\cos x|, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \cot x \, dx = \ln|\sin x|, \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos x} \, dx = \frac{1}{2} \ln\left(\frac{1+\sin x}{1-\sin x}\right), \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin x} \, dx = \frac{1}{2} \ln\left(\frac{1-\cos x}{1+\cos x}\right), \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos^2 x} \, dx = \tan x, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin^2 x} \, dx = -\cot x, \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z}.$$

Hyperbolic:

$$\int \sinh x \, dx = \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \cosh x \, dx = \sinh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \tanh x \, dx = \ln \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \coth x \, dx = \ln|\sinh x|, \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh x} \, dx = \operatorname{Arctan}(\sinh x), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\cosh x} \, dx = 2 \operatorname{Arctan}(e^x), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh x} \, dx = \frac{1}{2} \ln\left(\frac{\cosh x - 1}{\cosh x + 1}\right), \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\sinh x}\,dx = \ln\left|\frac{e^x-1}{e^x+1}\right|, \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh^2 x}\,dx = \tanh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh^2 x}\,dx = -\coth x, \qquad \text{for } x \neq 0.$$

21.7 Trigonometric formulæ

The trigonometric formulæ are closely connected with circular movements. Thus $(\cos u, \sin u)$ are the coordinates of a point P on the unit circle corresponding to the angle u, cf. figure A.1. This geometrical interpretation is used from time to time.

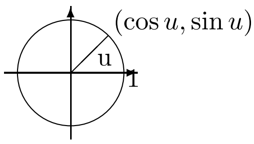


Figure 21.1: The unit circle and the trigonometric functions.

The fundamental trigonometric relation:

$$\cos^2 u + \sin^2 u = 1, \qquad \text{for } u \in \mathbb{R}.$$

Using the previous geometric interpretation this means according to *Pythagoras's theorem*, that the point P with the coordinates $(\cos u, \sin u)$ always has distance 1 from the origo $(0,0)$, i.e. it is lying on the boundary of the circle of centre $(0,0)$ and radius $\sqrt{1} = 1$.

Connection to the complex exponential function:

The *complex exponential* is for imaginary arguments defined by

$$\exp(\mathrm{i}\,u) := \cos u + \mathrm{i}\ \sin u.$$

It can be checked that the usual functional equation for exp is still valid for complex arguments. In other word: The definition above is extremely conveniently chosen.

By using the definition for $\exp(\mathrm{i}\,u)$ and $\exp(-\mathrm{i}\,u)$ it is easily seen that

$$\cos u = \frac{1}{2}(\exp(\mathrm{i}\,u) + \exp(-\mathrm{i}\,u)),$$

$$\sin u = \frac{1}{2i}(\exp(\mathrm{i}\,u) - \exp(-\mathrm{i}\,u)),$$

.

Moivre's formula: We get by expressing $\exp(\mathrm{i}nu)$ in two different ways:

$$\exp(\mathrm{i}nu) = \cos nu + \mathrm{i}\,\sin nu = (\cos u + \mathrm{i}\,\sin u)^n.$$

Example 21.1 If we e.g. put $n = 3$ into Moivre's formula, we obtain the following typical application,

$$\begin{aligned}\cos(3u) + \mathrm{i}\,\sin(3u) &= (\cos u + \mathrm{i}\,\sin u)^3\\ &= \cos^3 u + 3\mathrm{i}\,\cos^2 u\cdot\sin u + 3\mathrm{i}^2\cos u\cdot\sin^2 u + \mathrm{i}^3\sin^3 u\\ &= \{\cos^3 u - 3\cos u\cdot\sin^2 u\} + \mathrm{i}\{3\cos^2 u\cdot\sin u - \sin^3 u\}\\ &= \{4\cos^3 u - 3\cos u\} + \mathrm{i}\{3\sin u - 4\sin^3 u\}\end{aligned}$$

When this is split into the real- and imaginary parts we obtain

$$\cos 3u = 4\cos^3 u - 3\cos u, \qquad \sin 3u = 3\sin u - 4\sin^3 u. \quad \diamondsuit$$

Addition formulæ:

$$\sin(u+v) = \sin u\,\cos v + \cos u\,\sin v,$$

$$\sin(u-v) = \sin u\,\cos v - \cos u\,\sin v,$$

$$\cos(u+v) = \cos u\,\cos v - \sin u\,\sin v,$$

$$\cos(u-v) = \cos u\,\cos v + \sin u\,\sin v.$$

Products of trigonometric functions to a sum:

$$\sin u\,\cos v = \frac{1}{2}\sin(u+v) + \frac{1}{2}\sin(u-v),$$

$$\cos u\,\sin v = \frac{1}{2}\sin(u+v) - \frac{1}{2}\sin(u-v),$$

$$\sin u\,\sin v = \frac{1}{2}\cos(u-v) - \frac{1}{2}\cos(u+v),$$

$$\cos u\,\cos v = \frac{1}{2}\cos(u-v) + \frac{1}{2}\cos(u+v).$$

Sums of trigonometric functions to a product:

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right),$$

$$\sin u - \sin v = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right),$$

$$\cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right),$$

$$\cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right).$$

Formulæ of halving and doubling the angle:

$$\sin 2u = 2\sin u\,\cos u,$$

$$\cos 2u = \cos^2 u - \sin^2 u = 2\,\cos^2 u - 1 = 1 - 2\,\sin^2 u,$$

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}} \qquad \text{followed by a discussion of the sign,}$$

$$\cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}} \qquad \text{followed by a discussion of the sign,}$$

21.8 Hyperbolic formulæ

These are very much like the trigonometric formulæ, and if one knows a little of Complex Function Theory it is realized that they are actually identical. The structure of this section is therefore the same as for the trigonometric formulæ. The reader should compare the two sections concerning similarities and differences.

The fundamental relation:

$$\cosh^2 x - \sinh^2 x = 1.$$

Definitions:

$$\cosh x = \frac{1}{2}\left(\exp(x)+\exp(-x)\right), \qquad \sinh x = \frac{1}{2}\left(\exp(x)-\exp(-x)\right).$$

"Moivre's formula":

$$\exp(x) = \cosh x + \sinh x.$$

This is trivial and only rarely used. It has been included to show the analogy.

Addition formulæ:

$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y),$$

$$\sinh(x-y) = \sinh(x)\cosh(y) - \cosh(x)\sinh(y),$$

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y),$$

$$\cosh(x-y) = \cosh(x)\cosh(y) - \sinh(x)\sinh(y).$$

Formulæ of halving and doubling the argument:

$$\sinh(2x) = 2\sinh(x)\cosh(x),$$

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x) = 2\cosh^2(x) - 1 = 2\sinh^2(x) + 1,$$

$$\sinh\left(\frac{x}{2}\right) = \pm\sqrt{\frac{\cosh(x)-1}{2}} \qquad \text{followed by a discussion of the sign,}$$

$$\cosh\left(\frac{x}{2}\right) = \sqrt{\frac{\cosh(x)+1}{2}}.$$

Inverse hyperbolic functions:

$$\operatorname{Arsinh}(x) = \ln\left(x+\sqrt{x^2+1}\right), \qquad x \in \mathbb{R},$$

$$\operatorname{Arcosh}(x) = \ln\left(x+\sqrt{x^2-1}\right), \qquad x \geq 1,$$

$$\operatorname{Artanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \qquad |x| < 1,$$

$$\operatorname{Arcoth}(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right), \qquad |x| > 1.$$

21.9 Complex transformation formulæ

$$\cos(\mathrm{i}x) = \cosh(x), \qquad \cosh(\mathrm{i}x) = \cos(x),$$

$$\sin(\mathrm{i}x) = \mathrm{i}\sinh(x), \qquad \sinh(\mathrm{i}x) = \mathrm{i}\sin x.$$

21.10 Taylor expansions

The generalized binomial coefficients are defined by

$$\binom{\alpha}{n} := \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{1\cdot 2\cdots n},$$

with n factors in the numerator and the denominator, supplied with

$$\binom{\alpha}{0} := 1.$$

The Taylor expansions for *standard functions* are divided into *power like* (the radius of convergency is finite, i.e. = 1 for the standard series) and*exponential like* (the radius of convergency is infinite).
Power like:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1,$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x| < 1,$$

$$(1+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j, \qquad n \in \mathbb{N}, x \in \mathbb{R},$$

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \qquad \alpha \in \mathbb{R}\setminus\mathbb{N}, |x| < 1,$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n}, \qquad |x| < 1,$$

$$\mathrm{Arctan}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \qquad |x| < 1.$$

Exponential like:

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \qquad x \in \mathbb{R}$$

$$\exp(-x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} x^n, \qquad x \in \mathbb{R}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}, \qquad x \in \mathbb{R},$$

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}, \qquad x \in \mathbb{R},$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}, \qquad x \in \mathbb{R},$$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}, \qquad x \in \mathbb{R}.$$

21.11 Magnitudes of functions

We often have to compare functions for $x \to 0+$, or for $x \to \infty$. The simplest type of functions are therefore arranged in an hierarchy:

1) logarithms,

2) power functions,

3) exponential functions,

4) faculty functions.

When $x \to \infty$, a function from a higher class will always dominate a function form a lower class. More precisely:

A) A *power function* dominates a *logarithm* for $x \to \infty$:

$$\frac{(\ln x)^\beta}{x^\alpha} \to 0 \qquad \text{for } x \to \infty, \quad \alpha,\, \beta > 0.$$

B) An *exponential* dominates a *power function* for $x \to \infty$:

$$\frac{x^\alpha}{a^x} \to 0 \qquad \text{for } x \to \infty, \quad \alpha,\, a > 1.$$

C) The *faculty function* dominates an *exponential* for $n \to \infty$:

$$\frac{a^n}{n!} \to 0, \qquad n \to \infty, \quad n \in \mathbb{N}, \quad a > 0.$$

D) When $x \to 0+$ we also have that a *power function* dominates the *logarithm:*

$$x^\alpha \ln x \to 0-, \qquad \text{for } x \to 0+, \quad \alpha > 0.$$

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