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Real Functions in Several Variables: Volume IV

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Leif Mejlbro

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Contents

Preface

The topic of this series of books on *"Real Functions in Several Variables"* is very important in the description in e.g. *Mechanics* of the real 3-dimensional world that we live in. Therefore, we start from the very beginning, modelling this world by using the coordinates of $\mathbb{R}^3$ to describe e.g. a motion in space. There is, however, absolutely no reason to restrict ourselves to $\mathbb{R}^3$ alone. Some motions may be rectilinear, so only $\mathbb{R}$ is needed to describe their movements on a line segment. This opens up for also dealing with $\mathbb{R}^2$, when we consider plane motions. In more elaborate problems we need higher dimensional spaces. This may be the case in *Probability Theory* and *Statistics*. Therefore, we shall in general use $\mathbb{R}^n$ as our abstract model, and then restrict ourselves in examples mainly to $\mathbb{R}^2$ and $\mathbb{R}^3$.

For rectilinear motions the familiar *rectangular coordinate system* is the most convenient one to apply. However, as known from e.g. *Mechanics*, circular motions are also very important in the applications in engineering. It becomes natural alternatively to apply in $\mathbb{R}^2$ the so-called *polar coordinates* in the plane. They are convenient to describe a circle, where the rectangular coordinates usually give some nasty square roots, which are difficult to handle in practice.

Rectangular coordinates and polar coordinates are designed to model each their problems. They supplement each other, so difficult computations in one of these coordinate systems may be easy, and even trivial, in the other one. It is therefore important always in advance carefully to analyze the geometry of e.g. a domain, so we ask the question: Is this domain best described in rectangular or in polar coordinates?

Sometimes one may split a problem into two subproblems, where we apply rectangular coordinates in one of them and polar coordinates in the other one.

It should be mentioned that in *real life* (though not in these books) one cannot always split a problem into two subproblems as above. Then one is really in trouble, and more advanced mathematical methods should be applied instead. This is, however, outside the scope of the present series of books.

The idea of polar coordinates can be extended in two ways to $\mathbb{R}^3$. Either to *semi-polar* or *cylindric coordinates*, which are designed to describe a cylinder, or to *spherical coordinates*, which are excellent for describing spheres, where rectangular coordinates usually are doomed to fail. We use them already in daily life, when we specify a place on Earth by its longitude and latitude! It would be very awkward in this case to use rectangular coordinates instead, even if it is possible.

Concerning the contents, we begin this investigation by modelling point sets in an n-dimensional Euclidean space E^n by $\mathbb{R}^n$. There is a subtle difference between E^n and $\mathbb{R}^n$, although we often identify these two spaces. In E^n we use *geometrical methods* without a coordinate system, so the objects are independent of such a choice. In the coordinate space $\mathbb{R}^n$ we can use ordinary calculus, which in principle is not possible in E^n. In order to stress this point, we call E^n the "abstract space" (in the sense of calculus; not in the sense of geometry) as a warning to the reader. Also, whenever necessary, we use the colour black in the "abstract space", in order to stress that this expression is theoretical, while variables given in a chosen coordinate system and their related concepts are given the colours blue, red and green.

We also include the most basic of what mathematicians call *Topology*, which will be necessary in the following. We describe what we need by a function.

Then we proceed with limits and continuity of functions and define continuous curves and surfaces, with parameters from subsets of $\mathbb{R}$ and $\mathbb{R}^2$, resp..

Continue with (partial) differentiable functions, curves and surfaces, the chain rule and Taylor's formula for functions in several variables.

We deal with maxima and minima and extrema of functions in several variables over a domain in $\mathbb{R}^n$. This is a very important subject, so there are given many worked examples to illustrate the theory.

Then we turn to the problems of integration, where we specify four different types with increasing complexity, *plane integral, space integral, curve (or line) integral and surface integral*.

Finally, we consider *vector analysis*, where we deal with vector fields, Gauß's theorem and Stokes's theorem. All these subjects are very important in theoretical Physics.

The structure of this series of books is that each subject is *usually* (but not always) described by three successive chapters. In the first chapter a brief theoretical theory is given. The next chapter gives some practical guidelines of how to solve problems connected with the subject under consideration. Finally, some worked out examples are given, in many cases in several variants, because the standard solution method is seldom the only way, and it may even be clumsy compared with other possibilities.

I have as far as possible structured the examples according to the following scheme:

A *Awareness*, i.e. a short description of what is the problem.

D *Decision*, i.e. a reflection over what should be done with the problem.

I *Implementation*, i.e. where all the calculations are made.

C *Control*, i.e. a test of the result.

This is an ideal form of a general procedure of solution. It can be used in any situation and it is not linked to Mathematics alone. I learned it many years ago in the Theory of Telecommunication in a situation which did not contain Mathematics at all. The student is recommended to use it also in other disciplines.

From high school one is used to immediately to proceed to **I**. *Implementation*. However, examples and problems at university level, let alone situations in real life, are often so complicated that it in general will be a good investment also to spend some time on the first two points above in order to be absolutely certain of what to do in a particular case. Note that the first three points, **ADI**, can always be executed.

This is unfortunately not the case with **C** *Control*, because it from now on may be difficult, if possible, to check one's solution. It is only an extra securing whenever it is possible, but we cannot include it always in our solution form above.

I shall on purpose not use the logical signs. These should in general be avoided in Calculus as a shorthand, because they are often (too often, I would say) misused. Instead of $\wedge$ I shall either write "and", or a comma, and instead of $\vee$ I shall write "or". The arrows $\Rightarrow$ and $\Leftrightarrow$ are in particular misunderstood by the students, so they should be totally avoided. They are not telegram short hands, and from a logical point of view they usually do not make sense at all! Instead, write in a plain language what you mean or want to do. This is difficult in the beginning, but after some practice it becomes routine, and it will give more precise information.

When we deal with multiple integrals, one of the possible pedagogical ways of solving problems has been to colour variables, integrals and upper and lower bounds in blue, red and green, so the reader by the colour code can see in each integral what is the variable, and what are the parameters, which do not enter the integration under consideration. We shall of course build up a hierarchy of these colours, so the order of integration will always be defined. As already mentioned above we reserve the colour black for the theoretical expressions, where we cannot use ordinary calculus, because the symbols are only shorthand for a concept.

The author has been very grateful to his old friend and colleague, the late Per Wennerberg Karlsson, for many discussions of how to present these difficult topics on real functions in several variables, and for his permission to use his textbook as a template of this present series. Nevertheless, the author has felt it necessary to make quite a few changes compared with the old textbook, because we did not always agree, and some of the topics could also be explained in another way, and then of course the results of our discussions have here been put in writing for the first time.

The author also adds some calculations in MAPLE, which interact nicely with the theoretic text. Note, however, that when one applies MAPLE, one is forced first to make a geometrical analysis of the domain of integration, i.e. apply some of the techniques developed in the present books.

The theory and methods of these volumes on "Real Functions in Several Variables" are applied constantly in higher Mathematics, Mechanics and Engineering Sciences. It is of paramount importance for the calculations in *Probability Theory*, where one constantly integrate over some point set in space.

It is my hope that this text, these guidelines and these examples, of which many are treated in more ways to show that the solutions procedures are not unique, may be of some inspiration for the students who have just started their studies at the universities.

Finally, even if I have tried to write as careful as possible, I doubt that all errors have been removed. I hope that the reader will forgive me the unavoidable errors.

Leif Mejlbro
March 21, 2015

Introduction to volume IV, Curves and Surfaces

This is the fourth volume in the series of books on *Real Functions in Several Variables.* Its topic is composed of differentiable curves and their tangents, differentiable surfaces and their tangent surfaces.

A curve is here defined as a continuous and piecewise C^1-function $\mathbf{f} : I \to \mathbb{R}^n$ of an interval I into either $\mathbb{R}^2$ or $\mathbb{R}^3$ for our purposes, though any dimension would do. It makes sense to talk of only continuous curves, but the problem is, that they include the so-called "space filling curves", i.e. a continuous curves, which e.g. sweeps through every point in $\mathbb{R}^2$, or even $\mathbb{R}^3$. By a modification of the construction one is able to even define a curve with no double points, where its graph has a positive area or volume, although the curve is felt to be of dimension 1, because its parametric description is in a 1-1 correspondence with e.g. $\mathbb{R}$. In order to avoid these anomalies we restrict ourselves to continuous piecewise C^1-curves, where the set of points, where f is not of class C^1, does not have positive area or volume.

We must carefully distinguish between the curve with its parametric description and the range of the curve in space, which may be a smooth 1-dimensional set in space, while the curve itself may not be smooth. The reason is that if the parameter is interpreted as the time t, then the curve $f : I \to \mathbb{R}^n$ also gives us some information of the velocity of a particle which is following the curve in time. The range may be smooth, while the particle may have some discontinuities in its velocity. The interpretation above as the description of a path of a particle is what makes the parametric description of a curve so important in Mechanics.

A C^1-curve has a tangent at every point, where $\mathbf{f}'(t) \neq \mathbf{0}$. If instead $\mathbf{f}'(t) = \mathbf{0}$, then the curve may be smooth at this point, but it may also have a bend, or even a cusp. Therefore, the case where $\mathbf{f}'(t) = \mathbf{0}$ always requires a special treatment.

Once we have introduced the C^1-curves, one would believe that the generalisation to C^1-surfaces should be straightforward. It almost is. A continuous surface is a function $\mathbf{f} : D \to \mathbb{R}^3$ (or to higher dimensional spaces $\mathbb{R}^n$), where $D \subseteq \mathbb{R}^2$ is some connected domain. Again one must avoid the space filling surfaces, which is done by the extra requirement that the curve is also of class C^1 with the exception of some isolated points or points lying on some "nice curves". The latter requirement is hard to make precise, so instead we appeal to figures in each situation.

The above was the first obstacle. The second one is, what is a tangent plane? Clearly, in most cases we can introduce the so-called parameter curves, where one of the parameters is held fixed, while the other one varies. This gives us two continuous piecewise C^1-curves through the point under consideration. If they both have a tangent vector $\neq \mathbf{0}$, and these are not parallel, then they span a plane, which is called the tangent plane at this point. If the tangent plane lies in $\mathbb{R}^3$, then it defines a normal vector, perpendicular to the tangent plane in the point of the surface under consideration. We give various descriptions of the tangent plane – either by a parametric description, or by an equation which should be fulfilled. Note that the approximating polynomial P_1 of at most degree 1 again enters the possible description of the tangent plane. In practical examples we shall benefit from the chain rule already described in Volume III.

If the surface is given by an equation $f(\mathbf{x}) = 0$, then the normal at a point $\mathbf{x}$ has the direction $\triangledown f(\mathbf{x})$, provided that this gradient is $\neq \mathbf{0}$. This result holds in all dimensions.

13 Differentiable curves and surfaces, and line integrals

13.1 Introduction

In this volume we shall consider the important geometrical objects, *differentiable curves* and *differentiable surfaces*. It must be emphasized that we make a distinction between the parametric description of a curve and the image of a curve. Two different curves (in their parametric descriptions) may have the same image in space. In this way we can describe how e.g. a particle geometrically can run through a given curve at different speed.

Similarly for differentiable surfaces, where two different parametric descriptions may have the same geometric image.

Finally, we analyze the line integrals in several variables. Usually two different line integrals of the same integrand may not be equal, even when the end points of the integration curves are the same. We shall therefore investigate under which conditions this is fulfilled, i.e. when the value of the line integral of a given integrand only depends on the end points of the integration curves. There are indeed some pitfalls here, which should be avoided.

13.2 Differentiable curves

In general, a *continuous curve* $\mathcal{C}$ is uniquely specified by its parametric description, i.e. there is given a continuous function $\mathbf{r} : I \to \mathbb{R}^n$ of an interval $I \subseteq \mathbb{R}$ into the real n-space $\mathbb{R}^n$.

This definition looks indeed very innocent, had it not been for the Italian mathematician *Peano*, who shortly before 1900 constructed a continuous curve

$$\mathbf{r} : [0,1] \to [0,1] \times [0,1],$$

which mapped the interval $[0,1]$ onto the square $[0,1] \times [0,1]$, i.e. the curve runs through every point in the unit square.

Peano's construction contained a lot of double points. However, a couple of years later the Canadian mathematician Osgood modified the curve to a space filling curve without double points. More precisely, for every given $\varepsilon \in]0,1[$ Osgood constructed a continuous curve

$$\mathbf{r}_\varepsilon : [0,1] \to [0,1] \times [0,1]$$

without double points, such that the area of the image of the curve satisfied the estimate

$$|\mathbf{r}_\varepsilon([0,1])| \geq 1 - \varepsilon,$$

where $\varepsilon > 0$ can be chosen as small as you like.

These examples are not convenient to use in practical applications, so instead we introduce:

Definition 13.1 *A curve $\mathcal{C}$ of the parametric description*

$$\mathcal{C} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \mathbf{r}(t) \text{ for } t \in I\}$$

*is called a C^k-*curve*, if the function $\mathbf{r} : I \to \mathbb{R}^n$, where I is an interval, is a C^k-function.*

A curve is from now on by definition always continuous. We shall for convenience also introduce the following extension of the definition above.

Definition 13.2 *A (continuous) curve* $\mathcal{C}$ *given by the parametric function* $\mathbf{r} : I \to \mathbb{R}^n$*, where* $I = [a, b]$*, is called a* piecewise C^k-curve*, if there exist points*

$$a = t_0 < t_1 < \cdots < t_p = b,$$

such that the restriction of $\mathbf{r}$ *to each of the subintervals* $[t_0, t_1], [t_1, t_2], \cdots, [t_{p-1}, t_p]$ *are all* C^k*-curves.*

A broken line, e.g. a polygon, is an example of a piecewise C^∞-curve.

Since we define a curve by its parametric description, it is not hard to give examples, where $\mathbf{r}_1 : I \to \mathbb{R}^n$ is a C^∞-curve, while $\mathbf{r}_2 : I \in \mathbb{R}^n$ is not, even if the images of $\mathbf{r}_1$ and $\mathbf{r}_2$ coincide in $\mathbb{R}^n$. Therefore, one should never rely only on the geometrical shape of a curve. Such a picture may give some information, but in the deeper analysis one should *always* consider the function $\mathbf{r} : I \to \mathbb{R}^n$, which defines the parametric description.

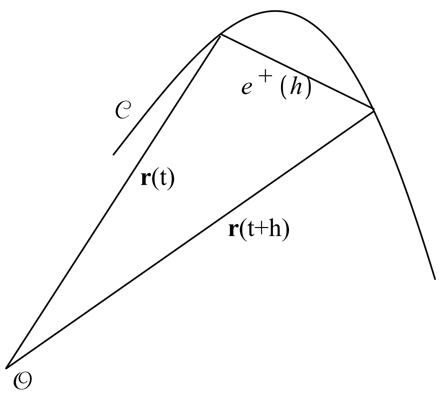


Figure 13.1: Construction of the tangent to a curve.

Let $\mathbf{r} : I \to \mathbb{R}^n$ be a C^1-curve, and fix a point $\mathbf{r}(t)$, $t \in I$, on the curve. Let h be small in absolute value, and consider the neighbouring curve point $\mathbf{r}(t + h)$. We shall assume that t, $t + h \in I$ for all h sufficiently small, and that also $\mathbf{r}(t + h) \neq \mathbf{r}(t)$ for h sufficiently small. Then $\mathbf{r}(t + h) - \mathbf{r}(t) \neq \mathbf{0}$ defines a direction, i.e. a unit vector with its foot point at $\mathbf{r}(t)$. We shall use the notation

$$\mathbf{e}^+(t, h) = \mathbf{e}^+(h) = \frac{\mathbf{r}(t + h) - \mathbf{r}(\mathbf{t})}{\|\mathbf{r}(t + h) - \mathbf{r}(t)\|}, \qquad \text{whenever } h > 0,$$

and

$$\mathbf{e}^-(t, h) = \mathbf{e}^-(h) = \frac{\mathbf{r}(t + h) - \mathbf{r}(\mathbf{t})}{\|\mathbf{r}(t + h) - \mathbf{r}(t)\|}, \qquad \text{whenever } h < 0,$$

so $\|\mathbf{e}^+(h)\| = \|\mathbf{e}^-(h)\| = 1$. If the limit

$$\lim_{h\to 0+} \mathbf{e}^+(h) = \mathbf{e}^+$$

exists, we call the half-line from $\mathbf{r}(t)$ of direction $\mathbf{e}^+$ the *positive half-tangent* of the curve at $\mathbf{r}(t)$.

Similarly, if

$$\lim_{h\to 0-} \mathbf{e}^-(h) = \mathbf{e}^-$$

exists, then the half-line from $\mathbf{r}(t)$ of direction $\mathbf{e}^-$ is called the *negative half-tangent* of the curve at $\mathbf{r}(t)$.

If t is an end point of the interval I, there is only one half-tangent.

Assume that both $\mathbf{e}^+$ and $\mathbf{e}^-$ exist at $\mathbf{r}(t)$. Then we have three possibilities:

1) If $\mathbf{e}^+ = -\mathbf{e}^-$, the two half-tangents point in opposite directions. Then can be joined together into a straight line, which is called the *tangent* of the curve at $\mathbf{r}(t)$, and $\mathbf{e}^+$ is called the *unit tangent vector* of the curve at this point.

2) If $\mathbf{e}^+ = \mathbf{e}^-$, then the two half-tangents coincide at $\mathbf{r}(t)$. We say that the curve has a *cusp* at $\mathbf{r}(t)$.

3) Finally, if $\mathbf{e}^+ \neq \pm\mathbf{e}^-$, then we say that the curve has a *bend* at $\mathbf{r}(t)$.

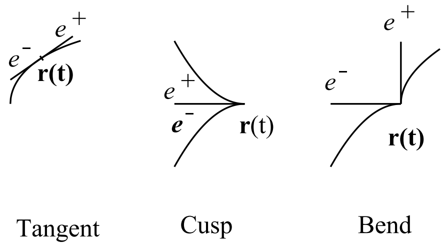


Figure 13.2: A tangent, a cusp and a bend in $\mathbb{R}^2$.

We assume that $\mathbf{r}$ is differentiable and that $\mathbf{r}'(t) \neq \mathbf{0}$. Then

$$\mathbf{r}(t+h) - \mathbf{r}(t) = \{\mathbf{r}'(t) + \varepsilon(h)\}\, h,$$

hence

$$\|\mathbf{r}(t+h) - \mathbf{r}(t)\| = \|\mathbf{r}'(t) + \varepsilon(h)\|\; |h|.$$

Here, $\varepsilon(h) \to \mathbf{0}$ for $h \to 0$, so $\|\varepsilon(h)\| < \|\mathbf{r}'(t)\|$ for $|h|$ sufficiently small. This implies that $\mathbf{r}(t+h) \neq \mathbf{r}(t)$ for $h \neq 0$ sufficiently small. Then for such $h > 0$,

$$\mathbf{e}^+(h) = \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{\|\mathbf{r}(t+h) - \mathbf{r}(t)\|} = \frac{\mathbf{r}'(t) + \varepsilon(h)}{\|\mathbf{r}'(t) + \varepsilon(h)\|} \cdot \frac{h}{|h|} \to \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \mathbf{e}^+ \quad \text{for } h \to 0+\,.$$

Similarly, for $h < 0$ numerically sufficiently small,

$$\mathbf{e}^+(h) = \frac{\mathbf{r}'(t) + \varepsilon(h)}{\|\mathbf{r}'(t) + \varepsilon(h)\|} \cdot \frac{h}{|h|} \to -\frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \mathbf{e}^- \quad \text{for } h \to 0-,$$

so by comparison, $\mathbf{e}^- = -\mathbf{e}^+$ both exist and are of opposite directions, and we are in case 1) above.

We conclude that if $\mathbf{r}'(t) \neq \mathbf{0}$, then the curve has a tangent at $\mathbf{r}(t)$ of unit tangent direction

$$\mathbf{e}^+ = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}.$$

When t is fixed and $\mathbf{r}'(t) \neq \mathbf{0}$, we therefore get the following parametric description of the tangent of the curve at the point $\mathbf{r}(t)$,

$$\mathbf{x} = \mathbf{r}(t) + v\,\mathbf{r}'(t), \qquad v \in \mathbb{R} \text{ and fixed } t \in I.$$

The point $\mathbf{r}(t)$, corresponding to the parametric value $v = 0$, is called the *point of contact* of the tangent. In the special case of dimension 3 we also get an equation of the *normal plane* of a differentiable curve in the space $\mathbb{R}^3$ at the point of contact $\mathbf{r}(t)$, by using that the tangent is perpendicular to the normal plan at $\mathbf{r}(t)$, i.e. if $\mathbf{x}$ lies in the normal plane, then $\mathbf{r}'(t)$ and $\mathbf{x} - \mathbf{r}(t)$ are perpendicular to each other, so

$$(\mathbf{x} - \mathbf{r}(t)) \cdot \mathbf{r}'(t) = 0.$$

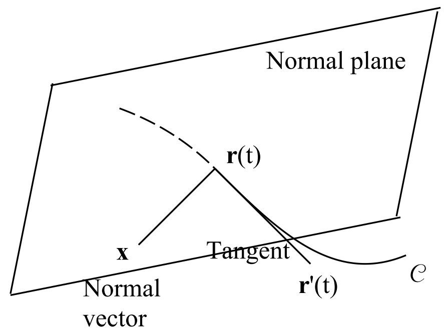


Figure 13.3: The tangent of a differentiable curve $\mathcal{C}$ is perpendicular to the normal plane.

If $\mathbf{r}'(t) = \mathbf{0}$, we say that the point $\mathbf{r}(t)$ is a *singular point* of the curve $\mathcal{C}$. Let $\mathbf{r}(t)$ be a singular point of a C^2-curve $\mathcal{C}$. Then by Taylor's formula,

$$\mathbf{r}(t+h) - \mathbf{r}(t) = \left\{ \frac{1}{2}\,\mathbf{r}''(t) + \varepsilon(h) \right\} h^2.$$

If $\mathbf{r}''(t) \neq \mathbf{0}$, then it follows that $\mathbf{e}^+ = \mathbf{e}^-$, so we have a cusp. Therefore, if a differentiable C^2-curve $\mathcal{C}$ has indeed a *tangent* at a *singular point*, then both

$$\mathbf{r}'(t) = \mathbf{0} \qquad \text{and} \qquad \mathbf{r}'(t) = \mathbf{0}$$

at this point. This condition is of course not sufficient for the existence of a tangent at a singular point.

We shall proceed by showing how we in practice compute the tangent of the graph of a C^1-function

$$g : I \to \mathbb{R}, \qquad I \times \mathbb{R} \subseteq \mathbb{R}^2,$$

where I is some real interval.

The graph can be considered as a curve $\mathcal{C}$ in $\mathbb{R}^2$. In rectangular coordinates we use the parametric description

$$\mathbf{r}(x) = (x, g(x)), \qquad \text{so } \mathbf{r}'(x) = (1, g'(x)) \neq (0,0),$$

and the graph $\mathcal{C}$ of g has a tangent at every point, where the function is of (local) class C^2. The parametric description of the tangent becomes

$$(x,y) = (x_0, g(x_0)) + v \cdot (1, g'(x_0)) = (x_0 + v, g(x_0) + v\, g'(x_0)), \qquad v \in \mathbb{R},$$

or, in each of the coordinates,

$$x = x_0 + v, \qquad y = g(x_0) + v \cdot g'(x_0).$$

The parameter v can be eliminated by using that $v = x - x_0$. When this is done, the equation of the tangent is written in the more familiar way

$$y = g(x_0) + (x - x_0)\, g'(x_0), \qquad x \in \mathbb{R}.$$

We note that the right hand side of this equation is also the *approximating polynomial* $P_1(x)$ of at most first degree, so the equation of the tangent of the graph can also be written

$$y = P_1(x), \qquad x \in \mathbb{R}.$$

We shall in the following give some simple examples which show how we use the theory above in practice.

Example 13.1 Given the C^∞-curve $\mathcal{C}$ in 3-space of the parametric description

$$(x,y,z) = \mathbf{r}(t) = \left(t, t^2, t^3\right), \qquad t \in \mathbb{R}.$$

We shall find the tangent and the normal plane at the curve point $(-1,1,-1) \in \mathbb{R}^3$, corresponding to the value $t = -1$ of the parameter.

First, by a differentiation,

$$\mathbf{r}'(t) = \left(1, 2t, 3t^2\right) \neq (0,0,0) \qquad \text{for } t \in \mathbb{R}.$$

Since $\mathbf{r}'(t) \neq \mathbf{0}$, the curve $\mathcal{C}$ has a tangent at all its points. In particular, at $(-1,1,-1)$, i.e. for $t = -1$,

$$\mathbf{r}'(-1) = (1,-2,3).$$

This implies that a parametric description of the tangent is given by

$$(x,y,z) = (-1,1,-1) + v(1,-2,3) = (-1+v, 1-2v, -1+3v), \qquad \text{for } v \in \mathbb{R}.$$

The curve for $t = -1, \dots, 0$ and the tangent at $(-1,1,-1)$ for $v = 0, \dots, 0.3$ are depicted in Figure 13.4.

The normal plane is given by

$$(\mathbf{x} - \mathbf{r}(t)) \cdot \mathbf{r}'(t) = 0,$$

i.e.

$$0 = ((x,y,z) - (-1,1,-1)) \cdot (1,-2,3) = (x+1, y-1, z+1) \cdot (1,-2,3) = (x+1) - 2(y-1) + 3(z+1).$$

Depending on the actual application one would either keep this form, in which $(x_0, y_0, z_0) = (-1, 1-1)$ still occurs, which sometimes may be convenient, or reduce to the equation

$$x - 2y + 3z = -6. \qquad \Diamond$$

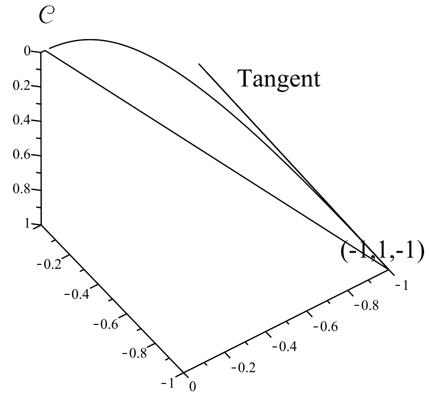


Figure 13.4: The curve and the tangent of Example 13.1.

Example 13.2 We shall here give an example of a *cusp* at a singular point of a C^∞-curve. Consider

$$\mathcal{C}: \quad (x,y,z) = \mathbf{r}(t) = \left(t^2, t^3, t^4\right), \qquad t \in \mathbb{R}.$$

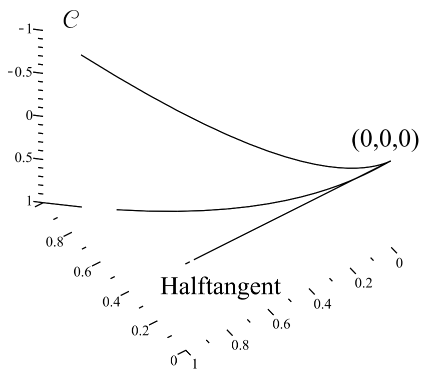


Figure 13.5: The curve and the tangent of Example 13.2.

Then by differentiation,

$$\mathbf{r}'(t) = \left(2t, 3t^2, 4t^4\right), \qquad \text{and} \qquad \mathbf{r}''(t) = \left(2, 6t, 12t^2\right).$$

It is obvious that $t = 0$ is defining the only *singular point* $(0,0,0)$, so $\mathcal{C}$ has a tangent at every other point of the curve. Since $\mathbf{r}''(t) \neq \mathbf{0}$ for all $t \in \mathbb{R}$, we must have a cusp at $(0,0,0)$. The two coinciding half-tangents are the positive x-axis. Cf. Figure 13.5. $\Diamond$

Example 13.3 We shall here give an example of a curve $\mathcal{C}$, where the tangent exists at the singular point. The construction is very simple and illustrates the difference between a curve as a function and a curve, represented by its graph alone.

Let the underlying graph be the parable of equation

$$y = x^2, \qquad x \in \mathbb{R}.$$

Then it is obvious that this parable has a tangent at all of its points. To get a *curve* $\mathcal{C}$, i.e. a function, with a singular point, we only change the parameter to $x = t^3$, $t \in \mathbb{R}$. Then $\mathcal{C}$ has the parametric description

$$(x, y) = \mathbf{r}(t) = \left(t^3, t^6\right), \qquad t \in \mathbb{R}.$$

The underlying graph is still the parable with tangents at all of its points, but since $\mathbf{r}'(t) = \left(3t^2, 6t^5\right) = (0, 0)$ for $t = 0$, we see that $(0, 0)$ is a singular point.

We note how we can introduce singular points by a simple change of the parameter. $\Diamond$

When $\mathcal{C}$ is a curve in $\mathbb{R}^2$ or in $\mathbb{R}^3$, we may interpret $\mathcal{C}$ as the path, which a particle goes through with *velocity* $\mathbf{r}'(t)$ and *acceleration* $\mathbf{r}''(t)$ of the particle at the point $\mathbf{r}(t)$ of the curve. This is the usual application of curves in e.g. Mechanics.

Example 13.4 It is customary in *Fluid Mechanics* to consider a function $f(x,y,z,t)$ in the three space variables (x,y,z) and the time variable t as a function in time alone of a given (fluid) particle. In other words, given $f(x,y,z,t)$ we want to consider $(x(t),y(t),z(t))$ as the path of this fluid particle. We obtain, using the *chain rule*,

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}\,\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\,\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\,\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{\partial f}{\partial t},$$

which is correct, and nevertheless confusing, because the variable t enters the equation in two different ways, represented by the symbols

$$\frac{\mathrm{d}f}{\mathrm{d}t} \qquad \text{and} \qquad \frac{\partial f}{\partial t}.$$

They are clearly not equal, so how do we explain this?

The trick is to introduce a new time variable τ, whenever we are considering the fluid particle, and keep t when we are not. This means that the particle is in 4-space described by the curve

$$(x(\tau),y(\tau),z(\tau)), \qquad \text{where } t(\tau)=\tau.$$

When we then differentiate with respect to τ, we get by the chain rule that

$$\frac{\mathrm{d}f}{\mathrm{d}\tau} = \frac{\partial f}{\partial x}\,\frac{\mathrm{d}x}{\mathrm{d}\tau} + \frac{\partial f}{\partial y}\,\frac{\mathrm{d}y}{\mathrm{d}\tau} + \frac{\partial f}{\partial z}\,\frac{\mathrm{d}z}{\mathrm{d}\tau} + \frac{\partial f}{\partial t}\,\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{\partial f}{\partial x}\,\frac{\mathrm{d}x}{\mathrm{d}\tau} + \frac{\partial f}{\partial y}\,\frac{day}{da\tau} + \frac{\partial f}{\partial z}\,\frac{daz}{\mathrm{d}\tau} + \frac{\partial f}{\partial t},$$

because $t(\tau)=\tau$. Using this simple trick we see the difference between

$$\frac{\mathrm{d}f}{\mathrm{d}\tau} \qquad \text{and} \qquad \frac{\partial f}{\partial t}.$$

Then note that

$$\left(\frac{\mathrm{d}x}{\mathrm{d}\tau},\frac{\mathrm{d}y}{\mathrm{d}\tau},\frac{\mathrm{d}z}{\mathrm{d}\tau}\right) := \mathbf{v}(\tau)$$

is the velocity of the particle, so we can write

$$\frac{\mathrm{d}f}{\mathrm{d}\tau} = \mathbf{v}\cdot\bigtriangledown f + \frac{\partial f}{\partial t},$$

where $\bigtriangledown f$ is the gradient of f *with respect only to the space variables* (x,y,z).

Once all this has been realized, we may shift back to the old notation without τ, writing t instead of τ, so

$$\frac{\mathrm{d}f}{\mathrm{d}\tau} = \frac{\partial f}{\partial x}\,\frac{\mathrm{d}x}{\mathrm{d}\tau} + \frac{\partial f}{\partial y}\,\frac{\mathrm{d}y}{\mathrm{d}\tau} + \frac{\partial f}{\partial z}\,\frac{\mathrm{d}z}{\mathrm{d}\tau} + \frac{\partial f}{\partial t},$$

and

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \mathbf{v}\cdot\bigtriangledown f + \frac{\partial f}{\partial t}.$$

Here, $\dfrac{\mathrm{d}f}{\mathrm{d}t} = \dfrac{\mathrm{d}f}{\mathrm{d}\tau}$, is sometimes also written $\dfrac{\mathrm{D}f}{\mathrm{D}t}$ in the literature, and it is called the *substantial derivative*, while $\dfrac{\partial f}{\partial t}$ denotes the ordinary partial derivative with respect to time t.

We obtain from the above the unexpected result that even if the motion is *stationary*, i.e. $\dfrac{\partial f}{\partial t} = 0$, the particle may still feel some independence in time, which is represented by the socalled *convective term* $\mathbf{v} \cdot \bigtriangledown f$.

Another consequence of the above is that when *Newton's second law* is applied, we must find the acceleration of the particle by taking the *substantial derivative* of the velocity $\mathbf{v}$. This means that we in the rule of differentiation above replace f by $\mathbf{v}$, getting

$$\frac{da\mathbf{v}}{\mathrm{d}t} \left(= \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}\tau} \right) = (\mathbf{v} \cdot \bigtriangledown)\mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}.$$

Hence, a fluid may be in a stationary condition, i.e. $\dfrac{\partial \mathbf{v}}{\partial t} = \mathbf{0}$, (in the ordinary time variable t) and yet the fluid particles are subjected to an acceleration, if the differential $d\mathbf{v}(\mathbf{x}, \mathbf{v}) = (\mathbf{v} \cdot \bigtriangledown)\mathbf{v}$ is not equal to the zero vector. We here tacitly change to the other time variable τ, which is bound to the particle. $\Diamond$

13.3 Level curves

Cf. also Section 5.2 in Volume II. Let f be a C^1-function in two variables, and let c be some constant. It was mentioned in Section 5.2 that the equation

$$f(x, y) = c$$

usually describes a *level curve* of the function f.

A level curve needs not be a continuous curve. It may instead be the union of several continuous curves,, which are then called *branches.* An obvious example is given by the level curves on a map describing e.g. a hilly landscape.

In order not to make it too complicated for us in the following we shall here only consider the given level curve in a neighbourhood of some given point (u_0, v_0) on this level curve, where $f(u_0, v_0) = c$. In general, when (u_0, v_0) is not singular (meaning that this point does only lie on one of the possible branches), the neighbourhood can be chosen so small that it does not contain points from other branches of the level curve.

In order to proceed we are forced to apply the *implicit function theorem.* The proof of this is fairly long and complicated, as well as tedious, so we shall here only mention the main result, which we are going to apply here in the following.

Theorem 13.1 *Let $f \in C^1$ in a neighbourhood of the point (u_0, v_0), where $f(u_0, v_0) = c$, and assume that $\bigtriangledown f(u, v) \neq 0$ everywhere in this neighbourhood. Then the level curve equation $f(u_0, v_0) = c$ describes a C^1-curve $\mathcal{C}$ in (possibly another) neighbourhood of (u_0, v_0), where $\mathcal{C}$ is the graph of a C^1-function in one variable only.*

Furthermore, if $f'_y(u_0, v_0) \neq 0$, then $\mathcal{C}$ can be described by an equation of the form $y = Y(x)$, where Y is a C^1-function on some interval I, which has u_0 as an interior point. In this case we may locally write

$$f(x, Y(x)) = c \qquad \text{for } x \in I.$$

If instead $f'_x(u_0, v_0) \neq 0$, then $\mathcal{C}$ can be described by an equation of the form $x = X(y)$, where X is a C^1-function on some interval J, which has v_0 as an interior point. In this case we may locally write

$$f(X(y), y)) = c \qquad \text{for } y \in J.$$

Let us assume that $f \in C^1$ fulfils the assumptions of the *implicit function theorem* above in a neighbourhood of some point (u_0, v_0). If $f'_y(u_0, v_0) \neq 0$, we know that there exists a function $Y : I \to \mathbb{R}$, $Y \in C^1(I)$ such that the level curve is locally described by

$$f(x, Y(x)) = c.$$

We do not know, however, how $Y(x)$ is explicitly defined, with the exception of the value $Y(u_0) = v_0$. But since $Y \in C^1$, we still can differentiate the equation above with respect to x, giving by the chain rule,

$$0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dY}{dx} = (1, Y'(x)) \cdot \left(f'_x(x, Y(x)), f'_y(x, Y(x))\right).$$

Solving the first equation with respect to $Y'(x)$ we get

$$(13.1) \qquad Y'(u) = -\frac{f'_x(u, Y(u))}{f'_y(u, Y(u))}, \qquad \text{from which} \qquad Y'(u_0) = -\frac{f'_x(u_0, Y(u_0))}{f'_y(u_0, Y(u_0))}$$

Thus, $Y(u_0)$ and $Y'(u_0)$ can be explicitly found, so we can find the tangent of the curve $\mathcal{C}$ at this point (u_0, v_0). In fact, the tangent is described by

$$Y(u_0) + (x - u_0) Y'(u_0), \qquad x \in \mathbb{R},$$

which is also the first approximation of $Y(x)$ from the expansion point u_0,

$$Y(x) \simeq Y(u_0) + (x - u_0) Y'(u_0).$$

If furthermore $f \in C^2$, then also $Y \in C^2(I)$, and we can find $Y''(u_0)$ by a differentiation of (13.1), etc.. Clearly, the computations normally blow up very fast, because we have to differentiate a quotient where the denominator $f'_x(x, Y(x))$ needs not be simple. In other words, the method may in principle be applied, but it is in practice only of limited value, because the computations almost from the very beginning become overwhelming.

If instead $f'_x(u, v) \neq 0$ in this neighbourhood, then we analogously show the existence of a locally unique function $x = X(y)$, where

$$(X'(y), 1) \cdot \nabla f(X(y), y) = 0, \qquad y \in J,$$

and

$$X'(y) = -\frac{f'_y(X(y), y)}{f'_x(X(y), y)}, \qquad y \in J.$$

A general result is that since $\nabla f(u, v) \neq \mathbf{0}$, this gradient is always perpendicular to the level curve.

If both $f'_x(u, v) \neq 0$ and $f'_y/u, v) \neq 0$, then we easily derive from the above that

$$Y'(x)X'(y) = \left\{-\frac{f'_x(x, Y(x))}{f'_y(x, Y(x))}\right\} \cdot \left\{\frac{f'_y(X(y), y)}{f'_x(X(y), y)}\right\} = 1,$$

because $(x, Y(x)) = (X(y), y)$ are two descriptions of the same point. This result is recognized as the rule of the derivative of the inverse function,

$$Y'(x)X'(y) = 1, \qquad \text{or} \qquad X'(y) = \frac{1}{Y'(x)},$$

provided that $Y'(x) \neq 0$ and $X'(y) \neq 0$.

13.4 Differentiable surfaces

When we introduce the concept of surfaces, we would expect that we should copy the definition og a curve and extend this to allowing two parameters. This is indeed the case.

Definition 13.3 *Let $E \subseteq \mathbb{R}^2$ be a domain, and let $\mathbf{r} : E \to \mathbb{R}^3$ be a C^n-function. Then $\mathbf{r}$ defines a C^n surface $\mathcal{F}$ by*

$$\mathcal{F} = \left\{\mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} = \mathbf{r}(u, v),\ (u, v) \in E\right\}.$$

If $\mathbf{r} \in C^\infty$, then $\mathcal{F}$ is C^∞, and if $\mathbf{r} \in C^0$, then $\mathcal{F}$ is a continuous, or C^0, surface.

One should in the first analysis always avoid the general C^0-surfaces, because they may have some strange properties. It is not hard to construct a C^0-surface, i.e. a continuous vector function $\mathbf{r} : [0, 1]^2 \to \mathbb{R}^3$, such that $\mathbf{r}\left([0, 1]^2\right) = \mathbb{R}^3$, i.e. the C^0-surface is the full space $\mathbb{R}^3$. In fact, we already know that this is even possible for a C^0-curve. We avoid such pathological surfaces by restricting ourselves to at least C^1-surfaces (almost everywhere; see below). They will be called differentiable surfaces in the following.

In practical applications usually only C^∞-surfaces are needed.

We shall clearly need that some C^n-surfaces can be put together to form a *piecewise* C^n-surface. This allows us to describe the surface of e.g. a cube, or more generally, a polyhedron, or a cylinder, etc.. We shall not give the precise definition of a piecewise C^n-surface, but leave the problem to the reader's intuition.

Given a C^1-surface $\mathcal{F}$, the analogue of the tangent of a C^1-curve must be a *tangent plane*, which we shall now introduce.

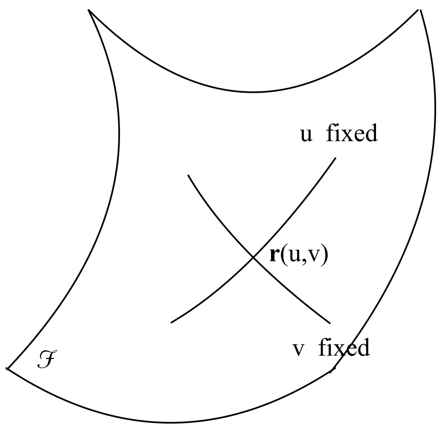


Figure 13.6: Parameter curves of $\mathcal{F}$.

Let $\mathbf{r}(u, v) \in \mathcal{F}$ be a point of a C^1-surface $\mathcal{F}$.

If we fix one of the parameters u, v, and let the other vary, we get the two C^1-curves

$$\mathbf{x} = \mathbf{r}(t, v), \qquad v \text{ fixed and } t \in I, \qquad (t, v) \in E,$$

$$\mathbf{x} = \mathbf{r}(u, t), \qquad u \text{ fixed and } t \in J, \qquad (u, t) \in E,$$

which are called the two *parameter curves* of $\mathcal{F}$ through the point $\mathbf{r}(u, v)$. Cf. Figure 13.6.

Let us assume that both I and J are open intervals. In order to find the tangents of these curves at $\mathbf{r}(u, v)$ we differentiate with respect to t, so $\mathbf{r}'_u(u, v)$ and $\mathbf{r}'_v(u, v)$ are the vectors, which specify the tangents, provided that none of them is the zero vector.

If $\mathbf{r}'_u(u, v)$, $\mathbf{r}'_v(u, v) \neq \mathbf{0}$ are not parallel, then they span a plane with foot point at $\mathbf{r}(u, v)$, i.e. a plane of the parametric description in the new parameters (s, t),

$$\mathbf{r}(u, v) + s\,\mathbf{r}'_u(u, v) + t\,\mathbf{r}'_v(u, v), \qquad \text{for } (s, t) \in \mathbb{R}^2.$$

When this plane exists, we call it the *tangent plane of* $\mathcal{F}$ *at the point* $\mathbf{r}(u, v)$.

Hence, the tangent plane at $\mathbf{r}(u, v)$ of a C^1-surface $\mathcal{F}$ exists, if and only if the two vectors $\mathbf{r}'(u, v)$ and $\mathbf{r}'_v(u, v)$ are linearly independent.

Write

$$\mathbf{N}(u, v) = \mathbf{r}'_u(u, v) \times \mathbf{r}'_v(u, v), \qquad \text{for } (u, v) \in E.$$

Then $\mathbf{N}(u, v) \neq \mathbf{0}$, if and only if $\mathbf{r}'_u(u, v)$ and $\mathbf{r}'_v(u, v)$ are linearly independent, so the tangent plane exists, whenever $\mathbf{N}(u, v) \neq \mathbf{0}$. In this case $\mathbf{N}(u, v)$ is perpendicular to $\mathbf{r}'_u(u, v)$ and $\mathbf{r}'(u, v)$, which generate the tangent plane, so we call $\mathbf{N}(u, v)$ the *normal vector* of the surface $\mathcal{F}$. Cf. Figure 13.7.

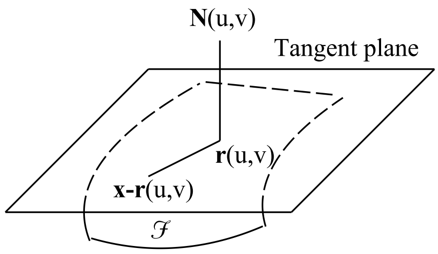


Figure 13.7: Tangent plane and normal vector of a surface $\mathcal{F}$.

We conclude that the C^1-surface $\mathcal{F}$ has a tangent plane at the point $\mathbf{r}(u, v) \in \mathcal{F}$, if and only if $\mathbf{N}(u, v) \neq \mathbf{0}$.

Given $\mathbf{N}(u, v) \neq \mathbf{0}$ by the construction above. We then get an alternative description of the tangent plane by using the orthogonality, so we get the following equation of the tangent plane,

$$\mathbf{N}(u, v) \cdot \{\mathbf{x} - \mathbf{r}(u, v)\} = 0, \qquad \text{for } \mathbf{x} = (x, y, z) \in \mathbb{R}^3.$$

This equation states that the normal vector $\mathbf{N}(u, v)$ is perpendicular to any tangent vector $\mathbf{x} - \mathbf{r}(u, v)$ from the fixed point $\mathbf{r}(u, v)$ of the surface. Cf. Figure 13.7.

We shall in the following restrict ourselves to piecewise C^1-surfaces, for which $\mathbf{N} \neq \mathbf{0}$, with possibly some minor exceptions of isolated points, or points on C^1-curves, lying on the surface. The latter occurs typically, when two C^1-surfaces are glued together along such a C^1-curve. Consider e.g. a cube, where the normal vector does not exist on the edges, but clearly elsewhere.

Once the normal vector $\mathbf{N} \neq \mathbf{0}$ is specified, it *locally* defines an *orientation* of the surface. A surface has *locally* two sides. The side, where $\mathbf{N}$ points outwards, is called the *positive side*. The other one, where $\mathbf{N}$ is pointing inwards, is called the negative side of the surface.

Since $\mathbf{N}(u,v) = \mathbf{r}_u'(u,v) \times \mathbf{r}_v'(u,v)$, one may also express this by saying that the vectors $(\mathbf{r}_u', \mathbf{r}_v', \mathbf{N})$ taken in this order define a righthanded screw, because one can put one's right hand in such a position that $\mathbf{r}'u$ is pointing along the thumb, $\mathbf{r}_v'$ along the index finger, and $\mathbf{N}$ along the middle finger. It follows from this description that if one interchanges the parameters u and v, then the normal will point in the opposite direction, and the orientation is changed as well.

We shall later need the *unit normal vector*, which is denoted by

$$\mathbf{n} = \frac{\mathbf{N}}{\|\mathbf{N}\|}, \qquad \text{provided that } \mathbf{N} \neq \mathbf{0}.$$

Remark 13.1 It should be mentioned that the orientation in general only is defined locally. It is not difficult to construct a surface $\mathcal{F}$, which (of course) locally has two sides, but globally only one, and hence no global orientation. The simplest example is the so-called *Möbius strip*. Take a strip of paper, twist one end 180° and then glue the two ends together. By following the middle line of the strip on can move the (local) normal vector continuously inside the strip, until it points in the opposite direction, so there is only one side of the surface and there cannot be a global orientation. ♢

Example 13.5 In order to exercise the procedure of determining the tangent plane of a surface we consider the following C^∞-function,

$$\mathbf{r}(u,v) = \left(u^2+v, u^3+v^2, u+v^2\right), \qquad (u,v) \in \mathbb{R}^2.$$

We shall find the tangent plane at the point $\mathbf{r}(1,2) = (3,5,5)$. The surface in the neighbourhood of this point is shown on Figure 13.8.

We get by partial differentiations,

$$\mathbf{r}_u'(u,v) = \left(2u, 3u^2, 1\right) \qquad \text{and} \qquad \mathbf{r}_v'(u,v) = (1, 2v, 2v).$$

The normal vector is computed by using the following method known from *Linear Algebra* by using a formal determinant,

$$\mathbf{N}(u,v) = \det\begin{pmatrix} \mathbf{r}_u' \\ \mathbf{r}_v' \\ \text{“}\mathbf{e}_i\text{“} \end{pmatrix} = \begin{vmatrix} 2u & 3u^2 & 1 \\ 1 & 2v & 2v \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{vmatrix} = \left(6u^2v - 2v, 1+4uv, 4uv-3u^2\right).$$

The tangent plane has in general the equation

$$\mathbf{N}(u,v) \cdot \{\mathbf{x} - \mathbf{r}(u,v)\} = 0,$$

so by insertion,

$$(8,-7,5) \cdot \{(x,y,z) - (3,5,5)\} = 8(x-3) - 7(y-5) + 5(z-5) = 0.$$

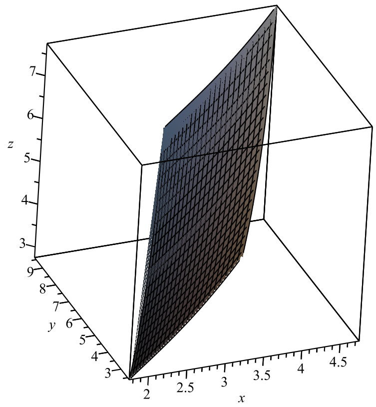


Figure 13.8: The surface $\mathcal{F}$ of Example 13.5 in a neighbourhood of the point (3,5,5).

In many applications, where (x, y, z) lies in a neighbourhood of (3,5,5), one would for numerical reasons keep this equation,

$$8(x-3) - 7(y-5) + 5(z-5) = 0,$$

though one may of course reduce it to

$$8x - 7y + 5z = 14. \qquad \Diamond$$

13.5 Special C^1-surfaces

Some surfaces are considered over and over again, so we collect here the descriptions in rectangular coordinates of the most important cases.

1) *The graph of a C^1-function in two variables.*
The *parametric description* is given by

$$\mathbf{r}(x,y) = (x,y,Z(x,y)) \qquad \text{for } (x,y) \in E \subseteq \mathbb{R}^2.$$

The *normal vector* is in this parametric description given by

$$\mathbf{N}(x,y) = \left(-Z_x'(x,y), -Z_y'(x,y), 1\right) = \mathbf{e}_z - \nabla Z(x,y).$$

Clearly, $\mathbf{N})x,y) \neq \mathbf{0}$ everywhere, and the angle between $\mathbf{N}$ and the z-axis is $\operatorname{Arccos}\left(\|\mathbf{N}\|^{-1}\right)$.
The *tangent plane* at (x,y) is spanned by the vectors

$$\mathbf{r}_x'(x,y) = (1,0,Z_x'(x,y)) \qquad \text{and} \qquad \mathbf{r}_y'(x,y) = \left(0,1,Z_y'(x,y)\right).$$

An *equation of the tangent plane* is given by

$$z = P_1(x,y),$$

where $P_1(x,y)$ is the approximating polynomial of $Z(x,y)$ of at most degree 1 at the point $(x,y,Z(x,y))$.

2) *A cylindric surface with generators parallel to the z-axis.*
The *parametric description* has the form

$$\mathbf{r}(t,z) = (X(t),Y(t),z), \qquad t \in I, \quad z \in J(t),$$

where I and $J(t)$ are intervals in $\mathbb{R}$.
The *normal vector* is

$$\mathbf{N}(t,z) = (Y'(t), -X'(t), 0), \qquad t \in I, \quad z \in J(t).$$

Its *generating curve* $\mathcal{L}$ in the (x,y)-plane has the parametric description

$$\mathbf{r}(t) = (X(t),Y(t),0), \qquad t \in I.$$

3) *A surface of revolution with the z-axis as its axis.*
The *parametric description* is

$$\mathbf{r}(t,\varphi) = (P(t)\cos\varphi, P(t)\sin\varphi, Z(t)), \qquad t \in I, \quad \varphi \in [0,2\pi[,$$

and the *normal vector* is

$$\mathbf{N}(t,\varphi) = P(t)(-Z'(t)\cos\varphi, -Z'(t)\sin\varphi, P'(t)), \qquad t \in I, \quad \varphi \in [0,2\pi[.$$

The *meridian curve* $\mathcal{M}$ is in semi-polar coordinates (ϱ,φ,z) given by

$$\varrho = P(t) \quad \text{and} \quad z = Z(t), \qquad \text{for } t \in I.$$

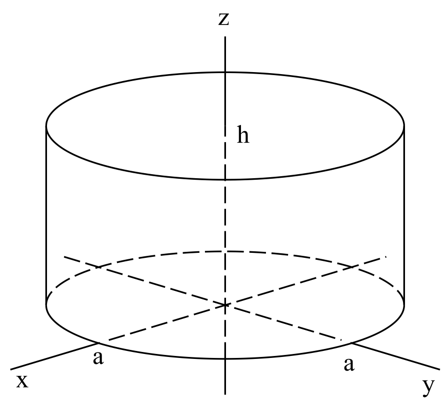


Figure 13.9: The cylinder of revolution of Example 13.6 with a chosen coordinate system.

Example 13.6 We shall as a simple exercise find the outward unit normal vector $\mathbf{n}$ of the surface of a cylinder of revolution of radius a and height h. We choose the coordinate sysem as in Figure 13.9. The piecewise C^1-surface is actually composed of three C^∞-surfaces.
The plane surface at height $h > 0$ has at every point the outward unit normal vector $\mathbf{n} = (0, 0, 1)$.

The plane surface in the (x, y)-plane has at every point the outward unit normal vector $\mathbf{n} = (0, 0, -1)$. Note that the normal vector with this orientation is pointing away from the cylinder.

The third curved surface is described by the function

$$\mathbf{r}(\varphi, z) = (a \cos \varphi, a \sin \varphi, z), \qquad (\varphi, z) \in [0, 2\pi[\times [0, h].$$

Then by differentiation

$$\mathbf{r}'_\varphi(\varphi, z) = (-a \sin \varphi, a \cos \varphi, 0) \qquad \text{and} \qquad \mathbf{r}'_z(\varphi, z) = (0, 0, 1),$$

so the normal vector is

$$\mathbf{N}(\varphi, z) = \mathbf{r}'_\varphi(\varphi, z) \times \mathbf{r}'_z(\varphi, z) = \begin{vmatrix} -a \sin \varphi & a \cos \varphi & 0 \\ 0 & 0 & 1 \\ \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \end{vmatrix} = (a \cos \varphi, a \sin \varphi, 0) = (x, y, 0).$$

At any point (x, y, z) on the curved surface the vector $(x, y, 0)$ clearly points away from the cylinder. Hence the outward unit normal vector is here given by

$$\mathbf{n}(\varphi, z) = (\cos \varphi, \sin \varphi, 0) = \left(\frac{x}{a}, \frac{y}{a}, 0 \right), \qquad \varphi \in [0, 2\pi[,$$

where (x, y, z) is assumed to lie on the curved surface. Clearly, we could alternatively have obtained this result by elementary geometry by considering Figure 13.9. $\Diamond$

Example 13.7 It has previously been shown that the ellipsoid of the equation

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1, \qquad a,\, b,\, c>0,$$

has the parametric description

$$\mathbf{r}(\theta,\varphi)=(x,y,z)=(a\sin\theta\cos\varphi, b\sin\theta\sin\varphi, c\cos\theta), \qquad \theta\in[0,\pi], \quad \varphi\in[0,2\pi[.$$

We shall in this example find the tangent plane at any given point of the ellipsoid, which clearly exists. However, using the particular parametric description above we shall see that the computed normal vector $\mathbf{N}(\theta,\varphi)=\mathbf{0}$ at some points, so we shall see how we can handle this problem.

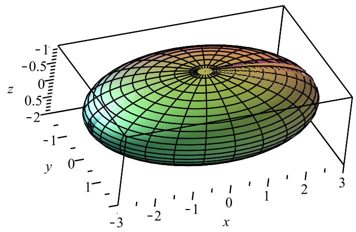


Figure 13.10: The ellipsoid of Example 13.7 with $a=3$, $b=2$ and $c=1$.

By differentiation,

$$\mathbf{r}'_\theta(\theta,\varphi)=\left(\frac{\partial x}{\partial\theta},\frac{\partial y}{\partial\theta},\frac{\partial z}{\partial\theta}\right)=(a\cos\theta\cos\varphi, b\cos\theta\sin\varphi, -c\sin\theta),$$

$$\mathbf{r}'_\varphi(\theta,\varphi)=\left(\frac{\partial x}{\partial\varphi},\frac{\partial y}{\partial\varphi},\frac{\partial z}{\partial\varphi}\right)=(-a\sin\theta\sin\varphi, b\sin\theta\cos\varphi, 0).$$

Then the normal vector becomes

$$\begin{aligned}\mathbf{N}(\theta,\varphi) &= \mathbf{r}'_\theta(\theta,\varphi)\times\mathbf{r}'_\varphi(\theta,\varphi)=\begin{vmatrix} a\cos\theta\cos\varphi & b\cos\theta\sin\varphi & -c\sin\theta \\ -a\sin\theta\sin\varphi & b\sin\theta\cos\varphi & 0 \\ \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z\end{vmatrix} \\ &= \left(bc\sin^2\theta\cos\varphi, ac\sin^2\theta\sin\varphi, ab\cos\theta\sin\theta\right) \\ &= abc\sin\theta\left(\frac{1}{a}\,\sin\theta\cos\varphi, \frac{1}{b}\,\sin\theta\sin\varphi, \frac{1}{c}\,\cos\theta\right) \\ &= abc\sin\theta\left(\frac{x}{a^2},\frac{y}{b^2},\frac{z}{c^2}\right).\end{aligned}$$

The vector $\left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}\right)$ is always $\neq \mathbf{0}$. However, the common factor $abc \sin\theta$ is 0 for $\theta = 0$ or $\theta = \pi$, corresponding to the two points $(0, 0, \pm c)$, where we clearly have the corresponding unit normal vectors $(0, 0, \pm 1)$, cf. Figure 3.10, while the computed normal vector $\mathbf{N}(\theta, \varphi)$ in this particular chosen coordinate system i $\mathbf{0}$ for $\theta = 0$ or $\theta = \pi$.

In other words, a surface may have a unit normal vector $\mathbf{n}$, $\|\mathbf{n}\| = 1$, even if we get $\mathbf{N} = \mathbf{0}$ at the same point. This shows that $\mathbf{N}$ depends on the chosen coordinate system, while $\mathbf{n}$ does not!

To resolve this problem we note that we obtain another normal vector $\tilde{\mathbf{N}}$ by removing the common factor $abc \sin\theta$, which is not identical zero. Letting (ξ, η, ζ) denote a fixed point on the ellipsoidal surface, corresponding to the parameters (θ, φ), we get

$$\tilde{\mathbf{N}}(\theta, \varphi) = \left(\frac{\xi}{a^2}, \frac{\eta}{b^2}, \frac{\zeta}{c^2}\right), \qquad \text{where } \frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} = 1.$$

The equation of the tangent plane at this point (ξ, η, ζ) on the surface is then given by

$$\begin{aligned} 0 &= (x-\xi, y-\eta, z-\zeta) \cdot \left(\frac{\xi}{a^2}, \frac{\eta}{b^2}, \frac{\zeta}{c^2}\right) \\ &= \frac{x\xi}{a^2} - \frac{\xi^2}{a^2} + \frac{y\eta}{b^2} - \frac{\eta^2}{b^2} + \frac{z\zeta}{c^2} - \frac{\zeta}{c^2} = \frac{x\xi}{b^2} + \frac{y\eta}{b^2} + \frac{z\zeta}{c^2} - 1, \end{aligned}$$

so the equation of the tangent plane at any point (ξ, η, ζ) on the ellipsoidal surface is then

$$\frac{x\xi}{b^2} + \frac{y\eta}{b^2} + \frac{z\zeta}{c^2} = 1,$$

where the new parameters are the usual (x, y, z).

We note that in order to find an equation of the tangent plane at a point (ξ, η, ζ) on the ellipsoidal surface we just replace one of the x-s, y-s and z-s in the original equation by (ξ, η, ζ). $\Diamond$

13.6 Level surfaces

We shall in the following assume that $f \in C^1$ in three variables. We consider the equation

$$f(x,y,z) = c$$

for some constant c. Let (ξ,η,ζ) be a point, such that

$$(13.2) \qquad f(\xi,\eta,\zeta) = c \qquad \text{and} \qquad \nabla f(\xi,\eta,\zeta) \neq \mathbf{0}.$$

Using a version of the *Implicit function theorem* we can choose a *neighbourhood* of (ξ,η,ζ), such that (13.2) in this neighbourhood describes a surface $\mathcal{F}_c$, which is the graph of some C^1-function in two variables.

Since by assumption $\nabla f(\xi,\eta,\zeta) \neq \mathbf{0}$, at least one of the partial derivatives is $\neq 0$ at this point. Let us for a moment assume that $f_z'(\xi,\eta,\zeta) \neq 0$. Then in this neighbourhood the surface $\mathcal{F}_c$ can be described by a function

$$z = Z(x,y)$$

in two variables, where Z is a C^1-function in some open set containing the point $\xi,\eta)$. This implies that in this neighbourhood, (13.2) can be written in the form

$$f(x,y,Z(x,y)) = c.$$

This equation is then differentiated partially with respect to x and y. Thus, by the *chain rule*,

$$\frac{\partial}{\partial x}\{f(x,y,Z(x,y))\} = \frac{\partial f}{\partial x}\cdot 1 + \frac{\partial f}{\partial y}\cdot 0 + \frac{\partial f}{\partial z}\cdot\frac{\partial Z(x,y)}{\partial x} = (1,0,Z_x'(x,y))\cdot\nabla f(x,y,Z(x,y)) = 0,$$

and

$$\frac{\partial}{\partial y}\{f(x,y,Z(x,y))\} = \frac{\partial f}{\partial x}\cdot 0 + \frac{\partial f}{\partial y}\cdot 1 + \frac{\partial f}{\partial z}\cdot\frac{\partial Z(x,y)}{\partial y} = \left(0,1,Z_y'(x,y)\right)\cdot\nabla f(x,y,Z(x,y)) = 0.$$

Since $(1,0,Z_x'(x,y))$ and $\left(0,1,Z_y'(x,y)\right)$ are linearly independent tangent vectors, and $\nabla f(x,y,Z(x,y))$ is perpendicular to both of them, we conclude that $\nabla f(x,y,Z(x,y))$ is a *normal vector* to the surface at the point $(x,y,Z(x,y))$.

Summing up we get

Theorem 13.2 The tangent plane of a level surface. *A level surface $\mathcal{F}_c$ is given by the equation $f(x,y,z)=c$. Assume that the point $(\xi,\eta,\zeta)\in\mathcal{F}_c$ satisfies $\nabla f(\xi,\eta,\zeta)\neq(0,0,0)$. Then the tangent plane af (ξ,η,ζ) is given by the equation*

$$(x-\xi,y-\eta,z-\zeta)\cdot\nabla f(\xi,\eta,\zeta) = 0,$$

which can also be written

$$P_1(x,y,z) = c,$$

where P_1 is the approximating polynomial of at most degree 1 of the function $f(x,y,z)$ with the expansion point (ξ,η,ζ).

We derived previously,

$$0 = (1, 0, Z'_x(x,y)) \cdot \nabla f(x,y,Z(x,y)) = f'_x(x,y,Z(x,y)) + f'_z(x,y,Z(x,y)) \cdot Z'_x(x,y),$$

and

$$0 = \big(0, 1, Z'_y(x,y)\big) \cdot \nabla f(x,y,Z(x,y)) = f'_y(x,y,Z(x,y)) + f'_z(x,y,Z(x,y)) \cdot Z'_y(x,y),$$

from which, since $f'_z(x,y,Z(x,y)) \neq 0$ by assumption,

$$Z'_x(x,y) = -\frac{f'_x(x,y,Z(x,y))}{f'_z(x,y,Z(x,y))}, \qquad \text{and} \qquad Z'_y(x,y) = -\frac{f'_y(x,y,Z(x,y))}{f'_z(x,y,Z(x,y))}.$$

If $f'_x(x,y,z)) \neq 0$, then analogously the surface is described (locally) as a C^1-function $x = X(y,z)$, such that

$$X'_y(y,z) = -\frac{f'_y(X(y,z),y,z)}{f'_x(X(y,z),y,z)}, \qquad \text{and} \qquad X'_z(y,z) = -\frac{f'_z(X(y,z),y,z)}{f'_x(X(y,z),y,z)}.$$

Finally, if $f'_y(x,y,z) \neq 0$, then we can find a (locally defined) C^1-function $y = Y(x,z)$, such that

$$Y'_x(x,z) = -\frac{f'_x(x,Y(x,z),z)}{f'_y(x,Y(x,z),z)}, \qquad \text{and} \qquad Y'_z(x,z) = -\frac{f'_y(x,Y(x,z),z)}{f'_y(x,Y(x,z),z)}.$$

Then assume that all three partial derivatives f'_x, f'_y, f'_z are $\neq 0$ in a neighbourhood of a point (x,y,z) of a level surface $\mathcal{F}_c$. Then all six equations above exist, and we get by some multiplications,

Theorem 13.3 Theorem of local solution. *Let $f \in C^1$, and let $(x,y,z) \in \mathcal{F}_c$ be point on the level surface given by $f(x,y,z) = c$. Assume that $f'_x(x,y,z)$, $f'_y(x,y,z)$, $f'_z(x,y,z)$ are all $\neq 0$, so locally any variable can be expressed as a C^1-function in the other two variables,*

$$x = X(y,z), \qquad y = Y(x,z), \qquad z = Z(x,y).$$

Then

$$X'_y(y,z) \cdot Y'_x(x,z) = 1, \qquad X'_z(y,z) \cdot Z'_x(x,y) = 1, \qquad Y'_z(x,z) \cdot Z'_y(x,y) = 1,$$

and

$$X'_y(y,z) \cdot Y'_z(x,z) \cdot Z'_x(x,y) = -1 \qquad \textit{and} \qquad X'_z(y,z) \cdot Y'_x(x,z) \cdot Z'_y(x,y) = -1.$$

Example 13.8 We consider a *thermodynamical system* of volume V, temperature T and pressure p. We assume that these variables satisfy an equation of the type $f(p,V,T) = 0$, where f is some C^1-function. When we apply the *Theorem of local solution* above, then we get in the thermodynamical notation,

$$\left(\frac{\partial p}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_p \left(\frac{\partial T}{\partial p}\right)_V = -1, \quad \text{and} \quad \left(\frac{\partial T}{\partial p}\right)_V \left(\frac{\partial p}{\partial T}\right)_V = 1, \ \text{or} \ \left(\frac{\partial T}{\partial p}\right)_V = \frac{1}{\left(\frac{\partial p}{\partial T}\right)_V}. \quad \Diamond$$

14 Examples of tangents (curves) and tangent planes (surfaces)

14.1 Examples of tangents to curves

Example 14.1 *Find in each of the following cases an equation or a parametric description of the tangent to the given curve at the given point.*

1) The curve is given by $x^3 - y^3 + 2x - 3y + 1 = 0$ *and the point is* $(1,1)$.

2) The curve is given by $x^y - y^x = 0$ *and the point is* $(2,4)$.

3) The curve is given by $\mathbf{r}(t) = (\cos t, \sin t, e^t)$ *and the point is* $(1,0,1)$.

4) The curve is given by $\mathbf{r}(t) = (t - \sin t, 1 - cost)$*, and the point is* $\left(\frac{\pi}{2} - 1, 1\right)$, [cf. **Example 14.3.3**]

5) The curve is given by $x^{\frac{2}{3}} + y^{\frac{2}{3}} = \frac{5}{4}$ *and the point is* $\left(\frac{1}{8}, 1\right)$.

6) The curve is given by $\mathbf{r}(t) = \left(\ln t, \cos(t-1), 2t^4 - t^2\right)$ *and the point is* $(0,1,1)$.

7) The curve is given by $\mathbf{r}(t) = (\text{Arcsin } t, \text{Arctan}(2t), \text{Arccot}(2t))$ *and the point is* $\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{6}\right)$.

8) The curve is given by $\mathbf{r}(t) = (2\sin t, -\cos t, 3t)$ *and the point is* $(0,1,3\pi)$.

A Find the tangent to a curve at a point.

D First check if the point lies on the curve. Find the slope of the curve at the point. Write down the equation of the tangent. Note that the case where the curve is given by an equation is treated differently from a curve given a parametric description.

I 1) When we put $(x, y) = (1, 1)$ into the equation we get $1^3 - 1^3 + 2 - 3 + 1 = 0$, proving that $(1, 1)$ lies on the curve.

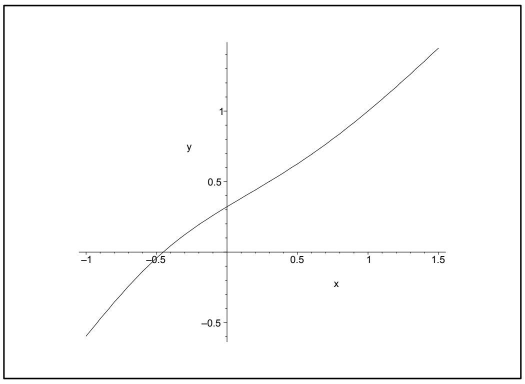


Figure 14.1: The curve in 1).

When we differentiate the equation of the curve with respect to x we get

$$0 = -(3y^2 + 3)\frac{\mathrm{d}y}{\mathrm{d}x} + 3x^2 + 2.$$

When we here put $(x, y) = (1, 1)$, we get $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{5}{6}$, so the equation of the tangent becomes

$$y - 1 = \frac{5}{6}(x - 1).$$

2) By putting $(x, y) = (2, 4)$ into the equation of the curve we get $2^4 - 4^2 = 0$, proving that $(2, 4)$ lies on the curve. When we differentiate the equation of the curve with respect to x, we get

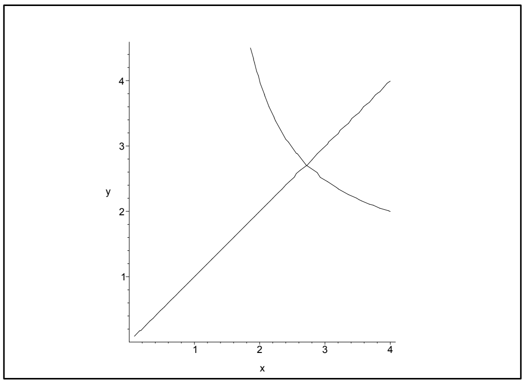


Figure 14.2: The curve in 2); note that the curve contains a self intersection.

$$\begin{aligned} 0 &= \frac{d}{dx}(x^y - y^x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{y \ln x}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x \ln y}\right) = x^y \frac{\mathrm{d}}{\mathrm{d}x}(y \ln x) - y^x \frac{\mathrm{d}}{\mathrm{d}x}(x \ln y) \\ &= x^y \left\{\ln x \cdot \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x}\right\} - y^x \left\{\ln y + \frac{x}{y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x}\right\}. \end{aligned}$$

When we put $(x, y) = (2, 4)$, we get

$$0 = 16\left\{\ln 2 \cdot \frac{dy}{dx} + 2\right\} - 16\left\{2\ln 2 + \frac{1}{2}\,\frac{dy}{dx}\right\} = 16\left\{(\ln 2 - \ dfrac12)\,\frac{dy}{dx} - (2\ln 2 - 2)\right\},$$

hence

$$\frac{dy}{dx} = \frac{2\ln 2 - 2}{\ln 2 - \frac{1}{2}} = 4 \cdot \frac{\ln 2 - 1}{2\ln 2 - 1}.$$

The equation of the tangent becomes

$$y - 4 = -4 \cdot \frac{1 - \ln 2}{2\ln 2 - 1}\,(x - 2).$$

3) It is immediately seen that $\mathbf{r}(0) = (1, 0, 1)$, so the point lies on the curve. Furthermore,

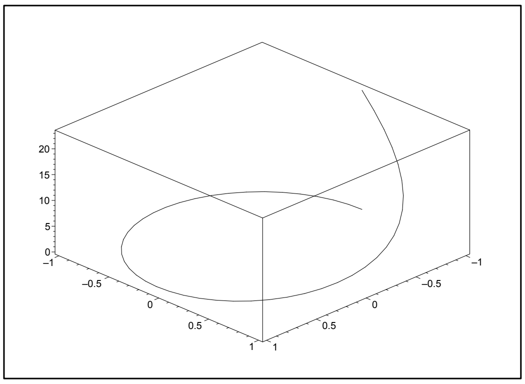


Figure 14.3: The curve in 3).

$$\mathbf{r}'(t) = (-\sin t, \cos t, e^t), \qquad \mathbf{r}'(0) = (0, 1, 1),$$

so a parametric description of the tangent is

$$(x(u), y(u), z(u)) = \mathbf{r}(0) + u \cdot \mathbf{r}'(0) = (1, 0, 1) + u(0, 1, 1), \qquad u \in \mathbb{R}.$$

4) It follows immediately that $\mathbf{r}\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2} - 1, 1\right)$, proving that the point lies on the curve corresponding to the value of the parameter $t = \frac{\pi}{2}$. Furthermore,

$$\mathbf{r}'(t) = (1 - \cos t, \sin t), \qquad \mathbf{r}'\left(\frac{\pi}{2}\right) = (1, 1).$$

A parametric description of the tangent is

$$(x(u), y(u)) = \left(\frac{\pi}{2}, 1\right) + u \cdot (1, 1), \qquad u \in \mathbb{R}.$$

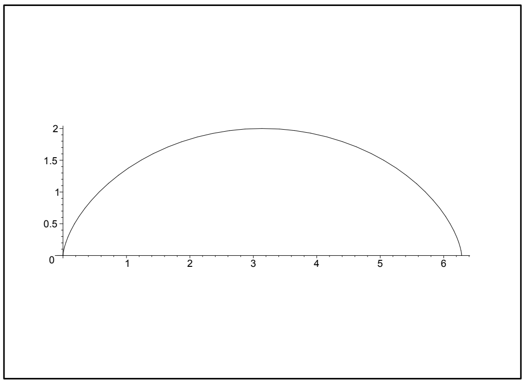


Figure 14.4: The curve in 4).

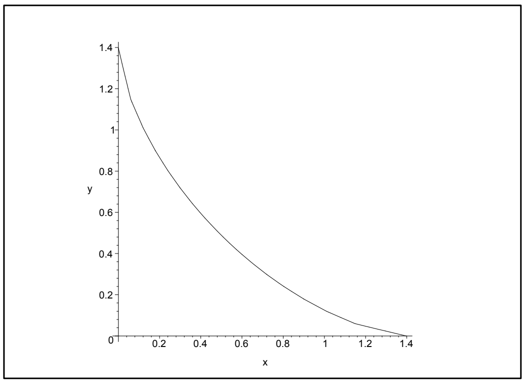


Figure 14.5: The curve in 5).

5) It follows from

$$\left(\frac{1}{8}\right)^{\frac{2}{3}}+1^{\frac{2}{3}}=\left(\frac{1}{2}\right)^{3\cdot\frac{2}{3}}+1=\frac{1}{4}+1=\frac{5}{4},$$

that the point $\left(\frac{1}{8},1\right)$ lies on the curve.
If we put $f(x,y)=x^{2/3}+y^{2/3}$, then

$$\bigtriangledown f(x,y)=\frac{2}{3}\left(\frac{1}{\sqrt[3]{x}},\frac{1}{\sqrt[3]{y}}\right)\qquad\text{for } x\neq 0 \text{ and } y\neq 0,$$

and hence

$$\bigtriangledown f\left(\frac{1}{8},1\right)=\frac{2}{3}(2,1),$$

which indicates the direction of the *normal* of the curve at the point. The direction of the tangent is then perpendicular to the normal $\bigtriangledown f$, e.g. $(1,-2)$.

A parametric description of the tangent is

$$(x(t), y(t)) = \left(\frac{1}{8}, 1\right) + t\,(1, -2) = \left(t + \frac{1}{8}, -2t + 1\right), \qquad t \in \mathbb{R}.$$

This implies that $t = x - \dfrac{1}{8}$, so $y = 1 - 2t = 1 - 2x + \dfrac{1}{4} = \dfrac{5}{4} - 2x$. Finally, we can write the equation of the tangent

$$y + 2x = \frac{5}{4}.$$

ALTERNATIVELY the equation $f(x, y) = \dfrac{5}{4}$ is differentiated with respect to x. Then

$$\frac{2}{3}\,x^{-\frac{1}{3}} + \frac{2}{3}\,y^{-\frac{1}{3}}\,\frac{dy}{dx} = 0 \qquad \text{for } x \neq 0 \text{ and } y \neq 0.$$

At the point $\left(\dfrac{1}{8}, 1\right)$ we find the slope

$$\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{\frac{1}{3}} = -2,$$

so the equation of the tangent becomes

$$y-1=-2\left(x-\frac{1}{8}\right)=-2x+\frac{1}{4},$$

i.e.

$$y+2x=\frac{5}{4}.$$

REMARK. The methods fail when either $x=0$ or $y=0$. This is in accordance with the fact that we have cusps in the corresponding points of the curve. $\Diamond$

6) Putting $t=1$ we get

$$\mathbf{r}(1)=(\ln 1,\cos(1-1),2\cdot 1^4-1^2)=(0,1,1),$$

so $(0,1,1)$ lies on the curve. Furthermore,

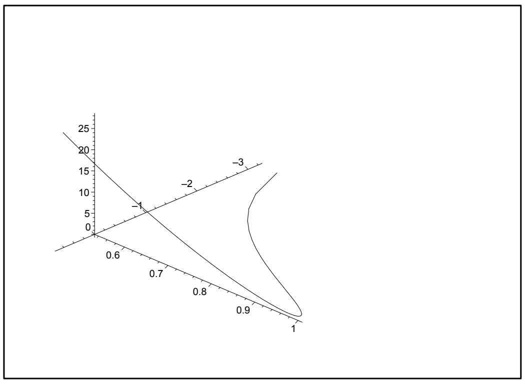


Figure 14.6: The curve in 6).

$$\mathbf{r}'(t)=\left(\frac{1}{t},-\sin(t-1),8t^3-2t\right),\qquad \mathbf{r}'(1)=(1,0,6),$$

so a parametric description of the tangent is

$$(x(u),y(u),z(u))=\mathbf{r}(1)+u\,\mathbf{r}'(1)=(0,1,1)+u\,(1,0,6).$$

7) If we choose $t=\dfrac{\sqrt{3}}{2}$, we get by insertion,

$$\mathbf{r}\left(\frac{\sqrt{3}}{2}\right)=\left(\operatorname{Arcsin}\left(\frac{\sqrt{3}}{2}\right),\operatorname{Arctan}(\sqrt{3}),\operatorname{Arccot}(\sqrt{3})\right)=\left(\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{6}\right),$$

so $\left(\dfrac{\pi}{3},\dfrac{\pi}{3},\dfrac{\pi}{6}\right)$ lies on the curve. Furthermore,

$$\mathbf{r}'(t)=\left(\frac{1}{\sqrt{1-t^2}},\frac{2}{1+4t^2},-\frac{2}{1+4t^2}\right),$$

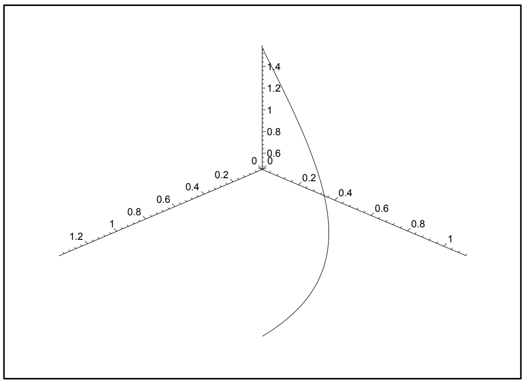


Figure 14.7: The curve in 7).

so

$$\mathbf{r}'\left(\frac{\sqrt{3}}{2}\right) = \left(\frac{1}{\sqrt{1-\frac{3}{4}}}, \frac{2}{1+4\cdot\frac{3}{4}}, -\frac{2}{1+4\cdot\frac{3}{4}}\right) = \left(2, \frac{1}{2}, -\frac{1}{2}\right),$$

and a parametric description of the tangent is

$$(x(u), y(u), z(u)) = \left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{6}\right) + u\,(4, 1, -1).$$

8) If we choose $t = \pi$ we see that

$$\mathbf{r}(\pi) = (2\sin\pi, -\cos\pi, 3\pi) = (0, 1, 3\pi),$$

so $(0, 1, 3\pi)$ lies on the curve. Furthermore,

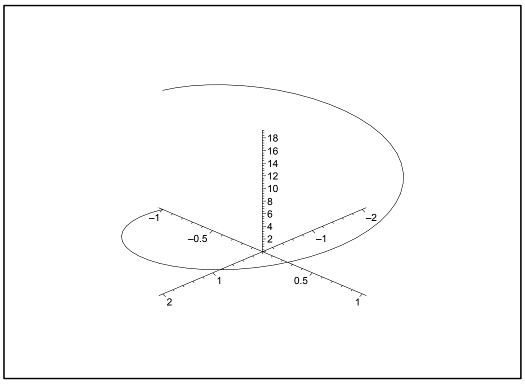


Figure 14.8: The curve in **Example 14.1.8**.

$$\mathbf{r}'(t) = (2\cos t, \sin t, 3)$$

where

$$\mathbf{r}'(\pi) = (2\cos\pi, \sin\pi, 3) = (-2, 0, 3).$$

A parametric description of the tangent is

$$(x(u), y(u), z(u)) = (0, 1, 3\pi) + u\,(-2, 0, 3).$$

Example 14.2 *A curve is given by the parametric description*

$$x = a\{\ln(1 + \sin t) - \ln \cos t - \sin t\}, \quad y = -a \cos t, \qquad t \in \left[0, \frac{\pi}{2}\right[.$$

1) Prove that

$$\frac{dx}{dt} = \frac{a\,\sin^2 t}{\cos t},$$

and find the direction of the tangent of the curve in the point $P(t)$ corresponding to the value $t > 0$ of the parameter.

2) Find an equation of a parametric description of the tangent at $P(t)$.

3) Finally find the length of the straight line from $P(t)$ to the intersection of the tangent with the X axis,

A Tangent of a curve, which is given by a parametric description.

D Follow the guidelines of the text.

I

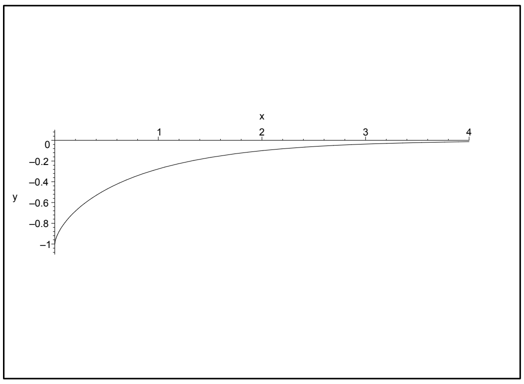


Figure 14.9: The curve in **Example 14.2**

1) By a differentiation,

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= a\left(\frac{\cos t}{1+\sin t} + \frac{\sin t}{\cos t} - \cos t\right) = a\left(\frac{\cos t(1-\sin t)}{1-\sin^2 t} + \frac{\sin t}{\cos t} - \cos t\right)\\ &= a\left(\frac{\cos t(1-\sin t)}{\cos^2 t} + \frac{\sin t}{\cos t} - \cos t\right) = a\left(\frac{1-\sin t}{\cos t} + \frac{\sin t}{\cos t} - \cos t\right)\\ &= a\left(\frac{1}{\cos t} - \cos t\right) = a \cdot \frac{1-\cos^2 t}{\cos t} = a \cdot \frac{\sin^2 t}{\cos t}.\end{aligned}$$

This gives us the direction of the tangent

$$\mathbf{r}'(t) = \left(\frac{dx}{dt}, \frac{dy}{day}\right) = a\left(\frac{\sin^2 t}{\cos t}, \sin t\right) = a\tan t\,(\sin t, \cos t)$$

for $t \in \left]0, \frac{\pi}{2}\right[$.

Note that the latter form is very convenient, because it immediately gives us the vector $(\sin t, \cos t)$ of the direction.

2) From the result of 1) we get a parametric description of the tangent at the point $P(t)$,

$$(x(u), y(u)) = a(\ln(1+\sin t) - \ln\cos t - \sin t, -\cos t) + au\,(\tan t)\,(\sin t, \cos t),$$

where $t \in \left]0, \frac{\pi}{2}\right[$, and where one may put the factor $(\tan t)$ into the parameter u, hence obtaining an equivalent (and simpler) solution without the factor $(\tan t)$.

When we apply the variant above, we get

$$\begin{cases} au \cdot \sin t \cdot \tan t & = & x - a\{\ln(1+\sin t) - \ln\cos t - \sin t\}, \\ au \cdot \sin t & = & y + a\cos t, \end{cases}$$

hence in particular,

$$\begin{aligned} &\tan t \cdot (y + a\cos t) = \tan t \cdot y + a\sin t = au\,\sin t \cdot \tan t \\ &\qquad = x - a\ln(1+\sin t) + a\ln\cos t + a\sin t. \end{aligned}$$

The equation of the tangent is obtained by a reduction,

(14.1) $\tan t \cdot y = x - a\ln(1+\sin t) + a\ln\cos t,$

which can also be written in one of the two forms

$$\begin{cases} y = \cot t \cdot \{x - a\ln(1+\sin t) + a\ln\cos t\}, \\ \sin t \cdot y = \cos t \cdot \{x - a\ln(1+\sin t) + a\ln\cos t\}. \end{cases}$$

3) The intersection point of the tangent with the X axis is found by putting $y = 0$ into the equation of the tangent and then solve the equation. Here we have two variants.

a) If we use the parametric description we get

$$y(u) = -a \cdot \cos t + au \cdot \sin t = 0,$$

which is fulfilled for $u = \cot t$. Using this u we get

$$x(u) = a\{\ln(1+\sin t) - \ln\cos t - \sin t + \sin t\} = a\{\ln(1+\sin t) - \ln\cos t\}.$$

Since $P(t)$ has the coordinates $(x(0), y(0))$ from the equation of the tangent, the length of the straight line on the tangent between $P(t)$ and the intersection point with the X axis is

$$\begin{aligned} L &= \sqrt{\{x(0)-x(u)\}^2 - \{y(0)\}^2} \\ &= \sqrt{\{\ln(1+\sin t)-\ln\cos t-\sin t-\ln(1+\sin t)+\ln\cos t\}^2 + \{-\cos t\}^2} \\ &= a\sqrt{\sin^2 t + \cos^2 t} = a. \end{aligned}$$

b) If we instead use the equation of the tangent, it follows from (14.1) that

$$0 = x - a\ln(1+\sin t) + a\ln\cos t,$$

so the abscissa of the intersection point is given by

$$x = a\{\ln(1+\sin t) - \ln\cos t\},$$

thus $x - x_0 = -\sin t$, and

$$L = a\sqrt{\{x-x_0\}^2 + y_0^2} = a\sqrt{\sin^2 t + \cos^2 t} = a.$$

Example 14.3 *There are below given some plane curves by a parametric description*

$$\mathbf{x} = \mathbf{r}(t), \qquad t \in I.$$

Explain in each case why the curve is a C^1-curve with $\mathbf{r}'(0) = \mathbf{0}$. Then check whether the curve has a cusp at the point $\mathbf{r}(0)$. If the curve does not *have a cusp, then find another parametric description of the curve,*

$$\mathbf{x} = \mathbf{R}(u), \qquad u \in J,$$

such that $u = 0$ corresponds to $t = 0$, and such that $\mathbf{R}'(0) \neq \mathbf{0}$.

1) $\mathbf{r}(t) = (t^2, t^3)$ *for* $t \in \mathbb{R}$.

2) $\mathbf{r}(t) = (t^3, t^6)$ *for* $t \in \mathbb{R}$.

3) $\mathbf{r}(t) = (t - \sin t, 1 - \cos t)$ *for* $t \in [-\pi, \pi]$. [Cf. **Example 14.1.4**].

4) $\mathbf{r}(t) = \left(\cos^3 t, \sin^3 t\right)$ *for* $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

5) $\mathbf{r}(t) = \left(t^3, \sin^2 t\right)$ *for* $t \in \left[-\frac{\pi}{2}, \frac{pi}{2}\right]$.

A C^1-curves with or without cusps.

D Follow the guidelines. One may also sketch the curve.

I 1) The coordinate functions of $\mathbf{r}(t) = (t^2, t^3)$, $t \in \mathbb{R}$ are clearly C^∞-functions in t.

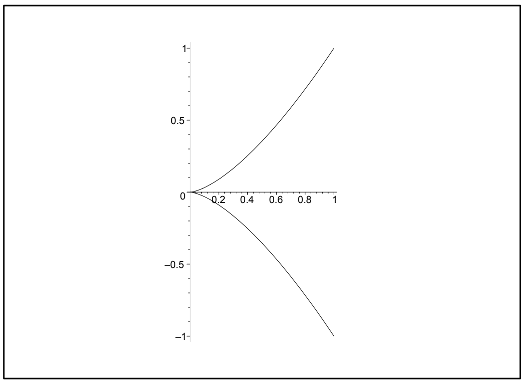


Figure 14.10: The curve in 1).

Then by a differentiation,

$$\mathbf{r}'(t) = (2t, 3t^2) \qquad \text{where } \mathbf{r}'(0) = \mathbf{0}.$$

The figure indicates that we have a cusp. However, one must never in situations like this trust the figure 100 %. An argument is needed! Now, $\frac{dx}{dt}$ changes its sign, when t goes through 0, while $\frac{dy}{dt}$ does not change its sign, and it goes also faster towards 0 than $\frac{dx}{dt}$. We therefore conclude that we indeed have a cusp for $t = 0$

It is here possible to eliminate t, and one gets

$$x = y^{2/3}, \qquad y \in \mathbb{R}.$$

REMARK. If instead one tries to express y by x, then we get the more confused expression $|y| = x^{3/2}$, due to the fact that the square root occurs latently. Always be careful, whenever the *square root* enters a problem. "If one can handle the square root, then one can handle anything inside mathematics."

2) The coordinate functions of $\mathbf{r}(t) = (t^3, t^6)$, $t \in \mathbb{R}$, are clearly C^∞-functions in t.

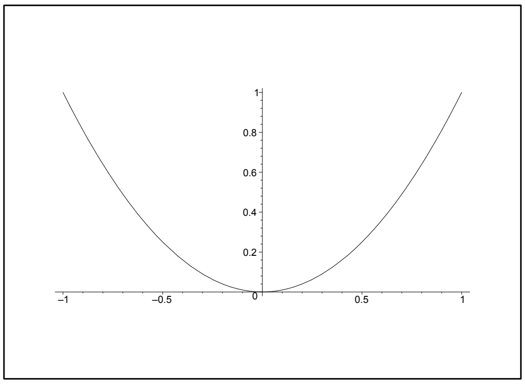


Figure 14.11: The curve in 2).

We get by a differentiation

$$\mathbf{r}'(t) = (3t^2, 6t^5) \qquad \text{where } \mathbf{r}'(0) = \mathbf{0}.$$

It follows immediately that $y = t^6 = (t^3)^2 = x^2$, so the curve is a parabola, which does not have a cusp.

An obvious alternative parametric description is $u = t^3$, by which $u = 0$ for $t = 0$, and

$$\mathbf{R}'(u) = (u, u^2), \qquad u \in \mathbb{R},$$

where

$$\mathbf{R}'(u) = (1, 2u) \quad \text{and} \quad \mathbf{R}'(0) = (1, 0) \neq \mathbf{0}.$$

3) Clearly, the coordinate functions of

$$\mathbf{r}(t) = (t - \sin t, 1 - \cos t), \qquad t \in [-\pi, \pi],$$

are C^∞-functions in $t \in]-\pi, \pi[$.
The curve is a part of the cycloid with a cusp at $\mathbf{0}$. We get by a differentiation,

$$\mathbf{r}'(t) = (1 - \cos t, \sin t) \qquad \text{where } \mathbf{r}'(0) = \mathbf{0}.$$

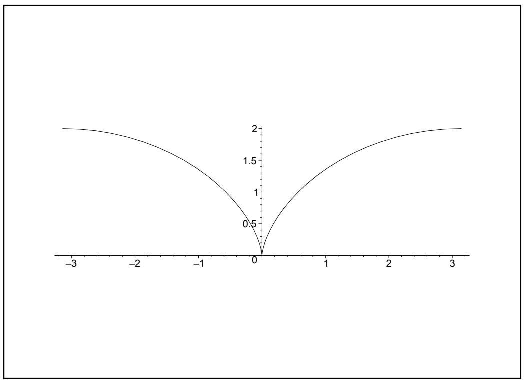


Figure 14.12: The curve in 3).

Since $\dfrac{dx}{dt}$ does not change its sign, while $\dfrac{\mathrm{d}y}{\mathrm{d}t}$ changes its sign when we pass through $t = 0$, and since

$$1 - \cos t = \frac{1}{2}\,t^2 + t^2\varepsilon(t)$$

tends faster towards zero than $\sin t = t + t\varepsilon(t)$ for $t \to 0$, we conclude that the curve has a cusp.

4) The coordinate functions of $\mathbf{r}(t) = (\cos^3 t, \sin^3 t)$, $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, are clearly C^∞-functions in the open interval $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.

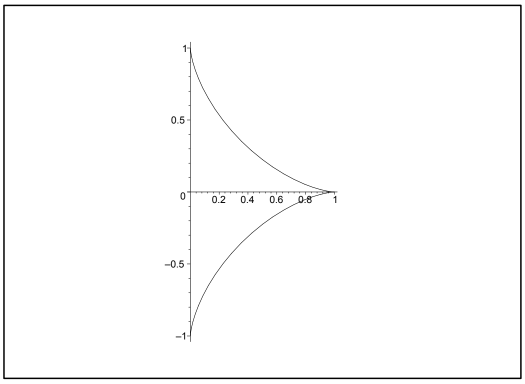


Figure 14.13: The curve in 4).

The figure indicate that we may have a cusp for $t = 0$, i.e. at the point $(1, 0)$.

Then by a differentiation,

$$\mathbf{r}'(t) = 3(-\cos^2 t \sin t, \sin^2 t \cos t) \qquad \text{where } \mathbf{r}'(0) = \mathbf{0}.$$

Since $\frac{\mathrm{d}x}{\mathrm{d}t}$ changes its sign, while $\frac{\mathrm{d}y}{\mathrm{d}t}$ does not for $t \to 0$, and since $\frac{\mathrm{d}y}{\mathrm{d}t}$ tends faster towards 0 than $\frac{\mathrm{d}x}{dat}$ for $t \to 0$, we conclude again that the curve has a cusp.

5) The coordinate functions of $\mathbf{r}(t) = (t^3, \sin^2 t)$ are clearly C^∞-functions in the interval $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.

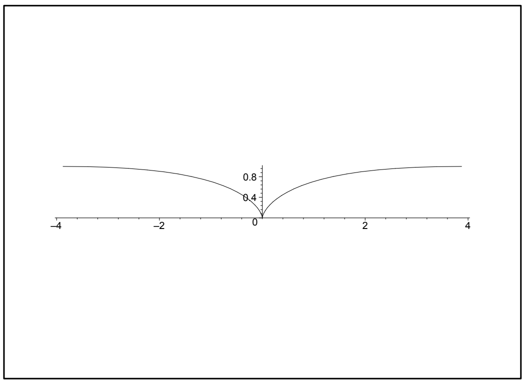


Figure 14.14: The curve in 5).

Then by a differentiation,

$$\mathbf{r}'(t) = (3t^2, 2\sin t \cos t) = (3t^2, \sin 2t) \qquad \text{where } \mathbf{r}'(0) = \mathbf{0}.$$

The curve has a cusp for $t = 0$, because $\frac{\mathrm{d}y}{\mathrm{d}t}$ changes its sign, while $\frac{\mathrm{d}x}{\mathrm{d}t}$ does not, when we pass through $t = 0$, and because $\frac{\mathrm{d}x}{\mathrm{d}t} = 3t^2$ goes faster towards zero for $t \to 0$ than $\frac{\mathrm{d}y}{dat} \approx 2t$.

Remark. Since $\sin t \approx t$ for small t, the curve lies in the neighbourhood of $t = 0$ close to

$$\tilde{\mathbf{r}}(t) = (t^3, t^2) \qquad \text{for } |t| \text{ small.}$$

Cf. 1). $\Diamond$

Example 14.4 *A space curve $\mathcal{K}$ is given by the parametric description*

$$\mathbf{r}(r) = \left(t^2, e^{2t}, 4 + t^3\right), \qquad t \in \mathbb{R}.$$

1) Find a parametric description of the tangent to $\mathcal{K}$ at the point $\mathbf{r}(2)$.

2) Prove that this tangent intersects the Y axis at some point $(0, \beta, 0)$, and find β.

A Tangent to a space curve.

D Standard methods.

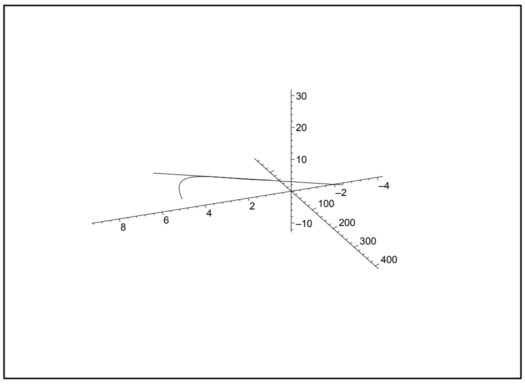


Figure 14.15: The curve $\mathcal{K}$ for $t \in [1, 3]$ and its tangent at the point $\mathbf{r}(2)$. Notice the different scales on the axes.

I 1) We get by a differentiation,

$$\mathbf{r}'(t) = \left(2t, 2e^{2t}, 3t^2\right), \quad \text{hence} \quad \mathbf{r}'(2) = \left(4, 2e^4, 12\right).$$

Since $\mathbf{r}(2) = \left(4, e^4, 12\right)$, the parametric description of the tangent to $\mathcal{K}$ at the point $\mathbf{r}(2)$ is

$$\begin{aligned}(x(t), y(t), z(t)) &= \mathbf{r}(2) + t \cdot \mathbf{r}'(2) \\ &= \left(4, e^4, 12\right) + t\left(4, 2e^4, 12\right) \\ &= \left(4(t+1), e^4(2t+1), 12(t+1)\right).\end{aligned}$$

2) It follows immediately that $x(t) = z(t) = 0$ for $t = -1$, hence

$$(x(-1), y(-1), z(-1)) = \left(0, -e^4, 0\right) = (0, \beta, 0),$$

and we conclude that

$$\beta = -e^4.$$

14.2 Examples of tangent planes to a surface

Example 14.5 *Find in each of the following cases an equation of the tangent plane to the given surface at the given point.*

1) *The surface is given by* $\mathbf{r}(u,v) = (2u, u^2+v, v^2)$*, and the point is* $(0,1,1)$.

2) *The surface is given by* $\mathbf{r}(u,v) = u^2 - v^2, u+v, u^2+4v)$*, and the point is* $\left(-\frac{1}{4}, \frac{1}{2}, 2\right)$.

3) *The surface is given by* $xy - 2xz + xyz = 6$*, and the point is* $(-3,2,1)$.

4) *The surface is given by* $\mathbf{r}(u,v) = (u+v, u^2+v^2, u^3+v^3)$ *for* $u > v$*, and the point is* $(0,2,0)$

5) *The surface is given by* $\cos x - \cos y + \sin z = 1$*, and the point is* $\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}\right)$.

6) *The surface is given by* $z^2 + 2xz + 2yz + 4y = 0$*, and the point is* $\left(1, \frac{1}{2}, -1\right)$.

7) *The surface is given by* $z = \text{Arctan}(xy)$*, and the point is* $\left(1, 1, \frac{\pi}{4}\right)$.

A Tangent plane to a given surface.

D First check that the given point lies on the surface.

If the surface is given by a parametric description, then calculate the normal vector

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}.$$

If the surface instead is given implicitly by $f(x,y,z) = c$, then it can be considered as a level surface, so its normal vector is $\bigtriangledown f(x,y,z)$.

I 1) Clearly, the point corresponds to the values of the parameters $u = 0$ and $v = 1$.

Then by partial differentiations

$$\frac{\partial \mathbf{r}}{\partial u} = (2, 2u, 0) \qquad \text{and} \qquad \frac{\partial \mathbf{r}}{\partial v} = (0, 1, 2v),$$

so the normal vector is

$$\frac{\partial \mathbf{r}}{\partial u}(0,1) \times \frac{\partial \mathbf{r}}{\partial v}(0,1) = (2,0,0) \times (0,1,2) = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 2 & 0 & 0 \\ 0 & 1 & 2 \end{vmatrix} = (0, -4, 2).$$

The tangent plane is given by

$$0 \cdot (x-0) - 4(y-1) + 2(z-1) = 0,$$

hence by a rearrangement,

$$2y - z = 1.$$

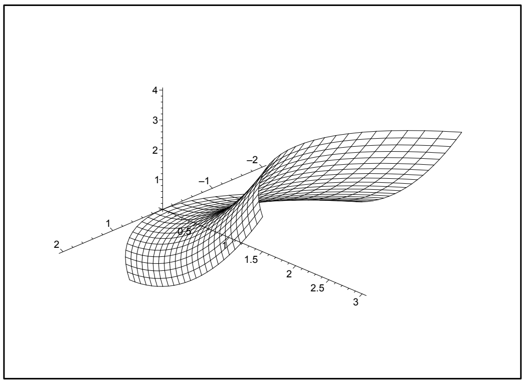


Figure 14.16: The surface in 1).

2) We first check that $\left(-\frac{1}{4}, \frac{1}{2}, 2\right)$ lies on the surface. We that solve the equations

$$\begin{cases} u^2 - v^2 & = & -\dfrac{1}{4}, \\ u + v & = & \dfrac{1}{2}, \\ u^2 + 4v & = & 2. \end{cases}$$

When we divide the second equation into the first one we get

$$u-v=-\frac{1}{2} \qquad \text{and} \qquad u+v=\frac{1}{2},$$

thus $u=0$ and $v=\frac{1}{2}$. These values solve the first two equations, and then we see by insertion that the third one also holds. Thus we have proved that the point $\left(-\frac{1}{4},\frac{1}{2},2\right)$ lies on the surface corresponding to $(u,v)=\left(0,\frac{1}{2}\right)$.

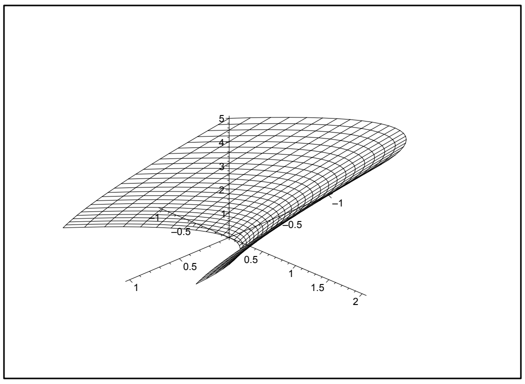


Figure 14.17: The surface in 2).

Then we get by partial differentiation,

$$\frac{\partial \mathbf{r}}{\partial u}=(2u,1,2u), \qquad \frac{\partial \mathbf{r}}{\partial u}\left(0,\frac{1}{2}\right)=(0,1,0),$$

$$\frac{\partial \mathbf{r}}{ddv}=(-2v,1,4), \qquad \frac{\partial \mathbf{r}}{\partial v}\left(0,\frac{1}{2}\right)=(-1,1,4).$$

The normal vector is

$$\frac{\partial \mathbf{r}}{\partial u}\left(0,\frac{1}{2}\right)\times\frac{\partial \mathbf{r}}{\partial v}\left(0,\frac{1}{2}\right)=(0,1,0)\times(-1,1,4)=\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 0 & 1 & 0 \\ -1 & 1 & 4 \end{vmatrix}=(4,0,1).$$

The tangent plane is given by

$$4\left(x+\frac{1}{4}\right)+0\cdot\left(y-\frac{1}{2}\right)+1\cdot(z-2)=0,$$

which is reduced to

$$4x+z=1.$$

3) This is an implicitly given surface. Putting $(x,y,z)=(-3,2,1)$ into the left hand side of the equation we get

$$-3\cdot 2-2(-3)\cdot 1+3\cdot 2\cdot 1=-6+6+6=6,$$

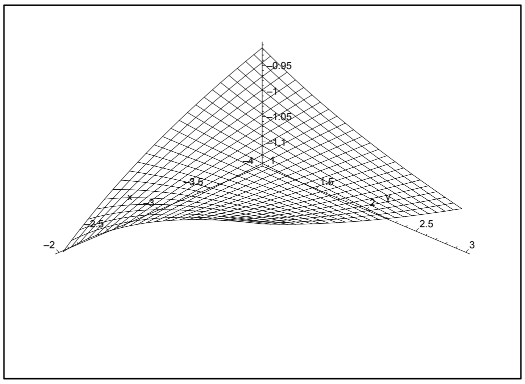


Figure 14.18: The surface in 3).

so the point lies on the surface.

The gradient is given by

$$\bigtriangledown f(x,y,z) = (y-2z, x+3z, -2x+3y),$$

hence

$$\bigtriangledown f(-3,2,1) = (2-2, -3+3, 606) = (0,0,12).$$

A normal vector is $(0,0,1)$, and the tangent plane is given by

$$z = 1.$$

4) If $(0,2,0)$ lies on the surface we must have

$$\begin{cases} u+v=0, \\ u^2+v^2=2, \\ u^3+v^3=0, \end{cases}$$

where the solution should also satisfy $u > v$. It is easily seen that $(u,v) = (1,-1)$ is a solution.
Remark. In the present formulation of the example one does not have to check whether there are other possible values of the parameters, which can be used instead. For completeness we prove that there actually are no other values. This follows from

$$0 = (u+v)^2 = (u^2+v^2) + 2uv = 2 + 2uv,$$

thus $uv = -1$. This implies that u and v are solutions of the equation of second degree

$$\lambda^2 - (u+v)\lambda + uv = \lambda^2 - 0 \cdot \lambda - 1 = \lambda^2 - 1 = 0,$$

i.e. $\lambda = \pm 1$. Since $u > v$, we see that $(u,v) = (1,-1)$ is the only solution. $\Diamond$.

Now,

$$\frac{\partial \mathbf{r}}{\partial u} = (1, 2u, 3u^2), \qquad \frac{\partial \mathbf{r}}{\partial u}(1,-1) = (1,2,3),$$

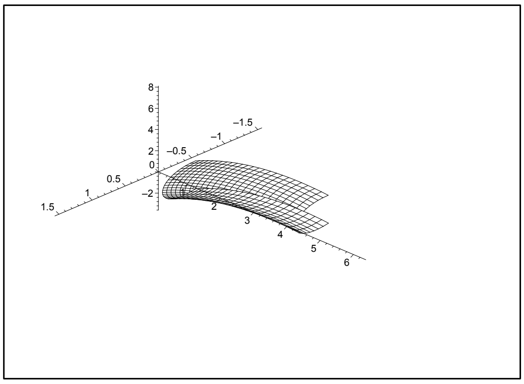


Figure 14.19: The surface of 4).

$$\frac{\partial \mathbf{r}}{\partial v} = (1, 2v, 3v^2), \qquad \frac{\partial \mathbf{r}}{\partial v}(1,-1) = (1,-2,3).$$

A normal vector is

$$\frac{\partial \mathbf{r}}{\partial u}(1,-1) \times \frac{\partial \mathbf{r}}{\partial v} = (1,2,3) \times (1,-2,3) = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & 2 & 3 \\ 1 & -2 & 3 \end{vmatrix} = 4(3,0,-1).$$

The equation of the tangent plane is

$$z = 3x.$$

5) When $(x, y, z) = \left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}\right)$ is put into the left hand side of the equation, we get

$$f\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}\right) = \cos\frac{\pi}{3} - \cdot\frac{\pi}{3} + \sin\frac{\pi}{2} = 1,$$

proving that the point lies on the surface.

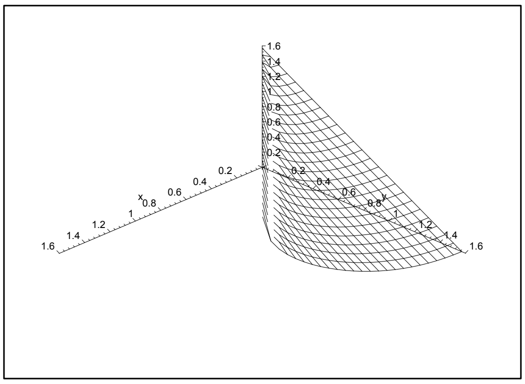


Figure 14.20: The surface in 5).

The gradient is

$$\nabla f(x, y, z) = (-\sin x, \sin y, \cos z)$$

and

$$\nabla f\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}\right) = \left(-\sin\frac{\pi}{3}, \sin\frac{\pi}{3}, \cos\frac{\pi}{2}\right) = -\sin\frac{\pi}{3} \cdot (1, -1, 0).$$

A convenient normal of the surface at the point is therefore $(1, -1, 0)$, and a tangent plane is

$$0 = (1, -1, 0) \cdot \left(x - \frac{\pi}{3}, y - \frac{\pi}{3}, z - \frac{\pi}{2}\right) = x - \frac{\pi}{3} - \left(y - \frac{\pi}{3}\right) = x - y,$$

i.e.

$$y = x.$$

6) When $(x, y, z) = \left(1, \frac{1}{2}, -1\right)$ is put into the left hand side of the equation we get

$$(-1)^2 + 2 \cdot 1 \cdot (-1) + 2 \cdot \frac{1}{2} \cdot (-1) + 4 \cdot \frac{1}{2} = 1 - 2 - 1 + 2 = 0,$$

proving that the point $\left(1, \frac{1}{2}, -1\right)$ lies on the surface.

It follows from

$$\nabla F = (2z, 2z + 4, 2z + 2x + 2y),$$

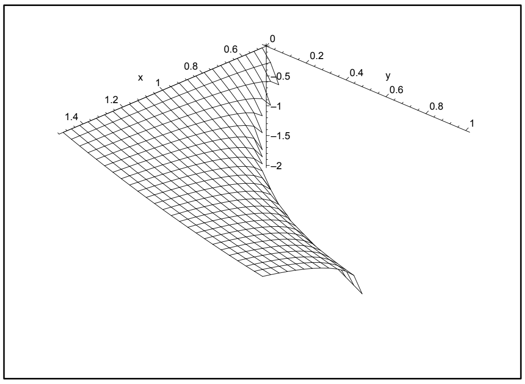


Figure 14.21: The surface in 6).

that

$$\nabla F\left(1,\frac{1}{2},-1\right) = (-2,-2+4,-2+2+1) = (-2,2,1)$$

is perpendicular to the surface at the point $\left(1,\frac{1}{2},-1\right)$.

The tangent equation becomes

$$\begin{aligned} 0 &= \nabla F\left(1,\frac{1}{2},-1\right)\cdot\left(x-1,y-\frac{1}{2},z+1\right) \\ &= -2(x-1)+2\left(y-\frac{1}{2}\right)+(z+1) \\ &= -2x+2y+z+2. \end{aligned}$$

7) The equation is equivalent to

$$F(x,y,z) = \text{ Arctan}(x,y) - z = 0.$$

If we put $(x,y,z) = \left(1,1,\frac{\pi}{4}\right)$, we get

$$F\left(1,1,\frac{\pi}{4}\right) = \text{ Arctan } 1 - \frac{\pi}{4} = 0,$$

and $\left(1,1,\frac{\pi}{4}\right)$ lies on the surface.

Now

$$\nabla F = \left(\frac{y}{1+(xy)^2},\frac{x}{1+(xy)^2},-1\right),$$

and

$$\nabla F\left(1,1,\frac{\pi}{4}\right) = \frac{1}{2}(1,1,-2),$$

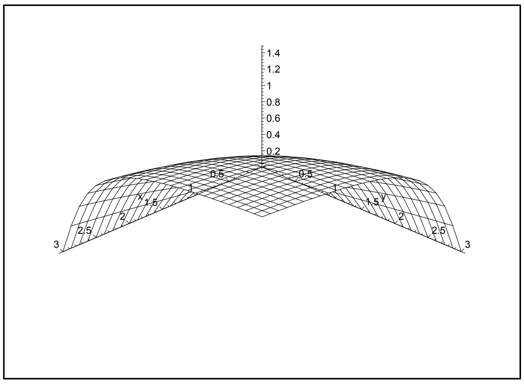


Figure 14.22: The surface of 7).

so the equation of the tangent plane becomes

$$\begin{aligned} 0 &= 2 \bigtriangledown F\left(1,1,\frac{\pi}{4}\right) \cdot \left(x-1, y-1, z-\frac{\pi}{4}\right) \\ &= (1,1,-2) \cdot \left(x-1, y-1, z-\frac{\pi}{4}\right) \\ &= x-1+y-1-2z+\frac{\pi}{2}, \end{aligned}$$

hence by a rearrangement,

$$x+y-2z = 2-\frac{\pi}{2}.$$

Example 14.6 *Let $\mathcal{F}$ be an hyperboloid with two nets, given by*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

Consider $\mathcal{F}$ as a level surface and find an equation of the tangent plane of $\mathcal{F}$ through the point (ξ, η, ζ).

A Tangent plan for a level surface.

D Find the gradient and set up the equation of the tangent plane. We shall need that (ξ, η, ζ) lies on the surface.

I If we put

$$F(x,y,z) = \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2},$$

we see that the hyperboloid $\mathcal{F}$ can be considered as the level surface $F(x,y,z) = 1$.

Let $(\xi, \eta, \zeta) \in \mathcal{F}$, i.e.

$$\frac{\xi^2}{a^2} - \frac{\eta^2}{b^2} - \frac{\zeta^2}{c^2} = 1.$$

The gradient in (ξ, η, ζ) is

$$\nabla F(\xi, \eta, \zeta) = 2\left(\frac{\xi}{a^2}, -\frac{\eta}{b^2}, -\frac{\zeta}{c^2}\right),$$

so the equation of the tangent plane is

$$0 = \frac{1}{2} \nabla F(\xi, \eta, \zeta) \cdot (x - \xi, y - \eta, z - \zeta) = \frac{\xi x - \xi^2}{a^2} - \frac{\eta y - \eta^2}{b^2} - \frac{\zeta z - \zeta^2}{c^2},$$

hence by a rearrangement

$$\frac{\xi x}{a^2} - \frac{\eta y}{b^2} - \frac{\zeta z}{c^2} = \frac{\xi^2}{a^2} - \frac{\eta^2}{b^2} - \frac{\zeta^2}{c^2} = 1.$$

Thus the equation of the tangent plane at $(\xi, \eta, \zeta) \in \mathcal{F}$ is

$$\frac{\xi x}{a^2} - \frac{\eta y}{b^2} - \frac{\zeta z}{c^2} = 1.$$

Example 14.7 *Let $\mathcal{F}$ be that part of the surface of equation $xyz = 1$ which lies in the first octant.*

1) Find an equation of the tangent plane $\mathcal{T}$ of $\mathcal{F}$ at the point (ξ, η, ζ) on the surface.

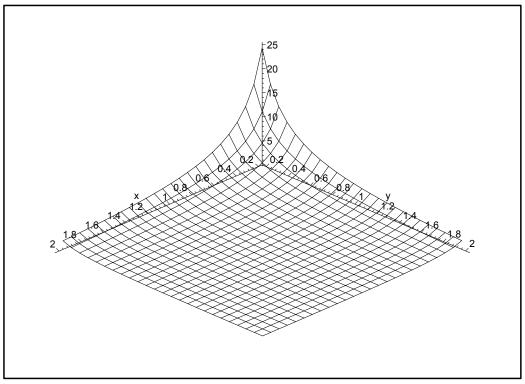
2) Find the intersection points between $\mathcal{T}$ and the coordinate axes.

3) Find the volume of the tetrahedron, which is bounded by $\mathcal{T}$ and the three coordinate planes.

A Tangent plane of a level surface.

D Find the gradient and the the tangent plane.

I 1) When $F(x, y, z) = xyz$, the $\mathcal{F}$ is the level surface $F(x, y, z) = 1$ in the first octant.



If $(\xi, \eta, \zeta) \in \mathcal{F}$, then

$$\eta\zeta = \frac{1}{\xi}, \qquad \zeta\xi = \frac{1}{\eta}, \qquad \xi\eta = \frac{1}{\zeta},$$

and the gradient is here

$$\bigtriangledown F(\xi, \eta, \zeta) = (\eta\zeta, \zeta\xi, \xi\eta) = \left(\frac{1}{\xi}, \frac{1}{\eta}, \frac{1}{\zeta}\right).$$

The equation of the tangent plane becomes

$$0 = \bigtriangledown F(\xi, \eta, \zeta) \cdot (x - \xi, y - \eta, z - \zeta) = \frac{x}{\xi} - 1 + \frac{y}{\eta} - 1 + \frac{z}{\zeta} - 1,$$

i.e.

$$\frac{x}{\xi} + \frac{y}{\eta} + \frac{z}{\zeta} = 3,$$

or alternatively,

$$x\eta\zeta + y\xi\zeta + z\xi\eta = 3.$$

2) The X axis is characterized by $y = 0$ and $z = 0$, so the intersection point is $(3\xi, 0, 0)$.

We get analogously $(0, 3\eta, 0)$ on the Y axis and $(0, 0, 3\zeta)$ on the Z axis.

3) The volume is by the method of sections given by $\int_0^{3\zeta} A(z)\,dz$, where $A(z)$ denotes the area of a triangle which is cut out of the tetrahedron by a plane, parallel to the XY plane at the height z.

At the height $z = 0$ we have

$$A(0) = \frac{9}{2}\,\xi\eta.$$

By the similarity we then get in general

$$A(z) = \frac{9}{2}\,\xi\eta\left(1 - \frac{z}{3\zeta}\right)^2, \qquad z \in [0, 3\zeta].$$

Finally, by insertion and calculation of the integral we get the volume

$$\begin{aligned} V &= \int_0^{3\zeta} A(z)\,\mathrm{d}z = \frac{9}{2}\,\xi\eta \int_0^{3\zeta}\left(1 - \frac{z}{3\zeta}\right)^2 \mathrm{d}z \qquad \left[t = \frac{z}{3\zeta}\right] \\ &= \frac{9}{2}\,\xi\eta \cdot 3\zeta \int_0^1 (1-t)^2\,\mathrm{d}t = \frac{27}{2}\,\xi\eta\zeta \int_0^1 u^2\,\mathrm{d}u \\ &= \frac{27}{2} \cdot 1 \cdot 13 = \frac{9}{2}. \end{aligned}$$

We note that the volume is constant $= \frac{9}{2}$, no matter which point $(\xi, \eta, \zeta) \in \mathcal{F}$ we have chosen.

Example 14.8 *The surface* $\mathcal{F}$ *is given by the equation* $x^4 + y^4 + 2z^2 = 19$. *Find an equation of the tangent plane of* $\mathcal{F}$ *through the point* $(2, 1, -1)$.

A Tangent plane.

D First check that $(2, 1, -1)$ lies on the surface. Then find a normal.

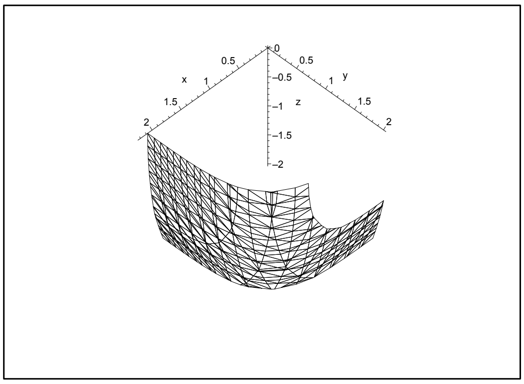


Figure 14.23: Part of the surface in the neighbourhood of the point $(2, 1, -1)$.

I It follows from

$$2^4 + 1^4 + 2(-1)^2 = 16 + 1 + 2 = 19,$$

that $(2,1,-1)$ lies on $\mathcal{F}$. Then

$$\mathbf{N}(x,y,z) = (4x^3, 4y^3, 4z) = 4(x^3, y^3, z),$$

and

$$\mathbf{N}(2,1,-1) = 4(8,1,-1),$$

so the equation of the tangent plane is

$$0 = \frac{1}{4}\mathbf{N}(2,1,-1)\cdot(x-2, y-1, z+1) = 8(x-2) + (y-1) - (z+1),$$

hence by a reduction

$$8x + y - z = 18.$$

Example 14.9 *The surface $\mathcal{F}$ is given by the equation*

$$x^3 + y^3 + z^3 + 6 = 0.$$

Prove that the straight line given by $x = 1$ and $y = 1$ intersects $\mathcal{F}$ in a single point P, and then find an equation of the tangent plane of $\mathcal{F}$ at P.

A Tangent plane.

D Either exploit the geometric meaning of ∇F, or find two tangents the cross products of which gives a normal.

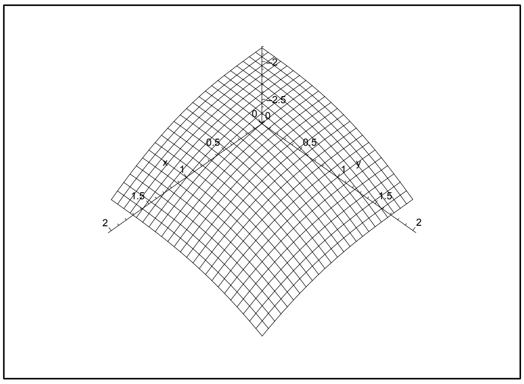


Figure 14.24: The surface $\mathcal{F}$ in the neighbourhood of P. Note that the coordinate system is translated along the Z axis, so the origo does not occur on the figure.

I By putting $x = y = 1$ we get

$$0 = 1 + 1 + z^3 + 6 = z^3 + 8 = (z+2)(z^2 - 2z + 4) = (z+2)\{(z-1)^2 + 3\}.$$

It follows that $z = -2$ is the only *real* solution, so the line of the parametric description $\mathbf{r}(z) = (1, 1, z)$ does only intersect the surface in the point P: $(1, 1, -2) \in \mathcal{F}$.

We shall find the tangent plane in two ways.

First variant. If we put $F(x, y, z) = x^3 + y^3 + z^3 + 6$, then ∇F indicates the direction in which the increase of F is largest, i.e. ∇F is a normal to the surface $F(x, y, z) = C$ at the surface point.

We find

$$\nabla F = 3(x^2, y^2, z^2), \qquad \nabla F(1, 1, -2) = 3(1, 1, 4),$$

so an equation of the tangent plane is e.g. given by

$$\begin{aligned} 0 &= \frac{1}{3} \nabla F(1, 1, -2) \cdot (x-1, y-1, z+2) \\ &= (1, 1, 4) \cdot (x-1, y-1, z+2) \\ &= x - 1 + y - 1 + 4z + 8 = x + y + 4z + 6, \end{aligned}$$

hence by a rearrangement

$$x+y+4z=-6.$$

Second variant. A parametric description of the surface $\mathcal{F}$ is in a neighbourhood of the point $(1,1,-2)$ given by

$$\mathbf{r}(u,v)=\left(u,v,-\sqrt[3]{6+u^3+v^3}\right), \qquad (u,v)\in K((1,1,-2);r).$$

The parameter curves have the tangents

$$\frac{\partial \mathbf{r}}{\partial u}=\left(1,0,-\left\{\frac{u}{\sqrt[3]{6+u^3+v^3}}\right\}^2\right), \qquad \frac{\partial \mathbf{r}}{\partial v}=\left(0,1,-\left\{\frac{v}{\sqrt[3]{6+u^3+v^3}}\right\}^2\right).$$

At the point $(1,1,-2)$ we get the tangents

$$\mathbf{r}_u'(1,1,-2)=\left(1,0,-\frac{1}{4}\right) \quad \text{and} \quad \mathbf{r}_v'(1,1,-2)=\left(0,1,-\frac{1}{4}\right).$$

Then a parametric description of the tangent plane is given by

$$\begin{aligned}\mathbf{r}(u,v) &= (x,y,z)=(1,1,-2)+u\left(1,0,-\frac{1}{4}\right)+v\left(0,1,-\frac{1}{4}\right)\\ &= \left(u+1,v+1,-\frac{1}{4}(u+v+8)\right), \qquad (u,v)\in\mathbb{R}^2.\end{aligned}$$

When we eliminate u and v we get

$$z=-\frac{1}{4}(u+v+8)=-\frac{1}{4}\{(u+1)+(v+1)+6\}=-\frac{1}{4}(x+y+6),$$

hence by a rearrangement

$$x+y+4z=-6.$$

Remark. Here we might alternatively have found the normal instead,

$$\mathbf{N}(1,1,-2)=\mathbf{r}_u'(1,1,-2)\times\mathbf{r}_v'(1,1,-2).$$

The details are left to the reader. $\Diamond$

Example 14.10 *Given the function*

$$f(x,y,z) = x^2 - 4y^2 + 4x + 8y - z, \qquad (x,y,z) \in \mathbb{R}^3,$$

and the level surface $\mathcal{F}$ *of the equation*

$$f(x,y,z) = 12.$$

1) Indicate the type of the surface and its top point.

2) Find $\bigtriangledown f(2,0,0)$*, and set up an equation of the tangent plane of* $\mathcal{F}$ *at the point* $(2,0,0)$*.*

A Level surface and tangent plane.

D Rewrite the equation of the level surface to one of the generic forms. The tangent plane is found by the standard method.

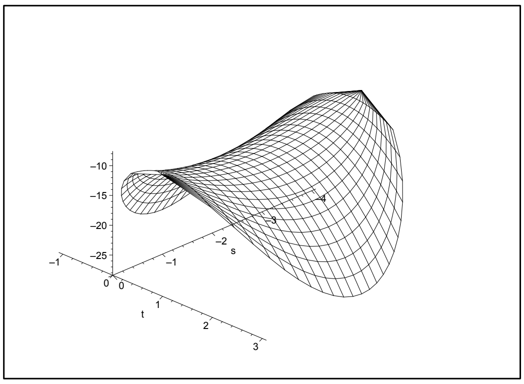


Figure 14.25: Part of the level surface $\mathcal{F}$.

I 1) The equation of the level surface

$$f(x,y,z) = x^2 - 4y^2 + 4x + 8y - z = (x+2)^2 - 4(y-1) - z = 12$$

is rewritten as

$$z + 12 = (x+2)^2 - 4(y-1)^2,$$

and we see that $\mathcal{F}$ is an hyperbolic paraboloid of top point $(-2,1,-12)$.

2) It follows from

$$\bigtriangledown f(x,y,z) = (2x+4, -8y+8, -1)$$

that the normal of the surface at the point $(2,0,0)$ is

$$\bigtriangledown f(2,0,0) = (8,8,-1).$$

Since

$$f(2,0,0) = 4 - 4\cdot 0^2 + 4 \cdot 2 + 8 \cdot 0 - 0 = 12,$$

we see that $(2,0,0)$ lies on the surface.

The equation of the tangent plane is

$$0 = \bigtriangledown f(2,0,0) \cdot (x-2, y-0, z-0) = (8,8,-1) \cdot (x-2, y, z) = 8x - 16 + 8y - z,$$

so by a rearrangement,

$$z = 8x + 8y - 16.$$

Example 14.11 *Let α be a constant. Consider the surface $\mathcal{F}_\alpha$ given by the equation*

$$x^2 + 4y^2 - z^2 + \alpha\, z^4 = \alpha + \alpha^2.$$

Let $\mathcal{T}_\alpha$ denote the tangent plane of $\mathcal{F}_\alpha$ at the point $\left(\alpha, \frac{1}{2}, 1\right)$ on the surface.

1) *Find an equation of $\mathcal{T}_\alpha$.*

2) *Check if α can be chosen such that the point $(1, 1, 0)$ belongs to the plane $\mathcal{T}_\alpha$.*

3) *Prove that there exists a value of α, for which $\mathcal{F}_\alpha$ is a conic section and indicate its type and centrum.*

A Tangent plane; conic section.

D Check if $\left(\alpha, \frac{1}{2}, 1\right)$ lies on the surface. Find a normal and prove that this can always be chosen $\neq \mathbf{0}$. Set up an equation of the tangent plane. Insert $(1, 1, 0)$ and solve with respect to α. Finally, find α, such that every term is at most a square.

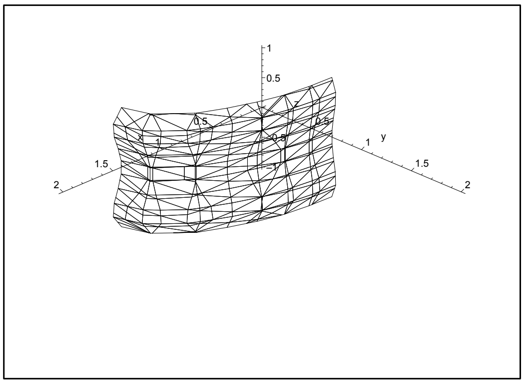


Figure 14.26: The surface $\mathcal{F}_\alpha$ for $\alpha = 1$ in the neighbourhood of the point $(1, 1, 0)$, cf. 2).

I 1) If we put $(x, y, z) = \left(\alpha, \frac{1}{2}, 1\right)$, then

$$\alpha^2 + 4 \cdot \frac{1}{4} - 1 + \alpha = \alpha^2 + \alpha,$$

and the point $\left(\alpha, \frac{1}{2}, 1\right)$ lies on the surface $\mathcal{F}_\alpha$.

In general a normal is given by

$$\nabla F = (2x, 8y, -2z + 4\alpha z^3) = 2(x, 4y, -z + 2\alpha z^3).$$

By removing the factor 2 and inserting $\left(\alpha, \frac{1}{2}, 1\right)$ we get

$$\mathbf{n} = (\alpha, 2, -1 + 2\alpha) \neq \mathbf{0} \qquad \text{for every } \alpha.$$

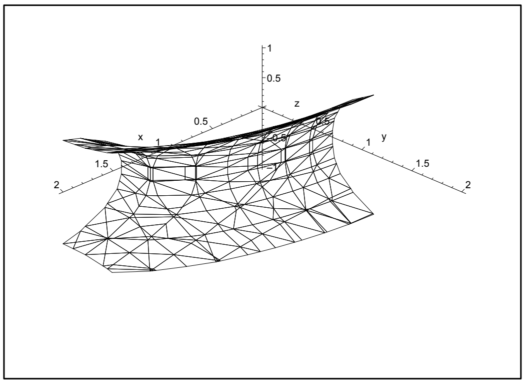


Figure 14.27: The surface $\mathcal{F}_\alpha$ for $\alpha = -2$ in the neighbourhood of the point $(1,1,0)$, cf. 2).

Since $\mathbf{n}$ is perpendicular to the tangent plane, the equation of the tangent plane becomes

$$\begin{aligned} 0 &= \mathbf{n}\cdot\left(x-\alpha, y-\frac{1}{2}, z-1\right) \\ &= \alpha(x-\alpha)+2\left(y-\frac{1}{2}\right)+(-1+2\alpha)(z-1) \\ &= \alpha x+2y+(2\alpha-1)z-\alpha^2-1+1-2\alpha \\ &= \alpha x+2y+(2\alpha-1)z-\alpha^2-2\alpha, \end{aligned}$$

which is rewritten as

$$\alpha x+2y+(2\alpha-1)z=\alpha^2+2\alpha.$$

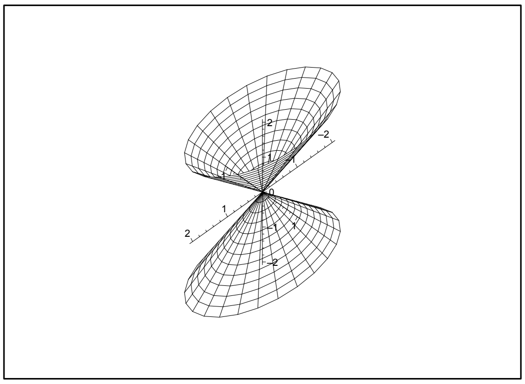


Figure 14.28: The conic section in 3), i.e. for $\alpha = 0$.

2) If we put $(x,y,z)=(1,1,0)$ into the equation of the tangent plane, we get the following equation of second order in α,

$$\alpha^2+2\alpha=\alpha+2+0, \qquad \text{i.e.} \qquad \alpha^2+\alpha-2=0,$$

the roots of which are $\alpha = 1$ and $\alpha = -2$. For these values of α the point $(1,1,0)$ lies in $\mathcal{T}_\alpha$.

3) When we consider conic sections, the corresponding equation must at most contain terms of degree two. We are therefore forced to put $\alpha = 0$, corresponding to

$$x^2 + 4y^2 - z^2 = 0, \qquad \text{i.e.} \qquad z^2 = x^2 + 4y^2.$$

This is the equation of a cone of centrum $(0,0,0)$ and the Z axis as axis.

Example 14.12 *Given the function*

$$f(x,y,z) = \ln\left(16 - x^2 - 2y^2 - 4z^2\right), \qquad (x,y,z) \in A,$$

where A is given by the inequality

$$x^2 + 2y^2 + 4z^2 < 16.$$

1. *Indicate the boundary ∂A and explain why ∂A is a conic section; find the name of this and indicate its half axes.*

2. *Check if A is*

1) open,

2) closed,

3) star shaped.

3. *Find the gradient $\bigtriangledown f(x,y,z)$.*

Let $\mathcal{F}$ be that level surface of f, which contains the point $(2,1,1)$.

4. *Find the value of $f(x,y,z)$ for $(x,y,z) \in \mathcal{F}$.*

5. *Find an equation of the tangent plane of $\mathcal{F}$ at the point $(2,1,1)$.*

A Conic section; level surface; tangent plane.

D Follow the guidelines of the text.

I 1) It follows from the continuity of the function that the equation of the boundary ∂A is

$$x^2 + 2y^2 + 4z^2 = 16,$$

hence by a norming (division by 16),

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{2\sqrt{2}}\right)^2 + \left(\frac{z}{2}\right)^1 = 1.$$

This normed form shows that ∂A is a conic section, in fact an ellipsoid of centrum $(0,0,0)$ and half axes 4, $2\sqrt{2}$ and 2 along the X axis, the Y axis and the Z axis respectively.

2) Since a polynomial is continuous and since we have "sharp" inequality signs, we conclude that A is open. Hence A is an open ellipsoid, therefore convex, and thus also star shaped. The set A is clearly not closed.

3) the gradient of f in A is

$$\nabla f(x,y,z)=\frac{1}{16-x^2-2y^2-4z^2}\,(-2x,-4y,-8z).$$

4) By insertion we see that if $(x,y,z)\in\mathcal{F}$, then

$$f(x,y,z)=f(2,1,1)\ln(16-4-2-4)=\ln 6.$$

5) We get at the point $(2,1,1)$ that

$$\nabla f(2,1,1)=\frac{1}{6}\,(-4,-4,-8)=-\frac{2}{3}\,(1,1,2).$$

The gradient is always perpendicular to the tangent plane, so an equation of the tangent plane of $\mathcal{F}$ at the point $(2,1,1)$ is given by

$$\begin{aligned}0 &= -\frac{3}{2}\,\nabla f(2,1,1)\cdot(x-2,y-1,z-1)\\ &= (1,1,2)\cdot(x-2,y-1,z-1)\\ &= x-2+y-1+2z-2\\ &= x+y+2z-5,\end{aligned}$$

thus after a rearrangement,

$$x+y+2z=5.$$

15 Formulæ

Some of the following formulæ can be assumed to be known from high school. It is highly recommended that one *learns most of these formulæ in this appendix by heart.*

15.1 Squares etc.

The following simple formulæ occur very frequently in the most different situations.

$$
\begin{array}{ll}
(a+b)^2 = a^2+b^2+2ab, & a^2+b^2+2ab=(a+b)^2, \\
(a-b)^2 = a^2+b^2-2ab, & a^2+b^2-2ab=(a-b)^2, \\
(a+b)(a-b) = a^2-b^2, & a^2-b^2=(a+b)(a-b), \\
(a+b)^2 = (a-b)^2+4ab, & (a-b)^2=(a+b)^2-4ab.
\end{array}
$$

15.2 Powers etc.

Logarithm:

$$
\begin{array}{rcll}
\ln|xy| &=& \ln|x|+\ln|y|, & x,y\neq 0, \\
\ln\left|\dfrac{x}{y}\right| &=& \ln|x|-\ln|y|, & x,y\neq 0, \\
\ln|x^r| &=& r\ln|x|, & x\neq 0.
\end{array}
$$

Power function, fixed exponent:

$$(xy)^r = x^r\cdot y^r, x,y>0 \quad \text{(extensions for some } r\text{)},$$

$$\left(\frac{x}{y}\right)^r = \frac{x^r}{y^r}, x,y>0 \quad \text{(extensions for some } r\text{)}.$$

Exponential, fixed base:

$$a^x\cdot a^y = a^{x+y}, \quad a>0 \quad \text{(extensions for some } x,\, y\text{)},$$
$$(a^x)^y = a^{xy}, a>0 \quad \text{(extensions for some } x,\, y\text{)},$$

$$a^{-x} = \frac{1}{a^x}, a>0, \quad \text{(extensions for some } x\text{)},$$

$$\sqrt[n]{a} = a^{1/n}, a\geq 0, \qquad n\in\mathbb{N}.$$

Square root:

$$\sqrt{x^2} = |x|, \qquad x\in\mathbb{R}.$$

Remark 15.1 It happens quite frequently that students make errors when they try to apply these rules. They must be mastered! In particular, as one of my friends once put it: "If you can master the square root, you can master everything in mathematics!" Notice that this innocent looking square root is one of the most difficult operations in Calculus. Do not forget the *absolute value!* $\Diamond$

15.3 Differentiation

Here are given the well-known rules of differentiation together with some rearrangements which sometimes may be easier to use:

$$\{f(x) \pm g(x)\}' = f'(x) \pm g'(x),$$

$$\{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x) = f(x)g(x)\left\{\frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}\right\},$$

where the latter rearrangement presupposes that $f(x) \neq 0$ and $g(x) \neq 0$.
If $g(x) \neq 0$, we get the usual formula known from high school

$$\left\{\frac{f(x)}{g(x)}\right\}' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

It is often more convenient to compute this expression in the following way:

$$\left\{\frac{f(x)}{g(x)}\right\} = \frac{d}{dx}\left\{f(x) \cdot \frac{1}{g(x)}\right\} = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2} = \frac{f(x)}{g(x)}\left\{\frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}\right\},$$

where the former expression often is *much easier* to use in practice than the usual formula from high school, and where the latter expression again presupposes that $f(x) \neq 0$ and $g(x) \neq 0$. Under these assumptions we see that the formulæ above can be written

$$\frac{\{f(x)g(x)\}'}{f(x)g(x)} = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)},$$

$$\frac{\{f(x)/g(x)\}'}{f(x)/g(x)} = \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}.$$

Since

$$\frac{d}{dx}\ln|f(x)| = \frac{f'(x)}{f(x)}, \qquad f(x) \neq 0,$$

we also name these the *logarithmic derivatives.*

Finally, we mention the rule of **differentiation of a composite function**

$$\{f(\varphi(x))\}' = f'(\varphi(x)) \cdot \varphi'(x).$$

We first differentiate the function itself; then the insides. This rule is a 1-dimensional version of the so-called *Chain rule.*

15.4 Special derivatives.

Power like:

$$\frac{d}{dx}\left(x^{\alpha}\right) = \alpha \cdot x^{\alpha-1}, \qquad \text{for } x > 0, \text{ (extensions for some } \alpha).$$

$$\frac{d}{dx}\ln|x| = \frac{1}{x}, \qquad \text{for } x \neq 0.$$

Exponential like:

$$\frac{d}{dx}\exp x = \exp x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}(a^x) = \ln a \cdot a^x, \qquad \text{for } x \in \mathbb{R} \text{ and } a > 0.$$

Trigonometric:

$$\frac{d}{dx}\sin x = \cos x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\cos x = -\sin x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\tan x = 1 + \tan^2 x = \frac{1}{\cos^2 x}, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, p \in \mathbb{Z},$$

$$\frac{d}{dx}\cot x = -(1 + \cot^2 x) = -\frac{1}{\sin^2 x}, \qquad \text{for } x \neq p\pi, p \in \mathbb{Z}.$$

Hyperbolic:

$$\frac{d}{dx}\sinh x = \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\cosh x = \sinh x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\tanh x = 1 - \tanh^2 x = \frac{1}{\cosh^2 x}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\coth x = 1 - \coth^2 x = -\frac{1}{\sinh^2 x}, \qquad \text{for } x \neq 0.$$

Inverse trigonometric:

$$\frac{d}{dx}\text{ Arcsin } x = \frac{1}{\sqrt{1-x^2}}, \qquad \text{for } x \in \,]-1,1\,[,$$

$$\frac{d}{dx}\text{ Arccos } x = -\frac{1}{\sqrt{1-x^2}}, \qquad \text{for } x \in \,]-1,1\,[,$$

$$\frac{d}{dx}\text{ Arctan } x = \frac{1}{1+x^2}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\text{ Arccot } x = \frac{1}{1+x^2}, \qquad \text{for } x \in \mathbb{R}.$$

Inverse hyperbolic:

$$\frac{d}{dx}\text{ Arsinh } x = \frac{1}{\sqrt{x^2+1}}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\text{ Arcosh } x = \frac{1}{\sqrt{x^2-1}}, \qquad \text{for } x \in \,]1,+\infty\,[,$$

$$\frac{d}{dx}\text{ Artanh } x = \frac{1}{1-x^2}, \qquad \text{for } |x| < 1,$$

$$\frac{d}{dx}\text{ Arcoth } x = \frac{1}{1-x^2}, \qquad \text{for } |x| > 1.$$

Remark 15.2 The derivative of the trigonometric and the hyperbolic functions are to some extent exponential like. The derivatives of the inverse trigonometric and inverse hyperbolic functions are power like, because we include the logarithm in this class. $\Diamond$

15.5 Integration

The most obvious rules are dealing with linearity

$$\int \{f(x)+\lambda g(x)\}\,dx = \int f(x)\,dx + \lambda \int g(x)\,dx, \qquad \text{where } \lambda \in \mathbb{R} \text{ is a constant},$$

and with the fact that differentiation and integration are "inverses to each other", i.e. modulo some arbitrary constant $c \in \mathbb{R}$, which often tacitly is missing,

$$\int f'(x)\,dx = f(x).$$

If we in the latter formula replace $f(x)$ by the product $f(x)g(x)$, we get by reading from the right to the left and then differentiating the product,

$$f(x)g(x) = \int \{f(x)g(x)\}'\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx.$$

Hence, by a rearrangement

The rule of partial integration:

$$\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx.$$

The differentiation is moved from one factor of the integrand to the other one by changing the sign and adding the term $f(x)g(x)$.

Remark 15.3 This technique was earlier used a lot, but is almost forgotten these days. It must be revived, because MAPLE and pocket calculators apparently do not know it. It is possible to construct examples where these devices cannot give the exact solution, unless you first perform a partial integration yourself. $\lozenge$

Remark 15.4 This method can also be used when we estimate integrals which cannot be directly calculated, because the antiderivative is not contained in e.g. the catalogue of MAPLE. The idea is by a succession of partial integrations to make the new integrand smaller. $\lozenge$

Integration by substitution:

If the integrand has the special structure $f(\varphi(x))\cdot\varphi'(x)$, then one can change the variable to $y=\varphi(x)$:

$$\int f(\varphi(x))\cdot\varphi'(x)\,dx = \text{“}\int f(\varphi(x))\,d\varphi(x)\text{”} = \int_{y=\varphi(x)} f(y)\,dy.$$

Integration by a monotonous substitution:

If $\varphi(y)$ is a *monotonous* function, which maps the y-interval *one-to-one* onto the x-interval, then

$$\int f(x)\,dx = \int_{y=\varphi^{-1}(x)} f(\varphi(y))\varphi'(y)\,dy.$$

Remark 15.5 This rule is usually used when we have some "ugly" term in the integrand $f(x)$. The idea is to put this ugly term equal to $y = \varphi^{-1}(x)$. When e.g. x occurs in $f(x)$ in the form $\sqrt{x}$, we put $y = \varphi^{-1}(x) = \sqrt{x}$, hence $x = \varphi(y) = y^2$ and $\varphi'(y) = 2y$. $\Diamond$

15.6 Special antiderivatives

Power like:

$$\int \frac{1}{x}\,dx = \ln|x|, \qquad \text{for } x \neq 0. \quad \text{(Do not forget the numerical value!)}$$

$$\int x^\alpha\,dx = \frac{1}{\alpha+1}x^{\alpha+1}, \qquad \text{for } \alpha \neq -1,$$

$$\int \frac{1}{1+x^2}\,dx = \text{ Arctan } x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{1-x^2}\,dx = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|, \qquad \text{for } x \neq \pm 1,$$

$$\int \frac{1}{1-x^2}\,dx = \text{ Artanh } x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{1-x^2}\,dx = \text{ Arcoth } x, \qquad \text{for } |x| > 1,$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \text{ Arcsin } x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = -\text{ Arccos } x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \text{ Arsinh } x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \ln(x+\sqrt{x^2+1}), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{x}{\sqrt{x^2-1}}\,dx = \sqrt{x^2-1}, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \text{ Arcosh } x, \qquad \text{for } x > 1,$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \ln|x+\sqrt{x^2-1}|, \qquad \text{for } x > 1 \text{ eller } x < -1.$$

There is an error in the programs of the pocket calculators TI-92 and TI-89. The *numerical signs* are missing. It is obvious that $\sqrt{x^2-1} < |x|$ so if $x < -1$, then $x + \sqrt{x^2-1} < 0$. Since you cannot take the logarithm of a negative number, these pocket calculators will give an error message.

Exponential like:

$$\int \exp x \, dx = \exp x, \qquad \text{for } x \in \mathbb{R},$$

$$\int a^x \, dx = \frac{1}{\ln a} \cdot a^x, \qquad \text{for } x \in \mathbb{R}, \text{ and } a > 0, a \neq 1.$$

Trigonometric:

$$\int \sin x \, dx = -\cos x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \cos x \, dx = \sin x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \tan x \, dx = -\ln|\cos x|, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \cot x \, dx = \ln|\sin x|, \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos x} \, dx = \frac{1}{2} \ln\left(\frac{1+\sin x}{1-\sin x}\right), \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin x} \, dx = \frac{1}{2} \ln\left(\frac{1-\cos x}{1+\cos x}\right), \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos^2 x} \, dx = \tan x, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin^2 x} \, dx = -\cot x, \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z}.$$

Hyperbolic:

$$\int \sinh x \, dx = \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \cosh x \, dx = \sinh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \tanh x \, dx = \ln \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \coth x \, dx = \ln|\sinh x|, \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh x} \, dx = \text{Arctan}(\sinh x), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\cosh x} \, dx = 2\,\text{Arctan}(e^x), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh x} \, dx = \frac{1}{2} \ln\left(\frac{\cosh x - 1}{\cosh x + 1}\right), \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\sinh x}\,dx = \ln\left|\frac{e^x-1}{e^x+1}\right|, \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh^2 x}\,dx = \tanh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh^2 x}\,dx = -\coth x, \qquad \text{for } x \neq 0.$$

15.7 Trigonometric formulæ

The trigonometric formulæ are closely connected with circular movements. Thus $(\cos u, \sin u)$ are the coordinates of a point P on the unit circle corresponding to the angle u, cf. figure A.1. This geometrical interpretation is used from time to time.

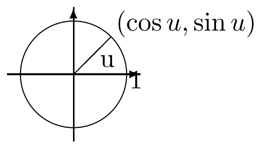


Figure 15.1: The unit circle and the trigonometric functions.

The fundamental trigonometric relation:

$$\cos^2 u + \sin^2 u = 1, \qquad \text{for } u \in \mathbb{R}.$$

Using the previous geometric interpretation this means according to *Pythagoras's theorem*, that the point P with the coordinates $(\cos u, \sin u)$ always has distance 1 from the origo $(0,0)$, i.e. it is lying on the boundary of the circle of centre $(0,0)$ and radius $\sqrt{1} = 1$.

Connection to the complex exponential function:

The *complex exponential* is for imaginary arguments defined by

$$\exp(\mathrm{i}\,u) := \cos u + \mathrm{i}\ \sin u.$$

It can be checked that the usual functional equation for exp is still valid for complex arguments. In other word: The definition above is extremely conveniently chosen.

By using the definition for $\exp(\mathrm{i}\,u)$ and $\exp(-\mathrm{i}\,u)$ it is easily seen that

$$\cos u = \frac{1}{2}(\exp(\mathrm{i}\,u) + \exp(-\mathrm{i}\,u)),$$

$$\sin u = \frac{1}{2i}(\exp(\mathrm{i}\,u) - \exp(-\mathrm{i}\,u)),$$

.

Moivre's formula: We get by expressing $\exp(\mathrm{i}nu)$ in two different ways:

$$\exp(\mathrm{i}nu) = \cos nu + \mathrm{i}\,\sin nu = (\cos u + \mathrm{i}\,\sin u)^n.$$

Example 15.1 If we e.g. put $n = 3$ into Moivre's formula, we obtain the following typical application,

$$\begin{aligned}\cos(3u) + \mathrm{i}\,\sin(3u) &= (\cos u + \mathrm{i}\,\sin u)^3\\ &= \cos^3 u + 3\mathrm{i}\,\cos^2 u\cdot\sin u + 3\mathrm{i}^2\cos u\cdot\sin^2 u + \mathrm{i}^3\sin^3 u\\ &= \{\cos^3 u - 3\cos u\cdot\sin^2 u\} + \mathrm{i}\{3\cos^2 u\cdot\sin u - \sin^3 u\}\\ &= \{4\cos^3 u - 3\cos u\} + \mathrm{i}\{3\sin u - 4\sin^3 u\}\end{aligned}$$

When this is split into the real- and imaginary parts we obtain

$$\cos 3u = 4\cos^3 u - 3\cos u, \qquad \sin 3u = 3\sin u - 4\sin^3 u. \quad \diamond$$

Addition formulæ:

$$\sin(u+v) = \sin u\,\cos v + \cos u\,\sin v,$$

$$\sin(u-v) = \sin u\,\cos v - \cos u\,\sin v,$$

$$\cos(u+v) = \cos u\,\cos v - \sin u\,\sin v,$$

$$\cos(u-v) = \cos u\,\cos v + \sin u\,\sin v.$$

Products of trigonometric functions to a sum:

$$\sin u\,\cos v = \frac{1}{2}\sin(u+v) + \frac{1}{2}\sin(u-v),$$

$$\cos u\,\sin v = \frac{1}{2}\sin(u+v) - \frac{1}{2}\sin(u-v),$$

$$\sin u\,\sin v = \frac{1}{2}\cos(u-v) - \frac{1}{2}\cos(u+v),$$

$$\cos u\,\cos v = \frac{1}{2}\cos(u-v) + \frac{1}{2}\cos(u+v).$$

Sums of trigonometric functions to a product:

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right),$$

$$\sin u - \sin v = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right),$$

$$\cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right),$$

$$\cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right).$$

Formulæ of halving and doubling the angle:

$$\sin 2u = 2\sin u\,\cos u,$$

$$\cos 2u = \cos^2 u - \sin^2 u = 2\,\cos^2 u - 1 = 1 - 2\,\sin^2 u,$$

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}} \qquad \text{followed by a discussion of the sign,}$$

$$\cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}} \qquad \text{followed by a discussion of the sign,}$$

15.8 Hyperbolic formulæ

These are very much like the trigonometric formulæ, and if one knows a little of Complex Function Theory it is realized that they are actually identical. The structure of this section is therefore the same as for the trigonometric formulæ. The reader should compare the two sections concerning similarities and differences.

The fundamental relation:

$$\cosh^2 x - \sinh^2 x = 1.$$

Definitions:

$$\cosh x = \frac{1}{2}\left(\exp(x) + \exp(-x)\right), \qquad \sinh x = \frac{1}{2}\left(\exp(x) - \exp(-x)\right).$$

"Moivre's formula":

$$\exp(x) = \cosh x + \sinh x.$$

This is trivial and only rarely used. It has been included to show the analogy.

Addition formulæ:

$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y),$$

$$\sinh(x-y) = \sinh(x)\cosh(y) - \cosh(x)\sinh(y),$$

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y),$$

$$\cosh(x-y) = \cosh(x)\cosh(y) - \sinh(x)\sinh(y).$$

Formulæ of halving and doubling the argument:

$$\sinh(2x) = 2\sinh(x)\cosh(x),$$

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x) = 2\cosh^2(x) - 1 = 2\sinh^2(x) + 1,$$

$$\sinh\left(\frac{x}{2}\right) = \pm\sqrt{\frac{\cosh(x)-1}{2}} \qquad \text{followed by a discussion of the sign,}$$

$$\cosh\left(\frac{x}{2}\right) = \sqrt{\frac{\cosh(x)+1}{2}}.$$

Inverse hyperbolic functions:

$$\operatorname{Arsinh}(x) = \ln\left(x + \sqrt{x^2+1}\right), \qquad x \in \mathbb{R},$$

$$\operatorname{Arcosh}(x) = \ln\left(x + \sqrt{x^2-1}\right), \qquad x \geq 1,$$

$$\operatorname{Artanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \qquad |x| < 1,$$

$$\operatorname{Arcoth}(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right), \qquad |x| > 1.$$

15.9 Complex transformation formulæ

$$\cos(\mathrm{i}x) = \cosh(x), \qquad \cosh(\mathrm{i}x) = \cos(x),$$
$$\sin(\mathrm{i}x) = \mathrm{i}\,\sinh(x), \qquad \sinh(\mathrm{i}x) = \mathrm{i}\,\sin x.$$

15.10 Taylor expansions

The generalized binomial coefficients are defined by

$$\binom{\alpha}{n} := \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{1\cdot 2\cdots n},$$

with n factors in the numerator and the denominator, supplied with

$$\binom{\alpha}{0} := 1.$$

The Taylor expansions for *standard functions* are divided into *power like* (the radius of convergency is finite, i.e. $= 1$ for the standard series) and*exponential like* (the radius of convergency is infinite).
Power like:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1,$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x| < 1,$$

$$(1+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j, \qquad n \in \mathbb{N}, x \in \mathbb{R},$$

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \qquad \alpha \in \mathbb{R}\setminus\mathbb{N}, |x| < 1,$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n}, \qquad |x| < 1,$$

$$\operatorname{Arctan}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \qquad |x| < 1.$$

Exponential like:

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \qquad x \in \mathbb{R}$$

$$\exp(-x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} x^n, \qquad x \in \mathbb{R}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}, \qquad x \in \mathbb{R},$$

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}, \qquad x \in \mathbb{R},$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}, \qquad x \in \mathbb{R},$$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}, \qquad x \in \mathbb{R}.$$

15.11 Magnitudes of functions

We often have to compare functions for $x \to 0+$, or for $x \to \infty$. The simplest type of functions are therefore arranged in an hierarchy:

1) logarithms,
2) power functions,
3) exponential functions,
4) faculty functions.

When $x \to \infty$, a function from a higher class will always dominate a function form a lower class. More precisely:

A) A *power function* dominates a *logarithm* for $x \to \infty$:

$$\frac{(\ln x)^\beta}{x^\alpha} \to 0 \qquad \text{for } x \to \infty, \quad \alpha, \beta > 0.$$

B) An *exponential* dominates a *power function* for $x \to \infty$:

$$\frac{x^\alpha}{a^x} \to 0 \qquad \text{for } x \to \infty, \quad \alpha, a > 1.$$

C) The *faculty function* dominates an *exponential* for $n \to \infty$:

$$\frac{a^n}{n!} \to 0, \qquad n \to \infty, \quad n \in \mathbb{N}, \quad a > 0.$$

D) When $x \to 0+$ we also have that a *power function* dominates the *logarithm:*

$$x^\alpha \ln x \to 0-, \qquad \text{for } x \to 0+, \quad \alpha > 0.$$

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