Real Functions in Several Variables: Volume III

Differentiable Functions in Several Variables

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2nd edition

ISBN 978-87-403-0909-6

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Preface

The topic of this series of books on *"Real Functions in Several Variables"* is very important in the description in e.g. *Mechanics* of the real 3-dimensional world that we live in. Therefore, we start from the very beginning, modelling this world by using the coordinates of $\mathbb{R}^3$ to describe e.g. a motion in space. There is, however, absolutely no reason to restrict ourselves to $\mathbb{R}^3$ alone. Some motions may be rectilinear, so only $\mathbb{R}$ is needed to describe their movements on a line segment. This opens up for also dealing with $\mathbb{R}^2$, when we consider plane motions. In more elaborate problems we need higher dimensional spaces. This may be the case in *Probability Theory* and *Statistics*. Therefore, we shall in general use $\mathbb{R}^n$ as our abstract model, and then restrict ourselves in examples mainly to $\mathbb{R}^2$ and $\mathbb{R}^3$.

For rectilinear motions the familiar *rectangular coordinate system* is the most convenient one to apply. However, as known from e.g. *Mechanics*, circular motions are also very important in the applications in engineering. It becomes natural alternatively to apply in $\mathbb{R}^2$ the so-called *polar coordinates* in the plane. They are convenient to describe a circle, where the rectangular coordinates usually give some nasty square roots, which are difficult to handle in practice.

Rectangular coordinates and polar coordinates are designed to model each their problems. They supplement each other, so difficult computations in one of these coordinate systems may be easy, and even trivial, in the other one. It is therefore important always in advance carefully to analyze the geometry of e.g. a domain, so we ask the question: Is this domain best described in rectangular or in polar coordinates?

Sometimes one may split a problem into two subproblems, where we apply rectangular coordinates in one of them and polar coordinates in the other one.

It should be mentioned that in *real life* (though not in these books) one cannot always split a problem into two subproblems as above. Then one is really in trouble, and more advanced mathematical methods should be applied instead. This is, however, outside the scope of the present series of books.

The idea of polar coordinates can be extended in two ways to $\mathbb{R}^3$. Either to *semi-polar* or *cylindric coordinates*, which are designed to describe a cylinder, or to *spherical coordinates*, which are excellent for describing spheres, where rectangular coordinates usually are doomed to fail. We use them already in daily life, when we specify a place on Earth by its longitude and latitude! It would be very awkward in this case to use rectangular coordinates instead, even if it is possible.

Concerning the contents, we begin this investigation by modelling point sets in an n-dimensional Euclidean space E^n by $\mathbb{R}^n$. There is a subtle difference between E^n and $\mathbb{R}^n$, although we often identify these two spaces. In E^n we use *geometrical methods* without a coordinate system, so the objects are independent of such a choice. In the coordinate space $\mathbb{R}^n$ we can use ordinary calculus, which in principle is not possible in E^n. In order to stress this point, we call E^n the "abstract space" (in the sense of calculus; not in the sense of geometry) as a warning to the reader. Also, whenever necessary, we use the colour black in the "abstract space", in order to stress that this expression is theoretical, while variables given in a chosen coordinate system and their related concepts are given the colours blue, red and green.

We also include the most basic of what mathematicians call *Topology*, which will be necessary in the following. We describe what we need by a function.

Then we proceed with limits and continuity of functions and define continuous curves and surfaces, with parameters from subsets of $\mathbb{R}$ and $\mathbb{R}^2$, resp..

Continue with (partial) differentiable functions, curves and surfaces, the chain rule and Taylor's formula for functions in several variables.

We deal with maxima and minima and extrema of functions in several variables over a domain in $\mathbb{R}^n$. This is a very important subject, so there are given many worked examples to illustrate the theory.

Then we turn to the problems of integration, where we specify four different types with increasing complexity, *plane integral, space integral, curve (or line) integral and surface integral*.

Finally, we consider *vector analysis*, where we deal with vector fields, Gauß's theorem and Stokes's theorem. All these subjects are very important in theoretical Physics.

The structure of this series of books is that each subject is *usually* (but not always) described by three successive chapters. In the first chapter a brief theoretical theory is given. The next chapter gives some practical guidelines of how to solve problems connected with the subject under consideration. Finally, some worked out examples are given, in many cases in several variants, because the standard solution method is seldom the only way, and it may even be clumsy compared with other possibilities.

I have as far as possible structured the examples according to the following scheme:

A *Awareness*, i.e. a short description of what is the problem.

D *Decision*, i.e. a reflection over what should be done with the problem.

I *Implementation*, i.e. where all the calculations are made.

C *Control*, i.e. a test of the result.

This is an ideal form of a general procedure of solution. It can be used in any situation and it is not linked to Mathematics alone. I learned it many years ago in the Theory of Telecommunication in a situation which did not contain Mathematics at all. The student is recommended to use it also in other disciplines.

From high school one is used to immediately to proceed to **I**. *Implementation*. However, examples and problems at university level, let alone situations in real life, are often so complicated that it in general will be a good investment also to spend some time on the first two points above in order to be absolutely certain of what to do in a particular case. Note that the first three points, **ADI**, can always be executed.

This is unfortunately not the case with **C** *Control*, because it from now on may be difficult, if possible, to check one's solution. It is only an extra securing whenever it is possible, but we cannot include it always in our solution form above.

I shall on purpose not use the logical signs. These should in general be avoided in Calculus as a shorthand, because they are often (too often, I would say) misused. Instead of $\wedge$ I shall either write "and", or a comma, and instead of $\vee$ I shall write "or". The arrows $\Rightarrow$ and $\Leftrightarrow$ are in particular misunderstood by the students, so they should be totally avoided. They are not telegram short hands, and from a logical point of view they usually do not make sense at all! Instead, write in a plain language what you mean or want to do. This is difficult in the beginning, but after some practice it becomes routine, and it will give more precise information.

When we deal with multiple integrals, one of the possible pedagogical ways of solving problems has been to colour variables, integrals and upper and lower bounds in blue, red and green, so the reader by the colour code can see in each integral what is the variable, and what are the parameters, which do not enter the integration under consideration. We shall of course build up a hierarchy of these colours, so the order of integration will always be defined. As already mentioned above we reserve the colour black for the theoretical expressions, where we cannot use ordinary calculus, because the symbols are only shorthand for a concept.

The author has been very grateful to his old friend and colleague, the late Per Wennerberg Karlsson, for many discussions of how to present these difficult topics on real functions in several variables, and for his permission to use his textbook as a template of this present series. Nevertheless, the author has felt it necessary to make quite a few changes compared with the old textbook, because we did not always agree, and some of the topics could also be explained in another way, and then of course the results of our discussions have here been put in writing for the first time.

The author also adds some calculations in MAPLE, which interact nicely with the theoretic text. Note, however, that when one applies MAPLE, one is forced first to make a geometrical analysis of the domain of integration, i.e. apply some of the techniques developed in the present books.

The theory and methods of these volumes on "Real Functions in Several Variables" are applied constantly in higher Mathematics, Mechanics and Engineering Sciences. It is of paramount importance for the calculations in *Probability Theory*, where one constantly integrate over some point set in space.

It is my hope that this text, these guidelines and these examples, of which many are treated in more ways to show that the solutions procedures are not unique, may be of some inspiration for the students who have just started their studies at the universities.

Finally, even if I have tried to write as careful as possible, I doubt that all errors have been removed. I hope that the reader will forgive me the unavoidable errors.

Leif Mejlbro
March 21, 2015

Introduction to volume III, Differentiable Functions in Several Variables

This is the third volume in the series of books on *Real Functions in Several Variables*. Its topic is differential functions. The idea of differentiability goes back to the technique of approximation of a problem by linearizing it. Consider a differentiable function $f : A \to \mathbb{R}$, $A \subseteq \mathbb{R}$, in only one variable. When we want to describe the behaviour of f in the neighbourhood of a point $x_0 \in A$, we may approximately describe the graph of f by its tangent at the point $(x_0, f(x_0))$, i.e. the line given by the equation

$$y = f(x_0) + f'(x_0) \cdot (x - x_0) = f(x_0) + f'(x_0)\, h,$$

where we have introduced the new variable $h := x - x_0$, which is actually used on the tangent.

It is tempting to extend this model to higher dimensions. If $f : A \to \mathbb{R}$ is a differentiable function in two variables (x, y) (whatever "differentiable" means in this case; it has not been defined yet), then it would be natural to approximate $f(x, y)$ instead by approximating the graph of f at a given point by its tangent plane at this point. The tangent plane should be 2-dimensional, so the points of the tangent plane are specified by the chosen point $\mathbf{x} = (x, y) \in A$ and the two coordinates $\mathbf{h} = (h_1, h_2)$ "living on" the approximating plane. Therefore, it is natural to expect that the function is a function in two sets of variables, $(\mathbf{x}, \mathbf{h}) \in A \times \mathbb{R}^2$.

The program above clearly needs a lot of tidying, where we first must deviate from the general idea. In the first section we make the definitions precise and show that the differentiability in higher dimensions has most of its properties in common with differentiability in one dimension. We also introduce differentiable vector functions, at the approximating polynomial of degree 1 in the coordinates. The latter is closely connected with the equation of the tangent (hyper)plane of the graph, but it also opens up for other generalizations later on.

Then follows a section on the *chain rule*, which describes how one differentiates a composite function in several variables. This section is fairly technical, and the author has had many discussions with his late colleague, Per Wennerberg Karlsson, of how to present the matter in the best way.

9 Differentiable functions in several variables

9.1 Differentiability

9.1.1 The gradient and the differential

We shall first consider the well-known case of a differentiable function in one variable. The reason is that we then are able to analyze how to proceed with the generalization to differentiable functions in several variables.

When $f : A \to \mathbb{R}$, $A \subseteq \mathbb{R}$ is a function in just one variable, there are two equivalent ways to introduce differentiability of f. The first method, known from high school, requires that the *difference quotient* at x below has a well-defined limit for $h \to 0$, i.e.

$$(9.1) \qquad \frac{f(x+h)-f(x)}{h} \to a \qquad \text{for } h \to 0.$$

The second method, which here may be obtained from (9.1), when we multiply by h, requires that the increase of the function f at the point x satisfies

$$(9.2) \qquad f(x+h) - f(x) = ah + \varepsilon(h)|h|,$$

where a is some constant, and where $\varepsilon(h)$ denotes some function, for which $\varepsilon(h) \to 0$ for $h \to 0$. Since we can redefine $\varepsilon(h)$ and build in the sign of h, we may just write $\varepsilon(h)h$ instead of $\varepsilon(h)|h|$.

Let us turn to functions in several variables, like $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^n$ and $n \geq 2$. It follows immediately that we cannot generalize (9.1), because the pair (x, h) in one dimensional should be replaced by the pair of vectors $(\mathbf{x}, \mathbf{h})$. A generalization of (9.1) would require that we should have a *vector* $\mathbf{h}$ in the denominator, and that is not possible.

Fortunately, (9.2) is easy to generalize.

Definition 9.1 Differentiability. *Let $A \subseteq \mathbb{R}^n$ be an open set, and let $f : A \to \mathbb{R}$ be a function on A. We call f* differentiable *at the point $\mathbf{x} \in A$, if for all $\mathbf{h}$, for which $\mathbf{x} + \mathbf{h} \in A$,*

$$f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) = \mathbf{a} \cdot \mathbf{h} + \varepsilon(\mathbf{h})\|\mathbf{h}\|,$$

where the vector $\mathbf{a}$ is independent of$\mathbf{h}$ is some function, for which $\varepsilon(\mathbf{h}) \to 0$ for $\mathbf{h} \to \mathbf{0}$.

The interpretation of this definition of differentiability at $\mathbf{x} \in A$ is, that the increase (decrease) of the function,

$$\Delta f := f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}),$$

behaves locally as a linear function $\mathbf{a} \cdot \mathbf{h}$ in the increase $\mathbf{h}$ of the variable, plus a term $\varepsilon(\mathbf{h})\|\mathbf{h}\|$, which tends faster towards 0 for $\mathbf{h} \to \mathbf{0}$ than the linear function $\mathbf{a} \cdot \mathbf{h}$.

In particular, $\Delta f \to 0$ for $\mathbf{h} \to \mathbf{0}$, so we get the result:

A differentiable function at $\mathbf{x} \in A$ is also continuous at $\mathbf{x} \in A$.

Let $A \subseteq \mathbb{R}^n$, $n \geq 2$, be an open set. If a function $f : A \to \mathbb{R}$ is differentiable at every point $\mathbf{x} \in A$, we call it *differentiable in* A, or just *differentiable*.

If $f : A \to \mathbb{R}$ is differentiable at $\mathbf{x} \in A$, i.e.

$$f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) = \mathbf{a} \cdot \mathbf{h} + \varepsilon(\mathbf{h})\|\mathbf{h}\|,$$

then the vector $\mathbf{a}$ is uniquely determined at $\mathbf{x}$. In fact, assume that also

$$f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) = \mathbf{a}_1 \cdot \mathbf{h} + \varepsilon(\mathbf{h})\|\mathbf{h}\|.$$

Then by subtraction,

$$0 = (\mathbf{a} - \mathbf{a}_1) \cdot \mathbf{h} + \varepsilon(\mathbf{h})\|\mathbf{h}\|.$$

Choosing $\mathbf{h} = \lambda\,(\mathbf{a} - \mathbf{a}_1)$, we get

$$0 = \lambda\,\|\mathbf{a} - \mathbf{a}_1\|^2 + \varepsilon\,(\lambda\,(\mathbf{a} - \mathbf{a}_1)) \cdot |\lambda|\;\|\mathbf{a} - \mathbf{a}_1\|\,,$$

where the latter term tends faster towards 0 than λ for $\lambda \to 0$. This is only possible, if $\|\mathbf{a} - \mathbf{a}_1\|^2 = 0$, and we conclude that $\mathbf{a}_1 = \mathbf{a}$, and the uniqueness of $\mathbf{a}$ is proved.

In general, the vector $\mathbf{a}$ depends on $\mathbf{x} \in A$, so $\mathbf{a} = \mathbf{a}(\mathbf{x})$ is a *vector field*. We call it the *gradient* of f and denote it by

$$\mathbf{a} = \mathbf{grad}\; f(\mathbf{x}) = \triangledown f(\mathbf{x}),$$

where "$\triangledown$" reads *"nabla"*.

Remark. In the 1800s, when the gradient was introduced, the mathematicians needed a name for its shorthand notation $\triangledown$. At that time one had just started the excavations of ruins in the Middle East, and Assyrian became fashionable. The inverted triangle $\triangledown$ resembled an Assyrian harp as shown on the bas reliefs, and its name in Assyrian was "nabla" as read on the cuneiform tablets. $\lozenge$

The gradient is therefore defined by the increase of the function in the following way,

$$\begin{aligned} \Delta f &= f(\mathbf{x}+\mathbf{x}) \\ &= \mathbf{h} \cdot \triangledown f(\mathbf{x}) + \varepsilon(\mathbf{h})\|\mathbf{h}\|, \qquad \text{where } \varepsilon(\mathbf{h}) \to 0 \text{ for } \mathbf{h} \to \mathbf{0}. \end{aligned}$$

Here we should strictly speaking more correctly write $\varepsilon(\mathbf{x}, \mathbf{h})$, because this ε-function also depends on the point $\mathbf{x} \in A$. However, we shall only consider it for fixed $\mathbf{x} \in A$, so we leave out the $\mathbf{x}$ in the notation.

The linear part of the increase Δf of the function is called the *differential* of f and denoted $\mathrm{d}f$. When the domain A of f is open in $\mathbb{R}^n$, then the differential is a function in $2n$ variables. More specific,

$$\mathrm{d}f(\mathbf{x}, \mathbf{h}) = \mathbf{h} \cdot \triangledown f(\mathbf{x}).$$

We note that if $n = 1$, then $\triangledown f(x) = f'(x)$, so the gradient is equal to the differential quotient in this case. Furthermore, its differential is (in one variable)

$$\mathrm{d}f(x, h) = f'(x)\,h = \triangledown f(x)\,h,$$

so the gradient $\triangledown f$ in n-dimensional space is a replacement of the derivative $f'(x)$, when $n = 1$.

This extension $\triangledown f$, inherits the same *rules of computation* as the derivative f'. We mention the following, where we assume that $A \subseteq \mathbb{R}^n$ is open:

1) Let α, β be constants, and f, $g : A \to \mathbb{R}$ differential functions. Then

$$\nabla(\alpha f + \beta g) = \alpha \nabla f + \beta \nabla g.$$

2) If f, $g : A \to \mathbb{R}$ are differentiable functions, then

$$\nabla(fg) = f \nabla g + g \nabla f.$$

3) If α is a constant, then

$$\nabla \alpha = \mathbf{0}.$$

These rules of computation are proved in the same way as for the derivative of functions in one variable.

In order to become familiar with a new concept it is customary in practice always to start by considering polynomials of first and second degree in the coordinates.

1) A polynomial of first degree in the rectangular coordinates is written

$$f(\mathbf{x}) = a + \mathbf{b} \cdot \mathbf{x}, \qquad \text{for } \mathbf{x} \in \mathbb{R}^n,$$

where $a \in \mathbb{R}$ is a constant, and $\mathbf{b} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ is a constant vector. The increase of the function is written

$$\Delta f := f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) = a + \mathbf{b} \cdot (\mathbf{x} + \mathbf{h}) - a - \mathbf{b} \cdot \mathbf{x} = \mathbf{b} \cdot \mathbf{h},$$

so we only get the linear term in $\mathbf{h}$ and no ε-function. We conclude that

$$\nabla f = \mathbf{b} \qquad \text{and} \qquad \mathrm{d}f(\mathbf{x}, \mathbf{h}) = \mathbf{b} \cdot \mathbf{h}.$$

We mention the special case, when $n = 2$, in which case we have

$$f(x, y) = a + bx + ct \qquad \text{and} \qquad \nabla f(x, y) = (b, c).$$

2) Then we consider a special polynomial of second degree in the coordinates, namely

$$f(\mathbf{x}) = \mathbf{x} \cdot \mathbf{x} \qquad \text{for } \mathbf{x} \in \mathbb{R}^n.$$

The increase is here

$$\begin{aligned} \Delta f &= f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) = (\mathbf{x} + \mathbf{h}) \cdot (\mathbf{x} + \mathbf{h}) - \mathbf{x} \cdot \mathbf{x} \\ &= \mathbf{x} \cdot \mathbf{x} + 2\mathbf{h} \cdot \mathbf{x} + \mathbf{h} \cdot \mathbf{h} - \mathbf{x} \cdot \mathbf{x} = 2\mathbf{x} \cdot \mathbf{h} + \|\mathbf{h}\|^2. \end{aligned}$$

Since $\varepsilon(\mathbf{h}\|\mathbf{h}\| = \|\mathbf{h}\|^2$, we see that $\varepsilon(\mathbf{h}) = \|\mathbf{h}\| \to 0$ for $\|\mathbf{h}\| \to 0$, so

$$\nabla f = 2\mathbf{x} \qquad \text{and} \qquad \mathrm{d}f(\mathbf{x}, \mathbf{h}) = 2\mathbf{x} \cdot \mathbf{h}.$$

When $n = 2$ we have

$$f(x, y) = x^2 + y^2 \qquad \text{and} \qquad \nabla f(x, y) = (2x, 2y).$$

Concerning applications in Physics we here just mention that the gradient enters *Fourier's law*

$$\mathbf{q} = -\lambda \nabla T,$$

where $\mathbf{q}$ denotes the density of the heat flow, and T is the temperature, and λ is the constant of the heat conductivity.

We find the same mathematical structure in *Fick's first law of diffusion*, and in *Ohm's law* for an electric current.

9.1.2 Partial derivatives

We derived previously a vector field, the gradient $\bigtriangledown f$, of a differentiable function. We shall next find the coordinates of this gradient.

As usual, let the domain $A \subseteq \mathbb{R}^n$ of f be an open set. Choose a (fixed) point $\mathbf{x} = (x_1, \dots, x_n) \in A$, and introduce the auxiliary function

$$f_1(t) := f(t, x_2, \dots, x_n).$$

If f_1 is differentiable for $t = x_1$, we call its derivative $f_1'(x_1)$ the *partial derivative* of $f(\mathbf{x})$ with respect to the first variable x_1. More specifically,

$$f_1'(x_1) = \lim_{h \to 0} \frac{f_1(x_1 + h) - f_1(x_1)}{h} = \lim_{h \to 0} \frac{f(x_1 + h, x_2, \dots, x_n) - f(x_1, x_2, \dots, x_n)}{h}.$$

In this construction we have confined $\mathbf{h}$ to the special vectors of the form $\mathbf{h} = (h, 0, \dots, 0)$, in which case the problem of taking the limit has become 1-dimensional, so we can use (9.1), known from high school.

Even if the partial derivative of f exists with respect to x_1, we cannot be sure that the function f itself is differentiable. Let us for the time being assume that f is differentiable at $\mathbf{x}$. Then the first coordinate of $\bigtriangledown f$ at $\mathbf{x}$ is indeed the partial derivative $f_1'(\mathbf{x})$ introduced above.

In fact, let $\mathbf{h} = (h, 0, \dots, 0)$. Then

$$\begin{aligned} f_1(x_1+h) - f_1(x_1) &= f(x_1+h, x_2, \dots, x_n) - f(x_1, x_2, \dots, x_n) \\ &= (h, 0, \dots, 0) \cdot \bigtriangledown f(\mathbf{x}) + \varepsilon(h)h = h \cdot \{(\bigtriangledown f(\mathbf{x}))_1 + \varepsilon(h)\}, \end{aligned}$$

where $(\bigtriangledown f(\mathbf{x}))_1$ denotes the first rectangular coordinate of $\bigtriangledown f(\mathbf{x})$. When $h \to 0$, then it follows that the auxiliary function f_1 is differentiable at $t = x_1$, and its derivative is the first coordinate $(\bigtriangledown f(\mathbf{x}))_1$ of the gradient at $\mathbf{x}$, and we have proved that

$$f_1'(x_1) = (\bigtriangledown f(\mathbf{x}))_1 = \bigtriangledown f(\mathbf{x}) \cdot \mathbf{e}_1.$$

An analogous analysis gives us the partial derivative of f with respect to the j-th coordinate x_j, for $j = (1), 2, \dots, n$.

We shall of course not use the auxiliary function $f_j'(x_1)$ as our notation for the partial derivative of f with respect to x_j. Instead we write one of the following possibilities,

$$f_{x_j}'(\mathbf{x}), \qquad \frac{\partial f}{\partial x_j}(\mathbf{x}), \qquad D_j f(\mathbf{x}).$$

We shall often leave out the variable $\mathbf{x}$ and just write

$$f_{x_j}', \qquad \frac{\partial f}{\partial x_j} \qquad \text{or} \qquad D_j f.$$

In the frequently considered case of $\mathbb{R}^3$, i.e. when $n = 3$, we usually write

$$f_x', f_y', f_z', \qquad \text{or} \qquad \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \qquad \text{or} \qquad D_x f\, D_y f, D_z f.$$

Similarly for $n = 2$, where the z-coordinate does not appear.

Since the coordinates of the gradient are the partial derivatives, we immediately get

Theorem 9.1 *Let $A \subseteq \mathbb{R}^n$ be an open set. Assume that $f : A \to \mathbb{R}$ is differentiable. Then all its partial derivatives exist, and the gradient is given by*

$$\bigtriangledown f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right).$$

It follows from Theorem 9.1 that when f is differentiable (and thus the gradient exists), then the gradient is unique. On the other hand, one must be aware of strange phenomena like all partial derivatives of f exist at a point, and yet f is not differentiable, so the gradient does not exist. A simple illustrative example is given by the function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0), \\ \\ 0, & (x,y) = (0,0). \end{cases}$$

We have in Chapter 2 shown that $f(x,y)$ is not continuous at $(0,0)$. If one has forgotten this, just restrict the function to the line $y = 2x$, $x \neq 0$, on which $f(x, 2x) = 1 \to 1 \neq 0$ for $x \to 0$. The function f has nevertheless partial derivatives at $(0,0)$, because the restriction to the x-axis is

$$f(x,0) = 0 \quad \text{for all } x \in \mathbb{R}, \qquad \text{with } \frac{\partial f}{\partial x}(0,0) = 0,$$

and the restriction to the y-axis is

$$f(0,y) = 0 \quad \text{for all } y \in \mathbb{R}, \qquad \text{with } \frac{\partial f}{\partial y}(0,0) = 0.$$

In order to obtain a positive result we mention without the long proof (it consists of two pages) of the following theorem.

Theorem 9.2 *Let $A \subseteq \mathbb{R}^n$ be open, and let $f : A \to \mathbb{R}$ be a given function. Assume that all the partial derivatives of f exist in a whole neighbourhood of $\mathbf{x} \in A$ and are all continuous,* (this means that we can find an open ball $B(\mathbf{x}, r) \subseteq A$, in which all derivatives of f exist and are continuous) *then f is even* differentiable *at $\mathbf{x}$.*

In most cases we prove the differentiability of a function f by applying Theorem 9.2 in the following way: First we calculate all the partial derivatives in a neighbourhood of the given point $\mathbf{x} \in A$, and then we show that they are all continuous.

It is of course not hard to show that the continuity of the partial derivatives fail in the case of the function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0), \\ \\ 0, & (x,y) = (0,0). \end{cases}$$

The following theorem is a generalization of a well-known result from the theory of real functions in one variable, namely that if f is differentiable, and f' is zero everywhere in an interval, then f is a constant. The trick in the proof is to use this 1-dimensional theorem repeatedly.

Theorem 9.3 *Given an open domain A in $\mathbb{R}^n$, and assume that $f : A \to \mathbb{R}$ is differentiable of gradient $\bigtriangledown f = \mathbf{0}$ everywhere in A. Then f is constant in A.*

Sketch of proof. First note that the gradient in the formulation of Theorem 9.3 is used as a shorthand for the generalization of the derivative in one dimension. In order to apply the corresponding theorem in one dimension we of course use the partial derivatives instead. We shall use that since the open domain A is open and connected, we can to any two points $\mathbf{a}$, $\mathbf{b} \in A$ find a step line connecting them. This is a continuous curve lying totally in A with $\mathbf{a}$ as starting point and $\mathbf{b}$ as final point and consisting only of axiparallel line segments, on each of which just one coordinate varies. We can exploit this, because then we can locally formulate the problem by the partial derivative with respect to this variable.

The gradient was assumed to be $\mathbf{0}$ everywhere in A, i.e. $\bigtriangledown f = \mathbf{0}$. Then along each of the afore mentioned axiparallel line segments, the restriction f_1 of f is an ordinary function in one variable, for which $f_1' = 0$. It follows from the 1-dimensional result that f_1 is constant on this line segment. This is true for all axiparallel line segments of the step line, and as f is also continuous, then constant must be the same on all line segments. In particular, $f(\mathbf{a}) = f(\mathbf{b})$. As $\mathbf{a}$, $\mathbf{b} \in A$ were chosen arbitrarily, we finally conclude that f is constant on A. $\square$

We here add the proof of the result that if $f,\, g : A \to \mathbb{R}$ are both differentiable, then

$$\bigtriangledown(fg) = f \bigtriangledown g + g \bigtriangledown f.$$

When we look at each coordinate separately, the proof is straightforward. In fact,

$$(\bigtriangledown(fg))_j := \frac{\partial(fg)}{\partial x_j} = f\,\frac{\partial g}{\partial x_j} + g\,\frac{\partial f}{\partial x_j} = f(\bigtriangledown g)_j + g(\bigtriangledown f)_j = (f \bigtriangledown g + g \bigtriangledown f)_j,$$

where the lower index j indicates the j-th coordinate.

We include an important observation on functions defined by an integral of variable upper and lower bounds and with an extra variable in the integrand which in the integration process is considered as a parameter for the time being. Let us for example consider the following integral

$$G(x,y,z) = \int_z^y f(t,x)\,\mathrm{d}t,$$

which will illustrate the principle. We shall often in the following volumes meet such functions, so that is why we here premise a remark to the effect that they will be at hand later on, when they are needed.

Assume that the integrand f is *continuous*. Then it has an *antiderivative* $F(t,x)$, which satisfies $F_t'(t,x) = f(t,x)$. Then we use the main theorem of differential and integration calculus in one variable to get

$$G(x,y,z) = F(y,x) - F(z,x).$$

We then turn to the problem of finding $\bigtriangledown G$. Clearly, y and z are the easy variables, because the partial derivatives are straightforward,

$$G_y'(x,y,z) = F_y'(y,x) - 0 = f(y,x),$$

$$G_z'(x,y,z) = 0 - F_z'(z,x) = -f(z,x).$$

The variable x enters here only the integrand, so one would expect that

$$G_x'(x,y,z) = \int_z^y f_x'(t,x)\,\mathrm{d}t.$$

This is true, *if we furthermore assume that the partial derivative f_x' of the integrand is continuous!* So when both $f(t,x)$ and $f_x'(t,x)$ are continuous, the gradient of $G(x,y,z)$ given as the integral above is

$$\bigtriangledown G = \left(\int_z^y f_x'(t,x)\,\mathrm{d}t, f(y,x), -f(z,x)\right).$$

Remark 9.1 We have of course here chosen a purely mathematical notation. In the applications in e.g. Physics this notation may sometimes be ambiguous, so one is forced to modify the notation in order to make it more precise. Let us consider a thermodynamic system. In this we have the following possible variables, the volume V, the pressure p, the temperature T and the entropy S. The ambiguity of the previous notation occurs because the system is totally described by just *two* of these four variables. This means that a notation like $\frac{\partial V}{\partial p}$ is not unique, unless one also makes it precise, if the state of the system is determined by (p,T), or by (p,S). One usually adds an index, like for instance $\left(\frac{\partial V}{\partial p}\right)_T$, or $\left(\frac{\partial V}{\partial p}\right)_S$, resp.. Here, $\left(\frac{\partial V}{\partial p}\right)_T$ means the partial derivative of the volume with respect to the pressure, provided that the temperature T is kept constant. Similarly, $\left(\frac{\partial V}{\partial p}\right)_S$ means that the entropy is kept constant. This change of notation makes it easier in Thermodynamics to formulate many results than if we instead had only used the pure mathematical notation. We mention here the so-called *Maxwell relation*, which in the physical notation becomes

$$\left(\frac{\partial T}{\partial p}\right)_p = \left(\frac{\partial V}{\partial S}\right)_p .$$

The reader can easily imagine the problems in only using the mathematical notations, because then we had to add a comment on that the entropy S is kept constant on the left hand side of the equation, while we on the right hand side of the equation instead keep the pressure fixed. $\Diamond$

9.1.3 Differentiable vector functions

A vector function $\mathbf{f} : A \to \mathbb{R}^m$, where $A \subseteq \mathbb{R}^n$, is called *differentiable*, if all its coordinate functions are differentiable. This means more precisely that

$$f_i(\mathbf{x}+\mathbf{h}) - f_i(\mathbf{x}) = \mathbf{h} \cdot \bigtriangledown f_i(\mathbf{x}) + \varepsilon_i(\mathbf{h})\|\mathbf{h}\|, \qquad \text{where } \varepsilon(\mathbf{h}) \to 0 \text{ for } \mathbf{h} \to \mathbf{0}, \quad \text{for } i = 1, \dots, m.$$

Combining all coordinates we have

$$\mathbf{f}(\mathbf{x}+\mathbf{h}) - \mathbf{f}(\mathbf{x}) = (\mathbf{h} \cdot \bigtriangledown)\mathbf{f}(\mathbf{x}) + \varepsilon(\mathbf{h})\|\mathbf{h}\|, \qquad \text{where } \varepsilon(\mathbf{h}) \to \mathbf{0} \quad \text{for } \mathbf{h} \to \mathbf{0}.$$

We define the *differential of the vector function*,

$$\mathrm{d}\mathbf{f}(\mathbf{x},\mathbf{h}) = (\mathbf{h} \cdot \bigtriangledown)\mathbf{f}(\mathbf{x}),$$

by all its coordinates,

$$(\mathbf{h} \cdot \bigtriangledown)\mathbf{f}(\mathbf{x}) = (\mathbf{h} \cdot \bigtriangledown f_1(\mathbf{x}), \dots, \mathbf{h} \cdot \bigtriangledown f_m(\mathbf{x})) = (\,\mathrm{d}f_1(\mathbf{x}), \dots, \,\mathrm{d}f_m(\mathbf{x}))\,.$$

If we here choose $\mathbf{f}$ as the identity map, i.e. $\mathbf{f}(\mathbf{x}) := \mathbf{x}$, then $\mathbf{f}(\mathbf{x}+\mathbf{h}) - \mathbf{f}(\mathbf{x}) = \mathbf{h}$, so the differential becomes

$$\mathrm{d}\mathbf{f}(\mathbf{x},\mathbf{h}) = \mathbf{h}.$$

When we write $\mathbf{x}$ instead of $\mathbf{f}$ we get the strictly speaking incorrect, though very practical notation, namely $\mathrm{d}\mathbf{x} = \mathbf{h}$, and hence in general,

$$\mathrm{d}f = \,\mathrm{d}\mathbf{x} \cdot \bigtriangledown f \qquad \text{in one dimension,}$$

$$\mathrm{d}\mathbf{f} = (\,\mathrm{d}\mathbf{x} \cdot \bigtriangledown)\mathbf{f} \qquad \text{in several dimensions.}$$

All information on the mn partial derivatives of $\mathbf{f}$ is collected in the so-called *functional matrix* $\mathbb{D}\,\mathbf{f}$, which is defined by

$$\mathbb{D}\,\mathbf{f}(\mathbf{x}) := \begin{pmatrix} \dfrac{\partial f_1}{\partial x1}(\mathbf{x}) & \cdots & \dfrac{\partial f_1}{\partial x_n}(\mathbf{x}) \\ \vdots & & \vdots \\ \dfrac{\partial f_m}{\partial x_1}(\mathbf{x}) & \dots & \dfrac{\partial f_m}{\partial x_n}(\mathbf{x}) \end{pmatrix},$$

and we get by using some *Linear Algebra* that the differential can be written as a *matrix product*,

$$\mathrm{d}\mathbf{f}(\mathbf{x},\mathbf{h}) = (\mathbb{D}\,\mathbf{f}(\mathbf{x}))\,\mathbf{h}, \qquad \text{or for short } \mathrm{d}\mathbf{f} = (\mathbb{D}\,\mathbf{f})\,\mathbf{h},$$

where $\mathbf{h}$ should be written as an $(n \times 1)$-column.

9.1.4 The approximating polynomial of degree 1

Let us return to the definition of the differentiability of a function $f : A \to \mathbb{R}$, i.e.

$$f(\mathbf{x}) = f(\mathbf{x}_0) + (\mathbf{x} - \mathbf{x}_0) \cdot \triangledown f(\mathbf{x}_0) + \varepsilon(\mathbf{x} - \mathbf{x}_0) \|\mathbf{x} - \mathbf{x}_0\|,$$

where $\mathbf{x}_0 \in A$ is the chosen point, and where we have written the increment as $\mathbf{h} = \mathbf{x} - \mathbf{x}_0$. Since $\varepsilon(\mathbf{x} - \mathbf{x}_0) \to 0$ for $\mathbf{x} \to \mathbf{x}_0$, it follows that the approximation by a polynomial of degree 1 in a neighbourhood of $\mathbf{x}_0 \in A$ is given by

$$f(\mathbf{x}) \simeq P_1(\mathbf{x}),$$

where we have defined

$$P_1(\mathbf{x}) := f(\mathbf{x}_0) + (\mathbf{x} - \mathbf{x}_0) \cdot \triangledown f(\mathbf{x}_0).$$

We call this $P_1(\mathbf{x})$ the *approximating polynomial of at most degree 1* of the function f at the *point of expansion* $\mathbf{x}_0$.

Remark 9.2 It is important to keep the variable in the form $\mathbf{x} - \mathbf{x}_0 = (x_1 - x_{01}, \dots, x_n - x_{0n})$, and not to reduce it to a function in $\mathbf{x}$ alone. The reason is that we in the applications only use the approximating polynomial *in the neighbourhood of* $\mathbf{x}_0$, where $\mathbf{x} - \mathbf{x}_0$ is small. $\Diamond$

We mention for later references the structures of the approximating polynomials for $n = 2$ and $n = 3$,

$$P_1(x, y) = f(x_0, y_0) + f'_x(x_0, y_0)(x - x_0) + f'_y(x_0, y_0)(y - y_0), \qquad \text{for } n = 2,$$

$$\begin{aligned} P_1(x, y, z) &= f(x_0, y_0, z_0) + f'_x(x_0, y_0, z_0)(x - x_0) + f(x_0, y_0, z_0) + f'_y(x_0, y_0, z_0)(y - y_0) \\ &\quad + f(x_0, y_0, z_0) + f'_z(x_0, y_0, z_0)(z - z_0), \qquad \text{for } n = 3. \end{aligned}$$

As a simple application we consider the function $f(x, y)$ in two variables given by

$$f(x, y) = \exp\left(x^2 - y^2\right) \qquad \text{for } (x, y) \in \mathbb{R}^2,$$

where we shall find the approximating polynomial of degree 1 derived from the point of expansion $(x_0, y_0) = (1, -1)$.

We first calculate

$$f'_x(x, y) = 2x \exp\left(x^2 - y^2\right) \qquad \text{and} \qquad f'_y(x, y) = -2y \exp\left(x^2 - y^2\right).$$

Then we compute all the necessary constants,

$$f(1, -1) = 1, \qquad f'_x(1, -1) = 2, \qquad f'_y(1, -1) = 2.$$

By insertion the approximating polynomial becomes

$$P_1(x, y) = 1 + 2(x - 1) + 2(y + 1),$$

which is reasonable useful, when $(x - 1, y + 1)$ is small. As an example we get

$$P_1(0.95, -1.02) = 0.86, \qquad \text{in comparison with } f(0.95, -1.02) = 0.87118\cdots.$$

If we instead "reduce" $P_1(x, y)$ to a polynomial in (x, y), then we use $(0, 0)$ as expansion point for an approximation, which is only reasonable at a point $(1, -1)$ far away. The result $P_1(x, y) = 1 + 2x + 2y$ looks of course nicer, but we lose the important information that it can only be used for $(x - 1, y + 1)$ small. Therefore: *Always keep the variable in the form* $\mathbf{x} - \mathbf{x}_0$*, where* $\mathbf{x}_0$ is the point of expansion.

9.2 The chain rule

9.2.1 The elementary chain rule

As usual we start with the 1-dimensional case in order to find out in what direction we should go, when we generalize to the case of higher dimensions.

The elementary chain rule. *Let $f : A \to \mathbb{R}$ and $X : B \to A$ be two differentiable functions, each in one variable. Then the composite function $F := f \circ X : B \to \mathbb{R}$ is also differentiable, and*

$$\frac{dF}{du} = \frac{df}{dx}(X(u))\,\frac{dX}{du}(u),$$

which is also written

$$F'(u) = f'(X(u))X'(u).$$

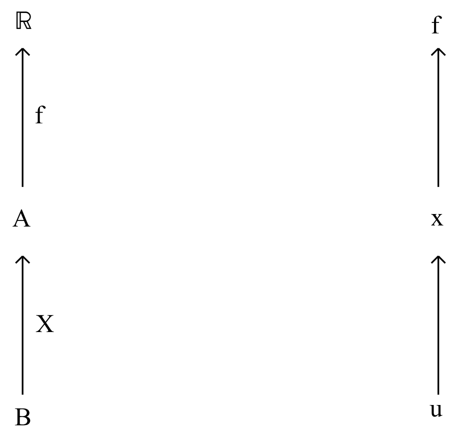

Figure 9.1: The elementary chain rule. The composite function is $F = f \circ X : B \to \mathbb{R}$ (the tree to the left), so first we map $u \in B$ into $x = X(u) \in A$, which is then mapped into

$$f(x) = f(X(u)) = (f \circ X)(u).$$

To the right we have indicated the three levels. We shall differentiate f on the highest level with respect to $u \in B$ on the lowest level, through $x \in A$ in the middle level.

Proof. Obviously, the composite function $F := f \circ X : B \to \mathbb{R}$ is well-defined. We shall prove that it is also differentiable.

Let $u_0 \in B$. Then $x_0 = X(u_0) \in A$, and we can find an open neighbourhood $B_1 \subseteq B$ of u_0, such that $x = X(u) \in A$ for all $u \in B_1$. We may of course in the following assume that $B_1 = B$.

Let Δu denote an increment of $u \in B$, such that also $u + \Delta u \in B$. We have assumed that X is differentiable, so

$$X(u+\Delta u) - X(u) := \Delta X \to 0 \qquad \text{for } \Delta u \to 0 \text{ and } u,\, u+\Delta u \in B,$$

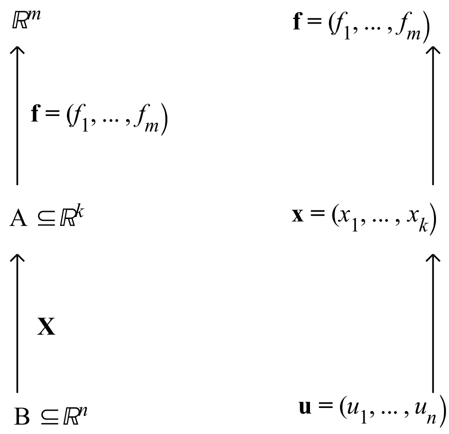

Figure 9.2: The general scheme of the chain rule. We shall differentiate the vector function $\mathbf{f}$ at the highest level with respect to $\mathbf{u}$ on the lowest level via $\mathbf{x}$ on the middle level. Only the middle level $\mathbf{x}$ will be in contact with both the upper level $\mathbf{f}$ and the lower level $\mathbf{u}$.

and also

$$\frac{\Delta X}{\Delta u} \to X'(u) \qquad \text{for } \Delta u \to 0,$$

which can be written in the form (after a rearrangement)

$$X(u + \Delta u) = X(u) + X'(u)\Delta u + \varepsilon(\Delta u)\Delta u, \qquad \text{where } \varepsilon(\Delta u) \to 0 \text{ for } \Delta u \to 0.$$

We also assumed that the function f is differentiable in A, so

$$f(x + \Delta x) - f(x) := \Delta f \to 0 \qquad \text{for } \Delta x \to 0 \text{ and } x + \Delta x \in A,$$

and

$$\frac{\Delta f}{\Delta x} \to f'(x) \qquad \text{for } \Delta x \to 0,$$

and

$$f(x + \Delta x) = f(x) + f'(x)\Delta x + \varepsilon(\Delta x)\Delta x.$$

Using that $F(u) := f(X(u))$, and that $X(u + \Delta u) \in A$ for u, $u + \Delta u \in B$, we get

$$\begin{aligned}
\frac{\Delta F}{\Delta u} &= \frac{1}{\Delta u}\{F(u + \Delta u) - F(u)\} = \frac{1}{\Delta u}\{f(X(u + \Delta u)) - f(X(u))\} \\
&= \frac{1}{\Delta u}\{f(X(u) + \Delta X) - f(X(u))\} \\
&= \frac{1}{\Delta u}\{f(X(u)) + f'(X(u))\Delta X + \varepsilon(\Delta X)\Delta X - f(X(u))\} \\
&= f'(X(u)) \cdot \frac{\Delta X}{\Delta u} + \varepsilon(\Delta X) \to f'(X(u)) \cdot X'(u) \qquad \text{for } \Delta u \to 0,
\end{aligned}$$

and the elementary chain rule is proved. □

We shall in the following generalize this elementary chain rule to the higher dimensional case as described schematically on Figure 9.2. We still keep the arrows, but later we shall exclude them, because we shall always calculate the derivatives from below, i.e. in the upward direction. First we note that the vector function $\mathbf{f}(\mathbf{x})$ is a function of the vector $\mathbf{x}$, which again is a function of the vector $\mathbf{u}$. Clearly, at head on approach is doomed to fail, so we shall first analyze a couple of simpler cases, before we show the chain rule in general. The chain rule may at the first glance seem very technical. It is, however, important in the practical applications.

9.2.2 The first special case

We first consider the case where $m = k = 1$ and $n > 1$. In the following we shall only consider the trees to the right in Figure 9.1 and Figure 9.2.

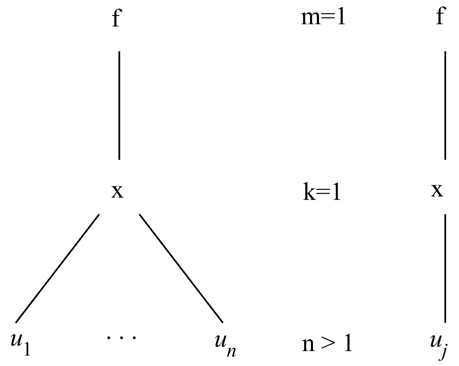

Figure 9.3: The chain rule in the first special case. The subtree, where we only differentiate with respect to one variable u_j is shown to the right.

When we confine ourselves to the partial derivatives of the composite function with respect to u_j, it follows from the tree at the right hand side of Figure 9.3 that when all the other u-variables are considered as parameters, then we have reduced the problem to the elementary case of the one-dimensional chain rule, so if we write $F = f \circ X$, we get

$$\frac{\partial F}{\partial u_j}(\mathbf{u}) = \frac{\mathrm{d}f}{\mathrm{d}x}(X(u))\frac{\partial X}{\partial u_j}(u), \qquad \text{for } j = 1, \dots, n.$$

Collecting all the coordinate functions in one equation, we get the following

First special case of the chain rule. *If $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}$, and $X : B \to A$, where $B \subseteq \mathbb{R}^n$, and $F = f \circ X : B \to \mathbb{R}$, then*

$$F(\mathbf{u}) = f(X(\mathbf{u})) \qquad \textit{and} \qquad \nabla F(\mathbf{u}) = f'(X(\mathbf{u})) \nabla X(\mathbf{u}).$$

One particular case will be useful in the following, namely when

$$F(u) = f\left(\sqrt{u_1^2 + \cdots + u_n^2}\right)$$

only depends on the distance from $\mathbf{0}$ in the $\mathbf{u}$-space. If $\mathbf{u} \neq \mathbf{0}$, we put

$$X(\mathbf{u}) = \|\mathbf{u}\| = \sqrt{u_1^2 + \cdots + u_n^2} \qquad \text{where } \frac{\partial X}{\partial u_1} = \frac{u_1}{\|\mathbf{u}\|} \text{ etc.,}$$

so

$$\nabla X(\mathbf{u}) = \frac{\mathbf{u}}{\|\mathbf{u}\|}.$$

When $F(\mathbf{u}) = f(\|\mathbf{u}\|)$ and $\mathbf{u} \neq \mathbf{0}$, we get by the chain rule above that

$$\triangledown F(\mathbf{u}) = f'(\|\mathbf{u}\|)\frac{\mathbf{u}}{\|\mathbf{u}\|}.$$

In other words, the gradient of F is in this special case equal to the derivative f' of f, multiplied by a unit vector, which is directed away from origo.

9.2.3 The second special case

This case is also easy. We choose $m > 1$ and $k = n = 1$, so we get the tree on Figure 9.4.

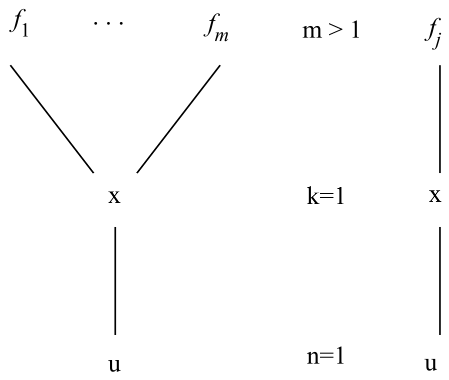

Figure 9.4: The chain rule in the second special case. The subtree, where we only differentiate one function f_j is shown to the right.

The j-th coordinate function $F_j(u) = f_j(X(u))$ is differentiated in the following way, according to the elementary chain rule,

$$\frac{\mathrm{d}F_j}{\mathrm{d}u}(u) = \frac{\mathrm{d}f_j}{\mathrm{d}x}(X(u))\,\frac{\mathrm{d}X}{\mathrm{d}u}(u), \qquad \text{for } j = 1, \dots, m.$$

Putting all coordinate functions together we obtain:

Second special case of the chain rule. *If* $\mathbf{f} : A \to \mathbb{R}^m$*, where* $A \subseteq \mathbb{R}$*, and* $X : B \to A$*, where* $B \subseteq \mathbb{R}$*, and* $\mathbf{F} = \mathbf{f} \circ X : B \to \mathbb{R}$*, then*

$$\mathbf{F}(u) = \mathbf{f}(X(u)) \qquad \textit{and} \qquad \mathbf{F}'(u) = \mathbf{f}'(X(u)).$$

9.2.4 The third special case

This is the most complicated special case, where $k > 1$, while $m = n = 1$. The tree is shown to the left of Figure 9.5 with the general case to the left, and the special case of $k = 2$ to the right.

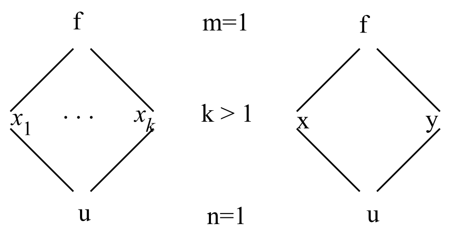

Figure 9.5: The chain rule in the third special case. The subtree, where we only have two variables, x and y, is shown to the right.

In order to avoid a mess of indices in the proof we shall only prove this special case for $k = 2$, where we use (x, y) instead of (x_1, x_2). We shall therefore consider the composite function

$$F(u) = f(X(u), Y(u)).$$

Once the chain rule has been proved in this special case, it is easy to generalize.

Before we prove the chain rule in this case, we make some preparations. If the variable u is given an increment Δu, then we put

$$X(u + \Delta u) := X(u) + \Delta X \qquad \text{and} \qquad Y(u + \Delta u) := Y(u) + \Delta Y.$$

We assume of course that $X(u)$ and $Y(u)$ are differentiable, so

$$\Delta X \to 0 \quad \text{and} \quad \Delta Y \to 0 \qquad \text{for } \Delta u \to 0,$$

and

$$\frac{\Delta X}{\Delta u} \to X'(u) \quad \text{and} \quad \frac{\Delta Y}{\Delta u} \to Y'(u) \qquad \text{for } \Delta u \to 0.$$

Furthermore, we assume that the function f is differentiable at the point (x, y). This means that

$$f(x+\Delta x, y+\Delta y) = f(x, y) + f'_x(x, y)\Delta x + f'_y(x, y)\Delta y + \varepsilon(\Delta x, \Delta y)\sqrt{(Deltax)^2+(\Delta y)^2},$$

where $\varepsilon(\Delta x, \Delta y) \to 0$ for $(\Delta x, \Delta y) \to (0, 0)$, i.e. for $\sqrt{(\Delta x)^2 + (\Delta y)^2} \to 0$.

Then we have to put all things together, so we shall compute the differential quotient of the composite function $F(u) = F(X(u), Y(u))$ and use the above to reformulate this expression.

We get

$$\frac{\Delta F}{\Delta u} = \frac{1}{\Delta u}\{f(X(u+\Delta u),Y(u+\Delta u)) - f(X(u),Y(u))\}.$$

Then insert $X(u+\Delta u) = X(u)+\Delta X$ and $Y(u+\Delta u) = Y(u)+\Delta Y$ to get

$$\begin{aligned}\frac{\Delta F}{\Delta u} &= \frac{1}{\Delta u}\{f(X(u)+\Delta X, Y(u)+\Delta Y) - f(X(u),Y(u))\}\\ &= \frac{1}{\Delta u}\left\{f_x'(X(u),Y(u))\Delta X + f_y'(X(u),Y(u))\Delta Y\right\} + \frac{1}{\Delta u}\,\varepsilon(\Delta X,\Delta Y)\sqrt{(\Delta X)^2+(\Delta Y)^2}\\ &= f_x'(X(u),Y(u))\frac{\Delta X}{\Delta u} + f_y'(X(u),Y(u))\frac{\Delta Y}{\Delta u} \pm \varepsilon(\Delta X,\Delta Y)\sqrt{\left(\frac{\Delta X}{\Delta u}\right)^2+\left(\frac{\Delta Y}{\Delta u}\right)^2},\end{aligned}$$

where the $\pm$ indicates the sign of Δu.

Then by taking the limit $\Delta u \to 0$,

$$F'(u) = \lim_{\Delta u\to 0}\frac{\Delta F}{\Delta u} = f_x'(X(u),Y(u))X'(u) + f_y'(X(u),Y(u))Y'(u),$$

because

$$\sqrt{\left(\frac{\Delta X}{\Delta u}\right)^2+\left(\frac{\Delta Y}{\Delta u}\right)^2} \to \sqrt{(X'(u))^2+(Y'(u))^2} \qquad \text{is finite for } \Delta u \to 0,$$

and $\varepsilon(\Delta X,\Delta Y) \to 0$ for $\Delta u \to 0$.

Summing up, we have proved the chain rule for $k = 2$ and $m = n = 1$:

Third special case of the chain rule for $k = 2$. *If* $f : A \to \mathbb{R}$, *where* $A \subseteq \mathbb{R}^2$, *and* $(X,Y) : B \to A$, *where* $B \subseteq \mathbb{R}$, *and* $F = f \circ (X,Y) : B \to \mathbb{R}$, *then*

$$F(u) = f(X(u), Y(u)) \qquad \textit{and} \qquad F'(u) = \left[f'_x(x,y)X'(u) + f'_y(x,y)Y'(u)\right]_{x=X(u),\, y=Y(u)}.$$

In practice we first compute the partial derivatives $f'_x(x,y)$ and $f'_y(x,y)$, and then the ordinary derivatives $X'(u)$ and $Y'(u)$, for finally to insert $x = X(u)$ and $y = Y(u)$. A short way of writing this formula is

$$\frac{\mathrm{d}F}{\mathrm{d}u} = \frac{\partial f}{\partial x}\,\frac{\mathrm{d}x}{\mathrm{d}u} + \frac{\partial f}{\partial y}\,\frac{\mathrm{d}y}{\mathrm{d}u}.$$

This version of the chain rule is often used, when the function which should be differentiated, is fairly complicated. We illustrate this by considering the function

$$F(u) = \operatorname{Arctan}\left(\sqrt{\frac{e^u - \sin u}{e^u + \sin u}}\right), \qquad \text{for } u \in \mathbb{R}.$$

The trick is to write $F(u) = f(X(u), Y(u))$ as a composite function. Here one would choose

$$f(x,y) = \operatorname{Arctan}\left(\sqrt{\frac{x}{y}}\right) \qquad \text{for } (x,y) \in \mathbb{R}_+^2,$$

and

$$X(u) = e^u - \sin u \qquad \text{and} \qquad Y(u) = e^u + \sin u \qquad\qquad \text{for } u \in \mathbb{R}.$$

(Note that $X(u)$, $Y(u) > 0$ for $u \in \mathbb{R}$.)

Then

$$f'_x(x,y) = \frac{1}{1 + \dfrac{x}{y}} \cdot \frac{1}{2\sqrt{xy}} = \frac{y}{2(x+y)\sqrt{xy}}, \qquad f'_y(x,y) = \frac{1}{1 + \dfrac{x}{y}} \cdot \frac{-\sqrt{x}}{2y\sqrt{y}} = \frac{-x}{2(x+y)\sqrt{xy}},$$

while

$$X'(u) = e^u - \cos u \qquad \text{and} \qquad Y'(u) = e^u + \cos u.$$

By insertion of $X(u) = e^u - \sin u$ and $Y(u) = e^u + \sin u$ we get

$$f'_x(x,y) = \frac{e^u + \sin u}{2 \cdot 2e^u\sqrt{e^{2u} - \sin^2 u}}, \quad \text{and} \quad f'_y(x,y) = \frac{-e^u + \sin u}{2 \cdot 2e^u\sqrt{e^{2u} - \sin^2 u}},$$

so

$$\begin{aligned}
F'(u) &= \left[f'_x(x,y)X'(u) + f'_y(x,y)Y'(u)\right]_{x=X(u),\, y=Y(u)} \\
&= \frac{1}{4e^u\sqrt{e^{2u} - \sin^2 u}}\left\{(e^u + \sin u)\,(e^u - \cos u) - (e^u - \sin u)\,(e^u + \cos u)\right\} \\
&= \frac{1}{4e^u\sqrt{e^{2u} - \sin^2 u}}e^u(-\cos u + \sin u + \sin u - \cos u) = \frac{\sin u - \cos u}{2\sqrt{e^{2u} - \sin^2 u}}.
\end{aligned}$$

If we here apply MAPLE, we write

$$\frac{\mathrm{d}}{\mathrm{d}u}\arctan\left(\sqrt{\frac{e^u - sin(u)}{e^u + \sin(u)}}\right),$$

which produces the following

$$\frac{1}{2}\cdot\frac{\dfrac{e^u - cos(u)}{e^u + \sin(u)} - \dfrac{(e^u - \sin(u))\,(e^u + \cos(u))}{(e^u + \sin(u))^2}}{\sqrt{\dfrac{e^u - \sin(u)}{e^u + \sin(u)}}\left(1 + \dfrac{e^u - \sin(u)}{e^u + \sin(u)}\right)}$$

which clearly needs to be reduced.

Without going into details we mention that if $k > 2$, then we just copy the proof above to get

Third special case of the chain rule for $k > 2$. *If $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^k$, and $\mathbf{X} : B \to A$, where $B \subseteq \mathbb{R}^k$, and $F = f \circ \mathbf{X} : B \to \mathbb{R}$, then $F(u) = f(\mathbf{X}(u))$, and*

$$F'(u) = \left[f'_{x_1}(\mathbf{x})X'_1(u) + \cdots + f'_{x_k}(\mathbf{x})X'_k(u)\right]_{\mathbf{x}=\mathbf{X}(u)} = \bigtriangledown f(\mathbf{X}(u)) \cdot \mathbf{X}'(u).$$

The latter equation follows from that the first result actually is a scalar product.

An important application occurs, when we shall differentiate a function, which is given by an integral, in which the upper and lower bounds are differentiable functions in the variable under consideration, as well as the integrand. Let us consider

$$g(x) = \int_{Z(x)}^{Y(x)} f(t,x)\,\mathrm{d}t, \qquad x \in I,$$

where I is an interval. We define a function $G(x,y,z)$ in three variables by

$$G(x,y,z) := \int_z^y f(t,x)\,\mathrm{d}t,$$

and then note that

$$g(x) = G(x, Y(x), Z(x)).$$

We have previously found that

$$G'_x(x,y,z) = \int_z^y f'_x(t,z)\,\mathrm{d}t, \qquad G'_y(x,y,z) = f(y,x), \qquad G'_z(x,y,z) = -f(z,x),$$

so we get by the chain rule for $k = 3$ and $m = n = 1$ and $X(x) = x$, that

$$\begin{aligned}\frac{\mathrm{d}g}{\mathrm{d}x} &= \frac{\partial G}{\partial x}\cdot\frac{\mathrm{d}X}{\mathrm{d}x} + \frac{\partial G}{\partial y}\cdot\frac{\mathrm{d}Y}{\mathrm{d}x} + \frac{\partial G}{\partial z}\cdot\frac{\mathrm{d}Z}{\mathrm{d}x}\\ &= \int_{Z(x)}^{Y(x)} f'_x(t,x)\ \mathrm{d}t + f(Y(x),x)Y'(x) - f(Z(x),x)Z'(x).\end{aligned}$$

This rule is valid, when the functions f, f'_x, Y' and Z' are all continuous.

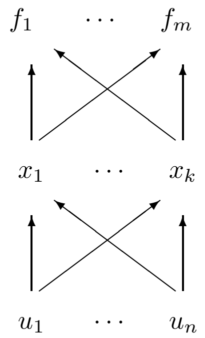

Figure 9.6: The general diagram of the chain rule.

9.2.5 The general chain rule

In the general case we have the situation as described on Figure 9.6. By fixing the index $r \in \{1, \dots, m\}$ in the upper layer and $j \in \{1, \dots, n\}$ in the lower layer we reduce the complicated scheme of Figure 9.6 to Figure 9.7, which we recognize as the diagram for the third special case of the chain rule in Section 9.2.4. Therefore, the general chain rule follows by gluing all cases together of $r \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$.

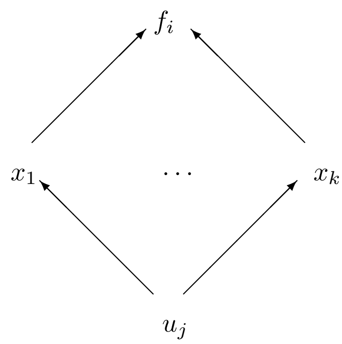

Figure 9.7: The reduced diagram of the chain rule.

The general chain rule. *Given the composite function* $\mathbf{F}(\mathbf{u}) = \mathbf{f}(\mathbf{X}(\mathbf{u}))$, *where the coordinate functions are given by*

$$F_r(u_1, \dots, u_n) = f_r(\mathbf{X}(u_1, \dots, u_n)).$$

If $\mathbf{f}$ *and* $\mathbf{X}$ *are differentiable, then so is* $\mathbf{F} = \mathbf{f} \circ \mathbf{X}$, *and we get for each coordinate function* F_r *and each variable* u_j *that*

$$\frac{\partial F_r}{\partial u_j}(\mathbf{u}) = \frac{\partial f_r}{\partial x_1}(\mathbf{X}(\mathbf{u}))\, \frac{\partial X_1}{\partial u_j}(\mathbf{u}) + \cdots + \frac{\partial f_r}{\partial x_k}(\mathbf{X}(\mathbf{u}))\, \frac{\partial X_k}{\partial u_j}(\mathbf{u}),$$

for $r \in \{1, \dots, m\}$ *and* $j \in \{1, \dots, n\}$.
If we use the functional matrix differential operator $\mathbb{D}$ *defined by*

$$\mathbb{D}\mathbf{f} = \left\{ \frac{\partial f_r}{\partial u_j} \right\}_{r=1,\dots,m;j=1,\dots,n},$$

then the general chain rule can also be written in the following matrix notation,

$$\mathbb{D}(\mathbf{f} \circ \mathbf{X})(\mathbf{u}) = \mathbb{D}\mathbf{f}(\mathbf{X}(\mathbf{u}))\mathbb{D}\mathbf{X}(\mathbf{u}).$$

Clearly, all the previously obtained special cases are obtained by putting (at least) two of the numbers m, k, n equal to 1 (and trivially replace "∂" by "d", when we have got only one variable.

A frequent application consists in the change from rectangular coordinates in the plane to polar coordinates. So given the function $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^2$, we shall consider partial differentiations with respect to the polar coordinates (ϱ, ϱ) of

$$F(\varrho, \varphi) = f(x, y) = f(\varrho \cos \varphi, \varrho \sin \varphi).$$

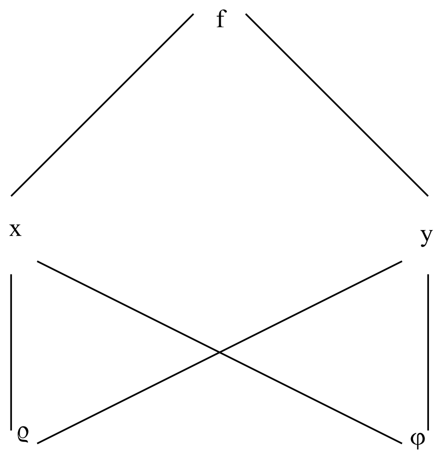

Figure 9.8: The diagram for partial differentiation in polar coordinates in the plane instead of in rectangular coordinates.

We get by the chain rule,

$$\frac{\partial F}{\partial \varrho} = \frac{\partial f}{\partial x}\,\frac{\partial x}{\partial \varrho} + \frac{\partial f}{\partial y}\,\frac{\partial y}{\partial \varrho} = \frac{\partial f}{\partial x}(\varrho\cos\varphi, \varrho\sin\varphi)\cos\varphi + \frac{\partial f}{\partial y}(\varrho\cos\varphi, \varrho\sin\varphi)\sin\varphi,$$

and

$$\frac{\partial F}{\partial \varphi} = \frac{\partial f}{\partial x}\,\frac{\partial x}{\partial \varphi} + \frac{\partial f}{\partial y}\,\frac{\partial y}{\partial \varphi} = \frac{\partial f}{\partial y}(\varrho\cos\varphi, \varrho\sin\varphi)(-\varrho\sin\varphi) + \frac{\partial f}{\partial y}(\varrho\cos\varphi, \varrho\sin\varphi)\,\varrho\cos\varphi,$$

or with an understandable shorthand,

$$\frac{\partial F}{\partial \varrho} = \frac{\partial f}{\partial x}\cos\varphi + \frac{\partial f}{\partial y}\sin\varphi, \qquad \frac{\partial F}{\partial \varphi} = -\frac{\partial f}{\partial x}\,\varrho\sin\varphi + \frac{\partial f}{\partial y}\,\varrho\cos\varphi,$$

where we first differentiate f with respect to x and y, and then insert $x = \varrho\cos\varphi$, $y = \varrho\sin\varphi$ into the result.

This is an example of the general principle in practical applications. We shall usually not bother with whether we are considering $\mathbf{f}$ or $\mathbf{F}$, and we shall usually in the first calculation leave out the variables.

The method is that we *differentiate through all variables on the middle level and finally add all these results*,

$$\frac{\partial f_r}{\partial u_j} = \frac{\partial f_r}{\partial x_1}\,\frac{\partial x_1}{\partial u_j} + \cdots + \frac{\partial f_r}{\partial x_k}\,\frac{\partial x_k}{\partial u_j},$$

where the blue variables from the middle level are added,

Note that the symbol of a partial differential quotient is a *notation* and *not* a fraction, so one cannot just cancel all the blue contributions. An furthermore, in the final result, only the variables $\mathbf{u}$ (and not $\mathbf{x}$) should occur.

9.3 Directional derivative

We shall sometimes need the derivative of a function $f(\mathbf{x})$ in a specific direction from a given point $\mathbf{x}$. Let $\mathbf{e}$ be a unit vector starting at the point $\mathbf{x}$ and pointing in the direction, in which we want to find the derivative. Take the restriction of f to a line segment through $\mathbf{x}$ in the direction $\mathbf{e}$, i.e. we define

$$F(t) = f(\mathbf{x} + t\mathbf{e}), \qquad \text{for } t \in I,$$

where $I \subseteq \mathbb{R}$ is some open interval, for which $0 \in I$. Then $F(t)$, $t \in I$, is an ordinary function in t, while the right hand side $f(\mathbf{x} + t\mathbf{e})$ is a composite function,

$$F = f \circ \mathbf{X}, \qquad \text{where } \mathbf{X}(t) = \mathbf{x} + t\mathbf{e} = (x_1 + te_1, \dots, x_n + te_n)\,.$$

Then by the chain rule,

$$F'(t) = \frac{\partial f}{\partial x_1}\frac{\mathrm{d}x_1}{\mathrm{d}t} + \cdots + \frac{\partial f}{\partial x_n}\frac{\mathrm{d}x_n}{\mathrm{d}t} = \bigtriangledown f(\mathbf{x}) \cdot \mathbf{X}'(t) = \mathbf{e} \cdot \bigtriangledown f(\mathbf{x} + t\mathbf{e}).$$

Consider in particular $F'(0)$. The interpretation of $F'(0)$ is that it gives a measure of the variation of f, when one moves a small distance from $\mathbf{x}$ in the direction of $\mathbf{e}$. We call $F'(0)$ the *directional derivative* of f in the direction of $\mathbf{e}$, and we shall use the notation

$$f'(\mathbf{x}, \mathbf{e}) \; (= F'(0)) = \mathbf{e} \cdot \bigtriangledown f(\mathbf{x}).$$

Since $\mathbf{e}$ is a unit vector, we clearly have the inequalities

$$-\| \bigtriangledown f(\mathbf{x})\| \le f'(\mathbf{x}, \mathbf{e}) \le \| \bigtriangledown f(\mathbf{x})\|.$$

We obtain equality to the left, when $\mathbf{e}$ is pointing in the opposite direction of $\bigtriangledown f(\mathbf{x})$, and similarly equality to the right, when the unit vector $\mathbf{e}$ points in the same direction as $\bigtriangledown f(\mathbf{x}$. In particular, the gradient $\bigtriangledown f(\mathbf{x}$ is pointing in the direction from the point $\mathbf{x}$, in which the function $f(\mathbf{x})$ obtains its biggest increase.

If the unit vector $\mathbf{e}$ is chosen as one of the vectors of the orthonormal basis, $\mathbf{e}_j$, then the directional derivative is equal to the partial derivative with respect to x_j, i.e.

$$\frac{\partial f}{\partial x_j} = \mathbf{e}_j \cdot \bigtriangledown f(\mathbf{x}),$$

which we have also seen previously.

We mention in this connection a slightly different problem, the solution of which is derived from the above. Given two different points $\mathbf{x}_0$, $\mathbf{x}_1 \in A$. We shall find the directional derivative of $f(\mathbf{x})$ in the direction from $\mathbf{x}_0$ towards $\mathbf{x}_1$.

We shall only find the *unit vector*, which points from $\mathbf{x}_0$ towards $\mathbf{x}_1$. This is clearly

$$\mathbf{e} := \frac{\mathbf{x}_1 - \mathbf{x}_0}{\|\mathbf{x}_1 - \mathbf{x}_0\|},$$

so the directional derivative of f at $\mathbf{x}_0$ in the direction from $\mathbf{x}_0$ towards $\mathbf{x}_1$ is given by

$$\frac{\mathbf{x}_1 - \mathbf{x}_0}{|\mathbf{x}_1 - \mathbf{x}_0\|} \cdot \bigtriangledown f\,(\mathbf{x}_0)\,.$$

9.4 C^n-functions

We have previously introduced the partial derivatives of *first order*

$$\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_k}$$

of a function $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^k$, whenever they do exist. One may be tempted to check if these (at most) k partial derivatives of first order again are differentiable functions with respect to the variables $\mathbf{x} = (x_1, \dots, x_k)$. When this is the case, we call the results *partial derivatives of second order.*

Assume that $\dfrac{\partial f}{\partial x_i}$ has a partial derivative with respect to the variable x_j. We shall then use one of the following notations for this partial derivative of second order:

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}(\mathbf{x})\right), \qquad \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x}), \qquad f''_{x_i x_j}, \quad \text{or} \quad D_j D_i f(\mathbf{x}).$$

The symbol closest to the function f is always applied first. However, we mention that some authors prefer to write in the opposite order

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}),$$

so here the order of differentiation follows the way this symbol is read. In practice this will not cause any trouble, because we shall see in the following that under very mild assumptions, which are always met in the rest of this series of books, we have

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i},$$

no matter which interpretation we have chosen.

If $x_j = x_i$, we also write

$$\frac{\partial^2 f}{\partial x_i^2}(\mathbf{x}), \qquad \text{or} \qquad f''_{x_i^2}(\mathbf{x}). \qquad \text{or} \qquad D_i^2 f(\mathbf{x})$$

for the corresponding partial derivative of second order.

The extension from partial derivatives of order 1 to partial derivatives of order 2 is the biggest one. Once we have understood this step, it is obvious how to introduce partial derivatives of order n, whenever they exist. Also, the notation of the partial derivatives of order n,

$$\frac{\partial^n}{\partial x_{i_1} \cdots \partial x_{i_n}}(\mathbf{x}), \qquad f^{(n)}_{x_{i_n} \cdots x_{i_1}}(\mathbf{x}), \qquad \text{or} \qquad D_{i_1} \cdots D_{i_n} f(\mathbf{x}),$$

is easy to understand.

When the dimensions are $n = 2$ or 3, then we use the notation (x, y) or (x, y, z) for the variables, instead of (x_1, x_2) or (x_1, x_2, x_3).

Assume that all possible partial derivatives of f of order n exist in A and that they are all continuous in A. Then we say that f has *continuous partial derivatives of order* n in A, and we write

$$f \in C^n(A).$$

When this is true for all n, we write $f \in C^\infty(A)$. Then it is natural to add $f \in C^0(A)$ to mean that f is *continuous* in A.

Often the open domain A is tacitly understood, in which case we just write $f \in C^n$, or $f \in C^\infty$, or $f \in C^0$.

The importance of the class $C^n(A)$ follows from the following theorem,

Theorem 9.4 Interchange of the order of differentiation. *Assume that $A \subseteq \mathbb{R}^k k$ is open and that $f \in C^2(A)$. Then*

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x}) \qquad \text{for } \mathbf{x} \in A \text{ and } i, j \in \{1, \dots, k\}.$$

Theorem 9.4 is only formulated for $n = 2$, but if e.g. $f \in C^3(A)$, then every $\dfrac{\partial f}{\partial x_i} \in C^2(A)$, and we may apply Theorem 9.4 with f replaced by $\dfrac{\partial f}{\partial x_i}$, and then use induction to obtain the general result.

The proof of Theorem 9.4 is fairly long and tedious, for which reason it is not given here.

The formulation of Theorem 9.4 above gives us a hint of that there may exist examples of functions, for which the partial derivatives of e.g. second order exist, and yet there may exist points in which the order of differentiation is essential. This is indeed true! Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(x,y) = \sqrt{(x^2+2(x+y)^2)\,(y^2+(x-y)^2)} = \sqrt{3x^4-2x^3y+4xy^3+4y^4}.$$

If $(x,y) \neq (0,0)$, then

$$f_x'(x,y) = \frac{6x^3-3x^2y+2y^3}{\sqrt{3x^4-2x^3y+4xy^3+4y^4}}, \qquad f_y'(x,y) = \frac{-x^3+6xy^2+8y^3}{\sqrt{3x^4-2x^3y+4xy^3+4y^4}}.$$

It follows that even $f \in C^\infty\left(\mathbb{R}^2 \setminus \{(0,0)\}\right)$, so we shall only investigate the point $(0,0)$. We get in particular from the above,

$$f_x'(0,y) = y \quad \text{for } y \neq 0, \qquad \text{and} \qquad f_y'(x,0) = -\frac{x}{\sqrt{3}} \quad \text{for } x \neq 0,$$

Since we have the restriction $f(x,0) = \sqrt{3}\,x^2$ for $x \in \mathbb{R}$, we get $f_x'(x,0) = 2\sqrt{3}\,x$, hence

$$f_x'(0,0) = 0,$$

Since we have the restriction $f(0,y) = 2y^2$ for $y \in \mathbb{R}$, we get $f_y'(0,y) = 4y$, hence

$$f_y'(0,0) = 0.$$

Summing up, we have

$$f_x'(0,y) = y \quad \text{for } y \in \mathbb{R}, \qquad \text{and} \qquad f_y'(x,0) = -\frac{x}{\sqrt{3}} \quad \text{for } x \in \mathbb{R},$$

from which we get

$$f_{xy}''(0,y) = 1 \quad \text{for } y \in \mathbb{R} \qquad \text{and} \qquad f_{yx}''(x,0) = -\frac{1}{\sqrt{3}} \quad \text{for } x \in \mathbb{R}.$$

Then

$$f_{xy}''(0,0) = 1 \neq -\frac{1}{\sqrt{3}} = f_{yx}''(0,0),$$

proving that the order of differentiation cannot be interchanged at the point $(0,0)$, although the partial derivatives of second order clearly exist in all of $\mathbb{R}^2$.

We have above only considered the class $C^n(A)$, when A was an open set. In many applications we also need to talk of $C^n(\tilde{A})$, when $\tilde{A}$ is not open. We introduce the following

Definition 9.2 *Let $A \subseteq \mathbb{R}^k$ be a nonempty set, and let $f : A \to \mathbb{R}$ be a function. We say that $f \in C^n(A)$ is n times continuously differentiable in A, if there exists an extension $\tilde{f} : \tilde{A} \to \mathbb{R}$ of f to an open set $\tilde{A}$, such that $\tilde{f} \in C^n(\tilde{A})$ and $\tilde{f}(\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x} \in A$.*
If $\mathbf{f} : A \to \mathbb{R}^m$ is a vector function, we say that $\mathbf{f} \in C^n(A)$, if all its coordinate functions $f_i \in C^n(A)$.

9.5 Taylor's formula

9.5.1 Taylor's formula in one dimension

We shall often need a method to approximate a C^n-function in a neighbourhood of a point, using a polynomial as approximation. There are some possibilities, of which we here choose the most well-known, namely *Taylor's formula.* As usual we shall start with the 1-dimensional case and then derive the general results in n dimensions.

Usually one uses *Rolle's theorem* or the *mean value theorem* to prove Taylor's formula, but as we shall see below, this is not necessary, if we only assume that the function is n times continuously differentiable.

Let $I \subseteq \mathbb{R}$ be an open interval, which contains $0 \in I$, and let $F : I \to \mathbb{R}$ be a $C^n(I)$-function. This means that

$$F(t),\, F'(t),\, \dots ,\, F^{(n)}(t), \qquad t \in I,$$

all exist and are continuous. In particular, in a neighbourhood of $0 \in I$,

$$F^{(n)}(t) = F^{(n)}(0) + \varepsilon(t), \qquad \text{where } \varepsilon(t) \to 0 \text{ for } t \to 0,$$

a relation which we shall need below.

Another preparation for the proof is the following observation that by a partial integration for $h \in I$,

$$\begin{aligned}\int_0^h \frac{(h-t)^{k-1}}{(k-1)!}\, F^{(k)}(t)\; \mathrm{d}t &= \left[-\frac{(h-t)^k}{k!}\, F^{(k)}(t)\right]_0^h + \int_0^h \frac{(h-t)^k}{k!}\, F^{(k+1)}(t)\, \mathrm{d}t \\ &= \frac{h^k}{k!}\, F^{(k)}(0) + \int_0^h \frac{(h-t)^k}{k!}\, F^{(k+1)}(t)\, \mathrm{d}t \qquad \text{for } k = 1, \dots, n-1,\end{aligned}$$

because $F \in C^n(I)$.

Using this result repeatedly we obtain by induction

$$\begin{aligned}F(h) - F(0) &= \int_0^h 1 \cdot F'(t)\, \mathrm{d}t = \frac{h^1}{1!}\, F'(0) + \int_0^h \frac{(h-t)^1}{1!}\, F''(t)\, \mathrm{d}t \\ &= \frac{1}{1!}\, h^1 F'(0) + \frac{1}{2!}\, h^2\, F''(0) + \cdots + \frac{1}{(n-1)!},h^{n-1}\, F^{(n-1)}(0) + \int_0^h \frac{(h-t)^{n-1}}{(n-1)!}\, F^{(n)}(t)\, \mathrm{d}t.\end{aligned}$$

Since $F^{(n)}(t) = F^{(n)}(0) + \varepsilon(t)$, where $\varepsilon(t) \to 0$ for $t \to 0$ in I, we finally get for the remainder term,

$$\int_0^h \frac{(h-t)^{n-1}}{(n-1)!}\, F^{(n)}(t)\, \mathrm{d}t = \int_0^h \frac{(h-t)^{n-1}}{(n-1)!} \left\{F^{(n)}(0) + \varepsilon(t)\right\}\, \mathrm{d}t = \frac{h^n}{n!}\, F^{(0)} + h^n \cdot \tilde{\varepsilon}(h),$$

where we have used the estimates

$$\left|\int_0^h \frac{(h-t)^{n-1}}{(n-1)!}\, \varepsilon(t)\, \mathrm{d}t\right| \le \int_0^{|h|} |h|^{n-1} \max_{-|h| \le t \le |h|} |\varepsilon(t)|\, \mathrm{d}t = |h|^{n-1} \cdot |h| \cdot \varepsilon_1(h) = |h|^n \cdot \varepsilon_1(h),$$

where $\varepsilon_1(h) \to 0$ for $h \to 0$. It follows that

$$\int_0^h \frac{(h-t)^{n-1}}{(n-1)!}\, \varepsilon(t)\, \mathrm{d}t = h^n \cdot \tilde{\varepsilon}(h), \qquad \text{where } \varepsilon_1(h) \to 0 \text{ for } h \to 0.$$

Summing up se see after a rearrangement that we have proved

Taylor's formula in one dimension. *Let $I \subseteq \mathbb{R}$ be an open interval, where $0 \in I$, and let $F : I \to \mathbb{R}$ be a $C^n(I)$-function. Then*

$$F(h) = F(0) + \frac{F'(0)}{1!}\,h + \frac{F''(0)}{2!}\,h^2 + \cdots + \frac{F^{(n)}(0)}{n!}\,h^n + h^n\varepsilon(h), \qquad \textit{for } h \in I,$$

where $\varepsilon(h) \to 0$ for $h \to 0$.

Here,

$$P_n(h, F) = P_n(h) = F(0) + \frac{F'(0)}{1!}\,h + \frac{F''(0)}{2!}\,h^2 + \cdots + \frac{F^{(n)}(0)}{n!}\,h^n$$

is a polynomial of at most degree n in h. We call it the *approximating polynomial* of $F(h)$ of degree at most n, and n, which is determined by the corresponding remainder term above, is called the *order of expansion*.

Note that we may have come across an expansion of order n, where the approximating polynomial actually is of degree $< n$. The only requirement is that $F^{(n)}(0) = 0$, a possibility, which cannot be excluded.

If the *point of expansion* is not 0, but instead some $t_0 \in I$, then we just introduce the new variable $\tau = t - t_0$, and the point of expansion for τ becomes $\tau_0 = 0$, and we can use the above to get

$$F(t) = F\left(t_0\right) + \frac{F'\left(t_0\right)}{1!}\left(t - t_0\right) + \frac{F''\left(t_0\right)}{2!}\left(t - t_0\right)^2 + \cdots + \frac{F^{(n)}\left(t_0\right)}{n!}\left(t - t_0\right)^n + \left(t - t_0\right)^n \varepsilon\left(t - t_0\right),$$

where $F \in C^n(I)$, and where $\varepsilon\left(t - t_0\right) \to 0$ for $t \to t_0$.

Before we in detail discuss Taylor's formula in several variables we show, how we get from the one dimensional version above to Taylor's formula in n dimensions. The idea is simple. Let $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}^k$ is open, be a C^n-function. Let $\mathbf{x} \in A$, and let $\mathbf{x} + \mathbf{h} \in A$ be a neighbouring point, such that the (closed) line segment between $\mathbf{x}$ and $\mathbf{x} + \mathbf{h}$ lies in A. If $\mathbf{h} \neq \mathbf{0}$, put $\mathbf{h} = h\mathbf{e}$, where $h > 0$ and $\mathbf{e}$ is a unit vector. We define

$$(9.3) \qquad F(t) := f(\mathbf{x} + t\mathbf{e}), \qquad \text{for } t \in I,$$

where I is an open interval containing $[0, h]$. Then $F(0) = f(\mathbf{x})$ and $F(h) = f(\mathbf{x} + h\mathbf{e}) = f(\mathbf{x} + \mathbf{e})$, and $F(t)$ is a $C^n(I)$-function, so we can apply Taylor's formula in one dimension om $F(t)$, proved above. We shall of course in the differentiation of (9.3) above use the chain rule on the right hand side. We shall first consider the simple case, when $n = 1$.

9.5.2 Taylor expansion of order 1

We apply the third special case of the chain rule to get

$$F'(t) = \frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{x} + t\mathbf{e}) = \frac{\partial f}{\partial x_1}(\mathbf{x} + t\mathbf{e}) e_1 + \cdots + \frac{\partial f}{\partial x_k}(\mathbf{x} + t\mathbf{e}) = \mathbf{e} \cdot \bigtriangledown f(\mathbf{x} + t\mathbf{e}) \qquad \text{for } t \in I.$$

Put $n = 1$ and $t = h$ into Taylor's formula to get

$$F(h) = f(\mathbf{x} + \mathbf{h}) = f(\mathbf{x}) + h\mathbf{e} \cdot \bigtriangledown f(\mathbf{x} + t\mathbf{e}) = f(\mathbf{x}) + \mathbf{h} \cdot \bigtriangledown f(\mathbf{x}) + \varepsilon(\mathbf{h})\|\mathbf{h}\|,$$

where the ε-function depends on both the length $h > 0$ and the direction $\mathbf{e}$, so if $f \in C^1(A)$ and $\mathbf{x} \in A$, then

$$f(\mathbf{x} + \mathbf{h}) = f(\mathbf{x}) + \mathbf{h} \cdot \nabla f(\mathbf{x}) + \varepsilon(\mathbf{h})\|\mathbf{h}\|.$$

This was fairly easy and only illustrates that $f \in C^1(I)$ is continuously differentiable.

One would of course expect that the situation becomes more complicated for $n > 1$, and so it is! In order to get the general idea we shall therefore in the next section confine ourselves to the case where $n = 2$ and just $k = 2$, and only briefly at the end of the section mention the result, when $k = 3$.

9.5.3 Taylor expansion of order 2 in the plane

We shall proceed with the Taylor expansion of second order, $n = 2$, in several variables. We shall start with he simplest case, where we have only two variable (x, y). Let $A \subseteq \mathbb{R}^2$ be an open and non-empty set in the plane, and let $f \in C^2(A)$ be a twice continuously differentiable function of A. We shall find for a given point $(x, y) \in A$ and a small increment (h_x, h_y) an expression of $f(x + h_x, y + h_y)$ in a neighbouring point $(x + h_x, y + h_y)$, where we use f and its first and second partial derivatives at the given point (x, y).

The set A is open, at $(x, y) \in A$, so there exists an $r > 0$, such that the open disc $B((x, y), r) \subseteq A$. If therefore the increment (h_x, h_y) is small $\left(h_x^2 + h_y^2 < r^2 \text{ will be sufficient}\right)$, then the closed line segment between (x, y) and $(x + h_x, y + h_y)$ is totally contained in A, so we can take the restriction of f to this line segment and apply the previously developed theory.

Let $h := \sqrt{h_x^2 + h_y^2}$. Then there are constants $\alpha\,(=\cos\varphi)$ and $\beta\,(=\sin\varphi)$, such that

$$h_x = \alpha\, h \qquad \text{and} \qquad h_y = \beta\, h,$$

and the restriction of f to the line segment is written

$$F(t) = f(x+\alpha\, t, y+\beta\, t), \qquad \text{for } t \in I \supset [0,h],$$

where

$$F(h) = f\left(x+h_x, y+h_y\right) \qquad \text{and} \qquad F(0) = f(x,y).$$

We get by the chain rule,

$$F'(t) = \alpha\, f_x'(x+\alpha\, t, y+\beta\, t) + \beta\, f_y'(x+\alpha\, t, y+\beta\, t).$$

The assumption that $f \in C^2(A)$ secures that the right side of this equation is again continuously differentiable. Hence, once more by the chain rule (third special case)

$$\begin{aligned} F''(t) \;\; &= \;\; \alpha\left\{\alpha\, f_{xx}''(x+\alpha\, t, y+\beta\, t) + \beta\, f_{xy}''(x+\alpha\, t, y+\beta\, t)\right\} \\ &\quad\;\; +\beta\left\{\alpha\, f_{yx}''(x+\alpha\, t y + beta\, t) + \beta\, f_{yy}''(x+\alpha\, t, y+\beta\, t)\right\}. \end{aligned}$$

Since $f \in C^2(A)$, the order of differentiation can be interchanged, so $f_{yx}'' = f_{xy}''$, and we reduce the expression above to

$$F''(t) = \alpha^2 f_{xx}''(x+\alpha\, t, y+\beta\, t) + 2\alpha\beta f_{xy}''(x+\alpha\, t, y+\beta\, t) + \beta^2 f_{yy}''(x+\alpha\, t, y+\beta\, t).$$

Summing up we have

$$F(0) = f(x,y), \quad F'(0) = \alpha f_x'(x,y) + \beta f_y'(x,y), \quad F''(0) = \alpha^2 f_{xx}''(x,y) + 2\alpha\beta f_{xy}''(x,y) + \beta^2 f_{yy}''(x,y),$$

hence by insertion,

$$\begin{aligned} f(x+\alpha\, h, y+\beta h) &= F(h) = F(0) + F'(0)h + \frac{1}{2}\, F''(0)h^2 + \varepsilon(h)h^2 \\ &= f(x,y) + \alpha\, h f_x'(x,y) + \beta\, h f_y'(x,y) \\ &\quad + \frac{1}{2!}\left\{(\alpha h)^2 f_{xx}''(x,y) + 2(\alpha h)(\beta h) f_{xy}''(x,y) + (\beta h)^2 f_{yy}''(x,y)\right\} + \varepsilon(\alpha\, h, \beta\, h)h^2 \\ &= f(x,y) + h_x f_x'(x,y) + h_y f_y'(x,y) + \frac{1}{2!}\left\{h_x^2 f_{xx}''(x,y) + 2h_x h_y f_{xy}''(x,y) + h_y^2 f_{yy}''(x,y)\right\} \\ &\quad + \varepsilon\left(h_x, h_y\right)\left(h_x^2 + h_y^2\right), \end{aligned}$$

where $\varepsilon\left(h_x, h_y\right) \to 0$ for $\left(h_x, h_y\right) \to (0,0)$.

Summing up, we have proved

Taylor's formula for $n = 2$ and $k = 2$. *Assume that $A \subseteq \mathbb{R}^2$ is open and non-empty, and that $f \in C^2(A)$. If $(x,y) \in A$, then*

$$\begin{aligned} f\left(x+h_x, y+h_y\right) &= f(x,y) + h_x f_x'(x,y) + h_y f_y'(x,y) \\ &\quad + \frac{1}{2!}\left\{h_x^2 f_{xx}''(x,y) + 2h_x h_y f_{xy}''(x,y) + h_y^2 f_{yy}''(x,y)\right\} \\ &\quad + \varepsilon\left(h_x, h_y\right) \cdot \left(h_x^2 + h_y^2\right), \end{aligned}$$

where $\varepsilon\left(h_x, h_y\right) \to 0$ for $\left(h_x, h_y\right) \to (0,0)$, and where A contains the closed line segment between $(x,y) \in A$ and $\left(x+h_x, y+h_y\right) \in A$.

We note that the *differential*

$$\mathrm{d}f(\mathbf{x},\mathbf{h}) = h_x f_x'(x,y) + h_y f_y'(x,y) = \left(h_x, h_y\right) \cdot \left(f_x'(x,y), f_y'(x,y)\right) = \mathbf{h} \cdot \bigtriangledown f(x,y)$$

enters the expression above. It is therefore tempting to introduce the *second differential* $d^2 f$ of the function f by collecting all terms, which contain two partial derivatives of f,

$$f^2(\mathbf{x},\mathbf{h}) := h_x^2 f_{xx}''(x,y) + 2h_x h_y f_{xy}''(x,y) + h_y^2 f_{yy}''(x,y).$$

We shall see below, that this is really a convenient definition.

If we here put

$$f_1(\mathbf{x}) = \mathbf{h} \cdot \bigtriangledown f(\mathbf{x}) = \mathrm{d}f(\mathbf{x},\mathbf{h}),$$

then

$$\mathrm{d}^2 f(\mathbf{x},\mathbf{h}) = \mathbf{h} \cdot \bigtriangledown f_1(\mathbf{x}) = (\mathbf{h} \cdot \bigtriangledown)(\mathbf{h} \cdot \bigtriangledown) f(\mathbf{x}), \qquad \text{for } f \in C^2(A).$$

In general, we define by induction the p-th differential of a function $f \in C^n(A)$ by

$$\mathrm{d}^p f(\mathbf{x},\mathbf{h}) = \mathbf{h} \cdot \bigtriangledown_{p-1}(\mathbf{x},\mathbf{h}) = \cdots = (\mathbf{h} \cdot \bigtriangledown)^p f(\mathbf{x}), \qquad \text{for } p = 1, \dots, n,$$

where the differential operator $\mathbf{h} \cdot \bigtriangledown$ operates $p\,(\leq n)$ times.

Once we have seen this structure, we can immediately extend this construction to $A \subseteq \mathbb{R}^k$, where $k \geq 2$, and

$$\mathbf{h} \cdot \bigtriangledown := h_1 \frac{\partial}{\partial x_1} + \cdots + h_k \frac{\partial}{\partial x_k},$$

so we obtain in general,

Taylor's formula for $f \in C^n(A)$, $A \subseteq \mathbb{R}^k$ open. *If $\mathbf{x}$, $\mathbf{x}+\mathbf{h} \in A$ are chosen, such that the closed line segment between $\mathbf{x}$ and $\mathbf{x}+\mathbf{h}$ is totally contained in A, then*

$$f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) + df(\mathbf{x},\mathbf{h}) + \frac{1}{2}\, d^2 f(\mathbf{x},\mathbf{h}) + \cdots + \frac{1}{n!}\, d^n f(\mathbf{x},\mathbf{h}) + \varepsilon(\mathbf{h})\|\mathbf{h}\|^n,$$

where $\varepsilon(\mathbf{h}) \to 0$ for $\mathbf{h} \to \mathbf{0}$.

In general, $\mathrm{d}^p f(\mathbf{x},\mathbf{h})$ is computed in the following way,

$$\mathrm{d}^p f(\mathbf{x},\mathbf{h}) = (\mathbf{h}\cdot\bigtriangledown)^p f(\mathbf{x},\mathbf{h}),$$

where

$$(\mathbf{h}\cdot\bigtriangledown)^p = \left(h_1\,\frac{\partial}{\partial x_1}+\cdots+h_k\,\frac{\partial}{\partial x_k}\right)^p$$

is calculated as an ordinary polynomial in the operators $\dfrac{\partial}{\partial x_j}$ of constant coefficients h_j.

We mention in particular for $k=3$,

$$\mathrm{d}f(\mathbf{x},\mathbf{h}) = h_x f_x'(x,y,x) + h_y f_y'(x,y,z) + h_z f_z'(x,y,z)$$

and

$$\begin{aligned}\mathrm{d}^2 f(\mathbf{x},\mathbf{h}) &= h_x^2 f_{xx}''(x,y,z) + h_y^2 f_{yy}''(x,y,z) + h_z^2 f_{zz}''(x,y,z)\\ &\quad +2h_xh_y f_{xy}''(x,y,z) + 2h_yh_z f''yz(x,y,z) + 2h_zh_x f_{zx}''(x,y,z)\end{aligned}$$

for $f\in C^2(A)$ and $A\subseteq\mathbb{R}^3$ an open set. Hence, in three variables,

$$\begin{aligned}&f\left(x+h_x,y+h_y,z+h_z\right)\\ &\quad = f(x,y,z)+\frac{1}{1!}\left\{h_x f_x'(x,y,z)+h_y f_y'(x,y,z)+h_z f_z'(x,y,z)\right\}\\ &\qquad +\frac{1}{2!}\left\{h_x^2 f_{xx}''(x,y,z)+h_y^2 f_{yy}''(x,y,z)+h_z^2 f_{zz}''(x,y,z)\right\}\\ &\qquad +\left\{h_xh_y f_{xy}''(x,y,z)+h_yh_z f_{yz}''(x,y,z)+h_zh_x f_{zx}''(x,y,z)\right\}\\ &\qquad +\varepsilon\left(h_x,h_y,h_z\right)\left(h_x^2+h_y^2+h_z^2\right).\end{aligned}$$

In the applications in e.g. Physics, one rarely goes beyond the order $n=2$ of the expansion. Also, the dimensions are usually $k=2$ or $k=3$, so we have above covered the most important cases for the applications. And yet we have still the possibility of extending Taylor's formula to $k>3$ and $n>2$, which is of importance in the next section.

9.5.4 The approximating polynomial

Assume that $f\in C^n(A)$. Then by Taylor's formula,

$$f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x})+\frac{1}{1!}(\mathbf{h}\cdot\bigtriangledown)f(\mathbf{x},\mathbf{h})+\cdots+\frac{1}{n!}(\mathbf{h}\cdot\bigtriangledown)^n f(\mathbf{x},\mathbf{h})+\varepsilon(\mathbf{h})\|\mathbf{h}\|^n.$$

If we remove the remainder term $\varepsilon(\mathbf{h})\|\mathbf{h}\|^n$, we get a polynomial in $\mathbf{h}$ of at most degree n. We call it the *approximating polynomial of at most degree n in the variable* $\mathbf{h}$, i.e.

$$P_n(\mathbf{x},\mathbf{h}) = f(\mathbf{x})+\frac{1}{1!}(\mathbf{h}\cdot\bigtriangledown)f(\mathbf{x},\mathbf{h})+\cdots+\frac{1}{n!}(\mathbf{h}\cdot\bigtriangledown)^n f(\mathbf{x},\mathbf{h})$$

where $\mathbf{x}$ is the *expansion point*. It is an approximation of $f(\mathbf{x})$ in the neighbourhood of $\mathbf{x}$, because

$$|f(\mathbf{x}+\mathbf{h}) - P_n(\mathbf{x}), \mathbf{h}| = \varepsilon(\mathbf{h})\|\mathbf{h}\|^n, \qquad \text{where } \varepsilon(\mathbf{h}) \to 0 \text{ for } \mathbf{h} \to \mathbf{0},$$

i.e. the error is of the size $\varepsilon(\mathbf{h})\|\mathbf{h}\|^n$. One may write this

$$f(\mathbf{x}+\mathbf{h}) \simeq P_n(\mathbf{x}, \mathbf{h}).$$

In practice we denote the expansion point by $\mathbf{x}_0$, and then write $\mathbf{x} = \mathbf{x}_0 + \mathbf{h}$, so the increment is $\mathbf{h} = \mathbf{x} - \mathbf{x}_0$.

We then write

$$f(\mathbf{x}) \simeq P_n\left(\mathbf{x}_0, \mathbf{x} - \mathbf{x}_0\right) \qquad \text{in a neighbourhood of } \mathbf{x}_0.$$

Of particular importance are the cases, where $k = 2$ and $k = 3$, and the order of expansion is 2, because this is the most commonly used approximations in Physics. We therefore explicitly give the approximating polynomials of order 2 below.

Assume that $A \subseteq \mathbb{R}^2$. Then we write the variable in the form $\mathbf{x} = (x, y)$. Let (x_0, y_0) denote the expansion point. Then the approximal polynomial of at most degree 2 of $f \in C^2(A)$ is given by

$$\begin{aligned}
P_2\left(\mathbf{x}_0, \mathbf{x} - \mathbf{x}_0\right) &= P_2\left(\left(x_0, y_0\right), \left(x - x_0, y - y_0\right)\right) \\
&= f\left(x_0, y_0\right) + \left(x - x_0\right) f_x'\left(x_0, y_0\right) + \left(y - y_0\right) f_y'\left(x_0, y_0\right) \\
&\quad + \frac{1}{2}\left(x - x_0\right)^2 f_{xx}''\left(x_0, y_0\right) + \left(x - x_0\right)\left(y - y_0\right) f_{xy}''\left(x_0, y_0\right) + \frac{1}{2}\left(y - y_0\right)^2 f_{yy}''\left(x_0, y_0\right).
\end{aligned}$$

We must for numerical reasons keep $h = x - x_0$ and $k = y - y_0$ as the natural variables and not "reduce" the polynomial, using x and y as the variables.

Similarly in three dimensions, if $f \in C^2(A)$, where $A \subseteq \mathbb{R}^3$. In this case,

$$\begin{aligned}
P_2\left(\mathbf{x}_0, \mathbf{x} - \mathbf{x}_0\right) &= P_2\left(\left(x_0, y_0, z_0\right), \left(x - x_0, y - y_0, z - z_0\right)\right) \\
&= f\left(x_0, y_0, z_0\right) + \left(x - x_0\right) f_x'\left(x_0, y_0, z_0\right) + \left(y - y_0\right) f_y'\left(x_0, y_0, z_0\right) + \left(z - z_0\right) f_z'\left(x_0, y_0, z_0\right) \\
&\quad + \frac{1}{2}\left(x - x_0\right)^2 f_{xx}''\left(x_0, y_0, z_0\right) + \frac{1}{2}\left(y - y_0\right)^2 f_{yy}''\left(x_0, y_0, z_0\right) + \frac{1}{2}\left(z - z_0\right)^2 f_{zz}''\left(x_0, y_0, z_0\right) \\
&\quad + \left(x - x_0\right)\left(y - y_0\right) f_{xy}''\left(x_0, y_0, z_0\right) + \left(y - y_0\right)\left(z - z_0\right) f_{yz}''\left(x_0, y_0, z_0\right) \\
&\qquad + \left(z - z_0\right)\left(x - x_0\right) f_{zx}''\left(x_0, y_0, z_0\right),
\end{aligned}$$

where we keep $(x - x_0, y - y_0, z - z_0)$ as our variables.

In practice the notation $P_2\left(\left(x_0, y_0\right), \left(x - x_0, y - y_0\right)\right)$ and $P_2\left(\left(x_0, y_0, z_0\right), \left(x - x_0, y - y_0, z - z_0\right)\right)$ are too clumsy, so we just write $P_2(x, y)$ and $P_2(x, y, z)$ instead, where we tacitly assume the point of expansion, (x_0, y_0), resp. (x_0, y_0, z_0).

We shall below illustrate the principle in a concrete example, in which we also demonstrate an alternative, using results from Chapter 12. This alternative is sometimes more easy to apply.

Consider the function

$$f(x, y) = \exp\left(x^2 - y^2\right) \qquad \text{for } (x, y) \in \mathbb{R}^2,$$

where we choose the expansion point $(1, -1)$.

First method. We compute the first and second partial derivatives of $f(x,y)$ and then compute their values at the expansion point $(1,-1)$, where

$$
\begin{aligned}
&f(x,y) = \exp\left(x^2-y^2\right), && f(1,-1)=1,\\
&f'_x(x,y) = 2x\exp\left(x^2-y^2\right), && f'_x(1,-1)=2,\\
&f'_y(x,y) = -2y\exp\left(x^2-y^2\right), && f'_y(1,-1)=2,\\
&f''_{xx}(x,y) = \left(2+4x^2\right)\exp\left(x^2-y^2\right), && f''_{xx}(1,-1)=6,\\
&f''_{xy}(x,y) = -4xy\exp\left(x^2-y^2\right), && f''_{xy}(1,-1)=4,\\
&f''_{yy}(x,y) = \left(-2+4y^2\right)\exp\left(x^2-y^2\right), && f''_{yy}(1,-1)=2,
\end{aligned}
$$

so the approximating polynomial in $(x-1,y+1)$ of at most degree 2 is

$$
\begin{aligned}
P_2(x,y) &= f(1,-1)+f'_x(1,-1)(x-1)+f'_y(1,-1)(y+1)\\
&\quad +\frac{1}{2}f''_{xx}(1,-1)(x-1)^2+f''_{xy}(1,-1)(x-1)(y+1)+\frac{1}{2}f''_{yy}(1,-1)(y+1)^2\\
&= 1+2(x-1)+2(y-1)+3(x-1)^2+4(x-1)(y+1)+(y+1)^2,
\end{aligned}
$$

where the polynomial should not be reduced further.

Second method. When $(1,-1)$ is the expansion point, we introduce $x=1+h$ and $y=-1+k$, or $h=x-1$ and $k=y+1$, where (h,k) are the new variables, which should be kept small in the approximations.

Then,

$$x^2-y^2=(1+h)^2-(-1+k)^2=2h+2k+h^2-k^2,$$

which for small (h,k) behaves like $\sim 2h+2k$ of first degree, while the remainder terms

$$h^2-k^2=\varepsilon(h,k)\sqrt{h^2+k^2}.$$

We know already, cf. Chapter 12, that

$$e^t=1+t+\frac{1}{2}\,t^2+\cdots,$$

where the dots indicate terms of degree >2, i.e. of the type $\varepsilon(t)t^2$.

If we put $t=2h+2k+h^2-k^2$, then clearly $t^3=\varepsilon(h,k)\left(h^2+k^2\right)$, so by an expansion of order 2,

$$
\begin{aligned}
\exp\left(x^2-y^2\right) &= 1+\left(2h+2k+h^2-k^2\right)+\frac{1}{2}\left(2h+2k+h^2-k^2\right)^2+\cdots\\
&= 1+2h+2k+3h^2+4hk+k^2+\cdots,
\end{aligned}
$$

where the dots indicate terms of degree >2.

Finally, $h = x-1$ and $k = y+1$, so

$$P_2(x,y) = 1+2(x-1)+2(y+1)+3(x-1)^2+4(x-1)(y+1)+(y+1)^2.$$

If we want to find an approximation of $f(0.95,-1.02)\,[=0.87118\ldots]$, and no computer or pocket calculator is at hand, then we use the approximate polynomial $P_2(x,y)$ with $x-1=-0.05$ and $y+1=-0.02$, and we get by insertion,

$$P_2(0.95,-1.02) = 1+2(-0.5)+2(-0.02)+3(-0.05)^2+4(-0.05)(-0.02)+(-0.02)^2 = 0.8719,$$

which is a fairly good approximation of $f(0.92,-1.02)$.

Then we consider the following case in $\mathbb{R}^3$,

$$f(x,y,z) = y\ln x + z^2 e^y \qquad \text{for } x>0,$$

where the expansion point is chosen as $(1,0,1)$.

First method. We compute

$$\begin{array}{ll}
f(x,yt,z) = y\ln x + z^2e^y, & f(1,0,1)=1, \\[2mm]
f_x'(x,y,z) = \dfrac{y}{x}, & f_x'(1,0,1)=0, \\[2mm]
f_y'(x,y,z) = \ln x + z^2e^y, & f_y'(1,0,1)=1, \\[2mm]
f_z'(x,y,z) = 2ze^y, & f_z'(1,0,1)=2, \\[2mm]
f_{xx}''(x,y,z) = -\dfrac{y}{x^2}, & f_{xx}''(1,0,1)=0, \\[2mm]
f_{yy}''(x,y,z) = z^2e^y, & f_{yy}''(1,0,1)=1, \\[2mm]
f_{zz}''(x,y,z) = 2e^y, & f_{zz}''=2, \\[2mm]
f_{xy}''(x,y,z) = \dfrac{1}{x}, & f_{xy}''(1,0,1)=1, \\[2mm]
f_{yz}''(x,y,z) = 2ze^y, & f_{yz}''(1,0,1)=2, \\[2mm]
f_{zx}''(x,y,z) = 0, & f_{zx}''(1,0,1)=0,
\end{array}$$

so the approximate polynomial $P_2(x,y,z)$ of at most degree 2 is

$$P_2(x,y,z) = 1+y+2(z-1)+\frac{1}{2}\,y^2+(z-1)^2+(x-1)y+2y(z-1).$$

Second method. Since the expansion point is $(1,0,1)$, we put $x=1+h$, $y=k$ and $z=1+p$. Then

$$\begin{aligned}
f(x,y,z) &= k\ln(1+h)+(1+p)^2e^k = k\{h+\cdots\}+\left(1+2p+p^2\right)\left(1+k+\frac{1}{2}\,k^2+\cdots\right) \\
&= hk+1+k+\frac{1}{2}\,k^2+2p+2kp+p^2+\cdots\,,
\end{aligned}$$

where the dots indicate terms of degree > 2. Hence,

$$\begin{aligned} P_2(x,y,z) &= 1+k+2p+hk+2kp+\frac{1}{2}\,k^2+p^2 \\ &= 1+y+2(z-1)+(x-1)y+2y(z-1)+\frac{1}{2}\,y^2+(z-1)^2. \end{aligned}$$

At last we show that even if $f(x,y)$ is a polynomial, the approximating polynomial $P_2(x,y)$ is not necessarily the same polynomial. If we choose

$$f(x,y)=x^2y \qquad \text{for } (x,y)\in\mathbb{R}^2,$$

then $f(x,y)$ is a monomial of degree $2+1=3$, so one would expect that $P_2(x,y)$ would be different from $f(x,y)$, no matter the point of expansion. This is obvious for the point of expansion $(0,0)$, because the approximating polynomial of at most degree 2 in this case is 0. Then let us consider the expansion point $(1,2)$. Then $x=1+h$ and $y=2+k$, hence by insertion,

$$f(x,y)=x^2y=(1+h)^2y=\left(1+2h+h^2\right)(2+k)=2+4h+k+2h^2+2hk+h^2k.$$

The approximation of second order is then obtained by removing all terms of degree > 2, which in the present case is h^2k, so when the expansion point is $(1,2)$, we get

$$P_2(x,y)=2+4h+k+2h^2+2hk=2+4(x-1)+(y-2)+2(x-1)^2+2(x-1)(y-2)\neq x^2y,$$

because we are missing the term $h^2k=(x-1)^2(y-2)$.

10 Some useful procedures

10.1 Introduction

We mention two procedures, which are relevant for this book, namely *The chain rule* and *The directional derivative*. For some reason these simple procedures are felt very difficult, the first time one meets them, so the reader should be careful here.

10.2 The chain rule

Problem 10.1 *Let* $\mathbf{f} = (f_1, \dots, f_m)$ *be a differentiable vector function in the* k *real variables* $x_1, \dots,$ x_k*, and assume that these again are differentiable functions in the* n *variables* $u_1, \dots, u_n$*.*
Find the (partial) derivatives of $(f_1, \dots, f_m)$ *after* $u_1, \dots, u_n$*.*

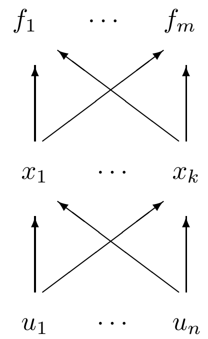

Figure 10.1: The general diagram of the chain rule.

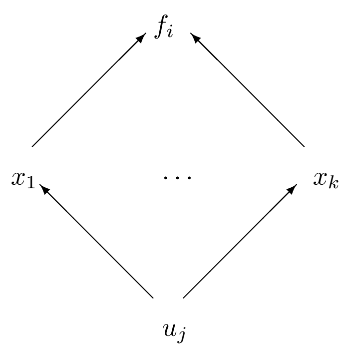

Figure 10.2: The reduced diagram of the chain rule.

Procedure.

1) Sketch the general diagram as in Figure 10.1, and reduce the i-th $\mathbf{f}$-coordinate and the j-th $\mathbf{u}$-coordinate as shown on Figure 10.2.

2) "Pull the differentiation apart" in the following way with k specimens (i.e. the number of $\mathbf{x}$-coordinates) on the right hand side

$$\frac{\partial f_i}{\partial u_j} = \frac{\partial f_i}{\partial} \frac{\partial}{\partial u_j} + \cdots + \frac{\partial f_i}{\partial} \frac{\partial}{\partial u_j}.$$

3) The empty places are then filled in with all the variables $x_1, \dots, x_k$ from the layer in the middle,

$$\frac{\partial f_i}{\partial u_j} = \frac{\partial f_i}{\partial x_1} \frac{\partial x_1}{\partial u_j} + \cdots + \frac{\partial f_i}{\partial x_k} \frac{\partial x_k}{\partial u_j}.$$

4) Repeat this process for every relevant i and j.

Remark 10.1 If one of the layers of differentiations only contains one variable, then ∂ is replaced by d, i.e. one writes $\dfrac{d\cdots}{d\cdots}$ instead of $\dfrac{\partial\cdots}{\partial\cdots}$. $\Diamond$

10.3 Calculation of the directional derivative

Geometric interpretation: Assume that $f(\mathbf{x})$ is a differentiable function. Then the *directional derivative*

$$f'(\mathbf{x};\mathbf{e})$$

of f at the point $\mathbf{x}$ and in the direction $\mathbf{e}$ indicates how much $f(\mathbf{x})$ increases (decreases) per unit in the direction $\mathbf{e}$. By a *direction* we shall always understand a *unit vector* $\mathbf{e}$, i.e. $\|\mathbf{e}\| = 1$. In this case we have

$$f'(\mathbf{x};\mathbf{e}) = \mathbf{e} \cdot \bigtriangledown f(\mathbf{x}).$$

Typical problems are:

Problem 10.2 *Let a* unit vector $\mathbf{e}$ *be given.*
Find the directional derivative $f'(\mathbf{x};\mathbf{e})$ *of* f *in the direction* $\mathbf{e}$.

Procedure.

1) Calculate the gradient $\bigtriangledown f(\mathbf{x})$.

2) Calculate the inner product $f'(\mathbf{x};\mathbf{e}) = \mathbf{e} \cdot \bigtriangledown f(\mathbf{x})$.

Problem 10.3 *Given two points* $\mathbf{x}_1$ *and* $\mathbf{x}_2$.
Find the directional derivative of $f(\mathbf{x})$ *at* $\mathbf{x}_1$ *in the direction of* $\mathbf{x}_2$.

Procedure.

1) Calculate the gradient $\nabla f(\mathbf{x})$ i $\mathbf{x}_1$.

2) Find the directional vector $\mathbf{e}$ from $\mathbf{x}_1$ to $\mathbf{x}_2$:

$$\mathbf{e} = \frac{\mathbf{x}_2 - \mathbf{x}_1}{\|\mathbf{x}_2 - \mathbf{x}_1\|},$$

(Do not forget to find the norm of the vector).

3) The directional derivative is given by

$$f'(\mathbf{x}_1;\mathbf{e}) = \mathbf{e} \cdot \nabla f(\mathbf{x}_1) = \frac{1}{\|\mathbf{x}_2 - \mathbf{x}_1\|}\,(\mathbf{x}_2 - \mathbf{x}_1) \cdot \nabla f(\mathbf{x}_1).$$

Remark 10.2 The definition is extremely simple. Nevertheless it is causing students a lot of trouble for some reason which is not understood by me. I have taken the consequence to include this section here. Note that when we norm $\nabla f(\mathbf{x})$, we obtain the direction in which the function $f(\mathbf{x})$ has its highest increase. $\Diamond$

10.4 Approximating polynomials

The most common case is given by the following problem. Other cases are obtained by suitable modifications of it.

Problem 10.4 *Find the approximating polynomial of at most* second degree *for a function* $f(x,y)$ *in two variables from the point* (x_0,y_0) *in its open domain.*

There are here several possible methods, which all have there advantages and disadvantages, so one cannot say that one particular method is always the easiest one to use. However, when the student encounters this problem for the first time, her or his preference will probably without doubt be the following

A. Standard procedure

1) Start by explaining *(text)*, why f is a C^2-function in the neighbourhood of (x_0,y_0).

2) Calculate the following equations:

$$\begin{array}{ll}
\text{order zero:} & \left\{ \; f(x,y) = \cdots, \qquad f(x_0,y_0) = \cdots, \right. \\[2mm]
\text{first order:} & \left\{ \begin{array}{ll} f'_x(x,y) = \cdots, & f'_x(x_0,y_0) = \cdots, \\[2mm] f'_y(x,y) = \cdots, & f'_y(x_0,y_0) = \cdots, \end{array} \right. \\[2mm]
\text{second order:} & \left\{ \begin{array}{ll} f''_{xx}(x,y) = \cdots, & f''_{xx}(x_0,y_0) = \cdots, \\[2mm] f''_{xy}(x,y) = \cdots, & f''_{xy}(x_0,y_0) = \cdots, \\[2mm] f''_{yy}(x,y) = \cdots, & f''_{yy}(x_0,y_0) = \cdots. \end{array} \right.
\end{array}$$

3) Insert the values of the column to the right into the formula

$$\begin{aligned}
P_2(x,y) \;=\; & f(x_0,y_0) + \frac{1}{1!}\left\{ f'_x(x_0,y_0)\cdot(x-x_0) + f'_y(x_0,y_0)\cdot(y-y_0) \right\} \\
& + \frac{1}{2!}\left\{ f''_{xx}(x_0,y_0)\cdot(x-x_0)^2 + 2\, f''_{xy}(x_0,y_0)\cdot(x-x_0)(y-y_0) + f''_{yy}(x_0,y_0)\cdot(y-y_0)^2 \right\}.
\end{aligned}$$

Remark 10.3 Since the approximating polynomial is the best description of $f(x,y)$ *in a neighbourhood of* (x_0,y_0), the right variables here are always $x-x_0$ and $y-y_0$, and *not* x and y. Therefore, one shall *not* reduce the expression further to a polynomial in x and y alone, because we by doing this will obtain an expression with a higher numerical uncertainty! $\Diamond$

B. Taylor expansions

1) Explain *(text)* that $f(x,y)$ is composed of *standard functions*, for which a Taylor expansion already is known from e.g. Chapter 12.

2) Reset the problem to zero, i.e. change the variables to

$$(h,k) = (x-x_0, y-y_0), \qquad (x,y) = (x_0+h, y_0+k).$$

Then we get $f(x,y) = F(h,k)$ in the new variables (h,k).

3) Apply convenient *standard expansions* in $F(h,k)$. We list all the usual standard expansions for t small, which should be known.

$$\begin{array}{ll} \dfrac{1}{1-t} = 1+t+t^2+\cdots\,, & \dfrac{1}{1+t} = 1-t+t^2-\cdots\,, \\ (1+t)^\alpha = 1+\alpha t+\dfrac{\alpha(\alpha-1)}{2}\,t^2+\cdots\,, & \ln(1+t) = t-\dfrac{1}{2}\,t^2+\cdots\,, \\ \text{Arctan}\, t = t-\cdots\,, & e^t = 1+t+\dfrac{1}{2}\,t^2+\cdots\,, \\ \sin t = t-\cdots\,, & \cos t = 1-\dfrac{1}{2}\,t^2+\cdots\,, \\ \sinh t = t+\cdots\,, & \cosh t = 1+\dfrac{1}{2}\,t^2+\cdots\,, \end{array}$$

where the dots indicate terms of the type $t^2\varepsilon(t)$.

4) Calculate $F(h,k)$, where every term which contains *at least* three factors of the type h, k, is symbolized by $\cdots$ (of the type $t^2\varepsilon(t)$).

5) One obtains the approximating polynomial by deleting the dots and then change variables back to

$$(h,k) = (x-x_0, y-y_0).$$

Remark 10.4 One should always be very careful to rewrite to one of the ten standard functions above. We have for example (for $\alpha = 1/2$)

$$\begin{aligned} \sqrt{25+t} &= 5\left(1+\frac{t}{25}\right)^{\frac{1}{2}} = 5\left\{1+\frac{1}{2}\,\frac{t}{25}-\frac{1}{8}\,\frac{t^2}{625}+\cdots\right\} \\ &= 5+\frac{1}{10}\,t-\frac{1}{1000}\,t^2+\cdots\,, \qquad t \text{ small.} \qquad \Diamond \end{aligned}$$

The following example shows that Taylor expansions may be easier to use than the standard procedure:

Example 10.1 *Find the approximating polynomial of at most second degree for the function*

$$f(x,y) = \exp(x-y^2)\,\text{Arctan}(x+2y)\,\cos(x^2+4y)$$

from the point $(x_0,y_0) = (2,-1)$.

It is obvious that the standard procedure will give us a mess of calculations! Let us therefore turn to the method of using known Taylor expansions.

1. The function is a product of standard functions, where we know the Taylor expansions.

2. The change of variables is here

$$(h,k) = (x-2, y+1), \quad \text{i.e.} \quad (x,y) = (2+h, -1+k).$$

In particular we get for $t = h+2k$ that

$$\text{Arctan}(x+2y) = \text{Arctan}(h+2k) = (h+2k)+\cdots\,,$$

so it suffices to expand the other factors of only *first degree*, since one degree is used in the factor $\text{Arctan}(h+2k)$.

3. and 4. Since

$$\begin{aligned}\exp(x-y^2) &= \exp(1+h+2k-k^2) = e\cdot\exp(h+k)\cdot\exp(-k^2)\\ &= e\cdot\{1+(h+2k)+\cdots\}\cdot\{1+\cdots\} = e+e\cdot(h+2k)+\cdots\,,\end{aligned}$$

and

$$\cos(x^2+4y) = \cos(4h+4k+k^2) = 1+\cdots\,,$$

we get

$$\begin{aligned}f(x,y) &= \exp(1+h+2k-k^2)\,\text{Arctan}(h+2k)\,\cos(4h+4k+k^2)\\ &= \{e+e\cdot(h+2k)+\cdots\}\cdot\{(h+2k)+\cdots\}\cdot\{1+\cdots\}\\ &= e\cdot(h+2k)+e(h+2k)^2+\cdots\,.\end{aligned}$$

5. The approximating polynomial is obtained by deleting the dots and then use the inverse transformation of variables,

$$\begin{aligned}P_2(x,y) &= e(h+2k)+e(h+2k)^2\\ &= e\{(x-2)+2(y+1)+(x-2)^2+4(x-2)(y+1)+4(y+)^2\}.\end{aligned}$$

It should be noted that there also exist examples where the standard procedure is the easiest one, so it is impossible to say in advance that one method is always better that the other one. ◊

11 Examples of differentiable functions

11.1 Gradient

Example 11.1 *Assume that the function $f : A \to \mathbb{R}$, $A \subseteq \mathbb{R}^k$, satisfies*

$$|f(\mathbf{x}) - f(\mathbf{u})| \le a\|\mathbf{x}-\mathbf{u}\|^{c+1}, \qquad \mathbf{x} \in K(\mathbf{u}; b),$$

where $\mathbf{u}$ is a fixed point in the open domain A of f and where b is so small that $K(\mathbf{u}; b) \subset A$. Prove that f is differentiable at the point $\mathbf{u}$ with the gradient $\mathbf{0}$.

A Differentiability; gradient.

D Analyze the definition of differentiability.

I If we put $\mathbf{x} = \mathbf{u} + \mathbf{h}$, then $\mathbf{h} = \mathbf{x} - \mathbf{u}$, and the assumption of the example can be written

$$|f(\mathbf{u}+\mathbf{h}) - f(\mathbf{u})| \le a\|\mathbf{h}\|^{1+c} = \mathbf{0}\cdot\mathbf{h} + \tilde{\varepsilon}(\mathbf{h})\cdot\|\mathbf{h}\|,$$

where $\tilde{\varepsilon}(\mathbf{h}) = a\|\mathbf{h}\|^c \to 0$ for $\mathbf{h} \to \mathbf{0}$. This shows that there exist a function $\varepsilon(\mathbf{h})$ with $|\varepsilon(\mathbf{h})| \le \tilde{\varepsilon}(\mathbf{h})$, such that

$$f(\mathbf{u}+\mathbf{h}) - f(\mathbf{u}) = \mathbf{0}\cdot\mathbf{h} + \varepsilon(\mathbf{h})\cdot\|\mathbf{h}\|.$$

According to the definition, f is differentiable at $\mathbf{u}$ and its gradient is

$$\bigtriangledown f(\mathbf{u}) = \mathbf{0}.$$

Example 11.2 *Let $P(x,y)$ be an* homogeneous *polynomial of degree n in two variables. Prove that*

$$xP_x'(x,y) + yP_y'(x,y) = n\,P(x,y).$$

Formulate and prove an analogous theorem for an homogeneous polynomial of degree n in k variables.

A Homogeneous polynomials.

D Split $P(x,y)$ into its parts and differentiate.

I A typical term in $P(x,y)$ is of the form

$$P_k(x,y) = a_k x^k y^{n-k},$$

from which we get

$$\begin{aligned} x\left(P_k\right)_x' + y\left(P_k\right)_y' &= a_k k x \cdot c^{k-1} y^{n-k} + a_k(n-k) s^k y \cdot y^{n-k-1} \\ &= a_k\,k\,x^k y^{n-k} + a_k(n-k)x^k y^{n-k} \\ &= n a_k x^k y^{n-k} = n\,P_k(x,y). \end{aligned}$$

Since differentiation and multiplication by x (or by y) are linear operations, it follows by adding all such terms that we have for any homogeneous polynomial $P(x,y)$ of degree n that

$$xP_x'(x,y) + yP_y'(x,y) = n\,P(x,y).$$

In general it follows that if $P(x_1, \dots, x_m)$ is an *homogeneous* polynomial of degree n in m variables, then

$$x_1 P'_{x_1}(\mathbf{x}) + \cdots + x_m P'_{x_m}(\mathbf{x}) = \sum_{j=1}^{m} x_j P'_{x_j}(\mathbf{x}) = n\, P(\mathbf{x}).$$

In fact, $P(\mathbf{x})$ is built up by linear combinations of terms of the form

$$Q(\mathbf{x}) = x_1^{k_1} x_1^{k_2} \cdots x_m^{k_m}, \quad k_1, \dots, k_m \geq 0 \text{ og } k_1 + \cdots + k_m = n,$$

where

$$\sum_{j=1}^{m} x_j Q'_{x_j}(\mathbf{x}) = \sum_{j=1}^{m} k_j x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m} = (k_1 + \cdots + k_m) Q(\mathbf{x}) = n\, Q(\mathbf{x}).$$

This holds for every term in any homogeneous polynomial $P(\mathbf{x})$, and then it follows by the linearity that it also holds for $P(\mathbf{x})$ itself.

Example 11.3 *Find in each of the following cases the gradient of the given function in two variables.*

1) $f(x, y) = \text{ Arctan } \dfrac{x}{y}$, *for* $y \neq 0$.

2) $f(x, y) = \text{ Arctan } \dfrac{y}{x}$, *for* $x \neq 0$.

3) $f(x, y) = \ln \dfrac{3 + xy}{4 + \sin y}$, *for* $(x, y) \in \mathbb{R}^2$, $3 + xy > 0$.

4) $f(x, y) = \ln \sqrt{x^2 + y^2}$, *for* $(x, y) \neq (0, 0)$.

A Gradients.

D Differentiate.

I 1) When $f(x, y) = \text{Arctan } \dfrac{x}{y}$, $y \neq 0$, we get

$$\frac{\partial f}{\partial x} = \frac{1}{1 + \left(\dfrac{x}{y}\right)^2} \cdot \frac{1}{y} = \frac{y}{x^2 + y^2}, \qquad \frac{\partial f}{\partial y} = \frac{1}{1 + \left(\dfrac{x}{y}\right)^2} \cdot \left(-\frac{x}{y^2}\right) = -\frac{x}{x^2 + y^2},$$

hence

$$\bigtriangledown f(x, y) = \left(\frac{y}{x^2 + y^2}, -\frac{x}{x^2 + y^2} \right), \qquad y \neq 0.$$

2) Remark. One might be misled to believe that this result can be derived from 1), but it turns up that this is not the case. $\Diamond$

After the warning in the remark above we calculate as above for $f(x,y) = \text{Arctan}\,\dfrac{y}{x}$, $x \neq 0$, that

$$\frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(-\frac{y}{x^2}\right) = \frac{y}{x^2+y^2}, \qquad \frac{\partial f}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2},$$

so

$$\bigtriangledown f(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right), \qquad x \neq 0.$$

3) When $xy > -3$, the function is defined an of class C^∞, so

$$\frac{\partial f}{\partial x} = \frac{y}{3+xy}, \qquad \frac{\partial f}{\partial y} = \frac{x}{3+xy} - \frac{\cos y}{4+\sin y},$$

and

$$\bigtriangledown f(x,y) = \left(\frac{y}{3+xy}, \frac{x}{3+xy} - \frac{\cos y}{4+\sin y}\right), \qquad \text{for } xy > -3.$$

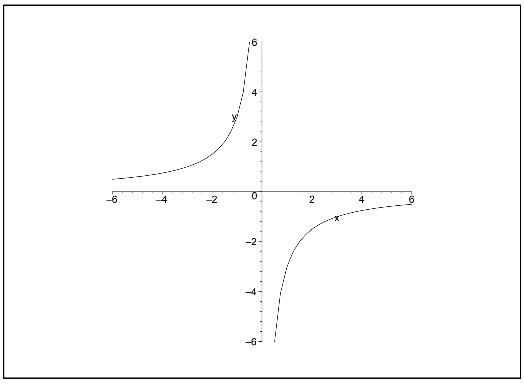

Figure 11.1: The domain of 3).

4) When $f(x,y) = \ln\sqrt{x^2+y^2} = \frac{1}{2}\ln(x^2+y^2)$, $(x,y) \neq (0,0)$, we get

$$\frac{\partial f}{\partial x} = \frac{x}{x^2+y^2}, \qquad \frac{\partial f}{\partial y} = \frac{y}{x^2+y^2},$$

hence

$$\bigtriangledown f(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right).$$

If we use MAPLE, then start with

with(VectorCalculus):

Let $[x,y]$ specify the rectangular coordinate system, and then proceed in the following way:

1) $\text{Gradient}\left(\arctan\left(\frac{x}{y}\right), [x, y]\right)$

$$\left(\frac{1}{y\left(1+\frac{x^2}{y^2}\right)}\right)\overline{e}_x - \frac{x}{y^2\left(1+\frac{x^2}{y^2}\right)}\overline{e}_y$$

2) $\text{Gradient}\left(\arctan\left(\frac{y}{x}\right), [x, y]\right)$

$$-\frac{y}{x^2\left(1+\frac{y^2}{x^2}\right)}\overline{e}_x + \left(\frac{1}{x\left(1+\frac{y^2}{x^2}\right)}\right)\overline{e}_y$$

3) $\text{Gradient}\left(\ln\left(\frac{3+x\cdot y}{4+\sin(y)}\right), [x, y]\right)$

$$\left(\frac{y}{xy+3}\right)\overline{e}_x + \left(\frac{\left(\frac{x}{4+\sin(y)} - \frac{(xy+3)\cos(y)}{(4+\sin(y))^2}\right)(4+\sin(y))}{xy+3}\right)\overline{e}_y$$

4) Gradient$\left(\ln\left(\sqrt{x^2+y^2}\right), [x,y]\right)$

$$\left(\frac{x}{x^2+y^2}\right)\overline{e}_x + \left(\frac{y}{x^2+y^2}\right)\overline{e}_y$$

Clearly, 1)–3) need some reductions, which we shall not give here, because this is not the main subject.

Example 11.4 *Find in each of the following cases the gradient of the given function in three variables.*

1) $f(x,y,z) = (x+y)(y+z)(z+x)$*, for* $(x,y,z) \in \mathbb{R}^3$.

2) $f(x,y,z) = x\,3^{y+xz}$*, for* $(x,y,z) \in \mathbb{R}^3$.

3) $f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$*, for* $(x,y,z) \neq (0,0,0)$.

4) $f(x,y,z) = \exp\left(x^2 - y + z\right)$*, for* $(x,y,z) \in \mathbb{R}^3$.

5) $f(x,y,z) = x\,\tan(yz^2) + \cos(x^3 z)$*, for* $yz^2 \neq \left(p+\frac{1}{2}\right)\pi$*,* $p \in \mathbb{Z}$.

A Gradients.

D Differentiate.

I 1) It follows from $f(x,y,z) = (x+y)(y+z)(z+x)$ that

$$\frac{\partial f}{\partial x} = (x+z)(y+z) + (x+y)(y+z) = (2x+y+z)(y+z) = (x+y+z)^2 - x^2.$$

In this case it follows from the symmetry that we can simply interchange the letters in order to get

$$\frac{\partial f}{\partial y} = (x+y+z)^2 - y^2, \qquad \frac{\partial f}{\partial z} = (x+y+z)^2 - z^2,$$

hence

$$\nabla f = \left((x+y+z)^2 - x^2, (x+y+z)^2 - y^2, (x+y+z)^2 - z^2\right).$$

2) It follows from

$$f(x,y,z) = x\,3^{y+xz} = x\exp\{(y+xz)\ln 3\}$$

that

$$\frac{\partial f}{\partial x} = 3^{y+xz} + x\,3^{y+xz} z\,\ln 3 = e^{y+xz}(1 + xz\ln 3),$$

$$\frac{\partial f}{\partial y} = x\,e^{y+xz}\ln 3, \qquad \frac{\partial f}{\partial z} = x^2\ln 3 \cdot 3^{y+xz},$$

and accordingly,

$$\nabla f(x,y,z) = 3^{y+xz}\left(1 + xz\ln 3, x\ln 3, x^2\ln 3\right).$$

3) When

$$f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}, \qquad (x,y,z) \neq (0,0,0),$$

we get

$$\frac{\partial f}{\partial x} = -\frac{1}{2} \cdot \frac{2x}{\left(\sqrt{x^2+y^2+z^2}\right)^3} = -\frac{x}{\left(\sqrt{x^2+y^2+z^2}\right)^3},$$

and by the symmetry, analogous expressions for $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial z}$, so

$$\bigtriangledown f = -\frac{(x,y,z)}{\left(\sqrt{x^2+y^2+z^2}\right)^3}, \qquad (x,y,z) \neq (0,0,0).$$

Remark. If we introduce the notation

$$\mathbf{r} = (x,y,z), \qquad r = \sqrt{x^2+y^2+z^2},$$

then this *important* result can be written in the short form

$$\bigtriangledown r = -\frac{\mathbf{r}}{r^3}. \qquad \Diamond$$

4) When $f(x,y,z) = \exp(x^2-y+z)$, then

$$\frac{\partial f}{\partial x} = \exp(x^2-y+z) \cdot 2x,$$
$$\frac{\partial f}{\partial y} = \exp(x^2-y+z) \cdot (-1),$$
$$\frac{\partial f}{\partial z} = \exp(x^2-y+z),$$

hence

$$\bigtriangledown f(x,y,z) = \exp(x^2-y+z)\,(2x,-1,1).$$

5) We see that the function

$$f(x,y,z) = x\,\tan(yz^2) + \cos(x^3 z)$$

is defined and of class C^∞, when $yz^2 \neq \dfrac{\pi}{2} + p\pi$, $p \in \mathbb{Z}$. Then by a differentiation

$$\frac{\partial f}{\partial x} = \tan(yz^2) - 3x^2 z \sin(x^3 z),$$
$$\frac{\partial f}{\partial y} = xz^2\{1+\tan^2(yz^2)\} = \frac{xz^2}{\cos^2(yz^2)},$$
$$\frac{\partial f}{\partial z} = 2xyz\{1+\tan^2(yz^2)\} - x^3\sin(x^3 z) = \frac{2xyz}{\cos^2(yz^2)} - x^3\sin(x^3 z),$$

and accordingly in the given domain,

$$\bigtriangledown f = \left(\tan(yz^2) - 3x^2 z\sin(x^3 z), \frac{xz^2}{\cos^2(yz^2)}, \frac{2xyz}{\cos^2(yz^2)} - x^3\sin(x^3 x)\right).$$

In MAPLE we first write

with(VectorCalculus)

Then specify the rectangular coordinate system by writing $[x, y, z]$ and proceed in the following way:

1) Gradient$((x+y)\cdot(y+z)\cdot(z+x), [x,y,z])$

$$((y+z)(z+x)+(x+y)(y+z))\overline{e}_x + ((y+z)(z+x)+(x+y)(z+x))\overline{e}_y \\ +((x+y)(z+x)+(x+y)(y+z))\overline{e}_z$$

2) Gradient$(x\cdot 3^{y+x\cdot z}, [x,y,z])$

$$\left(3^{xz+y}+x3^{xz+y}z\ln(3)\right)\overline{e}_x + \left(x3^{xz+y}\ln(3)\right)\overline{e}y + \left(x^2 3^{xz+y}\ln(3)\right)\overline{e}_z$$

3) Gradient$\left(\dfrac{1}{\sqrt{x^2+y^2+z^2}}, [x,y,z]\right)$

$$-\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}}\overline{e}_x - \frac{y}{\left(x^2+y^2+z^2\right)^{3/2}}\overline{e}_y - \frac{z}{\left(x^2+y^2+z^2\right)^{3/2}}\overline{e}_z$$

4) Gradient$\left(e^{x^2-y+z}, [x,y,z]\right)$

$$2xe^{x^2-y+z}\overline{e}_x - e^{x^2-y+z}\overline{e}_y + \left(e^{x^2-y+z}\right)\overline{e}_z$$

5) Gradient$\left(x\cdot\tan\left(y\cdot z^2\right)+\cos\left(x^3\cdot z\right), [x,y,z]\right)$

$$\left(\tan\left(yz^2\right)-3\sin\left(x^3z\right)x^2z\right)\overline{e}_x + \left(x\left(1+\tan\left(yz^2\right)^2\right)z^2\right)\overline{e}_y \\ +\left(2x\left(1+\tan\left(yz^2\right)^2\right)yz-\sin\left(x^3z\right)x^3\right)\overline{e}_z$$

The MAPLE results should also here be reduced.

Example 11.5 *In some of the cases where it is not possible to decide only by using the rules of calculation whether a given function of several variables is differentiable at some given point, one may try instead to use the definition directly in the following way.*
Use restrictions to see if the partial derivatives exist at the point. When this is the case, then insert the values into the definition of differentiability, in which the ε function occurs; then check if this ε function has the required property.
Use this procedure to prove the following claims:

- *In 1)–3) the function is not differentiable at* $(0,0)$.
- *In 4)–5) the function is differentiable at* $(0,0)$ *with the gradient zero.*

1) $f(x,y)=\sqrt{x^2+y^2}$.

2) $f(x,y)=|x+y|$.

3)

$$f(x,y)=\begin{cases}\dfrac{x^3}{x^2+y^2}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0).\end{cases}$$

4) $f(x,y)=\sqrt{x^4+y^4}$.

5) $f(x,y)=|x^2-y^2|$.

A Gradients by using the definition.

D Follow the given description.

I First note that if f is differentiable, then

$$f(\mathbf{x}+\mathbf{h})-f(\mathbf{x})=\mathbf{h}\cdot\bigtriangledown f(\mathbf{x})+\varepsilon(\mathbf{h})\|\mathbf{h}\|,$$

where $\varepsilon(\mathbf{h})\to 0$ for $\mathbf{h}\to\mathbf{0}$.

1) Here,

$$\frac{\partial f}{\partial x}=\frac{x}{\sqrt{x^2+y^2}},\qquad \frac{\partial f}{\partial y}=\frac{y}{\sqrt{x^2+y^2}},$$

hence

$$\frac{\partial f}{\partial x}(x,0)\to\begin{cases}1 & \text{for } x\to 0+,\\ -1 & \text{for } x\to 0-,\end{cases}\qquad \frac{\partial f}{\partial y}(0,y)\to\begin{cases}1 & \text{for } y\to 0+,\\ -1 & \text{for } y\to 0-.\end{cases}$$

Then

$$\begin{aligned}\varepsilon(x,y) &= \frac{1}{\sqrt{x^2+y^2}}\left\{f(x,y)-f(0,0)-x\,\frac{\text{“}\partial f\text{”}}{\partial x}-y\,\frac{\text{“}\partial f\text{”}}{\partial y}\right\}\\ &= \begin{cases}1-\dfrac{x}{\sqrt{x^2+y^2}}-\dfrac{y}{\sqrt{x^2+y^2}} & \text{for } x>0,\ y>0,\\ 1+\dfrac{x}{\sqrt{x^2+y^2}}-\dfrac{y}{\sqrt{x^2+y^2}} & \text{for } x<0,\ y>0,\\ 1+\dfrac{x}{\sqrt{x^2+y^2}}+\dfrac{y}{\sqrt{x^2+y^2}} & \text{for } x<0,\ y<0,\\ 1-\dfrac{x}{\sqrt{x^2+y^2}}+\dfrac{y}{\sqrt{x^2+y^2}} & \text{for } x>0,\ y<0.\end{cases}\end{aligned}$$

By using polar coordinates we see that these expressions do not tend to zero in the given domains, when $(x,y) \to (0,0)$. The function is accordingly not differentiable at $(0,0)$.

2) Here

$$\frac{\partial f}{\partial x}(x,0) = \begin{cases} 1 & \text{for } x > 0, \\ -1 & \text{for } x < 0, \end{cases} \qquad \frac{\partial f}{\partial y}(0,y) = \begin{cases} 1 & \text{for } y > 0, \\ -1 & \text{for } y < 0, \end{cases}$$

so

$$\varepsilon(x,y) = \frac{1}{\sqrt{x^2+y^2}}\{|x+y| - |x| - |y|\},$$

which does not tend towards zero for $(x,y) \to (0,0)$. [Try e.g. $y = -x$.]

3) Here,

$$\frac{\partial f}{\partial x}(x,0) = 1 \qquad \text{and} \qquad \frac{\partial f}{\partial y}(0,y) = 0,$$

so

$$\varepsilon(x,y) = \frac{1}{\sqrt{x^2+y^2}}\left\{\frac{x^3}{x^2+y^2} - x\right\} = -\frac{xy^2}{\left(\sqrt{x^2+y^2}\right)^3} = -\cos\varphi \cdot \sin^2\varphi$$

in polar coordinates. This expression does not tend to 0 for $\varrho = \sqrt{x^2+y^2} \to 0$.

4) Here

$$\frac{\partial f}{\partial x}(x,0) = \frac{\partial}{\partial x}(x^2) = 2x \to 0 \qquad \text{for } x \to 0,$$

and analogously

$$\frac{\partial f}{\partial y}(0,y) = 2y \to 0 \qquad \text{for } y \to 0,$$

hence

$$\varepsilon(x,y) = \frac{1}{\sqrt{x^2+y^2}}\left\{\sqrt{x^4+y^4}-0-0\right\} = \varrho\sqrt{\cos^4\varphi+\sin^4\varphi} \to 0$$

for $\varrho \to 0$.
Hence, the function is differentiable at $\mathbf{0}$ and

$$\bigtriangledown f(\mathbf{0}) = \mathbf{0}.$$

5) Here $f(x,0) = x^2$, so

$$\frac{\partial f}{\partial x}(x,0) = 2x \to 0 \qquad \text{for } x \to 0,$$

and $f(0,y) = y^2$, and thus

$$\frac{\partial f}{\partial y}(0,y) = 2y \to 0 \qquad \text{for } y \to 0.$$

Then

$$\varepsilon(x,y) = \frac{|x^2-y^2|}{\sqrt{x^2+y^2}} = \frac{\varrho^2}{\varrho}|\cos^2\varphi - \sin^2\varphi| = \varphi|\cos 2\varphi| \to 0$$

for $\varrho \to 0$, and we conclude that the function is differentiable at $\mathbf{0}$ and

$$\bigtriangledown f(0) = \mathbf{0}.$$

Example 11.6 *Find in each of the following cases the gradient of the given function* $f : \mathbb{R}^3 \to \mathbb{R}$. *The vector* $\mathbf{a}$ *is constant.*

1) $f(\mathbf{x}) = \mathbf{x} \cdot \mathbf{a}$.

2) $f(\mathbf{x}) = (\mathbf{x} \cdot \mathbf{a})^2$.

3) $f(\mathbf{x}) = \|\mathbf{x} \times \mathbf{x}\|$.

4) $f(\mathbf{x}) = \mathbf{x} \times (\mathbf{x} \times \mathbf{a}) \cdot \mathbf{a}$.

A Gradients.

D Calculate the expressions and then differentiate.

I 1) Since $f(\mathbf{x}) = \mathbf{x} \cdot \mathbf{a} = x_1a_1 + x_2a_2 + x_3a_3$, it follows that

$$\bigtriangledown f(\mathbf{x}) = \mathbf{a}.$$

2) We get from $f(\mathbf{x}) = (\mathbf{x} \cdot \mathbf{a})^2 = \{x_1a_1 + x_2a_2 + x_3a_3\}^2$ that

$$\frac{\partial f}{\partial x_i} = 2a_i(x_1a_1 + x_2a_2 + x_3a_3) = 2a_i\,(\mathbf{x} \cdot \mathbf{a}),$$

so we get as expected,

$$\bigtriangledown f(\mathbf{x}) = 2(\mathbf{x} \cdot \mathbf{a})\,\mathbf{a}.$$

3) Since $f(\mathbf{x}) = \|\mathbf{x} \times \mathbf{x}\| = 0$, we get $\bigtriangledown f(\mathbf{x}) = \mathbf{0}$.

4) First calculate

$$\mathbf{x} \times \mathbf{a} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ x_1 & x_2 & x_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = (x_2a_3 - x_3a_2, x_3a_1 - x_1a_3, x_1a_2 - x_2a_1)\,,$$

whence

$$\begin{aligned} \mathbf{x} \times (\mathbf{x} \times \mathbf{a}) &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ x_1 & x_2 & x_3 \\ x_2a_3 - x_3a_2 & x_3a_1 - x_1a_3 & x_1a_2 - x_2a_1 \end{vmatrix} \\ &= (x_2(x_1a_2 - x_2a_1) - x_3(x_3a_1 - x_1a_3))\,\mathbf{e}_1 \\ &\quad + (x_3(x_2a_3 - x_3a_2) - x_1(x_1a_2 - x_2a_1))\,\mathbf{e}_2 \\ &\quad + (x_1(x_3a_1 - x_1a_3) - x_2(x_2a_3 - x_3a_2))\,\mathbf{e}_3. \end{aligned}$$

We conclude that

$$\begin{aligned} \mathbf{x} \times (\mathbf{x} \times \mathbf{a}) \cdot \mathbf{a} &= a_1(x_1x_2a_2 - x_2^2a_1 - x_3^2a_1 + x_1x_3a_3) \\ &\quad + a_2(x_2x_3a_3 - x_3^2a_3 - x_1^2a_2 + x_1x_2a_1) \\ &\quad + a_3(x_1x_3a_1 - x_1^2a_3 - x_2^2a_3 + x_2x_3a_2) \\ &= -x_1^2(a_2^2 + a_3^2) - x_2^2(a_1^2 + a_3^2) - x_3^2(a_1^2 + a_2^2) \\ &\quad + 2x_1x_2a_1a_2 + 2x_1x_3a_1a_3 + 2x_2x_3a_2a_3, \end{aligned}$$

which can be further reduced. This is, however, not necessary here, because we shall only need the derivatives in the following,

$$\begin{aligned}\frac{\partial f}{\partial x_1} &= -2x_1(a_1^2+a_3^2)+2x_2a_1a_2+2x_3a_1a_3\\ &= -2x_1(a_1^2+a_2^2+a_3^2)+2a_1(x_1a_1+x_2a_2+x_3a_3)\\ &= -2x_1\|\mathbf{a}\|^2+2a_1\mathbf{a}\cdot\mathbf{x}),\end{aligned}$$

$$\frac{\partial f}{\partial x_2} = -2x_2\|\mathbf{a}\|^2+2a_2(\mathbf{a}\cdot\mathbf{x}),$$

$$\frac{\partial f}{\partial x_3} = -2x_3\|\mathbf{a}\|^2+2a_2(\mathbf{a}\cdot\mathbf{x}).$$

These are the coordinates of $\bigtriangledown f$, so all things put together we finally get

$$\bigtriangledown f(\mathbf{x}) = -2(\mathbf{a}\cdot\mathbf{a})\,\mathbf{x}+2(\mathbf{x}\cdot\mathbf{a})\,\mathbf{a}.$$

We start in MAPLE by declaring

with(LinearAlgebra):
with(VectorCalculus):

Then proceed in the following way: First declare the vectors $\mathbf{x}$, $\mathbf{a}\in\mathbb{R}^3$,

$X := < x, y, z >$:

$A := < a, b, c >$:

In order not to involve complex conjugation in the dot product, $\mathbf{x}\cdot\mathbf{a}$, we write

DotProduct(map(conjugate),X)

1) Gradient(DotProduct(map(conjugate,X),A),[x, y, z])

$(a)\overline{e}_x+(b)\overline{e}_y+(c)\overline{e}_z$

2) Reuse the above and then proceed with
Gradient$\big($(DotProduct(map(conjugate,X),A))^2, [x, y, z]$\big)$

$2(ax+by+cz)a\overline{e}_x+2(ax+by+cz)b\overline{e}_y+2(ax+by+cz)c\overline{e}_z$

3) This is trivially $\mathbf{0}$.

4) DotProduct(map(conjugate,X &x (X &x A)),A)

$$(y(-ay+bx)-z(az-cx))a+(-x(-ay+bx)+z(-bz+cy))b$$
$$+(x(az-cx)-y(-bz+cy))c.$$

Example 11.7 *Let A denote the point set where we have removed the coordinate axes from the plane $\mathbb{R}^2$, i.e.*

$$A = \{(x, y) \mid xy \neq 0\}.$$

We define a function $f : A \to \mathbb{R}$ by putting $f(x, y)$ equal to the number of the quadrant, which (x, y) belongs to. Find ∇f.

A Gradient.

D Use that f is constant on every connected component of A.

I The task is now trivial, because $f(x, y)$ is constant on each of the four open quadrants, where it is defined, hence $\nabla f = \mathbf{0}$.

11.2 The chain rule

Example 11.8 . *Use the chain rule to calculate the derivative of the function $F(u) = f(\mathbf{X}(u))$, i.e. without finding $F(u)$ explicitly, in the following cases:*

1) $f(x,y) = xy$, *where* $\mathbf{X}(u) = (e^u, \cos u)$, $u \in \mathbb{R}$.

2) $f(x,y) = e^{xy}$, *where* $\mathbf{X}(u) = (3u^2, u^3)$, $u \in \mathbb{R}$.

3) $f(x,y) = x^3 + y^3 - 3xy$, *where* $\mathbf{X}(u) = \left(u^2, \dfrac{3u}{1+u}\right)$, $u > -1$.

4) $f(x,y) = y^x$, *where* $\mathbf{X}(u) = (\sin u, u^3)$, $u > 0$.

5) $f(x,y) = y\,e^x$, *where* $\mathbf{X}(u) = (\operatorname{Arctan}(1+u), e^u)$, $u \in \mathbb{R}$.

6) $f(x,y) = y \sin x$, *where* $\mathbf{X}(u) = \left(-u, \sqrt{1+u^2}\right)$, $u \in \mathbb{R}$.

A The chain rule.

D Start by formulating the general chain rule. No matter the formulation we shall nevertheless also calculate $F(u)$ and find the derivative in the usual way, so that it is possible to compare the two methods.

I The task is to insert (correctly) into the chain rule,

$$F'(u) = \frac{\partial f}{\partial x}\frac{dx}{du} + \frac{\partial f}{\partial y}\frac{dy}{du},$$

where x and y are the coordinates of $\mathbf{X} = (x,y)$.

1) When $f(x,y) = xy$ and $(x,y) = (e^u, \cos u)$, we get

$$F'(u) = y\frac{dx}{du} + x\frac{dy}{du} = \cos u \cdot e^u - e^u \sin u = e^u(\cos u - \sin u).$$

TEST. By insertion we also have

$$F(u) = e^u \cos u,$$

so

$$F'(u) = e^u(\cos u - \sin u).$$

We see that we get the same result, and in this case the application of the chain rule is not easier than the traditional method. ♢

In MAPLE we declare

$$f := (x,y) \to x \cdot y$$

$$(x,y) \to x\,y$$

$$X := u \to (e^u, \cos(u))$$

$$u \to (e^u, \cos(u))$$

Then

$$\frac{d}{du} f(X(u))$$

$$e^u \cos(u) - e^u \sin(u)$$

2) When $f(x,y) = e^{xy}$ and $(x,y) = (3u^2, u^3)$, we get by the chain rule,

$$F'(u) = e^{xy}\, y\, \frac{\mathrm{d}x}{\mathrm{d}y} + e^{xy}\, x\, \frac{\mathrm{d}y}{\mathrm{d}u} = e^{3u^5} \cdot u^3 \cdot 6u + e^{3u^5} \cdot 3u^2 \cdot 3u^2 = 15u^4 \exp(3u^5).$$

TEST. By insertion we get

$$F(u) = e^{xy} = \exp(3u^5),$$

so by a differentiation,

$$F'(u) = 15u^4 \exp(3u^5).$$

We see that the two results agree, and also that the direct method is easier to apply in this case than the chain rule. ♢

In MAPLE we declare

$$f := (x,y) \to e^{x \cdot y}$$

$$u \to e^{xy}$$

$$X := u \to \left(3u^2, u^3\right)$$

$$u \to \left(3u^2, u^3\right)$$

Then

$$\frac{d}{du}, f(X(u))$$

$$15u^4 e^{3u^5}$$

3) When $f(x,y) = x^3 + y^3 - 3xy$ and $(x,y) = \left(u^2, \dfrac{3u}{1+u}\right)$, $u > -1$, we get

$$\begin{aligned}
F'(u) &= (3x^2 - 3y)\frac{dx}{du} + (3y^2 - 3x)\frac{dy}{du} \\
&= 3\left(u^4 - \frac{3u}{1+u}\right)2u + 3\left(\frac{9u^2}{(1+u)^2} - u^2\right)\cdot\frac{3(1+u)-3u}{(1+u)^2} \\
&= 6u^2\left(u^3 - \frac{3}{1+u}\right) + 9\,\frac{u^2}{(1+u)^2}\left\{\frac{9}{(1+u)^2} - 1\right\} \\
&= \frac{81u^2}{(1+u)^4} - \frac{9u^2}{(1+u)^2} - \frac{18u^2}{1+u} + 6u^5 \\
&= 6u^5 + \frac{9u^2}{(1+u)^4}\left\{9 - (1+u)^2 - 2(1+u)^3\right\} \\
&= 6u^5 + \frac{9u^2}{(1+u)^4}\{9 - 1 - 2u - u^2 - 2 - 6u - 6u^2 - 2u^3\} \\
&= 6u^5 - \frac{9u^2}{(1+u)^4}\{2u^3 + 7u^2 + 8u - 6\}.
\end{aligned}$$

TEST. By insertion we get

$$F(u) = u^6 + \frac{27u^3}{(1+u)^3} - \frac{9u^3}{1+u},$$

hence

$$\begin{aligned}
F'(u) &= 6u^5 - \frac{27u^2}{1+u} + \frac{9u^3}{(1+u)^3} + \frac{81u^2}{(1+u)^3} - \frac{81u^3}{(1+u)^4} \\
&= 6u^5 + \frac{9u^2}{(1+u)^4}\{-3(1+u)^3 + u(1+u)^2 + 9(1+u) - 9u\} \\
&= 6u^5 - \frac{9u^5}{(1+u)^4}\{3 + 9u + 9u^2 + 3u^3 - u - 2u^2 - u^3 - 9 - 9u + 9u\} \\
&= 6u^5 - \frac{9u^2}{(1+u)^4}\{2u^3 + 7u^2 + 8u - 6\}.
\end{aligned}$$

The two results agree. This time the two methods are more comparable in effort than in the previous ones. ♢

In MAPLE we declare

$$f := (x,y) \to x^3 + y^3 - 3x\cdot y$$

$$(x,y) \to x^3 + y^3 - 3xy$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}u}\, f(X(u))$$

$$6u^5 + \frac{81u^2}{(1+u)^3} - \frac{81u^3}{(1+u)^4} - \frac{27u^2}{1+u} + \frac{9u^3}{(1+u)^2}$$

4) When $f(x,y) = y^x$ and $(x,y) = (\sin u, u^3)$, $u > 0$, we get

$$\frac{\partial f}{\partial x} = \ln y \cdot y^x, \qquad \frac{\partial f}{\partial y} = x\, y^{x-1}, \qquad \frac{\mathrm{d}x}{\mathrm{d}u} = \cos u, \qquad \frac{\mathrm{d}y}{\mathrm{d}u} = 3u^2,$$

so

$$\begin{aligned} F'(u) &= \frac{\partial f}{\partial x}\,\frac{dx}{du} + \frac{\partial f}{\partial y}\,\frac{\mathrm{d}y}{\mathrm{d}u} \\ &= \{\ln y \cdot y^x\} \cos u + xy^{x-1} \cdot 3u^2 \\ &= \ln(u^3) \cdot u^{3\sin u} \cos u + \sin u \cdot u^{3(\sin u - 1)} \cdot 3u^2 \\ &= 3 \ln u \cdot u^{3\sin u} \cos u + 3u^{3\sin u - 1} \sin u. \end{aligned}$$

TEST. We get by insertion

$$F(u) = u^{3\sin u} = \exp(3\sin u \cdot \ln u), \qquad u > 0,$$

hence

$$F'(u) = u^{3\sin u}\{3\ln u \cdot \cos u + 3\,\frac{1}{u}\,\sin u\} = 3\ln u \cdot u^{3\sin u} \cos u + 3u^{2\sin u - 1} \sin u.$$

The two results agree. $\Diamond$

In MAPLE we declare

$$f := (x,y) \to y^x$$

$$(x,y) \to y^x$$

$$X := u \to \left(\sin(u), u^3\right)$$

$$u \to \left(\sin(u), u^3\right)$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}u}\, f(X(u))$$

$$\left(u^3\right)^{\sin(u)} \left(\cos(u)\ln\left(u^3\right) + \frac{3\sin(u)}{u}\right)$$

5) When $f(x,y) = y\,e^x$ and $(x,y) = (\operatorname{Arctan}(1+u), e^u)$, we get

$$\frac{\partial f}{\partial x} = y\,e^x, \quad \frac{\partial f}{\partial y} = e^x, \quad \frac{dx}{du} = \frac{1}{1+(1+u)^2}, \quad \frac{\mathrm{d}y}{\mathrm{d}u} = e^u,$$

hence

$$\begin{aligned} F'(u) &= \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}u} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}u} \\ &= e^u \cdot e^{\operatorname{Arctan}(1+u)} \cdot \frac{1}{1+(1+u)^2} + e^{\operatorname{Arctan}(1+u)} \cdot e^u \\ &= \left\{1 + \frac{1}{1+(1+u)^2}\right\} \exp(u + \operatorname{Arctan}(1+u)). \end{aligned}$$

Test. By insertion,

$$F(u) = e^u e^{\operatorname{Arctan}(1+u)} = \exp\{u + \operatorname{Arctan}(1+u)\},$$

so

$$F'(u) = \left\{1 + \frac{1}{1+(1+u)^2}\right\} \exp\{u + \operatorname{Arctan}(1+u)\}.$$

The two results agree. ◊

In MAPLE we declare

$$f := (x,y) \to y \cdot e^x$$

$$(x,y) \to y\,e^x$$

$X := u \to (\arctan(1+u), e^u)$

$$u \to (\arctan(1+u), e^u)$$

Then

$\dfrac{\mathrm{d}}{\mathrm{d}u} f(X(u))$

$$e^u e^{\arctan(1+u)} + \frac{e^u \arctan(1+u)}{1+(1+u)^2}$$

6) When $f(x,y) = y \sin x$ and $(x,y) = (-u, \sqrt{1+u^2})$, we get

$$\begin{aligned} F'(u) &= \frac{\partial f}{\partial x}\frac{dx}{du} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}u} \\ &= y\,\cos x \cdot (-1) + \sin x \cdot \frac{u}{\sqrt{1+u^2}} \\ &= -\sqrt{1+u^2}\cdot \cos(-u) + \sin(-u)\cdot \frac{u}{\sqrt{1+u^2}} \\ &= -\sqrt{1+u^2}\cdot \left\{\cos u + \frac{u\,\sin u}{1+u^2}\right\}. \end{aligned}$$

TEST. By insertion,

$$F(u) = -\sqrt{1+u^2}\cdot \sin u,$$

hence

$$\begin{aligned} F'(u) &= -\sqrt{1+u^2}\cdot \cos u - \frac{u}{\sqrt{1+u^2}}\cdot \sin u \\ &= -\sqrt{1+u^2}\cdot \left\{\cos u + \frac{u\,\sin u}{1+u^2}\right\}. \end{aligned}$$

The two results agree. $\Diamond$

In MAPLE we declare

$f := (x,y) \to y \cdot \sin(x)$

$$(x,y) \to y\,\sin(x)$$

$X := u \to \left(-u, \sqrt{1+u^2}\right)$

$$u \to \left(-u, \sqrt{1+u^2}\right)$$

Then

$\dfrac{\mathrm{d}}{\mathrm{d}u} f(X(u))$

$$-\frac{\sin(u)u}{\sqrt{u^2+1}} - \sqrt{u^2+1}\cos(u)$$

Remark. From a pedagogical point of view it is very inconvenient that the usual method is easier to apply in all cases than the chain rule. It is therefore here not very convincing that the chain rule is a practical device in some situations, where the usual calculation becomes messy. The reader is referred to **Example 11.13**, where the direct calculation is not possible, and yet the result can be obtained by using the chain rule instead. $\diamond$

Example 11.9 *Calculate the partial derivatives of the function $F(u,v) = f(\mathbf{X}(u,v))$ by means of the chain rule, i.e. without finding $F(u,v)$ explicitly, in the following cases:*

1) $f(x,y) = x^2 y$, $\mathbf{X}(u,v) = (u+v, uv)$, *where* $(x,y) \in \mathbb{R}^2$.

2) $f(x,y) = \dfrac{x}{x+y}$, $\mathbf{X}(u,v) = (u^2+v^2, 2uv)$, *where* $(u,v) \in \mathbb{R}_+^2$.

3) $f(x,y) = \text{ Arctan } \dfrac{y}{x}$, $\mathbf{X}(u,v) = (u^2 - uv + v^2, 2uv)$, *where* $(u,v) \neq (0,0)$.

4) $f(x,y) = \text{ Arctan}(x+y^2)$, $\mathbf{X}(u,v) = (u, \exp(u \sin v))$, *where* $(u,v) \in \mathbb{R}^2$.

5) $f(x,y) = x \cos y$, $\mathbf{X}(u,v) = \sqrt{1+u^2+v^2} \cdot \text{Arcsin } u$, *where* $|u| < 1$ *and* $v \in \mathbb{R}$.

6) $f(x,y) = x \sinh y$, $\mathbf{X}(u,v) = (u^3 v, \ln u + \ln v)$, *where* $(u,v) \in \mathbb{R}_+^2$.

A Partial derivatives of composite functions by the chain rule.

D Set up the chain rule. Then differentiate in each case and insert. In spite of the text we shall nevertheless check the result by using the traditional method in the test.

I The chain rule is written in two versions,

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v},$$

where one should be very careful to insert the right coordinates. Whenever f and x and y are present, we first calculate in the intermediate coordinates x and y, and then afterwards we put $x = x(u,v)$ and $y = y(u,v)$. Therefore, in the rough workings we obtain a mixed result in which both x and y as well as u and v occur. Then x and y are eliminated in the next step.

1) When $f(x,y) = x^2 y$ and $(x,y) = \mathbf{X}(u,v) = (u+v, uv)$, then

$$\frac{\partial f}{\partial x} = 2xy \qquad \text{and} \qquad \frac{\partial f}{\partial y} = x^2,$$

and

$$\frac{\partial x}{\partial u} = 1, \quad \frac{\partial y}{\partial u} = v, \qquad \frac{\partial x}{\partial v} = 1, \quad \frac{\partial y}{\partial v} = u,$$

so

$$\frac{\partial F}{\partial u} = 2xy \cdot 1 + x^2 \cdot v = 2(u+v)uv + (u+v)^2 v = v(u+v)(3u+v),$$

and

$$\frac{\partial F}{\partial v} = 2xy \cdot 1 + x^2 \cdot u = 2(u+v)uv + (u+v)^2 u = u(u+v)(u+3v).$$

TEST. We get by insertion

$$F(u,v) = (u+v)^2 uv,$$

thus

$$\frac{\partial F}{\partial u} = 2(u+v)v + (u+v)^2 v = v(u+v)(3u+v),$$

and

$$\frac{\partial F}{\partial v} = \frac{2u(v-u)}{(u+v)^3}.$$

The results agree. $\Diamond$

In MAPLE we declare

$$f := (x, y) \to x^2 \cdot y$$

$$(x, y) \to x^2 y$$

$$X := (u, v) \to (u + v, u \cdot v)$$

$$(u, v) \to (u + v, uv)$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}u} f(X(u, v))$$

$$2(u + v)uv + (u + v)^2 v$$

and

$$\frac{\mathrm{d}}{\mathrm{d}v} f(X(u, v))$$

$$2(u + v)uv + (u + v)^2 u$$

2) Given

$$f(x, y) = \frac{x}{x + y}, \qquad \mathbf{X}(u, v) = \left(u^2 + v^2, 2uv\right), \quad (u, v) \in \mathbb{R}_+^2.$$

Then clearly $x + u = u^2 + v^2 + 2uv = (u + v)^2 > 0$ for $(u, v) \in \mathbb{R}_+^2$, so the composite function $f(\mathbf{X}(u, v))$ is defined and of class C^∞ for $(u, v) \in \mathbb{R}_+^2$. We find

$$\frac{\partial f}{\partial x} = \frac{1}{x + y} - \frac{x}{(x + y)^2} = \frac{y}{(x + y)^2} = \frac{2uv}{(u + v)^4}$$

and

$$\frac{\partial f}{\partial y} = -\frac{x}{(x + y)^2} = -\frac{u^2 + v^2}{(u + v)^4}.$$

Furthermore,

$$\frac{\partial x}{\partial u} = 2u, \qquad \frac{\partial y}{\partial u} = 2v, \qquad \frac{\partial x}{\partial v} = 2v, \qquad \frac{\partial y}{\partial v} = 2v,$$

so

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\,\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\,\frac{\partial y}{\partial u} = \frac{2uv}{(u + v)^4} \cdot 2u - \frac{u^2 + v^2}{(u + v)^4} \cdot 2v = 2v \cdot \frac{u - v}{(u + v)^3},$$

and

$$\frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\,\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\,\frac{\partial y}{\partial v} = \frac{2uv}{(u+v)^4}\cdot 2v - \frac{u^2+v^2}{(u+v)^4}\cdot 2u = 2u\cdot\frac{v-u}{(u+v)^3}.$$

In MAPLE we declare

$$f := (x,y) \to \frac{x}{x+y}$$

$$(x,y) \to \frac{x}{x+y}$$

$$X := (u,v) \to \left(u^2+v^2, 2u\cdot v\right)$$

$$(u,v) \to \left(u^2+v^2, 2uv\right)$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}u}\, f(X(u,v))$$

$$\frac{2u}{u^2+2uv+v^2} - \frac{\left(u^2+v^2\right)(2u+2v)}{\left(u^2+2uv+v^2\right)^2}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}v}\, f(X(u,v))$$

$$\frac{2v}{u^2+2uv+v^2} - \frac{\left(u^2+v^2\right)(2u+2v)}{\left(u^2+2uv+v^2\right)^2}$$

where both results can be reduced further.

3) Consider

$$f(x,y) = \operatorname{Arctan}\frac{y}{x}, \qquad \mathbf{X}(u,v) = (u^2-uv+v^2, 2uv), \quad (u,v) \neq (0,0).$$

We first check that the composite function is defined (and of class C^∞, where it is defined). Here we shall just check that $x \neq 0$ for $(u,v) \neq (0,0)$. Now

$$x(u,v) = u^2-uv+v^2 = \left(u-\frac{1}{2}\,v\right)^2 + \frac{3}{4} \neq 0 \quad \text{for } (u,v) \neq (0,0).$$

Therefore, $f(\mathbf{X}(u,v))$ is defined and of class C^∞ for $(u,v) \neq (0,0)$.

In the calculations we shall need x^2+y^2 expressed by u and v. We see that

$$\begin{aligned} x^2+y^2 &= (u^2-uv+v^2)^2 + 4u^2v^2 \\ &= u^4+u^2v^2+v^4-2u^3v+2u^2v^2-2uv^3+4u^2v^2 \\ &= u^4-2u^3v+7u^2v^2-2uv^3+v^4. \end{aligned}$$

Remark. It is not worth trying the variant

$$x(u,v) = u^2 - uv + v^2 = \frac{u^3+v^3}{u+v} \qquad \text{for } u \neq -v,$$

because the following expressions are very complicated. $\Diamond$

Then by a calculation,

$$\frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}, \qquad \frac{\partial f}{\partial y} = \frac{x}{x^2+y^2},$$

and

$$\frac{\partial x}{\partial u} = 2u - v, \quad \frac{\partial y}{\partial u} = 2v, \qquad \frac{\partial x}{\partial v} = 2v - u, \quad \frac{\partial y}{\partial v} = 2u,$$

so

$$\begin{aligned}\frac{\partial F}{\partial u} &= -\frac{y}{x^2+y^2} \cdot (2u-v) + \frac{x}{x^2+y^2} \cdot 2v = \frac{-2uv(2u-v)+(u^2-uv+v^2)\cdot 2v}{u^4-2u^3v+7u^2v^2-2uv^3+v^4} \\ &= \frac{2v(-2u^2+uv+u^2-uv+v^2)}{u^4-2u^3v+7u^2v^2-2uv^3+v^4} = \frac{2v(v^2-u^2)}{u^4-2u^3v+7u^2v^2-2uv^3+v^4},\end{aligned}$$

and

$$\begin{aligned}\frac{\partial F}{\partial v} &= -\frac{y}{x^2+y^2} \cdot (2v-u) + \frac{x}{x^2+y^2} \cdot 2u = \frac{-2uv(2v-u)+(u^2-uv+v^2)2u}{u^4-2u^3v+7u^2v^2-2uv^3+v^4} \\ &= \frac{2u(-2v^2+uv+u^2-uv+v^2)}{u^4-2u^3v+7u^2v^2-2uv^3+v^4} = \frac{2u(u^2-v^2)}{u^4-2u^3v+7u^2v^2-2uv^3+v^4}.\end{aligned}$$

Test. We get by insertion that

$$(11.1) \quad F(u,v) = \text{ Arctan}\left(\frac{2uv}{u^2-uv+v^2}\right) = F(v,u),$$

thus

$$\begin{aligned}\frac{\partial F}{\partial u} &= \frac{1}{1+\left\{\frac{2uv}{u^2-uv+v^2}\right\}^2} \cdot \left\{\frac{2v(u^2-uv+v^2)-2uv(2u-v)}{(u^2-uv+v^2)^2}\right\} \\ &= \frac{2v(u^2-uv+v^2-2u^2+uv)}{(u^2-uv+v^2)^2+4u^2v^2} = \frac{2v(v^2-u^2)}{u^4-2u^3v+7u^2v^2-2uv^3+v^4}.\end{aligned}$$

Due to the symmetry of (11.1) we obtain $\dfrac{\partial F}{\partial v}$ by interchanging u and v.
The results agree. $\Diamond$

Remark. One may wonder why there is given no attempt to reduce the denominator $u^4-2u^3v+7u^2v^2-2uv^3+v^4$ as a product of factors $u-av$ of first degree. The reason is that a then must satisfy the equation

$$a^4 - 2a^3 + 7a^2 - 2a + 1 = 0,$$

of fourth degree and with $\pm$ integers as coefficients. It can be proved that the only *possible* rational roots must be of the form $a = \pm\dfrac{1}{1} = \pm 1$, and it is easily seen that none of these possibilities satisfies the equation. The problem is therefore to solve an equation of fourth degree without any rational solutions, and such a procedure is not commonly known in Calculus courses. $\Diamond$

In MAPLE we declare

$$f := (x, y) \to \arctan\left(\frac{y}{x}\right)$$

$$(x, y) \to \arctan\left(\frac{y}{x}\right)$$

$$X := (u, v) \to \left(u^2 - u \cdot v + v^2, 2u \cdot v\right)$$

$$(u, v) \to \left(u^2 - uv + v^2, 2uv\right)$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}u}\,f(X(u,v))$$

$$\frac{\dfrac{2v}{u^2+v^2-uv}-\dfrac{4u^2v}{\left(u^2+v^2-uv\right)^2}}{1+\dfrac{4u^2v^2}{\left(u^2+v^2-uv\right)^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}v}\,f(X(u,v))$$

$$\frac{\dfrac{2u}{u^2+v^2-uv}-\dfrac{4uv^2}{\left(u^2+v^2-uv\right)^2}}{1+\dfrac{4u^2v^2}{\left(u^2+v^2-uv\right)^2}}$$

4) When $f(x,y) = \operatorname{Arctan}(x+y^2)$ and $(x,y) = \mathbf{X}(u,v) = \left(u, e^{u\sin v}\right)$, we get

$$\frac{\partial f}{\partial x} = \frac{1}{1+(x+y^2)^2} \qquad \text{and} \qquad \frac{\partial f}{\partial y} = \frac{2y}{1+(x+y^2)^2},$$

and

$$\frac{\partial x}{\partial u} = 1, \quad \frac{\partial y}{\partial u} = \sin v \cdot e^{u\sin v}, \qquad \frac{\partial x}{\partial v} = 0, \quad \frac{\partial y}{\partial v} = u\cos v \cdot e^{u\sin v},$$

hence

$$\begin{aligned}\frac{\partial F}{\partial u} &= \frac{1}{1+(x+y^2)^2}\cdot 1 + \frac{2y}{1+(x+y^2)^2}\cdot \sin v \cdot e^{u\sin v}\\ &= \frac{1}{1+(u+e^{2u\sin v})^2}\cdot\left(1+2\sin v\cdot e^{2u\sin v}\right),\end{aligned}$$

and

$$\frac{\partial F}{\partial v} = \frac{1}{1+(x+y^2)^2}\cdot 0 + \frac{2y}{1+(x+y^2)^2}\cdot e^{u\sin v}\cdot u\cos v = \frac{2u\cos v\cdot e^{2u\sin v}}{1+(u+e^{2u\sin v})^2}.$$

Test. We get by insertion,

$$F(u,v) = \operatorname{Arctan}(u+e^{2u\sin v}),$$

hence

$$\frac{\partial F}{\partial u} = \frac{1+2\sin v e^{2u\sin v}}{1+(u+e^{2u\sin v})^2} \quad \text{and} \quad \frac{\partial F}{\partial v} = \frac{2u\cos v\cdot e^{2u\sin v}}{1+(u+e^{2u\sin v})^2}.$$

The results agree. $\Diamond$

In MAPLE we declare

$$f := (x,y) \to \arctan\left(x+y^2\right)$$

$$(x,y) \to \arctan\left(x+y^2\right)$$

$$X := (u,v) \to \left(u, e^{u\cdot\sin(v)}\right)$$

$$(u,v) \to \left(u, e^{u\sin(v)}\right)$$

Then

$$\frac{\mathrm{d}}{dau}\, f(X(u,v))$$

$$\frac{1+2\left(e^{u\sin(v)}\right)^2 \sin(v)}{1+\left(u+\left(e^{u\sin(v)}\right)^2\right)^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}v}\, f(X(u,v))$$

$$\frac{2\left(e^{u\sin(v)}\right)^2 u\cos(v)}{1+\left(u+\left(e^{u\sin(v)}\right)^2\right)^2}$$

5) When $f(x,y) = x\cos y$ and $(x,y) = \mathbf{X}(u,v) = (\sqrt{1+u^2+v^2}, \text{Arcsin}\ u)$, it follows that the composite function is defined and of class C^∞ for $|u|<1$ and $v \in \mathbb{R}$. Then,

$$\frac{\partial f}{\partial x} = \cos y \qquad \text{and} \qquad \frac{\partial f}{\partial y} = -x\sin y,$$

as well as

$$\frac{\partial x}{\partial u} = \frac{u}{\sqrt{1+u^2+v^2}}, \qquad \frac{\partial y}{\partial u} = \frac{1}{\sqrt{1-u^2}},$$

$$\frac{\partial x}{\partial v} = \frac{v}{\sqrt{1+u^2+v^2}}, \qquad \frac{\partial y}{\partial v} = 0.$$

We get accordingly,

$$\begin{aligned}\frac{\partial F}{\partial u} &= \frac{\cos(\text{Arcsin}\ u)\cdot u}{\sqrt{1+u^2+v^2}} - \frac{\sqrt{1+u^2+v^2}\cdot\sin(\text{Arcsin}\ u)}{\sqrt{1-u^2}} = \frac{u\sqrt{1-u^2}}{\sqrt{1+u^2+v^2}} - \frac{u\sqrt{1+u^2+v^2}}{\sqrt{1-u^2}} \\ &= u\left\{\sqrt{\frac{1-u^2}{1+u^2+v^2}} - \sqrt{\frac{1+u^2+v^2}{1-u^2}}\right\}\end{aligned}$$

and

$$\frac{\partial F}{\partial v} = \frac{\cos(\text{Arcsin}\ u)\cdot v}{\sqrt{1+u^2+v^2}} + 0 = +v\sqrt{\frac{1-u^2}{1+u^2+v^2}}.$$

Test. We get by insertion,

$$F(u,v) = \sqrt{1+u^2+v^2}\cdot\cos(\text{Arcsin}\ u) = +\sqrt{1+u^2+v^2}\cdot\sqrt{1-u^2},$$

hence

$$\frac{\partial F}{\partial u} = \frac{u\sqrt{1-u^2}}{\sqrt{1+u^2+v^2}} - \frac{u\sqrt{1+u^2+v^2}}{\sqrt{1-u^2}} = u\left\{\sqrt{\frac{1-u^2}{1+u^2+v^2}} - \sqrt{\frac{1+u^2+v^2}{1-u^2}}\right\},$$

and

$$\frac{\partial F}{\partial v} = v\sqrt{\frac{1-u^2}{1+u^2+v^2}}.$$

The results agree. $\Diamond$

In MAPLE we declare

$$f := (x,y) \to x \cdot \cos(y)$$

$$(x,y) \to x\,\cos(y)$$

$$X := (u,v) \to \left(\sqrt{1+u^2+v^2}, \arcsin(u)\right)$$

$$(u,v) \to \left(\sqrt{1+u^2+v^2}, \arcsin(u)\right)$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}u}\, f(X(u,v))$$

$$\frac{\sqrt{-u^2+1}}{\sqrt{u^2+v^2+1}} - \frac{\sqrt{u^2+v^2+1}\,u}{\sqrt{-u^2+1}}$$

$$\frac{\mathrm{d}}{\mathrm{d}v}\, f(X(u,v))$$

$$\frac{\sqrt{-u^2+1}\,v}{\sqrt{u^2+v^2+1}}$$

6) When $f(x,y) = x\sinh y$ and $(x,y) = \mathbf{X}(u,v) = \left(u^3 v, \ln u + \ln v\right)$, $(u,v) \in \mathbb{R}_+^2$, then the composition of the functions is defined and of class C^∞. From

$$\frac{\partial f}{\partial x} = \sinh y, \qquad \frac{\partial f}{\partial y} = x\cosh y,$$

and

$$\frac{\partial x}{\partial u} = 3u^2 v, \quad \frac{\partial y}{\partial u} = \frac{1}{u}, \qquad \frac{\partial x}{\partial v} = u^3, \quad \frac{\partial y}{\partial v} = \frac{1}{v},$$

follows that

$$\frac{\partial F}{\partial u} = \sinh y \cdot 3u^2 v + x \cdot \cosh y \cdot \frac{1}{u},$$

and

$$\frac{\partial F}{\partial v} = \sinh y \cdot u^3 + x \cdot \cosh y \cdot \frac{1}{v}.$$

Since $y(u,v) = \ln u + \ln v$, we have

$$\sinh y = \frac{1}{2}\left(uv - \frac{1}{uv}\right) \quad \text{and} \quad \cosh y = \frac{1}{2}\left(uv + \frac{1}{uv}\right).$$

Then by insertion,

$$\begin{aligned}
\frac{\partial F}{\partial u} &= 3u^2 v \cdot \frac{1}{2}\left(uv - \frac{1}{uv}\right) + u^3 v \cdot \frac{1}{2}\left(uv + \frac{1}{uv}\right) \cdot \frac{1}{u} \\
&= \frac{3}{2}\,u^3 v^2 - \frac{3}{2}\,u + \frac{1}{2}\,u^3 v^2 + \frac{1}{2}\,u \\
&= 2u^3 v - u,
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial F}{\partial v} &= \frac{1}{2}\left(uv - \frac{1}{uv}\right) \cdot u^3 + u^3 v \cdot 12\left(uv + \frac{1}{uv}\right) \cdot 1v \\
&= \frac{1}{2}\,u^4 v - \frac{1}{2}\,\frac{u^2}{v} + \frac{1}{2}\,u^4 v + \frac{1}{2}\,\frac{u^2}{v} \\
&= u^4 v.
\end{aligned}$$

TEST. We get by insertion

$$F(u,v) = u^3 v \cdot 12 \left(uv - \frac{1}{uv} \right) = \frac{1}{2}\, u^4 v^2 - \frac{1}{2}\, u^2,$$

hence

$$\frac{\partial F}{\partial u} = 2u^3 v - u \quad \text{and} \quad \frac{\partial F}{\partial v} = u^4 v.$$

The results agree. $\Diamond$

In MAPLE we declare

$$f := (x,y) \rightarrow x \cdot \sinh(y)$$

$$(x,y) \rightarrow x \sinh(y)$$

$$X := (u,v) \rightarrow \big(u^3 \cdot v, \ln(u) + \ln(v) \big)$$

$$(u,v) \rightarrow \big(u^3 v, \ln(u) + \ln(v) \big)$$

Then

$$\frac{\mathrm{d}}{< dau}\, f(X(u,v))$$

$$3u^2 v \sinh(\ln(u) + \ln(v)) + u^2 v \cosh(\ln(u) + \ln(v))$$

$$\frac{\mathrm{d}}{\mathrm{d}v}\, f(X(u,v))$$

$$u^3 \sinh(\ln(u) + \ln(v)) + u^3 \cosh(\ln(u) + \ln(v))$$

Since sinh and cosh are built up by the exponential function, the latter two results can be reduced.

REMARK. All these examples are very simple because they should train the reader to use a new method. Unfortunately, in all the chosen examples the usual method is easier to apply; but there exist examples, like e.g. **Example 11.13**, where the chain rule is the most efficient one. However, in the previous two examples, **Example 11.8** and **Example 11.9** we must admit that the chain rule is more difficult to apply. $\Diamond$

Example 11.10 *It can be proved that the differential equation*

$$\frac{dw}{du} = w^2 + u^2, \qquad u \in \mathbb{R},$$

among its solutions has

$$w = X(u), \quad u \in \mathbb{R}, \qquad \text{where } X(0) = 1,$$

and

$$w = Y(u), \quad u \in \mathbb{R}, \qquad \text{where } Y(0) = 2.$$

Let

$$F(u) = f(X(u), Y(u)), \qquad f(x,y) = \ln(1 + xy^2).$$

Find the derivative $F'(0)$.

A The chain rule.

D Since the functions $X(u)$ and $Y(u)$ cannot be found explicitly by elementary methods, we shall try the chain rule instead.

Remark. The non-linear differential equation above is a so-called Ricatti equation. Such equations cannot be solved in general *unless one knows one solution.* It can be proved that the equation then can be completely solved. Therefore, one usually says that the Ricatti equation can only be solved by guessing. This is not true. There exist some special cases, in which the Ricatti equation can be completely solved without knowing a solution in advance. The considered equation is actually of this type, but since its solution lies far beyond what can be mentioned here, we shall not solve it. $\Diamond$

I First note that for $xy^2 > -1$,

$$\frac{\partial f}{\partial x} = \frac{y^2}{1+xy^2} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{2xy}{1+xy^2}.$$

Then

$$\frac{\mathrm{d}X}{\mathrm{d}u} = X(u)^2 + u^2 \quad \text{and} \quad \frac{\mathrm{d}Y}{\mathrm{d}u} = Y(u)^2 + u^2,$$

so when we apply the chain rule we get

$$\begin{aligned} F'(u) &= \frac{\partial f}{\partial x}\frac{\mathrm{d}X}{\mathrm{d}u} + \frac{\partial f}{\partial y}\frac{\mathrm{d}Y}{\mathrm{d}u} = \frac{y^2}{1+xy^2}\left\{X(u)^2+u^2\right\} + \frac{2xy}{1+xy^2}\left\{Y(u)^2+u^2\right\} \\ &= \frac{Y(u)^2}{1+X(u)Y(u)^2}\left\{X(u)^2+u^2\right\} + \frac{2X(u)Y(u)}{1+X(u)Y(u)^2}\left\{Y(u)^2+u^2\right\}. \end{aligned}$$

Now $X(0) = 1$ and $Y(0) = 2$, so $X(u)Y(u)^2 > -1$ in an interval around $u = 0$, and $F'(0)$ is defined. We get the value by inserting the values of the calculations above.

$$F'(0) = \frac{4}{1+1\cdot 4}\{1+0\} + \frac{2\cdot 1\cdot 2}{1+1\cdot 4}\{4+0\} = \frac{4}{5} + \frac{4}{5}\cdot 4 = 4.$$

Example 11.11 *Let u and w denote two functions in two variables. We assume that they fulfil the differential equations*

$$a\,\frac{\partial w}{\partial t} = -\frac{\partial u}{\partial z}, \qquad b\,\frac{\partial u}{\partial t} = -\frac{\partial w}{\partial z}, \qquad (z,t) \in \mathbb{R}^2.$$

We also consider two C^1-functions F, $G : \mathbb{R} \to \mathbb{R}$, and we put

$$u(z,t) = F(x+ct) + G(z-ct),$$

$$w(z,t) = \gamma\{F(z+ct) - G(z-ct)\}.$$

Prove that one can choose the constants c and γ such that the differential equations are satisfied.

A System of partial differential equations.

D Insert the given functions and find c and γ.

I By partial differentiation we get

$$\frac{\partial w}{\partial t} = \gamma\{cF'(z+ct) + cG'(z-ct)\} = c\gamma\{F'(z+ct) + G'(z-ct)\},$$

$$\frac{\partial w}{\partial z} = \gamma\{F'(z+ct) - G'(z-ct)\},$$

and

$$\frac{\partial u}{\partial z} = F'(z+ct) + G'(x-ct),$$

$$\frac{\partial u}{\partial t} = c\,F'(z+ct) - c\,G'(z-ct) = c\{F'(z+ct) - G'(z-ct)\}.$$

It follows from the equation $a\,\dfrac{\partial w}{\partial t} = -\dfrac{\partial u}{\partial z}$ that

$$ac\gamma\{F'(z+ct) + G'(z-ct)\} = -\{F'(z+ct) + G'(z-ct)\}.$$

Since F and G are arbitrary, we get $ac\gamma = -1$.

Then it follows from the equation $b\,\dfrac{\partial u}{\partial t} = -\dfrac{\partial w}{\partial z}$ that

$$bc\{F'(z+ct) - G'(z-ct)\} = -\gamma\{F'(z+ct) - G'(z-ct)\}.$$

Since F and G are arbitrary, we get $bc = -\gamma$.

Then solve the system of two equations

$$ac\gamma = -1 \qquad \text{and} \qquad bc = -\gamma$$

in c and γ for given a, $b > 0$ by eliminating γ, i.e. $-abc^2 = -1$, and then accordingly

$$c = +\frac{1}{\sqrt{ab}},$$

where we have chosen the sign $+$, such that $c > 0$. If we instead choose the sign $-$, we interchange F and G.

By the choices above of c we get $\gamma = -bc = -\sqrt{\dfrac{b}{a}}$, thus

$$c = \frac{1}{\sqrt{ab}} \qquad \text{and} \qquad \gamma = -\sqrt{\frac{b}{a}}.$$

The system has the solutions

$$\begin{cases} u(z,t) = F\left(z + \dfrac{t}{\sqrt{ab}}\right) + G\left(z - \dfrac{t}{\sqrt{ab}}\right), \\ w(z,t) = -\sqrt{\dfrac{b}{a}}\left\{F\left(z + \dfrac{t}{\sqrt{ab}}\right) - G\left(z - \dfrac{t}{\sqrt{ab}}\right)\right\}. \end{cases}$$

These solutions are valid for any C^1-functions F, $G : \mathbb{R} \to \mathbb{R}$.

Remark 1. If F and G are of class C^2, then the functions are solutions of the wave equation. $\Diamond$

Remark 2. The reason why the example is placed here is that one latently applies the chain rule in a very simple version when we calculate the dd derivative. However, this cannot be clearly seen due to all the other messages in the example. $\Diamond$

Example 11.12 *Let $P_n(x,y,z)$ be an homogeneous polynomial of degree n in three variables. Consider P_n as a function of the spherical coordinates (r,θ,φ). Prove by using the result of* **Example 11.2** *that*

$$r\,\frac{\partial P_n}{\partial r} = n\,P_n.$$

A Homogeneous polynomial in $\mathbb{R}^3$.

D Apply **Example 11.2**.

I We have according to **Example 11.2**,

$$x\,\frac{\partial P_n}{\partial x} + y\,\frac{\partial P_n}{\partial y} + z\,\frac{\partial P_n}{\partial z} = n\,P_n.$$

Then note that

$$r\,\frac{\partial x}{\partial r} = r\,\frac{\partial}{\partial r}\{r\sin\theta\cos\varphi\} = r\sin\theta\cos\varphi = x \quad \text{for } r>0,$$

and analogously for the other rectangular coordinate functions, so

$$r\,\frac{\partial x}{\partial r} = x, \quad r\,\frac{\partial y}{\partial r} = y, \quad r\,\frac{\partial z}{\partial r} = z, \qquad \text{for } r>0.$$

Then we get by the chain rule

$$r\,\frac{\partial P_n}{\partial r} = r\,\frac{\partial x}{\partial r}\,\frac{\partial P_n}{\partial x} + r\,\frac{\partial y}{\partial r}\,\frac{\partial P_n}{\partial y} + r\,\frac{\partial z}{\partial r}\,\frac{\partial P_n}{\partial P_n}\partial z = x\,\frac{\partial P_n}{\partial x} + y\,\frac{\partial P_n}{\partial y} + z\,\frac{\partial P_n}{\partial z} = n\,P_n.$$

Example 11.13 *Given the functions*

$$X(u) = \ln(2+u), \quad u>-2, \quad \textit{and} \quad f(x,y) = y^3e^x, \quad (x,u)\in\mathbb{R}^2,$$

and a C^1-function $Y(x)$, $x\in A$, of which we only know that

$$0\in A \qquad Y(0)=\pi, \qquad Y'(0)=2.$$

Considering the composite function $F(u) = f(X(u),Y(u))$ we shall find the derivative $F'(0)$.

A Determination of a derivative, where we apparently are missing some information.

D Analyze the chain rule.

I We get by the chain rule

$$\begin{aligned}
F'(u) &= \frac{\partial f}{\partial x}\,\frac{dX}{du} + \frac{\partial f}{\partial y}\,\frac{dY}{du} = y^3e^x\cdot\frac{1}{2+u} + 3y^2e^x\cdot Y'(u)\\
&= Y(u)^3\cdot(2+u)\cdot\frac{1}{2+u} + 3Y(u)^2\cdot(2+u)\cdot Y'(u)\\
&= Y(u)^3 + 3Y(u)^2\cdot Y(u)\cdot(2+u).
\end{aligned}$$

Putting $u=0$ we get

$$F'(0) = Y(0)^3 + 3Y(0)^2\cdot Y'(0)\cdot 2 = \pi^3 + 6\pi^2\cdot 2 = \pi^3 + 12\pi^2 = \pi^2(12+\pi).$$

Example 11.14 . *Find the derivative of the function*

$$F(u) = \text{Arcsin}\left(\frac{\sin u \cos u}{\sqrt{2+\cos^2 u}}\right), \qquad u \in \mathbb{R}$$

by putting $F(u) = f(X(u), Y(u))$, *where*

$$f(x,y) = \text{Arcsin}\left(\frac{x}{\sqrt{y}}\right).$$

A The chain rule.

D Identify $X(u)$ and $Y(u)$, and use the chain rule.

I We shall clearly choose

$$x = X(u) = \sin u \cdot \cos u, \quad \text{and} \quad y = Y(u) = 2 + \cos^2 u.$$

First calculate

$$\frac{\mathrm{d}x}{\mathrm{d}u} = \cos^2 u - \sin^2 u = \cos 2y,$$

$$\frac{\mathrm{d}y}{\mathrm{d}u} = -2\sin u \cdot \cos u = -\sin 2u,$$

together with

$$\frac{\partial f}{\partial x} = \frac{1}{\sqrt{1 - \dfrac{x^2}{y}}} \cdot \frac{1}{\sqrt{y}} = \frac{1}{\sqrt{y - x^2}} = \frac{1}{\sqrt{2 + \cos^2 u - \sin^2 u \cdot \cos^2 u}} = \frac{1}{\sqrt{2+\cos^2 u}},$$

and

$$\frac{\partial f}{\partial y} = \frac{1}{\sqrt{1 - \dfrac{x^2}{y}}} \cdot \left(-\frac{1}{2}\,\frac{x}{y\sqrt{y}}\right) = -\frac{x}{2y} \cdot \frac{1}{\sqrt{y-x^2}} = -\frac{\sin u \cdot \cos u}{4 + 2\cos^2 u} \cdot \frac{1}{\sqrt{2+\cos^2 u}}.$$

Then by the chain rule

$$\begin{aligned} F'(u) &= \frac{\partial f}{\partial x} \cdot \frac{\mathrm{d}x}{\mathrm{d}u} + \frac{\partial f}{\partial y} \cdot \frac{\mathrm{d}y}{\mathrm{d}u} \\ &= \frac{\cos^2 u - \sin^2 u}{\sqrt{2+\cos^2 u}} - \frac{\sin u \cos u}{4+2\cos^2 u} \cdot \frac{1}{\sqrt{2+\cos^2 u}}\,(-2\sin u \cos u) \\ &= \frac{1}{\sqrt{2+\cos^2 u}}\left\{\cos^2 u - \sin^2 u + \frac{2\sin^2 u \cos^2 u}{4+2\cos^2 u}\right\} \\ &= \frac{1}{\sqrt{2+\cos^2 u}}\left\{\cos^2 u - \frac{2\sin^2 u}{2+\cos^2 u}\right\}. \end{aligned}$$

C By the traditional calculation we get

$$\begin{aligned}
F'(u) &= \frac{1}{\sqrt{1-\dfrac{\sin^2 u\cos^2 u}{2+\cos^2 u}}}\left\{\frac{\cos^2 u-\sin^2 u}{\sqrt{2+\cos^2 u}}-\frac{1}{2}\cdot\frac{\sin u\cos u\cdot(-2\sin u\cos^2 u)}{(2+\cos^2 u)\sqrt{2+\cos^2 u}}\right\}\\
&= \frac{1}{\sqrt{2+\cos^2 u-\sin^2 u\cos^2 u}}\left\{\cos^2 u-\sin^2 u+\frac{\sin^2 u\cos^2 u}{2+\cos^2 u}\right\}\\
&= \frac{1}{\sqrt{2+\cos^2}}\left\{\cos^2 u+\sin^2 u\cdot\frac{-2-\cos^2 u+\cos^2 u}{2+\cos^2 u}\right\}\\
&= \frac{1}{\sqrt{2+\cos^2 u}}\left\{\cos^2 u-\frac{2\sin^2 u}{2+\cos^2 u}\right\}.
\end{aligned}$$

The two results agree.

In MAPLE we declare

$$f := (x,y)\to \arcsin\left(\frac{x}{\sqrt{y}}\right)$$

$$(x,y)\to \arcsin\left(\frac{x}{\sqrt{y}}\right)$$

$$X := u\to \left(\sin(u)\cdot\cos(u), 2+\cos(u)^2\right)$$

$$u\to \left(\sin(u)\cdot\cos(u), 2+\cos(u)^2\right)$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}u}f(X(u))$$

$$\frac{\dfrac{\cos(u)^2}{\sqrt{2+\cos(u)^2}}-\dfrac{\sin(u)^2}{\sqrt{2+\cos(u)^2}}+\dfrac{\sin(u)^2\cos(u)^2}{(2+\cos(u)^2)^{3/2}}}{\sqrt{1-\dfrac{\sin(u)^2\cos(u)^2}{2+\cos(u)^2}}}$$

This expression can be reduced.

11.3 Directional derivative

Example 11.15 *Find in each of the following cases the directional derivative of the given function* $f : \mathbb{R}^3 \to \mathbb{R}$ *in the point given by the index 0 in the direction of the vector* $\mathbf{v}$.

1) $f(x,y,z) = x + 2xy - 3y^2$, $(x_0,y_0,z_0) = (1,2,1)$, $\mathbf{v} = (3,4,0)$.

2) $f(x,y,z) = ze^x\cos(\pi y)$, $(x_0,y_0,z_0) = (0,-1,1)$, $\mathbf{v} = (-1,2,1)$.

3) $f(x,y,z) = x^2 + 2y^2 + 3z^2$, $(x_0,y_0,z_0) = (1,1,0)$, $\mathbf{v} = (1,-1,2)$.

4) $f(x,y,z) = xy + yz + xz$, $(x_0,y_0,z_0) = (1,2,3)$, $\mathbf{v} = (1,1,1)$.

A Directional derivative.

D Insert into the formula

$$f'\left(\mathbf{x};\frac{\mathbf{v}}{|\mathbf{v}|}\right) = \frac{\mathbf{v}}{|\mathbf{v}|}\cdot \nabla f(\mathbf{x}),$$

where we *must* remember to norm $\mathbf{v}$.

I 1) Here,

$$\nabla f(x,y) = (1+2y, 2x-6y) \qquad \text{and} \qquad |\mathbf{v}| = \sqrt{3^2+4^2} = 5,$$

so

$$\begin{aligned} f'\left((1,2);\left(\frac{3}{5},\frac{4}{5}\right)\right) &= \frac{1}{5}\,(3,4)\cdot(1+4, 2-12) = \frac{1}{5}\,(3,4)\cdot(5,-10) \\ &= (3,4)\cdot(1,-2) = -5. \end{aligned}$$

2) Here,

$$\nabla f(x,y,z) = \left(ze^x\cos(\pi y), -\pi ze^x\sin(\pi y), e^x\cos(\pi z)\right),$$

and $|\mathbf{v}| = \sqrt{1+4+1} = \sqrt{6}$, so

$$\begin{aligned} f'\left((0,-1,1); \frac{1}{\sqrt{6}}(-1,2,1)\right) &= \frac{1}{\sqrt{6}}(-1,2,1)\cdot(1\cdot e^0\cdot(-1),0,-1) \\ &= \frac{1}{\sqrt{6}}\{(-1)^2+0-1\} = 0. \end{aligned}$$

3) Here

$$\nabla f(x,y,z) = (2x,4y,6z) \qquad \text{and} \qquad |\mathbf{v}| = \sqrt{6},$$

so

$$f'\left((1,1,0); \frac{1}{\sqrt{6}}(1,-1,2)\right) = \frac{1}{\sqrt{6}}(1,-1,2)\cdot(2,4,0) = \frac{1}{\sqrt{6}}(2-4) = -\frac{2}{\sqrt{6}} = -\sqrt{\frac{2}{3}}.$$

4) Here

$$\nabla f(x,y,z) = (y+z, x+z, x+y) \qquad \text{and} \qquad |\mathbf{v}| = \sqrt{3},$$

so

$$f'\left((1,2,3); \frac{1}{\sqrt{3}}(1,1,1)\right) = \frac{1}{\sqrt{3}}\cdot 2[x+y+z]_{(x,y,z)=(1,2,3)} = \frac{12}{\sqrt{3}} = 4\sqrt{3}.$$

Example 11.16 *Find in each of the following cases the directional derivative of the function f at the point given by the index 0 in the direction of the point given by the index 1.*

1) $f(x,y,z) = xyz + \dfrac{x}{y} + \dfrac{y}{z} + \dfrac{z}{x}$ *defined for* $xyz \neq 0$ *from* $(x_0,y_0,z_0) = (1,-1,1)$ *to* $(x_1,y_1,z_1) = (3,1,2)$.

2) $f(x,y,z) = 2x^3y - 3y^2z$ *defined in* $\mathbb{R}^3$ *from* $(x_0,y_0,z_0) = (1,2,-1)$ *to* $(x_1,y_1,z_1) = (3,-1,5)$.

3) $f(x,y,z) = x\ln(1+e^{yz})$ *defined in* $\mathbb{R}^3$ *from* $(x_0,y_0,z_0) = (1,1,0)$ *to* $(x_1,y_1,z_1) = (0,0,-1)$.

A Directional derivative.

D Calculate $\nabla f(x_0,y_0,z_0)$ and find the unit vector $\mathbf{e}$.

I 1) Here

$$\nabla f = \left(yz + \frac{1}{y} - \frac{z}{x^2}, xz - \frac{x}{y^2} + \frac{1}{z}, xy - \frac{y}{z^2} + \frac{1}{x}\right),$$

so

$$\nabla f(1,-1,1) = (-1-1-1, 1-1+1, -1+1+1) = (-3,1,1).$$

Furthermore,

$$\mathbf{v} = (3,1,2)_{(}1,-1,1) = (2,2,1), \qquad \text{where } |\mathbf{v}| = \sqrt{2^2+2^2+1^2} = 3,$$

so

$$f'\left((1,-1,1);\frac{1}{3}(2,2,1)\right) = \frac{1}{3}(2,2,1)\cdot(-3,1,1) = \frac{1}{3}\{-6+2+1\} = -1.$$

2) Here

$$\nabla f = \left(6x^2y, 2x^3-6yz, -3y^2\right),$$

so

$$\nabla f(1,2,-1) = (6\cdot 1^2\cdot 2, 2-6\cdot 2\cdot(-1), -3\cdot 2^2) = (12,14,-12).$$

Furthermore,

$$\mathbf{v} = (3,-1,5)-(1,2,-1) = (2,-3,6), \qquad \text{where } |\mathbf{v}| = \sqrt{2^2+3^2+6^2} = 7,$$

so

$$f'\left((1,2,-1);\frac{1}{7}(2,-3,6)\right) = \frac{1}{7}(2,-3,6)\cdot(12,14,-12) = \frac{1}{7}\{-48-42\} = -\frac{90}{7}.$$

3) Here

$$\nabla f = \left(\ln(1+e^{yz}), \frac{xze^{yz}}{1+e^{yz}}, \frac{xye^{yz}}{1+e^{yz}}\right),$$

so

$$\nabla f(1,1,0) = \left(\ln 2, 0, \frac{1}{2}\right).$$

Furthermore,

$$\mathbf{v} = (0,0,-1)-(1,1,0) = (-1,-1,-1), \qquad \text{where } |\mathbf{v}| = \sqrt{3},$$

so

$$f'\left((1,1,0);\frac{1}{\sqrt{3}}(-1,-1,-1)\right) = -\frac{1}{\sqrt{3}}\left(\frac{1}{2}+\ln 2\right).$$

Example 11.17 *Given the function*

$$f(x,y,z) = \text{ Arctan}\left(x+\frac{1}{y}\right)+\sinh\left(z^2-1\right), \qquad y<0.$$

Find the direction in which the directional derivation of f at the point $(1,-1,1)$ is smallest, and indicate this minimum.

A Directional derivative.

D First calculate $\triangledown f(1,-1,1)$. Then conclude that the direction must be

$$\mathbf{e} = -\frac{\triangledown f}{\| \triangledown f\|}.$$

I We get by differentiation

$$\triangledown f = \left(\frac{1}{1+\left(x+\dfrac{1}{y}\right)^2}, \frac{-\dfrac{1}{y^2}}{1+\left(x+\dfrac{1}{y}\right)^2}, 2z\cosh(z^2-1)\right),$$

hence

$$\triangledown f(1,-1,1) = (1,-1,2) \qquad \text{where} \qquad \| \triangledown f(1,-1,1)\| = \sqrt{6}.$$

Using the direction

$$\mathbf{e} = -\frac{1}{\sqrt{6}}(1,-1,2) = -\frac{\triangledown f(1,-1,1)}{\| \triangledown f(1,-1,1)\|}$$

we get the directional derivative

$$f'((1,-1,1);\mathbf{e}) = \mathbf{e}\cdot \triangledown f(1,-1,1) = -\frac{\| \triangledown f(1,-1,1)\|^2}{\| \triangledown f(1,-1,1)\|} = -\| \triangledown f(1,-1,1)\| = -\sqrt{6}.$$

Example 11.18 *Let f be a C^1-function of two variables. We sketch from a fixed point (x_0, y_0) in any direction the corresponding directional derivative of f at the point (x_0, y_0). Prove that we by this procedure obtain two circles which are tangent to each other at the point (x_0, y_0), and find the centres of these circles.*

A "Theoretical" example concerning the directional derivative.

D Without loss of generality we may assume that $(x_0, y_0) = (0, 0)$. Calculate

$$f'(\mathbf{0}; \mathbf{e})\mathbf{e} \qquad \text{or} \qquad |f'(\mathbf{0}; \mathbf{e}|\mathbf{e}$$

for every unit vector $\mathbf{e}$.

I We can obviously assume that $(x_0, y_0) = (0, 0)$.

Then let

$$\nabla f(\mathbf{0}) = \left(\frac{\partial f}{\partial x}(\mathbf{0}), \frac{\partial f}{\partial y}(\mathbf{0}) \right) := (a, b).$$

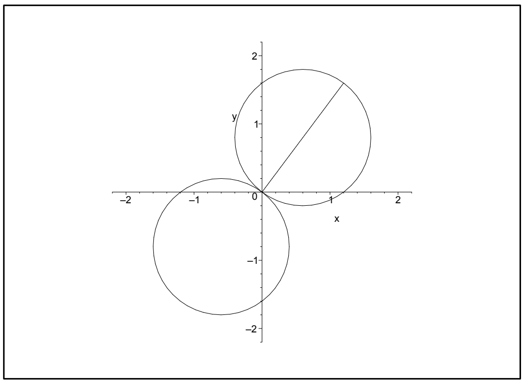

Figure 11.2: The sketched diameter is $\nabla f(0, 0)$.

Any unit vector can be written in the form

$$\mathbf{e}(\varphi) = (\cos\varphi, \sin\varphi), \qquad \varphi \in [0, 2\pi[,$$

so

$$(11.2)\quad f'(\mathbf{0}; \mathbf{e}(\varphi))\mathbf{e}(\varphi) = (a\cos\varphi + b\sin\varphi)(\cos\varphi, \sin\varphi) = (x(\varphi), y(\varphi)),$$

where

$$x(\varphi) = a\cos^2\varphi + b\sin\varphi\cos\varphi = \frac{1}{2}\{a\cos 2\varphi + b\sin 2\varphi\} + \frac{a}{2}$$

and

$$y(\varphi) = a\sin\varphi\cos\varphi + b\sin^2\varphi = \frac{1}{2}\{a\sin 2\varphi - b\cos 2\varphi\} + \frac{b}{2}.$$

Hence

$$\left\{x(\varphi) - \frac{a}{2}\right\}^2 + \left\{y(\varphi) - \frac{b}{2}\right\}^2 = \frac{1}{4}\{a^2 + b^2\},$$

and the centre lies at the point

$$\left(\frac{a}{2},\frac{b}{2}\right)=\frac{1}{2}\,\bigtriangledown f(\mathbf{0}),$$

and the radius is $\frac{1}{2}|\bigtriangledown f(\mathbf{0})|$.

We have above calculated the signed directional derivative. If we instead interpret(11.2) as

$$|f'(\mathbf{0};\mathbf{e}(\varphi))|\mathbf{e}(\varphi),$$

then we obtain the cirle which is the mirror image in $\mathbf{0}$.

Example 11.19 *Find the directional derivative of the function*

$$f(x,y,z)=y\sqrt{1+x^2z^2},\qquad (x,y,z)\in\mathbb{R}^3,$$

at the point $(\sqrt{2},1,2)$ *in the direction towards the point* $(\sqrt{2},2,2+\sqrt{3})$.

A Directional derivative.

D Find the unit vector in the direction and apply the formula of the directional derivative.

I The direction is

$$\mathbf{v}=(\sqrt{2},2,2+\sqrt{3})-(\sqrt{2},1,2)=(0,1,\sqrt{3})=2\left(0,\frac{1}{2},\frac{\sqrt{3}}{2}\right),$$

hence $\|\mathbf{v}\|=2$, and $\mathbf{e}=\left(0,\frac{1}{2},\frac{\sqrt{3}}{2}\right)$. Then the directional derivative is

$$\begin{aligned}
&f'\left((\sqrt{2},1,2)\,;\,\left(0,\frac{1}{2},\frac{\sqrt{3}}{2}\right)\right)\\
&\qquad=0\cdot\frac{\partial f}{\partial x}(\sqrt{2},1,2)+\frac{1}{2}\,\frac{\partial f}{\partial y}(\sqrt{2},1,2)+\frac{\sqrt{3}}{2}\,\frac{\partial f}{\partial z}(\sqrt{2},1,2)\\
&\qquad=\frac{1}{2}\left[\sqrt{1+x^2z^2}\right]_{(\sqrt{2},1,2)}+\frac{\sqrt{3}}{2}\left[\frac{x^2yz}{\sqrt{1+x^2z^2}}\right]_{(\sqrt{2},1,2)}\\
&\qquad=\frac{1}{2}\sqrt{1+2\cdot 4}+\frac{\sqrt{3}}{2}\cdot\frac{2\cdot 1\cdot 2}{\sqrt{1+2\cdot 4}}=\frac{3}{2}+\frac{2\sqrt{3}}{3}=\frac{9+4\sqrt{3}}{6}.
\end{aligned}$$

Example 11.20 *Given the function*

$$f(x,y,z) = 2x + 2y^2z + xy^2z, \qquad (x,y,z) \in \mathbb{R}^3.$$

Find $(\bigtriangledown f)(1,-1,2)$*, and then the unit vector* $\mathbf{e}$*, for which*

$$f'((1,-1,2);\mathbf{e})$$

is as large as possible.

A Gradient and directional derivative.

D Just calculate.

I The gradient is

$$\bigtriangledown f = (2 + y^2z, 4yz + 2xyz, 2y^2 + xy^2),$$

hence

$$(\bigtriangledown f)(1,-1,2) = (2 + (-1)^2 \cdot 2, -4 \cdot 2 + 2 \cdot 1(-1) \cdot 2, 2 + 1) = (4,-12,3)$$

where the maximum is

$$\| \bigtriangledown f(1,-1,2)\| = \sqrt{16+144+9} = \sqrt{169} = 13 = f'((1,-1,2);\mathbf{e})$$

obtained for

$$\mathbf{e} = \left(\frac{4}{13}, -\frac{12}{13}, \frac{3}{13}\right).$$

11.4 Partial derivatives of higher order

Example 11.21 *Find in each of the following cases the first and the second differential for the function* f *at the point which is indicated with the index 0.*

1) $f(x,y) = x\exp(y^2-1)$ *in* $\mathbb{R}^2$ *from* $(x_0,y_0) = (1,1)$.

2) $f(x,y) = \operatorname{Arctan}(x+y) + \ln(1+x)$ *for* $x > -1$ *from* $(x_0,y_0) = (0,1)$.

3) $f(x,y) = (x^2+y^2)\ln(x^2+y^2)$ *in* $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ *from* $(x_0,y_0) = (0,1)$.

4) $f(x,y) = \sqrt{x^2+y^2}$ *i* $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ *from* $(x_0,y_0) = (3,4)$.

A First and second differential.

D First calculate the partial derivatives.

I It is obvious that $f(x,y)$ is of class C^∞ in the domain in all four cases.

1) The partial derivatives are here

$$\frac{\partial f}{\partial x} = \exp(y^2-1), \qquad \frac{\partial f}{\partial y} = 2xy\exp(y^2-1),$$

$$\frac{\partial^2 f}{\partial x^2} = 0, \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 2y\exp(y^2-1),$$

$$\frac{\partial^2 f}{\partial y^2} = 2x\exp(y^2-1) + 4xy^2\exp(y^2-1) = 2x(1+2y^2)\exp(y^2-1),$$

so

$$\mathrm{d}f((1,1),\mathbf{h}) = \bigtriangledown f(1,1)\cdot\mathbf{h} = (1,2)\cdot(h_x,h_y) = h_x + 2h_y = \text{“}\mathrm{d}x + 2\,\mathrm{d}y\text{”}$$

and

$$\begin{aligned} \mathrm{d}^2 f((1,1);\mathbf{h}) &= \frac{\partial^2 f}{\partial x^2}(1,1)\,h_x^2 + 2\,\frac{\partial^2 f}{\partial x \partial y}(1,1)\,h_x h_y + \frac{\partial^2 f}{\partial y^2}(1,1)\,h_y^2 \\ &= 0\cdot h_x^2 + 2\cdot 2h_x h_y + 2(1+2)h_y^2 \\ &= 4h_x h_y + 6h_y^2 = \text{“}4\,\mathrm{d}x\,\mathrm{d}y + 6(\,\mathrm{d}y)^2\text{”}. \end{aligned}$$

The differentiations are easy in MAPLE,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\cdot e^{y^2-1}\right)$$

$$e^{y^2-1}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(x\cdot e^{y^2-1}\right)$$

$$2xye^{y^2-1}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\,\frac{\mathrm{d}}{\mathrm{d}x}\left(x\cdot e^{y^2-1}\right)$$

$$0$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\,\frac{\mathrm{d}}{\mathrm{d}y}\left(x\cdot e^{y^2-1}\right)$$

$$2ye^{y^2-1}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\,\frac{\mathrm{d}}{\mathrm{d}y}\left(x\cdot e^{y^2-1}\right)$$

$$2xe^{y^2-1}+4x^2e^{y^2-1}$$

2) Here

$$\frac{\partial f}{\partial x}=\frac{1}{1+(x+y)^2}+\frac{1}{1+x}, \qquad \frac{\partial f}{\partial x}(0,1)=\frac{1}{2}+1=\frac{3}{2},$$

$$\frac{\partial f}{\partial y}=\frac{1}{1+(x+y)^2}, \qquad \frac{\partial f}{\partial y}(0,1)=\frac{1}{2},$$

$$\frac{\partial^2 f}{\partial x^2}=-\frac{2(x+y)}{\{1+(x+y)^2\}^2}-\frac{1}{(1+x)^2}, \qquad \frac{\partial^2 f}{\partial x^2}(0,1)=-\frac{2}{2^2}-1=-\frac{3}{2},$$

$$\frac{\partial^2 f}{\partial x}\partial y=-\frac{2(x+y)}{\{1+(x+y)^2\}^2}, \qquad \frac{\partial^2 f}{\partial x\partial y}(0,1)=-\frac{1}{2},$$

$$\frac{\partial^2 f}{\partial y^2}=-\frac{2(x+y)}{\{1+(x+y)^2\}^2}, \qquad \frac{\partial^2 f}{\partial y^2}(0,1)=-\frac{1}{2}.$$

Then by insertion,

$$\mathrm{d}f((0,1);\mathbf{h})=\frac{3}{2}\,h_x+\frac{1}{2}\,h_y=\text{“}\frac{3}{2}\,\mathrm{d}y+\frac{1}{2}\,\mathrm{d}y\text{”},$$

and

$$\mathrm{d}^2 f((0,1);\mathbf{h}) = -\frac{3}{2}\,h_x^2 - h_x h_y - \frac{1}{2}h_y^2 = \text{“}-\frac{3}{2}(\,\mathrm{d}x)^2 - \,\mathrm{d}x\,\mathrm{d}y - \frac{1}{2}(\,\mathrm{d}y)^2\text{”}.$$

The differentiations are easy in MAPLE,

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arctan(x+y)+\ln(1+x))$$

$$\frac{1}{1+(x+y)^2}+\frac{1}{1+x}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}(\arctan(x+y)+\ln(1+x))$$

$$\frac{1}{1+(x+y)^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\,\frac{\mathrm{d}}{\mathrm{d}x}(\arctan(x+y)+\ln(1+x))$$

$$-\frac{2x+2y}{\left(1+(x+y)^2\right)^2}-\frac{1}{(1+x)^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\,\frac{\mathrm{d}}{\mathrm{d}y}(\arctan(x+y)+\ln(1+x))$$

$$-\frac{2x+2y}{\left(1+(x+y)^2\right)^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\,\frac{\mathrm{d}}{\mathrm{d}y}(\arctan(x+y)+\ln(1+x))$$

$$-\frac{2x+2y}{\left(1+(x+y)^2\right)^2}$$

3) Here

$$\frac{\partial f}{\partial x} = 2x\ln(x^2+y^2)+2x, \qquad \frac{\partial f}{\partial x}(0,1)=0,$$

$$\frac{\partial f}{\partial y} = 2y\ln(x^2+y^2)+2y, \qquad \frac{\partial f}{\partial y}(0,1)=2,$$

$$\frac{\partial^2 f}{\partial x^2} = 2\ln(x^2+y^2)+\frac{4x^2}{x^2+y^2}+2; \qquad \frac{\partial^2 f}{\partial x^2}(0,1)=2$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{4xy}{x^2+y^2}, \qquad \frac{\partial^2 f}{\partial x\partial y}(0,1)=0,$$

$$\frac{\partial^2 f}{\partial y^2} = 2\ln(x^2+y^2)+\frac{4y^2}{x^2+y^2}+2; \qquad \frac{\partial^2 f}{\partial y^2}(0,1)=6.$$

We see that

$$\mathrm{d}f((0,1);\mathbf{h}) = 2h_y = \text{ “}2\,\mathrm{d}y\text{”}$$

and

$$\mathrm{d}^2 f((0,1);\mathbf{h}) = 2h_x^2 + 6h_y^2 = \text{ “}2(\,\mathrm{d}x)^2 + 6(\,\mathrm{d}y)\text{”}.$$

The differentiations are easy in MAPLE,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(x^2+y^2\right)\cdot\ln\left(x^2+y^2\right)\right)$$

$$2x\ln\left(x^2+y^2\right)+2x$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\left(x^2+y^2\right)\cdot\ln\left(x^2+y^2\right)\right)$$

$$2y\ln\left(x^2+y^2\right)+2y$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\,\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(x^2+y^2\right)\cdot\ln\left(x^2+y^2\right)\right)$$

$$2\ln\left(x^2+y^2\right)+\frac{4x^2}{x^2+y^2}+2$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\,\frac{\mathrm{d}}{\mathrm{d}y}\left(\left(x^2+y^2\right)\cdot\ln\left(x^2+y^2\right)\right)$$

$$\frac{4yx}{x^2+y^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\,\frac{\mathrm{d}}{\mathrm{d}y}\left(\left(x^2+y^2\right)\cdot\ln\left(x^2+y^2\right)\right)$$

$$2\ln\left(x^2+y^2\right)+\frac{4y^2}{x^2+y^2}+2$$

4) Here

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}, \qquad \frac{\partial f}{\partial x}(3,4) = \frac{3}{5},$$

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}, \qquad \frac{\partial f}{\partial y}(3,4) = \frac{4}{5},$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{\sqrt{x^2+y^2}} - \frac{x^2}{(\sqrt{x^2+y^2})^3} = \frac{y^2}{(\sqrt{x^2+y^2})^3}, \qquad \frac{\partial^2 f}{\partial x^2}(3,4) = \frac{16}{125},$$

$$\frac{\partial^2 f}{\partial x\partial y} = -\frac{xy}{(\sqrt{x^2+y^2})^3}, \qquad \frac{\partial^2 f}{\partial x\partial y}(3,4) = -\frac{12}{125},$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{x^2}{(\sqrt{x^2+y^2})^3}, \qquad \frac{\partial^2 f}{\partial y^2}(3,4) = \frac{9}{125},$$

hence

$$\mathrm{d}f((3,3);\mathbf{h}) = \frac{3}{5}\,h_x + \frac{4}{5}\,h_y = \text{“}\frac{3}{5}\,\mathrm{d}x + \frac{4}{5}\,\mathrm{d}y\text{”},$$

and

$$\mathrm{d}^2 f((3,3);\mathbf{h}) = \frac{16}{125}\,h_x^2 - \frac{24}{125}\,h_x h_y + \frac{9}{125}\,h_y^2.$$

The differentiations are easy in MAPLE,

$$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x^2+y^2}$$

$$\frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\sqrt{x^2+y^2}$$

$$\frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\,\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x^2+y^2}$$

$$-\frac{x^2}{(x^2+y^2)^{3/2}} + \frac{1}{\sqrt{x^2+y^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\,\frac{\mathrm{d}}{\mathrm{d}y}\sqrt{x^2+y^2}$$

$$-\frac{yx}{(x^2+y^2)^{3/2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\,\frac{\mathrm{d}}{\mathrm{d}y}\sqrt{x^2+y^2}$$

$$-\frac{y^2}{(x^2+y^2)^{3/2}} + \frac{1}{\sqrt{x^2+y^2}}$$

Example 11.22 *Let the function* $f : \mathbb{R}^2 \to \mathbb{R}$ *be given by*

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

1) Prove that f has partial derivatives of first order at every point of the plane.

2) Prove that the mixed derivatives f''_{xy} and f''_{yx} both exist at the point $(0,0)$, though

$$f''_{xy}(0,0) \neq f''_{yx}(0,0).$$

3) Find $f''_{xy}(x,y)$ for $(x,y) \neq (0,0)$, and prove that this function does not have any limit for $(x,y) \to (0,0)$.

A Partial derivatives of first and second order.

D Discuss the existence of f'_x and f'_y; then calculate $f''_{xy}(0,0)$ and $f_{yx}(0,0)$ *at the point* $(0,0)$. Finally, calculate $f''_{x,y}(x,y)$ in general and switch to polar coordinates.

I 1) When $(x,y) \neq (0,0)$, we see that $f(x,y)$ is a quotient of two polynomials where the denominator is > 0. Accordingly the partial derivatives of $f(x,y)$ exist of any order when $(x,y) \neq (0,0)$. We get for $(x,y) \neq (0,0)$ that

$$\frac{\partial f}{\partial x} = \frac{y^3}{x^2+y^2} - \frac{2x^2y^3}{(x^2+y^2)^2} = \frac{y^3(y^2-x^2)}{(x^2+y^2)^2}$$

and

$$\frac{\partial f}{\partial y} = \frac{3xy^2}{x^2+y^2} - \frac{2xy^4}{(x^2+y^2)^2} = \frac{xy^2(3x^2+y^2)}{(x^2+y^2)^2}.$$

We find at $(0,0)$

$$f(x,0) - f(0,0) = 0 = f(0,y) - f(0,0),$$

so we conclude that

$$\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0.$$

Summarizing we see that the partial derivatives of first order exist everywhere in $\mathbb{R}^2$.

2) Then it follows from the expressions of $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ that

$$\frac{\partial f}{\partial x}(0,y) - \frac{\partial f}{\partial x}(0,0) = \frac{y^5}{y^4} - 0 = y,$$

and

$$\frac{\partial f}{\partial y}(x,0) - \frac{\partial f}{\partial y}(0,0) = 0.$$

We conclude that

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{y \to 0} \frac{1}{y}\left\{\frac{\partial f}{\partial x}(0,y) - \frac{\partial f}{\partial x}(0,0)\right\} = \lim_{y\to 0}\frac{y}{y} = 1$$

and

$$\frac{\partial^2 f}{\partial y}\partial x(0,0) = \lim_{x \to 0} \frac{1}{x}\left\{\frac{\partial f}{\partial y}(x,0) - \frac{\partial f}{\partial y}(0,0)\right\} = 0,$$

so both

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = 1 \qquad \text{and} \qquad \frac{\partial^2 f}{\partial y \partial x}(0,0) = 0$$

exist and yet they are different.

3) It follows from 1) that

$$\frac{\partial f}{\partial x} = \frac{y^3(y^2-x^2)}{(x^2+y^2)^2} = \frac{y^5 - y^3x^2}{(x^2+y^2)^2},$$

so

$$\begin{aligned}\frac{\partial^2 f}{\partial x \partial y} &= \frac{5y^4 - 3y^2x^2}{(x^2+y^2)^2} - 2 \cdot \frac{2y}{(x^2+y^2)^3} \cdot (y^5 - y^3x^2) \\ &= \frac{y^2}{(x^2+y^2)^2}\,(5y^2 - 3x^2) - \frac{4y^4}{(x^2+y^2)^3} \cdot (y^2 - x^2).\end{aligned}$$

When we switch to polar coordinates $x = \varrho\cos\varphi$, $y = \varrho\sin\varphi$, we get

$$\begin{aligned}
f''_{xy}(x,y) &= \frac{\varrho^2\sin^2\varphi}{\varrho^4}(5\varrho^2\sin^2\varphi - 3\varrho^2\cos^2\varphi) - \frac{\varrho^4\sin^4\varphi}{\varrho^6}(\varrho^2\sin^2\varphi - \varrho^2\cos^2\varphi) \\
&= \sin^2\varphi(5\sin^2\varphi - 3\cos^2\varphi) - \sin^4\varphi(\sin^2\varphi - \cos^2\varphi) \\
&= \sin^2\varphi\left(4\sin^2\varphi - 4\cos^2\varphi + \sin^2\varphi + \cos^2\varphi + \left(\frac{1-\cos 2\varphi}{2}\right)^2\cos 2\varphi\right) \\
&= \sin^2\varphi\left(-4\cos 2\varphi + 1 + \frac{1}{4}\left\{1 - 2\cos 2\varphi + \cos^2 2\varphi\right\}\cos 2\varphi\right).
\end{aligned}$$

This expression is not constant in φ (the latter factor is a polynomial of third degree in $\cos 2\varphi$), hence the limit does not exist when $\varrho \to 0$, and there are no further conditions on φ.

There is no point here to show the calculations in MAPLE, because the main issue is to give an example where both $f''_{xy}(0,0)$ and f''_{yx} exist without being equal. MAPLE can be used, but only following the same procedure as above.

Example 11.23 *Find in each of the following cases the partial derivatives of first and second order of the given function* $f : \mathbb{R}^2 \to \mathbb{R}$.

1) $f(x,y) = \sin(x^2y^3)$.

2) $f(x,y) = \sin(\cos(2x-3y))$.

3) $f(x,y) = \sqrt{1+x^2+y^2}$.

4) $f(x,y) = \ln(1+\cos^2(xy))$.

5) $f(x,y) = \exp(x+xy-2y)$.

6) $f(x,y) = \text{ Arctan}(x-y)$.

A Partial derivatives of first and second order of C^∞-functions.

D Differentiate.

I 1) When $f(x,y) = \sin(x^2y^3)$, then

$$\frac{\partial f}{\partial x} = 2xy^3\cos(x^2y^3) \quad \text{and} \quad \frac{\partial f}{\partial y} = 3x^2y^2\cos(x^2y^3),$$

whence

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= 2y^3\cos(x^2y^3) - 4x^2y^6\sin(x^2y^3),\\
\frac{\partial^2 f}{\partial x\partial y} &= \frac{\partial^2 f}{\partial y\partial x} = 6xy^2\cos(x^2y^3) - 6x^3y^5\sin(x^2y^3)\\
\frac{\partial^2 f}{\partial y^2} &= 6x^2y\cos(x^2y^3) - 9x^4y^4\sin(x^2y^3).
\end{aligned}$$

Here, MAPLE is easy to apply,

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin\left(x^2\cdot y^3\right)$$

$$2\cos\left(x^2y^3\right)xy^3$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\sin\left(x^2\cdot y^3\right)$$

$$3\cos\left(x^2y^3\right)x^2y^2$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{d}{dx}\sin\left(x^2\cdot y^3\right)$$

$$-4\sin\left(x^2y^3\right)x^2y^6 + 2\cos\left(x^2y^3\right)y^3$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}y}\sin\left(x^2\cdot y^3\right)$$

$$-6\sin\left(x^2y^3\right)x^3y^5 + 6\cos\left(x^2y^3\right)xy^2$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\frac{\mathrm{d}}{\mathrm{d}y}\sin\left(x^2\cdot y^3\right)$$

$$-9\sin\left(x^2y^3\right)x^4y^4+6\cos\left(x^2y^3\right)x^2y$$

2) When $f(x,y)=\sin(\cos(2x-3y))$, then

$$\frac{\partial f}{\partial x}=-2\sin(2x-3y)\cdot\cos(\cos(2x-3y)),$$

$$\frac{\partial f}{\partial y}=3\sin(2x-3y)\cdot\cos(\cos(2x-3y)),$$

whence

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= -4\cos(2x-3y)\cos(\cos(2x-3y))-4\sin^2(2x-3y)\sin(\cos(2x-3y)),\\ \frac{\partial^2 f}{\partial x\partial y} &= \frac{\partial^2 f}{\partial y\partial x}=6\cos(2x-3y)\cos(\cos(2x-3y))+6\sin^2(2x-3y)\sin(\cos(2x-3y)),\\ \frac{\partial^2 f}{\partial y^2} &= -9\cos(2x-3y)\cos(\cos(2x-3y))-9\sin^2(2x-3y)\sin(\cos(2x-3y)).\end{aligned}$$

The partial differentiations are easy in MAPLE,

$$\frac{\mathrm{d}}{dax}\sin(\cos(2x-3y))$$

$$-2\cos(\cos(2x-3y))\sin(2x-3y)$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\sin(\cos(2x-3y))$$

$$3\cos(\cos(2x-3y))\sin(2x-3y)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}\sin(\cos(2x-3y))$$

$$-4\sin(\cos(2x-3y))\sin(2x-3y)^2-4\cos(\cos(2x-3y))\cos(2x-3y)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}y}\sin(\cos(2x-3y))$$

$$6\sin(\cos(2x-3y))\sin(2x-3y)^2+6\cos(\cos(2x-3y))\cos(2x-3y)$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\frac{\mathrm{d}}{\mathrm{d}y}\sin(\cos(2x-3y))$$

$$-9\sin(\cos(2x-3y))\sin(2x-3y)^2-9\cos(\cos(2x-3y))\cos(2x-3y)$$

3) When $f(x,y)=\sqrt{1+x^2+y^2}$, then

$$\frac{\partial f}{\partial x}=\frac{x}{\sqrt{1+x^2+y^2}},\qquad \frac{\partial f}{\partial y}=\frac{y}{\sqrt{1+x^2+y^2}},$$

whence

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{1}{\sqrt{1+x^2+y^2}}-\frac{x^2}{(\sqrt{1+x^2+y^2})^3}=\frac{1+y^2}{(\sqrt{1+x^2+y^2})^3},\\
\frac{\partial^2 f}{\partial x\partial y} &= \frac{\partial^2 f}{\partial y\partial x}=-\frac{xy}{(\sqrt{1+x^2+y^2})^3},\\
\frac{\partial^2 f}{\partial y^2} &= \frac{1+x^2}{(\sqrt{1+x^2+y^2})^3}.
\end{aligned}$$

This is also easy in MAPLE,

$$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{1+x^2+y^2}$$

$$\frac{x}{\sqrt{x^2+y^2+1}}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\sqrt{1+x^2+y^2}$$

$$\frac{y}{\sqrt{x^2+y^2+1}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\,\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{1+x^2+y^2}$$

$$-\frac{x^2}{(x^2+y^2+1)^{3/2}}+\frac{1}{\sqrt{x^2+y^2+1}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\,\frac{\mathrm{d}}{\mathrm{d}y}\sqrt{1+x^2+y^2}$$

$$-\frac{yx}{(x^2+y^2+1)^{3/2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\,\frac{\mathrm{d}}{\mathrm{d}y}\sqrt{1+x^2+y^2}$$

$$-\frac{y^2}{(x^2+y^2+1)^{3/2}}+\frac{1}{\sqrt{x^2+y^2+1}}$$

4) When $f(x,y) = \ln(1+\cos^2(xy))$, then

$$\frac{\partial f}{\partial x} = -\frac{2y\sin(xy)\cos(xy)}{1+\cos^2(xy)} = -\frac{y\sin(2xy)}{1+\cos^2(xy)},$$

$$\frac{\partial f}{\partial y} = -\frac{2x\sin(xy)\cos(xy)}{1+\cos^2(xy)} = -\frac{x\sin(2xy)}{1+\cos^2(xy)},$$

and accordingly

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= -\frac{2y^2\cos(2xy)}{1+\cos^2(xy)} + \frac{y\sin(2xy)}{\{1+\cos^2(xy)\}^2}\{-2\cos(xy)\sin(xy)\}y \\
&= -2y^2\,\frac{(\cos^2(xy)-\sin^2(xy))(1+\cos^2(xy)) - 2\sin^2(xy)\cos^2(xy)}{\{1+\cos^2(xy)\}^2} \\
&= -2y^2\frac{\cos^2(xy)+\cos^4(xy)-\sin^2(xy)-2\sin^2(xy)\cos^2(xy)}{\{1+\cos^2(xy)\}^2} \\
&= -2y^2\frac{-2\cos^2(xy)+4\cos^4(xy)-\sin^2(xy)}{\{1+\cos^2(xy)\}^2} \\
&= 2y^2\frac{1+\cos^2(xy)-4\cos^4(xy)}{\{1+\cos^2(xy)\}^2}.
\end{aligned}$$

Due to the symmetry in x and y we by interchanging the letters

$$\frac{\partial^2 f}{\partial y^2} = 2x^2\frac{1+\cos^2(xy)-4\cos^4(xy)}{\{1+\cos^2(xy)\}^2}.$$

Finally,

$$\begin{aligned}\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x} &= -\frac{\sin(2xy)}{1+\cos^2(xy)}-\frac{2xy\cos(2xy)}{1+\cos^2(xy)}\\ &\qquad +\frac{x\sin(2xy)}{\{1+\cos^2(xy)\}^2}\{-2\cos(xy)\sin(xy)\cdot y\}\\ &= -\frac{\sin(2xy)+2xy\cos(2xy)}{1+\cos^2(xy)}-\frac{xy\sin^2(2xy)}{\{1+\cos^2(xy)\}^2}.\end{aligned}$$

The calculations in MAPLE are easy, but the results need to be tidied up,

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(1+\cos(x\cdot y)^2\right)$$

$$-\frac{2\cos(xy)\sin(xy)y}{1+\cos(xy)^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\ln\left(1+\cos(x\cdot y)^2\right)$$

$$-\frac{2\cos(xy)\sin(xy)x}{1+\cos(xy)^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(1+\cos(x\cdot y)^2\right)$$

$$\frac{2\sin(xy)^2y^2}{1+\cos(xy)^2}-\frac{2\cos(xy)^2y^2}{1+\cos(xy)^2}-\frac{4\cos(xy)^2\sin(xy)^2y^2}{\left(1+\cos(xy)^2\right)^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}y}\ln\left(1+\cos(x\cdot y)^2\right)$$

$$\frac{2\sin(xy)^2yx}{1+\cos(xy)^2}-\frac{2\cos(xy)^2yx}{1+\cos(xy)^2}-\frac{2\cos(xy)\sin(xy)}{1+\cos(xy)^2}-\frac{4\cos(xy)^2\sin(xy)xy}{\left(1+\cos(xy)^2\right)^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\frac{d}{dy}\ln\left(1+\cos(x\cdot y)^2\right)$$

$$\frac{2\sin(xy)^2x^2}{1+\cos(xy)^2}-\frac{2\cos(xy)^2x^2}{1+\cos(xy)^2}-\frac{4\cos(xy)^2\sin(xy)^2x^2}{\left(1+\cos(xy)^2\right)^2}$$

5) When $f(x,y)=\exp(x+xy-2y)$, then

$$\begin{aligned}\frac{\partial f}{\partial x} &= (1+y)\exp(x+xy-2y)=(1+y)f(x,y),\\ \frac{\partial f}{\partial y} &= (x-2)\exp(x+xy-2y)=(x-2)f(x,y),\end{aligned}$$

whence

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= (1+y)^2 f(x,y) = (1+y)^2 \exp(x+xy-2y), \\
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial^2 f}{\partial y \partial x} = 1 \cdot f(x,y) + (1+y)(x-2)f(x,y) \\
&= (x+xy-2y-1)\exp(x+xy-2y), \\
\frac{\partial^2 f}{\partial y^2} &= (x-2)^2 f(x,y) = (x-2)^2 \exp(x+xy-2y).
\end{aligned}$$

Easy in MAPLE,

$$\frac{\mathrm{d}}{\mathrm{d}x}\, e^{x+x\cdot y-2y}$$

$$(y+1)e^{xy+x-2y}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\, e^{x+x\cdot y-2y}$$

$$(x-2)e^{xy+x-2y}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\,\frac{\mathrm{d}}{\mathrm{d}x}\, e^{x+x\cdot y-2y}$$

$$(y+1)^2 e^{xy+x-2y}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\,\frac{\mathrm{d}}{\mathrm{d}y}\, e^{x+x\cdot y-2y}$$

$$e^{xy+x-2y} + (x-2)(y+1)e^{xy+x-2y}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\,\frac{\mathrm{d}}{\mathrm{d}y}\, e^{x+x\cdot y-2y}$$

$$(x-2)^2 e^{xy+x-2y}$$

6) When $f(x,y) = \operatorname{Arctan}(x-y)$, we get

$$\frac{\partial f}{\partial x} = \frac{1}{1+(x-y)^2}, \qquad \frac{\partial f}{\partial y} = -\frac{1}{1+(x-y)^2},$$

hence

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= -\frac{2(x-y)}{\{1+(x-y)^2\}^2}, \\
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial^2 f}{\partial y \partial x} = \frac{2(x-y)}{1+(x-y)^2\}^2}, \\
\frac{\partial^2 f}{\partial y^2} &= -\frac{2(x-y)}{\{1+(x-y)^2\}^2}.
\end{aligned}$$

In MAPLE,

$$\frac{\mathrm{d}}{\mathrm{d}x} \arctan(x-y)$$

$$\frac{1}{1+(x-y)^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}y} \arctan(x-y)$$

$$-\frac{1}{1+(x-y)^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{\mathrm{d}}{\mathrm{d}x} \arctan(x-y)$$

$$\frac{-2x+2y}{\left(1+(x-y)^2\right)^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{\mathrm{d}}{\mathrm{d}y} \arctan(x-y)$$

$$\frac{2x-2y}{\left(1+(x-y)^2\right)^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}y} \frac{\mathrm{d}}{\mathrm{d}y} \arctan(x-y)$$

$$\frac{-2x+2y}{\left(1+(x-y)^2\right)^2}$$

Example 11.24 *Prove in each of the following cases that the given function f satisfies the given differential equation everywhere in its domain. In some of the cases there occur some constants α, β, γ; check if these can be chosen freely. Note that the variables are not x, y or z in all cases.*

1) *Prove that the function $\ln\sqrt{x^2+y^2}$, defined in $\mathbb{R}^2\setminus\{\mathbf{0}\}$, fulfils the differential equation*

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0.$$

2) *Prove that the function $e^{\alpha x}\cos(\alpha y)$, defined in $\mathbb{R}^2$, fulfils the differential equation*

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0.$$

3) *Prove that the function $e^{-t}(\cos x+\sin y)$, defined in $\mathbb{R}^3$, fulfils the differential equation*

$$\frac{\partial f}{\partial t}=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}.$$

4) *Prove that the function $\sin(\alpha x)\sin(\beta y)\sin(\gamma\sqrt{\alpha^2+\beta^2}\,t)$, defined in $\mathbb{R}^3$, fulfils the differential equation*

$$\frac{1}{\gamma^2}\,\frac{\partial^2 f}{\partial t^2}=\frac{\partial^2 t}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}.$$

5) *Prove that the function $\dfrac{1}{\sqrt{x^2+y^2+z^2}}$, defined in $\mathbb{R}^3\setminus\{\mathbf{0}\}$, fulfils the differential equation*

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}=0.$$

6) *Prove that the function $t^\alpha\exp\left(-\dfrac{r^2}{4t}\right)$, defined for $t>0$, fulfils the differential equation*

$$r^2\frac{\partial f}{\partial t}=\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right).$$

A Partial differential equations.

D Differentiate the given function and put it into the differential equation.

I 1) When $f(x,y)=\ln\sqrt{x^2+y^2}=\dfrac{1}{2}\ln(x^2+y^2)$, we get

$$\frac{\partial f}{\partial x}=\frac{x}{x^2+y^2},\qquad \frac{\partial f}{\partial y}=\frac{y}{x^2+y^2},$$

hence

$$\frac{\partial^2 f}{\partial x^2}=\frac{1}{x^2+y^2}-\frac{2x^2}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2},\qquad \frac{\partial^2 f}{\partial y^2}=\frac{x^2-y^2}{(x^2+y^2)^2}.$$

Then by insertion

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2}{\partial y^2}=\frac{y^2-x^2}{(x^2+y^2)^2}+\frac{x^2-y^2}{(x^2+y^2)^2}=0,$$

and the equation is fulfilled.

2) Here

$$\frac{\partial f}{\partial x}=\alpha e^{\alpha x}\cos(\alpha y), \qquad \frac{\partial f}{\partial y}=-\alpha e^{\alpha x}\sin(\alpha y),$$

and

$$\frac{\partial^2 f}{\partial x^2}=\alpha^2 e^{\alpha x}\cos(\alpha y), \qquad \frac{\partial^2 f}{\partial y^2}=-\alpha^2 e^{\alpha x}\cos(\alpha y).$$

Then by insertion into the differential equation

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=\alpha^2\left\{e^{\alpha x}\cos(\alpha y)-e^{\alpha x}\cos(\alpha y)\right\}=0.$$

The equation is satisfied, and we can choose any α.

3) Here

$$\frac{\partial f}{\partial t} = -e^{-t}(\cos x + \sin y).$$

Then

$$\frac{\partial^2 f}{\partial x^2} = -e^{-t}\cos x \quad \text{and} \quad \frac{\partial^2 f}{\partial y^2} = -e^{-t}\sin y,$$

and the differential equation is fulfilled

4) Here

$$\frac{1}{\gamma^2}\,\frac{\partial^2 f}{\partial t^2} = -(\alpha^2+\beta^2)f(x,y,t),$$

and

$$\frac{\partial^2 f}{\partial x^2} = -\alpha^2 f(x,y,t), \qquad \frac{\partial^2 f}{\partial y^2} = -\beta^2 f(x,y,t),$$

and the differential equation is fulfilled.

We must require that $\gamma \neq 0$. Note that when $\gamma = 0$, then $f(x,y,t) \equiv 0$, while $\dfrac{1}{\gamma^2}\,\dfrac{\partial^2 f}{\partial t^2}$ is not defined.

5) When $f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$, we have

$$\frac{\partial f}{\partial x} = -\frac{x}{(x^2+y^2+x^2)^{3/2}},$$

and

$$\frac{\partial^2 f}{\partial x^2} = -\frac{1}{(x^2+y^2+z^2)^{3/2}} + \frac{3x^2}{(x^2+y^2+z^2)^{5/2}} = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}.$$

Due to the symmetry we get by interchanging the letters

$$\frac{\partial^2 f}{\partial y^2} = \frac{-x^2+2y^2-z^2}{(x^2+y^2+z^2)^{5/2}}, \qquad \frac{\partial^2 f}{\partial z^2} = \frac{-x^2-y^2+2z^2}{(x^2+y^2+z^2)^{5/2}},$$

thus

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2} = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}+\frac{-x^2+2y^2-z^2}{(x^2+x^2+z^2)^{5/2}}+\frac{-x^2-y^2+2z^2}{(x^2++y^2+z^2)^{5/2}} = 0.$$

The equation is satisfied.

6) When $f(r,t) = t^\alpha \exp\left(-\dfrac{r^2}{4t}\right)$, we get

$$\frac{\partial f}{\partial t} = \alpha t^{\alpha-1}\exp\left(-\frac{r^2}{4t}\right) + t^\alpha \exp\left(-\frac{r^2}{4t}\right)\cdot\frac{r^2}{4t^2} = \left\{\alpha t^{\alpha-1}+\frac{1}{4}\,r^2 t^{\alpha-2}\right\}\exp\left(-\frac{r^2}{4t}\right),$$

and accordingly

$$(11.3)\quad r^2\frac{\partial f}{\partial t} = \left\{\alpha r^2 t^{\alpha-1}+\frac{1}{4}\,r^4 t^{\alpha-2}\right\}\exp\left(-\frac{r^2}{4t}\right).$$

Furthermore,

$$\frac{\partial f}{\partial r} = t^{\alpha}\exp\left(-\frac{r^2}{4t}\right)\cdot\left(-\frac{r}{2t}\right) = -\frac{1}{2}\,t^{\alpha-1}r\exp\left(-\frac{r^2}{4t}\right),$$

so

$$r^2\frac{\partial f}{\partial r} = -\frac{1}{2}\,t^{\alpha-1}r^3\exp\left(-\frac{r^2}{4t}\right)$$

and

$$\begin{aligned}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) &= -\frac{3}{2}\,t^{\alpha-1}r^2\exp\left(-\frac{r^2}{4t}\right)+\frac{1}{4}\,r^2t^{\alpha-2}\exp\left(-\frac{r^2}{4t}\right)\\ (11.4)\qquad &= \left\{-\frac{3}{2}\,t^{\alpha-1}r^2+\frac{1}{4}\,r^4t^{\alpha-2}\right\}\exp\left(-\frac{r^2}{4t}\right).\end{aligned}$$

By comparison we see that (11.3) and (11.4) only equals each other when $\alpha = -\frac{3}{2}$, corresponding to the fact that only

$$f(r,t) = t^{-3/2}\exp\left(-\frac{r^2}{4t}\right),\qquad t>0,$$

of the given set of functions are solutions of

$$r^2\,\frac{\partial f}{\partial t} = \frac{\partial}{\partial r}\left(r^2\,\frac{\partial f}{\partial r}\right).$$

It is not obvious how to apply MAPLE in these cases. One must apparently apply the command "evala", and yet the expression is not always fully reduced. We only show the first three cases.

$$\mathrm{evala}\left(\frac{\mathrm{d}}{dax}\,\frac{\mathrm{d}}{\mathrm{d}x}\,\ln\left(\sqrt{x^2+y^2}\right)+\frac{\mathrm{d}}{\mathrm{d}y}\,\frac{\mathrm{d}}{\mathrm{d}y}\,\ln\left(\sqrt{x^2+y^2}\right)\right)$$

$$0$$

$$\mathrm{evala}\left(\frac{\mathrm{d}}{\mathrm{d}x}\,\frac{\mathrm{d}}{\mathrm{d}x}\,(e^{\alpha\cdot x}\cdot\cos(\alpha\cdot y))+\frac{\mathrm{d}}{\mathrm{d}y}\,\frac{\mathrm{d}}{\mathrm{d}y}\,(e^{\alpha\cdot x}\cdot\cos(\alpha\cdot y))\right)$$

$$0$$

$$\mathrm{evala}\left(\frac{\mathrm{d}}{\mathrm{d}x}\,\frac{d}{dx}\left(e^{-t}\cdot(\cos(x)+\sin(y))\right)+\frac{\mathrm{d}}{\mathrm{d}y}\,\frac{\mathrm{d}}{\mathrm{d}y}\left(e^{-t}\cdot(\cos(x)+\sin(y))\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-t}\cdot(\cos(x)+\sin(y))\right)\right)$$

$$-e^{-t}\cos(x)-e^{-t}\sin(y)+e^{-t}(\cos(x)+\sin(y))$$

which of course is 0 after an inspection. But we did not expect that we should repeat the reduction.

Example 11.25 *A C^2-function f in two variables satisfies the partial differential equation*

$$\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} = 0.$$

Introduce the new variables $u = x + y$ and $v = x - y$, and prove that the function

$$f(u,v) = f\left(\frac{u+v}{2}, \frac{u-v}{2}\right)$$

fulfils the equation

$$\frac{\partial^2 g}{\partial u \partial v} = 0.$$

Furthermore, prove that it follows from

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

that

$$\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} = 0.$$

A Transform of the variables in partial differential equations.

D Follow the given guidelines.

I When

$$f(u,v) = f\left(\frac{u+v}{2}, \frac{u-v}{2}\right), \qquad x = \frac{u+v}{2}, \quad y = \frac{u-v}{2},$$

then

$$(11.5) \quad \frac{\partial g}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2}\frac{\partial f}{\partial y} = \frac{1}{2}\left\{\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right\},$$

hence

$$\frac{\partial^2 g}{\partial u \partial v} = \frac{1}{2}\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial x}{\partial u} - \frac{1}{2}\frac{\partial^2 f}{\partial y^2} \cdot \frac{\partial y}{\partial u} = \frac{1}{4}\left\{\frac{\partial^2}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}\right\} = 0$$

using the assumption.

Assume that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

We perform the following calculation

$$\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u} = \frac{1}{2}\left\{\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right\},$$

thus

$$\begin{aligned}\frac{\partial^2 g}{\partial u^2} &= \frac{1}{2}\left\{\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial x}{\partial u}+\frac{\partial^2 f}{\partial y\partial x}\cdot\frac{\partial y}{\partial u}+\frac{\partial^2 f}{\partial x\partial y}\cdot\frac{\partial x}{\partial u}+\frac{\partial^2 f}{\partial y^2}\cdot\frac{\partial y}{\partial u}\right\}\\ &= \frac{1}{4}\left\{\frac{\partial^2 f}{\partial x^2}+2\frac{\partial^2 f}{\partial x\partial y}+\frac{\partial^2 f}{\partial y^2}\right\}=\frac{1}{2}\,\frac{\partial^2 f}{\partial x\partial y}.\end{aligned}$$

Finally, we get from (11.25)

$$\begin{aligned}\frac{\partial^2 g}{\partial v^2} &= \frac{1}{2}\left\{\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial x}{\partial v}+\frac{\partial^2 f}{\partial y\partial x}\cdot\frac{\partial y}{\partial v}+\frac{\partial^2 f}{\partial x\partial y}\cdot\frac{\partial x}{\partial v}+\frac{\partial^2 f}{\partial y^2}\cdot\frac{\partial y}{\partial v}\right\}\\ &= \frac{1}{4}\left\{\frac{\partial^2 f}{\partial x^2}-2\frac{\partial^2 f}{\partial x\partial y}+\frac{\partial^2 f}{\partial y^2}\right\}=-\frac{1}{2}\,\frac{\partial^2 f}{\partial x\partial y},\end{aligned}$$

so by adding,

$$\frac{\partial^2 g}{\partial u^2}+\frac{\partial^2 g}{\partial v^2}=0.$$

This is a well-known trick in the theory of partial differential equations.

11.5 Taylor's formula for functions of several variables

Example 11.26 *Given the function*

$$f(x,y) = \exp(x + xy - 2y), \qquad (x,y) \in \mathbb{R}^2.$$

Find the approximating polynomial of at most second degree $P(x,y)$ and $Q(x,y)$ from the points of expansion $(0,0)$ and $(1,1)$ respectively. Calculate the values $P\left(\frac{1}{2},\frac{1}{2}\right)$ and $Q\left(\frac{1}{2},\frac{1}{2}\right)$; compare these with the value $f\left(\frac{1}{2},\frac{1}{2}\right)$ found on e.g. a pocket calculator.

A Approximating polynomials.

D Differentiate and apply a formula.

I For $f(x,y) = \exp(x + xy - 2y)$ we get

$$\frac{\partial f}{\partial x} = (1+y)\exp(x + xu - 2y), \qquad \frac{\partial f}{\partial y} = (x-2)\exp(x+xy-2y),$$

and

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= (1+y)^2 \exp(x+xy-2y),\\
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial^2 f}{\partial y \partial x} = (x+xy-2y-1)\exp(x+xy-2y),\\
\frac{\partial^2 f}{\partial y^2} &= (x-2)^2 \exp(x+xy-2y).
\end{aligned}$$

1) When the point of expansion is $(0,0)$ we get the coefficients

$$f(0,0) = 1, \quad f'_x(0,0) = 1, \quad f'_y(0,0) = -2,$$

$$f''_{xx}(0,0) = 1, \quad f''_{xy}(0,0) = f''_{yx}(0,0) = -1, \quad f''_{yy}(0,0) = 4,$$

and accordingly,

$$\begin{aligned}
P(x,y) &= f(0,0) + f'_x(0,0)\cdot x + f'_y(0,0)\cdot y\\
&\quad + \frac{1}{2}\left\{f''_{xx}(0,0)\cdot x^2 + 2f''_{xy}(0,0)\cdot xy + f''_{yy}(0,0)\cdot y^2\right\}\\
&= 1 + x - 2y + \frac{1}{2}x^2 - xy + 2y^2.
\end{aligned}$$

ALTERNATIVELY,

$$\exp(t) = 1 + t + \frac{1}{2}t^2 + \cdots,$$

so if we write $t = x - 2y + xy$ and include every term of higher degree than 2 in the dots, we get

$$\begin{aligned}\exp(x+xy-2y) &= 1+\{x-2y+xy\}+\frac{1}{2}\{x-2y+xy\}^2+\cdots\\ &= 1+x-2y+xy+\frac{1}{2}(2-2y)^2+\cdots\\ &= 1+x-2y+xy+\frac{1}{2}\,x^2-2xy+2y^2+\cdots\\ &= 1+x-2y+\frac{1}{2}\,x^2-xy+2y^2+\cdots\,.\end{aligned}$$

As mentioned above the dots indicate the terms of higher degree than 2. We get the wanted approximating polynomial by deleting the dots, i.e.

$$P(x,y) = 1+x-2y+\frac{1}{2}\,x^2-xy+2y^2.$$

2) When the point of expansion is $(1,1)$ we get the coefficients

$$f(1,1)=1, \quad f_x'(1,1)=2, \quad f_y'(1,1)=-1,$$
$$f_{xx}''(1,1)=4, \quad f_{xy}''(1,1)=f_{yx}''(1,1)=-1, \quad f_{yy}''(1,1)=1,$$

so

$$\begin{aligned}Q(x,y) &= f(1,1)+f_x'(1,1)(x-1)+f_y'(1,1)(y-1)\\ &\quad +\frac{1}{2}f_{xx}''(1,1)(x-1)^2+f_{xy}''(1,1)(x-1)(y-1)+\frac{1}{2}f_{yy}''(1,1)(y-1)^2\\ &= 1+2(x-1)-(y-1)+2(x-1)^2-(x-1)(y-1)+\frac{1}{2}(y-1)^2.\end{aligned}$$

Remark. The variables in $Q(x,y)$ *ought to be* $(x-1,y-1)$ and *not* (x,y). The reason is that the approximating polynomial $Q(x,y)$ supplies us with the best approximation in the *neightbourhood* of the point $(1,1)$, which means that for numerical reasons should *not* expand from the fairly distant point $(0,0)$. $\Diamond$

The polynomial can also in this case be found *alternatively*. Since the point of expansion is $(1,1)$, we introduce the new variables $(h,k)=(x-1,y-1)$, which are small in the neighbourhood of $(1,1)$. Hence, $(x,y)=(h+1,k+1)$. Then

$$\begin{aligned}\exp(x+xy-2y) &= \exp(h+1+(h+1)(k+1)-2(k+1))\\ &= \exp(1+h+1+h+k+hk-2-2k)=\exp(2h-k+hk)\\ &= 1+\{2h-k+hk\}+\frac{1}{2!}\{2h-k+hk\}^2+\cdots\\ &= 1+2h-k+hk+\frac{1}{2}(2h-k)^2+\cdots\\ &= 1+2h-k+hk+2h^2-2hk+\frac{1}{2}\,k^2+\cdots\,,\end{aligned}$$

where the dots as usual indicate terms of higher degree. Thus

$$\begin{aligned}Q(x,y) &= 1+2h-k+2h^2-hk+\frac{1}{2}\,k^2\\ &= 1+2(x-1)-(y-1)+2(x-1)^2-(x-1)(y-1)+\frac{1}{2}(y-1)^2.\end{aligned}$$

3) We evaluate

$$\begin{aligned}P\left(\frac{1}{2},\frac{1}{2}\right) &= 1+\frac{1}{2}-2\cdot\frac{1}{2}+\frac{1}{2}\left(\frac{1}{2}\right)^2-\frac{1}{2}\cdot\frac{1}{2}+2\cdot\left(\frac{1}{2}\right)^2\\ &= 1+\frac{1}{2}-1+\frac{1}{8}-\frac{1}{4}+\frac{1}{2}=\frac{7}{8}=0.875,\end{aligned}$$

and

$$\begin{aligned}Q\left(\frac{1}{2},\frac{1}{2}\right) &= 1+2\cdot\left(-\frac{1}{2}\right)-\left(-\frac{1}{2}\right)+2\left(-\frac{1}{2}\right)^2-\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)+\frac{1}{2}\left(-\frac{1}{2}\right)^2\\ &= 1-1+\frac{1}{2}+\frac{1}{2}-\frac{1}{4}+\frac{1}{8}=\frac{7}{8}=0.875.\end{aligned}$$

Finally, we get by using a pocket calculator

$$f\left(\frac{1}{2},\frac{1}{2}\right)=\exp\left(\frac{1}{2}+\frac{1}{4}-1\right)=\exp\left(-\frac{1}{4}\right)\approx 0.779.$$

The approximations have a relatively large error (approx. 12 %). This is caused by the fact that the point $\left(\frac{1}{2},\frac{1}{2}\right)$ is fairly distant from both points of expansions.

Example 11.27 *Let $f\in C^2(A)$, where A is an open subset of $\mathbb{R}^2$. Prove that for $(x,y)\in A$ and $|h|$ sufficiently small,*

$$4h^2 f''_{xy}(x,y)=\{f(x+h,y+h)+f(x-h,y-h)-f(x+h,y-h)-f(x-h,y+h)\}+\varepsilon(h),$$

where $\dfrac{\varepsilon(h)}{h^2}\to 0$ for $h\to 0$. When we neglect $\varepsilon(h)$ we get an approximative expression of $f''_{xy}(x,y)$, which can be applied in numerical calculations.
Set up analogous formulæ for $f''_{xx}(x,y)$ and $f''_{yy}(x,y)$.

A Approximating polynomials.

D Calculate the approximating polynomial for $f(x+h,y+k)$. Replace (h,k) by $(\pm h,\pm h)$ (all four combinations) and compare.

I We know already that

$$f(x+h,y+k)=f+f'_x\cdot h+f'_y\cdot k+\frac{1}{2}\left\{f''_{xx}h^2+2f''_{xy}hk+f''_{yy}k^2\right\}+\varepsilon(h,k),$$

where $\varepsilon(h,k)/(h^2+k^2)\to 0$ for $(h,k)\to 0$, and where we have used the shorthand f, f'_x, etc. instead of the total expression $f(x,y)$, $f'_x(x,y)$, etc. in all details.

By successively replacing (h,k) by (h,h), $(-h,-h)$, $(h,-h)$ and $(-h,h)$ we get

$$f(x+h,y+h)=f+f'_x\cdot h+f'_y\cdot h+\frac{1}{2}f''_{xx}\cdot h^2+f''_{xy}\cdot h^2+\frac{1}{2}f''_{yy}\cdot h^2+\varepsilon_1(h),$$

$$f(x-h,y-h)=f-f'_x\cdot h-f'_y\cdot h+\frac{1}{2}f''_{xx}\cdot h^2+f''_{xy}\cdot h^2+\frac{1}{2}f''_{yy}\cdot h^2+\varepsilon_2(h),$$

$$f(x+h,y-h) = f + f'_x \cdot h - f'_y \cdot h + \frac{1}{2} f''_{xx} \cdot h^2 - f''_{xy} \cdot h^2 + \frac{1}{2} f''_{yy} \cdot h^2 + \varepsilon_3(h),$$

$$f(x-h,y+h) = f - f'_x \cdot h + f'_y \cdot h + \frac{1}{2} f''_{xx} \cdot h^2 - f''_{xy} \cdot h^2 + \frac{1}{2} f''_{yy} \cdot h^2 + \varepsilon_4(h),$$

where $\dfrac{\varepsilon_i(h)}{h^2} \to 0$ for $h \to 0$.

It follows that

$$\begin{aligned} &f(x+h,y+h) + f(x-h,y-h) - f(x+h,y-h) - f(x-h,y+h) \\ &\qquad = 0 \cdot f + 0 \cdot f'_x \cdot h + 0 \cdot f'_y \cdot h + 0 \cdot f''_{xx} \cdot h^2 + 4 f''_{xy} \cdot h^2 + 0 \cdot f''_{yy} \cdot h^2 + \varepsilon(h), \end{aligned}$$

hence by a rearrangement

$$4 f''_{xy}(x,y) h^2 = \{f(x+h,y+h) + f(x-h,y-y) - f(x+h,y-h) - f(x-h,y+h)\} + \varepsilon(h),$$

where $\dfrac{\varepsilon(h)}{h^2} \to 0$ for $h \to 0$, and the claim is proved.

Remark. This formula is useful in numerical calculations of $f''_{xy}(x,y)$, when we know the values of $f(x+mh,y+nh)$, m, $n \in \mathbb{Z}$. $\Diamond$

If we instead put $k = 0$, we get

$$f(x+h,y) = f(x,y) + f_x'(x,y)h + \frac{1}{2}\, f_{xx}''(x,y)h^2 + \varepsilon_1(h),$$

$$f(x-h,y) = f(x,y) - f_x'(x,y)h + \frac{1}{2}\, f_{xx}''(x,y)h^2 + \varepsilon_2(h),$$

hence by adding,

$$f(x+h,y) + f(x-h,y) = 2f(x,y) + f_{xx}''(x,y)\cdot h^2 + \varepsilon(h),$$

and by a rearrangement,

$$h^2 f_{xx}''(x,y) = \{f(x+h,y) - 2f(x,y) + f(x-h,y)\} + \varepsilon(h).$$

Analogously,

$$h^2 f_{yy}''(x,y) = \{f(x,y+h) - 2f(x,y) + f(x,y-h)\} + \varepsilon(h),$$

where $\dfrac{\varepsilon(h)}{h^2} \to 0$ for $h \to 0$.

Example 11.28 *Find the approximating polynomial of at most second degree of the given functions in the given points of expansion:*

1) *The function* $\ln\{(x+1)^2+(y-1)^2\}$*, defined in* $\mathbb{R}^2 \setminus \{(-1,1)\}$*, from the point* $(0,0)$.

2) *The function* $\sqrt{x^2+y^2}$*, defined in* $\mathbb{R}^2 \setminus \{(0,0)\}$ *from the point* $(3,4)$.

3) *The function* $\operatorname{Arctan}\,\dfrac{y}{x}$*, defined for* $x>0$ *from the point* $(1,\sqrt{3})$.

4) *The function* $\sqrt[5]{x^2+2y^3}$*, defined for* $x^2+2y^3>0$ *from* $(4,2)$.

5) *The function* $x^3+xy-12x-6y$*, defined in* $\mathbb{R}^2$ *from* $(1,3)$.

6) *The function* $\sqrt{x^2+y^2+z^2}$*, defined in* $\mathbb{R}^3 \setminus \{(0,0,0)\}$ *from the point* $(3,6,6)$.

7) *The function* $\sin(x-y)+z(x+y)-2x+1$*, defined in* $\mathbb{R}^3$ *from* $(0,0,1)$.

8) *The function* $(\cosh x)\cdot\sin(x-y-2z)$*, defined in* $\mathbb{R}^3$ *from* $\left(0,\dfrac{\pi}{2},0\right)$.

A Approximating polynomials of at most second degree.

D Use preferably the standard method, i.e. differentiate and apply a formula. Note the standard scheme in each case.

In some cases it is possible instead to use standard Taylor series.

I 1) The function $f(x,y)=\ln\{(x+1)^2+(y-1)^2\}$ is of class C^∞ in the given domain, and

$$f(x,y)=\ln\{(x+1)^2+(y-1)^2\}, \qquad f(\mathbf{0})=\ln 2,$$

$$\frac{\partial f}{\partial x}=\frac{2(x+1)}{(x+1)^2+(y-1)^2}, \qquad \frac{\partial f}{\partial x}(\mathbf{0})=1,$$

$$\frac{\partial f}{\partial y}=\frac{2(y-1)}{(x+1)^2+(y-1)^2}, \qquad \frac{\partial f}{\partial y}(\mathbf{0})=-1,$$

$$\frac{\partial^2 f}{\partial x^2}=\frac{1}{(x+1)^2+(y-1)^2}-\frac{4(x+1)^2}{\{(x+1)^2+(y-1)^2\}^2}, \qquad \frac{\partial^2 f}{\partial x^2}(\mathbf{0})=0,$$

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}=-\frac{4(x+1)(y-1)}{\{(x+1)^2+(y-1)^2\}^2}, \qquad \frac{\partial^2 f}{\partial x\partial y}(\mathbf{0})=1,$$

$$\frac{\partial^2 f}{\partial y^2}=\frac{2}{(x+1)^2+(y-1)^2}-\frac{4(y-1)^2}{\{(x+1)^2+(y-1)^2\}^2}, \qquad \frac{\partial^2 f}{\partial y^2}(\mathbf{0})=0.$$

The coefficients of the approximating polynomial are the numbers in the right hand column. We get by insertion,

$$\begin{aligned}P_2(x,y) &= f(\mathbf{0})+\left\{\frac{\partial f}{\partial x}(\mathbf{0})\cdot(x-0)+\frac{\partial f}{\partial y}(\mathbf{0})\cdot(y-0)\right\}\\ &\quad+\frac{1}{2!}\left\{\frac{\partial^2 f}{\partial x^2}(\mathbf{0})(x-0)^2+2\frac{\partial^2 f}{\partial x\partial y}(\mathbf{0})(x-0)(y-0)+\frac{\partial^2 f}{\partial y^2}(\mathbf{0})(y-0)^2\right\}\\ &= \ln 2+x-y+\frac{1}{2}\cdot 2xy=\ln 2+x-y+xy.\end{aligned}$$

2) The function is of course also defined at $(0,0)$, but it is only of class C^∞ in $\mathbb{R}\setminus\{(0,0)\}$. Using the same procedure as before we get for the point $(3,4)$,

$$f(x,y)=\sqrt{x^2+y^2}, \qquad f(3,4)=5,$$

$$\frac{\partial f}{\partial x}=\frac{x}{\sqrt{x^2+y^2}}, \qquad \frac{\partial f}{\partial x}(3,4)=\frac{3}{5},$$

$$\frac{\partial f}{\partial y}=\frac{y}{\sqrt{x^2+y^2}}, \qquad \frac{\partial f}{\partial y}(3,4)=\frac{4}{5},$$

$$\frac{\partial^2 f}{\partial x^2}=\frac{y^2}{(\sqrt{x^2+y^2})^3}, \qquad \frac{\partial^2 f}{\partial x^2}(3,4)=\frac{16}{125},$$

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}=-\frac{xy}{(\sqrt{x^2+y^2})^3}, \qquad \frac{\partial^2 f}{\partial x\partial y}(3,4)=-\frac{12}{125},$$

$$\frac{\partial^2 f}{\partial y^2}=\frac{x^2}{(\sqrt{x^2+y^2})^3}, \qquad \frac{\partial^2 f}{\partial y^2}(3,4)=\frac{9}{125}.$$

By choosing $(x_1,y_1)=(x-x_0,y-y_0)=(x-3,y-4)$ as our new variables we get

$$\begin{aligned}P_2(x,y) &= 5+\frac{3}{5}(x-3)+\frac{4}{5}(y-4)\\ &\quad+\left\{\frac{16}{125}(x-3)^2-\frac{12}{125}\cdot 2(x-3)(y-4)+\frac{9}{125}(y-4)^2\right\}\\ &= 5+\frac{3}{5}(x-3)+\frac{4}{5}(y-4)+\frac{8}{125}(x-3)^2-\frac{12}{125}(x-3)(y-4)+\frac{9}{250}(y-4)^2,\end{aligned}$$

which can be reduced to

$$P_2(x,y)=5+\frac{3}{5}(x-3)+\frac{4}{5}(y-4)+\frac{1}{250}\{4(x-3)-3(y-4)\}^2.$$

3) The function is of class C^∞ in the given domain (and of course also defined for $x<0$; but this

case is not at all relevant here). We have as before

$$f(x,y) = \text{ Arctan } \frac{y}{x}, \qquad f(1,\sqrt{3}) = \frac{\pi}{3},$$

$$\frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}, \qquad \frac{\partial f}{\partial x}(1,\sqrt{3}) = -\frac{\sqrt{3}}{4},$$

$$\frac{\partial f}{\partial y} = \frac{x}{x^2+y^2}, \qquad \frac{\partial f}{\partial y}(1,\sqrt{3}) = \frac{1}{4},$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{2xy}{(x^2+y^2)^2}, \qquad \frac{\partial^2 f}{\partial x^2}(1,\sqrt{3}) = \frac{\sqrt{3}}{8},$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}, \qquad \frac{\partial^2 f}{\partial x \partial y}(1,\sqrt{3}) = \frac{1}{8},$$

$$\frac{\partial^2 f}{\partial y^2} = -\frac{2xy}{(x^2+y^2)^2}, \qquad \frac{\partial^2 f}{\partial y^2}(1,\sqrt{3}) = -\frac{\sqrt{3}}{8}.$$

The approximating polynomial from $(1,\sqrt{3})$ is

$$\begin{aligned} P_2(x,y) \quad &= \quad \frac{\pi}{3} - \frac{\sqrt{3}}{4}(x-1) + \frac{1}{4}(y-\sqrt{3}) \\ &\quad + \frac{\sqrt{3}}{16}(x-1)^2 + \frac{1}{8}(x-1)(y-\sqrt{3}) - \frac{\sqrt{3}}{16}(y-1)^2, \end{aligned}$$

which can be reduced to

$$\begin{aligned} P_2(x,y) \quad &= \quad \frac{\pi}{3} - \frac{\sqrt{3}}{4}(x-1) + \frac{1}{4}(y-\sqrt{3}) \\ &\quad + \frac{\sqrt{3}}{16}\{(x-1)+\sqrt{3}(y-\sqrt{3})\}\left\{(x-1)-\frac{1}{\sqrt{3}}(y-\sqrt{3})\right\}. \end{aligned}$$

4) We see that when $x^2+2y^3>0$, then the function is of class C^∞. We calculate as before,

$$f(x,y) = (x^2+2y^3)^{1/5}, \qquad f(4,2) = 2,$$

$$\frac{\partial f}{\partial x} = \frac{2}{5}x(x^2+2y^3)^{-4/5}, \qquad \frac{\partial f}{\partial x}(4,2) = \frac{1}{10},$$

$$\frac{\partial f}{\partial y} = \frac{6}{5}\,y^2(x^2+2y^3)^{-4/5}, \qquad \frac{\partial f}{\partial y}(4,2) = \frac{3}{10},$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{2}{5}\,(x^2+2y^3)^{-4/5} - \frac{16}{25}\,x^2(x^2+2y^3)^{-9/5}, \qquad \frac{\partial^2 f}{\partial x^2}(4,2) = \frac{1}{200},$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = -\frac{48}{25}\,xy^2(x^2+2y^3)^{-9/5}, \qquad \frac{\partial^2 f}{\partial x \partial y}(4,2) = -\frac{3}{50},$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{12}{5}\,y(x^2+2y^3)^{-4/5} - \frac{144}{25}\,y^4(x^2+2y^3)^{-9/5}, \qquad \frac{\partial^2 f}{\partial y^2}(4,2) = \frac{3}{25}.$$

The approximating polynomial from $(4,2)$ is

$$P_2(x,y) = 2 + \frac{1}{10}(x-4) + \frac{3}{10}(y-2) + \frac{1}{400}(x-4)^2 - \frac{3}{50}(x-4)(y-2) + \frac{3}{50}(y-2)^2.$$

5) When one is asked to find the approximating polynomial for

$$f(x,y) = x^3 + xy^2 - 12x - 6y$$

of at most second degree from $(1,3)$, it is tempting just to remove the terms $x^3 + xy^2$, which are of third degree. This is, however, not the right procedure, because the point of expansion is *not* $(0,0)$, but translated to $(1,3)$.

In order to explain what is going on we shall first apply the rather elaborate standard procedure, for later to give an alternative. We get by the standard procedure,

$$
\begin{array}{ll}
f(x,y) = x^3 + xy^2 - 12x - 6y, & f(1,3) = -20, \\[2mm]
\dfrac{\partial f}{\partial x} = 3x^2 + y^2 - 12, & \dfrac{\partial f}{\partial x}(1,3) = 0, \\[2mm]
\dfrac{\partial f}{\partial y} = 2xy - 6, & \dfrac{\partial f}{\partial y}(1,3) = 0, \\[2mm]
\dfrac{\partial^2 f}{\partial x^2} = 6x, & \dfrac{\partial^2 f}{\partial x^2}(1,3) = 6, \\[2mm]
\dfrac{\partial^2 f}{\partial x \partial y} = 2y, & \dfrac{\partial^2 f}{\partial x \partial y}(1,3) = 6, \\[2mm]
\dfrac{\partial^2 f}{\partial y^2} = 2x, & \dfrac{\partial^2 f}{\partial y^2}(1,3) = 2.
\end{array}
$$

Then the approximating polynomial is

$$P_2(x,y) = -20 + 3(x-1)^2 + 6(x-1)(y-3) + (y-3)^2,$$

where we of course use $(x-1, y-3)$ as the new (and more correct) variables.

ALTERNATIVELY we start by introducing $(x_1, y_1) = (x-1, y-3)$ as our new variables, i.e. $(x,y) = (x_1+1, y_1+3)$. Then by insertion,

$$
\begin{aligned}
f(x,y) &= x^3 + xy^2 - 12x - 6y \\[2mm]
&= (x_1+1)^3 + (x_1+1)(y_1+3)^2 - 12(x_1+1) - 6(y_1+3) \\[2mm]
&= x_1^3 + 3x_1^2 + 3x_1 + 1 + (x_1+1)(y_1^2 + 6y_1^2 + 9) - 12x_1 - 12 - 6y_1 - 18 \\[2mm]
&= x_1^3 + 3x_1^2 - 9x_1 - 6y_1 - 29 + x_1y_1^2 + 6x_1y_1 + 9x_1 + y_1^2 + 6y_1 + 9 \\[2mm]
&= -20 + 3x_1^2 + 6x_1y_1 + y_1^2 + \left\{x_1^3 + x_1y_1^2\right\}.
\end{aligned}
$$

The approximative polynomial from $(1,3)$ is then obtained by deleting all terms of degree > 2 in (x_1, y_1), thus

$$
\begin{aligned}
P_2(x,y) &= -20 + 3x_1^2 + 6x_1y_1 + y_1^2 \\
&= -20 + 3(x-1)^2 + 6(x-1)(y-3) + (y-3)^2.
\end{aligned}
$$

6) The function is of class C^∞ for $(x,y,z) \neq (0,0,0)$. We use the same method as before, only

supplied with an extra variable. (Notice the systematics).

$$f(x,y,z)=\sqrt{x^2+y^2+z^2}, \qquad f(3,6,6)=9,$$

$$\frac{\partial f}{\partial x}=\frac{x}{\sqrt{x^2+y^2+z^2}}, \qquad \frac{\partial f}{\partial x}(3,6,6)=\frac{1}{3},$$

$$\frac{\partial f}{\partial y}=\frac{y}{\sqrt{x^2+y^2+z^2}}, \qquad \frac{\partial f}{\partial y}(3,6,6)=\frac{2}{3},$$

$$\frac{\partial f}{\partial z}=\frac{x}{\sqrt{x^2+y^2+z^2}}, \qquad \frac{\partial f}{\partial z}(3,6,6)=\frac{2}{3},$$

$$\frac{\partial^2 f}{\partial x^2}=\frac{y^2+z^2}{(\sqrt{x^2+y^2+z^2})^3}, \qquad \frac{\partial^2 f}{\partial x^2}(3,6,6)=\frac{8}{81},$$

$$\frac{\partial^2 f}{\partial y^2}=\frac{x^2+z^2}{(\sqrt{x^2+y^2+z^2})^3}, \qquad \frac{\partial^2 f}{\partial y^2}(3,6,6)=\frac{5}{81},$$

$$\frac{\partial^2 f}{\partial z^2}=\frac{x^2+y^2}{(\sqrt{x^2+y^2+z^2})^3}, \qquad \frac{\partial^2 f}{\partial z^2}(3,6,6)=\frac{5}{81},$$

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}=-\frac{xy}{(\sqrt{x^2+y^2+z^2})^3}, \qquad \frac{\partial^2 f}{\partial x\partial y}(3,6,6)=-\frac{2}{81},$$

$$\frac{\partial^2 f}{\partial x\partial z}=\frac{\partial^2 f}{\partial z\partial x}=-\frac{xz}{(\sqrt{x^2+y^2+z^2})^3}, \qquad \frac{\partial^2 f}{\partial x\partial z}(3,6,6)=-\frac{2}{81},$$

$$\frac{\partial^2 f}{\partial y\partial z}=\frac{\partial^2 f}{\partial z\partial y}=-\frac{yz}{(\sqrt{x^2+y^2+z^2})^3}, \qquad \frac{\partial^2 f}{\partial y\partial z}(3,6,6)=-\frac{4}{81}.$$

From this we get the approximating polynomial from $(3,6,6)$,

$$\begin{aligned}P_2(x,y,z) &= 9+\frac{1}{1!}\left\{\frac{1}{3}(x-6)+\frac{2}{3}(y-6)+\frac{2}{3}(z-6)\right\}\\
&\quad+\frac{1}{2!}\left\{\frac{8}{81}(x-3)^2+\frac{5}{81}(y-6)^2+\frac{5}{81}(z-6)^2\right\}\\
&\quad-\frac{2}{2!}\left\{\frac{2}{81}(x-3)(y-6)+\frac{4}{81}(y-6)(z-6)+\frac{2}{81}(z-6)(x-3)\right\}\\
&= 9+\frac{1}{3}(x-3)+\frac{2}{3}(y-6)+\frac{2}{3}(z-6)+\frac{4}{81}(x-3)^2+\frac{5}{162}(y-6)^2\\
&\quad+\frac{5}{162}(z-6)^2-\frac{2}{81}(x-3)(y-6)-\frac{4}{81}(y-6)(z-z)-\frac{2}{81}(z-6)(x-3).\end{aligned}$$

7) Using the same method as above we get

$$f(x,y,z)=\sin(x-y)+z(x+y)-2x+1, \qquad f(0,0,1)=1,$$

$$\frac{\partial f}{\partial x}=\cos(x-y)+z-2, \qquad \frac{\partial f}{\partial x}=0,$$

$$\frac{\partial f}{\partial y}=-\cos(x-y)+z, \qquad \frac{\partial f}{\partial y}(0,0,1)=0,$$

$$\frac{\partial f}{\partial z}=x+y, \qquad \frac{\partial f}{\partial z}(0,0,1)=0,$$

$$\frac{\partial^2 f}{\partial x^2}=-\sin(x-y), \qquad \frac{\partial^2 f}{\partial x^2}(0,0,1)=0,$$

$$\frac{\partial^2 f}{\partial y^2}=-\sin(x-y), \qquad \frac{\partial^2 f}{\partial y^2}(0,0,1)=0,$$

$$\frac{\partial^2 f}{\partial z^2}=0, \qquad \frac{\partial^2 f}{\partial z^2}(0,0,1)=0,$$

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}=\sin(x-y), \qquad \frac{\partial^2 f}{\partial x\partial y}(0,0,1)=0,$$

$$\frac{\partial^2 f}{\partial x\partial z}=\frac{\partial^2 f}{\partial z\partial x}=1, \qquad \frac{\partial^2 f}{\partial x\partial z}(0,0,1)=1,$$

$$\frac{\partial^2 f}{\partial y\partial z}=\frac{\partial^2 f}{\partial z\partial y}=1, \qquad \frac{\partial^2 f}{\partial y\partial z}(0,0,1)=1.$$

Accordingly, the approximating polynomial from $(0,0,1)$ is

$$P_2(x,y,z)=1+\frac{1}{1!}\cdot 0+\frac{2}{2!}\{(x-0)(z-1)+(y-0)(z-1)\}=10(x+y)(z-1).$$

We note that thee natural parameters are here $(x,y,z-1)$.

Alternatively we exploit that $(x-0)-(y-0)=x-y$ is the approximating polynomial for $\sin(x-y)$ of at most second degree, and since the rest is a polynomial of second degree in (x,y,z), we get

$$P_2(x,y,z)=x-y+z(x+y)-2x+1=z(x+y)-(x+y)+1=1+(x+y)(z-1).$$

8) We first rewrite the expression and use series expansions,

$$\begin{aligned}
f(x,y,z) &= \cosh x \cdot \sin(x-y-2z) = \cosh x \cdot \sin\left(x-\left(y-\frac{\pi}{2}\right)-2z-\frac{\pi}{2}\right) \\
&= -\cosh x \cdot \cos\left(x-\left(y-\frac{\pi}{2}\right)-2z\right) \\
&= -\left\{1+\frac{1}{2}x^2+\cdots\right\}\left\{1-\frac{1}{2}\left[x-\left(y-\frac{\pi}{2}\right)-2z\right]^2+\cdots\right\} \\
&= -1-\frac{1}{2}x^2+\frac{1}{2}\left\{x-\left(y-\frac{\pi}{2}\right)-2z\right\}^2+\cdots \\
&= -1+\frac{1}{2}\left(y-\frac{\pi}{2}\right)^2+4z^2-x\left(y-\frac{\pi}{2}\right)+2\left(y-\frac{\pi}{2}\right)z-2xz+\cdots,
\end{aligned}$$

where the dots denote terms of higher degree. The approximating polynomial is obtained by removing these dots:

$$\begin{aligned}
P_2(x,y,z) &= -1-\frac{1}{2}x^2+\frac{1}{2}\left\{x-\left(y-\frac{\pi}{2}\right)-2z\right\}^2 \\
&= -1-\frac{1}{2}\left\{2x-\left(y-\frac{\pi}{2}\right)-2z\right\}\left\{\left(y-\frac{\pi}{2}\right)+2z\right\} \\
&= -1+\frac{1}{2}\left(y-\frac{\pi}{2}\right)^2+4z^2-x\left(y-\frac{\pi}{2}\right)+2\left(y-\frac{\pi}{2}\right)z-2xz.
\end{aligned}$$

ALTERNATIVELY we get by the standard method,

$$f(x,y,z)=\cosh x\cdot\sin(x-y-2z), \qquad f\left(0,\frac{\pi}{2},0\right)=-1,$$

$$\frac{\partial f}{\partial x}=\sinh x\sin(x-y-2z)+\cosh x\cos(x-y-2z), \qquad \frac{\partial f}{\partial x}\left(0,\frac{\pi}{2},0\right)=0,$$

$$\frac{\partial f}{\partial y}=-\cosh x\cos(x-y-2z), \qquad \frac{\partial f}{\partial y}\left(0,\frac{\pi}{2},0\right)=0,$$

$$\frac{\partial f}{\partial z}=-2\cosh x\cos(x-y-2z), \qquad \frac{\partial f}{\partial z}\left(0,\frac{\pi}{2},0,\right)=0,$$

$$\frac{\partial^2 f}{\partial x^2}=2\sinh x\cos(x-y-2z), \qquad \frac{\partial^2 f}{\partial x^2}\left(0,\frac{\pi}{2},0\right)=0,$$

$$\frac{\partial^2 f}{\partial y^2}=\cosh x\sin(x-y-2z), \qquad \frac{\partial^2 f}{\partial y^2}\left(0,\frac{\pi}{2},0\right)=1,$$

$$\frac{\partial^2 f}{\partial z^2}=-4\cosh x\sin(x-y-2z), \qquad \frac{\partial^2 f}{\partial z^2}\left(0,\frac{\pi}{2},0\right)=4,$$

$$\frac{\partial^2 f}{\partial x\partial y}=-\sinh x\cos(x-y-2z)+\cosh x\sin(x-y-2z), \qquad \frac{\partial^2 f}{\partial x\partial y}\left(0,\frac{\pi}{2},0\right)=-1,$$

$$\frac{\partial^2 f}{\partial y\partial z}=-2\cosh x\sin(x-y-2z), \qquad \frac{\partial^2 f}{\partial y\partial z}\left(0,\frac{\pi}{2},0\right)=2,$$

$$\frac{\partial^2 f}{\partial z\partial x}=-2\sinh x\cos(x-y-2z)+2\cosh x\sin(x-y-2z), \qquad \frac{\partial^2 f}{\partial z\partial x}\left(0,\frac{\pi}{2},0\right)=-2.$$

Hence, the approximating polynomial from $\left(0,\frac{\pi}{2},0\right)$ is

$$\begin{aligned}P_2(x,y,z) &= -1+\frac{1}{1!}\{0\}+\frac{1}{2!}\left\{\left(y-\frac{\pi}{2}\right)^2+4z^2\right\}\\ &\quad+\frac{2}{2!}\left\{-x\left(y-\frac{\pi}{2}\right)+2\left(y-\frac{\pi}{2}\right)z-2xz\right\}\\ &= -1+\frac{1}{2}\left(y-\frac{\pi}{2}\right)^2+4z^2-x\left(y-\frac{\pi}{2}\right)+2z\left(y-\frac{\pi}{2}\right)-2xz.\end{aligned}$$

Example 11.29 *Find approximating values of the following expressions by using the approximating polynomials of at most second degree from* **Example 11.28**. *Compare with the values which we get by using a pocket calculator instead.*

1) The length L of the diagonal in a rectangle of edge lengths 2.9 and 4.2.

2) The length L of the space diagonal in a rectangular box of edge lengths 3.03 and 5.98 and 6.01.

3) $\sqrt[5]{3.8^2 + 2 \cdot 2.1^3}$.

A Approximating values.

D Identify the corresponding function f. Apply the approximations found in **Example 11.28**. Compare the results with a calculation on a pocket calculator.

I 1) By using a pocket calculator we find that the length is

$$L = \sqrt{2.9^2 + 4.2^2} \approx 5.103\,920.$$

The corresponding function is $f(x,y) = \sqrt{x^2+y^2}$, expanded from $(3,4)$.

According to **Example 11.28.2** the approximation is given by

$$\begin{aligned}
(11.6)\quad P_2(z,y) &= 5+\frac{3}{5}(x-3)+\frac{4}{5}(y-4)+\frac{8}{125}(x-3)^2-\frac{12}{125}(x-3)(y-4)+\frac{9}{250}(y-4)^2 \\
(11.7)\quad &= 5+\frac{3}{5}(x-3)+\frac{4}{5}(y-4)+\frac{1}{250}\{4(x-3)-3(y-4)\}^2.
\end{aligned}$$

Since $x-3 = -\dfrac{1}{10}$ and $y-4 = \dfrac{1}{5} = \dfrac{2}{10}$, it follows from (11.6) that

$$\begin{aligned}
P_2(2,9;4,2) &= 5+\frac{6}{10}\left(-\frac{1}{10}\right)+\frac{8}{10}\cdot\frac{2}{10}+\frac{64}{1000}\left(-\frac{1}{10}\right)^2-\frac{96}{1000}\left(-\frac{1}{10}\right)\frac{2}{10}+\frac{36}{1000}\left(\frac{2}{10}\right)^2 \\
&= 5-\frac{6}{100}+\frac{16}{100}+\frac{1}{100\,000}(64+192+144) \\
&= 5+\frac{1}{10}+\frac{400}{100\,000} = 5.104.
\end{aligned}$$

If we instead use (11.7), we get by somewhat simpler calculations,

$$\begin{aligned}
P_2(2,9;4,2) &= 5+\frac{6}{10}\left(-\frac{1}{10}\right)+\frac{8}{10}\cdot\frac{2}{10}+\frac{4}{1000}\left\{-\frac{4}{10}-3\cdot\frac{2}{10}\right\}^2 \\
&= 5+\frac{1}{10}+\frac{4}{1000} = 5.104.
\end{aligned}$$

By comparison we see that the relative error is $< 1.6 \cdot 10^{-3}\%$.

2) A calculation on a pocket calculator shows that the length is

$$L = \sqrt{3.03^2+5.98^2+6.01^2} \approx 9.003\,410.$$

The corresponding function is $f(x,y,z) = \sqrt{x^2+y^2+z^2}$, expanded from the point $(3,6,6)$. According to **Example 11.28.6** the corresponding approximation is given by

$$\begin{aligned}
P_2(x,y,z) &= 9+\frac{1}{3}(x-3)+\frac{2}{3}(y-6)+\frac{2}{3}(x-6)+\frac{4}{81}(x-3)^2+\frac{5}{162}(y-6)^2+\frac{5}{162}(z-6)^2 \\
&\quad -\frac{2}{81}(x-3)(y-6)-\frac{4}{81}(y-6)(z-6)-\frac{2}{81}(z-6)(x-3).
\end{aligned}$$

When r $(x, y, z) = (3.03; 5.98; 6.01)$, we have $x - 3 = \frac{3}{100}$ and $y - 6 = -\frac{2}{100}$ and $z - 6 = \frac{1}{100}$. Then we get the approximate value by insertion,

$$\begin{aligned}
P_2(3.03; 5.98; 6.01) &= 9 + \frac{1}{3} \cdot \frac{3}{100} + \frac{2}{3}\left(-\frac{2}{100}\right) + \frac{2}{3} \cdot \frac{1}{100} \\
&\quad + \frac{4}{81}\left(\frac{3}{100}\right)^2 + \frac{5}{162}\left(-\frac{2}{100}\right)^2 + \frac{5}{162}\left(\frac{1}{100}\right)^2 \\
&\quad - \frac{2}{81} \cdot \frac{3}{100}\left(-\frac{2}{100}\right) - \frac{4}{81}\left(-\frac{2}{100}\right) \cdot \frac{1}{100} - \frac{2}{81}\left(\frac{1}{100}\right)\left(\frac{3}{100}\right) \\
&= 9 + \frac{1}{300}(3 - 4 + 2) + \frac{1}{162 \cdot 10\,000}(2 \cdot 4 \cdot 9 + 5 \cdot 405 + 24 + 16 - 12) \\
&= 9 + \frac{1}{300} + \frac{1}{1\,620\,000}(72025 + 28) \\
&= 9 + \frac{1}{300} + \frac{125}{162 \cdot 10\,000} \\
&= 9 + \frac{1}{900}\left(1 + \frac{5}{216}\right) = 9 + \frac{221}{64800} \\
&\approx 9.003\,410 \quad (!).
\end{aligned}$$

The error is invisible here, in particular because the value found on a pocket calculator is also an approximate value.

3) We get by means of a pocket calculator

$$\sqrt[5]{3.8^2 + 2 \cdot 2.21^3} \approx 2.,011\,883.$$

The corresponding function is $f(x, y) = \sqrt[5]{x^2 + 2y^3}$, expanded from the point $(4, 2)$.

We get from **Example 11.28.4** the approximation

$$P_2(x, y) = 2 + \frac{1}{10}(x - 4) + \frac{3}{10}(y - 2) + \frac{1}{400}(x - 4)^2 - \frac{3}{50}(x - 4)(y - 2) + \frac{3}{50}(y - 2)^2.$$

Since $x - 4 = -\frac{2}{10}$ and $y - 2 = \frac{1}{10}$, it follows by insertion that

$$\begin{aligned}
P_2(3.8; 2.1) &= 2 - \frac{2}{100} + \frac{3}{100} + \frac{1}{400} \cdot \frac{4}{100} - \frac{3}{50}\left(-\frac{2}{100}\right) + \frac{3}{50} \cdot \frac{1}{100} \\
&= 2 + \frac{1}{100} + \frac{1}{10\,000}(1 + 12 + 6) = 2 + \frac{1}{100} + \frac{19}{10\,000} = 2.0119.
\end{aligned}$$

A comparison shows that this is a very accurate approximation.

Example 11.30 *A function $f \in C^\infty(\mathbb{R}^2)$ satisfies the equations*

$$f(x,0) = e^x, \qquad f'_y(x,y) = 2y\, f(x,y).$$

Find the approximating polynomial of at most second degree for the function f with $(0,0)$ as the point of expansion.

A Approximating polynomial from apparently very vague assumptions.

D Find the constants by using the definition of partial differentiability.

I Since $f \in C^\infty$, we are allowed to interchange the order of the differentiations, whenever it is necessary. By using the standard method we get

$$\begin{aligned}
&f(x,0) = e^x, && f(0,0) = 1\\
&f'_x(x,0) = e^x, && f'_x(0,0) = 1,\\
&f'_y(x,y) = 2y\, f(x,y), && f'_y(0,0) = 0,\\
&f''_{xx}(x,0) = e^x, && f''_{xx}(0,0) = 1,\\
&f''_{xy}(x,y) = 2y\, f'_x(x,y), && f''_{xy}(0,0) = 0,\\
&f''_{yy}(x,y) = 2f(x,y) + 4y^2 f(x,y), && f''_{yy}(0,0) = 2.
\end{aligned}$$

The approximating polynomial is

$$P_2(x,y) = 1 + 1\cdot x + 0\cdot y + \frac{1}{2}\cdot 1\cdot x^2 + 0\cdot xy + \frac{1}{2}\cdot 2y^2 = 1 + x + \frac{1}{2}x^2 + y^2.$$

C It is actually possible to determine $f(x,y)$ uniquely from the given information. In fact, if we divide the latter equation by $f(x,y) \neq 0$, then

$$\frac{f_y'(x,y)}{f(x,y)} = \frac{\partial}{\partial y} \ln|f(x,y)| = 2y.$$

When we integrate with respect to y we get with some arbitrary function $\varphi(x)$ in x that $\ln|f(x,y)| = y^2 + \varphi(x)$. Hence there exists a function $\Phi(x)$, such that

$$f(x,y) = \Phi(x) \cdot \exp(y^2).$$

We put $y = 0$. Then it follows from the former of the given equations that

$$f(x,0) = e^x = \Phi(x).$$

Hence

$$\begin{aligned} f(x,y) &= \exp(x+y^2) = 1 + \{x+y^2\} + \frac{1}{2}\{x+y^2\}^2 + \cdots \\ &= 1 + x + y^2 + \frac{1}{2}x^2 + \cdots . \end{aligned}$$

It follows immediately that the approximating polynomial is

$$P_2(x,y) = 1 + x + \frac{1}{2}x^2 + y^2,$$

and we have tested our result. $\Diamond$

Example 11.31 *Indicate on a figure the domain of the function*

$$f(x,y) = \ln\left\{\left(4y - y^2 - x\right)\sqrt{x}\right\}.$$

Then find the approximating polynomial of at most first degree for f at the point of expansion $(2,1)$.

A Domain and approximating polynomial.

D Check where $f(x,y)$ is defined.

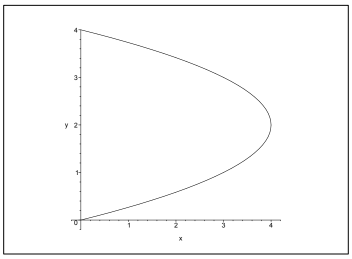

Figure 11.3: The domain of $f(x,y)$.

I The logarithm is only defined on the set of positive numbers, so $(4y - y^2 - x)\sqrt{x}$ must be defined and positive. In particular, $x > 0$ and $4y - y^2 - x > 0$, so

$$0 < 4y - y^2 - x = 4 - (4 - 4y + y^2) - x = 4 - (y-2)^2 - x,$$

and thus

$$0 < x < 4 - (y-2)^2.$$

The domain is bounded of the Y axis and the parabola of the equation $x = 4 - (y-2)^2$.

By the rearrangement

$$f(x,y) = \ln\left(4y - y^2 - x\right) + \frac{1}{2}\ln x \qquad \text{for } (x,y) \in D,$$

we get

$$f(2,1) = \ln(4-1-2) + \frac{1}{2}\ln 2$$

and

$$\frac{\partial f}{\partial x} = -\frac{1}{4y - y^2 - x} + \frac{1}{2}\cdot\frac{1}{x}, \qquad \frac{\partial f}{\partial x}(2,1) = -1 + \frac{1}{4} = -\frac{3}{4},$$

and

$$\frac{\partial f}{\partial y} = \frac{4-2y}{4y - y^2 - x}, \qquad \frac{\partial f}{\partial y}(2,1) = 2,$$

hence

$$P_1(x,y) = \frac{1}{2}\ln 2 - \frac{3}{4}(x-2) + 2(y-1).$$

Example 11.32 *It is well-known that an equation like*

$$f(x,y)=0$$

under suitable circumstances can be solved with respect to one of its variables, and one has e.g. $y = Y(x)$*, and then a differentiation of* $f(x,y)=0$ *with respect to* x *gives a formula of the derivative:*

$$Y'(x)=-\frac{f_x'(x,Y(x))}{f_y'(x,Y(x))}.$$

Prove by a similar procedure the formula

$$Y''(x)=-\frac{f_{yy}''(x,Y(x))\{Y'(x)\}^2+2f_{xy}''(x,Y(x))Y'(x)+f_{xx}''(x,Y(x))}{f_y'(x,Y(x))}.$$

This formula holds under the assumptions that the denominator is different from zero, and that both f *and* Y *are* C^2*-functions.*

A Implicit given function.

D Differentiate $f(x,Y(x))=0$ twice with respect to x.

I Under the given assumptions we get by an implicit differentiation (i.e. in fact the chain rule) that

$$\begin{aligned}0 &= \frac{\mathrm{d}}{\mathrm{d}x}\,f(x,Y(x))\\ &= f_x'(x,Y(x))\,\frac{\mathrm{d}x}{\mathrm{d}x}+f_y'(x,Y(x))\,\frac{\mathrm{d}Y}{\mathrm{d}x}\\ &= f_y'(x,Y(x))\cdot Y'(x)+f_x'(x,Y(x)),\end{aligned}$$

hence by another differentiation

$$\begin{aligned}0 &= f_y'(x,Y(x))\,Y''(x)+f_{xy}''(x,Y(x))\,Y'(x)+f_{yy}''(x,Y(x))\,\{Y'(x)\}^2\\ &\qquad +f_{xx}''(x,Y(x))+f_{xy}''(x,Y(x))\,Y'(x)\\ &= f_y'(x,Y(x))\,Y''(x)+f_{xx}''(x,Y(x))+2f_{xy}''(x,Y(x))\,Y'(x)+f_{yy}''(x,Y(x))\,\{Y'(x)\}^2.\end{aligned}$$

When we divide by $f_y'(x,Y(x))\neq 0$ and rearrange we obtain the searched formula.

Example 11.33 *Given the function*

$$f(x,y) = y^3 \cos x + y + x - 2, \qquad (x,y) \in \mathbb{R}^2.$$

1. *Solve the equation* $f(0,y) = 0$.

Then we get the information that the equation $f(x,y) = 0$ *in a neighbourhood of the point* $(0,1)$ *defines* y *uniquely as a function of* x, *i.e.* $y = Y(x)$.

2. *Find* $Y(0)$, *and then find* $Y'(0)$ *and* $Y''(0)$ *by using the formulæ from* **Example 11.32**. *Find the approximating polynomial of at most second degree for* Y *with the point of expansion* $x_0 = 0$.

A Implicit given function.

D Use the guidelines.

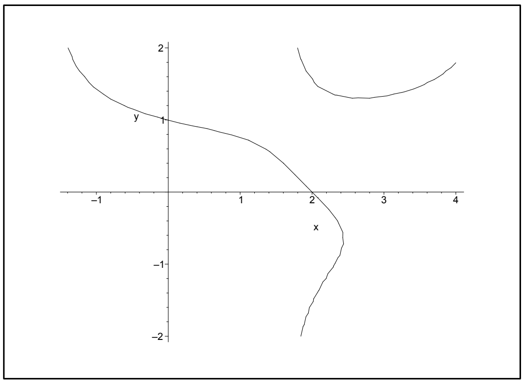

Figure 11.4: The graph of the equation $y^3 \cos x + y + x - 2 = 0$.

I 1) First solve the equation

$$0 = f(0,y) = y^3 + y - 2.$$

It is obvious that $y = 1$ is a solution. Since

$$f(0,y) = y^3 + y - 2 = y^3 - y + 2(y-1) = (y-1)(y^2+2),$$

it follows that $y = 1$ is the only real solution.

2) Then clearly $Y(0) = 1$. Furthermore,

$$f_x'(x,y) = -y^3 \sin x + 1, \qquad f_x(0,1) = 1,$$

$$f_y'(x,y) = 3y^2 \cos x + 1, \qquad f_y'(0,1) = 4,$$

$$f_{xx}''(x,y) = -y^3 \cos x, \qquad f_{xx}''(0,1) = -1,$$

$$f_{xy}''(x,y) = -3y^2 \sin x, \qquad f_{xy}''(0,1) = 0,$$

$$f_{yy}''(x,y) = 6y \cos x, \qquad f_{yy}''(0,1) = 6.$$

Using the formulæ of **Example 11.32** we get

$$Y'(0) = -\frac{f_x'(0,1)}{f_y'(0,1)} = -\frac{1}{4}$$

and

$$\begin{aligned} Y''(0) &= -\frac{f_{yy}''(0,1)\,\{Y'(0)\}^2 + 2f_{xy}''(0,1)\cdot Y'(0) + f_{xx}''(0,1)}{f_y'(0,1)} \\ &= -\frac{6\cdot\left(-\frac{1}{4}\right)^2 + 2\cdot 0\cdot\left(-\frac{1}{4}\right) - 1}{4} = -\frac{\frac{6}{16}-1}{4} = -\frac{1}{4}\left(\frac{3}{8}-1\right) = \frac{5}{32}. \end{aligned}$$

We get in particular the approximating polynomial of at most second degree,

$$P_2(x) = Y(0) + \frac{1}{1!}\,Y'(0)\cdot(x-x_0) + \frac{1}{2!}\,Y''(0)\cdot(x-x_0)^2 = 1 - \frac{1}{4}\,x + \frac{5}{64}\,x^2.$$

It is seen on the figure that the approximation is very accurate in the neighbourhood of $(0,1)$.

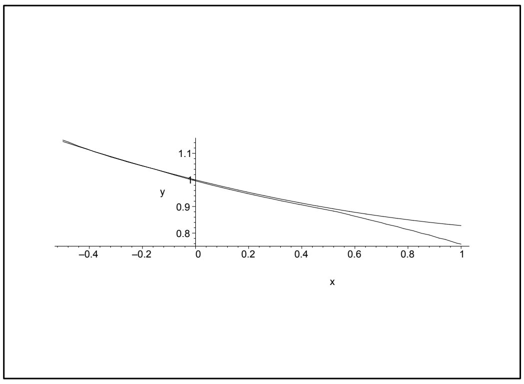

Figure 11.5: The graphs of $f(x,y)=0$ and the approximating polynomial from $(0,1)$.

Example 11.34 *Write Taylor's formula for a C^2-function f, where we choose successively the vector of increase (h_x,h_y) as*

$$(h,0), \qquad (0,h), \qquad (-h,0) \quad \textit{or} \quad (0,-h).$$

Explain why $\bigtriangledown^2 f(x,y)$ is a measure of how much $f(x,y)$ deviates from the average of the values of the function in the four neighbouring points. Prove in particular that an harmonic *function f approximately fulfils*

$$f(x,y)=\frac{1}{4}\{f(x+h,y)+f(x,y+h)+f(x-h,y)+f(x,y-h)\}.$$

Derive an analogous result in the case where one consider the four neighbouring points for which (h_x,h_y) is equal to

$$(h,h), \qquad (h,-h), \qquad (-h,h) \quad \textit{or} \quad (-h,-h).$$

A Taylor's formula; approximation of the average.

D Start by writing down Taylor's formula, and then make the analysis from this.

I Taylor's formula is

$$\begin{aligned} f\left(x+h_x,y+h_y\right) &= f(x,y)+h_x f_x'(x,y)+h_y f_y'(x,y) \\ &\quad +\frac{1}{2}\left\{h_x^2 f_{xx}''(x,y)+2h_xh_y f_{xy}''(x,y)+h_y^2 f_{yy}''(x,y)\right\} \\ &\quad +\varepsilon\left(h_x,h_y\right)\cdot\left(h_x^2+h_y^2\right), \end{aligned}$$

where $\varepsilon\left(h_x,h_y\right)\to 0$ for $(h_x,h_y)\to(0,0)$.

We get in particular,

$$f(x+h,y)=f(x,y)+h\,f_x'(x,y)+\frac{1}{2}\,h^2 f_{xx}''(x,y)+\varepsilon(h)\,h^2,$$

$$f(x-h,y)=f(x,y)-h\,f_x'(x,y)+\frac{1}{2}\,h^2 f_{xx}''(x,y)+\varepsilon(h)\,h^2,$$

$$f(x,y+h)=f(x,y)+h\,f_y'(x,y)+\frac{1}{2}\,h^2 f_{yy}''(x,y)+\varepsilon(h)\,h^2,$$

$$f(x,y-h)=f(x,y)-h\,f_y'(x,y)+\frac{1}{2}\,h^2 f_{yy}''(x,y)+\varepsilon(h)\,h^2.$$

The average is

$$\begin{aligned}\mathcal{M}f((x,y);h) &= \frac{1}{4}\{f(x+h,y)+f(x-h,y)+f(x,y+h)+f(x,y-h)\}\\ &= f(x,y)+\frac{1}{4}\,h^2\left\{f_{xx}''(x,y)+f_{yy}''(x,y)\right\}+\varepsilon(h)\,h^2\\ &= f(x,y)+\frac{h^2}{4}\,\bigtriangledown^2\,f(x,y)+\varepsilon(h)\,h^2.\end{aligned}$$

Then by a rearrangement

$$f(x,y)=\mathcal{M}f((x,y);h)-\frac{1}{4}\,h^2\;\bigtriangledown^2\,f(x,y)+\varepsilon(h)\,h^2,$$

so in this sense $\bigtriangledown^2 f(x,y)$ is a measure of the deviation of the average from the value of the function.

If f is harmonic then $\bigtriangledown^2 f(x,y)=0$, so

$$f(x,y)=\mathcal{M}f((x,y);h)+\varepsilon(h)\,h^2,$$

and we see that the average is a good approximation.

If we instead choose $(h_x,h_y)=(\pm h,\pm h)$ with all four possible combinations of the sign, then by letting f_x' etc. be a shorthand of $f_x'(x,y)$, etc.,

$$f(x+h,y+h)=f(x,y)+h\{f_x'+f_y'\}+\frac{1}{2}\,h^2\{f_{xx}''+2f_{xy}''+f_{yy}''\}+\varepsilon(h)\,h^2,$$

$$f(x-h,y-h)=f(x,y)-h\{f_x'+f_y'\}+\frac{1}{2}\,h^2\{f_{xx}''+2f_{xy}''+f_{yy}''\}+\varepsilon(h)\,h^2,$$

$$f(x+h,y-h)=f(x,y)+h\{f_x'-f_y'\}+\frac{1}{2}\,h^2\{f_{xx}''-2f_{xy}''+f_{yy}''\}+\varepsilon(h)\,h^2,$$

$$f(x-h,y+h)=f(x,y)-h\{f_x'-f_y'\}+\frac{1}{2}\,h^2\{f_{xx}''-2f_{xy}''+f_{yy}''\}+\varepsilon(h)\,h^2.$$

Here the average is

$$\begin{aligned}\tilde{\mathcal{M}}f((x,y);h) &= \frac{1}{4}\{f(x+h,y+h)+f(x-h,y-h)+f(x+h,y-h)+f(x-,y+h)\}\\ &= f(x,y)+\frac{1}{2}\,h^2\,\bigtriangledown^2\,f(x,y)+\varepsilon(h)\cdot h^2,\end{aligned}$$

hence by a rearrangement

$$f(x,y) = \tilde{\mathcal{M}} f((x,y);h) - \frac{1}{2}\, h^2 \bigtriangledown^2 f(x,y) + \varepsilon(h)\, h^2.$$

We get the same conclusion as above, since the only difference is the factor $\frac{1}{2}$ instead of $\frac{1}{4}$.

If f is harmonic, we also get in this case that

$$f(x,y) = \tilde{\mathcal{M}} f((x,y);h) + \varepsilon(h)\, h^2,$$

and we see again that the average is a very good approximation.

Example 11.35 *Find the approximating polynomial of at most second degree of the function*

$$f(x,y) = x \sinh(x+2y), \qquad (x,y) \in \mathbb{R}^2$$

expanded from the point $(x,y) = (2,-1)$.

A Approximating polynomial.

D Either use Taylor's formula or known series expansions.

I First method. First calculate

$$\begin{array}{ll} f(x,y) = x \sinh(x+2y), & f(2,-1) = 0, \\ f_x'(x,y) = \sinh(x+2y) + x\cosh(x+2y), & f_x'(2,-1) = 2, \\ f_y'(x,y) = 2x\cosh(x+2y), & f_y'(2,-1) = 4, \\ f_{xx}''(x,y) = 2\cosh(x+2y) + x\sinh(x+2y), & f_{xx}''(2,-1) = 2, \\ f_{xy}''(x,y) = 2\cosh(x+2y) + 2x\sinh(x+2y), & f_{xy}''(2,-1) = 2, \\ f_{yy}''(x,y) = 4x\sinh(x+2y), & f_{yy}''(2,-1) = 0. \end{array}$$

By means of the second column we get the coefficients of the Taylor expansion, hence

$$\begin{aligned} P_2(x,y) &= 0+\frac{1}{1!}\{2(x-2)+4(y+1)\}+\frac{1}{2!}\{2(x-2)^2+2\cdot 2(x-2)(y+1)+0\} \\ &= 2(x-2)+4(y+1)+(x-2)^2+2(x-2)(y+1). \end{aligned}$$

Second method. First change variables by putting $x = 2+\xi$ and $y = -1+\eta$. Then by insertion followed by known series expansions, in which terms of higher order are written as dots,

$$\begin{aligned} f(x,y) &= x\sinh(x+2y) = (2+\xi)\sinh(\xi+2\eta) \\ &= (2+\xi)\{(\xi+2\eta)+\cdots\} \\ &= 2\xi+4\eta+\xi^2+2\xi\eta+\cdots \\ &= 2(x-2)+4(y+1)+(x-2)^2+2(x-2)(y+1)+\cdots, \end{aligned}$$

hence

$$P_2(x,y) = 2(x-2)+4(y+1)+(x-2)^2+2(x-2)(y+1).$$

Remark. Of numerical reasons one shall *always* in examples of approximating polynomials use the variables $\mathbf{x}-\mathbf{x}_0$, here $(x-2, y+1)$, because the expansion is bound to the point $\mathbf{x}_0$, here $(2,-1)$. Many textbooks erroneously "reduce" further to the variables (x,y). $\Diamond$

Example 11.36 *Find the approximating polynomial of at most second degree for the function*

$$g(x,y) = \sqrt{4-2x^2-y^2}, \qquad 2x^2+y^2<4,$$

with the point of expansion $(1,1)$.

A Approximating polynomial.

D Either use Taylor's formula, or rewrite $g(x,y)$ as some known function for which we know the Taylor series.

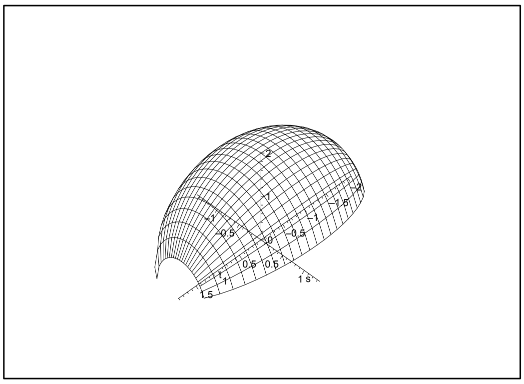

Figure 11.6: Part of the graph of $g(x,y)$.

I If we put $z = g(x,y) = \sqrt{4-2x^2-y^2} \geq 0$, it follows by a squaring and a rearrangement that the equation of the surface can also be written

$$\left(\frac{x}{\sqrt{2}}\right)^2 + \left(\frac{y}{2}\right)^2 + \left(\frac{z}{2}\right)^2 = 1, \qquad z \geq 0,$$

i.e. the graph is the upper half of an ellipsoidal surface of centre $(0,0,0)$ and half axes $\sqrt{2}$, 2 and 2.

First method. Clearly, the function $g(x,y)$ is of class C^∞ in the domain, where the point $(1,1)$ lies. Then calculate

$$g(x,y) = \sqrt{4-2x^2-y^2}, \qquad g(1,1) = 1,$$

$$g'_x(x,y) = -\frac{2x}{\sqrt{4-2x^2-y^2}}, \qquad g'_x(1,1) = -2,$$

$$g'_y(x,y) = -\frac{y}{\sqrt{4-2x^2-y^2}}, \qquad g'_y(1,1) = -1,$$

$$g''_{xx}(x,y) = -\frac{2}{\sqrt{4-2x^2-y^2}} - \frac{4x^2}{(\sqrt{4-2x^2-y^2})^3}, \quad g''_{xx}(1,1) = -6,$$

$$g''_{xy}(x,y) = -\frac{2xy}{(\sqrt{4-2x^2-y^2})^3}, \qquad g''_{xy}(1,1) = -2,$$

$$g''_{yy}(x,y) = -\frac{1}{\sqrt{4-2x^2-y^2}} - \frac{y^2}{(\sqrt{4-2x^2-y^2})^3}, \quad g''_{yy}(1,1) = -2.$$

Then the approximating polynomial is according to Taylor's formula and the right hand column

$$\begin{aligned} P_2(x,y) &= 1-2(x-1)-(y-1)+\frac{1}{2}\left\{-6(x-1)^2-2\cdot 2(x-1)(y-1)-2(y-1)^2\right\} \\ &= 1-2(x-1)-(y-1)-3(x-1)^2-2(x-1)(y-1)-(y-1)^2. \end{aligned}$$

Second method. First introduce some new variables by $x = 1+\xi$ and $y = 1+\eta$. Then by insertion and introduction of a known series expansion for $\sqrt{1+t}$, where the dots as usual indicate terms of higher order,

$$\begin{aligned} g(x,y) &= \sqrt{4-2(1+\xi)^2-(1+\eta)^2} = \sqrt{1-4\xi+2\xi^2-2\eta-\eta^2} \\ &= 1-\frac{1}{2}\left(4\xi+2\eta+2\xi^2+\eta^2\right) - \frac{1}{8}(4\xi+2\eta+\cdots)^2+\cdots \\ &= 1-2\xi-\eta-\xi^2-\frac{1}{2}\eta^2-\frac{1}{8}\left(16\xi^2+16\xi\eta+4\eta^2\right)+\cdots \\ &= 1-2\xi-\eta-3\xi^2-2\xi\eta-\eta^2+\cdots, \end{aligned}$$

and we conclude that the approximating polynomial is

$$\begin{aligned} P_2(x,y) &= 1-2\xi-\eta-3\xi^2-2\xi\eta-\eta^2 \\ &= 1-2(x-1)-(y-1)-3(x-1)^2-2(x-1)(y-1)-(y-1)^2. \end{aligned}$$

Example 11.37 *Find the approximating polynomial of at most second degree of the function*

$$f(x,y) = \ln x + \exp(xy-2), \qquad (x,y) \in \mathbb{R}_+ \times \mathbb{R},$$

expanded from the point $(x,y) = (1,2)$.

A Approximating polynomial.

D Either calculate the Taylor coefficients, or use some known series expansions.

I First method. *The standard method.* Clearly, $f \in C^\infty(\mathbb{R}_+ \times \mathbb{R})$ and $(1,2) \in \mathbb{R}_+ \times \mathbb{R}$. Then by differentiation,

$$\begin{array}{ll}
f(x,y) = \ln x + \exp(xy-2), & f(1,2) = 1, \\[2mm]
f_x'(x,y) = \dfrac{1}{x} + y\,\exp(xy-2), & f_x'(1,2) = 3, \\[2mm]
f_y'(x,y) = x\,\exp(xy-2), & f_y'(1,2) = 1, \\[2mm]
f_{xx}''(x,y) = -\dfrac{1}{x^2} + y^2\exp(xy-2), & f_{xx}''(1,2) = 3, \\[2mm]
f_{xy}''(x,y) = f_{yx}''(x,y) = (1+xy)\exp(xy-2), & f_{xy}''(1,2) = 3, \\[2mm]
f_{yy}''(x,y) = x^2\exp(xy-2), & f_{yy}''(1,2) = 1.
\end{array}$$

The approximating polynomial of at most second degree is

$$\begin{aligned}
P_2(x,y) &= f(1,2) + f_x'(1,2)\cdot(x-1) + f_y'(1,2)\cdot(y-2) \\
&\quad + \frac{1}{2}\left\{f_{xx}''(1,2)\,(x-1)^2 + 2f_{xy}''(1,2)\,(x-1)(y-2) + f_{yy}''(1,2)\,(y-2)^2\right\} \\
&= 1+3(x-1)+(y-2)+\frac{3}{2}(x-1)^2+3(x-1)(y-2)+\frac{1}{2}(y-2)^2.
\end{aligned}$$

Second method. *Suitable series expansions of known standard functions.* First rewrite $f(x,y)$ as a function of the translated variables $(x-1,y-2)$, which are zero at the point of expansion $(1,2)$. Then

$$\begin{aligned}
f(x,y) &= \ln x + \exp(xy-2) \\
&= \ln(1+(x-1))+\exp\{(x-1)(y-2)+2x+y-4\} \\
&= \ln\{1+(x-1)\}+\exp\{(x-1)(y-2)+2(x-)+(y-2)\} \\
&= \ln\{1+(x-)\}+\exp\{(x-1)(y-2)\}\cdot\exp\{2(x-1)\}\cdot\exp(y-2).
\end{aligned}$$

By means of known series expansions for $\ln(1+t)$ and $\exp(t)$, where we remove all terms of degree higher than 2 in $x-1$ and $y-2$, we get

$$\begin{aligned}
f(x,y) &= \ln\{1+(x-1)\}+\exp\{(x-1)(y-2)\}\cdot\exp\{2(x-1)\}\cdot\exp(y-2)\\
&= (x-1)-\frac{1}{2}\,(x-1)^2+\cdots\\
&\quad +\{1+(x-1)(y-2)+\cdots\}\{1+2(x-1)+2(x-1)^2+\cdots\}\{1+(y-2)+\frac{1}{2}(y-2)^2+\cdots\}\\
&= (x-1)-\frac{1}{2}(x-1)^2+(x-1)(y-2)+1+(y-2)+\frac{1}{2}(y-2)^2\\
&\qquad +2(x-1)+2(x-1)(y-2)+2(x-1)^2+\cdots\\
&= \frac{3}{2}(x-1)^2+3(x-1)(y-2)+\frac{1}{2}(y-2)^2+3(x-1)+(y-2)+1+\cdots,
\end{aligned}$$

and we conclude that

$$P_2(x,y) = 1+3(x-1)+(y-2)+\frac{3}{2}(x-1)^2+3(x-1)(y-2)+\frac{1}{2}(y-2)^2.$$

Example 11.38 *1) Sketch the domain D of*

$$f(x,y) = \sqrt{2+x-y} + \ln\left(4-x^2-y^2\right).$$

2) Check whether D is open or closed or none of the kind.

3) Find the approximating polynomial of at most first degree for f with $(1,-1)$ as point of expansion.

4) Find the domain E of the vector field

$$\mathbf{V}(x,y) = \left(\sqrt{2+x-y}, \sqrt{y} + \ln\left(4-x^2-y^2\right)\right).$$

A Domains, open and closed sets, approximating polynomial.

D Standard task.

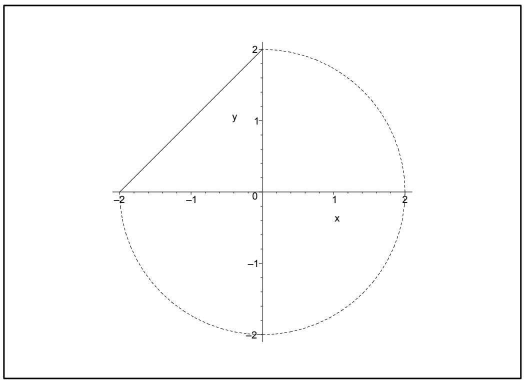

Figure 11.7: The domain D.

I 1) The function $f(x,y) = \sqrt{2+x-y} + \ln\left(4-x^2-y^2\right)$ is defined for

$$2+x-y \geq 0 \qquad \text{and} \qquad 4-x^2-y^2 > 0,$$

hence for

$$y \leq x+2 \qquad \text{and} \qquad x^2+y^2 < 4 = 2^2.$$

2) The set D is neither open nor closed.

3) The approximating polynomial from $(1,-1)$.

First variant. It follows from

$$f(x,y) = \sqrt{2+x-y} + \ln\left(4-x^2-y^2\right), \qquad f(1,-1) = 2 + \ln 2,$$

$$f_x'(x,y) = \frac{1}{2}\,\frac{1}{\sqrt{2+x-y}} - \frac{2x}{4-x^2-y^2}, \qquad f_x'(1,-1) = -\frac{3}{4},$$

$$f_y'(x,y) = -\frac{1}{2}\,\frac{1}{\sqrt{2+x-y}} - \frac{2y}{4-x^2-y^2}, \qquad f_y'(1,-1) = \frac{3}{4},$$

that

$$P_1(x,y) = 2 + \ln 2 - \frac{3}{4}(x-1) + \frac{3}{4}(y+1).$$

Second variant. If we put $x = x_1 + 1$ and $y = y_1 - 1$, then we get by series expansions,

$$\begin{aligned}
f(x,y) &= \sqrt{2+x-y}+\ln(4-x^2-y^2)\\
&= \sqrt{2+x_1+1-y_1+1}+\ln(4-\{x_1+1\}^2-\{y_1-1\}^2)\\
&= \sqrt{4+x_1-y_1}+\ln(2-2x_1+2y_1-x_1^2-y_1^2)\\
&= 2\sqrt{1+\frac{x_1}{4}-\frac{y_1}{4}}+\ln 2+\ln\left(1-x_1+y_1-\frac{1}{2}\,x_1^2-\frac{1}{2}\,y_1^2\right)\\
&= 2\left\{1+\frac{1}{2}\left(\frac{x_1}{4}-\frac{y_1}{4}\right)+\cdots\right\}+\ln 2+\left\{-x_1+y_1-\frac{1}{2}\,x_1^2-\frac{1}{2}\,y_1^2\right\}+\cdots\\
&= 2+\ln 2+\frac{1}{4}\,x_1-\frac{1}{4}\,y_1-x_1+y_1+\cdots,
\end{aligned}$$

where the dots as usual indicate terms of higher order. We conclude that

$$P_1(x,y) = 2+\ln 2-\frac{3}{4}\,x_1+\frac{3}{4}\,y_1 = 2+\ln 2-\frac{3}{4}\,(x-1)+\frac{3}{4}\,(y+1).$$

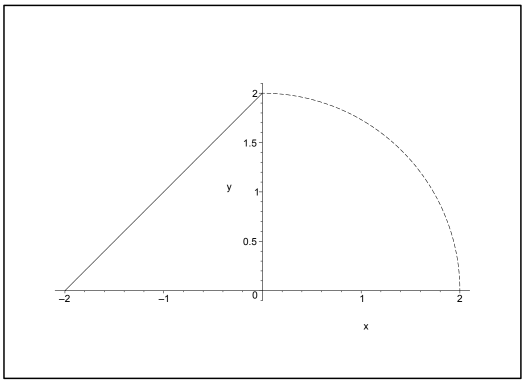

Figure 11.8: The domain E of the vector field $\mathbf{V}$.

4) The domain E of the vector field consists of the points in D, for which $\sqrt{y}$ is also defined, so we must also require that $y \geq 0$.

Example 11.39 *1) Sketch the domain D of*

$$f(x,y)=e^{x+y}+\ln\left(4-x^2-4y^2\right).$$

2) Check if D is open or closed of none of the kind.

3) Find the approximating polynomial of at most second degree for f with $(0,0)$ as point of expansion.

A Domain and approximating polynomial for a function.

D Analyze where each subfunction is defined. Then the approximating polynomial is either found by means of known series expansions or by calculating the Taylor coefficients.

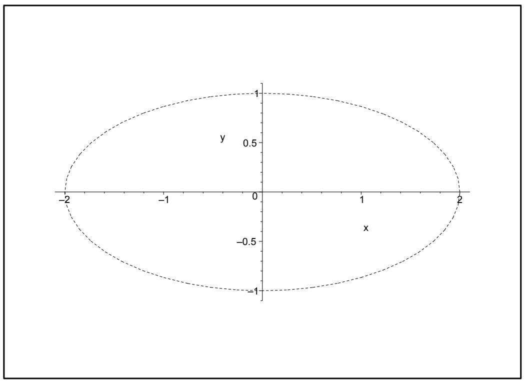

Figure 11.9: The domain D is the open ellipsoidal disc.

I 1) The function e^{x+y} is defined for every $(x,y)\in\mathbb{R}^2$.
The function $\ln\left(4-x^2-4y^2\right)$ is defined, if and only if $4-x^2-4y^2>0$, i.e. if and only if

$$\left(\frac{x}{2}\right)^2+\left(\frac{y}{1}\right)^2<1.$$

The domain is the open ellipsoidal disc of centrum $(0,0)$ and half axes 2 and 1, cf. the figure.

2) As mentioned above in 1), the set D is open.

3) **First variant.** *Known series expansions.*
Let $(x,y)\in K(\mathbf{0};1)\subset D$, and let dots denote terms of higher degree than 2. Then

$$\begin{aligned}f(x,y) &= e^{x+y}+2\ln 2+\ln\left(1-\frac{1}{4}\left(x^2+4y^2\right)\right)\\ &= 1+\frac{1}{1!}(x+y)+\frac{1}{2!}(x+y)^2+\cdots+2\ln 2-\frac{1}{4}\left(x^2+4y^2\right)+\cdots\\ &= 1+2\ln 2+x+y+\frac{1}{2}x^2+xy+\frac{1}{2}y^2-\frac{1}{4}x^2-y^2+\cdots\\ &= 1+2\ln 2+x+y+\frac{1}{4}x^2+xy-\frac{1}{2}y^2+\cdots.\end{aligned}$$

The approximating polynomial of at most second degree from $(0,0)$ is

$$P_2(x,y)=1+2\ln 2+x+y+\frac{1}{4}x^2+xy-\frac{1}{2}y^2.$$

Second variant. *Taylor expansion.*
We get by successive differentiation

$$\begin{aligned}
&f(x,y)=e^{x+y}+\ln(4-x^2-4y^2), && f(0,0)=1+\ln 4=1+2\ln 2,\\
&f_x'(x,y)=e^{x+y}-\frac{2x}{4-x^2-yy^2}, && f_x'(0,0)=1,\\
&f_y'(x,y)=e^{x+y}-\frac{8y}{4-x^2-4y^2}, && f_y'(0,0)=1,\\
&f_{xx}''(x,y)=e^{x+y}-\frac{2}{4-x^2-4y^2}-\frac{4x^2}{(4-x^2-4y^2)^2}, && f_{xx}''(0,0)=1-\frac{2}{4}=\frac{1}{2},\\
&f_{yy}''(x,y)=e^{x+y}-\frac{8}{4-x^2-4y^2}-\frac{64y^2}{(4-x^2-4y^2)^2}, && f_{yy}''(0,0)=1-\frac{8}{4}=-1,\\
&f_{xy}''(x,y)=e^{x+y}-\frac{16xy}{(4-x^2-4y^2)^2}, && f_{xy}''(0,0)=1.
\end{aligned}$$

Hence

$$P_2(x,y)=1+2\ln 2+x+y+\frac{1}{4}\,x^2-\frac{1}{2}\,y^2+xy.$$

REMARK. The expressions of the second derivative may occur in several variants:
a)

$$f''_{xx}(x,y) = e^{x+y} - \frac{8+2x^2-8y^2}{(4-x^2-4y^2)^2}, \quad f''_{xx}(0,0) = 1 - \frac{8}{16} = \frac{1}{2},$$

$$f''_{yy}(x,y) = e^{x+y} - \frac{32-8x^2+32y^2}{(4-x^2-4y^2)^2}, \quad f''_{yy}(0,0) = 1 - \frac{32}{16} = -1,$$

together with the more elegant version, where dots denote terms which will become zero by the insertion of $(x,y) = (0,0)$:
b)

$$f''_{xx}(x,y) = e^{x+y} - \frac{2}{4-x^2-4y^2} + \cdots, \quad f''_{xx}(0,0) = 1 - \frac{2}{4} = \frac{1}{2},$$

$$f''_{yy}(x,y) = e^{x+y} - \frac{8}{4-x^2-4y^2} + \cdots, \quad f''_{yy}(0,0) = 1 - \frac{8}{4} = -1,$$

$$f''_{xy}(x,y) = e^{x+y} + \cdots, \qquad f''_{xy}(0,0) = 1.$$

Example 11.40 *Given the function*

$$f(x,y) = e^{xy} + (2-x)e^y - 2ey, \qquad (x,y) \in \mathbb{R}^2.$$

Find the approximating polynomial of at most second degree for f with $(1,1)$ as point of expansion.

A Approximating polynomial.

D The function is clearly of class C^∞. Either calculate the Taylor coefficients, or use known series expansions.

I First method. *Calculation of the Taylor coefficients.*
We get by mechanical computations,

$$f(x,y) = e^{xy} + (2-x)e^y - 2ey, \qquad f(1,1) = 0,$$

$$f'_x(x,y) = y\,e^{xy} - e^y, \qquad f'_x(1,1) = 0,$$

$$f'_y(x,y) = x\,e^{xy} + (2-x)e^y - 2e, \qquad f'_y(1,1) = 0,$$

$$f''_{xx}(x,y) = y^2 e^{xy}, \qquad f''_{xx}(1,1) = e,$$

$$f''_{xy}(x,y) = e^{xy} + xy\,e^{xy} - e^y, \qquad f''_{xy}(1,1) = e,$$

$$f''_{yy}(x,y) = x^2 e^{xy} + (2-x)e^y, \qquad f''_{yy}(1,1) = 2e.$$

Then the approximating polynomial of at most second degree for f from $(1,1)$ is

$$\begin{aligned} P_2(x,y) &= f(1,1) + \frac{1}{1!}\left\{f'_x(1,1)\cdot(x-1) + f'_y(1,1)\cdot(y-1)\right\} \\ &\quad + \frac{1}{2!}\left\{f''_{xx}(1,1)\cdot(x-1)^2 + f''_{xy}(1,1)\cdot(x-1)(y-1) + f''_{yy}(1,1)\cdot(y-1)^2\right\} \\ &= \frac{e}{2}(x-1)^2 + e(x-1)(y-1) + e(y-1)^2. \end{aligned}$$

Second method. *Application of known series expansions.*

When we translate

$$x_1 = x-1, \quad y_1 = y-1, \quad \text{i.e.} \quad x = x_1+1, \quad y = y_1+1,$$

to the point of expansion and use known series expansions up to the second degree (and where terms of higher degrees are indicated by dots) we get

$$\begin{aligned} f(x,y) &= e^{xy} + (2-x)e^{y} - 2ey \\ &= \exp((x_1+1)(y_1+1)) + (1-x_1)\exp(y_1+1) - 2e(y_1+1) \\ &= \exp(1+x_1+y_1+x_1y_1) + e(1-x_1)\exp(y_1) - 2e - 2ey_1 \\ &= e\left\{\exp(x_1+y_1)\cdot\exp(x_1y_1) + (1-x_1)\exp(y_1) - 2 - 2y_1\right\}, \end{aligned}$$

and hence

$$\begin{aligned} f(x,y) &= e\left(\left\{1+x_1+y_1+\frac{1}{2}\,(x_1+y_1)^2+\cdots\right\}\{1+x_1y_1+\cdots\}\right. \\ &\qquad \left. +(1-x_1)\left\{1+y_1+\frac{1}{2}\,y_1^2+\cdots\right\}-2-2y_1\right) \\ &= e\left\{1+x_1+y_1+\frac{1}{2}\,x_1^2+x_1y_1+\frac{1}{2}\,y_1^2+x_1y_1+\cdots\right. \\ &\qquad \left. +1+y_1+\frac{1}{2}\,y_1^2-x_1-x_1y_1+\cdots-2-2y_1\right\} \\ &= e\left\{\frac{1}{2}\,x_1^2+x_1y_1+y_1^2+\cdots\right\}. \end{aligned}$$

The dots indicate terms of higher degree, so we conclude that the approximating polynomial of at most second degree with $(1,1)$ as point of expansion is

$$P_2(x,y)=\frac{e}{2}\,x_1^2+ex_1y_1+ey_1^2=\frac{e}{2}\,(x-1)^2+e(x-1)(y-1)+e(y-1)^2.$$

Example 11.41 *Given the function*

$$f(x,y)=\sqrt{1-2x-y}+\ln(1-2y+x), \qquad (x,y)\in D.$$

1) Find the domain D.

2) Sketch D.

3) Check if D is

a) open,
b) closed,
c) bounded,
d) star shaped.

4) Find the approximating polynomial ofat most second degree for f with the point of expansion $(0,0)$.

A Domain; approximating polynomial.

D Analyze each part of the function separately and take the intersections of all these domains. Then use either known series expansions, or calculate the Taylor coefficients.

I 1.–3. The function is defined when

$$1-2x-y\geq 0 \qquad \text{and} \qquad 1-2y+x>0,$$

i.e. when

$$y\leq 1-2x \qquad \text{og} \qquad y<\frac{1}{2}\,(x+1),$$

or written in another way,

$$x\leq\frac{1}{2}\,(1-y) \qquad \text{and} \qquad x>2y-1.$$

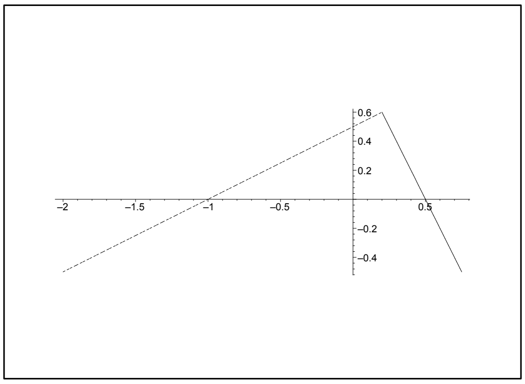

Figure 11.10: The domain D is the angular space inclusive the fully drawn boundary curve and exclusive the dotted boundary curve. The domain in unbounded downwards.

Since the lines intersect at $\left(\dfrac{1}{5}, \dfrac{3}{5}\right)$, the domain can be written

$$D = \left\{ (x,y) \;\middle|\; y < \frac{3}{5},\, 2y-1 < x \le \frac{1}{2}\,(1-y) \right\}.$$

Note that D is the intersection of an open and a closed half plane.

We see immediately that

1) D is not open, because a part of the boundary, though not the total boundary, lies in D,
2) D is not closed, because a part of the boundary, though not the total boundary, lies outside D,
3) D is not bounded. The whole of the negative Y axis lies in D.
4) Since D is the intersection of two convex sets, it is itself convex and therefore also starshaped with respect to any point in D.

4. We have here two variants.

First variant. *The standard method.* It follows from the computations

$$f(x,y) = \sqrt{1-2x-y} + \ln(1-2y+x), \qquad f(0,0) = 1,$$

$$f'_x(x,y) = -\frac{1}{\sqrt{1-2x-y}} + \frac{1}{1-2y+x}, \qquad f'_x(0,0) = 0,$$

$$f'_y(x,y) = -\frac{1}{2}\,\frac{1}{\sqrt{1-2x-y}} - \frac{2}{1-2y+x}, \qquad f'_y(0,0) = -\frac{5}{2},$$

$$f''_{xx}(x,y) = \frac{1}{(1-2x-y)^{3/2}} - \frac{1}{(1-2y+x)^2}, \qquad f''_{xx}(0,0) = -2,$$

$$f''_{xy}(x,y) = -\frac{1}{2}\,\frac{1}{(1-2x-y)^{3/2}} - \frac{4}{(1-2y+x)^2}, \quad f''_{yy}(0,0) = \frac{17}{4},$$

that the approximating polynomial of at most second degree from $(0,0)$ is

$$\begin{aligned} P_2(x,y) &= f(0,0)+f_x'(0,0)\cdot x+f_y'(0,0)\cdot y \\ &\qquad +\frac{1}{2}\left\{f_{xx}''(0,0)\cdot x^2+2f_{xy}''(0,0)\cdot xy+f_{yy}''(0,0)\cdot y^2\right\} \\ &= 1-\frac{5}{2}\,y-x^2+\frac{3}{2}\,xy-\frac{17}{8}\,y^2. \end{aligned}$$

Second variant. *Known series expansions.* It is well-known that

$$\sqrt{1+t}=1+\binom{\frac{1}{2}}{1}t+\binom{\frac{1}{2}}{2}t^2+\cdots=1+\frac{1}{2}\,t-\frac{1}{8}\,t^2+\cdots,$$

and

$$\ln(1+u)=u-\frac{1}{2}\,u^2+\cdots.$$

If we put

$$t=-(2x+y)=-2x-y \qquad \text{and} \qquad u=x-2y,$$

then both t and u are for the first degree in (x,y), and the approximating polynomial of at most second degree is

$$\begin{aligned}P_2(x,y) &= 1+\frac{1}{2}\,t-\frac{1}{8}\,t^2+u-\frac{1}{2}u^2\\ &= 1-\frac{1}{2}\,(2x+y)-\frac{1}{8}\,(2x+y)^2+x-2y-\frac{1}{2}\,(x-2y)^2\\ &= 1-x-\frac{1}{2}\,y+x-2y-\frac{1}{8}\,(4x^2+4xy+y^2)-\frac{1}{2}\,(x^2-4xy+4y^2)\\ &= 1-\frac{5}{2}\,y-x^2+\frac{3}{2}\,xy-\frac{17}{8}\,y^2.\end{aligned}$$

Example 11.42 *1) Sketch the domain D of the function*

$$f(x,y)=\ln\left(4-x^2-y^2\right)-\sqrt{5-4x}+y^2.$$

2) Check if D is open or closed or none of the kind.

3) Compute the gradient $\bigtriangledown f$.

4) Find the approximating polynomial of at most first degree for the function f, when the point $(1,\sqrt{2})$ is used as point of expansion.

A Domain, gradient, approximating polynomial.

D Treat every subfunction separately. The approximating polynomial can then be found in several ways.

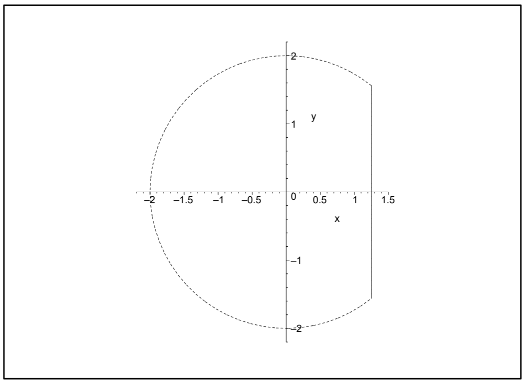

Figure 11.11: The domain D.

I 1) The function $\ln\left(4-x^2-y^2\right)$ is defined for $4-x^2-y^2>0$, i.e. for $x^2+y^2<4=2^2$, which describes the open disc of centre $(0,0)$ and radius 2.

The function $\sqrt{5-4x}$ is defined for $5-4x \geq 0$, i.e. in the closed half space $x \leq \dfrac{5}{4}$.

Now y^2 is defined for every $(x,y) \in \mathbb{R}^2$, so the domain D is the intersection of the two sets mentioned above,

$$D = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; x^2+y^2 < 2^2,\, x \leq \frac{5}{4} \right\}.$$

2) The set D is neither open (a part of the boundary, $x = \dfrac{5}{4}$, lies in D) nor closed (another part, the circular arc, does not lie in D).

3) The gradient is calculated straight away,

$$\nabla f(x,y) = \left(-\frac{2x}{4-x^2-y^2} + \frac{2x}{\sqrt{5-4x}}, -\frac{2y}{4-x^2-y^2} + 2y \right).$$

Note that

$$\nabla f(1,\sqrt{2}) = \left(-\frac{2}{1} + \frac{2}{1}, -\frac{2\sqrt{2}}{1} + 2\sqrt{2} \right) = \mathbf{0},$$

so $(1,\sqrt{2})$ is a stationary point of f.

4) **First variant.** The approximating polynomial of at most first degree with $(1,\sqrt{2})$ as point of expansion is according to 3) given by,

$$\begin{aligned} P_1(x,y) &= f(1,\sqrt{2}) + \nabla f(1,\sqrt{2}) \cdot (x-1, y-\sqrt{2}) = f(1,\sqrt{2}) + 0 \\ &= \ln(4-1-2) - \sqrt{5-4} + 2 = 1. \end{aligned}$$

Second variant. If we put $x = s+1$ and $y = t+\sqrt{2}$, it follows by insertion and by using known series expansions that

$$\begin{aligned} f(x,y) &= \ln\left(4-x^2-y^2\right) - \sqrt{5-4x} + y^2 \\ &= \ln\left(4-(s+1)^2-(t+\sqrt{2})^2\right) - \sqrt{5-4(s+1)} + (t+\sqrt{2})^2 \\ &= \ln\left(1-2s-2\sqrt{2}\,t-s^2-t^2\right) - \sqrt{1-4s} + 2 + 2\sqrt{2}\,t + t^2 \\ &= -2s - 2\sqrt{2}\,t + \cdots - \left\{ 1 - \frac{1}{2} \cdot 4s + \cdots \right\} + 2 + 2\sqrt{2}\,t + \cdots \\ &= 1 + \cdots, \end{aligned}$$

where the dots as usual denote terms of degree ≥ 2.

The approximating polynomial of at most first degree from $(1,\sqrt{2})$ is therefore the constant

$$P_1(x,y) = 1.$$

Remark. There is nothing unusual in the fact that the approximating polynomial of at most first degree is a constant, i.e. a degerenated polynomial of degree zero. $\Diamond$

12 Formulæ

Some of the following formulæ can be assumed to be known from high school. It is highly recommended that one *learns most of these formulæ in this appendix by heart.*

12.1 Squares etc.

The following simple formulæ occur very frequently in the most different situations.

$$\begin{array}{ll} (a+b)^2 = a^2+b^2+2ab, & a^2+b^2+2ab = (a+b)^2, \\ (a-b)^2 = a^2+b^2-2ab, & a^2+b^2-2ab = (a-b)^2, \\ (a+b)(a-b) = a^2-b^2, & a^2-b^2 = (a+b)(a-b), \\ (a+b)^2 = (a-b)^2+4ab, & (a-b)^2 = (a+b)^2-4ab. \end{array}$$

12.2 Powers etc.

Logarithm:

$$\begin{array}{rcll} \ln|xy| &=& \ln|x|+\ln|y|, & x,y\neq 0, \\ \ln\left|\dfrac{x}{y}\right| &=& \ln|x|-\ln|y|, & x,y\neq 0, \\ \ln|x^r| &=& r\ln|x|, & x\neq 0. \end{array}$$

Power function, fixed exponent:

$$(xy)^r = x^r\cdot y^r,\, x,y>0 \quad \text{(extensions for some } r\text{)},$$

$$\left(\frac{x}{y}\right)^r = \frac{x^r}{y^r},\, x,y>0 \qquad \text{(extensions for some } r\text{)}.$$

Exponential, fixed base:

$$a^x\cdot a^y = a^{x+y}, \quad a>0 \quad \text{(extensions for some } x,\, y\text{)},$$
$$(a^x)^y = a^{xy},\, a>0 \qquad \text{(extensions for some } x,\, y\text{)},$$

$$a^{-x} = \frac{1}{a^x},\, a>0, \qquad \text{(extensions for some } x\text{)},$$

$$\sqrt[n]{a} = a^{1/n},\, a\geq 0, \qquad n\in\mathbb{N}.$$

Square root:

$$\sqrt{x^2} = |x|, \qquad x\in\mathbb{R}.$$

Remark 12.1 It happens quite frequently that students make errors when they try to apply these rules. They must be mastered! In particular, as one of my friends once put it: "If you can master the square root, you can master everything in mathematics!" Notice that this innocent looking square root is one of the most difficult operations in Calculus. Do not forget the *absolute value!* $\Diamond$

12.3 Differentiation

Here are given the well-known rules of differentiation together with some rearrangements which sometimes may be easier to use:

$$\{f(x) \pm g(x)\}' = f'(x) \pm g'(x),$$

$$\{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x) = f(x)g(x)\left\{\frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}\right\},$$

where the latter rearrangement presupposes that $f(x) \neq 0$ and $g(x) \neq 0$.
If $g(x) \neq 0$, we get the usual formula known from high school

$$\left\{\frac{f(x)}{g(x)}\right\}' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

It is often more convenient to compute this expression in the following way:

$$\left\{\frac{f(x)}{g(x)}\right\} = \frac{d}{dx}\left\{f(x) \cdot \frac{1}{g(x)}\right\} = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2} = \frac{f(x)}{g(x)}\left\{\frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}\right\},$$

where the former expression often is *much easier* to use in practice than the usual formula from high school, and where the latter expression again presupposes that $f(x) \neq 0$ and $g(x) \neq 0$. Under these assumptions we see that the formulæ above can be written

$$\frac{\{f(x)g(x)\}'}{f(x)g(x)} = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)},$$

$$\frac{\{f(x)/g(x)\}'}{f(x)/g(x)} = \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}.$$

Since

$$\frac{d}{dx}\ln|f(x)| = \frac{f'(x)}{f(x)}, \qquad f(x) \neq 0,$$

we also name these the *logarithmic derivatives.*

Finally, we mention the rule of **differentiation of a composite function**

$$\{f(\varphi(x))\}' = f'(\varphi(x)) \cdot \varphi'(x).$$

We first differentiate the function itself; then the insides. This rule is a 1-dimensional version of the so-called *Chain rule.*

12.4 Special derivatives.

Power like:

$$\frac{d}{dx}(x^\alpha) = \alpha \cdot x^{\alpha-1}, \qquad \text{for } x > 0, \ \text{(extensions for some } \alpha).$$

$$\frac{d}{dx}\ln|x| = \frac{1}{x}, \qquad \text{for } x \neq 0.$$

Exponential like:

$$\frac{d}{dx}\exp x = \exp x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\left(a^x\right) = \ln a \cdot a^x, \qquad \text{for } x \in \mathbb{R} \text{ and } a > 0.$$

Trigonometric:

$$\frac{d}{dx}\sin x = \cos x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\cos x = -\sin x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\tan x = 1 + \tan^2 x = \frac{1}{\cos^2 x}, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, p \in \mathbb{Z},$$

$$\frac{d}{dx}\cot x = -\left(1 + \cot^2 x\right) = -\frac{1}{\sin^2 x}, \qquad \text{for } x \neq p\pi, p \in \mathbb{Z}.$$

Hyperbolic:

$$\frac{d}{dx}\sinh x = \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\cosh x = \sinh x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\tanh x = 1 - \tanh^2 x = \frac{1}{\cosh^2 x}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\coth x = 1 - \coth^2 x = -\frac{1}{\sinh^2 x}, \qquad \text{for } x \neq 0.$$

Inverse trigonometric:

$$\frac{d}{dx}\ \text{Arcsin}\ x = \frac{1}{\sqrt{1 - x^2}}, \qquad \text{for } x \in \,]-1,1\,[,$$

$$\frac{d}{dx}\ \text{Arccos}\ x = -\frac{1}{\sqrt{1 - x^2}}, \qquad \text{for } x \in \,]-1,1\,[,$$

$$\frac{d}{dx}\ \text{Arctan}\ x = \frac{1}{1 + x^2}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\ \text{Arccot}\ x = \frac{1}{1 + x^2}, \qquad \text{for } x \in \mathbb{R}.$$

Inverse hyperbolic:

$$\frac{d}{dx}\ \text{Arsinh}\ x = \frac{1}{\sqrt{x^2 + 1}}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\ \text{Arcosh}\ x = \frac{1}{\sqrt{x^2 - 1}}, \qquad \text{for } x \in \,]1,+\infty\,[,$$

$$\frac{d}{dx}\ \text{Artanh}\ x = \frac{1}{1 - x^2}, \qquad \text{for } |x| < 1,$$

$$\frac{d}{dx}\ \text{Arcoth}\ x = \frac{1}{1 - x^2}, \qquad \text{for } |x| > 1.$$

Remark 12.2 The derivative of the trigonometric and the hyperbolic functions are to some extent exponential like. The derivatives of the inverse trigonometric and inverse hyperbolic functions are power like, because we include the logarithm in this class. $\Diamond$

12.5 Integration

The most obvious rules are dealing with linearity

$$\int \{f(x) + \lambda g(x)\}\, dx = \int f(x)\, dx + \lambda \int g(x)\, dx, \qquad \text{where } \lambda \in \mathbb{R} \text{ is a constant,}$$

and with the fact that differentiation and integration are "inverses to each other", i.e. modulo some arbitrary constant $c \in \mathbb{R}$, which often tacitly is missing,

$$\int f'(x)\, dx = f(x).$$

If we in the latter formula replace $f(x)$ by the product $f(x)g(x)$, we get by reading from the right to the left and then differentiating the product,

$$f(x)g(x) = \int \{f(x)g(x)\}'\, dx = \int f'(x)g(x)\, dx + \int f(x)g'(x)\, dx.$$

Hence, by a rearrangement

The rule of partial integration:

$$\int f'(x)g(x)\, dx = f(x)g(x) - \int f(x)g'(x)\, dx.$$

The differentiation is moved from one factor of the integrand to the other one by changing the sign and adding the term $f(x)g(x)$.

Remark 12.3 This technique was earlier used a lot, but is almost forgotten these days. It must be revived, because MAPLE and pocket calculators apparently do not know it. It is possible to construct examples where these devices cannot give the exact solution, unless you first perform a partial integration yourself. $\diamond$

Remark 12.4 This method can also be used when we estimate integrals which cannot be directly calculated, because the antiderivative is not contained in e.g. the catalogue of MAPLE. The idea is by a succession of partial integrations to make the new integrand smaller. $\diamond$

Integration by substitution:

If the integrand has the special structure $f(\varphi(x)) \cdot \varphi'(x)$, then one can change the variable to $y = \varphi(x)$:

$$\int f(\varphi(x)) \cdot \varphi'(x)\, dx = \text{“}\int f(\varphi(x))\, d\varphi(x)\text{”} = \int_{y=\varphi(x)} f(y)\, dy.$$

Integration by a monotonous substitution:

If $\varphi(y)$ is a *monotonous* function, which maps the y-interval *one-to-one* onto the x-interval, then

$$\int f(x)\,dx = \int_{y=\varphi^{-1}(x)} f(\varphi(y))\varphi'(y)\,dy.$$

Remark 12.5 This rule is usually used when we have some "ugly" term in the integrand $f(x)$. The idea is to put this ugly term equal to $y = \varphi^{-1}(x)$. When e.g. x occurs in $f(x)$ in the form $\sqrt{x}$, we put $y = \varphi^{-1}(x) = \sqrt{x}$, hence $x = \varphi(y) = y^2$ and $\varphi'(y) = 2y$. $\Diamond$

12.6 Special antiderivatives

Power like:

$$\int \frac{1}{x}\,dx = \ln|x|, \qquad \text{for } x \neq 0. \quad \text{(Do not forget the numerical value!)}$$

$$\int x^\alpha\,dx = \frac{1}{\alpha+1}x^{\alpha+1}, \qquad \text{for } \alpha \neq -1,$$

$$\int \frac{1}{1+x^2}\,dx = \text{ Arctan } x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{1-x^2}\,dx = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|, \qquad \text{for } x \neq \pm 1,$$

$$\int \frac{1}{1-x^2}\,dx = \text{ Artanh } x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{1-x^2}\,dx = \text{ Arcoth } x, \qquad \text{for } |x| > 1,$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \text{ Arcsin } x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = -\text{ Arccos } x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \text{ Arsinh } x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \ln(x+\sqrt{x^2+1}), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{x}{\sqrt{x^2-1}}\,dx = \sqrt{x^2-1}, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \text{ Arcosh } x, \qquad \text{for } x > 1,$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \ln|x+\sqrt{x^2-1}|, \qquad \text{for } x > 1 \text{ eller } x < -1.$$

There is an error in the programs of the pocket calculators TI-92 and TI-89. The *numerical signs* are missing. It is obvious that $\sqrt{x^2-1} < |x|$ so if $x < -1$, then $x + \sqrt{x^2-1} < 0$. Since you cannot take the logarithm of a negative number, these pocket calculators will give an error message.

Exponential like:

$$\int \exp x\, dx = \exp x, \qquad \text{for } x \in \mathbb{R},$$

$$\int a^x\, dx = \frac{1}{\ln a} \cdot a^x, \qquad \text{for } x \in \mathbb{R}, \text{ and } a > 0, a \neq 1.$$

Trigonometric:

$$\int \sin x\, dx = -\cos x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \cos x\, dx = \sin x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \tan x\, dx = -\ln|\cos x|, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \cot x\, dx = \ln|\sin x|, \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos x}\, dx = \frac{1}{2} \ln\left(\frac{1+\sin x}{1-\sin x}\right), \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin x}\, dx = \frac{1}{2} \ln\left(\frac{1-\cos x}{1+\cos x}\right), \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos^2 x}\, dx = \tan x, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin^2 x}\, dx = -\cot x, \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z}.$$

Hyperbolic:

$$\int \sinh x\, dx = \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \cosh x\, dx = \sinh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \tanh x\, dx = \ln \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \coth x\, dx = \ln|\sinh x|, \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh x}\, dx = \operatorname{Arctan}(\sinh x), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\cosh x}\, dx = 2 \operatorname{Arctan}(e^x), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh x}\, dx = \frac{1}{2} \ln\left(\frac{\cosh x - 1}{\cosh x + 1}\right), \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\sinh x}\,dx = \ln\left|\frac{e^x-1}{e^x+1}\right|, \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh^2 x}\,dx = \tanh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh^2 x}\,dx = -\coth x, \qquad \text{for } x \neq 0.$$

12.7 Trigonometric formulæ

The trigonometric formulæ are closely connected with circular movements. Thus $(\cos u, \sin u)$ are the coordinates of a point P on the unit circle corresponding to the angle u, cf. figure A.1. This geometrical interpretation is used from time to time.

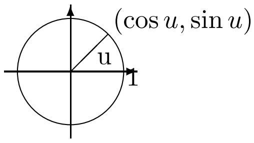

Figure 12.1: The unit circle and the trigonometric functions.

The fundamental trigonometric relation:

$$\cos^2 u + \sin^2 u = 1, \qquad \text{for } u \in \mathbb{R}.$$

Using the previous geometric interpretation this means according to *Pythagoras's theorem*, that the point P with the coordinates $(\cos u, \sin u)$ always has distance 1 from the origo $(0,0)$, i.e. it is lying on the boundary of the circle of centre $(0,0)$ and radius $\sqrt{1} = 1$.

Connection to the complex exponential function:

The *complex exponential* is for imaginary arguments defined by

$$\exp(\mathrm{i}\,u) := \cos u + \mathrm{i}\ \sin u.$$

It can be checked that the usual functional equation for exp is still valid for complex arguments. In other word: The definition above is extremely conveniently chosen.

By using the definition for $\exp(\mathrm{i}\,u)$ and $\exp(-\mathrm{i}\,u)$ it is easily seen that

$$\cos u = \frac{1}{2}(\exp(\mathrm{i}\,u) + \exp(-\mathrm{i}\,u)),$$

$$\sin u = \frac{1}{2i}(\exp(\mathrm{i}\,u) - \exp(-\mathrm{i}\,u)),$$

.

Moivre's formula: We get by expressing $\exp(\mathrm{i}nu)$ in two different ways:

$$\exp(\mathrm{i}nu) = \cos nu + \mathrm{i}\,\sin nu = (\cos u + \mathrm{i}\,\sin u)^n.$$

Example 12.1 If we e.g. put $n = 3$ into Moivre's formula, we obtain the following typical application,

$$\begin{aligned}\cos(3u) + \mathrm{i}\,\sin(3u) &= (\cos u + \mathrm{i}\,\sin u)^3\\ &= \cos^3 u + 3\mathrm{i}\cos^2 u \cdot \sin u + 3\mathrm{i}^2 \cos u \cdot \sin^2 u + \mathrm{i}^3 \sin^3 u\\ &= \{\cos^3 u - 3\cos u \cdot \sin^2 u\} + \mathrm{i}\{3\cos^2 u \cdot \sin u - \sin^3 u\}\\ &= \{4\cos^3 u - 3\cos u\} + \mathrm{i}\{3\sin u - 4\sin^3 u\}\end{aligned}$$

When this is split into the real- and imaginary parts we obtain

$$\cos 3u = 4\cos^3 u - 3\cos u, \qquad \sin 3u = 3\sin u - 4\sin^3 u. \quad \Diamond$$

Addition formulæ:

$$\sin(u+v) = \sin u\,\cos v + \cos u\,\sin v,$$

$$\sin(u-v) = \sin u\,\cos v - \cos u\,\sin v,$$

$$\cos(u+v) = \cos u\,\cos v - \sin u\,\sin v,$$

$$\cos(u-v) = \cos u\,\cos v + \sin u\,\sin v.$$

Products of trigonometric functions to a sum:

$$\sin u\,\cos v = \frac{1}{2}\sin(u+v) + \frac{1}{2}\sin(u-v),$$

$$\cos u\,\sin v = \frac{1}{2}\sin(u+v) - \frac{1}{2}\sin(u-v),$$

$$\sin u\,\sin v = \frac{1}{2}\cos(u-v) - \frac{1}{2}\cos(u+v),$$

$$\cos u\,\cos v = \frac{1}{2}\cos(u-v) + \frac{1}{2}\cos(u+v).$$

Sums of trigonometric functions to a product:

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right),$$

$$\sin u - \sin v = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right),$$

$$\cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right),$$

$$\cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right).$$

Formulæ of halving and doubling the angle:

$$\sin 2u = 2\sin u\,\cos u,$$

$$\cos 2u = \cos^2 u - \sin^2 u = 2\,\cos^2 u - 1 = 1 - 2\,\sin^2 u,$$

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}} \qquad \text{followed by a discussion of the sign,}$$

$$\cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}} \qquad \text{followed by a discussion of the sign,}$$

12.8 Hyperbolic formulæ

These are very much like the trigonometric formulæ, and if one knows a little of Complex Function Theory it is realized that they are actually identical. The structure of this section is therefore the same as for the trigonometric formulæ. The reader should compare the two sections concerning similarities and differences.

The fundamental relation:

$$\cosh^2 x - \sinh^2 x = 1.$$

Definitions:

$$\cosh x = \frac{1}{2}\left(\exp(x) + \exp(-x)\right), \qquad \sinh x = \frac{1}{2}\left(\exp(x) - \exp(-x)\right).$$

"Moivre's formula":

$$\exp(x) = \cosh x + \sinh x.$$

This is trivial and only rarely used. It has been included to show the analogy.

Addition formulæ:

$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y),$$

$$\sinh(x-y) = \sinh(x)\cosh(y) - \cosh(x)\sinh(y),$$

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y),$$

$$\cosh(x-y) = \cosh(x)\cosh(y) - \sinh(x)\sinh(y).$$

Formulæ of halving and doubling the argument:

$$\sinh(2x) = 2\sinh(x)\cosh(x),$$

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x) = 2\cosh^2(x) - 1 = 2\sinh^2(x) + 1,$$

$$\sinh\left(\frac{x}{2}\right) = \pm\sqrt{\frac{\cosh(x)-1}{2}} \qquad \text{followed by a discussion of the sign,}$$

$$\cosh\left(\frac{x}{2}\right) = \sqrt{\frac{\cosh(x)+1}{2}}.$$

Inverse hyperbolic functions:

$$\operatorname{Arsinh}(x) = \ln\left(x + \sqrt{x^2+1}\right), \qquad x \in \mathbb{R},$$

$$\operatorname{Arcosh}(x) = \ln\left(x + \sqrt{x^2-1}\right), \qquad x \geq 1,$$

$$\operatorname{Artanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \qquad |x| < 1,$$

$$\operatorname{Arcoth}(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right), \qquad |x| > 1.$$

12.9 Complex transformation formulæ

$$\cos(\mathrm{i}x) = \cosh(x), \qquad \cosh(\mathrm{i}x) = \cos(x),$$
$$\sin(\mathrm{i}x) = \mathrm{i}\,\sinh(x), \qquad \sinh(\mathrm{i}x) = \mathrm{i}\,\sin x.$$

12.10 Taylor expansions

The generalized binomial coefficients are defined by

$$\binom{\alpha}{n} := \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{1\cdot 2\cdots n},$$

with n factors in the numerator and the denominator, supplied with

$$\binom{\alpha}{0} := 1.$$

The Taylor expansions for *standard functions* are divided into *power like* (the radius of convergency is finite, i.e. $= 1$ for the standard series) and*exponential like* (the radius of convergency is infinite).
Power like:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1,$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x| < 1,$$

$$(1+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j, \qquad n \in \mathbb{N}, x \in \mathbb{R},$$

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \qquad \alpha \in \mathbb{R}\setminus\mathbb{N}, |x| < 1,$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \qquad |x| < 1,$$

$$\mathrm{Arctan}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \qquad |x| < 1.$$

Exponential like:

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \qquad x \in \mathbb{R}$$

$$\exp(-x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} x^n, \qquad x \in \mathbb{R}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}, \qquad x \in \mathbb{R},$$

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}, \qquad x \in \mathbb{R},$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}, \qquad x \in \mathbb{R},$$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}, \qquad x \in \mathbb{R}.$$

12.11 Magnitudes of functions

We often have to compare functions for $x \to 0+$, or for $x \to \infty$. The simplest type of functions are therefore arranged in an hierarchy:

1) logarithms,

2) power functions,

3) exponential functions,

4) faculty functions.

When $x \to \infty$, a function from a higher class will always dominate a function form a lower class. More precisely:

A) A *power function* dominates a *logarithm* for $x \to \infty$:

$$\frac{(\ln x)^\beta}{x^\alpha} \to 0 \qquad \text{for } x \to \infty, \quad \alpha,\, \beta > 0.$$

B) An *exponential* dominates a *power function* for $x \to \infty$:

$$\frac{x^\alpha}{a^x} \to 0 \qquad \text{for } x \to \infty, \quad \alpha,\, a > 1.$$

C) The *faculty function* dominates an *exponential* for $n \to \infty$:

$$\frac{a^n}{n!} \to 0, \qquad n \to \infty, \quad n \in \mathbb{N}, \quad a > 0.$$

D) When $x \to 0+$ we also have that a *power function* dominates the *logarithm:*

$$x^\alpha \ln x \to 0-, \qquad \text{for } x \to 0+, \quad \alpha > 0.$$

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