Examples of Sequences

Leif Mejlbro

Leif Mejlbro

Examples of Sequences

Calculus 3c-1

Calculus 3c-1 – Examples of Sequences

ISBN 978-87-7681-375-8

Contents

Preface

Here follows a collection of *sequences*, including sequences, which satisfy some simple difference equations. The reader is also referred to *Calculus 3b*. Since my aim also has been to demonstrate some solution strategy I have as far as possible structured the examples according to the following form

A *Awareness*, i.e. a short description of what is the problem.

D *Decision*, i.e. a reflection over what should be done with the problem.

I *Implementation*, i.e. where all the calculations are made.

C *Control*, i.e. a test of the result.

This is an ideal form of a general procedure of solution. It can be used in any situation and it is not linked to Mathematics alone. I learned it many years ago in the Theory of Telecommunication in a situation which did not contain Mathematics at all. The student is recommended to use it also in other disciplines.

One is used to from high school immediately to proceed to **I**. *Implementation*. However, examples and problems at university level are often so complicated that it in general will be a good investment also to spend some time on the first two points above in order to be absolutely certain of what to do in a particular case. Note that the first three points, **ADI**, can always be performed.

This is unfortunately not the case with **C** *Control*, because it from now on may be difficult, if possible, to check one's solution. It is only an extra securing whenever it is possible, but we cannot include it always in our solution form above.

I shall on purpose not use the logical signs. These should in general be avoided in Calculus as a shorthand, because they are often (too often, I would say) misused. Instead of $\wedge$ I shall either write "and", or a comma, and instead of $\vee$ I shall write "or". The arrows $\Rightarrow$ and $\Leftrightarrow$ are in particular misunderstood by the students, so they should be totally avoided. Instead, write in a plain language what you mean or want to do.

It is my hope that these examples, of which many are treated in more ways to show that the solutions procedures are not unique, may be of some inspiration for the students who have just started their studies at the universities.

Finally, even if I have tried to write as careful as possible, I doubt that all errors have been removed. I hope that the reader will forgive me the unavoidable errors.

Leif Mejlbro
13th May 2008

1 Sequences in General

Example 1.1 *Check if the sequence*

$$a_n = \frac{n}{n+1} - \frac{n+1}{n}$$

is convergent or divergent. Find its limit, if it is convergent.

Here we have several possibilities:

1st variant. If the numerator and the denominator in both fractions are divided by n, it follows by the rules of calculations that

$$a_n = \frac{n}{n+1} - \frac{n+1}{n} = \frac{1}{1+\frac{1}{n}} - \left(1+\frac{1}{n}\right) \to \frac{1}{1+0} - (1+0) = 0 \quad \text{for } n \to \infty.$$

2nd variant. If we remove 1 from both fractions we get

$$a_n = \frac{n}{n+1} - \frac{n+1}{n} = \left(1 - \frac{1}{n+1}\right) - \left(1+\frac{1}{n}\right) = -\frac{1}{n+1} - \frac{1}{n} \to 0 \quad \text{for } n \to \infty.$$

3rd variant. If everything is put on the same fraction line, we get s

$$a_n = \frac{n}{n+1} - \frac{n+1}{n} = \frac{n^2-(n+1)^2}{(n+1)n} = -\frac{2n+1}{(n+1)n} = -\frac{2+\frac{1}{n}}{n+1} \to 0 \quad \text{for } n \to \infty.$$

It is seen in all three variants that the sequence is convergent and its limit is 0. $\Diamond$

Example 1.2 *Check if the sequence*

$$a_n = \frac{n^2}{n+1} - \frac{n^2+1}{n}$$

is convergent or divergent. In case of convergence, find its limit.

1st variant. (Does not work, but it illustrates the problem). If we reduce by n in the numerator and the denominator in the two fractions, we get

$$a_n = \frac{n^2}{n+1} - \frac{n^2+1}{n} = \frac{n}{1+\frac{1}{n}} - n - \frac{1}{n} \to \infty - \infty - 0.$$

This is an *illegal type of convergence* and nothing can be concluded in this way.

2nd variant. (The elegant variant). Add $0 = -1+1$ to the first numerator and apply that $n^2-1 = (n+1)(n-1)$:

$$\begin{aligned} a_n &= \frac{n^2}{n+1} - \frac{n^2+1}{n} = \frac{(n^2-1)+1}{n+1} - \left\{n + \frac{1}{n}\right\} = \frac{(n+1)(n-1)}{n+1} + \frac{1}{n+1} - n - \frac{1}{n} \\ &= n - 1 + \frac{1}{n+1} - n - \frac{1}{n} = -1 + \frac{1}{n+1} - \frac{1}{n} \to -1+0-0 = -1 \quad \text{for } n \to \infty. \end{aligned}$$

3rd variant. (Brute force). Put everything on the same fraction line and reduce,

$$\begin{aligned} a_n &= \frac{n^2}{n+1} - \frac{n^2+1}{n} = \frac{n^3-(n+1)(n^2+1)}{(n+1)n} = \frac{n^3 - \{n^3+n^2+n+1\}}{(n+1)n} \\ &= -\frac{n^2+n+1}{n^2+n} = -1 - \frac{1}{n^2+n} \to -1 \quad \text{for } n \to \infty. \end{aligned}$$

The latter calculation can of course be performed more or less elegant. ◊

Example 1.3 *Check if the sequence*

$$a_n = \cos\frac{n\pi}{2}$$

is convergent or divergent. Find the limit in case of convergence.

It follows from

$$a_{n+4} = \cos\frac{(n+4)\pi}{2} = \cos\frac{n\pi}{2} = a_n,$$

that the values

$$a_1 = 0, \quad a_2 = -1, \quad a_3 = 0, \quad a_4 = 1,$$

are repeated cyclically, i.e. they all occur infinitely often. Thus we have four candidates of the limit, but since any possible limit is unique, it does not exist in this case, and the sequence is divergent. ◊

Example 1.4 . *Check if the sequence*

$$a_n = n^{(-1)^n}$$

is convergent or divergent. Find the limit in case of convergence.

Since the subsequence

$$a_{2n} = (2n)^{(-1)^{2n}} = 2n$$

is divergent, the "bigger sequence" (a_n) (it contains more elements) must also be divergent. $\Diamond$

Example 1.5 . *Check if the sequence*

$$a_n = \frac{a^n}{n}, \qquad a \in \mathbb{R},$$

is convergent or divergent. Find the limit in case of convergence.

This sequence contains a parameter, and the question of convergence depends on the the size of the parameter.

1) If $|a| > 1$, it follows from the magnitudes that

$$|a_n| = \frac{1}{n}|a|^n \to \infty \qquad \text{for } n \to \infty.$$

(The exponential function "dominates" the power function in n). In this case we have divergence.

2) If $|a| \le 1$, we get the estimate

$$|a_n - 0| = |a_n| = \frac{1}{n}|a|^n \le \frac{1}{n} \to 0 \qquad \text{for } n \to \infty.$$

It follows immediately from the definition that (a_n) is convergent and that its limit is 0. $\Diamond$

Example 1.6 . *Check if the sequence*

$$a_n = \ln(n^2+1) - 2\ln n$$

is convergent or divergent. Find the limit in case of convergence.

The type of convergence is "$\infty - \infty$, so we first apply the functional equation of the logarithm. Thus

$$a_n = \ln(n^2+1) - 2\ln n = \ln\left(\frac{n^2+1}{n^2}\right) = \ln\left(1+\frac{1}{n^2}\right).$$

Then follow at least two variants.

1st variant. Since ln is continuous on på $\mathbb{R}_+$, and $1+\dfrac{1}{n^2} \to 1$ for $n \to \infty$, we can interchange ln and the limit,

$$\lim_{n\to\infty} a_n = \ln\left(\lim_{n\to\infty}\left\{1+\frac{1}{n^2}\right\}\right) = \ln 1 = 0,$$

and it follows that the sequence is convergent towards the limit 0.

2nd variant. According to Taylor's formula,

$$\ln(1+t) = t + t\varepsilon(t).$$

We get by putting $t = 1/n^2$,

$$a_n = \ln\left(1+\frac{1}{n^2}\right) = \frac{1}{n^2} + \frac{1}{n^2}\varepsilon\left(\frac{1}{n}\right) \to 0 + 0\cdot 0 = 0 \quad \text{for } n \to \infty,$$

hence the sequence is convergent and its limit is 0. $\Diamond$

Example 1.7 . *Check if the sequence*

$$a_n = \frac{(-1)^n}{n} + \frac{1+(-1)^n}{2}$$

is convergent or divergent. Find the limit in case of convergence.

Due to the change of sign $(-1)^n$ a good strategy would be to consider odd and even indices separately. Thus we shall consider the two subsequences,

$$a_{2n+1} = \frac{(-1)^{2n+1}}{2n+1} + \frac{1+(-1)^{2n+1}}{2} = -\frac{1}{2n+1} \to 0 \qquad \text{for } n \to \infty,$$

and

$$a_{2n} = \frac{(-1)^{2n}}{2n} + \frac{1+(-1)^{2n}}{2} = \frac{1}{2n} + 1 \to 1 \qquad \text{for } n \to \infty.$$

It follows that we have two different candidates of the limit, and since a limit is always unique, we conclude that it does not exist and the sequence is divergent. $\Diamond$

Example 1.8 . *Check if the sequence*

$$a_n = \frac{1}{n}\sin^5 n$$

is convergent or divergent. Find the limit if the sequence is convergent.

This example is trying to pull the reader's leg, because one is persuaded to concentrate on the mysterious term $\sin^5 n$, which apparently cannot be controlled.

Notice that we always have $|\sin x| \le 1$, so

$$|a_n - 0| = |a_n| = \frac{1}{n}|\sin^5 n| \le \frac{1}{n} \to 0 \qquad \text{for } n \to \infty,$$

and we conclude that

$$|a_n - 0| \to 0 \qquad \text{for } n \to \infty,$$

The sequence is convergent according to the definition and its limit is 0. $\Diamond$

Example 1.9 . *Check if the sequence*

$$a_n = \sqrt{n+1} - \sqrt{n}$$

is convergent or divergent. Find its limit if it is convergent.

This example is of the type "$\infty - \infty$". It follows from

$$(a+b)(a-b) = a^2 - b^2,$$

that

$$a - b = \frac{a^2 - b^2}{a+b}.$$

Then putting $a = \sqrt{n+1}$ and $b = \sqrt{n}$ we find

$$\begin{aligned} a_n &= \sqrt{n+1} - \sqrt{n} = a - b = \frac{a^2 - b^2}{a+b} = \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} \\ &= \frac{1}{\sqrt{n+1} + \sqrt{n}} \to 0 \qquad \text{for } n \to \infty, \end{aligned}$$

thus the sequence is convergent and its limit is 0. $\Diamond$

Example 1.10 . *Check if the sequence*

$$a_n = \left(\frac{2n-3}{3n+7}\right)^4$$

is convergent or divergent. Find its limit if it is convergent.

The function $f(x) = x^4$ is continuous and independent of n, and the "inner part" converges,

$$\frac{2n-3}{3n+7} = \frac{2-\frac{3}{n}}{3+\frac{7}{n}} \to \frac{2-0}{3+0} = \frac{2}{3} \qquad \text{for } n \to \infty.$$

In this case f and the limit can be interchanged, hence the sequence is convergent, and the limit is

$$\lim_{n\to\infty} a_n = f\left(\lim_{n\to\infty} \frac{2n-3}{3n+7}\right) = f\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^4 = \frac{16}{81}.$$

In practice, the following shorter version is also accepted,

$$a_n = \left(\frac{2n-3}{3n+7}\right)^4 = \left(\frac{2-\frac{3}{n}}{3+\frac{7}{n}}\right) \to \left(\frac{2}{3}\right)^4 = \frac{16}{81} \qquad \text{for } n \to \infty,$$

which is correct, as long as the *exponent is a constant*, i.e. it does not depend on n. $\Diamond$

Example 1.11 . *Check if the sequence*

$$a_n = \frac{n^{4/3}\cos\left(n!\pi/(\sqrt{2})^n\right)}{n+1}$$

is convergent or divergent. Find the limit in case of convergence.

We first rewrite a_n in the following way,

$$a_n = \frac{n^{4/3}}{n+1}\cos\left(n!\pi/(\sqrt{2})^n\right).$$

The first factor tends to ∞,

$$\frac{n^{4/3}}{n+1} = \frac{\sqrt[3]{n}}{1+\frac{1}{n}} \to \infty \qquad \text{for } n \to \infty,$$

which, however, in general is not sufficient, because $\cos x$ during the limit might lie close to 0, so we get the type of convercence "$\infty \cdot 0$".

Notice that if we only consider *even* indices, then we get rid of the square root. By the chance of parameter $n \to 2n$ we get

$$\cos\left(\frac{(2n)!}{2^n}\pi\right) = \cos\left(\frac{1\cdot 2\cdot 3\cdot 4\cdots 2n}{1\cdot 2\cdot 1\cdot 2\cdots 2}\pi\right) = 1 \qquad \text{for } n \geq 2,$$

because $(2n)!/2^n$ is *even* for $n \geq 2$. This proves that we have for $n \geq 2$,

$$a_{2n} = \frac{(2n)^{4/3}}{2n+1} \cos\left(\frac{(2n)!}{2^n}\pi\right) = \frac{(2n)^{4/3}}{2n+1} \to \infty \qquad \text{for } n \to \infty,$$

and the sequence is divergent. $\Diamond$

Example 1.12 . *Check if the sequence*

$$a_n = \cot\frac{1}{n} - n$$

is convergent or divergent. Find the limit in case of convergence

This is a tricky example, in which one must

1) replace $1/n$ by $x = 1/n$, i.e. $x \to 0+$ for $n \to \infty$,

2) apply that $\cot x = \cos x/\sin x$, followed by putting everything on the same fraction line,

3) apply Taylor's formula in both the numerator and the denominator, followed by some reduction,

4) finally take the limit $x \to 0+$.

The details of this program look like the following:

$$\begin{aligned} a_n &= \cos\frac{1}{n} - n = \cot x - \frac{1}{x} = \frac{\cos x}{\sin x} - \frac{1}{x} = \frac{x\cos x - \sin x}{x \sin x} \\ &= \frac{x\left\{1 - \frac{1}{2}x^2 + x^2\varepsilon(x)\right\} - \left\{x - \frac{1}{6}x^3 + x^3\varepsilon(x)\right\}}{x\{x + x\varepsilon(x)\}} = \frac{x - \frac{1}{2}x^3 - x + \frac{1}{6}x^3 + x^3\varepsilon(x)}{x^2\{1+\varepsilon(x)\}} \\ &= -\frac{1}{3}\,x \cdot \frac{1+\varepsilon(x)}{1+\varepsilon(x)} \to -\frac{1}{3}\cdot 0 \cdot \frac{1+0}{1+0} = 0 \qquad \text{for } x \to 0+\,. \end{aligned}$$

We conclude that the sequence is convergent with the limit 0. $\Diamond$

Example 1.13 . *Check if the given sequence is convergent or divergent. Find the limit in case of convergence.*

$$a_n = \frac{3^n + (-2)^n}{3^{n+1} + (-2)^{n+1}}, \qquad n \in \mathbb{N}.$$

When we estimate expressions consisting of two terms the trick is to put the numerically larger and dominating term outside the expression as a factor. We get by using this principle in both the numerator and the denominator that

$$a_n = \frac{3^n + (-2)^n}{3^{n+1} + (-2)^{n+1}} = \frac{3^n}{3^{n+1}} \cdot \frac{1 + \left(-\frac{2}{3}\right)^n}{1 + \left(-\frac{2}{3}\right)^{n+1}} = \frac{1}{3} \cdot \frac{1 + \left(-\frac{2}{3}\right)^n}{1 + \left(-\frac{2}{3}\right)^{n+1}}.$$

Now $\left|-\frac{2}{3}\right| < 1$, so $\left(-\frac{2}{3}\right)^n \to 0$ and $\left(-\frac{2}{3}\right)^{n+1} \to 0$ for $n \to \infty$ (standard sequences), hence according to the rules of calculations,

$$a_n = \frac{1}{3} \cdot \frac{1 + \left(-\frac{2}{3}\right)^n}{1 + \left(-\frac{2}{3}\right)^{n+1}} \to \frac{1}{3} \cdot \frac{1+0}{1+0} = \frac{1}{3} \qquad \text{for } n \to \infty.$$

The sequence is convergent and its limit is

$$\lim_{n\to\infty} a_n = \frac{1}{3}. \qquad \Diamond$$

Example 1.14 . *Check if the sequence*

$$a_n = \frac{n(n+2)}{n+1} - \frac{n^3}{n^2+1}$$

is convergent or divergent. Find the limit in case of convergence.

The type of convergence is "$\infty - \infty$". We note that both terms behave approximately as n, so we subtract n from both terms:

$$\begin{aligned} a_n &= \frac{n(n+2)}{n+1} - \frac{n^3}{n^2+1} = \left\{\frac{n(n+2)}{n+1} - n\right\} - \left\{\frac{n^3}{n^2+1} - n\right\} = \frac{n^2+2n-n^2-n}{n+1} - \frac{n^3-n^3-n}{n^2+1} \\ &= \frac{n}{n+1} + \frac{n}{n^2+1} = 1 - \frac{1}{n+1} + \frac{1}{n+\frac{1}{n}} \to 1 - 0 + 0 = 1 \qquad \text{for } n \to \infty. \end{aligned}$$

We see that the sequence is convergent with the limit 1.

A simpler variant is obtained if we immediately see that

$$\frac{n(n+2)}{n+1} = \frac{n^2+2n+1-1}{n+1} = n+1 - \frac{1}{n+1},$$

where we use that $n^2+2n+1 = (n+1)^2$. $\Diamond$

Example 1.15 . *Check if the sequence*

$$a_n = \sqrt[3]{n^3+1} - n$$

is convergent or divergent. Find the limit in case of convergence.

The type is "$\infty - \infty$". In this case the trick $a - b = (a^2 - b^2)/(a+b)$ does not work. However, we succeed by a small modification. First notice that the cubic is removed by taking the third power, i.e. we start by considering $a^3 - b^3$ where $a = \sqrt[3]{n^3+1}$ amd $b = n$. Then

$$1 = (n^3+1) - n^3 = a^3 - b^3 = (a-b)(a^2+ab+b^2),$$

hence by a rearrangement,

$$a_n = \sqrt[3]{n^3+1} - n = a - b = \frac{a^3-b^3}{a^2+ab+b^2} = \frac{1}{(\sqrt[3]{n^3+1})^2 + n\sqrt[3]{n^3+1} + n^2} \to 0$$

for $n \to \infty$, and we see that the sequence is convergent with the limit 0.

Alternatively one applies Taylor's formula on $\sqrt[3]{1+x}$, i.e.

$$(1+x)^{1/3} = 1 + \binom{1/3}{1} x + x\varepsilon(x) = 1 + \frac{1}{3}\,x + x\varepsilon(x),$$

where $\varepsilon(x) \to 0$ for $x \to 0$. By a small rearrangement, in which we put $x = 1/n^3 \to 0$ for $n \to \infty$ we get

$$\begin{aligned} a_n &= \sqrt[3]{n^3+1} - n = n\sqrt[3]{1+\frac{1}{n^3}} - n = n\left\{1 + \frac{1}{3}\,\frac{1}{n^3} + \frac{1}{n^3}\,\varepsilon\left(\frac{1}{n}\right)\right\} - n \\ &= \frac{1}{3}\,\frac{1}{n^2} + \frac{1}{n^2}\,\varepsilon\left(\frac{1}{n}\right) \to 0 \text{ for } n \to \infty, \end{aligned}$$

and the sequence is convergent with the limit 0. $\Diamond$

Example 1.16 . *Check if the sequence*

$$a_n = (2^n + 3^n)^{1/n}$$

is convergent or divergent. Find the limit in case of convergence.

The trick in case of expressions with several terms is always to put the dominating term as a factor. Here, $3^n \gg 2^n$, hence

$$(1)\quad a_n = (2^n + 3^n)^{1/n} = 3\left\{1 + \left(\frac{2}{3}\right)^n\right\}^{1/n}$$

We get from $1 < 1 + \left(\frac{2}{3}\right)^n < 2$,

$$3 < a_n = 3\left\{1 + \left(\frac{2}{3}\right)^n\right\}^{1/n} < 3\sqrt[n]{2} \to 3 \qquad \text{for } n \to \infty.$$

Then all terms a_n lie between 3 and a sequence which converges towards 3, hence (a_n) is convergent with the limit 3.

In a **variant** we can instead take the logarithm of (1),

$$\ln a_n = \ln 3 + \frac{1}{n} \ln\left(1 + \left(\frac{2}{3}\right)^n\right) \to \ln 3 + 0 \cdot 0 = \ln 3 \qquad \text{for } n \to \infty,$$

which shows that (a_n) is convergent with the limit 3. $\Diamond$

Example 1.17 *Let (a_n) be some real sequence which is convergent with the limit a, and let the sequence(b_n) be given by*

$$b_n = \left(1 + \frac{a_n}{n}\right)^n, \qquad n \in \mathbb{N}.$$

Prove that (b_n) is convergent and find its limit.

(Hint: One may apply Taylor's formula for $\ln(1+x)$*.)*

Taylor's formula for $\ln(1+x)$ gives

$$\ln(1+x) = x - \frac{1}{2}x^2 + x^2\varepsilon(x).$$

Since $a_n \to a$ for $n \to \infty$, there exists an $N \in \mathbb{N}$, such that

$$\left|\frac{a_n}{n}\right| < 1 \qquad \text{for } n \geq N.$$

By putting $x = a_n/n$ it follows from Taylo's formula for $n \geq N$ that

$$\begin{aligned}
\ln b_n &= n \ln\left(1 + \frac{a_n}{n}\right) \\
&= n\left\{\frac{a_n}{n} - \frac{a_n^2}{2n^2} + \frac{a_n^2}{n^2}\,\varepsilon\left(\frac{a_n}{n}\right)\right\} \\
&= a_n - \frac{1}{n}\,a_n^2 + \frac{1}{n}\,\frac{a_n^2}{n^2}\,\varepsilon\left(\frac{a_n}{n}\right) \\
&\to a - 0 + 0 = a \qquad \text{for } n \to \infty.
\end{aligned}$$

Since exp is continuous, we finally conclude that

$$b_n = \exp(\ln b_n) \to \exp a = e^a \qquad \text{for } n \to \infty,$$

and (b_n) is convergent with the limit e^a. $\Diamond$

2 Summable sequences

Example 2.1 . *Given a real sequence (a_n). Define another sequence (b_n) by*

$$b_n = \frac{1}{n}\left\{a_1 + \cdots + a_n\right\}, \qquad n \in \mathbb{N}.$$

Prove that if (a_n) is convergent with the limit a, then (b_n) is also convergent with the limit a.
Give an example of a divergent sequence (a_n), for which the corresponding sequence (b_n) is convergent.

We say that a sequence (a_n) is *summable*, if its corresponding sequence (b_n) defined as above is convergent. We shall prove that if (a_n) is convergent, then (a_n) is also summable. Then we shall construct an example of a summable sequence (a_n), which is not convergent. Hence, there are more summable sequences that convergent ones.

1) Assume that $a_n \to a$ for $n \to \infty$. This means that one to *every* $\varepsilon > 0$ can find some $N = N(\varepsilon) \in \mathbb{N}$, such that

(2) $|a - a_n| < \dfrac{\varepsilon}{2}$ for every $n \geq N(\varepsilon)$.

Then

$$|a - b_n| = \left|a - \frac{1}{n}(a_1 + \cdots + a_n)\right| = \frac{1}{n}\left|(a - a_1) + (a - a_1) + \cdots + (a - a_n)\right| \leq \frac{1}{n}\sum_{k=1}^{n}|a - a_k|.$$

If $n > N(\varepsilon)$, we split the sum in the following way

$$\begin{aligned} |a - b_n| &\leq \frac{1}{n}\sum_{k=1}^{N(\varepsilon)}|a - a_k| + \frac{1}{n}\sum_{k=N(\varepsilon)+1}^{n}|a - a_k| \leq \frac{1}{n}\sum_{k=1}^{N(\varepsilon)}|a - a_k| + \frac{n - N(\varepsilon)}{n}\cdot\frac{\varepsilon}{2} \\ &\leq \frac{1}{n}\sum_{k=1}^{N(\varepsilon)}|a - a_k| + \frac{\varepsilon}{2}, \end{aligned}$$

since the $n - N(\varepsilon)$ terms of the latter sum are all $< \varepsilon/2$ by (2).

Since $N(\varepsilon)$ is fixed (corresponding to the given $\varepsilon > 0$), the sum is

$$\sum_{k=1}^{N(\varepsilon)}|a - a_k|$$

i.e. a constant, which is independent of n. Thus, there exists an $N_1 \geq N(\varepsilon)$, such that

$$\frac{1}{n}\sum_{k=1}^{N(\varepsilon)}|a - a_k| < \frac{\varepsilon}{2} \qquad \text{for ethvert } n \geq N_1.$$

As a conclusion we get that we to *every* $\varepsilon > 0$ can find an $N_1 \in \mathbb{N}$, such that

$$|a - b_n| \leq \frac{1}{n}\sum_{k=1}^{N(\varepsilon)}|a - a_k| + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \text{for ethvert } n \geq N_1.$$

This is precisely the definition of $b_n \to a$ for $n \to \infty$.

2) The classical example of a divergent sequence (a_n), for which (b_n) is convergent, is

$$a_n = (-1)^{n-1}, \qquad \text{where } b_{2n-1} = \frac{1}{2n-1} \text{ og } b_{2n} = 0.$$

Clearly, $b_n \to 0$ for $n \to \infty$.

A slightly more "wild" example is

$$a_{2n-1} = \sqrt{n} \quad \text{and} \quad a_{2n} = -\sqrt{n}.$$

In this case,

$$b_{2n-1} = \frac{\sqrt{n}}{2n-1} \quad \text{og} \quad b_{2n} = 0,$$

thus $b_n \to 0$ for $n \to \infty$.

Remark 2.1 . It follows from

$$nb_n = a_1 + a_2 + \cdots + a_n,$$

that $b_1 = a_1$ and

$$a_n = nb_n - (n-1)b_{n-1} \qquad \textit{for } n \geq 2.$$

Example 2.2 *We define for a real sequence (a_n) another sequence (b_n) by*

$$b_n = \frac{1}{n}\{a_1 + \cdots + a_n\}, \qquad n \in \mathbb{N}.$$

Prove that if $a_n \to \infty$ for $n \to \infty$, then $b_n \to \infty$ for $n \to \infty$.
Give an example of a sequence (a_n) which does not *tend towards ∞ for n tending towards ∞, for which the corresponding sequence (b_n) fulfils $b_n \to \infty$ for $n \to \infty$.*

First note that if $|a_n| \leq c$, then also $|b_n| \leq c$. It follows that if $b_n \to \infty$, then (a_n) must be unbounded.

Assume that $a_n \to \infty$ for $n \to \infty$. This means that we to *every* $c > 0$ can find an $N = N(c) \in \mathbb{N}$, such that (e.g.)

$$a_n > 3c \qquad \text{for every } n \geq N(c).$$

If so, we have for $n > N(c)$ that

$$b_n = \frac{1}{n}\sum_{k=1}^{N(c)} a_k + \frac{1}{n}\sum_{k=N(c)+1}^{n} a_k > \frac{1}{n}\sum_{k=1}^{N(c)} a_k + \frac{n-N(c)}{n} \cdot 3c.$$

To avoid that the finite sum is negative we choose $N_1 > 3N(c)$, such that

$$\left|\frac{1}{n}\sum_{k=1}^{N(c)} a_k\right| < c \quad \left[\text{and trivially } \frac{3N(c)}{n} < 1\right] \quad \text{for } n \geq N_1.$$

Then for every $n \geq N_1$,

$$b_n > \frac{1}{n}\sum_{k=1}^{N(c)} a_k + \frac{n-N(c)}{n}\cdot 3c > -c+3c-c = c.$$

Since for every $c>0$ we can choose N_1, such that

$$b_n > c \qquad \text{for every } n \geq N_1,$$

we conclude that $b_n \to \infty$ for $n \to \infty$.

By the introducing remark, $b_n \to \infty$ implies that (a_n) in unbounded. We note that an unbounded sequence does not necessarily tend towards ∞. Choose e.g.

$$a_{2n} = 2n \qquad \text{and} \qquad a_{2n-1} = 0.$$

Then (a_n) is unbounded, and it does not tend towards ∞. We note that $b_1 = 0$ and

$$b_{2n} = \frac{1}{2n}(2+4+\cdots+2n) = \frac{1}{n}(1+2+\cdots+n) = \frac{1}{n}\cdot\frac{n(n+1)}{2} = \frac{n+1}{2} \to \infty,$$

and

$$b_{2n+1} = \frac{1}{2n+1}(2+4+\cdots+2n) = \frac{n(n+1)}{2n+1} \to \infty,$$

thus. $b_n \to \infty$.

We have here used that

$$(3)\quad 1+2+\cdots+n = \frac{1}{2}\,n(n+1), \qquad n \in \mathbb{N}.$$

For completeness, we see that this is true for $n = 1, 2, 3$. If (3) holds for some $n \in \mathbb{N}$, then we get for the following term by using (3) that

$$1+2+\cdots+n+(n+1) = \frac{1}{2}\,n(n+1)+(n+1) = \frac{1}{2}\,(n+1)(n+2),$$

which we recognize as (3) where n has been replaced by $n+1$. Then (3) follows in general by induction (the *boot strap principle*), because if (3) holds for some n, then it also holds for the following term, etc.. Since (3) is true for $n = 1$ (in the beginning), we see that (3) is true for all $n \in \mathbb{N}$. $\Diamond$

3 Recursively given sequences

Example 3.1 *Let the sequence* (a_n) *be recursively given by*

$$a_1 = \frac{1}{\sqrt{2}}, \qquad a_{n+1} = \sqrt{a_n + \frac{1}{2}}, \qquad n \in \mathbb{N}.$$

Prove that (a_n) *is convergent and find the limit.*

We shall first find the *possible* limit.

Assume that the sequence is convergent, $a_n \to a$ for $n \to \infty$. Since taking the square root of nonnegative numbers is a continuous function, we get by taking the limit in

$$a_{n+1} = \sqrt{a_n + \frac{1}{2}} > 0,$$

that

$$a = \sqrt{a + \frac{1}{2}} \geq 0, \qquad \text{i.e.} \qquad a^2 = a + \frac{1}{2} \text{ where } a \geq 0.$$

This equation of degree two has the roots $a = (1 \pm \sqrt{3})/2$. Since $a > 0$, the only *possible* limit is

$$a = \frac{1 + \sqrt{3}}{2}.$$

It does not yet follow that the sequence actually *is* convergent. We continue in the following way.

A sequence (a_n) is convergent, if it is (weakly) increasing and bounded from above.

1) **The sequence is bounded from above.**
Obviously, $a_1 = 1/\sqrt{2} < (1+\sqrt{3})/2$ [the possible limit]
If $a_n < (1+\sqrt{3})/2$, then it follows for the next element that also

$$a_{n+1} = \sqrt{a_n + \frac{1}{2}} < \sqrt{1 + \frac{\sqrt{3}}{2}} = \frac{1+\sqrt{3}}{2},$$

where we have used that

$$\left(\frac{1+\sqrt{3}}{2}\right)^2 = \frac{1}{4}\left\{1+3+2\sqrt{3}\right\} = 1 + \frac{\sqrt{3}}{2}.$$

Then it follows by induction that (a_n) is bounded from above.

Alternatively it follows from the assumption $a_n < 3/2$ that

$$a_{n+1} = \sqrt{a_n + \frac{1}{2}} < \sqrt{\frac{3}{2} + \frac{1}{2}} = \sqrt{2} < \frac{3}{2},$$

hence by induction, $a_n < 3/2$, and it is bounded from above.

2) **The sequence is increasing**.
Firstly,

$$a_2 = \sqrt{\frac{1}{\sqrt{2}} + \frac{1}{2}} > \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = a_1, \quad \text{i.e. } a_2 > a_1.$$

Then assume that $a_n > a_{n-1}$ for some $n \geq 2$. (This is at least true for $n = 2$). Then

$$a_{n+1} - a_n = \sqrt{a_n + \frac{1}{2}} - \sqrt{a_{n-1} + \frac{1}{2}} = \frac{a_n - a_{n-1}}{\sqrt{a_n + \frac{1}{2}} + \sqrt{a_{n-1} + \frac{1}{2}}} > 0,$$

and it follows by induction that (a_n) is increasing.

3) We have now proved that (a_n) is convergent.
We showed in the beginning that $a = (1+\sqrt{3})/2$ is the only *possible* limit
Since the limit exists, we must have the limit $a = (1+\sqrt{3})/2$. $\Diamond$

Example 3.2 . *Let the sequence* (a_n) *be recursively given by*

$$a_1 = 1, \qquad a_{n+1} = \sqrt{3a_n}, \qquad n \in \mathbb{N}.$$

Prove that (a_n) *is convergent, and find its limit.*

If $a_n \to a$ for $n \to \infty$, it follows by taking the limit in the recursion formula,

$$a = \sqrt{3a}, \qquad \text{or } a^2 = 3a \text{ by a squaring.}$$

We use here that taking the square root is a continuous operation, so the limit and the square root can be interchanged. Thus we conclude that $a = 0$ and $a = 3$ are the only *possible* limits. We shall prove that the sequence indeed *is* convergent. (Our assumption above).

1) **The sequence is bounded.**

   If $a \geq 1$, then $a_{n+1} \geq \sqrt{3} \geq 1$, i.e. (a_n) is bounded from below. It follows in particular that if the limit a exists, then we must have $a \neq 0$, thus $a = 3$ is the only possible limit.

   If $a_n < 3$, then $a_{n+1} = \sqrt{3a_n} < \sqrt{3 \cdot 3} = 3$, and (a_n) is also bounded from above.

2) **The sequence is increasing.**

   In fact,

$$a_{n+1} - a_n = \sqrt{3a_n} - \sqrt{3a_{n-1}} = \sqrt{3} \cdot \{\sqrt{a_n} - \sqrt{a_{n-1}}\}$$

   shows that if $a_n > a_{n-1} \geq 1$, then also $\sqrt{a_n} > \sqrt{a_{n-1}}$, hence $a_{n+1} - a_n > 0$, and we have $a_{n+1} > a_n$.

   From $a_2 = \sqrt{3} > 1 = a_1$ follows that $a_2 > a_1$. Then by induction,

$$1 = a_1 < a_2 < \cdots < a_n < a_{n+1} < \cdots .$$

3) **Conclusion**.

   An increasing bounded sequence (a_n) is convergent, so the given sequence is convergent. The only possible limits were $a = 0$ and $a = 3$. But since all $a_n \geq 1$, we can exclude $a = 0$, and

$$\lim_{n \to \infty} a_n = 3. \qquad \Diamond$$

Example 3.3 *Define the sequence* (a_n) *by*

$$a_1 = k, \qquad a_{n+1} = \frac{1}{2}\left(a_n + \frac{p}{a_n}\right), \qquad n \in \mathbb{N},$$

where $k > 0$ *and* $p > 0$ *are any positive numbers. Prove that* $a_n^2 \geq p$ *for every* $n \geq 2$ *and that the sequence* $a_2, a_3, a_4, \cdots$, *is weakly decreasing. Then prove that* (a_n) *is convergent with the limit* $\sqrt{p}$. *Calculate* a_3 *with 4 decimals i the case* $k = 2$, $p = 3$, *and compare the result with the value on the pocket calculator of* $\sqrt{3}$. *Finally, describe the connection between the sequence and Newton's iteration method of solution of the equation* $x^2 - p = 0$.

1) **Proof of** $a_n^2 \geq p$ **for every** $n \geq 2$**.**

If we put $f(t) = \frac{1}{2}\left(t + \frac{p}{t}\right)$, $t > 0$, then

$$a_{n+1} = f(a_n) \geq \min_{t>0} f(t) \qquad \text{for } n \geq 1.$$

Since

$$f'(t) = \frac{1}{2}\left(1 - \frac{p}{t^2}\right) = 0 \qquad \text{for } t = \sqrt{p} > 0,$$

and $f(t) \to \infty$ for $t \to 0+$ and for $t \to \infty$, we must have that $t = \sqrt{p}$ corresponds to a minimum, hence

$$a_{n+1} \geq f(\sqrt{p}) = \frac{1}{2}\left(\sqrt{p} + \frac{p}{\sqrt{p}}\right) = \sqrt{p} \qquad \text{for } n \geq 1,$$

and $a_n \geq \sqrt{p}$, and thus $a_n^2 \geq p$ for all $n \geq 2$.

2) **Proof of the claim that** (a_n) **is weakly decreasing for** $n \geq 2$**.**

We first prove that $a_2 \geq a_3$. This follows from

$$\begin{aligned} a_2 - a_3 &= \frac{1}{2}\left(a_1 + \frac{p}{a_1}\right) - \frac{1}{2}\left(a_2 + \frac{p}{a_2}\right) = \frac{1}{2}(a_1 - a_2) + \frac{p}{2}\left(\frac{1}{a_1} - \frac{1}{a_2}\right) = \frac{a_1 - a_2}{2a_2}\left\{a_2 - \frac{p}{a_1}\right\} \\ &= \frac{1}{2a_2}\,\frac{1}{2}\left(k - \frac{p}{k}\right)\frac{1}{2}\left(k - \frac{p}{k}\right) = \frac{1}{8a_2}\left(k - \frac{p}{k}\right)^2 \geq 0. \end{aligned}$$

Then *assume* that $a_{n-1} \geq a_n$, i.e. $a_{n-1} - a_n \geq 0$ for $n \geq 3$. This is true for $n = 3$, according to our first result. then

$$\begin{aligned} a_n - a_{n+1} &= \frac{1}{2}\left(a_{n-1} + \frac{p}{a_{n-1}}\right) - \frac{1}{2}\left(a_n + \frac{p}{a_n}\right) = \frac{1}{2}(a_{n-1} - a_n) - \frac{p}{2}\left(\frac{1}{a_n} - \frac{1}{a_{n-1}}\right) \\ &= \frac{1}{2}(a_{n-1} - a_n) \cdot \frac{a_n a_{n-1} - p}{a_n a_{n-1}}. \end{aligned}$$

By 1. we have $a_n \geq \sqrt{p}$ and $\sqrt{a_{n-1}} \geq \sqrt{p}$, so $a_n a_{n-1} - p \geq 0$.

By the inductions assumption we get $a_{n-1} - a_n \geq 0$.

We see that we also have $a_n - a_{n+1} \geq 0$, hence $a_n \geq a_{n+1}$.

Then by induction, at $a_{n-1} \geq a_n$ for all $n \geq 2$, and the sequence is weakly decreasing.

3) Since (a_n) is bounded from below and weakly decreasing, it is convergent.

4) The function $f(t) = \frac{1}{2}\left(t + \frac{p}{t}\right)$ is continuous, hence we can find the **limit value** by taking the limit in the recursion formula, i.e. replace a_{n+1} and a_n by the limit value a. We get the equation

$$a = \frac{1}{2}\left(a + \frac{p}{a}\right), \qquad \text{i.e.} \qquad a^2 = p \text{ ved omordning.}$$

Since every $a_n \geq \sqrt{p}$, we have $a \geq \sqrt{p}$, it is in particular positive. It therefore follows that $a = \sqrt{p}$.

5) When $k = 2$ and $p = 3$, we get

$$a_1 = 2 \qquad \text{og} \qquad a_{n+1} = \frac{1}{2}\left(a_n + \frac{3}{a_n}\right).$$

Hence

$$a_2 = 1,750000, \quad a_3 = 1,732143, \quad a_4 = 1,732051.$$

By a comparison with the value on a pocket calculator we see that a_3 agrees on the first 3 decimals with $\sqrt{3}$ and a_4 agrees on the first 6 decimals with $\sqrt{3}$.

6) **Comparison with the Newton-Raphson iteration method.**
Let $F(x) = x^2 - p$, then $F'(x) = 2x$, hence

$$g(x) = x - \frac{F(x)}{F'(x)} = x - \frac{x^2 - p}{2x} = \frac{1}{2}\left(x + \frac{p}{x}\right).$$

The iteration formula becomes

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{p}{a_n}\right),$$

which is precisely the considered recursive sequence. $\Diamond$

Example 3.4 *Let the sequence (a_n) be given by $a_1 = 0$, $a_2 = 1$, and each of the following terms as the arithmetical mean of the two preceding terms:*

$$a_3 = \frac{1}{2}, \quad a_4 = \frac{3}{4}, \quad a_5 = \frac{5}{8}, \dots, a_n = \frac{1}{2}\{a_{n-1} + a_{n-2}\}, \dots.$$

Prove by induction that

$$a_n = \frac{2}{3} \cdot \frac{(-1)^{n-2}}{2^{n-1}} + \frac{2}{3}.$$

Then prove that the sequence (a_n) is convergent with the limit $2/3$.

For $n = 1$ we get $\frac{2}{3} \cdot \frac{(-1)^{n-2}}{2^{n-1}} + \frac{2}{3} = -\frac{2}{3} + \frac{2}{3} = 0 = a_1$.
For $n = 2$ we get $\frac{2}{3} \cdot \frac{(-1)^{n-2}}{2^{n-1}} = \frac{1}{3} + \frac{2}{3} = 1 = a_2$.
For $n = 3$ we get $\frac{2}{3} \cdot \frac{(-1)^{n-2}}{2^{n-1}} = -\frac{1}{6} + \frac{2}{3} = \frac{1}{2} = a_3$.

Assume that

$$a_{n-2} = \frac{2}{3} \cdot \frac{(-1)^n}{2^{n-3}} + \frac{2}{3} \quad \text{og} \quad a_{n-1} = \frac{2}{3} \cdot \frac{(-1)^{n-1}}{2^{n-2}} + \frac{2}{3}$$

for some $n \geq 3$. This is at least true for $n = 3$ and $n = 4$. Then by an addition,

$$a_n = \frac{1}{2}(a_{n-1} + a_{n-2}) = \frac{2}{3} + \frac{2}{3} \cdot \frac{1}{2}\left\{\frac{(-1)^n}{2^{n-3}} + \frac{(-1)^{n-1}}{2^{n-2}}\right\} = \frac{2}{3} + \frac{2}{3} \cdot \frac{(-1)^n}{2^{n-1}},$$

which has the same structure, and the formula follows by induction.

Since

$$\left|a_n - \frac{2}{3}\right| = \frac{2}{3} \cdot \frac{1}{2^{n-1}} = \frac{4}{3} \cdot \frac{1}{2^n} \to 0 \qquad \text{for } n \to \infty,$$

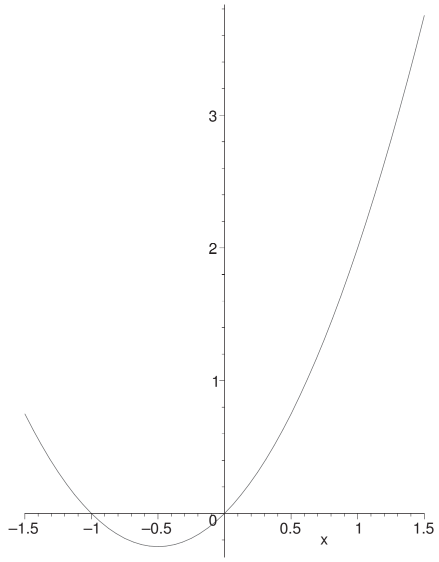
it follows by the definition that $a_n \to \frac{2}{3}$.

Example 3.5 *Consider the recursively given sequence (a_n), where $a_1 = c \in \mathbb{R}$, $a_{n+1} = f(a_n)$ for $f(x) = x + x^2$. The function f has the fix point $x_0 = 0$. Show graphically that the fix point is attractive for some values of $c < 0$, and repelling for every $c > 0$.*

It follows from the equation $f(x_0) = x_0$, i.e. $x_0^2 = 0$, that $x_0 = 0$, hence $x_0 = 0$ is a fix point. Since

$$f(x) = x^2 + x = \left(x + \frac{1}{2}\right) - \frac{1}{4},$$

it is easy to sketch the graph.



It is difficult to sketch on the figure in MAPLE the lines which shows the convergence, so this is left to the reader. We see that x_0 is attractive for $c \in [-1, 0]$ and repelling for $c \in \mathbb{R} \setminus [-1, 0]$

We shall now prove these claims.

1) If $c > 0$, then $f(c) = c^2 + c > c$, and $f(c)$ moves away from $x_0 = 0$, thus the point is repelling.

2) If r $c < -1$, then $f(c) = c^2 + c = |c|(|c| - 1) > 0$, and we are back in case 1..

3) If either $c = 0$ or $c = -1$, then $f(c) = 0$. Since trivially $f(0) = 0$ in all the following iterations, it follows that $x_0 = 0$ is attractive for these values of c.

4) Finally, if $-1 < c < 0$, then $f(c) = c + c^2 = c(1 + c) < 0$, and $f(c) > c$, hence

$$c < f(c) < 0,$$

and we conclude that the fix point is attractive for $c \in]-1, 0[$. $\Diamond$

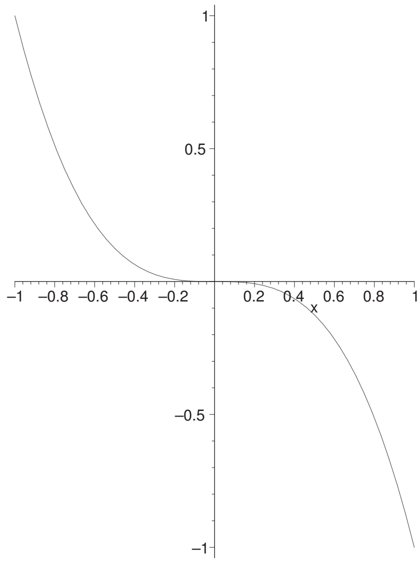
Example 3.6 *Consider the recursively given sequence* (a_n) *where*

$$a_1 = c, \qquad a_{n+1} = -a_n^3.$$

1) Compute for $c = -1$ *the terms* $a_2, \dots, a_5$*, and give a graphical discription.*

2) The same for $c = -1/2$.

Again, it is difficult to give all necessary details on a figure in MAPLE-figure, so these additions are left to the reader.



1) If $c = -1$, then

$$a_1 = -1, \quad a_2 = 1, \quad a_3 = -1, \quad a_4 = 1, \quad a_5 = -1,$$

and in general $a_n = (-1)^n$.

2) If $c = -1/2$, then

$$a_1 = -\frac{1}{2}, \quad a_2 = \frac{1}{2^3}, \quad a_3 = -\frac{1}{2^9}, \quad a_4 = \frac{1}{2^{27}}, \quad a_5 = -\frac{1}{2^{81}},$$

and in general

$$a_n = (-1)^n 2^{-(3^{n-1})}, \qquad n \in \mathbb{N}. \qquad \Diamond$$

Example 3.7 *Let the sequence* (a_n) *be recursively given by*

$$a_1 = 100, \qquad a_{n+1} = 2\sqrt{a_n} - 1.$$

1) Prove by induction that $a_n > 1$ *for every* $n \in \mathbb{N}$.

2) Prove by induction that the sequence (a_n) *is decreasing.*

3) Finally, prove that the sequence (a_n) *is convergent, and find its limit value.*

1) *Assume* that $a_n > 1$. Then $a_{n+1} = 2\sqrt{a_n} - 1 > 2 \cdot 1 - 1 = 1$, and we see that $a_n > 1$ implies that also the successor satisfies $a_{n+1} > 1$. Since $a_1 = 100 > 1$, the claim follows by induction, hence (a_n) is bounded from below.

2) We get by insertion, $a_2 = 2\sqrt{100} - 1 = 19 < 100 = a_1$, thus $a_2 < a_1$.
Assume that $a_{n-1} > a_n$ (this is true for $n = 2$). When n is replaced by $n+1$, we get

$$a_n - a_{n+1} = (2\sqrt{a_{n-1}} - 1) - (2\sqrt{a_n} - 1) = 2(\sqrt{a_{n-1}} - \sqrt{a_n}) > 0,$$

hence $a_n > a_{n+1}$. We conclude by induction that

$$a_1 > a_2 > a_3 > \cdots > a_{n-1} > a_n > a_{n+1} > \cdots \geq 1.$$

3) Since (a_n) is decreasing and bounded from below, it follows that (a_n) is convergent. Denote the limit value by a. By taking the limit of the recursion formula we get

$$0 = \lim_{n\to\infty} a_{n+1} - \lim_{n\to\infty} 2\sqrt{a_n} + 1 = a - 2\sqrt{a} + 1 = (\sqrt{a}-1)^2,$$

hence $\sqrt{a} = 1$, and thus $\lim_{n\to\infty} a_n = a = 1$.

Remark 3.1 If we consider the function $f(x) = 2\sqrt{x} - 1$, where $f'(x) = \dfrac{1}{\sqrt{x}}$, then $f'(x_0) = f'(1) = 1$, corresponding to a limiting case, in which one usually can say nothing about the convergence. This is also demonstrated by an iteration on a pocket calculator, because the approximation becomes slower the closer one is to $a = 1$,

$$a_5 = 3,26904, \quad a_{10} = 1,69909, \quad a_{15} = 1,39299, \quad a_{20} = 1,26997. \qquad \Diamond$$

Example 3.8 *Find the smallest positive solution of the equation*

$$\cos x = \frac{1}{1 + 25 \cdot 10^{-6}/\sin x}$$

with four decimals by using a convenient iteration method. Then comment on the convergence.

Remark 3.2 This is a very vicious example! $\Diamond$

The difficulty of this example apparently stems from the denominator on the right hand side. First we rewrite in the following way

$$(4) \quad \cos x = \frac{1}{1 + 25 \cdot 10^{-6}/\sin x} = \frac{40000 \sin x}{40000 \sin x + 1} = 1 - \frac{1}{40000 \sin x + 1}.$$

It is easily seen that $x > 0$ must be small, hence by Taylor's formula

$$\sin x \approx x \qquad \text{og} \qquad \cos x \approx 1 - \frac{x^2}{2},$$

and (4) can approximatively be written

$$1 - \frac{x^2}{2} \approx 1 - \frac{1}{40000x + 1}, \quad \text{i.e.} \quad x^2(40000x + 1) \approx 2.$$

We must in particular have $40000x \gg 1$, thus we have in the first iteration (the starting value)

$$x \approx 1/\sqrt[3]{20000} \approx 0,03684.$$

Then the method is reduced to the well-known *regula falsi*, i.e. insert

$$x_1 = 0,03683, \qquad x_2 = 0,03684, \qquad x_3 = 0,03685$$

into (4) and compare,

n	x_n	$\cos x_n$	$1-\dfrac{1}{40000\sin x+1}$	Left side relation right side
1	0,03683	0,999321852	0,999321513	$>$
2	0,03684	0,999321484	0,999321697	$<$
3	0,03685	0,999321116	0,999321881	$<$

We are just inside the range of the accuracy of the pocket calculator, because the factor 40000 gives an error of rounding off which is 40000 times bigger than usual (i.e. 40000 times 10^{-12}).

It is quite ironical that our first approximation by using Taylor's formula in fact gives the best approximation with four decimals,

$$x \approx 0,03684,$$

which should be compared the the interpolation of the table,

$$x \approx 0,036836.$$

If we instead apply the Newton-Raphson iteration on

$$F(x) = \cos x + \frac{1}{40000\sin x+1} - 1$$

where

$$F'(x) = -\sin x - \frac{\cos x}{(40000\sin x+1)^2}$$

we get the auxiliary function

$$g(x) = x - \frac{F(x)}{F'(x)} = x + \frac{\cos x - 1 + \dfrac{1}{40000\sin x+1}}{\sin x + \dfrac{\cos x}{(40000\sin x+1)^2}}.$$

Even if we choose the value $x_0 = 0,037$ as our start [where we have an eye to the value 0,03684] the iteration is extremely slow,

$$x_1 = 0,036991, \quad x_2 = 0,036982, \quad x_3 = 0,036974.$$

If we instead rewrite (4) to

$$40000\cos x\cdot\sin x + \cos x = 40000\sin x,$$

i.e. to

$$F(x) = 20000\sin 2x - 40000\sin x + \cos x = 0,$$

then

$$F'(x) = 40000\cos 2x - 40000\cos x - \sin x,$$

and we get by the Newton-Raphson iteration

$$x_{n+1} = x_n - \frac{20000\sin 2x_n - 40000\sin x_n + \cos x_n}{40000\cos 2x_n - 40000\cos x_n - \sin x_n}$$

with starting value $x_0 = 0,037$, the first values

$$x_1 = 0,036837, \qquad x_2 = 0,036836, \qquad x_3 = 0,036836.$$

Example 3.9 *Given the function*

$$F(x)=\cos x+\frac{1}{\cosh x}.$$

1) Write down the Newton-Raphson iteration formula for the solution of the equation $F(x)=0$.

2) Apply a programmable pocket calculator to the Newton-Raphson iteration and find the first four positive zeros of $F(x)=0$ by choosing the starting values

$$x_0^{(1)}=\frac{\pi}{2},\quad x_0^{(2)}=\frac{3\pi}{2},\quad x_0^{(3)}=\frac{5\pi}{2},\quad x_0^{(4)}=\frac{7\pi}{2}.$$

Apply 3 iterations and use 5 decimals.

1) Since

$$F'(x)=-\sin x-\frac{\sinh x}{\cosh^2 x}=-\frac{\sin x\cdot\cosh^2 x+\sinh x}{\cosh^2 x},$$

the Newton-Raphson iteration is written

$$x_{n+1}=x_n-\frac{F(x_n)}{F'(x_n)}=x_n+\frac{\cos x_n\cdot\cosh^2 x_n+\cosh x_n}{\sin x_n\cdot\cosh^2 x_n+\sinh x_n}.$$

2) The requested iterations give with 5 decimals,

n	1	2	3	4
$x_0^{(n)}$	1,57080	4,71239	7,85398	10,99560
$x_1^{(n)}$	1,86265	4,69410	7,85476	10,99550
$x_2^{(n)}$	1,87507	4,69409	7,85476	10,99550
$x_3^{(n)}$	1,87510	4,69409	7,85476	10,99550

Example 3.10 *Given the function*

$$F(x)=e^x\sin x-1,\qquad x\in\left[0,\frac{\pi}{2}\right].$$

Prove that the equation $F(x)=0$ has precisely one solution α, and find this by an iteration in two steps.

Since both e^x and $\sin x$ are increasing in $\left[0,\frac{\pi}{2}\right]$, and $F(x)$ is continuous with $F(0)=-1<0$ and $F\left(\frac{\pi}{2}\right)-1>0$, there exists precisely one zero $\alpha\in\left]0,\frac{\pi}{2}\right[$.

Since

$$F'(x)=e^x(\sin x+\cos x),$$

it follows by **Newton-Raphson iteration** that

$$\alpha_{n+1} = \alpha_n - \frac{F(\alpha_n)}{F'(\alpha_n)} = \alpha_n - \frac{e^{\alpha_n}\sin\alpha_n - 1}{e^{\alpha_n}(\sin\alpha_n + \cos\alpha_n)}.$$

The next question depends on the choice of starting value. In the specific case, however, the process is fairly robust.

If we choose $\alpha_0 = 1$, then we get successively

$$\alpha_1 = 0,657, \qquad \alpha_2 = 0,591, \qquad \alpha_3 = 0,5885.$$

If we instead choose $\alpha_0 = 0,5$ then we also get $\alpha_3 = 0,5885$.

4 Sequences of functions

Example 4.1 *Prove that the sequence of functions*

$$f_n(x) = \frac{x}{x^2+1} + \frac{x}{x^2+4} + \cdots + \frac{x}{x^2+n^2},$$

where $f_n : [-1,1] \to \mathbb{R}$*, is pointwise convergent on the interval* $[-1,1]$*.*
Hint. *Apply the* General principle of convergence.
Apply a programmable pocket calculator to sketch the graph of $f_n(x)$ *for some large* n*.*
It can be proved that f_n *converges uniformly on the interval* $[-1,1]$ *towards the function*

$$f(x) = \begin{cases} \dfrac{1}{2}\left[\pi\dfrac{e^{2\pi x}+1}{e^{2\pi x}-1} - \dfrac{1}{x}\right] & \textit{for } x \neq 0, \\[2ex] 0 & \textit{for } x = 0. \end{cases}$$

For any $x \in [-1,1]$ we have

$$|f_{n+m}(x) - f_n(x)| = \left| \frac{x}{x^2+(n+1)^2} + \cdots + \frac{x}{x^2+(n+m)^2} \right| \leq \sum_{j=1}^{m} \frac{1}{(n+j)^2}.$$

It can be proved from the *Theory of Fourier Series* that $\sum_{n=1}^{\infty} \dfrac{1}{n^2} = \dfrac{\pi^2}{6}$ is convergent. This means that to any $\varepsilon > 0$ there exists an $N = N(\varepsilon) \in \mathbb{N}$, such that

$$\sum_{j=1}^{m} \frac{1}{(n+j)^2} \leq \sum_{j=1}^{\infty} \frac{1}{(n+j)^2} = \sum_{j=n+1}^{\infty} \frac{1}{j^2} < \varepsilon \quad \text{for alle } n \geq N(\varepsilon).$$

By insertion we get that $|f_{n+m}(x) - f_n(x)| < \varepsilon$ for $n \geq N(\varepsilon)$, not just pointwisely, but even uniformly.

Since every f_n, $n \in \mathbb{N}$ is an odd function, we shall only sketch the graph of f_n for $x \in [0,1]$.
That the limit function $f(x)$ is precisely the given function can either be shown by a formula from **Complex Function Theory** or by some clever application of a **Fourier series**. Note that

$$f(x) = \frac{1}{2}\left\{\pi \coth(\pi x) - \frac{1}{x}\right\} \qquad \text{for } x \neq 0. \qquad \diamond$$

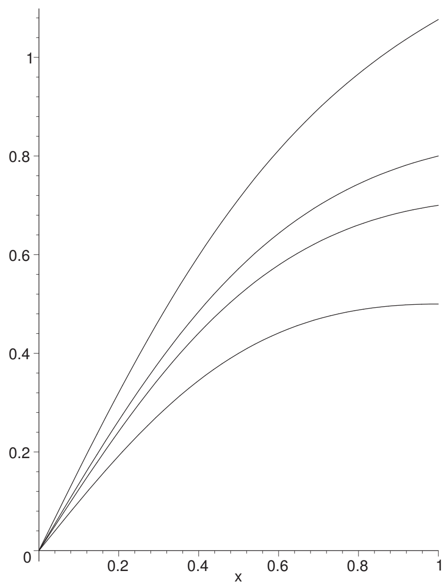
Example 4.2 *Prove that the sequence of functions* $f_n : \mathbb{R} \to \mathbb{R}$*, defined by*

$$f_n(x) = \frac{1+x^4}{(1+x^2)(n+x^4)}, \qquad n \in \mathbb{N},$$

converges uniformly towards the function 0.

First variant. Since $0 < \dfrac{1+x^4}{n+x^4} \leq 1$, and $\sqrt{y} \geq y$ for $y \in]0,1]$, we get

$$\frac{1+x^4}{n+x^4} \leq \sqrt{\frac{1+x^4}{n+x^4}}, \qquad \text{where we have put } y = \frac{1+x^4}{n+x^4}.$$



Hence we have the estimate

$$0 < f_n(x) = \frac{1+x^4}{(1+x^2)(n+x^4)} \le \frac{\sqrt{1+x^4}}{1+x^2} \cdot \frac{1}{\sqrt{n+x^4}} \le 1 \cdot \frac{1}{\sqrt{n}} \to 0 \quad \text{for } n \to \infty,$$

independent of $x \in \mathbb{R}$, so (f_n) converges uniformly towards 0 over $\mathbb{R}$.

Second variant. Put $u = x^2$. Then by a decomposition with respect to u,

$$\begin{aligned} 0 < f_n(x) &= \frac{1+u^2}{(1+u)(n+u^2)} = \frac{2}{n+1} \cdot \frac{1}{1+u} + \frac{1}{1+u}\left\{\frac{1+u^2}{n+u^2} - \frac{2}{n+1}\right\} \\ &= \frac{2}{n+1} \cdot \frac{1}{1+u} + \frac{1}{1+u}\left\{\frac{u^2-1}{n+u^2} + 2\left[\frac{1}{n+u^2} - \frac{1}{n+1}\right]\right\} \\ &= \frac{2}{n+1} \cdot \frac{1}{1+u} + \frac{u-1}{n+u^2} + \frac{2}{1+u} \cdot \frac{1-u^2}{(n+u^2)(n+1)} \\ &= \frac{2}{n+1} \cdot \frac{1}{1+u} + \frac{u-1}{n+u^2}\left\{1 - \frac{2}{n+1}\right\}. \end{aligned}$$

Since $\dfrac{u-1}{n+u^2}$ has its maximum for $u = \sqrt{n+1}$, and since $1 - \dfrac{2}{n+1} \in [0,1[$, we get the estimate

$$0 < f_n(x) < \frac{1}{n+1} + \frac{\sqrt{n+1}-1}{n+n+1} \cdot 1 < \frac{2}{n+1} + \frac{\sqrt{n+1}}{2n+1} \to 0 \quad \text{for } n \to \infty,$$

independently of $u \in \mathbb{R}_+ \cup \{0\}$, hence also independently of $x \in \mathbb{R}$. This proves that (f_n) converges uniformly towards 0 all over $\mathbb{R}$.

Example 4.3 *Find a sequence of C^1-functions $f_n : [0,1] \to \mathbb{R}$, which converges uniformly on $[0,1]$ towards a C^1-function $f : [0,1] \to \mathbb{R}$, and for which the sequence of derivatives (f_n') is pointwise convergent, but* not *converging towards the function f'.*

The best strategy must be first to choose some convenient pointwisely convergent sequence of functions (f_n'), where the limit function is not continuous, and then integrate the terms of this sequence from 0.

It is well-known that $g_n(x) = x^n$ is pointwise convergent towards the discontinuous function

$$g(x) = \begin{cases} 0 & \text{for } x \in [0,1[, \\ 1 & \text{for } x = 1. \end{cases}$$

Choose

$$f_n(x) = \int_0^x g_n(t)\,dt = \frac{x^{n+1}}{n+1}.$$

It is then obvious that

$$f_m'(x) = x^n = g_n(x) \to g(x) \qquad \text{for } n \to \infty.$$

It follows from the estimate

$$|f_n(x) - 0| = \frac{x^{n+1}}{n+1} \le \frac{1}{n+1} \to 0 \quad \text{for } n \to \infty, \quad \text{alle } x \in [0,1],$$

that $f_n \to 0$ uniformly. It is obvious that $f(x) = 0$ is a differentiable function and $f'(x) = 0$, which is $\neq \lim f_n'(x) = g(x)$.

By using the example above (one introduces a singularity at $x = 1$) it is possible to construct a sequence of functions (f_n), which converges uniformly towards 0 in $[0, 1]$ where

$$\lim_{n\to\infty} f_n'(x) \neq 0 \qquad \text{for ethvert } x \in [0,1] \cap \mathbb{Q}.$$

The construction, however, uses series which formally have not yet been introduced.

Example 4.4 *For every* $n \in \mathbb{N}$ *we put*

$$f_n(x) = nxe^{-nx^2}, \qquad x \in \mathbb{R}.$$

Find $\lim_{n\to\infty} f_n(x)$, *and prove that*

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)\,dx.$$

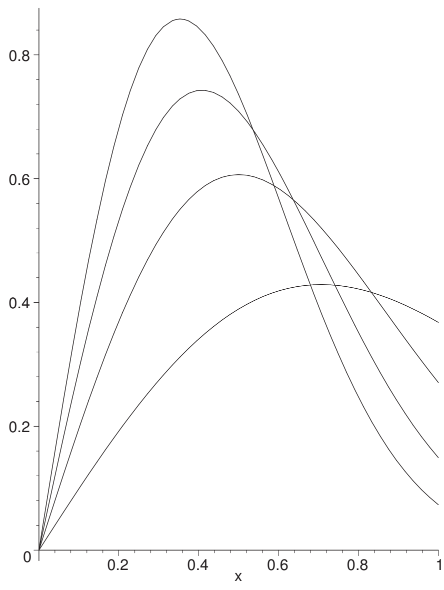
1) **Pointwise convergence**. What is here obvious? When $x = 0$ is fixed, then

$$f_n(0) = 0 \to 0 = f(0) \qquad \text{for } n \to \infty.$$

If instead $x \neq 0$ (fast), it follows by the magnitudes that

$$f_n(x) = x \cdot \frac{n}{\left(e^{x^2}\right)^2} \to 0 \quad \text{for } n \to \infty, \text{ da } e^{x^2} > 1.$$

As a conclusion we get that (f_n) is pointwisely convergent with the limit function



$$f(x) = \lim_{n\to\infty} f_n(x) = 0.$$

2) **The integrals**. It follows immediately that

$$\int_0^1 \lim_{n\to\infty} f_n(x)\,dx = \int_0^1 f(x)\,dx = \int_0^1 0\,dx = 0.$$

Then we get by the substitution $u = nx^2$, $du = 2nxdx$, that

$$\int_0^1 f_n(x)\,dx = \int_0^1 nxe^{-nx^2}\,dx = \frac{1}{2}\int_0^n e^{-u}\,du = \frac{1}{2}\left(1-e^{-n}\right).$$

Finally, by taking the limit,

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \lim_{n\to\infty}\frac{1}{2}\left(1-e^{-n}\right) = \frac{1}{2} \neq 0 = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx.$$

Remark 4.1 All functions $f_n(x)$ are continuous. Since the integration and the limit *cannot* be interchanged, the convergence of (f_n) can *never* be uniform. $\Diamond$

Example 4.5 *Let* $f_n : \mathbb{R} \to \mathbb{R}$ *be given by*

$$f_n(x) = \frac{x^{2n}}{1+x^{2n}}, \qquad n \in \mathbb{N}.$$

1) Prove that the sequence (f_n) *is pointwise convergent, but not uniformly convergent.*

2) Prove that the sequence (f_n) *is uniformly convergent in the interval* $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

1) This example is tricky, because one must split it up into three cases,

(a) $|x| < 1$, (b) $|x| = 1$, (c) $|x| > 1$.

In the remaining part of this question we keep x *fixed* in the given domain.

(a) If $|x| < 1$, then

$$|f_n(x) - 0| = f_n(x) = \frac{x^{2n}}{1+x^{2n}} \le x^{2n} \to 0 \qquad \text{for } n \to \infty.$$

(b) If $|x| = 1$, i.e. $x = \pm 1$, then $f_n(\pm 1) = \frac{1}{2}$.

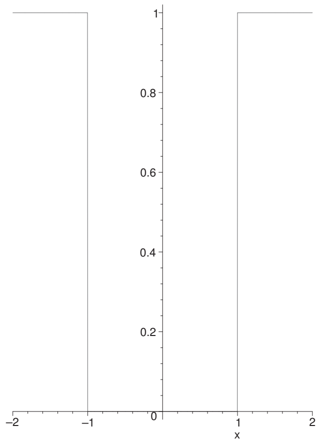
(c) If $|x| > 1$, then

$$f_n(x) = \frac{x^{2n}}{1+x^{2n}} = 1 - \frac{1}{1+x^{2n}} \to 1 - 0 = 1 \qquad \text{for } n \to \infty.$$

Summarizing we see that (f_n) is pointwise convergent with the limit function

$$f(x) = \begin{cases} 0 & \text{for } |x| < 1, \\ \frac{1}{2} & \text{for } |x| = 1, \\ 1 & \text{for } |x| > 1. \end{cases}$$

Since every f_n is continuous and the limit function f is *not* continuous [cf. the figure], it follows that the sequence is not uniformly convergent.



2) **Get rid of x!** When $|x| \le \frac{1}{2}$, we get the estimate

$$|f_n(x) - 0| = f_n(x) = \frac{x^{2n}}{1+x^{2n}} \le x^{2n} \le \left(\frac{1}{2}\right)^{2n} \to 0 \quad \text{for } n \to \infty.$$

For *every* $\varepsilon > 0$ there *exists* an N, such that for *every* $n \geq N$ and *every* $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$,

$$|f_n(x) - 0| \leq x^{2n} \leq \left(\frac{1}{2}\right)^{2n} < \varepsilon \qquad [\text{for } n \geq N(\varepsilon)],$$

i.e. (f_n) converges *uniformly* towards 0 on $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

Example 4.6 *Prove that the sequence of functions*

$$f_n(x) = \frac{x}{1+x^n}, \qquad x \in [0, \infty[,$$

is pointwise convergent, and find its limit function.
Check if the convergence is uniform.

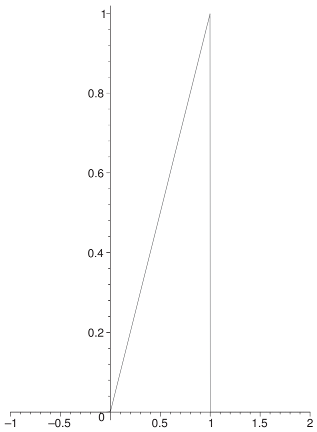
1) If $x \in [0,1[$ is kept fixed, then x^n tends to 0 for $n \to \infty$, hence

$$f_n(x) = \frac{x}{1+x^n} \to x \quad \text{for } n \to \infty, \quad \text{when } x \in [0,1[.$$

2) If $x = 1$, er $f_n(1) = \frac{1}{2}$ for every $n \in \mathbb{N}$, then $f_n(1) \to \frac{1}{2}$ for $n \to \infty$.

3) If $x > 1$ is kept fixed, then x^n tends towards ∞ for $n \to \infty$, hence

$$f_n(x) = \frac{x}{1+x^n} \to 0 \quad \text{for } n \to \infty, \quad \text{når } x \in \,]1, \infty[.$$



As a conclusion we see that (f_n) is pointwise convergent and its limit function is

$$f(x) = \begin{cases} x & \text{for } x \in [0,1[, \\ \frac{1}{2} & \text{for } x = 1, \\ 0 & \text{for } x \in \,]1, \infty[. \end{cases}$$

Since every $f_n(x)$ is continuous on $[0,\infty[$, and $f(x)$ is *not* continuous, it follows that the convergence *cannot be uniform.*

Example 4.7 *Prove that the sequence of functions* (f_n)*, which is given by*

$$f_n(x) = \frac{1}{(1+x^2)^n}, \qquad n \in \mathbb{N},$$

is pointwise convergent, and find its limit function.
Check if the convergence is uniform in the interval $[0,\infty[$*, in the interval* $]0,\infty[$*, and in the interval* $[1,\infty[$*, respectively.*

Pointwise convergence. What is obvious? For $x=0$ we have

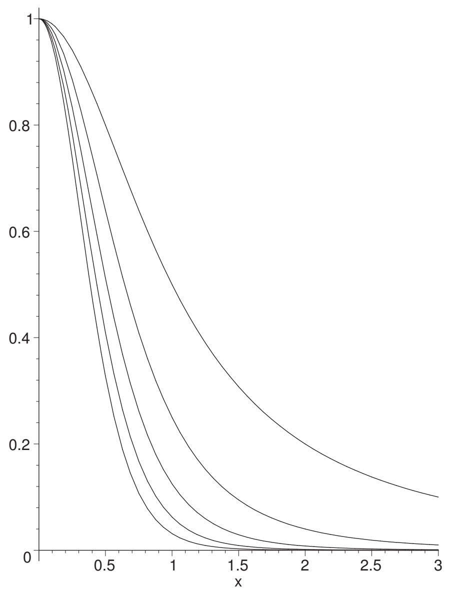
$$f_n(0) = \frac{1}{(1+0^2)^n} = \frac{1}{1^n} = 1 \to 1 = f(0) \qquad \text{for } n \to \infty.$$

Let $x \neq 0$ be *fixed.* If we put $1+x^2 = a > 1$ (a fixed number), we see that

$$f_n(x) = \frac{1}{(1+x^2)^n} = \frac{1}{a^n} \to 0 \qquad \text{for } n \to \infty.$$

We conclude that we have *pointwise convergence* and the limit function is

$$f(x) = \begin{cases} 1 & \text{for } x = 0, \\ 0 & \text{for } x \neq 0. \end{cases}$$



Uniform convergence in $[0,\infty[$**?** This is not possible, because every f_n is continuous, while the limit function is clearly discontinuous in 0.

Uniform convergence i $]0,\infty[$**?** This is a really tricky question, because the limit function $f(x) = 0$ is continuous in $]0,\infty[$. We shall nevertheless prove that the convergence is *not* uniform!

First note that the range of every f_n is $]0,1[$, i.e. $f_n(]0,\infty[) =]0,1[$.

If we choose $x_n = \sqrt{\sqrt[n]{2}-1} > 0$, we get

$$|f_n(x_n) - 0| = f_n(x_n) = \frac{1}{(1+\{\sqrt[n]{2}-1\})^n} = \frac{1}{2}.$$

Thus, we shall *always* obtain the value $\frac{1}{2}$ for every f_n, and since the constant $\frac{1}{2}$ "not can be made as small as possible", the convergence *cannot be uniform*.

Uniform convergence in $[1,\infty[$**?** In this case the limit function is again $f(x) = 0$. Since we are far away from the discontinuity at $x = 0$, it will be reasonable to get rid of x by an estimation: For $x \geq 1$ we have $1 + x^2 \geq 2$, thus

$$|f_n(x) - 0| = f_n(x) = \frac{1}{(1+x^2)^n} \leq \frac{1}{2^n} \to 0 \qquad \text{for } n \to \infty,$$

and we conclude that the convergence is uniform in $[1,\infty[$.

Remark 4.2 The above will always be accepted. A more careful solution is the following.

1) To *every* $\varepsilon > 0$ we *choose* $N = N(\varepsilon)$, such that

$$\frac{1}{2^N} \leq \varepsilon \qquad \left(\text{choose } N \geq \frac{\ln(1/\varepsilon)}{\ln 2} \text{ fixed}\right).$$

2) Since $(1/2^n)$ is decreasing for increasing n, we have

$$\frac{1}{2^n} \leq \frac{1}{2^N} \leq \varepsilon \qquad \text{for every } n \geq N(\varepsilon) \quad [\text{independently of } x].$$

3) For every $x \geq 1$ we have

$$\frac{1}{(1+x^2)^n} \leq \frac{1}{2^n} \leq \varepsilon \qquad \text{for every } n \geq N(\varepsilon) \quad [\text{independently of } x].$$

Finally we summarize the above:

To every $\varepsilon > 0$ there exists an $N = N(\varepsilon)$ [which does not depend on x], such that for every $n \geq N$ and every $x \geq 1$,

$$|f_n(x) - 0| \leq \varepsilon,$$

which means that we have uniform convergence on $[1,\infty[$.

Example 4.8 . *Prove that the sequence of functions* (g_n), *given by*

$$g_n(x) = \frac{1+ne^x}{x+n}, \qquad x \in [0,1],$$

converges uniformly towards the function e^x, $x \in [0,1]$.
Find the limit function $\lim_{n\to\infty} \int_0^1 g_n(x)\,dx$.

Let $x \in [0,1]$. The difference between $g_n(x)$ and the possible limit function e^x is given by

$$g_n(x) - e^x = \frac{1+ne^x}{x+n} - e^x = \frac{1}{x+n} - \frac{x}{x+n}\,e^x.$$

This gives For $x \in [0,1]$ the following estimate,

$$|g_n(x) - e^x| \le \frac{1}{x+n} + \frac{xe^x}{x+n} \le \frac{1}{0+n} + \frac{1\cdot e^1}{0+n} = \frac{1+e}{n},$$

because *a fraction with a positive numerator and positive denominator is made bigger, if we increase the numerator and the denominator is replaced by a smaller positive constant.*

Since the right hand side $\dfrac{1+e}{n} \to 0$ for $n \to \infty$, is *independent of* $x \in [0,1]$, it follows that (g_n) converges *uniformly* towards e^x in the interval $[0,1]$.

Since the convergence is *uniform*, and the interval of integration $[0,1]$ is bounded, we can interchange the limit process and the integration,

$$\lim_{n\to\infty}\int_0^1 g_n(x)\,dx = \lim_{n\to\infty}\int_0^1 \frac{1+ne^x}{x+n}\,dx = \int_0^1 \lim_{n\to\infty}\left(\frac{1+ne^x}{x+n}\right)dx = \int_0^1 e^x\,dx = e-1.$$

Remark 4.3 Note that none of the integrals

$$\int_0^1 \frac{1+ne^x}{x+n}\,dx, \qquad n\in\mathbb{N},$$

can be expressed by elementary functions. $\Diamond$

Example 4.9 *Let* $f_n : [0,1] \to$ *be given by*

$$f_n(x) = nx(1-x)^n, \qquad n\in\mathbb{N}.$$

1) Prove that the sequence (f_n) *is pointwise convergent, and find* $\lim_{n\to\infty} f_n(x)$.

2) Prove that

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx.$$

3) Find for every $n\in\mathbb{N}$ *the maximum of* $f_n(x)$.

4) Prove that (f_n) *is* not *uniformly convergent.*

1) If $x=0$, then $f_n(0)=0$, så $f(0)=0$. If $x\in\,]0,1]$, then $a=1-x\in[0,1[$, i.e.

$$f_n(x) = (1-a)\cdot na^n \to 0 \qquad \text{for } n\to\infty$$

according to the magnitudes of the functions.
It follows that (f_n) converges pointwise towards 0 in $[0,1]$.

2) We get by a partial integration that

$$\begin{aligned}\int_0^1 f_n(x)\,dx &= n\int_0^1 x(1-x)^n\,dx = \left[-\frac{n}{n+1}\,x(1-x)^{n+1}\right]_0^1 + \frac{n}{n+1}\int_0^1 (1-x)^{n+1}\,dx\\ &= 0+\frac{n}{(n+1)(n+2)}\left[-(1-x)^{n+2}\right]_0^1 = \frac{n}{(n+1)(n+2)}.\end{aligned}$$

Alternatively change the variable $t=1-x$, by which we get

$$\begin{aligned}\int_0^1 f_n(x)\,dx &= n\int_0^1 x(1-x)^n\,dx = n\int_0^1 (1-t)t^n\,dt = n\int_0^1 \left\{t^n - t^{n+1}\right\}dt\\ &= n\left\{\frac{1}{n+1}-\frac{1}{n+2}\right\} = \frac{n}{(n+1)(n+2)}\end{aligned}$$

There are other alternatives, which the reader may find himself.

It follows that

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \lim_{n\to\infty}\frac{n}{(n+1)(n+2)} = \lim_{n\to\infty}\frac{1}{\left(1+\dfrac{1}{n}\right)\cdot(n+2)} = 0$$
$$= \int_0^1 0\,dx = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx.$$

Alternatively one may use that

$$\frac{n}{(n+1)(n+2)} = \frac{2}{n+2} - \frac{1}{n+1} \to 0 \qquad \text{for } n\to\infty.$$

Since we now can interchange the limit process and the integration, we may *erroneously* jump to the wrong conclusion that the convergence should be uniform. The last two questions of the example show that this is not the case.

3) We get by differentiation

$$f_n'(x) = n(1-x)^n - n^2x(1-x)^{n-1} = n(1-x)^{n-1}(1-x-nx) = n(1-x)^{n-1}\{1-(n+1)x\}.$$

Since $f_n(0) = f_n(1) = 0$, and $f_n(x) > 0$ for $0 < x < 1$, the continuous function $f_n(x)$ must have a maximum.

Since $f_n \in C^\infty$, and since $f_n'(x) = 0$ is only fulfilled for $x = \dfrac{1}{n+1}$ in $]0,1[$, this value corresponds to the unique maximum. The value of the function here is

$$f_n\left(\frac{1}{n+1}\right) = n\cdot\frac{1}{n+1}\left(1-\frac{1}{n+1}\right)^n = \left(1-\frac{1}{n+1}\right)^{n+1}.$$

4) We see that

$$(5)\quad \lim_{n\to\infty} f_n\left(\frac{1}{n+1}\right) = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^{n+1} = \frac{1}{e} \neq 0.$$

Alternatively, $\ln(1-x) = -x + x\varepsilon(x)$ according to Taylor's formula. Putting $x = \dfrac{1}{n+1}$ we get

$$\ln\left\{\left(1-\frac{1}{n+1}\right)^{n+1}\right\} = (n+1)\ln\left(1-\frac{1}{n+1}\right)$$
$$= (n+1)\left\{-\frac{1}{n+1}+\frac{1}{n+1}\varepsilon\left(\frac{1}{n+1}\right)\right\} 0 - 1 + \varepsilon\left(\frac{1}{n+1}\right) \to -1$$
for $n\to\infty$.

Now exp is continuous, hence

$$f_n\left(\frac{1}{n+1}\right) = \left(1-\frac{1}{n+1}\right)^{n+1} = \exp\left(-1+\varepsilon\left(\frac{1}{n+1}\right)\right) \to \frac{1}{e} \quad \text{for } n\to\infty.$$

According to (5) we can find an $N \in \mathbb{N}$, such that

$$f_n\left(\frac{1}{n+1}\right) \geq \frac{1}{2} \cdot \frac{1}{e} > 0 \qquad \text{for all } n \geq N.$$

Since $\dfrac{1}{2e}$ is a constant (it cannot be made as small as we wish), the convergence *cannot* be uniform.

Example 4.10 *Given the function*

$$F(x) = e^x \sin x - 1, \qquad x \in \left[0, \frac{\pi}{2}\right].$$

from Example 3.10. Let $f_n : \left]0, \frac{\pi}{2}\right] \to \mathbb{R}$ *be given by*

$$f_n(x) = x - \frac{1}{(e^x \sin x)^n}, \qquad n \in \mathbb{N}.$$

1) Express by means of α *the largest set* $A \subseteq \left]0, \frac{\pi}{2}\right]$*, for which the sequence* (f_n) *is pointwise convergent in* A*. Find the limit function.*

2) Check if the sequence is also uniformly convergent on A*. Does there exist a largest set* $B \subseteq \left]0, \frac{\pi}{2}\right]$*, such that the sequence is uniformly convergent on* B*?*

1) Since $e^x \sin x = F(x) + 1 > 0$ for $x \in \left]0, \frac{\pi}{2}\right]$, the sequence of functions can also be written

$$f_n(x) = x - \left\{\frac{1}{F(x)+1}\right\}^n, \qquad x \in \left]0, \frac{\pi}{2}\right], \quad n \in \mathbb{N}.$$

Since (q^n) is convergent for $-1 < q \leq 1$, and since $F(x) + 1 > 0$, we get the condition

$$0 < \frac{1}{1 + F(x)} \leq 1, \qquad \text{i.e. } F(x) \geq 0.$$

According to Example 3.10 this is true for $x \in A = \left[\alpha, \frac{\pi}{2}\right]$, because $F(\alpha) = 0$, and because $F(x)$ is increasing.

If $x = \alpha$, then $F(\alpha) + 1 = 1$, hence

$$f_n(\alpha) = \alpha - 1.$$

If $\alpha < x \leq \frac{\pi}{2}$, then $0 < \dfrac{1}{F(x)+1} < 1$, hence $\left\{\dfrac{1}{F(x)+1}\right\}^n \to 0$ for $n \to \infty$. We conclude that the limit function is

$$f(x) = \begin{cases} \alpha - 1 & \text{for } x = \alpha, \\ x & \text{for } x \in \left]\alpha, \frac{\pi}{2}\right]. \end{cases}$$

2) Since every function f_n is continuous in A, and the limit function f is *not* continuous in A, the sequence *cannot* be uniformly convergent in A.

There does not exist any largest set B, on which (f_n) is uniformly convergent. In fact, for every $\varepsilon \in \left]0, \frac{\pi}{2} - \alpha\right[$ we have that (f_n) is uniformly convergent in $\left[\alpha + \varepsilon, \frac{\pi}{2}\right]$, and the smallest set which contains all these intervals is $\left]\alpha, \frac{\pi}{2}\right]$, on which (f_n) is *not* uniformly convergent.

ALTERNATIVELY, choose a sequence (y_n), such that

$$\frac{1}{1+F(y_n)} = \frac{1}{\sqrt[n]{2}}, \qquad \text{i.e. } F(y_n) = \sqrt[n]{2} - 1 \quad (\to 0 \text{ for } n \to \infty).$$

Then $y_n \to \alpha+$, and

$$f_n(y_n) = y_n - \frac{1}{2} \to \alpha - \frac{1}{2} \qquad \text{for } n \to \infty,$$

hence $(f_n(y_n))$ neither converges towards α nor towards $\alpha - 1$.

5 Linear difference equations

Example 5.1 *1) Find the complete solution of the difference equation*

$$x_k + x_{k-1} = 0, \qquad k \geq 1.$$

2) Find a particular solution of the difference equation

$$(6)\quad x_k + x_{k-1} = 4, \qquad k \geq 1,$$

and then find the total solution of (6), for which $x_0 = 1$.

A. Linear difference equation of first order.

D. Find the solution of the homogeneous equation and then a particular solution.

I. 1) The characteristic polynomial $R+1$ has the root $R = -1$, thus the complete solution of the homogeneous equation is given by

$$x_k = c \cdot (-1)^k, \qquad k \in \mathbb{N}_0, \quad c \in \mathbb{R}.$$

2) By inspection we see that the constant solution $x_k = 2$ is one solution of (6). Hence the complete solution is

$$x_k = 2 + c \cdot (-1)^k, \qquad k \in \mathbb{N}_0, \quad c \in \mathbb{R}.$$

For $k = 0$ we get $x_0 = 1 = 2 + c$, i.e. $c = -1$, and the wanted solution becomes

$$x_k = 2 + (-1)^{k+1}, \qquad k \in \mathbb{N}_0,$$

i.e.

$$x_k = \begin{cases} 1 & \text{for } k \text{ even,} \\ 3 & \text{for } k \text{ odd,} \end{cases} \qquad k \in \mathbb{N}_0.$$

Example 5.2 *Find a particular solution of the difference equation*

$$x_k - x_{k-1} = 4, \qquad k \geq 1,$$

and then find the complete solution.

A. Difference equation of first order.

D. Guess some particular solution.

I. Put $x_k = 4k$. Then we get by insertion (i.e. we are testing this sequence) that

$$x_k - x_{k-1} = 4k - 4(k-1) = 4,$$

and we have shown that $x_k = 4k$, $k \in \mathbb{N}$, is a particular solution.

The characteristic polynomial $R - 1$ has the root $R = 1$, hence the total solution is given by

$$x_k = 4k + c, \qquad k \in \mathbb{N}_0, \quad c \in \mathbb{R}.$$

Example 5.3 *Find that solution of the difference equation*

$$x_k + x_{k-1} = k, \qquad k \geq 1,$$

for which $x_0 = 2$.

A. Linear, inhomogeneous difference equation of first order.

D. Guess a particular solution.

I. If we guess on $x_k = \alpha k + \beta$, then we get by insertion that

$$\alpha k + \beta + \alpha(k-1) + \beta = 2\alpha k = (2\beta - \alpha),$$

which is equal to the variable k for $\alpha = \dfrac{1}{2}$ and $\beta = \dfrac{1}{4}$.

Since the characteristic polynomial $R+1$ has the root $R = -1$, the complete solution is given by

$$x_k = \frac{1}{2}\,k + \frac{1}{4} + c \cdot (-1)^k, \qquad k \in \mathbb{N}_0, \quad c \in \mathbb{R}.$$

For $k = 0$ we get

$$c = x_0 - \frac{1}{4} = 2 - \frac{1}{4} = \frac{7}{4},$$

thus the wanted solution is

$$x_k = \frac{1}{2}\,k + \frac{1}{4} + \frac{7}{4} \cdot (-1)^k, \qquad k \in \mathbb{N}_0.$$

Example 5.4 *Find that solution of the difference equation*

$$x_k - x_{k-1} = k, \qquad k \geq 1,$$

for which $x_2 = 4$.

A. Linear, inhomogeneous difference equation of first order.

D. Solve the corresponding homogeneous equation. Then guess a particular solution.

I. The characteristic polynomial $R - 1$ has the root $R = 1$. Thus, the complete solution of the homogeneous equation is given by

$$x_k = c, \qquad k \in \mathbb{N}_0, \quad c \in \mathbb{R}.$$

Since already $x_k = 1$ is a solution of the homogeneous equation and since $k \cdot 1$ occurs on the right hand side of the equation, we guess in analogy with the method of solving differential equations on a solution of the structure

$$x_k = \alpha k^2 + \beta k, \qquad k \in \mathbb{N}_0.$$

We get by insertion,

$$x_k - x_{k-1} = \alpha\left\{k^2 - (k-1)^2\right\} + \beta\{k-(k-1)\} = \alpha(2k-1)+\beta = 2\alpha k + (\beta-\alpha).$$

This expression is equal to the variable k, if $\alpha = \beta = \dfrac{1}{2}$, thus the complete solution is

$$x_k = \frac{1}{2}\,k^2 + \frac{1}{2} + c, \qquad k \in \mathbb{N}_0, \quad c \in \mathbb{R}.$$

If $k=2$, we get the condition

$$4 = \frac{1}{2}\cdot 2^2 + \frac{1}{2}\cdot 2 + c = 3 + c,$$

from which $c = 1$, and the solution is

$$x_k = \frac{1}{2}\,k^2 + \frac{1}{2}\,k + 1, \qquad k \in \mathbb{N}_0.$$

Example 5.5 *At some given time we have 1000 bacteria in a culture of bacteria . We assume in general that the number of bacteria is increased by 250 % every second hour. How many bacteria will there be after 24 hours?*

A. Exponential growth (difference equation). Notice that since we are looking at $12\cdot 2 = 24$ hours, we shall find x_{12}.

D. There are some problems here with the interpretation of the text. If the *increase* really is 250 %, then the corresponding difference equation becomes

$$x_k = x_{k-1} + \frac{5}{2}\,x_{k-1} = \frac{7}{2}\,x_{k-1}.$$

If the meaning instead is that the increase is 250 % of the previous value at time $(k-1)2$, then the difference equation becomes

$$x_k = \frac{5}{2}\,x_{k-1}.$$

Since there is some linguistic uncertainty in the original text (I do not remember where I found this example), we shall as an exercise go through the solving of both equations.

I. We have $x_0 = 1000$.

1) In the first interpretation we get

$$x_{12} = \left(\frac{7}{2}\right)^{12}\cdot 1000 \approx 3\,379\,220\,508.$$

2) In the second interpretation we get

$$x_{12} = \left(\frac{5}{1}\right)^{12}\cdot 1000 \approx 59\,604\,645.$$

Example 5.6 *A man pays every quarter 600 euro into his account. The interest is 11 % p.a., where the interest is added every quarter. When will there be 25 000 euro on the account?*

A. Difference equation for savings.

D. Write down the model of difference equation and then solve this equation.

I. Let x_k denote the capital at the k-th quarter after his payment. Then $x_0 = 600$, and since the interest per quarter is $\frac{11}{4}$ %, we get the equation

$$x_k = 600 + \left\{1 + \frac{11}{400}\right\} x_{k-1}, \qquad k \in \mathbb{N}.$$

This is then rewritten as the difference equation

$$x_k - \frac{411}{400}\, x_{k-1} = 600.$$

The solution of the corresponding equation is

$$x_k = c \cdot \left(\frac{411}{400}\right)^k, \qquad k \in \mathbb{N}_0, \quad c \in \mathbb{R}.$$

Then we guess a particular solution of the structure $x_k = \alpha$, $k \in \mathbb{N}_0$. We get by insertion

$$x_k - \frac{411}{400}\,x_{k-1} = -\frac{11}{400}\,\alpha,$$

which is equal to 600 for

$$\alpha = -\frac{240\,000}{11}.$$

Thus the complete solution is

$$x_k = -\frac{240\,000}{11} + c \cdot \left(\frac{411}{400}\right)^k, \qquad k \in \mathbb{N}_0, \quad c \in \mathbb{R}.$$

For $k = 0$ we get

$$c = 600 + \frac{240\,000}{11} = \frac{246\,600}{11},$$

so the solution becomes

$$x_k = -\frac{240\,000}{11} + \frac{246\,600}{11} \cdot \left(\frac{411}{400}\right)^k, \qquad k \in \mathbb{N}_0.$$

Finally, we shall find the smallest $k \in \mathbb{N}_0$, for which $x_k \geq 25\,000$. Hence we shall find the smallest $k \in \mathbb{N}_0$, for which

$$\frac{246\,600}{11} \cdot \left(\frac{411}{400}\right)^k \geq 25\,000 + \frac{240\,000}{11} = \frac{515\,000}{11}.$$

This is rearranged as

$$\left(\frac{411}{400}\right)^k \geq \frac{515\,000}{246\,000} = \frac{2575}{1233},$$

from which

$$k \geq \frac{\ln\left(\dfrac{2575}{1233}\right)}{\ln\left(\dfrac{411}{400}\right)} \approx \frac{0,7364}{0,0271} \approx 27,15.$$

Hence, we see that after 28 quarters, corresponding to 7 years, we get $x_k \geq 25\,000$ for the first time.

C. Weak control. There is paid $4 \cdot 600 = 2400$ euro per year, hence 16 800 euro in 7 years, so the result looks reasonable. considering the high (and today unrealistic) interest.

Example 5.7 *In the following examples we shall deal with annuity loans. The background is in general the following:*
A loan on G_0 euro is repaid with a fixed payment of A euro per settling period, and the interest to be paid of the debt is r per settling period (where we give r as a usual fraction, and not in %). Let G_n denote the remaining debt after n settling periods.

1) Prove that G_n satisfies

$$G_n - (1+r)G_{n-1} = -A, \qquad n \geq 1,$$

and then find G_n.

2) Establish the condition that the debt is repaid.

A. Annuity loans.

D. Analyze the situation at the end of the n-th settling period in order to find the difference equation of the problem. Then solve this difference equation.

I. 1) The remaining debt G_n at the n-th settling period is equal to the remaining debt G_{n-1} at the $(n-1)$-th settling period, plus the interest, $\times G_{n-1}$, and minus the payment A, i.e.

$$G_n = (1+r)G_{n-1} - A,$$

which we rearrange as the solution

$$G_n - (1+r)G_{n-1} = -A, \qquad n \geq 1.$$

Here we guess a particular solution of the form $G_n = \alpha$. We get by insertion (i.e. testing this solution) that

$$G_n - (1+r)G_{n-1} = \alpha - (1+r)\alpha = -r\alpha = -A,$$

and a particular solution is the constant sequence

$$G_n = \alpha = \frac{A}{r}, \qquad n \in \mathbb{N}_0.$$

Since the corresponding homogeneous equation has the solution $c\cdot(1+r)^n$, the complete solution of the inhomogeneous equation is given by

$$G_n = \frac{A}{r} + c\cdot(1+r)^n, \qquad n \in \mathbb{N}_0, \quad c \in \mathbb{R}.$$

For $n = 0$ we get the condition

$$G_0 = \frac{A}{r} + c, \qquad \text{i.e.} \qquad c = G_0 - \frac{A}{r},$$

and the wanted solution becomes

$$G_n = \frac{A}{r} + \left(G_0 - \frac{A}{r}\right)\cdot(1+r)^n, \qquad n \in \mathbb{N}_0.$$

2) The debt will be paid back, if and only if G_n at some time becomes ≤ 0. This means that $G_0 - \dfrac{A}{r} < 0$, hence

$$A > r \cdot G_0,$$

which is only expressing the reasonable fact that the payment must be bigger than the interest in one settling period of the original loan.

Example 5.8 *Let G_0 and A be given, and assume that A is sufficiently large to assure that the loan will be paid. Find a formula for the number of settling periods which is needed to pay the debt (thus the smallest number n of settling periods for which $G_n \leq 0$.)*

A. A continuation of Example 5.7.

D. Apply the solution of Example 5.7.

I. According to Example 5.7 the remaining debt G_n is given by

$$G_n = \frac{A}{r} + \left(G_0 - \frac{A}{r} \right) \cdot (1+r)^n, \qquad n \in \mathbb{N}_0,$$

where we must assume that $G_0 - \dfrac{A}{r} < 0$, i.e. $A > r \cdot G_0$.
We shall find the smallest $n \in \mathbb{N}_0$, for which $G_n \leq 0$, i.e. the smallest $n \in \mathbb{N}_0$, for which

$$\left(\frac{A}{r} - G_0 \right) \cdot (1+r)^n \geq \frac{A}{r}, \quad \text{i.e.} \quad (1+r)^n \geq \frac{A}{A - r \cdot G_0}.$$

We get by taking the logarithm that

$$n \geq \frac{\ln A - \ln(A - r \cdot G_0)}{\ln(1+r)}.$$

Example 5.9 *Let the number of settling periods be fixed to N. Find a formula for the (constant) payment, by which the debt is paid in precisely N settling periods.*

A. A continuation of Example 5.8.

D. Apply the solution of Example 5.8.

I. If $A > r \cdot G_0$, we see from Example 5.8 that we shall find A, such that

$$N = \frac{\ln A - \ln(A - r \cdot G_0)}{\ln(1+r)} = -\frac{\ln \left(\dfrac{A - r \cdot G_0}{A} \right)}{\ln(1+r)} = -\frac{\ln \left(1 - \dfrac{rG_0}{A} \right)}{\ln(1+r)},$$

which we rewrite as

$$\ln \left(1 - \frac{rG_0}{A} \right) = -N \ln(1+r) = \ln \left\{ (1+r)^{-N} \right\}.$$

We get from here the condition

$$(1+r)^{-N} = 1 - \frac{rG_0}{A}, \quad \text{i.e.} \quad \frac{rG_0}{A} = 1 - (1+r)^{-N},$$

hence

$$A = G_0 \cdot \frac{r}{1-(1+r)^{-N}} = G_0 \cdot \frac{r(1+r)^N}{(1+r)^N - 1} = G_0 \cdot \frac{(1+r)^{N+1} - (1+r)^N}{(1+r)^N - 1}.$$

Example 5.10 *Assume that G_0 is 100 000 euro and that the annual interest is 9 % and that there are 4 settling periods (quarters) per year. When is the loan repaid if the payment is 3 000 euro per quarter? And when is the loan repaid, if we double the payment?*

A. A continuation of the Examples 5.7–5.9.

D. Apply the results from Examples 5.7–5.9.

I. First calculate

$$G_0 = 100\,000, \qquad r = \frac{9}{400} \quad \text{and} \quad A = 3\,000.$$

We get

$$rG_0 = \frac{9}{400} \cdot 100\,000 = 9 \cdot 250 = 2\,250 < 3\,000 = A,$$

which guarantees that the loan will be repaid with 3 000 euro in payment per settling period, cf. Example 5.7.

It follows from Example 5.9 that

$$n \geq \frac{\ln A - \ln(A - rG_0)}{\ln(1+r)} = \frac{\ln 3000 - \ln 750}{\ln\left(1 + \frac{9}{400}\right)} = \frac{\ln 4}{\ln \frac{409}{400}} \approx 62,3,$$

thus the loan is repaid after 63 settling periods, corresponding to $15\,\frac{3}{4}$ years.

When A is doubled to 6 000 euro, we get $A - rG_0 = 3\,750$, hence

$$n \geq \frac{\ln A - \ln(A - rG_0)}{\ln(1+r)} = \frac{\ln \frac{6000}{3750}}{\ln \frac{409}{400}} \approx 21,12,$$

and the loan is repaid after (only) 22 settling periods, corresponding to $5\,\frac{1}{2}$ år.

Example 5.11 *Assume that G_0 is 100 000 euro and that the annual interest is 9 % and that the payments are quarterly. How big shall we choose the payment, if the loan is repaid after 20 and 30 years, respectively?*

A. A continuation of the Examples 5.7-5.10.

D. Use the previous results from Example 5.9.

I. Here $G_0 = 100\,000$ and $r = \frac{9}{400}$. In the first case, $N = 4 \cdot 20 = 80$, and in the second case is $N = 4 \cdot 30 = 120$. It follows from Example 5.9 that

$$A = \frac{rG_0(1+r)^N}{(1+r)^N - 1} = \frac{rG_0}{1-(1+r)^{-N}}.$$

1) If $N = 80$, then

$$A = \frac{\frac{9}{400} \cdot 100\,000}{1 - \left(\frac{400}{409}\right)^{80}} = \frac{2250}{1 - \left(\frac{400}{409}\right)^{80}} = 2\,706,38 \text{ euro.}$$

2) Når $N = 120$, er

$$A = \frac{2250}{1 - \left(\frac{400}{409}\right)^{120}} = 2\,417,40 \text{ euro}$$

Example 5.12 *Find that solution of the difference equation*

$$x_k + 2x_{k-1} = \cos\left(\frac{\pi}{2}\,k\right), \qquad k \geq 1,$$

for which $x_0 = 0$.

A Linear, inhomogeneous difference equation of first order.

D. Start by finding the solution of the homogeneous equation. Then analyze the right hand side (guess a complex solution).

I. The characteristic polynomial $R+2$ has the root $R=-2$, so the complete solution of the homogeneous equation is given by

$$x_k = c\cdot(-2)^k, \qquad k \in \mathbb{N}_0, \quad c \in \mathbb{R}.$$

Now, $\cos\left(\frac{\pi}{2}\,k\right) = \mathrm{Re}\{i^k\}$, so if we insert $x_k = \alpha\, i^k$, we get

$$x_k + 2x_{k-1} = \alpha\, i^k + 2\alpha\, i^{k-1} = \alpha(1-2i)i^k,$$

which is equal to i^k, if

$$\alpha = \frac{1}{1-2i} = \frac{1+2i}{5}.$$

Thus, a particular solution is

$$x_k = \mathrm{Re}\left\{\frac{1+2i}{5}\,i^k\right\} = \frac{1}{5}\,\mathrm{Re}\left\{i^k + 2\,i^{k+1}\right\} = \frac{1}{5}\cos\left(k\cdot\frac{\pi}{2}\right) + \frac{2}{5}\cos\left((k+1)\,\frac{\pi}{2}\right),$$

and the complete solution becomes

$$x_k = \frac{1}{5}\cos\left(k\cdot\frac{\pi}{2}\right) + \frac{2}{5}\cos\left((k+1)\,\frac{\pi}{2}\right) + c\cdot(-2)^k.$$

It follows from the initial condition that

$$x_0 = 0 = \frac{1}{5} + 0 + c, \qquad \text{i.e.} \qquad c = -\frac{1}{5}.$$

The wanted solution becomes

$$\begin{aligned} x_k &= \frac{1}{5}\left\{(-2)^k + \cos\left(\frac{\pi}{2}\,k\right) + 2\cos\left(\frac{\pi}{2}\,(k+1)\right)\right\} \\ &= \frac{1}{5}\left\{(-2)^k + \cos\left(\frac{\pi}{2}\,k\right) - 2\sin\left(\frac{\pi}{2}\,k\right)\right\}, \qquad k \in \mathbb{N}_0. \end{aligned}$$

Example 5.13 *Let $a_n = 1 + 2 + \cdots + n$, $n \geq 1$. Find a difference equation which is fulfilled by a_n. Then find a formula for a_n.*

A. Establishment and solution of a difference equation.

D. Look at $a_n - a_{n-1}$.

I. Obviously,

$$a_n - a_{n-1} = n, \qquad n \geq 1.$$

The corresponding homogeneous equation has the constant solution

$$a_n = c, \qquad n \in \mathbb{N}_0, \quad c \in \mathbb{R}.$$

Then we guess the structure $a_n = \alpha n^2 + \beta n$. By insertion,

$$\begin{aligned} a_n - a_{n-1} &= \alpha\left\{n^2 - (n-1)^2\right\} + \beta\{n - (n-1)\} \\ &= \alpha(2n-1) + \beta = 2\alpha n + (\beta - \alpha) = n, \end{aligned}$$

thus $\alpha = \dfrac{1}{2}$ and $\beta = \alpha = \dfrac{1}{2}$. The complete solution becomes

$$a_n = \frac{1}{2}\,n^2 + \frac{1}{2}\,n + c = \frac{1}{2}\,n(n+1) + c, \quad n \in \mathbb{N}_0, \quad c \in \mathbb{R}.$$

Since $a_1 = 1 = \dfrac{1}{2} \cdot 1 \cdot 2 + c = 1 + c$, we have $c = 0$, and the searched solution is

$$a_n = \frac{1}{2}\,n(n+1), \qquad n \in \mathbb{N}_0.$$

Example 5.14 *Let $a_n = 1 + 2^2 + \cdots + n^2$, $n \geq 1$. Find a difference equation which is fulfilled by a_n. Then find a formula for a_n.*

A. Establish a (simple) difference equation.

D. Find such a difference equation for a_n, and solve it.

I. It is immediately seen that

$$a_n - a_{n-1} = n^2.$$

The corresponding homogeneous equation has the constant sequence $a_n = c$, $c \in \mathbb{R}$, as a solution. Then we guess a particular solution of the form

$$a_n = \alpha n^3 + \beta n^2 + \gamma n,$$

hence by insertion,

$$\begin{aligned} a_n - a_{n-1} &= \alpha\left\{n^3 - (n-)^3\right\} + \beta\left\{n^2 - (n-1)^2\right\} + \gamma\{n - (n-1)\} \\ &= \alpha\left\{3n^2 - 3n + 1\right\} + \beta\{2n - 1\} + \gamma \\ &= 3\alpha n^2 + (-3\alpha + 2\beta)n + (\alpha - \beta + \gamma). \end{aligned}$$

This expression is equal to n^2, if and only if

$$\begin{cases} 3\alpha = 1, \\ -3\alpha + 2\beta = 0, \\ \alpha - \beta + \gamma = 0, \end{cases} \qquad \text{i.e.} \qquad \begin{cases} \alpha = \frac{1}{3}, \\ \beta = \frac{1}{2}, \\ \gamma = \beta - \alpha = \frac{1}{6}. \end{cases}$$

A particular solution is then

$$a_n = \frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{1}{6} n = \frac{1}{6} n(2n^2 + 3n + 1) = \frac{1}{6} n(2n+1)(n+1).$$

Since $a_1 = \dfrac{1}{6} \cdot 1 \cdot (2+3+1) = 1$, we see that this is in fact the wanted solution, so

$$a_n = \frac{1}{6} n(n+1)(2n+1), \qquad n \in \mathbb{N}.$$

The complete solution is of course

$$a_n = \frac{1}{6} n(n+1)(2n+1) + c, \qquad n \in \mathbb{N}, \quad c \in \mathbb{R}.$$

Example 5.15 *Find the complete solution of the following difference equations*

$$
\begin{array}{lll}
(1) & x_k - 5x_{k-1} + 6x_{k-2} = 0, & k \geq 2, \\
(2) & x_k - 6x_{k-1} + 9x_{k-2} = 0, & k \geq 2, \\
(3) & x_k + 2x_{k-1} + 2x_{k-2} = 0, & k \geq 2.
\end{array}
$$

A. Linear homogeneous difference equations of second order.

D. Find the roots of the characteristic polynomials and apply the solution formula.

I. 1) The characteristic polynomial $R^2 - 5R + 6$ has the simple roots $R = 2$ and $R = 3$, hence the complete solution is

$$x_k = c_1 \cdot 2^k + c_2 \cdot 3^k, \qquad k \in \mathbb{N}_0, \quad c_1,\, c_2 \in \mathbb{R}.$$

2) The characteristic polynomial $R^2 - 6R + 9$ has the double root $R = 3$, hence the complete solution is

$$x_k = c_1 \cdot 3^k + c_2 \cdot k \cdot 3^k, \qquad k \in \mathbb{N}_0, \quad c_1,\, c_2 \in \mathbb{R}.$$

3) The characteristic polynomial $R^2 + 2R + 2$ has the two complex conjugated roots

$$R = -1 \pm i = \sqrt{2}\,\exp\left(\pm i\,\frac{3\pi}{4}\right),$$

hence the complete solution is given by

$$x_k = c_1(\sqrt{2})^k \cos\left(\frac{3\pi}{4}\,k\right) + c_2(\sqrt{2})^k \sin\left(\frac{3\pi}{4}\,k\right), \qquad k \in \mathbb{N}_0,$$

where c_1, $c_2 \in \mathbb{R}$ are arbitrary constants.

Example 5.16 *Find in each of the following cases that solution of the difference equation which also satisfies the given initial condition.*

$$
\begin{array}{llll}
(1) & x_k - 7x_{k-1} + 10x_{k-2} = 0, & k \geq 2, & x_0 = 3,\ x_1 = 15. \\
(2) & 9x_k + 12x_{k-1} + 4x_{k-2} = 0, & k \geq 2, & x_0 = 1,\ x_1 = 4. \\
(3) & x_k + 4x_{k-2} = 0, & k \geq 2, & x_0 = x_1 = 1.
\end{array}
$$

A. Linear homogeneous difference equations of second order with given initial conditions.

D. Find the roots of the characteristic polynomials and apply some convenient solution formula. Then insert into the initial conditions.

I. 1) The characteristic polynomial $R - 7R + 10$ has the two simple roots $R = 2$ and $R = 5$, hence the complete solution is

$$x_k = c_1 \cdot 2^k + c_2 \cdot 5^k, \qquad k \in \mathbb{N}_0, \quad c_1,\, c_2 \in \mathbb{R}.$$

It follows from the initial conditions that

$$\begin{cases} x_0 = 3 = c_1 + c_2, \\ x_1 = 15 = 2c_1 + 5c_2, \end{cases} \qquad \text{i.e.} \qquad \begin{cases} 2c_1 + 2c_2 = 6, \\ 2c_1 + 5c_2 = 15, \end{cases}$$

thus $3c_2 = 15 - 6 = 9$, i.e. $c_2 = 3$ and $c_1 = 0$. The wanted solution is

$$x_k = 3 \cdot 5^k, \qquad k \in \mathbb{N}_0.$$

2) The characteristic polynomial $9R + 12R + 4 = (3R+2)^2$ has the double root $R = -\dfrac{2}{3}$, hence the complete solution is

$$x_k = c_1 \left(-\frac{2}{3}\right)^k + c_2 \cdot k \left(-\frac{2}{3}\right)^k, \qquad k \in \mathbb{N}_0, \quad c_1,\, c_2 \in \mathbb{R}.$$

It follows from the initial conditions that

$$x_0 = 1 = c_1 \qquad \text{and} \qquad x_1 = 4 = -\frac{2}{3}\,(c_1 + c_2),$$

thus $c_1 = 1$ and $c_1 + c_2 = -6$, i.e. $c_2 = -7$. The wanted solution is

$$x_k = (1 - 7k) \cdot \left(-\frac{2}{3}\right)^k, \qquad k \in \mathbb{N}_0.$$

3) The characteristic polynomial $R^2 + 4$ has the two complex conjugated roots

$$R = \pm 2i = 2\,\exp\left(\pm i\,\frac{\pi}{2}\right).$$

Hence, the complete solution is

$$x_k = c_1 \cdot 2^k \cos\left(\frac{\pi}{2}\,k\right) + c_2 \cdot 2^k \sin\left(\frac{\pi}{2}\,k\right), \qquad k \in \mathbb{N}_0,$$

where $c_1,\, c_2 \in \mathbb{R}$ are arbitrary constants.
It follows from the initial conditions that

$$x_0 = 1 = c_1 \quad \text{and} \quad x_1 = 1 = 2c_2, \quad \text{i.e.} \quad c_2 = \frac{1}{2}.$$

Thus, the wanted solution becomes

$$x_k = 2^k \left\{ \cos\left(\frac{\pi}{2}\,k\right) + \frac{1}{2}\,\sin\left(\frac{\pi}{2}\,k\right) \right\}, \qquad k \in \mathbb{N}_0.$$

Example 5.17 *Find the complete solution of the difference equation*

$$x_k + 4x_{k-1} + 4x_{x-2} = 7, \qquad k \geq 2.$$

A. Linear inhomogeneous difference equation of second order.

D. Find the roots of the characteristic polynomial and apply the solution formula when solving the homogeneous equation. Finally, guess the structure of a particular solution and apply the linearity.

I. We guess a particular solution as a constant sequence, $x_k = c$. It is seen by insertion that $x_k = \frac{7}{9}$, $k \in \mathbb{N}_0$, is a particular solution.

The characteristic polynomial $R^2 + 4R + 4 = (R+2)^2$ has the double root $R = -2$, hence the complete solution becomes

$$x_k = \frac{7}{9} + c_1(-2)^k + c_2 k(-2)^k, \qquad k \in \mathbb{N}_0, \quad c_1,\, c_2 \in \mathbb{R}.$$

Example 5.18 *Find that solution of the difference equation*

$$x_k + 2x_{k-1} + 2x_{k-2} = 5^k, \qquad k \geq 2,$$

for which $x_0 = x_1 = \frac{1}{5}$.

A. Linear inhomogeneous difference equation of second order with initial conditions.

D. Find the roots of the characteristic polynomial and apply a solution formula for the homogeneous equation. Then guess the structure of a particular solution and exploit the linearity. Insert finally into the initial conditions and find the constants.

I. The characteristic polynomial $R^2 + 2R + 2$ has the two complex conjugated roots

$$R = -1 \pm i = \sqrt{2}\,\exp\left(\pm \frac{3\pi}{4}\, i\right).$$

The complete solution of the homogeneous equation is

$$x_k = c_1(\sqrt{2})^k \cos\left(\frac{3\pi}{4}\,k\right) + c_2(\sqrt{2})^k \sin\left(\frac{3\pi}{4}\,k\right), \qquad k \in \mathbb{N}_0,$$

where $c_1,\, c_2 \in \mathbb{R}$ are arbitrary constants.

Then we guess a particular solution of the structure $x_k = \alpha \cdot 5^k$. We get by insertion

$$x_k + 2x_{k-1} + 2x_{k-2} = \alpha\left(5^k + 2\cdot 5^{k-1} + 2\cdot 5^{k-2}\right) = \frac{\alpha}{25}\,(25+10+2)\cdot 5^k = \frac{37}{25}\,\alpha \cdot 5^k.$$

This expression is equal to 5^k, if $\alpha = \frac{25}{37}$, thus the complete solution becomes

$$x_k = \frac{25}{37}\,5^k + c_1(\sqrt{2})^k \cos\left(\frac{3\pi}{4}\,k\right) + c_2(\sqrt{2})^k \sin\left(\frac{3\pi}{4}\,k\right), \qquad k \in \mathbb{N}_0,$$

where c_1 and $c_2 \in \mathbb{R}$ are arbitrary constants.

It follows from the initial conditions that

$$x_0 = \frac{1}{5} = \frac{25}{37} + c_1,$$

hence

$$c_1 = \frac{1}{5} - \frac{25}{37} = \frac{37-125}{185} = -\frac{88}{185},$$

and

$$x_1 = \frac{1}{5} = \frac{125}{37} - c_1 \frac{\sqrt{2}}{\sqrt{2}} + c_2 \frac{\sqrt{2}}{\sqrt{2}} = \frac{125}{37} - c_1 + c_2,$$

whence

$$c_2 = \frac{1}{5} + \frac{1}{5} - \frac{25}{37} - \frac{125}{37} = \frac{2}{5} - \frac{150}{37} = \frac{74-750}{185} = -\frac{676}{185}.$$

The wanted solution is therefore

$$x_k = \frac{25}{37} \cdot 5^k - \frac{88}{185}(\sqrt{2})^k \cos\left(\frac{3\pi}{4}\,k\right) - \frac{676}{185}(\sqrt{2})^k \sin\left(\frac{3\pi}{4}\,k\right), \qquad k \in \mathbb{N}_0.$$

Example 5.19 *Find that solution of the difference equation*

$$x_k + 3x_{k-1} + 2x_{k-2} = 3^k, \qquad k \geq 2,$$

for which $x_0 = 0$ *and* $x_1 = 0$.

A. Linear inhomogeneous difference equation of second order with initial conditions.

D. Find the roots of the characteristic polynomial and apply a solution formula for the complete solution of the homogeneous equation. Then guess the structure of a particular solution and exploit the linearity. Insert finally into the final conditions and find the constants.

I. The characteristic polynomial $R^2 + 3R + 2$ has the two simple roots $R = -1$ and $R = -2$, hence the complete solution of the homogeneous difference equation becomes

$$x_k = c_1(-1)^k + c_2(-2)^k, \qquad k \in \mathbb{N}_0, \quad c_1, \, c_2 \in \mathbb{R}.$$

Then we guess on a particular solution of the structure $x_k = \alpha \cdot 3^k$. By insertion into the equation we get

$$\alpha \left\{ 3^k + 3^k + 2 \cdot 3^{k-2} \right\} = \alpha \left(2 + \frac{2}{9} \right) 3^k = \frac{20}{9} \, \alpha \cdot 3^k,$$

which is equal to 3^k for $\alpha = \dfrac{9}{20}$. Thus the complete solution is

$$x_k = \frac{9}{20} \cdot 3^k + c_1(-1)^k + c_2(-2)^k, \qquad k \in \mathbb{N}_0, \quad c_1, \, c_2 \in \mathbb{R}.$$

Then we get by the final conditions,

$$\begin{cases} x_0 = 0 = \dfrac{9}{20} + c_1 + c_2, \\[2ex] x_1 = 0 = \dfrac{27}{20} - c_1 - 2c_2, \end{cases}$$

whence

$$c_2 = \frac{27}{20} + \frac{9}{20} = \frac{36}{20} = \frac{9}{5},$$

and

$$c_1 = -\frac{9}{20} - c_1 = -\frac{9}{20} - \frac{9}{5} = -\frac{45}{20} = -\frac{9}{4}.$$

The wanted solution is therefore

$$x_k = \frac{9}{20} \cdot 3^k - \frac{9}{4}\,(1-)^k + \frac{9}{5}\,(-2)^k, \qquad k \in \mathbb{N}_0.$$

Example 5.20 *Find the complete solution of the difference equations*

(1) $\quad x_k - x_{k-2} = \sin\left(k \cdot \frac{\pi}{2}\right), \qquad k \geq 2,$

(2) $\quad x_k - x_{k-2} = \cos\left(k \cdot \frac{\pi}{2}\right), \qquad k \geq 2.$

A. Linear inhomogeneous difference equations of second order.

D. Find the roots of the characteristic equation and apply a solution formula for the solution of the homogeneous difference equation. Then try to make a complex guess of a particular solution.

I. Since $\sin\left(k \cdot \frac{\pi}{2}\right) = \mathrm{Im}\left(i^k\right)$, and $\cos\left(k \cdot \frac{\pi}{2}\right) = \mathrm{Re}\left(i^k\right)$, we can solve both problems at the same time, until we at last are forced to split into the real and the imaginary part.

The characteristic polynomial $R^2 - 1$ has the two simple roots $R = \pm 1$, hence the complete solution of the homogeneous equation becomes

$$x_k = c_1 + c_2(-1)^k, \qquad k \in \mathbb{N}_0, \quad c_1,\, c_2 \in \mathbb{R}.$$

Next insert $x_k = \alpha\, i^k$. We get

$$x_k - x_{k-2} = \alpha\left\{i^k - i^{k-2}\right\} = 2\alpha\, i^k,$$

which is equal to i^k for $\alpha = \frac{1}{2}$.

1) The complete solution of

$$x_k - x_{k-2} = \sin\left(k \cdot \frac{\pi}{2}\right) = \mathrm{Im}\left(i^k\right)$$

is

$$x_k = \mathrm{Im}\left(\frac{1}{2}\, i^k\right) + c_1 + c_2(-1)^k = \frac{1}{2}\sin\left(k \cdot \frac{\pi}{2}\right) + c_1 + c_2(-1)^k, \qquad k \in \mathbb{N}_0, \quad c_1,\, c_2 \in \mathbb{R}.$$

2) The complete solution of

$$x_k - x_{k-2} = \cos\left(k \cdot \frac{\pi}{2}\right) = \mathrm{Re}\left(i^k\right)$$

is

$$x_k = \mathrm{Re}\left(\frac{1}{2}\, i^k\right) + c_1 + c_2(-1)^k = \frac{1}{2}\cos\left(k \cdot \frac{\pi}{2}\right) + c_1 + c_2(-1)^k, \qquad k \in \mathbb{N}_0, \quad c_1,\, c_2 \in \mathbb{R}.$$

Example 5.21 *Find that solution of the difference equation*

$$x_k - 6x_{k-1} + 9x_{k-2} = 3 \cdot 2^k + 17 \cdot 4^k, \qquad k \geq 2,$$

for which $x_0 = 40$ *and* $x_1 = 400$.

A. Linear inhomogeneous difference equation of second order with initial conditions.

D. Solve the characteristic equation; guess a particular solution.

I. The characteristic polynomial $R^2 - 6R + 9 = (R-3)^2$ has the double root $R = 3$, hence the homogeneous equation has the complete solution

$$x_k = c_1\, 3^k + c_2\, k \cdot 3^k, \qquad k \in \mathbb{N}_0, \quad c_1,\, c_2 \in \mathbb{R}.$$

By guessing the structure $x_k = a \cdot 2^k + b \cdot 4^k$, we get by insertion (i.e. checking this possible solution)

$$\begin{aligned} x_k - 6x_{k-1} + 9x_{k-2} &= a \cdot 2^k + b \cdot 4^4 - 6a \cdot 2^{k-1} - 6b \cdot 4^{k-1} + 9a \cdot 2^{k-2} + 9b \cdot 4^{k-2} \\ &= a\left(1 - 3 + \frac{9}{4}\right) 2^k + b\left(1 - \frac{3}{2} + \frac{9}{16}\right) 4^k = \frac{a}{4} \cdot 2^k + \frac{b}{16} \cdot 4^k. \end{aligned}$$

This expression is equal to $3 \cdot 2^k + 17 \cdot 4^4$, if $a = 12$ and $b = 16 \cdot 17 = 272$, hence the complete solution becomes

$$x_k = 12 \cdot 2^k + 272 \cdot 4^k + c_1 \cdot 3^k + c_2\, k \cdot 3^k, \qquad k \in \mathbb{N}_0, \quad c_1,\, c_2 \in \mathbb{R}.$$

By insertion into the conditions we get

$$x_0 = 40 = 12 + 272 + c_1 = 284 + c_1,$$

thus $c_1 = -244$, and

$$x_1 = 400 = 24 + 1088 + 3c_1 + 3c_2 = 1112 - 732 + 3c_2 = 380 + 3c_2,$$

i.e. $c_2 = \dfrac{20}{3}$. The solution is

$$x_k = 12 \cdot 2^k + 272 \cdot 4^k - 244 \cdot 3^k + \frac{20}{3}\, k \cdot 3^k, \qquad k \in \mathbb{N}_0.$$

Example 5.22 *Find the complete solution of the difference equations*

(1) $x_k - 2x_{k-1} + x_{k-2} - 2x_{k-3} = 0, \qquad k \geq 3,$

(2) $x_k - 2x_{k-1} - x_{k-2} + 2x_{k-3} = 0, \qquad k \geq 3,$

(3) $x_k - x_{k-4} = 0, \qquad k \geq 4.$

A. Two linear homogeneous difference equations of third order, and an homogeneous difference equation of fourth order.

D. Find the roots of the characteristic polynomials and then apply a solution formula.

I. 1) The characteristic polynomial

$$R^3 - 2R^2 + R - 2 = (R-2)(R^2+1)$$

has the simple roots $R = 2$ and $R = \pm i = \exp\left(\pm i\,\frac{\pi}{2}\right)$, thus all (real) solutions are

$$x_k = c_1 \cdot 2^k + c_2 \cos\left(k \cdot \frac{\pi}{2}\right) + c_3 \sin\left(k \cdot \frac{\pi}{2}\right), \qquad k \in \mathbb{N}_0, \quad c_1, c_2, c_3 \in \mathbb{R}.$$

2) The characteristic polynomial

$$R^3 - 2R^2 - R + 2 = (R-2)(R^2-1) = (R-2)(R-1)(R+1)$$

has the three simple roots $R = 2$ and $R = \pm 1$, hence the complete solution is

$$x_k = c_1 \cdot 2^k + c_2 + c_3\,(-1)^k, \qquad k \in \mathbb{N}_0, \quad c_1,\, c_2,\, c_3 \in \mathbb{R}.$$

3) The characteristic polynomial

$$R^4 - 1 = (R-1)(R+1)(R-i)(R+i)$$

has the four simple roots $R = \pm 1$ and $R = \pm i$, hence all (real) solutions are

$$x_k = c_1 0 c_2\,(-1)^k + c_3\,\cos\left(k \cdot \frac{\pi}{2}\right) + c_4\,\sin\left(k \cdot \frac{\pi}{2}\right), \qquad k \in \mathbb{N}_0,$$

where $c_1,\, c_2,\, c_3,\, c_4 \in \mathbb{R}$ are arbitrary constants.

Example 5.23 *Find that solution of the difference equation*

$$x_k - x_{k-1} + 4x_{k-2} - 4x_{k-3} = 0, \qquad k \geq 3,$$

for which $x_0 = -1$, $x_2 = 2$, $x_4 = 4$.

A. Linear homogeneous difference equation of third order with three (non-successive) conditions.

D. Find the roots of the characteristic polynomial and apply the solution formula. Finally, insert into the conditions.

I. The characteristic polynomial

$$R^3 - R^2 + 4R - 4 = (R-1)(R^2+4)$$

has the three simple roots $R = 1$ and $R = \pm 2i$. Hence, the complete solution is

$$x_k = c_1 + c_2\,2^k \cos\left(k \cdot \frac{\pi}{2}\right) + c_3\,2^k \sin\left(k \cdot \frac{\pi}{2}\right).$$

Then by insertion into the conditions,

$$\left\{ \begin{array}{lllllll} x_0 & = & -1 & = & c_1 & + & c_2, \\ x_1 & = & 2 & = & c_1 & & \qquad + \quad 2c_3, \\ x_4 & = & 4 & = & c_1 & + & 16c_2, \end{array} \right.$$

from which we get $15c_2 = 5$, thus $c_2 = \dfrac{1}{3}$, and $c_1 = -\dfrac{4}{3}$. Then finally

$$2c_3 = 2 + \frac{4}{3}, \qquad \text{i.e.} \qquad c_3 = 1 + \frac{2}{3} = \frac{5}{3}.$$

The wanted solution is

$$x_k = -\frac{4}{3} + \frac{1}{3} \cdot 2^k \cos\left(k \cdot \frac{\pi}{2}\right) + \frac{5}{3} \cdot 2^k \sin\left(k \cdot \frac{\pi}{2}\right), \qquad k \in \mathbb{N}_0.$$

Example 5.24 *Find the complete solution of the difference equation*

$$x_k - 2x_{k-1} + x_{k-2} - 2x_{x-3} = 4, \qquad k \geq 3.$$

A. Linear inhomogeneous difference equation of fourth order. The corresponding homogeneous difference equation is treated in Example 5.22 (1).

D. Find the roots of the characteristic polynomial; then guess a solution of the inhomogeneous equation.

I. The characteristic polynomial

$$R^3 - 2R^2 + R - 2 = (R-2)(R^2+1)$$

has the three simple roots $R = 2$ and $R = \pm i$. It is seen by inspection that $x_k = -2$, $k \in \mathbb{N}_0$, is a particular solution. Hence the complete solution is

$$x_k = -2 + c_1 \cdot 2^k + c_2 \cos\left(k \cdot \frac{\pi}{2}\right) + c_3 \sin\left(k \cdot \frac{\pi}{2}\right), \qquad k \in \mathbb{N}_0,$$

where c_1, c_2, $c_3 \in \mathbb{R}$ are arbitrary constants.

Example 5.25 *Find the complete solution of the difference equation*

$$x_k - 2x_{k-1} - x_{k-2} + 2x_{k-3} = -2, \qquad k \geq 3.$$

A. Linear inhomogeneous difference equation of third order. The corresponding homogeneous equation was dealt with in Example 5.22 (2).

D. Find the roots of the characteristic polynomial; then guess a solution of the inhomogeneous equation.

I. The characteristic polynomial

$$R^3 - 2r^2 - R + 2 = (R-2)(R-1)(R+1)$$

has the three simple roots $R = 2$ and $R = \pm 1$. Since already $R = 1$ is a root, corresponding to the solution $x_k = c$, $k \in \mathbb{N}_0$, of the homogeneous equation, our guess must be modified, so we guess a particular solution of the form $x_k = a \cdot k$. We get by insertion,

$$\begin{aligned} x_k - 2x_{k-1} - x_{k-2} + 2x_{k-3} &= a\{k - 2(k-1) - (k-2) + 2(k-3)\} \\ &= a\{k - 2k + 2 - k + 2 + 2k - 6\} = -2a, \end{aligned}$$

which is equal to -2 for $a = 1$. Hence, the complete solution is given by

$$x_k = k + c_1 + c_2(-1)^k + c_3 \cdot 2^k, \qquad k \in \mathbb{N}_0, \quad c_1,\, c_2,\, c_3 \in \mathbb{R}.$$

Example 5.26 *Find the complete solution of the difference equations*

(1) $\quad x_k + x_{k-2} - 2x_{x-k} = \cos\left(k \cdot \frac{\pi}{2}\right), \qquad k \geq k.$

(2) $\quad x_k + x_{k-2} - 2x_{x-4} = \sin\left(k \cdot \frac{\pi}{2}\right), \qquad k \geq 4.$

A. Linear inhomogeneous difference equation of fourth order with two different right hand sides.

D. Find the roots of the characteristic polynomial. Then make a complex guess, when the right hand side is replaced by i^k. Then the two questions are answered by taking the real and the imaginary part of the complex solution.

I. The characteristic polynomial

$$R^4 + R^2 - 2 = (R^2+2)(R^2-1)$$

has the four simple roots $R = \pm 1$ and $R = \pm i\sqrt{2}$, hence the homogeneous equation has the complete solution,

$$x_k = c_1 + c_2(-1)^k + c_3(\sqrt{2})^k \cos\left(k \cdot \frac{\pi}{2}\right) + c_4(\sqrt{2})^k \sin\left(k \cdot \frac{\pi}{2}\right), \qquad k \in \mathbb{N}_0,$$

where c_1, c_2, c_3, $c_4 \in \mathbb{R}$ are arbitrary constants.

If we guess of a complex solution $x_k = \alpha\, i^k$, we get by insertion

$$\begin{aligned} x_k + x_{k-2} - 2x_{k-4} &= \alpha\left\{i^k + i^{k-2} - 2i^{k-4}\right\} \\ &= \alpha\, i^k\,(1-1-2) = -2\alpha\, i^k, \end{aligned}$$

which is equal to i^k for $\alpha = -\frac{1}{2}$. Hence the complex equation has the complete solution

$$x_k = -\frac{1}{2}\, i^k + c_1 + c_2(-1)^k + c_3(\sqrt{2})^k \cos\left(k \cdot \frac{\pi}{2}\right) + c_4(\sqrt{2})^k \sin\left(k \cdot \frac{\pi}{2}\right), \qquad k \in \mathbb{N}_0,$$

where c_1, c_2, c_3, $c_4 \in \mathbb{R}$ are arbitrary constants.

1) By taking the real part of the solution of the complex solution we get the solution of (1),

$$x_k = -\frac{1}{2}\cos\left(k \cdot \frac{\pi}{2}\right) + c_1 + c_2(-1)^k + c_3(\sqrt{2})^k \cos\left(k \cdot \frac{\pi}{2}\right) + c_4(\sqrt{2})^k \sin\left(k \cdot \frac{\pi}{2}\right),$$

where $k \in \mathbb{N}_0$, and where c_1, c_2, c_3, $c_4 \in \mathbb{R}$ are arbitrary constants.

2) By taking the imaginary part of the complex solution, we get the solution of (2),

$$x_k = -\frac{1}{2}\sin\left(k \cdot \frac{\pi}{2}\right) + c_1 + c_2(-1)^k + c_3(\sqrt{2})^k \cos\left(k \cdot \frac{\pi}{2}\right) + c_4(\sqrt{2})^k \sin\left(k \cdot \frac{\pi}{2}\right),$$

where $k \in \mathbb{N}_0$, and where c_1, c_2, c_3, $c_4 \in \mathbb{R}$ are arbitrary constants.

Example 5.27 *Let the sequence* x_k *fulfil the difference equation*

$$x_k + 16 x_{k-4} = 17(-1)^k, \qquad k \geq 4,$$

where also $x_0 = 1$, $x_1 = -1$, $x_2 = 9$ *and* $x_3 = -1$. *Calculate* x_{400}.

A. Linear inhomogeneous difference equation of fourth order. We shall only find x_{400}.

D. Put $y_k = x_{4k}$, $y_{100} = x_{400}$, and set up another difference equation.

I. If we put $y_k = x_{4k}$, then we get the difference equation

$$y_k + 16 y_{k-1} = 16, \qquad k \geq 1,$$

where $y_0 = x_0 = 1$. It is immediately seen that $y_k = 1$, $k \in \mathbb{N}_0$, is a solution of the inhomogeneous equation, which also satisfies the condition. We therefore conclude that this is the wanted solution, hence

$$y_{100} = x_{400} = 1.$$

Example 5.28 *The so-called* Fibonacci numbers F_n *are defined as the elements of the sequence which satisfies*

$$F_n = F_{n-1} + F_{n-2}, \qquad n \geq 2,$$

and the initial conditions $F_0 = 0$, $F_1 = 1$.
Find a general formula of the n-tt Fibonacci number F_n.

The Fibonacci numbers were mentioned for the first time in a book from 1202. They were the solution of the following problem:
Assume that any couple of rabbits which are more than one month of age bear a new couple of rabbits at the end of each month. If one starts with one pair and none of the rabbits die, how many couples of rabbits are there after n *months?*

A. Linear homogeneous difference equation of second order.

D. Find the roots of the characteristic polynomial.

I. The characteristic equation $R^2 = R + 1$, or $R^2 - R - 1 = 0$, has the roots $R = \frac{1}{2}\{1 \pm \sqrt{5}\}$, thus the complete solution is

$$F_n = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n, \qquad n \in \mathbb{N}_0, \quad c_1,\, c_2 \in \mathbb{R}.$$

It follows from the initial conditions that $F_0 = 0 = c_1 + c_2$ and

$$F_1 = 1 = c_1 \cdot \frac{1+\sqrt{5}}{2} + c_2 \cdot \frac{1-\sqrt{5}}{2} = \frac{1}{2}\left(c_1 + c_2\right) + \frac{\sqrt{5}}{2}\left(c_1 - c_2\right).$$

Hence

$$\begin{cases} c_1 + c_2 = 0, \\ c_1 - c_2 = \dfrac{2}{\sqrt{5}}, \end{cases} \qquad \text{i.e.} \qquad \begin{cases} c_1 = \dfrac{1}{\sqrt{5}}, \\ c_2 = -\dfrac{1}{\sqrt{5}}, \end{cases}$$

and the solution is

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n, \qquad n \in \mathbb{N}_0.$$

Example 5.29 *We consider in the usual plane n lines where each pair intersects each other, and where no three lines intersect at the same point. Let A_n denote the number of domains which the plane is divided in by these lines.*

1) Calculate A_2 and A_3.

2) What is the connection between A_n and A_{n-1}? Use the result to find a formula for A_n for every $n \geq 2$.

A. The setup of a difference equation.

D. Analyze A_2 and A_3, and the general case.

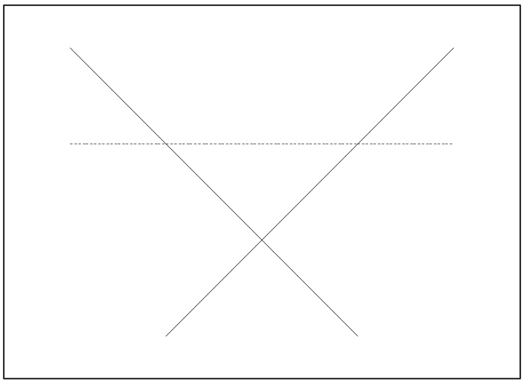


Figure 1: The case of 2 lines and 3 lines.

I. 1) Clearly, $A_2 = 4$, and it follows by the figure that $A_3 = 7$.

2) Given $n-1$ lines satisfying the conditions of the text. When we add the n-th line we get under the same assumptions as above $n-1$ intersection points with the other lines, so the new line is divided into n segments. Each of these segments will provide us with a new subdomain, so the connection between A_{n-1} and A_n becomes

$$A_n = A_{n-1} + n, \qquad \text{i.e.} \qquad A_n - A_{n-1} = n.$$

3) Clearly, all solutions of the homogeneous difference equation are the constant sequences $A_n = c$. Then we guess of a particular solution of the form

$$A_n = \alpha\, n^2 + \beta\, n,$$

from which we get by insertion

$$A_n - A_{n-1} = \alpha\left\{n^2 - (n-1)^2\right\} + \beta\{n - (n-1)\} = 2n\alpha + (\beta - \alpha).$$

This expression is equal to n for $\alpha = \dfrac{1}{2} = \beta$, so the complete solution becomes

$$A_n = \frac{1}{2}\, n^2 + \frac{1}{2}\, n + c = \frac{1}{2}\, n(n+1) + c, \qquad n \in \mathbb{N}, \quad c \in \mathbb{R}.$$

It follows from the condition

$$A_2 = 4 = \frac{1}{2} \cdot 2 \cdot (2+1) + c = 3 + c,$$

that $c = 1$, so the final solution becomes

$$A_n = \frac{1}{2}\, n(n+1) + 1, \qquad n \in \mathbb{N}.$$

Example 5.30 *We shall by a binary sequence of length n understand n binary numbers in a given order, where the binary numbers are 0 and 1. Find the number A_n of all such sequences, which do* not *contain two successive zeros.*

Let B_n denote the number of sequences which satisfies the condition above and which also end by 1. Similarly, let C_n denote the number of sequences satisfying the condition above and ending on 0.

1) Find A_1, A_2 and A_3.

2) What is the connection between A_{n+1} and B_n and C_n?

3) What is the connection between A_{n-1} and B_n?

4) Find by means of (2) and (3) a difference equation for A_n, and find A_n.

A. Setup of a difference equation. The situation is the same as in Example 5.31, although we her have a more complicated equation. By a comparison between the results of the two examples we see that we obtain the same solution.

D. Analyze the given situations.

I. 1) We get by counting all possible sequences that

a) If $n = 1$, we have the sequences 0, 1, which are both of the desired type, hence $A_1 = 2$.

b) If $n = 2$, then we have four possibilities

00, 01, 10, 11,

hog which the latter three satisfy the claim, hence $A_2 = 3$.

c) If r $n = 3$, then we have eight possibilities,

000, 001, <u>010</u>, <u>011</u>, 100, <u>101</u>, <u>110</u>, <u>111</u>,

of which the five underlined sequences fulfil the criterion, hence $A_3 = 5$.

2) It follows immediately that

$$A_{n+1} = 2B_n + C_n.$$

In fact, after a sequence from $\mathcal{B}_n$ can we choose both 0 and 1 (two possibilities), while we after a sequence from $\mathcal{C}_n$ are forced to choose 1, so we have only one possibility.

3) Clearly,

$$A_{n-1} = B_n.$$

In fact, every sequence from $\mathcal{A}_{n-1}$ can always be followed by 1 without destroying the condition. The number must therefore be the same.

4) Before we can set up a difference equation for A_n, we have to eliminate C_n. Every sequence from $\mathcal{C}_n$ has 0 as its last term, so the second last digit must necessarily be 1. This means that

$$C_n = B_{n-1}.$$

We have the three equations

$$A_{n+1} = 2B_n + C_n, \qquad A_{n-1} = B_n, \qquad C_n = B_{n-1}.$$

Hence,

$$A_{n+1} = 2B_n + C_n = 2B_n + B_{n-1} = 2A_{n-1} + A_{n-2},$$

thus by a shift of index and a rearrangement,

$$A_n - 2A_{n-2} - A_{n-3} = 0, \qquad n \geq 3.$$

This is a linear homogeneous difference equation of third order. The characteristic polynomial $R^3 - 2R - 1$ has the root $R = -1$, so

$$R^3 - 2R - 1 = (R+1)(R^2 - R - 1).$$

Hence we find the three simple roots

$$R=-1 \quad \text{and} \quad R=\frac{1}{2}\pm\frac{\sqrt{5}}{2}=\frac{1}{2}\,(1\pm\sqrt{5}).$$

Thus, the complete solution is

$$A_n=c_n(-1)^n+c_2\left(\frac{1+\sqrt{5}}{2}\right)^n+c_3\left(\frac{1-\sqrt{5}}{2}\right)^n, \qquad n\in\mathbb{N},$$

where c_1, c_2, $c_3\in\mathbb{R}$ are arbitrary constants.

We get the following conditions from (1),

$$\begin{aligned}
A_1 &= 2=-c_1+c_2\cdot\frac{1+\sqrt{5}}{2}+c_3\cdot\frac{1-\sqrt{5}}{2},\\
A_2 &= 3=+c_1+c_2\cdot\left(\frac{1+\sqrt{5}}{2}\right)^2+c_3\cdot\left(\frac{1-\sqrt{5}}{2}\right)^2,\\
A_3 &= 5=-c_1?c_2\cdot\left(\frac{1+\sqrt{5}}{2}\right)^3+c_3\cdot\left(\frac{1-\sqrt{5}}{2}\right)^3,
\end{aligned}$$

hence

$$\begin{aligned}
2 &= -c_1+\frac{1}{2}\,(c_2+c_3)+\frac{\sqrt{5}}{2}\,(c_2-c_3),\\
3 &= +c_1+\frac{3}{2}\,(c_2+c_3)+\frac{\sqrt{5}}{2}\,(c_2-c_3),\\
5 &= -c_1+2\,(c_2+c_3)+\sqrt{5}\,(c_1-c_3),
\end{aligned}$$

and thus

$$c_1=0, \qquad c_2=\frac{5+3\sqrt{5}}{10}, \quad \text{og} \quad c_3=\frac{5-3\sqrt{5}}{10}.$$

The solution is

$$\begin{aligned}
A_n &= \frac{5+3\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n+\frac{5-3\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n\\
&= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2}-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+2}, \qquad n\in\mathbb{N}.
\end{aligned}$$

Example 5.31 *A coin is thrown n times. For each throw we notice whether we have got heads or tails, and the total result can be described by a sequence like $HHTH \dots TT$. There are in total 2^n such sequences. The problem is to find the probability of that one in such a sequence does* not *get two tails successively.*

Let A_n denote the number of sequences with the desired property, and let p_n denote the unknown probability.

1) Prove that A_n fulfils the difference equation

$$A_{n+1} = A_n + A_{n-1}, \qquad n \geq 2.$$

2) Find p_n and $\lim_{n\to+\infty} p_n$.

A. Difference equation. If we write 0 instead of T, and 1 instead of H, the situation is analogous to the one in Example 5.30, and the solution A_n ought to be the same. Note that the difference equation here is simpler than the one in Example 5.30.

D. Analyze how we get the difference equation. Then solve the equation and find p_n.

I. 1) Let us consider an element from $\mathcal{A}_n$. Then we have two possibilities:

a) If the element terminates with a T, will the next element only be included, if the next throw is an H. We denote the number of elements by $A_{n,K}$.

b) If the element terminates with an H (and the number is $A_{n,H}$), then we can allow both H and T in the next throw, thus

$$A_{n+1} = A_{n,K} + 2A_{n,P} = (A_{n,K} + A_{n,P}) + A_{n,P} = A_n + A_{n,P}.$$

Then $A_{n,P}$ must be equal to A_{n-1}, because one after each element, which contributes to A_{n-1}, can allow H to be the next digit, when we have a contribution to $A_{n,P}$, and every contribution to $A_{n,P}$ is obtained in this way. Then

$$A_{n+1} = A_n + A_{n-1}, \qquad n \geq 2.$$

2) Then we get by a rearrangement that

$$A_n - A_{n-1} - A_{n-2} = 0, \qquad n \geq 3.$$

This is a linear, homogeneous difference equation of second order. The characteristic polynomial $R^2 - R - 1$ has the two simple roots

$$R = \frac{1 \pm \sqrt{5}}{2}.$$

Hence the complete solution is

$$A_n = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n, \qquad n \in \mathbb{N}_0, \quad c_1, c_2 \in \mathbb{R}.$$

By a counting of the possibilities we get

a) If $n = 1$, then we have the two possibilities H and T, hence $A_1 = 2$.

b) If $n=2$, then we have the four possibilities

$$HH, \qquad HT, \qquad TH, \qquad TT,$$

the first three of which satisfy the conditions, hence $A_2 = 3$.

Thus, we have the following equations for c_1 and c_2,

$$\begin{aligned} A_1 &= 2 = c_1 \cdot \frac{1+\sqrt{5}}{2} + c_2 \cdot \frac{1-\sqrt{5}}{2} = \frac{1}{2}\left(c_1+c_2\right) + \frac{\sqrt{5}}{2}\left(c_1-c_2\right), \\ A_2 &= 3 = c_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^2 + c_2 \cdot \left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{3}{2}\left(c_1+c_2\right) + \frac{\sqrt{5}}{2}\left(c_1-c_2\right), \end{aligned}$$

hence, $c_1 + c_2 = 1$ and $c_1 - c_2 = \dfrac{3}{\sqrt{5}}$, so

$$c_1 = \frac{1}{\sqrt{5}} \cdot \frac{\sqrt{5}+3}{2} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^2,$$

$$c_2 = \frac{1}{\sqrt{5}} \cdot \frac{\sqrt{5}-3}{2} = -\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^2,$$

This gives us the solution

$$A_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+2}, \qquad n \in \mathbb{N},$$

and thus

$$p_n = \frac{1}{2^n}\,A_n = \frac{4}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{4}\right)^{n+2} - \frac{4}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{4}\right)^{n+2}, \qquad n \in \mathbb{N}.$$

From $\sqrt{5} < 3$, follows that $\left|\dfrac{1 \pm \sqrt{5}}{4}\right| < 1$, so $\left(\dfrac{1 \pm \sqrt{5}}{4}\right)^n \to 0$ for $n \to +\infty$, and we have

$$\lim_{n \to +\infty} p_n = 0.$$